Complexity : a study of fractals and self-organized criticality

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COMPLEXITY: A STUDY OF FRACTALS AND SELF-ORGANIZED CRITICALITY

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“The scientist does not study nature because it is useful to do so. He studies it because he takes pleasure in it, and he takes pleasure in it because it is beautiful. If nature were not beautiful it would not be worth knowing, and life would not be worth living.”

HENRI POINCARÉ

Abstract

Over the past few decades, Complex Systems or Complexity has emerged as a new field of Science to study abundant complicated behaviours of systems with nonlinear interactions among many degrees of freedom. These systems can range from a very simple system like one-dimensional map (May R., 1976 *Nature* **261** 459) or a collective system with many (spatial) degrees of freedom like cellular model of sandpile (Bak P., Tang C., and Wiesenfeld K., 1987 *Phys. Rev. Lett.* **59** 381) in theoretical study to complicated natural systems like the Atmosphere (Peters O., Hertlein C., and Christensen K., 2001 *Phys. Rev. Lett.* **88** 018701) or the Earth's crust (Gutenberg B., and Richter C. F., 1955 *Nature* **176** 795). The emergent feature of these systems is the ubiquitous scale-invariance in temporal as well as spatial observables.

In this thesis, Complexity is looked at from two perspectives: Fractals and Self-Organized Criticality. They both share the same path from simplicity to complexity: The repeated application of simple microscopic interacting rules among elements of a physical system, as time evolves, gives rise to very complicated macroscopic structures observed.

This thesis comprises of two parts: first part is a study of Fractals, and second part is a study of Self-Organized Criticality.

In the first part, an idea of creating fractals by using the geometric arc as the basic element is presented. This approach of generating fractals, through the tuning of just three parameters, gives a universal way to obtain many different fractals including the classic ones. The fractals generated using this arc-fractal system are shown to possess a number of features, one of which is the ability to tile the space. Furthermore, by assuming that coastline formation is based purely on the processes of erosion and deposition, the arc-fractal system can also serve as a dynamical model of coastal morphology, with each level of its construction corresponding to the time evolution of the shape of the coastal features. Remarkably, the results indicate that the arc-fractal

system can provide an explanation on the origin of fractality in real coastline.

In the second part, high-accuracy moment analysis is performed to analyse the avalanche size, duration and area distribution of the Abelian Manna model. The model is studied on a vast number of lattices in different dimensions ranging from one to three, including the noninteger ones, with various detailed structures. It is found that for fractal lattices the scaling behaviours of the model depend on both the dimension and detailed structure of the lattices. Furthermore, the usual scaling law $D(2 - \tau) = 2$ is generalised to $D(2 - \tau) = d_w$ in which d_w is the fractal dimension of random walk on the lattice. For regular integer-dimensional lattices, the results provide strong support for establishing universality of exponents and moment ratios across different lattices. From this result, universality can be viewed as the robustness of the system against the perturbations to *dynamic rules* of the model and the *static (geometric) structures* of the lattice (also boundary conditions and aspect ratio). A good survey for the strength of corrections to scaling, which are notorious in the Manna universality class, is also provided. Various scaling relations are also confirmed. The universality of the model in integer dimensions below the upper critical one allows the coefficients of an ϵ -expansion to be determined, which well agrees with the expansion for the roughness and dynamical exponents of the quenched Edwards-Wilkinson equation. By rescaling the critical exponents by the lattice dimension and incorporating the random walker dimension, a remarkable relation is observed, satisfied by both regular and fractal lattices. In addition, a partially directed version of the Abelian Manna model is introduced and shows similar robust features of a self-organized critical system. An operator approach to analytically solve the model is developed which enables the exact determination of probability distribution of observables from first principle paving the path to (analytical) understanding of how power-law distribution arises and why scaling laws like the narrow joint distribution hold.

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¹There is a difference in the conventional way of writing people's names between the East and the West. To respect their names, I keep the names the way they are originally written. The SURNAME is written in block letters.

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HOAI NGUYEN HUYNH (NEIL HÎTÔ ÆVILÈNE)
Singapore, July 2012.

¹These two sentences are kept from the original template, see Colophon on page 322.

To my Parents

Kính tặng Ba Mẹ

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Summary

The work here is inspired by the pioneering work of Benoit Mandelbrot on Fractals ([Mandelbrot, 1983](#)) and Per Bak, Chao Tang and Kurt Wiesenfeld on Self-Organized Criticality ([Bak et al., 1987](#)). These two phenomena are paradigmatic examples of repeated application of simple rules leading to complicated behaviours and structures. The first one shows how complicated structures arise in Nature while the latter attempts to explain the ubiquity of $1/f$ noise. Both of them are manifestation of the so-called scale-invariance of space and time in natural phenomena.

Study of complexity, meanwhile, is rapidly developing in many forefront fields of Science. However, the official definition of “complexity” is continuously being revised ([Christensen and Moloney, 2005](#)). Yet, there is no unified framework available to study complexity. This work, therefore, is motivated toward the understanding at fundamental level of complexity by searching for insights in Fractals and [Self-Organized Criticality \(subject\) \(SOC\)](#).

Fractal and [self-organized criticality \(phenomenon\) \(SOC\)](#) are fascinating phenomena observed in Nature. They are examples of complexity arising from simplicity. The repeated application of simple microscopic interacting rules among elements of a physical system, as time evolves, gives rise to very complicated macroscopic structures observed.

Those emergent structures and behaviours look messy, disordered and complex; yet, they possess a general property — scale-invariance. Scale-invariance refers to patterns that are invariant regardless of scale, *i.e.* they are the same at any scale. In the case of fractal, this is the self-similar property of the fractal object. Scale-invariance is characterised by a power-law distribution. And in the case of [SOC](#), the distribution of the observables is of power-law type.

This scale-invariant behaviour is of great interest to physical and mathematical researchers because of its emergence and ubiquity. It can be observed everywhere:

patterns of tree branches, landscape (clouds, coastline, terrain, *etc...*), distribution of earthquake, rainfall, forest fire, and hospital waiting time, *etc...*

Understanding of the origin and properties of scale-invariance leads to understanding of how a large number of important natural phenomena work and ability to make use of them for human purpose (like earthquake, forest fire, *etc...*).

Even though this thesis consists of two parts — one on Fractals and one on Self-Organized Criticality — more weight will be given to the later (about four times more, in terms of number of pages). Besides the central theme of “repeated application of simple rules leading to complicated behaviours and structures”, the first part links to the second one by providing the fractal lattices generated from the arc-fractal system.

In general, critical phenomena play an important rôle in our understanding of complex systems in Nature. One of the key features of critical systems is the notion of universality (Stanley, 1999). It suggests common underlying mechanisms of seemingly different phenomena and also justifies the analysis of (over)simplified numerical models of otherwise much more complex natural systems. A huge number of numerical models have been proposed to study different features of critical systems. Traditionally, those systems require external fine tuning of control parameter to critical point. However, many models exhibit the features of the critical state without the need of tuning a control parameter, which is known as self-organized criticality (Bak *et al.*, 1987). When the concept of self-organized criticality was introduced, it gained immediate popularity on the one hand because it attempted to explain the prevalence of scaling and fractality in nature and on the other hand suggested that many features of very different phenomena would be common to all of them, by the power of universality. At times disputed (Christensen and Olami, 1992), it is now widely accepted that the predictive power of SOC lies with its universality; the elusive qualities of a critical point, normally confined to a very narrow range of a control parameter, are the norm in SOC models and shared between them. While models of traditional critical phenomena have been very well studied both analytically and numerically, and important results including exact ones have been obtained (*e.g.* Syôzi, 1951), little success has been achieved for self-organized critical phenomena. One example of such models is the Manna model (Manna, 1991b) which has so far defied any attempt for an analytical approach, but has been studied extensively numerically. On the other hand, while universality has been at the centre of the debate about SOC (*e.g.* Nakanishi and Sneppen, 1997; Ben-Hur and Biham, 1996), it comes as a great surprise that it has

focused mainly on its occurrence across different models (e.g. [Dhar, 2006](#)), whereas very little work has been done on establishing it within the same models but across different lattices (e.g. [Duarte, 1990](#); [Manna, 1991a](#); [Hu and Lin, 2003](#); [Azimi-Tafreshi et al., 2010](#)). The question of universality, therefore, seems to have been overlooked in the [SOC](#) literature on a very fundamental point: *Does a model display the same critical behaviour on different lattices of the same dimension?* The observation that critical phenomena and scaling displays universal features on different lattices has traditionally been one of the most important insights, enabling, in particular, exact results ([Syôzi, 1951](#); [Itzykson and Drouffe, 1991](#)).

During the last decade or so, it has become increasingly clear that [SOC](#) is far more elusive than originally envisaged. It is therefore all the more important to establish which features known from ordinary critical phenomena carry over to [SOC](#). Independence of certain observables from the underlying lattice is one such aspect to substantiate. If universality was not to be found for the same model on different lattices, then the notion of universality in [SOC](#) as a whole had to be revised.

In this thesis, the universality hypothesis in [SOC](#) is put to test by studying the [Abelian Manna model \(AMM\)](#) ([Manna, 1991b](#); [Dhar, 1999a](#)) on different one-, two- and three-dimensional lattices. Owing to their robust scaling behaviour, in recent years, attention has focused on the [AMM](#) and the Oslo Model ([Christensen et al., 1996](#)), which seem to be in the same universality class ([Nakanishi and Sneppen, 1997](#)). This is contrasted by the poor scaling behaviour of many other established [SOC](#) models, such as the [forest-fire model \(FFM\)](#) ([Pruessner and Jensen, 2002](#); [Grassberger, 2002](#)), the [Bak-Tang-Wiesenfeld \(BTW\)](#) model ([Bak et al., 1987](#); [Dorn et al., 2001](#)) or the [Olami-Feder-Christensen \(OFC\)](#) model ([Olami et al., 1992](#); [Christensen and Olami, 1992](#)).

Apart from probing universality, corrections to scaling can also be tested, with the ultimate aim of finding a lattice that is particularly suited for simulating the [AMM](#). This is of particular interest in one dimension, where (logarithmic) corrections, are known to be remarkably strong ([Dickman and Campelo, 2003](#)). In addition, rarely studied moment ratios are also reported which provide further support for the universality of the [AMM](#).

On the other hand, although extensive research has been performed for models on hypercubic lattices, far less work has been done on fractal lattices ([Kutnjak-Urbanc et al., 1996](#); [Lee et al., 2009](#)). It remains somewhat unclear what to conclude from the latter studies. Fractal lattices are important for the understanding of critical phenomena

for a number of reasons. First, results for critical exponents in lattices with noninteger dimensions might provide a means to determine the terms of their $\epsilon = 4-d$ expansion. Second, fractal lattices are particularly suitable for real-space renormalisation-group procedures, in particular those by Migdal (1975), Kadanoff (1976) and Carmona *et al.* (1998). Third, scaling relations that are derived in a straightforward fashion on hypercubic lattices can be put to test in a more general setting. In this work, we address the first and the third aspects by examining both numerically and analytically the scaling behaviour of the Abelian version of the Manna model (Manna, 1991b; Dhar, 1999a, 2006) on various different fractal lattices. It would also be of great interest to see if the results in fractal dimensions agree with those in integer dimensions in a systematic manner.

At the thesis level, several pieces of work exist tackling self-organized criticality from different angles. As one of the rare studies of SOC in noninteger dimensions, Daerden (2001) studied the BTW model on Sierpiński gasket and provided a description of wave topping in the model. His main focus is on statistical properties of waves, as well as last waves, and successive waves correlations. Kloster (2004) partly¹ studied a directed sandpile model with stochastic toppling rules (the Manna model), and provided a solution for its asymptotic behaviour in all (integer) dimensions. Pruessner (2004b) performed an extensive and comprehensive study of many different existing models in SOC. His main focus is on the Oslo model and its variants and he found that the Oslo model is in fact a discrete realisation of the quenched Edwards-Wilkinson equation. An exact solution of an anisotropic variant of the model is also obtained. Stapleton (2007) studied the directed versions of Oslo model in one dimension and obtained several exact results. Bonachela (2008) studied self-organized criticality at a general level focusing on its universal features. His work appears to settle several controversial topics in SOC like rôle of conservation, distinction between the Directed Percolation (DP) and Conserved Directed Percolation (C-DP) universality classes.

The purpose of this thesis is *not* to provide any sort of review of the field Self-Organized Criticality after a quarter of century of research. That is a meaningless task in the current context since there exist out there several excellently written books and reviews on the subject (*e.g.* Bak, 1996; Jensen, 1998; Christensen and Moloney, 2005; Sornette, 2006). The latest effort of Pruessner (2012) provides researchers inside and outside the field with a very comprehensive picture of its. The focus of this thesis is

¹Only a portion of this thesis is on Self-Organized Criticality.

an in-depth study of one of the most important models in SOC that displays all the expected features of a *true* self-organized critical model.

Highlights of the main results

Below is the list of key results obtained from the work done in this thesis.

- A novel idea of fractal generator.
- Tiling of planar fractals.
- Possible application of the arc-fractal system to modelling of coastal morphology.
- Zoo of fractal lattices for study of lattice models.
- Generalisation of the scaling relation for avalanche size $D(2 - \tau) = d_w$ for lattices of general dimensions including noninteger ones.
- Revelation of scaling behaviour of sandpile model in noninteger dimension which is different from integer dimension.
- Illustration of (expected but being ignored) universality of critical exponents and moment ratios of sandpile model across lattices of different structures which is one of the most important features of ordinary critical phenomena.
- Revelation of universal moment (leading) amplitude of avalanche area across lattices of different structures.
- A remarkable scaling relation of critical exponents in all dimensions including noninteger ones.
- Rôle of scaling against N especially in noninteger dimensions.
- Introduction of a model which provides access to determination of [probability density function \(PDF\)](#) from first principle, and also to understanding why scaling laws like $D(\tau - 1) = D_a(\tau_a - 1)$ hold.

To emphasise the significance of the current study, it needs to be remembered that fractal lattices were known before but the examples of them having the same dimension but different structures are close to none¹. On the other hand, few works in SOC

¹In fact, to my best knowledge, no such example exists.

literature address the question of universality of a model across many different lattices. The issue of the critical behaviours in many different dimensions including the noninteger ones has not been addressed before. The work in this thesis presents a thorough and systematic numerical study of Abelian Manna model, quoting many key features of one of the most important SOC models like avalanche exponents, moment ratios, moment amplitudes, particle density, *etc...*

Summary of chapters

Below is a summary of the thesis, chapter by chapter. The ones with “★” contain my published results or (yet to be published) original contribution.

Part I

- Chapter 1: This chapter provides an overview of fractal: what it is, what its properties are, and a brief touch of examples of fractal in Nature.
- Chapter 2★: This chapter introduces the arc-fractal system. The construction of the system is presented. The properties of the generated fractals are discussed. Comparison to the L -system — a universal fractal generator — is made. Possibility of generalising the fractal generator to multifractal generator is briefly touched. One of the properties of the fractals is emphasised, that is the invariant set of points of the generated fractal. For each fractal, this set constitutes the so-called lattice corresponding to that fractal, hence the fractal lattice. The fractal lattices make the link between the first and the second part of the thesis. These fractal lattices will be employed for study of AMM in Self-Organized Criticality.

This chapter is analytical with development and derivation of mathematics and proofs of several properties. Results have been published in (Huynh and Chew, 2011).

- Chapter 3★: This chapter presents a possible application of the arc-fractal system to modelling of coastal morphology which is an excellent example of fractal in Nature.

This chapter is numerical with computer simulation. Results have been published in (Huynh and Chew, 2011).

Part II

- Chapter 4: This chapter provides an overview of Self-Organized Criticality: the explanation of the term itself, what it is about, the simplest models, how it is related to study of complex systems and complexity, comparison against ordinary critical phenomena which have been well studied, what is self-organized criticality and what is *not*, basic and important features of self-organized criticality, current state of art of the field and open problems and challenges.
- Chapter 5: This chapter introduces the Abelian Manna model — one of the most important models in Self-Organized Criticality — which is the main model considered in this thesis. All the results reported here are centralised around this model. The definition of the model is provided, the properties and comparison among different versions are discussed. The observables in the model is defined and their expected behaviours are presented.
- Chapter 6*: This chapter presents the lattices employed in this study. This forms one of the major parts of the thesis. In SOC literature, to my best knowledge, no single work addressed the study in such an in-depth survey of a wide range of lattices in different dimensions (either regular integer dimensions or noninteger ones) and with various structures. The chapter starts with a discussion on general properties of lattices from the point of view of graphs. Several issues are addressed like *the* dimension of graphs or lattices and their aspect ratio. Analysis and representation of lattices are performed. Many lattices in various dimensions with different structures including the fractal ones are discussed. The reason why this chapter is focused in details is to provide a good and clear presentation (introduction) to the fractal lattices which were not known in the literature before, yet may provide means to investigation of many features of the models studied on them (see (Zhang *et al.*, 2012) for a recent study that makes use of some of these fractal lattices).

This chapter is analytical with analysis and representation of lattices in computer's language. Details have been partially published in (Huynh et al., 2010, 2011; Huynh and Pruessner, 2012).

- Chapter 7*: This chapter presents the details of numerical simulation performed in this study. Method and details of implementation are described. Techniques for analysing the data are discussed. Numerical results are presented, looking at various aspects of the main model introduced in Chapter 5.

This chapter is numerical with extensive and large-scale computer simulation. Results have been published in (Huynh et al., 2010, 2011; Huynh and Pruessner, 2012).

- Chapter 8*: This chapter present several analytical approaches to tackle the models of self-organized criticality. Several exact expressions are derived by mapping the sandpile model to random walk process. Analytical construction of the adjacency matrix of (fractal) lattices are discussed hoping for analytical treatment of the model on them. This chapter also includes a short review of few exactly solvable cases like directed models. Algebraic and operator approach is also discussed but a feasible development is far from being realised. A solvable model is introduced and the solving steps are developed, introducing pretty neat framework hoping to show the robust scaling behaviour of the model and self-organized criticality in general. Less general method like graph is also discussed and illustrated how it can be employed to tackle sandpile model. Mean-field approximation is also applied to solve a model.

This chapter is analytical with mathematical developments of the exact solutions. Most of the results have not been published. Some exact expressions by mapping to random walk are to be published in (Pruessner and Huynh, 2012). The exact solution of the partially directed model is to be published in (Huynh and Tran, 2012).

- Chapter 9*: This chapters discusses the universality — one of the utmost importances of self-organized criticality and critical phenomena in general — by comparing the critical behaviours of different models known in the literature, and AMM itself on different lattices considered in this study. Surprisingly, it is observed that the (unexpected) moment amplitude of avalanche area is also found to be universal. Different scaling relations in the model are discussed referring to underlying physical principles (and symmetries?). The comprehensive study of AMM on a wide range of lattices in different dimension and structures allows access to very accurate estimates of critical exponents. And remarkably, a seemingly new scaling relation between exponents of the model and dimension of the lattices is revealed posing a equation of the old ϵ -expansion in perturbative renormalisation group (RG) literature. The long debated problem of relation between Manna class (also absorbing state phase transition or C-DP) and DP is addressed.

This chapter is numerical with comparison against known results in literature. Some of results have been published in (Huynh and Pruessner, 2012).

Notes on presentation

Clarity and consistency of writing are important to both the author and readers. As a non-native English speaker, it is pretty hard for me to justify the way nonstandard terms are written (especially terms that arise in scientific contexts). Below are the general rules that I obey through the entire thesis.

Hyphen and dashes

Hyphen “-” is used in compound words. En-dash “–” is used in range like number ranges. Em-dash “—” is used in punctuation in sentences like explanation of terms. The following compound words are used in this thesis.

absorbing-state	long-range	short-range
arc-fractal	mean-field	single-rule
body-centred	multiple-rule	spatio-temporal
box-counting	nearest-neighbour	three-dimensional
coarse-grained	(interaction)	two-dimensional
cross-check	next-nearest-neighbour	two-scale
face-centred	one-dimensional	two-state
finite-size (scaling)	power-law	well-behaved
first-passage	random-access (memory)	well-defined
goodness-of-fit	random-number	well-established
high-accuracy	(generator)	well-known
honeycomb-shaped	random-rule	
integer-dimensional	scale-invariance	

Prefixes

Most words with prefix are written without hyphen. A few exceptions include the prefix “self-”, or words in which the last letter of the prefix and the first letter of the word are the same (*e.g.* “semi-inverse”).

Capitalisation

Words that are name of a field are capitalised like Complex Systems, Complexity, Physics, Mathematics, Science, Self-Organized Criticality, *etc...* Words that are used

as supplement to another noun are *not* capitalised like “physics literature”, “mathematics literature”, *etc.*...

“Nature”, “Earth”, “Atmosphere” are capitalised.

Japanese names like “kagomé”, “mitsubishi” or “futatsubishi” are *not* capitalised.

Exotic names

To respect the origin of the names, names with accents are written with proper orthography in their own language (*e.g.* Waław Sierpiński, Sven Lübeck).

Nonstandard words in scientific contexts

- “Sandpile” is treated as one word, not a compound one and hence written without hyphen.
- “Abelian” is capitalised because it is derived from name of the Norwegian mathematician Niels Henrik Abel.

Numbers

Unless being mathematical quantities, dates and years, numbers to indicate the amount of anything are written in word (*e.g.* one, two, three rather than 1, 2, 3).

The integral and fractional parts of a decimal number are separated by “.”. Parts in thousands are separated by “,”.

Inline fractions with explicit numbers like $1/2$, $1/3$ are written with “/” while ones with symbols like $\frac{1}{n}$ are written in fraction form, except $1/f$ noise which is used as a word.

Spellings and style

The spelling of the words follows British English. For example, “-ise”, “-yse” endings are used over “-ize”, “-yze” where appropriate. There is, however, one exception. That is the phrase “self-organized criticality” (also “self-organized critical” as adjective, but *not* “self-organized” alone) in which “-ize” ending, which is American English spelling, is used to pay respect to the original use of the term when Bak *et al.* (1987) published their seminal paper that gave birth to the entire field. Other examples include “behaviour” vs. “behavior”, “centre” vs. “center”, “honour” vs. “honor”.

The writing style is also British, for example words like “*e.g.*” or “*i.e.*” have comma “,” before but *not* after.

Open problems

At the end of each part of this thesis, a list of open problems is provided. These are the ideas or problems arising during the course of this research but have not been investigated in (great) details due to the timeframe and also scope of the thesis. The ideas or problems, however, are interesting and of important value to the literature. They will be pursued in future research. At some points in the thesis, the problems are briefly discussed and carry the label **OPEN PROBLEM**. The lists of open problems provide the reference back to where they are mentioned in the text.

Others

Throughout the text, symbol star “★” is used over asterisk “*”.

Keywords

This section provides a list of keywords that best describe the content of this thesis.

Fractal	Abelian Manna model	Operator approach
Arc	Sandpile model	Cluster size distribution
Coastline	Lattice	Directed model
Geomorphology	Universality	Random walk
Lindenmayer system	Moment	Dimension
Tiling	Critical exponent	Manna class
Pascal triangle	Finite-size scaling	Critical phenomena
Pseudorandom number	ϵ -expansion	Complex systems
Self-organized criticality	Scaling relation	Complexity

List of Acronyms

AMM	Abelian Manna model. xix , xxi , xxii , xxvii , xxx , xxxii , 60 , 64 , 67 , 70 , 88 , 101–103 , 144 , 191 , 193 , 209 , 221 , 250 , 297
ASM	Abelian sandpile model. xxii , 59 , 64 , 209 , 210 , 212–214
BTW	Bak-Tang-Wiesenfeld. xxvii , xxviii , 59–62 , 64–66 , 72 , 143 , 214 , 220 , 245
C-DP	Conserved Directed Percolation. xxviii , xxxii , 64 , 243
CCMM	continuous conserved Manna model. 243
CLG	conserved lattice gas. 219
CPU	central processing unit. xiii , 102–104 , 221 , 224 , 225
CTTP	conserved threshold transfer process. 219 , 224
DASM	directed Abelian sandpile model. 63
DCMM	discrete conserved Manna model. 243

DP	Directed Percolation. xv , xxviii , xxxii , 62 , 222 , 243 , 244
FES	fixed-energy sandpile. 221
FFM	forest-fire model. xxvii , 63 , 245
MC	Manna class. 219 , 243
OFC	Olami-Feder-Christensen. xxvii , 61 , 63
PDF	probability density function. xxix , 71 , 112–114 , 123 , 248 , 250
RAM	random-access memory. 102 , 105
RG	renormalisation group. xxxii , 63–65
RNG	random-number generator. 105
SASM	stochastic Abelian sandpile model. 185
SOC	Self-Organized Criticality (subject). xxv , xxxi , 55 , 56 , 58 , 61 , 65 , 137 , 142 , 143 , 245 , 246 , 248 , 251
SOC	self-organized criticality (phenomenon). xxv–xxx , 57 , 58 , 61–66 , 71 , 87 , 95 , 100 , 134 , 141 , 143 , 221 , 230 , 243 , 245–248 , 252
TAOM	totally asymmetric Oslo model. 208

List of Symbols

$[\cdot]$	quantity in the square bracket not being fitted with a goodness-of-fit better than 0.1. 127
a	avalanche area observable. 68
$\mathcal{A}$	adjacency matrix. 105
$\mathcal{A}^*$	reduced adjacency matrix. 106
$\mathfrak{a}$	particle addition operator in Sadhu and Dhar (2009) 's notation. 186
$A_n^{(x)}$	coefficient of leading term in n th moment of observable x . 119
a_x	normalisation metric factor in the probability density function of observable x . 71
$A_{x,n}$	amplitude of n th moment of observable x . 131
$B_n^{(x)}$	coefficient of second leading term in n th moment of observable x . 119
B	column vector with all entries being t except the first one being $2t$. 162
b	column vector with all entries being t . 155

b_x	finite-size scaling metric factor in the probability density function of observable x . 71
$C_n^{(x)}$	coefficient of third leading term in n th moment of observable x . 119
D	finite-size scaling exponent for avalanche size. 71
D_a	finite-size scaling exponent for avalanche area. 71
d_b	box dimension. 7
d_H	Hausdorff dimension. 8
d	dimension of a fractal object (Part I). 19
d	dimension of a lattice (Part II). 72
$D_{L,n}(s)$	same as $D_n(s)$ but for system with L elements. 208
$D_n(s)$	probability to have an avalanche of size s due to n charges on system of single element. 196
$D_n^{(x)}$	coefficient of leading correction term in n th moment ratio of observable x . 131
d_s	self-similarity dimension. 5
d_w	fractal dimension of random walk on a lattice. 130
$E_n^{(x)}$	coefficient of second leading correction term in n th moment ratio of observable x . 131

F_0	coefficient of leading term in particle density. 136
F_1	coefficient of correction term in particle density. 136
$g_n^{(x)}$	moment ratio of order n th for observable x . 121
$\mathcal{G}_x$	scaling function in the probability density function of observable x . 71
h	height variable at each site on a lattice. 66
I	index of rotation of an arc. 21
$\mathcal{I}$	identity operator on arc. 25
i	ordinal of a divided arc segment (Part I). 13
i	index of a site on a lattice (Part II). 66
L	linear size of a lattice. 71
M	sample size of a data set. 112
$\mathfrak{M}$	mirror inversion operator on arc. 25
m	level of iteration in the arc-fractal system (Part I). 14
m	level of iteration of a fractal lattice (Part II). 95
N	total number of sites on a lattice. 66
N_a	number of simultaneously active sites. 68

n	number of segments into which an arc is divided (Part I). 12
n	order of moment of an observable (Part II). 71
P_C	column vector whose elements are $P_C(C_i)$. 195
$P_C(C_i)$	probability to have stable configuration C_i of system of single element in the stationary state. 194
$P_{Cn}(C_i, s)$	probability to have stable configuration C_i of system of single element after <i>precisely</i> s topplings in the stationary state due to n charges. 194
$P_{Cn}(s)$	column vector whose elements are $P_{Cn}(C_i, s)$. 195
$P_{L,C}$	same as P_C but for system with L elements. 206
$\mathcal{P}^{(x)}(x)$	probability density function of the observable x . 71
q	goodness-of-fit. 119
q_i	number of neighbours of site i on a lattice. 66
$q_i^{(v)}$	number of virtual neighbours of site i on a lattice. 67
R	radius of an arc. 13
r	distance between centres of a daughter arc and its mother arc. 14
$\Re$	rotation operator on arc. 25

R'	radius of a replacing arc. 20
s	avalanche size observable. 68
s_i	scaling variable of the i th replaced arc segment with respect to its mother arc. 13
t	avalanche duration observable. 68
T_i	expected residence time of a random walker on a lattice starting at site i . 144
$T_{ij}^{(n)}(s)$	probability that configuration C_j is transformed to C_i after precisely s topplings due to n charges on system of single element. 194
$\mathbb{T}_{L,n}(s)$	same as $\mathbb{T}_n(s)$ but for system with L elements. 208
$\mathbb{T}_n$	sum of $\mathbb{T}_n(s)$ over all avalanche size s . 195
$\mathbb{T}_n(s)$	matrix whose elements are $T_{ij}^{(n)}(s)$. 195
$\mathbf{T}$	column vector of residence times of random walker starting from different sites on a lattice. 155
x	horizontal Cartesian coordinates. 13
x_0	lower cutoff in the probability distribution of observable x . 71
y	vertical Cartesian coordinates. 13
z	finite-size scaling exponent for avalanche duration. 71

α	opening angle of an arc (Part I). 12
α	distribution exponent for avalanche duration (Part II). 71
β_i	angle of the i th divided arc segment of an mother arc. 13
Δ	equal to $\pi - \phi$. 16
ϵ	size of covers used in determining dimension of an object (Part I). 19
γ	angle that the radial of a daughter arc makes with the radial of its mother arc. 16
$\mu_n^{(x)}$	scaling exponent of n th moment of observable x against linear system size L . 72
ω	orientation of the replacing arc. 12
ϕ	angle that the line connecting the centres of a daughter arc and its mother arc makes with a reference direction. 15
φ	difference between the azimuth angle that a daughter arc makes with a reference direction and the azimuth angle that its mother arc makes with the same reference direction. 17
$\varpi_{L,n}$	matrix of number of particles released from a system of L elements due to n charges. 205
τ	distribution exponent for avalanche size. 71
τ_a	distribution exponent for avalanche area. 71
θ	azimuth angle that an arc makes with a reference direction. 13
ϑ	angle that the line connecting the centres of a daughter arc and its mother arc makes with the radial of the daughter arc. 15

ξ	cluster size. 137
ζ	stationary particle density. 134

Publications

The following publications are produced from the work done in this thesis.

Journal articles

Hoai Nguyen HUYNH, Lock Yue CHEW and Gunnar PRUESSNER, "Abelian Manna model on two fractal lattices", *Physical Review E*, **82** 042103 (2010).

Hoai Nguyen HUYNH and Lock Yue CHEW, "Arc-fractal and the dynamics of coastal morphology", *Fractals - Complex Geometry, Patterns, and Scaling in Nature and Society*, **19** 141 (2011).

Hoai Nguyen HUYNH, Gunnar PRUESSNER and Lock Yue CHEW, "Abelian Manna model on various lattices in one and two dimensions ", *Journal of Statistical Mechanics: Theory and Experiment*, **2011** P09024 (2011).

Hoai Nguyen HUYNH and Gunnar PRUESSNER, "Abelian Manna model in three dimensions and below", *Physical Review E*, **85** 061133 (2012).

Gunnar PRUESSNER and **Hoai Nguyen HUYNH**, "Towards a field theory for stochastic sandpile model: periodic systems with a control parameter", *to be submitted to Journal of Physics A: Mathematical and Theoretical* (2012), unpublished.

Hoai Nguyen HUYNH and Chieu Minh TRAN, "Exact solution of a partially directed Abelian Manna model", *to be submitted to Journal of Physics A: Mathematical and Theoretical* (2012), unpublished.

International presentations

Hoai Nguyen HUYNH and Lock Yue CHEW, "A dynamical model of coastline formation", poster presented at *KAGI21*, Kyoto University, Kyoto, Japan, 22nd–27th

August 2009.

Hoai Nguyen HUYNH, "Abelian Manna model on various lattices of different dimensions before mean-field", talk given at *Complexity Workshop*, Feza Gürsey Institute, Istanbul, Turkey, 05th–10th September 2011.

Hoai Nguyen HUYNH, Gunnar PRUESSNER and Lock Yue CHEW, "Self-organized critical model on fractal lattices of dimension between 1 and 2", poster presented at *ECCS'11*, University of Vienna, Vienna, Austria, 12th–16th September 2011.

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Part I

Fractal

Chapter 1

A review of Fractals

“Clouds are not spheres, mountains are not cones, coastlines are not circles, bark is not smooth, nor does lightning travel in a straight line.” Mandelbrot (1983)

1.1 What is fractal?

Fractals, initiated by French mathematician Benoit Mandelbrot in late 1970s (Mandelbrot, 1983), is a branch of Mathematics that studies highly complicated objects with very fine structures at any scales. The main concepts of fractals lie in the noninteger dimension and the self-similarity properties. The noninteger dimension of an object arises from the fact that its measures in the two successive integer dimensions are infinity and zero respectively (see Sec. 1.2.2.3). Measure in the “correct” dimension of the object yields finite value (Hausdorff, 1918). Since the term “fractal” was coined by Mandelbrot (1967, 1983), many research efforts have been directed towards uncovering the fractality in nature as well as its related phenomena (e.g. Meakin, 1983; Rammal and Toulouse, 1983; Mandelbrot *et al.*, 1984; Niemeyer *et al.*, 1984; Pentland, 1984; Sawada *et al.*, 1986; Knizhnik *et al.*, 1988; Götze and Sjögren, 1992; Leland *et al.*, 1994). In general, fractals are complicated shapes with self-similar properties and fine structures at arbitrarily small scales which are not possible to be described by simple geometry. Such complex objects can be created mathematically by iterating a function. They are also found to approximate many natural objects such as clouds, tree branches, blood vessels, *etc.*... In theoretical studies, fractals can be generated by many different methods: iterated function systems (Barnsley, 1988), recursive operation (Edgar, 2004), Lindenmayer system (Lindenmayer, 1968; Prusinkiewicz and Hanan, 1989) or

Figure 1.1: An example of fractal showing first three steps of constructing Koch curve. The arrows show the segments to be replaced by the new equilateral triangles.

from chaotic dynamical systems like Lorenz attractor (Lorenz, 1963), Feigenbaum attractor (Feigenbaum, 1979), etc... The procedure below illustrates a classic example of how fractal can arise from recursive operation.

- Start with a line segment of unit length.
- Divide it into three equal segments; construct an equilateral triangle based on the middle one third; remove that middle one third base.
- Repeat the procedure with each of the new segments.

This procedure produces the well-known Koch curve (Fig. 1.1), name in honour of the Swedish mathematician Helge von Koch (Koch, 1904; Edgar, 2004).

1.2 Properties of fractal

1.2.1 Self-similarity of fractal

One of the striking features of fractal objects is the self-similarity, *i.e.* a part of the object is similar to the whole object itself. And this feature is preserved in the structure of

Figure 1.2: Image of Romanesco broccoli showing striking fractal pattern. Picture is taken from http://en.wikipedia.org/wiki/Romanesco_broccoli.

the object at any lengthscale which is also known as the (spatial) scale-invariant property. The similarity is sometimes exact, yet more often only approximate or statistical. Figure 1.2 illustrates the (approximate or statistical) self-similarity feature of the Romanesco broccoli, a beautiful vegetable famous for its fractal pattern. A small part of the broccoli resembles its entire shape.

Fractals with exact (strict) self-similarity property are generated using mathematical rules. One such example is the Cantor set in Fig. 1.3. The set is constructed by starting with a single line segment, dividing it into three equal pieces and removing the middle segment, and repeating the whole procedure for each remaining segment. Any small portion of the set is a scaled copy of the big set.

1.2.2 Dimension of fractal

There are several ways to calculate the dimension of an object. The three most common methods are box-counting dimension, similarity dimension and Hausdorff dimension. Similarity dimension is only applicable when dealing with strictly self-similar fractal objects. In general, box-counting dimension is widely used thanks to its simplicity and applicability to many different non-self-similar fractals. However, the most accurate

Figure 1.3: The Cantor set with exact (strict) self-similarity feature. Any small portion of the set is a scaled copy of the big set.

and mathematically rigorous method is the Hausdorff dimension which is based on the Hausdorff measure of the object. In most cases, we deal with regular fractal objects. Hence, the definitions (at least box dimension and Hausdorff dimension) above yield the same value for the measure of dimension. This value is referred to as *the fractal dimension* of the object (sometimes, Hausdorff dimension, to be precise).

1.2.2.1 Similarity dimension

Because of self-similarity property of fractals, it is natural to define a measure of dimension based on the self-similarity of the fractal. The idea is to break the object of interest into small pieces of the same shape (which is possible thanks to its self-similar nature) and consider the relation between the scaling factor of pieces and the number of them. To see how this work, we first consider the regular objects like a line segment, a square and a cube.

For a line segment of length L , breaking it into n (equal) pieces yields small segments of length $\frac{L}{n}$ which are no different from the original line segment being scaled down by a factor of $s = n$. In the case of a square of size $L \times L$, we can break it into n^2 pieces of linear length $\frac{L}{n}$ which are no different from the original square being scaled down by a factor of $s = n$. Similarly for the case of a cube, the number of small cubes is n^3 and the scaling factor is $s = n$. We see that the number of small divided pieces relates to the scaling factor s through an exponent. That exponent is defined to be the self-similarity dimension d_s of a self-similar object. Formally, we have

$$d_s = \frac{\ln N}{\ln s} \quad (1.1)$$

in which N is the number of small (equally) divided pieces and s is the scaling fac-

Figure 1.4: An example of a fractal that is not self-similar. The fractal can be constructed by first dividing a square region into nine equal squares, selecting and discarding one of the small squares at random; then repeating the process on one of the eight remaining small squares *ad infinitum*. The example is suggested by Chew Lock Yue (see [Strogatz, 2000](#), p. 410).

tor. In the three examples above, this definition is perfectly consistent with our intuitive perception of dimension of regular objects like line $d_s(\text{line}) = \frac{\ln n}{\ln n} = 1$, square $d_s(\text{square}) = \frac{\ln n^2}{\ln n} = 2$ and cube $d_s(\text{cube}) = \frac{\ln n^3}{\ln n} = 3$.

For the Koch curve in Fig. 1.1, we observe that it is a self-similar object and four small copies being scaled down by a factor of 3 make up the object. Hence, the self-similarity dimension of this Koch curve is

$$d_s(\text{Koch}) = \frac{\ln N}{\ln s} = \frac{\ln 4}{\ln 3} \approx 1.262. \quad (1.2)$$

1.2.2.2 Box dimension

The self-similarity dimension above is a very simple, yet convenient and consistent definition of dimension. However, its limitation lies in the name itself, *i.e.* it is only applicable to strictly self-similar objects. In general, fractal objects are not self-similar. An example of a fractal that is not self-similar is given in Fig. 1.4. In such case, the self-similarity dimension defined above cannot be employed to calculate the dimension of the object¹. Hence, we need to develop a new definition of dimension that can deal with these objects and at the same time is still consistent for regular objects, *i.e.* a more generalised measure of dimension.

At this point, we introduce the concept of a *cover*. The idea of this method is to cover the object with boxes of same size, or lengthscale ϵ . The minimum number of boxes

¹since there is no way to divide the object into small pieces that are (precisely) scaled versions of the original object

$N(\epsilon)$ needed is obtained as a function of the size of the boxes. The box dimension d_b is then obtained from that function. Formally, we have

$$d_b = \lim_{\epsilon \rightarrow 0} \frac{\ln N(\epsilon)}{\ln \frac{1}{\epsilon}}. \quad (1.3)$$

One can immediately verify that this definition of box dimension consistently yields $d_b(\text{line}) = 1$, $d_b(\text{square}) = 2$ and $d_b(\text{cube}) = 3$ for regular objects like line, square and cube. Now we proceed to calculate the box dimension of the fractal given in the example in Fig. 1.4. According to the rule of construction of the fractal, we cover the object at every level m th by boxes of size $\epsilon = \frac{1}{3^m}$ and the minimum of boxes need is $N = 8^m$. In the limit of vanishing size of the covers, one obtains the box dimension of the fractal

$$D_b(\text{random fractal}) = \lim_{\epsilon \rightarrow 0} \frac{\ln N(\epsilon)}{\ln \frac{1}{\epsilon}} = \lim_{m \rightarrow \infty} \frac{\ln 8^m}{\ln 3^m} = \frac{\ln 8}{\ln 3} \approx 1.893. \quad (1.4)$$

1.2.2.3 Hausdorff dimension

One limitation of the box dimension defined above is the use of covers of the *same* size. In many cases, the fractal object is a nontrivial one which makes the application of box dimension give a wrong result for the dimension of the object. This problem is overcome by the Hausdorff dimension which is a well-established mathematical definition and applies in all cases to all objects. The difference (and the reason why Hausdorff is superior) between the two definitions of dimension is the use of covers of different sizes in Hausdorff dimension, which apparently is more general than covers of the same size in box dimension.

The idea of Hausdorff dimension is based on the concept of Hausdorff measure. The Hausdorff measure is a general measure in an arbitrary s -dimensional space of an object¹. The familiar examples of Hausdorff measure include the length, area and volume which correspond to one-dimensional, two-dimensional and three-dimensional Hausdorff measure. The s -dimensional Hausdorff measure $H^{(s)}(F)$ of an object F can either be 0, ∞ or some finite value. It is 0 if the value of s is greater than the *true* dimension d of the object, and is ∞ if the value of s is smaller than the *true* dimension d of the object. It only stays finite if the value of s coincides with the *true* dimension d

¹The object is itself d -dimensional but has measure in any s -dimensional space.

of the object. This is easy to see in the examples of regular objects: a square ($d = 2$) or a cube ($d = 3$) has infinite length ($s = 1$) while a line ($d = 1$) has zero area ($s = 2$) or volume ($s = 3$). Hence, the value of s for which the Hausdorff measure $H^{(s)}(F)$ of an object F stays finite is the Hausdorff dimension d_H of that object.

Formally (see, for example, [Ott, 1993](#), pp 100–103), let F be a set in a d -dimensional Cartesian space. The diameter $|F|$ of F is defined to be the largest distance between any two points x and y in F

$$|F| = \sup_{x, y \in F} |x - y|. \quad (1.5)$$

Let S_i denote a countable collection of subsets of the Cartesian space such that the diameters ϵ_i of the S_i satisfies

$$0 < \epsilon_i \leq \delta \quad (1.6)$$

and such that the S_i are a covering of F , i.e. $F \subset \bigcup_i S_i$. We define the quantity

$$H_\delta^{(s)}(F) = \inf_{S_i} \sum_{i=1} \epsilon_i^s. \quad (1.7)$$

The s -dimensional Hausdorff measure of F is then defined as

$$H^{(s)}(F) = \lim_{\delta \rightarrow 0} H_\delta^{(s)}(F). \quad (1.8)$$

In general, it can be shown that $H^{(s)}(F)$ is $+\infty$ if s is less than some critical value and is zero if s is greater than that critical value. That critical value is called the Hausdorff dimension d_H of the set F .

In practice, Hausdorff dimension is rather difficult to apply because of the determination of the optimal set of covers in Eq. (1.7), i.e. the collection of covering sets S_i with diameter less than or equal to δ which minimises the sum in Eq. (1.7). Yet, most of the times, unless dealing with multifractal (see below), it is sufficient to obtain the fractal dimension of an object using the box dimension, since we are dealing with regular fractals. In these cases, the box dimension and Hausdorff dimension coincide.

1.2.3 Types of fractal

There are many different types of fractal. They include deterministic fractal and random fractal. As the name suggests, deterministic fractals are generated using a fixed, deterministic set of rules. Hence, the fractals are strictly self-similar. Random fractals

are generated with some degree of randomness in the rules. The fractal in Fig. 1.4 is an example of random fractal.

Fractals can also be classified based on their ramification. A fractal is said to be finitely ramified if any part of it can be isolated by removing a finite number of links with the other parts. Otherwise, the fractal is said to be infinitely ramified.

1.2.4 Multifractal

Where as fractal is considered to be a generalisation of regular objects in which non-integer dimension arises, one can also have a generalisation of fractal itself. Such generalisation leads to the so-called multifractal (Halsey *et al.*, 1986). *“A multifractal [...] is a fractal for which a probability measure on the fractal support is given. For example, if the fractal is the attractor of a map in a numerical experiment, the probability measure is given by the relative frequencies of the iterates, which are interpreted as probabilities [...]”* (Beck, 1995, pp 114–126). A multifractal is characterised by more general dimensions (e.g. Rényi dimension (Rényi, 1970; Mandelbrot, 1974; Hentschel and Procaccia, 1983; Grassberger, 1983)) that contain information about the probability distribution on the fractal.

The concept of multifractal is important because in Nature we are dealing with multifractal more than fractal, *i.e.* the fractal dimension is not sufficient to describe such objects. This can be perceived from the tendency of natural objects to grow spontaneously with a high degree of randomness, and the fact that different regions of the evolved objects possess different fractal properties (Stanley and Meakin, 1988), which leads to numerous scaling laws found in nature (Paladin and Vulpiani, 1987).

1.3 Fractal in Nature

In Nature, one can easily find many real examples of fractal. Coastline appears to be the most famous example fractal in Nature since it was first coined by Mandelbrot (1983) in 1970s when he studied self-similar properties of objects in Nature. Coastlines are usually thought of as one-dimensional curves but indeed they can have dimension of some value between 1 and 2. A smooth coastline should have dimension closer to 1 and a more irregular one closer to 2. Many studies have shown that the dimension can range from 1.02 for coastline of South Africa to 1.25 for West coast of Britain (Richardson, 1961).

Other examples in Nature include network of rivers, shape of clouds, terrain, fault

lines, ocean waves or even everyday things like tree branches, broccoli, *etc...* Fractals in Nature has triggered many studies in Science and inspired various applications in Engineering. One of such applications is to use fractal shape in making antenna (*e.g.* Puente *et al.*, 1996, 1998) to improve its performance.

Chapter 2

Arc-fractal system

In this chapter, we explore an idea of creating fractals by using the geometric arc as the basic element. This approach of generating fractals, through the tuning of just three parameters, gives a universal way to obtain many different fractals including the classic ones. The fractals generated using this arc-fractal system are shown to possess a number of features, one of which is the ability to tile the space.

A novel idea of generating fractals by making use of circular arcs is developed. Interestingly, this method is able to produce many classic fractals such as the Koch curve, the Heighway dragon, the Lévy dragon, the Sierpiński gasket and the Eisenstein fraction. In the literature, there exist other methods that are able to generate different types of fractals from a single approach like the arc-fractal presented here. These are the iterated function system (Barnsley and Demko, 1985; Barnsley, 1988) and the Lindenmayer system (Lindenmayer, 1968; Prusinkiewicz and Hanan, 1989; Prusinkiewicz and Lindenmayer, 1990). Since a fractal is typically self-similar, these methods have to preserve the self-similar structure of the fractal at all levels of construction. In the case of iterated function system, this is done by successively applying the function operators on the system. And in the case of the Lindenmayer system, it is done through repeating a sequence of symbols which determines the rule of construction.

A common method of constructing a fractal is to perform a set of recursive operations, which is somewhat similar to the operation of iterated function system. One famous example of this approach is the construction of the Koch curve (Edgar, 2004): start with a line segment of unit length, divide it into three equal segments, and replace the middle segment with two new equal segments which form the shape of an equilateral triangle (see Fig. 1.1). Inspired by this idea, we apply the technique *not* on

a line segment but on an arc segment. Remarkably, it is found that through defining the parameters of the arc and tuning those parameters, this procedure can generate many different types of fractals, including the five classic fractals mentioned above. We know that those fractals can be obtained by means of recursive operation as well, but the known examples use different basic objects, such as line for the Koch curve, and triangle for the Sierpinski gasket. In contrast, our method employs a single element of the geometric arc to build many fractals. This implies the intrinsic simplicity and universal quality of the arc-fractal approach, just like the well-known Lindenmayer system, or L -system.

The arc-fractal system is also shown to generate a nontrivial sequence of numbers that may be associated with pseudorandom numbers. This might be an interesting feature because although there are many algorithms available to generate pseudorandom numbers based on deterministic rules (Brent, 1974; Niederreiter, 1978; Rotenberg, 1960; Wichmann and Hill, 1982), none of them has been associated with deterministic fractal. In addition, like the L -system, the arc-fractal technique is able to create fractals that tile. The tiling of space with fractals is a topic of great interest in many fields (Lagarias and Wang, 1997; Lai, 2009). While tiling in art can be looked upon as a mode of decoration, tiling in mathematics reflects the basic symmetry of the tiling objects (Palagallo and Salcedo, 2008). Indeed, the problems of fractal tiling have been investigated extensively, with aperiodic tiling using the famous Penrose tiles (Penrose, 1974), or tiling using different basic prototypes (Fathauer, 2001, 2002).

2.1 Geometric representation

The basic procedure of generating arc-fractals begins with a simple arc of angle α . This arc is next divided into n segments (not necessarily of equal size). Subsequently, each segment is replaced by a new arc of some angle with the orientation either inward or outward (characterised by ω , see Fig. 2.1). By repeating this procedure *ad infinitum*, the arc-fractal is obtained. In our construction, three types of rules are applicable.

- (a) Single-rule: the same rule is applied at each level of the iteration.
- (b) Multiple-rule: a periodic repetition of different rules as the iteration progresses.
- (c) Random-rule: the rule at each level is determined by a random selection from a set of different rules.

For each type of rule, the following main variables are needed at each level of the iteration process.

- (a) α : the opening angle of the arc.
- (b) n : the number of dividing segments (and scales, if the segments are to be unequally divided).
- (c) ω : the orientation of the arc.

For example, let rule A be $\{\alpha \text{ is } \pi \text{ radian; divide into three segments; orientation is out-in-out}\}$ and rule B be $\{\alpha \text{ is } \pi \text{ radian; divide into three segments; orientation is in-out-in}\}$. Then, a multiple-rule is given by: at the odd level, apply rule A; at the even level, apply rule B. On the other hand, a random rule is given by: select either rule A or rule B at random at each level.

2.2 Algebraic representation

Since an arc is embedded in two-dimensional Euclidean space, it will be appropriate for us to present the arc-fractal system within the Cartesian X - Y plane. An arc in this plane will be represented by a set of five values $(x, y, R, \theta, \alpha)$, where (x, y) indicates the Cartesian coordinates of the centre of the arc; R the radius of the arc; θ the (azimuth) angle that the arc makes with a reference direction, which is chosen to coincide with the horizontal x -axis; and α the opening angle of the arc (see Fig. 2.2). OPEN PROBLEM

see list on page 51

Next, we shall introduce the scaling variable s_i . First, let us denote β_i , where $i = 1, \dots, n$, to be the angle of the i th divided segment of the mother arc. Then, s_i is defined as the ratio between β_i and the opening angle α of the initial mother arc. In other words, $\beta_i = s_i \alpha$. As $\sum_{i=1}^n \beta_i = \alpha$, we obtain $\sum_{i=1}^n s_i = 1$. In the case of equally divided arc, we have $s_i = \frac{1}{n}$.

As discussed in Sec. 2.1, the arcs at each level have two possible orientations — inward and outward. This will be represented by the parameter ω_i which shall take only two values: +1 (for inward arc) and -1 (for outward arc).

With these general definitions of the key variables, we can proceed to derive the recursive equations for the generation of the arc-fractal. Let us first define the vari-

(a) Definition of an arc

(b) Inward replacement

(c) Outward replacement

Figure 2.1: The basic idea of generating an arc in the arc-fractal system.

ables employed in the derivation of the recursive equation of the arc-fractal system (see Figs. 2.3 and 2.4, the superscript m refers to the arc at level m).

Symbol	Meaning
(x^m, y^m)	coordinates of the centre of the mother arc.
(x_i^{m+1}, y_i^{m+1})	coordinates of the centre of the i th daughter arc.
R^m	radius of the mother arc.
R_i^{m+1}	radius of the i th daughter arc.
r_i^m	distance between the centres of the i th daughter arc

Figure 2.2: The parameters that define an arc.

and mother arc.

α^m : opening angle of the mother arc.

α_i^{m+1} : opening angle of the i th daughter arc.

β_i : opening angle of the i th segment of the mother arc.

θ^m : azimuth angle that the mother arc makes with the reference direction.

θ_i^{m+1} : azimuth angle that the i th daughter arc makes with the reference direction.

ϕ_i^m : angle between the line that connects (x^m, y^m) to (x_i^{m+1}, y_i^{m+1}) and the reference direction.

ϑ_i^m : angle between the line that connects (x^m, y^m) to (x_i^{m+1}, y_i^{m+1})

Figure 2.3: Definition of angles and radii for the arc-fractal system.

and the radial R_i^{m+1} .

γ_i^m : angle that the radial R_i^{m+1} makes with the radial R^m .

Δ : equal to $\pi - \phi_i^m$.

s_i : scale of i th segment, equal to $\frac{\beta_i}{\alpha^m}$.

Note that all angles have positive values except for the azimuth angles, which take negative values when the angular displacement is clockwise and positive values when the displacement is counterclockwise. Furthermore, since there are two radial edges to choose from when the azimuth angles are to be determined with respect to an arc, the convention is to choose the first edge that is encountered when a counterclockwise angular displacement is made.

Figure 2.4: Angles for the inward and outward arc.

According to Fig. 2.3, one can ascertain the following relation:

$$\begin{pmatrix} x_i^{m+1} \\ y_i^{m+1} \\ R_i^{m+1} \\ \theta_i^{m+1} \\ \alpha_i^{m+1} \end{pmatrix} = \begin{pmatrix} x^m \\ y^m \\ f_i(R^m) \\ \theta^m \\ g_i^m(\alpha^m) \end{pmatrix} + \begin{pmatrix} r_i^m \cos \phi_i^m \\ r_i^m \sin \phi_i^m \\ 0 \\ \varphi_i^m \\ 0 \end{pmatrix}. \quad (2.1)$$

in which we have implicitly defined $\varphi_i^m = \theta_i^{m+1} - \theta_i^m$ and $\alpha_i^{m+1} = g_i^m(\alpha^m)$ reflects some prescribed rule that assigns value to the new arcs. To evaluate r_i^m and R_i^{m+1} , we employ the sine theorem. From Fig. 2.4, which illustrates the cases of inward and outward arc, we observe that

$$\frac{r_i^m}{\sin \gamma_i^m} = \frac{R^m}{\sin \vartheta_i^m}. \quad (2.2)$$

Since

$$\vartheta_i^m + \gamma_i^m + \frac{s_i \alpha^m}{2} = \pi, \quad (2.3)$$

we obtain the following results for two possible orientations of the new arc:

$$\vartheta_i^m = \begin{cases} \frac{\alpha_i^{m+1}}{2} \\ \pi - \frac{\alpha_i^{m+1}}{2} \end{cases} \Rightarrow \sin \vartheta_i^m = \sin \frac{\alpha_i^{m+1}}{2} \quad (2.4)$$

and

$$\gamma_i^m = \pi - \vartheta_i^m - \frac{s_i \alpha^m}{2} = \begin{cases} \pi - \frac{\alpha_i^{m+1}}{2} - \frac{s_i \alpha^m}{2} = \pi - \left(\frac{\alpha_i^{m+1}}{2} + \frac{s_i \alpha^m}{2} \right) \\ \pi - \pi + \frac{\alpha_i^{m+1}}{2} - \frac{s_i \alpha^m}{2} = \frac{\alpha_i^{m+1}}{2} - \frac{s_i \alpha^m}{2} \end{cases} \quad (2.5)$$

$$\Rightarrow \sin \gamma_i^m = \sin \left(\frac{\alpha_i^{m+1}}{2} \pm \frac{s_i \alpha^m}{2} \right) = \sin \left(\frac{\alpha_i^{m+1}}{2} + \omega_i \frac{s_i \alpha^m}{2} \right). \quad (2.6)$$

This explains why the two different orientations are simply presented by $\omega_i = \pm 1$ ($\omega_i = +1$ corresponds to the inward orientation, and $\omega_i = -1$ the outward orientation).

Substituting this result into Eq. (2.2), we obtain

$$r_i^m = \frac{R^m}{\sin \vartheta_i^m} \sin \gamma_i^m = \frac{R^m}{\sin \frac{\alpha_i^{m+1}}{2}} \sin \left(\frac{\alpha_i^{m+1}}{2} + \omega_i \frac{s_i \alpha^m}{2} \right). \quad (2.7)$$

A similar application of the sine theorem leads to

$$R_i^{m+1} = \frac{R^m}{\sin \frac{\alpha_i^{m+1}}{2}} \sin \frac{s_i \alpha^m}{2} = f_i(R^m). \quad (2.8)$$

From Fig. 2.3, we see that ϕ_i^m is the sum of θ^m and all β_k before the i th segment plus half of β_i . Thus,

$$\phi_i^m = \theta^m + \sum_{k=1}^{i-1} s_k \alpha^m + \frac{s_i \alpha^m}{2}. \quad (2.9)$$

In addition, Figs. 2.3 and 2.4 show that φ_i^m (implicitly defined in Eq. (2.1)) can be easily calculated through the angle Δ :

$$\begin{aligned} \varphi_i^m &= \theta_i^{m+1} - \theta^m = \mp \vartheta_i^m - \Delta - \theta^m = \mp \vartheta_i^m - (\pi - \phi_i^m) - \theta^m \\ &= \begin{cases} -\frac{\alpha_i^{m+1}}{2} - \pi + \phi_i^m - \theta^m \\ \pi - \frac{\alpha_i^{m+1}}{2} - \pi + \phi_i^m - \theta^m \end{cases} = -\frac{\alpha_i^{m+1}}{2} + \sum_{k=1}^{i-1} s_k \alpha^m + \frac{s_i \alpha^m}{2} - (1 + \omega_i) \frac{\pi}{2}. \end{aligned} \quad (2.10)$$

Note that the “ $\mp$ ” sign results from the definition of the azimuth angle given above.

Finally, by putting all the results together, we obtain the key mathematical prescription that enables the generation of the arc-fractal as follow

$$\left\{ \begin{array}{l} x_i^{m+1} = x^m + \frac{R^m}{\sin \frac{\alpha_i^{m+1}}{2}} \sin \left(\frac{\alpha_i^{m+1}}{2} + \omega_i \frac{s_i \alpha^m}{2} \right) \cos \left(\theta^m + \sum_{k=1}^{i-1} s_k \alpha^m + \frac{s_i \alpha^m}{2} \right) \\ y_i^{m+1} = y^m + \frac{R^m}{\sin \frac{\alpha_i^{m+1}}{2}} \sin \left(\frac{\alpha_i^{m+1}}{2} + \omega_i \frac{s_i \alpha^m}{2} \right) \sin \left(\theta^m + \sum_{k=1}^{i-1} s_k \alpha^m + \frac{s_i \alpha^m}{2} \right) \\ R_i^{m+1} = \frac{R^m}{\sin \frac{\alpha_i^{m+1}}{2}} \sin \frac{s_i \alpha^m}{2} \\ \theta_i^{m+1} = \theta^m + \sum_{k=1}^{i-1} s_k \alpha^m + \frac{s_i \alpha^m}{2} - \frac{\alpha_i^{m+1}}{2} - (1 + \omega_i) \frac{\pi}{2} \\ \alpha_i^{m+1} = g_i^m(\alpha^m) \end{array} \right. \quad (2.11)$$

It is important to note that α_i , ω_i and $n(s_i)$ are independent parameters defined at each level m , while R_i depends on α_i ; and s_i , θ_i and (x_i, y_i) depend on all the variables. Finally, the function g_i depends on the level m as well as the divided segment.

2.3 Properties of arc-fractal

Theorem. *In the case of equally divided segments with the same opening angle α throughout the construction process, the dimension d^1 of the arc-fractal is given by:*

$$d = \frac{\ln n}{\ln \frac{\sin \frac{\alpha}{2}}{\sin \frac{\alpha}{2n}}} \quad (2.12)$$

where n is the number of segments.

Proof. Equation (2.12) can be easily determined by employing the box dimension and using arcs of angle α , which are all of the same size, to cover the fractal.

At the zeroth level S_0 , the size ϵ of the cover is the length of the arc

$$\epsilon_0 = \alpha R, \quad (2.13)$$

¹These fractals are regular, hence any definition of dimension yields the same result.

where R is the radius of the arc. At the next level S_1 , the new arc has radius R' . Hence, the size of the cover is

$$\epsilon_1 = \alpha R', \quad (2.14)$$

with

$$R' = \frac{\sin \frac{\alpha}{2n}}{\sin \frac{\alpha}{2}} R, \quad (2.15)$$

according to Eq. (2.8). Hence the size of the cover reduces by

$$\frac{R'}{R} = \frac{\sin \frac{\alpha}{2n}}{\sin \frac{\alpha}{2}} \quad (2.16)$$

at the next level. This implies the following size for the cover at the level S_m

$$\epsilon_m = \alpha R \left(\frac{\sin \frac{\alpha}{2n}}{\sin \frac{\alpha}{2}} \right)^m. \quad (2.17)$$

Next, let us determine the length of the fractal at level m . As we go from level m th to level $(m+1)$ th, we divide the arc into n parts and replace each part by a new arc. Since the length of the fractal at $(m+1)$ th level is $n \frac{\sin \frac{\alpha}{2n}}{\sin \frac{\alpha}{2}}$ times that at m th level, the length of the fractal at S_m is

$$L_m = L_0 \left(n \frac{\sin \frac{\alpha}{2n}}{\sin \frac{\alpha}{2}} \right)^m = \alpha R \left(n \frac{\sin \frac{\alpha}{2n}}{\sin \frac{\alpha}{2}} \right)^m. \quad (2.18)$$

This means that the number of covers required at S_m is

$$N_m = \frac{L_m}{\epsilon_m} = n^m. \quad (2.19)$$

Based on the box dimension, we obtain the dimension of the arc-fractal as follow

$$d = \lim_{m \rightarrow \infty} \frac{\ln N_m}{\ln \frac{1}{\epsilon_m}} = \frac{\ln n}{\ln \frac{\sin \frac{\alpha}{2n}}{\sin \frac{\alpha}{2}}}. \quad (2.20)$$

2.3.1 Invariant set of points

The two end points of every arc, obtained during the construction of the arc-fractal, belong to the invariant set of points of the final fractal.

It is easy to observe that the operation of replacing the divided segment of the former arc by a new arc has actually left the end points of the arc unchanged throughout the construction. Thus one may surmise that the arc-fractal is a union of set of invariant points. This invariant set of points constitutes a lattice corresponding to the fractal. Some of the fractal lattices obtained from arc-fractal system are later used in Part II for the study of Self-Organized Criticality.

2.3.2 Generation of nontrivial sequences

*In the case of a constant opening angle α , where $\alpha = \frac{p}{q}\pi$ ($p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$), and $s_k = \frac{1}{n}$ ($k = 1, \dots, n$), one can define an index of rotation I , which provides a one-to-one association with each generated arc. By tracing along the arc-fractal at each level of construction, which is based on either single- or a **deterministic** multiple-rule, the series of index I is found to generated a nontrivial sequence of numbers.*

Let us consider the construction of an arc-fractal based on one single-rule as an example, with the arc being a semicircle, *i.e.* $\alpha = \pi$. At level m , we trace the curve by going counterclockwise from one arc element to the adjacent arc element and write down the sequence of indices (or label, refer to Appendix A.1 for details) of the arcs. The number of indices in the sequence at level m is n^m

$$I_1 I_2 \dots I_{n^m}. \quad (2.21)$$

This sequence is nontrivial since no apparent patterns are found and might be pseudorandom in lieu of the fact that at level m , cycling sequence (*i.e.* $I_1 I_2 \dots I_{n^m} \rightarrow I_2 I_3 \dots I_1 \rightarrow \dots \rightarrow I_{n^m} I_1 \dots I_{n^m-1}$) gives us n^m distinct sequences. This behaviour results from the fact that although the arc-fractal is self-similar, it is not repetitive. Thus, the choice of a different initial arc element will lead to a completely different sequence.

Let us take the “crab fractal”¹, so-called because it looks like a crab, to illustrate what we mean (see Fig. 2.5). As discussed in Appendix A.1, the labels for this system

¹The name “crab fractal” was given by me based on the intuition of the shape of the created fractal (Huynh and Chew, 2011). Later it is brought to my attention by Helmberg (2011) that the name was actually used before (Helmberg, 2007).

Figure 2.5: “Crab fractal” and its construction rule.

can be characterised by the following matrix

$$I = \begin{pmatrix} 11 & 7 & 3 \\ 12 & 8 & 4 \\ 1 & 9 & 5 \\ 2 & 10 & 6 \\ 3 & 11 & 7 \\ 4 & 12 & 8 \\ 5 & 1 & 9 \\ 6 & 2 & 10 \\ 7 & 3 & 11 \\ 8 & 4 & 12 \\ 9 & 5 & 1 \\ 10 & 6 & 2 \end{pmatrix}. \quad (2.22)$$

At the zeroth level, the arc is a horizontal semicircle, whose index of rotation is $i_0 = 1$. At level 1, there are three new semicircles, whose indices of rotation can be obtained by means of the matrix I above. Since the index of the initial semicircle is $i_0 = 1$, the indices at the next level is given by the first row of the matrix I , *i.e.* the sequence of matrix elements $I_{i_0,1}I_{i_0,2}I_{i_0,3}$. Thus, the indices of the semicircles at level 1 are given by (11, 7, 3). At level 2, three new semicircles are to be generated from each of the semicircles at level 1 labelled as 11, 7 and 3. The indices of these new semicircles are to be obtained in the same manner. For the first semicircle labelled by 11, the three

new semicircles are labelled as $I_{11,1}I_{11,2}I_{11,3}$, which is (9, 5, 1). Similarly, for the second semicircle, the sequence is $I_{7,1}I_{7,2}I_{7,3}$, which is (5, 1, 9). For the third semicircle, the sequence is $I_{3,1}I_{3,2}I_{3,3}$, which is (1, 9, 5). Putting them together, and because of the rule of construction of the crab fractal which requires the second semicircle to be flipped inward, we need to invert the second subsequence (refer to Eq. (A.21)). As a result, the sequence at level 2 is (9, 5, 1, $\underbrace{9, 1, 5}_{\substack{\text{this sequence} \\ \text{is inverted}}}$, 1, 9, 5) instead of (9, 5, 1, 5, 1, 9, 1, 9, 5). The growth of the sequence as the level increases is illustrated below

$$\begin{aligned}
 1 &\rightarrow (11, 7, 3) \\
 &\rightarrow (9, 5, 1, \underline{9, 1, 5}, 1, 9, 5) \\
 &\rightarrow (7, 3, 11, \underline{7, 11, 3}, 11, 7, 3, 7, 3, 11, \underline{3, 7, 11}, 3, 11, 7, 11, 7, 3, \underline{11, 3, 7}, 3, 11, 7) \\
 &\rightarrow \dots
 \end{aligned} \tag{2.23}$$

Now, if we take the sequence at level 2 and cycle it repeatedly, we observe that the resulting sequences are distinct from each other:

$$\begin{aligned}
 &(7, 3, 11, 7, 11, 3, 11, 7, 3, 3, 11, 7, 3, 7, 11, 7, 3, 11, 11, 7, 3, 11, 3, 7, 3, 11, 7) \\
 &(3, 11, 7, 11, 3, 11, 7, 3, 3, 11, 7, 3, 7, 11, 7, 3, 11, 11, 7, 3, 11, 3, 7, 3, 11, 7, 7) \\
 &\vdots \\
 &(7, 7, 3, 11, 7, 11, 3, 11, 7, 3, 3, 11, 7, 3, 7, 11, 7, 3, 11, 11, 7, 3, 11, 3, 7, 3, 11)
 \end{aligned} \tag{2.24}$$

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2.3.3 Tiling of arc-fractals

The arc-fractal can serve as the boundary of a two-dimensional piece that tiles the entire two-dimensional plane. Such two-dimensional piece is called “fractile”.

Tiling a plane with tiles that have a fractal boundary is of great interest and has been discussed extensively in the literature, such as tiles based on the Koch snowflakes or f -tilings based on the kite- or dart-shaped prototiles (Fathauer, 2001). Interestingly, the fractile whose boundary is constructed based on the arc-fractal approach can tile the plane too. A basic reason as to why the arc-fractal can tile results from the fact that an outward arc can be fitted perfectly to an inward arc of the same radius and

Figure 2.6: Tiling of arc-fractals.

opening angle. Another reason is due to the intrinsic property of sequence matching between the tiling pieces. The sequence mentioned here is a collection of orientation of successive arcs, which is parametrised by the index I_k as discussed in Sec. 2.3.2 above. This sequence is like a code of the boundary. If two arcs of the same radius fit perfectly together, they must possess an index I that is complementary to each other. In other words, if two fractiles are to fit together, their fitting boundary must possess complementary codes. And when a set of fractiles have these matching codes, these objects can fit together and tile the plane.

In Fig. 2.6 we can observe two examples of fractal boundaries generated by the arc-fractal systems that tile. One is the Eisenstein fraction, and the other is the Koch curve. Note that in the case of the Eisenstein fraction, the tiles are all of the same size. As for the Koch curve, we will need to rescale the arc-fractal one level up by $\frac{1}{\sqrt{3}}$ in order to tile with the Koch curve at the current level.

A proof that the Eisenstein fraction can indeed tile as shown is given below, while the case for the Koch curve can be similarly proven.

The strategy of our proof that the Eisenstein fraction, which is generated by our arc-fractal system, is a fractile is to show that four pieces of the Eisenstein fraction which tile together again form an Eisenstein fraction. In other words, the tile is self-similar. In our proof, we shall perform two types of operation on a sequence of labels at a particular level of construction $C = C_1 C_2 \dots C_n$. These operations are rotation and

Figure 2.7: Action of the rotation operator $\mathfrak{R}$.

mirror inversion. First, let us define the rotation operator $\mathfrak{R}$

$$\mathfrak{R}C = \mathfrak{R}(C_1 C_2 \dots C_n) = C'_1 C'_2 \dots C'_{n'} \quad (2.25)$$

where $C'_i \equiv C_i + 2 \pmod{12}$ (since $n = 3$ for the Eisenstein fraction). On the other hand, the mirror inversion operator $\mathfrak{M}$ is defined by

$$\mathfrak{M}C = \mathfrak{M}(C_1 C_2 \dots C_n) = C_n C_{n-1} \dots C_1 = \overline{C}. \quad (2.26)$$

Physically, operator $\mathfrak{R}$ rotates the segment associated with sequence C by an angle of $\frac{\pi}{3}$ counterclockwise (see Fig. 2.7), whilst operator $\mathfrak{M}$ inverses the labelling of a given sequence of a segment (see Fig. 2.8). Note that these two operators commute

$$\mathfrak{M}\mathfrak{R} = \mathfrak{R}\mathfrak{M}. \quad (2.27)$$

In addition, the following identities hold

$$\mathfrak{R}^6 = \mathfrak{I}, \quad (2.28a)$$

$$\mathfrak{M}^2 = \mathfrak{I}, \quad (2.28b)$$

where $\mathfrak{I}$ is the identity operator $\mathfrak{I}C = C$.

