

Unknown state and non-matching disturbance estimations with hybrid sliding mode observers

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**Unknown State and Non-matching Disturbance
Estimations with Hybrid Sliding Mode
Observers**

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A thesis submitted to the Nanyang Technological University
in fulfillment of the requirements for the degree of
Doctor of Philosophy

2012

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

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Date

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Zhou Yong

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strength to me.

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Abstract

This thesis presents a new perspective on the observer design and analysis for a class of nonlinear uncertain systems in which the uncertainties enter the systems through unknown-state-dependent distribution vectors, i.e., the systems have non-matching uncertainties in the observer sense. The main idea lies with exploiting the observabilities of the unknown states and uncertainties from the measurable outputs. This allows us to design appropriate robust terms to asymptotically track the uncertainties and thereby estimating the unmeasurable states so that precise control and monitoring of control systems can be achieved.

The first part of the thesis addresses the estimation problem of a single input single output (SISO) nonlinear Lipschitz system with the unknown input being non-matching in the observer sense. A hybrid observer that combines the high gain observer with a higher order sliding mode related nonlinear feedback term is proposed. For such a hybrid observer, the high gain feedback works to constrain the estimation error to within an invariant set regardless of the initial conditions, in which the sliding mode condition is satisfied. Then, the sliding mode feedback ensures that the sliding mode surface is reached in finite time and remained thereafter. As a result, the unknown input can be recovered from the sliding mode term after all states have converged to their true values. However, the identifiability of the unknown input is strictly related to the stability of the estimation dynamics on the sliding surface, which is only dependent on the structure of the nonlinear system and is difficult to be verified.

As an application example, the application of the proposed results to the series DC motor which is widely used due to its high ratio of torque per ampere of current, especially in the industrial applications that require high starting torque,

is studied. The non-matching flux related motor parameter of the series DC motor is time-varying because of magnetic saturation or imperfect manufacturing. Together with the effect of an unknown external load disturbance, they often limit the corresponding control system's performance. In order to overcome such limitations, the proposed robust hybrid sliding mode observer that is developed via the *Lie* derivatives transformation is applied. With the measurable current and input voltage, the non-matching parameter can be exactly estimated without filtering effect, and the identified flux related parameter is then used to enhance the speed estimation performance in the presence of the external disturbance. The expected estimation performance is demonstrated through a series of experimental results.

We explore how the proposed sliding mode observer design can also be applied to the rotor speed and position estimations of a surface-mounted permanent magnet synchronous motor (PMSM), in which the asymptotical stability property of the reduced order system is not satisfied. Unlike the conventional sliding mode observers, we treat the position related dynamics as new unknown system states instead of as a part of the system uncertainties, then the filtering/chattering effect on the position estimation can be completely avoided. Such methodology can be used to improve the accuracy of position estimation at low-speed situations when a one time calibration of the rotor position is available. The results are demonstrated through simulation studies.

The second part of the thesis focuses on the identifiability of a class of multi-input-multi-output (MIMO) nonlinear systems with non-matching unknown inputs, i.e., without satisfying the involutive condition, but the number of the measurement outputs is assumed to be one more than the number of the unknown inputs. We shall establish conditions for uniform observability of these uncertain systems as well as the identifiability of the unknown inputs. For this class of uncertain systems the original nonlinear system can be divided into two subsystems, of which one is a square subsystem with the matching unknown inputs appearing in the corresponding last equation, and the other subsystem has non-matching unknown inputs.

To handle such a nonlinear uncertain system, a high gain observer appended with multiple higher order sliding mode terms is proposed, where the nonlinear sliding mode feedbacks are designed to track the unknown inputs individually, and replace them with some nominal dynamics on the sliding mode surfaces. As a result, the uniform observability for the subsystem with the non-matching inputs can be guaranteed, and the high gain observer works to ensure that the remaining dynamics on the sliding mode surfaces is asymptotically stable. Therefore, the unknown inputs can be reconstructed after all the states have converged to their true values.

A class of more general MIMO nonlinear uncertain systems in which the unknown inputs or disturbances appear in both the state dynamics and the measurement outputs, is then considered. With one more output than the number of unknown inputs, it guarantees that at least one clean output signal can be achieved in the initial stage. Then, a recursive sliding mode observer with high gain feedback is developed, in which the sliding mode feedbacks with recursive structures are designed to ensure that the sliding mode surfaces are reached sequentially, and that the valuable signals on the measurement outputs are gradually extracted by cancelling the unknown inputs in sequence. Then, the high gain feedback works to guarantee the unknown inputs and the states can be identified asymptotically.

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Nomenclature

Algebraic Operators

| | |
|-------------------------------|--|
| \mathbf{I} | Identity matrix |
| \mathbf{x}^T | Transpose of vector \mathbf{x} |
| $\hat{\mathbf{x}}$ | The estimate of \mathbf{x} |
| \mathbf{A}^T | Transpose of matrix \mathbf{A} |
| \mathbf{A}^{-1} | Inverse of matrix \mathbf{A} |
| $\ \cdot\ $ | Euclidian norm of vector or its induced matrix norm |
| $\lambda_{min}(\cdot)$ | The minimum eigenvalue of a matrix |
| $\lambda_{max}(\cdot)$ | The maximum eigenvalue of a matrix |
| $\sigma(\cdot)$ | The condition number of matrix, given by $\sigma(\mathbf{A}) = \ \mathbf{A}\ \ \mathbf{A}^{-1}\ $ |
| $\text{sign}(\cdot)$ | The signum function |
| $\text{sat}(\cdot)$ | The saturation function |
| $ \cdot $ | The absolute value of a number |
| $L_{\mathbf{f}}h(\mathbf{x})$ | The <i>Lie</i> derivative, given by $\left[\frac{\partial h(\mathbf{x})}{\partial \mathbf{x}} \right] \mathbf{f}(\mathbf{x})$ |

Sets and Set Operators

| | |
|--------------------|--|
| \Re | Set of real numbers |
| \Re^n | Set of real vectors with n elements |
| $\Re^{n \times m}$ | Set of real matrices with n rows and m columns |

Abbreviations

| | |
|----------|------------------------------------|
| SISO | Single Input Single Output |
| BIBO | Bounded Input Bounded Output |
| MIMO | Multiple Inputs Multiple Outputs |
| LTI | Linear Time Invariant |
| HGO | High Gain Observer |
| SMO | Sliding Mode Observer |
| SMC | Sliding Mode Control |
| VSC | Varying Structure Control |
| HSMC | Higher-order Sliding Mode Control |
| HSMO | Higher-order Sliding Mode Observer |
| DC Motor | Direct Current Electric Motor |
| PM Motor | Permanent Magnet Motor |
| PMSM | Permanent Magnet Synchronous Motor |
| Back-EMF | The Back Electromotive Force |
| DSP | Digital signal processor |

Chapter 1

Introduction

1.1 Motivation

Real-time and accurate information of the system state is often necessary for effective control of a system or monitoring of a process. However, in practice, there are only partial state and/or input information available through the measurable outputs, plus the existence of system uncertainties or disturbances that are caused by parameter deviations or modeling errors. The lack of full state information and the presence of uncertainties often limit the performance of the controlled processes. For such situations, a robust observer with high estimation accuracy is required to estimate the unknown states and recover the uncertainties¹.

The sliding mode observer (SMO) has been proven to be an effective approach for handling uncertain systems, due to its insensitivity to the uncertainties and the capability of reconstructing the uncertainties based on the equivalent injection input concept. Essentially, the SMO works with a switching feedback mechanism to control the estimation system's trajectory towards a predefined manifold (namely the sliding surface) and staying on it thereafter.

In this context, the motivation for this research arises from the desire to design a robust estimation method for a class of nonlinear uncertain systems, so as to provide highly accurate and reliable estimates of the unmeasurable states as well

¹Through out the thesis, the terms 'uncertainties' or 'disturbances' or 'unknown inputs' are used interchangeably. They indicate the presence of unknown dynamics in the system.

as the unknown dynamics present in the systems.

1.2 Earlier Works

An observer is a dynamic system that models a real system in order to provide on-line estimations of its internal states, or the unknown dynamics itself, based on the measurements of the inputs and outputs of the real system.

1.2.1 Observers for Linear and Nonlinear Systems

The observer design was first presented by Luenberger in [1]- [3] for deterministic continuous-time linear time-invariant (LTI) systems, and later extended into time-varying systems, discrete-time systems, and time-delay systems [4]- [8]. In [9], the Luenberger observer with adapting parameters was proposed to estimate the states of an unknown linear system, and the convergence rates can be made arbitrarily fast by choosing large adaptive gains. Similar works based on Lyapunov stability theory have been reported in [10]- [12]. Meanwhile, the Kalman filter based algorithms can provide an optimal estimate with the minimum variance for a class of stochastic systems with white noise [13, 14].

In general, the linear estimation problem for deterministic linear systems with or without the white noise is almost solved with the development of Luenberger-like observer and Kalman filter based algorithms. Their successes and practical limitations have motivated further research and development of various extensions for handling uncertain linear and nonlinear dynamic systems, to provide accurate system state estimation as well as unknown parameter identification [15]- [19], so that the results can be applied to most practical systems.

Unlike linear systems, the observability of a nonlinear system is input-dependent, which means the nonlinear system may have singular or bad input that makes it unobservable [35].

The early attempt to design asymptotic observer for a nonlinear system through the coordinate transformation was reported in [20] - [22], in which the transformed dynamic system is linear and observable, and all the previous linear estimation

techniques can be applied by means of inverse mapping. The work in [23] presented the necessary and sufficient conditions for the existence of such linearization transformation for nonlinear systems with or without inputs. Moreover, the results in [24, 25] provided a general introduction to the geometric methods by linearizing the error dynamics based on exact nonlinear feedback or correction terms. However, these observer design methods are hampered by the requirement of strict existence conditions of the invertibility of the state transformation or the so-called Jacobian matrix, which is obtained from higher order *Lie* derivatives of the system output [26].

The Lyapunov-based approach has also been proposed as an effective observer design methodology for nonlinear systems [27]- [32]. The early attempt for nonlinear observer design based on Lyapunov function was Thau [27], in which the sufficient conditions for convergence were also addressed. Later, extensions to Thau's work for a class of Lipschitz nonlinear systems were developed based on the off-line solution of a Riccati equation [30, 31]. However, the feedback gain design based on Lyapunov method is not straight forward and only sufficient conditions are available.

In addition to the above mentioned results, the high gain observer, which can be treated as a kind of extended Luenberger observers, has been given full attention for its ability to handle nonlinear Lipschitz systems [33]- [37]. In [33], through the inverse mapping of *Lie* transformation, a nonlinear observer has been developed for a class of uniformly observable systems, and exponential convergence can be achieved in spite of large Lipschitz constants. Later on, the observability of a general nonlinear systems has been carefully addressed in [35], in which a nonlinear system with a triangular structure is proven to be uniformly observable for any arbitrary bounded input, and an exponential observer was developed for such nonlinear systems. After that, the work in [36] presented an explicit feedback gain design methodology for a special class of nonlinear system with triangular structure, which made the high gain observer easier to implement.

Although fruitful results have been reported for linear and nonlinear systems, the above mentioned observers are only suitable for the nominal dynamic sys-

tems. In other words, the presence of unknown dynamics in the system, such as variations of the system parameter, uncertainties in modeling or external disturbance, can result in poor estimation performance. Therefore, it is important and necessary to develop a robust observer for the identification of the unknown states as well as the system uncertainties.

In order to handle unknown system dynamics, many effective methods and technologies have been reported during the past few decades [38]- [43], and successfully implemented into industrial applications [44]- [50]. The early work on unknown input observer design was based on the geometric conditions to decouple the uncertainties from the nominal dynamics [38]. The work in [41] presented a comprehensive analysis on unknown input observability and reconstruction for LTI systems, and the corresponding necessary and sufficient conditions are also provided. Based on an adaptive observer design technique, two reduced-order input estimators have been proposed for LTI systems [43], which can also be applied to certain non-minimum phase systems.

1.2.2 Sliding Mode Observer for Uncertain Systems

The sliding mode control (SMC) has been established as a robust method for handling system uncertainties [51]- [56]. It forces the system trajectory to move along a predefined manifold and remain on it thereafter. Based on the same concept, the sliding mode based observer for the state and disturbance estimations of uncertain systems became an attractive research field in recent years, due to its advantages of high state-estimation accuracy, simplicity, robustness and the capability of reconstructing the uncertainties.

The early works based on Lyapunov method in this area were developed by Walcott and Zak for dynamic systems with bounded disturbance [57], and extended to a more general class of nonlinear systems in [58, 59].

The idea of sliding mode observer (SMO) design based on the equivalent control concept was first proposed by Utkin and Drakunov in [52, 55], and was applied into a class of nonlinear system with triangular structure without the knowledge of the input derivative, in which only the discontinuous term was fed back through

properly designed gains [61]. The result in [63] incorporated a sliding mode term into a high gain observer to realize a robust nonlinear observer for a class of nonlinear Lipschitz uncertain systems, then the unknown disturbance was replaced with nominal terms on the sliding surface, and reconstructed after all states have converged. Similar work can be found in [64]. However, these methods were only applicable to systems where the relative degree between the uncertainty and the measurement output is one, moreover, the undesirable chattering will degrade the estimation performance and that a low pass filter is required for the reconstruction of the uncertainty.

With the development of higher order sliding mode (HOSM) techniques, the relative degree restriction has been completely relaxed and a better estimation accuracy can be achieved with proper design [53,66]- [74]. As a result, the HOSM-based observers have received increasing attention in recent years [75]- [84], and the second-order sliding mode observers have been successfully applied in many applications [85]- [92], such as the suboptimal sliding mode observer with the first derivative of the output was employed to estimate the unknown velocity and torque in electrical drives systems [85,89]. Later on, with the modified super-twisting algorithm the observer was developed without differentiator in [86], and it has been successfully applied into electromechanical systems [87].

