Application of extended finite element method for plastic hinges and yield lines analysis

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**NANYANG
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UNIVERSITY**

**APPLICATION OF EXTENDED FINITE
ELEMENT METHOD FOR PLASTIC HINGES
AND YIELD LINES ANALYSIS**

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**SCHOOL OF CIVIL AND ENVIRONMENTAL
ENGINEERING**

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**APPLICATION OF EXTENDED FINITE
ELEMENT METHOD FOR PLASTIC HINGES
AND YIELD LINES ANALYSIS**

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SUMMARY

The application of extended finite element method (XFEM) formulation for nonlinear structural analyses is presented in this thesis. It aims to capture accurately the elastic response of a beam with an internal pin connection and the elasto-plastic response of a beam or a plate structure at a relatively low computational cost by utilizing the XFEM Timoshenko beam and Reissner-Mindlin plate elements.

In the XFEM formulation for an internal pin, a step function is employed in the enriched rotation approximation field and an absolute level set function is adopted in the enriched translation approximation field. The enrichment function for a plastic hinge is formulated by using Hermite function over the high gradient zone resulted from the plastic hinge. The Hermite function regularizes the discontinuous enrichment function for an internal pin to be a continuous function with a high gradient zone. The strain fields derived from the enriched displacement approximation fields remain continuous inside an element. As the absolute level set function is constructed by standard finite element shape functions, such an enrichment function is also called ‘local’ enrichment function. The *local* enrichment function is applied in the XFEM plate element in this thesis. However, it is found that such *local* enrichment function is not suitable for the formulation of a 9-node quadrilateral plate element. Hence a *global* enrichment function is proposed. The *global*

enrichment function is constructed on the structure level and independent of mesh scheme. Thus, it is applicable for both triangular and quadrilateral plate elements.

Shear locking mitigation method is one of the major concerns in the present XFEM formulation. Two methods, including reduced integration and assumed shear strain methods, are employed to control shear locking in this thesis. In the assumed shear strain method, the mixed interpolation of tensorial components (MITC) technique and the discrete shear gap (DSG) technique are adopted in the XFEM plate elements.

Numerical examples are given for different applications including an internal pin in a beam, a plastic hinge in a beam and a yield line in a plate. The numerical results show that the XFEM formulation is able to capture the discontinuous displacement over an internal pin connection and the locally high gradient displacement resulting from a plastic hinge or a yield line. It is also shown that shear locking can be controlled effectively by the reduced integration method and the assumed shear strain method.

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LIST OF SYMBOLS

w_h	deflection field
θ_h	rotation field
N_i	shape functions
ξ	natural coordinates for beam element
ε_j	the normal strain at the j^{th} layer in layered model
ε_m	membrane strain
ε_e	elastic strain
ε_p	plastic strain
γ	shear strain
χ	bending strain
e	assumed shear strain
z_j	distance between the reference surface and the j th layer
x, y	local coordinates (in CR formulation)
$\mathbf{e}_x, \mathbf{e}_y$	axis of local coordinate system
$\mathbf{X}$	global coordinates
ψ	global rotation (in CR formulation)
U, W	global translation (in CR formulation)
u_{eL}	local displacement (in CR formulation)
R	rotation matrix
$\mathbf{D}$	material matrix
k	shear correction factor
G	shear modulus
E	Young's modulus
ν	Poisson's ratio
$\mathbf{f}_L$	local internal force vector (in CR formulation)
$\mathbf{B}$	strain-displacement matrix
$\mathbf{K}_L$	local stiffness matrix (in CR formulation)
$\mathbf{f}_G$	global internal force vector (in CR formulation)
$\mathbf{K}_G$	global stiffness matrix (in CR formulation)
$\mathbf{T}$	transformation matrix (in CR formulation)
r, s	natural coordinates for plate element
β_x, β_y	rotation angle of cross-section in plate element
σ	stress
σ_0	yield strength
σ_e	effective stress
$\mathbf{P}$	deviatoric stress matrix

e_{rt}, e_{st}	assumed shear strain
h_i	interpolation function for assumed shear strain
$\mathbf{S}$	orientation matrix
ξ_0	parametric location of non-smoothness
S_{pp}	enrichment function for rotation field for an internal pin
H	step function
M_{pp}	interpolation function for rotation field for an internal pin
φ	level set function
F_{pp}	enrichment function for deflection field for an internal pin
L_{pp}	interpolation function for deflection field for an internal pin
A_i	global additional DOF for rotation field
B_{ui}, B_{wi}	global additional DOF for translation field
a_i	additional DOF for rotation field
b_i	additional DOF for deflection field
b_{ui}, b_{wi}	local additional DOF for translation field
u_{enrG}	displacement vector for enriched element in global coordinate system
u_{enrL}	displacement vector for enriched element in local coordinate system
l_{ns}	physical length of a high gradient part
l_e	the size of an enriched element
S_{ph}	enrichment function for rotation field for a plastic hinge
F_{ph}	enrichment function for deflection field for a plastic hinge
H_s	Hermite function in the enrichment for rotation field in beam element
H_w	Hermite function in the enrichment for deflection field in beam element
ξ^*	coordinate for high gradient part
I^*	node set of enriched nodes
I	node set of the nodes in the whole domain
M_{phi}	interpolation function for rotation field for a plastic hinge
L_{phi}	interpolation function for deflection field for a plastic hinge
S_{xx}	elastic modulus of UB section with respect to the strong principle axis
Z_{xx}	plastic modulus of UB section with respect to the strong principle axis
M_u	maximum plastic moment
ξ, η	coordinates for high gradient part in plate element
ω	parametric length of a high gradient part
H_t	Hermite function in the enrichment for deflection field in plate element
H_r	Hermite function in the enrichment for rotation field in plate element

M	Hermite matrix
P_ξ, P_η	cubic power basis vectors of ξ and η
G	Hermite geometry vector
<i>S</i>	enrichment function for rotation in plate element
<i>F</i>	enrichment function for deflection in plate element

CHAPTER 1 INTRODUCTION

1.1 Research background

Plastic hinge and yield line are popular concepts to describe the deformation when plastic bending occurs at a beam section and a plate section, respectively. At a cross-section in a beam, when the effective stress at the extreme fibers reaches the yield strength, the cross-section begins to yield while the middle fibers remain elastic. With increasing load, most of the fibers of the section will be yielded and the stiffness of the fully yielded cross-section approaches zero. A plastic hinge is formed when the bending moment reaches plastic moment M_p . A point at the extreme layer of the plate yields when the effective stress at that point reaches the yield strength of the material. With increasing loading, more material points in the plate reach yield. The yielded zone extends to the middle layer in the thickness direction and also spreads to the nearby area. A yield line forms when a continuous area has yielded and a plastic mechanism forms when sufficient yield lines coalesce together in the plate.

In some cases of one-way plate applications, a yield line can be abstracted to a plastic hinge for simplicity. From engineering point view, this simplification does not bring too much difference. However, difference can be found between the abstracted plastic hinge and its original form of yield line from viewpoint of plasticity. In an elasto-plastic plane stress application, plastic flow rule must be

considered. After applying the plastic flow rule, the stress components could change its value. If the yield line is abstracted to a plastic hinge, this plastic flow phenomenon cannot be captured.

Numerical methods for plastic hinge and yield line analyses are necessary when analyzing the behavior of a structure under ultimate loading, for example progressive collapse analyses of structures. When using finite element method (FEM) to analyze the elasto-plastic response of a structural member under bending behavior, users must pay attention to the computational model, since a plastic hinge or a yield line forms when the structural members collapse. The plastic hinge or the yield line leads to a high gradient zone in both rotation and deflection displacement fields. Rapid changes appear within a short range along the beam in both the rotation field and the gradient of the deflection field over a high gradient zone in a beam structure, as shown in Figure 1.1, while in a plate the rapid changes are normal to the yield line (Figure 1.2). The rapid changes of deformation make the displacement field non-smooth locally near a plastic hinge or a yield line. This non-smoothness resulting from a plastic hinge or a yield line is formed within a short range or a finite dimension. Such non-smoothness is often associated with the term ‘high gradient’ (Fries and Belytschko 2010). When standard finite element method (FEM) is employed for analyses with a locally high gradient displacement field, either a fine mesh is required to be generated before the analysis or a refinement algorithm is needed around the yielded area during the analysis. In order to avoid the

difficulty led by the locally high gradient displacement, researchers have proposed some special numerical models for FEM in plastic hinge analyses (Corradi and Poggi 1984; Izzuddin and Elnashai 1993a; Izzuddin and Elnashai 1993b) and yield line analyses (Bauer and Redwood 1987).

In this thesis, the extended finite element method (XFEM) (Belytschko and Black 1999; Moës et al. 1999) is applied for plastic hinge and yield line analyses. The XFEM formulation offers particular advantages to model non-smooth physical phenomena. In an XFEM analysis, special functions, which are able to describe the locally non-smooth displacement field based on *a priori* knowledge, are added into the displacement approximation fields. These special functions are called *enrichment functions*. By using the enrichment functions, XFEM makes it possible to reproduce a locally non-smooth displacement field in a structure. A coarse mesh can be employed in elasto-plastic analyses and it is unnecessary to refine the coarse mesh in the vicinity of the non-smoothness resulting from a plastic hinge or a yield line during an analysis.

Figure 1.1 A plastic hinge in a beam structure

Figure 1.2 A yield line in a plate structure

Similar to a plastic hinge, a sudden change appears in the rotation displacement field over a pin connection and a kink appears in the deflection field, as shown in Figure 1.3. Different from a plastic hinge, the width of a pin connection is usually idealized as zero in numerical modeling. Very often, perfect pin connections are prescribed in a numerical model as boundary or release conditions. In order to model a pin connection inside a beam structure, a specific mesh scheme is generated so that the pin connection is placed at the boundary of an element. Master and slave degrees of freedom (DOF) technique (Crisfield 1991b) can be employed.

Figure 1.3 A pin connection in a beam structure

1.2 The objective and scope of this thesis

The objective of this thesis is to propose a new numerical method, the XFEM formulation, to conduct plastic hinge and yield line analyses. The advantage of the present XFEM formulation over the standard FEM formulation is the avoidance of a fine mesh in the yielded part of the structure so that a large amount of computational effort can be saved. Compared with other plastic hinge models, the present formulation does not require additional moment-rotation relationship for yielded part. Compared with the yield line model (Bauer and Redwood 1987), the present XFEM formulation is able to capture the full process of plate yielding. Different from some other numerical methods, such as element-free Galerkin method (Belytschko et al. 1994), and meshless method (Liu and Gu 2005), the present XFEM formulation can be easily implemented in a standard finite element code.

Enrichment function is one of the characteristics of XFEM. In this thesis, two kinds of enrichment functions are adopted: discontinuous functions for an internal pin connection and regularized functions for a plastic hinge or a yield line. The regularized enrichment function can be further categorized as the local enrichment functions and the global enrichment functions. The former refer to those enrichment functions constructed on the element level by the standard shape functions of FEM, whereas the latter refer to those constructed on the structural level. Besides the enrichment functions, mitigating shear locking phenomenon is another objective in this thesis. Reduced integration method and

assumed shear strain method, including the MITC technique and the DSG technique, are used in the XFEM elements to circumvent shear locking phenomenon. As listed in Table 1.1, the XFEM formulation is implemented in beam and plate elements.

Table 1.1 The XFEM elements developed in the present thesis

	non-smoothness	enrichment		shear locking control
		local	global	
CR beam element	internal pin	discontinuous	---	RI*
beam element	plastic hinge	regularized	---	DSG
6-node plate element	yield line	regularized	regularized	RI/ MITC/DSG
9-node plate element	yield line	---	regularized	MITC/DSG

*RI: reduced integration

As the first trial to apply the XFEM formulation in plastic hinge and yield line analyses, the purpose of this thesis is to show the effectiveness of the enrichment functions in capturing the non-smooth displacement field due to a plastic hinge or a yield line, as well as mitigating shear locking phenomenon in the XFEM element. The XFEM formulations presented in this thesis have the following restrictions:

- An XFEM element is cut through by one non-smoothness zone only. However, one non-smoothness zone can cut through several elements in a particular mesh and there could exist several pieces of non-smoothness in a particular example. Based on this restriction, as one element is cut through by one piece of non-smoothness only, an enriched element has one set of additional DOF only. The cases with branched non-

smoothness strips, which are resulted from branched yield line (Figure 1.4), are not considered.

- The material nonlinearity and geometrical nonlinearity are not considered in the XFEM formulation simultaneously.

Figure 1.4: A branched yield line which is not within the scale of present study

According to this restriction, only geometrical nonlinearity is considered in the XFEM formulation for an internal pin connection, while in the XFEM formulation for plastic hinge analysis and yield line analysis, small deformation assumption is employed. It is noted that plastic hinges and yield lines appear in the yielded area resulting from bending action. In this thesis, only bending action is considered in the XFEM formulation while axial deformation and membrane action are excluded in the XFEM elements for plastic hinge and yield line analyses. Axial deformation is included in the XFEM formulation for an internal pin connection; however the material property for that formulation is linear elastic.

1.3 The novelty of the research in this thesis

The novelty of this thesis includes the following:

- The XFEM formulation is applied to capture the non-smooth displacement due to elasto-plastic bending behaviour
- The XFEM formulation for an internal pin connection is incorporated in a co-rotational frame for beam elements.
- The rotation and deflection displacement fields are enriched simultaneously, while the enrichment functions for the two displacement fields are different.
- Hermite functions are used in the regularized enrichment functions so that the strain field in an enriched element is continuous and the strain energy can be expressed naturally without introducing additional moment-curvature or moment-rotation relationship.
- Different from other XFEM formulations, the present enrichment functions are able to be activated automatically. Therefore, it is unnecessary to specify a particular criterion to activate the enrichment during analyses and tip enrichment functions are unnecessary for the present XFEM formulation.

1.4 The layout of this thesis

In Chapter 2, a literature review on the XFEM formulations is presented. The standard FEM formulation, on which the XFEM formulations are based, is

introduced in Chapter 3. In order to find a good enrichment function for a yield line, the XFEM formulation for an internal pin connection inside a beam element is provided in Chapter 4. In Chapter 5, a modification is made on the enrichment function in Chapter 4 to make it suitable for plastic hinge analyses. In Chapter 6, the idea of the enrichment function for a plastic hinge is extended further to the 2D space so that it is able to capture the non-smooth displacement resulted from a yield line. However, it is found that the enrichment function presented in Chapter 6 has several disadvantages. Hence, another enrichment function, which is constructed independent of the element shape functions, is presented in Chapter 7.

Several simple examples based on the XFEM formulations are illustrated in each chapter. These examples are used to show: (i) the effectiveness of the present XFEM formulation in capture the local non-smoothness resulted from plastic hinges and yield lines in displacement field, (ii) the stability of the present XFEM formulation and (iii) the potential ability of the present XFEM formulation for more complex applications in future research.

CHAPTER 2 LITERATURE REVIEW

2.1 The Ritz method and the finite element method

2.1.1 The Ritz method

The Ritz method is a widely used numerical method to find solutions of boundary value problems. The theoretical basis of the Ritz method is the principle of stationary potential energy. When using the Ritz method to find the displacement of a structure for a certain loading and boundary condition, a structure with infinite number of degrees of freedom (DOF) is abstracted to be a computational model with finite number of DOF. Instead of seeking the solution of the displacement of the structure the approach is to find an approximation of the displacement field for the discretized model. The displacement approximation is assumed *a priori*. For a single DOF problem, the approximated displacement field is a linear combination of a series of trial functions f_i and coefficients c_i :

$$u_h(x) = \mathbf{c}^T \mathbf{F} \quad (2.1)$$

where

$$\mathbf{c} = (c_1, c_2, \dots, c_i, \dots, c_n)^T \quad (2.2)$$

$$\mathbf{F} = (f_1, f_2, \dots, f_i, \dots, f_n)^T \quad (2.3)$$

In Equation (2.3), each f_i represents a deformation mode and all the deformation modes $f_1, f_2, \dots, f_n$ comprise a set, which is called the Ritz (trial) Space. In the Ritz space, each deformation mode is linearly independent of the others. In order to guarantee convergence of the solution, the Ritz space must be complete. The functions f_i can be chosen arbitrarily as long as they satisfy the essential boundary condition of the problem to be solved. The target of the Ritz method is to find the values of the coefficients c_i in Equation (2.2), which appportion the respective contribution of corresponding deformation modes f_i to the overall approximated displacement u_h . The total potential energy of the structure can then be expressed by the displacement approximation field:

$$\Pi = \int_{\Omega} \varepsilon D \varepsilon d\Omega - \sum_{i=1}^n P_i u_i - \int_S p ds - \int_{\Omega} q d\Omega \quad (2.4)$$

where ε is the strain field derived from the displacement approximation field u_h , D is the material property, P_i are the external loadings, u_i are the displacements corresponding to P_i , p is the surface force, S is the surface of the domain corresponding to p and q is the body force. The coefficients c_i of the displacement approximation field u_h can be solved by taking the first derivative of the total potential energy with respect to each of the coefficients c_i and set it to zero:

$$\frac{\partial \Pi}{\partial \mathbf{c}^T} = \left(\frac{\partial \Pi}{\partial c_1}, \frac{\partial \Pi}{\partial c_2}, \dots, \frac{\partial \Pi}{\partial c_i}, \dots, \frac{\partial \Pi}{\partial c_n} \right)^T = \mathbf{0}_{n \times 1} \quad (2.5)$$

The accuracy of the solution depends on the discrepancy between the trial functions f_i and the real displacement field for the computational model. If the exact solution is contained in the Ritz space, it can be reproduced in general. If the exact solution is not contained in the Ritz space, only an approximated solution can be obtained. The more trial functions are used in the linear combination, the more accurate solution u_h will yield.

2.1.2 The finite element method

In the Ritz method, the deformation modes f_i in the approximated displacement field u_h are defined over the whole physical domain of the boundary value problem. However, it is quite difficult to do so if the shape of the physical domain is complex. The FEM is more suitable than the Ritz method to solve the boundary value problem with a complex physical domain because the deformation modes are defined piecewise in FEM. As the most successful commercialized numerical approach, the finite element method can be regarded as a *piecewise* Ritz method (Zienkiewicz and Taylor 2005b).

Figure 2.1 The shape functions, nodal support and the elements

As shown in Figure 2.1, in an FEM analysis, a physical domain is divided into some compactly connected sub-domains. Each sub-domain is an element (element l, m, n, p, q, r, s, t). The scheme of a division of a physical domain is called a mesh scheme. One element connects to adjacent elements by sharing the common nodes along the edges shown in Figure 2.1, element l connects to element p by the common nodes: node i and node j , while element q connects to element l by the common nodes: node j and node k . The FEM is a piecewise Ritz method because the trial functions f_i are localized by a set of node-based functions. The node-based functions are only non-zero within a small area near its associated node and zero outside that area. In the FEM, the localized trial function associated with a node is called the *shape function* of the node. As shown in Figure 2.1, the shape function N_i , associated with node i , is non-zero in element l, m, n and p , but zero over the other elements in the physical domain. The small area where N_i is non-zero is called the support of node i . As shown in Figure 2.1, the area of element l, m, n and p is called the support of node i . The

nodal supports are overlapped (the shape functions N_i and N_j are overlapped in the area of element p and l).

The displacement of an arbitrary point is reproduced by a linear combination of shape functions which are non-zero at that point. As shown in Figure 2.1, the displacement at point A, u_A , can be reproduced by

$$u_A = N_g u_g + N_i u_i + N_j u_j + N_k u_k \quad (2.6)$$

where N_g , N_i , N_j , N_k are the shape functions associated with node g , i , j , k , respectively, and u_g , u_i , u_j and u_k are the coefficients of the linear combination. It can be seen that in the FEM, the displacement of an arbitrary point is interpolated by only a small part of the shape functions (localized trial functions). Thus, it is much easier to choose the trial functions f_i in the FEM. Furthermore, different from the Ritz method, the shape functions in the FEM are not required to satisfy the boundary condition. In FEM method, the essential boundary condition is satisfied by setting the boundary nodes to comply with the prescribed values at the global assembly level.

In order to guarantee monotonic convergence of the finite element solution, the displacement functions must be linear complete. In the meantime, an element in a mesh must be compatible with its adjacent elements. The accuracy of the FEM solution can be increased by a further refined mesh, if these two requirements are satisfied. The requirement of completeness implies that the displacement approximation field must be able to reproduce the rigid body

motion and constant strain deformation, while the requirement of compatibility implies that the displacement approximation field is continuous within the elements and between two connected elements.

2.2 Shear locking

2.2.1 Shear locking phenomenon

Locking phenomenon is a source of numerical unreliability in the FEM (Bathe 1995). Due to locking, some elements do not work well in some cases while they can provide acceptable results for other specific problems. Shear locking may appear in the analyses of bending-dominant cases when using displacement-based conforming thick beam / plate / shell elements.

Beam elements and plate elements are very widely used in finite element structural analyses. A beam element and a plate element abstract a 3D structural member to be a 1D and a 2D numerical model, respectively. Hence, a great amount of computational effort is saved by using beam elements and plate elements in a numerical analysis. By using Euler-Bernoulli theory for beam element or Kirchhoff theory for plate element, a complete and conforming beam / plate element is not difficult to be developed. The shape functions of an Euler-Bernoulli beam element or a Kirchhoff plate element can satisfy the requirement of completeness and compatibility very easily. However, the Euler-Bernoulli beam theory and the Kirchhoff plate theory are only valid for *thin*

structural members. When conducting an analysis for a beam or a plate with intermediate thickness, Timoshenko beam assumption and Reissner-Mindlin plate assumption must be used instead, in which the rotation of the cross-sections of structural members is regarded as an independent variable (Cook et al. 2002). However, when using displacement-based conforming thick beam / plate elements in an analysis, a fine mesh is required to obtain an acceptable result because of shear locking. As the mesh gets more refined, the ratio of the element size over the element thickness decreases, the shear locking phenomenon is somewhat alleviated but not completely overcome. The computational efficiency from the abstraction (a 3D structural member to be a 1D or a 2D numerical model) is counteracted by shear locking problem. Shear locking occurs when the thickness of beam elements or plate elements decreases. In an element with a very low thickness to length ratio, the parasitic shear strain energy will be unreasonably magnified and the elements behave more stiffly in shear deformation than expected (Huang 1987).

2.2.2 Reduced integration

A straightforward way to mitigate shear locking is to use a fine mesh. However, a fine mesh decreases the computational efficiency. An alternative way to alleviate shear locking in beam and plate elements is to use reduced integration (Zienkiewicz et al. 1971; Hughes et al. 1977; Hughes et al. 1978; Pugh et al. 1978). A “full” numerical integration is defined as the order that gives the exact matrices (i.e., the analytically integrated values) when the elements are

geometrically undistorted (Bathe 1995). Any integration scheme with a lower order than the full integration scheme is regarded as reduced integration. Although the full integration is strongly recommended by Bathe (1995), the reduced integration shows its own advantages. Besides the fact that computational effort can be saved by conducting reduced integration in analysis, this method is shown to be one of the most effective ways to remove shear locking.

Zienkiewicz *et al* (1971) first proposed the application of 2×2 integration scheme in an 8-node serendipity plate / shell element, which shows potential in solving shear locking problems. Later on reduced integration was also applied to solve other locking phenomena, such as volumetric locking (Naylor 1974; Doll et al. 2000; Reese et al. 2000) and membrane locking (Stolarski and Belytschko 1982).

Malkus and Hughes (1978) proves that a displacement-based 2-node beam element with reduced integration is equivalent to the mixed interpolated 2-node beam element, in which the displacement approximation fields are interpolated linearly and the shear strain field is assumed to be constant over the whole element. By using reduced integration scheme, Dvorkin *et al* (1988) developed a 3D 3-node Total Lagrangian beam element.

Although the application of reduced integration in beam element is proved successful, its application in plate element triggers zero energy modes (Parisch 1979). The zero energy modes (hourglass modes) are those deformation modes of an element which cannot be captured by reduced integration. A zero energy mode may appear in combination with another zero energy mode or a rigid body motion, and it could propagate in an assembly of elements. Only the elements without communicable zero energy modes are acceptable in finite element analyses. In linear analyses, an 8-node plane element with reduced integration has no communicable zero energy modes (Cook et al. 2002). However, it is reported by Borst and Nauta (1993) and Crisfield (1986) that in reinforced concrete analyses, the hourglass mode might propagate in an assembly of elements even in an 8-node plane element. In order to suppress the effect of zero energy modes, stabilization technique must be introduced into the finite element formulation with reduced integration (Kosloff and Frazier 1978; Belytschko et al. 1987; Liu et al. 1994; MacNeal 1994).

Besides the zero energy modes, the stability of the stiffness matrix is another concern. If the number of unknowns exceeds the number of independent relations supplied at all the integrating points, then the matrix must be singular (Zienkiewicz et al. 2005). In a 9-node plate element, as an example, there are totally 27 (3×9) DOF. However, a reduced integration (2×2 Gaussian integration scheme) can provide 20 (5×4) independent relations only. In a 2D 3-node beam element, there are 9 (3×3) DOF in one element, but only 4 (2×2)

independent relations can be provided by a reduced integration scheme. Therefore, the stiffness matrix is singular and ‘Escher’ mode could appear in structure response (Zienkiewicz and Taylor 2005b). In some cases, the ‘Escher’ mode could enlarge the displacement and destruct the numerical solution process. In addition, an unstable stiffness matrix also contains zero energy modes.

2.2.3 Assumed shear strain method

The assumed shear strain method is proved to be another effective approach to circumvent shear locking phenomenon. The first assumed shear strain element was proposed by Hughes and Tezduyar (Hughes and Tezduyar 1981) and MacNeal (1982). Park and his group (Park 1986; Park and Stanley 1986) developed the assumed natural strain element, in which the assumed strain components are constructed in natural coordinate system. The Mixed Interpolation of Tensorial Component (MITC) elements, proposed by Bathe and his research group (Dvorkin and Bathe 1984; Bucalem and Bathe 1997; Iosilevich et al. 1997; Bathe et al. 2000a; Bathe et al. 2000b; Lee and Bathe 2005; Lee et al. 2007; Kim and Bathe 2009; Lee and Bathe 2010; Bathe et al. 2011; Bathe and Lee 2011), are proved to be among the most successful series of assumed strain elements. The MITC technique was first proposed for four-node and eight-node quadrilateral shell elements (MITC4 and MITC8) by Dvorkin and Bathe (1984). Bucalem and Bathe (1993) applied the MITC technique in nine-node and sixteen-node shell elements, which are respectively

named as MITC9 and MITC16. Lee and Bathe (2004) proposed a series of triangular shell elements with the MITC technique (MITC3, MITC6a and MITC6b). Bathe *et al.* (1989) applied the MITC technique to plate elements and proposed MITC7 and MITC9 for quadratic triangular plate elements and quadratic quadrilateral plate elements.

In the MITC technique, the assumed shear strain field is interpolated by compatible shear strains at some ‘judiciously’(Lee and Bathe 2010) selected tying points. It is noted that the choices of the tying points in plate elements are different from those in shell elements. In a recent study (Lee and Bathe 2010), it is found that the MITC7 plate element performs better than the MITC6a shell element in plate bending tests, while the MITC9 shell element performs better or equal to the MITC9 plate element. The MITC plate elements (MITC7 and MITC9 plate) have a strong mathematical basis and have shown to be optimal in their convergence against the plate thickness in some numerical tests. However, both MITC9 and MITC7 plate elements have an internal node with rotation DOF only, which makes it difficult to extend the MITC plate elements to nonlinear analyses.

The Discrete Shear Gap (DSG) method is another assumed shear strain method. This method was originally proposed by Bletzinger *et al.* (2000). The term “shear gap” is defined as the deflection resulting from pure shear deformation

in a total deflection. The basic idea of the DSG method is to assume the transverse shear strain corresponding to the shear gap. One of the advantages of the DSG method is that this method can be easily applied to any displacement-based finite element with different shapes and order.

2.2.4 Enhanced assumed strain method

Besides the reduced integration method and the assumed strain method, the enhanced assumed strain method shows same potential to mitigate shear locking effectively. The enhanced strain method was first proposed by Simo and Rifai (1990). Different from the standard displacement-based finite element formulation, the enhanced assumed strain element is derived from a three field variational principle formulation (Hu-Washizu variational principle). The displacement, the strain and the stress in the element are assumed independently. The displacement is assumed as the standard displacement-based element. The strain field is enriched by an enhanced strain terms. The stress field is eliminated from the finite element formulation by introducing an orthogonality condition. Therefore, the basic unknowns for this formulation are the nodal displacements and the coefficients for the assumed strain field. Since it is unnecessary to guarantee the continuity of the assumed strain field between two adjacent elements, the unknowns associated with the assumed strain field can be eliminated on element level by static condensation. Hence, the basic

unknowns for the enhanced assumed strain method are the displacements as a standard displacement-based formulation.