Figure 2.8: Action of the mirror operator $\mathfrak{M}$.

It is important to note that operator $\mathfrak{M}$ does not change the segment. It only inverts the labelling of the segment (from one end to another). The flipped sequence $\overline{C}$ and the original sequence C are associated with the same segment. The difference occurs when that segment is to be connected to another segment. For example, two sequences $C = C_1 C_2 \dots C_n$ and $D = D_1 D_2 \dots D_n$ can join at $C_n D_1$ to make a new sequence $E = CD = C_1 C_2 \dots C_n D_1 D_2 \dots D_n$. The reason for introducing $\mathfrak{M}$ is that during the construction of the Eisenstein fraction, in addition to the requirement of rotation, there is an occasional need to invert the sequences in order to obtain a correct sequence of labels that runs around the boundary of the Eisenstein fraction.

Since the Eisenstein fraction is constituted from three pieces of crab fractal, we shall begin our proof by considering the crab fractal. At level m , let the label of our crab fractal be the sequence $C_1 C_2 \dots C_{3^m}$ (see Fig. 2.9). Then, the next level can be constructed by putting the object at the previous level together (after some scaling, see Fig. 2.10). Note that

$$C'_i \equiv C_i + 8 \pmod{12}, \quad (2.29a)$$

$$C''_i \equiv C_i + 4 \pmod{12}. \quad (2.29b)$$

While there is no problem with the first and third segments, one must be careful with the middle segment. Let us denote the sequence of segments by A_1 , A_2 and A_3 . It can be clearly seen that $A_3 = \mathfrak{R}^2 A_1$. If we were to take $A_2 = \mathfrak{R}^4 A_1$, the new sequence of the object at level $m+1$ would be $A_1 A_2 A_3 = C_1 \dots C_{3^n} C'_1 \dots C'_{3^n} C''_1 \dots C''_{3^n}$. But this sequence is incorrect because the sequence is $A_1 A_2 A_3 = C_1 \dots C_{3^n} C'_{3^n} \dots C'_1 C''_1 \dots C''_{3^n}$, as observed from Fig. 2.10. Thus, the correct way is to take $A_2 = \mathfrak{M} \mathfrak{R}^4 A_1$ in order to get the correct sequence at level $m+1$. In summary, the crab fractal at level $m+1$ is associated with the sequence A_1, A_2, A_3 , which obeys the following functional relations

Figure 2.9: Sequence of labels of the crab fractal at a particular level of construction.

Figure 2.10: Labelling at the next level obtained through putting the set of arcs at the previous level together.

$$A_2 = \mathfrak{M}\mathfrak{N}^4 A_1, \quad (2.30a)$$

$$A_3 = \mathfrak{R}^2 A_1. \quad (2.30b)$$

With a piece of Eisenstein fraction being made up of three pieces of crab fractal, it can be easily deduced that the sequence for the boundary of the Eisenstein fraction at level $m + 1$ is $A_1 A_2 A_3 \overline{A_1 A_2 A_3}$ (see Fig. 2.11).

Next, let us examine the situation when four pieces of the $(m + 1)$ th level Eisenstein fraction tile together. Clearly, we observe that the sequence A_i of one piece is complementary to the sequence A_j of another piece at the position where they are to tile together. This matching enables all the four pieces to fit together, and the boundary of the new object (see Fig. 2.12) has the following sequence

$$A_1 A_2 A_3 \overline{A_1 A_2 A_3} \overline{A_1 A_2 A_3} \overline{A_1 A_2 A_3} \overline{A_1 A_2 A_3} \overline{A_1 A_2 A_3} \overline{A_1 A_2 A_3} \overline{A_1 A_2 A_3}. \quad (2.31)$$

Now, we will show that this sequence is precisely the boundary of the Eisenstein

Figure 2.11: Labels for the boundary of the Eisenstein fraction at the level $m + 1$.

fraction at level $m + 2$. If this sequence is indeed the boundary of the Eisenstein fraction at level $m + 2$ (see Fig. 2.13), it must take the form

$$B_1 B_2 B_3 \overline{B_1 B_2 B_3}, \quad (2.32)$$

with $B_2 = \mathfrak{M}\mathfrak{R}^4 B_1$ and $B_3 = \mathfrak{R}^2 B_1$. Since $B_1 = A_1 A_2 A_3$, we have

$$\begin{aligned} B_2 &= \mathfrak{M}\mathfrak{R}^4 (A_1 A_2 A_3) \\ &= \mathfrak{M} (\mathfrak{R}^4 A_1 \mathfrak{R}^4 A_2 \mathfrak{R}^4 A_3) \\ &= \mathfrak{M} (\mathfrak{M} A_2 \mathfrak{M} \mathfrak{R}^8 A_1 \mathfrak{R}^4 \mathfrak{R}^2 A_1) \\ &= \mathfrak{M} (\mathfrak{M} A_2 \mathfrak{M} A_3 A_1) \\ &= \mathfrak{M} A_1 \mathfrak{M}^2 A_3 \mathfrak{M}^2 A_2 \\ &= \overline{A_1} A_3 A_2, \end{aligned} \quad (2.33)$$

where we have made use of Eqs. 2.30a and 2.30b as well as the fact that $\mathfrak{M} A_2 = \mathfrak{M}^2 \mathfrak{R}^4 A_1 = \mathfrak{R}^4 A_1$. Similarly, we found that

$$\begin{aligned} B_3 &= \mathfrak{R}^2 (A_1 A_2 A_3) \\ &= \mathfrak{R}^2 A_1 \mathfrak{R}^2 A_2 \mathfrak{R}^2 A_3 \\ &= A_3 \overline{A_1 A_2}, \end{aligned} \quad (2.34)$$

Figure 2.12: The tiling of four pieces of the Eisenstein fraction.

and

$$\overline{B_1} = \mathfrak{M}B_1 = \mathfrak{M}(A_1A_2A_3) = \overline{A_3A_2A_1}, \quad (2.35)$$

$$\overline{B_2} = \mathfrak{M}B_2 = \mathfrak{M}(\overline{A_1}A_3A_2) = \overline{A_2A_3A_1}, \quad (2.36)$$

$$\overline{B_3} = \mathfrak{M}B_3 = \mathfrak{M}(A_3\overline{A_1}A_2) = \overline{A_2A_1A_3}. \quad (2.37)$$

which yield precisely Eq. (2.31). This shows that four pieces of Eisenstein fraction can tile together to form an Eisenstein fraction, thus proving that the Eisenstein fraction is a fractile.

Figure 2.13: Labels for the boundary of the Eisenstein fraction at level $m+2$.

Table 2.1: The combinations of α , n , and ω that give the classic fractals.

Classic fractals	α	n	ω
Koch	$\frac{2\pi}{3}$	2	(+1; +1)
Heighway	π	2	(-1; +1)
Lévy	π	2	(-1; -1)
Sierpiński	π	3	(+1; -1; +1)
Eisenstein*	π	3	(-1; +1; -1)

*Eisenstein fraction is created by joining three pieces of crab fractals together.

2.4 Comparison to L -system

Let us now consider some interesting special cases of fractals generated by our arc-fractal system. These fractals are created by an equal division of arc segments. Furthermore, their opening angle α is held fixed throughout the construction process. Interestingly, through a judicious selection of both n and α , we are able to reproduce many different classic fractals in our construction. A selection of these fractals is summarised in Table 2.1.

This characteristic of the arc-fractal system, where one system is able to produce many different fractals, is shared by the L -system. Indeed, one may immediately realise that the invariant set of points generated by the arc-fractal system is the same as the set of end points generated by the L -system. This may lead one to think that the arc-fractal scheme and the L -system represent two different approaches to describe the same fractal object. While in the language of the L -system, one is required to define a rule with many variables in accordance with a sequence of symbols to construct the fractal, one need to define only three variables to generate the arc-fractal. In other words, the arc-fractal system is a simpler system. Nonetheless, the mathematical objects created by the arc-fractal system and those by the L -system converge to the same set even though the generating sets are different (see Appendix A.2 for a proof). In other words, the arc-fractal system and the L -system possess the same attractor. A comparison of the classic fractals generated by these two systems will be discussed next.

a. Koch curve: $\alpha = \frac{2\pi}{3}$, $n = 2$, orientation in-in.

In this case, Eq. (2.11) becomes

$$\begin{cases} x_i^{m+1} = x^m + \frac{2R^m}{\sqrt{3}} \cos \left[\theta^m + \frac{(2i-1)\pi}{6} \right] \\ y_i^{m+1} = y^m + \frac{2R^m}{\sqrt{3}} \sin \left[\theta^m + \frac{(2i-1)\pi}{6} \right] \\ R_i^{m+1} = \frac{R^m}{\sqrt{3}} \\ \theta_i^{m+1} = \theta^m + \frac{(2i-9)\pi}{6} \\ \alpha_i^{m+1} = \frac{2\pi}{3} \end{cases}, \quad (2.38)$$

where $i = 1, 2$. According to Sec. 2.3.1, the set of invariant points generated by this arc-fractal system is given by

$$(x^m + R^m \cos \theta^m, y^m + R^m \sin \theta^m) \quad (2.39)$$

and

$$\left(x^m + R^m \cos \left(\theta^m + \frac{2\pi}{3} \right), y^m + R^m \sin \left(\theta^m + \frac{2\pi}{3} \right) \right). \quad (2.40)$$

If we shift the coordinate system by $\Delta x = -\frac{1}{2}$ and $\Delta y = \frac{1}{2\sqrt{3}}$, the set of invariant points becomes

$$\left(x^m + R^m \cos \theta^m + \frac{1}{2}, y^m + R^m \sin \theta^m - \frac{1}{2\sqrt{3}} \right) \quad (2.41)$$

and

$$\left(x^m + R^m \cos \left(\theta^m + \frac{2\pi}{3} \right) + \frac{1}{2}, y^m + R^m \sin \left(\theta^m + \frac{2\pi}{3} \right) - \frac{1}{2\sqrt{3}} \right), \quad (2.42)$$

which belong precisely to the conventional Koch curve. For example, by letting $\theta^0 = \frac{\pi}{6}, x^0 = 0, y^0 = 0, R^0 = \frac{1}{\sqrt{3}}$, we obtain the two points: $(1, 0)$ and $(0, 0)$ at the zeroth level. Subsequent points at the higher level can be similarly determined from the formula and they are $\left(\frac{1}{2}, \frac{1}{2\sqrt{3}} \right), \left(\frac{2}{3}, 0 \right), \left(\frac{1}{3}, 0 \right), \text{etc.}$

Figure 2.14 shows the Koch curve generated by the arc-fractal system and L -system at level 6. Note the circular points in Fig. 2.14(b) correspond to the invariant set of points of the arc-fractal. Similar comments apply to the fractals below.

b. Heighway dragon: $\alpha = \pi, n = 2$, orientation out-in.

Figure 2.14: Koch curve at level 6.

Figure 2.15 shows the Heighway dragon generated by the arc-fractal system and the L -system at level 5.

Figure 2.15: Heighway dragon at level 5.

c. *Lévy dragon*: $\alpha = \pi$, $n = 2$, orientation out-out.

Figure 2.16 shows the Lévy dragon generated by the arc-fractal system and the L -system at level 5.

d. *Sierpiński gasket*: $\alpha = \pi$, $n = 3$, orientation in-out-in.

Figure 2.17 shows the Sierpiński gasket generated by the arc-fractal system and the L -system at level 5¹.

e. *Eisenstein fraction*: $\alpha = \pi$, $n = 3$, orientation out-in-out.

¹To be precise, this is only the pattern of the Sierpiński gasket but not the gasket itself.

Figure 2.16: Lévy dragon at level 5.

Figure 2.17: Sierpiński gasket at level 5.

Figure 2.18 shows the Eisenstein fraction generated by the arc-fractal system and the L -system at level 5.

2.5 Generalisation to multifractal

The deterministic rules employed to generate arc-fractals can be generalised to create multifractals (Halsey *et al.*, 1986) through introducing an element of stochasticity in the construction. One of the simplest multifractal is the two-scale Cantor set discussed by Halsey *et al.* (1986). A multifractal can be constructed by associating every scaling parameter l_i with a probability measure p_i . Thus, multifractals can be generated

Figure 2.18: Eisenstein fraction at level 5.

from the arc-fractal system by incorporating probability measures into the formalism (see Appendix A.3). In other words, by endowing the scaling parameters n , s and α with the respective probability measures $p_1(n)$, $p_2(s)$ and $p_3(\alpha)$, a multifractal can be constructed. The idea is that for the many arc elements that arise at each construction step, there is a probability $p_1(n)$ that these elements will be divided into n segments, a probability $p_2(s)$ that the divided arc segments will take a scale s , and a probability $p_3(\alpha)$ that the divided arc segments will have an opening angle α . The specific detail of the construction will be reported elsewhere. [OPEN PROBLEM](#) see list on page 51

2.6 More examples of arc-fractals

So far we have seen that arc-fractal can generate classic known fractals using single-rule, *i.e.* the same rule over every iteration. Here we illustrate that many more fractals can indeed be generated using the arc-fractal system with multiple-rule.

[OPEN PROBLEM](#) see list on page 51

2.6.1 Sierpiński carpet

The Sierpiński carpet is a well-known fractal in two dimensions which is the square version of the celebrated Sierpiński gasket. Using the arc-fractal system with multiple-rule, it is illustrated that this fractal can also be obtained using arcs.

Figures 2.19 and 2.20 show the first eight iterations (zeroth to seventh) in constructing the fractal. The procedure starts with an arc of opening angle $\alpha = \frac{3\pi}{2}$ at zeroth iteration $m = 0$. Next, the initial arc is divided into three equal segments. Each

Figure 2.19: First four iterations in constructing the Sierpiński carpet using arc-fractal system. The black dots indicate the beginning and end of each arc segment. They belong to the invariant set of points of the final fractal, see Sec. 2.3.1.

segment is then replaced by a new arc of angle $\alpha_1 = 2\theta^* + \frac{\pi}{2}$ ($\theta^* = \arcsin\left(\frac{1}{\sqrt{5}}\right)$) with orientations all in. Next, each new arc is divided into three segments with ratio $\frac{4\theta^*}{\pi + 4\theta^*} : \frac{\pi - 4\theta^*}{\pi + 4\theta^*} : \frac{4\theta^*}{\pi + 4\theta^*}$. The newly divided segments are respectively replaced by new arcs of angle π , $\frac{3\pi}{2}$ and π with orientation out, in and out. Next, the two new arcs of angle π are divided into two equal segments, each replaced by arcs of angle $\frac{3\pi}{2}$; while the new arc of angle $\frac{3\pi}{2}$ is kept as is (it could be viewed as being divided

Figure 2.20: Next four iterations in constructing the Sierpiński carpet using arc-fractal system. The black dots indicate the beginning and end of each arc segment. They belong to the invariant set of points of the final fractal, see Sec. 2.3.1.

into one segment and replaced by arc of the same angle $\frac{3\pi}{2}$). These steps constitute the first four iterations. At this stage, all the arcs are of angle $\frac{3\pi}{2}$. And the procedure repeats. This is a process with multiple-rule because three (in this case) different rules are applied at different iteration. However, it is still a deterministic process because the set of these three rules repeats. Hence, after every three iterations, the shape of the object looks statistically the same.

Figure 2.21: Eighth iteration in the construction of the Sierpiński carpet using arc-fractal system showing a clear (statistical) pattern of the Sierpiński carpet. The black dots indicate the beginning and end of each arc segment. They belong to the invariant set of points of the final fractal, see Sec. 2.3.1.

In the first few iterations, it is not obvious to recognise the pattern of the Sierpiński carpet. However, at eighth iteration, the pattern emerges and one can easily see it in Fig. 2.21.

Figure 2.22: First sixteen rows of the Pascal triangle.

2.6.2 Fractals from the Pascal triangle

There is a very surprising and interesting connection between the famous Pascal triangle and the Sierpiński gasket¹. As is well-known, the Pascal triangle is a triangle of numbers whose entries are the coefficients of the binomial expansion. This triangle indeed contains many rows of numbers with the first row containing a single entry 1. The number of entries contained in each row increases by one as one goes down to the next row. Each row begins and end with entries 1, making the two sides of the

¹I couldn't trace down to the original publication(s) on this matter. Yet, there are *many* useful links over the internet. <http://math.rice.edu/~lanius/fractals/pas2.html> is one of them.

Figure 2.23: First sixteen rows of the Pascal triangle with even sites (based on modulus 2) being shaded showing the pattern of the Sierpiński gasket at fourth iteration.

triangle filled with 1's. Besides the first and last entry 1, each of the remaining entries of a row is the sum of two adjacent entries (including the two 1's) in the previous row. Figure 2.22 shows the first sixteen rows of the Pascal triangle.

Patterns in Pascal triangle can be obtained by removing entries that satisfy certain rules. One such rule is the modulus rule. And surprisingly, the pattern based on modulus 2 yields the Sierpiński gasket! Figure 2.23 shows the Pascal triangle with even entries being shaded. One can easily recognise the pattern of the Sierpiński gasket.

One can certainly change the base of the modulus and obtain other patterns in the

Figure 2.24: First sixteen rows of the Pascal triangle with multiple-of-3 sites (modulus 3) being shaded showing fractal pattern.

Pascal triangle. One popular pattern is modulus 3 shown in Fig. 2.24.

One may now start wondering: What do all these have to do with the arc-fractal system? The answer is straightforward: The arc-fractal system can generate these fractals too! We have already seen how the Sierpiński gasket can be generated using single-rule with the appropriate values of the parameters. Patterns like the one in Fig. 2.24 are more complicated and hence require multiple-rule in the arc-fractal system. Nevertheless, this is possible and the steps are shown in Figs. 2.25 and 2.26.

Figure 2.25: First four iterations in constructing the Pascal triangle modulus 3 in Fig. 2.24 using arc-fractal system. The black dots indicate the beginning and end of each arc segment. They belong to the invariant set of points of the final fractal, see Sec. 2.3.1.

2.6.3 More arc-fractals

The arc-fractal system with the parameters are very rich in generating fractals. Many more fractals can be obtained by combining the rules and tuning the parameters. Some examples are shown in Figs. 2.27 and 2.28. Some of them are later used as fractal lattices in Part II for the study of Self-Organized Criticality.

Figure 2.26: Later four iterations in constructing the Pascal triangle modulus 3 in Fig. 2.24 using arc-fractal system. The black dots indicate the beginning and end of each arc segment. They belong to the invariant set of points of the final fractal, see Sec. 2.3.1.

(a) Quadratic Koch curve

(b) Semi-inverse square triadic Koch curve

(c) Another version of semi-inverse square triadic Koch curve

Figure 2.27: Some more fractals generated using arc-fractal system. Rules are based on the Koch curve and its variants (Addison, 1997).

(a) Crabarro 1

(b) Crabarro 2

Figure 2.28: Some more fractals generated using arc-fractal system. Rules are combinations of those used in Figs. 2.17 and 2.18 (see also Appendix B).

Chapter 3

Application of arc-fractal system

The arc-fractal system introduced in Chapter 2 finds its usage in generating the fractal lattices for studying lattice models (the sandpile model in the second part of this thesis is one of them), beside generating some nice-looking fractals. It can also be used in studying real physical objects like coastline. In this chapter, a heuristic model of coastal morphology is introduced based on some simple assumptions about the processes of erosion and deposition in coastline formation. The arc-fractal system can then serve as a dynamical model of coastal morphology, with each level of its construction corresponding to the time evolution of the shape of the coastal features. These results indicate that the arc-fractal system can provide an explanation on the origin of fractality in real coastlines.

3.1 A dynamical model of coastal morphology

In this section, we shall employ the arc-fractal system to model the morphological evolution of real coastline. The main idea is that coastal erosion and accretion can be modelled through the element of an arc.

In coastal studies, it has been shown that the two fundamental processes that shape the coastline are: (i) erosion by the sea waves, and (ii) accretion of sediment or sand (Bird, 2008; Sapoval *et al.*, 2004). In our model, we shall assume that the effect of the sea waves hitting the coast is a smooth curve that can be approximated as an arc, in a statistical sense.

While we will be investigating the idea of multifractal in the future, we have in this work explored into the effect of randomness in a model of coastline formation based on the arc-fractal approach. The study of coastal geomorphology, *i.e.* the natu-

ral shape of the coastline, requires an understanding on the processes that influence its time evolution as the coast interacts with its surrounding marine environment. In this respect, we have associated the fundamental processes of erosion and deposition, which are important drivers of coastline formation and evolution, with our arc-fractal model. With some basic assumptions, we shall show that our model can well illustrate the course of coastal evolution and how a fractal coastline is formed. We deem that our approach provides an explanation on the origin of fractality in real coastlines, which have yet to be attained by other models.

The size of this arc (its opening angle α and radius r) depends on how strong and how big the wave is, *i.e.* the power of the wave and the area in which it is activated. Since these characteristics of the wave vary along the coast, different parts of the coast will be eroded and deposited differently. We shall model this effect by arcs of different sizes. (The orientation of the arc reflects the physical result of deposition and erosion.) Thus, we shall represent the resulting coastline by an arc-fractal which is constructed by means of the following parametrisation: $\alpha = \alpha(i, m)$, $r = r(i, m)$ and $n = n(i, m)$, where i is the position along the coast and m is the evolution time.

Indeed, one can imagine that the construction of the arc-fractal at each stage according to the requirement above is equivalent to one time step in the evolution of the coastline. We thus anticipate arcs of smaller sizes at the later stages of the evolution. This implies that after a long period of evolution, the shape of the coastline changes less drastically at the longer length scales. This can be understood from the fact that the soil has become harder as time goes by and also the irregularity of the coasts has dampened the eroding power of the sea waves. The result of this coastal evolution is that the coastline will eventually adopt the morphology of a fractal.

3.2 Results and discussions

Figure 3.1 illustrates the evolution of a part of the coastline through the construction of the arc-fractal as described above. As an illustration, in our model, α and n are random variables obeying the uniform distribution. They are constrained within the range

$$\begin{cases} \frac{\pi}{3} \leq \alpha \leq \pi \\ 2 \leq n \leq 6 \end{cases} \quad (3.1)$$

Figure 3.1: Evolution of a part of the coastline.

with r dependent on α . Figures 3.2 and 3.3 demonstrate that this model is able to generate coastlines that are realistic looking.

Figure 3.2: A generated coastline in the form of an island.

The assumption that coastline formation is solely attributed to the process of erosion and accretion may be too simplistic to model the more complicated processes involved in the shaping of natural coastline. For example, the transportation of sediments along the coastline due to currents is a process that is too important to be ignored (Sapoval *et al.*, 2004). However, the idea of the model will be true in certain

Figure 3.3: More coastlines generated by the arc-fractal system.

coastal regions in the world, and it is especially fruitful in the context of an island or a coastal zone that is small relative to the size of a continent¹. Additional refinement to the model may require the introduction of more advanced components such as chaos theory and the concept of self-organized criticality (Baas, 2002). The current model can also be made more realistic by imposing deterministic rules or by incorporating information obtained from real data of natural factors that affect the sea waves like winds, nearshore sea floor, *etc...* Nevertheless, it is important to emphasise that the simplicity of the current model does capture the main features of generic coastal geomorphological evolution. **OPEN PROBLEM** see list on page 52

¹Thanks to Shigeo Yoden and Keiji Takemura for comments and discussions on this.

Summary I

A novel scheme to generate fractals based on an arc segment has been proposed in this work. We have shown that by using an arc element instead of a line segment in the conventional way of constructing fractals, various types of fractal objects can be obtained. This includes the five classic fractals which were discovered earlier by different methods. We have also discussed on the properties of the arc-fractal system, with its interesting features of generating nontrivial sequences and the tiling of two-dimensional space. Some of the fractals generated here are used as fractal lattices in Part II for the study of Self-Organized Criticality. In addition, we have extended the idea of the arc-fractal to the physical situation of coastline formation. The proposed model is simple and is based on the fundamental physical mechanisms of coastline formation — erosion and deposition. The results obtained are promising with the generated coastlines look realistic compared to the coastlines in the real world. Nonetheless, more sophisticated judgements are required to bring real data of natural factors into the modelling effort to make further improvement on the results. This effort will be reported in future publications. **OPEN PROBLEM** see list on page 52

Open problems

Below is a list of open problems that one can look further into.

- Qualifying the pseudorandom sequences generated by the arc-fractal system. see page 23
- Dimension of the arc-fractals with overlapping, multiple rules. see page 35
- Generalising the current arc-fractal system to higher-dimensional Euclidean space. see page 13

- A concrete method to generate multifractal based on the arc-fractal system. see page [35](#)
- Real model of coastline formation based on the arc-fractal system incorporating data on the processes taking place, *e.g.* the amount of deposition or erosion. see page [50](#)
- More applications of the arc-fractal system and its significance in other disciplines that are related to Fractal. see page [51](#)

Part II

Self-Organized Criticality

Chapter 4

A review of Self-Organized Criticality

“Who could ever calculate the path of a molecule? How do we know that the creations of worlds are not determined by falling grains of sand?”

—Victor Hugo, *Les Misérables*
(after Bak, 1996)

“Self-organized criticality is a new way of viewing nature. The basic picture is one where nature is perpetually out of balance, but organized in a poised state—the critical state—where anything can happen within well-defined statistical laws. The aim of the science of self-organized criticality is to yield insight into the fundamental question of why nature is complex, not simple as the laws of physics imply.

Self-organized criticality explains some ubiquitous patterns existing in nature that we view as complex. Fractal structure and catastrophic events are among those regularities. Applications range from the study of pulsars and black holes to earthquakes and the evolution of life. One intriguing consequence of the theory is that catastrophes can occur for no reason whatsoever. Mass extinction may take place without any external triggering mechanism such as a volcanic eruption or a meteorite hitting the earth (although the theory of course cannot rule that this has in fact occurred).”¹ (Bak, 1996)

¹precisely quoted

4.1 Self-organized criticality in general

Besides puzzles of complex spatial structures like fractal, as seen in the previous part of this thesis, scientists are also confronted with the abundance of scale-invariant spatio-temporal structures in Nature. In an attempt to provide explanation for this phenomenon, in 1987, Per Bak, Chao Tang and Kurt Wiesenfeld suggested the so-called Self-Organized Criticality ([Bak et al., 1987](#)). This is a theory about complex systems, with overwhelmingly large number of internal degrees of freedom interacting with each other, that self-organise into the critical state characterised by the ubiquitous power-laws. The fact that those systems self-organise is the absence of explicit tuning of any parameters to reach the critical state ([Tang and Bak, 1988](#)). That is different from classical critical phenomena in which tuning parameter like temperature is the key factor to drive the systems into the critical state.

Since being introduced by Bak, Tang and Wiesenfeld, Self-Organized Criticality has been the most popular subject within Complexity to explain the emergent natural phenomena. Originally started in Statistical Physics, a branch of Physics, SOC has gradually found its significance in other branches of Physics, and soon gone way beyond the realm of physics research. Nowadays, SOC is employed by researchers from many different disciplines in studies ranging from natural to social phenomena like earthquake, biological evolution, collapse of societies, *etc.*... Bak, Tang and Wiesenfeld's seminal paper on the subject has now been cited more than 3400¹ times by articles in a vast number of fields.

4.1.1 Notion of complexity

Self-Organized Criticality has become the most popular subject within Complexity or Complex Systems for it proposed models with very simple mechanisms, yet can generate many emergent behaviours found in natural phenomena. Complexity or the Science of complex systems is about studying systems that are complicated in structure with a huge number of individual components, each contributes to the so-called emergent behaviours of the entire systems. The emergent behaviours refer to the patterns or things that can be observed, measured at macroscopic level. There has not been a well-established definition of complex systems, yet it is widely accepted that complex systems possess some key characteristics ([West, 2011](#)) listed below.

¹Figure is taken from ISI Web of Knowledge, as of June 2012.

- Many components.
- Many individual actors/agents.
- Multiple spatial and temporal scales.
- Strongly coupled/interacting.
- Nonlinear.
- Sensitivity to boundary conditions (chaos).
- Emergent phenomena/multiple phases.
- Unintended consequences.
- Adaptive/evolving.
- Historically contingent/path dependent.
- Robust/resilient.
- Nonequilibrium.
- Underlying simplicity.
- Complicated vs. complex.

From this list, one sees that many features of complex systems are well captured by SOC, like many components interacting with one another giving rise to emergent phenomena which in this case are manifested by the existence of power-law behaviours characterising the critical state of the systems. In Nature, one can easily find many examples of such systems like the Atmosphere or Earth's crust. Table 4.1 illustrates the analogy between the rain events, earthquakes and the sandpile metaphor (Peters *et al.*, 2001; Christensen and Moloney, 2005).

4.1.2 Relation to classical critical phenomena

The term “self-organized criticality” is understood in the sense of two components: “self-organized” and “criticality”. “Criticality” refers to the critical state that the system evolves into. This critical state is perceived by the lack of a characteristic length scale, *i.e.* all length scales are relevant at the critical point. System in this state can generate events of arbitrary sizes, from very large to very small. A small perturbation to

4.2 Ubiquitous power-law and features of an SOC system

Table 4.1: Comparison between different natural systems like Atmosphere or Earth's crust and a simple sandpile model emphasising key features of a self-organized critical system.

Phenomenon	Rainfall	Earthquake	Sandpile
System	Atmosphere	Earth's crust	Granular pile
Driving	slow and constant energy input from the Sun (water is evaporated from the oceans)	slow and constant currents movement in the liquid core of the Earth	adding particles to random position
Energy storage	water vapour in the atmosphere	tension building up between the tectonic plates	column of particles at every site
Avalanche	bursts when the vapour condenses to water drops	sudden movements of the ground	chain updates of sites

the system can lead to a catastrophic event. In the context of ordinary critical phenomena, systems like the Ising model (Onsager, 1944) or percolation (Stauffer and Aharony, 1994) require a parameter to be tuned to some critical value in order for the lack of length scale (divergence of correlation length in the Ising model or (expected) cluster size in percolation) to take place. The fundamental difference between the critical state of systems in self-organized criticality and those in ordinary critical phenomena is the absence of any control parameter in the former. In other words, self-organized critical systems *reach the critical state without external fine tuning of control parameters*. In the case of Ising model, the parameter is the temperature of the system. In the case of percolation, the parameter is the probability that a site is occupied. There is, however, no such parameter available in SOC systems.

4.2 Ubiquitous power-law and features of an SOC system

In Nature, one can easily find many different systems that exhibit power-law behaviour but not all of them are considered SOC, *e.g.* diffusion limited aggregation, invasion percolation, turbulence *etc.*... Power-law distribution of event sizes is a necessary condition to signify the critical state of the system but not a sufficient one. There are many mechanisms that can give rise to power-law distribution in a system (Sornette, 2006, chapter 14) (also Pruessner, 2012, chapter 9). The following criteria must

be met by a system in order for it to be classified as an SOC system.

- Involve many individual elements interacting with one another through local interactions (lattice model).
- Driving mechanism (driven by slow constant energy input).
- Threshold dynamics for every individual elements (mechanism to accumulate, store energy in the system, and release once the threshold is reached).
- Dissipative mechanism (relaxation of the system so that it can get back to stable state after being excited, perturbed).
- Separation of timescale between driving and relaxation (relaxation takes place at a much faster timescale than that of driving).

4.3 Avalanche dynamics in sandpile models

A common feature of SOC systems is the existence of long-range correlation generated by short-range (nearest-neighbour) interaction through the spreading of chain updates of events, *i.e.* avalanche. In this section, we make a brief review of key models in SOC literature¹ that play important rôle in developing and shaping the entire field since the seminal work of Bak *et al.* (1987, 1988); Bak (1990).

Inspired by the original model of Bak *et al.* (1987), many authors have devised different types of model on lattices to further investigate the self-organized critical phenomena. These models involve discrete individuals or units, usually referred to as “particles”, and they constitute a group known as sandpile or granular models. This group of models is by far the most studied one in SOC literature.

4.3.1 Bak-Tang-Wiesenfeld model

The very first sandpile model is, of course, the celebrated model of Bak, Tang and Wiesenfeld. In their seminal paper, Bak *et al.* (1987) used a deterministic cellular automaton model of sandpile which was defined on a two-dimensional square lattice. Later, the model is improved by different authors making it easier to study both analytically and numerically. Dhar (1990) introduced his version of the model, pointing

¹A very well written description of the models and their key features can be found in (Pruessner, 2012).

out its Abelian property (see below). Dhar's version nowadays is considered a proper presentation of BTW model that allows identification of observables without ambiguity¹. This version is commonly known as **Abelian sandpile model (ASM)**. Throughout this work, we discuss **ASM** rather than **BTW** model.

On the lattice, an integer variable $h(x, y)$ is associated with each lattice point (x, y) . This integer variable has a global threshold $h_c = 4$ (the choice of this threshold indeed doesn't affect the dynamics of the model). If the variable at one site exceeds this threshold, that site will topple and the neighbouring sites are updated according to the following rule

$$\begin{aligned} h(x, y) &\rightarrow h(x, y) - 4, \\ h(x \pm 1, y) &\rightarrow h(x \pm 1, y) + 1, \\ h(x, y \pm 1) &\rightarrow h(x, y \pm 1) + 1. \end{aligned}$$

Once the system is in stable configuration, *i.e.* $h(x, y) \leq 4 \forall x, y$, a random small perturbation is applied. That is increasing $h(x, y)$ by one unit at some randomly chosen site (x, y) . It can be seen that the small perturbation will induce a contagious effect called avalanche. The picture is that the neighbours of (x, y) on receiving one unit from (x, y) may exceed the threshold $h_c = 4$ and therefore topple. And the dynamics continues. The Abelian property of the model is that when there are multiple active sites ($h(x, y) > h_c$), the order of sequential relaxations does not affect the dynamics of the model. The process from randomly adding one unit to the total relaxation of all sites is called an avalanche. There are four observables in the system, two of which, cluster size and lifetime, were defined by **Bak *et al.* (1987)** and the other two, toppling size and cluster radius, were defined soon later. In nowadays standard literature, the observables are termed avalanche size, duration, area, and radius. The avalanche size is the toppling size. It counts the total number of relaxation events during an avalanche. The avalanche duration is the number of parallel updating steps during an avalanche. The avalanche area is the cluster size as defined by **Bak *et al.* (1987)**. It is determined by the number of distinct sites visited by an avalanche. And the avalanche radius is the linear size of the cluster formed during an avalanche. The distributions of those observables follow power-law pattern with cutoff near the end (see Fig. 4.1 and more details in Sec. 5.4).

¹BTW model was defined based on height difference between neighbouring sites on the lattice, while Dhar's model on the absolute height itself.

Figure 4.1: Distribution of cluster sizes s and lifetimes t at critical state in two and three dimensions. Figures adapted from (Bak *et al.*, 1987).

4.3.2 Manna model

Manna model, originally published by Manna (1991b) to extend the universality class of BTW model, is a stochastic sandpile model. Similarly to BTW model, original model of Manna does not possess Abelian symmetry. Later Dhar (1999a,c) introduced the Abelian version of this model, *i.e.* the AMM, making it easier and more efficient to perform the numerical simulation. The emphasis of this model is on its stochastic dynamics in contrast to deterministic dynamics of BTW model.

In this model, the critical height at each lattice site is only 1 regardless of its number of nearest neighbours. On toppling, an unstable site releases its particles randomly and independently to its nearest neighbours.

4.3.3 Oslo model

Oslo model was inspired by a ricepile model performed by [Frette *et al.* \(1996\)](#) (see also [Christensen *et al.*, 1996](#); [Paczuski and Boettcher, 1996](#)) at the University of Oslo (hence the name). Similarly to Manna model, there are two versions of the Oslo model, with and without Abelian property. The original version generally does not possess Abelian property¹ and is well-defined in one dimension only. The simplified Oslo model (hereafter referred to as *the* Oslo model) comes in full Abelian property of the original version with simplified boundary conditions and is well-defined in any dimension. The Oslo model, which is also a stochastic sandpile model, however, is a more closely related to [BTW](#) model than the Manna model. The critical height at a site is chosen to be either $(q - 1)$ or $(2q - 2)$ (q is the number of nearest neighbours of that site) with equal probability. On toppling, an unstable site transfers one particle to each of its nearest neighbours and a new critical height is again chosen to be either $(q - 1)$ or $(2q - 2)$ with equal probability. This way, one sees that the distribution of particle in the Oslo is the same as in the [BTW](#) model. The stochasticity enters in the choice of new critical height after each toppling.

4.3.4 Other granular models

Other granular models include the [OFC](#) model and the Zhang model. [OFC](#) model is a model proposed by [Olami *et al.* \(1992\)](#). It is a nonconservative model that later invoked the rôle of conservation in [SOC](#). The Zhang model was first published by [Zhang \(1989\)](#) to extend the universality class of [BTW](#) model. It is a continuous energy model.

4.4 Developments and current state of art

In the early days, *i.e.* end of 1980s beginning of 1990s, the motivation for [SOC](#) was to explain (and understand) the ubiquitous scaling phenomena found in Nature. There have been several main themes of research in [SOC](#).

- Origin of $1/f$ noise (*e.g.* [Bak *et al.*, 1987](#); [Jensen *et al.*, 1989](#); [Bak, 1990](#); [Jensen, 1990](#); [Christensen *et al.*, 1991](#)).
- Origin of simple scaling, power-law (*e.g.* [Levy and Solomon, 1996](#); [Dickman *et al.*, 1998](#)).

¹It is, however, Abelian when not driven in bulk ([Dhar, 2004](#); [Pruessner, 2012](#)).

- Rôle of conservation, isotropy (e.g. Kadanoff *et al.*, 1989; Grinstein *et al.*, 1990; Christensen and Olami, 1992; Olami *et al.*, 1992; Tsuchiya and Katori, 1999a; Bonachela and Muñoz, 2009).
- Universality class of BTW and Manna model (e.g. Ben-Hur and Biham, 1996; Vespignani *et al.*, 1995; Chessa *et al.*, 1999; Lübeck, 2000).
- Proper scaling behaviour of BTW model (e.g. Ali and Dhar, 1995a,b; De Menech *et al.*, 1998; Priezhev and Sneppen, 1998; Tebaldi *et al.*, 1999).
- Abelian versus non-Abelian property (e.g. Milshtein *et al.*, 1998; Hughes and Paczuski, 2002; Pan *et al.*, 2005; Zhang *et al.*, 2005; Jo and Ha, 2010).
- Realisation of theoretical sandpile models (e.g. Nagel, 1992; Grumbacher *et al.*, 1993; Frette *et al.*, 1996).
- Numerical techniques in analysing SOC models (e.g. Dickman and Campelo, 2003; Pruessner, 2004b).
- Universality class of Manna model and DP (e.g. Mohanty and Dhar, 2002; Bonachela *et al.*, 2006; Bonachela and Muñoz, 2007; Mohanty and Dhar, 2007; Bonachela and Muñoz, 2008; Basu and Mohanty, 2009; Basu *et al.*, 2012).
- Analytical approaches to SOC (field theory (e.g. Paczuski *et al.*, 1994), mean-field theory (e.g. Tang and Bak, 1988; Flyvbjerg *et al.*, 1993; Vergeles *et al.*, 1997; Zapperi *et al.*, 1995; Vespignani and Zapperi, 1998), renormalisation group (e.g. Pietronero *et al.*, 1994; Diaz-Guilera, 1994; Vespignani *et al.*, 1995; Lin, 2010), branching process (e.g. Manna, 2009), Langevin description (e.g. Grinstein *et al.*, 1990; Paczuski and Bassler, 2000a; Vespignani *et al.*, 2000; Bonachela and Muñoz, 2009), operator approach (e.g. Dhar, 1999b,a, 2004, 2006; Sadhu and Dhar, 2009; Pruessner, 2004a)).
- Finding evidences of SOC in natural (e.g. earthquake (e.g. Turcotte, 1992), rainfall (e.g. Peters *et al.*, 2001), superconductor (e.g. Zaitsev, 1992)) and social (e.g. banking networks (e.g. Aleksiejuk *et al.*, 2002), epidemic spreading (e.g. Dorogovtsev *et al.*, 2008)) systems.

An enormous amount of numerical works has been done to study SOC. From the original sandpile model of Bak *et al.* (1987), different models have been proposed with

more and more properties: Dhar's [directed Abelian sandpile model \(DASM\)](#) ([Dhar and Ramaswamy, 1989](#)), Manna two-state model ([Manna, 1991b](#)), [OFC](#) earthquake model ([Olami *et al.*, 1992](#)), Drossel and Schwabl's [FFM](#) ([Drossel and Schwabl, 1992](#)), *etc...* (see, for example, ([Jensen, 1998](#); [Dhar, 2006](#)) for reviews).

The Abelian nature and directed toppling rule of Dhar's model helped him solve exactly the critical states. In this version, the toppling is performed in a certain direction, not to all nearest neighbours like in the original version. Furthermore, Dhar realised that the order of toppling, *i.e.* order of updating the nearest neighbour, does not affect the dynamics of the system. The [OFC](#) earthquake model introduced the nonconservation to [SOC](#) and the importance of conservation was discussed as a key factor to criticality of the model. The power-law behaviour depends strongly on the level of conservation. The rôle of conservation has also been discussed by [Bonachela and Muñoz \(2009\)](#). They suggest that the mechanism of power-law in nonconservative systems is different from that in conservative system. The [FFM](#) brought the attention to the link with percolation theory. A large number of methods in studying percolation ([Stauffer and Aharony, 1994](#)) can be applied to investigate [SOC](#) in [FFM](#) *etc...*

On the theoretical ground of [SOC](#), there have been several attempts. They comprise of a few main approaches: stochastic nonlinear equations, group theory, renormalisation group and field theory. The stochastic nonlinear equations approach was proposed by [Diaz-Guilera \(1992\)](#); [Hwa and Kardar \(1989\)](#). From the dynamic microscopic rules, they obtained a set of nonlinear equations with different sources of noise, which they called internal and external. And different correlations of the noise give rise to different critical behaviours. The group theory proposed by [Dhar \(1990\)](#); [Dhar and Majumdar \(1990\)](#); [Majumdar and Dhar \(1992\)](#) deals with the Abelian nature of the sandpile model, which enables it to be solved exactly. The method involves driving the sandpile models in terms of operators acting on a configuration space. The [RG](#) method introduced by [Pietronero *et al.* \(1994\)](#); [Vespignani *et al.* \(1995\)](#) took the spirit of [RG](#) theory in classical critical phenomena to apply to [SOC](#) systems. This [RG](#) scheme allows one to characterise the critical state and the scale-invariant dynamics in sandpile models and can be extended to other models as well. The attractive fixed point in the scheme clarifies the nature of self-organisation in [SOC](#) systems. Universality classes can be identified and the critical exponents can be computed analytically. And finally, [Vespignani and Zapperi \(1998\)](#) proposed a unified dynamical mean-field theory for stochastic [SOC](#) models.

There are also other approaches like mapping SOC to absorbing state phase transition (Dickman *et al.*, 1998; Vespignani *et al.*, 2000; Christensen *et al.*, 2004; Pruessner and Peters, 2006) or considering it as a branching process (Christensen *et al.*, 1993; Christensen and Olami, 1993; Paczuski *et al.*, 1994; Zapperi *et al.*, 1995; Manna, 2009).

As the nature of critical phenomena, classifying critical behaviours of different SOC systems is an important issue that has been attracting a large number of articles in literature. Universality classes of many models are still very much in dispute (for example, see (Bonachela and Muñoz, 2008)). Among them, the most severe one is between models with deterministic toppling rule (BTW model, also ASM, is the representative) and models with stochastic toppling rule (Manna two-state model, also AMM, is the representative). Different studies report different set of exponent values leading to claim that these two models are in the same class or not in the same class (Ben-Hur and Biham, 1996; Chessa *et al.*, 1999). Vespignani *et al.* (1995) using RG approach claimed that BTW and Manna models are in the same class. They showed that the exponents of the model converge to the same fixed point in their RG scheme. This finding was later supported by Chessa *et al.* (1999), who used moment analysis of the distributions of observables suggested by De Menech *et al.* (1998) to show that exponents of these models do not differ much. However, this argument was soon rejected by Lübeck (2000), who re-performed the moment analysis for the two models and claimed that the exponents differ significantly. On the other hand, it has been clearly established that directed percolation (percolation with preferred direction) and Manna model (also C-DP) constitute two different universality classes (Lübeck, 2004; Dornic *et al.*, 2005). Recently, Bonachela and Muñoz (2007) proposed a way to discriminate between these two classes. And many different lately proposed sandpile models are shown to belong to the C-DP class (Bonachela and Muñoz, 2008; Bonachela *et al.*, 2006).

On the other hand, fractal is one of the important features in SOC systems because of the scale-invariant property. The power-laws found in critical phenomena say that the dynamics of the system at all scales is the same and the details of the system can be captured by a few exponents. However, the other way around — how fractal structures affect SOC behaviours — was rarely studied. Kutnjak-Urbanc *et al.* (1996) studied the sandpile model on the Sierpiński gasket fractal by means of numerical simulation. They found that the avalanche size distribution shows power-law behaviour modulated by logarithmic oscillations. They suggested that the discrete scale-invariance of the underlying fractal lattice may be responsible for this behaviour. Daerden and Van-

derzande (1998); Daerden *et al.* (2001) performed further extensive simulations of the model and employed the RG for Potts model on Sierpiński gasket to calculate the exponents of the distribution functions of wave size. Recently, Lee *et al.* (2009) study Bak-Sneppen model (Bak and Sneppen, 1993), an extremal dynamics model that exhibits SOC, on the same fractal lattice by numerical simulation. They also observe periodic oscillation in the distributions.

Overall, the main themes of research in SOC centralise about determining critical behaviours in different types of model, identifying the universality classes of these models, and attempting to solve the models analytically with general approaches. Also, studying patterns of cluster formed during avalanches is interesting as this shows the fractal structures generated by very simple evolution rules (Ostojic, 2003; Dhar *et al.*, 2009). Other possible directions and unsolved open problems in the field were recently pointed out by Dhar (2006). They include analytical studying of time-dependent correlation functions in sandpile models. Logarithmic conformal field theory (Jeng, 2005; Moghimi-Araghi *et al.*, 2005) may also be useful in investigating avalanche exponents of undirected BTW model in two dimensions.

Chapter 5

Abelian Manna model

After Bak et al. (1987) introduced the BTW model in 1987, researchers in the field of Statistical Mechanics started looking into other models of similar type (sandpile models) but with different dynamical rules. Many of them have later become important models that contribute towards the understanding and advancement of Self-Organized Criticality (refer to Sec. 4.3). One of such models is the Manna model which was introduced by Manna (1991b) to extend the universality class of the BTW model. This model, along with Oslo model (Frette et al., 1996; Christensen et al., 1996; Paczuski and Boettcher, 1996), is one of the few true SOC models that display robust and solid scaling behaviours. Several years later, Dhar (1999a,c) made an improvement to the original Manna model, pointing out it is not Abelian. The Abelian version of Dhar allows the numerical study of the model more efficient while doesn't change its critical behaviour (Dickman and Campelo, 2003) (this allows access to analytical treatments, too, see Sec. 8.4.2 (also Pruessner, 2004a)). In this chapter, the Abelian Manna model is introduced, making the basis for further investigation later.

5.1 Definition

The Abelian Manna model (Manna, 1991b; Dhar, 1999a,c) is defined on a lattice $\mathcal{L}$ with N sites. Each site i on $\mathcal{L}$ has q_i neighbours, and is assigned to it a non-negative integer variable h_i which can be thought of as the number of particles at that site or the height of a stack of particles there. The threshold height at all sites is 1 above which a site is said to be active or unstable, otherwise it is stable. The system evolves as follows.

- *Driving*: When the system is in a stable or “quiescent” configuration, i.e. $h_i \leq 1$ for all sites, the system is charged by picking a site i at random and incrementing

Figure 5.1: Relaxation scheme of the Abelian Manna model on part of a one-dimensional chain lattice involving three sites. The top diagram shows the initial configuration with middle site being unstable. The (small) arrows show possible directions that the particles can move. The three bottom diagrams show possible configurations in the next time step. There are three possibilities: both particles go to the left site, one particle goes to left site and the other one goes to right site, and both particles go to the right site. In the isotropic version of the model, the possibilities take place with probability $1/4$, $1/2$ and $1/4$ respectively.

h_i by 1.

- *Relaxation:* Every unstable site i relaxes by transferring two particles to its neighbours, possibly rendering the receiving site unstable.

The recipient of each of the two particles is chosen randomly and independently. In general, the model can be anisotropic, *i.e.* the recipients can be chosen with some unevenly distributed weight, one is more probable than the other. However, in this study, we only focus on the isotropic Abelian Manna model, *i.e.* any neighbour of an unstable site i can be chosen with equal probability $\frac{1}{q_i}$. Unstable sites are updated in a random sequential order (see Appendix D.1). The relaxation scheme of AMM is illustrated in Fig. 5.1. By virtue of its Abelian nature the order of relaxations is irrelevant for (the statistics of) the final state of the lattice and the statistics of the avalanche size (see Sec. 5.2 below).

Each relaxation of a site constitutes a toppling, which in the bulk is conservative, *i.e.* the total $\sum_i h_i$ remains unchanged by bulk topplings. A (small) number of sites are considered dissipative boundary sites, which have $q_i^{(v)}$ “virtual” neighbours, that are included in the total count of neighbours q_i introduced above (but not in the number

of sites N). Virtual neighbours (or sites) never topple themselves, *i.e.* particles are lost from the system at receipt. Such sites therefore provide a dissipation mechanism, and are introduced here solely as a bookkeeping device. As a general principle, the number of virtual neighbours of sites in the lattices discussed below are always chosen so that the resulting q_i at a boundary site matches that of a corresponding site in the bulk. For the lattices considered in this study, q_i might take on different values, yet bulk sites and boundary sites do not differ in that respect. This point will be discussed in further details in Sec. 6.2.

5.2 Abelian property

The Abelian symmetry (see Sec. 4.3.1) does not exist in the original model (Manna, 1991b), but simplifies its implementation greatly. Originally, *all* particles were redistributed at toppling which were subject to parallel updates, so that all sites that were active at time t had toppled before sites activated by a toppling during that time step. In that version of the model, it takes one time unit to update N_a simultaneously active sites. In the Abelian version, this holds only for constant N_a (number of active sites remaining unchanged). Yet, the update rate for both versions is $\frac{1}{N_a}$ when averaged over suitable intervals (one time unit in the former, one update in the latter), although neither one is Poissonian. Despite these differences in the definition, Dickman and Campelo (2003) have shown that implementing parallel or sequential updating has no noticeable impact on the statistics. They also found that exponents derived from the Abelian variant of the Manna model coincide with those published for the original version. The differences in the rules of the two versions are shown in Table 5.1.

5.3 Observables

An avalanche is the totality of all topplings until the system is quiescent again. The size s of the avalanche is measured as the number topplings performed between driving and quiescence, so that $s = 0$ if no avalanche occurs after the driving. The area a of an avalanche is the number of distinct sites which received at least a particle during the avalanche. This includes the site charged by the driving, so that $a \geq 1$. The duration t of an avalanche is the number of parallel updates performed during the avalanche. That is, however, very difficult to implement in practice due to parallel tracking of many actions taking place simultaneously on the lattice. There is, fortunately, an al-

Table 5.1: Comparison between the Abelian and original version of the Manna model.

Original Manna Model (oMM) (Manna, 1991b)	Abelian Manna Model (AMM) (Dhar, 1999a,c)
<ul style="list-style-type: none"> • Stochastic (bulk) drive and relaxation • Driving: add a particle at random site i_0: $h_{i_0} \rightarrow h_{i_0} + 1$ • Toppling: for each site i with $h_i > h_c = 1$, distribute <i>all</i> particles randomly and independently among its nearest neighbours i', $h_{i'} \rightarrow h_{i'} + 1$ (h_i times) and reset $h_i \rightarrow 0$ 	<ul style="list-style-type: none"> • Stochastic (bulk) drive and relaxation • Driving: add a particle at random site i_0: $h_{i_0} \rightarrow h_{i_0} + 1$ • Toppling: for each site i with $h_i > h_c = 1$, distribute <i>two</i> particles randomly and independently among its nearest neighbours i', $h_{i'} \rightarrow h_{i'} + 1$ (2 times) and set $h_i \rightarrow h_i - 2$

ternative way to do this. The definition of time used to determine the duration t of an avalanche is based on the idea that each active site undergoes a Poissonian decay, *i.e.* all active sites topple with the same rate. This rate is chosen to be unity, so that on average time $\frac{1}{N_a}$ goes by until a site topples if N_a sites are active. This is the amount by which time advances each time a site topples. Each time a site is picked at random, exactly two of its particles are redistributed. This procedure is, of course, not a faithful representation of a true Poisson process, which has random waiting times, but it is an increasingly good approximation in the limit of large activity N_a (Liggett, 2005). Thanks to its Abelian nature (Dhar, 1999a), the model's evolution from quiescent state to quiescent state does not require a specific dynamics on the microscopic time scale, so the definition of time in the present case is to the same degree arbitrary as the dynamics in, say, model A (Hohenberg and Halperin, 1977)¹.

A concise summary of (common²) avalanche observables is given in Table 5.2. An illustration of these observables is provided in Fig. 5.2 showing (a possible³) evolution of the model on a one-dimensional chain lattice.

¹Thanks to Gunnar Pruessner for enlightening me on this matter of duration in sandpile models.

²One can indeed define as many observables as needed in a sandpile model. The three observable area a , duration t and size s are the more fundamental and commonly studied ones (area and duration are usually overlooked, though) in the literature.

³out of many, since the model is stochastic

Figure 5.2: Illustration of observables during an avalanche of AMM on one-dimensional chain lattice of length $L = 10$. The arrow on top of a particle shows where it will move to in the next time step. The values of the observables in the final step are the area, duration and size of that avalanche.

Table 5.2: Summary of (common) avalanche observables in sandpile models.

Observable	Symbol	Definition
Area	a	Total number of distinct sites visited
Duration	t	Total number of parallel updates
Size	s	Total number of topplings

5.4 Scaling behaviour

One of the reasons why the (Abelian) Manna model has been receiving much interest in SOC literature is its robust scaling behaviours with a very clean power-law distribution of events' size, making it a very good candidate for investigating scaling phenomena in general.

With the observables defined in Sec. 5.3 and following the ordinary critical phenomena, one can make the usual finite-size scaling assumption for observable x . Its PDF $\mathcal{P}^{(x)}(x)$ follows

$$\mathcal{P}^{(x)}(x; L) = a_x x^{-\tau_x} \mathcal{G}_x\left(\frac{x}{b_x L^{D_x}}\right) \quad (5.1)$$

asymptotically in large $x \gg x_0$ with lower cutoff x_0 , linear system size L , nonuniversal metric factors a_x and b_x and universal exponents τ_x and D_x . In what follows, the fractal dimension of avalanches is denoted by D_a (Lübeck, 2000) and the avalanche area exponent by τ_a . For historic reasons, the dynamical exponent is denoted by z (rather than D_t), the avalanche duration exponent by α (rather than τ_t), and finally the avalanche dimension by D (rather than D_s) and the avalanche size exponent by τ (rather than τ_s). The universal, dimensionless scaling function $\mathcal{G}_x$ of a dimensionless argument decays, for large arguments, faster than any power law, so that all moments

$$\langle x^n \rangle(L) = \int_0^\infty dx \mathcal{P}^{(x)}(x; L) x^n \quad (5.2)$$

exist for any finite system and moment order $n \geq 0$. Provided that $n + 1 - \tau_x > 0$, one can easily show (De Menech *et al.*, 1998; Christensen *et al.*, 2008) that gap scaling follows (see Sec. 7.3.2.2), so that

$$\langle x^n \rangle(L) = a_x (b_x L^{D_x})^{n+1-\tau_x} \int_0^\infty dy y^{n-\tau_x} \mathcal{G}_x(y) \quad (5.3)$$

exists asymptotically in large L , or, more accurately,

$$0 < \lim_{L \rightarrow \infty} \frac{\langle x^n \rangle (L)}{L^{D_x(n+1-\tau_x)}} < \infty. \quad (5.4)$$

One can define $\mu_n^{(x)}$ as the exponent that characterises the scaling of the n th moment of observable x against the linear system size L , identified as $\mu_n^{(x)} = D_x(n+1-\tau_x)$, which will be employed in the moment analysis later in Sec. 7.3.2. **OPEN PROBLEM** see list on page 250

Because of particle conservation in the bulk, every particle placed on the lattice anywhere can only leave through the boundary. On the way there, it performs an independent random walk, even when it occasionally rests (Nakanishi and Sneppen, 1997). Every move by a particle is caused by a site's toppling and the number of particle moves during an avalanche is therefore exactly twice the number of topplings (this argument is later made use of to calculate the exact first moment of avalanche size $\langle s \rangle$ in Sec. 8.1.1). At stationarity, one particle leaves the system per particle added and (half of) its trajectory length is its contribution to the various avalanches it has been part of. The average contribution per particle and thus per avalanche is the average trajectory length, which scales like L^2 independent of the dimensionality of the lattice (Itzykson and Drouffe, 1991) and many of the details of the boundary. This argument remains valid if avalanches of size 0 are excluded from the average, provided the probability of producing an avalanche of size 0 (*i.e.* hitting an empty site) does not converge to 1.

As a result $\langle s \rangle \propto L^2$ asymptotically, or here, $\langle s \rangle \propto N^{2/d}$ (since the size N of the lattice scales with its linear size L through the dimension d as $N \propto L^d$, see Sec. 6.1.2), *i.e.* $\mu_1^{(s)} = 2$ and under the assumption of gap scaling $2 = (2 - \tau)D$.

Furthermore, by defining

$$-\Sigma_x = D_x(\tau_x - 1) \quad (5.5)$$

the assumption of a sufficiently (Chessa *et al.*, 1999; Pruessner and Jensen, 2004) narrow joint probability density function (Jensen *et al.*, 1989; Christensen *et al.*, 1991; Lübeck, 2000) produces the scaling law $\Sigma := \Sigma_s = \Sigma_t = \Sigma_a$, *i.e.*

$$-\Sigma = D(\tau - 1) = z(\alpha - 1) = D_a(\tau_a - 1). \quad (5.6)$$

Finally, as in the BTW model (Christensen and Olami, 1993), in the Manna model

avalanches are often assumed to be compact (Ben-Hur and Biham, 1996; Chessa *et al.*, 1999), *i.e.* $D_a = d$ (supposedly up to the upper critical dimension¹, where we expect dangerous irrelevant fields to spoil this scaling relation).

¹Numerical results show that D_a takes mean-field value above the upper critical dimension $d_c = 4$ (Pruessner and Huynh, 2012). OPEN PROBLEM see list on page 249

Chapter 6

The lattices

Lattices play an important rôle in the development of a huge number of physics studies in the modern time. Many lattice models like the Ising model, the Potts model, or the XY model (e.g. [Onsager, 1944](#); [Potts, 1952](#)), etc... have been providing physicists with insights into the nature of many different phenomena like magnetisation and phase transition. In short, they are the discretisation of the continuous space in any dimension. This arises naturally when one enters the regime of studying by methods of computational physics. And some of the models are even exactly solvable. Sandpile models are no doubt lattice models and play very important rôle in understanding scaling phenomena in Nature. This chapter provides an overview of lattices — its definition, context of occurrence, properties — and the description of all the lattices employed in the study of Abelian Manna model described in [Chapter 5](#). This hopes to be a good reference for future studies of models on lattices of different dimensions with various structures, especially fractal ones¹.

6.1 General properties

The first and natural question about lattice is: *what is a lattice?* A lattice is a collection of points (called lattice sites) and the links between them telling the interactions between lattice sites. This definition makes a lattice no different from a graph which also has nodes (points) and links (connections between pair of points). In other words, a lattice is precisely a graph! But different research communities prefer using the term in different ways. Mathematicians are used to graphs, while physicists are more used to

¹See ([Zhang et al., 2012](#)) for a recent study that makes use of some of the fractal lattices introduced in this thesis.

lattices, some others are used to networks. But they all refer to the same thing: collection of isolated points with links showing possible interactions (connections) among them.

6.1.1 Lattice and graph

However, the way lattices are used is different from that of general graphs. Lattices are usually very regular and consist of repetition of a set of rules for the ways the sites are distributed or connected. In other words, lattices comprise a specific subset of graphs. In lattices, the geometrical drawing of the sites in space, *i.e.* their coordinates, is respected. The interactions between lattice sites are local, *i.e.* they are restricted by the presentation of the lattice itself (while in general graphs, one can draw a link between any pair of nodes, no matter where they are).

In general graphs, the number of links that a node has is called the degree of that node. This degree can vary unrestrictedly from node to node. In lattices, the number of links that a site has tells the number of nearest neighbours that the site has. However, the number of nearest neighbours of a site is usually constant or takes few different values throughout the entire lattice.

Another slight difference between general graphs and lattices is that a graph is the entire presentation of itself while a lattice may not be. What this means is that a node of a graph can only communicate with other nodes that are part of the graph while a site of a lattice is allowed to communicate with sites outside the lattice. One example is a lattice with dissipative boundary in which sites along the boundary of the lattice interact with “virtual sites” (not part of the lattice) and hence the information transferred there is lost from the system. One can, however, argue that those sites along the boundary of the lattice indeed correspond to the sinks of the equivalent graph but it is very unusual for the sinks of a graph to have an undirected communication with other nodes of the graph. Moreover, the number of nearest neighbours including virtual ones of a site of lattice allows one to do more things than the degree of a node of graph (see Sec. 6.2 below to have a better sense of this).

Finally, like graphs, lattices also have directed and undirected versions. In undirected version, two sites of a single link play equal rôle (communicate in a “democratic” manner) while in directed version, one site is not allowed to transfer information to but only to receive from the other site (one-way communication).

Being regular and uniform in the distribution and connection of the sites, the lat-

tices are good representation of the discretisation of the continuous space that they live in. This becomes clear when one refers to the simple chain lattice, square lattice or cubic lattice which (naturally) well present the one-, two- and three-dimensional Euclidean space. In many physics problems, like studying of a field in space, using the lattice representation, one can write down the equations describing physics at every lattice point in space¹. In many cases, these of equations are easier to solved than the continuous ones. This also naturally allows one to study the phenomena by computer simulation.

6.1.2 Dimension of lattice

Like graphs, a lattice has a measure of dimension too. But unlike graphs, whose the dimension is defined as the minimum integer number n such that the graph can be embedded into Euclidean n -space E_n with every edge of the graph being segments of a straight line of length of unity (Erdos *et al.*, 1965; Eppstein, 2005; Boza and Revuelta, 2007), the dimension of a lattice is a measure of compactness of the points (lattice sites) distributed on the lattice. The concept of dimension of lattices involves two quantities: the total number of sites N (volume) and the length L (linear extent) of the lattice. In this definition, one sees the importance of the uniform distribution of sites on the lattice (however, see further below). This definition of dimension of a lattice borrows the idea of generalised dimension of an arbitrary object when one studies fractals.

Recall in Sec. 1.2.2, we employ the concept of covers to define the dimension of an object, its dimension is defined to be

$$d = \lim_{\epsilon \rightarrow 0} \frac{\ln N(\epsilon)}{\ln \frac{1}{\epsilon}} \quad (6.1)$$

in which ϵ is the linear size² of the covers and $N(\epsilon)$ is the number of covers of size ϵ used to cover the object. In other words, turning this definition around, one observes that the size of the covers and the number of covers display a scaling relation

$$N(\epsilon) \propto \epsilon^d, \quad (6.2)$$

¹Certainly, one has to take care of the so-called lattice spacing in the discretisation to make the equations good approximation of the continuous space.

²Certainly this is not as general as Hausdorff dimension where covers may have various sizes. But this definition of box dimension is as good as Hausdorff dimension on applying to “regular” fractals which are the objects of investigation in this work.

which is a proportional relation rather than an exact one because some factor might have been ruled out in the limiting process in Eq. (6.1)

Following this idea, we see that the dimension of a lattice tells the scaling relation between its length L and total number of sites N . Importantly, in studying of lattice models, what is of the real interest is the system in the thermodynamics limit $N \rightarrow \infty$ or $L \rightarrow \infty$. And very often in study of critical phenomena, one employs finite-size scaling (Cardy, 1988), hence the lattices of varying size. In that sense, the dimension of the lattices becomes clear to be

$$d = \lim_{L \rightarrow \infty} \frac{\ln N(L)}{\ln L} \quad (6.3)$$

or the other way around

$$N(L) \propto L^d. \quad (6.4)$$

This definition, however, raises an issue of what the *length* of a lattice is. Naturally (and naïvely?), one would think that this length is simply the linear size of the lattice from one end to another end. This is true for regular and simple lattices like the chain lattice in one dimension, the square lattice in two dimensions or the cubic lattice in three dimension but not for more complicated ones. The issue can be clearly seen when one asks what the length of the lattice like the Archimedes lattice featured in Fig. 6.4(c) is. The situation becomes even worse for fractal lattices like the crab lattice featured in Fig. 6.7(c). **OPEN PROBLEM** see list on page 249

All these issues, however, boil down to a conceptual question: *what dimension is being referred to?* The definition of dimension of lattice given in Eq. (6.3) indeed only refers to the geometry of the lattice but *not* the structure of the lattice, per se. That definition takes the distribution of the sites in space but *not* the connection among them into account. However, with the idea of dimension being the measure of compactness of the sites on a lattice, it is acceptable to take the dimension defined in Eq. (6.3) as *the dimension of the lattice*¹. This dimension is sometime referred to as Hausdorff dimension of the lattice (for example in the field theoretic literature)². In that sense, also given that the site are equally distant from one another, the length of the lattice can be measured in terms of the distance between any pair of neighbouring sites on the lattice. This was actually employed in (Huynh *et al.*, 2010) to obtain the dimension of the two fractal lattices — the arrowhead and crab lattices — which is illustrated in

¹provided that the sites on the lattice are properly distributed and connected

²Thanks to Gunnar Pruessner for discussion on this matter (see also Huynh and Pruessner, 2012).

Figure 6.1: Sierpiński arrowhead and crab lattices at fourth iteration with definition of length L .

Fig. 6.1.

The idea to get the length of the lattice is to lay out a “ruler” which is a chain of sites that are equally separated by the same distance as those in the lattice itself. By doing this, it is easy to tell what the length of the lattice is: it is the length of the chain that is needed to measure the lattice. The regular lattices, however, create less trouble. Any good consistent measure of the length would still produce the correct dimension of the lattice on applying the formula in Eq. (6.3) (see illustrations in Figs. 6.3–6.5).

6.1.3 Aspect ratio

Being a geometrical object, lattices naturally have aspect ratio which in many cases has been reported to affect the behaviours of the system studied on them (*e.g.* Privman *et al.*, 1991). Like the length of a lattice, aspect ratio is well-defined for regular and simple lattices like two-dimensional square or three-dimension cubic lattices. However, it causes serious ambiguity in less simple lattices even in regular integer dimension like the triangular lattice, kagomé lattice and especially the honeycomb lattice.

One question for debate would be: what is a lattice of aspect ratio 1 : 1 (in the case of two-dimension planar lattices) like? This question has an absolutely trivial answer in the case of lattices with four-fold symmetry presented in Fig. 6.4. The issue arises when one looks at the lattices with six-fold symmetry presented in Fig. 6.5.

Triangular lattice in Fig. 6.5(a) clearly has structure of layer by layer. Each layer of that lattice is a simple chain. One naturally thinks that a piece of triangular lattice

of aspect ratio 1 : 1 has the same number of layers as the length of each layer. But isn't it that the lattice respects the equal distance among sites? What this mean is that the distance between two successive layers is indeed $\frac{\sqrt{3}}{2}$ times that between a pair of neighbouring sites in each layer. So, doesn't it mean that in order to have a piece of triangular lattice with aspect ratio 1 : 1, the number of layers should be $\frac{2}{\sqrt{3}}$ times the number of sites contained in each layer?

The issue with kagomé lattice is similar to that of the triangular lattice. The one with honeycomb is more troublesome. We now construct a so-called "brick lattice" like the ones in Fig. 6.2. It is clear that this brick lattice and the honeycomb lattice have precisely the same structure of connections among sites. And it is also clear that a piece of honeycomb lattice corresponding to a piece of brick lattice in square shape does not have aspect ratio 1 : 1, and vice versa. At this point, it is tempting to introduce the notion of "geometrical aspect ratio" and "topological aspect ratio". The former is simple. That is just the normal geometrical intuition of the lattice and can be obtained easily using the trick that was applied to get the length of the lattice in Fig. 6.1. The latter is defined to be the ratio $A : B$ in which A is the *smallest number of moves* a particle performs to get from the top layer down to the bottom layer of the lattice and B is the *smallest number of moves* a particle performs to get from the leftmost side to the rightmost side of the lattice. For simple lattices like the square lattice, these two quantities do not differ. But they do significantly for more complicate ones like the honeycomb lattice being discussed here.

However things might get simpler if we are concerned about the symmetry of the lattices. Obviously, the brick lattice possesses much less symmetry than the honeycomb lattice does. And in studying lattice models, one is bounded by symmetry considerations (maintain high symmetry) as well as isotropy (maintain as much of it as possible). Moreover, given that the finite lattices "pave" the way to an infinite lattice, where local topology should no longer matter, one should be in favour of the geometrical definition over the topological one. That is also precisely what we strictly follow in the entire study of Abelian Manna model on different lattices¹. However, the issue still remains open to some extent, and one has to perform careful studies on it to see whether it really plays any rôle in the critical behaviours of the models.

¹Thanks to Gunnar Pruessner for a very interesting discussion on this matter.

Figure 6.2: Illustration of brick and corresponding honeycomb lattices with different aspect ratios. The dashed boxes enclosing the lattices are the reference boxes with strict aspect ratio 1 : 1.

6.2 Lattices used in this study

This study aims at providing an in-depth and comprehensive survey of lattices in many different dimensions with various structures. These lattices include the popular ones like square lattice, triangular lattice, the lesser known ones like Archimedes $(4, 8^2)$ lattice, *etc.*... and the new ones like crab fractal lattice, Sierpiński tetrahedron lattice which can help provide a very rich picture about behaviours of the Abelian Manna model.

In the following, we describe the regular one-dimensional (Fig. 6.3), two-dimensional (Figs. 6.4 and 6.5), three-dimensional (Fig. 6.6) and fractal lattices (Fig. 6.7). In the figures of one-dimensional and two-dimensional lattices, links to adjacent sites are indicated by solid lines, those to virtual neighbours by dashed ones. As virtual neighbours are merely an accounting device, the positioning of the dashed lines in the figures is arbitrarily chosen to match that of bulk links. For fractal and three-dimensional lattices, for simplicity of presentation, only the real lattice sites are shown.

Very often, the name of a lattice is rather long which is quite inconvenient for presentation in several contexts. Hence, we introduce the so-called “lattice codes” in Table 6.1 as the short names of the lattices. The codes are written in all block letters but they are *not* abbreviation of the lattice names, and hence *not* included in the List of Acronyms in the early part of this thesis.