It should be mentioned that the traditional sliding mode techniques are only robust to uncertainties and disturbances satisfying the matching condition in that they enter the system via the same channels as the control inputs. In other words, these sliding mode observers are only suitable for uncertain systems with the uncertainties appearing only in the corresponding last dynamic equation. In [76], a traditional Luenberger observer with high order sliding mode differentiator was developed for a class of LTI systems with the unknown input satisfying the so-called strong observability or strong detectability condition. The work in [81] applied a higher order sliding mode observer to a SISO nonlinear system to identify the unknown disturbance, in which the relative degree between the unknown disturbance and measurable output is full order or higher order. Similar works have been extended into the MIMO nonlinear uncertain systems [62,78] based on

the involutive condition, with which the original uncertain system can be decomposed into two subsystems: the first subsystem possesses a strictly differential structure with the uncertainties in the last dynamic equation, and the other one has a nominal dynamics. However, the identifiability of the uncertainties are strongly related to the stability of the nominal subsystem which is difficult to verify.

Besides the restriction of the matching condition, the chattering effect remains as another challenging problem in sliding mode observer design, and in order to alleviate such shortcomings of the sliding mode observers, some researches based on the integration of SMO with other methodologies (i.e. hybrid sliding mode designs) have been reported [93]- [100].

1.3 Objectives and Contributions

The focus of this thesis is on robust hybrid sliding mode observer design techniques for unknown state estimation and disturbance identification of nonlinear uncertain systems, in which the unknown disturbances enter the system dynamics through the unknown-state dependent vectors or matrices, i.e., non-matching disturbances in the observer sense. The main emphasis lies with exploiting the observability of the system states from the measurement outputs, as well as the identifiability of the unknown disturbances.

In a controller design, the so-called matching condition of the disturbance means that it enters the system via the same channel as the control input, and its corresponding distribution vector is the same as the control input. However, with regards to observer design, the channels of feedback input can be arbitrarily chosen, and the corresponding distribution vector can be designed flexibly based on any known state/output. Therefore, the matching condition in the observer sense means that the distribution vectors of the uncertainties and disturbances are independent of any known state ².

In this thesis, we shall present a new perspective on the design and analysis

²Throughout the thesis, the concept of the non-matching in the observer sense is used for the system uncertainties or the disturbances or the unknown inputs.

of robust observers for unknown state estimation and non-matching uncertainties identification of nonlinear systems. The whole development relies on the idea that *"the unknown disturbances can be replaced by some nominal dynamics when the corresponding sliding mode happens"*. As a result, the observability of an uncertain system is related to the reachability of the sliding mode, as well as the stability of the remaining dynamics on the sliding surface.

The approaches to the robust observer designs in the thesis will be examined for both single-input-single-output and multi-input-multi-output nonlinear systems with the input being unknown. The major contributions of this thesis can be summarized as follows:

- (i) A hybrid nonlinear observer that combines a full-order high gain feedback with a higher-order sliding mode feedback is proposed. With the proposed observer, the high gain feedback works to constrain the estimation error to within an invariant set regardless of the initial conditions, in which the sliding mode condition is satisfied. Then the sliding mode feedback tracks the unknown input and ensures that the sliding mode surface is reached in finite time. Finally, both the unknown input and states can be identified when the remaining dynamics on the sliding surface is self asymptotically stable. However, it will be pointed out that the stability of the remaining dynamics is completely independent of the observer gains, and is only related to the original system structure.
- (ii) The proposed hybrid observer design approach is experimentally verified on a series DC motor. Based on the measurable current and input voltage, a robust hybrid observer is developed to identify a non-matching time varying parameter that affects the speed sensorless series DC motor through an unknown-speed dependent vector. The identified parameter is then used to enhance the speed estimation performance in the presence of external disturbance. The stability for the developed observers are carefully addressed, and experimental results are provided to demonstrate the expected estimation performance. Moreover, the sensitivity of the estimated parameter against variations in resistance and inductance is demonstrated by Monte

carlo simulations.

- (iii) The sliding mode observer design for rotor position and speed estimations of a surface-mounted permanent magnet synchronous motor (PMSM) is considered. By separating the rotor position related dynamics from the back electromotive forces (EMFs) and modeling them as new unknown system states, the filtering effect on the position estimation can be completely avoided. From the observer design point of view, the reduced-order estimation error dynamics is self-stable, but not asymptotically stable. Then, a one time rotor position calibration is required for accurate position estimation. The observability of speed and position estimations is carefully addressed, and the effectiveness of the proposed method is verified by simulation studies.
- (iv) Robust observer design for a class of MIMO nonlinear systems in which the number of measurement outputs is one more than the number of unknown inputs is considered. A new hybrid observer that combines a reduced-order high gain feedback with multi sliding mode terms is proposed based on the *Lie* derivatives transformation. The sliding mode feedbacks work to ensure the corresponding sliding mode surface will be reached individually and remained thereafter. Then, with all the sliding mode surfaces being reached, the high gain feedback designed based on the extra output guarantees that the remaining dynamics is asymptotically stable on the sliding surfaces. Therefore, all the unknown inputs and states can be exactly identified.
- (v) A general class of MIMO nonlinear systems in which the unknown inputs appear in both the dynamics of the states and the measurement outputs is also examined. A novel hybrid observer that combines higher order sliding mode observers with a reduced-order high gain feedback is proposed. The sliding mode observer is designed with recursive structures to ensure that the sliding mode surfaces are reached sequentially, and that the valuable signals on the measurement outputs are gradually extracted by cancelling the unknown inputs in sequence. Then, the reduced-order high gain feedback

designed based on the extra output works to guarantee that the unknown inputs and the states can be identified asymptotically on the sliding mode surfaces.

1.4 Organization of the Thesis

The rest of the thesis is organized as follows:

In **Chapter 2**, a comprehensive review of sliding mode observers is presented. The methodologies of the high gain observer theory and nonlinear transformation are also reviewed.

In **Chapter 3**, a new perspective on the design of robust observer for a class of SISO nonlinear systems with non-matching unknown input is presented. The identifiability of the unknown input is carefully addressed, and a hybrid observer design approach that integrates a full-order high gain feedback with sliding mode term is proposed.

In **Chapter 4**, a robust hybrid observer is developed to identify a non-matching time-varying parameter that affects the speed sensorless series DC motor via an unknown-speed dependent vector. Based on the measurable current and input voltage, the non-matching parameter can be exactly identified, as well as the unknown speed. Then, the identified parameter is applied into another observer to enhance the speed estimation performance in the presence of external disturbance. Experimental results are provided to demonstrate the effectiveness of the proposed results.

In **Chapter 5**, a novel perspective on the sliding mode observer design for speed sensorless estimation of a PMSM is proposed. Based on the idea that the chattering/filtering effect of the sliding mode observer only affects the reconstruction of unknown dynamics, but not the system states, the rotor position can be identified without filtering effect if its related dynamics are considered as new system states instead of a part of the unknown back-EMFs.

In **Chapter 6**, a class of MIMO nonlinear uncertain systems where the number

of measurement outputs is one more than the number of the unknown inputs is considered. A hybrid observer that combines a reduced-order high gain feedback with multiple sliding mode terms is proposed, in which the sliding mode feedbacks ensure that the corresponding sliding surfaces are reached individually and remained thereafter. Then the reduced-order high gain feedback guarantees the asymptotic stability of the reduced-order estimation dynamics on the sliding mode surface.

In **Chapter 7**, a class of more general MIMO nonlinear systems is considered, in which the non-matching unknown inputs appear in both the dynamics of states and the measurement outputs. A novel nonlinear observer that combines a reduced-order high gain feedback with a recursive sliding observer is developed. The number of measurable outputs is assumed to be more than the number of unknown inputs, so as to guarantee that at least one clean output signal can be extracted in the initial stages. Then the recursive sliding mode observers ensure that the sliding mode surfaces are reached sequentially and the valuable signals on the measurement outputs are gradually extracted as well. Finally, the high gain feedback guarantees the asymptotic stability of the remaining dynamics on the sliding surfaces.

In **Chapter 8**, the thesis is concluded. Several interesting research topics are also presented as possible future research directions.

Chapter 2

Sliding Mode Observer: A Review

In this chapter, a comprehensive review on existing sliding mode techniques will be conducted. The existing results of high gain observer theory will also be reviewed, to provide a theoretical framework for the development of new results in the remainder of the thesis.

The chapter is organized as follows: Section 2.1 presents an overview on the theory of sliding mode control, as well as sliding mode observer designs. In section 2.2, some existing sliding mode observer techniques will be reviewed according to the relative degree of the unknown disturbance. In section 2.3, the observability of nonlinear systems will be discussed, and brief review of a nonlinear state transformation is also presented. Then the high gain observer theory will be introduced. Section 2.4 concludes this chapter.

2.1 Sliding Mode Theory

The sliding mode control (SMC) concept was first derived from the varying structure control (VSC) theory in the early 1990's by Emelyanov and several co-researchers, in which a discontinuous control action is adopted to force the system trajectory onto a prescribed manifold, namely the sliding surface, regardless of

any bounded matching dynamics¹ [54].

In general, the whole trajectory of a sliding mode control system consists of two modes : the *reaching mode*, or called *nonsliding mode*, in which the system trajectory moves towards the sliding surface from anywhere; and the *sliding mode*, in which the trajectory reaches the sliding surface and stays on it thereafter. Therefore, the corresponding design procedure of sliding mode control can be divided into two steps: selecting a sliding mode surface, and designing a sliding control law to ensure the existence of sliding mode.

Consider the control problem of a continuous-time single-input-single-output system given by

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})u \\ y &= \sigma(\mathbf{x})\end{aligned}\tag{2.1}$$

where $\mathbf{x} \in \Re^n$ is the state, $u \in \Re$ is the input, $y \in \Re$ is the measurable output, and $\mathbf{f}, \mathbf{b} \in \Re^n$ are smooth functions.

Suppose the relative-degree r between the output σ and the input u is constant and known. Then the nonlinear system (2.1) can be expressed in the form of the measurement output as [105]

$$\sigma^{(r)} = h(\mathbf{x}) + g(\mathbf{x})u\tag{2.2}$$

where $g(\mathbf{x}) = L_{\mathbf{f}}^{(r)}\sigma(\mathbf{x})$, $h(\mathbf{x}) = L_{\mathbf{b}}L_{\mathbf{f}}^{(r-1)}\sigma(\mathbf{x})$, with the *Lie* derivative being defined as $L_{\mathbf{f}}\sigma(\mathbf{x}) = [\partial\sigma(\mathbf{x})/\partial\mathbf{x}]\mathbf{f}$. Therefore, the control input that could bring about the sliding mode in system (2.2) would be of the form

$$u = \begin{cases} u^+(\mathbf{x}) & \text{with } \sigma^{(r-1)} > 0 \\ u^-(\mathbf{x}) & \text{with } \sigma^{(r-1)} < 0 \end{cases}\tag{2.3}$$

and the functions $u^+(\mathbf{x}) \neq u^-(\mathbf{x})$.

Definition 2.1. [53, 69, 74] Consider a smooth dynamic system with a smooth output σ , the successive total time derivatives $\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}$ are continuous functions, and the set $\sigma = \dot{\sigma} = \dots = \sigma^{(r-1)} = 0$ is non-empty and consists of locally

¹The dynamics can include any nonlinear function, disturbance, and uncertainty.

Filippov trajectories. Then, the motion on the set $\sigma = \dot{\sigma} = \dots = \sigma^{(r-1)} = 0$ is said to exist in the r th-order sliding mode, and the r th derivative $\sigma^{(r)}$ is mostly supposed to be discontinuous or non-existent.

It is well-known that the intrinsic robustness of SMC lies with its switching control mechanism, but it may cause the undesirable chattering issues due to the limitation of bandwidth in real applications. Such phenomenon is dangerous in a control system since it may excite the high frequency un-modeled dynamics, which may degrade the system performance or even dominate the system's stability. In order to attenuate or remove the chattering effect, several interesting techniques to overcome the limitation have been proposed in the past two decades [53, 69, 78, 101].

With regards to the sliding mode observer (SMO) which is a numerical model implemented in the computer, the existing sliding mode control techniques can be seamlessly ported into the observer design. Then, the chattering issues are only related to the uncertainty of the estimation accuracy because of the requirement of a low-pass filter, but it has no effect on the state estimation accuracy. Moreover, the channels of feedback input can be arbitrarily chosen, and the corresponding distribution vector can be designed flexibly based on any known state or output.

2.2 Sliding Mode Observers

Based on Definition 2.1, it can be found that the sliding control law design (2.3) is strictly related to the order of the relative degree between the measurement output and the system input. In other words, with regards to the sliding mode observer design for uncertain systems, the feedback input design is related to the relative degree between the system output and the uncertainty/disturbance ².

In this section, existing sliding mode observers for a class of SISO nonlinear uncertain systems will be reviewed. Moreover, the chattering effect and the corresponding attenuation methodologies will also be included.

²In the following, the relative degree means it between the system uncertainty and the measurable output.

2.2.1 Standard Sliding Mode Observer

Let's consider the following uncertain system which is expressed in the form of the measurement output, as given by

$$\dot{x} = f(x, u) + d(t) \quad (2.4)$$

where $x \in \Re$ is measurable state, $u \in \Re$ is known system input, $f(\cdot)$ is a known continuous function, and $d(t)$ denotes the system uncertainty or disturbance, which is to be identified.