Further derivation (Simo and Rifai 1990) shows that the enhanced assumed transverse strain field for a 4-node quadrilateral plate element is equivalent to that in the MITC4 (Dvorkin and Bathe 1984) shell element. Following the work of Simo and Rifai (1990), several other authors (Hueck et al. 1994; Crisfield and Moita 1996; Freischlager and Schweizerhof 1996; Korelc and Wriggers 1996; Nagtegaal and Fox 1996; Wriggers and Korelc 1996; Wriggers and Reese 1996; Yeo and Lee 1996; de Sciarra 2002; Bischoff et al. 2004; Schwarze and Reese 2009; Witkowski 2009; Braess et al. 2010; Polat 2010; Caylak and Mahnken 2011; Schwarze and Reese 2011) have also developed similar element formulations for small strain applications.

2.3 The extended finite element method

2.3.1 Non-smooth approximation field

Many examples with non-smooth solution field can be found in the real world. Non-smoothness in an approximation field can be explained as a rapid change in the field quantity within a finite or zero length Δl . Three types of non-smoothness can be found in engineering application (Fries and Belytschko 2010), which are 1) a rapid change in the field quantity within a zero length ($\Delta l = 0$), 2) a rapid change in the field quantity within a finite length but can be

idealized as zero ($\Delta l \approx 0$) and 3) a rapid change in the field quantity within a finite length and the length *cannot* be idealized as zero ($\Delta l \neq 0$). The non-smoothness of type 1 and type 2 can be identified as a discontinuity while the non-smoothness of type 3 is associated with the term ‘high gradient’.

Two different kinds of discontinuities can be categorized: strong discontinuities and weak discontinuities (Sukumar et al. 2001). Strong discontinuities are those discontinuities in which the quantity of the approximation field is discontinuous, for example, when a crack cuts through an element, the displacement field normal to the interface of the crack is discontinuous, while weak discontinuities are those in which the gradient of the approximation field is discontinuous, for example, the gradient of a displacement field across an interface between two materials is discontinuous.

Despite the success of the standard FEM in a large scale of engineering application, its application in engineering analyses with non-smooth approximation field is not as satisfactory as expected. When the standard FEM is employed to model the non-smoothness, a remeshing technique is usually required during analyses or a fine mesh must be generated before analyses. No matter which option is chosen, a high cost in computational effort is inevitable.

Some meshfree methods is able to tackle the problems with non-smooth displacement such as the Smoothed Particle Hydrodynamics (SPH) (Gingold

and Monaghan 1977), the Element-free Galerkin method (EFGM) (Belytschko et al. 1994), the Reproducing Kernel Particle Method (RKPM) (Liu et al. 1995). In both of the Meshfree method, some nodes scatter the problem domain and these nodes do not form a mesh. The displacement field is approximated by interpolation of nodal values. The interpolation functions are associated with the nodes, which can contain a non-smooth part within its domain of definition. Despite their advantages in tackling problems with non-smooth displacement, the meshfree methods require more computational effort than standard FEM. Although mesh scheme and elements do not appear in meshfree methods, a grid is still needed for numerical integration. In the meantime, as some of the interpolation functions associated with the nodes not located on the physical boundary are not zero, the essential boundary condition must be set with additional computational effort.

It is found that by adding some *enrichment* into the approximation field, the solution of the numerical method can be significantly improved (Melenk and Babuska 1996) at a relatively low computational cost. In order to obtain the advantages of the meshfree method in capturing locally non-smooth displacement without increasing computational effort too much, Belytschko and Black (Belytschko and Black 1999) proposed the extended finite element method (XFEM). In the XFEM, the enrichment is introduced into the standard FEM formulation. Comparing with the Meshfree method, the XFEM formulation is easy to be implemented in the widely used FEM code.

2.3.2 The enrichment

In order to capture the non-smoothness in an approximation field with minimum effort, some enrichment is added into the trial function space. The solution field is enriched as

$$u_h = \sum_i N_i u_i + F a = u_{\text{FEM}} + u_{\text{enrichment}} \quad (2.7)$$

where u_h is the approximation field, N_i are the standard shape functions for the standard FEM analysis, u_i are the nodal value for the standard FEM analysis, F is the enrichment function, a is the additional DOF corresponding to the enrichment function and $u_{\text{enrichment}}$ is the enrichment for the approximation field. Some major research works on the Partition of Unity (PU) enriched formulations were reported in (Areias and Belytschko 2006; Rabczuk et al. 2007a; Rabczuk et al. 2010; Chau-Dinh et al. 2012). The PU is satisfied by the condition:

$$\sum_i N_i = 1 \quad (2.8)$$

or in a more general form:

$$\sum_i N_i F_i = F \quad (2.9)$$

where F is the function to be reproduced.

2.3.2.1 The enrichment function for a discontinuity

In numerical simulations, the enrichment function (F in Equation (2.7)) in an approximation field is selected by *a priori* knowledge. For a strong discontinuity, a step function is employed as the enrichment function, as shown in Figure 2.2. The discontinuity is represented by a sudden jump in the step function:

$$F(x) = \begin{cases} -1 & x < x_0 \\ 0 & x = x_0 \\ 1 & x > x_0 \end{cases} \quad (2.10)$$

Figure 2.2 The plot of a step function for strong discontinuities

For a weak discontinuity, an absolute level set function is employed as the enrichment function, as shown in Figure 2.3. The kink of the function represents the discontinuity in the gradient of the approximation field:

$$F(x) = |\varphi(x)| = |x - x_0| \quad (2.11)$$

where $\varphi(x)$ is the level set function.

Figure 2.3 The plot of level set function for weak discontinuities

In the first paper on the XFEM formulation by Belytschko and Black (1999), the role of discontinuous enrichment in virtual work principle is presented. The introduction of a discontinuous enrichment into a displacement approximation field can be regarded as adding a free traction boundary condition to the system.

Proof by Belytschko and Black (1999)

As shown in Figure 2.4, Ω represents a physical domain, Γ_t is the traction boundary, Γ_d is the displacement boundary and Γ_c is the crack. $\Gamma_c = \Gamma_{c^+} \cup \Gamma_{c^-}$.

Figure 2.4 A physical domain with a crack

The virtual work principle is expressed as

$$\int_{\Omega} \nabla_s \mathbf{v} : \boldsymbol{\sigma} d\Omega - \int_{\Gamma_t} \mathbf{v} \cdot \bar{\mathbf{t}} d\Gamma = 0, \text{ for } \forall \mathbf{v} \quad (2.12)$$

where $\mathbf{v}$ is an admissible infinitesimal displacement, $\boldsymbol{\sigma}$ is the stress field in the physical domain and $\bar{\mathbf{t}}$ is the external force applied on the physical domain, assuming that there is no body force. The admissible infinitesimal displacement function $\mathbf{v}$ is the virtual function. The test function is taken as the same as the virtual function. After applying divergence theorem on the first term of the left hand side in Equation (2.12), the virtual work principle is rewritten as

$$-\int_{\Omega-\Gamma_c} \mathbf{v} \cdot \nabla \cdot \boldsymbol{\sigma} d\Omega - \int_{\Gamma_t} \mathbf{v} \cdot (\bar{\mathbf{t}} - \boldsymbol{\sigma} \mathbf{n}) d\Gamma + \int_{\Gamma_{c^+}} \mathbf{v} \cdot \boldsymbol{\sigma} \mathbf{n} d\Gamma + \int_{\Gamma_{c^-}} \mathbf{v} \cdot \boldsymbol{\sigma} \mathbf{n} d\Gamma = 0, \text{ for } \forall \mathbf{v} \quad (2.13)$$

First, if $\mathbf{v}$ is chosen as zero on $\Gamma_t \cup \Gamma_c$, therefore

$$\int_{\Omega-\Gamma_c} \mathbf{v} \cdot \nabla \cdot \boldsymbol{\sigma} d\Omega = 0 \quad (2.14)$$

If $\mathbf{v}$ is next chosen to be non-zero on Γ_t , therefore

$$\int_{\Gamma_t} \mathbf{v} \cdot (\bar{\mathbf{t}} - \boldsymbol{\sigma} \mathbf{n}) d\Gamma = 0 \quad (2.15)$$

Hence, Equation (2.13) becomes

$$\int_{\Gamma_{c^+}} \mathbf{v} \cdot \boldsymbol{\sigma} \mathbf{n} d\Gamma + \int_{\Gamma_{c^-}} \mathbf{v} \cdot \boldsymbol{\sigma} \mathbf{n} d\Gamma = 0 \quad (2.16)$$

Finally, if $\mathbf{v}$ is chosen to be continuous and non-zero on Γ_{c^+} and zero on Γ_{c^-} ,

Equation (2.16) can be further simplified as

$$\int_{\Gamma_{c^+}} \mathbf{v} \cdot \boldsymbol{\sigma} \mathbf{n} d\Gamma = 0, \text{ for } \forall \mathbf{v} \quad (2.17)$$

Thus, it can be concluded that $\boldsymbol{\sigma} \mathbf{n} = 0$ on Γ_{c^+} , which means the traction on the crack faces is zero.

End of the proof

2.3.2.2 The enrichment function for a high gradient zone

Regularized enrichment functions (Benvenuti 2008; Benvenuti et al. 2008) are employed to approximate the non-smoothness within a short range. A transition function is used to connect the two parts near the high gradient zone, as shown in Figure 2.5. The functions used in the high gradient zone depend on the properties of the solution.

(a) High gradient in the quantity of approximation field

(b) High gradient in the first derivative of the approximation field
Figure 2.5 The plot of enrichment for high gradient zone

The role of the enrichment function for a high gradient zone is equivalent to the expansion of the Ritz space. As shown in Figure 2.6, the plate structure domain Ω is bounded by Γ . The traction $\bar{\mathbf{t}}$ is defined on the boundary Γ_t and the displacement $\bar{\mathbf{u}}$ defined on the boundary Γ_u . The sub-domain Ω_{ns} denotes the high gradient zone. Given the spaces

$$\mathcal{U}_0 = (\mathbf{u} \in C_0, \mathbf{u} = 0 \text{ on } \Gamma_u, \mathbf{u} \text{ is of high gradient within } \Omega_{ns}) \quad (2.18)$$

and

$$\mathcal{U} = (\mathbf{u} \in C_0, \mathbf{u} = \bar{\mathbf{u}} \text{ on } \Gamma_u, \mathbf{u} \text{ is of high gradient within } \Omega_{ns}) \quad (2.19)$$

The equilibrium equation is solved by finding $\mathbf{u} \in \mathcal{U}$,

$$\int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{v})^T \cdot \boldsymbol{\sigma}(\boldsymbol{\varepsilon}(\mathbf{u})) d\Omega - \int_{\Omega} \mathbf{g} \cdot \mathbf{v} d\Omega - \int_{\Gamma_t} \bar{\mathbf{t}} \cdot \mathbf{v} d\Gamma = 0, \quad \forall \mathbf{v} \in \mathcal{U}_0 \quad (2.20)$$

where $\boldsymbol{\sigma}$ is the Cauchy stress and $\mathbf{g}$ is the body force. Small strain assumption is employed and the strain $\boldsymbol{\varepsilon}$ is expressed as

$$\boldsymbol{\varepsilon} = \nabla_s \mathbf{u} \quad (2.21)$$

Figure 2.6 Domain definition

where ∇_s is the symmetric gradient operator.

$$\nabla_s \mathbf{u} = \frac{1}{2} (\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla) \quad (2.22)$$

2.3.3 The partition of unity condition

The enrichment is realized through the PU (Partition of Unity) condition proposed by Babuska (1994). PUFEM (Melenk and Babuska 1996; Babuska and Melenk 1997) was proposed based on the concept of PU condition. An enrichment based on the global domain is added into the approximation field, for example, harmonic polynomials for the Laplace and the Helmholtz equations, and holomorphic functions for linear elasticity problems.

In the XFEM, enrichment is introduced into the standard finite element displacement approximation field

$$u = \sum_i N_i (u_i + F a_i) = \sum_i N_i u_i + \sum_i F N_i a_i \quad (2.23)$$

where $N_i u_i$ is the standard finite element displacement approximation field, F is the enrichment function, which is able to describe the locally non-smooth displacement, and a_i is the additional DOF of node i corresponding to the enrichment function F .

Figure 2.7 The window functions for the localization of enrichment

Compared with PUFEM, the largest improvement of the XFEM is that the enrichment is only activated at the subdomain where discontinuities occur and is suppressed to zero elsewhere. The enrichment is localized by window functions: only the nodes supported by the elements, through which discontinuity passes, are enriched. Thus, the computational effort is reduced tremendously. An example of window functions is shown in Figure 2.7, where a discontinuity is located at x_0 . The 2-node element k is an enriched element, where the corresponding node i and node $i+1$ are the enriched nodes. The shape functions associated with node i and node $i+1$ are the window functions.

Due to the localization, three types of elements can be categorized: 1) enriched element, 2) blending elements and 3) standard element. Enriched elements are those elements with all its nodes enriched. Blending elements are those elements in which only a part of its nodes are enriched. Standard elements are those elements in which none of the nodes are enriched. As shown in Figure 2.7, Element k is the enriched element, Element $k-1$ and Element $k+1$ are the blending elements and Element $k-2$ is the standard element.

Although the enrichment is suppressed in most of the elements in the physical domain by localization, it is activated in blending elements. This activation of enrichment in blending elements leads to some problems in XFEM formulations (Fries 2008): first, the PU condition cannot be satisfied, which means that the enrichment function is not able to be reproduced in blending elements, for example in element $k-1$ in Figure 2.8, the summation of the interpolation functions for the enrichment is expressed as

$$\sum_{l=i-1}^i N_l F_l = N_i F \neq F \quad (2.24)$$

Second, some unwanted terms are introduced into the approximation field in blending elements. The unwanted terms cannot be eliminated by the approximation field of standard FEM (Chessa et al. 2003). For example, in element $k+1$ in Figure 2.8, the approximation field is written as

$$u = N_{i+1}u_{i+1} + N_{i+2}u_{i+2} + N_{i+1}Fa_{i+1} \quad (2.25)$$

The unwanted term comes from $N_{i+1}F$ and it cannot be eliminated by the standard finite element approximation N_{i+1} or N_{i+2} .

(a) The localized enrichment function for node i

(b) The localized enrichment function for node $i+1$

Figure 2.8 The localized enrichment

A straightforward way (Chessa et al. 2003) to eliminate the influence from the unwanted terms is to use enhanced assumed strain method (described in Section 2.2.3 in this thesis) proposed by Simo and Rifai (1990). Some enhanced higher order terms are assumed in the standard finite element approximation field to nullify the effect of unwanted terms resulting from the enrichment in blending elements.

Zi and Belytschko (2003) and Moës *et al* (2003) shifted enrichment functions in XFEM formulations to solve the problems in blending elements. A shifted enrichment function for a strong discontinuity is formulated as

$$F_i(x) = F(x) - F(x_i) \quad (2.26)$$

whereas for a weak discontinuity, the shifted enrichment function is formulated as

$$F = \sum_i N_i |\varphi_i| - \left| \sum_i N_i \varphi_i \right| \quad (2.27)$$

After localizing by window functions, the shifted enrichment function is suppressed automatically outside the enriched elements. Thus, the two problems from blending elements do not appear. An example of localized shifted enrichment function for a strong discontinuity is shown in Figure 2.9.

(a) The localized shifted enrichment function for node i

(b) The localized shifted enrichment function for node $i+1$
Figure 2.9 The localized shifted enrichment function

Fries (2008) proposed a way to correct the XFEM formulation in blending elements, in which the enrichment function is modified as

$$F_{\text{mod}} = F \cdot R \quad (2.28)$$

where F is the original enrichment function, R is a ramp function and can be expressed as

$$R = \sum_{i \in I^*} N_i \quad (2.29)$$

where I^* is the set of enriched nodes before correction. It can be seen from Figure 2.10 that the ramp function R only modifies the enrichment function in blending element and has no influence on the enrichment function in enriched elements.

Figure 2.10 The plot of the ramp function $R(x)$

The corrected XFEM formulation can solve the problems in blending elements effectively, for example, the approximation field in the blending element $k+1$ is modified as

$$u = u_{\text{FEM}} + N_{i+1}F_{\text{mod}}a_{i+1} + N_{i+2}F_{\text{mod}}a_{i+2} = u_{\text{FEM}} + N_{i+1}FN_{i+1}a_{i+1} + N_{i+2}FN_{i+1}a_{i+2} \quad (2.30)$$

It can be seen from Equation (2.30) that node $i+2$ becomes an enriched node in the modified XFEM approximation field. However, the value of the modified enrichment function is zero because of the well-known Kronecker condition $N_{i+1}(x_{i+2}) = 0$. On the other hand, the PU condition can be satisfied in the blending elements:

$$\sum_{l=i+1}^{i+2} N_l F_l = N_{i+1}F_{\text{mod}} + N_{i+2}F_{\text{mod}} = F_{\text{mod}} \quad (2.31)$$

while in element $k-1$ in Figure 2.8, the enrichment function is modified as

$$F_{\text{mod}} = FN_{i+1} \quad (2.32)$$

It is obvious that in the enriched element k , the enrichment function is kept the same form as the original enrichment function.

The disadvantage of this corrected XFEM formulation is that as the order of the enrichment function is increased by the modification, more integration points are required in numerical integration. On the other hand, more nodes are enriched, which means computational cost increases in the corrected XFEM. For example, node $i-1$ and node $i+2$, in Figure 2.10, are not enriched before correction, but they become enriched nodes after correction.

2.3.4 The XFEM in engineering applications

The XFEM is first proposed by Belytschko and Black (1999) and Moës *et al.* (1999) to model crack growth inside a plane element. A crack inside a plane element is regarded as a discontinuity with zero length. The step function, which generates a strong discontinuity in a displacement field, is added into the displacement approximation field as an enrichment function to model the crack development. Wells and Sluys (2001) employed a jump function as the enrichment function for cohesive crack. Moes and Belytschko (2002) adopted a jump function for the part of the cohesive crack not adjacent to its tip, and a branch function adjacent to the tip. Zi and Belytschko (2003) enriched all cracked elements including the elements containing the crack tip by a sign function. Mariani and Perego (2003) modelled the tip of a cohesive crack by using a cubic displacement discontinuity. Asadpoure *et al.* (2006) proposed near-tip enrichment functions to model a crack tip in orthotropic media. Arias and Belytschko (2006) adopted a hyperbolic tangent function as shear band enrichment, which varies rapidly in one direction and slowly in the other, to

model shear band growth. Rabczuk *et al.* (2007b) and Rabczuk and Samaniego (2008) studied the behaviour of a shear band by using a strong discontinuous enrichment function. In their work, a cohesive constitutive law is employed to simulate the dissipation energy in the shear band. Benvenuti *et al.* (2008) adopted a regularized model for discontinuity with a conspicuous finite length. Sukumar *et al.* (2001) introduced a level set function (Osher and Sethian 1988), which is discontinuous in a displacement gradient field, to model interface of two materials inside an element.

Areias and Belytschko (2005b) applied XFEM formulation in Mindlin-Reissner shell formulation to trace arbitrary crack propagations in shell structures with both geometrical and material nonlinearities. A modified enrichment function is used in their research due to the non-additive rotational DOF and a shift in location of discontinuity is needed when it passes through an element node. Anahid and Khoei (2008) applied XFEM formulation in elasto-plastic large deformation analyses, in which Total Lagrangian formulation is used with XFEM enrichment functions to capture the interface of two elasto-plastic material. Khoei *et al.* (2008) extended the work by Anahid and Khoei (2008) from two-dimensional to three-dimensional.

Up to now, many efforts have been done on engineering applications of the XFEM such as propagations of cracks (Stazi *et al.* 2003; Areias and Belytschko 2005b; Areias and Belytschko 2005a), crack growth with frictional contact

(Dolbow et al. 2001), dislocations (Belytschko and Gracie 2007), interfaces of two different materials (Sukumar et al. 2001; Hettich and Ramm 2006), delamination in shell structures (Nagashima and Suemasu 2010), functionally graded materials (Dolbow and Gosz 2002; Natarajan et al. 2011), multi-phase flow (Chessa and Belytschko 2003) and plastic hinges (Xu et al. 2012). More applications of XFEM can be found in the review by Abdelaziz and Hamouine (2008)

2.4 The special model for plastic hinge and yield line analyses

Plastic hinge analysis is commonly used in frame analyses and pile foundation analyses (Chiou et al. 2008). Corradi and Poggi (1984) proposed a refined finite element model for elastio-plastic frame analyses. In the refined finite element model, plastic strain is regarded as an independent variable. An additional relationship between axial force and bending moment is required to consider the N - M interaction. Izzuddin and Elnashai (1993a) (1993b) proposed two different types of plastic hinge models: concentrated plastic hinge model and distributed plastic hinge model. The concentrated model treats a plastic hinge as a point with zero length and extracts the strain energy in the plastic hinge by an explicit expression based on force-displacement relationship, such as moment-rotation and moment-curvature. The actual length of the plastic hinge is defined by users in advance for the program to find the plastic rotation or curvature.

Instead of input a single point as a plastic hinge, in the distributed plastic hinge model, a series of points are placed in the range of an expected yield zone. The program is able to detect the length of the plastic hinge by checking the stress state of each inserted points. Similar to the concentrated model, the relationship between the moment and rotation or moment and curvature is essential in the distributed model. In both of the two plastic hinge models, the relationship between the dissipation energy and the displacement is required to be expressed explicitly. Armero and Ehrlich (2006b) proposed a computational framework to solve numerical instability due to strain softening material. In their method, a strong discontinuous function is added into the axial and deflection displacement field. A cohesive law based on strain and stress resultant is employed. In the subsequent work (Ehrlich and Armero 2005; Armero and Ehrlich 2006a), they extended this idea from beam elements to plate elements.

Table 2.1 The comparison on the plastic hinge models and XFEM formulation

	concentrated model	distributed model	XFEM
location of the plastic hinge	☼	☼	☼
length of the plastic hinge	☼	--	☼
dissipation energy	☼	☼	--
non-smooth approximation	--	--	☼

☼: required

--: unnecessary

Compared with the existing plastic hinge model, the XFEM formulation provides a convenient way to model a plastic hinge. Table 2.1 lists the

comparison of the plastic hinge model and the XFEM formulation in terms of several computational aspects.

Compared with plastic hinge models, yield line models are much difficult to propose. Besides that the constitutive relationship is in 2D space in yield line model, it is not easy to find a relationship of moment-curvature or moment-rotation for a plate. The first yield line analysis was performed by Ingerslev (1923), in which an analysis for a simply supported rectangular plate is carried out by ‘normal moment method’. The normal moment method employs the equilibrium condition between loading and bending moment. The normal moment method is nearly abandoned as it is restricted to the particular case in which only yield lines of the same sign meet at a point (Quintas 2003). Johansen (Johansen 2004) proposed the work method for yield line analysis. In the work method, the principle of virtual work is applied to the collapse mechanism of certain yield patterns. Currently, the work method is the most popular approach for yield line analysis. Quintas (2003) proposed a new method for yield line analysis, which is called skew moment method. In the skew moment method, both bending moment and twisting moment are considered acting on a yield line. Quintas (2003) proved that the widely-used work method is a special case of the skew moment method. A numerical model based on the work method is proposed by Bauer and Redwood (Bauer and Redwood 1987). In this model, a plate is partitioned into several rigid body segments and rigid plastic model is used as the constitutive relationship.

Therefore, the elastic response of the plate at the initial stage is not available by using this model and it cannot simulate the full process of plate yielding. Two main advantages make it convenient to extend the XFEM formulation to yield line analysis. Firstly, the relationship between the dissipation energy and the displacement is unnecessary in analyses. The constitutive relationship for an elasto-plastic material can be employed directly inside a plastic hinge. The strain energy inside the plastic hinge is simulated in the same way as that in elastic parts. As it is shown in Section 2.3.2.2, the principle of virtual work is satisfied in the XFEM formulation. This advantage is magnified when applying the XFEM formulation in yield line analyses, since it is not easy to find an explicit expression of strain energy in terms of displacement variables in plate structures. Secondly, no additional criteria are needed to check when to introduce the XFEM formulation during an analysis. As the appearance of local non-smoothness in displacement field is a gradual process. It is not easy to specify a critical stage to introduce the XFEM formulation into analysis. The XFEM enrichment is added into the displacement approximation field at the beginning of the analyses. The XFEM formulation is able to be activated by itself when it is detected to be necessary. This feature ensures the avoidance of any additional technique for the tip close of a yield line. The close-up of the enrichment at yield line tip can be achieved by adjusting the value of the additional DOF, while the adjustment is also conducted by the program itself according to the equilibrium condition.

CHAPTER 3 THE FORMULATIONS FOR STANDARD FINITE ELEMENTS FOR ELASTO-PLASTIC ANALYSIS

3.1 Introduction

In this chapter, the formulations of the standard finite elements including a 2D 3-node co-rotational Timoshenko beam element with linear material property, a 2D 3-node Timoshenko beam element, a 6-node triangular Mindlin plate element and a 9-node quadrilateral Mindlin plate element, are presented. It is noted that for the last three elements, linear kinematics with elastic-perfectly-plastic material property are employed. Some assumptions are stated as follows:

- (1) The cross-section of a beam element or a plate element is initially a plane and it remains a plane after deformation
- (2) Transverse shear strain and the transverse shear stress are assumed uniformly distributed across the cross-section of an element.
- (3) The infinitesimal strain assumption is applied for all the elements.
- (4) The beam element is initially straight.
- (5) In the elasto-plastic constitutive relationship, transverse shear stress is excluded from the yield criteria and the flow rule.

In order to capture the nonlinear response of the structure, the Generalized Displacement Control method (Yang and Shieh 1990) is adopted.

3.2 The 2D 3-node beam element

In this section, a 2D 3-node beam element is introduced. As shown in Figure 3.1, there are 2 degrees of freedom (DOF) per node in the beam element, $\mathbf{u}_i = (w_i, \theta_i)^T$. The displacement approximation field w_h and θ_h is expressed as

$$w_h = \sum_{i=1}^3 N_i w_i \quad (3.1)$$

$$\theta_h = \sum_{i=1}^3 N_i \theta_i \quad (3.2)$$

$$N_1 = -0.5\xi(1-\xi), \quad N_2 = (1-\xi)(1+\xi), \quad N_3 = 0.5\xi(1+\xi) \quad (3.3)$$

The Lagrangian shape functions N_i and the natural coordinate system are plotted in Figure 3.2(a) and Figure 3.2(b), respectively.

Figure 3.1 The DOF of a 3-node beam element

(a) The Lagrangian shape function N_i

(b) The natural coordinate system

Figure 3.2 The Lagrangian shape function N_i and the natural coordinate system

In order to trace the entire yielding process more accurately, a layered model (in Figure 3.3) is adopted. The kinematic equations for the 3-node 2D beam element are expressed as

$$\varepsilon_j = z_j \chi = z_j \sum_i \frac{dN_i}{dx} \theta_i \quad (3.4)$$

$$\gamma = -\sum_i N_i \theta_i + \sum_i \frac{dN_i}{dx} w_i \quad (3.5)$$

where ε_j and γ are the normal strain and the shear strain for the j^{th} layer. χ is the bending strain, respectively. z_j is the distance between the reference surface and the middle surface of the j th layer.

Figure 3.3 The layered model for beam element

3.3 The 2D 3-node co-rotational beam element

3.3.1 The Co-rotational frame

In the co-rotational approach, a large deformation is separated into two parts: a rigid body motion and a pure deformation. As shown in Figure 3.4, (a) is the initial configuration, (c) is the deformed configuration and (b) is an intermediate configuration. The initial configuration (a) is transferred to an intermediate configuration (b) by a rigid body motion. From (b) to (c) there is a pure deformation. It should be noted that the intermediate configuration (b) is not physically unique.

Figure 3.4 Co-rotational approach for a 2D beam element

In order to express the rigid body motion of an element, two Cartesian coordinate systems are defined in the co-rotational approach: the global coordinate system OXY , which is unique for the whole computational model and local coordinate system oxy , which co-rotates with the element. The local x and y axes are associated with unit vectors $\mathbf{e}_x$ and $\mathbf{e}_y$, respectively, in each element. In the present formulation, the element coordinate system is tied to the mid-node of the element, as shown in (b). The initial directions of the axes, ${}^0\mathbf{e}_x$ and ${}^0\mathbf{e}_y$ are defined as:

$${}^0\mathbf{e}_x = \frac{{}^0\mathbf{X}_3 - {}^0\mathbf{X}_1}{|{}^0\mathbf{X}_3 - {}^0\mathbf{X}_1|}, \quad {}^0\mathbf{e}_y = \begin{pmatrix} -{}^0\mathbf{e}_{xY} & {}^0\mathbf{e}_{xX} \end{pmatrix}^T \quad (3.6)$$

where ${}^0\mathbf{X}_1$ and ${}^0\mathbf{X}_3$ are respectively the initial position vectors of node 1 and node 3 in an element; ${}^0\mathbf{e}_{xX}$ and ${}^0\mathbf{e}_{xY}$ are the X- and Y-components of ${}^0\mathbf{e}_x$, respectively. ${}^0\mathbf{e}_x$ and ${}^0\mathbf{e}_y$ are the initial local axes before deformation. The local axes in the deformed configuration (c) can be found by

$$\begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{pmatrix} = \begin{bmatrix} \cos\psi_2 & -\sin\psi_2 \\ \sin\psi_2 & \cos\psi_2 \end{bmatrix} \begin{pmatrix} {}^0\mathbf{e}_x \\ {}^0\mathbf{e}_y \end{pmatrix} \quad (3.7)$$

where ψ_2 is the global rotation displacement of node 2. The translation and rotation of a local coordinate system define the rigid body motion.