For the sake of presentation, we focus on the details of the figures of the one- and two-dimensional lattices. The three-dimensional and fractal ones follow the same manner. The figures show in particular the boundaries, whose structure we maintain during finite-size scaling (see Sec. 7.3.2.2), *i.e.* when we increase the lattice size, we make sure that the edges and corners of the larger lattice matched that of the smaller one. Initially we dismissed such issues as small corrections, but given the high accuracy with which we determine moments, changes in the boundaries become clearly visible, in particular in the mitsubishi lattice (Fig. 6.5(d)). In hindsight this is hardly surprising given the great importance of transport of particles from the conservative bulk to the dissipative boundaries (Paczuski and Bassler, 2000a,b).

The definition of linear size L varies from lattice to lattice. This is due to the way we index the sites of the lattice, the ease of implementation is given higher priority. However, the consistency of the linear size is always maintained. In most cases, the lattice has structure of layer(s) and its linear size L is simply the number of sites in each layer. For some simple lattices, the number of sites N is exactly the power d (dimension) of their linear (Euclidean) extension L . In more complicated cases, this may hold only approximately. In order to maintain a reasonably high symmetry of the finite lattices, which have to be thought of as being cut out of an (infinite) tiling of the plane, we have to make compromises. In one dimension ($d = 1$, see Table 6.2), N is a multiple of L for the first three lattices discussed. The fourth, the futatsubishi lattice, follows $N = 3L + 1$ for all sizes considered. This is because the linear size L of

Table 6.1: Code or short name of all the lattices employed in this study. The code standing alone itself can refer to the lattice, without the need of being accompanied by the word “lattice”. Regular lattices in integer dimensions are listed first, followed by fractal ones.

Lattice name	dimension d	Code
Simple chain	1	LINE
Rope ladder	1	LADD
Next-nearest-neighbour chain	1	NNN
Futatsubishi	1	FUTA
Square	2	SQUA
Jaggy square	2	JASQ
Archimedes ($4, 8^2$)	2	ARCH
Noncrossing diagonal square	2	NOCR
Triangular	2	TRIA
Kagomé	2	KAGO
Honeycomb	2	HONE
Mitsubishi	2	MIT
Simple cubic	3	SC
Body-centred cubic	3	BCC
Next-nearest-neighbour body-centred cubic	3	BCCN
Face-centred cubic	3	FCC
Next-nearest-neighbour face-centred cubic	3	FCCN
Semi-inverse square triadic Koch	$\frac{\ln 5}{\ln 3} \approx 1.465$	SSTK
Sierpiński arrowhead	$\frac{\ln 2}{\ln 3} \approx 1.585$	ARRO
Crab	$\frac{\ln 2}{\ln 3} \approx 1.585$	CRAB
Sierpiński tetrahedron	$\frac{\ln 4}{\ln 2} = 2$	SITE
Extended Sierpiński gasket	$1 + \frac{\ln 2}{\ln 3} \approx 2.585$	EXGA

this lattice is considered to be the number of “diamonds” in the chain (see description below). In two dimensions, $d = 2$, two different lengths L_x and L_y are introduced for the two different dimensions, defined in a way as natural as possible. The cutting of the finite two-dimensional lattices and the resulting shape of the boundaries is guided by the following principles, which clash and therefore hold only approximately: the number of sites in the lattice $N = L_x L_y$ has to be as close as possible to the power of 2 (*i.e.* 2^p), the number of sites in a simple square lattice. The ratio $\frac{L_x}{L_y}$ is to be held constant with increasing N . The (Euclidean) aspect ratio should be unity or very close, if that clashes with the shape of the boundaries. For each lattice, $m_{x,y}$ and $c_{x,y}$ are set such that $L_{x,y} \equiv c_{x,y} \pmod{m_{x,y}}$, to maintain the shape of the boundaries across different sizes. The parameters and resulting lattice sizes are listed in Table 6.3. The number L_y can normally be thought as a number of “layers” and L_x as the number of sites in a layer. It is clear that such a definition, necessary because of the mismatch of the symmetry of lattice to the square symmetry of the cutout, makes the linear sizes L_x and L_y rather poor fitting parameters (see below).

The size of each lattice is indicated exemplarily in the caption of its figure. The sizes of one- and two-dimensional lattices employed in this study are listed in Tables 6.2 and 6.3. Three-dimensional and fractal lattices follow in the same manner, their sizes are list listed in Tables 6.4 and 6.5. Further details are described below.

6.2.1 One-dimensional lattices

6.2.1.1 Simple chain (Fig. 6.3(a))

This lattice is the usual one-dimensional chain, where each site connects to two nearest neighbours. The leftmost and the rightmost sites have one virtual neighbour each.

6.2.1.2 Ladder rope (Fig. 6.3(b))

The shape of this lattice is that of a rope ladder. It is a simple extension of the simple chain (Fig. 6.3(a)), which eventually leads to the square lattice (Fig. 6.4(a)). Each site has three nearest neighbours, one on the left, one on the right and one below or above it. Four boundary sites have one virtual neighbour each.

Table 6.2: Sizes of the one-dimensional lattices used in this study. Only the seven sizes eventually used are listed. The (nominal) linear size L , corresponding to the examples given in Fig. 6.3, is shown in brackets.

Lattice	$N(L)$						
LINE	1024(1024)	2048(2048)	4096(4096)	8192(8192)	16384(16384)	32768(32768)	65536(65536)
LADD	2048(1024)	4096(2048)	8192(4096)	16384(8192)	32768(16384)	65536(32768)	131072(65536)
NNN	2048(1024)	4096(2048)	8192(4096)	16384(8192)	32768(16384)	65536(32768)	131072(65536)
FUTA	3073(1024)	6145(2048)	12289(4096)	24577(8192)	49153(16384)	98305(32768)	196609(65536)

Table 6.3: Sizes of the two-dimensional lattices used in this study. Only the seven sizes eventually used are listed. The number of sites N equals the product $L_x L_y$. The last row shows the number of sites N of the standard square lattice against which the lattices are compared.

Lattice	(c_x, m_x, c_y, m_y)	$L_x \times L_y$						
SQUA	(0, 1, 0, 1)	256×256	362×362	512×512	724×724	1024×1024	1448×1448	2048×2048
JASQ	(1, 2, 0, 1)	361×182	511×257	723×363	1023×513	1447×725	2047×1025	2895×1449
ARCH	(0, 4, 0, 2)	360×182	512×256	724×362	1024×512	1448×724	2048×1024	2896×1448
NOCR	(1, 2, 1, 2)	255×257	361×363	511×513	723×725	1023×1025	1447×1449	2047×2049
TRIA	(0, 1, 0, 1)	239×274	337×389	476×551	673×779	953×1100	1347×1557	1906×2201
KAGO	(1, 3, 0, 1)	412×159	583×225	826×317	1168×449	1651×635	2335×898	3301×1271
HONE	(1, 2, 0, 2)	337×194	475×276	675×388	953×550	1347×778	1903×1102	2695×1556
MIT5	(2, 3, 1, 2)	239×275	338×387	476×551	674×777	953×1101	1349×1555	1904×2203
standard		65536	131044	262144	524176	1048576	2096704	4194304

Table 6.4: Sizes of the three-dimensional lattices used in this study. Only the six sizes eventually used are listed. The (nominal) linear size L is shown in brackets. The relation between the number of sites N and the linear size L is given in Sec. 6.2.3.

Lattice	$N(L)$					
SC	5929741(181)	16777216(256)	47437928(362)	134217728(512)	379503424(724)	1073741824(1024)
BCC	5910191(144)	16855091(204)	47527775(288)	134342559(407)	379228599(575)	1072755125(813)
BCCN	5910191(144)	16855091(204)	47527775(288)	134342559(407)	379228599(575)	1072755125(813)
FCC	6004495(115)	16849134(162)	47721997(229)	134168063(323)	380524249(457)	1075842586(646)
FCCN	6004495(115)	16849134(162)	47721997(229)	134168063(323)	380524249(457)	1075842586(646)

Table 6.5: Sizes of the fractal lattices used in this study. Only the four sizes eventually used are listed. The (nominal) linear size L is shown in brackets together with the number of iterations m of the fractal lattice. Since the fractal lattices have different dimensions d , this value is also shown. The number of sites N and the linear size L as functions of number of iterations m are given in Sec. 6.2.5.

Lattice	d	$N(L, m)$			
SSTK	1.464..	3126(244, 5)	15626(730, 6)	78126(2188, 7)	390626(6562, 8)
ARRO	1.584..	730(65, 6)	2188(129, 7)	6562(257, 8)	19684(513, 9)
CRAB	1.584..	730(96, 6)	2188(192, 7)	6562(384, 8)	19684(768, 9)
SITE	2	4097(65, 6)	16385(129, 7)	65537(257, 8)	262145(513, 9)
EXGA	2.584..	71175(65, 6)	423378(129, 7)	2529651(257, 8)	15146838(513, 9)

6.2.1.3 Next-nearest-neighbour chain (Fig. 6.3(c))

Despite its triangular pattern, this is the simple chain (Fig. 6.3(a)) extended by allowing for next-nearest-neighbour interactions. This lattice was motivated by the observation (Hughes and Paczuski, 2002) that some models require such extensions in one dimension to prevent degeneracy. Alternatively, it can be seen as the first step towards a two-dimensional triangular lattice (Fig. 6.5(a)). Each site has four neighbours, with boundary sites having either one or two virtual neighbours.

6.2.1.4 Futatsubishi (Fig. 6.3(d))

In appreciation of the kagomé lattice discussed below, “futatsubishi” is a Japanese name, which translates to “two-diamond”, reflecting the shape of the lattice. The futatsubishi lattice is fully contained in the mitsubishi lattice discussed below (three of them meet at every vertex), or as a suitable slice of the square lattice (Fig. 6.4(a)) or the jagged lattice (Fig. 6.4(b)). On the present lattice, each site has either two or four neighbours in the bulk and two virtual neighbours at the boundary.

6.2.2 Two-dimensional lattices

6.2.2.1 Square (Fig. 6.4(a))

This is the standard square lattice, the most commonly used lattice that the SOC models have been studied on. Each site has four nearest neighbours. As indicated by the dashed lines in Fig. 6.4(a), edge sites have one virtual neighbour and corner sites have

Figure 6.3: The four one-dimensional lattices considered in this study. Sites are shown as filled circles, adjacency is indicated by solid lines. Dashed lines indicate links to virtual neighbours.

two, leading to a constant $q_i = 4$ (the same notation as used in Sec. 5.1) for all sites i .

6.2.2.2 Jagged (Fig. 6.4(b))

This lattice is a square lattice rotated by 45 degrees and fitted into a square shape which makes the boundary look jagged. The only difference to the square lattice above is therefore the boundary, where sites have either two (edge) or three (corner) virtual neighbours, producing, again, a constant $q_i = 4$ for all sites. There are various ways of cutting the lattice out of the bulk — we decided to maintain the left-right mirror symmetry, which results in three different boundaries. In hindsight, a slightly different choice would have resulted in a lattice of higher symmetry and an aspect ratio closer to unity. In this study, the perfect match of the universal features of the [AMM](#) on this lattice with those on all others is testament of the strong universal behaviour of the model (see Sec. 7.4).

6.2.2.3 Archimedes $(4, 8^2)$ (Fig. 6.4(c))

The name of this lattice is normally complemented by $(4, 8^2)$, which reflects the parameters for the general rule to construct these two-dimensional lattices ([Grunbaum and Shephard, 1987](#)). In the present case, every vertex is surrounded by one square and

 (a) The square lattice. $L_x = L_y = 6, N = 36$.

 (b) The jagged lattice. $L_x = 9, L_y = 4, N = 36$.

 (c) The Archimedes lattice. $L_x = 8, L_y = 4, N = 32$.

 (d) The noncrossing diagonal square lattice. $L_x = L_y = 5, N = 25$.

Figure 6.4: The four two-dimensional lattices with four-fold symmetry considered in this study.

two octagons. Each lattice site in the bulk has three neighbours. Along the boundary, one of them is replaced by a virtual neighbour, so that $q_i = 3$ throughout.

6.2.2.4 Noncrossing diagonal square (Fig. 6.4(d))

This lattice is based on a square lattice by adding alternate diagonals such that there are no crossing diagonals. Each site has either four or eight nearest neighbours. Boundary sites have either one, three or five virtual neighbours.

(a) The triangular lattice. $L_x = 5, L_y = 7, N = 35$.

(b) The kagomé lattice. $L_x = 10, L_y = 4, N = 40$.

(c) The honeycomb lattice. $L_x = 9, L_y = 4, N = 36$.

(d) The mitsubishi lattice. $L_x = 5, L_y = 7, N = 35$.

Figure 6.5: The four two-dimensional lattices with six-fold symmetry considered in this study.

6.2.2.5 Triangular (Fig. 6.5(a))

This lattice is the approximate square-shaped clipping of a tessellation of the plane by triangles. It is probably the second most frequently studied lattice in statistical mechanics. Owing to its six-fold symmetry which clashes with the four-fold symmetry

of the square, boundary conditions at the upper and the lower edge differ from those on the left and on the right. This is not the case for three of the four preceding two-dimensional lattices, but does similarly apply to the following three. In the present case, the number of virtual neighbours of sites along the edge varies between one and four, with a constant $q_i = 6$ throughout.

6.2.2.6 Kagomé (Fig. 6.5(b))

This lattice was first studied by Syôzi (1951, 1972). Its name is Japanese, referring to the pattern of holes (“me”) in a basket (“kago”). Each site in the bulk has four nearest neighbours. This lattice has a six-fold symmetry, which generates three different boundary conditions by the way we decided to cut it at top and bottom. Similar to the jagged lattice (Fig. 6.4(b)), in hindsight we may have picked slightly different boundaries.

6.2.2.7 Honeycomb (Fig. 6.5(c))

Similar to the triangular lattice, this lattice is a tiling by hexagons, leading to a honeycomb-shaped pattern. Each bulk site has three neighbours, $q_i = 3$, and the number of virtual neighbours along the edge is one everywhere, even when top and bottom edges differ from those on the left and on the right.

6.2.2.8 Mitsubishi (Fig. 6.5(d))

This is a Japanese name which translates to “three-diamond” reflecting the shape of the lattice (the naming is inspired by the logo of the famous Japanese company of the same name)¹. It is also known as “the diced lattice” (Syôzi, 1951, 1972). Each lattice site has either three or six nearest neighbours with a number of virtual neighbours at the boundary varying between one and four. Again, top and bottom edges differ from those left and right.

6.2.2.9 Relation among two-dimensional lattices

Some of the two-dimensional lattices are related by transforms, which are frequently used in the analysis of equilibrium critical phenomena (Syôzi, 1951, 1972; Baxter, 2007;

¹The name was suggested by me and encouraged by Gunnar Pruessner for its being the name of the company. Later it is discovered that the word itself has a meaning in the same manner as the word “kagomé”. This is main reason for the name to be used.

Loebl, 2010). Denoting the duality transform by $\mathfrak{D}$, the star-triangle transform by Δ and the decoration-iteration transform by δ , the following equalities hold between two-dimensional lattices (refer to Table 6.1 for the lattice codes) (Syôzi, 1972, p. 294).

$$\begin{aligned}\mathfrak{D}(\text{HONE}) &\equiv \text{TRIA}, \\ \mathfrak{D}(\text{KAGO}) &\equiv \text{MITS}, \\ \mathfrak{D}(\text{ARCH}) &\equiv \text{NOCR}, \\ \mathfrak{D}(\text{SQUA}) &\equiv \text{SQUA}, \\ \Delta(\text{MITS}) &\equiv \text{TRIA}, \\ \Delta(\delta(\text{HONE})) &\equiv \text{KAGO}.\end{aligned}$$

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6.2.3 Three-dimensional lattices

Five three-dimensional lattices are employed in this study. The three-dimensional lattices (Ashcroft and Mermin, 1976) are built upon the standard simple cubic (SC) lattice. The body-centred cubic (BCC) and face-centred cubic (FCC) lattices are also studied with next-nearest-neighbour interactions (BCCN and FCCN, respectively). The total number of sites N of all five lattices are chosen to be as close as possible to one another. Typically, six system sizes ranging from $N = 181^3$ to $N = 1024^3$ are used.

6.2.3.1 Simple cube (Fig. 6.6(a))

Simple cubic lattice is a standard lattice that is commonly used for study of model in three dimensions. It belongs to the family of generalised cubic lattices in arbitrary dimension d . On a d -dimensional lattice, each site has $2d$ nearest neighbours and there are d different types of boundary sites with $2d - 1$, $2d - 2$,... and d real nearest neighbours (sites which are part of the lattice, similar to two-dimensional lattices) respectively. Thus, a simple cubic lattice has 3 different types of boundary sites, namely face, edge and corner with 5, 4 and 3 real nearest neighbours respectively. In the bulk, each lattice sites has 6 nearest neighbours. The linear length L of this lattice is simply the number of sites along one of its edges. The total number of sites on this lattice is then $N = L^3$.

(a) The simple cubic lattice. $N = 27$.

(b) The body-centred cubic lattice. $N = 35$.

(c) The next-nearest-neighbour body-centred cubic lattice. $N = 35$.

(d) The face-centred cubic lattice. $N = 63$.

(e) The next-nearest-neighbour face-centred cubic lattice. $N = 63$.

Figure 6.6: The five three-dimensional lattices considered in this study.

6.2.3.2 Body-centred cube (Fig. 6.6(b))

Body-centred cubic lattice is obtained from a simple cubic lattice by adding an additional site at the centre of each elementary cube (formed by every nearest 8 lattice sites). We call the sites of the simple cubic lattice “principal sites”. If the distance between two nearest principal sites is a (also lattice spacing), the distance between an added body-centred site and its (equidistant) nearest neighbours is $a\frac{\sqrt{3}}{2}$. This distance, $a\frac{\sqrt{3}}{2} \approx 0.866a$, is shorter than the distance between two nearest principal sites. Thus, there is *no* interaction between principal sites. The distance between two nearest body-centred sites is also a , and thus, there is neither interaction between body-centred sites. Hence, on this lattice, there is only interaction between principal sites and body-centred sites. Each site on this lattice has 8 nearest neighbours. The linear length L of this lattice is the number of sites along one of its edges. The total number of sites on this lattice is $N = L^3 + (L - 1)^3$.

6.2.3.3 Next-nearest-neighbour body-centred cube (Fig. 6.6(c))

Next-nearest-neighbour body-centred cubic lattice is obtained from a body-centred cubic lattice by allowing additional interactions both between principal sites and between body-centred sites. Hence, each lattice site has 14 nearest neighbours. The linear length and total number of sites of this lattice is the same as the body-centred cubic lattice, namely $N = L^3 + (L - 1)^3$.

6.2.3.4 Face-centred cube (Fig. 6.6(d))

Face-centred cubic lattice is obtained from a simple cubic lattice by adding an additional site at the centre of each elementary square (formed by every nearest 4 lattice sites). If the distance between two nearest principal site is a , the distance between an added face-centred cubic site and its (equidistant) nearest neighbours is $a\frac{\sqrt{2}}{2}$. This distance, $a\frac{\sqrt{2}}{2} \approx 0.707a$, is shorter than the distance between two nearest principal sites. Thus, there is *no* interaction between principal sites. The distance between two face-centred sites intersecting elementary squares (they share a common edge) is also $a\frac{\sqrt{2}}{2}$. Thus, there is interaction between such face-centred sites. However, there is *no* interaction between face-centred sites of nonintersecting squares (no common edge) as the distance between two nearest such sites is also a . Each site on this lattice has 12

nearest neighbours. The linear length L of this lattice is the number of sites along one of its edges. The total number of sites on this lattice is $N = L^3 + 3L(L - 1)^2$.

6.2.3.5 Next-nearest-neighbour face-centred cube (Fig. 6.6(e))

Next-nearest-neighbour face-centred cubic lattice is obtained from a face-centred cubic lattice by allowing addition interactions both between principal sites and between face-centred sites of nonintersecting elementary squares. Hence, each lattice site has 18 nearest neighbours. The linear length and total number of sites of this lattice is the same as the face-centred cubic lattice, namely $N = L^3 + 3L(L - 1)^2$.

6.2.4 Periodic lattices

For the lattices described above, besides using open boundary conditions we also consider hypercubic lattices with periodic boundary conditions in $(d - 1)$ directions and open boundary condition in the remaining direction. In other words, these lattices are generalised tube. The purpose of studying the model on these lattices is to see how robust the critical behaviour depends on the boundary conditions which is one of the important feature of SOC systems. We study this in two and three dimensions.

6.2.5 Fractal lattices

The fractal lattices are where the first and second part of this thesis meet. With the arc-fractal system, a zoo of fractal lattices is generated. In this study, five of them are employed with detailed results produced. A similar amount of lattices is also studied but not in details and they are discussed in Sec. 6.3. The mathematical structures of some fractal lattices, especially the arrowhead and the crab, are provide in details in Appendix B.

6.2.5.1 Semi-inverse square triadic Koch (Fig. 6.7(a))

Semi-inverse square triadic Koch lattice is generated from the arc-fractal system using multiple-rule (Huynh and Chew, 2011) (also Sec. 2.6 and Fig. 2.27(b)). The name of the lattice is chosen due to the fact that it is similar to a so-called triadic Koch curve (Addison, 1997, pp. 16–19) but the elements are half inverse. Each lattice site has either 2, 3 or 4 nearest neighbours. At m th iteration of the lattice, the linear size is given by

(a) The semi-inverse square triadic Koch lattice. $m = 3, L = 28, N = 126$.

(b) The arrowhead lattice. $m = 4, L = 17, N = 82$.

(c) The crab lattice. $m = 4, L = 24, N = 82$.

(d) The Sierpiński tetrahedron lattice. $m = 2, L = 5, N = 34$.

(e) The extended Sierpiński gasket lattice. $m = 2, L = 5, N = 75$.

Figure 6.7: The five fractal lattices considered in this study.

$L = 3^m + 1$ and the total number of sites is $N = 5^m + 1$. Hence the dimension of the lattice is $d = \lim_{m \rightarrow \infty} \frac{\log N(m)}{\log L(m)} = \frac{\log 5}{\log 3} \approx 1.465$.

6.2.5.2 Arrowhead (Fig. 6.7(b))

This lattice is indeed the same as Sierpiński arrowhead described in (Mandelbrot, 1983) using different generator. In this case, it is generated through the arc-fractal system, with the number of segments $n = 3$ and the opening angle of the arc $\alpha = \pi$. The rule for orientating the arc at each iteration is “in-out-in”. At m th iteration of the lattice, the linear size is given by $L = 2^m + 1$ (the method to measure this length is discussed in Sec. 6.1.2) and the total number of sites is $N = 3^m + 1$. Hence the dimension of the lattice is $d = \lim_{m \rightarrow \infty} \frac{\log N(m)}{\log L(m)} = \frac{\log 3}{\log 2} \approx 1.585$. Asymptotically, one third of the sites have three nearest neighbours (called extended sites), while the remaining two thirds have two nearest neighbours (call normal sites). The detailed mathematical structure of this lattice can be found in Appendix B.

6.2.5.3 Crab (Fig. 6.7(c))

The name of this lattice is inspired by its overall shape looking like a crab. Similar to the arrowhead lattice above, the crab lattice is generated through the arc-fractal system with the number of segments $n = 3$ and the opening angle of the arc $\alpha = \pi$. The rule for orientating the arc at each iteration is “out-in-out”. At m th iteration of the lattice, the linear size is given by $L = 3 \times 2^{m-1}$ (the method to measure this length is discussed in Sec. 6.1.2) and the total number of sites is $N = 3^m + 1$. Hence the dimension of the lattice is $d = \lim_{m \rightarrow \infty} \frac{\log N(m)}{\log L(m)} = \frac{\log 3}{\log 2} \approx 1.585$. Asymptotically, one third of the sites have three nearest neighbours (called extended sites), while the remaining two thirds have two nearest neighbours (call normal sites). The detailed mathematical structure of this lattice can be found in Appendix B.

6.2.5.4 Sierpiński tetrahedron (Fig. 6.7(d))

Sierpiński tetrahedron lattice is the three-dimensional version of the well-known Sierpiński gasket. The Sierpiński gasket is constituted from equilateral triangles while Sierpiński tetrahedron is from equilateral tetrahedra. Each lattice site has 6 nearest neighbours. At m th iteration of the lattice, the linear size is $L = 2^m + 1$ and the total number of sites is $N = 2 \times 4^m + 2$. Hence the dimension of the lattice is

$d = \lim_{m \rightarrow \infty} \frac{\log N(m)}{\log L(m)} = \frac{\log 4}{\log 2} = 2$. This lattice is an interesting case because of its integer dimension and fractal structure. It provides a good example to compare against regular two-dimensional lattices.

6.2.5.5 Extended Sierpiński gasket (Fig. 6.7(e))

Extended Sierpiński gasket lattice is obtained by stacking copies of Sierpiński gasket lattice together. We employ as many layers of Sierpiński gasket as necessary in order to have square shape of the newly formed faces. For example, at m th iteration of the lattice, the linear size is $L = 2^m + 1$. Therefore we would stack $2^m + 1$ layers of Sierpiński gasket together, one on top of another. The shape of the lattice looks like a prism. From side view of that prism, we see square lattice. From top view of that prism, we see Sierpiński gasket lattice. Each site of one layer has two extra nearest neighbours to the corresponding site of the layer right below and above. Since each site on Sierpiński gasket lattice has 4 nearest neighbours, each site on this extended Sierpiński gasket lattice has 6 nearest neighbours. Each layer of Sierpiński gasket lattice has $N_1 = \frac{3(3^m + 1)}{2}$ sites, thus, total number of sites on the extended Sierpiński gasket lattice is $N = LN_1 = \frac{3(2^m + 1)(3^m + 1)}{2}$. Hence the dimension of the lattice is $d = \lim_{m \rightarrow \infty} \frac{\log N(m)}{\log L(m)} = \frac{\log 6}{\log 2} \approx 2.58$. This is an example of a way to extend planar fractal lattice embedded in two dimensions by stacking them together. Thus, it could be called $(d_f + 1)$ extension in which d_f is the fractal dimension of the planar fractal lattice.

6.3 The “unused” lattices

A number of lattices that were briefly studied but not in great details are also reported here. They are the crabbarro, Sierpiński gasket, Sierpiński carpet and the Koch snowflake lattices. They do not provide any firm conclusion for the study, yet it is interesting to include them in the discussion of the critical behaviours of the Abelian Manna model.

The crabbarro lattices (Figs. 6.8(a) and 6.8(b)) are hybrid between the arrowhead and crab lattices. In other words, they are generated through the arc-fractal system using multiple-rule which is the combination of the rules for the arrowhead and the crab. However, due to the nature of the combination, in the finite-size scaling study (Sec. 7.3.2.2), only either the odd or even iterations of the lattice can be used. This

Figure 6.8: Some fractal lattices that were not investigated in details in this study (I).

practically means that the size of the lattices required in order for a good behaviour to be seen is very large. Otherwise, the scaling behaviour is not robust and conclusive, sometimes inconsistent. Nevertheless, the detailed mathematical structure of these two lattices can be found in Appendix B.

The Koch snowflake lattice (Fig. 6.8(c)) might be an interesting one because it is essentially a two-dimensional triangular lattice with a fractal boundary of the Koch curve lattice. This was expected to produce some nontrivial behaviours of the model but it doesn't. Hence, further study on it has not been completed.

The Sierpiński gasket (Fig. 6.8(d)) is a popular lattice for study of model in noninte-

Figure 6.9: Some fractal lattices that were not investigated in details in this study (II).

ger dimension due to its simplicity and several nice features like exact renormalisation group calculation ([ben-Avraham and Havlin, 2000](#)). It is included here as a reference and also a check against existing literature. Some study of SOC was performed on this lattice by [Kutnjak-Urbanc *et al.* \(1996\)](#); [Daerden and Vanderzande \(1998\)](#); [Daerden \(2001\)](#); [Daerden *et al.* \(2001\)](#); [Vanderzande and Daerden \(2001\)](#); [Lee *et al.* \(2009\)](#).

Two different versions of Sierpiński carpet lattice (Figs. 6.9(a) and 6.9(b)) are introduced here treating the voids in the lattice differently. In the first version, without the voids, the coordination number (number of nearest neighbour interactions) is $q_i = 4$ for any site i on the lattice. Hence there are many interactions that are not the nearest-neighbour ones. This is to prevent the particles from diffusing fast through the lattice. In the second version, with the voids, the coordination number is 3 for those sites along the voids and 4 for all the others (including the ones along the outer boundary). With the constraint of time of this thesis, these two lattices have not been investigated. And the study of the model on them will be published elsewhere.

Chapter 7

Numerical study

7.1 Implementation of the model

Throughout this work we used the same implementation in C of the [AMM](#), which takes as an input the adjacency information of the various lattices, that are generated separately from running the actual Manna model. The full adjacency matrix is extremely sparse and it therefore makes little sense to store the information in that format. Rather, all sites are sequentially indexed and sites adjacent to a given site are listed by their index in a sequence (of varying length) (see Sec. 7.2). Negative indices indicate virtual neighbours. The adjacency information is then filled into a C struct for each site i , which contains a vector holding the sequence of indices of adjacent sites, their number, a flag whether the site has been hit by the currently running avalanche and finally the height h_i . Using indices rather than pointers to reference sites in our experience produces very fast code with strongly optimising compilers.

Storing the adjacency information in memory (rather than being implemented explicitly by rules), makes the code more flexible, but large lattices are comparatively expensive in terms of memory requirements (especially for three-dimensional lattices). Significant amounts of memory are also required for the stack of active sites, which holds every site i whose h_i exceeds 1. Sites are placed on the stack at the time when they make the transition from $h_i = 1$ to $h_i = 2$. Random sequential updating requires random access to that stack. Sites i picked from the stack topple only once and thus remain on the stack until $h_i \leq 1$. At the time when an avalanche is triggered all sites might hold one particle, so that the theoretical maximum number of sites exceeding the threshold at any one time simultaneously is $\frac{N+1}{2}$. Although this maximum is not

reached in practice, building in safeguards to protect smaller stacks from overflowing is computationally more costly than providing a stack as large as the theoretical maximum.

A second stack is required to keep track of all sites toppling during any one avalanche. Once a site is updated, a flag associated with it is changed, and the site's index is placed on the stack. It is an invariant that all sites with the flag raised are located on the stack. The area of an avalanche is the height of that stack at the end of the avalanche. The flags are reset by scanning through the stack.

Even when using memory lavishly, memory requirements for the [AMM](#) implementation as described above are rather modest compared to what any modern desktop computer has to offer. In one and two dimensions, large lattices are prohibitively large in terms of [central processing unit \(CPU\)](#) time, not in terms of memory, as the average avalanche size grows quadratically in the linear system size, *i.e.* in the (average) chemical distance to the open (dissipative) boundary. An additional, sometimes very noticeable constraint on systems with multiple cores or multiple logical cores (hyper-threading) is the memory bus, which can be alleviated only partly by reducing the memory requirements. In three dimensions, however, the constraint is the huge amount of memory to hold a large lattice. Therefore, computer nodes with large [random-access memory \(RAM\)](#) of up to 4GB per node are needed.

The output of the code described above is a string of moments (because we employ moment analysis in this study, see Sec. [7.3.2](#)), effectively subsamples, averaged over a number of avalanches (ranging from many millions to several ten thousand, sometimes several thousands due to very long time taken by one avalanche on large lattices in three dimensions), which we call “chunks” in the following. Typically, 100 to 10,000 chunks were generated for each lattice. The statistics of the chunks allows for an estimate of the statistical error, while a simple average (weighted by the chunk size if necessary) across chunks produces an unbiased, consistent estimate of the moments.

The sizes of the chunks were chosen so that a new chunk would be produced every ten to sixty minutes. In one dimension the linear size of the lattices spanned about three orders of magnitude, in two dimensions the square root of that. The average size and thus roughly the [CPU](#) time to produce a single avalanche grows quadratically in the linear size, ranging over six orders of magnitude in one dimension, and over three in two dimensions, see Table [7.1](#). The chunk sizes have to be adjusted accordingly. At the same time, the avalanche-avalanche correlation time increases like a power of

the linear extension, but *not* with the dynamical exponents z , which defines the link between microscopic and macroscopic time scale, but with L^{D-d} , which is a measure of the characteristic fluctuations of the AMM in the interface picture (Paczuski and Boettcher, 1996; Pruessner, 2003; Morand *et al.*, 2010).

The chunks were generated on the SCAN facility at Imperial College London,¹ which harvests CPU time from undergraduate computing facilities when not used by students, providing up to seven hundred and six (706) logical CPUs simultaneously, mostly in the form of Intel® Core2™ processors with 2.66GHz. In this setup, all I/O is done over the network, which in principle constrains the amount of output per process running. Given the drastic difference in CPU time required to generate a single avalanche, by far the most CPUs were assigned to the largest systems. In two dimensions, for example, the smallest four system sizes each had one logical CPU to themselves. As a result, the smallest systems (that we kept, see below) have sample sizes exceeding 10^9 avalanches, which leads to highly accurate estimates for their moments. In total, about 168, 413, 72, 447 and 38, 032 hours of CPU time were spent on generating the data in one, two and three dimensions, respectively, and 75, 456 hours for the fractal lattices.

For simplicity, checkpointing was implemented only right after a chunk was written out. The power cycling of the computer setup thus limited the amount of time available to generate a single chunk. While the original intention was to choose the chunk size such that correlations (which normally come in the form of anti-correlations) are negligibly small, this turned out to be unsustainable for the very large system sizes. However, uncorrelated chunks greatly facilitate the calculation of statistical errors, compared to, say, a full-blown resampling plan (Efron, 1982). To this end, chunks were merged during post-processing, as discussed below.

As each instance of the AMM was started with an empty lattice, a generous amount of chunks is dropped as transient from the set considered in the subsequent data analysis. This equilibration “time” is estimated by inspection of individual series of chunks, as a multiple of the time to “obvious” stationary. For the largest lattices in one dimension typically 10^5 avalanches (more for larger lattices) are rejected as transient and 10^6 retained for statistics in one dimension. In two dimensions, typical numbers are 6×10^6 as transient and 400×10^6 for statistics². As a rule of thumb, at least 3/2 times

¹Once again, I am very grateful to the collaboration with Gunnar Pruessner and the computing support of Andy Thomas and the SCAN team there.

²In hindsight, the lattices used in two dimensions appear rather small compared to one, as the CPU

Table 7.1: The amount of time spent on different lattices for computer simulation. The unit is CPU hour. The amount is the total over all cores running the jobs.

Lattice	Type	Simulation time	Total
LINE	Regular 1D	51,350	168,413
LADD	Regular 1D	54,299	
NNN	Regular 1D	7,943	
FUTA	Regular 1D	54,821	
SQUA	Regular 2D	9,530	72,447
JASQ	Regular 2D	9,408	
ARCH	Regular 2D	11,106	
NOCR	Regular 2D	9,802	
TRIA	Regular 2D	6,305	
KAGO	Regular 2D	10,881	
HONE	Regular 2D	7,842	
MIT	Regular 2D	7,573	
SC	Regular 3D	6,953	38,032
BCC	Regular 3D	8,805	
BCCN	Regular 3D	13,946	
FCC	Regular 3D	3,724	
FCCN	Regular 3D	4,604	
SSTK	Fractal	8,354	75,456
ARRO	Fractal	7,652	
CRAB	Fractal	7,120	
SITE	Fractal	10,186	
EXGA	Fractal	42,144	

the number of sites in the lattice avalanches are removed as transient. For smaller lattices a much larger fraction is taken, as for them, equilibration was often apparently reached within a single chunk. Equilibration can also be observed in the density of particles, which, at least in one dimension, initially grows almost perfectly linearly with only a minute amount of dissipation. Once a certain fraction of sites is occupied, the occupation density displays very little relative fluctuation, for large lattices of around 2×10^{-4} .

The transient serves the additional purpose of warming up the [random-number generator \(RNG\)](#), which was of course seeded uniquely in each instance, except for the mitsubishi lattice, whose one hundred and ninety (190) different seeds equal those used for honeycomb lattice. All results presented in the following are based on the Mersenne twister ([Matsumoto and Nishimura, 1998](#)), which has received some criticism for its correlations across differently seeded instances ([Marsaglia, 2005](#)). The independence of the present results from the [RNG](#) was tested by re-running a few setups with Marsaglia's KISS [RNG](#) ([Marsaglia, 1999](#))¹.

7.2 Representation of the lattices

All the lattices employed in this study (described in Sec. 6.2) are represented by their reduced adjacency matrix. This adjacency matrix is stored directly in [RAM](#) of each node performing the simulation making very quick access to the sites on the lattice.

7.2.1 Reduced adjacency matrix

As introduced earlier in Sec. 7.1, a general routine is written for the Abelian Manna model. A lattice is represented by its adjacency matrix which is to be read by that general routine. That way, we only need to specify the name of the file containing adjacency matrix in the driver of the Manna routine in order to simulate a lattice. This provides a very convenient way to run the simulation on big cluster with many computer nodes.

The adjacency matrix we use in the simulation is not quite the conventional one in which the matrix $\mathcal{A}$ for a lattice $\mathcal{L}$ with N sites has element $\mathcal{A}_{ij} = 1$ if site i and j are nearest neighbour of one another and $\mathcal{A}_{ij} = 0$ otherwise. But rather, we use it

time required is essentially determined by the linear extent, $N^{1/2}$.

¹This test was done and confirmed by Gunnar Pruessner.

Table 7.2: Reduced adjacency matrix of the futatsubishi lattice in Fig. 7.1(d). The first entry indicates the index of a site i . The remaining entries indicate *all* the neighbours of that site i , including the virtual ones (represented by -1).

First entry	Remaining entries	Number of nearest neighbours
0	1 2 -1 -1	4
1	0 3	2
2	0 3	2
3	1 2 4 5	4
4	3 6	2
5	3 6	2
6	4 5 7 8	4
7	6 9	2
8	6 9	2
9	7 8 -1 -1	4

in the reduced form. In this reduced form, each lattice site i is associated with row i th of a two-dimensional array $\mathcal{A}^*$ that contains all the nearest neighbours of i . For example, if site i has four nearest neighbours i_1, i_2, i_3 and i_4 , the entries are $\mathcal{A}_{i1}^* = i_1, \mathcal{A}_{i2}^* = i_2, \mathcal{A}_{i3}^* = i_3$ and $\mathcal{A}_{i4}^* = i_4$. If site i is a dissipative site, *i.e.* it is on the boundary and connects to virtual neighbours, the entry for each virtual neighbour it connects to is -1 . The number of virtual neighbours is chosen in such a way that the total number of neighbours a site has (both real and virtual ones) is the same as that when the site is in the bulk. We can imagine like we have an infinitely large lattice and we cut out a finite piece of it. The missing nearest neighbours of a site are now virtual neighbours of that site. Examples of reduced adjacency matrix are provided in Tables 7.2–7.4.

An advantage of the reduced adjacency matrix $\mathcal{A}^*$ over the original adjacency matrix $\mathcal{A}$ is the huge amount of memory saved. While the original adjacency matrix $\mathcal{A}$ has size $N \times N$, the reduced adjacency matrix $\mathcal{A}^*$ has size $N \times q_i^{max}$ where q_i^{max} is the maximum coordination number of $\mathcal{L}$. The lattice that has largest q_i^{max} is face-centred cube with $q_i^{max} = 18$.

Table 7.3: Reduced adjacency matrix of the mitsubishi lattice in Fig. 7.2(d). The first entry indicates the index of a site i . The remaining entries indicate *all* the neighbours of that site i , including the virtual ones (represented by -1).

First entry	Remaining entries	Number of nearest neighbours
0	1 5 -1	3
1	0 2 6 7 -1 -1	6
2	1 8 -1	3
3	4 8 -1	3
4	3 9 -1 -1 -1 -1	6
5	0 6 10 -1 -1 -1	6
6	5 1 11	3
7	1 8 11	3
8	7 9 2 3 12 13	6
9	4 8 14	3
10	5 11 15	3
11	10 12 6 7 16 17	6
12	11 8 18	3
13	8 14 18	3
14	9 13 19 -1 -1 -1	6
15	10 16 20 -1 -1 -1	6
16	15 11 21	3
17	11 18 21	3
18	17 19 12 13 22 23	6
19	14 18 24	3
20	21 15 -1	3
21	20 22 16 17 -1 -1	6
22	18 21 -1	3
23	24 18 -1	3
24	23 19 -1 -1 -1 -1	6

Table 7.4: Reduced adjacency matrix of the kagomé lattice in Fig. 7.3(d). The first entry indicate the index of a site i . The remaining entries indicate *all* the neighbours of that site i , including the virtual ones (represented by -1).

First entry	Remaining entries	Number of nearest neighbours
0	1 2 -1 -1	4
1	0 2 7 8	4
2	1 3 0 -1	4
3	2 4 5 -1	4
4	3 5 10 11	4
5	4 6 3 -1	4
6	5 -1 -1 -1	4
7	8 1 -1 -1	4
8	7 9 10 1	4
9	8 10 16 17	4
10	9 11 8 4	4
11	10 12 13 4	4
12	11 13 19 20	4
13	12 11 -1 -1	4
14	15 16 -1 -1	4
15	14 16 -1 -1	4
16	15 17 14 9	4
17	16 18 19 9	4
18	17 19 -1 -1	4
19	18 20 17 12	4
20	19 12 -1 -1	4

7.2.2 Labelling scheme for the sites

On a lattice of N sites, all sites are labelled sequentially $0, 1, \dots, N-1$. Since the simulation is implemented in C, the sequential index starts from 0 rather than 1. The strategy to label the sites is to stay as systematic for all lattices as possible. The easiest example is the simple chain lattice whose sites are already in the sequential order, one after the other.

Figure 7.1: Indexing of sites of one-dimensional lattices.

In general, the complexity of such labelling scheme increases in higher dimensions where many degrees of freedom (number of nearest neighbours) and the geometry of the lattices start coming in to play. It is clear that one-dimensional lattices are the easiest ones to deal with (see Fig. 7.1). Besides the trivial case of the simple chain lattice mentioned above, the sites of the ladder lattice is labelled in two layers, one followed by another. The labelling for the next-nearest-neighbour chain lattice stays the same as that of the simple chain lattice, only the new next-nearest-neighbour interactions are added. For the futatsubishi lattice, the labelling finishes for each diamond then moves on to the next one.

In two dimensions, the sites are labelled layer by layer (see Figs. 7.2 and 7.3). The straightforward and clear ones are the square and noncrossing square lattices which have the same labelling of sites. Somewhat less clear but still straightforward (at least by the way the boundaries are chosen) ones include the Archimedes, jagged, triangular, honeycomb and mitsubishi lattices. The most complicated one in two dimensions is the kagomé lattice.

In three dimensions, the term “layer” now refers to the two-dimensional layers. In each of those layers, the sites are labelled similarly to the two-dimensional lattices described above. This is trivial for the simple cubic lattice. However, the situation is much more complicated when the body-centred or face-centred sites are added in. The main issue is that the newly added sites make the number of sites in each layer vary and hence the layer-by-layer labelling very complicated. To deal with these lattices, two different methods¹ were employed. The first one is to treat the body-centred or face-

¹The first one was done by me, and the second one was done by Gunnar Pruessner. Both of them,

Figure 7.2: Indexing of sites of two-dimensional lattices with “clear” layer-by-layer structure.

centred sites as they belong to new cubic lattice (or lattices) immersed in the original one. The labelling is done first for the original cube, followed by the new one(s). This method is reflected in the formula of size N against linear size L of the body-centred or face-centred cubic lattice. In the case of the body-centred cubic lattice, $N = L^3 + (L - 1)^3$, it can be thought of as two cubic lattices, one of size $L \times L \times L$ and the other one $(L - 1) \times (L - 1) \times (L - 1)$, put together. In the case of the face-centred cubic lattice,

of course, give the same results. This serves as a cross-check to ensure no mistake is taking place in the study of three-dimensional lattices.

Figure 7.3: Indexing of sites of two-dimensional lattices with less “clear” layer-by-layer structure.

$N = L^3 + 3L(L-1)^2$, it can be thought of as four cubic lattices, one of size $L \times L \times L$ and the other three $L \times (L-1) \times (L-1)$, put together. The second method introduces the so-called “fake sites” which serve the purpose of making the number of sites in each layer constant. After the introduction of fake sites, the labelling follows the same procedure as in the simple cubic lattice. These two methods, of course, provide the same results and confirm each other.

For fractal lattices, thanks to the nature of fractals generated by the arc-fractal

system, the sites are naturally in sequence. The challenge, however, is the nearest-neighbour interactions which are very irregular. In such cases, the analysis of their structures in Appendix B is very useful. One can, however, employ a “quick-and-dirty” approach. That is to make copies of the lattice in the previous level to build it in the next level¹ (“copy-and-paste”). On doing this, the information of nearest-neighbour interactions is maintained, and only the new interactions (occurring due to joining of copies) need to be added. This approach is indeed employed for complicated fractal lattices like the Sierpiński tetrahedron or extended Sierpiński gasket lattices. In the extended Sierpiński gasket lattice, the “copy-and-paste” is done for each layer. After that, the sequential labelling of layer by layer is done in the same manner as the three-dimensional lattices.

7.3 Numerical techniques

In dealing with sandpile models, we are concerned with the probability distribution of events of different sizes, as described in Secs. 5.3 and 5.4. In numerical simulation, a huge amount of data is generated by the computers. Hence we need very good and useful techniques to analyse the data and extract relevant information about the model.

This section provides several standard and common techniques employed in analysing the data of sandpile models in general.

7.3.1 Data binning

Raw data always come with a lot of noise and often contain irrelevant information. Our task is how to extract the pieces of useful information and learn about the behaviours of the system. In the case of dealing with probability density function, data binning is a very good such technique.

In order to obtain the PDF of events, one collects a data set of sample size M and makes a histogram of them. The probability that an event of size x occurs is then given by the fraction of number of events of that size in the entire data set. Formally, we have

$$\mathcal{P}^{(x)}(x) = \frac{M(x)}{M}, \quad (7.1)$$

¹There is a subtle point here. This is very efficient for finitely ramified fractal lattices. For infinitely ramified fractal lattices like the Sierpiński carpet lattice, it turns out to be very cumbersome (yet, the only doable way). [OPEN PROBLEM](#) see list on page 249

in which $\mathcal{P}^{(x)}(x)$ is the (approximate) probability that an event of size x takes place, $M(x)$ is the number of events of size x in the data set and M is the size of the entire data set. The quantity $\mathcal{P}^{(x)}(x)$ defined this way tends to the *true* underlying PDF of the real system as the sample size approaches the thermodynamic limit $M \rightarrow \infty$.

A common problem, that is, however, usually encountered, is that the events of large (and very large) sizes are scattered. In other words, in plotting the graph of $\mathcal{P}^{(x)}(x)$ against x , the region of $x \gg 1$ is very sparse compared to the others. Hence, it is very hard to tell the likely behaviour of $\mathcal{P}^{(x)}(x)$ in that domain of large x . Especially, when the probability density function $\mathcal{P}^{(x)}(x)$ is expected to take on the power-law form $\mathcal{P}^{(x)}(x) \propto x^{-\tau_x}$ with some finite upper cutoff x_c (the (approximate) value of x beyond which $\mathcal{P}^{(x)}(x)$ is no longer a power-law but rather an exponential decay), this issue becomes very prominent. Not only masking the behaviour of $\mathcal{P}^{(x)}(x)$ around x_c , the noisy data there indeed also mask the value of x_c itself.

In this case, the idea is to perform a data binning. A data binning is indeed to get the average value of $\mathcal{P}^{(x)}(x)$ for some particular x , which is natural because it gives the likely value of $\mathcal{P}^{(x)}(x)$ from many different scattered data points. One advantage of data binning is that information about $\mathcal{P}^{(x)}(x)$ can be reliably extracted even when the data at hand is rather poor. The data binning is performed by grouping the data points into different bins. The bins are placed along the horizontal axis of the size x of events. The bins are labelled $j = 0, 1, \dots$, where a bin j covers the half-open interval $[b^j, b^{j+1})$, with bin parameter $b > 1$. Because probability density function $\mathcal{P}^{(x)}(x)$ of power-law form is usually presented on a double logarithmic scale, the bins are equally separated in that presentation. This is another advantage of data binning¹.

After the data binning, the “binned” distribution is obtained through the following transformation:

$$\overline{\mathcal{P}^{(x)}(\bar{x}^{(j)})} = \frac{M_j}{M \Delta \bar{x}^{(j)}}, \quad (7.2a)$$

$$\bar{x}^{(j)} = \sqrt{x_{min}^{(j)} x_{max}^{(j)}}, \quad (7.2b)$$

in which $x_{min}^{(j)} = \lceil b^j \rceil$ and $x_{max}^{(j)} = \lfloor b^{j+1} \rfloor$ are, respectively, the minimum and maximum integers in the interval $[b^j, b^{j+1})$, and $\Delta \bar{x}^{(j)} = x_{max}^{(j)} - x_{min}^{(j)} + 1$ is the number of integers in that interval; M_j is the number of data points in bin j . The relation in Eq. (7.2b)

¹There are, indeed, several ways of doing data binning. The method being described here is common and usually referred to as exponential or logarithmic binning. Another one is called power law binning (see (Pruessner, 2004b, pp. 91–93) for more details).

indicates that the “binned” variable $\bar{x}^{(j)}$ is the geometric mean of the avalanche sizes in bin j . One can indeed use a different relation and that shouldn’t cause any noticeable change in the resulting plot.

Figure 7.4 illustrates the probability of avalanche size s for the Abelian Manna model on the arrowhead lattices of different system sizes N before and after performing data binning. The plot after data binning shows a very clean and clear power-law behaviour from which one can estimate the exponent τ of the distribution $\mathcal{P}^{(s)}(s) \propto s^{-\tau}$. In general, when one is concerned with the PDF, the two observables duration t and size s require binning since they are not bounded, *i.e.* the avalanches can have arbitrarily large duration or size, while the observable area a does not so because it is controlled by the system size, *i.e.* $a \leq N$.

7.3.2 Moment analysis

With the data binning, one can see a very robust power-law behaviour of the PDF of observables in sandpile model. However, the estimation of the distribution exponent τ_x in $\mathcal{P}^{(x)}(x) \propto x^{-\tau_x}$ is not at all reliable. There are couple of reasons for this. The first one would be that for finite systems, the power-law distribution is very nice and clean but not a pure one, *i.e.* the power-law behaviour is irrelevant outside a certain domain (scaling region) of the distribution. In order to get the distribution exponent τ_x directly on the double logarithmic plot, one has to consistently determine the lower and upper cutoff of the scaling region. There is clearly no reliable way of doing this and that would render the exponent determined with a wild error bar. Another reason against the determination of exponent through directed measure of the slope the probability density function in the double logarithmic scale is that it depends on the size of the system. What one usually does is to estimate τ_x for different system sizes and then extrapolate to obtain the (presumed) value of τ_x for system of infinite size. In the early days of Self-Organized Criticality, this appeared to be the sole method used in the literature (*e.g.* Manna, 1991b; Kutnjak-Urbanc *et al.*, 1996).

However, it has been well-known from ordinary critical phenomena that a standard and well developed technique is available to determine the exponent in such situation, which is the moment analysis¹. The technique is discussed in details in this section. This is the main technique employed in analysing the data throughout this

¹It is a bit surprising that the technique was not used in SOC literature until introduced to the field by De Menech *et al.* (1998); Tebaldi *et al.* (1999).

(a) Probability distribution from raw histogram

(b) Probability distribution after data binning

Figure 7.4: The probability distribution of avalanche size s for the Abelian Manna model on the arrowhead lattice of different system sizes N before and after performing data binning. Various values of the bin parameter b in the range $1.1 \leq b \leq 1.5$ are used. The corresponding results do not differ much showing the consistency of the method and its insensitivity to the parameter (they actually coincide in the plot, causing the thickness of the curves). The distribution after data binning reveals a very clean and clear power-law behaviour with cutoff on finite-size systems (see further details in Sec. 7.3.2.2).

work.

One advantage of moment analysis is that it makes the numerical simulation of the model much more efficient by allowing direct storing of (a necessary number of) moments of the probability distribution rather than the entire histogram.

7.3.2.1 Moment and its statistical error

The object of moment analysis is, certainly, the moment of the observables avalanche area a , size s and duration t . While the moments can be easily determined using the definition in Eq. (5.2), the main concern is how to reliably determine their proper error bars.

The first moments of all three observables generally pose a problem, possibly because they were determined with an accuracy so high that it was virtually impossible to account for their corrections to scaling (see below). The statistical error of all moments is calculated on the basis of chunks whose size was chosen as to ensure their independence. This is tested by firstly calculating their autocorrelation function and, secondly, by successively increasing their size, *i.e.* merging them, to probe whether the statistical error derived on their basis is affected by this operation. In fact, for the largest system sizes (and thus smallest chunk sizes) tested, some correlations are visible (correlation length of about 0.7 chunks), and we decide to merge ten consecutive chunks throughout (for one-, two-, three-dimensional and fractal systems). It turns out, however, that only the statistical errors of the first moments are noticeably, yet still insignificantly, affected at all by this operation.

It is straightforward to determine estimates and their estimated variance on the basis of independent chunks. Denoting the observable in the i th chunk by c_i , an unbiased, consistent estimator (Brandt, 1999) of its population average $\langle c \rangle$ is

$$\frac{1}{M_c} \sum_{i=1}^{M_c} c_i \quad (7.3)$$

given a sample size of M_c . In the following, we will denote that estimate itself by $\langle c \rangle$. An unbiased, consistent estimator of the variance of this estimator is

$$\frac{1}{M_c - 1} \left[\frac{1}{M_c} \sum_{i=1}^{M_c} c_i^2 - \left(\frac{1}{M_c} \sum_{i=1}^{M_c} c_i \right)^2 \right]. \quad (7.4)$$

These estimates are used as input values for the estimation of the exponents, based on the scaling assumption discussed in the following. For completeness, covariances of the estimators for the averages of two observables c and c' , as used in Eq. (7.14), are estimated using

$$\text{cov}(c, c') = \frac{1}{M_c - 1} \left[\frac{1}{M_c} \sum_{i=1}^{M_c} c_i c'_i - \left(\frac{1}{M_c} \sum_{i=1}^{M_c} c_i \right) \left(\frac{1}{M_c} \sum_{i=1}^{M_c} c'_i \right) \right]. \quad (7.5)$$

As detailed below, the finite-size scaling of the moments of each observable was determined in the form

$$\langle x^n \rangle = \text{amplitude} \times L^{\mu_n^{(x)}} + \text{corrections} \quad (7.6)$$

where x stands for the size s , the area a or the duration t . The combined information of the scaling of a number of different moments n is then used to estimate the finite-size scaling exponents. Due to practical issues in analysing the data, even though we implemented the first eight moments, we use only moments 2 to 4 (i.e. $\mu_2^{(s)}$, $\mu_3^{(s)}$, $\mu_4^{(s)}$) when fitting the avalanche dimension (using $2 = (2 - \tau)D$ to determine τ) and moments 2 to 5 to fit exponents characterising the scaling of avalanche area and duration (all exponents are defined below) in regular integer dimensions. For fractal lattices, we use moments 2 to 5 for all exponents. Different moments estimated on the basis of a single Monte-Carlo simulation are not independent. To account for that, we (rather generously) multiply the statistical error of each moment by the square root of the number of moments considered simultaneously, as if each moment was determined independently from the others (Pruessner and Moloney, 2003). For example, the statistical error of the second, third and fourth moment of the avalanche size is multiplied by $\sqrt{3}$, before $\mu_2^{(s)}$, $\mu_3^{(s)}$ and $\mu_4^{(s)}$ respectively are determined. It seems that these correlations are frequently ignored in the literature. This procedure does *not* account for correlations in the estimates of moments of different observables and thus our results for the finite-size scaling exponents for different observables are not independent. Multiplying again by the square root of the number of different observables considered, may, however, seriously overestimate the impact of these correlations.

7.3.2.2 Finite-size scaling

Recall the scaling behaviour of the model in Sec. 5.4, its probability density function $\mathcal{P}^{(x)}(x)$ follows (Eq. (5.1))

$$\mathcal{P}^{(x)}(x; L) = a_x x^{-\tau_x} \mathcal{G}_x\left(\frac{x}{b_x L^{D_x}}\right), \quad (7.7)$$

whose moments are

$$\langle x^n \rangle(L) = \int_0^\infty dx \mathcal{P}^{(x)}(x; L) x^n = a_x (b_x L^{D_x})^{n+1-\tau_x} \int_0^\infty dy y^{n-\tau_x} \mathcal{G}_x(y) \quad (7.8)$$

asymptotically in large L , provided that $n + 1 - \tau > 0$.

To determine D_x (and τ_x , if independent), the leading order scaling of the moments according to Eq. (7.6) is estimated, fitting the resulting exponents $\mu_n^{(x)}$ against $D_x(n + 1 - \tau_x)$, without allowing for any further corrections. The results for integer dimensions are shown in Tables 7.6 and 7.8.

Since $\langle s \rangle \propto L^2$ (refer back to Sec. 5.4, $\mu_1^{(s)} = 2$ and under the assumption of gap scaling $2 = (2 - \tau)D$). This identity is used in the fitting of the avalanche dimension, i.e. $\mu_n^{(s)}$ for $n = 2, 3, 4$ is fitted against $2 + D(n - 1)$. At the same time, comparing the estimate for $\mu_1^{(s)}$ (Tables 7.6 and 7.8) to the exact value 2 allows us to assess the fitting procedure, in particular the form of the corrections discussed below.

The scaling law $-\Sigma = D(\tau - 1) = z(\alpha - 1) = D_a(\tau_a - 1)$ is also tested as a scaling hypothesis below. This is not a mathematical identity (Paczuski and Boettcher, 1996) (in fact, it seems to be broken in the original Manna Model (Lübeck, 2000)). The exponent Σ can be seen as a replacement for the exponents τ , α and τ_a , as $\mu_n^{(s)} = nD + \Sigma$, $\mu_n^{(t)} = nz + \Sigma$, $\mu_n^{(a)} = nD_a + \Sigma$ for $n > \tau - 1$, $n > \alpha - 1$ and $n > \tau_a - 1$ respectively.

Finally, the scaling relation $D_a = d$ for the assumption on the compactness of avalanches in the model is also tested and serves as another criterion to assess the fitting scheme and the quality of the exponents extracted (strictly, $D_a > d$ is prohibited). There is no mathematical proof for this feature, yet numerically it is well verified, see Tables 7.6 and 7.8.

The exponents $\mu_n^{(x)}$ characterise the asymptotic scaling of the moments in large L . It is widely known (Wegner, 1972), however, finite lattices suffer from finite-size corrections, which manifest themselves as sub-leading terms to be included on the right hand side of Eq. (7.6) (Chessa et al., 1999). *A priori*, the structure of the corrections is not known, yet they have a marked impact on the quality of the results, as they are

imposed when fitting the data. **OPEN PROBLEM** see list on page 250

Given that there is often no natural way of defining the linear extent of a lattice (see Sec. 6.1.2), we decide to replace L (as in Eq. (7.6)) by $N^{1/d}$. We consider a host of different fitting functions (with coefficients $A_n^{(x)}$, $B_n^{(x)}$ and $C_n^{(x)}$ to be fitted), such as

$$\langle x^n \rangle (N) = A_n^{(x)} N^{\mu_n^{(x)}/d}, \quad (7.9a)$$

$$\langle x^n \rangle (N) = A_n^{(x)} N^{\mu_n^{(x)}/d} + B_n^{(x)} N^{\mu_n^{(x)}/d - \omega_{x,n}/d}, \quad (7.9b)$$

$$\langle x^n \rangle (N) = A_n^{(x)} N^{\mu_n^{(x)}/d} + B_n^{(x)} N^{\mu_n^{(x)}/d - 1/4}, \quad (7.9c)$$

$$\langle x^n \rangle (N) = A_n^{(x)} N^{\mu_n^{(x)}/d} + B_n^{(x)} N^{\mu_n^{(x)}/d - 1/2} + C_n^{(x)} N^{\mu_n^{(x)}/d - 1}, \quad (7.9d)$$

and eventually settle for

$$\langle x^n \rangle (N) = A_n^{(x)} N^{\mu_n^{(x)}/d} + B_n^{(x)} N^{\mu_n^{(x)}/d - 1/4} + C_n^{(x)} N^{\mu_n^{(x)}/d - 1/2} \quad (7.10)$$

which yields particularly good estimates in regular integer dimensions. In particular, $\mu_1^{(s)} = 2$ is reproduced quite reliably. The quality of the estimates is assessed on the basis of the goodness-of-fit determined in the Levenberg-Marquardt least square fitting routine (Press *et al.*, 1992, and Appendix E.1). For the vast majority of moments and lattices, we could have dropped the last term in Eq. (7.10) arriving at Eq. (7.9c) and still achieve a goodness-of-fit (q -value) of more than 0.9. However, the first moments of the avalanche size, whose finite-size scaling exponent is the only exactly known one, is particularly poorly fitted without that term. For consistency, we decide to fit *all* moments using Eq. (7.10), achieving typically q -values of better than 0.9, suggesting that we overestimate the statistical errors. In all result tables, fits that has a q -value of less than 0.1 are marked as such.

For fractal dimensions, we employ

$$\langle x^n \rangle (N) = A_n^{(x)} N^{\mu_n^{(x)}/d} + B_n^{(x)} N^{\mu_n^{(x)}/d - 1} + C_n^{(x)} N^{\mu_n^{(x)}/d - 2} \quad (7.11)$$

which gives very high q -value. One can argue that for each fractal, we have only four system sizes, and hence, any fit with four free parameters would give high q -value. In fact, in most cases for fractals, we can indeed drop the last term in Eq. (7.11) without changing the estimated values much. And given that we use a systematic guess of the initial values (fitted results with less parameters are used to feed the next fit with one more added parameter, see later), we are confident that the fitting function in

Eq. (7.11) is good enough for fractal lattices and provides reasonable error bars (other fitting functions seem to provide pretty large error bars).

To reduce the impact of a possible dependence on the (arbitrary) choice of the initial values of the free parameters, the number of terms in the fitting function is increased successively starting from Eq. (7.9a), using the estimates of the parameters in the previous fit as the initial value of the same parameters in the next.

The simplicity of Eq. (7.10) means that we have to drop results for small system sizes, which suffer from stronger finite-size corrections, yet were determined with much greater accuracy than those of the bigger systems. This is a common theme in the present work: Moments in small system sizes were determined with such great accuracy that very many (*a priori* unknown) correction terms would have to be included to account for all such details. At the same time, it makes little sense to have almost as many free parameters in the fitting function as there are data points to fit. In order to retain the goodness-of-fit as a meaningful device to determine the quality of the fit, we therefore remove the smallest system sizes from the procedure, keeping only the largest ones listed in Tables 6.2–6.5¹.

Increasing the system size in order to suppress correction terms comes at the price of increased relative error if $\tau_x > 1$. According to Eq. (7.6) and Eq. (5.4), the variance of the n th moment has leading order $L^{D_x(2n+1-\tau_x)}$, *i.e.* the relative error scales like $L^{D_x(\tau_x-1)/2}$. Moreover, correlations are expected to die off after L^{D-d} , which reduces the number of effectively independent measurements with increasing system size.

Some of the fitting functions we tested include logarithmic corrections² as suggested by Dickman and Campelo (2003). However, these fits turn out to be quite volatile with a strong dependence on the initial guess. We aim for a fitting scheme that does well for *all* data, even at the expense of overly large statistical errors, rather than selecting it individually to accommodate each data set's special features and thus (possibly) introducing an undue bias in the results. We therefore decided against the inclusion of logarithmic corrections in the fitting function.

¹We keep the most number of system sizes for which the fit produces reasonably good results with as few number of free parameters as possible. It turns out that we need seven (largest) system sizes in one and two dimensions, six (largest) system sizes in three dimensions, and four (largest) system sizes in fractal dimensions.

²Thanks to an anonymous referee of (Huynh *et al.*, 2011) for reminding us of this.

7.3.2.3 Moment ratios

Besides making use of the moments themselves in extracting the critical exponents, one can also gain more knowledge about the critical behaviour of the model by studying the moment ratios. Ratios of products of moments (denoted as $g_n^{(x)}$), which (to leading order) are independent from L , characterise the scaling function $\mathcal{G}_x$, Eq. (5.1), directly. In equilibrium phase transition, the so-called Binder-cumulant (Binder, 1981a,b) is the best known such ratio, signalling the deviation from a Gaussian distribution of the order parameter around the critical point. There are many ways of constructing suitable moment ratios; assuming $\langle x^n \rangle \propto L^{D_x(n+1-\tau_x)}$, it is easy to see that any ratio of products of moments, which has the same number of moments (to cancel $\tau_x \neq 1$) and the same sum of orders of moments (to cancel D_x) in numerator and denominator leads to a nonscaling quantity (that is independent of the system size). Since the second moment is positive and bounded away from 0, traditionally moment ratios are formed by dividing by a power of it. Moreover, in many phase transitions, the order parameter follows a distribution with $\tau_x = 1$, which removes the constraint of having the same number of moments in the numerator and the denominator.

While the sets $\frac{\langle x^{n-m} \rangle \langle x^{n+m} \rangle}{\langle x^n \rangle^2}$ attracts by its simplicity and symmetry, the set

$$g_n^{(x)} = \frac{\langle x^n \rangle \langle x \rangle^{n-2}}{\langle x^2 \rangle^{n-1}} \quad (7.12)$$

has the particularly nice feature that $g_1^{(x)} = g_2^{(x)} = 1$ by definition, which fixes the metric factors a_x and b_x in Eq. (5.1) by *imposing* for $n = 1, 2$

$$g_n^{(x)} = \int_0^\infty dy y^{n-\tau_x} \mathcal{G}_x(y), \quad (7.13)$$

(see Eq. (5.3)) which is then consistent with Eq. (7.12) for all n .

The statistical error $\sigma(g_n^{(x)})$ of the estimator of Eq. (7.12) is to leading order in the

sample size given by

$$\begin{aligned} \sigma^2 \left(g_n^{(x)} \right) = & \left(g_n^{(x)} \right)^2 \left(\frac{\text{cov}(x^n, x^n)}{(\langle x^n \rangle)^2} + (n-2)^2 \frac{\text{cov}(x^1, x^1)}{(\langle x \rangle)^2} + (n-1)^2 \frac{\text{cov}(x^2, x^2)}{(\langle x^2 \rangle)^2} \right. \\ & - 2(n-2)(n-1) \frac{\text{cov}(x^2, x^1)}{\langle x \rangle \langle x^2 \rangle} - 2(n-1) \frac{\text{cov}(x^2, x^n)}{\langle x^2 \rangle \langle x^n \rangle} \\ & \left. + 2(n-2) \frac{\text{cov}(x^1, x^n)}{\langle x \rangle \langle x^n \rangle} \right), \end{aligned} \quad (7.14)$$

which matches perfectly (typically the first three or four significant digits) the error as found by the subsampling using chunks, *i.e.* determining $g_n^{(x)}$ for each chunk and estimating the error by the square root of its variance over the number of chunks. In Eq. (7.14) $\langle x^n \rangle$ strictly denotes the *estimator* of the n th moment and $\text{cov}(x^n, x^m)$ the estimated covariance of the n th and m th moment, see Eq. (7.5). Consistent with the preceding discussion, we use averages and statistical errors derived from chunks.

All lattices in two dimensions were set up with the intention of creating an aspect ratio of 1 : 1, which is trivial as long as the lattice has a four-fold symmetry. In particular for lattices without that symmetry, such as the triangular lattice (Fig. 6.5(a)), the kagomé lattice, the honeycomb lattice (Fig. 6.5(c)) and the mitsubishi lattice, but also, say, the jagged lattice (Fig. 6.4(b)), the aspect ratio might deviate slightly from unity and converge to 1 : 1 only with increasing system size. In any case, the aspect ratio might be more reasonably be defined using the Manhattan distance across the lattice¹ (refer back to Sec. 6.1.3). It is well known that universal scaling exponents are generally independent from the aspect ratio, whereas finite-size scaling functions are not (Privman *et al.*, 1991). Therefore, deviations of the moment ratios in particular in case of the lattices listed above are expected (but did in fact not materialise). Surprisingly, even when there is every reason to assume that no such problem can occur in one dimension, their moment ratios proved particularly difficult to analyse.

7.3.3 Data collapse

When dealing with finite-size scaling, the most commonly used technique is the so-called data collapse. The idea is to produce a collapse of data for different system sizes onto a single curve. In this study, since moment analysis proves to be a very reliable and efficient technique in extracting the critical exponents of the probability

¹The Manhattan distance between two points is defined to be the sum of the absolute differences of their coordinates (Krause, 1987).

Figure 7.5: Collapse of (binned) data for Sierpiński arrowhead lattice. The coincidence of different curves confirms the estimated values of the critical exponents obtained from moment analysis. The values of the exponents used are $\tau = 1.17$ and $D = 2.792$.

distributions, data collapse appears as a cross-check to moment analysis, *i.e.* we use the exponents obtained from moment analysis to see if the curves of PDF for different system sizes collapse onto a single one.

There is, however, no unique way of performing a data collapse. One way to produce a very good data collapse is to consider $\mathcal{P}^{(x)}(x) x^{\tau_x}$ against $\frac{x}{L^{D_x}}$ ¹. Figure 7.5 illustrates the collapse of five curves of (binned) probability distribution of avalanche size s of the Abelian Manna model on the arrowhead lattice of different system sizes.

The collapse of the curves or the choice of the $\mathcal{P}^{(x)}(x) x^{\tau}$ against $\frac{x}{L^{D_x}}$ is not a magic at all. Rather, it arises naturally when one looks at the *ansatz* for the scaling form of the model in Eq. (5.1)

$$\mathcal{P}^{(x)}(x; L) = a_x x^{-\tau_x} \mathcal{G}_x \left(\frac{x}{b_x L^{D_x}} \right). \quad (7.15)$$

What is being plotted in the graph is essentially the scaling function $\mathcal{G}_x$ when one realises that the horizontal axis $\frac{x}{L^{D_x}}$ is the argument of $\mathcal{G}_x$ and the vertical axis $\mathcal{P}^{(x)}(x) x^{\tau_x}$ is obtained by multiplying x^{τ_x} on both sides of Eq. (5.1). As a comparison to the original

¹This was used in (Huynh *et al.*, 2010) to produce a data collapse for distribution of avalanche size s of the Abelian Manna model on the arrowhead lattice.

plot of plain probability distribution $\mathcal{P}^{(x)}(x)$ against x , one observes that the horizontal axis is rescaled by a factor $\frac{1}{L^{D_x}}$ while the vertical one by a factor x^{τ_x} . The first one, controlled by the exponent D_x , is to align the curves vertically. The second one, controlled by the exponent τ_x , is to align the curves horizontally, making them level. Only with the *correct* values of τ_x and D_x , the curves are aligned properly, both in horizontal and vertical directions, and coincide.

Like data binning, in order to perform a data collapse, the entire histogram must be stored and would consume more memory than the moment analysis. Nevertheless, this technique gives a very good way to cross-check the moment analysis. All of them provide us with great confidence in the critical exponents extracted from the data.