A sliding mode observer can be designed as

$$\dot{\hat{x}} = f(\hat{x}, u) + u_r \quad (2.5)$$

where the sliding mode term u_r is given by

$$u_r = -\rho \text{sign}(\hat{x} - x) \quad (2.6)$$

with $\text{sign}(\cdot)$ being a discontinuous signum function, defined by

$$\text{sign}(\sigma) = \begin{cases} +1 & \text{if } \sigma > 0 \\ 0 & \text{if } \sigma = 0 \\ -1 & \text{if } \sigma < 0 \end{cases} \quad (2.7)$$

Then, the dynamics of the estimation error $e = \hat{x} - x$ can be described as

$$\dot{e} = f(\hat{x}, u) - f(x, u) - d(t) - \rho \text{sign}(e) \quad (2.8)$$

Suppose that the disturbance $d(t)$ is bounded, and the system state x is locally bounded, then the existence is ensured of a positive constant F , such that the following inequality holds:

$$|f(\hat{x}, u) - f(x, u) - d(t)| \leq F \quad (2.9)$$

Therefore, the sliding gain ρ can be chosen large enough, i.e., $\rho > F$, and it has

$$\begin{aligned} \frac{d}{dt}e^2 &= e\dot{e} \\ &\leq -\{\rho - |f(\hat{x}, u) - f(x, u) - d(t)|\}|e| \\ &\leq -(\rho - F)|e| \\ &< 0 \quad (\forall e \neq 0) \end{aligned}$$

The above inequality implies the estimation error e will asymptotically converge to zero provide that the sliding mode gain is chosen large enough, i.e., $\rho > F$. In other words, the first order sliding surface $e = \hat{x} - x = 0$ will be asymptotically reached and remained thereafter.

Once the sliding mode occurs, i.e., $\hat{x} = x, f(\hat{x}, u) = f(x, u)$, the unknown disturbance $d(t)$ can be recovered based on the equivalent input control concept, either by using a low-pass filter [52], or by using a small positive scalar δ [59], as follow:

$$\hat{d}(t) \approx \{\rho \text{sign}(e)\}_{eq} \approx \rho \frac{e}{(|e| + \delta)} \quad (2.10)$$

with $\{\cdot\}_{eq}$ denoting the equivalent signal obtained by the low-pass filter. Clearly, the accuracy of the disturbance estimation is strictly dependent on the parameter δ or the low-pass filter parameters.

In order to improve the estimation accuracy of the unknown disturbance by attenuating the chattering effect, some results in which the sliding mode feedback u_r is designed by replacing the switching function with a continuous function in a predefined boundary layer have been proposed [102, 103]. One common method is using the following saturation function:

$$\text{sat}(\sigma, \epsilon) = \begin{cases} \sigma/\epsilon & \text{if } |\sigma| \leq \epsilon \\ \text{sign}(\sigma) & \text{if } |\sigma| > \epsilon \end{cases} \quad (2.11)$$

where ϵ indicates the thickness of the boundary layer.

It can be found that inside the boundary layer, the signum function will be

approximated by a high gain linear feedback which is a continuous function. As a result, the sliding feedback u_r remains continuous everywhere and the unknown disturbance can be directly recovered without the filtering effect. However, the implementation of the boundary layer method will cause a trade-off between the robustness of observer and the estimation accuracy of disturbance because the Lyapunov stability cannot be guaranteed inside the boundary layer [104].

The super-twisting algorithm generates a continuous feedback on the sliding surface [53], which means the unknown disturbance can be recovered on the sliding surface without the filtering effect, as well as without sacrificing the robustness of the observer. Then, the sliding mode term u_r in (2.5) can be replaced by

$$\begin{aligned} u_r &= -\rho|e|^{1/2}\text{sign}(e) + v \\ \dot{v} &= \begin{cases} -u_r, & \text{if } |u_r| > \bar{u}_r \\ -M\text{sign}(e), & \text{if } |u_r| \leq \bar{u}_r \end{cases} \end{aligned} \quad (2.12)$$

with the positive parameters ρ, M, \bar{u}_r being properly chosen.

Once the sliding mode occurs, the unknown disturbance can be directly obtained from the sliding mode term, as

$$\hat{d}(t) \approx u_r \quad (2.13)$$

Remark 2.1. *Note that the sliding mode term u_r in (2.12) is a continuous function on the sliding surface $e = 0$, then the additional low-pass filter is completely avoided in the reconstruction of the unknown disturbance, without sacrificing the robustness of the observer. Moreover, the super-twisting algorithm belongs to the second order sliding mode family, which implies that the state estimation accuracy can be improved, as compared with other first order sliding mode observers.*

2.2.2 Second-order Sliding Mode Observer

In general, the second order sliding mode observers are used for handling uncertain systems in which the relative degree between the unknown disturbance and the measurement output is two. They are mainly designed based on the existing

second order sliding mode control techniques, such as the twisting-algorithm [53], the modified super-twisting algorithm [86], the terminal sliding mode algorithm [66], the sub-optimal sliding mode algorithm [101], etc.

Consider a nonlinear uncertain system in the form of

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, x_2, u) + d(t)\end{aligned}\tag{2.14}$$

where x_1 is the measurable state, u is the known system input, $f(\cdot)$ is a nominal continuous function, and $d(t)$ indicates the bounded unknown disturbance. Then, the purpose is to design a robust observer to estimate the unknown state x_2 , as well as the disturbance $d(t)$.

The sliding mode observer can be designed in the form of

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 \\ \dot{\hat{x}}_2 &= f(\hat{x}_1, \hat{x}_2, u) + u_r\end{aligned}\tag{2.15}$$

where the sliding mode term u_r can be designed based on the twisting algorithm [53], as given by

$$u_r = -\frac{\rho_1 + \rho_2}{2}\text{sign}(e_1) - \frac{\rho_1 - \rho_2}{2}\text{sign}(\dot{e}_1)\tag{2.16}$$

with the sliding gains $\rho_1 > \rho_2 > 0$ being chosen large enough, $e_1 = \hat{x}_1 - x_1$.

Suppose that the disturbance $d(t)$ is bounded, and the system states x_1 and x_2 are locally bounded, then the existence is ensured of a positive constant F , such that the following inequality holds:

$$|f(\hat{x}_1, \hat{x}_2, u) - f(x_1, x_2, u) - d(t)| \leq F\tag{2.17}$$

By defining a Lyapunov function $V = (\dot{e}_1)^2/2 + |e_1|(\rho_1 + \rho_2)/2$, which is continu-

ously differentiable except on $e_1 = 0$, it can be verified that

$$\begin{aligned}
 \dot{V} &= \dot{e}_1 \ddot{e}_1 + \frac{\rho_1 + \rho_2}{2} \dot{e}_1 \text{sign}(e_1) \\
 &= \dot{e}_1 [f(\hat{x}_1, \hat{x}_2, u) - f(x_1, x_2, u) - d(t)] - \frac{\rho_1 - \rho_2}{2} \dot{e}_1 \text{sign}(\dot{e}_1) \\
 &\leq \left[\frac{\rho_1 - \rho_2}{2} - F \right] |\dot{e}_1|
 \end{aligned} \tag{2.18}$$

which implies that it has $\dot{V} < 0$ if the sliding gains are chosen large enough, i.e., $(\rho_1 - \rho_2) > 2F$. In other words, the second order sliding surface $\dot{e}_1 = e_1 = 0$ will be reached in finite time and remained thereafter. Then, the unknown disturbance $d(t)$ can be recovered from the sliding mode term u_r through a low pass filter. Similar proof can be found in [126].

Obviously, the nonlinear term u_r can also be designed based on other second order sliding mode techniques, such as the sub-optimal sliding mode algorithm [70, 101], which has the form of

$$u_r = -\frac{\rho_1 + \rho_2}{2} \text{sign}(e_1 - e_1^*/2) + \frac{\rho_1 - \rho_2}{2} \text{sign}(e_1^*) \tag{2.19}$$

where e_1^* is the value of e_1 detected at the closest time in the past when \dot{e}_1 was 0.

Note that both the twisting algorithm in (2.16) and the sub-optimal algorithm in (2.19) require the information on the sign of the first time derivative of the system output. A modified super-twisting algorithm without any differentiator is developed in [86], and the second order sliding mode observer in (2.15) can be replaced by

$$\begin{aligned}
 \dot{\hat{x}}_1 &= \hat{x}_2 + z_1 \\
 \dot{\hat{x}}_2 &= f(\hat{x}_1, \hat{x}_2, u) + z_2
 \end{aligned} \tag{2.20}$$

and the correction variables z_1 and z_2 are output injections of the form

$$\begin{aligned}
 z_1 &= -\lambda |\hat{x}_1 - x_1| \text{sign}(\hat{x}_1 - x_1) \\
 z_2 &= -\rho \text{sign}(\hat{x}_1 - x_1)
 \end{aligned} \tag{2.21}$$

with λ and ρ being two positive and properly chosen parameters.

Remark 2.2. *The modified super-twisting algorithm based observer provides a ro-*

bust estimation for the system states without any differentiator, and the unknown disturbance can be reconstructed from the variable z_2 on the sliding surface.

Remark 2.3. *Unlike the first order sliding mode techniques, the second order (including higher order) sliding mode techniques guarantee that the corresponding sliding surface will be reached in finite time, but not exponential convergence. The stability analysis can be found in references [53, 86, 101, 126].*

2.2.3 Higher-order Sliding Mode Observer

To the best of our understanding, the finite time convergence of arbitrary order sliding mode techniques is still a challenging issue that has yet to be completely addressed. So far, there are only one or two families of higher-order ($r \geq 3$) sliding mode algorithms available based on the homogeneity properties that are proposed by Levant [69, 72].

Considering an r th-order SISO nonlinear uncertain system expressed in the form of measurement output, as described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_r &= f(\mathbf{x}, u) + d(t) \end{aligned} \tag{2.22}$$

where $\mathbf{x} = [x_1, x_2, \dots, x_r]^T \in \mathfrak{R}^r$, x_1 is the only measurable state, u is the known system input, the function $f(\cdot)$ is known and assumed to be locally bounded, and $d(t)$ represents the bounded unknown disturbance that has a relative degree r with respect to the measurement output x_1 . Our objective is to design a robust observer to estimate the unknown states x_2, \dots, x_r , as well as the disturbance $d(t)$.

By duplicating the nominal form of the given system (2.22) and replacing the uncertain part with a feedback term, a nonlinear observer can be designed in the

form of

$$\begin{aligned}
 \dot{\hat{x}}_1 &= \hat{x}_2 \\
 \dot{\hat{x}}_2 &= \hat{x}_3 \\
 &\vdots \\
 \dot{\hat{x}}_r &= f(\hat{\mathbf{x}}, u) + u_r
 \end{aligned} \tag{2.23}$$

where $\hat{\mathbf{x}} = [\hat{x}_1, \hat{x}_2, \dots, \hat{x}_r]^T \in \Re^r$ indicates the estimate of \mathbf{x} , and u_r can be designed based on an r th-order quasi-continuous sliding mode algorithm, as given by [72]

$$\begin{aligned}
 u_r &= -\rho \Psi_{r-1,r}(e_1, z_1, \dots, z_{r-1}), \quad i = 1, \dots, r-1 \\
 \varphi_{0,r} &= e_1, \quad N_{0,r} = |e_1|, \quad \Psi_{0,r} = \varphi_{0,r}/N_{0,r} = \text{sign}(e_1), \\
 \varphi_{i,r} &= z_i + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)} \Psi_{i-1,r}, \quad N_{i,r} = |z_i| + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)}, \\
 \Psi_{i,r} &= \varphi_{i,r}/N_{i,r}
 \end{aligned} \tag{2.24}$$

with $e_1 = \hat{x}_1 - x_1$, $\rho, \beta_1, \dots, \beta_{r-1}$ being positive parameters, and z_1, \dots, z_{r-1} represent the corresponding estimate of derivatives, i.e., $\dot{e}_1, \dots, e_1^{(r-1)}$, which are computed via the $(r-1)$ th-order sliding mode differentiator [69], as

$$\begin{aligned}
 \dot{z}_0 &= w_0, \\
 w_0 &= z_1 - a_0 M^{1/r} |z_0 - e_1|^{r-1/r} \text{sign}(z_0 - e_1) \\
 \dot{z}_1 &= w_1, \\
 w_1 &= z_2 - a_1 M^{1/(r-1)} |z_1 - w_0|^{(r-2)/(r-1)} \text{sign}(z_1 - w_0) \\
 &\vdots \\
 \dot{z}_{r-2} &= w_{r-2}, \\
 w_{r-2} &= z_{r-1} - a_{r-2} M^{1/2} |z_{r-2} - w_{r-3}|^{1/2} \text{sign}(z_{r-2} - w_{r-3}) \\
 \dot{z}_r &= -a_{r-1} M \text{sign}(z_{r-1} - w_{r-2})
 \end{aligned} \tag{2.25}$$

where the positive numbers a_0, \dots, a_{r-1} are chosen in advance, one possible choice with $r \leq 6$ is $a_{r-1} = 1.1, a_{r-2} = 1.5, a_{r-3} = 3, a_{r-4} = 5, a_{r-5} = 8, a_{r-6} = 12$. And the parameter M is chosen to satisfy $M \geq |f(\hat{\mathbf{x}}, u) - f(\mathbf{x}, u) - d(t) + \rho|$.

It has been proven in [69, 72] that the robust differentiator (2.25) ensures the derivatives of e_1 will be estimated in finite time, i.e., $e_1^{(i)} = z_i, i = 1, \dots, r-1$.

Then, the quasi-continuous sliding mode term u_r given in (2.24) ensures that the r th-order sliding surface $e_1^{(r-1)} = \dots = e_1 = 0$ will be reached and remained thereafter. Thus, the unknown states x_2, \dots, x_r can be exactly estimated by $\hat{x}_2, \dots, \hat{x}_r$, and the disturbance can be reconstructed based on the equivalent input injection concept, as

$$\hat{d}(t) \approx \{u_r\}_{eq} \quad (2.26)$$

2.3 Observability and High Gain Observer

2.3.1 Observability and Nonlinear Transformation

It is well-known that if a linear system is observable or detectable, it means that the initial states can be identified for any arbitrary control input. However, this is not true for nonlinear systems. In general, the observability of nonlinear systems is input-dependent, there may exist some "bad inputs", namely singular inputs, that make the system states unobservable. Therefore, it is important to verify whether the nonlinear system is observable, independent of input or not, before the observer design.

Consider a class of SISO nonlinear system in the form of

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})u(t) \\ y(t) &= h(\mathbf{x}) \end{cases} \quad (2.27)$$

where $\mathbf{x} = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$, the nonlinear functions $\mathbf{f}(\mathbf{x})$, $\mathbf{b}(\mathbf{x})$ are smooth known vector field in \mathbb{R}^n , $h(\mathbf{x})$ is a smooth function on \mathbb{R} , $u(t)$ is the system control input and $y(t)$ is the measurable output.