3.3.2 Displacements in the global and the element coordinate system

In the standard CR beam element, three DOF are defined at each node. The global displacements of an element $\mathbf{U}_{eG}$ are expressed as:

$$\mathbf{U}_{eG} = (U_1, W_1, \psi_1 | U_2, W_2, \psi_2 | U_3, W_3, \psi_3)^T \quad (3.8)$$

where $U_i, W_i (i = 1, 2, 3)$ are the two translational DOF and $\psi_i (i = 1, 2, 3)$ is the rotational DOF in the global X-Y coordinate system. The element displacement approximation $\mathbf{U}_{eh}$ is of the form

$$\mathbf{U}_{eh} = \begin{pmatrix} U_{eh} \\ W_{eh} \\ \psi_{eh} \end{pmatrix} = \begin{pmatrix} N_i U_i \\ N_i W_i \\ N_i \psi_i \end{pmatrix} \quad (3.9)$$

where N_i are the Lagrangian shape functions for a 3-node beam element and they are expressed in Equation (3.3); $\mathbf{U}_i = [U_i, W_i, \psi_i]$ is the nodal global displacement.

According to the definition of the local coordinate system in the present formulation, the local displacement of the mid-side node of each element is *zero during analyses*; therefore, the local displacement of an element $\mathbf{u}_{eL}$ can be expressed as:

$$\mathbf{u}_{eL} = (u_1, w_1, \theta_1 | u_3, w_3, \theta_3)^T \quad (3.10)$$

where u_i and w_i ($i = 1, 3$) are the two translational DOF in the local coordinate system and θ_i is the rotational DOF in the local coordinate system, as shown in Figure 3.5

Figure 3.5 The local coordinate system of the beam element

The local displacement approximation $\mathbf{u}_h$ is of the form:

$$\mathbf{u}_h = \begin{pmatrix} u_h \\ w_h \\ \theta_h \end{pmatrix} = N_1 \begin{pmatrix} u_1 \\ w_1 \\ \theta_1 \end{pmatrix} + N_3 \begin{pmatrix} u_3 \\ w_3 \\ \theta_3 \end{pmatrix} \quad (3.11)$$

where N_i is given in Equation (3.3). $\mathbf{u}_i = [u_i, w_i, \theta_i]^T$ is the nodal displacement in the local coordinate system.

During analyses, the local displacement for each element is obtained by filtering out the rigid body motion from the global displacement. The relationship between the local nodal and the global nodal displacement can be expressed by (Li 2007):

$$\begin{pmatrix} u_i \\ w_i \end{pmatrix} = \mathbf{R} \left(\begin{pmatrix} U_i \\ W_i \end{pmatrix} - \begin{pmatrix} U_2 \\ W_2 \end{pmatrix} \right) + (\mathbf{R} - {}^0\mathbf{R}) ({}^0\mathbf{X}_i - {}^0\mathbf{X}_2), \quad i=1,3 \quad (3.12)$$

$$\theta_i = \psi_i - \psi_2, \quad i=1,3 \quad (3.13)$$

where ${}^0\mathbf{X}_i$ is the position vector of node i in the initial configuration in the global coordinate system. $\mathbf{R} = [\mathbf{e}_x^T, \mathbf{e}_y^T]^T$ is the rotation matrix.

3.3.3 The stiffness matrix and the internal force vector

The strain in an arbitrary layer of an element $\boldsymbol{\varepsilon}_j$ is decoupled into three parts: membrane strain ε_m , shear strain γ and bending strain (curvature) χ . They are expressed as:

$$\boldsymbol{\varepsilon}_j = \begin{Bmatrix} \varepsilon \\ \gamma \end{Bmatrix} = \begin{Bmatrix} \varepsilon_m \\ \gamma \end{Bmatrix} + z_j \begin{Bmatrix} \chi \\ 0 \end{Bmatrix} \quad (3.14)$$

$$\varepsilon_m = \frac{\partial u_h}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial u_h}{\partial x} \right)^2 + \left(\frac{\partial w_h}{\partial x} \right)^2 \right] \quad (3.15)$$

$$\chi = -\frac{\partial \theta_h}{\partial x} \quad (3.16)$$

$$\gamma = -\theta_h + \frac{\partial w_h}{\partial x} \quad (3.17)$$

where z_j is the distance from the middle surface of the j^{th} layer to the reference surface, as shown in Figure 3.3.

The potential energy Π is expressed as:

$$\Pi = \frac{1}{2} \sum_{j=1}^n \int (\varepsilon_m + z_j \chi) D_{11} (\varepsilon_m + z_j \chi) dV + \frac{1}{2} \sum_{j=1}^n \int \gamma D_{22} \gamma dV - W \quad (3.18)$$

where n is the total number of layers in an element; $\mathbf{D} = \text{diag}(D_{11}, D_{22})$ is the consistent material matrix, for layers in which the material is elastic:

$$\mathbf{D} = \begin{pmatrix} E & 0 \\ 0 & kG \end{pmatrix} \quad (3.19)$$

In Equation (3.19), E is the Young's modulus, G the shear modulus and k is the shear correction factor for layered models and $k=5/6$.

The internal force with respect to the local coordinate system $\mathbf{f}_L$ is expressed as:

$$\mathbf{f}_L = \sum_{j=1}^n \int (\mathbf{B}_m + z_j \mathbf{B}_b)^T D_{11} (\varepsilon_m + z_j \chi) dV + \sum_{j=1}^n \int \mathbf{B}_\gamma^T D_{22} \gamma dV \quad (3.20)$$

where $\mathbf{B}_m$, $\mathbf{B}_b$ and $\mathbf{B}_\gamma$ are the strain-displacement matrices of the membrane part, the bending part and the shear part, respectively. The detailed expressions of the matrices $\mathbf{B}_m$, $\mathbf{B}_b$ and $\mathbf{B}_\gamma$ are included in Appendix A. The stiffness matrix in the local coordinate system $\mathbf{K}_L$ is expressed as:

$$\mathbf{K}_L = \sum_{j=1}^n \int \left(\mathbf{B}_m^T D_{11} \mathbf{B}_m + z_j^2 \mathbf{B}_b^T D_{11} \mathbf{B}_b + \frac{\partial \mathbf{B}_m^T}{\partial \mathbf{u}} D_{11} \varepsilon_m \right) dV + \sum_{j=1}^n \int \mathbf{B}_\gamma^T D_{22} \mathbf{B}_\gamma dV \quad (3.21)$$

The internal force vector with respect to the global coordinate system $\mathbf{f}_G$ is expressed as:

$$\mathbf{f}_G = \frac{\partial \Pi}{\partial \mathbf{u}_G^T} = \left(\frac{\partial \mathbf{u}_L}{\partial \mathbf{u}_G^T} \right)^T \frac{\partial \Pi}{\partial \mathbf{u}_L^T} = \mathbf{T}^T \mathbf{f}_L \quad (3.22)$$

where $\mathbf{T}$ is the transformation matrix. The stiffness matrix in the global coordinate system $\mathbf{K}_G$ is expressed as:

$$\mathbf{K}_G = \frac{\partial \mathbf{f}_G^T}{\partial \mathbf{u}_G^T} = \mathbf{T}^T \mathbf{K}_L \mathbf{T} + \frac{\partial \mathbf{T}^T}{\partial \mathbf{u}_G} \mathbf{f}_L \quad (3.23)$$

3.4 The formulations for plate elements

The plate element is initially flat and placed in the xy plane. There are three degrees of freedom (DOF) per node, which is $\mathbf{u}_i = (w_i, \theta_{xi}, \theta_{yi})^T$, ($i = 1, 2, \dots, n$), n is the number of nodes in an element, $n = 6$ for triangular element and $n = 9$ for quadrilateral element. w_i is the out-of-plane deflection at node i and θ_{xi} and θ_{yi} are the rotational angle with respect to x- and y-axis, respectively.

The displacement approximation field $\mathbf{u}_h$ is interpolated by nodal DOF $\mathbf{u}_i$, which is expressed as

$$\mathbf{u}_h = \sum_i N_i \mathbf{u}_i \quad (3.24)$$

where N_i are the standard shape functions for plate elements, $\mathbf{u}_i = (w_i, \theta_{xi}, \theta_{yi})^T$ and $\mathbf{u}_h = (w_h, \theta_{xh}, \theta_{yh})^T$.

Figure 3.6 The natural coordinate system for the 6-node and 9-node plate element

For the 6-node triangular plate element, the standard shape functions are:

$$N_1 = (1-r-s)(1-2r-2s), \quad N_2 = r(2r-1), \quad N_3 = s(2s-1)$$

$$N_4 = 4r(1-r-s), \quad N_5 = 4rs, \quad N_6 = 4s(1-r-s) \quad (3.25)$$

For the 9-node quadrilateral plate element, the standard shape functions are:

$$N_1 = \frac{1}{4}r(r-1)s(s-1), \quad N_2 = \frac{1}{4}r(r+1)s(s-1), \quad N_3 = \frac{1}{4}r(r+1)s(s+1),$$

$$N_4 = \frac{1}{4}r(r-1)s(s+1), \quad N_5 = -\frac{1}{2}(r-1)(r+1)s(s-1),$$

$$N_6 = -\frac{1}{2}r(r+1)(s-1)(s+1), \quad N_7 = -\frac{1}{2}(r-1)(r+1)s(s+1)$$

$$N_8 = -\frac{1}{2}r(r-1)(s-1)(s+1), \quad N_9 = (1-r)(1+r)(1-s)(1+s) \quad (3.26)$$

Figure 3.7 The layered model for plate elements

A layered model, as shown in Figure 3.7, is adopted in the present plate element so that the elasto-plastic behavior of a plate structure can be traced. The middle surface of an element is selected as the reference surface. The kinematic equation for Reissner-Mindlin theory is expressed as

$$\boldsymbol{\varepsilon} = -z_j \left(\frac{\partial \beta_{xh}}{\partial x}, \frac{\partial \beta_{yh}}{\partial y}, \frac{\partial \beta_{xh}}{\partial y} + \frac{\partial \beta_{yh}}{\partial x} \right)^T \quad (3.27)$$

$$\boldsymbol{\gamma} = \left(-\beta_{xh} + \frac{\partial w_h}{\partial x}, -\beta_{yh} + \frac{\partial w_h}{\partial y} \right)^T \quad (3.28)$$

where z_j is the distance between the reference surface and the middle surface of the j^{th} layer. β_{xh} and β_{yh} are the rotation angles of the lines normal to the undeformed neutral surface in the x-z and y-z planes, respectively, which can be written in matrix form as (Zienkiewicz and Taylor 2005b)

$$\begin{pmatrix} \beta_{xh} \\ \beta_{yh} \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \theta_{xh} \\ \theta_{yh} \end{pmatrix} \quad (3.29)$$

3.5 The elasto-plastic model

3.5.1 The elasto-plastic model for the beam element

An elastic-perfectly-plastic constitutive relationship is employed in the present formulation. Shear stress does not appear in the yield criteria, and only normal stress is used to check whether the material at a given Gaussian point has yielded (Owen and Hinton 1980). Therefore the following uniaxial elasto-plastic model (Figure 3.8) is used for each layer in beam elements:

$$\varepsilon = \varepsilon_e + \varepsilon_p \quad (3.30)$$

$$\sigma_n = \sigma_{n-1} + E d\varepsilon_e \quad (3.31)$$

where ε is the total normal strain of a layer in a cross-section; ε_e is the elastic strain and ε_p is the plastic strain; σ_n is the desired normal stress, σ_{n-1} is the normal stress at the latest equilibrium state and E is the Young's modulus of the material. $d\varepsilon_e$ is the elastic incremental strain.

Figure 3.8 The uniaxial elasto-plastic model for the beam element

3.5.2 The elasto-plastic model for plate elements

In the present plate elements, the out-of-plane stress components are not taken into account in the formulation (Owen and Hinton 1980). The J_2 yield criteria for plane stress cases can be simplified as

$$f(\sigma_x, \sigma_y, \tau_{xy}) = \left[\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 \right]^{1/2} - \sigma_0 = \sigma_e - \sigma_0 \quad (3.32)$$

where σ_e is the effective stress, and σ_0 is the yield strength.

The incremental plastic strain can be expressed as

$$d\boldsymbol{\varepsilon}_p = d\lambda \mathbf{P} \boldsymbol{\sigma} \quad (3.33)$$

where the matrix $\mathbf{P}$ and the vector $\boldsymbol{\sigma}$ are of the form:

$$\mathbf{P} = \frac{1}{3} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad \boldsymbol{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \quad (3.34)$$

The stress is updated by (Crisfield 1991a)

$$\boldsymbol{\sigma}_C = \mathbf{D}(\boldsymbol{\varepsilon}_C - \boldsymbol{\varepsilon}_{pC}) = \mathbf{D}(\boldsymbol{\varepsilon}_C - \boldsymbol{\varepsilon}_{pA} - d\boldsymbol{\varepsilon}_p) \quad (3.35)$$

where state C is the desired state, $\boldsymbol{\sigma}_C$ is the desired stress, $\boldsymbol{\sigma}_A$ is the stress state of the starting point of the current incremental step and $d\boldsymbol{\varepsilon}_p = \boldsymbol{\varepsilon}_{pC} - \boldsymbol{\varepsilon}_{pA}$ is the incremental plastic strain for the current step.

The desired stress state $\boldsymbol{\sigma}_C$ is expressed as (Simo and Hughes 1998)

$$\boldsymbol{\sigma}_C = \boldsymbol{\Xi} \mathbf{D} \boldsymbol{\sigma}_B \quad (3.36)$$

$$\Xi = (\mathbf{D}^{-1} + d\lambda \mathbf{P})^{-1} \quad (3.37)$$

where σ_B is the trial stress in which the incremental strain of the current step is assumed as elastic. Equation (3.36) establishes an expression of σ_C in which the plastic multiplier $d\lambda$ is the only unknown. The plastic multiplier $d\lambda$ can be obtained through a backward-Euler algorithm by solving a nonlinear equation whose physical meaning is the satisfaction of the yield criteria at stress state C

$$f(\sigma_C) = f(d\lambda) = 0 \quad (3.38)$$

The detail of the backward-Euler algorithm is presented in Appendix B.

3.6 Shear locking

Shear locking is one of the major obstacles in developing beam elements and plate elements. A compatible shear strain field introduces parasitic shear energy into the element in pure bending cases when the ratio of thickness to length is very small (Huang 1987). Reduced integration is used in the 2D 3-node co-rotational beam element. The reduced integration (2 Gaussian integration points for each element) in the beam element does not contain spurious mode (Noor and Peters 1981; Dvorkin et al. 1988). In the 2D 3-node beam element, the DSG technique is applied to mitigate shear locking. The assumed shear strain method, including the DSG technique (Bletzinger et al. 2000) and the MITC technique (Lee and Bathe 2004), are employed in the plate elements to control shear locking.

3.6.1 The DSG technique

In DSG technique (Bletzinger et al. 2000), a deflection is divided into two components: the component due to pure bending and the component due to shear deformation. The deflection associated with shear deformation is expressed as

$$\Delta w_\gamma = \int_{x_0}^x \gamma dx = w|_{x_0}^x - \int_{x_0}^x \theta dx = w(x) - w(x_0) - \Delta w_b(x) \quad (3.39)$$

where γ is shear strain, w is the total deflection, Δw_b is deflection due to pure bending and x_0 is the coordinate of a selected reference point in the element, at which the total deflection is assumed to be zero. The deflection due to shear deformation Δw_γ is the so-called shear gap and the shear gap at node i can be expressed as

$$\Delta w_{\gamma i} = w_i - w_0 - \Delta w_{b i} \quad (3.40)$$

After discretization, the shear gap can be rewritten as

$$\Delta w_\gamma = N_i \Delta w_{\gamma i} \quad (3.41)$$

Thus the assumed shear strain can be obtained by the first derivative of shear gap field with respect to the coordinate

$$e = \frac{\partial N_i}{\partial x} \Delta w_{\gamma i} \quad (3.42)$$

In the present plate elements, the shear strain field is assumed as

$$e_\eta = \frac{\partial \Delta w_{\eta t}}{\partial r}, \quad e_{\eta \zeta} = \frac{\partial \Delta w_{\eta st}}{\partial s} \quad (3.43)$$

where $\Delta w_{\eta t}$ is the shear gap obtained by integration of the compatible shear strain terms.

$$\Delta w_{rt} = N_i \Delta w_{rti} = N_i \int_{r_1}^{r_i} \gamma_{rt} dr, \quad \Delta w_{st} = N_i \Delta w_{sti} = N_i \int_{s_1}^{s_i} \gamma_{st} ds \quad (3.44)$$

where γ_{rt} and γ_{st} are compatible shear strains in curvilinear coordinate system.

Substituting Equation (3.44) into Equation (3.43) by using the kinematic equation for shear strain in Equation (3.28), the assumed shear strain can be expressed as

$$e_{rt} = \sum_i h_{ri} \beta_{ri} + \sum_i \frac{\partial N_i}{\partial r} w_i \quad (3.45)$$

$$e_{st} = \sum_i h_{si} \beta_{si} + \sum_i \frac{\partial N_i}{\partial s} w_i \quad (3.46)$$

where h_{ri} and h_{si} are the assumed shear strain interpolation functions.

In the 3-node beam element, the assumed shear strain interpolation functions h_i are expressed explicitly as

$$h_1 = -\frac{1}{2}\xi + \frac{1}{6}, \quad h_2 = \frac{2}{3}, \quad h_3 = \frac{1}{2}\xi + \frac{1}{6} \quad (3.47)$$

The assumed shear strain interpolation functions h_{ri} and h_{si} for the 9-node quadrilateral plate element can be expressed explicitly as

$$h_{r1} = \left(-\frac{1}{4}r + \frac{1}{12}\right)(s^2 - s), \quad h_{r2} = \left(\frac{1}{4}r + \frac{1}{12}\right)(s^2 - s), \quad h_{r3} = \left(\frac{1}{4}r + \frac{1}{12}\right)(s^2 + s),$$

$$h_{r4} = \left(-\frac{1}{4}r + \frac{1}{12}\right)(s^2 + s), \quad h_{r5} = \frac{1}{3}(s^2 - s), \quad h_{r6} = \left(\frac{1}{2}r + \frac{1}{6}\right)(1 - s^2),$$

$$h_{r7} = \frac{1}{3}(s^2 + s), \quad h_{r8} = \left(-\frac{1}{2}r + \frac{1}{6}\right)(1 - s^2), \quad h_{r9} = \frac{2}{3}(1 - s^2) \quad (3.48)$$

$$\begin{aligned} h_{s1} &= \left(-\frac{1}{4}s + \frac{1}{12}\right)(r^2 - r), \quad h_{s2} = \left(-\frac{1}{4}s + \frac{1}{12}\right)(r^2 + r), \quad h_{s3} = \left(\frac{1}{4}s + \frac{1}{12}\right)(r^2 + r), \\ h_{s4} &= \left(\frac{1}{4}s + \frac{1}{12}\right)(r^2 + r), \quad h_{s5} = \left(-\frac{1}{2}s + \frac{1}{6}\right)(1 - r^2), \quad h_{s6} = \frac{1}{3}(r + r^2), \\ h_{s7} &= \left(\frac{1}{2}s + \frac{1}{6}\right)(1 - r^2), \quad h_{s8} = \frac{1}{3}(r^2 - r), \quad h_{s9} = \frac{2}{3}(1 - r^2) \end{aligned} \quad (3.49)$$

On the other hand, the assumed shear strain interpolation functions h_{ri} and h_{si} for the 6-node quadrilateral plate element can be expressed explicitly as

$$\begin{aligned} h_{r1} &= \frac{2}{3} - r + 4rs - 3s + 2s^2, \quad h_{r2} = -\frac{1}{3} + r, \quad h_{r3} = -s + 2s^2 \\ h_{r4} &= \frac{2}{3} - 4rs, \quad h_{r5} = 4rs, \quad h_{r6} = 4s - 4rs - 4s^2 \end{aligned} \quad (3.50)$$

$$\begin{aligned} h_{s1} &= \frac{2}{3} - 3r + 4rs - s + 2r^2, \quad h_{s2} = -r + 2r^2, \quad h_{s3} = -\frac{1}{3} + s \\ h_{s4} &= -4rs + 4r - 4r^2, \quad h_{s5} = 4rs, \quad h_{s6} = \frac{2}{3} - 4rs \end{aligned} \quad (3.51)$$

3.6.2 The MITC technique

In the present 6-node triangular plate formulations, the MITC6a (Lee and Bathe 2004) is used to alleviate shear locking. The assumed shear strain field in MITC6a is expressed as

$$e_{rt} = a_1 + b_1r + c_1s + d_1rs + e_1r^2 + f_1s^2 \quad (3.52)$$

$$e_{st} = a_2 + b_2r + c_2s + d_2rs + e_2r^2 + f_2s^2 \quad (3.53)$$

where the coefficients a_i , b_i , c_i , d_i , e_i and f_i ($i = 1, 2$) are determined from sampling of shear strains at some selected tying points, as shown in Figure 3.9. The samplings of shear strains are obtained from compatible transverse shear strain field.

$$\begin{aligned}
a_1 &= \tilde{m}_{rt}^1 - \tilde{l}_{rt}^1, \quad b_1 = 2\tilde{l}_{rt}^1, \quad e_1 = 0, \quad a_2 = \tilde{m}_{st}^2 - \tilde{l}_{st}^2, \quad c_2 = 2\tilde{l}_{st}^2, \quad f_2 = 0, \\
c_1 &= 6e_{crt} - 3e_{cst} + 2\tilde{m}_{st}^3 - 2\tilde{m}_{rt}^3 - 4a_1 - b_1 + a_2, \\
b_2 &= -3e_{crt} + 6e_{cst} - 2\tilde{m}_{st}^3 + 2\tilde{m}_{rt}^3 + a_1 - 4a_2 - c_2, \\
e_2 &= 3e_{crt} - 6e_{cst} + 3\tilde{m}_{st}^3 - \tilde{l}_{st}^3 - 3\tilde{m}_{rt}^3 + \tilde{l}_{rt}^3 + b_1 + 3a_2 + c_2, \\
f_1 &= -6e_{crt} + 3e_{cst} - 3\tilde{m}_{st}^3 - \tilde{l}_{st}^3 + 3\tilde{m}_{rt}^3 + \tilde{l}_{rt}^3 + 3a_1 + b_1 + c_2, \\
d_1 &= -e_2, \quad d_2 = -f_1
\end{aligned} \tag{3.54}$$

where

$$\tilde{m}_{jt}^i = \frac{1}{2}(e_{1jt}^i + e_{2jt}^i), \quad \tilde{l}_{jt}^i = \frac{\sqrt{3}}{2}(e_{2jt}^i - e_{1jt}^i) \tag{3.55}$$

with $j = r, s$ and $i = 1, 2, 3$

In the MITC6a, the assumed transverse shear strain contains quadratic terms. As the MITC6b (Lee and Bathe 2004) is not discussed in this thesis, in the following part of the thesis, the term ‘MITC6’ specifically refers to the MITC6a interpolation scheme.

(a) The location of the tying points of the MITC6

(b) The sampling shear strain of the MITC6

Figure 3.9 The location of the tying points of the MITC6

The MITC9 is used to alleviate shear locking in the present 9-node quadrilateral element. In the MITC9, the assumed shear strain field $\tilde{e}_{rt}$ and $\tilde{e}_{st}$ are interpolated by the compatible shear strain at some selected tying points γ_{rti} and γ_{sti} ($i = 1, 2, 3 \dots, 6$). The locations of the tying points are shown in Figure 3.10.

(a) The tying points for γ_{rt}

(b) The tying points for γ_{st}

Figure 3.10 The location of the tying points of the MITC9

The assumed shear strain is expressed as

$$e_{rt} = l_{rti} \gamma_{rti} \quad (3.56)$$

$$e_{st} = l_{sti} \gamma_{sti} \quad (3.57)$$

where l_{rti} and l_{sti} are the Lagrangian interpolation functions for the assumed shear strain field, respectively. The Lagrangian interpolation function, l_{rti} and l_{sti} , can be expressed as

$$l_{rti} = \frac{(r - r_j)(s - s_J)(s - s_K)}{(r_i - r_j)(s_i - s_J)(s_i - s_K)} \quad (3.58)$$

$$l_{sti} = \frac{(r - r_j)(r - r_k)(s - s_J)}{(r_i - r_j)(r_i - r_k)(s_i - s_J)} \quad (3.59)$$

In Equation (3.58) and (3.59), $i \neq j \neq k$ and $i \neq J \neq K$.

The assumed shear strain in the global coordinate system can be obtained by

$$\begin{pmatrix} \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \mathbf{S}^{-1} \begin{pmatrix} \gamma_{rt} \\ \gamma_{st} \end{pmatrix} \quad (3.60)$$

where $\mathbf{S}$ is the orientation matrix at an integration point, which can be expressed as

$$\mathbf{S}^T = \begin{bmatrix} \frac{\mathbf{e}_r}{|\mathbf{e}_r|} & \frac{\mathbf{e}_s}{|\mathbf{e}_s|} \end{bmatrix}, \quad \mathbf{e}_r^T = \begin{pmatrix} \frac{\partial x_0}{\partial r} & \frac{\partial y_0}{\partial r} \end{pmatrix}, \quad \mathbf{e}_s^T = \begin{pmatrix} \frac{\partial x_0}{\partial s} & \frac{\partial y_0}{\partial s} \end{pmatrix} \quad (3.61)$$

In Equation (3.61), x_0 and y_0 are the x- and y-coordinates of the points with respect to the reference surface.

3.7 Closure

In this chapter, the formulations for the standard finite element are presented. The beam element and the plate elements will be used in the following chapters.

The XFEM formulation will be implemented in the 2D 3-node co-rotational beam element to model a perfect pin connection, while the XFEM formulation will be embedded in the 2D 3-node beam element and the plate elements to capture the high gradient displacement field resulting from a plastic hinge or a yield line. The layered model for the beam elements and the plate elements helps to trace the process of yielding in a cross-section more precisely. The reduced integration is employed to remove shear locking in the co-rotational beam element, while the DSG technique is adopted for the beam elements with linear kinematics. The MITC technique and the DSG technique are adopted to mitigate shear locking in plate elements.

CHAPTER 4 THE XFEM FORMULATION FOR A PIN CONNECTION INSIDE A BEAM ELEMENT

4.1 Introduction

In this chapter, a 2D co-rotational beam element with XFEM formulations is presented. The purpose of this chapter is to show the application of the XFEM formulation to model internal pin connections. This algorithm can be easily implemented in plastic hinge and yield line analyses by a small modification on the enrichment functions. The XFEM co-rotational beam element is developed to simulate a perfect pin connection inside a beam structure. A strong discontinuity appears in rotation field, while a weak discontinuity appears in deflection field. Since the rotation and deflection approximations are independent in Timoshenko theory, discontinuous enrichment functions are added into the rotation field and deflection field simultaneously. It has been shown in Section 2.3.2.1 that a discontinuous enrichment function introduces a zero-traction boundary condition into the governing equation. Therefore, the bending strain at the two sides of a perfect pin is eliminated to zero automatically by the equilibrium conditions.

4.2 The enrichment functions for a pin connection

The XFEM formulation is a numerical method based on *a priori* knowledge of non-smoothness in a displacement approximation field. In this chapter, the location of a perfect pin is regarded as *a priori* knowledge. The location of a perfect pin is prescribed in the natural coordinate system as a parametric location ξ_0 ($-1 \leq \xi_0 \leq 1$).

Over a pin connection, the rotational approximation shows a strong discontinuity and the deflection approximation shows a weak discontinuity. Hence, in the present formulation, the step function (Zi and Belytschko 2003) and absolute level set function (Moës et al. 2003) are used for the rotational and the translational DOF, respectively. In order to avoid the two issues resulting from blending elements (the violation of the PU condition and the appearance of unwanted strain terms), the enrichment functions are shifted.

In the following illustration, the parametric location of the perfect pin is prescribed as 0.5, as shown in Figure 4.1.