7.4 Numerical results

7.4.1 Avalanche exponents

7.4.1.1 Regular integer dimensions

After stripping off the transient, merging chunks as discussed above, deriving average moments and errors using the procedures described above, the scaling exponents $\mu_n^{(x)}$ are fitted using Eq. (7.10) for the three different observables, $x \in \{s, t, a\}$, avalanche size, duration and area respectively in one and two dimensions. In three dimensions, for avalanche size s and duration t , Eq. (7.10) apparently overestimates the error bars. Hence, in such cases, Eq. (7.9c) is used instead. Table 7.5 summarises the fitting procedure we employ in this study for different systems. As mentioned above, correlations between moments are taken into account by multiplying the statistical error of the moment by the square root of the number of moments considered. Allowing for no further corrections, $\mu_n^{(s)}$ are fitted for each of the different lattices separately against

$$\mu_n^{(s)} = 2 + D(n - 1) \quad (7.16)$$

for $n = 2, 3, 4$. The results are collected in Table 7.6. The scaling law $2 = (2 - \tau)D$ used in Eq. (7.16) is probed independently; $\mu_1^{(s)} = 2$ is a mathematical identity, but valid only *asymptotically* and the deviation of $\mu_1^{(s)}$ from 2 can therefore serve as an indicator to assess the quality of the fitting routines and the data and can help confirming that “asymptotia is reached”¹. The estimate for $\mu_1^{(s)}$ on the basis of Eq. (7.10) (Eq. (7.9c) in

¹The word “asymptotia” is suggested by Gunnar Pruessner.

three dimensions, see Table 7.5) is shown in Table 7.6 alongside the other finite-size scaling exponents. The exponent τ stated in Table 7.6 is derived from the estimate of D using $\tau = 2 - \frac{2}{D}$.

For all observables (size, area and duration), the first moments turn out to be problematic. According to Eq. (5.2), moments $n < \tau - 1$ remain finite in the thermodynamic limit, consistent with our observation that smaller moments generally require more correction terms. The average avalanche size is particularly difficult to handle, which was determined, due to the presence of anti-correlations (Pruessner, 2004a; Morand *et al.*, 2010), with incredible precision, typically with a relative error of the order 10^{-5} . We therefore decide to omit first moments from the determination of the finite-size scaling exponents throughout.

For $x \equiv t$ and $x \equiv a$, no scaling laws are used (even when $D_a = d$ is generally assumed to hold) and so the exponents $\mu_n^{(x)}$ are fitted against

$$\mu_n^{(t)} = z(n + 1 - \alpha), \quad (7.17a)$$

$$\mu_n^{(a)} = D_a(n + 1 - \tau_a) \quad (7.17b)$$

for $n = 2, 3, 4, 5$.

Fitting moments beyond $n = 5$ proves very difficult. We decide to drop all moments beyond the fourth for avalanche size and beyond the fifth for the avalanche area and duration, as $\mu_n^{(x)}$ become clearly dependent on the choice of the initial values of the fitting parameters. As mentioned above, in all fitting schemes used, we increase the number of free parameters successively and use the estimates of the fitting parameters of one scheme as the initial values for its extension. For example, we use $A_n^{(x)}$ and $\frac{\mu_n^{(x)}}{d}$ from a fit against Eq. (7.9a) as initial values in a fit against Eq. (7.9c), which in turn produce the initial values of $A_n^{(x)}$, $B_n^{(x)}$ and $\frac{\mu_n^{(x)}}{d}$ to fit with Eq. (7.10). We observe this procedure in all fitting schemes discussed below.

As the variance of the n th moment scales like $L^{D_x(2n+1-\tau_x)}$, its numerical estimate is increasingly affected by the floating point precision (double-extended throughout) — equivalently, the typical largest measurement of the n th moment scales like the n th power of the cutoff, L^{nD_x} , which for $n = 6$, $D_x = 2.25$ and $L = 2^{16}$ is 2^{216} . Given that the smallest event size is 0, this is to be compared to the 64 bits in the mantissa on a long double on the x86 architecture¹.

¹Thanks to Gunnar Pruessner for pointing out this to me.

Table 7.5: Summary of fitting procedures employed in this study for different systems.

Lattice	Number of systems	Observable	Number of moments	Fitting function
Regular 1D	7	s	$3 (\mu_2^{(s)} - \mu_4^{(s)})$	Eq. (7.10): $\langle x^n \rangle (N) = A_n^{(x)} N^{\frac{\mu_n^{(x)}}{d}} + B_n^{(x)} N^{\frac{\mu_n^{(x)}}{d} - \frac{1}{4}} + C_n^{(x)} N^{\frac{\mu_n^{(x)}}{d} - \frac{1}{2}}$
		t	$4 (\mu_2^{(t)} - \mu_5^{(t)})$	
		a	$4 (\mu_2^{(a)} - \mu_5^{(a)})$	
Regular 2D	7	s	$3 (\mu_2^{(s)} - \mu_4^{(s)})$	Eq. (7.10): $\langle x^n \rangle (N) = A_n^{(x)} N^{\frac{\mu_n^{(x)}}{d}} + B_n^{(x)} N^{\frac{\mu_n^{(x)}}{d} - \frac{1}{4}} + C_n^{(x)} N^{\frac{\mu_n^{(x)}}{d} - \frac{1}{2}}$
		t	$4 (\mu_2^{(t)} - \mu_5^{(t)})$	
		a	$4 (\mu_2^{(a)} - \mu_5^{(a)})$	
Regular 3D	6	s	$3 (\mu_2^{(s)} - \mu_4^{(s)})$	Eq. (7.9c): $\langle x^n \rangle (N) = A_n^{(x)} N^{\frac{\mu_n^{(x)}}{d}} + B_n^{(x)} N^{\frac{\mu_n^{(x)}}{d} - \frac{1}{4}}$
		t	$4 (\mu_2^{(t)} - \mu_5^{(t)})$	
		a	$4 (\mu_2^{(a)} - \mu_5^{(a)})$	
Fractal	4	s	$4 (\mu_2^{(s)} - \mu_5^{(s)})$	Eq. (7.11): $\langle x^n \rangle (N) = A_n^{(x)} N^{\frac{\mu_n^{(x)}}{d}} + B_n^{(x)} N^{\frac{\mu_n^{(x)}}{d} - 1} + C_n^{(x)} N^{\frac{\mu_n^{(x)}}{d} - 2}$
		t	$4 (\mu_2^{(t)} - \mu_5^{(t)})$	
		a	$4 (\mu_2^{(a)} - \mu_5^{(a)})$	

The exponent Σ_x can be derived either from the definition Eq. (5.5) through D_x and τ_x or by independently fitting $\mu_n^{(x)}$ against $nD_x + \Sigma_x$. We take that approach for $x \equiv t$ and $x \equiv a$. Using the three different observables size, duration and area, provides effectively estimates for $D(\tau - 1)$, $z(\alpha - 1)$ and $D_a(\tau_a - 1)$ respectively. In the case of $D(\tau - 1)$ this is in fact exactly $D - 2$, since we impose $D(2 - \tau) = 2$ when estimating D . The entry for Σ_s in Table 7.6 is therefore derived from the estimate for D . Except for τ , all other entries in Table 7.6 are based on fitting $\mu_n^{(x)}$ directly.

The fitting function Eq. (7.10) produces reliable, most consistent and robust results. And at the same time it produces very large q -factors. That fact might suggest that we were too generous either with estimating the statistical errors or with the number of free parameters. We therefore also tried Eq. (7.9c), which, however, gave partly inconsistent results. In particular estimates for $\mu_1^{(s)}$ deviated from the exact value 2 by about 30 standard deviations (although the relative error was only 3×10^{-3}). For the simple chain the avalanche size moments tested display a poor quality of fit using Eq. (7.9c), as do the first moments of the avalanche size for all lattices except for the next-nearest-neighbour chain lattice, the futatsubishi lattice and the Archimedes lattice (Fig. 6.4(c)). The results for the fits against Eq. (7.9c) are also summarised in Table 7.7. A bracket $[\cdot]$ indicates finite-size scaling exponents not being fitted with a goodness-of-fit better than 0.1.

In general, the futatsubishi lattice and the simple chain are particularly difficult to fit using whichever fitting function.

In order to extract the exponent of the sub-leading terms, the remainder $\langle x^n \rangle(N) - A_n^{(x)} N^{\mu_n^{(x)}/d}$ is fitted against

$$\tilde{B}_n^{(x)} N^{\mu_n^{(x)}/d - \omega_{x,n}/d} \quad (7.18)$$

determining $\omega_{x,n}$. This procedure is aiming much more at a qualitative result rather than a quantitative one and generated rather noisy estimates. The futatsubishi and the triangular lattices prove particularly difficult to handle. The data in $d = 1$ produce, unfortunately quite inconsistently $\omega_{s,n} \approx 0.28$, $\omega_{t,n} \approx 0.20$ and $\omega_{a,n} \approx 0.20$, while $d = 2$ produce, slightly more consistently $\omega_{s,n} \approx 0.23$, $\omega_{t,n} \approx 0.32$ and $\omega_{a,n} \approx 0.47$ fairly independent of lattice and n (but more reliably for large n and observables other than the avalanche size)¹. These exponents could in turn be used in Eq. (7.9b) to fit the data for $\mu_n^{(x)}$ at fixed $\omega_{x,n}$. The resulting overall estimates for the finite-size scaling exponents are also shown in Table 7.7.

¹We abandon doing this for three and fractal dimensions, though.

Table 7.6: Summary of all avalanche exponents in the seventeen different lattices in regular integer dimensions, using Eq. (7.10) (Eq. (7.9c) for s and t in three dimensions, see Table 7.5). The estimates for τ and $D(\tau - 1)$ are *not* determined by fitting the data, but through the scaling relation $D(2 - \tau) = 2$. The estimates for $\mu_1^{(s)}$ verify this scaling relation. The estimates in the last three columns should coincide under the narrow-joint-distribution assumption, Eq. (5.6). Estimates for different observables are *not* independent.

lattice	d	D	τ	z	α	D_a	τ_a	$\mu_1^{(s)}$	$-\Sigma_s$	$-\Sigma_t$	$-\Sigma_a$
LINE	1	2.27(2)	1.117(8)	1.450(12)	1.19(2)	0.998(4)	1.260(13)	2.000(4)	0.27(2)	0.27(3)	0.259(14)
LADD	1	2.24(2)	1.108(9)	1.44(2)	1.18(3)	0.998(7)	1.26(2)	1.989(5)	0.24(2)	0.26(5)	0.26(2)
NNN	1	2.33(11)	1.14(4)	1.48(11)	1.22(14)	0.997(15)	1.27(5)	1.991(11)	0.33(11)	0.3(2)	0.27(5)
FUTA	1	2.24(3)	1.105(14)	1.43(3)	1.16(6)	0.999(15)	1.24(5)	2.008(11)	0.24(3)	0.23(9)	0.24(5)
SQUA	2	2.748(13)	1.272(3)	1.52(2)	1.48(2)	1.992(8)	1.380(8)	1.9975(11)	0.748(13)	0.73(4)	0.76(2)
JASQ	2	2.764(15)	1.276(4)	1.54(2)	1.49(3)	1.995(7)	1.384(8)	2.0007(12)	0.764(15)	0.76(5)	0.77(2)
ARCH	2	2.76(2)	1.275(6)	1.54(3)	1.50(3)	1.997(10)	1.382(11)	2.001(2)	0.76(2)	0.78(6)	0.76(3)
NOCR	2	2.750(14)	1.273(4)	1.53(2)	1.49(2)	1.992(7)	1.381(8)	2.0005(12)	0.750(14)	0.75(4)	0.76(2)
TRIA	2	2.76(2)	1.275(5)	1.51(2)	1.47(3)	2.003(11)	1.388(12)	1.997(2)	0.76(2)	0.71(6)	0.78(3)
KAGO	2	2.741(13)	1.270(4)	1.53(2)	1.49(2)	1.993(8)	1.381(9)	1.9994(12)	0.741(13)	0.75(5)	0.76(2)
HONE	2	2.73(2)	1.268(6)	1.55(4)	1.51(4)	1.990(13)	1.376(14)	2.000(2)	0.73(2)	0.79(8)	0.75(3)
MIT5	2	2.75(2)	1.273(6)	1.54(3)	1.50(4)	1.999(12)	1.387(12)	1.998(2)	0.75(2)	0.77(7)	0.77(3)
SC	3	3.38(2)	1.408(3)	1.779(7)	1.784(9)	3.04(5)	1.45(4)	2.0057(5)	1.38(2)	1.395(16)	1.36(13)
BCC	3	3.36(2)	1.404(4)	1.777(8)	1.78(1)	2.99(2)	1.444(18)	2.0030(5)	1.36(2)	1.390(19)	1.33(6)
BCCN	3	3.38(3)	1.408(4)	1.776(9)	1.783(11)	3.01(3)	1.44(3)	2.0041(6)	1.38(3)	1.39(2)	1.32(7)
FCC	3	3.35(4)	1.402(8)	1.765(16)	1.78(2)	3.1(2)	1.48(14)	2.0035(11)	1.35(4)	1.37(4)	1.5(5)
FCCN	3	3.38(4)	1.408(7)	1.781(14)	1.787(18)	3.00(4)	1.44(3)	2.0051(8)	1.38(4)	1.40(3)	1.32(9)

Table 7.7: Comparison of overall estimates of scaling exponents in one and two dimensions using different fitting functions. Only the fits using Eq. (7.10), based on the data in Table 7.6 and shown in bold, are fully reliable. Entries for Eq. (7.9c) and Eq. (7.9b) are for comparison to other estimates only. Fits with a goodness of less than 0.1 are marked by [.]. The estimate for Σ , Eq. (5.6), is based on all estimates for $D(\tau - 1)$, $z(\alpha - 1)$ and $D_a(\tau_a - 1)$ in Table 7.6. Their correlation is taken into account by multiplying their respective error by $\sqrt{3}$.

d	function	D	τ	z	α	D_a	τ_a	$-\Sigma$
1	Eq. (7.10)	2.253(14)	1.112(6)	1.445(10)	1.18(2)	0.998(3)	1.259(11)	0.26(2)
1	Eq. (7.9c)	[2.265(4)]	[1.117(2)]	[1.449(2)]	1.172(3)	1.0000(6)	1.249(2)	0.249(3)
1	Eq. (7.9b)	[2.2520(3)]	[1.11188(11)]	[1.4632(6)]	[1.219(2)]	1.0000(8)	1.276(2)	[0.297(3)]
2	Eq. (7.10)	2.750(6)	1.273(2)	1.532(8)	1.4896(96)	1.995(3)	1.382(3)	0.761(13)
2	Eq. (7.9c)	2.7698(12)	1.2779(3)	1.5407(14)	1.498(2)	1.9990(5)	1.3843(6)	0.768(2)
2	Eq. (7.9b)	[2.7673(3)]	[1.27728(7)]	1.541(2)	1.501(2)	1.9985(6)	1.3853(6)	0.770(2)

Table 7.8: Overall estimates of scaling exponents in one, two and three dimensions. The fits are based on the data in Table 7.6. Entries for The estimate for Σ , Eq. (5.6), is based on all estimates for $D(\tau - 1)$, $z(\alpha - 1)$ and $D_a(\tau_a - 1)$ in Table 7.6. Their correlation is taken into account by multiplying their respective error by $\sqrt{3}$.

d	D	τ	z	α	D_a	τ_a	$-\Sigma$
1	2.253(14)	1.112(6)	1.445(10)	1.18(2)	0.998(3)	1.259(11)	0.26(2)
2	2.750(6)	1.273(2)	1.532(8)	1.4896(96)	1.995(3)	1.382(3)	0.761(13)
3	3.370(11)	1.407(2)	1.777(4)	1.783(5)	3.003(14)	1.442(12)	1.380(13)

Overall, Table 7.6 provides very strong support for universality across different lattices. Under the assumption that universality holds, estimates for exponents gained from different lattices can be taken together to produce an overall estimate. The result of that procedure is shown in Table 7.8.

7.4.1.2 Fractal dimensions

For fractal lattices we use Eq. (7.11)

$$\langle x^n \rangle (N) = A_n^{(x)} N^{\mu_n^{(x)}/d} + B_n^{(x)} N^{\mu_n^{(x)}/d-1} + C_n^{(x)} N^{\mu_n^{(x)}/d-2} \quad (7.19)$$

for all observables.

Unlike the regular integer dimensions, on fractal lattices, we don't have the exact scaling relation $D(2 - \tau) = 2$ for the exponents of avalanche size s . Rather, this relation should be generalised to $D(2 - \tau) = d_w$ in which d_w is the fractal dimension of random walk on that lattice (Huynh *et al.*, 2010). Unfortunately, for many of the fractal lattices considered in this study, none of them is known with exact value of d_w . The fractal dimension of random walk on the lattice d_w is estimated using various techniques like renormalisation calculation of first-passage time, see Sec. 8.1.3. Hence, in this case, the exponent τ for avalanche size s is estimated directly from moment analysis (same as done for avalanche area a and duration t). The product $D(2 - \tau)$ is then compared against the calculated exponent d_w to assess the quality of the fit.

We do not encounter any problem in fitting the fifth moment of avalanche size $\langle s^5 \rangle$ which poses serious problem in integer dimensions. However, the first moment $\langle x \rangle$ of all observables still produces very poor fit. Table 7.9 summarises the estimate critical exponents on five fractal lattices considered in this study.

One could see the slight mismatches between the calculated d_w and the estimated $D(2 - \tau)$ in Table 7.9. They seem to be caused by finite-size corrections, which are further suppressed at seventh iteration and above. In the presence of strong finite-size corrections and high-accuracy measurements of the moments, the estimates for D and τ are sensitive to the choice of the fitting function, which is constrained by the number of data points, *i.e.* system sizes available, but needs to contain as many correction terms as possible to account for the accurate data. Our choice reflects the desire to reduce the sensitivity of the estimate on the initial values.

It is important to note that there is a level of ambiguity in the finite-size scaling in fractal lattices, because due to its highly irregular nature, there is *a priori* no unique way of increasing the lattice size of a fractal (Pruessner *et al.*, 2001). At a given level of iterations, in order to increase the lattice size further, one might either proceed by iterating the fractal or use the given fractal to tessellate the hypercubic lattice of appropriate (embedding) dimension. One might argue that finite-size scaling is of course sensitive to that choice and, as a result, generates *asymptotically* the exponents either of the fractal lattice or of the embedding space. However, in ordinary critical phenomena, there are cases (Pruessner *et al.*, 2001) where the (effective) critical point and even the scaling functions change with the level of iteration m . In the current context that translates to, for example, the amplitude $A_{x,n} = A_n^{(x)} + \frac{B_n^{(x)}}{N} + \frac{C_n^{(x)}}{N^2}$ in Eq. (7.11) to acquire a dependence on m , which might distort the resulting estimates. The exponents derived above can thus be seen only as effective exponents of a fractal lattice.

7.4.2 Universal moment ratios

Similar to the plain moments, one has to allow for corrections when fitting moment ratios (with coefficients $D_n^{(x)}$ and $E_n^{(x)}$ to be fitted). Most two-dimensional lattices (except noncrossing diagonal square lattice (Fig. 6.4(d)) and mitsubishi lattice) produce consistent results with a goodness-of-fit of greater than 0.1 with

$$g_n^{(x)} + D_n^{(x)} N^{-0.25} \quad (7.20)$$

but in order to capture all lattices and for consistency with the above we decide to add a further correction, finally fitting against

$$g_n^{(x)} + D_n^{(x)} N^{-0.25} + E_n^{(x)} N^{-0.5}. \quad (7.21)$$

Table 7.9: Avalanche exponents of five fractal lattices.

Lattice	d_w	D	τ	z	α	D_a	τ_a	$\mu_1^{(s)}$	$-\Sigma_s$	$-\Sigma_t$	$-\Sigma_a$
SSTK ^a	2.552..	2.94(3)	1.13(2)	1.817(17)	1.21(2)	1.466(5)	1.273(11)	2.551(6)	0.37(6)	0.38(4)	0.399(17)
ARRO ^b	2.322..	2.793(2)	1.173(2)	1.673(1)	1.280(2)	1.5847(3)	1.2985(6)	2.3103(4)	0.484(5)	0.468(3)	0.473(1)
CRAB ^c	2.578..	3.020(5)	1.151(4)	1.837(3)	1.237(4)	1.5847(8)	1.279(2)	2.5655(12)	0.456(11)	0.435(7)	0.443(3)
SITE ^d	2.584..	3.232(6)	1.211(4)	1.870(4)	1.357(4)	1.9975(9)	1.339(2)	2.5533(6)	0.682(14)	0.667(8)	0.677(3)
EXGA ^e	2.321..	3.352(4)	1.312(3)	1.835(3)	1.581(3)	2.5895(6)	1.3915(8)	2.3000(2)	1.046(10)	1.066(6)	1.014(2)
$^a d = \frac{\ln 5}{\ln 3}, ^b d = \frac{\ln 3}{\ln 2}, ^c d = \frac{\ln 3}{\ln 2}, ^d d = \frac{\ln 4}{\ln 2}, ^e d = 2, \text{ } ^f d = 1 + \frac{\ln 3}{\ln 2}.$											

In contrast to the finite-size scaling exponents $\mu_n^{(x)}$ of the moments considered above, all moment ratios are fitted as if they were independent, *i.e.* considering them simultaneously may be misleading as they are correlated and these correlations have not been accounted for.

The results are shown in Tables 7.10–7.12. In general, two-dimensional lattices are much better than one- and three-dimensional ones. The observable most easily fitted is the area size distribution (which might be caused by the plain (leading) amplitudes $A_n^{(a)}$ being universal, see Sec. 9.3; some difficulties in three dimensions, though). The only two-dimensional lattice that displays low goodness-of-fit throughout is the triangular lattice, while the honeycomb lattice has a single poorly fitting ratio. In one dimension, the picture is reversed. There is hardly any reasonably fitted moment ratio. In three dimensions, the body-centred cubic lattice shows low goodness-of-fit for all moment ratios except $g_5^{(t)}$ and $g_6^{(t)}$. The face-centred cubic lattice and its nearest-neighbour version seem to produce very good and reliable ratios across all moments. Fits which produce a goodness of less than 0.1 are marked in Tables 7.10–7.12 again by $[\cdot]$. Results are rather noisy for the highest moment ratios, which might suggest an explanation for the slight inconsistencies, which are not covered by the statistical error, for example for $g_6^{(a)}$ in noncrossing diagonal square lattice and mitsubishi lattice. Yet, in one dimension, it is the lower order moment ratios that are most difficult to handle.

In any event, together with the avalanche exponents, the moment ratios presented here, which are normally ignored in the literature,¹ provide very strong support for universality in regular lattices. Under that assumption, one can obtain the overall estimates for all (available) lattices in the same dimension. Table 7.13 lists the overall moment ratios based on all lattices considered in this study. We refrain from stating an overall estimate, where not at least two lattices produce reliable estimates as shown in Tables 7.10–7.12. Those that show low quality of fit are marked as above. There is, again, a certain sensitivity to the fitting function. Eq. (7.20) gives slightly incompatible results, with remarkably small error bars. Given what has been said about the goodness-of-fit, we place our confidence in the results presented in Table 7.13.

Since moment ratio is expected to be a quantity displaying universality which only holds for regular lattices in integer dimensions, it doesn't make sense to compare the moment ratios for fractal lattices. Hence, even though the data is available for fractal lattices, we decide to omit it from this work.

¹Thanks to Gunnar Pruessner for making me aware of this.

Table 7.10: Estimates for the moment ratios in one dimension as defined in Eq. (7.12) obtained by fitting the relevant ratios against Eq. (7.21). Fits with a goodness of less than 0.1 are marked by $[\cdot]$.

lattice	d	x	$g_3^{(x)}$	$g_4^{(x)}$	$g_5^{(x)}$	$g_6^{(x)}$
LINE	1	s	[1.482(4)]	[2.68(2)]	[5.40(11)]	[11.3(5)]
LADD	1	s	[1.453(6)]	[2.48(3)]	[4.43(15)]	[7.2(6)]
NNN	1	s	[1.479(13)]	[2.61(7)]	[5.0(3)]	[9.4(13)]
FUTA	1	s	[1.412(10)]	[2.26(6)]	[3.5(2)]	[3.58(97)]
LINE	1	t	[1.472(3)]	[2.63(2)]	[5.34(7)]	[11.9(3)]
LADD	1	t	[1.455(4)]	[2.52(2)]	[4.84(9)]	[9.8(4)]
NNN	1	t	[1.470(9)]	[2.58(5)]	[5.0(2)]	10.3(8)
FUTA	1	t	[1.437(6)]	[2.43(3)]	[4.44(13)]	[8.2(5)]
LINE	1	a	1.3318(6)	1.961(2)	3.037(5)	4.839(11)
LADD	1	a	1.3301(11)	1.957(4)	3.029(9)	4.83(2)
NNN	1	a	1.340(2)	[1.990(7)]	[3.11(2)]	[5.00(4)]
FUTA	1	a	1.332(2)	1.962(6)	3.038(15)	4.85(3)

7.4.3 Particle density

The particle density ζ is also a quantity of interest in sandpile models. It is defined as the average number of particles at a site. In this case, since each site can contain at most one particle, ζ equals the portion of occupied sites on the lattice. The particle density itself is a nonuniversal quantity, which provides another means to determine whether the system has reached its stationary state. It has been argued (Vespignani and Zapperi, 1997; Dickman *et al.*, 1998) that it is a temperature-like parameter, which controls an absorbing state phase transition underlying SOC.

Our interest in the particle density did not arise until a very late stage after starting of this work¹. Hence, in the flow of this work, in one and two dimensions, the estimates of particle density are reconstructed “forensically”, based on the configurations of the lattices we retained at checkpointing (we had very little data for the kagomé lattice) in the stationary state. For three-dimensional and fractal lattices, the measure of particle density is done properly in the same manner as how the avalanche observables are collected. Little data is available in the literature to compare to, except $\zeta \approx 0.9488$ for

¹Thanks to an anonymous referee of (Huynh *et al.*, 2011) for reminding us of this.

Table 7.11: Estimates for the moment ratios in two dimensions as defined in Eq. (7.12) obtained by fitting the relevant ratios against Eq. (7.21). Fits with a goodness of less than 0.1 are marked by $[\cdot]$.

lattice	d	x	$g_3^{(x)}$	$g_4^{(x)}$	$g_5^{(x)}$	$g_6^{(x)}$
SQUA	2	s	1.825(3)	4.35(2)	12.25(14)	39.0(8)
JASQ	2	s	1.830(3)	4.38(2)	12.44(14)	39.9(8)
ARCH	2	s	1.821(4)	4.32(3)	12.1(2)	37.86(99)
NOCR	2	s	1.828(3)	4.36(2)	12.30(14)	39.2(8)
TRIA	2	s	[1.830(5)]	[4.37(3)]	[12.3(2)]	[39.1(11)]
KAGO	2	s	1.832(3)	4.40(3)	12.6(2)	40.9(9)
HONE	2	s	1.829(5)	4.38(4)	12.5(2)	40.2(11)
MIT	2	s	1.820(5)	4.31(4)	12.0(2)	37.4(11)
SQUA	2	t	2.116(2)	5.95(2)	19.90(13)	75.7(9)
JASQ	2	t	2.117(2)	5.96(2)	20.06(13)	77.0(9)
ARCH	2	t	2.115(3)	5.94(2)	19.9(2)	76.1(11)
NOCR	2	t	2.114(2)	5.93(2)	19.78(13)	74.9(9)
TRIA	2	t	2.113(4)	[5.93(3)]	[19.8(2)]	[75.0(13)]
KAGO	2	t	2.116(3)	5.96(2)	20.04(14)	76.95(96)
HONE	2	t	2.110(4)	5.89(3)	19.6(2)	74.4(12)
MIT	2	t	2.110(3)	5.90(3)	19.7(2)	74.8(12)
SQUA	2	a	1.7501(11)	3.709(5)	8.69(2)	21.66(7)
JASQ	2	a	1.7496(10)	3.710(5)	8.70(2)	21.72(7)
ARCH	2	a	1.7503(14)	3.712(7)	8.70(3)	21.711(98)
NOCR	2	a	1.7517(10)	3.718(5)	8.72(2)	21.78(7)
TRIA	2	a	1.749(2)	3.710(8)	8.70(3)	21.73(11)
KAGO	2	a	1.7500(10)	3.713(5)	8.71(2)	21.78(7)
HONE	2	a	[1.747(2)]	3.698(9)	8.66(3)	21.62(11)
MIT	2	a	1.748(2)	3.703(8)	8.67(3)	21.62(11)

the one-dimensional Manna Model on the simple chain (Dickman *et al.*, 2001; Sadhu and Dhar, 2009), which is compatible with our results in Table 7.14. The estimates quoted in (Lübeck, 2004), on the other hand, are based on the original, non-Abelian

Table 7.12: Estimates for the moment ratios in three dimensions as defined in Eq. (7.12) obtained by fitting the relevant ratios against Eq. (7.21). Fits with a goodness of less than 0.1 are marked by $[\cdot]$.

lattice	d	x	$g_3^{(x)}$	$g_4^{(x)}$	$g_5^{(x)}$	$g_6^{(x)}$
SC	3	s	[2.366(11)]	[7.54(11)]	[26.9(9)]	[92(7)]
BCC	3	s	[2.388(12)]	[7.97(13)]	[32.0(11)]	[142(10)]
BCCN	3	s	[2.375(13)]	[7.72(14)]	[29.2(12)]	118(10)
FCC	3	s	2.357(26)	7.6(3)	29(3)	120(2)
FCCN	3	s	2.38(2)	7.8(2)	30.4(17)	128(14)
SC	3	t	4.152(9)	25.82(15)	200(2)	1798(34)
BCC	3	t	4.168(8)	26.05(14)	203(2)	1824(33)
BCCN	3	t	[4.158(9)]	[25.87(16)]	200(2)	1795(35)
FCC	3	t	4.13(2)	25.5(4)	195(5)	1727(67)
FCCN	3	t	4.203(15)	26.5(3)	209(4)	1908(56)
SC	3	a	2.335(6)	7.32(6)	26.8(4)	[108(3)]
BCC	3	a	2.332(7)	[7.35(6)]	27.3(4)	114(3)
BCCN	3	a	[2.333(7)]	[7.34(6)]	[27.2(4)]	[112(3)]
FCC	3	a	2.311(14)	7.15(12)	[25.9(9)]	[103(6)]
FCCN	3	a	2.332(11)	7.318(98)	26.9(7)	110(5)

Manna model and thus deviate from our estimates markedly.

The asymptotic particle densities (in the thermodynamic limit) are surprisingly difficult to fit. Eventually, we decide for the fitting function (with coefficients F_0 and F_1 to be fitted) $\zeta(N) = F_0 + F_1 N^{-\epsilon}$, which turns out to be the best one to be able to handle particle density, quoting F_0 in Table 7.14. The exponent ϵ varies from lattice to lattice and is typically of the order 0.7 in one dimension and 0.45 in two dimensions, with $F_1 < 0$. The goodness-of-fit for this scheme is above 0.98 in one and two dimensions. However, the goodness-of-fit is practically vanishing in three and fractal dimensions which might be due to the fact that the particle density was measured with extreme precision and incredibly small error bar.

Table 7.13: Overall estimates for the moment ratios defined in Eq. (7.12), based on the data presented in Tables 7.10–7.12, if enough data is available.

d	x	$g_3^{(x)}$	$g_4^{(x)}$	$g_5^{(x)}$	$g_6^{(x)}$
1	s	—	—	—	—
1	t	—	—	—	—
1	a	[1.3320(5)]	1.961(2)	3.035(4)	4.838(9)
2	s	1.8273(14)	4.363(10)	12.32(6)	[39.3(3)]
2	t	2.11423(98)	5.939(8)	19.88(6)	75.8(4)
2	a	1.7501(4)	3.711(2)	8.699(8)	21.72(3)
3	s	2.373(16)	7.76(17)	30.0(14)	121(8)
3	t	[4.164(6)]	[25.99(9)]	[201.4(12)]	1811(18)
3	a	2.331(4)	7.30(5)	27.1(3)	113(2)

7.4.4 Cluster size distribution

Like percolation (Stauffer and Aharony, 1994), the two states of a site — empty and occupied — allow the lattice to be divided into patches or clusters. And naturally, one would be interested in the distribution of the size of those clusters. In SOC literature, no work seem to study this observable. Unlike the ordinary observables like avalanche area a , duration t or size s , the cluster size ξ has its distribution right after every avalanche, *i.e.* after each avalanche one looks at the configuration of lattice and sees a distribution of sizes of clusters $P(\xi)$ and that distribution is averaged by many avalanches in the stationary state. **OPEN PROBLEM** see list on page 249

A cluster is defined to be a group of connected occupied sites, *i.e.* the sites i in state $h_i = 1$ that are neighbours of one another. The size ξ of a cluster is simply the number of sites in that cluster. Figure 7.6 illustrates a snapshot of the configuration of the Abelian Manna model on honeycomb lattice after an avalanche in the stationary state.

What can be observed in Fig. 7.6 is that on the lattice, there is a very large connected cluster occupying the majority of occupied sites. On excluding this large cluster, the distribution of the remaining clusters (of occupied sites, $h = 1$) seems to follow a power law distribution. Figure 7.7 shows the (raw histogram) distribution of sizes ξ of occupied clusters on Honeycomb lattice of different sizes. The data is rather noisy but its

Table 7.14: Asymptotic particle density ζ in the stationary state, also the average density of (singly) occupied sites. $\bar{q}$ denotes the average number of neighbours across all sites and $\overline{q^{(v)}}$ the average number of virtual neighbours among sites which have at least one virtual neighbour (both asymptotic in the thermodynamic limit). The (Hausdorff) dimension d and random walker dimension d_w of the lattice are also shown for reference.

lattice	d	d_w	$\bar{q}$	$\overline{q^{(v)}}$	ζ
LINE	1	2	2	1	0.9488(5)
LADD	1	2	3	1	0.8775(4)
NNN	1	2	4	1.5 ^a	0.8757(4)
FUTA	1	2	2.667 ^b	2	0.8845(3)
SQUA	2	2	4	1	0.7170(4)
JASQ	2	2	4	2	0.7171(6)
ARCH	2	2	3	1	0.7659(6)
NOCR	2	2	6	2	0.6782(5)
TRIA	2	2	6	2	0.6937(4)
KAGO	2	2	4	1.623 ^c	0.7347(9)
HONE	2	2	3	1	0.7532(4)
MIT	2	2	3	1.943 ^d	0.6978(5)
SC	3	2	6	1	[0.622325(1)]
BCC	3	2	8	4	[0.600620(2)]
BCCN	3	2	14	5	[0.581502(1)]
FCC	3	2	12	4	[0.589187(3)]
FCCN	3	2	18	5	[0.566307(3)]
SSTK	1.465 ^e	2.552 ^g	3	1	[0.8435(2)]
ARRO	1.585 ^f	2.322 ^g	2.333 ⁱ	1	[0.862(2)]
CRAB	1.585 ^f	2.578 ^g	2.333 ⁱ	1	[0.8794(6)]
SITE	2 ^g	2.584 ^g	6	3	[0.7427(3)]
EXGA	2.585 ^h	2.321 ^g	6	2	[0.65640(8)]

^a3/2; ^b8/3; ^c $(4 + 8\sqrt{3})/11$; ^d $(3 + 5\sqrt{3})/6$

^e $\ln 5/\ln 3$; ^f $\ln 3/\ln 2$; ^g $\ln 4/\ln 2$; ^h $1 + \ln 3/\ln 2$; ⁱ7/3

Figure 7.6: Snapshot of Abelian Manna model on honeycomb lattice of size 85×48 , see Sec. 6.2 for the size of a lattice. The filled sites are occupied and the hollow ones are empty. Sites in touch are nearest neighbours of one another.

trend of power-law distribution can be observed.¹

After all, the results presented here hardly provides any firm conclusion on the cluster size observable ξ . However, the preliminary results invoke some interesting features of the system that are worth further investigations. Detailed study of this is not done in the scope of this work and this remains an open problem.

This seems to be an interesting and a very subtle phenomenon. The fact that a

¹One can, however, ask in the thermodynamic limit whether there is just one single connected cluster (of occupied sites, $h = 1$).

Figure 7.7: Cluster size distribution on Honeycomb lattice.

power-law exists in the cluster distribution is not at all trivial. One would ask: Why should the naïve “brute force” definition of a cluster (presumably, two sites are in the same cluster if they are both occupied and nearest neighbours?), give rise to a scale-invariant distribution? What happens in one dimension? What does it mean physically? How are the exponents of the cluster size distribution related to the exponents characterising the Manna model?

In a general model without clear distinction of two states, is there a suitably general definition of “cluster”? Or is there a general definition of “cluster” that covers all models? Or what does it mean for two sites to be in the same cluster.¹

7.4.5 Scaling relations

Scaling relations were already discussed in Sec. 7.4.1, but it should be further stressed the rôle of precisely determining the critical exponents of a model in understanding its critical behaviours. Scaling relations are of fundamental importance to critical phenomena, as they point to underlying symmetries (for example, Ward-Takahashi identities (Ward, 1950; Peskin and Schroeder, 1995)). In SOC, several well-known relations are the scaling of the first moment of avalanche size $D(2 - \tau)$ (which was used to derive τ in Sec. 7.4.1) and the relation between exponents of different observables $D_x(\tau_x - 1) = D_y(\tau_y - 1)$. While the former one is well understood based on conservation and diffusion, the latter is merely based on the assumption of narrow joint distribution of observables (Jensen *et al.*, 1989; Christensen *et al.*, 1991; Lübeck, 2000) and has not been supported by any rigorous theory. There is also another scaling relation $D_a = d$ (a hyperscaling relation since it involves dimension d) which is usually assumed to hold on hypercubic lattices (Ben-Hur and Biham, 1996; Chessa *et al.*, 1999).

With all the critical exponents determined, one can immediately verify the above scaling relations for different avalanche observables. From hypercubic lattices, it is well-known (Nakanishi and Sneppen, 1997) that the first moment of the avalanche size is given by the expected number of moves that a random walker performs on the given lattice before it reaches the boundary and leaves, *i.e.* by its residence time. This is essentially because of bulk conservation: in the stationary state one particle leaves the system for every particle added (avalanche attempt), and the average number of moves it performs during its residency is exactly twice the average number of topplings occurring in the system per particle added, which is the avalanche size. Regardless of the

¹Thanks to Gunnar Pruessner for suggesting these very interesting questions.

specifics of the boundary, *i.e.* regardless of whether only two sites are dissipative or all sites along the perimeter, the first moment normally scales with the linear size of the lattices squared, $D(2 - \tau) = 2$, *independent of the dimension of the hypercubic lattice*. This is easily understood, as the time and thus the total number of moves performed by a random walker scale quadratically in the linear distance traversed. In the present case, this is very well verified from the numerical results of $\mu_1^{(s)}$ (in three dimensions, the error bars couldn't cover the exact results, but the deviations are very small, though). For the fractal lattices, it is obvious that $D(2 - \tau)$ is not equal to 2 and has in fact changed to $D(2 - \tau) = d_w$, where d_w is the fractal dimension of random walk on the lattice. The scaling law $D(2 - \tau) = d_w$ remains true for any lattice regardless of dimension and microscopic details.

The scaling relations $D_x(\tau_x - 1) = D_y(\tau_y - 1)$ and $D_a = d$ are shown to be true with great numerical reliability. They justify the assumptions made in the earlier days of SOC on narrow joint distribution of observables and the compactness of avalanches, extending them to general cases including fractal lattices. This fact provides very firm ground and motivation for developing theoretical understanding of the critical behaviours of the model.

Chapter 8

Analytical approaches

In the self-organized criticality literature, one can find many pieces of work that exactly solve the models. Some success has been achieved for deterministic models like BTW using the theory of Schramm–Löwner evolution (SLE, also stochastic Löwner evolution or Schramm–Löwner equation, etc...) (Azimi-Tafreshi et al., 2010), conformal field theory (Jeng, 2005; Moghimi-Araghi et al., 2005). Many authors have successfully solved the directed versions of different sandpile models (e.g. Dhar and Ramaswamy, 1989; Maslov and Zhang, 1995; Tsuchiya and Katori, 1999b; Paczuski and Bassler, 2000a; Kloster et al., 2001; Pruessner, 2004a). But no single work exists for solving the undirected version of the original Manna model (Manna, 1991b) or the Abelian Manna model (Dhar, 1999a,c). This model remains as one of the major challenges in Self-Organized Criticality.

In one of the rare compilations of advances in SOC, Jensen (1998) presented a brief overview of analytical techniques in SOC after ten years of research. These techniques include mean-field theory, Langevin equations, dynamically driven renormalisation group and Dhar’s algebraic operator approach. Fourteen years later, Pruessner (2012) makes a second attempt to provide an overview of the field again after a quarter of century of research. In the present work, no attempt is made to provide any sort of reviewing all the developed analytical approaches throughout the years. Rather, what are presented here are the analytical calculations of the system studied as complementary to numerical results presented in Sec. 7.4, focusing on fractal lattices which are a lack in the literature. Several attempts are made to develop (and extend) algebraic operator approach towards solving the (Abelian) Manna model. Some other approaches like graph and mean-field approximation are also discussed.

8.1 Mapping to random walk

As discussed in details in Secs. 7.4.1 and 7.4.5, the particles in the Abelian Manna model perform random walks on the lattice. Making use of that relation, some exact results for avalanche size can be obtained. [OPEN PROBLEM](#) see list on page 249

8.1.1 Exact calculation of first moment of avalanche size $\langle s \rangle$

By mapping the process in AMM to random walk, one can calculate the first moment of avalanche size $\langle s \rangle$ which is half the ensemble average residence time of particles on the lattice. This residence time is calculated using the first-passage time method (ben-Avraham and Havlin, 2000).

On a general lattice $\mathcal{L}$ of N sites (labelled $1, \dots, N$) with coordination number q_i at each site i , the expected residence time T_i of a random walker starting at a site (the amount of time after which the walker escapes $\mathcal{L}$) is

$$T_i = \sum_{j=1}^{q_i} \frac{1}{q_i} (t + T_{\mathcal{A}_{ij}^*}) = t + \frac{1}{q_i} \sum_{j=1}^{q_i} T_{\mathcal{A}_{ij}^*}. \quad (8.1)$$

In the above, t is the time taken by the walker to go from one lattice site to any of its nearest neighbours; usually we take $t = 1$. $\mathcal{A}_{ij}^*$ indicates the j th nearest neighbour of site i (in some arbitrary labelling order). In other words, $\mathcal{A}_{ij}^*$ is an element of the reduced adjacency matrix $\mathcal{A}^*$ that was introduced in Sec. 7.2.1. For open boundary condition of the model, sites outside the lattice are represented by -1 , i.e. $\mathcal{A}_{ij}^* = -1$ if the j th nearest neighbour of i is a virtual one. In such cases, we set $T_{-1} = 0$.

The first moment of avalanche size on the lattice would then be

$$\langle s \rangle = \frac{1}{2N} \sum_{i=1}^N T_i. \quad (8.2)$$

We will now proceed to calculate $\langle s \rangle$ for particular lattices with specific (reduced) adjacency matrix $\mathcal{A}^*$.

8.1.1.1 Simple chain

For the simple chain lattice, the reduced adjacency matrix $\mathcal{A}^*$ is simply

$$\mathcal{A}_{LINE}^* = \begin{pmatrix} 2 & -1 \\ 1 & 3 \\ \vdots & \vdots \\ N-1 & -1 \end{pmatrix}, \quad (8.3)$$

in other words $\mathcal{A}_{i1}^* = i - 1$, $\mathcal{A}_{i2}^* = i + 1 \forall i = 2, \dots, N - 1$ and $\mathcal{A}_{11}^* = 2$, $\mathcal{A}_{12}^* = -1$ and $\mathcal{A}_{N1}^* = N - 1$, $\mathcal{A}_{N2}^* = -1$.

We can then write down the equations to calculate the residence time on this lattice

$$T_i = t + \frac{1}{2}T_{i-1} + \frac{1}{2}T_{i+1} \quad (i = 1, \dots, N) \quad (8.4)$$

with boundary conditions $T_0 = T_{N+1} = 0$.

Eq. (8.4) can be easily solved by rearranging

$$2T_i = 2t + T_{i-1} + T_{i+1} \quad (8.5a)$$

$$\Rightarrow T_i - T_{i+1} = 2t + T_{i-1} - T_i \quad (8.5b)$$

and introducing new variable $D_i = T_i - T_{i+1}$ for which we have a new equivalent equation

$$D_i = 2t + D_{i-1}. \quad (8.6)$$

The solution to Eq. (8.6) is given by

$$D_i = D_0 + \sum_{j=0}^{i-1} 2t = D_0 + 2it. \quad (8.7)$$

Hence we have

$$T_i - T_{i+1} = D_i = D_0 + 2it \quad (8.8)$$

whose solution is given by

$$T_i = T_0 - \sum_{j=0}^{i-1} (D_0 + 2jt) = T_0 - iD_0 - (i-1)it = T_0 - iT_0 + iT_1 - i^2t + it. \quad (8.9)$$

With the boundary condition $T_0 = 0$, Eq. (8.9) becomes

$$T_i = iT_1 - i^2t + it. \quad (8.10)$$

And the other boundary condition $T_{N+1} = 0$ helps solve for T_1

$$T_{N+1} = (N+1)T_1 - (N+1)^2t + (N+1)t = 0 \quad (8.11a)$$

$$\Rightarrow T_1 = Nt. \quad (8.11b)$$

Hence the general solution to Eq. (8.4) is

$$T_i = -ti^2 + t(N+1)i. \quad (8.12)$$

Therefore the first moment of avalanche size in this case is

$$\begin{aligned} \langle s \rangle &= \frac{1}{2N} \sum_{i=1}^N T_i = \frac{1}{2N} \sum_{i=1}^N [-ti^2 + t(N+1)i] \\ &= \frac{1}{2N} \left[-t \frac{N(N+1)(2N+1)}{6} + t(N+1) \frac{N(N+1)}{2} \right] \\ &= \frac{(N+1)(N+2)}{12} t \end{aligned} \quad (8.13)$$

which, by setting $t = 1$, gives

$$\langle s \rangle (N, \text{LINE}) = \frac{(N+1)(N+2)}{12}. \quad (8.14)$$

8.1.1.2 Rope ladder

On a rope ladder lattice, we label the sites from 1 to L for one side and then $L + 1$ to $2L$ for the other side (see Fig. 7.1(b)). The reduced adjacency matrix $\mathcal{A}^*$ would then read

$$\mathcal{A}_{LADD}^* = \begin{pmatrix} 2 & L+1 & -1 \\ 1 & 3 & L+2 \\ \vdots & \vdots & \vdots \\ L-1 & 2L & -1 \\ L+2 & 1 & -1 \\ L+1 & L+3 & 2 \\ \vdots & \vdots & \vdots \\ 2L-1 & L & -1 \end{pmatrix}. \quad (8.15)$$

The equations to calculate the residence time on this lattice are

$$T_1 = t + \frac{1}{3}T_2 + \frac{1}{3}T_{L+1}, \quad (8.16a)$$

$$T_i = t + \frac{1}{3}T_{i-1} + \frac{1}{3}T_{i+1} + \frac{1}{3}T_{i+L} \quad (2 \leq i \leq L-1), \quad (8.16b)$$

$$T_L = t + \frac{1}{3}T_{L-1} + \frac{1}{3}T_{2L}, \quad (8.16c)$$

$$T_{L+1} = t + \frac{1}{3}T_{L+2} + \frac{1}{3}T_1, \quad (8.16d)$$

$$T_i = t + \frac{1}{3}T_{i-1} + \frac{1}{3}T_{i+1} + \frac{1}{3}T_{i-L} \quad (L+2 \leq i \leq 2L-1), \quad (8.16e)$$

$$T_{2L} = t + \frac{1}{3}T_{2L-1} + \frac{1}{3}T_L. \quad (8.16f)$$

By symmetry of the lattice, we have

$$T_i = T_{i+L} \quad (1 \leq i \leq L). \quad (8.17)$$

Hence the above system of equations can be reduced to a single equation

$$T_i = \frac{3}{2}t + \frac{1}{2}T_{i-1} + \frac{1}{2}T_{i+1} \quad (1 \leq i \leq L) \quad (8.18)$$

with boundary conditions $T_0 = T_{L+1} = 0$. This equation has precisely the same form

as Eq. (8.4), hence the general solution in this case is

$$T_i = -\frac{3}{2}ti^2 + \frac{3}{2}t(L+1)i. \quad (8.19)$$

Therefore the first moment of avalanche size in this case is

$$\begin{aligned} \langle s \rangle &= \frac{1}{2 \times 2L} \sum_{i=1}^{2L} T_i = \frac{1}{2L} \sum_{i=1}^L T_i = \frac{1}{2L} \sum_{i=1}^L \left[-\frac{3}{2}ti^2 + \frac{3}{2}t(L+1)i \right] \\ &= \frac{(L+1)(L+2)}{8}t. \end{aligned} \quad (8.20)$$

In term of total number of sites $N = 2L$, we have

$$\langle s \rangle = \frac{\left(\frac{N}{2} + 1\right)\left(\frac{N}{2} + 2\right)}{8}t = \frac{(N+2)(N+4)}{32}t \quad (8.21)$$

which, by setting $t = 1$, gives

$$\langle s \rangle (N, \text{LADD}) = \frac{(N+2)(N+4)}{32}. \quad (8.22)$$

8.1.1.3 NNN chain

The next-nearest-neighbour chain lattice (Fig. 6.3(c)) is a simple chain lattice with next-nearest-neighbour interactions. Hence, the equations for this lattice are similar to those of simple chain lattice with extra terms. From Fig. 7.1(c), one can easily write down for a lattice of N sites (here we start with index $i = 1$ rather than $i = 0$ as implemented on computer, hence there is shift of 1 in the indices)

$$T_1 = t + \frac{1}{4}T_2 + \frac{1}{4}T_3, \quad (8.23a)$$

$$T_2 = t + \frac{1}{4}T_1 + \frac{1}{4}T_3 + \frac{1}{4}T_4, \quad (8.23b)$$

$$T_i = t + \frac{1}{4}T_{i-2} + \frac{1}{4}T_{i-1} + \frac{1}{4}T_{i+1} + \frac{1}{4}T_{i+2} \quad (3 \leq i \leq N-2), \quad (8.23c)$$

$$T_{N-1} = t + \frac{1}{4}T_{N-3} + \frac{1}{4}T_{N-2} + \frac{1}{4}T_N, \quad (8.23d)$$

$$T_N = t + \frac{1}{4}T_{N-2} + \frac{1}{4}T_{N-1}. \quad (8.23e)$$

Two extra terms $\frac{1}{4}T_{i-2}$ and $\frac{1}{4}T_{i+2}$ in Eq. (8.23c) as compared to Eq. (8.4) arise due to the next-nearest-neighbour interactions.

Applying the same technique as in solving the simple chain case, we have Eq. (8.23c) become

$$T_i - T_{i-1} + T_i - T_{i-2} = 4t + T_{i+1} - T_i + T_{i+2} - T_i. \quad (8.24)$$

Adding $T_{i-1} - T_{i-1}$ on the left and $T_{i+1} - T_{i+1}$ on the right side of Eq. (8.24), we have

$$T_i - T_{i-1} + T_i - T_{i-1} + T_{i-1} - T_{i-2} = 4t + T_{i+1} - T_i + T_{i+2} - T_{i+1} + T_{i+1} - T_i, \quad (8.25)$$

which, by introducing $D_i = T_i - T_{i-1}$, can be rewritten

$$2D_i + D_{i-1} = 4t + 2D_{i+1} + D_{i+2}. \quad (8.26)$$

Adding $D_i - D_i$ on the left and $D_{i+1} - D_{i+1}$ on the right side of Eq. (8.26), we have

$$3D_i + D_{i-1} - D_i = 4t + 3D_{i+1} + D_{i+2} - D_{i+1}, \quad (8.27)$$

which, by introducing $E_i = D_i - D_{i-1}$, can be rewritten

$$4t + E_{i+2} + 3E_{i+1} + E_i = 0. \quad (8.28)$$

Eq. (8.28) is an inhomogeneous difference equation and can be turned into a homogeneous one by introducing $F_i = E_i - E_{i-1}$

$$F_{i+3} + 3F_{i+2} + F_{i+1} = 0, \quad (8.29)$$

which admits general solution

$$F_i = a_1 \left(\frac{-3 + \sqrt{5}}{2} \right)^i + a_2 \left(\frac{-3 - \sqrt{5}}{2} \right)^i \quad (8.30)$$

in which the parameters a_1 and a_2 are to be determined from boundary conditions in Eqs. 8.23a and 8.23b as well as Eqs. 8.23d and 8.23e.

The desired T_i is then given by

$$T_i = T_0 + \sum_{j=1}^i D_j = T_0 + \sum_{j=1}^i \left[D_1 + \sum_{k=2}^j \left(E_2 + \sum_{l=3}^k F_l \right) \right] \quad (8.31)$$

which requires very complicated calculation¹ and hence stays unknown in this work.

To be explicit, the first moment of avalanche size on the next-nearest-neighbour chain lattice of size N is given by

$$\begin{aligned} \langle s \rangle (N, \text{NNN}) &= \frac{1}{2N} \sum_{i=1}^N \left(T_0 + \sum_{j=1}^i D_j \right) \\ &= \frac{1}{2N} \sum_{i=1}^N \left[T_0 + \sum_{j=1}^i \left(D_1 + \sum_{k=2}^j \left\{ E_2 + \sum_{l=3}^k \left[a_1 \left(\frac{\sqrt{5}-3}{2} \right)^l + a_2 \left(\frac{3+\sqrt{5}}{-2} \right)^l \right] \right\} \right) \right]. \end{aligned} \quad (8.32)$$

8.1.1.4 Futatsubishi

Considering a futatsubishi lattice of $N = 3k + 1$ with k “diamonds” (see Fig. 6.3(d)), from the labelling in Fig. 7.1(d), one can write down the equations (here we start with index $i = 1$ rather than $i = 0$ as implemented on computer, hence there is shift of 1 in the indices)

$$T_1 = t + \frac{1}{4}T_2 + \frac{1}{4}T_3, \quad (8.33a)$$

$$T_{3i+2} = t + \frac{1}{2}T_{3i+1} + \frac{1}{2}T_{3i+4} \quad (i = 0, \dots, k-1), \quad (8.33b)$$

$$T_{3i+3} = t + \frac{1}{2}T_{3i+1} + \frac{1}{2}T_{3i+4} \quad (i = 0, \dots, k-1), \quad (8.33c)$$

$$T_{3i+1} = t + \frac{1}{4}T_{3i-1} + \frac{1}{4}T_{3i} + \frac{1}{4}T_{3i+2} + \frac{1}{4}T_{3i+3} \quad (i = 1, \dots, k-1), \quad (8.33d)$$

$$T_{3k+1} = t + \frac{1}{4}T_{3k-1} + \frac{1}{4}T_{3k}. \quad (8.33e)$$

From the equations above, one realises that $T_{3i+2} = T_{3i+3}$ so that Eq. (8.33d) can be rewritten

$$\begin{aligned} T_{3i+1} &= t + \frac{1}{2}T_{3i-1} + \frac{1}{2}T_{3i+2} \\ &= t + \frac{1}{2} \left(t + \frac{1}{2}T_{3i-2} + \frac{1}{2}T_{3i+1} + t + \frac{1}{2}T_{3i+1} + \frac{1}{2}T_{3i+4} \right) \\ &= 2t + \frac{1}{4}T_{3i-2} + \frac{1}{2}T_{3i+1} + \frac{1}{4}T_{3i+4} \end{aligned} \quad (8.34)$$

¹The procedure to obtain the explicit expression for the avalanche size on this lattice is doable but horribly lengthy. This remains an open task.

which can be rearranged to be

$$T_{3i+1} = 4t + \frac{1}{2}T_{3i-2} + \frac{1}{2}T_{3i+4} \quad (i = 1, \dots, k-1). \quad (8.35)$$

By introducing $\tilde{T}_i = T_{3i+1}$, we have Eq. (8.35) become

$$\tilde{T}_i = 4t + \frac{1}{2}\tilde{T}_{i-1} + \frac{1}{2}\tilde{T}_{i+1} \quad (i = 1, \dots, k-1) \quad (8.36)$$

which has precisely the same form as Eq. (8.4) and, hence, admits solution

$$\tilde{T}_i = \tilde{T}_0 - i\tilde{T}_0 + i\tilde{T}_1 - 4i^2t + 4it. \quad (8.37)$$

Therefore, we have

$$T_{3i+1} = T_1 - iT_1 + iT_4 - 4i^2t + 4it \quad (i = 1, \dots, k-1) \quad (8.38)$$

which requires two boundary conditions to determine T_1 and T_4 (Eq. (8.38) is self-consistent on substituting $i = 1$).

The left boundary condition gives

$$\begin{cases} T_1 = t + \frac{1}{4}T_2 + \frac{1}{4}T_3 \\ T_2 = T_3 = t + \frac{1}{2}T_1 + \frac{1}{2}T_4 \end{cases} \Rightarrow 3T_1 - T_4 - 6t = 0, \quad (8.39)$$

while the right boundary condition gives

$$\begin{cases} T_{3k+1} = t + \frac{1}{4}T_{3k-1} + \frac{1}{4}T_{3k} \\ T_{3k-1} = T_{3k} = t + \frac{1}{2}T_{3k-2} + \frac{1}{2}T_{3k+1} \end{cases} \Rightarrow 3T_{3k+1} - T_{3k-2} - 6t = 0. \quad (8.40)$$

From Eqs. 8.38–8.40, we have

$$\begin{cases} 3T_1 - T_4 = 6t \\ (1 - 2k)T_1 + (2k + 1)T_4 = 8(k^2 - 1)t + 6t \end{cases} \quad (8.41)$$

which yields $T_1 = (2k + 1)t$ and $T_4 = 3(2k - 1)t$. Hence, we have

$$T_{3i+1} = (2k + 1)t + 4kit - 4i^2t \quad (8.42)$$

which is actually true for $i = 0, \dots, k$ (not restricted to $i = 1, \dots, k-1$ like the beginning of the calculation), and subsequently

$$T_{3i+2} = T_{3i+3} = t + \frac{1}{2}T_{3i+1} + \frac{1}{2}T_{3i+4} = 4kt + 4(k-1)it - 4i^2t \quad (i = 0, \dots, k-1). \quad (8.43)$$

The first moment of avalanche size in this case is given by

$$\begin{aligned} \langle s \rangle &= \frac{1}{2N} \left(\sum_{i=0}^k T_{3i+1} + \sum_{i=0}^{k-1} T_{3i+2} + \sum_{i=0}^{k-1} T_{3i+3} \right) \\ &= \frac{1}{2N} \left(\sum_{i=0}^k T_{3i+1} + 2 \sum_{i=0}^{k-1} T_{3i+2} \right) \\ &= \frac{1}{2N} \left[(2k+1)(k+1)t + 4kt \frac{k(k+1)}{2} t - 4 \frac{k(k+1)(2k+1)}{6} t \right. \\ &\quad \left. + 2 \times 4k^2t + 2 \times 4(k-1) \frac{(k-1)k}{2} t - 2 \times 4 \frac{(k-1)k(2k-1)}{6} t \right] \\ &= \frac{1}{2N} (2k^3 + 6k^2 + 5k + 1) t \\ &= \frac{(k+1)(2k^2 + 4k + 1)}{2N} t \end{aligned} \quad (8.44)$$

which, by setting $t = 1$ and replacing the number of diamonds k by the total number of sites $N = 3k + 1$, gives

$$\langle s \rangle (N, \text{FUTA}) = \frac{(N+2)(2N^2 + 8N - 1)}{54N}. \quad (8.45)$$

8.1.1.5 Periodic hypercubic

One can easily realise that the difference equations in calculating the residence time T_i of a random walker on the lattice in one dimension correspond to ordinary differential equations. Similarly, in two (or higher) dimensions, these equations would turn into partial differential equations and hence in general cannot be solved. However, in some special cases, by making use of the symmetry of the system, these equations can be solved exactly. One of such cases is the lattices with periodic boundary conditions.

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In this study, we consider general hypercubic lattices in d dimensions (square lattice in two dimensions) with periodic boundary conditions in $(d-1)$ dimensions and open boundary condition in the remaining dimension. In two dimensions, this is simply a cylinder or tube (see Fig. 8.1).

Figure 8.1: A two-dimensional square lattice with periodic boundary condition in one dimension and open boundary condition in the other. The lattice looks like a tube or cylinder. Drawing courtesy of Gunnar Pruessner.

We now calculate the first moment of avalanche size for this lattice in two dimensions — a cylinder. Let's call x the direction in which open boundary condition is applied and y the direction for periodic boundary condition. In other words, size of the lattice in x -direction tells the height of the cylinder, while the size in y -direction characterises its diameter. Due to symmetry of the lattice, sites along y -direction are all equivalent.

We consider the lattice with L sites in x -direction and L' sites in y -direction, hence $N = LL'$ sites in total. Since all sites in y -direction are equivalent, we can use just one single index to label the sites in x -direction, say $i = 1, \dots, L$. Similarly to previous calculations, we have the equations for the residence time of a random walker on the lattice

$$T_i = t + \frac{1}{4}T_{i-1} + \frac{1}{4}T_{i+1} + \frac{1}{4}T_i + \frac{1}{4}T_i \quad (8.46)$$

or

$$T_i = 2t + \frac{1}{2}T_{i-1} + \frac{1}{2}T_{i+1} \quad (8.47)$$

which has the same form as Eq. (8.4) and admits solution

$$T_i = -2ti^2 + 2t(L+1)i. \quad (8.48)$$

The first moment of avalanche size on this lattice is

$$\begin{aligned}
 \langle s \rangle &= \frac{1}{2N} \sum_{i=1}^L L' T_i = \frac{1}{2L} \sum_{i=1}^L [-2ti^2 + 2t(L+1)i] \\
 &= \frac{1}{2L} \left[-2t \frac{L(L+1)(2L+1)}{6} + 2t(L+1) \frac{L(L+1)}{2} \right] \\
 &= \frac{(L+1)(L+2)}{6} t
 \end{aligned} \tag{8.49}$$

which is independent of L' , the size of the lattice in direction with periodic boundary condition.

In d dimensions, Eq. (8.46) generalises to

$$T_i = t + \frac{1}{2d} T_{i-1} + \frac{1}{2d} T_{i+1} + \frac{2d-2}{2d} T_i \tag{8.50}$$

which reduces to

$$T_i = dt + \frac{1}{2} T_{i-1} + \frac{1}{2} T_{i+1}. \tag{8.51}$$

Hence, the first moment of avalanche size $\langle s \rangle$ of a d -dimensional lattice with periodic boundary conditions in $(d-1)$ dimensions and open boundary condition in the remaining one is given by

$$\langle s \rangle (L, \text{periodic } d\text{-dimensional}) = \frac{d}{12} (L+1)(L+2). \tag{8.52}$$

On two-dimensional square lattice with open boundary condition, the first moment of avalanche size was indeed obtained by [Dhar \(1990\)](#)

$$\begin{aligned}
 \langle s \rangle &= \frac{1}{L^2(L+1)^2} \sum_{m,n} \cot^2 \frac{m\pi}{2(L+1)} \cot^2 \frac{n\pi}{2(L+1)} \left(\sin^2 \frac{m\pi}{2(L+1)} \right. \\
 &\quad \left. + \sin^2 \frac{n\pi}{2(L+1)} \right)
 \end{aligned} \tag{8.53}$$

where the summation over m, n extends over all *odd* integers $1 \leq m \leq L$ and $1 \leq n \leq L$.

All the calculated first moments $\langle s \rangle$ here compare very well against the numerical simulation results. This is another criterion to justify the numerical work done in this study.

8.1.2 Numerical calculation of first moment of avalanche size $\langle s \rangle$ on general lattices

The calculations in Sec. 8.1.1 somehow restrict to one-dimensional lattices or one with symmetry properties like periodic boundary conditions. In fact, in general, one can employ the conventional adjacency matrix (*not* the reduced one introduced earlier in this study) to compute the exact first moment of avalanche size $\langle s \rangle$ on a general lattice (still, by mapping to random walk process).

The idea is to write down the adjacency matrix $\mathcal{A}$ with element $\mathcal{A}_{ij}$ being 1 if sites i and j are neighbours and 0 otherwise. The diagonal of this matrix contains all zero-elements, because site i is not considered a neighbour of itself. In every row of this matrix, every nonzero entry is then replaced by $-\frac{1}{n}$ where n is the number of nonzero entries in that row. The next thing to do is to turn all the elements in the diagonal into 1. Now the adjacency matrix $\mathcal{A}$ has been turn into a so-called transition matrix $\mathcal{T}$. The equations in Sec. 8.1.1 can be simply written in matrix form as

$$\mathcal{T}\mathbf{T} = \mathbf{b} \quad (8.54)$$

in which the column vector $\mathbf{T}$ consists of the residence times of random walkers starting from different sites on the lattice

$$\mathbf{T} = \begin{pmatrix} T_1 \\ \vdots \\ T_N \end{pmatrix}, \quad (8.55)$$

and $\mathbf{b}$ is a column vector with all entries being t .

$$\mathbf{b} = \begin{pmatrix} t \\ \vdots \\ t \end{pmatrix}. \quad (8.56)$$

The solution $\mathbf{T}$ is then given by

$$\mathbf{T} = \mathcal{T}^{-1}\mathbf{b} \quad (8.57)$$

for which the inverse of the matrix $\mathcal{T}$ can be found by making it upper-triangular. The first moment of avalanche size is then half the average of all entries of T . This way, one realises the importance (out of many, see, for example, (Dhar, 2006) for the relation between adjacency matrix, lattice Laplacian and two-point correlation function) of the adjacency matrix and its analytical construction will be discussed in Sec. 8.2.

As an example, we consider the arrowhead lattice at second iteration $m = 2$ (see Fig. 8.2) for which the adjacency matrix is

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad (8.58)$$

and the reduce adjacency matrix is given in Table 8.1.

Converting the adjacency matrix in Eq. (8.58) into transition matrix, we have

Figure 8.2: The arrowhead lattice at second iteration $m = 2$. The index here starts with 1 rather than 0 as compared to representation of the lattice in simulation.

Table 8.1: Reduced adjacency matrix of the arrowhead in Fig. 8.2. The index here starts with 1 rather than 0 as compared to representation of the lattice in simulation. The virtual sites are still -1 .

Site index	Neighbours	Number of neighbours
1	2 -1	2
2	1 3	2
3	2 4 8	3
4	3 5	2
5	4 6	2
6	5 7	2
7	6 8	2
8	7 9 3	3
9	8 10	2
10	9 -1	2

Table 8.2: Comparison between first moment obtained from Eq. (8.57) and numerical simulation for the arrowhead lattice. The very slight mismatch between the two values that is not covered by the error bar of the simulated one might be due to the equilibration time before the system enters stationary state. That, however, does not affect the estimated exponents at all.

Iteration m	Linear size L	Total size N	Calculated $\langle s \rangle$	Simulated $\langle s \rangle$
6	65	730	5073.283..	5073.526 ± 0.117
7	129	2188	24473.968..	24473.917 ± 1.119
8	257	6562	119771.485..	119773.322 ± 5.376
9	513	19684	591162.597..	591133.563 ± 27.257

Eq. (8.57) become

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix}^{-1} \begin{pmatrix} t \\ t \\ t \\ t \\ t \\ t \\ t \\ t \\ t \\ t \end{pmatrix} \quad (8.59)$$

in which the diagonal entries have been turned to 1 from 0 and the adjacency entries have been turned to $-\frac{1}{n}$ from 1.

Table 8.2 compares the results obtained using this approach against the numerical results.

8.1.3 Calculation of d_w on fractal lattices

Besides calculation of first moment of avalanche size $\langle s \rangle$, one can also study the avalanche size observable by extracting the fractal dimension of random walk on the lattice using renormalisation calculation. The idea is to compare the first-passage time of the same site on the lattice of two different sizes and see how it scales. The exact renormalisation calculation on the celebrated Sierpiński gasket has been performed in (ben-Avraham and Havlin, 2000, pp. 61–62). Here the method is applied to other less studied fractal lattices. However, due to the nature of these fractal lattices, the calculation is performed exactly to single edge connecting two sites on the lattice, *i.e.* not coarse-grained renormalisation-like calculation. The idea is illustrated below.

8.1.3.1 Sierpiński arrowhead lattices

We first perform the calculation for the arrowhead lattice shown in Fig. 8.3.

We label the sites by $i = 1, 2, \dots$ from the middle of the chain towards one end and denote $T_i^{(m)}$ ($i = 1, 2, \dots$) as the average escape time from site i (exit through the end site, m is the number of iterations of the lattice). Thanks to the symmetry of the lattice, we only need to trace half the number of sites on the lattice. The size of the lattice at m th iteration is $(3^m + 1)$ but we don't have to trace the end site (where the walker exits) so we have i run from 1 to $\frac{3^m + 1}{2} - 1 = \frac{3^m - 1}{2}$.

We also, same as before, denote t as the time taken to go from one site to one of its nearest neighbours. The equations to solve for $T_i^{(m)}$ are then straightforward. We first define $E^{(m)}$ as the set of extended sites (see Appendix B) among i and denote i^* as its partner (On the chain, each site i has two normal nearest neighbours $(i - 1)$ and $(i + 1)$. If i is extended, it has the third nearest neighbour i^* which is called “partner”. This partner is known from the adjacency matrix). Now, if the walker is at site i , we have two situations.

- If $i \notin E^{(m)}$, the walker has only two options to choose from: go to $(i - 1)$ or go to $(i + 1)$. The probability for each choice is $1/2$. Then we have

$$\begin{aligned} T_i^{(m)} &= \frac{1}{2} \left(t + T_{i-1}^{(m)} \right) + \frac{1}{2} \left(t + T_{i+1}^{(m)} \right) \\ &= t + \frac{1}{2} T_{i-1}^{(m)} + \frac{1}{2} T_{i+1}^{(m)} \quad (i \notin E^{(m)}). \end{aligned} \quad (8.60)$$

- If $i \in E^{(m)}$, the walker has up to three options to choose from: go to $(i - 1)$, go

Figure 8.3: First-passage time on the arrowhead lattice.

to $(i + 1)$ or go to i^* . The probability for each choice is $1/3$. Then we have

$$\begin{aligned} T_i^{(m)} &= \frac{1}{3} (t + T_{i-1}^{(m)}) + \frac{1}{3} (t + T_{i+1}^{(m)}) + \frac{1}{3} (t + T_{i^*}^{(m)}) \\ &= t + \frac{1}{3} T_{i-1}^{(m)} + \frac{1}{3} T_{i+1}^{(m)} + \frac{1}{3} T_{i^*}^{(m)} \quad (i \in E^{(m)}). \end{aligned} \quad (8.61)$$

We realise that when $i = \frac{3^m + 1}{2} - 1$, the equation for T_i involves $T_{\frac{3^{m+1}}{2}}$ which is indeed 0. When $i = 1$, the equation for T_1 involves $T_{i-1} = T_0$ which is not defined above. But thanks to the symmetry of the lattice, we know that the equation reads

$$\begin{aligned} T_1^{(m)} &= \frac{1}{2} (t + T_1^{(m)}) + \frac{1}{2} (t + T_2^{(m)}) \\ &= t + \frac{1}{2} T_1^{(m)} + \frac{1}{2} T_2^{(m)} \end{aligned} \quad (8.62a)$$

Figure 8.4: Special sites (hollow circles) on the arrowhead lattice in calculation of the first-passage time.

$$\rightarrow T_1^{(m)} = 2t + T_2^{(m)}. \quad (8.62b)$$

Another special case is for those sites i of the form $i = 3^k$ ($k = 1, 2, \dots, m-1$) (hollow circles in Fig. 8.4).