Definition 2.2. (*Observability*) [35, 104] The system given by (2.27) is said to be uniformly observable if, for any pair of initial states $(\mathbf{x}_0, \bar{\mathbf{x}}_0)$ with $\mathbf{x}_0 \neq \bar{\mathbf{x}}_0$, and for all admissible inputs $u(t)$ during the time $t \in [0, T]$, it has $y(\mathbf{x}(\mathbf{x}_0, u, t), u) \neq y(\mathbf{x}(\bar{\mathbf{x}}_0, u, t), u)$.

Definition 2.3. (*Observability for any input*) [35, 104] The system given by (2.27) is said to be uniformly observable if, for any input $u(\cdot) \in \mathbb{R}$ and for any

pair of initial states $(\mathbf{x}_0, \bar{\mathbf{x}}_0)$ with $\mathbf{x}_0 \neq \bar{\mathbf{x}}_0$, there exists a time $t \geq 0$ such that $y(\mathbf{x}_0, u(\cdot), t) \neq y(\bar{\mathbf{x}}_0, u(\cdot), t)$.

The above definitions imply that the uniform observability of a system holds when the initial states can be recovered based on the data of the measurable output as well as its derivatives [33]. In order to analyze the observability of the nonlinear system in (2.27), a nonlinear state transformation is defined as

$$\begin{aligned} \mathbf{x} \rightarrow \tilde{\mathbf{x}} &= [\tilde{x}_1 \ \tilde{x}_2 \ \cdots \ \tilde{x}_n]^T \\ &= \Phi(\mathbf{x}) = \begin{bmatrix} h(\mathbf{x}) & L_{\mathbf{f}}h(\mathbf{x}) & \cdots & L_{\mathbf{f}}^{(n-1)}h(\mathbf{x}) \end{bmatrix}^T \end{aligned} \quad (2.28)$$

where the *Lie* derivative is defined as $L_{\mathbf{f}}h(\mathbf{x}) = [\partial h(\mathbf{x})/\partial \mathbf{x}] \mathbf{f}$, and it has

$$\begin{aligned} \dot{\tilde{x}}_1 &= L_{\mathbf{f}}h(\mathbf{x}) + L_{\mathbf{b}}h(\mathbf{x})u \triangleq \tilde{x}_2 + L_{\mathbf{b}}h(\mathbf{x})u \\ \dot{\tilde{x}}_2 &= L_{\mathbf{f}}^{(2)}h(\mathbf{x}) + L_{\mathbf{b}}L_{\mathbf{f}}h(\mathbf{x})u \triangleq \tilde{x}_3 + L_{\mathbf{b}}L_{\mathbf{f}}h(\mathbf{x})u \\ \vdots &= \vdots \\ \dot{\tilde{x}}_{n-1} &= L_{\mathbf{f}}^{(n-1)}h(\mathbf{x}) + L_{\mathbf{b}}L_{\mathbf{f}}^{(n-2)}h(\mathbf{x})u \triangleq \tilde{x}_n + L_{\mathbf{b}}L_{\mathbf{f}}^{(n-2)}h(\mathbf{x})u \\ \dot{\tilde{x}}_n &= L_{\mathbf{f}}^nh(\mathbf{x}) + L_{\mathbf{b}}L_{\mathbf{f}}^{(n-1)}h(\mathbf{x})u \end{aligned} \quad (2.29)$$

Therefore, the original nonlinear system in (2.27) can be transformed into

$$\begin{cases} \dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \alpha(\tilde{\mathbf{x}}) + \gamma(\tilde{\mathbf{x}})u \\ y = \mathbf{C}\tilde{\mathbf{x}} \end{cases} \quad (2.30)$$

with

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{n \times n}, \quad \mathbf{C} = [1 \ 0 \ \cdots \ 0] \in \mathbb{R}^n \\ \alpha(\tilde{\mathbf{x}}) &= [0 \ 0 \ \cdots \ 0 \ L_{\mathbf{f}}^nh(\mathbf{x})]^T \in \mathbb{R}^n \\ \gamma(\tilde{\mathbf{x}}) &= [L_{\mathbf{b}}h(\mathbf{x}) \ L_{\mathbf{b}}L_{\mathbf{f}}h(\mathbf{x}) \ \cdots \ L_{\mathbf{b}}L_{\mathbf{f}}^{(n-2)}h(\mathbf{x}) \ L_{\mathbf{b}}L_{\mathbf{f}}^{(n-1)}h(\mathbf{x})]^T \in \mathbb{R}^n \end{aligned}$$

Lemma 2.1. [35, 104] For a general nonlinear system in the form of (2.30),

suppose that the vectors $\alpha(\tilde{\mathbf{x}})$ and $\gamma(\tilde{\mathbf{x}})$ have the triangular structures, as given by

$$\begin{aligned}\alpha(\tilde{\mathbf{x}}) &= [\alpha_1(\tilde{x}_1) \quad \alpha_2(\tilde{x}_1, \tilde{x}_2) \quad \cdots \quad \alpha_n(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)]^T \in \mathbb{R}^n \\ \gamma(\tilde{\mathbf{x}}) &= [\gamma_1(\tilde{x}_1) \quad \gamma_2(\tilde{x}_1, \tilde{x}_2) \quad \cdots \quad \gamma_n(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)]^T \in \mathbb{R}^n\end{aligned}\tag{2.31}$$

Then, the nonlinear system in (2.30) is uniformly observable for any arbitrary bounded input u .

Compared the dynamics of the system given in (2.30) with Lemma 2.1, it is clear that the transformed system (2.30) is uniform observable if the vector $\gamma(\tilde{\mathbf{x}})$ has the triangular structure, i.e., $L_{\mathbf{b}}L_{\mathbf{f}}^{(i-1)}h(\mathbf{x}) = \gamma_i(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_i)$, $i = 1, \dots, n$. Furthermore, the original nonlinear system given in (2.27) is also uniformly observable in the case when the mapping $\Phi(\mathbf{x})$ in (2.28) is a diffeomorphism $\forall \mathbf{x}$.

The work in [35] has proved that the triangular structure condition in Lemma (2.1) is a sufficient condition, but not necessary, to ensure the uniform observability of the system for any system input. In the following subsection, a high gain observer for handling the state estimation of a nonlinear Lipschitz system with triangular structures will be reviewed.

2.3.2 High Gain Observer

The high gain observer provides one of the most effective techniques to estimate the unknown states of Lipschitz systems, and has been widely used due to its explicit feedback gain design [33, 36]. It was first proposed in [33] to achieve the exponential convergence of the estimation error dynamics, and was highlighted in [36] by providing an explicit feedback gain design methodology.

Now, we shall demonstrate the high gain observer design for the nonlinear transformed system given by (2.30).

Assumption 2.1. *For the transformed system (2.30), the vectors $\alpha(\tilde{\mathbf{x}})$ and $\gamma(\tilde{\mathbf{x}})$ are assumed to satisfy the triangular structure condition in (2.31) in Lemma 2.1,*

and their components are global Lipschitz functions with respect to $\tilde{\mathbf{x}}$, such that

$$\begin{aligned}\|\alpha_i(\tilde{\mathbf{x}}) - \alpha_i(\tilde{\mathbf{x}}')\| &\leq l_{\alpha i} \|\tilde{\mathbf{x}} - \tilde{\mathbf{x}}'\| \\ \|\gamma_i(\tilde{\mathbf{x}}) - \gamma_i(\tilde{\mathbf{x}}')\| &\leq l_{\gamma i} \|\tilde{\mathbf{x}} - \tilde{\mathbf{x}}'\|\end{aligned}\quad (2.32)$$

hold with some positive Lipschitz constants $l_{\alpha i}, l_{\gamma i}$, $i = 1, \dots, n$.

Consider the transformed nonlinear system (2.30) satisfying Assumption 2.1, a high-gain observer can be designed in the form of

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \alpha(\hat{\mathbf{x}}) + \gamma(\hat{\mathbf{x}})u - \mathbf{S}_\theta^{-1}\mathbf{C}^T(\mathbf{C}\hat{\mathbf{x}} - y) \quad (2.33)$$

where \mathbf{S}_θ is the unique solution of the Lyapunov algebraic equation

$$\theta\mathbf{S}_\theta + \mathbf{A}^T\mathbf{S}_\theta + \mathbf{S}_\theta\mathbf{A} - \mathbf{C}^T\mathbf{C} = 0 \quad (2.34)$$

with θ being a positive tuning parameter, and the explicit solution of (2.34) can be obtained as [36]

$$\mathbf{S}_\theta(i, j) = \frac{(-1)^{i+j} C_{i+j-2}^{j-1}}{\theta^{i+j-1}}, \quad 1 < i, j < n, \quad \text{with } C_n^r = \frac{n!}{(n-r)!r!} \quad (2.35)$$

Furthermore, \mathbf{S}_θ is symmetric positive definite matrix for any $\theta > 0$.

Defining the estimation error $\mathbf{e} = \hat{\mathbf{x}} - \tilde{\mathbf{x}}$, the corresponding dynamics can be obtained from (2.30) and (2.33) as

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{S}_\theta^{-1}\mathbf{C}^T\mathbf{C})\mathbf{e} + \alpha(\hat{\mathbf{x}}) - \alpha(\tilde{\mathbf{x}}) + \gamma(\hat{\mathbf{x}})u - \gamma(\tilde{\mathbf{x}})u \quad (2.36)$$

Lemma 2.2. *For the transformed system (2.30) satisfying Assumption 2.1, there exists $\theta_0 > 0$ such that $\forall \theta > \max\{\theta_0, 1\}$, the high gain observer (2.33) ensures that the estimation error \mathbf{e} will asymptotically converge to zero.*

Proof. For ease of analysis, the following equalities are given [64]

$$\begin{aligned}\Delta_\theta &= \text{diag}(1, \frac{1}{\theta}, \dots, \frac{1}{\theta^{n-1}}), & \mathbf{S}_\theta &= \frac{1}{\theta} \Delta_\theta \mathbf{S}_1 \Delta_\theta \\ \Delta_\theta \mathbf{A} \Delta_\theta^{-1} &= \theta \mathbf{A}, & \mathbf{C} \Delta_\theta &= \mathbf{C} \Delta_\theta^{-1} = \mathbf{C}\end{aligned}$$

Let $\xi = \Delta_\theta \mathbf{e}$, a Lyapunov function can be defined as $V = \xi^T \mathbf{S}_1 \xi$, in which \mathbf{S}_1 is a constant matrix of \mathbf{S}_θ with $\theta = 1$. Then, it has

$$\begin{aligned} \dot{V} &= 2\xi^T \mathbf{S}_1 \Delta_\theta \dot{\mathbf{e}} \\ &= 2\xi^T \mathbf{S}_1 \Delta_\theta (\mathbf{A} - \mathbf{S}_\theta^{-1} \mathbf{C}^T \mathbf{C}) \mathbf{e} + 2\xi^T \mathbf{S}_1 \Delta_\theta [\alpha(\hat{\tilde{\mathbf{x}}}) - \alpha(\tilde{\mathbf{x}}) + \gamma(\hat{\tilde{\mathbf{x}}})u - \gamma(\tilde{\mathbf{x}})u] \\ &= 2\theta \xi^T \mathbf{S}_1 \mathbf{A} \xi - 2\theta \xi^T \mathbf{C}^T \mathbf{C} \xi + 2\xi^T \mathbf{S}_1 \Delta_\theta [\alpha(\hat{\tilde{\mathbf{x}}}) - \alpha(\tilde{\mathbf{x}}) + \gamma(\hat{\tilde{\mathbf{x}}})u - \gamma(\tilde{\mathbf{x}})u] \end{aligned} \quad (2.37)$$

Based on the equation in (2.34), it can be obtained that

$$2\xi^T \mathbf{S}_1 \mathbf{A} \xi = \xi^T \mathbf{S}_1 \mathbf{A} \xi + \xi^T \mathbf{A}^T \mathbf{S}_1 \xi = -\xi^T \mathbf{S}_1 \xi + \xi^T \mathbf{C}^T \mathbf{C} \xi \quad (2.38)$$

With Assumption 2.1 of vectors $\alpha(\cdot)$ and $\gamma(\cdot)$, and $\theta > 1$, it can be deduced that

$$\|\Delta_\theta [\alpha(\hat{\tilde{\mathbf{x}}}) - \alpha(\tilde{\mathbf{x}})]\| \leq \sum_{i=1}^n \frac{1}{\theta^{i-1}} |\alpha_i(\hat{\tilde{\mathbf{x}}}) - \alpha_i(\tilde{\mathbf{x}})| \leq \sum_{i=1}^n \frac{1}{\theta^{i-1}} l_{\alpha i} \|\mathbf{e}\| \leq l_\alpha \|\xi\| \quad (2.39)$$

$$\|\Delta_\theta [\gamma(\hat{\tilde{\mathbf{x}}}) - \gamma(\tilde{\mathbf{x}})]\| \leq \sum_{i=1}^n \frac{1}{\theta^{i-1}} |\gamma_i(\hat{\tilde{\mathbf{x}}}) - \gamma_i(\tilde{\mathbf{x}})| \leq \sum_{i=1}^n \frac{1}{\theta^{i-1}} l_{\gamma i} \|\mathbf{e}\| \leq l_\gamma \|\xi\| \quad (2.40)$$

where $l_\alpha = \sup\{l_{\alpha 1}, \dots, l_{\alpha n}\}$ and $l_\gamma = \sup\{l_{\gamma 1}, \dots, l_{\gamma n}\}$ denote the maximum Lipschitz constants of $\alpha_i(\cdot)$ and $\gamma_i(\cdot)$, respectively. In fact, for the transformed system in (2.30), it has $l_\alpha = l_{\alpha n}$.