(a) An internal pin inside an element
element 2

(b) The element with a pin inside
Figure 4.1 The location of a pin

The shifted enrichment function for the strong discontinuity (S_{pp}), as shown in Figure 4.2, is of the form

$$S_{ppi} = H(\xi - \xi_0) - H(\xi_i - \xi_0) \quad (4.1)$$

where ξ_i is the natural coordinate of node i : $\xi_1 = -1$, $\xi_2 = 0$ and $\xi_3 = 1$. $H(\xi)$ is a step function

$$H(\xi) = \begin{cases} 1 & \dots \xi > 0 \\ 0 & \dots \xi = 0 \\ -1 & \dots \xi < 0 \end{cases} \quad (4.2)$$

(a) S_{pp1} and S_{pp2}

(b) S_{pp3}

Figure 4.2 Enrichment function (S_{ppi}) for rotational DOF in pin connection

The Lagrangian shape functions, which are used as the window functions, are shown in Figure 4.3, where element 1 and element 3 are the blending elements which are adjacent to the enriched element (element 2), $I^* = (3, 4, 5)$. It should be noted that node 3, node 4 and node 5 are the enriched nodes, which are indicated by ■, the conventional nodes are indicated by ● and the pin connection is indicated by ○.

Figure 4.3 The window functions for an enriched element with a perfect pin

The interpolation functions for strong discontinuity are constructed by the multiplication of Lagrangian shape functions and strong discontinuous enrichment

$$M_{ppi} = N_i S_{ppi}, \quad i = 1, 2, 3 \quad (4.3)$$

The plot of the interpolation functions M_{ppi} is shown in Figure 4.4.

Figure 4.4 The interpolation shape function M_{ppi}

The shifted weak discontinuous enrichment function (F_{pp}) (Moës et al. 2003) is expressed as

$$F_{pp} = N_i |\varphi_i| - |\varphi_i N_i| \quad (4.4)$$

In Equation (4.4), φ_i is the value of the absolute level set function for node i ($i = 1, 2, 3$). The enrichment function F_{pp} is shown in Figure 4.5.

Figure 4.5 The enrichment for translational DOF in a pin connection (F_{pp})

The interpolation functions for weak discontinuity are constructed by the multiplication of the Lagrangian shape functions and the weak discontinuous enrichment function

$$L_{ppi} = N_i F_{pp} \quad (4.5)$$

The plot of the interpolation function L_{ppi} is shown in Figure 4.6.

Figure 4.6 The interpolation function L_{ppi}

4.3 The enriched displacement field

In the present formulation, the both translational and rotational DOF are enriched. Therefore, three additional DOF per node are added to the nodal global displacement vector. The global displacement vector for an enriched element is of the form:

$$\mathbf{U}_{\text{enrG}} = \left\{ \mathbf{U}_{\text{enrG1}}^T \quad \mathbf{U}_{\text{enrG2}}^T \quad \mathbf{U}_{\text{enrG3}}^T \right\}^T \quad (4.6)$$

In Equation (4.6), $\mathbf{U}_{\text{enrGi}}^T = (U_i, W_i, \psi_i, A_i, B_{Ui}, B_{Wi})$, $i = 1, 2, 3$. A_i , B_{Ui} and B_{Wi} are the additional DOF corresponding to the nodal global displacement $\mathbf{U}_i$. The displacement approximation in the global coordinate system is of the form:

$$\psi_h = N_i \psi_i + M_i A_i \quad (4.7)$$

$$\begin{pmatrix} U_h \\ W_h \end{pmatrix} = \begin{pmatrix} N_i U_i \\ N_i W_i \end{pmatrix} + \begin{pmatrix} L_i B_{Ui} \\ L_i B_{Wi} \end{pmatrix} \quad (4.8)$$

where M_i and L_i are the enriched interpolation functions for the rotational and the translational DOF, respectively; I is the node set of the nodes in the whole domain. I^* is the node set of enriched nodes and $I^* \subseteq I$.

Two additional DOF per node are added into the local displacement vector

$$\mathbf{u}_{\text{enrL}} = \{u_1, w_1, \theta_1, a_1, b_{w1} \mid a_2, b_{w2} \mid u_3, w_3, \theta_3, a_3, b_{w3}\}^T \quad (4.9)$$

The local displacement approximation has the form

$$\theta_h = N_i \theta_i + M_i a_i \quad (4.10)$$

$$\begin{pmatrix} u_h \\ w_h \end{pmatrix} = \begin{pmatrix} N_i u_i \\ N_i w_i \end{pmatrix} + \begin{pmatrix} 0 \\ L_i b_{wi} \end{pmatrix} \quad (4.11)$$

where a_i is the additional DOF for the nodal local discontinuous rotation for node i ; b_{wi} is the additional DOF for the nodal local discontinuous transverse deflection for node i . The approximation of translation displacement along local x -axis is not enriched. However, in order to avoid ill-conditioning of the assembled stiffness matrix, a finite real number is placed in the diagonal cells corresponding to b_{wi} . In the present formulation, it is assigned by the largest value among the other diagonal cells.

As co-rotational technique is used for the present beam element, a relationship between the global additional DOF and the local additional DOF should be established and the physical meaning of the additional DOF should be clarified.

In the XFEM formulation, a displacement approximation is a summation of a continuous component and a discontinuous component. The continuous approximation is constructed by interpolation of the nodal displacement $\mathbf{U}_{ci} = [U_i, W_i, \psi_i]$ with continuous shape function N_i and the discontinuous approximation is constructed by interpolation of the nodal global displacement $\mathbf{U}_{di} = [A_i, B_{Ui}, B_{Wi}]$ with some discontinuous functions M_i and L_i . The additional DOF can be regarded as the discontinuous nodal displacement. Hence the relationship between the global additional DOF and the local additional DOF can be established in a similar way as the continuous nodal displacement

$$a_i = A_i \quad (4.12)$$

$$\begin{pmatrix} b_{ui} \\ b_{wi} \end{pmatrix} = \mathbf{R} \begin{pmatrix} B_{Ui} \\ B_{Wi} \end{pmatrix} \quad (4.13)$$

where $\mathbf{R}$ is the rotation matrix.

4.4 The stiffness matrix and the internal force vector

Since the layered model is used in the beam formulation, as shown in Figure 3.3, the strain in the j^{th} layer in an enriched element is given by Equation (3.14), in which the three strain components ε_m , γ and χ should be modified as follows:

$$\varepsilon_m = \varepsilon_{mc} + \varepsilon_{md} = \frac{\partial u_h}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial u_h}{\partial x} \right)^2 + \left(\frac{\partial w_h}{\partial x} \right)^2 \right] \quad (4.14)$$

$$\gamma = \gamma_c + \gamma_d = -\theta_h + \frac{\partial w_h}{\partial x} \quad (4.15)$$

$$\chi = \chi_c + \chi_d = -\frac{\partial \theta_h}{\partial x} \quad (4.16)$$

where u_h , w_h and θ_h are given in Equation (4.10) and (4.11). ε_{mc} , γ_c and χ_c are the membrane strain, shear strain and curvature due to continuous displacement; ε_{md} , γ_d and χ_d are the membrane strain, shear strain and curvature due to discontinuous displacement. They can be expressed of the form:

$$\begin{aligned} \varepsilon_{mc} &= \mathbf{B}_{mc} \mathbf{u}_{\text{enrL}}, \quad \varepsilon_{md} = \mathbf{B}_{md} \mathbf{u}_{\text{enrL}}, \\ \gamma_c &= \mathbf{B}_{\gamma c} \mathbf{u}_{\text{enrL}}, \quad \gamma_d = \mathbf{B}_{\gamma d} \mathbf{u}_{\text{enrL}}, \\ \chi_c &= \mathbf{B}_{bc} \mathbf{u}_{\text{enrL}}, \quad \chi_d = \mathbf{B}_{bd} \mathbf{u}_{\text{enrL}} \end{aligned} \quad (4.17)$$

where $\mathbf{u}_{\text{enrL}}$ is the local displacement vector for an XFEM element, $\mathbf{B}_m$, $\mathbf{B}_\gamma$, and $\mathbf{B}_b$ are the strain-displacement matrices for membrane, shear and bending part, the subscript d refers to discontinuous displacement and c to continuous displacement. Detailed expressions of $\mathbf{B}_m$, $\mathbf{B}_\gamma$ and $\mathbf{B}_b$ are shown in Appendix A.

The potential energy is expressed as

$$\begin{aligned} \Pi &= \frac{1}{2} \sum_{j=1}^n \int \left[(\varepsilon_{mc} + z_j \chi_c) D_{11} (\varepsilon_{mc} + z_j \chi_c) + \gamma_c D_{22} \gamma_c \right] dV \\ &+ \frac{1}{2} \sum_{j=1}^n \int \left[(\varepsilon_{mc} + z_j \chi_c) D_{11} (\varepsilon_{md} + z_j \chi_d) + \gamma_c D_{22} \gamma_d \right] dV \\ &+ \frac{1}{2} \sum_{j=1}^n \int \left[(\varepsilon_{md} + z_j \chi_d) D_{11} (\varepsilon_{mc} + z_j \chi_c) + \gamma_d D_{22} \gamma_c \right] dV \\ &+ \frac{1}{2} \sum_{j=1}^n \int \left[(\varepsilon_{md} + z_j \chi_d) D_{11} (\varepsilon_{md} + z_j \chi_d) + \gamma_d D_{22} \gamma_d \right] dV - W \end{aligned} \quad (4.18)$$

The internal force can be expressed as

$$\mathbf{f}_L = \begin{pmatrix} \mathbf{f}_{Lc} \\ \mathbf{f}_{Ld} \end{pmatrix} \quad (4.19)$$

$$\mathbf{f}_{Lc} = \sum_{j=1}^n \int (\mathbf{B}_{mc} + z_j \mathbf{B}_{mc})^T D_{11} (\varepsilon_m + z_j \chi) + \mathbf{B}_{\gamma c}^T D_{22} \gamma dV \quad (4.20)$$

$$\mathbf{f}_{Ld} = \sum_{j=1}^n \int \left[(\mathbf{B}_{md} + z_j \mathbf{B}_{bd})^T D_{11} (\varepsilon_m + z_j \chi) + \mathbf{B}_{\gamma d}^T D_{22} \gamma \right] dV \quad (4.21)$$

The stiffness matrix is the Hessian matrix of the potential energy

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{cc} & \mathbf{K}_{cd} \\ \mathbf{K}_{dc} & \mathbf{K}_{dd} \end{bmatrix} \quad (4.22)$$

$$\mathbf{K}_{cc} = \sum_{j=1}^n \int (\mathbf{B}_{mc} + z_j \mathbf{B}_{bc})^T D_{11} (\mathbf{B}_{mc} + z_j \mathbf{B}_{bc}) + \mathbf{B}_{\gamma c}^T D_{22} \mathbf{B}_{\gamma c} dV \quad (4.23)$$

$$\mathbf{K}_{cd} = \sum_{j=1}^n \int (\mathbf{B}_{mc} + z_j \mathbf{B}_{bc})^T D_{11} (\mathbf{B}_{md} + z_j \mathbf{B}_{bd}) + \mathbf{B}_{\gamma c}^T D_{22} \mathbf{B}_{\gamma d} dV \quad (4.24)$$

$$\mathbf{K}_{dc} = \sum_{j=1}^n \int (\mathbf{B}_{md} + z_j \mathbf{B}_{bd})^T D_{11} (\mathbf{B}_{mc} + z_j \mathbf{B}_{bc}) + \mathbf{B}_{\gamma d}^T D_{22} \mathbf{B}_{\gamma c} dV \quad (4.25)$$

$$\mathbf{K}_{dd} = \sum_{j=1}^n \int (\mathbf{B}_{md} + z_j \mathbf{B}_{bd})^T D_{11} (\mathbf{B}_{md} + z_j \mathbf{B}_{bd}) + \mathbf{B}_{\gamma d}^T D_{22} \mathbf{B}_{\gamma d} dV \quad (4.26)$$

It can be seen that the sub-matrix $\mathbf{K}_{cc}$ is the stiffness matrix for an ordinary element and it is symmetric. Since $\mathbf{K}_{cd} = \mathbf{K}_{dc}^T$ and $\mathbf{K}_{dd}$ is symmetric, the stiffness matrix of an enriched element is also *symmetric*.

The internal force and the stiffness of an enriched element in the global coordinate system are the Jacobian matrix and the Hessian matrix of the potential energy with respect to the global displacement. By using chain rule, they can be obtained as

$$\mathbf{f}_G = \frac{\partial \Pi}{\partial \mathbf{u}_{\text{enrG}}}^T = \mathbf{T}^T \mathbf{f}_L \quad (4.27)$$

$$\mathbf{K}_G = \frac{\partial \mathbf{f}_G}{\partial \mathbf{u}_{\text{enrG}}}^T = \mathbf{T}^T \mathbf{K}_L \mathbf{T} + \frac{\partial \mathbf{T}^T}{\partial \mathbf{u}_{\text{enrG}}} \mathbf{f}_L \quad (4.28)$$

$$\mathbf{T} = \frac{\partial \mathbf{u}_{\text{enrL}}}{\partial \mathbf{u}_{\text{enrG}}}^T \quad (4.29)$$

where $\mathbf{T}$ is the transformation matrix with additional DOF.

4.5 Numerical examples

In this section, two examples are tested with the same cross-section (UB section W12-12×12, 253.0 kg/m) and loading condition (a point load is applied at the middle of the beam), but with different locations of an internal pin. The geometrical properties of UB section W12-12×12 (253.0 kg/m) are shown in Figure 4.7: $D = 356.4$ mm, $B = 319.3$ mm, $T = 39.6$ mm, $t = 24.4$ mm. The area of the cross-section $A = 323$ cm². The second moment of area $I_{xx} = 68494$ cm⁴. The reference point load $F_0 = 2.0262 \times 10^5$ kN. The material properties for the beams are: Young's modulus $E = 210$ GPa and shear modulus $G = 80.77$ GPa.

Figure 4.7 The cross-section for the examples

The deformed configuration and rotation approximation are compared with the result from ANSYS beam3 element (ANSYS 2011). Master and slave DOF (Crisfield 1991b) technique is used in ANSYS analyses, which requires the meshes to conform to the pin in the beam. In the ANSYS analyses, a total of 6 elements with equal length are used. The result from ANSYS is regarded as a reference solution to the problem. A total of five elements with equal length are used for XFEM analyses ($1.2 \text{ m} \times 5$) for each example. For both examples with the internal pin, two formulations are tested, viz. the formulation with both strong and weak discontinuities (S+W) and with strong discontinuity only (S).

4.5.1 Example 1: A fixed-fixed beam with a pin at the one-third point

In this example, a pin is located at the one-third point of the beam, as shown in Figure 4.8, with the parametric location of the pin at element level $\xi_0 = 0.33$. The equilibrium path (the deflection at the loading point (w) versus the loading factor (λ)) is shown in Figure 4.9. The rotation displacement and the deflection along the beam are shown in Figure 4.10(a) and (b), respectively.

Figure 4.8 A fixed-fixed beam with a pin at the one-third point

Figure 4.9 The equilibrium path for Example 1

(a) The rotation field

(b) The deflection field

Figure 4.10 The displacement field for Example 1

As shown in Figure 4.10(a), the discontinuity in rotation cannot be captured if only the strong discontinuity is considered, while a combination of strong and weak enrichment for translational approximation can capture the behaviour of the discontinuity accurately. It is also found from Figure 4.10(a) that the

rotation at the right side of the pin does not match the prediction from ANSYS. This could be due to the removal of rigid body motion. In Equation (4.12), it is assumed that the local discontinuous rotation is equal to the global discontinuous rotation. This assumption may not be accurate for large deformation problems. However, the XFEM formulation has shown its ability to capture the discontinuous displacement over an internal pin connection.

4.5.2 Example 2: A fixed-fixed beam with a pin at the middle point

Figure 4.11 A fixed-fixed beam with a pin at the middle point

A pin is located at the middle of the beam, as shown in Figure 4.11, with the parametric location of the pin $\xi_0 = 0.0$. The equilibrium path (the deflection at the loading point (w) versus the loading factor (λ)) is shown in Figure 4.12. The rotation and the deflection fields along the beam are shown in Figure 4.13(a) and (b), respectively.

Figure 4.12 The equilibrium path for Example 2

(a) The rotation field

(b) The deflection field
Figure 4.13 The displacement field for Example 2

In this example, as shown in Figure 4.12 and Figure 4.13, the formulation with both strong and weak discontinuous enrichments can capture the discontinuous phenomenon in deflection adequately; however, the formulation, with strong discontinuity only can locate the jump in rotation but not the kink in deflection. Similar to Example 1 in this chapter, it is also found from Figure 4.13(a) that the rotation displacement at the enriched element is still slightly different from the ANSYS prediction. This could be due to the inability of Equation (4.12) to completely remove rigid body motion in a discontinuous rotation field.

From the numerical examples shown in this section, one can see that the XFEM formulation works well in the analyses involving internal pins if both strong and weak discontinuities are simultaneously added to the rotation and translation approximation fields.

4.6 Closure

In this chapter, an XFEM co-rotational beam element is presented. A strong discontinuous function and a weak discontinuous function are introduced into the rotation and deflection displacement field, respectively, to model a perfect pin connection inside a beam element. In the local displacement approximation field, only the deflection DOF and the rotational DOF are enriched, which means that only the DOF associated with bending action are enriched. Hence, it is unnecessary to distinguish the normal strain resulting from bending action from the total normal strain. The bending strain and bending stress are automatically suppressed to zero, according to the proof by Belytschko and Black. In the meantime, the enrichment has no influence on the membrane strain and the membrane stress, which comes from tension-compression action in analyses. The separately enriching technique solves the problem of applying natural boundary condition on bending action at a perfect pin connection in a *graceful* way.

Furthermore, as shifted enrichment functions are used, the enrichment functions are zero at the element nodes. Thus, the discontinuous displacement field has no contribution to rigid body motion. The co-rotation frame for the standard beam element can be easily applied in the XFEM beam element.

CHAPTER 5 THE XFEM FORMULATION

FOR PLASTIC HINGE ANALYSIS

5.1 Introduction

In this chapter, the XFEM formulation for a plastic hinge inside a beam element is introduced. The XFEM beam element is based on the small deformation theory. Regularized enrichment functions are introduced to capture the formation of plastic hinges and trace the behaviour of a beam after a plastic mechanism is formed. The enrichment functions are regularized by Hermite function so that they are C_1 continuous over the whole domain of an enriched element. The C_1 continuity guarantees the derivability of the enriched displacement approximation field. Therefore, the strain energy in the non-smooth part resulting from a plastic hinge can be calculated as it is in the smooth part, which avoids the explicit expression on the relationship between the strain energy and the displacement variables.

5.2 The enrichment functions for plastic hinges

As the rotation displacement field varies rapidly over a plastic hinge, the rotation field changes its sign within a small length. Meanwhile, there is a round transition part rather than a kink over a plastic hinge in the deflection field. Hence, regularized enrichment functions instead of discontinuous enrichment functions are used to model the non-smoothness in displacement

field due to a plastic hinge. Different from the non-smooth displacement field resulting from an internal pin connection, the non-smooth displacement in this chapter is continuous but with a local high gradient zone. A regularized step function (Benvenuti et al. 2008) and a regularized absolute level set function (Xu et al. 2012) are proposed as the enrichment functions for the rotation and the deflection displacement approximation field, respectively. Both of the enrichments are regularized by the Hermite function over the high gradient zone. Therefore, both of the enrichment functions are C_1 continuous inside an enriched element and C_0 continuous over the whole domain.

The location of a plastic hinge and the size of the high gradient part resulting from the plastic hinge are regarded as *priori* knowledge. A parametric location of the high gradient part ($-1 \leq \xi_0 \leq 1$) and a parametric length of the high gradient part ($0 \leq \omega_{ns} \leq 1$) are defined in the natural coordinate system of the related element. It is noted that a high gradient is resulting from a plastic hinge but it is not equivalent to a plastic hinge.

In the case of a plastic hinge inside a beam element, three sub-domains can be divided: the non-smooth part (the high gradient zone) and the two parts beside the non-smooth part (the smooth part). A parametric length of the high gradient part (ω) is defined in the formulation, which depends on the size of the enriched element (l_e) and the physical length of the high gradient part (l_{ns}) such that

$$\omega = \frac{l_{ns}}{l_e}, \quad 0 < l_{ns} < l_e \quad (5.1)$$

As shown in Figure 5.1, the parametric length of the high gradient part (ω) is prescribed as 0.2 and the parametric location of the high gradient part is prescribed as 0.5.

(a) A plastic hinge inside an element

(b) The element with a plastic hinge inside
Figure 5.1 An element with a plastic hinge inside

In the two smooth parts, the enrichment functions used for a perfect pin connection (Equation (4.1) and (4.4)) are adopted for both rotation and translation displacement approximation. In the non-smooth part, the Hermite function is used to connect the two enrichment functions over the two smooth parts.

The enrichment S_{ph} for rotational DOF and the enrichment F_{ph} for translational DOF are expressed by

$$S_{ph}(\xi) = \begin{cases} H(\xi - \xi_0) - H(\xi_k - \xi_0) \cdots & \xi \geq \xi_0 + 0.5\omega \\ H_s & \cdots \xi_0 - 0.5\omega < \xi < \xi_0 + 0.5\omega \\ H(\xi - \xi_0) - H(\xi_k - \xi_0) \cdots & \xi \leq \xi_0 - 0.5\omega \end{cases} \quad (5.2)$$

$$F_{ph}(\xi) = \begin{cases} H_w & \cdots \xi_0 - 0.5\omega < \xi < \xi_0 + 0.5\omega \\ \sum_{k=1}^3 N_k |\varphi_k| - \left| \sum_{k=1}^3 \varphi_k N_k \right| & \cdots \text{otherwise} \end{cases} \quad (5.3)$$

The plot of S_{ph} and F_{ph} are shown in Figure 5.2 and Figure 5.3, respectively.

Figure 5.2 The enrichment function of rotational DOF for a plastic hinge (S_{ph})

Figure 5.3 The enrichment function for translational DOF in a plastic hinge (F_{ph})

In Equation (5.2) and (5.3), H_s and H_w are the Hermite functions for rotational and translational DOF in the high gradient part, respectively. They are constructed by the expression

$$H_s = \mathbf{G}_s \cdot \mathbf{M} \cdot \mathbf{P} \quad (5.4)$$

$$H_w = \mathbf{G}_w \cdot \mathbf{M} \cdot \mathbf{P} \quad (5.5)$$

where $\mathbf{G}_s$ and $\mathbf{G}_w$ are the Hermite geometry vector for rotational and translational DOF, respectively; $\mathbf{M}$ is the Hermite matrix and $\mathbf{P}$ is the cubic power basis vector.

$$\mathbf{G}_s = \begin{pmatrix} S_{ph}(\xi_1) & S_{ph}(\xi_2) & \frac{\partial S_{ph}(\xi_1)}{\partial \xi^*} & \frac{\partial S_{ph}(\xi_2)}{\partial \xi^*} \end{pmatrix} \quad (5.6)$$

$$\mathbf{G}_w = \begin{pmatrix} F_{ph}(\xi_1) & F_{ph}(\xi_2) & \frac{\partial F_{ph}(\xi_1)}{\partial \xi^*} & \frac{\partial F_{ph}(\xi_2)}{\partial \xi^*} \end{pmatrix} \quad (5.7)$$

$$\frac{\partial S_{\text{ph}}(\xi_i)}{\partial \xi^*} = \frac{\partial S_{\text{ph}}(\xi)}{\partial \xi} \frac{\partial \xi}{\partial \xi^*} \bigg|_{\xi=\xi_i} = \frac{\partial S_{\text{ph}}(\xi_i)}{\partial \xi} \omega, \quad i=1,2 \quad (5.8)$$

$$\frac{\partial F_{\text{ph}}(\xi_i)}{\partial \xi^*} = \frac{\partial F_{\text{ph}}(\xi)}{\partial \xi} \frac{\partial \xi}{\partial \xi^*} \bigg|_{\xi=\xi_i} = \frac{\partial F_{\text{ph}}(\xi_i)}{\partial \xi} \omega, \quad i=1,2 \quad (5.9)$$

$$\xi_1 = \xi_0 - 0.5\omega, \quad \xi_2 = \xi_0 + 0.5\omega \quad (5.10)$$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad (5.11)$$

$$\mathbf{P} = (1 \quad \xi^* \quad \xi^{*2} \quad \xi^{*3})^T \quad (5.12)$$

where $\xi^* : [0,1]$ is the parent coordinate system for a high gradient part.

Figure 5.4 The window functions for enriched element with a plastic hinge

The Lagrangian shape functions of nodes are used as window functions for the localization of the enrichment, as shown in Figure 5.4, where element 1 and

element 3 are the blending elements and element 2 is the enriched element, $I^* = (3, 4, 5)$. The part in between the two dash lines is the high gradient part. Node 3, 4 and 5 are the enriched nodes indicated by ■. The unenriched nodes 1, 2, 6 and 7 are indicated by ●.

The interpolation functions for rotational and translational DOF are constructed by the multiplication of the enrichments and the window functions.

$$M_{\phi i} = N_i S_{\phi h} \quad (5.13)$$

$$L_{\phi i} = N_i F_{\phi h} \quad (5.14)$$

The plots of interpolation functions $M_{\phi i}$ and $L_{\phi i}$ are shown in Figure 5.5 and Figure 5.6, respectively.

Figure 5.5 The interpolation functions ($M_{\phi i}$)

Figure 5.6 The interpolation functions ($L_{\phi i}$)

Comparisons between the interpolation functions M_{ppi} and M_{phi} as well as L_{ppi} and L_{phi} are shown in Figure 5.7 and in Figure 5.8, respectively, in which the dotted curve is the part resulting from Hermite function.

Figure 5.7 The comparison between M_{ppi} and M_{phi}

Figure 5.8 The comparison between L_{ppi} and L_{phi}

5.3 The enriched displacement field

As shown in Figure 5.9, there are 4 DOF per node in the XFEM formulation for a plastic hinge, $\mathbf{u}_i = (w_i, \theta_i, a_i, b_i)^T$. The displacement approximation field is expressed as

$$w_h = \sum_i N_i w_i + \sum_i L_{phi} b_i \quad (5.15)$$

$$\theta_h = \sum_i N_i \theta_i + \sum_i M_{phi} a_i \quad (5.16)$$

where b_i and a_i are the additional DOF corresponding to the enrichment.

Figure 5.9 The DOF of an enriched beam element

The strain fields in the XFEM formulation are expressed as:

$$\varepsilon_j = z_j \chi = -z_j \frac{d\theta_h}{dx} \quad (5.17)$$

$$\gamma = -\theta_h + \frac{dw_h}{dx} \quad (5.18)$$

In order to alleviate shear locking phenomena, the DSG method is used to reconstruct the assumed shear strain in the XFEM beam element:

$$e = -h_i \left(\theta_i + S_{phi} a_i \right) + \frac{dN_i}{dx} \left(w_i + F_{phi} b_i \right) \quad (5.19)$$

where h_i are the interpolation functions for the assumed shear strain. The explicit expression of h_i is shown in Equation (3.47)

5.4 Numerical examples

In this section, three examples with the same geometric properties but different boundary conditions are tested. The beams are with UB section W12-12×12 (as shown in Figure 4.7). This cross-section is a Class 1 section according to Eurocode 3 (2005), in which a full plastic hinge can be formed with large rotation capacity without local buckling. The geometrical properties are the

same as those for section 4.5. The section modulus of this UB section, with respect to the strong principal axis is $S_{xx} = 3844 \text{ cm}^3$ and the plastic modulus is $Z_{xx} = 4502 \text{ cm}^3$. The elastic-perfectly-plastic material is used. The yield strength of steel is $\sigma_0 = 235 \text{ MPa}$. The maximum plastic moment without hardening effect is $M_u = Z_{xx} \times \sigma_0 = 1.058 \times 10^3 \text{ kNm}$.

Three cases are conducted for each example.

Case one: five standard elements with equal length ($1.2 \text{ m} \times 5$).

Case two: five elements with equal length and with enrichments where necessary ($1.2 \text{ m} \times 5$). All the physical lengths of the yielded zones are assumed to be 0.4 m .

Case three: thirty standard elements with equal length ($0.5 \text{ m} \times 30$).

For each case, 5 layers are used in each element. In Case one, a coarse mesh is used for elasto-plastic analyses, while in Case two a coarse mesh with XFEM formulation is used. Finally, in Case three a fine mesh is used. The results from XFEM formulation (Case two) are compared with those from Case one and Case three. The difference between Case one result and Case two result shows the benefit from the XFEM formulation. The results from Case three are regarded as the limit of the beam theory. The comparison on the total number of DOF for each case is listed in Table 5.1.

Table 5.1 The comparison on the total DOF used in each case

	FEM coarse mesh Case 1	FEM fine mesh Case 3	XFEM Case 2		
			clamped-pin supported beam	pin-pin supported beam	clamped- clamped beam
standard DOF	22	122	22	22	22
additional DOF	0	0	12	6	18
total DOF	22	122	34	28	40

5.4.1 Example 1: A beam with one end clamped and the other supported by pin on roller

A beam is clamped at one end and supported by pin on roller on the other end, as shown in Figure 5.10. The beam is applied by a concentrated load ($F_0 = 500$ kN). According to beam theory, the initial yielding occurs at a load factor $\lambda_0 = 1.93$ and the ultimate load factor $\lambda_u = 2.16$.