Those sites are special because their partner are outside the range we trace (on the other half of the lattice). But, again, thanks to the symmetry of the lattice, we know that these equations read

$$T_i^{(m)} = \frac{1}{3} (t + T_{i-1}^{(m)}) + \frac{1}{3} (t + T_{i+1}^{(m)}) + \frac{1}{3} (t + T_i^{(m)}) \quad (T_{i^*}^{(m)} = T_i^{(m)}) \quad (8.63a)$$

$$\rightarrow \frac{2}{3} T_i^{(m)} = t + \frac{1}{3} T_{i-1}^{(m)} + \frac{1}{3} T_{i+1}^{(m)}. \quad (8.63b)$$

In summary, we have a system of $\frac{3^m - 1}{2}$ linear equations with the same number

of unknowns T_i to be solved. The equations are

$$\left\{ \begin{array}{l} T_1^{(m)} = 2t + T_2^{(m)} \\ T_i^{(m)} = t + \frac{1}{2}T_{i-1}^{(m)} + \frac{1}{2}T_{i+1}^{(m)} \quad (i \notin E^{(m)}) \\ T_i^{(m)} = t + \frac{1}{3}T_{i-1}^{(m)} + \frac{1}{3}T_{i+1}^{(m)} + \frac{1}{3}T_{i^*}^{(m)} \quad \left(i \in E^{(m)} \setminus \left\{ 3^k \right\}_{k=1}^{m-1} \right) \\ \frac{2}{3}T_i^{(m)} = t + \frac{1}{3}T_{i-1}^{(m)} + \frac{1}{3}T_{i+1}^{(m)} \quad \left(i \in \left\{ 3^k \right\}_{k=1}^{m-1} \right) \\ T_{\frac{3^m-1}{2}} = t + \frac{1}{2}T_{\frac{3^m-1}{2}-1} \end{array} \right. \quad (8.64)$$

Since this is a linear system, we can write in terms of matrix

$$\tilde{\mathcal{T}}^{(m)} \mathbf{T}^{(m)} = \mathbf{B}^{(m)} \quad (8.65)$$

where $\tilde{\mathcal{T}}^{(m)}$ is the coefficient matrix¹ of size $\frac{3^m-1}{2} \times \frac{3^m-1}{2}$ and its elements are (m is the number of iterations of the lattice)

$$\left\{ \begin{array}{l} \tilde{\mathcal{T}}_{11}^{(m)} = 1; \tilde{\mathcal{T}}_{12}^{(m)} = -1 \\ \tilde{\mathcal{T}}_{i,i-1}^{(m)} = \tilde{\mathcal{T}}_{i,i+1}^{(m)} = -\frac{1}{2}; \tilde{\mathcal{T}}_{i,i}^{(m)} = 1 \quad (i \notin E^{(m)}) \\ \tilde{\mathcal{T}}_{i,i-1}^{(m)} = \tilde{\mathcal{T}}_{i,i+1}^{(m)} = \tilde{\mathcal{T}}_{i,i^*}^{(m)} = -\frac{1}{3}; \tilde{\mathcal{T}}_{i,i}^{(m)} = 1 \quad \left(i \in E^{(m)} \setminus \left\{ 3^k \right\}_{k=1}^{m-1} \right) \\ \tilde{\mathcal{T}}_{i,i-1}^{(m)} = \tilde{\mathcal{T}}_{i,i+1}^{(m)} = -\frac{1}{3}; \tilde{\mathcal{T}}_{i,i}^{(m)} = \frac{2}{3} \quad \left(i \in \left\{ 3^k \right\}_{k=1}^{m-1} \right) \\ \tilde{\mathcal{T}}_{\frac{3^m-1}{2}, \frac{3^m-1}{2}-1}^{(m)} = -\frac{1}{2}; \tilde{\mathcal{T}}_{\frac{3^m-1}{2}, \frac{3^m-1}{2}}^{(m)} = 1 \end{array} \right. \quad (8.66)$$

and all the other elements are 0. $\mathbf{B}^{(m)}$ is a constant column vector of size $\frac{3^m-1}{2}$ and the elements are

$$\left\{ \begin{array}{l} B_1^{(m)} = 2 \\ B_i^{(m)} = 1 \quad \left(i = 2, 3, \dots, \frac{3^m-1}{2} \right) \end{array} \right. \quad (8.67)$$

¹It is indeed the transition matrix mentioned in Sec. 8.1.2 but in this case we make use of the symmetry of the lattice in the calculation. Hence, it takes a slightly different form.

Table 8.3: First-passage time on the arrowhead lattice at different levels.

m	1	2	3	4	5	6	7	8
$T_1^{(m)}$	2	22	121.167	576.187	2812.12	13951.0	69606.0	347852
$T_{(3^{m-1}+1)/2}^{(m)}$	NA	20	99.1667	455.020	2235.94	11138.9	55655.0	278246
$T_{3^{m-1}}^{(m)}$	NA	16	75.5000	354.551	1703.60	8396.91	41803.7	208770
$T_{3^{m-1}+1}^{(m)}$	NA	9	71.6667	351.636	1700.82	8394.17	41800.9	208768

And $\mathbf{T}^{(m)}$ is the variable column vector of size $\frac{3^m - 1}{2}$

$$\mathbf{T}^{(m)} = \begin{pmatrix} T_1^{(m)} \\ T_2^{(m)} \\ \vdots \\ T_{\frac{3^m-1}{2}}^{(m)} \end{pmatrix}. \quad (8.68)$$

The solution to this system is

$$\mathbf{T}^{(m)} = \left(\tilde{\mathcal{J}}^{(m)} \right)^{-1} \mathbf{B}^{(m)}. \quad (8.69)$$

In the solution $\mathbf{T}^{(m)}$, we focus on the 4 special sites $T_1^{(m)}$, $T_{\frac{3^{m-1}+1}{2}}^{(m)}$, $T_{3^{m-1}}^{(m)}$ and $T_{3^{m-1}+1}^{(m)}$.

Now we turn to the result of $\mathbf{T}^{(m)} = \left(\tilde{\mathcal{J}}^{(m)} \right)^{-1} \mathbf{B}^{(m)}$. At $m = 1$, we only have $T_1^{(1)} = 2t$. The result for $m = 1, 2, \dots, 8$ are given in Table 8.3 (in unit of t).

We define new variable $S_i^{(m)}$ as the ratio between $T_i^{(m)}$ and $T_i^{(m-1)}$

$$S_i^{(m)} = \frac{T_i^{(m)}}{T_i^{(m-1)}}. \quad (8.70)$$

We have the following Table 8.4. From that table and Fig. 8.5, it appears and is conjectured that the ratios $S_i^{(m)}$ converge to $S^* = 5$.

On going from m to $m + 1$ iterations, the lattice is rescaled by a factor of 2, while the escape time $T_i^{(m)}$ rescales as $T_i^{(m)} \rightarrow T_i^{(m+1)} = S_i^{(m+1)} T_i^{(m)} = 5 T_i^{(m)}$.

Therefore, from

$$T \propto L^{d_w} \quad (8.71)$$

Table 8.4: Ratio of first-passage time on the arrowhead lattice at two successive levels.

m	2	3	4	5	6	7	8
$S_1^{(m)}$	11	5.5076	4.7553	4.8806	4.9610	4.9893	4.9974
$S_{(3^{m-1}+1)/2}^{(m)}$	NA	4.9583	4.5884	4.9139	4.9817	4.9965	4.9995
$S_{3^{m-1}}^{(m)}$	NA	4.7187	4.6960	4.8049	4.9289	4.9785	4.9941
$S_{3^{m-1}+1}^{(m)}$	NA	7.9630	4.9065	4.8369	4.9354	4.9798	4.9943

Figure 8.5: Convergence of ratio of first-passage time $S_i^{(m)}$ against m on the arrowhead lattice.

Figure 8.6: Sierpiński gasket at second iteration $m = 2$.

we conjecture that the fractal dimension of random walk on this lattice is

$$d_w = \frac{\log 5}{\log 2} \approx 2.32. \quad (8.72)$$

8.1.3.2 Sierpiński gasket

Here we perform the calculation on the Sierpiński gasket lattice in the same way as done for the arrowhead lattice above and see how they compare to one another. Figure 8.6 shows the Sierpiński gasket at second iteration $m = 2$. The letters label the average escape time from that node.

Denoting t as the time taken to go from one site to one of its nearest neighbours,

we can straight away write down the equations

$$\left\{ \begin{array}{l} T = t + C \\ A = t + \frac{1}{4}C + \frac{1}{4}D + \frac{1}{4}E + \frac{1}{4}F \\ B = t + \frac{1}{2}F + \frac{1}{2}G \\ C = \frac{4}{3}t + \frac{1}{3}A + \frac{1}{3}D + \frac{1}{3}T \\ D = t + \frac{1}{2}A + \frac{1}{2}C \\ E = t + \frac{1}{4}A + \frac{1}{4}F + \frac{1}{4}G \\ F = t + \frac{1}{4}A + \frac{1}{4}B + \frac{1}{4}E + \frac{1}{4}G \\ G = t + \frac{1}{4}B + \frac{1}{4}E + \frac{1}{4}F \end{array} \right. . \quad (8.73)$$

Seven linear equations, seven unknowns, the above system can be easily solved for which the solution yields

$$\left\{ \begin{array}{l} T = 25t \\ A = 20t \\ B = 15t \\ C = 24t \\ D = 23t \\ E = 13t \\ F = 16t \\ G = 12t \end{array} \right. . \quad (8.74)$$

We see that the solution of T , A and B at second iteration $m = 2$ of the lattice is 5 times that at first iteration $m = 1$ (calculated in (ben-Avraham and Havlin, 2000, p. 62)) and 25 times that at zeroth iteration $m = 0$ (a simple triangle with 3 sites). We can see a perfect scaling behaviour of random walker on Sierpiński gasket.

If we define $S_i^{(m)}$ for this Sierpiński gasket, we have $S_i^{(m)} = 5$ ($m = 2, 3, \dots$). At this point, a question arises: *Why does the escape time of random walk on these two lattices scale in different manners but eventually, converges to the same (conjectured) limit?* The answer may be that in the Sierpiński gasket, the three apexes of the triangle are absolutely equivalent to each other. But that's not the case for the Sierpiński arrowhead. This

Table 8.5: First-passage time on the crab fractal lattice at different levels.

m	1	2	3	4	5	6	7	8
$T_1^{(m)}$	2	20	142.833	993.098	6328.89	38615.1	232861	1390804
$T_{(3^{m-1}+1)/2}^{(m)}$	NA	18	132.667	914.990	5869.26	36067.9	217324	1298488
$T_{3^{m-1}}^{(m)}$	NA	14	120.000	752.500	4845.25	29871.9	178794	1068693
$T_{3^{m-1}+1}^{(m)}$	NA	8	100.000	690.500	4657.25	29305.9	177094	1063591

Table 8.6: Ratio of first-passage time on the crab fractal lattice at two successive levels.

m	2	3	4	5	6	7	8
$S_1^{(m)}$	10	7.1417	6.9528	6.3729	6.1014	6.0303	5.9727
$S_{(3^{m-1}+1)/2}^{(m)}$	NA	7.3704	6.8969	6.4146	6.1452	6.0254	5.9749
$S_{3^{m-1}}^{(m)}$	NA	8.5714	6.2708	6.4389	6.1652	5.9854	5.9772
$S_{3^{m-1}+1}^{(m)}$	NA	12.500	6.9050	6.7448	6.2925	6.0430	6.0058

distinction disappears in the large system size limit. One more reason is that the Sierpiński arrowhead at first iteration $m = 1$ is a very much different structure from itself at higher iterations. At first iteration $m = 1$, the Sierpiński arrowhead is simply a one-dimensional chain without any extra interaction between sites; hence it doesn't scale with a 1.58-dimensional lattice. Because high iterations of the Sierpiński arrowhead includes the first iteration $m = 1$, the effect of one-dimensional chain only vanishes for sufficient large m .

8.1.3.3 Crab fractal lattice

The method above is also applied to calculate fractal dimension of random walk on the crab fractal lattice. The results are given in Table 8.5.

We have the result for the ratios $S_i^{(m)}$ in Table 8.6. From that table and Fig. 8.5, it appears and is conjectured that the ratios $S_i^{(m)}$ converge to $S^* = 6$. OPEN PROBLEM see list on page 250

On going from m to $(m + 1)$ iterations, the lattice is rescaled by a factor of 2, while the escape time $T_i^{(m)}$ rescales as $T_i^{(m)} \rightarrow T_i^{(m+1)} = S_i^{*(m+1)} T_i^{(m)} = 6T_i^{(m)}$. Therefore,

Figure 8.7: Convergence of ratio of first-passage time $S_i^{(m)}$ against m on the crab fractal lattice.

we conjecture that the fractal dimension of random walk on this lattice is

$$d_w = \frac{\log 6}{\log 2} \approx 2.58. \quad (8.75)$$

Of the three fractal lattices with the same (Hausdorff) dimension, one notices that they have different structures and fractal dimension of random walk d_w . Table 8.7 summarises and compares the three lattices.

Table 8.7: Comparison of first-passage time among the three fractal lattices of the same dimension d .

Fractal lattice	Arrowhead	Crab	Gasket
(Hausdorff) dimension d	$\frac{\ln 3}{\ln 2} \approx 1.58$	$\frac{\ln 3}{\ln 2} \approx 1.58$	$\frac{\ln 3}{\ln 2} \approx 1.58$
Number of nearest neighbours	2 or 3	2 or 3	4
Number of dissipative sites	2	2	3
Random walk dimension d_w	$\frac{\ln 5}{\ln 2} \approx 2.32$	$\frac{\ln 6}{\ln 2} \approx 2.58$	$\frac{\ln 5}{\ln 2} \approx 2.32$

8.1.3.4 Sierpiński tetrahedron

Similarly, we perform the calculation on the Sierpiński tetrahedron lattice. Figure 8.8 shows the Sierpiński gasket at second iteration $m = 2$. The letters label the average escape time from that node.

One can write down the equations

$$\left\{ \begin{array}{l} T = t + A \\ A = \frac{3}{2}t + \frac{1}{2}B + \frac{1}{4}C + \frac{1}{4}T \\ B = \frac{3}{2}t + \frac{1}{2}A + \frac{1}{2}C \\ C = t + \frac{1}{6}A + \frac{1}{3}B + \frac{1}{6}D + \frac{1}{3}E \\ D = t + \frac{1}{6}C + \frac{1}{3}E + \frac{1}{3}F \\ E = \frac{6}{5}t + \frac{1}{5}C + \frac{1}{5}D + \frac{1}{5}F + \frac{1}{5}G + \frac{1}{5}H \\ F = \frac{6}{5}t + \frac{1}{5}D + \frac{1}{5}E + \frac{1}{5}G + \frac{1}{5}H \\ G = t + \frac{1}{3}E + \frac{1}{3}F + \frac{1}{3}H \\ H = t + \frac{1}{3}E + \frac{1}{3}F + \frac{1}{3}G \end{array} \right. . \quad (8.76)$$

The solutions to the above system yield $T = 36t$ which is 6 times that at first iteration and 36 times that at zeroth iteration (a simple tetrahedron with 4 sites). This shows that the fractal dimension of random walk on this lattice is

$$d_w = \frac{\log 6}{\log 2} \approx 2.58. \quad (8.77)$$

8.2 Construction of adjacency matrix

We have seen in Sec. 8.1.2 (see also [Dhar, 2006](#)) how the knowledge of the adjacency matrix can help in obtaining analytical results on a lattice. In this section, the analytical construction of adjacency matrix for Sierpiński arrowhead lattice is presented. This hopes to facilitate analytical approach to solving the sandpile model on this lattice (which does not materialise in this work). [OPEN PROBLEM](#) see list on page 249

Recall the construction of the lattice: the Sierpiński arrowhead lattice at level m th

Figure 8.8: Sierpiński tetrahedron at second iteration $m = 2$.

is made of three copies at level $(m - 1)$ th and there is one new extended interaction between copy 1 and 3. So if we denote $\mathcal{A}^{(m)}$ to be the adjacency matrix for the lattice at level m th, the adjacency matrix for the lattice at level $(m + 1)$ th looks something like

$$\mathcal{A}^{(m+1)} = \begin{pmatrix} \mathcal{A}^{(m)} & \mathcal{O} & \mathcal{B}^T \\ \mathcal{O} & \mathcal{A}^{(m)} & \mathcal{O} \\ \mathcal{B} & \mathcal{O} & \mathcal{A}^{(m)} \end{pmatrix} \quad (8.78)$$

where $\mathcal{O}$ indicates zero-element matrix, $\mathcal{B}$ indicates the interaction between copy 1 and 3 (that's why in the block matrix representation above, $\mathcal{B}$ is in column 1 and row 3). The adjacency matrix is symmetric so we put $\mathcal{B}^T$ in the top right corner. $\mathcal{A}^{(m)}$, $\mathcal{B}$ and $\mathcal{O}$ are all square matrices of the same size. Details are to be presented in the following.

Next observation is about the matrix $\mathcal{B}$. $\mathcal{B}$ is merely a zero-element matrix with one nonzero element somewhere. The position (row and column) of this nonzero element in $\mathcal{B}$ is determined by the label of sites involved in the new extended interaction. This could be confusing, let's look at Fig. 8.9.

Figure 8.9: Labels of sites of arrowhead lattice.

The sites on the lattice at level $(m+1)$ th are labelled $1, \dots, 3^{(m+1)} + 1$. The nonzero element in matrix $\mathcal{B}$ represents the interaction between site $\frac{3^m + 3}{2}$ and site $\frac{5 \times 3^m + 1}{2}$. Thus, its position can be easily traced out.

In this part, we will need the tensor product $\otimes$ (see (Hinrichsen, 2000) for its detailed explanations)

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ a_2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}. \quad (8.79)$$

Following Eq. (8.78), we will break the adjacency matrix into sum of three terms: $\mathcal{A}^{(m)}$, $\mathcal{B}$ and $\mathcal{B}^T$. It is good to note that, $\mathcal{A}^{(m)}$ is in fact not the precise adjacency matrix for the Sierpiński arrowhead lattice at level m th but the lattice less the last site. In other words, $\mathcal{A}^{(m)}$ is the adjacency matrix for the Sierpiński arrowhead lattice at level m th without the last site. The last site can be easily put back to form the full adjacency matrix because it does not change the structure of the lattice.

We would write

$$\mathcal{A}^{(m+1)} = \mathcal{J}_3 \otimes \mathcal{A}^{(m)} + \mathcal{J}_3 \otimes \mathcal{E}^{(m)} + \left(\mathcal{J}_3 \otimes \mathcal{E}^{(m)} \right)^T, \quad (8.80)$$

with the “frame matrices” $\mathcal{J}_3$ and $\mathcal{J}_3$ (see later). Indeed, we still need two more terms. They are for the joint between copies 1-2 and copies 2-3 and the transpose, of course. The correct expression is

$$\mathcal{A}^{(m+1)} = \mathcal{J}_3 \otimes \mathcal{A}^{(m)} + \mathcal{J}_3 \otimes \mathcal{E}^{(m)} + \left(\mathcal{J}_3 \otimes \mathcal{E}^{(m)} \right)^T + \mathcal{K}_3 \otimes \mathcal{F}^{(m)} + \left(\mathcal{K}_3 \otimes \mathcal{F}^{(m)} \right)^T. \quad (8.81)$$

$\mathcal{E}^{(m)}$ is the matrix $\mathcal{B}$ in Eq. (8.78). $\mathcal{F}^{(m)}$ is merely a zero-element matrix with one nonzero element in the top right corner. Before we can proceed, we need to clarify what $\mathcal{E}^{(m)}$ looks like. As indicated above, $\mathcal{E}^{(m)}$ is merely a zero-element matrix with one nonzero element. $\mathcal{A}^{(m)}$ and $\mathcal{E}^{(m)}$ are both of size $3^m \times 3^m$. In $\mathcal{A}^{(m+1)}$, that nonzero element is in row $\frac{5 \times 3^m + 1}{2}$ and column $\frac{3^m + 3}{2}$. So it can be easily deduced that in $\mathcal{E}^{(m)}$, that nonzero element is in row $\frac{3^m + 1}{2}$ and column $\frac{3^m + 3}{2}$. We know that element in row $\frac{3^m + 1}{2}$ and column $\frac{3^m + 1}{2}$ is precisely at the centre of $\mathcal{E}^{(m)}$. So that nonzero element is just one column to the right of the centre of $\mathcal{E}^{(m)}$

$$\mathcal{E}^{(m)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \boxed{\text{centre}} & \boxed{\text{nonzero}} & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (8.82)$$

And we have

$$\mathcal{E}^{(m+1)} = \mathcal{L}_3 \otimes \mathcal{E}^{(m)}, \quad (8.83)$$

and also

$$\mathcal{F}^{(m+1)} = \mathcal{M}_3 \otimes \mathcal{F}^{(m)}. \quad (8.84)$$

The “frame matrices” are

$$\begin{aligned} \mathcal{J}_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{J}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \mathcal{K}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ \mathcal{L}_3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{M}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (8.85)$$

The adjacency matrix $\mathcal{A}^{(m)}$ is such that element $A_{jk}^{(m)} = 1$ if sites j and k interact with each other and $A_{jk}^{(m)} = 0$ otherwise. This way, we have the initial matrices

$$\mathcal{A}^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{E}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{F}^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (8.86)$$

In summary, to construct the adjacency matrix for Sierpiński arrowhead lattice, we iterate the followings

$$\mathcal{A}^{(m+1)} = \mathcal{J}_3 \otimes \mathcal{A}^{(m)} + \mathcal{J}_3 \otimes \mathcal{E}^{(m)} + \left(\mathcal{J}_3 \otimes \mathcal{E}^{(m)} \right)^T + \mathcal{K}_3 \otimes \mathcal{F}^{(m)} + \left(\mathcal{K}_3 \otimes \mathcal{F}^{(m)} \right)^T, \quad (8.87)$$

$$\mathcal{E}^{(m+1)} = \mathcal{L}_3 \otimes \mathcal{E}^{(m)}, \quad (8.88)$$

$$\mathcal{F}^{(m+1)} = \mathcal{M}_3 \otimes \mathcal{F}^{(m)}. \quad (8.89)$$

Let's see some examples.

$$\begin{aligned}
 \mathcal{A}^{(2)} &= \mathcal{J}_3 \otimes \mathcal{A}^{(1)} + \mathcal{J}_3 \otimes \mathcal{E}^{(1)} + \left(\mathcal{J}_3 \otimes \mathcal{E}^{(1)} \right)^T + \mathcal{K}_3 \otimes \mathcal{F}^{(1)} + \left(\mathcal{K}_3 \otimes \mathcal{F}^{(1)} \right)^T \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
 &\quad + \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right)^T + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
 &\quad + \left(\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)^T \\
 &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{8.90}
 \end{aligned}$$

$$\mathcal{E}^{(2)} = \mathcal{L}_3 \otimes \mathcal{E}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (8.91)$$

$$\mathcal{F}^{(2)} = \mathcal{M}_3 \otimes \mathcal{F}^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (8.92)$$

The addition of one more row and one more column to $\mathcal{A}^{(2)}$ to form the full adjacency matrix for Sierpiński arrowhead lattice at level 2 is straightforward.

8.3 Directed model on fractal lattices

As discussed earlier, in the literature, a number of results exist for exactly solving directed models (see (Pruessner, 2012) for a detailed discussion on these models, their features and classifications). The reason why directed models are easy to handle (much easier than undirected ones) is that in those models one does not have to worry about

backward avalanches. In undirected models, on going avalanche spreads out the entire system isotropically in all directions (of interaction between sites) and can interfere with local activities at a region and then induces further activities. An analytical approach to these models has to be able to capture all these “looping” processes. It, however, needs to be pointed out that no work exists to address the problem of directed models on fractal lattices. In this section, we discussed several issues of directed Abelian Manna model (and other directed models in general) on fractal lattices. Several preliminary results are presented.

One of the features of directed model is the scaling of average avalanche size against the linear size of the lattice $\langle s \rangle \propto L$. Below we discuss the model on the arrowhead and gasket lattice.

8.3.1 Directed arrowhead lattice

We first consider the directed Abelian Manna model on the arrowhead lattice. On this lattice, there is no consistent direction because of the way the sites are located. However, thanks to the labelling of the sites and the string-like structure of the lattice, the most natural way is to follow the sequence of labelling of the sites, *i.e.* particles can only go in the direction from one site to another site that has higher index than the current one (where the particle is residing).

We denote $\langle s \rangle_m$ to be the average avalanche size on the lattice at m th iteration without the last site. The average avalanche size on the entire lattice is simply $\langle s \rangle'_m = \langle s \rangle_m + 1$. We obviously have

$$\langle s \rangle_0 = 1, \langle s \rangle'_0 = 2, \quad (8.93)$$

$$\langle s \rangle_1 = 3, \langle s \rangle'_1 = 4 \quad (8.94)$$

as the lattice at these iterations is just a simple one-dimensional chain. At second iteration (Fig. 8.10), we have the first extended interaction (see Appendix B) between site 3 and 8.

In this model, a particle starting off at site 1 definitely goes through site 2 then 3. At site 3, it has two choices of equal probability: 1/2 to go to site 4 and another 1/2 to go to site 8. If it goes to site 4, the avalanche size is straightforward like in one-dimensional case $\langle s \rangle_1 + \langle s \rangle_1 + \langle s \rangle_1 + 1$. If it goes to site 8, the avalanche size is then $\langle s \rangle_1 + 1 + \langle s \rangle_0 + 1$.

Figure 8.10: Direction on the arrowhead lattice at second iteration for directed model. The arrows show the allowed directions that a particle at a site can move.

Because of the probable choice, the average avalanche size is

$$\begin{aligned}\langle s \rangle_2 &= \langle s \rangle_1 + \frac{1}{2}(\langle s \rangle_1 + \langle s \rangle_1) + \frac{1}{2}(1 + \langle s \rangle_0) \\ &= 2 \langle s \rangle_1 + 1 = 7,\end{aligned}\tag{8.95a}$$

$$\langle s \rangle'_2 = \langle s \rangle_2 + 1 = 8.\tag{8.95b}$$

At third iteration (Fig. 8.11), the story is still very much the same, we evaluate the sum

$$\langle s \rangle_2 + \frac{1}{2}(\langle s \rangle_2 + \langle s \rangle_2) + \frac{1}{2}(1 + \langle s \rangle_0 + \langle s \rangle_1).\tag{8.96}$$

However, in the above expression, we have overcounted half of the first copy (the part from site 7 to site 9). We need to offset that and the equation now reads

$$\begin{aligned}\langle s \rangle_3 &= (\langle s \rangle_2 - \frac{1}{2} \langle s \rangle_1) + \frac{1}{2}(\langle s \rangle_2 + \langle s \rangle_2) + \frac{1}{2}(1 + \langle s \rangle_0 + \langle s \rangle_1) \\ &= 2 \langle s \rangle_2 + 1 = 15,\end{aligned}\tag{8.97a}$$

$$\langle s \rangle'_3 = \langle s \rangle_3 + 1 = 16.\tag{8.97b}$$

At fourth iteration (Fig. 8.12), the equation is

$$\begin{aligned}\langle s \rangle_4 &= \left[\langle s \rangle_3 - \frac{1}{2}(\langle s \rangle_1 + \langle s \rangle_2) \right] + \frac{1}{2}(\langle s \rangle_3 + \langle s \rangle_3) + \frac{1}{2}(1 + \langle s \rangle_0 + \langle s \rangle_1 + \langle s \rangle_2) \\ &= 2 \langle s \rangle_3 + 1 = 31,\end{aligned}\tag{8.98a}$$

Figure 8.11: Direction on the arrowhead lattice at third iteration for directed model. The arrows show the allowed directions that a particle at a site can move.

$$\langle s \rangle'_4 = \langle s \rangle_4 + 1 = 32. \quad (8.98b)$$

The terms in the first square bracket are for going from site 1 to 28 offset the choice of $1/2$ going from 15 to 28. Once the particle is at 28, it is straight forward that going from 28 to 54 is $\langle s \rangle_3$ and going from 55 to 81 is $\langle s \rangle_3$ as well. The terms in the last bracket are fairly straightforward too. This way, we have the general pattern

$$\begin{aligned} \langle s \rangle_m &= \left[\langle s \rangle_{m-1} - \frac{1}{2} \sum_{j=1}^{m-2} \langle s \rangle_j \right] + \frac{1}{2} (\langle s \rangle_{m-1} + \langle s \rangle_{m-1}) + \frac{1}{2} \left(1 + \sum_{j=0}^{m-2} \langle s \rangle_j \right) \\ &= 2 \langle s \rangle_{m-1} + \frac{1 + \langle s \rangle_0}{2} = 2 \langle s \rangle_{m-1} + 1, \end{aligned} \quad (8.99a)$$

$$\langle s \rangle'_m = \langle s \rangle_m + 1 = 2 \langle s \rangle_{m-1} + 2 = 2(\langle s \rangle_{m-1} + 1) = 2 \langle s \rangle'_{m-1}. \quad (8.99b)$$

This together with $\langle s \rangle'_0 = 2$ lead to

$$\langle s \rangle'_m = 2 \times 2^m. \quad (8.100)$$

Figure 8.12: Direction on the arrowhead lattice at fourth iteration for directed model. The arrows show the allowed directions that a particle at a site can move.

We know that the linear size of the lattice is

$$L_m = 2^m + 1. \quad (8.101)$$

So the average avalanche size is linear in L

$$\langle s \rangle \propto L. \quad (8.102)$$

This result (average avalanche size $\langle s \rangle$ being linear in linear system size L) is consistent with what is expected for a directed model. However, paying a careful look at the movement of the particles on this directed lattice, one realises that the introduction of direction indeed revokes the fractal structure of the lattice itself producing trivial behaviours that is not much of interest.

To see that, let's have a look at the directed arrowhead lattice at fourth iteration in Fig. 8.12. Recall in Sec. 6.2.5 (also Appendix B), majority (two-thirds) of the sites on the arrowhead lattice are “regular”, having only two neighbours. That means a particle at any of these sites has only one direction to go in the next step. The remaining one-third of the sites are “extended”, having three neighbours. That means a particle at any of these sites has two possible directions to go in the next step. On a directed lattice, particles always start from a fixed site. In the case of directed arrowhead, we start from the apex at site 1. We now calculate the probability that the particle passes through the other two apexes (at site 42 and 82).

Following the arrows, to pass through site 42, a particle must follow the path

$$1 \xrightarrow{p_3} 28 \xrightarrow{1} 29 \xrightarrow{1} 30 \xrightarrow{3/4} 35 \xrightarrow{1} 36 \xrightarrow{1} 37 \xrightarrow{1} 38 \xrightarrow{1} 39 \xrightarrow{1/2} 40 \xrightarrow{1} 41 \xrightarrow{1} 42$$

in which the straight arrow “ $\rightarrow$ ” indicates a single move between two sites while the wavy arrow “ $\rightsquigarrow$ ” indicates a move (possibly) involving more sites. The number on top of an arrow indicates the probability that the move takes place. Hence, starting from site 1, the probability for a particle to pass through site 42 is

$$p_4 = \left(1 - \frac{p_3}{2}\right) \times \frac{3}{4} \times \frac{1}{2} \quad (8.103)$$

in which p_3 is the probability to go from site 1 to site 15. Hence, the term $\left(1 - \frac{p_3}{2}\right)$ is the probability to go from first site to last site on arrowhead lattice at third iteration $m = 3$ without escaping at middle site 15¹. If the particle takes the shortcut from site 15 straight to site 68, the entire upper part of the lattice is not visited at all. The probability p_3 can be calculated by considering the path

$$1 \xrightarrow{1} 2 \xrightarrow{1} 3 \xrightarrow{3/4} 8 \xrightarrow{1} 9 \xrightarrow{1} 10 \xrightarrow{1} 11 \xrightarrow{1} 12 \xrightarrow{1/2} 13 \xrightarrow{1} 14 \xrightarrow{1} 15$$

which gives

$$p_3 = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}. \quad (8.104)$$

Hence, we have

$$p_4 = \left(1 - \frac{3}{16}\right) \times \frac{3}{8}. \quad (8.105)$$

¹This is subtle. A particle on arrowhead lattice at third iteration would definitely go to last site with probability unity. But that piece of lattice embedded in fourth iteration has a “leaked site” 15. This leaked site provides a “shortcut” to reach the end of the lattice faster. These shortcuts are the ones that make up the fractal structure of the arrowhead lattice.

Going to next iterations, from the structure of the lattice, one can write down

$$p_{m+1} = \left(1 - \frac{p_m}{2}\right) \times p_m \quad (8.106)$$

which admits the limit

$$\lim_{m \rightarrow \infty} p_m = 0. \quad (8.107)$$

This fact simply means that on a very large lattice, the top apex is not reached at all! It further implies that the big shortcut is not taken and the path taken by the particles in the directed model on this lattice is a trivial one (A nontrivial one expects a finite flux of particle through this shortcut because the shortcut makes up the fractal structure of the lattice. Without it, the fractal structure is forfeited.). Yet, it is still interesting to point out the outcome: the small shortcuts shorten the path of a particle and at the same time block its access to bigger shortcuts.

8.3.2 Directed gasket lattice

We have seen in the previous section the problems with directed model on the arrow-head lattice. Since there is no consistent way of defining a preferred direction on that lattice due to the geometrical location of the sites, the most natural way is to follow the sequence of labelling of sites thanks to its string-like structure. On the Sierpiński gasket lattice, one can define the directed model in the sense of existing directed models on normal square lattices: *every move takes the particle closer to the boundary or dissipative sites where it can leave the system*. That way, the gasket has two dissipative sites at the bottom and the particles enter at the top site. All arrows pointing downwards are obvious. The horizontal ones are chosen in direction such that they lead a particle to the nearest site where it can go down. In general, the direction of the arrows is chosen in such a way that it's always pointing down if possible, otherwise, it points to the nearest site to go down (see Fig. 8.13).

On this lattice, the particles start at the site on top (top apex). Denote $\langle s \rangle_m$ to be the average avalanche area on lattice at m th iteration. We can split $\langle s \rangle_m$ into two components: the going-down d_m and the horizontal h_m . d_m is the total number of down-going moves and h_m is the total number of horizontal moves for the entire lattice at m th iteration.

Denoting L_m to be the linear size of the lattice at m th iteration, that is the number

Figure 8.13: Direction on the Sierpiński gasket lattice at fourth iteration for directed model. The arrows show the allowed directions that a particle at a site can move.

of sites along one side of the gasket, we have

$$L_m = 2^m + 1. \quad (8.108)$$

It is clear that (see Fig. 8.13)

$$d_m = L_m - 1. \quad (8.109)$$

The remaining task is to calculate h_m . We do that iteratively.

We have $h_0 = 0$, no horizontal move occurs. Next, we have

$$h_1 = h_0 + \frac{1}{2^1} \frac{L_1 - 1}{2} + 2 \times \frac{1}{2^2} \frac{L_1 - 3}{2}. \quad (8.110)$$

Similarly,

$$h_2 = h_1 + h_0 + \frac{1}{2^2} \frac{L_2 - 1}{2} + 2 \times \frac{1}{2^2} \frac{L_2 - 3}{2} + 2 \times \frac{1}{2^3} \frac{L_2 - 5}{2}, \quad (8.111a)$$

$$h_3 = h_2 + h_1 + h_0 + \frac{1}{2^3} \frac{L_3 - 1}{2} + 2 \times \frac{1}{2^3} \frac{L_3 - 3}{2} + 2 \times \frac{1}{2^3} \frac{L_3 - 5}{2} + 2 \times \frac{1}{2^4} \frac{L_3 - 9}{2}, \quad (8.111b)$$

$\vdots$

$$h_m = \sum_{j=0}^{m-1} h_j + \frac{1}{2^m} \frac{L_m - 1}{2} + 2 \sum_{j=1}^{L_{m-1}-1} \frac{1}{2^m} \frac{L_m - (2j + 1)}{2} + 2 \times \frac{1}{2^{m+1}} \frac{L_m - (2^m + 1)}{2}. \quad (8.111c)$$

We have

$$\sum_{j=1}^{L_{m-1}-1} \frac{L_m - (2j + 1)}{2} = \sum_{k=1}^{L_{m-1}-2} k = \frac{(L_{m-1} - 1)(L_{m-1} - 2)}{2} = \frac{2^{m-1}(2^{m-1} - 1)}{2} \quad (8.112a)$$

$$\Rightarrow h_m = \sum_{j=0}^{m-1} h_j + \frac{1}{2^m} \frac{2^m}{2} + \frac{2}{2^m} \frac{2^{m-1}(2^{m-1} - 1)}{2} = \sum_{j=0}^{m-1} h_j + \frac{2^m}{4}. \quad (8.112b)$$

In explicit form

$$\begin{aligned} h_m &= \sum_{j=0}^{m-1} h_j + \frac{2^m}{4} \\ &= \frac{2^m}{4} + \sum_{j=0}^{m-2} h_j + h_{m-1} = \frac{2^m}{4} + \sum_{j=0}^{m-2} h_j + \sum_{j=0}^{m-2} h_j + \frac{2^{m-1}}{4} \\ &= \frac{2^m}{4} + \frac{2^{m-1}}{4} + 2 \sum_{j=0}^{m-2} h_j = \frac{2^m}{4} + \frac{2^{m-1}}{4} + 2 \left(\sum_{j=0}^{m-3} h_j + h_{m-2} \right) \\ &= \frac{2^m}{4} + \frac{2^{m-1}}{4} + 2 \times \frac{2^{m-2}}{4} + 4 \sum_{j=0}^{m-3} h_j = \dots = \frac{2^m}{4} + \sum_{j=1}^{m-1} 2^j \frac{2^{m-1-j}}{4} \\ &= \frac{2^m}{4} \left(1 + \frac{m-1}{2} \right) = \frac{(m+1)2^m}{8} = \frac{(m+1)(L_m - 1)}{8}. \end{aligned} \quad (8.113)$$

Therefore,

$$\langle s \rangle_m = d_m + h_m = L_m - 1 + \frac{(m-1)(L_m - 1)}{8} = \frac{(m+7)(L_m - 1)}{8} \propto L_m. \quad (8.114)$$

So the average avalanche size is linear in L

$$\langle s \rangle \propto L. \quad (8.115)$$

Again, this result is consistent with what is expected for a directed model. However, same as the arrowhead lattice, this gasket lattice also produces trivial behaviours for the directed model. The same problem arises here when we have a vanishing flux of particles through one of the two apexes at the bottom. The behaviour of the model is then no different from that on a one-dimensional lattice.

8.3.3 Directed model on fractal lattices

In the above two sections, we have discussed the directed model on two fractal lattices — the Sierpiński arrowhead and gasket lattices. Even though they both produce expected results of the directed model, *i.e.* $\langle s \rangle \propto L$, the behaviours are trivial because of vanishing flux of particles through certain parts of the lattice. This kind of behaviours is not in favour because the nontrivial structures (especially the fractal structures, which make the lattice a fractal lattice!) of the lattice are there but ignored by the particles, making the lattice a trivial one. The question is then: *what cause this failure of directed model on a fractal lattice?*

The problem is clearly the vanishing flux of particles. The reason for this can be seen from the structure of the lattice itself. One notices that on the arrowhead or gasket lattice, several sites are more special than the others because they are at the “bottlenecks” of the lattice. By bulk conservation, the particles eventually must pass through (one of) these bottlenecks. Due to the iteration of the fractal lattices, some bottlenecks tend to have more probability to be passed through than the others. Eventually, some bottlenecks are not reached by the particles at all, blocking their access to many parts of the lattice (as these sites are the sole entrances to those parts). This fact is related to the so-called ramification of the fractal. The Sierpiński arrowhead or gasket is finitely ramified because the fractal can be disjointed by removing a finite number of points (the bottlenecks in the above). This fact seems to lead to the requirement of a fractal lattice to be infinitely ramified (also having interfaces) in order for the directed model to be defined on it¹.

It remains an open problem that what is meant by direction on a general lattice,

¹Sierpiński carpet lattice is an example of this. However, it has not been investigated in this work.

whether the directed model can be defined on lattices like the arrowhead or the gasket and still produces consistent but nontrivial behaviours. It would be of great interest to see what kind of critical behaviours are present there on fractal lattices for directed model as a comparison to undirected model for which the results have been reported in Sec. 7.4.1.2. **OPEN PROBLEM** see list on page 249

8.4 Algebra and operator

8.4.1 Operator approach to the Abelian Manna model on general lattice

Sadhu and Dhar (2009) presented an operator approach to stochastic Abelian sand-pile model (SASM). But they only showed details for models on the one-dimensional chain lattice and obtained the steady state. Here, the formulation of the Abelian Manna model is written down for general lattices. And lattice transformations can then be hopefully employed to show relations between steady state on different lattices. From there, conclusion about scaling behaviours on different lattices can be drawn.

The Abelian Manna model is defined on a lattice $\mathcal{L}$ with N sites. At each site i , a non-negative integer height variable h_i is defined. The threshold height at all sites is $h_i^c = 2$ and a site is unstable if $h_i \geq h_i^c = 2$. If all sites are stable, the system is perturbed by charging a site at random $h_i \rightarrow h_i + 1$. Each site on $\mathcal{L}$ has coordination number q_i and a set of α_i^{max} lists $E_{\alpha,i}$ with $\alpha = 1, 2, \dots, \alpha_i^{max}$ (Sadhu and Dhar's notation). If a site is unstable, it relaxes by transferring $h_i^c = 2$ particles to its nearest neighbours. Each of those two particles is transferred to a site in the list $E_{\alpha,i}$ which is chosen with probability $p_{\alpha,i}$ from the set $\{E_{1,i}, E_{2,i}, \dots, E_{\alpha_i^{max},i}\}$. If a site occurs more than once in the list, many particles are added to that site.

A site i has q_i nearest neighbours: $i_1, i_2, \dots, i_{q_i}$. The list $E_{\alpha,i}$ in fact, has two elements $E_{\alpha,i} = \{i_j, i_k\}$ ($j, k = 1, 2, \dots, q_i$). The number α_i^{max} is the number of all possibilities of picking two elements (not distinct) from the set $\{i_j\}_{j=1}^{q_i}$. That is $\alpha_i^{max} = q_i^2 - C_{q_i}^2$. We also have

$$q_i^2 = q_i + 2C_{q_i}^2 \left(= q_i + 2 \frac{q_i!}{(q_i - 2)!2!} = q_i + q_i(q_i - 1) = q_i^2 \right). \quad (8.116)$$

The right hand side says that q_i is the number of lists $E_{\alpha,i}$ with duplicate element and $2C_{q_i}^2$ is the number of lists $E_{\alpha,i}$ with distinct elements (but degenerate to $C_{q_i}^2$ because permutation is doubly counted).

It follows that

$$p_{\alpha,i} = \begin{cases} \frac{1}{q_i^2} (E_{\alpha,i} = \{i_j, i_j\}, j = 1, 2, \dots, q_i) \\ \frac{2}{q_i^2} (E_{\alpha,i} = \{i_j, i_k\}_{j \neq k}, j, k = 1, 2, \dots, q_i) \end{cases}. \quad (8.117)$$

And the normalisation of probability $p_{\alpha,i}$ is

$$q_i \frac{1}{q_i^2} + C_{q_i}^2 \frac{2}{q_i^2} = 1. \quad (8.118)$$

Note that the factor 2 in Eq. (8.116) is because we distinguish $E_{\alpha,i} = \{i_j, i_k\}$ and $E_{\alpha,i} = \{i_k, i_j\}$.

The particle addition operator $\mathbf{a}$ charges the system in stable configuration C at site i and relaxes the system until a new stable configuration is reached. Formally

$$\mathbf{a}_i |C\rangle = \sum_{C'} P_i(C'|C) |C'\rangle \quad (8.119)$$

in which $P_i(C'|C)$ is the probability for the system charged at site i to transit from stable configuration C to new stable configuration C' .

The operators obey the equation

$$\mathbf{a}_i^2 = \sum_{\alpha=1}^{\alpha_i^{max}} p_{\alpha,i} \mathbf{a}^{E_{\alpha,i}} \quad (1 \leq i \leq N) \quad (8.120)$$

in which the notation $\mathbf{a}^E$ means

$$\mathbf{a}^E = \prod_{x \in E} \mathbf{a}_x \quad (8.121)$$

for any list E and

$$\mathbf{a}_i = \mathbf{i} \quad (8.122)$$

which is an identity operator for sites i outside the lattice.

For one-dimensional chain lattice, $q_i = 2 \forall i$, $E_{1,i} = \{i-1, i-1\}$, $E_{2,i} = \{i-1, i+1\}$, $E_{3,i} = \{i+1, i+1\}$, Eq. (8.120) then reads together with Eq. (8.117)

$$\mathbf{a}_i^2 = \frac{1}{4} \mathbf{a}_{i-1}^2 + \frac{2}{4} \mathbf{a}_{i-1} \mathbf{a}_{i+1} + \frac{1}{4} \mathbf{a}_{i+1}^2 = \frac{1}{4} (\mathbf{a}_{i-1} + \mathbf{a}_{i+1})^2. \quad (8.123)$$

For general lattice,

$$\begin{aligned}
 \mathfrak{a}_i^2 &= \frac{1}{q_i^2} \mathfrak{a}_{i_1}^2 + \frac{1}{q_i^2} \mathfrak{a}_{i_2}^2 + \cdots + \frac{1}{q_i^2} \mathfrak{a}_{i_{q_i}}^2 + \frac{2}{q_i^2} \mathfrak{a}_{i_1} \mathfrak{a}_{i_2} + \cdots \\
 &= \frac{1}{q_i^2} \sum_{j=1}^{q_i} \mathfrak{a}_{i_j}^2 + \frac{2}{q_i^2} \sum_{\substack{j,k=1 \\ j \neq k}}^{q_i} \mathfrak{a}_j \mathfrak{a}_k \\
 &= \frac{1}{q_i^2} \left(\sum_{j=1}^{q_i} \mathfrak{a}_{i_j} \right)^2
 \end{aligned} \tag{8.124}$$

in which the last equality is made by using the identity

$$\left(\sum_{i=1}^n \mathfrak{a}_i \right)^2 = \sum_{i=1}^n \mathfrak{a}_i^2 + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^n \mathfrak{a}_i \mathfrak{a}_j. \tag{8.125}$$

The equations for the eigenvalues have the same form

$$a_i^2 = \frac{1}{q_i^2} \left(\sum_{j=1}^{q_i} a_{i_j} \right)^2 \tag{8.126a}$$

$$\rightarrow |a_i| = \left| \frac{1}{q_i} \sum_{j=1}^{q_i} a_{i_j} \right| \tag{8.126b}$$

$$\rightarrow q_i |a_i| = \left| \sum_{j=1}^{q_i} a_{i_j} \right| \leq \sum_{j=1}^{q_i} |a_{i_j}|. \tag{8.126c}$$

Equation (8.126a) can be reduced to linear form

$$\eta_i a_i = \frac{1}{q_i} \sum_{j=1}^{q_i} a_{i_j} \tag{8.127}$$

in which $\eta_i = \pm 1$. And boundary condition is set by Eq. (8.122)

$$a_i = 1 \tag{8.128}$$

for sites i outside the lattice. We can then solve for $a_i \forall i$.

One question arises here: Can showing that the model on different lattices has same

stationary state (or Jordan block structure, see (Sadhu and Dhar, 2009)) show that the scaling behaviour (critical exponents) is the same? Equation (8.124) suggests that the operators act directly on the geometry and topology of the lattice. There could be transformations that transform, say, Eq. (8.127) for one lattice to that for another lattice. Let's look at an example between one-dimensional chain lattice and one-dimensional ladder lattice. We have for one-dimensional chain lattice $q_i = 2$ and $(i_1, i_2) = (i - i, i + 1)$. And

$$a_i^2 = \frac{1}{4} (a_{i-1} + a_{i+1})^2. \quad (8.129)$$

For the ladder lattice (upper row labelled $1, 2, \dots, L$ and lower row labelled $L + 1, L + 2, \dots, 2L$), things are a bit more complicated.

$$\left\{ \begin{array}{l} a_1^2 = \frac{1}{9} (i + a_2 + a_{L+1})^2 \\ a_i^2 = \frac{1}{9} (a_{i-1} + a_{i+1} + a_{i+L})^2 \quad (2 \leq i \leq L - 1) \\ a_L^2 = \frac{1}{9} (a_{L-1} + i + a_{2L})^2 \\ a_{L+1}^2 = \frac{1}{9} (i + a_{L+2} + a_1)^2 \\ a_i^2 = \frac{1}{9} (a_{i-1} + a_{i+1} + a_{i-L})^2 \quad (L + 2 \leq i \leq 2L - 1) \\ a_{2L}^2 = \frac{1}{9} (a_{2L-1} + i + a_L)^2 \end{array} \right. \quad (8.130)$$

If we denote a_i to be eigenvalue of a_i for $1 \leq i \leq L$ and b_i to be eigenvalue of a_i for $L + 1 \leq i \leq 2L$, we can write the eigenvalue equations

$$\left\{ \begin{array}{l} a_i = \pm \frac{1}{3} (a_{i-1} + a_{i+1} + b_i) \\ b_i = \pm \frac{1}{3} (b_{i-1} + b_{i+1} + a_i) \\ a_0 = a_{L+1} = b_0 = b_{L+1} = 1 \end{array} \right. \quad (8.131)$$

Now defining $c_i = a_i + b_i$ and temporarily ignoring the sign “ $\pm$ ”, we have

$$a_i + b_i = \frac{1}{3} (a_{i-1} + b_{i-1} + a_{i+1} + b_{i+1} + a_i + b_i) \quad (8.132a)$$

$$\rightarrow c_i = \frac{1}{3} (c_{i-1} + c_{i+1} + c_i) \quad (8.132b)$$

or

$$c_i = \frac{1}{2}(c_{i-1} + c_{i+1}) \quad (8.132c)$$

which is precisely of the same form as Eq. (8.127) for the line lattice.

These developments, however, are incomplete and require much further investigation to see whether the universality of the Abelian Manna model can be shown by some lattice transformations (like the ones in (Syôzi, 1972)).

8.4.2 A solvable model

Operator approach (Dhar *et al.*, 1995; Dhar, 1999a, 2004, 2006) has yielded some success in tackling sandpile models, mostly deterministic ones. For stochastic model like Abelian Manna model or Oslo model, only the directed version has been successfully dealt with (Pruessner, 2004b; Stapleton and Christensen, 2006). Some attempts have been made for the undirected version, yet its exact solution is still far from being achieved (Dhar, 1999b, 2004; Stilck *et al.*, 2004; Sadhu and Dhar, 2009).

Here, a partially directed Abelian Manna model is introduced on one-dimensional chain lattice. The lattice is a simple one-dimensional chain with mixture of directed and undirected sites. Undirected site has two nearest neighbours to the left and right while directed site has only one nearest neighbour to the right. The system is charged at fixed leftmost site which is a directed site. The undirected sites are uniformly distributed among directed sites. The proportion of undirected sites is tuned from 0 to 1/2 to observe the change in avalanche exponents $D_a, \tau_a, z, \alpha, D, \tau$. An algebraic approach is developed to exactly solve the model.

8.4.2.1 The model

A. Definition

The model is defined on a partially directed one-dimensional chain lattice with N sites. Unlike normal one-dimensional chain lattice, this partially directed lattice is a mixture of directed and undirected sites. Each undirected site has two nearest neighbours to the left and right while each directed site has only one nearest neighbour to the right (see Fig. 8.14). The undirected sites are uniformly distributed among directed sites with density ρ_u , i.e. after every κ directed sites we place an undirected site ($\kappa = 1, 2, \dots; \rho_u = \frac{1}{\kappa + 1}$, see Fig. 8.14). The rule for update of dynamics is the same as described in Sec. 5.1. Like other directed models, the system is charged at fixed left-

Figure 8.14: A sample of partially directed one-dimensional chain lattice with $N = 20$, $\kappa = 3$ and $\rho_u = \frac{1}{4}$. Sites $i = 4, 8, 12, 16, 20$ are undirected. Last site $i = 20$ has a virtual neighbour to its right (not shown).

most site $i = 1$ which is a directed site. The rightmost site is dissipative where particles can leave the system. This site can be either directed or undirected depended on total number of sites on the lattice and proportion of undirected sites. For simplicity, the total number of sites N is always a multiple of $(\kappa + 1)$. Hence, the last site is always undirected. The observables are the same as defined in Sec. 5.3. Here, we solely focus on the avalanche size s . The avalanche area a can also be considered by extending the approach.

B. Abelian property

The model is essentially the regular Abelian Manna model. The difference only comes from the structure of underlying lattice by modifying its nearest-neighbour interactions¹. The Abelian property of the model allows one to entirely relax the left part of the system and proceed to the right part with multiple particles injected (due to number of particle released from the left part).

8.4.2.2 Scaling behaviours from numerical results

From numerical simulation, we know that the model is well-behaved. It displays robust simple scaling (power-law distribution) for all three observable avalanche size s , area a and duration t . Their probability distributions is shown Fig. 8.15.

Moment analysis on the model reveals two types of behaviour. The system exhibits type I (trivial or deterministic) when $\rho_u = 0$, *i.e.* all the sites are directed. In this case, particles just come in pair to travel across the lattice from the first site on the left and leave the system at the last site on the right. The probability density function for all observables is then simply a Kronecker delta, $\mathcal{P}^{(x)}(x) = \delta_{x,N}$. When the density of undirected sites rises up to $\rho_u = \frac{1}{2}$, the system exhibits type II (non-trivial or stochastic). In this case, the probability distributions of the observables are power-law as described in Sec. 5.4 with critical exponents $D_a = 1$, $\tau_a = 3/2$, $z = 1$, $\alpha = 3/2$, $D = 3/2$ and $\tau = 4/3$. The exponents for avalanche size are the same as those for other

¹Indeed, there is no directed model. The model is fixed with general rules. The direction or anisotropy comes from the underlying lattice. Yet, the driving maybe different. We only drive the system at certain sites in the “directed model” (model on directed lattice).

 (a) Avalanche area $\mathcal{P}^{(a)}(a)$

 (b) Avalanche size $\mathcal{P}^{(s)}(s)$

 (c) Avalanche duration $\mathcal{P}^{(t)}(t)$

Figure 8.15: Probability density function of event sizes in the partially directed AMM for $\rho_u = 1/2$. The plots show robust power-law behaviour of the probability distributions. Two system sizes of $N = 1000$ and $N = 2000$ are used in the plots.

Figure 8.16: A basic element in the system of partially directed Abelian Manna model in one dimension with two sites. The arrows indicate the direction(s) in which particles can move.

directed stochastic model in one dimension (Paczuski and Bassler, 2000a; Kloster *et al.*, 2001; Pruessner, 2004a). It is interesting to know how type I flows to type II when ρ_u changes from 0 to $1/2$. [OPEN PROBLEM](#) see list on page 249

8.4.2.3 Algebraic approach and exact solution

A. Sandpile dynamics of basic element — two-site system

In this system, a directed site has only one nearest neighbour so it behaves in a trivial manner by transferring two particles to that neighbour on toppling. Therefore it would be sufficient to consider a basic element system consisting of two sites¹: left site is directed and right site is undirected. Figure 8.16 illustrates a basic element with two sites.

For undirected site, when it topples each of the two particles can go to left neighbour with probability p_l and right neighbour with probability p_r ($p_l + p_r = 1$). We can keep p_l and p_r general but for simplicity we boil them down to $p_l = p_r = \frac{1}{2}$.

a. Single-charge dynamics

A site can have two stable states $h = 0$ and $h = 1$, therefore, for two-site system, we have four stable configurations labelled C_1 , C_2 , C_3 and C_4 . We now illustrate the toppling dynamics of this system on receiving single charge at the left site and relaxing. $C_i = (a, b)$ refers to the stable configuration $h_1 = a$ and $h_2 = b$ with a and b being either 0 or 1.

In the above, the symbol in the bracket means the configuration obtained after certain number of topplings. A star “*” means that the configuration is charged once at the left site.

From the diagram above, it can be seen that if we start out with an empty lattice then stable configurations C_2 and C_4 are forbidden. A single charge will bring C_1 to C_3 and C_3 to either C_1 or C_3 . In other words, C_1 and C_3 form their own attractor; C_2 and C_4 form their own attractor.

¹The maximum density of undirected sites considered here is $\rho_{u,max} = \frac{1}{2}$, and the sites are uniformly distributed. Hence, two undirected sites are not allowed to stay next to one another.

Figure 8.17: Relaxation scheme of partially directed AMM for system of two sites. The arrow on top of a particle shows its possible direction(s) to move in the next time step. In the diagrams above, the left site is directed hence the arrows of the particles at that site point to one direction only, while the right site is undirected hence the arrows are bidirected. The numbers on top of the bold arrows indicate the probability that the configuration before them transforms to the one after. At the end of each process, the achieved configuration is indicated. This may not be a stable one. In some cases, an unstable one is reached, and the process continues. A configuration with a $\star$ means being charged (number of times depends on the number of $\star$'s).

Figure 8.18: The two main configurations of the system starting from an empty lattice. See more details in the caption of Fig. 8.17. The configurations have also been renamed.

Because we start from an empty lattice, we always end up in either C_1 or C_3 . So for simplicity, from now on we only consider two configuration $C_1 = (0,0)$ and $C_3 = (1,0)$ (renamed to C_1 and C_2 below).

The diagram in Fig. 8.18 tells us that charging C_1 , we get C_2 after 0 topplings with probability 1; charging C_2 , we get C_2^* after 2 topplings with probability $1/4$, C_2 after 2 topplings with probability $1/2$ and C_1 after 2 topplings with probability $1/4$.

Denoting $T_{ij}^{(1)}(s)$ ¹ (superscript “1” is for single charge) as probability that C_j is transformed to C_i after *precisely* s topplings ($i, j = 1, 2$), from the toppling scheme above, we have

$$\left\{ \begin{array}{l} T_{11}^{(1)}(s) = 0 \\ T_{12}^{(1)}(s) = \delta_{s0} \\ T_{12}^{(1)}(s = 2 + 2k) = \frac{1}{4} \frac{1}{4^k} \text{ (0 otherwise) } . \\ T_{22}^{(1)}(s = 2 + 2k) = \frac{1}{2} \frac{1}{4^k} \text{ (0 otherwise) } \\ k = 0, 1, \dots \end{array} \right. \quad (8.133)$$

Denoting $P_C(C_i)$ as probability to have stable configuration C_i and $P_{C1}(C_i, s)$ ²

¹ $T_{ij}^{(n)}(s)$

² $P_{Cn}(C_i, s)$

(subscript “1” after C is for single charge) as probability to have C_i after *precisely* s topplings in the stationary state, we have

$$P_C(C_i) = \lim_{s \rightarrow \infty} \sum_{s'=0}^s \sum_{j=1}^2 T_{ij}^{(1)}(s') P_C(C_j). \quad (8.134)$$

We can write in matrix form

$$\begin{pmatrix} P_C(C_1) \\ P_C(C_2) \end{pmatrix} = \lim_{s \rightarrow \infty} \sum_{s'=0}^s \begin{pmatrix} T_{11}^{(1)}(s') & T_{12}^{(1)}(s') \\ T_{21}^{(1)}(s') & T_{22}^{(1)}(s') \end{pmatrix} \begin{pmatrix} P_C(C_1) \\ P_C(C_2) \end{pmatrix}. \quad (8.135)$$

We also have

$$\begin{pmatrix} P_{C1}(C_1, s) \\ P_{C1}(C_2, s) \end{pmatrix} = \begin{pmatrix} T_{11}^{(1)}(s) & T_{12}^{(1)}(s) \\ T_{21}^{(1)}(s) & T_{22}^{(1)}(s) \end{pmatrix} \begin{pmatrix} P_C(C_1) \\ P_C(C_2) \end{pmatrix}. \quad (8.136)$$

We can write in terms of symbols¹

$$P_{C1}(s) = \mathbb{T}_1(s) P_C \quad (8.137)$$

and

$$P_C = \lim_{s \rightarrow \infty} \sum_{s'=0}^s P_{C1}(s') = \lim_{s \rightarrow \infty} \sum_{s'=0}^s \mathbb{T}_1(s') P_C. \quad (8.138)$$

Defining²

$$\mathbb{T}_1 = \lim_{s \rightarrow \infty} \sum_{s'=0}^s \mathbb{T}_1(s'), \quad (8.139)$$

we have

$$\boxed{P_C = \mathbb{T}_1 P_C}. \quad (8.140)$$

In details, the matrices are

$$\mathbb{T}_1(0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (8.141a)$$

¹ $P_{Cn}(s), \mathbb{T}_n(s), P_C$

² $\mathbb{T}_n$

$$\mathbb{T}_1(1+2k) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (8.141b)$$

$$\mathbb{T}_1(2+2k) = \begin{pmatrix} 0 & \frac{1}{4} \frac{1}{4^k} \\ 0 & \frac{1}{2} \frac{1}{4^k} \end{pmatrix}, \quad (8.141c)$$

$$k = 0, 1, \dots,$$

$$\mathbb{T}_1 = \lim_{s \rightarrow \infty} \sum_{s'=0}^s \mathbb{T}_1(s') = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & \frac{2}{3} \end{pmatrix}. \quad (8.142)$$

P_C can then be found to be

$$P_C = \begin{pmatrix} P_C(C_1) \\ P_C(C_2) \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}. \quad (8.143)$$

Next task is to find the probability distribution of avalanche size. Denote $D_1(s)$ ¹ as probability to have avalanche of size s due to single charge (subscript “1” is for single charge), we have

$$D_1(s) = \sum_{j=1}^2 \sum_{i=1}^2 T_{ij}^{(1)}(s) P_C(C_j) = \sum_{i=1}^2 P_{C1}(C_i, s). \quad (8.144)$$

Up to this point, we introduce the bra-ket notation.

$$\langle 0| = (0 \ 0), \quad |0\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\langle e_1| = (1 \ 0), \quad |e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \langle e_2| = (0 \ 1), \quad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\langle 1| = P_C(C_1) \langle e_1| = \left(\frac{1}{4} \ 0\right), \quad |1\rangle = P_C(C_1) |e_1\rangle = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix},$$

¹ $D_n(s)$

$$\begin{aligned}\langle 2| &= P_C(C_2) \langle e_2| = \begin{pmatrix} 0 & \frac{3}{4} \end{pmatrix}, & |2\rangle &= P_C(C_2) |e_2\rangle = \begin{pmatrix} 0 \\ \frac{3}{4} \end{pmatrix}, \\ \langle I| &= \langle e_1| + \langle e_2| = (1 \quad 1), & |I\rangle &= |e_1\rangle + |e_2\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ \mathbb{I} &= |e_1\rangle \langle e_1| + |e_2\rangle \langle e_2| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \mathbb{O} &= |0\rangle \langle 0| = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

With these notations, we can rewrite things

$$\mathbf{P}_C = |P_C\rangle = |1\rangle + |2\rangle \quad (8.145)$$

and

$$\boxed{D_1(s) = \langle I | \mathbb{T}_1(s) | P_C \rangle}. \quad (8.146)$$

So single-charge dynamics for basic element system is fully solved. From Eq. (8.146), we obtain the avalanche size distribution

$$D_1(0) = \langle I | \mathbb{T}_1(0) | P_C \rangle = \frac{1}{4}, \quad (8.147a)$$

$$D_1(1 + 2k) = \langle I | \mathbb{T}_1(1 + 2k) | P_C \rangle = 0, \quad (8.147b)$$

$$D_1(2 + 2k) = \langle I | \mathbb{T}_1(2 + 2k) | P_C \rangle = \frac{9}{4^{k+2}}. \quad (8.147c)$$

The stationary stable configuration distribution is

$$P_C(C_1) = \frac{1}{4}, \quad (8.148a)$$

$$P_C(C_2) = \frac{3}{4}. \quad (8.148b)$$

b. Multiple-charge dynamics

We have successfully solved the single-charge dynamics of two-site system. Now

we proceed to multiple-charge dynamics. We need to consider this because when we deal with large system, each two-site system can receive multiple particles from systems before it.

Thanks to the Abelian property of the model, we know that considering two simultaneous charges is the same as considering one charge after the other. The diagram in Fig. 8.19 illustrates this.

We still obtain back the states like in the case of single-charge. There are still only two stable configurations $C_1 = (0, 0)$ and $C_2 = (1, 0)$ in the stationary state.

Assume that we start out with stable configuration C_k . After the first charge is applied, we have C_k transformed to C_j after *precisely* s_1 topplings with probability $T_{jk}^{(1)}(s_1)$. Then the second charge is applied, we have C_j transformed to C_i after *precisely* s_2 topplings with probability $T_{ij}^{(1)}(s_2)$. As a whole, C_k is transformed to C_i via C_j after *precisely* s_1 -and- s_2 charges with probability $T_{ik}^{(2)}(s_1, s_2) = T_{ij}^{(1)}(s_1)T_{jk}^{(1)}(s_2)$. We need to sum over all possible pairs (s_1, s_2) with $s_1 + s_2 = s$ and intermediate configuration C_j to find the probability $T_{ik}^{(2)}(s)$ that C_k is transformed to C_i after *precisely* s topplings (due to 2 charges). We have

$$T_{ik}^{(2)}(s) = \sum_{s_1+s_2=s} \sum_{j=1}^2 T_{ij}^{(1)}(s_1)T_{jk}^{(1)}(s_2) = \sum_{s'=0}^s \sum_{j=1}^2 T_{ij}^{(1)}(s')T_{jk}^{(1)}(s-s'). \quad (8.149)$$

In matrix form

$$\mathbb{T}_2(s) = \sum_{s'=0}^s \mathbb{T}_1(s')\mathbb{T}_1(s-s'). \quad (8.150)$$

For multiple charges, we can easily generalise

$$\mathbb{T}_n(s) = \sum_{s'=0}^s \mathbb{T}_{n-1}(s')\mathbb{T}_1(s-s'). \quad (8.151)$$

Figure 8.19: Relaxation of system after receiving two consecutive charges. See more details in the caption of Fig. 8.17.

We can show that

$$\begin{aligned}
 \mathbb{T}_2 &= \lim_{s \rightarrow \infty} \sum_{s'=0}^s \mathbb{T}_2(s') \\
 &= \lim_{s \rightarrow \infty} \sum_{s'=0}^s \sum_{s''=0}^{s'} \mathbb{T}_1(s'') \mathbb{T}_1(s - s'') \\
 &= \left[\lim_{s \rightarrow \infty} \sum_{s'=0}^s \mathbb{T}_1(s') \right] \left[\lim_{s \rightarrow \infty} \sum_{s'=0}^s \mathbb{T}_1(s') \right] \\
 &= \mathbb{T}_1^2.
 \end{aligned} \tag{8.152}$$

And for multiple charges

$$\mathbb{T}_n = \mathbb{T}_1^n. \tag{8.153}$$

Similar expression for P_{Cn} can be deduced

$$P_{Cn}(s) = \mathbb{T}_n(s) P_C. \tag{8.154}$$

The more important one is

$$\boxed{P_C = \mathbb{T}_n P_C = \mathbb{T}_1^n P_C}. \tag{8.155}$$

The distribution of stable configurations due to multiple charges does not change. So P_C is the global stationary state distribution of stable configurations.

Now we are interested in the distribution of avalanche size due to multiple (simultaneous) charges $D_n(s)$. Equation (8.146) can be generalised to

$$\boxed{D_n(s) = \langle I | \mathbb{T}_n(s) | P_C \rangle}. \tag{8.156}$$

$|P_C\rangle$ is always ready as $|P_C\rangle = P_C = \begin{pmatrix} 1 & 3 \\ 4 & 4 \end{pmatrix}^T$, the only obstacle now is $\mathbb{T}_n(s)$ which is defined recursively

$$\mathbb{T}_n(s) = \sum_{s'=0}^s \mathbb{T}_{n-1}(s') \mathbb{T}_1(s - s'). \tag{8.157}$$

We progress with several basic information:

- $\mathbb{T}_0(s) = \delta_{s0} \mathbb{I}$: this simply means that if the system is not charged, the only possible avalanche size is $s = 0$.

- $\mathbb{T}_1(0) = |e_2\rangle \langle e_1|$; $\mathbb{T}_1(1 + 2k) = |0\rangle \langle 0|$; $\mathbb{T}_1(2 + 2k) = \frac{1}{2 \times 4^k} \left(\frac{1}{2} |e_1\rangle + |e_2\rangle \right) \langle e_2|$: these relations can be verified easily.
- $\mathbb{T}_{n>1}(0) = |0\rangle \langle 0|$: this simply means that it is not possible to charge the system more than one time without inducing any toppling.