Now, by substituting the above inequalities into (2.37), we have

$$\begin{aligned} \dot{V} &\leq -\theta \xi^T \mathbf{S}_1 \xi - \theta \|\mathbf{C} \xi\|^2 + 2\|\xi^T \mathbf{S}_1\| (l_\alpha \|\xi\| + l_\gamma \bar{u} \|\xi\|) \\ &\leq -\theta \xi^T \mathbf{S}_1 \xi + 2(l_\alpha + l_\gamma \bar{u}) \|\xi^T \mathbf{S}_1\| \|\xi\| \\ &\leq -[\theta - 2\sigma(\mathbf{S}_1)(l_\alpha + l_\gamma \bar{u})] V \\ &\triangleq -[\theta - \theta_0] V \end{aligned} \quad (2.41)$$

where $\theta_0 \triangleq 2\sigma(\mathbf{S}_1)(l_\alpha + l_\gamma \bar{u})$, with \bar{u} denoting the maximum of the modulus of the system input u , and $\sigma(\mathbf{S}_1)$ denoting the condition number of matrix \mathbf{S}_1 .

It is clear that by choosing $\theta > \{\theta_0, 1\}$, the derivative of Lyapunov function is negative, i.e., $\dot{V} < 0$, for $V \neq 0$. This implies that both ξ and \mathbf{e} will asymptotically converge to zero. In other words, the unknown states of the system (2.30) can be exactly estimated by the observer (2.33), i.e., $\hat{\tilde{\mathbf{x}}} = \tilde{\mathbf{x}}$. ■

In order to construct the observer for the original system given by (2.27), it is necessary to assume that the nonlinear transformation $\Phi(\mathbf{x})$ in (2.28) is a diffeomorphism function with respect to \mathbf{x} . In other words, the original system can be obtained by differentiating the state transformation of $\tilde{\mathbf{x}} = \Phi(\mathbf{x})$ with respect to time, i.e.,

$$\dot{\mathbf{x}} = \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]^{-1} \dot{\tilde{\mathbf{x}}} = \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]^{-1} [\mathbf{A}\tilde{\mathbf{x}} + \alpha(\tilde{\mathbf{x}}) + \gamma(\tilde{\mathbf{x}})u] = \mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})u(t) \quad (2.42)$$

Therefore, by performing the inverse mapping of $\hat{\tilde{\mathbf{x}}} = \Phi(\hat{\mathbf{x}})$ for the high gain observer in (2.30), we can obtain the estimate state $\hat{\mathbf{x}}$ of the original state \mathbf{x} as [64]

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{\mathbf{x}=\hat{\mathbf{x}}}^{-1} \dot{\hat{\tilde{\mathbf{x}}}} = \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{\mathbf{x}=\hat{\mathbf{x}}}^{-1} \left[\mathbf{A}\hat{\tilde{\mathbf{x}}} + \alpha(\hat{\tilde{\mathbf{x}}}) + \gamma(\hat{\tilde{\mathbf{x}}})u - \mathbf{S}_\theta^{-1}\mathbf{C}^T(\mathbf{C}\hat{\tilde{\mathbf{x}}} - y) \right] \\ &= \mathbf{f}(\hat{\mathbf{x}}) + \mathbf{b}(\hat{\mathbf{x}})u(t) + \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{\mathbf{x}=\hat{\mathbf{x}}}^{-1} \mathbf{S}_\theta^{-1}\mathbf{C}^T(y - h(\hat{\mathbf{x}})) \end{aligned} \quad (2.43)$$

It can be seen that the high gain observer provides an exponential convergence of the state estimation for Lipschitz nonlinear systems, and the feedback gain design is straightforward for any large Lipschitz constant by selecting a large tuning parameter θ . However, in most practical systems, there may exist system modeling error or uncertain dynamics due to the effect of parameter variations or unknown disturbances. For such situations, the high gain observer is not robust enough, and another well-known issue for high gain observer is the measurement noise which will be enlarged by the high gain feedback and deteriorate the state estimation performance.

2.4 Summary

This chapter provided an introduction to the sliding mode observer techniques, as well as high gain observer design methodologies. It seems that both the high gain observer and the sliding mode observer have their own characteristics strengths and weaknesses, which can be summarized as follows:

High gain observer:

- (i) For a Lipschitz nonlinear system, the design of HGO requires the system to be uniformly observable and with the triangular structures as shown in (2.31).
- (ii) The feedback gain design is straightforward and independent of the initial conditions.
- (iii) However, HGO is sensitive to system uncertainties, as well as measurement noise.

Sliding Mode Observer:

- (iv) SMO is robust to any bounded matching disturbance or uncertainty, and provides a mechanism to reconstruct it on the sliding surface.
- (v) The design of SMO requires the system dynamics to be in the form of a differentiator, and the last state dynamics in the system should be at least locally bounded. In other words, the feedback gain design may be related to the initial conditions.

In the next chapter, we shall develop a hybrid observer that integrates high gain feedback with sliding mode techniques for the unknown state estimation and input identification of a class of uncertain nonlinear SISO systems with non-matching uncertainties. The stability analysis and the observer design procedure will be carefully addressed.

Chapter 3

State and Unknown Input

Estimation: HGO plus HSMO

In this Chapter, we shall consider the estimation problem of a class of single-input-single-output (SISO) nonlinear Lipschitz systems with non-matching unknown input or disturbance, in which the distribution vector of the uncertainty may include the unknown states. A hybrid nonlinear observer structure that combines a high gain feedback with a higher order sliding mode term is proposed. The high gain feedback works to constrain the estimation error to within an invariant set and the sliding mode term will asymptotically track the uncertainty if the system satisfies strict structural assumptions. Furthermore, with the higher order sliding mode, the chattering effect will be effectively attenuated without sacrificing the robustness, and the system uncertainty can then be recovered without filtering effect.

The chapter is organized as follows: Section 3.1 introduces some existing sliding mode techniques. In section 3.2, we present the system description and problem formulation. In section 3.3, a hybrid observer is proposed for a class of uncertain nonlinear Lipschitz systems, and both the stability analysis and the design procedure are carefully addressed. In section 3.4, based on a numerical example, some simulation results are presented to illuminate the effectiveness of the proposed estimator. Section 3.5 concludes this chapter.

3.1 Introduction

Unlike linear systems, the observability of a nonlinear system is related to the system inputs, which means there may exist some singular inputs that make the system unobservable [35]. Thus, it is necessary to discuss the observability of nonlinear systems with respect to the system inputs before the observer design.

In [33], it has been proven that a nonlinear single-input-single-output (SISO) system with triangular structure is uniformly observable for any bounded known input, and an exponential convergence high gain observer is proposed. Later on, the work in [36] presented an explicit feedback gain design methodology for a special class of nonlinear Lipschitz systems with triangular structures, which made the high gain observer easy to implement. However, these methods are only applicable for nominal nonlinear systems with triangular structures, of which the observability of the unknown states is clearly guaranteed.

The sliding mode based observer for state estimation of nonlinear uncertain systems has been an active research field in the last few decades. This is due to its insensitivity to uncertainties and the capability of reconstructing the uncertainties based on the equivalent injection input concept. The early works based on the Lyapunov method in this area were developed by Walcott and Zak for dynamic systems with bounded disturbance [57], and extended to a more general class of nonlinear systems in [59, 60].

The idea of sliding mode observer (SMO) design based on the equivalent control concept was first proposed by Utkin and Drakunov in [52, 55]. It was later employed into a particular nonlinear system with triangular input form [61], in which only the discontinuous term was fed back through properly defined gains. In [62], a robust SMO for nonlinear systems subject to unknown input was developed. The result in [64] incorporated a sliding mode term into a high gain observer (HGO) to realize a robust nonlinear observer for a class of nonlinear Lipschitz systems, and the unknown disturbance can be replaced with nominal terms on the sliding surface. Similar works can be found in [63]. For the above methods, they are only applicable for systems where the relative degree between the uncertainties and system output is one, moreover, the undesirable chattering

will degrade the estimation performance and that a low pass filter is required for the reconstruction of the uncertainties.

Higher order sliding mode (HOSM) can remove the relative degree restriction and achieve a better sliding accuracy [53] - [80]. Based on this idea, HOSM-based observers have received increasing attention in recent years, especially the second order sliding mode observers (2SMOs). In [85,89], a suboptimal second order sliding mode observer was adopted to estimate the velocity and torque in electrical drives systems. A robust modified super-twisting observer without differentiator was proposed in [86], and successfully applied into electromechanical systems [87]. In the meantime, in [76], a traditional Luenberger observer with high order sliding mode differentiator was designed for linear time invariant systems with high order of relative degree of the unknown input with respect to the system output. The results have been extended to nonlinear systems based on *Lie* derivative transformation [81], where the relative degree between the unknown disturbance and measurable output is full order or higher order, i.e., the uncertainty or disturbance can only appear in the last dynamic equation. Such restriction is required because the sliding mode is only robust to matching uncertainty.

The problem of non-matching uncertainty remains a challenging problem both in the controller design and the observer design. For a controller, the matching uncertainty means that it enters the system via the same channel as the control input, and they have the same distribution vector. On the other hand, with regards to observer design, the channel of the feedback input can be chosen arbitrarily and the corresponding distribution vector can be designed based on any known state. The matching uncertainty in the observer sense means that its distribution vector is independent of any unknown state.

In this chapter, we shall consider the state estimation of a class of nonlinear Lipschitz systems with non-matching uncertainty in the observer sense, and with higher order sliding mode technique, the system uncertainty can be recovered without filtering effect.

3.2 Preliminaries

3.2.1 System Description

In this chapter, the following class of uniformly observable single-input-single-output (SISO) systems is considered:

$$\begin{cases} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \alpha(\mathbf{x}, u) + \mathbf{P}(\mathbf{x})d(t) \\ y &= \mathbf{C}\mathbf{x} \end{cases} \quad (3.1)$$

where $\mathbf{x} = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$, and

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{(n-1) \times 1} & \mathbf{I}_{(n-1) \times (n-1)} \\ \mathbf{0}_{1 \times 1} & \mathbf{0}_{1 \times (n-1)} \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^n$$

are constant matrices; the nonlinear functions $\alpha(\mathbf{x}, u)$ and $\mathbf{P}(\mathbf{x})$ are smooth vector fields on \mathbb{R}^n ; u is the system input; y is the measurable output; and $d(t) \in \mathbb{R}$ denotes the lumped system uncertainty which may include parameters' deviations and unknown disturbance.

Assumption 3.1. *The smooth vectors $\alpha(\mathbf{x})$ and $\mathbf{P}(\mathbf{x})$ have the following triangular structures, $1 \leq r < n$.*

$$\alpha(\mathbf{x}) = [0, \dots, 0, \alpha_r(x_1, \dots, x_r), \dots, \alpha_n(x_1, x_2, \dots, x_n)]^T \quad (3.2)$$

$$\mathbf{P}(\mathbf{x}) = [0, \dots, 0, 1, p_{r+1}(x_1, \dots, x_{r+1}), \dots, p_n(x_1, x_2, \dots, x_n)]^T \quad (3.3)$$

Moreover, the known functions $\alpha_i(\mathbf{x}) \triangleq \alpha_i(x_1, \dots, x_i)$, $p_i(\mathbf{x}) \triangleq p_i(x_1, \dots, x_i)$; $i = r, \dots, n$, are global Lipschitz functions with respect to (w.r.t.) \mathbf{x} .

Assumption 3.2. *The modulus of distribution vector $\mathbf{P}(\mathbf{x})$ is upper bounded w.r.t. its arguments on \mathbb{R}^n .*

Assumption 3.3. *The modulus of lumped system uncertainty $d(t)$ is upper bounded by \bar{d} .*

Assumption 3.4. *The differential of function $\alpha_r(\mathbf{x})$ is Lipschitz function w.r.t. its arguments on \mathbb{R}^n .*

Assumption 3.5. *The modulus of the first time derivative of $d(t)$ is bounded by \bar{d}' .*

Remark 3.1. *For ease of analysis, the system input u in vector $\alpha(\cdot)$ is omitted. This is reasonable in the following two cases: First, the distribution vector of system input u is constant or independent of any unknown state, then it will be completely canceled and have no effect on the estimation property. Second, the relative degree between the system input u and the system output y is higher than or equal to r , then it can be treated as a coefficient of vector $\alpha(\cdot)$ in the form of (3.2).*

In Assumption 3.1, the system given by (3.1) is assumed to satisfy the triangular structure, and such structure is proven in [33] to be one of the sufficient conditions, but not necessary, to ensure uniform observability for any unknown input. Assumption 3.2 and Assumption 3.3 are general and necessary to ensure that the system uncertainty is trackable.

Assumption 3.4 and Assumption 3.5 can be conservative, they are required to remove the filtering effect without sacrificing the robustness of the observer. Such assumptions will be satisfied in some practical systems, or at least satisfied locally almost everywhere. The system uncertainty is assumed to be smooth, this can be true when it comes from the system parameters' variations, such as due to the slowly changing temperature or operating conditions.

The distribution vector $\mathbf{P}(\mathbf{x})$ is assumed to be normalized with its first non-zero component, which means the relative degree between the system uncertainty $d(t)$ and the system output y is r . In the case when $r = n$, it is well-known that the considered system has a complete differentiator structure, and a traditional r th-order sliding mode differentiator in [69] can be used to handle it. Therefore, in this chapter, we only consider a general case of $r < n$.

3.2.2 Problem Formulation

The problem considered in this chapter is to design a robust observer to guarantee that the estimates $\hat{\mathbf{x}}$ and $k_r u_r$ will respectively track the system state \mathbf{x} and the system uncertainty $d(t)$ within a small bounded set, i.e.,

$$\lim_{t \rightarrow \infty} \|\hat{\mathbf{x}} - \mathbf{x}\| \leq \epsilon_1 \quad (3.4)$$

$$\lim_{t \rightarrow \infty} \|k_r u_r - d(t)\| \leq \epsilon_2 \quad (3.5)$$

where ϵ_1 and ϵ_2 are two constant values. Furthermore, the asymptotic stability conditions will be carefully addressed, i.e., $\epsilon_1 = \epsilon_2 = 0$, if an additional assumption (*Assumption 3.6*) is satisfied.