Figure 5.10 A beam with one end clamped and the other supported by pin on roller

The mid-point deflection (w) versus load factor (λ) is plotted for the three cases as shown in Figure 5.11. The deflection and the rotation fields corresponding to a mid-span deflection of 0.2 m are shown in Figure 5.12(a) and (b), respectively.

It is found that the XFEM formulation is able to predict the ultimate load close to the limit of the beam theory. The XFEM model is softer than the numerical model with a fine mesh when the deflection is less than 0.05. As the Ritz trial space is expanded by the enrichment function, XFEM provides a more flexible stiffness.

Figure 5.11 The equilibrium path of example 1

(a) The deflection field

(b) The rotation field
Figure 5.12 The displacement field for Example 1

From Figure 5.12, it can be found that the rotation displacement from the FEM coarse mesh is very different when compared with the beam theory limit while the XFEM rotation field is quite close to the beam theory limit, which implies that the present XFEM formulation is able to capture the high gradient displacement near a plastic hinge, especially for rotation displacement.

5.4.2 Example 2: A pin-pin supported beam applied by a mid-point load

A pin-pin supported beam, as shown in Figure 5.13, is tested in this example. The beam is applied by a point load ($F_0 = 500$ kN) at the mid-point. According to beam theory, the initial yielding occurs when load factor $\lambda_0 = 1.20$ and the ultimate load factor $\lambda_u = 1.41$. The deflection at the mid-point (w) versus the loading factor (λ) is plotted for the three cases as shown in Figure 5.14.

Figure 5.13 A pin-pin supported beam applied by a mid-point load

The deflection and the rotation fields at the stage when the deflection of the mid-point is equal to 0.2 m are shown in Figure 5.15(a) and (b), respectively.

Figure 5.14 The equilibrium path of Example 2

Similar to Example 1 in this chapter, the present XFEM is able to provide an ultimate load close to the beam theory limit. The displacement approximation fields from the XFEM formulation, especially the rotation field, are quite close to the results from the FEM fine mesh. The improvement from the XFEM formulation is obvious in this example.

(a) The deflection field

(b) The rotation field

Figure 5.15 The displacement field for Example 2

5.4.3 Example 3: A clamped beam applied by a mid-point load

A fully clamped beam is tested, as shown in Figure 5.16. The beam is applied by a point load ($F_0 = 500$ kN) at the mid-point. According to beam theory, the initial yielding occurs at a load factor $\lambda_0 = 2.41$ and the ultimate load factor $\lambda_u =$

2.81. The mid-point (w) deflection versus the load factor (λ) is plotted for the three cases as shown in Figure 5.17. The deflection and the rotation fields corresponding to the mid-span deflection of 0.12m are shown in Figure 5.18(a) and (b), respectively.

Figure 5.16 A clamped beam applied by a mid-point load

Figure 5.17 The equilibrium path of Example 3

Figure 5.17 shows that XFEM is able to predict an ultimate load close to the beam theory limit and the XFEM model is even softer than the numerical model with a fine mesh when the deflection is less than 0.06.

Again the XFEM formulation is able to reproduce the rotation displacement field as accurately as the FEM formulation with a fine mesh, while the FEM formulation with a coarse mesh fails to do so.

(a) The deflection field

(b) The rotation field

Figure 5.18 The displacement field for Example 3

5.5 Closure

In this chapter, the regularized enrichment function for a plastic hinge inside a beam element is introduced. Hermite function is employed in the non-smooth part resulting from a plastic hinge. The enrichment function over the smooth part of an enriched element is as the same shape as that for a perfect pin connection. Since Hermite function is used, the regularized enrichment functions for the rotation and deflection field are C_1 continuous over an enriched element. Hence, the strain field is continuous inside the enriched element.

It can be seen from the numerical examples that the XFEM formulation is able to provide the same ultimate loading level as the FEM formulation with a fine mesh. However, the computational effort is greatly saved by the XFEM formulation. The reproduced rotation approximation field from the XFEM formulation is even better than that from the FEM formulation with a fine mesh. The kinks in the rotation approximation field from the FEM analysis is smoothened by the XFEM formulation.

In this chapter, the XFEM formulation with regularized enrichment function performs well in capture the locally non-smooth displacement resulting from a plastic hinge and shows a potential in its application in yield line analyses. In the next chapter, regularized enrichment functions will be introduced in plate elements for yield line analysis. The enrichment constructed on the element level for plate elements to model a yield line is introduced. The enrichment shares the same origin as that for a plastic hinge presented in this chapter.

CHAPTER 6 THE XFEM PLATE ELEMENTS

WITH LOCAL ENRICHMENT FUNCTION

6.1 Introduction

In this chapter, an XFEM formulation embedded in plate elements to capture the non-smooth displacement field resulting from a yield line is presented. The XFEM formulation is based on a C_0 plate element including transverse shear deformations. Both the rotation and the translation displacement approximation fields are enriched by *local* enrichment functions. The enrichment functions are called ‘local’, because they are constructed on the element level by the element shape functions. The enrichment functions employed in this chapter are extensions of those in Chapter 5. A Hermite surface is constructed within the high gradient zone. It is noted that the high gradient zone *should not* be regarded as a yield line or a yield area (Xu et al. in press), because there is no technique in the present formulation to prevent a point outside the high gradient be yielded and in reality, a yielded area may not always produce non-smoothness. The enrichment functions are shifted to avoid the problems in blending elements, as mentioned in Chapter 4.

6.2 The enrichment functions

Before the enrichments are formulated, some assumptions on the high gradient zone are stated as follows:

- (1) A high gradient zone is assumed to be a band or a strip with a constant width along its longitudinal direction.
- (2) The centre line of a high gradient zone is assumed to be perpendicular to the direction of the maximum gradient of the displacement approximation field.
- (3) The width of a high gradient zone does not increase during a finite element analysis.
- (4) An enriched element is cut by one high gradient zone only, which means for each enriched element only a set of enrichment is added into its displacement approximation. However, a node can be enriched by many sets of additional DOF.

(a) a high gradient zone in the global coordinate

(b) the high gradient zone in the local coordinate system
Figure 6.1 A high gradient zone in a triangular element

In the present formulation, a high gradient zone is described by a function, $\varphi(r, s) = 0$, with a parametric width ($\omega \geq 0.0$) as *a priori* information. As shown in Figure 6.1, an enriched element is divided into two parts: the high gradient zone and the remaining parts of the element. The parametric width (ω) depends on the size of the enriched element (l_e) and the physical width of the transition range (l_{ns}) such that

$$\omega = \frac{l_{ns}}{l_e} \quad (6.1)$$

An example of the high gradient zone in a triangular element is shown in Figure 6.1. In the triangular element, the parametric length of the high gradient zone (ω) is prescribed as 0.4 and the parametric location of it is prescribed as $\varphi = r - 0.5 = 0$. A regularized shifted enrichment is constructed over the domain outside the high gradient zone. The Hermite functions (H_t for translation and H_r

for rotation) are constructed in the high gradient zone based on the 4 interpolation points, 8 tangent vectors and 4 second derivatives. A parent coordinate system (ζ, η) for high gradient zone, as shown in Figure 6.2, is established so that the Hermite functions can be expressed in a consistent way.

Figure 6.2 The parent coordinate system for a high gradient zone

The enrichment S for the rotation approximation is expressed by

$$S_i(r, s) = R(\varphi) - H(\varphi_i) = \begin{cases} H_r - H(\varphi_i) & \cdots |\varphi| < 0.5\omega \\ H(\varphi) - H(\varphi_i) & \cdots \text{otherwise} \end{cases} \quad (6.2)$$

where $R(\varphi)$ is the regularized enrichment for the *rotation displacement* approximation field before shifting, $H(\varphi)$ is the step function and $H(\varphi_i)$ ($i = 1, 2, \dots, n$) is the nodal value of the step function at node i and n is the total number of nodes in a plate element, namely, $n = 6$ for the 6-node plate element and $n = 9$ for the 9-node plate element.

$$H(\varphi) = \begin{cases} -1 & \varphi < 0 \\ 0 & \varphi = 0 \\ 1 & \varphi > 0 \end{cases} \quad (6.3)$$

The regularized shifted enrichment for the *translation displacement* approximation field used in the present formulation is of the form:

$$F(\varphi) = \begin{cases} H_t & |\varphi| < 0.5\omega \\ N_k |\varphi_k| - |\varphi_k| N_k & \text{otherwise} \end{cases} \quad (6.4)$$

where N_k are the standard shape functions for the plate elements.

In Equation (6.2) and (6.4), the Hermite functions for rotation and translation approximations, H_r and H_t , are constructed by the matrix expression as:

$$H_r = \mathbf{P}_\xi^T \cdot \mathbf{M}^T \cdot \mathbf{G}_r \cdot \mathbf{M} \cdot \mathbf{P}_\eta \quad (6.5)$$

$$H_t = \mathbf{P}_\xi^T \cdot \mathbf{M}^T \cdot \mathbf{G}_t \cdot \mathbf{M} \cdot \mathbf{P}_\eta \quad (6.6)$$

where $\mathbf{G}_r$ and $\mathbf{G}_t$ are the Hermite geometry vectors for rotational and translational DOFs, respectively. $\mathbf{M}$ is the Hermite matrix. $\mathbf{P}_\xi$ and $\mathbf{P}_\eta$ are the cubic power basis vectors of ξ and η , respectively.

$$\mathbf{G}_r = \begin{pmatrix} H(\varphi_{00}) & H(\varphi_{01}) & H_{,\eta}(\varphi_{00}) & H_{,\eta}(\varphi_{01}) \\ H(\varphi_{10}) & H(\varphi_{11}) & H_{,\eta}(\varphi_{10}) & H_{,\eta}(\varphi_{11}) \\ H_{,\xi}(\varphi_{00}) & H_{,\xi}(\varphi_{01}) & H_{,\xi\eta}(\varphi_{00}) & H_{,\xi\eta}(\varphi_{01}) \\ H_{,\xi}(\varphi_{10}) & H_{,\xi}(\varphi_{11}) & H_{,\xi\eta}(\varphi_{10}) & H_{,\xi\eta}(\varphi_{11}) \end{pmatrix} = \begin{pmatrix} H(\varphi_{00}) & H(\varphi_{01}) & 0 & 0 \\ H(\varphi_{10}) & H(\varphi_{11}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (6.7)$$

$$\mathbf{G}_t = \begin{pmatrix} F(\varphi_{00}) & F(\varphi_{01}) & F_{,\eta}(\varphi_{00}) & F_{,\eta}(\varphi_{01}) \\ F(\varphi_{10}) & F(\varphi_{11}) & F_{,\eta}(\varphi_{10}) & F_{,\eta}(\varphi_{11}) \\ F_{,\xi}(\varphi_{00}) & F_{,\xi}(\varphi_{01}) & F_{,\xi\eta}(\varphi_{00}) & F_{,\xi\eta}(\varphi_{01}) \\ F_{,\xi}(\varphi_{10}) & F_{,\xi}(\varphi_{11}) & F_{,\xi\eta}(\varphi_{10}) & F_{,\xi\eta}(\varphi_{11}) \end{pmatrix} \quad (6.8)$$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \mathbf{P}_\xi = (1 \quad \xi \quad \xi^2 \quad \xi^3)^T, \mathbf{P}_\eta = (1 \quad \eta \quad \eta^2 \quad \eta^3)^T \quad (6.9)$$

In Equation (6.7) and Equation (6.8), $S_{,\xi} = \partial S / \partial \xi$ refers to the partial derivative. It is obvious that $\partial S / \partial \xi = \partial S / \partial \eta = \partial^2 S / \partial \xi \partial \eta = 0$ outside the high gradient zone.

The first derivatives of the enrichment functions S_i and F with respect to the element parent coordinate r and s over the high gradient zone domain are of the form

$$\begin{pmatrix} \frac{\partial S_i}{\partial r} \\ \frac{\partial S_i}{\partial s} \end{pmatrix} = \frac{\partial S_i}{\partial \mathbf{r}} = \frac{\partial R}{\partial \mathbf{r}} = H_{r,r}, \quad \begin{pmatrix} \frac{\partial F}{\partial r} \\ \frac{\partial F}{\partial s} \end{pmatrix} = \frac{\partial F}{\partial \mathbf{r}} = H_{t,r} \quad (6.10)$$

$$H_{r,r} = \mathbf{P}_{\xi,r}^T \cdot \mathbf{M}^T \cdot \mathbf{G}_r \cdot \mathbf{M} \cdot \mathbf{P}_\eta + \mathbf{P}_\xi^T \cdot \mathbf{M}^T \cdot \mathbf{G}_r \cdot \mathbf{M} \cdot \mathbf{P}_{\eta,r} \quad (6.11)$$

$$H_{t,r} = \mathbf{P}_{\xi,r}^T \cdot \mathbf{M}^T \cdot \mathbf{G}_t \cdot \mathbf{M} \cdot \mathbf{P}_\eta + \mathbf{P}_\xi^T \cdot \mathbf{M}^T \cdot \mathbf{G}_t \cdot \mathbf{M} \cdot \mathbf{P}_{\eta,r} \quad (6.12)$$

and

$$\mathbf{P}_{\xi,r} = \frac{\partial \xi}{\partial \mathbf{r}} \begin{pmatrix} 0 \\ 1 \\ 2\xi \\ 3\xi^2 \end{pmatrix}, \quad \mathbf{P}_{\eta,r} = \frac{\partial \eta}{\partial \mathbf{r}} \begin{pmatrix} 0 \\ 1 \\ 2\eta \\ 3\eta^2 \end{pmatrix} \quad (6.13)$$

where chain rule is applied.

The local enrichment implemented in the 6-node plate element is illustrated in this section. The particular case that $\varphi = r - 0.5 = 0$ and $\omega = 0.4$ of a high

gradient zone in a triangular element, as shown in Figure 6.1(a) is taken as an example. The plot of the local enrichment for rotational DOF is shown in Figure 6.3.

(a) The 3D plot of R

(b) The contour of R

Figure 6.3 The plot of R

The local enrichment for translational DOF, F , is shown in Figure 6.4.

(a) The 3D plot of F

(b) The contour of F_2

Figure 6.4 The plot of F

The plots of the first derivative of the local enrichments with respect to natural coordinates are shown in Figure 6.5 to Figure 6.7.

(a) The 3D plot of R_r

(b) The contour of R_r

Figure 6.5 The plot of R_r

(a) The 3D plot of F_r

(b) The contour of F_r

Figure 6.6 The plot of F_r

Figure 6.7 The plot of $R_{,s}, F_{,s}$

It should be noted that a variation of the width of the high gradient zone could change the shape of the local enrichment for translational DOF. Two examples are shown in Figure 6.8 and Figure 6.9, respectively. In these two examples, the locations of the high gradient zone are the same as that shown in Figure 6.1(a) while the width is $\omega = 0.5$ and $\omega = 0.6$, respectively. It could be seen that the Hermite function is flat (of a constant value) when the width of the high gradient zone is $\omega = 0.5$, while the Hermite function is convex when the width of the high gradient zone is $\omega = 0.6$. It is also shown in Figure 6.4(a) that the sign of curvature in the Hermite function along the r direction for $\omega = 0.4$ is different from that of the case for $\omega = 0.6$. The influence from the width of a high gradient zone on numerical results is studied in the numerical examples in Section 6.5.

Figure 6.8 The local enrichment for translational DOF with $\omega = 0.5$

Figure 6.9 The local enrichment for translational DOF with $\omega = 0.6$

6.3 The enriched plate element formulation

In the present XFEM plate element, there are six DOF per node in enriched elements, which is $\mathbf{u}_{\text{enri}} = (w_i, \theta_{xi}, \theta_{yi}, a_{xi}, a_{yi}, b_i)^T$. The variables a_{xi} and a_{yi} are the additional rotational DOF with respect to x- and y-axis, while b_i is the additional translational DOF.

The displacement field can be approximated by the sum of the smooth part and the non-smooth part:

$$w_h = \sum_{i=1}^6 N_i w_i + \sum_{i=1}^6 L_i b_i \quad (6.14)$$

$$\theta_{xh} = \sum_{i=1}^6 N_i \theta_{xi} + \sum_{i=1}^6 M_i a_{xi} \quad (6.15)$$

$$\theta_{yh} = \sum_{i=1}^6 N_i \theta_{yi} + \sum_{i=1}^6 M_i a_{yi} \quad (6.16)$$

where a_{xi} , a_{yi} and b_i are the additional DOF corresponding to non-smooth displacement approximation fields for rotations and deflections, L_i and M_i are the interpolation functions for the non-smooth translation and rotation displacement, respectively. It should be noted that in a small deformation analysis, the rotational DOF can be regarded as a vector as the translational DOF (Crisfield 1991b) so that Equation (6.16) is valid. L_i and M_i can be found by the multiplication of the non-smooth enrichment and window functions. In the present XFEM formulation, the standard shape functions for a 6-node triangular element are used as window functions for the localization of the enrichments:

$$M_i = N_i S_i \quad (6.17)$$

$$L_i = N_i F \quad (6.18)$$

where S_i and F are enrichments for rotation and translation displacement, respectively (Equation (6.2) and Equation (6.4)).

The bending strain ϵ_{benr} and the shear strain γ_{enr} at an arbitrary point in an enriched element can be expressed by

$$\epsilon_{\text{benr}} = -z \begin{pmatrix} \frac{\partial \beta_{\text{xh}}}{\partial x} \\ \frac{\partial \beta_{\text{yh}}}{\partial y} \\ \frac{\partial \beta_{\text{yh}}}{\partial x} + \frac{\partial \beta_{\text{xh}}}{\partial y} \end{pmatrix} \quad (6.19)$$

$$\gamma_{\text{enr}} = \begin{pmatrix} -\beta_{\text{xh}} + \frac{\partial w_{\text{h}}}{\partial x} \\ -\beta_{\text{yh}} + \frac{\partial w_{\text{h}}}{\partial y} \end{pmatrix} \quad (6.20)$$

where β_{xh} and β_{yh} are the rotation angles of the lines normal to the undeformed neutral surface in x - z and y - z plane:

$$\begin{pmatrix} \beta_{\text{xh}} \\ \beta_{\text{yh}} \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \theta_{\text{xh}} \\ \theta_{\text{yh}} \end{pmatrix} \quad (6.21)$$

6.4 Shear locking and numerical integration scheme

In the present XFEM formulation, the 6×6 Gaussian integration scheme is regarded as the full integration scheme for the present triangular XFEM element, if the in-plane variation of the tangential modulus of the material is not taken into account. However, shear locking occurs if full integration scheme is used. From numerical point view, shear locking could be a major concern in the present XFEM formulation. In this chapter, a reduced integration is used to control shear locking. Hence, in the present XFEM element, the 4×4 Gaussian integration scheme is employed in each sub-domain while in some special cases

listed in Appendix C and Appendix D, the 3×3 Gaussian integration scheme can be used in each sub-domain.

An enriched element is at least divided into two sub-domains. Hence, a 3×3 Gaussian integration scheme in both sub-domains provides 18 integration points ($5 \times 18 = 90$ constraints) in the element. On the other hand, in the present XFEM formulation, $6 \times 6 = 36$ constraints are sufficient to avoid matrix singularity due to numerical integration (Zienkiewicz and Taylor 2005a).

6.5 Numerical examples

As mentioned in Section 6.2, the shape of the local enrichment for translational DOF depends on the width of a high gradient zone. In this section, the influence of the width of a high gradient zone is studied. Different values of the width of a high gradient zone are assumed in each example.

6.5.1 A flat strip with two fully fixed ends

Figure 6.10 Example 1: A flat strip with two fully fixed ends

A flat rectangular plate is tested in this example (Figure 6.10). The length of the plate is $L = 5.0\text{m}$ and the width is $b = 1.0\text{m}$. The plate is fully fixed on the two short edges and free on the two long edges. The Young's modulus is $E = 1 \times 10^6 \text{kN/m}^2$. The Poisson's ratio is $\nu = 0.0$. The mesh scheme and the location of the three possible high gradient zones are shown in Figure 6.11. As a layered model is used in the present XFEM formulation, 5 layers are used for each element in this example. The deflection of point A, as shown in Figure 6.10, is investigated.

Figure 6.11 The mesh scheme and the locations of the three possible yield lines, unit: m

6.5.1.1 Study on shear locking and reduced integration

The effect of reduced integration is studied in this example with five different thicknesses. The yield strength σ_y and the reference loading q_0 is normalized according to the plate thickness (Table 6.1) so that for all the five cases, the upper bound theoretical failure load factor λ_u obtained from yield line pattern

analysis (Johansen 2004) is all equal to 0.625. The normalizations are as follows:

$$\sigma_{yi} = \frac{t_i}{t_1} \sigma_{y1} \quad (6.22)$$

$$q_{0i} = \left(\frac{t_i}{t_1} \right)^2 q_{01} \quad (6.23)$$

where σ_{y1} and q_{01} are the yield strength and the reference loading for Case 1 and σ_{yi} and q_{0i} are the yield strength and the reference loading for Case i . The term t_1 and t_i are the thickness for Case 1 and Case i , respectively.

Table 6.1 The three cases of example 1

	thickness t (m)	yield strength σ_y (kN/m ²)	reference loading q_0 (kN/m)	thickness/length ratio t / L
Case 1	0.5	0.4	6.4×10^{-2}	1 / 10
Case 2	0.25	0.2	8×10^{-3}	1 / 20
Case 3	0.125	0.1	1×10^{-3}	1 / 40
Case 4	0.05	0.04	6.4×10^{-5}	1 / 100
Case 5	0.01	0.008	5.12×10^{-7}	1 / 500

The results from full integration are shown in Figure 6.12. The results from reduced integration are shown in Figure 6.13 and with the result from conventional plate FE analysis with fine mesh ($10 \times 50 \times 2$ triangular elements) and coarse mesh ($1 \times 5 \times 2$ triangular elements). The parametric width is set as $\omega_1 = \omega_2 = \omega_3 = 0.5$.

Figure 6.12 The equilibrium path obtained for Example 1 by full integration

It can be seen from Figure 6.12 that the load level from full integration are little lower than that from the coarse mesh when the deflection of point A is less than 5×10^{-4} m. However, when the ratio of thickness/element size becomes even smaller, the equilibrium path goes up and shear locking occurs. Shear locking counteracts the benefit from XFEM formulation. It is no surprising to find from Figure 6.13 that the reduced integration used in the XFEM element is able to alleviate shear locking effectively, because the essence of reduced integration is the penalty method (Cook et al. 2002), which can eliminate shear locking.

Figure 6.13 The equilibrium path obtained for Example 1 by reduced integration

6.5.1.2 Study on the width of the high gradient zone

In this study, the material and geometric properties of Case 2 in Section 6.5.1 is used. The width of the high gradient zone (ω) (Equation (6.1)) varies. The deflection of point A is investigated. The results from XFEM formulation are compared with the results from conventional FEM using a fine mesh ($10 \times 50 \times 2$ elements) and a coarse mesh ($1 \times 5 \times 2$ elements).

First the width of the middle high gradient zone (ω_2) is studied. The width of the two side high gradient zones is fixed as $\omega_1 = \omega_3 = 0.5$. The width of the middle high gradient zone (ω_2) varies from 0.3 to 0.9. As shown in Figure 6.14, it can be seen in this example that the value of ω_2 has significant influence on

the ultimate load. With a greater value of ω_2 , a lower prediction of ultimate load is obtained.

Figure 6.14 The equilibrium path obtained for Example 1 with variation of ω_2

Secondly, the influence from the width of the two side high gradient zones (ω_1 , ω_3) is studied. The width of the two side high gradient zones varies from 0.1 to 0.9, while the width of the middle high gradient zone is fixed as $\omega_2 = 0.5$. It can be seen from Figure 6.15 that the result from the case $\omega_1 = \omega_3 = 0.1$ gives a better prediction than the other results and the smaller the width of side high gradient zone is assumed, the closer the result is to that from conventional finite elements in a fine mesh. It is also found that the initial stiffness of the computational model with $\omega_1 = \omega_3 = 0.9$ is much softer than the others. This is due to singularity of the stiffness matrix corresponding to enriched elements.

Figure 6.15 The equilibrium paths obtained for Example 1 with variation of ω_1 and ω_3

As shown in Section 6.2, the shape of the enrichment function presented in this chapter depends on the width ω . Hence, the ultimate loadings predicted by different ω are different from each other, which are shown in Figure 6.14 and Figure 6.15.

From Figure 6.14, it is found that the greater the width of the high gradient zone, the closer the prediction is to the yield line pattern analysis, while from Figure 6.15, the converse is true, that is, the smaller the width, the closer the prediction is to the yield line pattern analysis. This phenomenon is observed from numerical predictions. As it is mentioned in Section 6.2 the shape of the enrichment depends on the width of the high gradient zone, which will

eventually have great influence on the numerical results. In the middle high gradient zone, the localized effect of non-smooth displacement is not so dominant, while in the high gradient zone on the two sides, the localized effect is obvious. Hence, in the middle of the span, if the width of the high gradient zone is enlarged, the ultimate load decreases. On the other hand, the ultimate load decreases with decreasing width of the high gradient zone on the two sides.

6.5.2 An L-shaped plate with two fully fixed ends

Figure 6.16 Example 2: An L-shaped plate with two fully fixed ends, unit: m

An L-shaped flat plate is tested in this example. The plate is fully fixed on the two short edges. The geometrical property is shown in Figure 6.16. The thickness is $t = 0.15\text{m}$. The Young's modulus is $E = 1.0 \times 10^6 \text{kN/m}^2$. The Poisson's ratio is $\nu = 0.3$. The yield strength is $\sigma_y = 1.0 \text{kN/m}^2$. The reference load is $q_0 = -3 \times 10^{-3} \text{kN/m}$. The coarse mesh scheme and the locations of three possible high gradient zones are shown in Figure 6.17, while the fine mesh

scheme is shown in Figure 6.18. It can be seen that in this example, the non-smooth part 2 passes cross the edges of element 5 and 6 (in Figure 6.17). It should be noted that in the present XFEM formulation, the enrichments are constructed on the element level by the standard shape functions for the isoparametric 6-node triangular element and the distance-function, therefore, the enrichment for high gradient part 3 is only C_0 continuous along the common edge of element 5 and 6.

Figure 6.17 The coarse mesh scheme of Example 2

Figure 6.18 The fine mesh scheme of Example 2

In XFEM analysis, a reduced integration scheme with 3×3 Gaussian quadrature is used in each sub-domain. As a layered model is employed in the present formulation, 5 layers are used for each element. First, the width of the middle high gradient zone is fixed as $l_{ns2} = 1.0$ and the width of the two side high gradient zones ($l_{ns1} = l_{ns3}$) varies from 0.1 to 0.5. The deflection of point A, at the corner of the L shape plate, is investigated. The result from XFEM formulation is shown in Figure 6.19, compared with the result from standard FEM with coarse mesh (as shown in Figure 6.17) and the result from ANSYS. In ANSYS analysis, a total of 1000 SHELL93 (ANSYS 2011) elements, with 5 layers for each element, are used. The mesh scheme of the ANSYS analyses is shown in Figure 6.18. It can be seen from Figure 6.19 that a smaller value of l_{ns1} and l_{ns3} provides a lower prediction on the ultimate load.

Figure 6.19 The equilibrium paths obtained for Example 2 with variation of l_{ns1} and l_{ns3}

Figure 6.20 The equilibrium paths obtained for Example 2 with variation of l_{ns2}

The influence from the width of the middle high gradient zone is studied. The width of the two side high gradient zones is fixed as $l_{ns1} = l_{ns3} = 0.1$ and the width of the middle high gradient zone varies from 0.6 to 1.2. The result is shown in Figure 6.20. It can be seen that the predicted ultimate load decreases with increasing assumed value of l_{ns2} .

It can be found from Figure 6.19 and Figure 6.20 that the XFEM results are in between those from the FEM fine mesh and coarse mesh. This is an acceptable result, since it means the XFEM improves the result in the same mesh scheme as the FEM.

6.5.3 A square plate with roller supports at the four edges

A square plate supported by rollers on the four edges is tested in this example. The length of edge is 32m. The thickness is $t = 1.0\text{m}$. The Young's modulus is $E = 1.0 \times 10^{10}\text{N/m}^2$. The Poisson's ratio is $\nu = 0.2$. The yield strength is $\sigma_y = 1.0 \times 10^5 \text{ N/m}^2$. The reference loading is $q_0 = 1.0 \times 10^4 \text{ N/m}$. A quarter of the entire plate is modelled because of symmetry. The yield line is along the diagonal of the plate. In this example two cases are tested with different assumptions of the width of the yield line. In each case, four different sizes of yield line are assumed ($l_{ns1} = 3.2\text{m}$, $l_{ns2} = 4.8\text{m}$, $l_{ns3} = 6.4\text{m}$ and $l_{ns4} = 9.6\text{m}$). There are 5 layers in each element for XFEM analyses. The 3×3 Gaussian quadrature scheme is used in each sub-domain. The deflection of the centre point (point A in Figure 6.21) is investigated.