Apparently, $\mathbb{T}_n(1 + 2k) = |0\rangle \langle 0|$ for all n so we have

$$\begin{aligned} \mathbb{T}_n(2 + 2k) &= \mathbb{T}_{n-1}(0)\mathbb{T}_1(2 + 2k) + \sum_{i=0}^{k-1} \mathbb{T}_{n-1}(2 + 2i)\mathbb{T}_1(2k - 2i) \\ &\quad + \mathbb{T}_{n-1}(2 + 2k)\mathbb{T}_1(0). \end{aligned} \quad (8.158)$$

Let's do this step by step. First consider $n = 2$.

$$\begin{aligned} \mathbb{T}_2(2 + 2k) &= \mathbb{T}_1(0)\mathbb{T}_1(2 + 2k) + \sum_{i=0}^{k-1} \mathbb{T}_1(2 + 2i)\mathbb{T}_1(2k - 2i) + \mathbb{T}_1(2 + 2k)\mathbb{T}_1(0) \\ &= |e_2\rangle \langle e_1| \frac{1}{2 \times 4^k} \left(\frac{1}{2} |e_1\rangle + |e_2\rangle \right) \langle e_2| \\ &\quad + \sum_{i=0}^{k-1} \frac{1}{2 \times 4^i} \left(\frac{1}{2} |e_1\rangle + |e_2\rangle \right) \langle e_2| \frac{1}{2 \times 4^{k-i-1}} \left(\frac{1}{2} |e_1\rangle + |e_2\rangle \right) \langle e_2| \\ &\quad + \frac{1}{2 \times 4^k} \left(\frac{1}{2} |e_1\rangle + |e_2\rangle \right) \langle e_2| (|e_2\rangle \langle e_1|). \end{aligned} \quad (8.159)$$

Up to this point, we introduce four “fundamental matrices”

$$\boxed{\mathbb{A} = |e_1\rangle \langle e_1|, \mathbb{B} = |e_1\rangle \langle e_2|, \mathbb{C} = |e_2\rangle \langle e_1|, \mathbb{D} = |e_2\rangle \langle e_2|}. \quad (8.160)$$

We have the sum

$$\mathbb{A} + \mathbb{D} = \mathbb{I} \quad (8.161)$$

and the product summarised below (entries are result of matrix multiplication between

quantity in row and column)

$\downarrow \times \rightarrow$	A	B	C	D	
A	A	B	O	O	
B	O	O	A	B	·
C	C	D	O	O	
D	O	O	C	D	

(8.162)

We then have

$$\begin{aligned}
 \mathbb{T}_2(2 + 2k) &= \frac{1}{2 \times 4^k} \mathbb{C} \left(\frac{1}{2} \mathbb{B} + \mathbb{D} \right) + \sum_{i=0}^{k-1} \frac{1}{4^k} \left(\frac{1}{2} \mathbb{B} + \mathbb{D} \right) \left(\frac{1}{2} \mathbb{B} + \mathbb{D} \right) \\
 &\quad + \frac{1}{2 \times 4^k} \left(\frac{1}{2} \mathbb{B} + \mathbb{D} \right) \mathbb{C} \\
 &= \frac{1}{4^k} \left(\frac{1}{4} \mathbb{D} + \frac{1}{4} \mathbb{A} + \frac{1}{2} \mathbb{C} \right) + \sum_{i=0}^{k-1} \frac{1}{4^k} \left(\frac{1}{2} \mathbb{B} + \mathbb{D} \right) \\
 &= \frac{1}{4^k} \left(\frac{1}{4} \mathbb{I} + \frac{k}{2} \mathbb{B} + \frac{1}{2} \mathbb{C} + k \mathbb{D} \right).
 \end{aligned}
 \tag{8.163}$$

The coefficients in front of the fundamental matrices are the entries in the matrix of the operator. So we have

$$\mathbb{T}_2(2 + 2k) = \frac{1}{4^k} \begin{pmatrix} \frac{1}{4} & \frac{k}{2} \\ \frac{1}{2} & k + \frac{1}{4} \end{pmatrix}.
 \tag{8.164}$$

We move on and obtain

$$\mathbb{T}_3(2 + 2k) = \frac{1}{4^k} \left[\frac{k}{2} \mathbb{I} + \frac{k(2k-1)}{4} \mathbb{B} + \left(k + \frac{1}{4} \right) \mathbb{C} + \frac{k(2k-1)}{2} \mathbb{D} \right]
 \tag{8.165}$$

and

$$\begin{aligned}
 \mathbb{T}_4(2 + 2k) &= \frac{1}{4^k} \left(\frac{k(2k-1)}{4} \mathbb{I} + \frac{k(2k-1)(k-1)}{6} \mathbb{B} + k^2 \mathbb{C} \right. \\
 &\quad \left. + \frac{k(2k-1)(k-1)}{3} \mathbb{D} \right).
 \end{aligned}
 \tag{8.166}$$

We realise that $\mathbb{T}_n(2 + 2k)$ can be written as

$$\begin{aligned}\mathbb{T}_n(2 + 2k) &= \frac{1}{4^k} [a_n(k)\mathbb{A} + b_n(k)\mathbb{B} + c_n(k)\mathbb{C} + d_n(k)\mathbb{D}] \\ &= \frac{1}{4^k} \{a_n(k)\mathbb{I} + b_n(k)\mathbb{B} + c_n(k) + [d_n(k) - a_n(k)]\mathbb{D}\} \\ &= \frac{1}{4^k} \begin{pmatrix} a_n(k) & b_n(k) \\ c_n(k) & d_n(k) \end{pmatrix}.\end{aligned}\tag{8.167}$$

Substituting this form into the recursive equation

$$\begin{aligned}\mathbb{T}_n(2 + 2k) &= \mathbb{T}_{n-1}(0)\mathbb{T}_1(2 + 2k) + \sum_{i=0}^{k-1} \mathbb{T}_{n-1}(2 + 2i)\mathbb{T}_1(2k - 2i) \\ &\quad + \mathbb{T}_{n-1}(2 + 2k)\mathbb{T}_1(0)\end{aligned}\tag{8.168}$$

and simplifying, we obtain

$$\begin{cases} a_n(k) = b_{n-1}(k) \\ b_n(k) = 2 \sum_{i=0}^{k-1} \left(\frac{1}{2} a_{n-1}(i) + b_{n-1}(i) \right) \\ c_n(k) = d_{n-1}(k) \\ d_n(k) = a_n(k) + 2b_n(k) \\ b_0(k) = 0 \\ b_1(k) = \frac{1}{4} \end{cases}.\tag{8.169}$$

Solving the above system we get

$$\begin{cases} b_n(k) = \frac{1}{4} \frac{(2k)!}{(2k - n + 1)!(n - 1)!} = \frac{1}{4} ({}_{n-1}C_{2k}) \\ a_n(k) = b_{n-1}(k) = \frac{1}{4} ({}_{n-2}C_{2k}) \\ c_n(k) = \frac{1}{4} [({}_{n-3}C_{2k}) + 2 ({}_{n-2}C_{2k})] \\ d_n(k) = \frac{1}{4} [({}_{n-2}C_{2k}) + 2 ({}_{n-1}C_{2k})] \\ ({}_nC_m) = 0 \text{ if } m > n \end{cases}.\tag{8.170}$$

So we have

$$\mathbb{T}_n(2+2k) = \frac{1}{4 \times 4^k} \begin{pmatrix} (n-2)C_{2k} & (n-1)C_{2k} \\ (n-3)C_{2k} + 2(n-2)C_{2k} & (n-2)C_{2k} + 2(n-1)C_{2k} \end{pmatrix}. \quad (8.171)$$

Substituting everything in Eq. (8.156), we have the probability to have avalanche of size s due to n charges on a basic element

$$D_n(2+2k) = \langle I | \mathbb{T}_n(2+2k) | P_C \rangle = \frac{1}{4^{k+2}} [(n-3)C_{2k} + 6(n-2)C_{2k} + 9(n-1)C_{2k}]. \quad (8.172)$$

And this result has been confirmed by numerical simulation. A specific case is double-charge

$$D_2(2+2k) = \frac{1}{4^{k+2}} [6({}_0C_{2k}) + 9({}_1C_{2k})] = \frac{18k+6}{4^{k+2}}. \quad (8.173)$$

B. Path to sandpile dynamics of many-element system

We have fully solved the dynamics of basic-element system. Now we proceed to construct dynamics for many-element system by joining basic elements together to make bigger system. We add in the subscript of the quantities the new index with comma. For example, the avalanche size distribution $D_n(s)$ due to n charges on two-site system is now written as $D_{1,n}(s)$ which refers to system with only one basic element. For a lattice of size $2L$ with L basic elements, the avalanche size distribution (with single charge) is $D_{L,1}(s)$. One of the desired results toward the end is

$$D_{\infty,1}(s) = \lim_{L \rightarrow \infty} D_{L,1}(s) \quad (8.174)$$

and we expect

$$D_{\infty,1}(s) \propto s^{-\tau}. \quad (8.175)$$

There are three possible ways to form bigger system namely: $1 + (L-1)$, $(L-1) + 1$ and $L + L$. $L + L$ is apparently highly desired because that provides direct access to finite-size scaling but it is very difficult to handle. $1 + (L-1)$ turns out to be easiest because the number of terms is under control. When the basic element system relaxes, only three possibilities of 0, 1 or 2 particles are released meaning that the big system $(L+1)$ is charged 0, 1 or 2 times only.

Consider two-element system, for single charge, in first element, state C_j is transformed to state C_i after *precisely* s_1 topplings with probability $T_{ij}^{(1)}(s_1)$. For each of

that transformation, second element is charged $(\varpi_{1,1})_{ij}$ (see next) times and state C_l is transformed to state C_k after *precisely* s_2 topplings with probability $T_{kl}^{((\varpi_{1,1})_{ij})}(s_2)$. $\varpi_{1,1}$ ¹ is the matrix that represents the corresponding number of particles released by a single-element (two-site) system due to $n = 1$ charges (the first subscript 1 is for single-element system and the second subscript 1 is for single charge). We have

$$\varpi_{1,1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \quad (8.176)$$

In general, we have for n charges

$$\varpi_{1,n} = \begin{pmatrix} n & n+1 \\ n-1 & n \end{pmatrix}. \quad (8.177)$$

As a whole, C_j is transformed to C_i in first element after s_1 topplings and C_l is transformed to C_k in second element after s_2 topplings with probability $T_{ij}^{(1)}(s_1)T_{kl}^{((\varpi_{1,1})_{ij})}(s_2)$. We need to sum over all possible combination of $s_1 + s_2 = s$ and use a 4×4 matrix to represent the transformation probabilities in both elements. We would have

$$\begin{aligned} \mathbb{T}_{2,1}(s) = & \sum_{s_1+s_2=s} \begin{pmatrix} T_{11}^{(1)}(s_1)T_{11}^{(1)}(s_2) & T_{11}^{(1)}(s_1)T_{12}^{(1)}(s_2) & T_{12}^{(1)}(s_1)T_{11}^{(2)}(s_2) & T_{12}^{(1)}(s_1)T_{12}^{(2)}(s_2) \\ T_{11}^{(1)}(s_1)T_{21}^{(1)}(s_2) & T_{11}^{(1)}(s_1)T_{22}^{(1)}(s_2) & T_{12}^{(1)}(s_1)T_{21}^{(2)}(s_2) & T_{12}^{(1)}(s_1)T_{22}^{(2)}(s_2) \\ T_{21}^{(1)}(s_1)T_{11}^{(0)}(s_2) & T_{21}^{(1)}(s_1)T_{12}^{(0)}(s_2) & T_{22}^{(1)}(s_1)T_{11}^{(1)}(s_2) & T_{22}^{(1)}(s_1)T_{12}^{(1)}(s_2) \\ T_{21}^{(1)}(s_1)T_{21}^{(0)}(s_2) & T_{21}^{(1)}(s_1)T_{22}^{(0)}(s_2) & T_{22}^{(1)}(s_1)T_{21}^{(1)}(s_2) & T_{22}^{(1)}(s_1)T_{22}^{(1)}(s_2) \end{pmatrix}. \end{aligned} \quad (8.178)$$

For n charges on the two-element system, the matrix for the number of particles

¹ $\varpi_{L,n}$

released is

$$\varpi_{2,n} = \begin{pmatrix} n & n+1 & n+1 & n+2 \\ n-1 & n & n & n+1 \\ n-1 & n & n & n+1 \\ n-2 & n-1 & n-1 & n \end{pmatrix}. \quad (8.179)$$

In general, we would have for n charges on the $(L+1)$ -element system

$$\varpi_{L+1,n} = \varpi_{L,n} \otimes \mathbf{1} + \mathbf{1}^{\otimes L} \otimes \mathbf{p} = n\mathbf{1}^{\otimes(L+1)} + \varpi_{L+1,0} \quad (8.180)$$

with

$$\varpi_{L+1,0} = \mathbf{p} \otimes \mathbf{1}^{\otimes L} + \mathbf{1} \otimes \varpi_{L,0} = \varpi_{L,0} \otimes \mathbf{1} + \mathbf{1}^{\otimes L} \otimes \mathbf{p} = \sum_{i=0}^L \mathbf{1}^{\otimes i} \otimes \mathbf{p} \otimes \mathbf{1}^{\otimes(L-i)} \quad (8.181)$$

in which

$$\mathbf{1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (8.182)$$

and

$$\mathbf{p} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (8.183)$$

The matrix $\varpi_{L,n}$ tells us the number of particles transferred to the right boundary of the L -element system on charging it n times at the left boundary. Any negative element (introduced to serve the purpose of an iterative equation) of the matrix is to be set to 0. That just simply means that all particles are absorbed and the avalanche stopped somewhere before the right boundary.

Before going on, we generalise the bra-ket notation for many-element system¹

$$\mathbf{P}_{n,C} = \mathbf{P}_{1,C}^{\otimes n} = |P_{n,C}\rangle = |P_{1,C}\rangle^{\otimes n},$$

$$\langle I_n| = \langle I_1|^{\otimes n}, \quad |I_n\rangle = |I_1\rangle^{\otimes n},$$

$$\mathbb{I}_n = \mathbb{I}_1^{\otimes n}, \quad \mathbb{O}_n = \mathbb{O}_1^{\otimes n}.$$

¹ $\mathbf{P}_{L,C}$

Recall what we will need

$$\mathbb{T}_{1,0}(s) = \delta_{s0} \mathbb{I}_1 = \begin{pmatrix} \delta_{s0} & 0 \\ 0 & \delta_{s0} \end{pmatrix}, \quad (8.184a)$$

$$\mathbb{T}_{1,1}(0) = \mathbb{C} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (8.184b)$$

$$\mathbb{T}_{1,1}(1+2k) = \mathbb{O}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (8.184c)$$

$$\mathbb{T}_{1,1}(2+2k) = \frac{1}{4^k} \begin{pmatrix} 0 & \frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}, \quad (8.184d)$$

$$\mathbb{T}_{1,2}(0) = \mathbb{T}_{1,2}(1+2k) = \mathbb{O}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (8.184e)$$

$$\mathbb{T}_{1,2}(2+2k) = \frac{1}{4^k} \begin{pmatrix} \frac{1}{4} & \frac{k}{2} \\ \frac{1}{2} & k + \frac{1}{4} \end{pmatrix}. \quad (8.184f)$$

In general, we have

$$\begin{aligned} \mathbb{T}_{L,1}(s) = \sum_{s_1+s_2=s} & [(\mathbb{A}\mathbb{T}_{1,1}(s_1)\mathbb{A}) \otimes \mathbb{T}_{L-1,1}(s_2) + (\mathbb{A}\mathbb{T}_{1,1}(s_1)\mathbb{D}) \otimes \mathbb{T}_{L-1,2}(s_2) \\ & + (\mathbb{D}\mathbb{T}_{1,1}(s_1)\mathbb{A}) \otimes \mathbb{T}_{L-1,0}(s_2) + (\mathbb{D}\mathbb{T}_{1,1}(s_1)\mathbb{D}) \otimes \mathbb{T}_{L-1,1}(s_2)], \end{aligned}$$

$$\mathbb{T}_{L,2}(s) = \sum_{s_1+s_2=s} \mathbb{T}_{L,1}(s_1)\mathbb{T}_{L,1}(s_2),$$

$$D_{L,1}(s) = \langle I_L | \mathbb{T}_{L,1}(s) | P_{L,C} \rangle.$$

The following quantities are trivial¹

$$\begin{cases} \mathbb{T}_{L,0}(s) = \delta_{s0} \mathbb{I}_L \\ \mathbb{T}_{L,2}(0) = \mathbb{O}_L \\ \mathbb{T}_{L,n}(1+2k) = \mathbb{O}_L \end{cases} \quad (8.185)$$

As a result²

$$\begin{cases} D_{L,0}(s) = \delta_{s0} \\ D_{L,2}(0) = 0 \\ D_{L,n}(1+2k) = 0 \end{cases} \quad (8.186)$$

We need to determine $D_{L,1}(0)$ and $D_{L,1}(2+2k)$ and also $D_{L,2}(2+2k)$ as a check.

Further developments and results will be published in (Huynh and Tran, 2012).

8.4.2.4 Remarks

It should be first noted that the exponents (from numerical results) of the current model (at least for the density of undirected sites $\rho_u = \frac{1}{2}$) is the same as those of the [totally asymmetric Oslo model \(TAOM\)](#) (Pruessner, 2004a). This implies that universality-wise, the current model is not very different from [TAOM](#). In fact, it has been shown that any finite amount of anisotropy in a stochastic model would drive it to the class of [TAOM](#) (Pruessner and Jensen, 2003)³. An interesting feature of the current model is the variation of the density of the undirected for $\rho_u = 0$ to $\rho_u = \frac{1}{2}$ and an transition from type I to type II in the critical behaviour of the model.

Furthermore, the model allows an operator approach to exactly obtain the probability distribution of avalanche size from the first principle. Other works in the literature usually make some scaling assumptions and from there derive the critical exponent (Paczuski and Bassler, 2000a; Kloster *et al.*, 2001). Pruessner (2004a) assumes gap scaling (De Menech *et al.*, 1998) of the avalanche size and then from first and second moment, the critical exponents can be easily calculated. In the current operator approach, no assumption is made and this serves to be a good way to see at least why the probability density function is a power-law. A more refined approach⁴ could provide access to avalanche area and from there it could also be understood why scaling relation like

¹ $\mathbb{T}_{L,n}(s)$

² $D_{L,n}(s)$

³also private communication with Gunnar Pruessner

⁴to be published in (Huynh and Tran, 2012)

Figure 8.20: Illustration of a row of 1-state sites on the one-dimensional chain lattice within which an avalanche can take place. The system is charged at a site contained in that row. “Leftmark” and “rightmark” shows the two ends of the row at which the avalanche stops upon reaching.

$D_x(\tau_x - 1) = D_y(\tau_y - 1)$ (follows assumption of narrow joint distribution ([Jensen *et al.*, 1989](#); [Christensen *et al.*, 1991](#); [Lübeck, 2000](#))) holds.

8.5 Graph method

Some intuitive methods like graph can help in solving models on simple geometric structures like a simple chain. In this section, we apply this idea to the [ASM](#) ([Dhar, 1990, 1999a](#)) and the [AMM](#) ([Manna, 1991b](#); [Dhar, 1999a,c](#)) (also [Sec. 5.1](#)).

8.5.1 Abelian sandpile model on a one-dimensional chain

This model is defined with the following update rules.

- Choose a random site (called charged site) i : $h_i \rightarrow h_i + 1$.
- If any $h_i \geq 2$: $h_i \rightarrow h_i - 2$, $h_{i-1} \rightarrow h_{i-1} + 1$, $h_{i+1} \rightarrow h_{i+1} + 1$.

One observes that if avalanche approaches from one side of site i whose $h_i = 0$, then sites on the other side of i are not affected by that avalanche. That means, avalanche activities only take place between two nearest 0-state sites, *i.e.* $h_i = 0$, (to the left and right) of the charged site (at which we charge the system). In other words, an avalanche takes place on a continuous row of 1-state sites, *i.e.* $h_i = 1$, that contains the charged site (see [Fig. 8.20](#)).

Now, we represent an avalanche by a tree (or graph). In [Fig. 8.21](#), root of the tree is the charged site where the avalanche starts out. The nodes in the same vertical position refer to a same site. Each branch connecting two nodes represents the action of transferring one particle to the neighbour. Viewing the tree as a top-down flow, each node has at most two incoming branches from sites directly above it and at most

Figure 8.21: A (proposed) tree to present an avalanche in [ASM](#).

two outgoing branches to sites directly below it. In principle, each node should have precisely two outgoing branches (when one site topples, it transfers two particles, each to one of its two nearest neighbours). A missing outgoing branch means that the action of transferring that particle does not lead to further relaxation events.

This way, the total number of nodes in the tree is the avalanche size s , the number of columns in the tree is the avalanche area a , the number of rows in the tree is the avalanche duration t . Let's see some examples.

We start with the configuration: $(h_i, h_{i+1}, h_{i+2}, h_{i+3}, h_{i+4}) = (0, 1, 1, 1, 0)$. Sites $j \leq i - 1$ and $j \geq i + 5$ are not visited by an avalanche if we charge at site $i \leq k \leq i + 4$. If we charge at site i , nothing happens: $a = 1$, $s = 0$ and $t = 0$. If we charge at site $i + 1$, the avalanche goes as follow (see Fig. 8.22)

$$(0, 1, 1, 1, 0) \rightarrow (0, 2, 1, 1, 0) \rightarrow (1, 0, 2, 1, 0) \rightarrow (1, 1, 0, 2, 0) \rightarrow (1, 1, 1, 0, 1). \quad (8.187)$$

If we charge at site $i + 2$, the avalanche goes as follow (see Fig. 8.23)

$$(0, 1, 1, 1, 0) \rightarrow (0, 1, 2, 1, 0) \rightarrow (0, 2, 0, 2, 0) \rightarrow (1, 0, 2, 0, 1) \rightarrow (1, 1, 0, 1, 1). \quad (8.188)$$

If we charge at site $i + 3$, by symmetry, everything is the reflection of the case charging at site $i + 1$. The trees for these case are illustrated in Fig. 8.24.

Now, we look at the structure of the trees. First observation is that “cap-shape” is not allowed. A “cap-shape” like in Fig. 8.25(a) is the representation of $(0, 1, 1, 1, 0) \rightarrow (0, 1, 2, 1, 0) \rightarrow (0, 2, 0, 2, 0)$. But the configuration $(0, 2, 0, 2, 0)$ is unstable and therefore the avalanche goes on. Second observation is that “bend-shape” is also not allowed. A “bend-shape” like in Fig. 8.25(b) is the representation of $(0, 1, 1, 0) \rightarrow (0, 1, 2, 0) \rightarrow (0, 2, 0, 1) \rightarrow (1, 0, 2, 1)$. The last update is forbidden because particle cannot be generated during an ongoing avalanche.

By excluding the “cap-shape” and “bend-shape”, one can easily claim that only “full-block-shape” is allowed. A “full-block-shape” means that the graph has precisely

Figure 8.22: Illustration of the avalanche in Eq. (8.187). The arrow on top of a particle shows where it will move to in the next time step.

Figure 8.23: Illustration of the avalanche in Eq. (8.188). The arrow on top of a particle shows where it will move to in the next time step.

Figure 8.24: Trees to present an avalanche for charging configuration $(0, 1, 1, 1, 0)$.

Figure 8.25: Forbidden shapes of a tree in representing avalanches in [ASM](#).

Figure 8.26: A (proper) tree to present an avalanche in [ASM](#).

four corners, except for “straight-leg-shape”.

As mentioned earlier, an avalanche takes place on a continuous row of 1-state sites, *i.e.* $h_i = 1$, that contains the charged site. We denote R as the number of 1-state sites, *i.e.* $h_i = 1$, in that row; and label the sites $r = 1, 2, \dots, R$ from one end to another.

Now we charge the system at site r . The tree representation of the avalanche is given in Fig. 8.26.

From the tree, one can deduce that $a = R$, $s = r(R + 1 - r)$, $t = R$.

It is interesting to note that $a = t$, *i.e.* avalanche area and duration coincide in this model. The result $s = r(R + 1 - r)$ is from the fact that only “full-block-shape” is allowed; in that case, the graph is a rectangle, with length r and width $R + 1 - r$; s is just the number of sites on that rectangle, *i.e.* product of length and width.

By assuming that in stationary state, each site has probability n_0 to be in 0-state, *i.e.* $h_i = 0$ and probability n_1 to be in 1-state, $h_i = 1$, one can write down the probability to have an avalanche of area a , size s and duration t .

Probability to have a row of continuous R sites in 1-state, *i.e.* $h_i = 1$, and two sites in 0-state, *i.e.* $h_i = 0$, at two ends is $n_0^2 n_1^R$ (assume statistical independence of the $R + 2$ sites and ignore the probability to choose those R sites out of all sites on the lattice, for the time being). Probability to charge at site r is $\frac{1}{R}$. Charging at r produces $a = R$, $s = r(R + 1 - r)$, $t = R$. So one can have

$$\mathcal{P}^{(a)}(a) = \mathcal{P}^{(t)}(t) = A n_0^2 n_1^R \quad (8.189)$$

where A accounts for the probability to choose R sites out of all sites on the lattice. Avalanche size s is a bit more complicated. We have

$$\mathcal{P}^{(s)}(s) = B n_0^2 \sum_{\substack{i=1 \\ s|i}}^s \frac{n_1^{\left(\frac{s}{i}+i-1\right)}}{\frac{s}{i}+i-1} \quad (8.190)$$

where B accounts for the probability to choose R sites out of all sites on the lattice; symbol $s|i$ means that i is a divisor of s , i.e. $s, i, \frac{s}{i} \in \mathbb{N}$. Special case, if s is a prime number

$$\mathcal{P}^{(s)}(s) = B \frac{2n_0^2 n_1^s}{s}. \quad (8.191)$$

The result $s = r(R+1-r)$, however, is similar to Eq. (7) of [Ruelle and Sen \(1992\)](#). And this approach shall lead to the same results as theirs.

8.5.2 Abelian Manna model

While the Abelian sandpile model on one-dimensional chain lattice has some nice graph structures, the Abelian Manna model, unfortunately, generates very complicated ones due the fact that a pair of particles on toppling can jump back and forth endlessly on the lattice, generating very large avalanche size on a system of just three sites! This feature has defied any attempt to solve the model exactly. In this section, the dynamics of a system of three sites is considered using the same approach as in Sec. 8.4.2 (Fig. 8.17). Figures 8.27 and 8.28 illustrate this idea. [OPEN PROBLEM](#) see list on page 250

8.6 Mean-field approximation

Here, a heuristic mean-field approximation is presented.

Again, consider the Abelian BTW model ([ASM](#)) on a one-dimensional simple chain with the following update rules.

- Choose a random site (called charged site) i : $h_i \rightarrow h_i + 1$.
- If any $h_i \geq 2$: $h_i \rightarrow h_i - 2$, $h_{i-1} \rightarrow h_{i-1} + 1$, $h_{i+1} \rightarrow h_{i+1} + 1$.

Avalanche size s is defined as the number of relaxation within a single avalanche.

Assume that in stationary state, (normalised) density of sites in state $h_i = 0$ is n_0 and state $h_i = 1$ is n_1 . If we pick a site i that is in state $h_i = 0$, then we have avalanche

Figure 8.27: Relaxation scheme for three sites on charging first four different stable configurations at the middle site. The symbol in the bracket means the configuration. A star “*” means that the configuration is charged once at the middle site. A dagger “†” means the mirrored configuration (about the middle site, left site and right site exchange state). Operators standing for “*” and “†” generally don’t commute, i.e. $C_i^{*\dagger} \neq C_i^{\dagger*}$ (but they do in the toppling scheme of three-site system above).

Figure 8.28: Relaxation scheme for three sites on charging next four different stable configurations at the middle site. The symbol in the bracket means the configuration. A star “*” means that the configuration is charged once at the middle site. A dagger “†” means the mirrored configuration (about the middle site, left site and right site exchange state). Operators standing for “*” and “†” generally don’t commute, *i.e.* $C_i^{\star\dagger} \neq C_i^{\dagger\star}$ (but they do in the toppling scheme of three-site system above).

size $s = 0$. So $\mathcal{P}^{(s)}(s = 0) = n_0$. To have avalanche size $s = 1$, we require that chosen site to topple and its neighbours do not topple on being charged.

Using that sort of argument, we have similarly $\mathcal{P}^{(s)}(s = 1) = n_1 [\mathcal{P}^{(s)}(0) \mathcal{P}^{(s)}(0)] = n_1 n_0^2$. In general, we would have the form of the probability of avalanche size s

$$\mathcal{P}^{(s)}(s) = \sum_i \alpha_i n_0^{a_i} n_1^{b_i}. \quad (8.192)$$

Two conditions are

$$\begin{cases} n_0 + n_1 = 1 \\ \sum_{s=0}^{\infty} \mathcal{P}^{(s)}(s) = 1 \end{cases}. \quad (8.193)$$

Now, let's make a so-called "zeroth approximation". That is to ignore the presumed state of a "touched" site (site that has been visited by the avalanche). On processing that site, we assume it to be in state h_i^* . So when that site topples, we know for sure that it is in state $h_i = 0$. However, when that site is revisited by the avalanche, we still assume that it is in state $h_i = 0$ with probability n_0 and in state $h_i = 1$ with probability n_1 .

To have avalanche size $s = 2$, we require the chosen site i to topple and that one of its neighbours, say, the left one $(i - 1)$ topples inducing *precisely* one more relaxation while the other does not. That is the same as charging site $(i - 1)$ giving rise to $s = 1$ and charging site $i + 1$ giving rise to $s = 0$. We have

$$\mathcal{P}^{(s)}(s = 2) = n_1 [\mathcal{P}^{(s)}(0) \mathcal{P}^{(s)}(1) + \mathcal{P}^{(s)}(1) \mathcal{P}^{(s)}(0)] = 2n_1^2 n_0^3. \quad (8.194)$$

Following the same argument, we have

$$\begin{aligned} \mathcal{P}^{(s)}(s) &= n_1 [\mathcal{P}^{(s)}(0) \mathcal{P}^{(s)}(s - 1) + \mathcal{P}^{(s)}(1) \mathcal{P}^{(s)}(s - 2) + \dots \\ &\quad + \mathcal{P}^{(s)}(s - 2) \mathcal{P}^{(s)}(1) + \mathcal{P}^{(s)}(s - 1) \mathcal{P}^{(s)}(0)] \\ &= n_1 \sum_{i=0}^{s-1} \mathcal{P}^{(s)}(i) \mathcal{P}^{(s)}(s - 1 - i) = \alpha_s n_0^{s+1} n_1^s. \end{aligned} \quad (8.195)$$

The last equality is because we know that going from $\mathcal{P}^{(s)}(k)$ to $\mathcal{P}^{(s)}(k + 1)$, the power

of n_0 and n_1 increase by 1. And we determine the coefficients by

$$\alpha_s = \alpha_0 \alpha_{s-1} + \alpha_1 \alpha_{s-2} + \cdots + \alpha_{s-2} \alpha_1 + \alpha_{s-1} \alpha_0 = \sum_{i=0}^{s-1} \alpha_i \alpha_{s-1-i} \quad (8.196)$$

with $\alpha_0 = 1$.

It turns out that the solution to the above series is Eq. (3.26) in (Christensen and Moloney, 2005)

$$\alpha_s = \frac{1}{s+1} \frac{(2s)!}{(s!)^2}. \quad (8.197)$$

Using Stirling's formula $n! \approx \sqrt{2\pi n} n^n \exp -n$ for $n \gg 1$, we arrive at the probability distribution of avalanche size for large event

$$\mathcal{P}^{(s)}(s) \approx \frac{n_0}{\sqrt{\pi}} s^{-\frac{3}{2}} (4n_0 n_1)^s (s \gg 1). \quad (8.198)$$

This is the same as the result obtain in (Christensen and Moloney, 2005, p. 270).

Chapter 9

Universality and scaling relations

Universality is the key feature of critical systems, which justifies the analysis of (over-)simplified numerical models of otherwise much more complex natural systems. On the other hand, different models with different dynamical rules may produce the same critical behaviours and hence belong to the same universality class.

9.1 Other models in the same class

The (Abelian) Manna model belongs to a very big and common class of the same name — [Manna class \(MC\)](#)¹ — which was first identified and classified by [Ben-Hur and Biham \(1996\)](#)². In this class, there are many other models like Oslo model ([Frette et al., 1996](#); [Christensen et al., 1996](#); [Paczuski and Boettcher, 1996](#)), [conserved lattice gas \(CLG\)](#) ([Rossi et al., 2000](#)) (also [Jensen, 1990](#)), [conserved threshold transfer process \(CTTP\)](#) ([Mendes et al., 1994](#)) (also [Dickman et al., 2002](#); [Dickman, 2006](#)), Maslov-Zhang sandpile ([Maslov and Zhang, 1996](#); [Bonachela and Muñoz, 2008](#)) etc...³

A universality class is identified by comparing the critical exponents of different models⁴. The word “universality” refers to the fact that different models with different dynamics produce the same type of critical behaviours like sharing the same set of exponents.

Originally, the Manna model was devised in order to extend the universality class of the deterministic sandpile mode of [Bak et al. \(1987\)](#). Later it was shown that the

¹The name of the class is given to the model with simplest dynamics.

²Thanks to Gunnar Pruessner for pointing this to me.

³See more in ([Lübeck, 2004](#)) or ([Pruessner, 2012](#)).

⁴To be complete, one has to compare the scaling functions and other universal quantities like moment ratios as well.

two models belong to two different classes (Lübeck, 2000) with very different critical behaviour¹. The Oslo model, which is also a model with stochastic dynamics, was shown to belong to the same class as the Manna model (Nakanishi and Sneppen, 1997), making it tempting to think that the deterministic and stochastic models constitute two different universality classes. However, recently, Giometto and Jensen (2012) shows that the deterministic lattice gas model (Jensen, 1990) also belongs to the Manna class of many stochastic models. This is indeed not the first deterministic model in this class. The purely deterministic Burrridge-Knopoff train model (Burrridge and Knopoff, 1967) earlier studied by de Sousa Vieira (1992) was conjectured by Paczuski and Boettcher (1996) to be in the same universality class as the Oslo model, hence the Manna class.

Since the Manna model and the BTW model belong to two different classes, it would be interesting to see the behaviour of the mixture of these models and how the crossover between the two classes takes place. This, in fact, was studied by Černák (2006). By changing the density c of the sites with stochastic (Manna) rule, he found that the critical exponents of the model change as a function of c . The critical value of c at which the crossover takes place was found to be in the interval $0 < c < 0.001$ (see also Karmakar *et al.*, 2005). This result suggests that there are some fundamental features in the dynamics of the models in a universality class and perturbing these features would drive the model to another universality class. The question remaining is how to identify all these features for a particular class and how robust are those features against perturbations.

9.2 Avalanche exponents and moment ratios

As mentioned earlier in this thesis, the focus of this work on universality is not about how a model compares itself to others but rather its behaviours across lattices of different structures. This is another aspect of universality in which the model is robust with respect to detailed structures of the system. This feature is actually one of the most important results in ordinary critical phenomena which allows insights and exact solution of the models.

Exponents (Table 7.6) and, at least in two and three dimensions, moment ratios (Tables 7.10–7.12) are universal across different lattices. Subleading orders of moments are noisy, but still fairly consistent. There can be little doubt that the Abelian Manna model

¹The BTW model indeed does not exhibit simple scaling but rather multifractal scaling (Dhar, 2006).

displays all the hallmarks of a critical system as they are known from equilibrium critical phenomena. The exponents shown in Table 7.8 and the moment ratios in Table 7.13 (except those shown in brackets $[\cdot]$) are perfectly consistent across all our results and represent reliable, high accuracy estimates of the universal quantities characterising this universality class.

In the literature can be found a large number of estimates for the exponents in Table 7.8. Traditionally, the AMM is studied in two dimensions and therefore many more estimates are available in that dimension. Given that the AMM is in the same universality class as the Oslo Model (Christensen *et al.*, 1996; Nakanishi and Sneppen, 1997), there is a second source for comparison. What makes the comparison more complicated is the fact that many authors have studied variants of the Manna Model (in fact, the Abelian version studied here is a variant of the original model); for example Dickman and Campelo (2003) studied the Manna Model with height restrictions and Lübeck and Heger (2003a) studied its “fixed-energy sandpile (FES)” version (also Vespignani *et al.*, 2000). In three dimensions, our results also compare well with the literature (Ben-Hur and Biham, 1996; Lübeck, 2000; Pastor-Satorras and Vespignani, 2001) (also (Alava and Muñoz, 2002; Lübeck and Heger, 2003b) for absorbing state phase transitions), although some variability and discrepancy are observed in particular for z which may be explained by the use of slightly different model definitions (and dynamics) by other authors. Tables 9.1 and 9.2 collect a broad range of estimates across the literature¹, which nevertheless provide a perfectly consistent picture. The present work clearly improves on comprehensiveness and on accuracy, which for most estimates is improved by one digit. The fact that some estimates in the literature have even smaller error bars than ours might be partly due to our over-estimation of statistical errors but also due to other authors using models that are better behaved in one dimension, as in (Christensen, 2004; Pruessner, 2004b).

In this work, certain finite-size scaling features specific to SOC are confirmed as well: compactness of avalanches, $D_a = d$ is very strongly supported² (assumed to hold on regular lattices by Ben-Hur and Biham (1996); Chessa *et al.* (1999) and here extended to fractal lattices), as is the universality of Σ (Eq. (5.5)). It is reassuring that the asymptotics of the first moment of the avalanche size, $\mu_1^{(s)} = 2$, are recovered, validating our numerical schemes.

Although we invested more than twice as much CPU time (in absolute terms) on

¹Thanks to Gunnar Pruessner for pointing out these references to me.

²In two dimensions, the deviation of D_a from d is slightly bigger than one standard deviation, though.

Table 9.1: Comparison of the results in the present work to the estimates in one and two dimensions found in the literature. Many of the works quoted below have studied variants of the Manna Model. The values taken from (Christensen, 2004; Pruessner, 2004b) in one dimension and (Bonachela, 2008) in two dimensions are for the Oslo Model. The exponents marked as DP are those for the DP universality class. They are derived via scaling laws (Lübeck, 2004).

d	reference	D	τ	z	α	D_a	τ_a	$-\Sigma$
1	this work	2.253(14)	1.112(6)	1.445(10)	1.18(2)	0.998(3)	1.259(11)	0.26(2)
1	(Nakanishi and Sneppen, 1997)	2.2(1)	1.09(3)	1.47(7)				
1	(Dickman and Campelo, 2003)		1.11(2)		1.18(2)			
1	(Lübeck and Heger, 2003a)		1.11(2)	1.393(37)	1.17(3)			
1	(Bonachela, 2008)		1.11(5)		1.17(5)			
1	(Christensen, 2004)	2.2496(12)						
1	(Pruessner, 2004b)	2.2509(6)						
1	(Jensen, 1999) (DP)	2.328673(12)		1.580745(10)		1		
2	this work	2.750(6)	1.273(2)	1.532(8)	1.4896(96)	1.995(3)	1.382(3)	0.761(13)
2	(Manna, 1991b)	2.75	1.28(2)	1.55	1.47(10)			
2	(Chessa <i>et al.</i> , 1999)	2.73(2)	1.27(1)	1.50(2)	1.50(1)	2.02(2)	1.35(1)	
2	(Lübeck, 2000)	2.76(1)		1.54(1)		2.03(1)		0.75(4)
2	(Dickman and Campelo, 2003)		1.30(1)		1.55(4)			
2	(Lübeck and Heger, 2003a)		1.28(14)	1.533(24)	1.50(3)			
2	(Bonachela, 2008)		1.26(3)		1.48(3)			
2	(Voigt and Ziff, 1997) (DP)	2.979(2)		1.765(3)		2		

Table 9.2: Comparison of the results in the present work to the estimates in three dimensions found in the literature. Many of the works quoted below have studied variants of the Manna Model.

d	reference	D	τ	z	α	D_a	τ_a	$-\Sigma$
3	this work	3.370(11)	1.407(2)	1.777(4)	1.783(5)	3.003(14)	1.442(12)	1.380(13)
3	(Ben-Hur and Biham, 1996)	3.33	1.43	1.8				
3	(Lübeck, 2000)	3.302(10)		1.713(10)				
3	(Pastor-Satorras and Vespignani, 2001)	3.36(1)	1.41(1)	1.76(1)	1.78(2)			
3	(Lübeck and Heger, 2003a)		1.41(2)	1.823(23)	1.77(4)			

one-dimensional lattices, the results are significantly noisier than for two-dimensional lattices. Providing a sufficient number of correction terms still produces consistent data (Table 7.7), but the results for two-dimensional lattices are clearly superior. In fact, the error bar on the exponents for two-dimensional lattices is typically half that of one-dimensional lattices.

It is known that the Manna Model suffers from significant logarithmic corrections (Dickman and Campelo, 2003). Manna (1991b) himself noted a “considerable curvature” in what should have been a straight line in a double logarithmic plot. Similarly, Lübeck and Heger (2003a) found a surprising splitting of the Manna universality class in one dimension where there are mismatches in the exponents of the (Abelian) Manna model itself and the CTTTP (a variant of Manna model with restricted height (also Dickman, 2006)), which might also be due to the presence of significant corrections. It was also suggested by Lübeck and Heger (2003a) that the mismatches are due to the fact that CTTTP becomes deterministic and trivial in one dimension. But one should not restrict to the one-dimensional chain lattice, but also consider other one-dimensional lattices like the rope ladder, next-nearest-neighbour chain or futatsubishi to see if the discrepancy persists. **OPEN PROBLEM** see list on page 249

The results for the finite-size scaling exponents in Table 7.7 suggest that one-dimensional systems are more difficult to fit, with the alternative fitting functions Eq. (7.9c) and Eq. (7.9b) both clearly performing worse than in two dimensions. One might think that some of the problems are caused by having much higher accuracy in the estimates in one dimension (and thus requiring more correction term, as bad choices for the fitting function can no longer be hidden in a large statistical error), given that we spent, per lattice, typically about five times more CPU time than in two dimensions. The opposite is the case (probably because correlation times grow like a power-law of the linear extent which are very large in one dimension): Depending on the observable, relative errors are between a factor 2 and 10 worse in simple chain compared to square lattice. This holds similarly for other lattices, except for the next-nearest-neighbour chain lattice, on which we spent less than 1/6 of the CPU time we spent on the other one-dimensional lattices.

In general the relative statistical error vanishes like the inverse square root of the CPU time, so that the product of the two gives a measure of “efficiency” of a lattice¹. Using that measure, the simple chain is the most efficient, followed by next-nearest-

¹Thanks to Gunnar Pruessner for suggesting this.

neighbour chain, rope ladder and finally the futatsubishi lattice, which is about a factor 4 less efficient (4 times the CPU time is needed for results with similar relative error). This statement, however, is put in perspective, by noting that we used variety of different hardware throughout. The two-dimensional lattices fall roughly in three classes: noncrossing diagonal square lattice, kagomé lattice, jagged lattice, square lattice, followed by triangular lattice, Archimedes lattice and mitsubishi lattice, and finally the honeycomb lattice. The latter is clearly the worse (again by about a factor 4), while the triangular lattice is in the first group for some of the observables.

There is a caveat, however. Within a given amount of CPU time and for a given system size, the noncrossing diagonal square lattice produces a larger number of avalanches, which are typically much smaller than those for other lattices, because the fraction of virtual neighbours is about 1.5 times higher for the noncrossing diagonal square lattice than for, say, the honeycomb lattice, so that particles are dissipated more frequently. As a result, statistics are comparatively better for noncrossing diagonal square lattice, which, however, may pose higher demands on correction terms with generally larger amplitudes. We could, however, not identify a systematic behaviour in this respect. For example, the moments of jagged lattice are, within error, the same as for the square lattice, yet the latter has a much smaller average fraction of dissipative links. In fact, as discussed below in Sec. 9.3, the same *leading* order amplitudes are found for the avalanche area distribution across all lattices of the same dimension.

A similar analysis can be applied to the (asymptotic) average particle density ζ see Table 7.14. The simple chain has a very high density among one-dimensional lattices because of only two dissipative sites making it difficult for particles to escape. Similarly, the Archimedes lattice and the honeycomb lattice have comparatively high particle densities, as the average number of virtual neighbours among dissipative sites, $\overline{q^{(v)}}$, is only unity. However, the average number of neighbours $\overline{q}$ and $\overline{q^{(v)}}$ seem to have a combined effect on the particle density, which is lowest where both $\overline{q^{(v)}}$ and $\overline{q}$ are large. **OPEN PROBLEM** see list on page 249

The one-dimensional lattices perform particularly badly for the moment ratios. Essentially only those for the area size distribution can be fitted well and then produce fairly consistent results (except for the next-nearest-neighbour chain lattice). Originally we expected improved scaling behaviour with the introduction of next-nearest-neighbour interaction, as it prevents degeneracy issues (and, say conserved quantities) as they are sometimes observed on the simple chain (Hughes and Paczuski, 2002; Dhar

and Ramaswamy, 1989). The two-dimensional lattices, with the exception of the triangular lattice, generally behave much better, when fitting moment ratios. Given the importance of the boundary conditions, it is remarkable how well all universal quantities addressed in the present work are reproduced, even when some two-dimensional lattices like the kagomé lattice and the jagged lattice (Fig. 6.4(b)) have rather complicated boundaries (although the latter is the square lattice in the thermodynamic limit) and many of the lattices have an aspect ratio of unity only asymptotically.

As explained above, the fitting function is a *hypothesis* and ultimately has an impact on the results. As the number of free parameters increases so does the susceptibility of the result on the initial condition. The approach described above, using fairly large systems (with weaker corrections) with not too small error bars, in conjunction with simple fitting functions, the initial values of which are determined by those with fewer terms, seems to produce robust and reliable results. Comparing the results based on Eq. (7.10) and Eq. (7.9c) in Table 7.7 to those on the basis of Eq. (7.9b) indicates that the former are superior. An acceptable goodness-of-fit is reached for Eq. (7.9b) only for those exponents that coincide within a bit more than one standard deviation with the estimates based on Eq. (7.10). Eq. (7.9c), on the other hand, in summary (Table 7.7) coincides with Eq. (7.10), but some, individual finite-size scaling exponents, such as $\mu_1^{(s)}$, but also Σ_x and D_a , were estimated too poorly.

A few remarks should be paid to the fractal lattices as well, which were first studied at the beginning of this project and motivate the whole development afterwards. One surprising conclusion from those results is that in contrast to lattices in regular integer dimension, for two fractal lattices with the same dimension but different microscopic structures, the critical exponents D_x and τ_x can both be different, which suggests that on fractal lattices, the critical exponents depend not only on the dimension but also on the microscopic details of the lattice. The interesting case of the Sierpiński tetrahedron lattice also shows that lattice of integer dimension with fractal structure behaves very differently from the regular ones. Yet, surprisingly all these fractal lattices provide an important mean to investigate the critical behaviours of the model across dimensions, see Sec. 9.5 below.

9.3 Moment amplitudes

To our surprise, the amplitudes $A_n^{(a)}$ of the leading orders in Eq. (7.10) obtained when fitting the moments of the area distribution seem to be universal themselves¹. According to Eq. (5.3)

$$A_n^{(a)} = a_a b_a^{n+1-\tau_a} \int_0^\infty dy y^{n-\tau_a} \mathcal{G}_a(y), \quad (9.1)$$

which is universal provided the metric factors a_a (not to be confused with the area moment $\langle a^n \rangle$) and b_a are, which is not normally the case. Since $D_a = d$, however, the amplitude b_a of the cutoff $b_a L^{D_a}$ is dimensionless. It is therefore plausible to assume that it is universal. There is no reason, however, to assume that the same should hold for a_a — the argument that a_a is determined by normalisation does not hold as Eq. (5.1) applies only asymptotically, *i.e.* the fraction of small event sizes which do not follow simple scaling can, in principle, vary from lattice to lattice. Moreover, a_a is dimensionful since $\tau_a \neq 1$. Nonetheless, it turns out that it does not vary. As a result, to leading order, the moments of the avalanche areas in one dimension follow

$$\langle a^1 \rangle = [2.01(7)] N^{2-1.259(11)}, \quad (9.2a)$$

$$\langle a^2 \rangle = [1.21(5)] N^{3-1.259(11)}, \quad (9.2b)$$

$$\langle a^3 \rangle = [0.96(4)] N^{4-1.259(11)}, \quad (9.2c)$$

$$\langle a^4 \rangle = [0.83(4)] N^{5-1.259(11)}, \quad (9.2d)$$

$$\langle a^5 \rangle = [0.76(4)] N^{6-1.259(11)}, \quad (9.2e)$$

in two dimensions

$$\langle a^1 \rangle = [0.756(7)] N^{2-1.382(3)}, \quad (9.3a)$$

$$\langle a^2 \rangle = 0.217(3) N^{3-1.382(3)}, \quad (9.3b)$$

$$\langle a^3 \rangle = 0.109(3) N^{4-1.382(3)}, \quad (9.3c)$$

$$\langle a^4 \rangle = 0.066(3) N^{5-1.382(3)}, \quad (9.3d)$$

$$\langle a^5 \rangle = 0.045(2) N^{6-1.382(3)}, \quad (9.3e)$$

¹Thanks to Gunnar Pruessner for suggestion and discussion on this.

and in three dimensions

$$\langle a^1 \rangle = [0.202(4)]N^{2-1.4396(8)}, \quad (9.4a)$$

$$\langle a^2 \rangle = 0.0151(15)N^{3-1.4396(8)}, \quad (9.4b)$$

$$\langle a^3 \rangle = 0.0027(6)N^{4-1.4396(8)}, \quad (9.4c)$$

$$\langle a^4 \rangle = 0.00055(19)N^{5-1.4396(8)}, \quad (9.4d)$$

$$\langle a^5 \rangle = 0.00012(6)N^{6-1.4396(8)}, \quad (9.4e)$$

as a function of the number N of sites, independent of the lattice type. We need to qualify the statement, by pointing out that in one dimension the amplitudes across different lattices are rather noisy, and in two and three dimensions the amplitude of $\langle a^1 \rangle$ has a goodness-of-fit of just under 0.1 (we are using the $[\cdot]$ notation again here), while the other results in two and three dimensions all have a goodness-of-fit better than 0.5.

Remarkably, it is crucial to consider $\langle a^n \rangle$ as a function of N , as fitting against $L = \lambda N^{1/d}$ leads to different amplitudes, because λ varies from lattice to lattice. Had we fitted the area moments against a lattice-dependent multiple of $N^{1/d}$, such as the perceived linear extent of the lattice, then the multiplier λ would have shown in the resulting amplitude $A_n^{(a)}$, and so the apparently universal behaviour would not have come to light. **OPEN PROBLEM** see list on page 249

9.4 The effects of boundary condition

As listed in Sec. 6.2, we also consider lattices with periodic boundary conditions. Here, we discussed how the periodic boundary conditions affect the critical behaviours of the model, which is another aspect of universality.

9.4.1 Avalanche exponents

The avalanche exponents for the lattices with periodic boundary conditions in two and three dimensions described in Sec. 6.2.4 are listed in Table 9.3 (also Table 9.4 for the scaling exponent of the first moment of avalanche size $\mu_1^{(s)}$ and the derived exponents Σ_x defined in Eq. (5.5)). They show very good agreement with the ones in the open boundary condition case (Table 7.8). This means that the avalanche exponent is very robust against the geometry of the system, *i.e.* the detailed microscopic structures and

Table 9.3: Avalanche exponents of the periodic (cubic) lattices in two and three dimensions. The estimate for τ is obtained from D via the exact relation $D(2 - \tau) = 2$.

d	D	τ	z	α	D_a	τ_a
2	2.763(8)	1.276(2)	1.542(5)	1.497(8)	2.0006(12)	1.3799(19)
3	3.39(11)	1.410(19)	1.80(6)	1.79(8)	3.02(2)	1.449(17)

Table 9.4: Scaling exponent of the first moment of avalanche size $\mu_1^{(s)}$ and the derived exponents Σ_x defined in Eq. (5.5) whose estimates are obtained from D in Table 9.3 via the exact relation $D(2 - \tau) = 2$.

d	$\mu_1^{(s)}$	$-\Sigma_s$	$-\Sigma_t$	$-\Sigma_a$
2	2.0000(6)	0.763(8)	0.766(12)	0.760(4)
3	2.002(4)	1.39(11)	1.42(15)	1.35(5)

boundary condition (and also aspect ratio¹).

9.4.2 Moment ratios

The moment ratios for these lattices (listed in Table 9.5), however, show their dependence on the boundary conditions. Comparing to the moment ratios quoted in Table 7.13, the values here for lattices with periodic boundary conditions are much smaller. This is actually a known feature in critical phenomena (Privman *et al.*, 1991) that, in contrast to critical exponents (see above), the moment ratios depend on the geometrical features of the lattice like boundary conditions (and aspect ratio) even though both of them are very robust against local microscopic structures of the lattice.

9.4.3 Particle density

The particle density for these lattices is also listed in Table 9.6. Interestingly, lattices with periodic boundary conditions produce the same asymptotic value of particle density as compared to ones with open boundary conditions. At first, this fact would come as a surprise since we expect more particles to be held on the lattice because the number of dissipative sites where particles can leave the system has greatly reduced. However, what we are really after is the *asymptotic* stationary particle density, *i.e.* the density on

¹This is not firmly concluded in this work. Yet, the fact that we allow a fair amount of arbitrariness in the aspect ratio of the lattices used but still obtain consistent results for the exponents really suggests that. **OPEN PROBLEM** see list on page 249

Table 9.5: Moment ratios of the periodic (cubic) lattices in two and three dimensions. Again, $[\cdot]$ denotes fit with low goodness-of-fit q .

d	x	$g_3^{(x)}$	$g_4^{(x)}$	$g_5^{(x)}$	$g_6^{(x)}$
2	s	1.6507(19)	3.409(12)	8.14(6)	21.6(3)
	t	1.9433(16)	4.839(12)	14.07(7)	45.9(4)
	a	1.5080(5)	2.5692(18)	4.642(5)	8.687(14)
3	s	[1.923(10)]	[4.70(7)]	13.0(4)	38(2)
	t	3.448(12)	16.40(15)	92.7(16)	589(17)
	a	1.735(3)	3.565(16)	7.98(6)	18.8(2)

Table 9.6: Asymptotic stationary particle density of the periodic (cubic) lattices in two and three dimensions (also refer to Table 7.14).

d	$\bar{q}$	$\overline{q^{(v)}}$	ζ
2	4	1	[0.7169783(3)]
3	6	1	[0.6222970(3)]

an infinitely large system. On such infinitely large system, the particles practically do not see the boundary at all, and hence only local structures of the lattice would affect the density. So this is a surprise, but a consistent surprise. In fact, it is true that the density on *finite* lattices with periodic boundary conditions is *higher* than that on ones with open boundary conditions (see Table 9.7). However, in the *asymptotic* limit, both boundary conditions consistently produce the same value of density.

9.5 Scaling relations

The determination of many different avalanche exponents with high precision in this work does not only show the robustness and universality of the model against different (geometric) structures of the lattices but also allows the scaling laws to be tested, verified and generalised. In SOC, there are a number of scaling relations identified (Table 9.8). Among these relations, only the scaling relation involving the avalanche size s is well understood based on diffusion and random walk theory by making use of moment analysis, while the others are merely based on assumptions and need precise, reliable numerical validations.

Table 9.7: Comparison of density between (cubic) systems with open and periodic boundary conditions. Open *finite* systems has lower density than the periodic ones. Yet, they possess the *same* density in thermodynamic limit. Open systems in two dimensions have bigger error bars compared to others because the data were constructed “forensically” (see also Sec. 3.3 in (Huynh *et al.*, 2011)).

d	system size	ζ for open boundary	ζ for periodic boundary
2	65536 (256^2)	0.7113(5)	[0.71398662(16)]
	131044 (362^2)	0.7132(5)	[0.71478692(18)]
	262144 (512^2)	0.7141(6)	[0.71537729(12)]
	524176 (724^2)	0.71475(16)	[0.71581000(9)]
	1048576 (1024^2)	0.71544(16)	[0.71612660(7)]
	2096704 (1448^2)	0.71583(11)	[0.71635692(5)]
	4194304 (2048^2)	0.71614(7)	[0.71652425(5)]
	∞	0.7170(4)	[0.7169783(3)]
3	2097152 (128^3)	[0.61912463(15)]	[0.62115636(18)]
	5929741 (181^3)	[0.61999776(11)]	[0.62147926(13)]
	16777216 (256^3)	[0.62063744(9)]	[0.62171228(12)]
	47437928 (362^3)	[0.6211030(3)]	[0.62187931(8)]
	134217728 (512^3)	[0.6214397(3)]	[0.62199887(7)]
	379503424 (724^3)	[0.62168239(15)]	[0.62208431(5)]
	1073741824 (1024^3)	[0.6218185(14)]	[0.62214504(4)]
	∞	[0.622325(1)]	[0.6222970(3)]

With a zoo of fractal lattices, besides many regular lattices in integer dimensions, these scaling have been validated and generalised. In particular, the scaling relation between exponents of different observables based on the assumption of narrow joint distribution $D_x(\tau_x - 1) = D_y(\tau_y - 1)$ (Jensen *et al.*, 1989; Christensen *et al.*, 1991; Lübeck, 2000) and the scaling relation for the finite-size scaling exponent of avalanche area $D_a = d$ (Ben-Hur and Biham, 1996; Chessa *et al.*, 1999) are confirmed and illustrated to hold for both regular and fractal lattices. The scaling relation for avalanche size $D(2 - \tau) = 2$ (Dhar, 1990; Christensen and Olami, 1993; Nakanishi and Sneppen, 1997; Lübeck, 2000), however, does not hold for fractal lattices and has to be generalised to $D(2 - \tau) = d_w$ in which the fractal dimension of random walk d_w is generally greater

Table 9.8: List of scaling relations known (assumed) in sandpile models.

Scaling relations	Observables involved	Physical interpretation
$D(2 - \tau) = 2$ (generalised to $D(2 - \tau) = d_w$)	avalanche size s	diffusion, conservation (Dhar, 1990; Christensen and Olami, 1993; Lübeck, 2000)
$D_x(\tau_x - 1) = D_y(\tau_y - 1)$	all observables $x, y \in \{a, s, t\}$	narrow joint distribution (assumed) (Jensen <i>et al.</i> , 1989; Christensen <i>et al.</i> , 1991; Lübeck, 2000)
$D_a = d$	avalanche area a	compact avalanche (assumed) (Ben-Hur and Biham, 1996; Chessa <i>et al.</i> , 1999)

than 2. This relation remains true for any lattice regardless of its dimension and microscopic structures.

With a wide range of lattices studied in many different dimensions (below the upper critical dimensions $d_c = 4$ (Lübeck and Heger, 2003b)), one has access to the relation between the critical exponents and the (Hausdorff) dimension of the lattice. This is done by first considering an ϵ -expansion for integer dimensions (idea borrowed from ordinary critical phenomena) and then a scaling relation for *all* dimensions.

9.5.1 A numerical ϵ -expansion

All critical exponents in different dimensions (after the universality hypothesis in integer dimensions) are summarised in Table 9.9. Firstly, on regular lattices, a relation between D_x , τ_x and the dimension d can be obtained by fitting exponents against a proposed function $D_x = f_x(d)$ and $\tau_x = h_x(d)$. With six exponents six functions are to be determined, which, however, are related by scaling laws listed in Table 9.8.

Using

$$\tau = 2 - \frac{2}{D} \quad (9.5a)$$

$$D_a = d \quad (9.5b)$$

$$\tau_a = 1 + \frac{D - 2}{d} \quad (9.5c)$$

$$\alpha = 1 + \frac{D - 2}{z} \quad (9.5d)$$

Table 9.9: Exponents in all dimensions.

Lattice	d	d_w	D	τ	z	α	D_a	τ_a	$\mu_1^{(s)}$	$-\Sigma$
regular	1	2	2.253(14)	1.112(6)	1.445(10)	1.18(2)	0.998(3)	1.259(11)	1.996(3)	0.26(2)
SSTK	1.464.. ^a	2.552..	2.94(3)	1.13(2)	1.817(17)	1.21(2)	1.466(5)	1.273(11)	2.551(6)	0.40(3)
ARRO	1.584.. ^b	2.322..	2.7938(19)	1.1731(16)	1.6732(12)	1.2797(17)	1.5847(3)	1.2985(6)	2.3103(4)	0.4730(16)
CRAB	1.584.. ^b	2.578..	3.020(5)	1.151(4)	1.837(3)	1.237(4)	1.5847(8)	1.2793(17)	2.5655(12)	0.442(4)
regular	2	2	2.750(6)	1.273(2)	1.532(8)	1.4896(96)	1.995(3)	1.382(3)	1.9993(5)	0.761(13)
SITE	2 ^c	2.584..	3.232(6)	1.211(4)	1.870(4)	1.357(4)	1.9975(9)	1.3388(14)	2.5533(6)	0.676(5)
EXGA	2.584.. ^d	2.321..	3.352(4)	1.312(3)	1.835(3)	1.581(3)	2.5895(6)	1.3915(8)	2.3000(2)	1.020(3)
regular	3	2	3.370(11)	1.407(2)	1.777(4)	1.783(5)	3.003(14)	1.442(12)	2.0043(3)	1.380(13)
regular	4 ^e	2	4	1.5	2	2	4	1.5	2	2

^aln 5/ln 3. ^bln 3/ln 2. ^cFractal lattice. ^dln 6/ln 2, strongly anisotropic.^eAt upper critical dimension $d_c = 4$ ([Lübeck and Heger, 2003b](#)), exponents take mean-field value ([Lübeck, 2004](#)).

there are thus only two functions to determine, which are best expressed in terms of $\epsilon = 4 - d$ since $d_c = 4$ is the upper critical dimension (Lübeck and Heger, 2003b), where the exponents are known exactly. Writing

$$D = 4 - c_1^{(s)}\epsilon + c_2^{(s)}\epsilon^2 + \dots, \quad (9.6)$$

at most two amplitudes $c_i^{(s)}$ can reasonably be determined on the basis of the three data points available. A fit of D with only a linear term produces a very poor goodness-of-fit, which does not improve satisfactorily by including a term quadratic in ϵ . Omitting the quadratic gives

$$D = 4 - 0.654(6)\epsilon + 0.0079(10)\epsilon^3 \quad (9.7)$$

with goodness-of-fit $q \approx 0.095$ ($c_1^{(s)} = 0.60(4)$, $c_2^{(s)} = -0.05(3)$, $c_3^{(s)} = -0.019(7)$ with three terms). Similarly,

$$z = 2 - 0.239(4)\epsilon + 0.0056(6)\epsilon^3, \quad (9.8)$$

however with nearly vanishing goodness-of-fit.

These results appear to be in good agreement with the ϵ -expansion for the quenched Edwards-Wilkinson equation obtained by Le Doussal *et al.* (2002)¹ using a two-loop functional renormalisation group theory. They obtain for the roughness exponent²

$$\zeta' = \frac{\epsilon}{3}(1 + 0.14331\epsilon) \quad (9.9)$$

which gives the corresponding expansion for the finite-size scaling exponent D of the avalanche size in the Manna model

$$D = d + \zeta' = 4 - \epsilon + \frac{\epsilon}{3}(1 + 0.14331\epsilon) = 4 - \frac{2}{3}\epsilon + 0.04777\epsilon^2 + \dots \quad (9.10)$$

whose coefficient of leading order well compares against that of Eq. (9.7). Similarly, for the dynamical exponent

$$z = 2 - \frac{2}{9}\epsilon - 0.04321\epsilon^2 + \dots \quad (9.11)$$

¹Thanks to Gunnar Pruessner for pointing out this to me.

²In the literature of depinning transition, ζ is the symbol of the roughness exponent, which, however, clashes with the symbol for stationary particle density. Since depinning transition is *not* the main subject of interest in this thesis, ζ' is used instead to represent the roughness exponent.

whose coefficient of leading order also well compares against that of Eq. (9.8).

9.5.2 Relation between exponents and dimension

Attempting to unify the above ϵ -expansion obtained for regular lattices with the results for fractals with Hausdorff dimension d is bound to fail, which is immediately clear when comparing the exponents found for the fractal lattice SITE ($d = 2$, Table 7.9) with those for the regular two-dimensional lattices, or the ARRO with the CRAB lattice, Table 7.9, which have the same Hausdorff dimension. As is well understood, the basic scaling relation $D(2 - \tau) = 2$ is valid only for regular lattices and has to be generalised to $D(2 - \tau) = d_w$ with random walker dimension $d_w \geq 2$ (ben-Avraham and Havlin, 2000; Huynh *et al.*, 2010).

It turns out, however, that D is essentially a linear function of d and d_w

$$D = \rho d + \psi d_w \quad (9.12)$$

with the same coefficients ρ and ψ for both regular and fractal lattices, which can be extracted from the ϵ -expansion obtained above with $d_c = 4 = 2d_w$, because $d_w = 2$ on regular lattices, $\rho = c_1^{(s)}$ and $\psi = 2(1 - c_1^{(s)})$ (see Sec. 9.5.3), so that on the basis of Eq. (9.7) $\rho = 0.654(6)$ and $\psi = 0.692(12)$ or $\rho = 0.60(4)$ and $\psi = 0.80(8)$ depending on $c_1^{(s)}$. Fitting the data in Table 9.9 against Eq. (9.12) gives $\rho = [0.550(4)]$, $\psi = [0.822(3)]$, and a fitting with the constraint $\psi = 2(1 - \rho)$ from exactly known values of exponents at $d = d_c = 4$ gives $\rho = [0.6061(5)]$, $\psi = [0.7878(10)]$. Figure 9.1 shows $\frac{D}{d}$ as a function of $\frac{d_w}{d}$ for all lattices listed in Table 9.9. As expected from Eq. (9.12), fractal and regular lattices display essentially the same linear dependence $\frac{D}{d} = \rho + \psi \frac{d_w}{d}$. Above the upper critical dimension $d_c = 4$ regular lattices attain their mean-field values, $D = 4$, $\tau = 3/2$ (Lübeck, 2004) and $d_w = 2$, therefore following Eq. (9.12) with $\rho = 0$ and $\psi = 2$. One may wonder whether fractal lattices with Hausdorff dimension greater than $d_c = 4$ have correspondingly exponents $D = 2d_w$. **OPEN PROBLEM** see list on page 249

The choice of rescaling exponent D by dimension d of the lattice is not random, but rather a natural choice, given that we performed all fitting of $\mu_n^{(x)}$ against the number of sites N rather than the lattices' linear length L and multiplied the results by the Hausdorff dimension d . The gap exponents for the scaling in N is $\frac{D}{d}$, which, as it turns out, displays a very systematic dependence on $\frac{d_w}{d}$.

Figure 9.1: The data of Table 9.9 plotted in the form $\frac{D}{d}$ versus $\frac{d_w}{d}$ as suggested by Eq. (9.12). The dashed straight line is based on the estimates $\rho = [0.6061(5)]$ and $\psi = [0.7878(10)]$, the dotted line is the mean-field theory, $\rho = 0$, $\psi = 2$. Plot touched up by Gunnar Pruessner (see also [Huynh and Pruessner, 2012](#)).

Further investigation shows that $\frac{D}{d}$ fits very well to

$$\left(\frac{D}{d}\right)^2 (\tau - \tilde{\rho}) = \tilde{\psi} \quad (9.13)$$

with $\tilde{\rho} = 1.020(2)$ and $\tilde{\psi} = 0.481(3)$ for *all* lattices which results in

$$D = 4 - 0.658(5)\epsilon + 0.00962(13)\epsilon^2 + 0.00161(3)\epsilon^3 + \dots \quad (9.14)$$

using $D(2 - \tau) = d_w = 2$ for the regular ones.

The form of Eq. (9.13) was obtained by first fitting τ against $\left(\frac{D}{d}\right)^\kappa$, which gives a κ deviating from -2 by less than 2%. The coefficient $\tilde{\rho}$ and $\tilde{\psi}$ are then fitted according to Eq. (9.13). Figure 9.2 compares that relation to results for lattices in *all* dimensions. In the same manner, a similar relation can be obtained for z and α ,

$$\left(\frac{z}{d}\right)^{\frac{3}{2}} (\alpha - \tilde{\rho}) = \tilde{\psi} \quad (9.15)$$

with $[\tilde{\rho} = 0.936(2)]$ and $[\tilde{\psi} = 0.3768(12)]$.

The above results suggest that the scaling in N is more suitable for fractals than the scaling in L . We suspect this is related to L not capturing the chemical distance, which is the distance particles need to travel on the lattice, whereas L is measured as a Euclidean distance. By using d_w , which is sensitive to the chemical distance, and considering the scaling against N , which is a well-defined measure of the size for any lattice, we are able to determine the relations above. OPEN PROBLEM see list on page 249

9.5.3 Connection between the assumed ϵ -expansion and the presumed scaling relation involving dimension

Here we show how the ϵ -expansion in Eq. (9.6) (to leading linear order) and the relation in Eq. (9.12) relate to one another. First we show that the relation in Eq. (9.12) can be written in the form of a scaling relation involving the (Hausdorff) dimension of the lattice.

By substituting $d_w = D(2 - \tau)$ in Eq. (9.12), dividing both sides by ψ and rearranging, we have

$$D(2 - \tau) = -\frac{\rho}{\psi}d + \frac{1}{\psi}D \quad (9.16a)$$

$$\rightarrow D(2 - \frac{1}{\psi} - \tau) = -\frac{\rho}{\psi}d. \quad (9.16b)$$

Denoting $a = 2 - \frac{1}{\psi}$ and $b = \frac{\rho}{\psi}$, we have

$$D(\tau - a) = bd \quad (9.17)$$

which is a scaling relation between the exponents of the avalanche size and the dimension of the lattice.

Ignoring the fact that $D(\tau - 2) \neq 2$ for fractal dimensions at this point, we impose two equalities

$$D(\tau - 2) = -2 \quad (9.18)$$

and

$$D(\tau - a) = bd. \quad (9.19)$$

(a) Avalanche size exponents and fit, $\frac{D}{d} = \left(\frac{\tilde{\psi}}{\tau - \tilde{\rho}} \right)^{\frac{1}{2}}$

(b) Avalanche duration exponents and fit, $\frac{z}{d} = \left(\frac{\tilde{\psi}}{\alpha - \tilde{\rho}} \right)^{\frac{2}{3}}$

Figure 9.2: Fit of the exponents in all dimensions (on regular and fractal lattices) against Eq. (9.13). The symbols represent the data in Table 9.9, the dashed lines are the fits as described in the text, Eq. (9.13) and Eq. (9.15), respectively. Plots touched up by Gunnar Pruessner (see also Huynh and Pruessner, 2012).

These relations can be rewritten as

$$D\tau = 2D - 2 \quad (9.20a)$$

$$D\tau = aD + bd \quad (9.20b)$$

which lead to

$$2D - 2 = aD + bd \quad (9.21a)$$

$$\rightarrow (2 - a)D = 2 + bd \quad (9.21b)$$

$$\rightarrow D = \frac{2}{2 - a} + \frac{b}{2 - a}d. \quad (9.21c)$$

By using the values of D and τ at critical dimension $d = 4$ we have

$$4 \left(\frac{3}{2} - a \right) = 4b \quad (9.22a)$$

$$\rightarrow a = \frac{3}{2} - b, \quad (9.22b)$$

which on substituting into Eq. (9.21c) gives

$$D = \frac{2}{2 - a} + \frac{b}{2 - a}d = \frac{2}{2 + b - \frac{3}{2}} + \frac{b}{2 + b - \frac{3}{2}}d = \frac{4}{2b + 1} + \frac{2b}{2b + 1}d. \quad (9.23)$$

Substituting back $d = 4 - \epsilon$, we obtain

$$D = \frac{4}{2b + 1} + \frac{2b}{2b + 1}(4 - \epsilon) \quad (9.24a)$$

$$\rightarrow D = 4 - \frac{2b}{2b + 1}\epsilon. \quad (9.24b)$$

Next, we derive the expression for τ . Using the scaling relation in Eq. (9.18), we have

$$\tau = 2 - \frac{2}{D}. \quad (9.25)$$

And using D derived above in Eq. (9.24b), we arrive at

$$\begin{aligned} \tau &= 2 - \frac{2(2b + 1)}{8b + 4 - 2b\epsilon} = 2 - \frac{2b + 1}{4b + 2 - b\epsilon} = \frac{8b + 4 - 2b\epsilon - 2b - 1}{4b + 2 - b\epsilon} \\ &= \frac{6b + 3 - 2b\epsilon}{4b + 2 - b\epsilon} = \frac{6b + 3 - \frac{3}{2}b\epsilon - \frac{1}{2}b\epsilon}{4b + 2 - b\epsilon} = \frac{3}{2} - \frac{b\epsilon}{8b + 4 - 2b\epsilon}. \end{aligned} \quad (9.26)$$

So by imposing two scaling relations

$$\begin{cases} D(\tau - 2) = -2 \\ D(\tau - a) = bd \end{cases}, \quad (9.27)$$

and from the values of $D = 4$ and $\tau = 3/2$ at the upper critical dimension $d = 4$, we get the ϵ -expansions

$$\begin{cases} D = 4 - \frac{2b}{2b+1}\epsilon \\ \tau = \frac{3}{2} - \frac{b\epsilon}{8b+4-2b\epsilon} \end{cases}, \quad (9.28)$$

with $\epsilon = 4 - d$ for $d = 1, 2, 3, 4$ only.