Definition 3.1. *Matching/Non-matching condition in observer*

Consider the distribution vector $\mathbf{P}(\mathbf{x}) \in \mathbb{R}^n$ of the disturbance/uncertainty $d(t) \in \mathbb{R}$, the disturbance $d(t)$ is said to satisfy the matching condition in the observer sense, if and only if the vector $\mathbf{P}(\mathbf{x})$ does not include any unknown state of \mathbf{x} . Otherwise, $d(t)$ is called a non-matching disturbance in the observer sense.

According to the definition above, it can be seen that the nonlinear system described in (3.1) is with non-matching uncertainty in the observer sense, and to the best of our understanding, there are few effective existing works that can handle such a system. In [78], a higher order sliding mode observer is adopted for the states and unknown inputs estimation of a class of multi-input-multi-output nonlinear systems, for which the reduced-order dynamics are independent of the unknown inputs, i.e., $p_i(\cdot) = 0, \forall i = r + 1, \dots, n$. In other words, such unknown inputs can be considered to satisfy the matching condition in the observer sense.

In the following, a hybrid observer that combines a high gain feedback with higher order sliding mode term will be proposed for a class of nonlinear system with non-matching uncertainty in the observer sense, and the stability analysis and design procedure will be carefully addressed.

3.3 Main Results

3.3.1 Observer Structure

For the system (3.1) satisfying *Assumptions 1-5*, a hybrid estimator is designed in the form of

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \alpha(\hat{\mathbf{x}}) + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}}) + \mathbf{P}(\hat{\mathbf{x}})k_r u_r \\ \dot{u}_r &= \begin{cases} 0, & \text{if } u_r \geq \bar{u}_r \\ v, & \text{if } u_r < \bar{u}_r \end{cases}\end{aligned}\quad (3.6)$$

where \mathbf{L} is the linear feedback gain designed based on HGO [33, 36], as given by

$$\mathbf{L} = \mathbf{S}_\theta^{-1} \mathbf{C}^T = [\theta C_n^1 \quad \theta^2 C_n^2 \quad \cdots \quad \theta^n C_n^n]^T = [l_1 \quad l_2 \quad \cdots \quad l_n]^T \quad (3.7)$$

with \mathbf{S}_θ being a positive-definite symmetric matrix of parameter $\theta > 0$,

$$\mathbf{S}_\theta(i, j) = \frac{(-1)^{i+j} C_{i+j-2}^{j-1}}{\theta^{i+j-1}}, \quad 1 \leq i, j \leq n, \quad C_n^k = \frac{n!}{(n-k)!k!} \quad (3.8)$$

and k_r is a positive tuning parameter of nonlinear feedback u_r , which is designed as an integral function of the quasi-continuous $(r+1)$ th-order sliding mode term v that is given by [71]

$$\begin{aligned}v &= -\rho \Psi_{r,r+1}(e_1, z_1, \dots, z_r), \quad i = 1, \dots, r \\ \varphi_{0,r+1} &= e_1, \quad N_{0,r+1} = |e_1|, \quad \Psi_{0,r+1} = \varphi_{0,r+1}/N_{0,r+1} = \text{sign}(e_1), \\ \varphi_{i,r+1} &= z_i + \beta_i N_{i-1,r+1}^{(r+1-i)/(r-i+2)} \Psi_{i-1,r+1}, \quad N_{i,r+1} = |z_i| + \beta_i N_{i-1,r+1}^{(r+1-i)/(r-i+2)}, \\ \Psi_{i,r+1} &= \varphi_{i,r+1}/N_{i,r+1}\end{aligned}\quad (3.9)$$

where β_1, \dots, β_r are positive numbers, and z_1, \dots, z_r are computed via the r th-order sliding mode differentiator [69], as

$$\begin{aligned}
\dot{z}_0 &= w_0 = z_1 - a_0 M^{1/(r+1)} |z_0 - e_1|^{r/(r+1)} \text{sign}(z_0 - e_1) \\
\dot{z}_1 &= w_1 = z_2 - a_1 M^{1/r} |z_1 - w_1|^{(r-1)/r} \text{sign}(z_1 - w_1) \\
&\vdots \\
\dot{z}_{r-1} &= w_{r-1} = z_r - a_{r-1} M^{1/2} |z_{r-1} - w_{r-2}|^{1/2} \text{sign}(z_{r-1} - w_{r-2}) \\
\dot{z}_r &= -a_r M \text{sign}(z_r - w_{r-1})
\end{aligned} \tag{3.10}$$

with $e_1 = \hat{x}_1 - y$, and the positive numbers a_0, \dots, a_r are chosen in advance. One possible choice with $r \leq 6$ is [71]: $a_{r-1} = 1.1$, $a_{r-2} = 1.5$, $a_{r-3} = 3$, $a_{r-4} = 5$, $a_{r-5} = 8$, $a_{r-6} = 12$. The choice of the remaining observer parameters θ , k_r , \bar{u}_r , ρ and M will be discussed in the following subsections.

Remark 3.2. *For the nonlinear system (3.1) with r th-order relative degree of the uncertainty, the $(r+1)$ th-order sliding mode algorithm (3.9) will attenuate the chattering effect, and the uncertainty can be directly recovered without filtering effect. Similar technique can be found in the controller design [101]. Furthermore, in the case when $r = 1$, one can use the classical second order sliding mode algorithms to replace (3.9), such as the twisting algorithm [53], super-twisting algorithm [53], suboptimal sliding mode [85, 89], etc.*

3.3.2 High Gain Feedback Design

Define the estimation error $\mathbf{e} = [e_1, \dots, e_n]^T = \hat{\mathbf{x}} - \mathbf{x}$, then it can be obtained from (3.1) and (3.6) that

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{S}_\theta^{-1} \mathbf{C}^T \mathbf{C}) \mathbf{e} + \alpha(\hat{\mathbf{x}}) - \alpha(\mathbf{x}) + \mathbf{P}(\hat{\mathbf{x}}) k_r u_r - \mathbf{P}(\mathbf{x}) d(t) \tag{3.11}$$

Theorem 3.1. *Suppose system (3.1) satisfies Assumptions 1-3, with the proposed estimator in (3.6), there exists θ_0 such that $\forall \theta > \max\{\theta_0, 1\}$, the following inequality holds:*

$$\|\Delta_\theta \mathbf{e}\| \leq c_1 / \{(\theta - \theta_0) \theta^{r-1}\} \tag{3.12}$$

with c_1 and θ_0 being two positive numbers given by

$$\begin{aligned}\theta_0 &= 2\sigma(\mathbf{S}_1)(n-r+1)(l_\alpha + k_r \bar{u}_r l_{\mathbf{P}}) \\ c_1 &= 2\sigma(\mathbf{S}_1)(n-r+1)(\bar{d} + k_r \bar{u}_r) b_{\mathbf{P}}\end{aligned}\tag{3.13}$$

where l_α and $l_{\mathbf{P}}$ are the largest Lipschitz constants of functions $\alpha_i(\cdot)$ and $p_i(\cdot)$, respectively; $b_{\mathbf{P}}$ denotes the upper boundary of the modulus of vector $\mathbf{P}(\mathbf{x})$; \bar{u}_r and \bar{d} denote the upper boundaries of the modulus of u_r and $d(t)$, respectively; \mathbf{S}_1 is the matrix of \mathbf{S}_θ with $\theta = 1$; $\sigma(\mathbf{S}_1)$ denotes the condition number of matrix \mathbf{S}_1 .

Proof. For ease of analysis, the following equalities are defined [64]

$$\begin{aligned}\Delta_\theta &= \text{diag}(1, \frac{1}{\theta}, \dots, \frac{1}{\theta^{n-1}}), & \mathbf{S}_\theta &= \frac{1}{\theta} \Delta_\theta \mathbf{S}_1 \Delta_\theta, \\ \Delta_\theta \mathbf{A} \Delta_\theta^{-1} &= \theta \mathbf{A}, & \mathbf{C} \Delta_\theta &= \mathbf{C} \Delta_\theta^{-1} = \mathbf{C}, \\ \theta \mathbf{S}_\theta + \mathbf{A}^T \mathbf{S}_\theta + \mathbf{S}_\theta \mathbf{A} - \mathbf{C}^T \mathbf{C} &= 0\end{aligned}\tag{3.14}$$

Then, by setting $\xi = \Delta_\theta \mathbf{e}$ and defining a Lyapunov function $V = \xi^T \mathbf{S}_1 \xi$, it can be evaluated that

$$\begin{aligned}\dot{V} &= 2\xi^T \mathbf{S}_1 \dot{\xi} \\ &= 2\xi^T \mathbf{S}_1 \Delta_\theta [(\mathbf{A} - \mathbf{S}_\theta^{-1} \mathbf{C}^T \mathbf{C})\mathbf{e} + \alpha(\hat{\mathbf{x}}) - \alpha(\mathbf{x}) + \mathbf{P}(\hat{\mathbf{x}})k_r u_r - \mathbf{P}(\mathbf{x})d(t)] \\ &= 2\xi^T \mathbf{S}_1 \theta (\mathbf{A} - \mathbf{S}_1^{-1} \mathbf{C}^T \mathbf{C})\xi + 2\xi^T \mathbf{S}_1 \Delta_\theta [\alpha(\hat{\mathbf{x}}) - \alpha(\mathbf{x}) + \mathbf{P}(\hat{\mathbf{x}})k_r u_r - \mathbf{P}(\mathbf{x})d(t)] \\ &= 2\theta \xi^T \mathbf{S}_1 \mathbf{A} \xi - 2\theta \|\mathbf{C}\xi\|^2 + 2\xi^T \mathbf{S}_1 \Delta_\theta [\alpha(\hat{\mathbf{x}}) - \alpha(\mathbf{x}) + \mathbf{P}(\hat{\mathbf{x}})k_r u_r - \mathbf{P}(\mathbf{x})d(t)]\end{aligned}\tag{3.15}$$

With the equalities in (3.14), it has

$$2\xi^T \mathbf{S}_1 \mathbf{A} \xi = -\xi^T \mathbf{S}_1 \xi + \|\mathbf{C}\xi\|^2\tag{3.16}$$

Under Assumptions 3.1-3.3, vectors $\alpha(\cdot)$ and $\mathbf{P}(\cdot)$ are global Lipschitz functions and have the triangular structures in (3.2) and (3.3), for $\theta \geq 1$, it can be deduced

that

$$\begin{aligned}
\|\Delta_\theta[\alpha(\hat{\mathbf{x}}) - \alpha(\mathbf{x})]\| &\leq \sum_{i=r}^n \frac{1}{\theta^{i-1}} |\alpha_i(\hat{\mathbf{x}}) - \alpha_i(\mathbf{x})| \\
&\leq \sum_{i=r}^n l_{\alpha i} \left\| \frac{\bar{\mathbf{e}}_i}{\theta^{i-1}} \right\| \\
&\leq (n-r+1)l_\alpha \|\Delta_\theta \mathbf{e}\| \\
&= (n-r+1)l_\alpha \|\xi\|
\end{aligned} \tag{3.17}$$

where $\alpha_i(\mathbf{x}) = \alpha_i(x_1, \dots, x_i)$, $\alpha_i(\hat{\mathbf{x}}) = \alpha_i(\hat{x}_1, \dots, \hat{x}_i)$, $\bar{\mathbf{e}}_i \triangleq [e_1, \dots, e_i, 0, \dots, 0]^T \in \mathbb{R}^n$; $l_{\alpha i}$ denotes the Lipschitz constant of function $\alpha_i(\cdot)$, and $l_\alpha = \max\{l_{\alpha r}, \dots, l_{\alpha n}\}$ is the largest Lipschitz constant. Furthermore, we have

$$\|\Delta_\theta[\mathbf{P}(\hat{\mathbf{x}}) - \mathbf{P}(\mathbf{x})]k_r u_r\| \leq (n-r+1)k_r \bar{u}_r l_{\mathbf{P}} \|\Delta_\theta \mathbf{e}\| \tag{3.18}$$

$$\|\Delta_\theta \mathbf{P}(\mathbf{x})[k_r u_r - d(t)]\| \leq (n-r+1)(k_r \bar{u}_r + \bar{d})b_{\mathbf{P}}/\theta^{r-1} \tag{3.19}$$

Therefore, the Lyapunov function in (3.15) can be reduced into

$$\begin{aligned}
\dot{V} &= -\theta \xi^T \mathbf{S}_1 \xi - \theta \|\mathbf{C}\xi\|^2 + 2\xi^T \mathbf{S}_1 \Delta_\theta \{[\alpha(\hat{\mathbf{x}}) - \alpha(\mathbf{x})] + [\mathbf{P}(\hat{\mathbf{x}}) - \mathbf{P}(\mathbf{x})]k_r u_r \\
&\quad + \mathbf{P}(\mathbf{x})[k_r u_r - d(t)]\} \\
&\leq -\theta \xi^T \mathbf{S}_1 \xi + 2(n-r+1)(l_\alpha + k_r \bar{u}_r l_{\mathbf{P}}) \|\xi^T \mathbf{S}_1\| \|\xi\| \\
&\quad + 2(n-r+1)(k_r \bar{u}_r + \bar{d})b_{\mathbf{P}} \|\xi^T \mathbf{S}_1\|/\theta^{r-1} \\
&\leq -\sigma(\mathbf{S}_1)^{-1} \|\xi^T \mathbf{S}_1\| \{[\theta - 2\sigma(\mathbf{S}_1)(n-r+1)(l_\alpha + k_r \bar{u}_r l_{\mathbf{P}})] \|\xi\| \\
&\quad - 2\sigma(\mathbf{S}_1)(n-r+1)(k_r \bar{u}_r + \bar{d})b_{\mathbf{P}}/\theta^{r-1}\} \\
&= -\sigma(\mathbf{S}_1)^{-1} \|\xi^T \mathbf{S}_1\| [(\theta - \theta_0) \|\xi\| - c_1/\theta^{r-1}]
\end{aligned} \tag{3.20}$$

where $\theta_0 \triangleq 2\sigma(\mathbf{S}_1)(n-r+1)(l_\alpha + k_r \bar{u}_r l_{\mathbf{P}})$, $c_1 \triangleq 2\sigma(\mathbf{S}_1)(n-r+1)(\bar{d} + k_r \bar{u}_r)b_{\mathbf{P}}$.