Figure 6.21 Example 3: A square plate with roller supports at the four edges

The results from mesh 1 and mesh 2 with the result from yield line pattern analysis (Johansen 2004), which gives $\lambda_u = 0.0586$, are shown in Figure 6.22

and Figure 6.23, respectively. In both of the mesh patterns, the greater value of l_{ns} provides a lower ultimate load.

Figure 6.22 The equilibrium paths obtained for Example 3 using mesh1

Figure 6.23 The equilibrium paths obtained for Example 3 using mesh2

Again, it can be found that the results from different values of ω are different from one another. This is due to the dependence of enrichment function shape on the value of ω , as shown in Section 6.2.

Similar to Example 6.5.2, it can be found that the XFEM results are lower than the result from the FEM with coarse mesh and higher than that from the FEM with fine mesh. It means that the XFEM formulation is able to improve the results with the same mesh scheme as the FEM formulation.

6.6 Closure

In this chapter, the enrichment function based on the element level is presented. The enrichment is constructed by the standard finite element shape functions. The Hermite function is adopted in the high gradient zone to connect the two parts with smooth displacement field. The enrichment is embedded in the 6-node triangular element. The reduced integration is used to control shear locking.

Although the enrichment function shows its effectiveness in the numerical examples, as shown in Appendix E, it is found that this *local* enrichment function depends on the location and size of the non-smooth part inside an enriched element. This feature restricts its application in the 9-node quadrilateral element. The local enrichment function is discontinuous in the 9-

node quadrilateral element. Besides, the shape of the enrichment also depends on the shape of an element in physical space and the mesh pattern, which decreases the robustness of the enrichment. It is also shown in the numerical examples that the numerical results are influenced by the width of the high gradient zone and the difference in the results from two different assumed high gradient zone widths is quite obvious.

Apart from the construction of enrichment, the mitigation of shear locking is another concern. Although reduced integration shows its effectiveness in the numerical examples, the robustness of this method is not as satisfactory as expected. As there exist several possibilities to partition an element, the integration scheme is not uniform for each case. The shape of the enrichment depends on the mesh scheme. Hence, the mesh pattern also has influence on the polynomial terms in the enrichment, which indicates that the integration scheme also depends on the mesh scheme. In the next chapter, the application of the assumed natural strain method is introduced, which is independent of the mesh scheme.

CHAPTER 7 THE XFEM PLATE ELEMENTS

WITH GLOBAL ENRICHMENT FUNCTION

7.1 Introduction

In this chapter, another type of enrichment function is chosen to enrich the displacement approximation field. Different from the *local* enrichment functions introduced in Chapter 6, the enrichment functions employed in this chapter are constructed on the structure level, which is independent of the mesh scheme. Therefore, it is called *global*. Again, the location and the width of a high gradient zone are regarded as *a priori* information in this chapter.

The DOF of the XFEM plate elements, the expression of the displacement approximation field and the kinematic equation in this chapter are the same as those in Chapter 6. Hence, they are not introduced in this chapter.

The assumed shear strain method is employed to circumvent shear locking in the XFEM formulation in this chapter. Both the MITC technique and the DSG technique are implemented in the XFEM plate elements. The assumed shear strain field is constructed in the natural coordinate system.

7.2 The enrichments based on the structure level

The basic idea of the establishment of the enrichment functions is that the enrichment functions for both rotation and deflection displacement field should at least be C_1 continuous over the whole physical domain. The variation of rotation displacement and the gradient of deflection displacement are of high gradient in the direction perpendicular to the yield line, while they are smooth along the yield line direction. Thus, the enrichments for both of the enriched displacement approximation field are cylinder-shape functions.

In the present XFEM plate element, a high gradient zone is defined on the structure level by a level set function, $\varphi(X, Y) = 0$ and a width l_{ns} . An example of a high gradient zone with $\varphi = 0.8X - 0.6Y - 0.8 = 0$ and $l_{ns} = 1.5$ in a square domain $(X, Y) \in [0, 10] \times [0, 10]$ is shown Figure 7.1.

Figure 7.1 An example of high gradient zone

The enrichment function S for the rotation approximation field is expressed by

$$S_i(\varphi) = R(\varphi) - H(\varphi_i) \quad (7.1)$$

where $R(\varphi)$ is the regularized enrichment function for the *rotation displacement* approximation field before shifting, which can be expressed as

$$R = \begin{cases} -1 & \varphi < -0.5l_{ns} \\ \frac{3}{l_{ns}}\varphi - \frac{4}{l_{ns}^3}\varphi^3 & -0.5l_{ns} \leq \varphi \leq 0.5l_{ns} \\ 1 & \varphi \geq 0.5l_{ns} \end{cases} \quad (7.2)$$

$$H(\varphi_i) = \begin{cases} 1 & \varphi_i > 0 \\ 0 & \varphi_i = 0 \\ -1 & \varphi_i < 0 \end{cases} \quad (7.3)$$

$H(\varphi_i)$ ($i = 1, 2, \dots, n_{\text{node}}$) is the nodal value of the step at node i and φ_i is the nodal value of the level set function φ at node i . A plot of $R(\varphi)$ for the particular case is shown in Figure 7.2. Since the enrichment is shifted by step function,

the problems due to blending elements are excluded in the rotation displacement approximation field (Fries 2008).

The enrichment F for the *deflection displacement* approximation field used in the present formulation is of the form

$$F(\varphi) = \begin{cases} 1.0 - \frac{8}{l_{\text{ns}}^2} \varphi^2 + \frac{16}{l_{\text{ns}}^4} \varphi^4 & |\varphi| < 0.5l_{\text{ns}} \\ 0 & \text{otherwise} \end{cases} \quad (7.4)$$

A plot of $F(\varphi)$ for the particular example is shown in Figure 7.3. It could be seen that the $F(\varphi)$ is non-zero inside high gradient zone and zero outside high gradient zone. Hence PU condition is satisfied over the whole domain.

(a) The 3D plot of R

(b) The contour of R
Figure 7.2 The plot of R

(a) The 3D plot of F

(b) The contour of F
Figure 7.3 The plot of F

Different from the local enrichment introduced in the previous chapters, the shape of the global enrichment is independent of the location and the width of the high gradient zone. Hence, the implementation of the global enrichment in the 6-node element and the 9-node element are the same. Therefore, in the following derivation, the implementation of the global enrichment in the 6-node plate element and the 9-node plate element are introduced together.

7.3 The partial derivative of the enrichment function with respect to natural coordinates

As the enrichment function S (Equation (7.1)) and F (Equation (7.4)) are constructed on the structure level, the relationship between the enrichment function and the element natural coordinates (r and s) is not explicit. The relationship is used in kinematic equation (Equation (6.19) and (6.20)). Therefore, it is expressed in this section.

The partial derivative of the non-smooth displacement field is written as:

$$\frac{\partial \alpha}{\partial \mathbf{r}} = S \frac{\partial N_i}{\partial \mathbf{r}} \alpha_i + \frac{\partial S}{\partial \mathbf{r}} N_i \alpha_i \quad (7.5)$$

$$\frac{\partial b}{\partial \mathbf{r}} = F \frac{\partial N_i}{\partial \mathbf{r}} b_i + \frac{\partial F}{\partial \mathbf{r}} N_i b_i \quad (7.6)$$

where α and b are the non-smooth rotational displacement field and the deflection displacement field, respectively, and $\partial / \partial \mathbf{r} = (\partial / \partial r, \partial / \partial s)^T$. Since the enrichments S and F are constructed on the structure level, the first derivative of enrichment functions with respect to natural coordinate variables r and s can be obtained by the chain rule as follows

$$\frac{\partial S}{\partial \mathbf{r}} = \frac{\partial \mathbf{X}^T}{\partial \mathbf{r}} \frac{\partial \varphi}{\partial \mathbf{X}} \frac{\partial R}{\partial \varphi} \quad (7.7)$$

$$\frac{\partial F}{\partial \mathbf{r}} = \frac{\partial \mathbf{X}^T}{\partial \mathbf{r}} \frac{\partial \varphi}{\partial \mathbf{X}} \frac{\partial F}{\partial \varphi} \quad (7.8)$$

where $\partial\varphi / \partial\mathbf{X}$ is the normalized direction vector of φ . It is noted that the values of $\partial S / \partial\mathbf{r}$ and $\partial F / \partial\mathbf{r}$ depend on the values of the components in the matrix $\partial\mathbf{X}^T / \partial\mathbf{r}$, which are related to the size of an enriched element in a mesh pattern. This could be a source of dependence of the high gradient zone width (l_{ns}) on the element size. The dependence of width on mesh size is also reported in (Abbas et al. 2010).

7.4 Shear locking and assumed shear strain method

In this chapter, the MITC technique and DSG technique are applied in the XFEM plate elements to mitigate shear locking. In order to make the implementation of the MITC technique convenient in the XFEM formulation, the expressions of the assumed shear strain fields in the MITC technique are rewritten in the form of an interpolation of nodal displacement variables.

Firstly, an alternative expression of the assumed shear strain field from MITC9 is introduced. Recall the original expression of the MITC9 (Equation (3.56) and Equation (3.57)):

$$e_{\text{rt}} = l_{\text{rti}} \gamma_{\text{rti}}, \quad e_{\text{st}} = l_{\text{sti}} \gamma_{\text{sti}}$$

The compatible shear strain at the tying points (Figure 3.10) can be expressed by the nodal displacement variables as

$$\gamma_{\text{rti}} = -N_j(r_i, s_i) \beta_{\text{rj}} + \frac{\partial N_j(r_i, s_i)}{\partial r} w_j \quad (7.9)$$

$$\gamma_{sti} = -N_j(r_i, s_i) \beta_{sj} + \frac{\partial N_j(r_i, s_i)}{\partial s} w_j \quad (7.10)$$

where the index i refers to the tying point number ($i = 1, 2, 3, \dots, 6$) and the index j refers to the node number ($j = 1, 2, 3, \dots, 9$).

Substituting Equation (7.9) into Equation (3.56) and Equation (7.10) into Equation (3.57), the alternative expression of the assumed shear strain in MITC9 is expressed as

$$e_{ti} = -h_{ti} \beta_{ti} + h_{twi} w_i \quad (7.11)$$

$$e_{si} = -h_{si} \beta_{si} + h_{swi} w_i \quad (7.12)$$

in which, h_{ti} and h_{twi} are the interpolation functions for the nodal displacement variables β_{ti} and w_i in assumed shear strain field e_{ti} in MITC9 technique, while h_{si} and h_{swi} are the interpolation functions for the nodal displacement variables β_{si} and w_i in the assumed shear strain field e_{si} in MITC9 technique. The interpolation functions can be expressed explicitly as

$$\begin{aligned} h_{t1} &= \left(-\frac{1}{4}r + \frac{1}{12}\right)(s^2 - s), \quad h_{t2} = \left(\frac{1}{4}r + \frac{1}{12}\right)(s^2 - s), \quad h_{t3} = \left(\frac{1}{4}r + \frac{1}{12}\right)(s^2 + s), \\ h_{r4} &= \left(-\frac{1}{4}r + \frac{1}{12}\right)(s^2 + s), \quad h_{r5} = \frac{1}{3}(s^2 - s), \quad h_{r6} = \left(\frac{1}{2}r + \frac{1}{6}\right)(1 - s^2), \\ h_{r7} &= \frac{1}{3}(s^2 + s), \quad h_{r8} = \left(-\frac{1}{2}r + \frac{1}{6}\right)(1 - s^2), \quad h_{r9} = \frac{2}{3}(1 - s^2) \\ h_{s1} &= \left(-\frac{1}{4}s + \frac{1}{12}\right)(r^2 - r), \quad h_{s2} = \left(-\frac{1}{4}s + \frac{1}{12}\right)(r^2 + r), \quad h_{s3} = \left(\frac{1}{4}s + \frac{1}{12}\right)(r^2 + r), \\ h_{s4} &= \left(\frac{1}{4}s + \frac{1}{12}\right)(r^2 + r), \quad h_{s5} = \left(-\frac{1}{2}s + \frac{1}{6}\right)(1 - r^2), \quad h_{s6} = \frac{1}{3}(r + r^2), \end{aligned} \quad (7.13)$$

$$h_{s7} = \left(\frac{1}{2}s + \frac{1}{6} \right) (1-r^2), \quad h_{s8} = \frac{1}{3}(r^2 - r), \quad h_{s9} = \frac{2}{3}(1-r^2) \quad (7.14)$$

$$h_{rwi} = \frac{\partial N_i}{\partial r} \quad (7.15)$$

$$h_{swi} = \frac{\partial N_i}{\partial s} \quad (7.16)$$

It can be seen that the interpolation functions for the assumed shear strain field are exactly the same for MITC9 and DSG9. Thus, it is proved that the DSG technique is equivalent to the MITC technique in 9-node quadrilateral plate element. This equivalence is also reported in 4-node quadrilateral plate element in (Bletzinger et al. 2000).

The above derivation process of alternative expressions of the assumed shear strain field can also be applied in MITC6 and DSG6. However the resulting interpolation functions show that DSG6 is not equivalent to MITC6. The derivation process is not introduced in this thesis, while the interpolation functions are provided.

In MITC6, the alternative expression of the shear strain field is expressed as

$$e_{rt} = -h_{ri}\beta_{ri} - h_{rs}\beta_{si} + h_{rwi}w_i \quad (7.17)$$

$$e_{st} = -h_{si}\beta_{ri} - h_{ss}\beta_{si} + h_{swi}w_i \quad (7.18)$$

where

$$h_{\text{tr}1} = \frac{2}{3} - r - \frac{7}{3}s + \frac{4}{3}rs + \frac{5}{3}s^2, \quad h_{\text{tr}2} = -\frac{1}{3} + r - \frac{2}{3}s + \frac{1}{3}rs + \frac{2}{3}s^2,$$

$$h_{\text{tr}3} = -s + \frac{1}{3}rs + \frac{5}{3}s^2,$$

$$h_{\text{tr}4} = \frac{2}{3} - \frac{4}{3}rs - \frac{2}{3}s^2, \quad h_{\text{tr}5} = \frac{4}{3}s + \frac{2}{3}rs - \frac{2}{3}s^2, \quad h_{\text{tr}6} = \frac{8}{3}s - \frac{4}{3}rs - \frac{8}{3}s^2 \quad (7.19)$$

$$h_{\text{rs}1} = s - \frac{5}{3}rs - \frac{4}{3}s^2, \quad h_{\text{rs}2} = \frac{2}{3}s - \frac{5}{3}rs - \frac{1}{3}s^2, \quad h_{\text{rs}3} = \frac{1}{3}s - \frac{2}{3}rs - \frac{1}{3}s^2,$$

$$h_{\text{rs}4} = -\frac{4}{3}s + \frac{8}{3}rs + \frac{4}{3}s^2, \quad h_{\text{rs}5} = \frac{2}{3}rs - \frac{2}{3}s^2, \quad h_{\text{rs}6} = -\frac{2}{3}s + \frac{2}{3}rs + \frac{4}{3}s^2 \quad (7.20)$$

$$h_{\text{sr}1} = r - \frac{5}{3}rs - \frac{4}{3}r^2, \quad h_{\text{sr}2} = \frac{1}{3}r - \frac{2}{3}rs - \frac{1}{3}r^2, \quad h_{\text{sr}3} = \frac{2}{3}r - \frac{5}{3}rs - \frac{1}{3}r^2,$$

$$h_{\text{sr}4} = -\frac{2}{3}r - \frac{5}{3}rs - \frac{1}{3}r^2, \quad h_{\text{sr}5} = \frac{2}{3}rs - \frac{2}{3}r^2, \quad h_{\text{sr}6} = -\frac{4}{3}r + \frac{8}{3}rs + \frac{4}{3}r^2 \quad (7.21)$$

$$h_{\text{ss}1} = \frac{2}{3} - \frac{7}{3}r - s + \frac{4}{3}rs + \frac{5}{3}r^2, \quad h_{\text{ss}2} = -r + \frac{1}{3}rs + \frac{5}{3}r^2,$$

$$h_{\text{ss}3} = -\frac{1}{3} - \frac{2}{3}r + s + \frac{1}{3}rs + \frac{2}{3}r^2$$

$$h_{\text{ss}4} = \frac{8}{3}r - \frac{4}{3}rs - \frac{8}{3}r^2, \quad h_{\text{ss}5} = \frac{4}{3}r + \frac{2}{3}rs - \frac{2}{3}r^2, \quad h_{\text{ss}6} = \frac{2}{3} - \frac{4}{3}rs - \frac{2}{3}r^2 \quad (7.22)$$

$$h_{\text{rwi}} = \frac{\partial N_i}{\partial r}, \quad h_{\text{swi}} = \frac{\partial N_i}{\partial s} \quad (7.23)$$

In DSG6, alternative expressions of the shear strain field are expressed as

$$e_{\text{rt}} = -h_{\text{ri}}\beta_{\text{ri}} + h_{\text{twi}}w_i \quad (7.24)$$

$$e_{\text{st}} = -h_{\text{si}}\beta_{\text{si}} + h_{\text{swi}}w_i \quad (7.25)$$

where the interpolation functions are expressed explicitly as

$$h_{r1} = \frac{2}{3} - r + 4rs - 3s + 2s^2, \quad h_{r2} = -\frac{1}{3} + r, \quad h_{r3} = -s + 2s^2$$

$$h_{r4} = \frac{2}{3} - 4rs, \quad h_{r5} = 4rs, \quad h_{r6} = 4s - 4rs - 4s^2 \quad (7.26)$$

$$h_{s1} = \frac{2}{3} - 3r + 4rs - s + 2r^2, \quad h_{s2} = -r + 2r^2, \quad h_{s3} = -\frac{1}{3} + s$$

$$h_{s4} = -4rs + 4r - 4r^2, \quad h_{s5} = 4rs, \quad h_{s6} = \frac{2}{3} - 4rs \quad (7.27)$$

$$h_{rwi} = \frac{\partial N_i}{\partial r}, \quad h_{swi} = \frac{\partial N_i}{\partial s} \quad (7.28)$$

An obvious difference in the assumed transverse shear strain field between the MITC6 and the DSG6 is that, in the MITC6 technique, there are cross terms ($h_{rs}\beta_{si}$ in Equation (7.17) and $h_{sr}\beta_{ri}$ in Equation (7.18)). These cross terms come from the tying points on the Hypotenuse edge and at the centre of the triangle, while in the DSG6 method, these cross terms do not appear.

7.5 Numerical examples

In this section, five numerical examples are shown to verify the XFEM formulation in capturing the behaviour of a plate structure in elasto-plastic analyses. The performance of the assumed natural strain method is also investigated. As the MITC9 and the DSG9 provide the same transverse shear strain field, only the MITC9 XFEM is tested for quadrilateral plate element. In the meantime, both MITC6 and DSG6 interpolation schemes are tested. The

first three examples in this section are the same as those in Section 6.5. However, the items checked are different. The focus of the examples in this section is on the effectiveness of the assumed natural strain method in the XFEM formulation. In addition, the efficiency of the XFEM formulation is also shown in each example. The total number of DOF n_{DOF} , the total number of Gaussian points n_{Gauss} , The total number of the stiffness matrix updating n_s and the computational time t for the formulations, including the standard FEM with coarse and fine mesh, the MITC6 XFEM, the MITC9 XFEM and the DSG6 XFEM, are listed in each example.

7.5.1 Example 1: A flat plate with two fixed ends

Figure 7.4 Example 1: A flat strip with two fully fixed ends, unit: m

A flat rectangular plate is tested in this example, as shown in Figure 7.4. The length of the plate is $L = 5.0\text{m}$ and the width is $b = 1.0\text{m}$. The plate is fixed along the two short edges and free on the two long edges. The Young's modulus is $E = 1 \times 10^6 \text{kN/m}^2$, while the Poisson's ratio is $\nu = 0.0$. The deflection of point A (Figure 7.4) is investigated.

(a) A uniform mesh by 9-node element

(b) A uniform mesh by 6-node element

Figure 7.5 The uniform mesh of Example 1, unit: m

7.5.1.1 Study on shear locking

In this section, shear locking is studied. Coarse meshes with 5 undistorted 9-node quadrilateral elements and with 10 undistorted 6-node triangular elements are shown in Figure 7.5(a) and Figure 7.5(b), respectively. The locations of the three possible high gradient zones are also shown in Figure 7.5. Furthermore, five fibre layers are used for each element in the thickness direction. Five cases with different plate thicknesses are tested. The yield strength and the reference loading are scaled according to the thickness (as listed in Table 7.1) so that the theoretical ultimate load factors from the yield line analysis for the five cases are kept the same ($\lambda_u = 0.625$).

Firstly, the analysis with compatible XFEM elements is carried out. In the analysis, quadrilateral elements are used. The mesh scheme shown in Figure 7.5(a) is used for this analysis. The MITC9 technique is employed in the standard elements, whereas compatible shear strain field, derived from the enriched displacement field, is used in the enriched elements. The result is shown in Figure 7.6. It can be seen that the XFEM locked analysis provides an ultimate loading level even higher than the standard FEM with a coarse mesh, which implies that parasitic strain energy is introduced by the compatible XFEM formulation. Therefore, it is necessary to use the assumed shear strain method in the XFEM formulation to control shear locking.

Figure 7.6 The equilibrium path of Example 1 without locking control in enriched elements

The results obtained from the 9-node XFEM plate element are shown in Figure 7.7. In the same figure, the results obtained by using the yield line pattern analysis (Johansen 2004) ($\lambda_u = 0.625$) and those obtained by using the conventional thick plate elements from the same coarse mesh and from a very fine mesh with $10 \times 50 = 500$ elements are also plotted. It could be seen from Figure 7.7 and Figure 7.8 that the proposed XFEM clearly outperformed the conventional plate element and the MITC method is able to circumvent shear locking effectively.

The results from the 6-node XFEM plate element employing MITC6 and DSG6 assumed transverse shear strain are shown in Figure 7.8 and Figure 7.9, respectively. It can be seen that both MITC6 and DSG6 interpolation schemes can circumvent shear locking in XFEM formulation effectively. Also the XFEM formulation in the two 6-node triangular element with XFEM formulations can improve the numerical results obviously. Numerical results from two 6-node plate element with a coarse mesh are very close to corresponding results from standard plate elements with a fine mesh. The computational cost of the XFEM formulation and standard FEM formulation is compared in Table 7.2. It can be seen that all the three XFEM formulations are able to reduce the computational time without reduction in accuracy of the predicted ultimate loading level.

Table 7.1 Example 1 with five different plate thicknesses

	thickness t (m)	yield strength σ_y (kN/m ²)	reference loading q_0 (kN/m)	thickness/length ratio t / L
T1	0.5	0.4	6.4×10^{-2}	1 / 10
T2	0.25	0.2	8×10^{-3}	1 / 20
T3	0.125	0.1	1×10^{-3}	1 / 40
T4	0.05	0.04	6.4×10^{-5}	1 / 100
T5	0.01	0.008	5.12×10^{-7}	1 / 500

Figure 7.7 The equilibrium path of Example 1 (MITC9)

Figure 7.8 The equilibrium path of Example 1 (MITC6)

Figure 7.9 The equilibrium path of Example 1 (DSG6)

7.5.1.2 Comparison on computational cost

The computational costs, including the total number of DOF, Gaussian points, stiffness matrix updating and the computational time, are listed in Table 7.2.

The initial loading factor is taken as $\lambda_0 = 0.3$ for all the analyses. The analyses

stops when the deflection $w_A = -0.001\text{m}$. It can be seen from Table 7.2 that all the three XFEM formulations can save a large amount of computational costs.

Table 7.2 Comparison of the computational cost for each analysis in Example 1

	n_{DOF}	n_{Gauss}	n_s	t (second)
FEM coarse mesh (9 node)	33	450	787	83
FEM fine mesh (9 node)	2121	45000	741	9113
FEM coarse mesh (6 node)	33	300	735	50
FEM fine mesh (6 node)	2121	30000	798	7063
XFEM MITC6	180	5160	735	1091
XFEM MITC9	180	2700	764	1291
XFEM DSG6	180	5160	742	1173

7.5.1.3 Study on distorted mesh

In this section, distorted meshes, as shown in Figure 7.10 for 9-node quadrilateral plate elements and in Figure 7.11 for 6-node triangular plate elements, are used to test the robustness of the enrichment formulation with respect to angular distortion (Bathe 1995). The geometrical and material properties of the T1 case in Section 6.1.1 are used.

Figure 7.10 Distorted mesh of Example 1 (9-node element), unit: m

Figure 7.11 Distorted mesh of Example 1 (6-node element), unit: m

The result from the MITC9 XFEM plate element is shown in Figure 7.12. It is no surprise to find that the results from uniform mesh are better than those from distorted mesh. In the meantime, it can be seen that the XFEM formulation can improve the result in both uniform mesh and angular distorted mesh. It is no surprise to find that the XFEM formulation is able to improve numerical result in both uniform mesh and distorted mesh. As pointed out by Lee and Bathe

(1993), in a distorted mesh of standard FEM element, some of the higher order terms are eliminated. This could affect the displacement approximation field in the enriched element, since the interpolation functions are constructed by the multiplication of enrichment function and shape functions. Hence, some of the higher order terms in XFEM formulation could be eliminated due to element distortion. This can be explained as the reason that the XFEM distorted mesh gives a higher prediction on ultimate load than XFEM uniform mesh.

Figure 7.12 Equilibrium path of Example 1 by distorted mesh (MITC9)

It is also found that the result from XFEM with the distorted mesh is very close to that from FEM with the uniform mesh. This can be explained as that the missing terms (Bathe 1995) in the FEM uniform mesh are compensated by the enrichment in XFEM formulation. Hence the Ritz trial space for the XFEM with distorted mesh is analogous to the FEM with uniform mesh.

The result from MITC6 and DSG6 XFEM plate elements are shown in Figure 7.13 and Figure 7.14, respectively. A similar phenomenon can be found in the two triangular XFEM elements. Both types of triangular elements can improve the numerical results in distorted mesh case. However, in MITC6 XFEM elements, the result from XFEM distorted mesh is not so close to the result from the FEM uniform mesh as the MITC9 and DSG6 elements. This could be due to the cross terms in the assumed shear strain field. The cross terms definitely appear in the uniform mesh analysis, but are eliminated in the distorted mesh.

Figure 7.13 Equilibrium path of Example 1 by distorted mesh (MITC6)

Figure 7.14 The equilibrium path of Example 1 by distorted mesh (DSG6)

7.5.2 Example 2: An L-shaped plate with two fully fixed ends

An L-shaped flat plate is tested in this example, as shown in Figure 7.15. Four cases with different plate thicknesses are tested to show the effectiveness of DSG method in controlling shear locking. The plate is fully fixed on the two short edges. Geometrical properties are shown in Figure 7.15. The Young's modulus is $E = 1.0 \times 10^6 \text{ kN/m}^2$, while the Poisson's ratio is $\nu = 0.3$. The yield strength and the reference load varying according to thickness are listed in Table 7.3, so that the ultimate loads for the four cases are the same. The mesh scheme for the 9-node quadrilateral plate elements and the locations of three possible high gradient zones are shown in Figure 7.16, whereas the mesh scheme for the 6-node quadrilateral plate element is shown in Figure 7.17. The width of the high gradient zones in this example is chosen as ($l_{ns1} = l_{ns2} = 0.1\text{m}$,

$l_{ns3} = 0.6\text{m}$) and five layers are used for each element in the thickness direction.

The deflection of corner point A is investigated.

Figure 7.15 Example 2: An L-shaped plate with two fully fixed ends, unit: m

Table 7.3 Example 2 with four different plate thicknesses

	thickness t	yield strength σ_y	reference loading q_0
T1	0.15	1	-2.12×10^{-3}
T2	0.1	6.67×10^{-1}	-6.29×10^{-4}
T3	0.03	0.2	-1.70×10^{-5}
T4	0.01	6.67×10^{-2}	-6.29×10^{-7}

Figure 7.16 The mesh pattern of the L-shape plate (9-node element)

Figure 7.17 The mesh pattern of the L-shape plate (6-node element)

The result from compatible XFEM formulation is shown in Figure 7.18, together with the results from the FEM coarse mesh and the yield line pattern analysis. In the XFEM locked analysis, triangular elements are employed. The MITC6 technique is employed in the standard elements while compatible shear strain is used in the enriched elements. It can be seen that the XFEM locked analysis provide a prediction on ultimate loading factor even higher than the

FEM coarse mesh analysis. Parasitic strain energy is introduced into the compatible XFEM elements. Hence, it is necessary to alleviate shear locking in the XFEM formulation.