We can proceed to do the same for avalanche duration. However, we don't really have an analytical expression for $z(2 - \alpha)$, so we need to make use of the narrow joint distribution which gives

$$z(\alpha - 1) = D(\tau - 1) = D - d_w. \quad (9.29)$$

For integer dimensions, we have

$$z(\alpha - 1) = D - 2. \quad (9.30)$$

And for all dimensions, by the same observation as for avalanche size exponents that $\frac{z(\alpha - 1)}{d}$ is essentially linear in $\frac{z}{d}$, we have a similar expression to Eq. (9.17) for the avalanche duration exponents

$$z(\alpha - u) = vd, \quad (9.31)$$

with the same coefficients u and v for both regular and fractal lattices.

By using the values of the exponents D , z and α at the upper critical dimension $d_c = 4$, we have

$$2(2 - u) = 4v \quad (9.32a)$$

$$\rightarrow u = 2 - 2v. \quad (9.32b)$$

We then have

$$z + D - 2 = uz + vd \quad (9.33a)$$

$$\rightarrow (1 - u)z = 2 + vd - D \quad (9.33b)$$

$$\rightarrow z = \frac{2}{1 - u} + \frac{vd - D}{1 - u}. \quad (9.33c)$$

Using Eq. (9.24b) and substituting back $d = 4 - \epsilon$, we have

$$\begin{aligned} z &= \frac{2}{1 + v - 2} + \frac{v(4 - \epsilon) - 4 + \frac{2b}{2b + 1}\epsilon}{1 + 2v - 2} = \frac{2 + 4v - 4 - v\epsilon + \frac{2b}{2b + 1}}{2v - 1} \\ &= \frac{4v - 2 + \left(\frac{2b}{2b + 1} - v\right)\epsilon}{2v - 1} = 2 + \frac{2b - 2bv - v}{(2b + 1)(2v - 1)}\epsilon \\ &= 2 + \frac{2b(1 - v) - v}{(2b + 1)(2v - 1)}\epsilon. \end{aligned} \quad (9.34)$$

Next, we derive the expression for α , which, in same manner as above, is given by

$$\begin{aligned} \alpha &= 1 + \frac{D - 2}{z} = \frac{z + D - 2}{z} = \frac{2 + \frac{2b(1 - v) - v}{(2b + 1)(2v - 1)}\epsilon + 4 - \frac{2b}{2b + 1}\epsilon - 2}{z} \\ &= \frac{4 + \frac{1}{2b + 1} \left(\frac{2b(1 - v) - v}{2v - 1} - 2b \right) \epsilon}{z} = \frac{4 + \frac{(4b - 6bv - v)\epsilon}{(2b + 1)(2v - 1)}}{z} \\ &= \frac{16bv - 8b + 8v - 4 + (4b - 6bv - v)\epsilon}{(2b + 1)(2v - 1)} \frac{z}{8bv - 4b + 4v - 2 + (2b - 2bv - v)\epsilon} \\ &= \frac{16bv - 8b + 8v - 4 + (4b - 6bv - v)\epsilon}{8bv - 4b + 4v - 2 + (2b - 2bv - v)\epsilon} \\ &= \frac{16bv - 8b + 8v - 4 + (4b - 4bv - 2v)\epsilon + (1 - 2b)v\epsilon}{8bv - 4b + 4v - 2 + (2b - 2bv - v)\epsilon} \\ &= 2 + \frac{(1 - 2b)v\epsilon}{8bv - 4b + 4v - 2 + (2b - 2bv - v)\epsilon}. \end{aligned} \quad (9.35)$$

So by imposing two scaling relations

$$\begin{cases} z(\alpha - 1) = D - 2 \\ a(\alpha - u) = vd \end{cases}, \quad (9.36)$$

and from the values of $D = 4$, $z = 2$ and $\alpha = 2$ at the upper critical dimension $d = 4$, we get the ϵ -expansions

$$\begin{cases} z = 2 + \frac{2b(1-v) - v}{(2b+1)(2v-1)}\epsilon \\ \alpha = 2 + \frac{(1-2b)v\epsilon}{8bv - 4b + 4v - 2 + (2b - 2bv - v)\epsilon} \end{cases}, \quad (9.37)$$

with $\epsilon = 4 - d$ for $d = 1, 2, 3, 4$ only.

To recap, we have three different relations: the assumed ϵ -expansion (in leading linear order) for integer dimensions $D = 4 - c\epsilon$, the presumed scaling relation involving dimension of the lattice for *all* dimensions $D(\tau - a) = bd$ and the scaling relation for exponents of avalanche size in regular integer dimensions $D(2 - \tau) = 2$. Combining any two of them leads to the remaining one. Even though the first two are only observed numerically and hold in the leading linear order, they pose the question of the existence of a very general scaling relation that captures *any* lattice and is true in *all* dimensions. This presumed relation might be linked to some (unknown) underlying physics of this model (and self-organized critical phenomenon in general?). Yet, a more fundamental understanding of this fact has not been achieved in this work¹, other than the explanation provided at the end of Sec. 9.5.2. It is also worth noting that Eq. (9.12) can be rewritten in the form

$$\frac{D}{d}(\tau - a) = b \quad (9.38)$$

which has very similar form to the known scaling relation $D(2 - \tau) = 2$; the term $\frac{D}{d}$ is nothing other than the finite-size scaling exponent of avalanche size against the total number of sites N of the lattice (as compared to D , the finite-size scaling exponent of avalanche size against the (perceived) linear size L of the lattice).

9.6 Link to directed percolation

Another reason why the Manna model is an important model in non-equilibrium statistical physics is because of its link to the so-called absorbing-state phase transition

¹We agree with two anonymous referees of (Huynh and Pruessner, 2012) on this point. However, it is very difficult to revoke this robust and striking numerical fact.

(Lübeck, 2004; Ódor, 2004; Henkel *et al.*, 2008). In literature, the Manna universality class is commonly referred to as **C-DP** universality class to emphasise the behaviours of the universality class of absorbing-state phase transition with conserved field (conservation of particles). The distinction between **DP** and **C-DP** universality classes is a question of much debate in the literature. There are couple of reasons of this. A very tempting one is the fact that many exponents of the two classes are very close to one another. Another one is the coincidence of the upper critical dimension $d_c = 4$ for the two classes (Bonachela and Muñoz, 2008). Many times, different models which were believed to belong to one class are later shown to belong to the other (e.g. Maslov and Zhang, 1996; Mohanty and Dhar, 2002, 2007). These issues seem to be resolved and settled after works of Bonachela *et al.* (2006); Bonachela and Muñoz (2007, 2008) (also Hipke *et al.*, 2009) by the introduction of different types of (nontrivial) boundary conditions.

Recently, Basu *et al.* (2012) performed a long numerical simulation of the conserved Manna model (a version of the original Manna model without any driving or dissipative mechanisms, hence does *not* represent an **SOC** model) in order to address the issue of universality class between **MC** and **DP**. They consider the discrete and continuous versions of the model; and find that the critical exponents for different types of initial conditions are very different for both **discrete conserved Manna model (DCMM)** and **continuous conserved Manna model (CCMM)**, that the exponents for natural homogeneous initial conditions are very much similar to those of **DP**, and that the **CCMM** exhibits better (more similar) **DP** scaling behaviour than **DCMM** does. Their results are strongly against what has been known from previous studies of the model. However, that fact does not necessarily mean that the two classes **MC** and **DP** are the same. But rather, it implies that the model may have been misclassified (see above). These results, however, are remarkable and pose the question whether the **SOC** version of the Manna model, *i.e.* the (Abelian) Manna model considered in this thesis, is just a variant of **DP**. In that respect, the numerical results with high precision in the present work very much disprove that (at least) the (Abelian) Manna model belongs to **DP**.

To make a comparison between the two universality classes, Table 9.1 also contains the avalanche exponents for **DP**. The exponent D is derived through the identity $D = d + z - \frac{\beta}{\nu_{\perp}}$, where $-\frac{\beta}{\nu_{\perp}}$ is the finite-size scaling exponent of the activity in **DP**¹. The dynamical exponent z is sometimes used in the **DP** literature as what would have been

¹Voigt and Ziff (1997) follow a different notation. $\frac{2}{z}$ in their work corresponds to z here and their $2\frac{\eta}{z}$

in the present notation $\frac{2}{z}$. The exponent for D_a is based on the assumption of compact avalanches (Lübeck, 2004). Despite disputes in the literature (Bonachela and Muñoz, 2007, 2008) as discussed above, using avalanche exponents leaves little doubt that the Manna universality class differs from DP. As DP is normally performed with periodic boundaries and with different observables, there is, to our knowledge unfortunately no published work on the moment ratios we considered here. In fact, depending on the definition of the ensemble (Marro and Dickman, 1999; Lübeck and Heger, 2003a; Pruessner, 2007, 2008), some moment ratios in DP can be undefined.

to $\frac{\beta}{\nu_{\perp}}$ here.

Summary II

(Abelian) Manna model is one of the most important models in Self-Organized Criticality for many reasons. It is one of the few models that display robust simple scaling behaviour with *clean* power-law behaviour in the distribution of event sizes, and possess all the key features of a *true* SOC model¹: nontrivial robust scaling behaviours with simple rules on local interactions. At the same time, it is also the simplest model that belongs to an important class of models with absorbing-state phase transition which, together with the universality class of Directed Percolation, presents ubiquitously in natural phenomena. Furthermore, unlike deterministic models, this model, which is a stochastic one, exhibits interesting and nontrivial behaviours even in one dimension.

While the first part of this thesis deals with study of fractal, its second part is devoted entirely to study of this important model in SOC. Three main questions have been investigated in the second part on Self-Organized Criticality.

1. How different are the behaviours of SOC models (in this case, the Abelian Manna model) on (irregular) fractal lattices from those on regular lattices?
2. How robust and universal are the critical behaviours of *an* SOC model *itself* (in this case, the Abelian Manna model)?
3. Are there any relations between behaviours of SOC models (in this case, the Abelian Manna model) in different dimensions, including both the regular integer and irregular noninteger ones? If so, what are they?

For these questions, the later ones were motivated by the previous ones and the very first one itself was motivated by the study of fractal in Part I of this thesis.

For the first question, it is found that the Abelian Manna model still displays the critical behaviours as expected on regular lattices. However, two major conclusions

¹Oslo model is another model like this. Many other models in SOC literature either are not critical (e.g. FFM) or does not display simple scaling but rather multifractal scaling behaviour (e.g. BTW model).

can be drawn from that investigation are: First, the ordinary SOC scaling relation for the exponents of avalanche size $D(2 - \tau) = 2$ does not hold on fractal lattices. Rather, that relation must be generalised to $D(2 - \tau) = d_w$, *i.e.* replacing 2 on the right hand side by d_w — the fractal dimension of random walk on the lattice. This finding further validates the argument of mapping the process of redistributing particles in (Abelian) Manna model during an avalanche to the process of random walk on the lattice. This generalised scaling relation also holds for regular lattices in integer dimensions, since $d_w = 2$ in those cases. Second, with the arc-fractal system in Part I of this thesis, several lattices with the *same* fractal dimension can be generated, creating a very good setup to see how the behaviours of the model on these lattices compare to one another. It is found that for fractal lattices of the same dimension but different structures, the critical exponents are very different. This second finding motivates the second question of how robustly the model would behave against the microscopic structures of the lattices in the integer dimensions.

Although motivated by the findings in the first question, the second question arises also because of the surprisingly serious lack of evidence in the literature to claim the robustness and universality of an SOC model on lattices in the *same* dimension. It is known from ordinary critical phenomena that the observation that critical phenomena and scaling display universal features on different lattices has traditionally been one of the most important insights, enabling, in particular, exact results to be obtained. For the first time in SOC literature, a coherent (numerical) study of Abelian Manna model on a good collection of lattices in one, two and three dimensions is performed. The findings from this investigation confirm the Abelian Manna model as an SOC model that displays nontrivial, robust, reproducible and universal scaling behaviour on regular lattices in integer dimensions. Besides traditional universal quantities like critical exponents or moment ratios, the moment amplitude of avalanche area observable is also found to be surprisingly universal, a finding which, even though needs further consolidation, provides more insights into an challenging model in SOC. Particle density is also reported in this investigation and is qualitatively understood in terms of the parameters that characterise the lattices. With the current lack of analytical results for Manna model, and stochastic models in general, the findings from this investigation may provide some hints for future development¹.

The third question arises naturally once the first two have been investigated. With

¹In fact, it has motivated Gunnar Pruessner to come out with a field theory for the Abelian Manna model. **OPEN PROBLEM** see list on page 249

all the critical exponents reliably obtained in all dimensions, including integer and noninteger ones, the relation between them and the dimension can be addressed. It is found that there exists a remarkable relation between them. In integer dimensions, this relation is shown to be closely related to an assumed ϵ -expansion. In noninteger dimension, the relation is only revealed by considering the finite-size scaling exponent D_x of the observables against the total size N rather than the linear size L of the lattice. This fact immediately poses the question of the rôle of scaling against N for lattices in general dimensions. The relation found here systematically reconciles the behaviours found on lattices in integer and noninteger dimensions. It also suggests that there might exist (some) hyperscaling relation(s) in the SOC models similarly to those found in ordinary critical phenomena. These scaling relations might be underlain by some (unknown) physical principles which help our understanding of self-organized critical phenomena.

These investigations of the three central questions have undoubtedly provided a complete numerical picture about the Abelian Manna model below its (supposedly) upper critical dimension $d_c = 4$. Besides the numerical results, some analytical results have also been developed in this thesis. They include the exact calculation of first moment of avalanche size $\langle s \rangle$ and the exponent d_w by mapping the Abelian Manna model to random walk. These calculated results and the numerical ones validate one another. Some initial efforts are also made toward solving the Abelian Manna model, at least in one dimension. They include operator approach and graph method.

To recap, we list some outstanding problems in Self-Organized Criticality (as of the time of writing this thesis, and not in any particular order) and discuss where this thesis finds a place to stand in the current literature.

- Analytical determination of exponents.
- Proof of power-law distribution.
- Field theory and mean-field theory.
- Identification of universality classes.
- Exact solution to models.
- Models in noninteger dimensions.
- Application of renormalisation group.

- Corrections to scaling.
- Reliable numerical techniques.
- Directed models.
- Determination of critical dimension.
- Mapping onto other known processes.
- Scaling relations.

From the list above, we first see that the work here fills in the gap of models in noninteger dimensions with an extensive study on many different fractal lattices from which firm conclusions can be drawn. In terms of numerical study, this work provides a good survey of corrections to scaling which are notorious in the Manna universality class. This thesis also provides a different angle to look at universality issue by considering a *single* model on lattices of *different* structures. SOC requires interaction of many individual components and hence the lattice models arise naturally. The universality can then be viewed as robustness, resilience in the emergent behaviours (of the observables) against perturbation in the *dynamic* rules of the model and the *static* (geometrical) structure of the lattice (also boundary condition, aspect ratio, *etc...*). The unveiling of a relation between exponents and dimension of the lattice strongly suggests the existence of hyperscaling relation(s) in SOC which might provide us with further insights into the phenomenon. Finally, power-law distribution of event sizes is merely an observation and usually enters the equations as an assumption or *ansatz*. In SOC literature, no work so far exists for analytical determining the PDF of an observable. A solvable model and the framework for its solution developed in this thesis illustrate that its PDF can be obtained *exactly* from first principle which might help us see how it appears as a power-law observed in numerical simulation.

Open problems

From this work, many ideas have arisen. Some of them are partially realised, some are still totally blank. Here is a list of open problems that still need further investigations¹ and their appearance in the text, if mentioned or relevant to. They will be pursued in future research. Label “Ⓐ” indicates analytical and “Ⓝ” numerical study.

¹Some of them are suggested by Gunnar Pruessner.

- ④ Abelian Manna model in hyperdimensions at and above the upper critical dimension $d_c = 4$. see page [73](#)
- ④ Cluster distribution of occupied and unoccupied sites on the lattice in the stationary state. see page [137](#)
- ④ Generalised fractal lattices embedded in d -dimensional Euclidean space based on Sierpiński carpet. see page [112](#)
- ④ Fractal lattices in dimensions above 4. see page [235](#)
- ④ Systems with different aspect ratios and boundary conditions. see page [229](#)
- ④ Further investigation into universality of moment amplitude of avalanche area observable. see page [228](#)
- ④ Study of other models in Manna class on one-dimensional lattices different from a simple chain like ladder rope, next-nearest-neighbour chain or futat-subishi to see if the splitting of universality class in one dimension, which is due to the models becoming fully deterministic, still takes place. see page [224](#)
- ④④ A “proper” measure of length of irregular lattices. see page [77](#)
- ④④ Particle density as a function of dimension, and lattice structure. see page [225](#)
- ④④ A field theory for the Abelian Manna model. see page [246](#)
- ④④ Directed models on infinitely ramified fractal lattices like Sierpiński carpet. see page [185](#)
- ④④ More insight into the scaling relation the exponents and the dimension of the lattice. see page [237](#)
- ④④ The partially directed Abelian Manna model on the simple chain lattice in all regimes of density of undirected sites. see page [192](#)
- ④ Calculation of the moments of the area covered by a branching random walk. see page [144](#)
- ④ Exact calculation of moment ratios based on the spectrum of the lattice Laplacian of an adjacency matrix in dimensions greater than 4. see page [169](#)

- ④ Exact calculation of first moment of avalanche size on a lattice using first-passage time. see page [152](#)
- ④ Exact determination of the fractal dimension of random walk on fractal lattices. see page [167](#)
- ④ Analytical insights into corrections to scaling. see page [119](#)
- ④ Analytical insights into scaling function, lower and upper cutoff of the [PDF](#). see page [72](#)
- ④ Making use of transformations between two-dimensional lattices to show the robustness of critical exponents. see page [92](#)
- ④ Operator approach to the [AMM](#) based on the diagrams of toppling schemes. see page [214](#)

Outlook

Science exists in Human society as an undeniable part of evolution. Its rôle is to understand Nature through either experimental or theoretical studies. Throughout the history, many revolutionary achievements have shaped and forever changed the perception and understanding of Human about Nature.

In the modern time, in the early half of twentieth century, Quantum Mechanics and General Relativity have emerged and shown their importance in Physics and Natural Sciences in general. In the later half, the discovery of Chaos and Fractals have once again opened the door showing us more fascinating phenomena taking place in the surrounding world. As part of the flow, Self-Organized Criticality arises with the initial ambitious claim to be a theory of everything. Even though nowadays, SOC is shown not to be as strong as earlier claim, yet it is still relevant to a large amount of phenomena found in both natural process and laboratorial experiments. The key question that SOC addresses is the origin of ubiquitous scale-invariant behaviours found in Nature.

In his classic book, Jensen (1998) asks four important questions.¹

1. *Can we identify SOC as a well-defined distinct phenomenon* different from any other category of behaviour?
2. *Can we identify* a certain construction that can be called a theory of *self-organized critical systems*?
3. *Has SOC taught us anything* about the world that we did not know prior to BTW's seminal 1987 paper?
4. *Is there any predictive power in SOC* — that is, can we state the necessary and sufficient conditions a system must fulfill in order to exhibit SOC? And, if we are

¹precisely quoted

able to establish that a system belongs to the category of SOC systems, does that then actually help us to understand the behaviour of the system?

These questions were answered by Jensen (1998) himself and revisited again by Pruessner (2012) after additional years of research. And the answers to them are all positive. The key message to be learnt from SOC is that it exists as a robust phenomenon with unique features which is distinct from others.

In the current context of this thesis, it is more relevant to address the question: *Is it worth investigating SOC?* Even though SOC nowadays is not ubiquitous, it is still important for providing understanding of a wide range of long-range correlated behaviours and critical phenomena without tuning. After twenty-five years of Self-Organized Criticality, the field itself is no longer a “fashionable” field of research like it used to be in the first twenty years after the seminal work by Bak *et al.* (1987). The field now reduces to a mathematical problem rather than actively finding fascinating examples in Nature. Yet, fundamental understanding of the phenomenon is always utterly important. What is currently lacking is a proper theoretical framework that can explain the observations, produce the features and justify the assumptions in SOC. This is done by first identifying a group of good models that can facilitate the above realisation. And we look forward to the revival of the field or birth of a new field in a foreseeable future once the hurdles have been overcome. In that sense, the answer to the question above is: *Yes, it is worth investigating SOC.* And the ultimate question, however, to be addressed is: *What are the necessary and sufficient conditions for SOC?*

After all, in the spirit of Complexity and Complex Systems, both Fractal and Self-Organized Criticality boil down to the same idea: simple (local interaction) rules applied repeatedly lead to emergent, complicated patterns or behaviours both spatially and temporally, which is unifying theme of research in this thesis. It was said very early by Anderson (1972) that “more is different” with the meaning that the overall behaviours of a system are much richer than the individual behaviours of each of the components in that system. And this seems to be general features of many natural as well as social systems.

Finally, Stephen Hawking said¹ “the twenty-first century will be the century of Complexity.” Many theories like Chaos (Lorenz, 1963), Fractal (Mandelbrot, 1983) and Self-Organized Criticality (Bak *et al.*, 1987) are necessary in understanding the complex nature of the world surrounding us. Only then, can the solutions to difficult problems

¹in an interview in January, 2000

be found and the evolution takes place in a sustainable way.

Appendix A

Arc-fractal system

A.1 Labelling of the arcs

In this Appendix, we shall describe the rules for labelling the arc and also discuss on the basic features of the labels with respect to the arc-fractal system. We shall perform this by taking the case of single-rule as an example, with the arc being a semicircle $\alpha = \pi$.

For the arc-fractal system, there are $4n$ possible indices of rotation I . This fact will be proved later. Each index $I(1, 2, \dots, 4n)$ is associated with the azimuth angle θ via the relationship

$$\theta = (I-1)\frac{\pi}{2n}, \quad (\text{A.1})$$

with $0 \leq \theta < 2\pi$. For example, when $I = 1$, $\theta = 0$, the arc is a horizontal semicircle. Indeed, the index I corresponds to a semicircle which is oriented at an angle of $(I-1)\frac{\pi}{2n}$ with respect to the horizontal. Since by rotating the semicircle $4n$ times, each time by $\delta = \frac{\pi}{2n}$, we get back to the original semicircle, we shall take the modulus of I with respect to $4n$ (after which 0 is written as $4n$) implicitly in our following discussion.

Let us start at level 0 with a semicircle of $I = 1$ or $\theta = 0$ (see Fig. A.1). This semicircle shall serve as our reference arc. Note that in general, the reference arc can be of any orientation, not necessarily horizontal. At level 1, the semicircle is divided into n equal segments. Each segment is replaced by a new semicircle. From Fig. A.2, we observe that the first segment gives rise to a semicircle with $\theta_1 = (3n + 1)\frac{\pi}{2n}$. (Note that an addition of 2π is to ensure that θ is in the range $[0, 2\pi]$. In other words, the addition of 2π is equivalent to taking the modulus of I with respect to $4n$). This new semicircle has an index of rotation of $I = 3n + 2$ if it is outward. On the other hand, it has an index

Table A.1: Indices of rotation of semicircles in the next level (arising from a semicircle of index $I = 1$).

Index I of	First semicircle	Second semicircle	i th semicircle	n th semicircle
Inward	$n + 2$	$n + 4$	$n + 2i$	$3n$
Outward	$3n + 2$	$3n + 4$	$3n + 2i$	n

Figure A.1: Initial arc of index $I = 1$.

$I = n + 2$ if it is inward since $\theta_1 = (3n + 2)\frac{\pi}{2n} + \pi - 2\pi = (n + 2)\frac{\pi}{2n}$. Note that the inward case is simply a π rotation relative to the outward case. The second semicircle, which arises from the second segment, can be obtained via a counterclockwise rotation of $\frac{\pi}{n}$ with respect to the first semicircle. With $\theta_2 = \theta_1 + \frac{\pi}{n} = (3n + 1)\frac{\pi}{2n} + \frac{\pi}{n} = (3n + 3)\frac{\pi}{2n}$, the second semicircle has an index I of $3n + 4$ if it is outward. Otherwise, with $\theta_2 = (3n + 3)\frac{\pi}{2n} + \pi - 2\pi = (n + 3)\frac{\pi}{2n}$, it has an index of $n + 4$ if it is inward. In this way, we can determine the indices of rotation of all the semicircles in the subsequent levels, which are summarised in Table A.1.

Now, if the operation begins with some I other than $I = 1$, we can adjust the labelling by simply rotating the whole system counterclockwise by an angle of $(I - 1)\frac{\pi}{2n}$. This is equivalent to increasing every index by an amount $I - 1$, as illustrated in Table A.2.

Next, let us look at some interesting features of this labelling system. First, we observe that at level 2, there are $2n$ possible indices of rotation associated with the semicircles. Then, when we move on to level 3, we observe that there can be another set of $2n$ possible indices of rotation. We shall now show that the set at level 3 and

Figure A.2: Index of rotation of the semicircle at the next level.

Table A.2: Indices of rotation of semicircles in the next level (arising from a semicircle of general index I).

Index I of	First semicircle	Second semicircle	i th semicircle	n th semicircle
Inward	$I + n + 1$	$I + n + 3$	$I + n + 2i - 1$	$1 + 3n - 1$
Outward	$I + 3n + 1$	$I + 3n + 3$	$I + 3n + 2i - 1$	$1 + n - 1$

the previous set at level 2 can form a complete set of $4n$ possible indices of rotation (or positions of the semicircle) under certain circumstances. According to Table A.2, the set of possible indices at level 2 can be enumerated as follow

$$S^2 = \{I + n + 2i - 1, I + 3n + 2i - 1\}_{i=1}^n \quad (1 \leq I \leq 4n). \quad (\text{A.2})$$

Then, the i th semicircle at level 2 generates the level 3 set

$$S_i^3 = \{I + n + 2i - 1 + n + 2j - 1, I + n + 2i - 1 + 3n + 2j - 1, \\ I + 3n + 2i - 1 + n + 2j - 1, I + 3n + 2i - 1 + 3n + 2j - 1\}_{j=1}^n, \quad (\text{A.3})$$

where an increase of index of the form of $I' - 1$ has been performed as discussed above. After some simplification, we obtain the following result

$$S_i^3 \equiv \{I + 2n + 2(i + j) - 2, I + 4n + 2(i + j) - 2\}_{j=1}^n. \quad (\text{A.4})$$

The full set of possible indices at level 3 is basically the union of all S_i^3

$$S^3 = \bigcup_{i=1}^n S_i^3. \quad (\text{A.5})$$

Because i is any integer in $[1, n]$ and j is any integer in $[1, n]$, $i + j$ is any integer in $[2, 2n]$. This enables us to define the subset S_k^{s3} (superscript “ s ” means subset) in the following way

$$S_k^{s3} = \{I + 2n + 2k - 2, I + 4n + 2k - 2\} \quad (2 \leq k \leq 2n). \quad (\text{A.6})$$

We have

$$S^3 = \bigcup_{k=2}^{2n} S_k^{s3}. \quad (\text{A.7})$$

Since

$$I + 2n + 2(k + n) - 2 \equiv I + 4n + 2k - 2, \quad (\text{A.8a})$$

$$I + 2n + 2k - 2 \equiv I + 4n + 2(k + n) - 2, \quad (\text{A.8b})$$

we observe that S_k^{s3} is redundant. By trimming the set, we obtain

$$S_k^{s3'} = \{I + 2n + 2k - 2, I + 4n + 2k - 2\} \quad (2 \leq k \leq n + 1). \quad (\text{A.9})$$

Then, by performing an integer shift of 1 on k , we have

$$S_k^{s3'} = \{I + 2n + 2(k + 1) - 2, I + 4n + 2(k + 1) - 2\} \quad (1 \leq k \leq n), \quad (\text{A.10a})$$

or

$$S_k^{s3'} = \{I + 2n + 2k, I + 4n + 2k\} \quad (1 \leq k \leq n). \quad (\text{A.10b})$$

Therefore,

$$S^3 = \bigcup_{k=1}^n S_k^{s3'} = \{I + 2n + 2k, I + 4n + 2k\}_{k=1}^n. \quad (\text{A.11})$$

Next, let us consider two cases: n is even and n is odd.

Even n :

We will prove that when n is even, *i.e.* $n = 2m$

$$S^3 = \{l\}_{l=1}^{4n} \setminus S^2, \quad (\text{A.12})$$

where the symbol “ $\setminus$ ” means exclusion. Now, according to Eq. (A.2), we observe that the elements of S^2 is parametrised by $2i - 1$. This implies that its complementary set

will be parametrised by $2i$ instead. Thus we expect

$$\{l\}_{l=1}^{4n} \setminus S^2 = \{I + n + 2i, I + 3n + 2i\}_{i=1}^n. \quad (\text{A.13})$$

By using $n = 2m$, it is easy to show that

$$I + n + 2(i + m) \equiv I + 2n + 2i, \quad (\text{A.14a})$$

$$I + 3n + 2(i + m) \equiv I + 4n + 2i. \quad (\text{A.14b})$$

With m being just simple shift of the index i , we infer that

$$\{I + n + 2i, I + 3n + 2i\}_{i=1}^n \equiv \{I + 2n + 2i, I + 4n + 2i\}_{i=1}^n. \quad (\text{A.15})$$

By comparing this result against Eq. (A.11), we see that Eq. (A.12) is indeed satisfied for even n .

Odd n :

When n is odd, *i.e.* $n = 2m + 1$, we observe that

$$I + 2n + 2(i + m) \equiv I + 3n + 2i - 1, \quad (\text{A.16a})$$

$$I + 4n + 2(i + m) \equiv I + n + 2i - 1. \quad (\text{A.16b})$$

By comparing against Eq. (A.11) and noting that m represents a simple shift of index, we have

$$S^3 = \{I + n + 2i - 1, I + 3n + 2i - 1\}_{i=1}^n, \quad (\text{A.17})$$

i.e.

$$S^3 \equiv S^2, \quad (\text{A.18})$$

for odd n . In other words, the set of possible indices of rotation at level 2 is the same as that at level 3.

A further analysis of the above results on level 4 will lead to $S^4 \equiv S^2$ for both even and odd n . In general, we have $S^{2i} \equiv S^{2j}$ and $S^{2i-1} \equiv S^{2j-1}$ ($i, j = 1, 2, \dots, i \neq j$) for both cases. Indeed, we have

$$S^{2i} \cup S^{2i+1} \subseteq \{I\}_{I=1}^{4n}, \quad (\text{A.19})$$

where the equal sign occurs only when n is even.

Figure A.3: Semicircles at next level for odd n and even n .

The above analysis has determined the infimum set of possible indices of rotation of semicircles in the arc-fractal system. The set is precisely $4n$ possible indices I in the case of even n . On the other hand, the set is only $2n$ possible I in the case of odd n . Figure A.3 illustrates why this is so. When n is odd, there exists a middle arc parametrised by $m = \left\lfloor \frac{n}{2} \right\rfloor + 1$ ($\lfloor x \rfloor$ means integer part of x), whose index is the same as that of the initial arc. This does not occur when n is even because there is no such middle arc.

Now, let us go into the details on the notation employed to label the arcs. Recall that for the sake of convenience, we have called the divided arc segments $1, 2, \dots, n$ counterclockwise (following the positive azimuth angle). These segments will be replaced by arc with indices $I_{i,1}, I_{i,2}, \dots, I_{i,n}$, where i is the index of the current arc (see Fig. A.4). A certain rule that indicates the relationship between the indices at two consecutive levels of the arc-fractal system is characterised by the following $4n \times n$ matrix, which assumes that all the arcs are outward

$$I = \begin{pmatrix} I_{1,1} & I_{1,2} & \cdots & I_{1,n} \\ I_{2,1} & I_{2,2} & \cdots & I_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ I_{4n,1} & I_{4n,2} & \cdots & I_{4n,n} \end{pmatrix}. \quad (\text{A.20})$$

Since a particular arc may take the inward orientation, it will be necessary to invert the corresponding row of indices in such a case during the coding of the sequence of indices for the arc-fractal. Otherwise, we simply select the row of indices as the subsequence. The truth of this can be understood from the general results given in

Figure A.4: Indices of semicircles at next level (arising from a semicircle of general index i).

Table A.2, which is written below in terms of our new notation

$$I_{i,j} = i = 2n \pm n + 2j - 1. \quad (\text{A.21})$$

Note that the “+” sign correspond to the outward case while the “−” the inward case. An immediate consequence of this is the following condition between the indices

$$I_{i,j} = I_{i,j+1} + 2(n-1) + 2kn \pmod{4n} \quad (k = 0, 1), \quad (\text{A.22})$$

where $k = 0$ is selected when the two arcs are of opposite orientations, *i.e.* inward-outward, while $k = 1$ is selected when the two arcs are of the same orientation. For the case when the *same* rule is employed to construct each level of the fractal, the following condition holds:

$$1 + I_{i,j} = I_{i+1,j} \pmod{4n}. \quad (\text{A.23})$$

Finally, by means of Eq. (A.21), we can write down the sequence of indices at each level by knowing the indices at the previous level. For example, by knowing the index i_0 of the starting arc (at level $m = 0$), the subsequent sequence as m increases can be obtained in the manner as illustrated below

$$i_0 \rightarrow I_{i_0,1} I_{i_0,2} \dots I_{i_0,n} \rightarrow \underbrace{I_{I_{i_0,1},1} I_{I_{i_0,1},2} \dots I_{I_{i_0,1},n}}_{\text{from } I_{i_0,1}} \underbrace{I_{I_{i_0,2},1} I_{I_{i_0,2},2} \dots I_{I_{i_0,2},n}}_{\text{from } I_{i_0,2}} \rightarrow \dots \quad (\text{A.24})$$

Note that the number of digits in the sequence at level m is n^m .

Figure A.5: The vanishing of the area of the gap between arc-fractal system and the L -system.

A.2 The attracting set of arc-fractal system

In the case of equally divided segments and the same opening angle α throughout the construction process, we know that the invariant set of points generated by the arc-fractal is the same as that produced by the L -system. However, points in the invariant set of points in the arc-fractal system are connected by arcs, while those in the L -system are connected by straight lines. Thus, in order to prove that the two systems share the same attracting set, it will be sufficient to prove that the area between the arc and the straight line (or the “gap”) vanishes as the level $m \rightarrow \infty$. This idea is illustrated in Fig. A.5.

At the level m , the “gap” is given by

$$A_m = N_m a_m, \quad (\text{A.25})$$

where N_m is the number of elements and a_m is the area between the arc and the straight line for each element (see Fig. A.6). The area a_m can be simply determined as follow

$$a_m = \frac{\alpha}{2} R_m^2 - \frac{1}{2} R_m^2 \sin \alpha, \quad (\text{A.26})$$

with the first term being the area of the arc while the second term the area of the triangle. From Eq. (2.8), we know that

$$R_m = R_0 \left(\frac{\sin \frac{\alpha}{2n}}{\sin \frac{\alpha}{2}} \right)^m, \quad (\text{A.27})$$

where R_0 is the radius of the initial arc and n the number of divided segments in each

Figure A.6: The element of area a_m .

arc based on the construction rules. In addition, according to Eq. (2.19)

$$N_m = n^m. \quad (\text{A.28})$$

Putting these results together, the gap can be evaluated as follow

$$A_m = N_m a_m = n^m \frac{\alpha - \sin \alpha}{2} R_0^2 \left(\frac{\sin \frac{\alpha}{2n}}{\sin \frac{\alpha}{2}} \right)^{2m} = \frac{\alpha - \sin \alpha}{2} R_0^2 \left(\frac{n \sin^2 \frac{\alpha}{2n}}{\sin^2 \frac{\alpha}{2}} \right)^m. \quad (\text{A.29})$$

Let us take the Sierpiński gasket as an example. Since $\alpha = \pi$ and $n = 3$ for this case, we have

$$A_m = \frac{\pi - \sin \pi}{2} R_0^2 \left(\frac{3 \sin^2 \frac{\pi}{2 \times 3}}{\sin^2 \frac{\pi}{2}} \right)^m = \frac{\pi}{2} R_0^2 \left(3 \times \frac{1}{4} \right)^m \quad (\text{A.30a})$$

$$\Rightarrow \lim_{m \rightarrow \infty} A_m = \lim_{m \rightarrow \infty} \frac{\pi}{2} R_0^2 \left(\frac{3}{4} \right)^m = 0. \quad (\text{A.30b})$$

Thus, we observe that the gap vanishes as $m \rightarrow \infty$, *i.e.* the fractals generated by the arc-fractal system and the L -system converge to the same set, which is the fractal known as the Sierpiński gasket.

A.3 Multifractal analysis of arc-fractal

A.3.1 The multifractal

The construction of arc fractal is the replacement of arcs at each level. The following information is to be deduced at the next level m th of construction.

- Opening angle of arc α_i :
- Scaling s_i : is the portion of current arc length to be replaced at the next level, $0 \leq s_i \leq 1$. For the normal fractal, $\frac{s_i = 1}{n}$ where n is the number of segments into which each arc is divided.
- Radius of arc R_i .
- Centre of arc (x_i, y_i) .
- Starting angle position of arc θ_i^s : is defined based on the positive trigonometric azimuth angle.
- Rule of orientation of arc r_i : can be either -1 or $+1$ for the orientation in or out of the new arc relative to the current arc.
- Number of segments to be divided n_i .

Indeed, s_i 's are equivalent to n in the sense that $\sum_{i=1}^n s_i = 1$. α_i, s_i, r_i and n_i are independent variables and are functions of the level m . R_i depends on α_i, s_i and n while θ_i^s and (x_i, y_i) depend on all α_i, s_i, r_i and n as well as i itself.

A.3.2 Analysis

Follow [Halsey *et al.* \(1986\)](#), we perform the analysis for this multifractal and show that it is indeed a multifractal. As a standard procedure, we cover the multifractal by disjoint cells $\sigma_1, \sigma_2, \dots, \sigma_r$. At level m th, each cell will cover a complete arc. And the

Figure A.7: Cover of arc element.

smallest possible spherical ball (circle in two dimensions) to completely cover the cell σ_i has radius l_i equal the distance between two ends of an arc for $\alpha \leq \pi$ and equal the radius of an arc for $\alpha > \pi$.

Let p_i be the probability attributed to the cell σ_i , we have the generalised partition function

$$Z(\beta, \tau) = \inf_{\{\sigma\}} \sum_i^r \frac{p_i^\beta}{l_i^\tau} (\beta \leq 1, \tau \leq 0) \quad (\text{A.31a})$$

and

$$Z(\beta, \tau) = \sup_{\{\sigma\}} \sum_i^r \frac{p_i^\beta}{l_i^\tau} (\beta > 1, \tau > 0). \quad (\text{A.31b})$$

For a simple example, we assume that $n = 2$, $\alpha = \pi$ for all levels and arcs. The orientation parameter r does not play any rôle in the analysis (surely, it is set in such a way there is no overlapping of arcs). Because we set $n = 2$, the scaling will be set to s_1 and $s_2 = 1 - s_1$ for all levels and arcs. More freedom for the parameters requires a much more complicated analysis which will be presented later.

Now with each scaling parameter s_i , we associate a probability w_i ($\sum_i w_i = 1$). That

means on going from one level to the next one, each segment of the current arc will have two choices to be: s_1 with probability w_1 or s_2 with probability w_2 respectively. Corresponding to the scaling factor s_i , we have the cover size scaled by

$$a_i = \frac{l_i^{(m+1)}}{l^{(m)}} = \frac{R^{(m)} \sin \frac{s_i \alpha^{(m)}}{2}}{R^{(m)} \sin \frac{\alpha^{(m)}}{2}} = \frac{\sin \frac{s_i \alpha^{(m)}}{2}}{\sin \frac{\alpha^{(m)}}{2}} = \sin \frac{s_i \pi}{2}. \quad (\text{A.32})$$

Note that

$$\sin \frac{s\pi}{2} + \sin \frac{(1-s)\pi}{2} = 2 \sin \frac{\pi}{4} \cos \frac{(1-2s)\pi}{2} = \sqrt{2} \cos \frac{(1-2s)\pi}{4} \geq 1 \quad (\text{A.33})$$

because

$$\cos x \geq \frac{1}{\sqrt{2}} \forall x \in \left[-\frac{\pi}{4}; \frac{\pi}{4}\right]. \quad (\text{A.34})$$

This shows that the total length of the object increases as the level m increases, as a consequence, the dimension of the final (multi)fractal must be greater than (or equal, which is unlikely to be) 1.

Possible length-scale of the cover size at level m th is

$$l_i^{(m)} = a_1^{m-k} a_2^k, \quad (\text{A.35})$$

and the associated probability is

$$p_i^{(m)} = w_1^{m-k} w_2^k. \quad (\text{A.36})$$

The partition function can then be written

$$\begin{aligned} Z(\beta, \tau) \sum_i \frac{p_i^\beta}{l_i^\tau} &= \frac{w_1^{(m-k)\beta} w_2^{k\beta}}{\alpha_1^{(m-k)\tau} \alpha_2^{k\tau}} \\ &= \sum_i \binom{m}{k} \left(\frac{w_1^\beta}{\alpha_1^\tau}\right)^{m-k} \left(\frac{w_2^\beta}{\alpha_2^\tau}\right)^k \\ &= \left(\frac{w_1^\beta}{\alpha_1^\tau} + \frac{w_2^\beta}{\alpha_2^\tau}\right)^m. \end{aligned} \quad (\text{A.37})$$

As $m \rightarrow \infty$, in order for $Z(\beta, \tau)$ to stay finite, we require

$$\left(\frac{w_1^\beta}{\alpha_1^\tau} + \frac{w_2^\beta}{\alpha_2^\tau} \right)^m = 1 \quad (\text{A.38})$$

which leads to the solution $\tau = \tau(\beta)$. And the Rényi dimension follows

$$D(\beta) = \frac{\tau(\beta)}{\beta - 1}. \quad (\text{A.39})$$

Appendix B

Structure of fractal lattices

Here we describe the structure of several fractal lattices presented in Sec. [6.2.5](#).

B.1 Arrowhead lattice

Basically, on this lattice the sites will interact to each other in a chain like one-dimensional lattice. However, what makes the difference is that certain sites can interact with other site (not in chain, or sequence). The crucial thing is to locate those sites with extra interactions, which we call extended sites. In this section, the following questions will be addressed in order to have a well-defined lattice at level m th of the Sierpiński arrowhead construction.

- How many sites are there on the lattice?
- How many extended sites are there on the lattice?
- Where are the extended sites?

B.1.1 Number of sites on the lattice

Besides the conventional way, the construction of Sierpiński arrowhead can also be done by putting the fundamental constructing blocks together to form a bigger block.

This fundamental constructing block is indeed the level 1 of Sierpiński arrowhead (see Fig. [B.1](#). At this level, there are 4 sites on the lattice. At the next level, level 2, we will have the Sierpiński arrowhead consist of 3 constructing blocks. When those block

Figure B.1: Building block of the arrowhead lattice.

fit together, the sites at the ends degenerate. Therefore the number of sites at level 2 is

$$3(\text{blocks}) \times 4(\text{sites per block}) - 2(\text{degenerations}) = 10. \quad (\text{B.1})$$

Similarly, at level 3, there will be $3 \times 10 - 2 = 28$ sites.

We will prove by induction that the number of sites at level m is $3^m + 1$.

Assume the number of sites at level k is

$$N_k = 3^k + 1, \quad (\text{B.2})$$

by Eq. (B.1) the number of sites at level $k + 1$ is

$$N(k + 1) = 3(3^k + 1) - 2 = 3 \times 3^k + 3 - 2 = 3^{k+1} + 1. \quad (\text{B.3})$$

Therefore the number of sites at level m is

$$\boxed{N_m = 3^m + 1}. \quad (\text{B.4})$$

B.1.2 Number of extended sites on the lattice

Let's observe the first few levels of Sierpinski arrowhead in Fig. B.2.

The extra interactions only occur from level 2 onwards. And at this level, there is only one extra interaction which means that there are 2 extended sites. Each extra interaction involves 2 extended sites therefore the number of extended sites is twice the number of extra interactions. We observe that at the next level, level 3, the number of extra interactions will be 3 times that at level 2 because we put 3 building blocks

Figure B.2: First few levels of the arrowhead lattice. Extended links are represented by broken ones.

together plus another extra interaction between building block 1 and 3. That is

$$3(\text{blocks}) \times 1(\text{extra interaction per block}) + 1(\text{new extra interaction}) = 4. \quad (\text{B.5})$$

Similarly, at level 4, there will be $3 \times 4 + 1 = 13$ extra interactions.

We will again prove by induction that the number of extra interactions at level m is $\sum_{i=2}^m 3^{i-2}$.

Assume the number of extra interactions at level k is

$$X_k = \sum_{i=2}^k 3^{i-2}, \quad (\text{B.6})$$

by Eq. (B.1) the number of extra interactions at level $k+1$ is

$$X_{k+1} = 3 \sum_{i=2}^k 3^{i-2} + 1 = \sum_{i=2}^k 3^{i-1} + 3^{2-2} = \sum_{i=2}^{k+1} 3^{i-2}. \quad (\text{B.7})$$

Therefore the number of extra interactions at level m is

$$\boxed{X_m = \sum_{i=2}^m 3^{i-2}} \quad (\text{B.8})$$

and hence the number of extended sites is

$$2X_m = 2 \sum_{i=2}^m 3^{i-2}. \quad (\text{B.9})$$

B.1.3 Identifying the extended sites

The next concern is to locate the extended sites on the lattice. To do this, we number the site along the chain from one end to another end. So at level m , the sites will be labelled $i = 1, 2, 3, \dots, 3^m + 1$.

Because level k is built by putting 3 copies of level $k - 1$ together, the distance $j - i$ between 2 extended sites at level k is basically that at level $k - 1$ plus the new distance between 2 extended sites of the extra interaction between copy 1 and 3. Extra interaction is only introduced from level 2 onwards, and the extra interaction is indeed between copy 1 and 3 of level 1. And this extra interaction will be manifested in the forthcoming levels. So in order to determine the extended sites, it is adequate to examine the new extra interaction of level k .

From the pictures, it is easy to see that these two extended sites at level k are

$$\frac{3^{k-1} + 1}{2} + 1 \text{ and } 3^k + 1 - \frac{3^{k-1} + 1}{2}$$

or

$$\frac{3^{k-1} + 3}{2} \text{ and } \frac{5 \times 3^{k-1} + 1}{2}.$$

In particular, the new extra extended sites are, at level 2,

$$\frac{3^{2-1} + 1}{2} + 1 = 3 \text{ and } 3^2 + 1 - \frac{3^{2-1} + 1}{2} = 8$$

or $[3, 8]$; and, at level 3,

$$\frac{3^{3-1} + 1}{2} + 1 = 6 \text{ and } 3^3 + 1 - \frac{3^{3-1} + 1}{2} = 23$$

or $[6, 23]$.

At level 3, besides the new extended sites $[6, 23]$, there are 6 (3 pairs) others extended sites which are the manifestations of the new extra extended sites $[3, 8]$ at level 2. One pair is still $[3, 8]$ while the other two are of the form $[3 + \alpha_1, 8 + \alpha_1]$ and $[3 + \alpha_2, 8 + \alpha_2]$ where α_1 and α_2 are the shift factors. α_1 is for copy 2 and α_2 is for copy 3 of level 2. It easy to see that $\alpha_2 = 2\alpha_1$. And the shift factor α_1 is indeed the number of sites at previous level (level 2) (less 1): $\alpha_1 = 3^2$. Therefore, the extended sites at level 3 can be listed: $[3, 8]$, $[3 + 3^2, 8 + 3^2]$, $[3 + 2 \times 3^2, 8 + 2 \times 3^2]$ and $[6, 23]$ or we can write in column

form

$$\begin{pmatrix} [3, 8] \\ [3 + 3^2, 8 + 3^2] \\ [3 + 2 \times 3^2, 8 + 2 \times 3^2] \\ [6, 23] \end{pmatrix}.$$

At level 4, the extended sites can be listed

$$\begin{pmatrix} [3, 8] \\ [3 + 3^2, 8 + 3^2] \\ [3 + 2 \times 3^2, 8 + 2^2] \\ [6, 23] \end{pmatrix},$$

$$\begin{pmatrix} [3, 8] \\ [3 + 3^2, 8 + 3^2] \\ [3 + 2 \times 3^2, 8 + 2 \times 3^2] \\ [6, 23] \end{pmatrix} + [3^3, 3^3],$$

$$\begin{pmatrix} [3, 8] \\ [3 + 3^2, 8 + 3^2] \\ [3 + 2 \times 3^2, 8 + 2 \times 3^2] \\ [6, 23] \end{pmatrix} + [2 \times 3^3, 2 \times 3^3],$$

$$[15, 68]$$

or in column form

$$\begin{pmatrix} [3, 8] \\ [3 + 3^2, 8 + 3^2] \\ [3 + 2 \times 3^2, 8 + 2 \times 3^2] \\ [6, 23] \\ [3 + 3^3, 8 + 3^3] \\ [3 + 3^2 + 3^3, 8 + 3^2 + 3^3] \\ [3 + 2 \times 3^2 + 3^3, 8 + 2 \times 3^2 + 3^3] \\ [6 + 3^3, 23 + 3^3] \\ [3 + 2 \times 3^3, 8 + 2 \times 3^3] \\ [3 + 3^2 + 2 \times 3^3, 8 + 3^2 + 2 \times 3^3] \\ [3 + 2 \times 3^2 + 2 \times 3^3, 8 + 2 \times 3^2 + 2 \times 3^3] \\ [6 + 2 \times 3^3, 23 + 2 \times 3^3] \\ [15, 68] \end{pmatrix}.$$

And the next levels follow *etc...*

The question now is “What is the general formula to determine the extended sites at level m ?”

Now look at the number of extra interactions or equivalently number of pairs of extended sites at level m

$$X_n = \sum_{i=2}^m 3^{i-2}. \quad (\text{B.10})$$

We see that this expression has $m - 1$ terms. A tricky observation suggests that each term 3^{i-2} is indeed the number of pairs whose the distance between two extended sites (in term of numbering label) is

$$\frac{5 \times 3^{n-i+1} + 1}{2} - \frac{3^{n-i+1} + 3}{2}.$$

For convenience, we call this distance the size of that pair. So it is suggested that, at level m , there are 3^{m-1-i} pairs of size $\frac{5 \times 3^i + 1}{2} - \frac{3^i + 3}{2}$ ($i = 1, 2, \dots, m - 1$). The

extended sites in those pairs are

$$\frac{3^i + 3}{2} + \sum_{l=3}^{m-i+1} \alpha^{(l)} \text{ and } \frac{5 \times 3^i + 1}{2} + \sum_{l=3}^{m-i+1} \alpha^{(l)}$$

where $\alpha^{(l)}$ is the shift factor at level l , it can be either 0, 3^{l-1} or $2 \times 3^{l-1}$.

We can write $(0, 3^{l-1}, 2 \times 3^{l-1})$ as $(0 \times 3^{l-1}, 1 \times 3^{l-1}, 2 \times 3^{l-1})$ or simply $c \times 3^{l-1}$ where $c = 0, 1, 2$. This point suggests a representation in base-3. If we write the number in the range $k = 0, 1, 2, \dots, 3^{m-1-i} - 1$ in base 3

$$k^{(10)} = \overline{k_1 k_2 \dots k_{m-1}}^{(3)},$$

we will get the digit k_j to be the desired coefficient c .

For example, at level 3, for $i = 1$, $k = 0, 1, 2$ the extended sites of pair of size $\frac{5 \times 3^1 + 1}{2} - \frac{3^1 + 3}{2}$ are

$$\begin{aligned} & \left[\frac{3^1 + 3}{2} + 0 \times 3^2, \frac{5 \times 3^1 + 1}{2} + 0 \times 3^2 \right], \\ & \left[\frac{3^1 + 3}{2} + 1 \times 3^2, \frac{5 \times 3^1 + 1}{2} + 1 \times 3^2 \right], \\ & \left[\frac{3^1 + 3}{2} + 2 \times 3^2, \frac{5 \times 3^1 + 1}{2} + 2 \times 3^2 \right]; \end{aligned}$$

for $i = 2$, $k = 0$ the extended sites of pair of size $\frac{5 \times 3^2 + 1}{2} - \frac{3^2 + 3}{2}$ are

$$\left[\frac{3^2 + 3}{2}, \frac{5 \times 3^2 + 1}{2} \right].$$

At level 4, for $i = 1$, $k = 0, 1, \dots, 8$ which are $\overline{00}^3, \overline{01}^3, \dots, \overline{22}^3$. The extended sites of pair of size $\frac{5 \times 3^1 + 1}{2} - \frac{3^1 + 3}{2}$ are

$$\begin{aligned} & \left[\frac{3^1 + 3}{2} + 0 \times 3^2 + 0 \times 3^3, \frac{5 \times 3^1 + 1}{2} + 0 \times 3^2 + 0 \times 3^3 \right], \\ & \left[\frac{3^1 + 3}{2} + 0 \times 3^2 + 1 \times 3^3, \frac{5 \times 3^1 + 1}{2} + 0 \times 3^2 + 1 \times 3^3 \right], \\ & \vdots \end{aligned}$$

$$\left[\frac{3^1 + 3}{2} + 2 \times 3^2 + 2 \times 3^3, \frac{5 \times 3^1 + 1}{2} + 2 \times 3^2 + 2 \times 3^3 \right];$$

for $i = 2, k = 0, 1, 2$ the extended sites of pair of size $\frac{5 \times 3^2 + 1}{2} - \frac{3^2 + 3}{2}$ are

$$\left[\frac{3^2 + 3}{2} + 0 \times 3^3, \frac{5 \times 3^2 + 1}{2} + 0 \times 3^3 \right],$$

$$\left[\frac{3^2 + 3}{2} + 1 \times 3^3, \frac{5 \times 3^2 + 1}{2} + 1 \times 3^3 \right],$$

$$\left[\frac{3^2 + 3}{2} + 2 \times 3^3, \frac{5 \times 3^2 + 1}{2} + 2 \times 3^3 \right];$$

for $i = 3, k = 0$ the extended sites of pair of size $\frac{5 \times 3^3 + 1}{2} - \frac{3^3 + 3}{2}$ are

$$\left[\frac{3^3 + 3}{2}, \frac{5 \times 3^3 + 1}{2} \right].$$

And the next levels follow *etc...*

In the programming language, the formula is presented in the following way.

At level m ,

for integer i from 1 to $m - 1$,

for integer k from 0 to $3^{m-1-i} - 1$,

the extended sites will be

$$\frac{3^i + 3}{2} + \sum_{j=1}^{m-1} k_j^{(3)} 3^j \text{ and } \frac{5 \times 3^i + 1}{2} + \sum_{j=1}^{m-1} k_j^{(3)} 3^j$$

where $k_j^{(3)}$ means the j th digit of the base-3 representation of k

$$k^{(10)} = \overline{k_1 k_2 \dots k_{m-1}}^{(3)}. \quad (\text{B.11})$$

The next issue is how to determine $k_j^{(3)}$. This is done by realising that the digits in base- n representation of a number K in base 10 are the remainders of successive divisions of K by n . So we have

$$k_j^{(3)} = \text{Rem} \left\{ \left[\frac{K}{3^{n-1-j}} \right], 3 \right\}. \quad (\text{B.12})$$

Figure B.3: Crab fractal lattice (at level 4).

B.2 Crab lattice

Here, the lattice sites structure on crab fractal (Fig. B.3) will be determined.

Basically, just like in the case of Sierpiński arrowhead the sites will interact to each other in a chain like one-dimensional lattice. And the difference is that certain sites can interact with other sites (not in chain, or sequence). The crucial thing is to locate those sites with extra interactions, which we call extended sites. Again the following questions will be addressed in order to have a well-defined lattice at level n th of the crab fractal construction.

- How many sites are there on the lattice?
- How many extended sites are there on the lattice?
- Where are the extended sites?

B.2.1 Number of sites on the lattice

The construction of crab fractal can also be done by putting the fundamental constructing blocks together to form a bigger block.

This fundamental constructing block is indeed the level 1 of crab fractal (see Fig. B.4). At this level, there are 4 sites on the lattice. At the next level, level 2, we will have the crab fractal consisting of 3 constructing blocks. When those block fit together, the sites at the ends degenerate. Therefore the number of sites at level 2 is

$$3(\text{blocks}) \times 4(\text{sites per block}) - 2(\text{degenerations}) = 10. \quad (\text{B.13})$$

Figure B.4: Building block of the crab fractal lattice.

Figure B.5: First few levels of the crab fractal lattice. Extended links are represented by broken ones.

Similarly, at level 3, there will be $3 \times 10 - 2 = 28$ sites.

We will prove by induction that the number of sites at level m is $3^m + 1$.

Assume the number of sites at level k is

$$N_k = 3^k + 1, \quad (\text{B.14})$$

by Eq. (B.13) the number of sites at level $k + 1$ is

$$N(k + 1) = 3(3^k + 1) - 2 = 3 \times 3^k + 3 - 2 = 3^{k+1} + 1. \quad (\text{B.15})$$

Therefore the number of sites at level m is

$$\boxed{N_m = 3^m + 1}. \quad (\text{B.16})$$

B.2.2 Number of extended sites on the lattice

Let's observe the first few levels of crab fractal in Fig. B.5.

The extra interactions only occur from level 3 onwards. And at this level, there are 2 extra interactions which means that there are 4 extended sites. Each extra interaction

Figure B.6: Crab fractal lattice at level 5.

involves 2 extended sites therefore the number of extended sites is twice the number of extra interactions. Because of the symmetry of the fractal, we need only consider one side.

We observe that at the next level, level 4, the number of extra interactions will be 3 times that at level 2 because we put 3 building blocks together plus another 2 extra interaction between each pair of building blocks 1 and 2 and blocks 2 and 3. That is

$$3(\text{blocks}) \times 2(\text{extra interactions per block}) + 2(\text{sides}) \times 2(\text{new extra interactions}) = 10. \quad (\text{B.17})$$

At level 5 (see Fig. B.6), however, the number of extra interactions between block 1 and 2 is 3 and this number indeed increases as the level increases. So there will be $3 \times 10 + 2 \times 3 = 36$ extra interactions.

We will again prove by induction that the number of extra interactions at level m is $\sum_{i=1}^{m-2} 2i \times 3^{m-2-i}$.

Assume the number of extra interactions at level k is

$$X_k = \sum_{i=1}^{k-2} 2i \times 3^{k-2-i}, \quad (\text{B.18})$$

by Eq. (B.17) the number of extra interactions at level $k + 1$ is

$$\begin{aligned}
 X_{k+1} &= 3 \sum_{i=1}^{k-2} 2i \times 3^{k-2-i} + 2 \times (k + 1 - 2) \\
 &= \sum_{i=1}^{k-2} 2i \times 3^{k-1-i} + 2 \times (k - 1) \\
 &= \sum_{i=1}^{k-1} 2i \times 3^{k-1-i} \\
 &= \sum_{i=1}^{(k+1)-2} 2i \times 3^{(k+1)-2-i}.
 \end{aligned} \tag{B.19}$$

Therefore the number of extra interactions at level m is

$$X_m = \sum_{i=1}^{m-2} 2i \times 3^{m-2-i} \tag{B.20}$$

and hence the number of extended sites is

$$2X_m = 2 \sum_{i=1}^{m-2} 2i \times 3^{m-2-i}. \tag{B.21}$$

B.2.3 Identifying the extended sites

The next concern is to locate the extended sites on the lattice. To do this, we number the site along the chain from one end to another end. So at level m , the sites will be labelled $i : i = 1, 2, 3, \dots, 3^m + 1$.

Because level k is built by putting 3 copies of level $k - 1$ together, the distances $j - i$ between 2 extended sites at level k are basically those at level $k - 1$ together with the new distances between extended sites of the extra interaction between copies 1 and 2 as well as copies 2 and 3. Extra interaction is only introduced from level 2 onwards, and the extra interaction is indeed between copy 1 and 2 of level 1. And this extra interaction will be manifested in the forthcoming levels. So in order to determine the extended sites, it is adequate to examine the new extra interaction of level k .

Let us call the distance between the pair of extended sites the size of that pair. The size(s) at level 1 is 1, at level 2 is 1 (no extra interactions yet), at level 3 is 5, at level 4 is 13, at level 5 is 41 *etc...* Denote the largest distance at level k to be D_k . It is observed

(from the figure) that $D_{k+1} = 2D_k + 3D_{k-1}$. This is a recursive sequence. And the general solution is given by

$$D_k = \lambda_1 x_1^k + \lambda_2 x_2^k \quad (\text{B.22})$$

where x_1 and x_2 are roots of the quadratic equation

$$x^2 - 2x - 3 = 0 \quad (\text{B.23})$$

and λ_1, λ_2 are coefficients to be determined by initial values of the sequence.

So we have

$$D_k = \frac{3^k}{6} - \frac{(-1)^k}{2} \quad (k = 1, 2, \dots). \quad (\text{B.24})$$

This is the largest size of the pair of extended sites at level k .

From the pictures, it is easy to see that the extended sites at level m are

$$3^{m-1} + 1 + \sum_{j=1}^k D_j - D_{k+2} \text{ and } 3^{m-1} + 1 + \sum_{j=1}^k D_j$$

and its mirror

$$3^m + 2 - \left(3^{m-1} + 1 + \sum_{j=1}^k D_j \right) \text{ and } 3^m + 2 - \left(3^{m-1} + 1 + \sum_{j=1}^k D_j - D_{k+2} \right)$$

$$k = 1, 2, \dots, n-2$$

In particular, the new extra extended sites are, at level 3,

$$3^{3-1} + 1 + \sum_{j=1}^1 D_j - D_{1+2} = 6 \text{ and } 3^{3-1} + 1 + \sum_{j=1}^1 D_j = 11$$

and

$$3^3 + 2 - \left(3^{3-1} + 1 + \sum_{j=1}^1 D_j \right) = 18 \text{ and } 3^3 + 2 - \left(3^{3-1} + 1 + \sum_{j=1}^1 D_j - D_{1+2} \right) = 23$$

or [6, 11] and [18, 23];

at level 4,

$$3^{4-1} + 1 + \sum_{j=1}^1 D_j - D_{1+2} = 24 \text{ and } 3^{4-1} + 1 + \sum_{j=1}^1 D_j = 29,$$

$$3^{4-1} + 1 + \sum_{j=1}^2 D_j - D_{2+2} = 17 \text{ and } 3^{4-1} + 1 + \sum_{j=1}^2 D_j = 30$$

and

$$3^4 + 2 - \left(3^{4-1} + 1 + \sum_{j=1}^1 D_j \right) = 54 \text{ and } 3^3 + 2 - \left(3^{3-1} + 1 + \sum_{j=1}^1 D_j - D_{1+2} \right) = 59,$$

$$3^4 + 2 - \left(3^{4-1} + 1 + \sum_{j=1}^2 D_j \right) = 53 \text{ and } 3^3 + 2 - \left(3^{3-1} + 1 + \sum_{j=1}^2 D_j - D_{2+2} \right) = 66$$

or [24, 29], [17, 30], [54, 59], [53, 66].

The next task is to determine the shift factors in order to provide the full general formula for extended sites at level n .

Similar to the case of Sierpiński arrowhead, the same shift factor technique will be used here. And the result is given here.

In the programming language, the formula is presented in the following way.

At level m ,

for integer i from 1 to $m - 2$,

for integer r from 1 to i ,

for integer k from 0 to $3^{m-2-i} - 1$,

the extended sites will be

$$3^{i+1} + 1 + \sum_{s=1}^r D_s - D_{r+2} + \sum_{j=1}^{m-1} k_j^{(3)} 3^j \text{ and } 3^{i+1} + 1 + \sum_{s=1}^r D_s + \sum_{j=1}^{m-1} k_j^{(3)} 3^j$$

and

$$3^{i+2} - 3^{i+1} + 1 - \sum_{s=1}^r D_s + \sum_{j=1}^{m-1} k_j^{(3)} 3^j \text{ and } 3^{i+2} - 3^{i+1} + 1 - \sum_{s=1}^r D_s + D_{r+2} + \sum_{j=1}^{m-1} k_j^{(3)} 3^j$$

where $k_j^{(3)}$ means the j th digit of the base-3 representation of k

$$k^{(10)} = \overline{k_1 k_2 \cdots k_{m-1}}^{(3)}. \quad (\text{B.25})$$

The next issue is how to determine $k_j^{(3)}$. This is done by realising that the digits in base- n representation of a number K in base 10 are the remainders of successive divisions of K by n . So we have

$$k_j^{(3)} = \text{Rem} \left\{ \left[\frac{k}{3^{n-1-j}} \right], 3 \right\}. \quad (\text{B.26})$$

B.3 Crabarro lattices

The following lattices are produced by hybridising the arrowhead and the crab lattices. There are two of them called crabarroline 1 and 2 respectively. The crabarroline 1 is constructed by applying the arrowhead rule followed by the crab rule alternatively. The crabarroline 2 is constructed by applying the crab rule followed by the arrowhead rule alternatively. The arrowhead rule refers to the rule of constructing the arrowhead lattice, *i.e.* $\omega = (1; -1; 1)$ or “in-out-in”. The crab rule refers to the rule of constructing the crab lattice, *i.e.* $\omega = (-1; 1; -1)$ or “out-in-out”.

The lattice sites structure on the crabarroline 1 and 2 will be determined. The lattices are given below in Fig. B.7.

Basically, just like in the case of Sierpinski arrowhead or crab lattice the sites will interact to each other in a chain like 1-D lattice. And the difference is that certain sites can interact with other sites (not in chain, or sequence). The crucial thing is to locate those sites with extra interactions, which we call extended sites. Again the following questions will be addressed in order to have a well-defined lattice at level m th of the crabarrolines construction:

- How many sites are there on the lattice?
- How many extended sites are there on the lattice?
- Where are the extended sites?

(a) Crabarroline 1

(b) Crabarroline 2

Figure B.7: Hybridising lattices of arrowhead and crab.

B.3.1 Number of sites on the lattice

The number of sites on these two lattices is the same as that of the arrowhead and crab lattices. The rules above only change the lattice structures but not the total number of sites.

Figure B.8: First few levels of the crabarroline lattices.

It has been proven that the number of sites at level m is

$$N_m = 3^m + 1. \quad (\text{B.27})$$

B.3.2 Number of extended sites on the lattice

Let's observe the first few level of the crabarrolines fractal in Fig. B.8.

We see that for the crabarro 1 (A), on going from odd to even level, a basic block is replaced by an arrowhead (at second level); on going from even to odd level, a basic block is replaced by a crab (at second level). For crabarro 2 (C), on going from odd to even level, a basic block is replaced by a crab (at second level); on going from even to odd level, a basic block is replaced by an arrowhead (at second level).

We also observe that replacement with crab leads to new interaction. The "former" new interactions retain. Hence, from level m to level $m + 2$, the number of new interactions increases by 2 for both lattices.

Let $A^{(m)}$ denote the crabarroline 1 at m th iteration and $C^{(m)}$ the crabarroline 2 at m th iteration. We have 3 $A^{(m)}$'s constitute $C^{(m+1)}$ and 3 $C^{(m)}$'s constitute $A^{(m+1)}$.

Let $X_A^{(m)}$ and $X_C^{(m)}$ denote the number of extended interactions on $A^{(m)}$ and $C^{(m)}$

respectively. The observation above transforms into the equations below

$$X_A^{(m+1)} = 3X_C^{(m)} + E_A^{(m+1)}, \quad (\text{B.28a})$$

$$X_C^{(m+1)} = 3X_A^{(m)} + E_C^{(m+1)} \quad (\text{B.28b})$$

where $E^{(m)}$ is the number of new interactions at level m

$$E_A^{(m)} = m - 1 + \text{mod}(m, 2) \quad (m \geq 2, E_A^{(1)} = 0), \quad (\text{B.29a})$$

$$E_C^{(m)} = m - 2 - \text{mod}(m, 2) \quad (m \geq 2, E_C^{(1)} = 0). \quad (\text{B.29b})$$

Substituting Eq. (B.29) into Eq. (B.28) yields

$$X_A^{(m+1)} = 9X_A^{(m-1)} + 4m - 5 - 4\text{mod}(m, 2) \quad (m \geq 2, X_A^{(1)} = 0, X_A^{(2)} = 1), \quad (\text{B.30a})$$

$$X_C^{(m+1)} = 9X_C^{(m-1)} + 4m - 5 + 4\text{mod}(m, 2) \quad (m \geq 2, X_C^{(1)} = 0, X_C^{(2)} = 0). \quad (\text{B.30b})$$

A general form can be obtained for odd and even n

$$X_A^{(2k+1)} = \sum_{i=1}^k (8i - 5)9^{k-i} \quad (k \geq 0), \quad (\text{B.31a})$$

$$X_A^{(2k)} = 9^{k-1} + \sum_{i=1}^{k-1} (8i - 5)9^{k-1-i} \quad (k \geq 1), \quad (\text{B.31b})$$

$$X_C^{(2k+1)} = \sum_{i=1}^k (8i - 5)9^{k-i} \quad (k \geq 0), \quad (\text{B.31c})$$

$$X_C^{(2k)} = \sum_{i=1}^{k-1} (8i + 3)9^{k-1-i} \quad (k \geq 1), \quad (\text{B.31d})$$

and equivalently for general m (through observation)

$$X_A^{(m)} = \sum_{i=0}^{m-2} \left[3^i - \frac{1}{2} + \frac{(-1)^i}{2} \right], \quad (\text{B.32a})$$

$$X_C^{(m)} = \sum_{i=1}^{m-2} \left[3^i - \frac{1}{2} - \frac{(-1)^i}{2} \right]. \quad (\text{B.32b})$$

Table B.1 provides values of $X^{(n)}$ and $E^{(n)}$ for verifying the general expressions.

Taking $X_A^{(m)}$ as an example, we will prove by induction that the number of extra

Table B.1: Number of extended and new interactions of crabarroline lattices at different levels.

Level	$X_A^{(n)}$	$X_C^{(n)}$	$E_A^{(n)}$	$E_C^{(n)}$
1	0	0	0	0
2	1	0	1	0
3	3	3	3	0
4	12	11	3	2
5	38	38	5	2
6	119	118	5	4
7	361	361	7	4

interactions on crabarroline 1 at level m is $\sum_{i=0}^{m-2} \left[3^i - \frac{1}{2} + \frac{(-1)^i}{2} \right]$.

Assume the number of extra interactions at level k is

$$X_A^{(k)} = \sum_{i=0}^{k-2} \left[3^i - \frac{1}{2} + \frac{(-1)^i}{2} \right], \quad (\text{B.33})$$

by Eq. (B.28) the number of extra interactions at level $k+1$ is

$$\begin{aligned}
 X_A^{(k+1)} &= 3X_C^{(k)} + E_A^{(k+1)} = 3 \sum_{i=0}^{k-2} \left[3^i - \frac{1}{2} + \frac{(-1)^i}{2} \right] + k - 1 + \text{mod}(k, 2) \\
 &= \sum_{i=0}^{k-2} \left[3 \times 3^i - 3 \times \frac{1}{2} + 3 \times \frac{(-1)^i}{2} \right] + k - 1 + \text{mod}(k, 2) \\
 &= \sum_{i=1}^{k-1} \left[3^i - \frac{1}{2} - \frac{(-1)^i}{2} \right] - (k-1) - \sum_{i=1}^{k-1} (-1)^i + k - 1 + \text{mod}(k, 2) \\
 &= \sum_{i=0}^{(k+1)-2} \left[3^i - \frac{1}{2} + \frac{(-1)^i}{2} \right]. \quad (\text{B.34})
 \end{aligned}$$

Therefore the number of extra interactions on crabarroline 1 at level m is

$$X_m = \sum_{i=0}^{m-2} \left[3^i - \frac{1}{2} + \frac{(-1)^i}{2} \right] \quad (\text{B.35})$$

and hence the number of extended sites is

$$2X_m = 2 \sum_{i=0}^{m-2} \left[3^i - \frac{1}{2} + \frac{(-1)^i}{2} \right]. \quad (\text{B.36})$$

B.3.3 Identifying the extended sites

The next concern is to locate the extended sites on the lattice. To do this, we number the site along the chain from one end to another end. So at level n , the sites will be labelled $i = 1, 2, 3, \dots, 3^m + 1$.