Now, it can be concluded from inequality (3.20) that the variable $\xi = \Delta_\theta \mathbf{e}$ will be bounded after a transient process, provided that $\theta > \max\{\theta_0, 1\}$. In other words, it has $\|\Delta_\theta \mathbf{e}\| \leq c_1/\{(\theta - \theta_0)\theta^{r-1}\}$. \blacksquare

Corollary 3.1. *For the system (3.1) satisfying Assumptions 1-3, the proposed estimator in (3.6) will constraint the estimation error within an invariant set after a transient process and remain inside thereafter, provided that $\theta > \max\{\theta_0, 1\}$.*

Then, for the partial error dynamic $\bar{\mathbf{e}}_i = [e_1, \dots, e_i, 0, \dots, 0]^T \in \mathbb{R}^n, \forall i = 1, \dots, n$, it has

$$|e_i| \leq \|\bar{\mathbf{e}}_i\| \leq \theta^{i-1} \|\Delta_\theta \mathbf{e}\| \leq \frac{c_1 \theta^{i-r}}{\theta - \theta_0} \quad (3.21)$$

Proof. The above inequality can be readily obtained by substituting the results in Theorem 1. Then, the boundary for the estimation error $\mathbf{e} = \bar{\mathbf{e}}_n$ can be evaluated as

$$\lim_{t \rightarrow \infty} \|\hat{\mathbf{x}} - \mathbf{x}\| \leq c_1 \theta^{n-r} / (\theta - \theta_0) \triangleq \epsilon_1 \quad (3.22)$$

which implies that the estimation error will be restrained to within an invariant set for a given θ , even when the considered system in (3.1) is unstable. Furthermore, for the individual estimation error $e_i, i = 1, \dots, r$, it can be made arbitrary small by selecting a large θ . ■

From (3.22), it can be found that asymptotic stability property can be achieved with high gain feedback when the system (3.1) is without uncertainty, i.e., $\bar{d} = 0$, $\bar{u}_r = 0$, $c_1 = 0$. Such conclusion is similar to the results in [33, 36]. Besides, Corollary 3.1 illuminates the infectivity of the system uncertainty in a high gain observer.

3.3.3 Nonlinear Feedback Design

In this section, we consider the nonlinear feedback design under the condition $u_r < \bar{u}_r$, with proper designs of parameters k_r, ρ and M , the sliding surface $e_1^{(r)} = \dot{e}_1 = e_1 = \dots = 0$ will be reached.

Consider e_1 as the sliding variable, its $(r+1)$ th-order dynamics can be deduced from (3.11) as

$$\begin{aligned}
e_1^{(r+1)} &= e_{r+2} - \sum_{i=1}^{r+1} l_i e_1^{(r-i+1)} + \alpha_{r+1}(\hat{\mathbf{x}}) - \alpha_{r+1}(\mathbf{x}) + p_{r+1}(\hat{\mathbf{x}})k_r u_r - p_{r+1}(\mathbf{x})d(t) \\
&\quad + \alpha'_r(\hat{\mathbf{x}}) - \alpha'_r(\mathbf{x}) - d'(t) + k_r v \\
&= \phi(\mathbf{x}, \hat{\mathbf{x}}) + k_r v
\end{aligned} \tag{3.23}$$

with

$$\begin{aligned}
\phi(\mathbf{x}, \hat{\mathbf{x}}) &= e_{r+2} - \sum_{i=1}^{r+1} l_i e_1^{(r-i+1)} + \alpha_{r+1}(\hat{\mathbf{x}}) - \alpha_{r+1}(\mathbf{x}) + p_{r+1}(\hat{\mathbf{x}})k_r u_r - p_{r+1}(\mathbf{x})d(t) \\
&\quad + \alpha'_r(\hat{\mathbf{x}}) - \alpha'_r(\mathbf{x}) - d'(t)
\end{aligned}$$

where $e_1^{(i)}$ denotes the i th-order derivative of e_1 ; $\phi(\mathbf{x}, \hat{\mathbf{x}})$ represents the dynamics in the right side of equation (3.11). Clearly, the sliding mode surface is reachable if the modulus of the nonlinear function $\phi(\mathbf{x}, \hat{\mathbf{x}})$ can be proven to be bounded.

Corollary 3.2. *For the system (1) satisfying Assumptions 3.1-3.5, the proposed estimator given by (3.6) will ensure that the modulus of the nonlinear function $\phi(\mathbf{x}, \hat{\mathbf{x}})$ remains bounded after a transient process for a given $\theta > \max\{\theta_0, 1\}$. Furthermore, its upper boundary $\bar{\phi}$ can be represented by*

$$\begin{aligned}
\bar{\phi} &= c_1[(1 + \sum_{i=1}^{r+1} \kappa_i)\theta^2 + (l_{\alpha(r+1)} + l_{p(r+1)}k_r \bar{u}_r)\theta + l'_{\alpha r}] / (\theta - \theta_0) \\
&\quad + b_P(k_r \bar{u}_r + \bar{d}) + \bar{d}'
\end{aligned} \tag{3.24}$$

where κ_i , $i = 1, \dots, r+1$ are positive numbers that are dependent on the dimension n and the relative degree r , and independent of θ ; $l_{\alpha(r+1)}$, $l_{p(r+1)}$ and $l'_{\alpha r}$ denote the corresponding Lipschitz constants; \bar{d}'_r represents the upper boundary of the modulus of the first time derivative of $d(t)$.

Proof. As shown in Theorem 3.1 and Corollary 3.1, the proposed estimator in (3.6) will restrain the estimation error within an invariant boundary for a given $\theta > \{\theta_0, 1\}$, and the following inequalities hold after a transient process,

$$|e_i| \leq \|\bar{\mathbf{e}}_i\| \leq \theta^{i-r} c_1 / (\theta - \theta_0), \quad i = 1, \dots, n. \tag{3.25}$$

With Assumptions 3.1-3.5, it can be evaluated that

$$\begin{aligned}
|e_{r+2}| &\leq \theta^2 c_1 / (\theta - \theta_0) \\
|\alpha'_r(\hat{\mathbf{x}}) - \alpha'_r(\mathbf{x})| &\leq l'_{\alpha r} c_1 / (\theta - \theta_0) \\
|\alpha_{r+1}(\hat{\mathbf{x}}) - \alpha_{r+1}(\mathbf{x})| &\leq l_{\alpha(r+1)} \theta c_1 / (\theta - \theta_0) \\
|[p_{r+1}(\hat{\mathbf{x}}) - p_{r+1}(\mathbf{x})]k_r u_r| &\leq l_{p(r+1)} k_r \bar{u}_r \theta c_1 / (\theta - \theta_0)
\end{aligned} \tag{3.26}$$

where $l'_{\alpha r}$, $l_{\alpha(r+1)}$ and $l_{p(r+1)}$ denote the Lipschitz constants of functions $\alpha'_r(\cdot)$, $\alpha_{r+1}(\cdot)$ and $p_{r+1}(\cdot)$, respectively.

Based on the structures of vectors $\alpha(\cdot)$ and $\mathbf{P}(\cdot)$ in *Assumption 1*, and the feedback gains $l_i = \theta^i C_n^i$ in (3.7), one can find positive numbers κ_i such that

$$\begin{aligned}
|l_{r+1} e_1| &\leq \theta^{r+1} C_n^{r+1} \theta^{1-r} c_1 / (\theta - \theta_0) = \kappa_1 \theta^2 c_1 / (\theta - \theta_0) \\
|l_r e_1^{(1)}| &\leq \theta^r C_n^r |e_2| + \theta^r C_n^r |\theta C_n^1 e_1| \leq C_n^r (1 + C_n^1) \theta^2 c_1 / (\theta - \theta_0) = \kappa_2 \theta^2 c_1 / (\theta - \theta_0) \\
&\vdots \\
|l_1 e_1^{(r)}| &\leq \kappa_{r+1} \theta^2 c_1 / (\theta - \theta_0)
\end{aligned} \tag{3.27}$$

where κ_i , $i = 1, \dots, r+1$ are positive numbers dependent on the dimension n and relative degree r , but independent of the parameter θ .

Therefore, the nonlinear term $\phi(\mathbf{x}, \hat{\mathbf{x}})$ will be bounded with

$$\begin{aligned}
|\phi(\mathbf{x}, \hat{\mathbf{x}})| &\leq |e_{r+2}| + \sum_{i=1}^{r+1} |l_i e_1^{(r-i+1)}| + |\alpha_{r+1}(\hat{\mathbf{x}}) - \alpha_{r+1}(\mathbf{x})| + |\alpha'_r(\hat{\mathbf{x}}) - \alpha'_r(\mathbf{x})| \\
&\quad + |[p_{r+1}(\hat{\mathbf{x}}) - p_{r+1}(\mathbf{x})]k_r u_r| + |p_{r+1}(\mathbf{x})k_r u_r| + |p_{r+1}(\mathbf{x})d(t)| + |d'(t)| \\
&\leq c_1 [(1 + \sum_{i=1}^{r+1} \kappa_i) \theta^2 + (l_{\alpha(r+1)} + l_{p(r+1)} k_r \bar{u}_r) \theta + l'_{\alpha r}] / (\theta - \theta_0) + \mathbf{b}_{\mathbf{P}}(k_r \bar{u}_r + \bar{d}) + \bar{d}' \\
&\triangleq \bar{\phi}
\end{aligned} \tag{3.28}$$

It can be seen that the modulus of the nonlinear function $\phi(\mathbf{x}, \hat{\mathbf{x}})$ will be bounded for a given θ , and this boundary is completely independent of the sliding mode parameters ρ and M . ■

Theorem 3.2. *For the system (3.1) satisfying Assumptions 3.1-3.5, the proposed estimator given by (3.6) - (3.10) ensures that the sliding surface $e_1 = \dot{e}_1 = \dots = e_1^{(r)} = 0$ will be reached in finite time, provided that $u_r < \bar{u}_r$, and the sliding mode*

parameters ρ and M are chosen large enough.

Proof. Consider the $(r + 1)$ th-order sliding variable dynamics in (3.23), the modulus of nonlinear function $\phi(\mathbf{x}, \hat{\mathbf{x}})$ will be bounded by $\bar{\phi}$ for a given θ , as shown in Corollary 3.2. Then, the provided quasi-continuous $(r + 1)$ th-order sliding mode algorithm (3.9) with the r th-order sliding differentiator (3.10) will ensure that the sliding surface $e_1 = \dot{e}_1 = \dots = e_1^{(r)} = 0$ is reached in finite time by choosing large enough sliding parameters ρ and M . More details can be founded in [71]. ■

Once the sliding mode occurs, it can be deduced from (3.11) that

$$e_1 = e_2 = \dots = e_r = 0; \quad e_{r+1} = d(t) - k_r u_r \quad (3.29)$$

The above equations present the relationship between the nonlinear feedback term $k_r u_r$ and the system uncertainty $d(t)$ on the sliding surface. Clearly, the system uncertainty $d(t)$ will be directly and exactly recovered from $k_r u_r$ after all the system states have converged to their true values, i.e., $e_{r+1} = 0$, and this implies that the tuning parameter k_r must be chosen large enough such that $k_r > \bar{d}/\bar{u}_r$.

Note that the sliding mode surface is reachable under the condition $u_r < \bar{u}_r$, together with the sliding parameters ρ and M being chosen large enough. However, according to the relationship between e_{r+1} and $k_r u_r$ given in (3.29), in the case that the estimation error e_{r+1} is unstable or divergent, the nonlinear feedback u_r will also diverge and reach at \bar{u}_r after some time. In other words, one may choose the sliding parameters large enough to ensure that the sliding surface is reached in a finite time, but not remained thereafter if the estimation error e_{r+1} is divergent. So, it is significant and important to consider the stability of the reduced-order estimation error dynamics.

3.3.4 Reduced-order Dynamics

Let $\mathbf{e} = [\mathbf{e}_s, \mathbf{e}_d]^T$, $\mathbf{e}_s = [e_1, \dots, e_r]^T \in \mathbb{R}^r$, $\mathbf{e}_d = [e_{r+1}, \dots, e_n]^T \in \mathbb{R}^{n-r}$, where \mathbf{e}_d denotes the reduced order estimation error on the sliding surface, i.e., $\mathbf{e}_s = \mathbf{0}$.

Then, with the equations in (3.29), the estimation error dynamics of \mathbf{e}_d can be deduced from (3.11)

$$\begin{aligned} \dot{\mathbf{e}}_d = & \mathbf{A}_d \mathbf{e}_d + \alpha_d(\mathbf{x}_s, \hat{\mathbf{x}}_d) - \alpha_d(\mathbf{x}_s, \mathbf{x}_d) + \mathbf{P}_d(\mathbf{x}_s, \hat{\mathbf{x}}_d) k_r u_r - \mathbf{P}_d(\mathbf{x}_s, \mathbf{x}_d) k_r u_r \\ & - \mathbf{P}_d(\mathbf{x}_s, \mathbf{x}_d) e_{r+1} \end{aligned} \quad (3.30)$$

where $\mathbf{x} = [\mathbf{x}_s, \mathbf{x}_d]^T$, $\mathbf{x}_s = [x_1, \dots, x_r]^T \in \mathbb{R}^r$, $\mathbf{x}_d = [x_{r+1}, \dots, x_n]^T \in \mathbb{R}^{n-r}$ and

$$\mathbf{A}_d = \begin{bmatrix} \mathbf{0}_{(n-r-1) \times 1} & \mathbf{I}_{(n-r-1) \times (n-r-1)} \\ \mathbf{0}_{1 \times 1} & \mathbf{0}_{1 \times (n-r-1)} \end{bmatrix} \in \mathbb{R}^{(n-r)(n-r)}$$

$$\alpha_d(\mathbf{x}_s, \mathbf{x}_d) = [\alpha_{r+1}(\mathbf{x}_s, x_{r+1}), \alpha_{r+2}(\mathbf{x}_s, x_{r+1}, x_{r+2}), \dots, \alpha_n(\mathbf{x}_s, x_{r+1}, \dots, x_n)]^T \in \mathbb{R}^{n-r}$$

$$\mathbf{P}_d(\mathbf{x}_s, \mathbf{x}_d) = [p_{r+1}(\mathbf{x}_s, x_{r+1}), p_{r+2}(\mathbf{x}_s, x_{r+1}, x_{r+2}), \dots, p_n(\mathbf{x}_s, x_{r+1}, \dots, x_n)]^T \in \mathbb{R}^{n-r}$$

with $\hat{\mathbf{x}}_d$ denotes the estimated vector of \mathbf{x}_d on the sliding surface.