Figure 7.18 The equilibrium path of Example 2 without locking control in enriched elements

The results obtained from the MITC9 XFEM plate element are shown in Figure 7.19, together with the results from (a) corresponding conventional plate elements with a coarse mesh (Figure 7.16), (b) commercial software ANSYS with a fine mesh (500 SHELL93 elements) and (c) yield line pattern analysis (Johansen 2004). The yield line pattern analysis provides an upper bound solution, which is $\lambda_u = 3.0$. The results, as shown in Figure 7.19, illustrate that there is a clear advantage of XFEM when compared with conventional FEM. Besides, it is found that the numerical analysis provides a predicted ultimate

load even lower than the analytical results. It should be noted that both of the numerical and the analytical results are upper bound solutions. However, it is difficult to predict which one is a greater ‘upper bound’. For this particular case, the numerical model is softer than the analytical model.

Figure 7.19 The equilibrium path of Example 2 (MITC9)

Figure 7.20 The equilibrium path of Example 2 (MITC6)

Figure 7.21 The equilibrium path of Example 2 (DSG6)

The results from MITC6 and DSG6 XFEM plate elements are shown in Figure 7.20 and Figure 7.21, respectively. Although both of the XFEM plate elements are able to improve the numerical result, it can be seen that MITC6 XFEM plate elements provide almost the same ultimate load as the analytical model, while the numerical model from DSG6 XFEM plate elements is stiffer than the analytical model.

Table 7.4 The comparison of the computational cost for each analysis in Example 2

	n_{DOF}	n_{Gauss}	n_s	t (second)
FEM coarse mesh (9 node)	33	450	265	27.1
FEM fine mesh (9 node)	2121	45000	292	3015.5
FEM coarse mesh (6 node)	33	300	269	17.3
FEM fine mesh (6 node)	2121	30000	286	1965.2
XFEM MITC6	180	5160	271	477.2
XFEM MITC9	180	2700	280	1110.8
XFEM DSG6	180	5160	282	512

The computational costs for each analysis, including the total number of DOF, Gaussian integration points, stiffness matrix updating and the computational time, are listed in Table 7.4. In each analysis, the initial loading factor is taken as $\lambda_0 = 1.0$. The analyses stop when the deflection $w_A = -0.005\text{m}$. It is clear that the XFEM formulation is able to save the computational time tremendously without loss of accuracy.

7.5.3 Example 3: A square plate with roller supports at the four edges

(a) The mesh scheme by 9-node plate element

(b) The mesh scheme by 6-node plate element

Figure 7.22 Example 3: the square plate with roller support on four edges

A square plate supported by rollers on the four edges is tested in this example, as shown in Figure 7.22(a) for 9-node quadrilateral elements and in Figure 7.22(b) for 6-node triangular elements. The length of edge is 32. The Young's modulus is $E = 1.0 \times 10^{10}$, while the Poisson's ratio is $\nu = 0.2$. A UDL is applied onto the plate with the reference loading q_0 . Again, five layers are used in each element in the thickness direction. A quarter of the entire plate is modelled because of symmetry. Four quadrilateral elements or eight triangular elements are used for both the XFEM and the standard FEM coarse mesh analysis. The yield line is along the diagonal of the plate. In this example, the high gradient zone width is assumed as $l_{ns} = 4$. The deflection of centre point A is investigated.

7.5.3.1 Study on shear locking

As shown in Example 1 and 2 in this chapter, the compatible XFEM formulation introduces parasitic strain energy into analyses. Hence, assumed shear strain method is used to control shear locking in the XFEM formulation. The effectiveness of the MITC and the DSG technique is studied in this section. As listed in Table 7.5, four cases with different thicknesses are tested in this study. The yield strength and the reference load vary according to the thickness so that the ultimate load factors are kept the same for the four cases ($\lambda_0 = 0.0586$).

The results from MITC9 XFEM plate elements are shown in Figure 7.23 and they are compared with the results from (a) FEM coarse mesh shown in Figure 7.22, (b) yield line analysis (Johansen 2004), (c) FEM fine mesh ($20 \times 20 = 400$ elements) and (d) commercial software ANSYS in which a total of 400 SHELL93 elements are employed.

The results from MITC6 and DSG6 XFEM plate elements are shown in Figure 7.24 and Figure 7.25, respectively, with numerical results from corresponding standard FEM plate elements in fine mesh ($20 \times 20 \times 2 = 800$ elements) and coarse mesh, yield line analysis.

Table 7.5 Example 3 with three different plate thicknesses

	thickness t (m)	yield strength σ_y (kN/m ²)	reference loading q_0 (kN/m)	thickness/length ratio t/L
T1	2.00	2.00×10^5	8.00×10^4	1/16
T2	1.00	1.00×10^5	1.00×10^4	1/32
T3	0.50	5.00×10^4	1.25×10^3	1/64

Figure 7.23 The equilibrium path of Example 3 (MITC9)

It can be seen that the proposed MITC9 XFEM quadrilateral plate elements and MITC6 XFEM triangular plate element outperform the corresponding standard FEM plate elements and give a similar ultimate load predictions as the highly refined ANSYS SHELL93 element mesh. Furthermore, the XFEM method can alleviate shear locking in the present XFEM plate element effectively. However, from Figure 7.25 it is also found that DSG6 XFEM elements fail to circumvent shear locking in the XFEM formulation in this example, since the prediction of the ultimate load is quite different from each other for different thicknesses: the thinner the plate is, the higher is the ultimate load.

7.5.3.2 Comparison of computational cost

The computational costs of the three XFEM formulations are listed in Table 7.6. For all the analyses listed in Table 7.6, the initial loading factor is $\lambda_0 = 0.02$ and the analyses stops when the deflection $w_A = -0.1$. It can be found that all the three XFEM formulations are able to reduce the computational time without loss in accuracy of the ultimate loading prediction.

Figure 7.24 The equilibrium path of Example 3 (MITC6)

Figure 7.25 The equilibrium path of Example 3 (DSG6)

Table 7.6 The comparison of the computational effort in Example 3

	n_{DOF}	n_{Gauss}	n_s	t (second)
FEM coarse mesh (9 node)	75	360	104	7.9
FEM fine mesh (9 node)	5043	36000	116	853.5
FEM coarse mesh (6 node)	75	240	109	5.3
FEM fine mesh (6 node)	5043	24000	114	568.2
XFEM MITC6	132	4380	110	122.2
XFEM MITC9	150	5040	108	164.3
XFEM DSG6	132	4380	120	142.4

7.5.3.3 Study on integration scheme

In this section, MITC9 XFEM plate elements are used to check the influence from different integration schemes on numerical accuracy. A total of 4 cases with different numerical integration schemes for each part of the enriched element are tested. The geometrical and material properties of T3 case in Section 7.5.3.1 are used in this study. The Gaussian integration schemes with 4 by 4, 6 by 6, 8 by 8 and 10 by 10 Gaussian points for each part of the enriched

element are employed for each case. The result is shown in Figure 7.26. It can be seen from Figure 7.26 that the XFEM plate element is not sensitive to numerical integration in this example.

Figure 7.26 The equilibrium path of Example 3 by different integration scheme (MITC9)

7.5.4 Example 4: A cantilever square plate

A cantilever square plate of 16 units is tested in this example, as shown in Figure 7.27. The Young's modulus is $E = 1.0 \times 10^{10} \text{ kN/m}^2$, while the Poisson's ratio $\nu = 0.2$. A line load, $q = \lambda q_0$, is applied at the free end with three different thicknesses. The reference loading q_0 and the yield strength σ_0 vary according to the thickness (as listed in Table 7.7) so that the ultimate load factors (λ_u) for the three cases are the same from yield line analysis (Johansen 2004). The deflection of point A in Figure 7.27 is investigated. In the XFEM analysis and the FEM analysis with coarse mesh, only one 9-node quadrilateral element is

used, while two 6-node triangular elements are employed. In the FEM analysis with a fine mesh, a total of 400 (20×20) elements are used for 9-node quadrilateral elements and 800 ($20 \times 20 \times 2$) elements used for 6-node triangular elements. In the XFEM analysis, it is assumed that $l_{ns} = 4.0\text{m}$.

Figure 7.27 Example 4: a cantilever square plate, unit: m

Similar to Example 1 and 2, an analysis with compatible XFEM formulation is carried out first to show the necessity of the employment of assumed shear

strain method in enriched elements in this example. In the analysis, two triangular elements are used as shown in Figure 7.27(b). Both of the two triangular elements are enriched elements. The result is shown in Figure 7.28. Again, it shows that parasitic strain energy is introduced into the analysis by the compatible XFEM formulation so that the predicted ultimate loading is even higher than FEM coarse mesh (the same mesh scheme as the XFEM locked analysis) analysis.

Figure 7.28 The equilibrium path of Example 4 without shear locking control in enriched elements

The results from the MITC9 XFEM plate element are shown in Figure 7.29, with the results from the FEM coarse mesh and fine mesh. The result from yield line pattern analysis is also shown in Figure 7.29, which provides the ultimate load factor as $\lambda_u = 0.25$. It can be seen that the XFEM formulation provides a

better prediction on ultimate load factor than the standard FEM with a coarse mesh or even a fine mesh.

Table 7.7 Example 4 with three different plate thicknesses

	thickness t (m)	yield strength σ_0 (kN/m ²)	reference load q_0 (kN/m)	thickness/length (t/l)
T1	1	3.0×10^5	3.0×10^5	1/16
T2	0.1	3.0×10^4	300	1/160
T3	0.01	3.0×10^3	0.3	1/1600

The results from MITC6 and DSG6 XFEM plate elements are shown in Figure 7.30 and Figure 7.31, respectively, with the results from the corresponding standard plate element employing a fine mesh or a coarse mesh. It can be seen that both MITC6 and DSG6 are able to mitigate shear locking in XFEM analysis. Clearly, the XFEM formulation performs well in triangular elements.

A comparison of the computational effort from XFEM and FEM analyses is listed in Table 7.8. In each analysis, the initial loading factor is taken as $\lambda_0 = 0.1$. The analyses stop when the deflection $w_A = -0.3$.

Figure 7.29 The equilibrium path of Example 4 (MITC9)

Figure 7.30 The equilibrium path of Example 4 (MITC6)

Figure 7.31 The equilibrium path of Example 4 (DSG6)

Table 7.8 The comparison of the computational effort in Example 4

	n_{DOF}	n_{Gauss}	n_s	t (second)
FEM coarse mesh (9 node)	27	30	128	2.6
FEM fine mesh (9 node)	5043	36000	132	1033.0
FEM coarse mesh (6 node)	27	60	119	0.9
FEM fine mesh (6 node)	5043	24000	129	672.6
XFEM MITC6	54	1440	110	41.9
XFEM MITC9	54	720	119	52.2
XFEM DSG6	54	1440	181	61.0

7.5.5 Example 5: A cantilever L-shape plate

In this example, a cantilever L-shape plate is tested, as shown in Figure 7.32. The geometric properties of this example are shown in Figure 7.32. The Young's modulus is $E = 1.0 \times 10^6 \text{ kN/m}^2$, while the Poisson's ratio is $\nu = 0.3$. A line load, $q = \lambda q_0$, is applied at the free end (Figure 7.32) with four different thicknesses are tested. The reference loading q_0 and the yield strength σ_0 vary according to the thickness (as listed in Table 7.9) so that the ultimate load

factors (λ_u) for the three cases are kept the same from yield line analysis (Johansen 2004). The deflection of point A in Figure 7.32 is investigated.

The mesh scheme for the MITC9 XFEM analyses is shown in Figure 7.33 which is also used for the FEM analysis with a coarse mesh. The mesh scheme for the 6-node XFEM plate elements and the FEM analysis with a coarse mesh are shown in Figure 7.34. In the FEM analysis with a fine mesh, a total of 500 elements are used for 9-node quadrilateral elements and 1000 elements for 6-node triangular elements. In the XFEM analysis, it is assumed that $l_{ns} = 0.4m$.

Table 7.9 Example 5 with four different plate thicknesses

	thickness t (m)	yield strength σ_0 (kN/m ²)	reference load q_0 (kN/m ²)
T1	0.15	1	2.40×10^{-3}
T2	0.1	0.667	7.11×10^{-4}
T3	0.03	0.2	1.92×10^{-5}
T4	0.01	0.06667	7.11×10^{-7}

Figure 7.32 Example 5: A cantilever L-shape plate, unit: m

Figure 7.33 The mesh pattern of the cantilever L-shape plate (9-node element), unit: m

Figure 7.34 The mesh pattern of the cantilever L-shape plate (6-node element), unit: m

The results from the four cases by MITC9 XFEM plate elements are shown in Figure 7.35, with a comparison of results from MITC9 FEM with a coarse mesh and a fine mesh. Yield line analysis is also shown in Figure 7.35, which gives $\lambda_u = 0.938$.

The results from MITC6 and DSG6 XFEM plate elements are shown in Figure 7.36 and Figure 7.37, respectively, with the results from corresponding FEM analyses using a fine mesh and a coarse mesh.

Figure 7.35 The equilibrium path of Example 5 (MITC9)

Figure 7.36 The equilibrium path of Example 5 (MITC6)

Figure 7.37 The equilibrium path of Example 5 (DSG6)

It could be seen from Figure 7.35 that shear locking does not occur in MITC9 and MITC6 XFEM plate elements with decreasing plate thickness. The ultimate load factor prediction from XFEM formulation is even better than that from standard FEM with a fine mesh. Although the XFEM formulation with the DSG technique is able to improve the accuracy in numerical results, the DSG does not perform as well as the MITC6 technique in this example.

Table 7.10 The comparison of the computational effort in Example 5

	n_{DOF}	n_{Gauss}	n_s	t (second)
FEM coarse mesh (9 node)	27	30	128	2.6
FEM fine mesh (9 node)	5043	36000	132	1033.0
FEM coarse mesh (6 node)	27	60	119	0.9
FEM fine mesh (6 node)	5043	24000	129	672.6
XFEM MITC6	54	1440	110	41.9

XFEM MITC9	54	720	119	52.2
XFEM DSG6	54	1440	181	61.0

The comparison on the computational efforts is listed in Table 7.10. It can be seen from Table 7.10 that the three XFEM formulations is able to save computational effort tremendously.

7.6 Closure

The enrichment functions constructed on structural level are introduced in this chapter. They are called *global* enrichment functions. As the global enrichment is independent of nodal shape functions, the shape of the global enrichment is mesh-invariant. Consequently, the global enrichment function is easily to be implemented in both the 9-node quadrilateral and the 6-node triangular elements.

It is shown in examples that compatible enriched elements add parasitic strain energy into the XFEM formulation. The assumed natural strain method is employed to soften the XFEM plate elements. Both the MITC technique and the DSG technique are used. The MITC6 scheme shows a good performance in all the numerical examples, while the DSG method in 6-node plate elements fails to control shear locking in some examples. Compared with reduced integration method shown in Chapter 6, the assumed natural strain method is independent of numerical integration scheme.

CHAPTER 8 CONCLUSIONS AND FUTURE WORK

8.1 Conclusion

In this thesis, the application of the XFEM formulation in non-linear structure analyses is presented. First of all, standard finite elements are developed including a 2D 3-node Timoshenko beam element, a 2D 3-node co-rotational Timoshenko beam element, a 9-node Mindlin plate element and a 6-node Mindlin plate element. Subsequently, the XFEM formulation is embedded in these elements. Two main threads summarize the present work, viz, the enrichment in XFEM formulation and the mitigation of shear locking in thin plates.

8.1.1 The enrichment function

The employment of enrichment is the most important characteristic of the XFEM formulation. The step function and the absolute level set function are used as the enrichment function for rotation and deflection displacement approximation fields simultaneously to model an internal pin connection in an element. They show a good performance in capture the strong discontinuity and the weak discontinuity in the corresponding displacement fields. The application of the XFEM formulation in internal pin connection shows its advantage in decreasing the computational effort in mesh process. A structured

mesh scheme can be employed without considering the location of the internal pin connection. A co-rotational frame is embedded in the XFEM formulation for an internal pin. Since the enrichment functions for an internal pin are zero at element node, the discontinuous displacement is excluded from rigid body motion of an element. Hence, the co-rotational frame for a standard FEM beam element is still applicable for the XFEM beam element.

A regularized step function and a regularized absolute level set function are employed as the enrichment functions for a plastic hinge. The regularized enrichment functions are C_1 continuous inside an element so that the strain energy can be calculated directly from the displacement approximation field. Furthermore, a global enrichment function is proposed to improve the adaptability of XFEM in analyses. Different from a perfect pin, the bending strain and the bending stress in a plastic hinge are of a finite value along a beam. The XFEM formulation provides a natural way to capture the locally non-smooth displacement field resulting from a plastic hinge. The XFEM simulation of the yield process of a structure member can be conducted without introducing any constitutive relationship between plastic moment and displacement variables. This advantage is magnified when applying the XFEM formulation in yield line analyses, since it is not easy to find an explicit expression of strain energy in terms of displacement variables in plate structures. Furthermore, in the present XFEM formulation, the introduction of enrichment function is equivalent to expanding the Ritz solution space. The

terms introduced by the enrichment function have very little contribution to the solution in elastic stage. This is because the deformation modes from the standard elements are capable of reproducing the elastic response of structural members. Therefore, the enrichment is suppressed by equilibrium condition when the structure is in elastic stage. When some points in the structural members are yielded, a locally non-smooth displacement field develops gradually and the deformation modes from the standard elements are unable to reproduce such a displacement field with a locally high gradient. In this case, the additional displacement modes from the enrichment function are mobilized. This process is called the activation of enrichment and it is done automatically by the formulation. It should also be noted that the appearance of local non-smoothness in displacement field is a gradual process. It is not easy to specify a critical stage when the locally high gradient appears. However, the present XFEM formulation is able to avoid such a criterion, which is one of the advantages of the present study, compared with the enriched numerical formulations with a discontinuous enrichment and a cohesive material model to describe the dissipated energy in the discontinuity.

It is shown in the numerical examples in each chapter that the computational cost of an XFEM analysis is low compared with a standard FEM analysis with a uniform fine mesh, while the results from both analyses are quite close. The aim of the present work is achieved. Although both the *local* enrichment function and the *global* enrichment function can capture the local non-

smoothness in displacement field, it is recommended to use the global enrichment function. The *global* enrichment function can be embedded in both quadrilateral plate elements and triangular plate elements. The *global* enrichment function can also be applied in a plate element of any order: linear, quadratic, cubic and so on. Furthermore, the *global* enrichment function is free from distorted mesh, as it is not related to the standard shape functions of the element, although the lower order standard elements outside the enriched zone may lead to distortion sensitivity.

8.1.2 The shear locking mitigation method in the XFEM formulation

The shear locking mitigation is also studied in this thesis. Reduced integration, the MITC technique and the DSG technique are tried in the XFEM beam and plate elements.

It is found that the reduced integration is suitable for the XFEM beam element, since the topology of an XFEM beam element is relatively simple. However, in an XFEM plate element, it becomes complicated, as there are several ways that a high gradient zone can cut through an enriched element. The expression of local enrichment functions varies from case to case. Therefore, the full integration scheme is case-dependent and the order of reduced integration scheme cannot be uniquely defined.

The assumed natural strain method in circumventing shear locking is also presented. The MITC and the DSG techniques are applied in both the smooth displacement field and the non-smooth displacement field in the 6-node and 9-node XFEM plate elements. The original form of the MITC scheme is expressed as an interpolation of compatible shear strains at some selected points. However, the MITC schemes adopted in the present thesis is rewritten in terms of an interpolation of nodal displacement variables. The alternative expression of the MITC technique simplifies its implementation in XFEM elements. It is proved in this thesis that the MITC technique and the DSG technique provides exactly the same assumed shear strain field in the 9-node quadrilateral plate element. However, the assumed shear strain field is different in 6-node triangular plate element.

It is recommended to use the assumed natural strain method. The assumed natural strain method is easily to be applied in the XFEM plate element and the integration scheme is independent of the way the high gradient zone cuts through an enriched element. It is also found that the MITC technique performs better than the DSG technique in the 6-node XFEM plate element.

8.2 Future work

The XFEM formulation is based on the *a priori* knowledge of the actual physical phenomenon. In the present work, the location of a high gradient zone

and its width are regarded as *a priori* knowledge. This implementation is feasible for an analysis on a simple structure. However, in an analysis of a complex structure, this knowledge is unknown before analysis. A smart algorithm to locate the yield line and identify the high gradient zone can be a research topic. Besides, the width of the high gradient zone is, in general, varying during an elasto-plastic analysis. The variable width of the high gradient zone can be developed in future. Due to the changing width of the high gradient zone, the area of the high gradient zone becomes large and the integration points change their location gradually as loading increases. It is well known that in an elasto-plastic analysis the current stress is not only a function of total strain, but also of stress history. Therefore, a smart mapping technique is required to trace the stress history of an arbitrary point in an element.

In the present thesis, infinitesimal strain assumption is employed in each element. As a high gradient displacement appears in non-smooth displacement zone, this assumption can be invalid. In future work, the finite strain assumption can be used in the beam element and the plate elements to describe the strain in the element. As a first try, geometrical nonlinearity is not considered in the material nonlinear analyses, which means the geometrical nonlinearity and the material nonlinearity are decoupled. The coupling of these two nonlinearities is one of the research topics for the candidate in future research.

In the present work, it is assumed that there is only one high gradient zone in an enriched element. However, in real engineering application, this assumption may not be reasonable. Two or more high gradient zones could pass through an enriched element. Hence, a more advanced technique on the partition of an enriched element should be developed.

As the XFEM formulation is embedded in plate elements and has shown good performance in capturing non-smoothness in displacement field, future work should focus on the implementation of XFEM formulation in initially curved shell elements. Apart from shear locking, membrane locking is another obstacle in developing a robust shell element. According to the author's experience on shear locking mitigation in XFEM plate elements, membrane locking in XFEM shell elements can be circumvented by assumed natural strain method, for example the MITC technique.

Finally, the most exciting part of the XFEM plate elements is its application in reinforced concrete slabs. As cracks appear in RC slabs, local non-smoothness in the displacement field is more obvious than that in steel plates. It is believed that the most challenging point of the present XFEM plate elements in RC structure is not the XFEM formulation, but the concrete constitutive model. The present 6-node and 9-node XFEM plate elements can easily be applied to the analyses on RC slabs, if they have a robust 2D concrete constitutive model.

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APPENDIX A

$$\mathbf{B}_m = [\mathbf{B}_{m1} \quad \mathbf{B}_{m3}], \quad \mathbf{B}_\gamma = [\mathbf{B}_{\gamma1} \quad \mathbf{B}_{\gamma3}], \quad \mathbf{B}_b = [\mathbf{B}_{b1} \quad \mathbf{B}_{b3}]$$

$$\mathbf{B}_{mi} = \begin{bmatrix} \left(1 + \frac{\partial u}{\partial x}\right) \frac{\partial N_i}{\partial x} & \frac{\partial w_h}{\partial x} \frac{\partial N_i}{\partial x} & 0 \end{bmatrix}, \quad i = 1, 3$$

$$\mathbf{B}_{\gamma i} = \begin{bmatrix} 0 & \frac{\partial N_i}{\partial x} & -N_i \end{bmatrix}, \quad i = 1, 3$$

$$\mathbf{B}_{bi} = \begin{bmatrix} 0 & 0 & \frac{\partial N_i}{\partial x} \end{bmatrix}$$

$$\mathbf{B}_m^c = [\mathbf{B}_{m1}^c \quad \mathbf{B}_{m2}^c \quad \mathbf{B}_{m3}^c], \quad \mathbf{B}_\gamma^c = [\mathbf{B}_{\gamma1}^c \quad \mathbf{B}_{\gamma2}^c \quad \mathbf{B}_{\gamma3}^c], \quad \mathbf{B}_b^c = [\mathbf{B}_{b1}^c \quad \mathbf{B}_{b2}^c \quad \mathbf{B}_{b3}^c]$$

$$\mathbf{B}_{mi}^c = [\mathbf{B}_{mi} \quad \mathbf{0}_{1 \times 3}], \quad i = 1, 3; \quad \mathbf{B}_{m2}^c = [\mathbf{0}_{1 \times 3}]$$

$$\mathbf{B}_{\gamma i}^c = [\mathbf{B}_{\gamma i} \quad \mathbf{0}_{1 \times 3}], \quad i = 1, 3; \quad \mathbf{B}_{\gamma 2}^c = [\mathbf{0}_{1 \times 3}]$$

$$\mathbf{B}_{bi}^c = [\mathbf{B}_{bi} \quad \mathbf{0}_{1 \times 3}], \quad i = 1, 3; \quad \mathbf{B}_{b2}^c = [\mathbf{0}_{1 \times 3}]$$

$$\mathbf{B}_m^d = [\mathbf{B}_{m1}^d \quad \mathbf{B}_{m2}^d \quad \mathbf{B}_{m3}^d], \quad \mathbf{B}_\gamma^d = [\mathbf{B}_{\gamma1}^d \quad \mathbf{B}_{\gamma2}^d \quad \mathbf{B}_{\gamma3}^d], \quad \mathbf{B}_b^d = [\mathbf{B}_{b1}^d \quad \mathbf{B}_{b2}^d \quad \mathbf{B}_{b3}^d]$$

$$\mathbf{B}_{mi}^d = \begin{bmatrix} \mathbf{0}_{1 \times 3} & 0 & 0 & \frac{\partial w_h}{\partial x} \frac{\partial L_i}{\partial x} \end{bmatrix}, \quad i = 1, 3; \quad \mathbf{B}_{m2}^d = \begin{bmatrix} 0 & 0 & \frac{\partial w_h}{\partial x} \frac{\partial L_2}{\partial x} \end{bmatrix}$$

$$\mathbf{B}_{\gamma i}^d = \begin{bmatrix} \mathbf{0}_{1 \times 3} & -M_i & 0 & \frac{\partial L_i}{\partial x} \end{bmatrix}, \quad i = 1, 3; \quad \mathbf{B}_{\gamma 2}^d = \begin{bmatrix} -M_2 & 0 & \frac{\partial L_2}{\partial x} \end{bmatrix}$$

$$\mathbf{B}_{bi}^d = \begin{bmatrix} \mathbf{0}_{1 \times 3} & \frac{\partial M_i}{\partial x} & 0 & 0 \end{bmatrix}, \quad i = 1, 3; \quad \mathbf{B}_{b2}^d = \begin{bmatrix} \frac{\partial M_i}{\partial x} & 0 & 0 \end{bmatrix}$$

APPENDIX B

In this appendix, the backward-Euler algorithm is introduced in detail. The backward-Euler algorithm is adopted in the present XFEM plate elements as well as in the standard FEM plate elements for elasto-plastic analyses.

B.1 The backward-Euler algorithm

In the present plate elements, the out-of-plane stress components are not taken into account in the formulation ($\sigma_{zx} = \sigma_{zy} \equiv 0$). Hence, the J2 yield criterion can be simplified as (Crisfield 1991a):

$$f(\sigma_x, \sigma_y, \tau_{xy}, \kappa) = [\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2]^{1/2} - \sigma_0(\kappa) = \sigma_e - \sigma_0(\kappa) \quad (\text{B.1})$$

where σ_e is the effective stress, σ_0 is the yield strength and κ is the isotropic hardening parameter.

Figure B.1 The flow vector $\mathbf{a}$

According to the Prandtl-Reuss flow rule, the plastic part for incremental strain is:

$$d\boldsymbol{\varepsilon}_p = \begin{pmatrix} d\varepsilon_{px} & d\varepsilon_{py} & d\varepsilon_{pxy} \end{pmatrix}^T = d\lambda \mathbf{a} \quad (\text{B.2})$$

where $d\lambda$ is plastic strain rate multiplier and $\mathbf{a}$ is flow vector (Figure B.1) which can be expressed as:

$$\mathbf{a} = \frac{\partial f}{\partial \boldsymbol{\sigma}^T} = \begin{pmatrix} \frac{\partial f}{\partial \sigma_x} & \frac{\partial f}{\partial \sigma_y} & \frac{\partial f}{\partial \tau_{xy}} \end{pmatrix}^T \quad (\text{B.3})$$

In the present plate elements, plane-stress J_2 flow theory is employed, so that the incremental plastic strain can be further expressed as:

$$d\boldsymbol{\varepsilon}_p = \frac{d\lambda}{3} \begin{pmatrix} 2\sigma_x - \sigma_y \\ 2\sigma_y - \sigma_x \\ 6\tau_{xy} \end{pmatrix} = d\lambda \mathbf{P} \boldsymbol{\sigma} \quad (\text{B.4})$$

$$\mathbf{P} = \frac{1}{3} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \quad (\text{B.5})$$

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \quad (\text{B.6})$$

The stress $\boldsymbol{\sigma}$ can be updated by

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_n + \mathbf{D}_{ep} d\boldsymbol{\varepsilon}_{n+1} \quad (\text{B.7})$$

where $\mathbf{D}_{ep}$ is the consistent elasto-plastic material matrix, $\boldsymbol{\sigma}_n$ is the stress state satisfies the yield criteria $\boldsymbol{\sigma}_{n+1}$ is the desired stress state at the end of the current increment step and $d\boldsymbol{\varepsilon}_{n+1}$ is the incremental strain for the current incremental step.