Because level k is built by putting 3 copies of level $k - 1$ together, the distances $j - i$ between 2 extended sites at level k are basically that at level $k - 1$ together with the new distances between extended sites of the extra interaction between copies 1 and 2, copies 2 and 3 as well as copies 1 and 3 correspondingly. The new extra interactions will be manifested in the forthcoming levels. So in order to determine the extended sites, it is adequate to examine the new extra interaction of level k .

For crabarroline 1, there are $E_A^{(k)} = k - 1 + \text{mod}(k, 2)$ new interactions at level k . 1 one them is between copy 1 and copy 3 of level $k - 1$ whilst $\frac{k - 2 + \text{mod}(k, 2)}{2}$ of them are between copies 1 and 2 and the same number of them between copies 2 and 3.

For crabarroline 2, there are $E_C^{(k)} = n - 2 - \text{mod}(k, 2)$ new interactions at level k . Half of them are between copies 1 and 2 and the other half are between copies 2 and 3.

Let us define the following quantities

$$P_m = 3^m \ (m \geq 0), \quad (\text{B.37a})$$

$$A_{2q} = \frac{2}{5} [9^q - (-1)^q] + 1 \ (q \geq 0), \quad (\text{B.37b})$$

$$A_{2q+1} = \frac{6}{5} 9^q - \frac{(-1)^q}{5} + 1 \ (q \geq 0), \quad (\text{B.37c})$$

$$D_m = P_m + 2A_m - 2 \ (m \geq 1), \quad (\text{B.37d})$$

$$E_m = 5P_m - 2A_m + 2. \quad (\text{B.37e})$$

The form of sequence A_m defined above is indeed the solution of the recursive sequence

$$A_{m+2} = 4P_m - A_m + 2 \ (m \geq 0, A_0 = 1, A_1 = 2). \quad (\text{B.38})$$

By observations, the new pairs of extended sites at level m are, for crabarroline 1,

$$2P_{m-2} + A_{m-2} \text{ and } 2P_{m-2} + A_{m-2} + E_{m-2} \ (m \geq 2),$$

$$3P_{m-2} - A_{m-2-4k} + 2 \text{ and } 3P_{m-2} - A_{m-2-4k} + 2 + D_{m-2-4k}$$

and its mirror

$$6P_{m-2} + A_{m-2-4k} - D_{m-2-4k} \text{ and } 6P_{m-2} + A_{m-2-4k}$$

$$k = 0, 1, \dots, \left\lfloor \frac{m-3}{4} \right\rfloor \ (m \geq 3),$$

$$3P_{m-2} - P_{m-4k} - A_{m-4k} + 2 \text{ and } 3P_{m-2} - P_{m-4k} - A_{m-4k} + 2 + D_{m-4k}$$

and its mirror

$$6P_{m-2} + P_{m-4k} + A_{m-4k} - D_{m-4k} \text{ and } 6P_{m-2} + P_{m-4k} + A_{m-4k}$$

$$k = 1, 2, \dots, \left\lfloor \frac{m-1}{4} \right\rfloor \ (m \geq 5);$$

for crabarroline 2,

$$3P_{m-2} - A_{m-3-4k} + 2 \text{ and } 3P_{m-2} - A_{m-3-4k} + 2 + D_{m-3-4k}$$

and its mirror

$$6P_{m-2} + A_{m-3-4k} - D_{m-3-4k} \text{ and } 6P_{m-2} + A_{m-3-4k}$$

$$k = 0, 1, \dots, \left\lfloor \frac{m-4}{4} \right\rfloor \ (m \geq 4),$$

$$3P_{m-2} - P_{m-1-4k} - A_{m-1-4k} + 2 \text{ and } 3P_{m-2} - P_{m-1-4k} - A_{m-1-4k} + 2 + D_{m-1-4k}$$

and its mirror

$$6P_{m-2} + P_{m-1-4k} + A_{m-1-4k} - D_{m-1-4k} \text{ and } 6P_{m-2} + P_{m-1-4k} + A_{m-1-4k}$$

$$k = 1, 2, \dots, \left\lfloor \frac{m-2}{4} \right\rfloor \ (m \geq 6).$$

In particular, the new extra extended sites are, for crabarroline 1, at level 2,

$$[3, 8],$$

at level 3,

$$[8, 21], [9, 14], [15, 20],$$

at level 4,

$$[23, 60], [24, 41], [42, 59],$$

at level 5,

$$[66, 179], [71, 120], [125, 174], [78, 83], [162, 167];$$

crabarroline 2, at level 4,

$$[27, 32], [51, 56],$$

at level 5,

$$[78, 95], [150, 167],$$

at level 6,

$$[233, 282], [449, 498], [240, 245], [486, 491].$$

The next task is to determine the shift factors in order to provide the fully general formula for extended sites at level n .

Similar to the case of Sierpinski arrowhead or the crab lattices, the same shift factor technique will be used here. And the result is given here.

In the programming language, the formula is presented in the following way.

Crabarroline 1:

At level m ,

for integer i from $\text{mod}(m, 2)$ to $m - 2$,

for integer k from 0 to $3^{m-2-i} - 1$,

the extended sites will be

$$2P_i + A_i + \sum_{j=1}^{m-1} k_j^{(3)} 3^j \text{ and } 2P_i + A_i + E_i$$

for integer r from 0 to $\frac{i+2-3}{4}$,

the extended sites will be

$$3P_i - A_{i-4r} + 2 + \sum_{j=1}^{m-1} k_j^{(3)} 3^j \text{ and } 3P_i - A_{i-4r} + 2 + D_{i-4r} + \sum_{j=1}^{m-1} k_j^{(3)} 3^j,$$

$$6P_i + A_{i-4r} - D_{i-4r} + \sum_{j=1}^{m-1} k_j^{(3)} 3^j \text{ and } 6P_i + A_{i-4r} + \sum_{j=1}^{m-1} k_j^{(3)} 3^j;$$

for integer r from 1 to $\frac{i+2-1}{4}$

the extended sites will be

$$3P_i - P_{i+2-4k} - A_{i+2-4k} + 2 + \sum_{j=1}^{m-1} k_j^{(3)} 3^j \text{ and } 3P_i - P_{i+2-4k} - A_{i+2-4k} +$$

$$2 + D_{i+2-4k} + \sum_{j=1}^{m-1} k_j^{(3)} 3^j,$$

$$6P_i + P_{i+2-4k} + A_{i+2-4k} - D_{i+2-4k} + \sum_{j=1}^{m-1} k_j^{(3)} 3^j \text{ and } 6P_i + P_{i+2-4k} + A_{i+2-4k} + \sum_{j=1}^{m-1} k_j^{(3)} 3^j$$

where $k_j^{(3)}$ means the j th digit of the base-3 representation of k

$$k^{(10)} = \overline{k_1 k_2 \cdots k_{m-1}}^{(3)}. \quad (\text{B.39})$$

Crabarroline 2:

At level m ,

for integer i from $\text{mod}(m+1, 2)$ to $m-2$,

for integer k from 0 to $3^{m-2-i} - 1$,

for integer r from 0 to $\frac{i+2-4}{4}$,

the extended sites will be

$$3P_i - A_{i-1-4k} + 2 + \sum_{j=1}^{m-1} k_j^{(3)} 3^j \text{ and } 3P_i - A_{i-1-4k} + 2 + D_{i-1-4k} + \sum_{j=1}^{m-1} k_j^{(3)} 3^j,$$

$$6P_i + A_{i-1-4k} - D_{i-1-4k} + \sum_{j=1}^{m-1} k_j^{(3)} 3^j \text{ and } 6P_i + A_{i-1-4k} + \sum_{j=1}^{m-1} k_j^{(3)} 3^j;$$

for integer r from 1 to $\frac{i+2-2}{4}$,
the extended sites will be

$$\begin{aligned}
 & 3P_i - P_{i+1-4k} - A_{i+1-4k} + 2 + \sum_{j=1}^{m-1} k_j^{(3)} 3^j \text{ and } 3P_i - P_{i+1-4k} - A_{i+1-4k} + \\
 & 2 + D_{i+1-4k} + \sum_{j=1}^{m-1} k_j^{(3)} 3^j, \\
 & 6P_i + P_{i+1-4k} + A_{i+1-4k} - D_{i+1-4k} + \sum_{j=1}^{m-1} k_j^{(3)} 3^j \text{ and } 6P_i + P_{i+1-4k} + A_{i+1-4k} + \sum_{j=1}^{m-1} k_j^{(3)} 3^j
 \end{aligned}$$

where $k_j^{(3)}$ means the j th digit of the base-3 representation of k

$$k^{(10)} = \overline{k_1 k_2 \cdots k_{m-1}}^{(3)}. \quad (\text{B.40})$$

The next issue is how to determine $k_j^{(3)}$. This is done by realising that the digits in base- n representation of a number K in base-10 are the remainders of successive divisions of K by n . So we have

$$k_j^{(3)} = \text{Rem} \left\{ \left[\frac{k}{3^{n-1-j}} \right], 3 \right\}. \quad (\text{B.41})$$

B.4 Snowflake lattice

Snowflake lattice is constructed by filling a Koch snowflake. Its linear length is the length of single Koch curve. The total number of sites on the lattice is sum of three filled Koch curve plus the central triangle.

At m th iteration, denote L_m as the linear length of the single Koch curve, N_m as the total number of sites on the lattice, K_m as the number of sites on the filled Koch curve. At $(m+1)$ th iteration, we need to fill the Koch curve with a triangle of linear size $L_m - 2$ whose number of sites is $\frac{(L_m - 2)(L_m - 1)}{2}$. Therefore

$$K_{m+1} = 4(K_m - 1) + 1 + \frac{(L_m - 2)(L_m - 1)}{2} = 4K_m - 3 + \frac{3^m(3^m - 1)}{2}. \quad (\text{B.42})$$

Then we have, with $K_1 = 5$,

$$K_i = 4^{m-1}K_1 - 3 \sum_{j=0}^{m-2} 4^j + \sum_{j=1}^{m-1} \frac{3^j(3^j - 1)}{2} 4^{m-1-j} = \frac{9^m + 4 \times 4^m + 5^m}{10} + 1, \quad (\text{B.43})$$

total number of sites on the lattice

$$\begin{aligned} N_m &= 3(K_m - 1) + \frac{(L_m - 3)(L_m - 2)}{2} \\ &= 3 \frac{9^m + 4 \times 4^m + 5 \times 3^m}{10} + \frac{(3^m - 2)(3^m - 1)}{2} \\ &= \frac{4 \times 9^m + 6 \times 4^m}{5} + 1, \end{aligned} \quad (\text{B.44})$$

its linear length

$$L_m = 3^m + 1, \quad (\text{B.45})$$

and dimension

$$D = \lim_{m \rightarrow \infty} \frac{\ln N_m}{\ln L_m} = 2. \quad (\text{B.46})$$

The lattice has dimension 2 while its boundary has dimension of the Koch curve

$$D_b = \frac{\ln 4}{\ln 3} \approx 1.26. \quad (\text{B.47})$$

Appendix C

Recursive sequences

Very often, we deal with objects or quantities that are defined in a recursive manner. In such cases, the knowledge of recursive sequences is very useful. Here we describe some known results for linear recursive sequences.

C.1 Linear homogeneous recurrence relations with constant coefficients

Given a sequence $\{a_k\}_{k=0}$, one commonly encountered linear recurrence relation is of the following type

$$a_k = c_1 a_{k-1} + c_2 a_{k-2} + \cdots + c_n a_{k-n} = \sum_{i=1}^n c_i a_{k-i} \quad (\text{C.1})$$

in which the coefficients c_i are all constant. This is a linear relation because an entry a_k is a linear combination of the other entries.

A technique to solve Eq. (C.1) is through the so-called characteristic polynomial, which is defined as

$$p(x) = x^n - c_1 x^{n-1} - c_2 x^{n-2} - \cdots - c_n \quad (\text{C.2})$$

which generally admits n roots $x_1, x_2, \dots, x_n$.

If all n roots x_i of $p(x)$ are distinct, the solution to the recurrence relation in Eq. (C.1)

is given by

$$a_k = b_1 x_1^k + b_2 x_2^k + \cdots + b_n x_n^k = \sum_{i=1}^n b_i x_i^k \quad (\text{C.3})$$

where the coefficients b_i are to be determined from the initial conditions of the sequence.

C.2 Nonhomogeneous recurrence relations

The relation in Eq. (C.1) is of homogeneous type because there is no free coefficient. With the presence of a free coefficient (or even a term that is k -dependent), the relation becomes nonhomogeneous. Generally, we have

$$a_k = c_0(k) + c_1 a_{k-1} + c_2 a_{k-2} + \cdots + c_n a_{k-n} = c_0(k) + \sum_{i=1}^n c_i a_{k-i}. \quad (\text{C.4})$$

If the free coefficient $c_0(k)$ is a constant, the nonhomogeneous relation can be solved through a homogeneous relation which is obtained by subtracting Eq. (C.4) from a similar equation for index $k + 1$

$$\begin{aligned} a'_k &= a_{k+1} - a_k \\ &= c_1(a_k - a_{k-1}) + c_2(a_{k-1} - a_{k-2}) + \cdots + c_n(a_{k-n+1} - a_{k-n}) \\ &= \sum_{i=1}^n c_i(a_{k-i+1} - a_{k-i}) \\ &= \sum_{i=1}^n c_i a'_{k-i}. \end{aligned} \quad (\text{C.5})$$

If the free coefficient $c_0(k)$ is *not* a constant, one has to find a particular solution to Eq. (C.4) to combine with the solution of Eq. (C.4) without the term $c_0(k)$ to obtain the its general solution.

C.3 List of summation formulas

Together with dealing with recursive sequences, one also usually deals with summation formulas. Below is a list of commonly encountered summation formulas. Some of them are used in the solution of the partially directed Abelian Manna model in Sec. 8.4.2 and determining the structure of the fractal lattices in Appendix B.

- Linear sum

$$\sum_{i=0}^n i = \frac{n(n+1)}{2}. \quad (\text{C.6})$$

- Square sum

$$\sum_{i=0}^n i^2 = \frac{n(n+1)(2n+1)}{6}. \quad (\text{C.7})$$

- Cubic sum

$$\sum_{i=0}^n i^3 = \left(\frac{n(n+1)}{2} \right)^2. \quad (\text{C.8})$$

- Quartic sum

$$\sum_{i=0}^n i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}. \quad (\text{C.9})$$

- Geometric sum

$$\sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x}. \quad (\text{C.10})$$

- Arithmetic-geometric sum

$$\sum_{i=0}^{n-1} ix^i = \frac{x - nx^n + (n-1)x^{n+1}}{(1-x)^2}. \quad (\text{C.11})$$

Appendix D

Abelian sandpile algorithm¹

D.1 Abelian Manna model

Here, we describe the algorithm for the Abelian Manna model on a general lattice that was employed in this study.

Generally, we need to specify the value of the following arguments to feed in the main routine function.

- n : number of iterations of lattice.
- T : number of avalanches to be triggered.

In the initialisation step, the followings are set.

- Z : height profile of sand (empty).
- A : list of active sites (empty).
- N : histogram of avalanche size (empty).

```
// MAIN LOOP
For avalanche 1 to T
{
    Initialise avalanche size s=0
    Choose a random site rs on the lattice
    Initialise trackers tr1=0 and tr2=1 (tr1 indicates available position
```

¹Thanks to Gunnar Pruessner for discussion on this.

in the middle of the queue in A, tr2 indicates available position right after the queue in A. {tr1,tr2}={0,1} means that A is empty)
 Increase Z at rs by 1
 If rs was empty, its Z is now 1 and move on, otherwise add it to A at position tr2 (make sure that its Z is less than 2 before update).
 Shift the tracker tr2 to next position

// AVALANCHE LOOP

Do this while A is not empty

{

Randomly pick a site in A (A is only filled from first position to right before tr2, the remaining is empty) and name it current site cs

Relax cs by decreasing its Z by 2

Increase avalanche size s by 1

If cs becomes in-active after toppling, remove it from A and assign the tracker tr1 to be the position of cs in A to indicate that this position is now available (a hole in the middle of the queue)
 Choose 2 nearest neighbours (they can be the same) (the choice is different for normal and extended sites). For each of these neighbours nb update as follow

{

Add 1 grain to nb by increasing its Z by 1

If nb becomes active after adding

{

If nb is not in A yet

{

If tr1 is not 0 (a hole is available in the queue), then add nb to A at that position

Otherwise, add nb to the end of A and shift the tracker tr2 to next position

}

}

```

    }
    If after updating neighbours, no new active site is added to A, move
        the last site in the queue in A to the hole in the middle of the
        queue (if available) and shift the tracker tr2 backward 1 position
        to indicate that one active site toppled, became in-active and left
        A
    }
    After transient time, update the histogram
}

```

D.2 Abelian BTW model

The algorithm for Abelian sandpile model on a general lattice is not very from that of [AMM](#) described about. The only differences are the threshold at each site (equal the number of nearest neighbours of that site, instead of 2) and the way the particles are distributed when an unstable site topples (each of its nearest neighbours receives one particle, while the number of particles at the site itself reduces by the number of its nearest neighbours). Due to the Abelianness, the implementations of the stack of active sites in the two models are the same.

Appendix E

Data analysis

E.1 Levenberg-Marquardt nonlinear fitting

In this work, we deal extensively with data fitting to extract the exponents of the model. Hence a description of good, reliable fitting method is important. In general, we use not-so simple fitting model, hence the least-square fitting is not a suitable choice. Rather, we employ nonlinear fitting method to allow for general behaviour of the fitting model. One particular such method is the Levenberg-Marquardt nonlinear fitting (Marquardt, 1963; Press *et al.*, 1992).

Given an initial guess for the set of fitted parameters $\mathbf{a}$, the recommended Marquardt recipe is as follows.

- Compute $\chi^2(\mathbf{a})$.
- Pick a modest value for λ , say $\lambda = 0.001$.
- (†) Solve the linear equations

$$\sum_{l=0}^{M-1} \alpha'_{kl} \delta_{al} = \beta_k \quad (\text{E.1})$$

for $\delta \mathbf{a}$ and evaluate $\chi^2(\mathbf{a} + \delta \mathbf{a})$.

- If $\chi^2(\mathbf{a} + \delta \mathbf{a}) \geq \chi^2(\mathbf{a})$, increase λ by a factor of 10 (or any other substantial factor) and go back to (†).
- If $\chi^2(\mathbf{a} + \delta \mathbf{a}) < \chi^2(\mathbf{a})$, decrease λ by a factor of 10, update the trial solution $\mathbf{a} \leftarrow \mathbf{a} + \delta \mathbf{a}$, and go back to (†).

E.2 Propagation of errors

The propagation of errors is summarised in Table E.1 for different relations between the combined quantities.

Table E.1: Summary of combined errors.

Relation between Z and (A, B)	Relation between errors ΔZ and $(\Delta A, \Delta B)$
$Z = A + B$	$(\Delta Z)^2 = (\Delta A)^2 + (\Delta B)^2$
$Z = A - B$	$(\Delta Z)^2 = (\Delta A)^2 + (\Delta B)^2$
$Z = AB$	$\left(\frac{\Delta Z}{Z}\right)^2 = \left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2$
$Z = \frac{A}{B}$	$\left(\frac{\Delta Z}{Z}\right)^2 = \left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2$
$Z = A^n$	$\frac{\Delta Z}{Z} = n \frac{\Delta A}{A}$
$Z = \ln A$	$\Delta Z = \frac{\Delta A}{A}$
$Z = e^A$	$\frac{\Delta Z}{Z} = \Delta A$

E.3 Weighted mean value

Very often, we estimate a quantity y by making n independent measurements y_i . The estimated value of y is then simply the arithmetic mean of the n measured values. This is, however, only true under the assumption that each measurement was obtained under the *same* condition, *i.e.* the *same* amount of uncertainty. In general, each of the n measurements involve an uncertainty σ_i . We have to account for this uncertainty by assigning appropriate weight to each data points. The weight for each measurement is given by

$$w_i = \frac{1}{\sigma_i^2}. \quad (\text{E.2})$$

The weighted mean value of the n independent measurements y_i is then equal to

$$\bar{y} = \frac{\sum_{i=1}^n w_i y_i}{\sum_{i=1}^n w_i} \quad (\text{E.3})$$

and the error of the estimated weighted mean value is given by

$$\sigma = \sqrt{\frac{1}{\sum_{i=1}^n w_i}}. \quad (\text{E.4})$$

This method produces the same result as fitting the data set against a free parameter in Levenberg-Marquardt fitting described above.

References

- Addison, P. S., 1997, *Fractals and chaos: an illustrated course* (Institute of Physics, Philadelphia PA, USA), ISBN: 0750303999, URL <http://www.crcpress.com/product/isbn/9780750304009>. xviii, 44, 95
- Alava, M., and M. A. Muñoz, 2002, *Interface depinning versus absorbing-state phase transitions*, Phys. Rev. E **65**, 026145. 221
- Aleksiejuk, A., J. A. Holyst, and G. Kossinets, 2002, *Self-organized criticality in a model of collective bank bankruptcies*, Int. J. Mod. Phys. C **13**, 333. 62
- Ali, A. A., and D. Dhar, 1995a, *Breakdown of simple scaling in Abelian sandpile models in one dimension*, Phys. Rev. E **51**, R2705–R2708. 62
- Ali, A. A., and D. Dhar, 1995b, *Structure of avalanches and breakdown of simple scaling in the abelian sandpile model in one dimension*, Phys. Rev. E **52**, 4804-4816. 62
- Anderson, P. W., 1972, *More is different*, Science **177**, 393-396. 252
- Ashcroft, N. W., and N. D. Mermin, 1976, *Solid state physics* (Holt, Rinehart and Winston, New York, USA), ISBN: 0030839939, URL <http://www.amazon.com/Solid-State-Physics-Neil-Ashcroft/dp/0030839939>. 92
- Azimi-Tafreshi, N., H. Dashti-Naserabadi, S. Moghimi-Araghi, and P. Ruelle, 2010, *The Abelian sandpile model on the honeycomb lattice*, J. Stat. Mech. **2010**, P02004. xxvii, 143
- Baas, A. C. W., 2002, *Chaos, fractals and self-organization in coastal geomorphology: simulating dune landscapes in vegetated environments*, Geomorphology **48**, 309-328. 50
- Bak, P., 1990, *Self-organized criticality*, Physica A **163**, 403-409. 58, 61

- Bak, P., 1996, *How nature works: the science of self-organized criticality* (Copernicus, New York, USA), ISBN: 0387947914, URL <http://www.amazon.com/How-Nature-Works-Self-organized-Criticality/dp/0387947914>. xxviii, 54
- Bak, P., and K. Sneppen, 1993, *Punctuated equilibrium and criticality in a simple model of evolution*, Phys. Rev. Lett. **71**, 4083-4086. 65
- Bak, P., C. Tang, and K. Wiesenfeld, 1987, *Self-organized criticality: an explanation of $1/f$ noise*, Phys. Rev. Lett. **59**, 381-384. i, xix, xxv, xxvi, xxvii, xxxiv, 55, 58, 59, 60, 61, 62, 66, 219, 252
- Bak, P., C. Tang, and K. Wiesenfeld, 1988, *Self-organized criticality*, Phys. Rev. A **38**, 364-374. 58
- Barnsley, M. F., 1988, *Fractals everywhere* (Academic Press, Boston, USA), ISBN: 0120790629, URL <http://www.amazon.com/Fractals-Everywhere-Michael-F-Barnsley/dp/0120790610>. 2, 11
- Barnsley, M. F., and S. Demko, 1985, *Iterated function systems and the global construction of fractals*, P. Roy. Soc. Lond. A Mat. **399**, 243. 11
- Basu, M., U. Basu, S. Bondyopadhyay, P. K. Mohanty, and H. Hinrichsen, 2012, *Fixed-energy sandpiles belong generically to directed percolation*, Phys. Rev. Lett. **109**, 015702. 62, 243
- Basu, U., and P. K. Mohanty, 2009, *Active-absorbing-state phase transition beyond directed percolation: A class of exactly solvable models*, Phys. Rev. E **79**, 041143. 62
- Baxter, R. J., 2007, *Exactly solved models in statistical mechanics* (Dover Publications, Mineola NY, USA), ISBN: 9780486462714, URL <http://www.amazon.com/Exactly-Solved-Models-Statistical-Mechanics/dp/0486462714>. 91
- Beck, C., 1995, *Thermodynamics of chaotic systems: an introduction* (Cambridge University Press, New York, USA), ISBN: 0521484510, URL <http://www.amazon.com/Thermodynamics-Chaotic-Systems-Introduction-Cambridge/dp/0521484510>. 9
- ben-Avraham, D., and S. Havlin, 2000, *Diffusion and reactions in fractals and disordered systems* (Cambridge University Press, Cambridge, UK), ISBN: 0521622786, URL <http://www.cambridge.org/aus/catalogue/catalogue.asp?isbn=9780521622783>. 100, 144, 159, 166, 235

- Ben-Hur, A., and O. Biham, 1996, *Universality in sandpile models*, Phys. Rev. E **53**, R1317-R1320. [xxvi](#), [62](#), [64](#), [73](#), [141](#), [219](#), [221](#), [223](#), [231](#), [232](#)
- Binder, K., 1981a, *Critical properties from Monte Carlo coarse graining and renormalization*, Phys. Rev. Lett. **47**, 693–696. [121](#)
- Binder, K., 1981b, *Finite size scaling analysis of Ising model block distribution functions*, Z. Phys. B **43**, 119–140. [121](#)
- Bird, E. C. F., 2008, *Coastal geomorphology: an introduction* (Wiley, Chichester, UK), 2 edition, ISBN: 0470517298, URL <http://as.wiley.com/WileyCDA/WileyTitle/productCd-0470517298.html>. [46](#)
- Bonachela, J. A., 2008, *Universality in self-organized criticality*, Ph.D. thesis, Departamento de Electromagnetismo y Física de la Materia & Institute Carlos I for Theoretical and Computational Physics, University of Granada, Granada, Spain, URL <http://hera.ugr.es/tesisugr/17706312.pdf>. [xv](#), [xxviii](#), [222](#)
- Bonachela, J. A., and M. A. Muñoz, 2007, *How to discriminate easily between directed-percolation and Manna scaling*, Physica A **384**, 89-93. [62](#), [64](#), [243](#), [244](#)
- Bonachela, J. A., and M. A. Muñoz, 2008, *Confirming and extending the hypothesis of universality in sandpiles*, Phys. Rev. E **78**, 041102. [62](#), [64](#), [219](#), [243](#), [244](#)
- Bonachela, J. A., and M. A. Muñoz, 2009, *Self-organization without conservation: true or just apparent scale-invariance?*, J. Stat. Mech. **2009**, P09009. [62](#), [63](#)
- Bonachela, J. A., J. J. Ramasco, H. Chate, I. Dornic, and M. A. Muñoz, 2006, *Sticky grains do not change the universality class of isotropic sandpiles*, Phys. Rev. E **74**, 050102(R). [62](#), [64](#), [243](#)
- Boza, L., and M. Revuelta, 2007, *The dimension of a graph*, Electronic Notes in Discrete Mathematics **28**, 231 - 238, 6th Czech-Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applications, 6th Czech-Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applications. [76](#)
- Brandt, S., 1999, *Data analysis: statistical and computational methods for scientists and engineers* (Springer, New York, USA), 3rd ed. edition, ISBN: 0387984984, URL <http://www.springer.com/physics/theoretical%2C+mathematical+%26+computational+physics/book/978-0-387-98498-8>. [116](#)

- Brent, R. P., 1974, *Gaussian pseudo-random number generator*, Commun. ACM **17**, 704. [12](#)
- Burridge, R., and L. Knopoff, 1967, *Model and theoretical seismicity*, Bull. Seismol. Soc. Am. **57**, 341, <http://www.bssaonline.org/content/57/3/341.full.pdf+html>. [220](#)
- Cardy, J. L., 1988, *Finite-size scaling* (North-Holland, Amsterdam, The Netherlands), ISBN: 0444871098, URL <http://www.amazon.com/Finite-Size-Scaling-John-L-Cardy/dp/0444871098>. [77](#)
- Carmona, J. M., U. M. B. Marconi, J. J. Ruiz-Lorenzo, and A. Tarancón, 1998, *Critical properties of the Ising model on Sierpinski fractals: A finite-size scaling-analysis approach*, Phys. Rev. B **58**, 14387–14396. [xxviii](#)
- Černák, J., 2006, *Inhomogeneous sandpile model: Crossover from multifractal scaling to finite-size scaling*, Phys. Rev. E **73**, 066125. [220](#)
- Chessa, A., H. E. Stanley, A. Vespignani, and S. Zapperi, 1999, *Universality in sandpiles*, Phys. Rev. E **59**, R12-R15. [62](#), [64](#), [72](#), [73](#), [118](#), [141](#), [221](#), [222](#), [231](#), [232](#)
- Christensen, K., 2004, *On self-organized criticality in one dimension*, Physica A **340**, 527–534, Proceedings of the symposium Complexity and Criticality: in memory of Per Bak (1947–2002), Copenhagen, Denmark, Aug 21–23, 2003. [xv](#), [221](#), [222](#)
- Christensen, K., A. Corral, V. Frette, J. Feder, and T. Jossang, 1996, *Tracer dispersion in a self-organized critical system*, Phys. Rev. Lett. **77**, 107–110. [xxvii](#), [61](#), [66](#), [219](#), [221](#)
- Christensen, K., N. Farid, G. Pruessner, and M. Stapleton, 2008, *On the scaling of probability density functions with apparent power-law exponents less than unity*, Eur. Phys. J. B **62**, 331–336. [71](#)
- Christensen, K., H. Flyvbjerg, and Z. Olami, 1993, *Self-organized critical forest-fire model - mean-field theory and simulation results in 1 to 6 dimensions*, Phys. Rev. Lett. **71**, 2737–2740. [64](#)
- Christensen, K., H. C. Fogedby, and H. J. Jensen, 1991, *Dynamical and spatial aspects of sandpile cellular automata*, J. Stat. Phys. **63**, 653–684. [61](#), [72](#), [141](#), [209](#), [231](#), [232](#)
- Christensen, K., and N. R. Moloney, 2005, *Complexity and criticality*, volume 1 of *Imperial College Press Advanced Physics Texts* (Imperial College Press, London, UK), ISBN:

- 9781860945045, URL <http://www.worldscibooks.com/physics/p365.html>. xxv, xxviii, 56, 218
- Christensen, K., N. R. Moloney, O. Peters, and G. Pruessner, 2004, *Avalanche behavior in an absorbing state Oslo model*, Phys. Rev. E **70**, 067101. 64
- Christensen, K., and Z. Olami, 1992, *Scaling, phase-transitions, and nonuniversality in a self-organized critical cellular-automaton model*, Phys. Rev. A **46**, 1829-1838. xxvi, xxvii, 62
- Christensen, K., and Z. Olami, 1993, *Sandpile models with and without an underlying spatial structure*, Phys. Rev. E **48**, 3361-3372. 64, 72, 231, 232
- Daerden, F., 2001, *Sandpile model on fractal lattices. a case study: the Sierpiński gasket*, Ph.D. thesis, Limburgs Universitair Centrum, URL http://bascoe.oma.be/archives/thesis_daerden.pdf. xxviii, 100
- Daerden, F., V. B. Priezhev, and C. Vanderzande, 2001, *Waves in the sandpile model on fractal lattices*, Physica A **292**, 43-54. 65, 100
- Daerden, F., and C. Vanderzande, 1998, *Sandpiles on a Sierpinski gasket*, Physica A **256**, 533-546. 64, 100
- De Menech, M., A. L. Stella, and C. Tebaldi, 1998, *Rare events and breakdown of simple scaling in the Abelian sandpile model*, Phys. Rev. E **58**, R2677-R2680. 62, 64, 71, 114, 208
- Dhar, D., 1990, *Self-organized critical state of sandpile automaton models*, Phys. Rev. Lett. **64**, 1613-1616. 58, 63, 154, 209, 231, 232
- Dhar, D., 1999a, *The Abelian sandpile and related models*, Physica A **263**, 4-25. xxvii, xxviii, 60, 62, 66, 69, 143, 189, 209
- Dhar, D., 1999b, *Some results and a conjecture for Manna's stochastic sandpile model*, Physica A **270**, 69-81. 62, 189
- Dhar, D., 1999c, *Studying self-organized criticality with exactly solved models*, URL <http://arxiv.org/abs/cond-mat/9909009>, (unpublished). 60, 66, 69, 143, 209
- Dhar, D., 2004, *Steady state and relaxation spectrum of the Oslo rice-pile model*, Physica A **340**, 535-543. 61, 62, 189

- Dhar, D., 2006, *Theoretical studies of self-organized criticality*, Physica A **369**, 29-70. xxvii, xxviii, 62, 63, 65, 156, 169, 189, 220
- Dhar, D., and S. N. Majumdar, 1990, *Abelian sandpile model on the Bethe lattice*, J. Phys. A: Math. Gen. **23**, 4333-4350. 63
- Dhar, D., and R. Ramaswamy, 1989, *Exactly solved model of self-organized critical phenomena*, Phys. Rev. Lett. **63**, 1659-1662. 63, 143, 225
- Dhar, D., P. Ruelle, S. Sen, and D. N. Verma, 1995, *Algebraic aspects of abelian sandpile models*, J. Phys. A: Math. Gen. **28**, 805-831. 189
- Dhar, D., T. Sadhu, and S. Chandra, 2009, *Pattern formation in growing sandpiles*, Europhys. Lett. **85**, 48002. 65
- Diaz-Guilera, A., 1992, *Noise and dynamics of self-organized critical phenomena*, Phys. Rev. A **45**, 8551-8558. 63
- Diaz-Guilera, A., 1994, *Dynamic renormalization group approach to self-organized critical phenomena*, Europhys. Lett. **26**, 177-182. 62
- Dickman, R., 2006, *Critical exponents for the restricted sandpile*, Phys. Rev. E **73**, 036131. 219, 224
- Dickman, R., M. Alava, M. A. Muñoz, J. Peltola, A. Vespignani, and S. Zapperi, 2001, *Critical behavior of a one-dimensional fixed-energy stochastic sandpile*, Phys. Rev. E **64**, 056104. 135
- Dickman, R., and J. M. M. Campelo, 2003, *Avalanche exponents and corrections to scaling for a stochastic sandpile*, Phys. Rev. E **67**, 066111. xxvii, 62, 66, 68, 120, 221, 222, 224
- Dickman, R., T. Tomé, and M. J. de Oliveira, 2002, *Sandpiles with height restrictions*, Phys. Rev. E **66**, 016111. 219
- Dickman, R., A. Vespignani, and S. Zapperi, 1998, *Self-organized criticality as an absorbing state phase transition*, Phys. Rev. E **57**, 5095-5105. 61, 64, 134
- Dorn, P. L., D. S. Hughes, and K. Christensen, 2001, *On the avalanche size distribution in the BTW model*, preprint from http://www.cmth.ph.ic.ac.uk/kim/papers/preprints/preprint_btw.pdf, accessed 19 Oct 2010, (unpublished). xxvii

- Dornic, I., H. Chate, and M. A. Muñoz, 2005, *Integration of Langevin equations with multiplicative noise and the viability of field theories for absorbing phase transitions*, Phys. Rev. Lett. **94**, 100601. 64
- Dorogovtsev, S. N., A. V. Goltsev, and J. F. F. Mendes, 2008, *Critical phenomena in complex networks*, Rev. Mod. Phys. **80**, 1275-1335. 62
- Drossel, B., and F. Schwabl, 1992, *Self-organized critical forest-fire model*, Phys. Rev. Lett. **69**, 1629-1632. 63
- Duarte, J. A. M. S., 1990, cited in (Manna, 1990) as private communication for studying the BTW model on a triangular lattice. xxvii
- Edgar, G. A., 2004, *Classics on fractals* (Westview Press, Boulder CO, USA), ISBN: 0813341531, URL <http://www.amazon.com/Classics-On-Fractals-Studies-Nonlinearity/dp/0201587017>. 2, 3, 11
- Efron, B., 1982, *The jackknife, the bootstrap and other resampling plans*, CBMS-NSF Regional Conference Series in Applied Mathematics (No. 38) (SIAM, Philadelphia PA, USA), ISBN: 0898711797, URL www.cambridge.org/9780898711790. 103
- Eppstein, D., 2005, *The lattice dimension of a graph*, Eur. J. Combin. **26**, 585. 76
- Erdos, P., F. Harary, and W. T. Tutte, 1965, *On the dimension of a graph*, URL http://www.renyi.hu/~p_erdos/1965-09.pdf. 76
- Fathauer, R., 2001, *Fractal tilings based on kite- and dart-shaped prototiles*, Comput. Graph. UK **25**, 323. 12, 23
- Fathauer, R., 2002, *Fractal tilings based on v-shaped prototiles*, Comput. Graph. UK **26**, 635, 28th International Conference on Computer Graphics and Interactive Techniques, LOS ANGELES, CALIFORNIA, AUG 12-17, 2001. 12
- Feigenbaum, M. J., 1979, *The universal metric properties of nonlinear transformations*, J. Stat. Phys. **21**, 669. 3
- Flyvbjerg, H., K. Sneppen, and P. Bak, 1993, *Mean-field theory for a simple-model of evolution*, Phys. Rev. Lett. **71**, 4087-4090. 62
- Frette, V., K. Christensen, A. Malthesorensen, J. Feder, T. Jossang, and P. Meakin, 1996, *Avalanche dynamics in a pile of rice*, Nature **379**, 49-52. 61, 62, 66, 219

- Giometto, A., and H. J. Jensen, 2012, *Connecting the microdynamics to the emergent macrovariables: Self-organized criticality and absorbing phase transitions in the deterministic lattice gas*, Phys. Rev. E **85**, 011128. [220](#)
- Götze, W., and L. Sjögren, 1992, *Relaxation processes in supercooled liquids*, Rep. Prog. Phys. **55**, 241. [2](#)
- Grassberger, P., 1983, *Generalized dimensions of strange attractors*, Phys. Lett. A **97**, 227. [9](#)
- Grassberger, P., 2002, *Critical behaviour of the Drossel-Schwabl forest fire model*, New J. Phys. **4**, 17. [xxvii](#)
- Grinstein, G., D. H. Lee, and S. Sachdev, 1990, *Conservation-laws, anisotropy, and self-organized criticality in noisy nonequilibrium systems*, Phys. Rev. Lett. **64**, 1927-1930. [62](#)
- Grumbacher, S. K., K. M. McEwen, D. A. Halverson, D. T. Jacobs, and J. Lindner, 1993, *Self-organized criticality - an experiment with sandpiles*, Am. J. Phys. **61**, 329. [62](#)
- Grunbaum, B., and G. C. Shephard, 1987, *Tilings and patterns* (W.H. Freeman, New York, USA), ISBN: 0716711931, URL <http://www.amazon.com/TILINGS-PATTERNS-Branko-Gruenbaum/dp/0716711931>. [88](#)
- Gutenberg, B., and C. F. Richter, 1955, *Magnitude and energy of earthquakes*, Nature **176**, 795. [i](#)
- Halsey, T. C., M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, 1986, *Fractal measures and their singularities - the characterization of strange sets*, Phys. Rev. A **33**, 1141. [9](#), [34](#), [263](#)
- Hausdorff, F., 1918, *Dimension und äußeres Maß*, Math. Ann. **79**, 157. [2](#)
- Helmberg, G., 2007, *Getting acquainted with fractals* (Walter de Gruyter, New York, USA), ISBN: 3110190923, URL <http://www.degruyter.com/view/product/178800>. [21](#)
- Helmberg, G., 2011, *The crab: a connected fractile of infinite connectivity*, Fractals **19**, 367. [21](#)
- Henkel, M., H. Hinrichsen, and S. Lübeck, 2008, *Non-equilibrium phase transitions: absorbing phase transitions*, volume 1 of *Theoretical and Mathematical Physics*

- (Springer Netherlands, Dordrecht, The Netherlands), ISBN 9781402087653, URL <http://www.springer.com/physics/theoretical%2C+mathematical+%26+computational+physics/book/978-1-4020-8764-6>. 243
- Hentschel, H., and I. Procaccia, 1983, *The infinite number of generalized dimensions of fractals and strange attractors*, Physica D **8**, 435. 9
- Hinrichsen, H., 2000, *Non-equilibrium critical phenomena and phase transitions into absorbing states*, Adv. Phys. **49**, 815. 171
- Hipke, A., S. Lübeck, and H. Hinrichsen, 2009, *Absorbing boundaries in the conserved Manna model*, J. Stat. Mech. **2009**, P07021. 243
- Hohenberg, P. C., and B. I. Halperin, 1977, *Theory of dynamic critical phenomena*, Rev. Mod. Phys. **49**, 435–479. 69
- Hu, C.-K., and C.-Y. Lin, 2003, *Universality in critical exponents for toppling waves of the BTW sandpile model on two-dimensional lattices*, Physica A **318**, 92-100. xxvii
- Hughes, D., and M. Paczuski, 2002, *Large scale structures, symmetry, and universality in sandpiles*, Phys. Rev. Lett. **88**, 054302. 62, 87, 225
- Huynh, H. N., and L. Y. Chew, 2011, *Arc-fractal and the dynamics of coastal morphology*, Fractals **19**, 141. xxx, 21, 95
- Huynh, H. N., L. Y. Chew, and G. Pruessner, 2010, *Abelian Manna model on two fractal lattices*, Phys. Rev. E **82**, 042103. xxxi, xxxii, 77, 123, 130, 235
- Huynh, H. N., and G. Pruessner, 2012, *Abelian Manna model in three dimensions and below*, Phys. Rev. E **86**, 061133. xxiii, xxxi, xxxii, 77, 236, 238, 242
- Huynh, H. N., G. Pruessner, and L. Y. Chew, 2011, *The Abelian Manna model on various lattices in one and two dimensions*, J. Stat. Mech. **2011**, P09024. xvi, xxxi, xxxii, 120, 134, 231
- Huynh, H. N., and C. M. Tran, 2012, *Exact solution of a partially directed Abelian Manna model*, to be submitted to J. Phys. A, (unpublished). xxxii, 208
- Hwa, T., and M. Kardar, 1989, *Dissipative transport in open systems - an investigation of self-organized criticality*, Phys. Rev. Lett. **62**, 1813-1816. 63

- Itzykson, C., and J.-M. Drouffe, 1991, *Statistical field theory*, volume 1 (Cambridge University Press, Cambridge, UK), 1st paperback ed. edition, ISBN: 0521408059, URL <http://www.cambridge.org/knowledge/isbn/item1140707/Statistical%20Field%20Theory>. xxvii, 72
- Jeng, M., 2005, *Conformal field theory correlations in the Abelian sandpile model*, Phys. Rev. E **71**, 016140. 65, 143
- Jensen, H. J., 1990, *Lattice gas as a model of $1/f$ noise*, Phys. Rev. Lett. **64**, 3103–3106. 61, 219, 220
- Jensen, H. J., 1998, *Self-organized criticality: emergent complex behavior in physical and biological systems*, Cambridge Lecture Notes in Physics (Cambridge University Press, Cambridge, UK), ISBN: 0521483719, URL <http://www.cambridge.org/9780521483711>. xxviii, 63, 143, 251, 252
- Jensen, H. J., K. Christensen, and H. C. Fogedby, 1989, *$1/f$ noise, distribution of lifetimes, and a pile of sand*, Phys. Rev. B **40**, 7425–7427. 61, 72, 141, 209, 231, 232
- Jensen, I., 1999, *Low-density series expansions for directed percolation: I. A new efficient algorithm with applications to the square lattice*, J. Phys. A: Math. Gen. **32**, 5233–5249. 222
- Jo, H.-H., and M. Ha, 2010, *Universality classes and crossover behaviors in non-Abelian directed sandpiles*, Phys. Rev. E **82**, 041101. 62
- Kadanoff, L. P., 1976, *Notes on Migdal’s recursion formulas*, Ann. Phys. **100**, 359. xxviii
- Kadanoff, L. P., S. R. Nagel, L. Wu, and S. M. Zhou, 1989, *Scaling and universality in avalanches*, Phys. Rev. A **39**, 6524–6537. 62
- Karmakar, R., S. S. Manna, and A. L. Stella, 2005, *Precise toppling balance, quenched disorder, and universality for sandpiles*, Phys. Rev. Lett. **94**, 088002. 220
- Kloster, M., 2004, *Self-organized criticality, competitive evolution and analysis of gene-expression data*, Ph.D. thesis, Princeton University, URL <http://www.princeton.edu/physics/academics/graduate-program/theses/theses-from-2004/M.Klosterthesis.pdf>. xxviii
- Kloster, M., S. Maslov, and C. Tang, 2001, *Exact solution of a stochastic directed sandpile model*, Phys. Rev. E **63**, 026111. 143, 192, 208

- Knizhnik, V. G., A. M. Polyakov, and A. B. Zamolodchikov, 1988, *Fractal structure of 2d - quantum-gravity*, Mod. Phys. Lett. A **3**, 819. [2](#)
- Koch, H. v., 1904, *Sur une courbe continue sans tangente, obtenue par une construction geometrique elementaire*, Ark. Mat. **1**, 681. [3](#)
- Krause, E. F., 1987, *Taxicab geometry: an adventure in non-Euclidean geometry* (Dover Publications, New York, USA), ISBN: 0486252027, URL <http://www.amazon.com/Taxicab-Geometry-Adventure-Non-Euclidean/dp/0486252027>. [122](#)
- Kutnjak-Urbanc, B., S. Zapperi, S. Milosevic, and H. E. Stanley, 1996, *Sandpile model on the Sierpinski gasket fractal*, Phys. Rev. E **54**, 272-277. [xxvii](#), [64](#), [100](#), [114](#)
- Lagarias, J., and Y. Wang, 1997, *Integral self-affine tiles in R-n. II. Lattice tilings*, J. Fourier Anal. Appl. **3**, 83. [12](#)
- Lai, P.-J., 2009, *Concise methods to generate fractal tilings*, Int. J. Nonlin. Sci. Num. **10**, 585. [12](#)
- Le Doussal, P., K. J. Wiese, and P. Chauve, 2002, *Two-loop functional renormalization group theory of the depinning transition*, Phys. Rev. B **66**, 174201. [234](#)
- Lee, K. E., J. Y. Sung, M. Y. Cha, S. E. Maeng, Y. S. Bang, and J. W. Lee, 2009, *Coevolutionary extremal dynamics on gasket fractal*, Phys. Lett. A **373**, 4260-4263. [xxvii](#), [65](#), [100](#)
- Leland, W., M. S. Taqqu, W. Willinger, and D. V. Wilson, 1994, *On the self-similar nature of ethernet traffic (extended version)*, IEEE ACM T. Network. **2**, 1. [2](#)
- Levy, M., and S. Solomon, 1996, *Power laws are logarithmic Boltzmann laws*, Int. J. Mod. Phys. C **7**, 595-601. [61](#)
- Liggett, T. M., 2005, *Stochastic interacting systems: contact, voter and exclusion processes*, volume 324 of *Grundlehren der mathematischen Wissenschaften* (Springer-Verlag, Berlin, Germany), ISBN: 978-3-540-65995-2, URL <http://www.springer.com/mathematics/probability/book/978-3-540-65995-2>. [69](#)
- Lin, C.-Y., 2010, *Renormalization group approach to the Manna sandpile*, Phys. Rev. E **81**, 021112. [62](#)

- Lindenmayer, A., 1968, *Mathematical models for cellular interactions in development. Parts I and II*, J. Theor. Biol. **18**, 280. 2, 11
- Loebl, M., 2010, *Discrete mathematics in statistical physics: introductory lectures* (Vieweg+Teubner Verlag, Wiesbaden, Germany), ISBN: 9783834893291, URL <http://www.ams.org/bookstore-getitem/item=VWALM-11>. 92
- Lorenz, E. N., 1963, *Deterministic nonperiodic flow*, J. Atmos. Sci. **20**, 130. 3, 252
- Lübeck, S., 2000, *Moment analysis of the probability distribution of different sandpile models*, Phys. Rev. E **61**, 204-209. 62, 64, 71, 72, 118, 141, 209, 220, 221, 222, 223, 231, 232
- Lübeck, S., 2004, *Universal scaling behavior of non-equilibrium phase transitions*, Int. J. Mod. Phys. B **18**, 3977-4118. xv, 64, 135, 219, 222, 233, 235, 243, 244
- Lübeck, S., and P. C. Heger, 2003a, *Universal finite-size scaling behavior and universal dynamical scaling behavior of absorbing phase transitions with a conserved field*, Phys. Rev. E **68**, 056102. 221, 222, 223, 224, 244
- Lübeck, S., and P. C. Heger, 2003b, *Universal scaling behavior at the upper critical dimension of nonequilibrium continuous phase transitions*, Phys. Rev. Lett. **90**, 230601. 221, 232, 233, 234
- Majumdar, S. N., and D. Dhar, 1992, *Equivalence between the abelian sandpile model and the $q \rightarrow 0$ limit of the Potts-model*, Physica A **185**, 129-145. 63
- Mandelbrot, B., 1967, *How long is coast of britain - statistical self-similarity and fractional dimension*, Science **156**, 636. 2
- Mandelbrot, B. B., 1974, *Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier*, J. Fluid Mech. **62**, 331. 9
- Mandelbrot, B. B., 1983, *The fractal geometry of nature* (W. H. Freeman, New York, USA), updated and augmented ed., revised ed. of: fractals. c1977. edition, ISBN: 0716711869, URL <http://www.amazon.com/Fractal-Geometry-Nature-Benoit-Mandelbrot/dp/0716711869>. xxv, 2, 9, 97, 252
- Mandelbrot, B. B., D. E. Passoja, and A. J. Paullay, 1984, *Fractal character of fracture surfaces of metals*, Nature **308**, 721. 2

- Manna, S. S., 1990, *Large-scale simulation of avalanche cluster distribution in sand pile model*, J. Stat. Phys. **59**, 509-521. [307](#)
- Manna, S. S., 1991a, *Critical exponents of the sand pile models in two dimensions*, Physica A **179**, 249-268. [xxvii](#)
- Manna, S. S., 1991b, *Two-state model of self-organized criticality*, J. Phys. A: Math. Gen. **24**, L363-L369. [xxvi](#), [xxvii](#), [xxviii](#), [60](#), [63](#), [66](#), [68](#), [69](#), [114](#), [143](#), [209](#), [222](#), [224](#)
- Manna, S. S., 2009, *Branching process in a stochastic extremal model*, Phys. Rev. E **80**, 021132. [62](#), [64](#)
- Marquardt, D., 1963, *An algorithm for least-squares estimation of nonlinear parameters*, J. Soc. Ind. Appl. Math. **11**, 431-441. [298](#)
- Marro, J., and R. Dickman, 1999, *Nonequilibrium phase transitions in lattice models*, Collection Alea-Saclay: Monographs and Texts in Statistical Physics (Cambridge University Press, New York, USA), ISBN: 0521480620, URL <http://www.cambridge.org/asia/catalogue/catalogue.asp?isbn=9780521019460>. [244](#)
- Marsaglia, G., 1999, *Random numbers for C: The END?*, newsgroup posting 20 Jan 1999, 9:00am, accessed 10 May 2011, URL <https://groups.google.com/group/sci.math.num-analysis/msg/eb4ddde782b17051>. [105](#)
- Marsaglia, G., 2005, *Mersenne Twister*, newsgroup posting 14 Jul 2005, 12:04am, accessed 10 May 2011, URL http://groups.google.com/group/sci.crypt/browse_thread/thread/305c507efbe85be4. [105](#)
- Maslov, S., and Y. C. Zhang, 1995, *Exactly solved model of self-organized criticality*, Phys. Rev. Lett. **75**, 1550-1553. [143](#)
- Maslov, S., and Y. C. Zhang, 1996, *Self-organized critical directed percolation*, Physica A **223**, 1-6. [219](#), [243](#)
- Matsumoto, M., and T. Nishimura, 1998, *Mersenne twister: a 623-dimensionally equidistributed uniform pseudorandom number generator*, ACM Trans. Model. Comp. Sim. **8**, 3. [105](#)
- May, R. M., 1976, *Simple mathematical models with very complicated dynamics*, Nature **261**, 459-467. [i](#)

- Meakin, P., 1983, *Formation of fractal clusters and networks by irreversible diffusion-limited aggregation*, Phys. Rev. Lett. **51**, 1119. [2](#)
- Mendes, J. F. F., R. Dickman, M. Henkel, and M. C. Marques, 1994, *Generalized scaling for models with multiple absorbing states*, J. Phys. A: Math. Gen. **27**, 3019-3028. [219](#)
- Migdal, A. A., 1975, *Recursion equations in gauge field theories*, Sov. Phys. JETP **42**, 413. [xxviii](#)
- Milstein, E., O. Biham, and S. Solomon, 1998, *Universality classes in isotropic, Abelian, and non-Abelian sandpile models*, Phys. Rev. E **58**, 303-310. [62](#)
- Moghim-Araghi, S., M. A. Rajabpour, and S. Rouhani, 2005, *Abelian sandpile model: A conformal field theory point of view*, Nucl. Phys. B **718**, 362-370. [65](#), [143](#)
- Mohanty, P. K., and D. Dhar, 2002, *Generic sandpile models have directed percolation exponents*, Phys. Rev. Lett. **89**, 104303. [62](#), [243](#)
- Mohanty, P. K., and D. Dhar, 2007, *Critical behavior of sandpile models with sticky grains*, Physica A **384**, 34-38. [62](#), [243](#)
- Morand, J., G. Pruessner, and K. Christensen, 2010, *Correlations of avalanche sizes in the Oslo model*, I don't have the access to this reference but thanks to Gunnar Pruessner for making me aware of it, (unpublished). [103](#), [125](#)
- Nagel, S. R., 1992, *Instabilities in a sandpile*, Rev. Mod. Phys. **64**, 321-325. [62](#)
- Nakanishi, H., and K. Sneppen, 1997, *Universal versus drive-dependent exponents for sandpile models*, Phys. Rev. E **55**, 4012-4016. [xxvi](#), [xxvii](#), [72](#), [141](#), [220](#), [221](#), [222](#), [231](#)
- Niederreiter, H., 1978, *Quasi-monte carlo methods and pseudo-random numbers*, B. Am. Math. Soc. **84**, 957. [12](#)
- Niemeyer, L., L. Pietronero, and H. J. Wiesmann, 1984, *Fractal dimension of dielectric-breakdown*, Phys. Rev. Lett. **52**, 1033. [2](#)
- Ódor, G., 2004, *Universality classes in nonequilibrium lattice systems*, Rev. Mod. Phys. **76**, 663-724. [243](#)
- Olami, Z., H. J. S. Feder, and K. Christensen, 1992, *Self-organized criticality in a continuous, nonconservative cellular automaton modeling earthquakes*, Phys. Rev. Lett. **68**, 1244-1247. [xxvii](#), [61](#), [62](#), [63](#)

- Onsager, L., 1944, *Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition*, Phys. Rev. **65**, 117–149. [57](#), [74](#)
- Ostojic, S., 2003, *Patterns formed by addition of grains to only one site of an abelian sandpile*, Physica A **318**, 187-199. [65](#)
- Ott, E., 1993, *Chaos in dynamical systems* (Cambridge University Press, New York, USA), ISBN: 0521432154, URL <http://www.amazon.com/Chaos-Dynamical-Systems-Edward-Ott/dp/0521010845>. [8](#)
- Paczuski, M., and K. E. Bassler, 2000a, *Theoretical results for sandpile models of self-organized criticality with multiple topplings*, Phys. Rev. E **62**, 5347. [62](#), [81](#), [143](#), [192](#), [208](#)
- Paczuski, M., and K. E. Bassler, 2000b, *Theoretical results for sandpile models of soc with multiple topplings*, URL <http://arxiv.org/abs/cond-mat/0005340v2>, (unpublished). [81](#)
- Paczuski, M., and S. Boettcher, 1996, *Universality in sandpiles, interface depinning, and earthquake models*, Phys. Rev. Lett. **77**, 111-114. [61](#), [66](#), [103](#), [118](#), [219](#), [220](#)
- Paczuski, M., S. Maslov, and P. Bak, 1994, *Field theory for a model of self-organized criticality*, Europhys. Lett. **27**, 97-102. [62](#), [64](#)
- Paladin, G., and A. Vulpiani, 1987, *Anomalous scaling laws in multifractal objects*, Phys. Rep. **156**, 147. [9](#)
- Palagallo, J., and M. Salcedo, 2008, *Symmetries of fractal tilings*, Fractals **16**, 69. [12](#)
- Pan, G. J., D. M. Zhang, Z. H. Li, H. Z. Sun, and Y. P. Ying, 2005, *Critical behavior in non-Abelian deterministic directed sandpile*, Phys. Lett. A **338**, 163-168. [62](#)
- Pastor-Satorras, R., and A. Vespignani, 2001, *Reaction-diffusion system with self-organized critical behavior*, Eur. Phys. J. B **19**, 583-587. [221](#), [223](#)
- Penrose, R., 1974, *The role of aesthetics in pure and applied mathematical research*, Bull. Inst. Math. Appl. **10**, 266. [12](#)
- Pentland, A. P., 1984, *Fractal-based description of natural scenes*, IEEE T. Pattern Anal. **6**, 661. [2](#)

- Peskin, M. E., and D. V. Schroeder, 1995, *An introduction to quantum field theory* (Westview Press, New York, USA), ISBN: 0201503972, URL <http://www.amazon.com/Introduction-Quantum-Theory-Frontiers-Physics/dp/0201503972>. 141
- Peters, O., C. Hertlein, and K. Christensen, 2001, *A complexity view of rainfall*, Phys. Rev. Lett. **88**, 018701. i, 56, 62
- Pietronero, L., A. Vespignani, and S. Zapperi, 1994, *Renormalization scheme for self-organized criticality in sandpile models*, Phys. Rev. Lett. **72**, 1690-1693. 62, 63
- Potts, R. B., 1952, *Some generalized order-disorder transformations*, Math. Proc. Cambridge **48**, 106-109, http://journals.cambridge.org/article_S0305004100027419. 74
- Press, W. H., S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, 1992, *Numerical recipes in c* (Cambridge University Press, New York, USA), 2nd edition, ISBN: 0521431085, URL <http://www.nrbook.com/a/bookcpdf.php>. 119, 298
- Priezzhev, V. B., and K. Sneppen, 1998, *Multiple scaling in a one-dimensional sandpile*, Phys. Rev. E **58**, 6959-6963. 62
- Privman, V., P. C. Hohenberg, and A. Aharony, 1991, *Universal critical point amplitude relations*, in *Phase transitions and critical phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, New York, USA), volume 14, chapter 1, pp. 1-134, ISBN: 0122203143, URL <http://www.amazon.com/Phase-Transitions-Critical-Phenomena-14/dp/0122203143>. 78, 122, 229
- Pruessner, G., 2003, *Oslo rice pile model is a quenched Edwards-Wilkinson equation*, Phys. Rev. E **67**, 030301. 103
- Pruessner, G., 2004a, *Exact solution of the totally asymmetric Oslo model*, J. Phys. A: Math. Gen. **37**, 7455-7471. 62, 66, 125, 143, 192, 208
- Pruessner, G., 2004b, *Studies in self-organised criticality*, Ph.D. thesis, Imperial College London, 180 Queen's Gate, London SW7 2BZ, UK, URL http://www.ma.imperial.ac.uk/~pruess/publications/thesis_final/. xv, xxviii, 62, 113, 189, 221, 222
- Pruessner, G., 2007, *Equivalence of conditional and external field ensembles in absorbing-state phase transitions*, Phys. Rev. E **76**, 061103. 244
- Pruessner, G., 2008, *Block scaling in the directed percolation universality class*, New J. Phys. **10**, 113003. 244

- Pruessner, G., 2012, *Self-organised criticality: theory, models and characterisation* (Cambridge University Press, Cambridge, UK), ISBN: 9780521853354, URL <http://www.cambridge.org/catalogue/catalogue.asp?isbn=9780521853354>. xxviii, 57, 58, 61, 143, 175, 219, 252
- Pruessner, G., and H. N. Huynh, 2012, *Towards a field theory for stochastic sandpile model: periodic systems with a control parameter*, to be submitted to J. Phys. A, (unpublished). xxxii, 73
- Pruessner, G., and H. J. Jensen, 2002, *A solvable non-conservative model of self-organized criticality*, Europhys. Lett. **58**, 250-256. xxvii
- Pruessner, G., and H. J. Jensen, 2003, *Anisotropy and universality: The Oslo model, the rice pile experiment, and the quenched Edwards-Wilkinson equation*, Phys. Rev. Lett. **91**, 244303. 208
- Pruessner, G., and H. J. Jensen, 2004, *Efficient algorithm for the forest fire model*, Phys. Rev. E **70**, 066707. 72
- Pruessner, G., D. Loison, and K. D. Schotte, 2001, *Monte Carlo simulation of an Ising model on a Sierpinski carpet*, Phys. Rev. B **64**, 134414. 131
- Pruessner, G., and N. R. Moloney, 2003, *Numerical results for crossing, spanning and wrapping in two-dimensional percolation*, J. Phys. A: Math. Gen. **36**, 11213-11228. 117
- Pruessner, G., and O. Peters, 2006, *Self-organized criticality and absorbing states: Lessons from the Ising model*, Phys. Rev. E **73**, 025106. 64
- Prusinkiewicz, P., and J. Hanan, 1989, *Lindenmayer systems, fractals, and plants* (Springer-Verlag, New York, USA), ISBN: 0387970924, URL <http://www.amazon.com/Lindenmayer-Systems-Fractals-Lecture-Biomathematics/dp/0387970924>. 2, 11
- Prusinkiewicz, P., and A. Lindenmayer, 1990, *The algorithmic beauty of plants* (Springer-Verlag, New York, USA), ISBN: 0387972978, URL <http://algorithmicbotany.org/papers/#abop>. 11
- Puente, C., J. Romeu, R. Pous, X. Garcia, and F. Benítez, 1996, *Fractal multiband antenna based on the Sierpinski gasket*, Electron. Lett. **32**, 1. 10

- Puente, C., J. Romeu, R. Pous, J. Ramis, and A. Hijazo, 1998, *Small but long Koch fractal monopole*, Electron. Lett. **34**, 9. [10](#)
- Rammal, R., and G. Toulouse, 1983, *Random-walks on fractal structures and percolation clusters*, J. Phys. Lett. Paris **44**, L13. [2](#)
- Rényi, A., 1970, *Probability Theory* (North-Holland, Amsterdam, The Netherlands), ISBN: 0486458679, URL <http://www.amazon.com/Probability-Theory-Dover-Books-Mathematics/dp/0486458679>. [9](#)
- Richardson, L. F., 1961, *The problem of contiguity: an appendix to statistic of deadly quarrels*, Gen. Syst. **6**, 139-187. [9](#)
- Rossi, M., R. Pastor-Satorras, and A. Vespignani, 2000, *Universality class of absorbing phase transitions with a conserved field*, Phys. Rev. Lett. **85**, 1803-1806. [219](#)
- Rotenberg, a., 1960, *A new pseudo-random number generator*, J. ACM **7**, 75. [12](#)
- Ruelle, P., and S. Sen, 1992, *Toppling distributions in one-dimensional abelian sandpiles*, J. Phys. A: Math. Gen. **25**, L1257-L1264. [214](#)
- Sadhu, T., and D. Dhar, 2009, *Steady state of stochastic sandpile models*, J. Stat. Phys. **134**, 427-441. [xxxviii](#), [62](#), [135](#), [185](#), [188](#), [189](#)
- Sapoval, B., A. Baldassarri, and A. Gabrielli, 2004, *Self-stabilized fractality of seacoasts through damped erosion*, Phys. Rev. Lett. **93**, 098501. [46](#), [48](#)
- Sawada, Y., A. Dougherty, and J. P. Gollub, 1986, *Dendritic and fractal patterns in electrolytic metal deposits*, Phys. Rev. Lett. **56**, 1260. [2](#)
- Sornette, D., 2006, *Critical phenomena in natural sciences: chaos, fractals, self-organization and disorder: concepts and tools* (Springer-Verlag, Berlin, Germany), ISBN: 9783540331827, URL <http://www.amazon.com/Critical-Phenomena-Natural-Sciences-Selforganization/dp/3540308822>. [xxviii](#), [57](#)
- de Sousa Vieira, M., 1992, *Self-organized criticality in a deterministic mechanical models*, Phys. Rev. A **46**, 6288-6293. [220](#)
- Stanley, H. E., 1999, *Scaling, universality, and renormalization: Three pillars of modern critical phenomena*, Rev. Mod. Phys. **71**, S358-S366. [xxvi](#)

- Stanley, H. E., and P. Meakin, 1988, *Multifractal phenomena in physics and chemistry*, Nature **335**, 405. [9](#)
- Stapleton, M., 2007, *Self-organized criticality and non-equilibrium statistical mechanics*, Ph.D. thesis, Imperial College London. [xxviii](#)
- Stapleton, M. A., and K. Christensen, 2006, *One-dimensional directed sandpile models and the area under a Brownian curve*, J. Phys. A: Math. Gen. **39**, 9107. [189](#)
- Stauffer, D., and A. Aharony, 1994, *Introduction to percolation theory* (Taylor & Francis, London, UK), ISBN: 9780748402533, URL <http://www.taylorandfrancis.com/books/details/9780748402533/>. [57](#), [63](#), [137](#)
- Stilck, J. F., R. Dickman, and R. R. Vidigal, 2004, *Series expansion for a stochastic sandpile*, J. Phys. A: Math. Gen. **37**, 1145-1157. [189](#)
- Strogatz, S. H., 2000, *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering*, Studies in nonlinearity (Westview Press, Cambridge MA, USA), 1st pbk. print edition, ISBN: 0738204536, URL <http://www.amazon.com/Nonlinear-Dynamics-And-Chaos-Applications/dp/0738204536>. [xvii](#), [6](#)
- Syôzi, I., 1951, *Statistics of Kagome Lattice*, Prog. Theor. Phys. **6**, 306–308. [xxvi](#), [xxvii](#), [91](#)
- Syôzi, I., 1972, *Transformation of ising models*, in *Phase transitions and critical phenomena*, edited by C. Domb and M. S. Green (Academic Press, London, UK), volume 1, chapter 7, pp. 269–329, ISBN: 0122203011, URL <http://www.amazon.com/Phase-Transitions-Critical-Phenomena-results/dp/0122203011>. [91](#), [92](#), [189](#)
- Tang, C., and P. Bak, 1988, *Mean-field theory of self-organized critical phenomena*, J. Stat. Phys. **51**, 797-802. [55](#), [62](#)
- Tebaldi, C., M. De Menech, and A. L. Stella, 1999, *Multifractal scaling in the Bak-Tang-Wiesenfeld sandpile and edge events*, Phys. Rev. Lett. **83**, 3952-3955. [62](#), [114](#)
- Tsuchiya, T., and M. Katori, 1999a, *Effect of anisotropy on the self-organized critical states of Abelian sandpile models*, Physica A **266**, 358-361. [62](#)
- Tsuchiya, T., and M. Katori, 1999b, *Exact results for the directed Abelian sandpile models*, J. Phys. A: Math. Gen. **32**, 1629-1641. [143](#)

- Turcotte, D. L., 1992, *Fractals, chaos, self-organized criticality and tectonics*, Terra Nova **4**, 4-12. [62](#)
- Vanderzande, C., and F. Daerden, 2001, *Dissipative Abelian sandpiles and random walks*, Phys. Rev. E **63**, 030301(R). [100](#)
- Vergeles, M., A. Maritan, and J. R. Banavar, 1997, *Mean-field theory of sandpiles*, Phys. Rev. E **55**, 1998-2000. [62](#)
- Vespignani, A., R. Dickman, M. A. Muñoz, and S. Zapperi, 2000, *Absorbing-state phase transitions in fixed-energy sandpiles*, Phys. Rev. E **62**, 4564-4582. [62](#), [64](#), [221](#)
- Vespignani, A., and S. Zapperi, 1997, *Order parameter and scaling fields in self-organized criticality*, Phys. Rev. Lett. **78**, 4793-4796. [134](#)
- Vespignani, A., and S. Zapperi, 1998, *How self-organized criticality works: A unified mean-field picture*, Phys. Rev. E **57**, 6345-6362. [62](#), [63](#)
- Vespignani, A., S. Zapperi, and L. Pietronero, 1995, *Renormalization approach to the self-organized critical behavior of sandpile models*, Phys. Rev. E **51**, 1711-1724. [62](#), [63](#), [64](#)
- Voigt, C. A., and R. M. Ziff, 1997, *Epidemic analysis of the second-order transition in the Ziff-Gulari-Barshad surface-reaction model*, Phys. Rev. E **56**, R6241-R6244. [222](#), [243](#)
- Ward, J. C., 1950, *An Identity in Quantum Electrodynamics*, Phys. Rev. **78**, 182-182. [141](#)
- Wegner, F. J., 1972, *Corrections to Scaling Laws*, Phys. Rev. B **5**, 4529-4536. [118](#)
- West, G., 2011, *Size matters: The complexity, simplicity and unity of living systems from cells to cities*, Lecture at Feza Gürsey Institute – Imperial College London International Summer School and Research Workshop on Complexity, Istanbul, Turkey, 5th – 10th September 2011. [55](#)
- Wichmann, B. A., and I. D. Hill, 1982, *An efficient and portable pseudo-random number generator*, Appl. Stat. - J. Roy. St. C **31**, 188. [12](#)
- Zaitsev, S. I., 1992, *Robin-hood as self-organized criticality*, Physica A **189**, 411-416. [62](#)
- Zapperi, S., K. B. Lauritsen, and H. E. Stanley, 1995, *Self-organized branching processes: mean-field theory for avalanches*, Phys. Rev. Lett. **75**, 4071-4074. [62](#), [64](#)

- Zhang, D. M., G. J. Pan, H. Z. Sun, Y. P. Ying, and R. Li, 2005, *Moment analysis of different stochastic directed sandpile model*, Phys. Lett. A **337**, 285-291. [62](#)
- Zhang, Y. C., 1989, *Scaling theory of self-organized criticality*, Phys. Rev. Lett. **63**, 470. [61](#)
- Zhang, Y.-W., G. Tang, K. Han, Z.-P. Xun, Y.-Y. Xie, and Y. Li, 2012, *Numerical simulations of dynamic scaling behavior of the etching model on fractal substrates*, Acta Phys. Sin. **61**, 020511. [xxxi](#), [74](#)

Colophon

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