Figure B.2 The backward-Euler algorithm

The backward Euler return algorithm is used in the present element. As shown in Figure B.2, the desired stress σ_C can be expressed by the following relationship:

$$\sigma_C = \mathbf{D}(\epsilon_C - \epsilon_{pC}) = \mathbf{D}(\epsilon_C - \epsilon_{pA} - d\epsilon_p) = \sigma_B - d\lambda \mathbf{D} \mathbf{a}_C \quad (\text{B.8})$$

where σ_B is the stress state of the starting point of the current incremental step, σ_B is the trial stress in which the incremental strain of the current step is assumed as elastic and $d\epsilon_p = \epsilon_{pC} - \epsilon_{pA}$ is the incremental plastic strain for the current step.

Substituting Equation (B.3) into Equation (B.8), the desired stress state σ_C can be expressed as

$$(\mathbf{I} + d\lambda \mathbf{D} \mathbf{P}) \sigma_C = \sigma_B \quad (\text{B.9})$$

where $\mathbf{I}$ is three-order identity matrix. In the above formulation, all the variables are known except the plastic multiplier $d\lambda$. In order to find the value of $d\lambda$, a

function g is constructed and Newton iteration is performed on the yield criterion with the desired stress state:

$$g = g(d\lambda) = \sigma_{eC}^2 - \sigma_{0C}^2 = 0 \quad (B.10)$$

$$\frac{dg}{d(d\lambda)} = \frac{d\sigma_{eC}^2}{d(d\lambda)} - \frac{d\sigma_{0C}^2}{d(d\lambda)} \quad (B.11)$$

$$\sigma_{eC}^2 = \sigma_{xC}^2 + \sigma_{yC}^2 - \sigma_{xC}\sigma_{yC} + 3\tau_{xyC}^2 = \frac{1}{4}(\sigma_{xC} + \sigma_{yC})^2 + \frac{3}{4}(\sigma_{xC} - \sigma_{yC})^2 + 3\tau_{xyC}^2 \quad (B.12)$$

Equation (B.9) can be expanded as:

$$\left[\frac{Ed\lambda}{3(1-\nu^2)}(2-\nu)+1 \right] \sigma_{xC} + \frac{Ed\lambda}{3(1-\nu^2)}(-1+2\nu)\sigma_{yC} = \sigma_{xB} \quad (B.13)$$

$$\frac{Ed\lambda}{3(1-\nu^2)}(-1+2\nu)\sigma_{xC} + \left[\frac{Ed\lambda}{3(1-\nu^2)}(2-\nu)+1 \right] \sigma_{yC} = \sigma_{yB} \quad (B.14)$$

$$\left(\frac{Ed\lambda}{1+\nu} + 1 \right) \tau_{xyC} = \tau_{xyB} \quad (B.15)$$

therefore,

$$\sigma_{xC} + \sigma_{yC} = (\sigma_{xB} + \sigma_{yB}) / \left[\frac{Ed\lambda}{3(1-\nu)} + 1 \right] \quad (B.16)$$

$$\sigma_{xC} - \sigma_{yC} = (\sigma_{xB} - \sigma_{yB}) / \left(\frac{Ed\lambda}{1+\nu} + 1 \right) \quad (B.17)$$

$$\tau_{xyC} = \tau_{xyB} / \left(\frac{Ed\lambda}{1+\nu} + 1 \right) \quad (B.18)$$

and σ_{eC}^2 can be expressed by the components of σ_B as:

$$\sigma_{eC}^2 = \frac{(\sigma_{xB} + \sigma_{yB})^2}{4 \left[\frac{Ed\lambda}{3(1-\nu)} + 1 \right]^2} + \frac{3 \left[(\sigma_{xB} - \sigma_{yB})^2 + 4\tau_{xyB}^2 \right]}{4 \left(\frac{Ed\lambda}{1+\nu} + 1 \right)^2} \quad (B.19)$$

$$\sigma_{eC} = \sqrt{\frac{(\sigma_{xB} + \sigma_{yB})^2}{4\left[\frac{Ed\lambda}{3(1-\nu)} + 1\right]^2} + \frac{3\left[(\sigma_{xB} - \sigma_{yB})^2 + 4\tau_{xyB}^2\right]}{4\left(\frac{Ed\lambda}{1+\nu} + 1\right)^2}} \quad (B.20)$$

and the first term on the right hand side of Equation (B.11) is of the form:

$$\frac{d\sigma_{eC}^2}{d(d\lambda)} = -\frac{E}{6(1-\nu)} \frac{(\sigma_{xB} + \sigma_{yB})^2}{\left[\frac{Ed\lambda}{3(1-\nu)} + 1\right]^3} - \frac{E}{1+\nu} \frac{3\left[(\sigma_{xB} - \sigma_{yB})^2 + 4\tau_{xyB}^2\right]}{2\left(\frac{Ed\lambda}{1+\nu} + 1\right)^3} \quad (B.21)$$

The second term on the right hand side of Equation (B.11) is of the form:

$$\frac{d\sigma_{0C}^2}{d(d\lambda)} = \frac{4}{3}\kappa\left(\sigma_{0A} + \kappa\frac{2}{3}\sigma_{eC}d\lambda\right)\left(\sigma_{eC} + \frac{d\sigma_{eC}}{d(d\lambda)}d\lambda\right) \quad (B.22)$$

where

$$\frac{d\sigma_{eC}}{d(d\lambda)} = \frac{d(\sigma_{eC}^2)^{1/2}}{d(d\lambda)} = \frac{1}{2\sigma_{eC}} \frac{d\sigma_{eC}^2}{d(d\lambda)} \quad (B.23)$$

B.2 The elasto-plastic consistent matrix

By differential Equation (B.8),

$$d\boldsymbol{\sigma}_C = \mathbf{D}\left[d\boldsymbol{\varepsilon}_C - d(d\lambda)\mathbf{P}\boldsymbol{\sigma}_C - d\lambda\mathbf{P}d\boldsymbol{\sigma}_C\right] \quad (B.24)$$

therefore,

$$d\boldsymbol{\sigma}_C = (\mathbf{I} + d\lambda\mathbf{PD})^{-1} \mathbf{D}\left[d\boldsymbol{\varepsilon}_C - \mathbf{P}\boldsymbol{\sigma}_C d(d\lambda)\right] \quad (B.25)$$

By differential Equation (B.10):

$$dg = \left(\frac{\partial g}{\partial \boldsymbol{\sigma}_C^T}\right)^T d\boldsymbol{\sigma}_C - \frac{\partial g}{\partial (d\lambda)} d(d\lambda) = 0 \quad (B.26)$$

where

$$\frac{\partial g}{\partial \boldsymbol{\sigma}_c^T} = \frac{\partial \sigma_{ec}^2}{\partial \boldsymbol{\sigma}_c^T} - \frac{\partial \sigma_{0c}^2}{\partial \boldsymbol{\sigma}_c^T} = 3\mathbf{P}\boldsymbol{\sigma}_c \quad (\text{B.27})$$

$$\frac{\partial g}{\partial (d\lambda)} = 0 \quad (\text{B.28})$$

For an elastic-perfectly-plastic analysis,

$$\boldsymbol{\sigma}_c^T \mathbf{P} d\boldsymbol{\sigma}_c = 0 \quad (\text{B.29})$$

The physical meaning is that the flow vector $\mathbf{a}$ and the incremental stress are perpendicular to each other, as shown in Figure B.1.

Substituting Equation (B.25) into Equation (B.29):

$$d(d\lambda) = \frac{\boldsymbol{\sigma}_c^T \mathbf{P} \boldsymbol{\Xi} d\boldsymbol{\varepsilon}_c}{\boldsymbol{\sigma}_c^T \mathbf{P} \boldsymbol{\Xi} \mathbf{P} \boldsymbol{\sigma}_c} \quad (\text{B.30})$$

where

$$\boldsymbol{\Xi} = (\mathbf{D}^{-1} + d\lambda \mathbf{P})^{-1} \quad (\text{B.31})$$

Since $\mathbf{D}$ and $\mathbf{P}$ are both symmetric matrix, $\boldsymbol{\Xi}$ is also a symmetric matrix.

Substituting Equation (B.30) into Equation (B.25):

$$d\boldsymbol{\sigma}_c = \left(\boldsymbol{\Xi} - \frac{\boldsymbol{\Xi} \mathbf{P} \boldsymbol{\sigma}_c \boldsymbol{\sigma}_c^T \mathbf{P} \boldsymbol{\Xi}}{\boldsymbol{\sigma}_c^T \mathbf{P} \boldsymbol{\Xi} \mathbf{P} \boldsymbol{\sigma}_c} \right) d\boldsymbol{\varepsilon}_c \quad (\text{B.32})$$

Hence the elasto-plastic consistent matrix

$$\mathbf{D}_{ep} = \boldsymbol{\Xi} - \frac{\boldsymbol{\Xi} \mathbf{P} \boldsymbol{\sigma}_c \boldsymbol{\sigma}_c^T \mathbf{P} \boldsymbol{\Xi}}{\boldsymbol{\sigma}_c^T \mathbf{P} \boldsymbol{\Xi} \mathbf{P} \boldsymbol{\sigma}_c} \quad (\text{B.33})$$

APPENDIX C

In Chapter 6, reduced integration scheme is employed in the 6-node XFEM plate element with local enrichment function. In this appendix, a proof on the convergence rate of reduced integration scheme for the smooth part is provided for the 6-node plate element. In the XFEM formulation in Chapter 6, the location of a high gradient zone can be expressed as a line on the element level in the parent coordinate system:

$$\varphi = ar + bs + c = 0 \quad (C.1)$$

where it is assumed that the coefficient b is always non-negative, if $b = 0.0$, the coefficient a is positive and the coefficient a and b are normalized so that $a^2 + b^2 = 1$

The enrichment for the translational displacement field outside the high gradient zone is expressed as:

$$F = \sum_{k=1}^6 N_k |\varphi_k| - \left| \sum_{k=1}^6 \varphi_k N_k \right| = \sum_{k=1}^6 N_k |\varphi_k| - |ar + bs + c| \quad (C.2)$$

where φ_k is the nodal value of the function φ :

$$\begin{aligned} \varphi_1 &= c, \quad \varphi_2 = a + c, \quad \varphi_3 = b + c \\ \varphi_4 &= 0.5a + c, \quad \varphi_5 = 0.5a + 0.5b + c, \quad \varphi_6 = 0.5b + c \end{aligned} \quad (C.3)$$

The expression of the first term in Equation (C.2) depends on the cases that the line passes through the element.

The perimeter of the triangle element is divided into 6 segments: s_1 , s_2 , s_3 , s_4 , s_5 and s_6 , as shown in Figure C.1. There are 12 cases for this line passing through the element, which are s_1s_3 , s_1s_4 , s_1s_5 , s_1s_6 , s_2s_3 , s_2s_4 , s_2s_5 , s_2s_6 , s_3s_5 , s_3s_6 , s_4s_5 and s_4s_6 . The notation ‘ s_1s_3 ’ refers to the case where the centre line of high gradient zone cuts through segments s_1 and s_3 , and so on. The polynomial terms appear and the complete polynomial order of the enrichment function F together with the corresponding minimal integration scheme needed to guarantee the convergence of smooth part in each case are shown in Table C.1.

Figure C.1 The line segment of an element

The enrichment S for rotation displacement field outside the high gradient zone is a constant and the degree of complete polynomial of S is 0. Hence, it can be concluded that the 3×3 Gaussian integration scheme in non Hermitian area is acceptable for cases in which the centre line of high gradient zone passes

through the hypotenuse of the triangle element, and the 4×4 Gaussian integration scheme in non Hermitian area is acceptable for cases in which the centre line of high gradient zone passes through the two catheti of the element.

Table C.1 The polynomials in the enrichment F in smooth part and the integration scheme required

Case	terms appear in F						degree of complete polynomial in interpolation function	minimal order of integration scheme
	1	r	s	r^2	rs	s^2		
s1s3	✓	✓	✓	✓	✓		3	3×3
s1s4	✓	✓	✓	✓	✓		3	3×3
s1s5	✓	✓	✓	✓	✓	✓	4	4×4
s1s6	✓	✓	✓	✓	✓	✓	4	4×4
s2s3	✓	✓	✓	✓			3	3×3
s2s4	✓	✓	✓	✓	✓		3	3×3
s2s5	✓	✓	✓	✓	✓	✓	4	4×4
s2s6	✓	✓	✓	✓	✓	✓	4	4×4
s3s6	✓	✓	✓		✓	✓	3	3×3
s4s6	✓	✓	✓		✓	✓	3	3×3
s3s5	✓	✓	✓			✓	3	3×3
s4s5	✓	✓	✓		✓	✓	3	3×3

APPENDIX D

In Chapter 6, reduced integration scheme is employed in the 6-node XFEM plate element. In this section, a proof on the convergence rate of the reduced integration adopted over high gradient zone in the formulation in Chapter 6 is provided. For general cases, a 4×4 Gaussian integration is used over high gradient zone. However it is found that in the following four special cases, a 3×3 Gaussian integration scheme could be used over the high gradient zone.

D.1 The yield line parallel to one of the catheti of the triangular element in the parent coordinate system

Assume that the yield line is parallel to the s – axis and the location of the yield line can be expressed as

$$\varphi = r - c = 0 \quad (\text{D.1})$$

Therefore, the enrichment for the non-Hermitian area is written as

$$F = \sum_{k=1}^6 N_k |\varphi_k| - \left| \sum_{k=1}^6 \varphi_k N_k \right| = c(1 - 2r) + (r - 2r^2) + 4|0.5 - c|(r - r^2) - |r - c| \quad (\text{D.2})$$

The enrichment is independent of the variable s , so

$$F(\varphi_{00}) = F(\varphi_{01}), \quad F(\varphi_{10}) = F(\varphi_{11}), \quad \frac{\partial F}{\partial s} = 0 \quad (\text{D.3})$$

According to the definition of r – s coordinate system, $\partial r / \partial \eta = 0$. Hence,

$$\begin{Bmatrix} \frac{\partial F}{\partial \xi} \\ \frac{\partial F}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial s}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial s}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial F}{\partial r} \\ \frac{\partial F}{\partial s} \end{Bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial s}{\partial \xi} \\ 0 & \frac{\partial s}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial F}{\partial r} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial r}{\partial \xi} \frac{\partial F}{\partial r} \\ 0 \end{Bmatrix} \quad (\text{D.4})$$

In this special case $\partial r / \partial \xi$ is a constant, and $\partial F / \partial r$ is independent of s , so

$$\frac{\partial F(\varphi_{00})}{\partial \xi} = \frac{\partial F(\varphi_{01})}{\partial \xi}, \quad \frac{\partial F(\varphi_{10})}{\partial \xi} = \frac{\partial F(\varphi_{11})}{\partial \xi} \quad (\text{D.5})$$

The second derivative of F is expressed as:

$$\frac{\partial^2 F}{\partial \xi \partial \eta} = \partial \left(\frac{\partial r}{\partial \xi} \frac{\partial F}{\partial r} \right) / \partial \eta = \frac{\partial r}{\partial \xi} \frac{\partial s}{\partial \eta} \frac{\partial^2 F}{\partial r \partial s} = 0 \quad (\text{D.6})$$

It should be noted that in the above derivation, the relation $\partial^2 r / \partial \xi \partial \eta = r_{00} - r_{10} - r_{10} + r_{11} = 0$ is used.

$$\frac{\partial^2 F}{\partial r \partial s} = \sum_{k=1}^6 \frac{\partial^2 N_k}{\partial r \partial s} \big| \varphi_k \big| = 0 \quad (\text{D.7})$$

In the above derivation, the following relations are used:

$$|\varphi_1| = |\varphi_3| = |\varphi_6| = c, \quad |\varphi_4| = |\varphi_5| = |0.5 - c|, \quad |\varphi_2| = 1 - c \quad (\text{D.8})$$

Set $F(\varphi_{00}) = F(\varphi_{01}) = a$, $F(\varphi_{10}) = F(\varphi_{11}) = b$, $\partial F(\varphi_{00}) / \partial \xi = \partial F(\varphi_{01}) / \partial \xi = e$,

$\partial F(\varphi_{01}) / \partial \xi = \partial F(\varphi_{11}) / \partial \xi = f$. Hence, $\mathbf{G}_t$ is of the form:

$$\mathbf{G}_t = \begin{bmatrix} a & a & e & e \\ b & b & f & f \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{D.9})$$

and $\mathbf{M}^T \mathbf{G}_t \mathbf{M}$ is of the form:

$$\mathbf{M}^T \cdot \mathbf{G}_t \cdot \mathbf{M} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3a+3b-2e-f & 0 & 0 & 0 \\ 2a-2b+e+f & 0 & 0 & 0 \end{bmatrix} \quad (\text{D.10})$$

The terms appear in $\mathbf{H}_t$ are shown in Table D.1 with underlines. It can be seen that the degree of the complete polynomial is 0, and in this case, the degree of the complete polynomial of the interpolation function for additional translational DOF is 2.

Table D.1 The polynomial terms in $\mathbf{H}_t$ when the yield line is parallel to one of the catheti

		<u>1</u>				--- 0
		<u>ξ</u>	<u>η</u>			--- 1
	<u>ξ^2</u>	<u>$\xi\eta$</u>	<u>η^2</u>			--- 2
<u>ξ^3</u>	<u>$\xi^2\eta$</u>	<u>$\xi\eta^2$</u>	<u>η^3</u>			--- 3
	<u>$\xi^3\eta$</u>	<u>$\xi^2\eta^2$</u>	<u>$\xi\eta^3$</u>			--- 4
		<u>$\xi^3\eta^2$</u>	<u>$\xi^2\eta^3$</u>			--- 5
			<u>$\xi^3\eta^3$</u>			--- 6

D.2 the yield line parallel to the hypotenuse of the triangle element in the natural coordinate

Figure D.1 The yield line parallel to the hypotenuse

In this case, the centre line of the yield line is parallel to the hypotenuse, as shown in Figure D.1. The location of the yield line can be expressed as

$$\varphi = r + s - c = 0 \quad (\text{D.11})$$

Therefore, the enrichment for the non-Hermitian area is written as

$$\begin{aligned} F &= \sum_{k=1}^6 N_k |\varphi_k| - \left| \sum_{k=1}^6 \varphi_k N_k \right| \\ &= c(1 - 2r - 2s) + (r + s)(2r + 2s - 1) + 4|0.5 - c|(r + s)(1 - r - s) - |r + s - c| \end{aligned} \quad (\text{D.12})$$

Set $r + s = \rho$, the enrichment function F can be rewritten as:

$$F = c(1 - 2\rho) + \rho(2\rho - 1) + 4|0.5 - c|\rho(1 - \rho) - |\rho - c| \quad (\text{D.13})$$

Since $\rho_{00} = \rho_{10}$, $\rho_{01} = \rho_{11}$, set

$$F(\varphi_{00}) = F(\varphi_{10}) = a, \quad F(\varphi_{01}) = F(\varphi_{11}) = b \quad (\text{D.14})$$

$$\begin{pmatrix} \frac{\partial F}{\partial \xi} \\ \frac{\partial F}{\partial \eta} \end{pmatrix} = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial s}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial s}{\partial \eta} \end{bmatrix} \begin{pmatrix} \frac{\partial F}{\partial r} \\ \frac{\partial F}{\partial s} \end{pmatrix} = \begin{bmatrix} r_{10} + \sqrt{2}\omega_e\eta & -s_{00} - \sqrt{2}\omega_e\eta \\ \sqrt{2}\omega_e\xi & \sqrt{2}(1-\xi)\omega_e \end{bmatrix} \begin{pmatrix} \frac{\partial F}{\partial r} \\ \frac{\partial F}{\partial s} \end{pmatrix} \quad (\text{D.15})$$

Because of the symmetry of variables ξ and η in the expression of F , $\partial F/\partial r = \partial F/\partial s$. On the other hand, $r_{10} = s_{00}$. Therefore, $\partial F/\partial \xi = 0$. Furthermore, $\partial^2 F/\partial \xi \partial \eta = 0$.

Set

$$\left. \frac{\partial F}{\partial \xi} \right|_{\substack{\rho=\rho_{10} \\ \rho=\rho_{00}}} = \left. \frac{\partial F}{\partial \eta} \right|_{\substack{\rho=\rho_{10} \\ \rho=\rho_{00}}} = -2c + 4\rho - 1 + 4|0.5 - c|(1 - 2\rho) - 1 = e \quad (\text{D.16})$$

$$\left. \frac{\partial F}{\partial \xi} \right|_{\substack{\rho=\rho_{11} \\ \rho=\rho_{01}}} = \left. \frac{\partial F}{\partial \eta} \right|_{\substack{\rho=\rho_{11} \\ \rho=\rho_{01}}} = -2c + 4\rho - 1 + 4|0.5 - c|(1 - 2\rho) + 1 = f \quad (\text{D.17})$$

and

$$\sqrt{2}\omega_e = l \quad (\text{D.18})$$

Substitute Equation (D.16), Equation (D.17) and Equation (D.18) into Equation (D.15):

$$\begin{pmatrix} \frac{\partial F}{\partial \xi} \\ \frac{\partial F}{\partial \eta} \end{pmatrix}_{00} = \begin{pmatrix} 0 \\ 2le \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial F}{\partial \xi} \\ \frac{\partial F}{\partial \eta} \end{pmatrix}_{01} = \begin{pmatrix} 0 \\ 2lf \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial F}{\partial \xi} \\ \frac{\partial F}{\partial \eta} \end{pmatrix}_{10} = \begin{pmatrix} 0 \\ le \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial F}{\partial \xi} \\ \frac{\partial F}{\partial \eta} \end{pmatrix}_{11} = \begin{pmatrix} 0 \\ 2lf \end{pmatrix} \quad (\text{D.19})$$

Hence, $\mathbf{G}_t$ is of the form:

$$\mathbf{G}_t = \begin{bmatrix} a & b & 2le & 2lf \\ a & b & le & 2lf \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{D.20})$$

and $\mathbf{M}^T \mathbf{G}_t \mathbf{M}$ is of the form:

$$\mathbf{M}^T \mathbf{G}_t \mathbf{M} = \begin{bmatrix} a & 2le & 3(b-a)-2l(2e+f) & 2(a-b)+2l(e+f) \\ 0 & 0 & 0 & 0 \\ 0 & -3le & 6le & -3le \\ 0 & 2le & -4le & 2le \end{bmatrix} \quad (\text{D.21})$$

The terms appear in $\mathbf{H}_t$ are shown in Table D.2 with underlines. It can be seen that the degree of the complete polynomial is 0, and in this case, the degree of the complete polynomial of the interpolation function for additional translational DOF is 2.

Table D.2 The polynomial terms in $\mathbf{H}_t$ when the yield line is parallel to the hypotenuse

		<u>1</u>				--- 0
	ξ		<u>η</u>			--- 1
	ξ^2	$\xi\eta$		<u>η^2</u>		--- 2
ξ^3		<u>$\xi^2\eta$</u>	$\xi\eta^2$		<u>η^3</u>	--- 3
	<u>$\xi^3\eta$</u>	<u>$\xi^2\eta^2$</u>		$\xi\eta^3$		--- 4
		<u>$\xi^3\eta^2$</u>	<u>$\xi^2\eta^3$</u>			--- 5
			<u>$\xi^3\eta^3$</u>			--- 6

D.3 The yield line perpendicular to the hypotenuse and passing through one of the corner points

It is shown in Figure D.2 that the yield line is perpendicular to the hypotenuse and passes through one of the corner points. The location of the yield line is expressed as

$$\varphi = r - s - 1 = 0 \quad (\text{D.22})$$

Figure D.2 The yield line perpendicular to the hypotenuse (1)

The enrichment for the non-Hermitian area is written as

$$F = \sum_{k=1}^6 N_k |\varphi_k| - \left| \sum_{k=1}^6 \varphi_k N_k \right| = 1 - r - s - |1 - r - s| \quad (\text{D.23})$$

Therefore, $F(\varphi_{00}) = F(\varphi_{01}) = 0$, $\partial F(\varphi_{00}) / \partial \zeta = \partial F(\varphi_{01}) / \partial \zeta = 0$, $\partial F(\varphi_{00}) / \partial \eta = \partial F(\varphi_{01}) / \partial \eta = 0$ and $\partial^2 F(\varphi_{00}) / \partial \zeta \partial \eta = \partial^2 F(\varphi_{01}) / \partial \zeta \partial \eta = 0$.

Table D.3 The polynomial terms in $\mathbf{H}_t$ when the yield line is perpendicular to the hypotenuse and passes through one of the corner point

			1				---	0	
		ξ		η				---	1
	ξ^2		$\xi\eta$		η^2			---	2
ξ^3		$\xi^2\eta$		$\xi\eta^2$		η^3		---	3
	$\xi^3\eta$		$\xi^2\eta^2$		$\xi\eta^3$			---	4
		$\xi^3\eta^2$		$\xi^2\eta^3$				---	5
			$\xi^3\eta^3$					---	6

Hence, the terms in the Hermite function are shown in Table D.3 with underlines. It can be seen that there is no complete polynomial in the Hermite function in this case.

D.4 The yield line perpendicular to the hypotenuse and passing through the opposite corner point

As shown in Figure D.3, the yield line is perpendicular to the hypotenuse and the centre of the high gradient zone passes through the right angle.

Figure D.3 The yield line perpendicular to the hypotenuse (2)

The location of the yield line is expressed as:

$$\varphi = r - s = 0 \quad (\text{D.24})$$

The enrichment for the non-Hermitian area is written as

$$F = \sum_{k=1}^6 N_k |\varphi_k| - \left| \sum_{k=1}^6 \varphi_k N_k \right| = r - 4rs + s - |r - s| \quad (\text{D.25})$$

Therefore, $F(\varphi_{00}) = F(\varphi_{10}) = 0$ and the constant term does not appear in the Hermite function in this case and there is no complete polynomial in the Hermite function in this case.

Finally, it can be concluded that in all the above cases, the 3×3 Gaussian integration scheme is sufficient for the convergence condition (Zienkiewicz and Taylor 2005a).

APPENDIX E

The *local* enrichment function is introduced in Section 6.2 for the 6-node XFEM plate element. However, it is found that the *local* enrichment function is not applicable for the 9-node XFEM plate element. The construction of the local enrichment function for the 9-node plate element is shown below to explain its failure in the 9-node plate element.

Figure E.1 A quadrilateral element with a high gradient zone

A particular case that $\varphi = r - s = 0$ and $\omega = 0.3$ of a high gradient zone in a quadrilateral element, as shown in Figure E.1, is taken as an example. The plot of the local enrichment for rotational DOF, R , is shown in Figure E.2.

(a) The 3D plot of R in a 9-node quadrilateral element

(b) The contour of R in a 9-node quadrilateral element

Figure E.2 The plot of R in a 9-node quadrilateral element

The plot of the *local* enrichment function for translational DOF in a 9-node quadrilateral element is shown in Figure E.3.

(a) The 3D plot of F in a 9-node quadrilateral element

(b) The contour of F in a 9-node quadrilateral element

Figure E.3 The plot of R in a 9-node quadrilateral element

Two problems can be found from Figure E.3. First, the local enrichment function for translational DOF in a 9-node quadrilateral element is not continuous at the interface between the high gradient zone and the smooth part. Secondly, the gradient of the local enrichment function F along the yield line

direction is not zero. The first problem is not acceptable for the present XFEM formulation, because the discontinuity of the enrichment is equivalent to a zero traction boundary condition. The source of this discontinuity can be explained as follows: along the interface (η -direction in Figure E.1), the absolute level set function has three stationary points, which means that the highest order of η in the absolute level set function is 4. However, in the Hermite function, the order of η is up to 3. Therefore, the Hermite function is not able to reproduce the curve of the absolute level set function at their interface. Furthermore, it could be found that the fourth order term η^4 comes from the term r^2s^2 in the standard 9-node shape functions. The term r^2s^2 only appears in a 9-node plate element and disappears in an 8-node plate element. Therefore, by using an absolute level set function constructed by the standard shape function of an 8-node element, the problem of discontinuity at the interface could be solved. Hence, Equation (6.4) is modified as

$$F(\varphi) = \begin{cases} H_t & |\varphi| < 0.5\omega \\ N_k |\varphi_k| - |\varphi_k N_k| & \text{otherwise} \end{cases}, \quad k = 1, 2, \dots, 8 \quad (\text{E.1})$$

where N_k is the standard shape functions for an 8-node plate element. After the modification, the plot of F is shown in Figure E.4.

(a) The 3D plot of the modified F in a 9-node quadrilateral element

(b) The contour of the modified F in a 9-node quadrilateral element

Figure E.4 The plot of the modified F in a 9-node quadrilateral element

After modification, the problem of the discontinuity at the interface of high gradient zone and smooth part is solved. However, the problem of the high gradient along the yield line direction still appears in the 9-node quadrilateral element. Hence, the *local* enrichment function in the 9-node quadrilateral element is not further discussed in this thesis.