Similarly, the dynamics of \mathbf{e}_d in (3.30) can also be rewritten as

$$\begin{aligned} \dot{\mathbf{e}}_d = & \mathbf{A}_d \mathbf{e}_d + \alpha_d(\mathbf{x}_s, \hat{\mathbf{x}}_d) - \alpha_d(\mathbf{x}_s, \mathbf{x}_d) + \mathbf{P}_d(\mathbf{x}_s, \hat{\mathbf{x}}_d) d(t) - \mathbf{P}_d(\mathbf{x}_s, \mathbf{x}_d) d(t) \\ & - \mathbf{P}_d(\mathbf{x}_s, \hat{\mathbf{x}}_d) e_{r+1} \end{aligned} \quad (3.31)$$

Based on the dynamics in (3.31), it can be seen that both the high gain feedback and nonlinear feedback terms disappear, which means the stability of \mathbf{e}_d is completely independent of the high gain feedback parameters and nonlinear feedback parameters, and is only related to the original system structure and the system uncertainty. In other words, one needs to evaluate the stability of the reduced order dynamics for a given system before the estimator design. Meanwhile, in the case that the distribution vector $\mathbf{P}(\cdot)$ includes only the partial state \mathbf{x}_s , i.e., $\mathbf{P}_d(\mathbf{x}_s, \hat{\mathbf{x}}_d) = \mathbf{P}_d(\mathbf{x}_s) = \mathbf{P}_d(\mathbf{x}_s, \mathbf{x}_d)$, the system uncertainty $d(t)$ will be canceled, and the stability of \mathbf{e}_d is only dependent on the nominal structure of the original system.

Since the dynamics of \mathbf{e}_d in (3.31) is only involved with the original system structure, but independent of the feedback gains, the stability of the reduced-order estimation error dynamics can be classified into the following three cases:

- 1) The reduced-order dynamics of \mathbf{e}_d in (3.31) is self-asymptotically stable, i.e., $\lim_{t \rightarrow \infty} \|\hat{\mathbf{x}}_d - \mathbf{x}_d\| = 0$.
- 2) The reduced-order dynamics of \mathbf{e}_d in (3.31) is self-stable, but not asymptotically stable, i.e., $\lim_{t \rightarrow \infty} \|\hat{\mathbf{x}}_d - \mathbf{x}_d\| = \varepsilon_3$, with ε_3 being a positive constant.
- 3) The reduced-order dynamics of \mathbf{e}_d in (3.31) is unstable/divergent, i.e., it is not self-stable.

For case 1, which is equivalent to Assumption 3.6, it is quite conservative and not all systems will satisfy this condition. Moreover, this assumption may be difficult to check due to the existence of system uncertainty. However, there are some practical systems that satisfy this requirement, such as the non-matching parameter identification problem of a series DC motor that will be fully discussed in Chapter 4.

For case 2, it requires that the reduced-order estimation error \mathbf{e}_d is self-bounded, which is general and easy to check for a specified application system, such as the speed and position estimation problem of a permanent magnet synchronous motor (PMSM) that will be discussed in Chapter 5.

For case 3, it means that the estimation error dynamics on the sliding surface is divergent. Here, a numerical example is used to illuminate this case.

Consider the following system:

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0.5x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d(t) \\ y &= [1 \ 0] \mathbf{x} \end{aligned} \quad (3.32)$$

where $\mathbf{x} = [x_1, x_2]^T \in \mathbb{R}^2$, and $d(t)$ denotes the unknown disturbance. It can be seen that this numerical system satisfies Assumptions 3.1-3.5, and the system uncertainty satisfies the matching condition in the observer sense, which can be considered as a special case of the result presented in this chapter. Then, the corresponding reduced-order dynamics on the sliding surface can be readily obtained by the formula in (3.31), as

$$\dot{e}_2 = 0.5e_2 \quad (3.33)$$

with $\mathbf{e} = [e_1, e_2]^T = [\hat{x}_1 - x_1, \hat{x}_2 - x_2]^T$. Clearly, the dynamics of e_2 on the sliding surface is unstable and divergent.

Assumption 3.6. (*Strictly conservative system*)

For the system given in (3.1), it is strictly conservative if the reduced order dynamics of \mathbf{e}_d in (3.31) is asymptotically stable, i.e., $\lim_{t \rightarrow \infty} \|\hat{\mathbf{x}}_d - \mathbf{x}_d\| = 0$.

Theorem 3.3. For the system (3.1) satisfying Assumptions 3.1-3.6, the proposed estimator given by (3.6)-(3.10) ensures that the unknown states and the system uncertainty will be asymptotically identified in finite time, provided that $\theta > \max\{\theta_0, 1\}$, and the parameters k_r , \bar{u}_r , ρ and M are properly chosen.

Proof. It has been proven in Theorem 3.2 that the sliding mode surface will be reached under the condition $u_r < \bar{u}_r$, where the nonlinear feedback u_r satisfies equation (3.29), i.e., $u_r = [d(t) - e_{r+1}]/k_r$. After that, Assumption 6 ensures that the reduced order dynamics asymptotically converge to zero, in other words, the modulus of the estimation error e_{r+1} will asymptotically decrease on the sliding surface. Thus, one can choose the parameter \bar{u}_r ($\bar{u}_r > \bar{d}$) large enough to guarantee the condition $u_r < \bar{u}_r$ is always satisfied, then the sliding mode surface will be reached and remained thereafter.

After the reduced order estimation error reaches zero, i.e., $\mathbf{e}_d = \hat{\mathbf{x}}_d - \mathbf{x}_d = 0$, the unknown states and the system uncertainty can be obtained from (3.29) as: $\hat{\mathbf{x}} = \mathbf{x}$, $d(t) = k_r u_r$. ■

Corollary 3.3. For the system (3.1) satisfying Assumptions 3.1-3.5, and that the reduced-order estimation error on the sliding surface is bounded by a constant ε_3 , then, the proposed estimator given by (3.6)-(3.10) ensures that the sliding surface will be reached and remained thereafter, i.e., $\hat{\mathbf{x}}_s - \mathbf{x}_s = 0$. In other words, the proposed estimator can only guarantee the partial system states \mathbf{x}_s being exactly estimated.

Proof. Similarly to Theorem 3.3, as the sliding surface will be reached under the condition $u_r < \bar{u}_r$, where the nonlinear feedback u_r satisfies the equation (3.29), i.e., $k_r u_r = [d(t) - e_{r+1}]$. After that, as the reduced-order dynamics is bounded

with ε_3 , i.e., $|e_{r+1}| \leq \|\mathbf{e}_d\| \leq \varepsilon_3$. Thus, one can choose the parameter \bar{u}_r and k_r large enough to guarantee the condition $u_r < \bar{u}_r$ is always satisfied, then the sliding surface will be remained forever. ■

Corollary 3.4. *For the system (3.1) satisfying Assumptions 3.1-3.5, and that the reduced-order estimation error \mathbf{e}_d is divergent on the sliding surface, then, the proposed estimator given by (3.6)-(3.10) can only ensure that the sliding surface will be reached in a finite time, but may not remained forever in the case when e_{r+1} is divergent.*

Proof. According to Theorem 3.2, under the condition $u_r < \bar{u}_r$ and with properly chosen sliding gains, the sliding surface will be reached in a finite time. Then, equation (3.29) holds, i.e., $k_r u_r = [d(t) - e_{r+1}]$. Therefore, in the case when e_{r+1} is divergent, the condition $u_r < \bar{u}_r$ will not be satisfied once $|e_{r+1}| = |d(t) - k_r u_r| > \bar{d} + k_r \bar{u}_r$. As a result, the partial state \mathbf{x}_d failed to be identified. ■

Based on the above analyses, and the stability of \mathbf{e}_s on the sliding surface, the observability of the unknown input and the states of the system given by (3.1) can be summarized as

- For case 1, the reduced-order dynamics of \mathbf{e}_d is asymptotically stable, i.e., Assumption 3.6 is satisfied. All system states and the unknown disturbance can be asymptotically identified with the proposed hybrid observer.
- For case 2, the reduced-order dynamics of \mathbf{e}_d is stable, but not asymptotically stable. Then, only the partial system state \mathbf{e}_s can be identified with the proposed hybrid observer.
- For case 3, the reduced-order dynamics of \mathbf{e}_d is divergent. Then, all system states and the unknown input may fail to be identified with the proposed observer.

3.3.5 Estimator Parameter Design Procedure

The stability of the reduced order estimation error dynamics is only related to the original system structure and the system uncertainty, but independent of the

feedback gains. Moreover, it may affect the sliding mode property. Therefore, it is necessary to check the stability of the reduced order dynamics through equation (3.31). The estimator parameter design procedure can be classified into two types, as given below:

Type I: The reduced-order dynamics of \mathbf{e}_d is divergent

- 1) Remove the nonlinear feedback, i.e., $k_r = 0$.
- 2) Choose high gain feedback parameter θ , such that $\theta > \max\{\theta_0, 1\}$.
- 3) Compute high gain feedback \mathbf{L} , as $\mathbf{L} = [\theta C_n^1 \quad \theta^2 C_n^2 \quad \dots \quad \theta^n C_n^n]^T$.

Type II: Assumption 3.6 is satisfied or the reduced-order dynamics of \mathbf{e}_d is stable

- 1) Choose the tuning parameter k_r and the upper boundary \bar{u}_r , such that $k_r \bar{u}_r > \bar{d}$.
- 2) Choose high gain feedback parameter θ , such that $\theta > \max\{\theta_0, 1\}$.
- 3) Compute high gain feedback \mathbf{L} , as $\mathbf{L} = [\theta C_n^1 \quad \theta^2 C_n^2 \quad \dots \quad \theta^n C_n^n]^T$.
- 4) Choose the sliding mode parameters ρ and M to be sufficiently large.

Remark 3.3. *For a class of well-defined nonlinear systems that can be transformed into the system given in (3.1) through a nonlinear transformation (such as the Lie transformation), and that all the Assumptions mentioned in this Chapter are satisfied, then a hybrid estimator can be designed by inverse transformation of the proposed estimator in (3.6)-(3.10). Furthermore, Assumption 3.6 can be verified in the original domain.*

3.4 Simulation Results

In this section, we present the simulation results of the numerical example given by (3.32), which belongs to case 3. For the other two cases, they will be carefully illuminated in Chapter 4 and Chapter 5, respectively.

Based on the observer structure given by (3.6)-(3.10), the proposed hybrid estimator for the numerical system of (3.32) can be designed in the form of

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{\mathbf{x}} + \begin{bmatrix} 0 \\ 0.5\hat{x}_2 \end{bmatrix} + \begin{bmatrix} 2\theta \\ \theta^2 \end{bmatrix} (y - \hat{x}_1) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} k_r u_r \quad (3.34)$$

where $\hat{\mathbf{x}} = [\hat{x}_1, \hat{x}_2]^T$, and u_r is given by

$$\begin{aligned} \dot{u}_r &= \begin{cases} 0, & \text{if } u_r \geq \bar{u}_r \\ v, & \text{if } u_r < \bar{u}_r \end{cases} \\ v &= -\rho[z_1 + |e_1|^{1/2}\text{sign}(z_0)]/(|z_1| + |e_1|^{1/2}) \\ \dot{z}_0 = w_0 &= z_1 - 1.5M^{1/2}|z_0 - e_1|^{1/2}\text{sign}(z_0 - e_1), \quad \dot{z}_1 = -1.1M\text{sign}(z_1 - w_0) \end{aligned} \quad (3.35)$$

The simulation parameters are chosen as: $d(t) = 80\sin(5t)$, $x_1(0) = 10$, $x_2(0) = 10$; $\hat{x}_1(0) = 0$, $\hat{x}_2(0) = 0$; $\theta = 1.5$, $\bar{u}_r = 50$, $\rho = 150$, $M = 150$. The tuning parameter k_r is set to $k_r = 2.5$ and $k_r = 0$ to indicate the proposed estimator with and without nonlinear feedback, respectively. The simulation results with $k_r = 2.5$ are shown in Figure 3.1.

From Figure 3.1c and Figure 3.1d, it can be seen that the sliding surface is reached and remained during the first 4 *seconds*, $e_1 = \hat{x}_1 - x_1 = 0$, because the nonlinear parameters are chosen large enough, i.e., $k_r\bar{u}_r > \|d(t) + e_2\|$. However, as the estimation error e_2 is divergent on the sliding surface, it causes the nonlinear feedback u_r to be divergent and clipped by \bar{u}_r after 4 *second*, where the sliding mode property is lost, as shown in Figure 3.1d.

From Figure 3.2, which is for the case without the nonlinear feedback, the proposed estimator works as a high gain observer, and it will constrain the estimation error to within an invariant boundary, as shown in Figure 3.2c. Comparing the estimation error of state x_2 in both Figure 3.2c and Figure 3.1d, one can find that the estimation error may be larger after 4 *second* in Figure 3.1d, which tallies with the analysis result on high gain feedback in Theorem 3.1. In other words, the nonlinear feedback may enlarge the size of the invariant set in the case when the reduced order dynamics is divergent on the sliding surface. Therefore, the

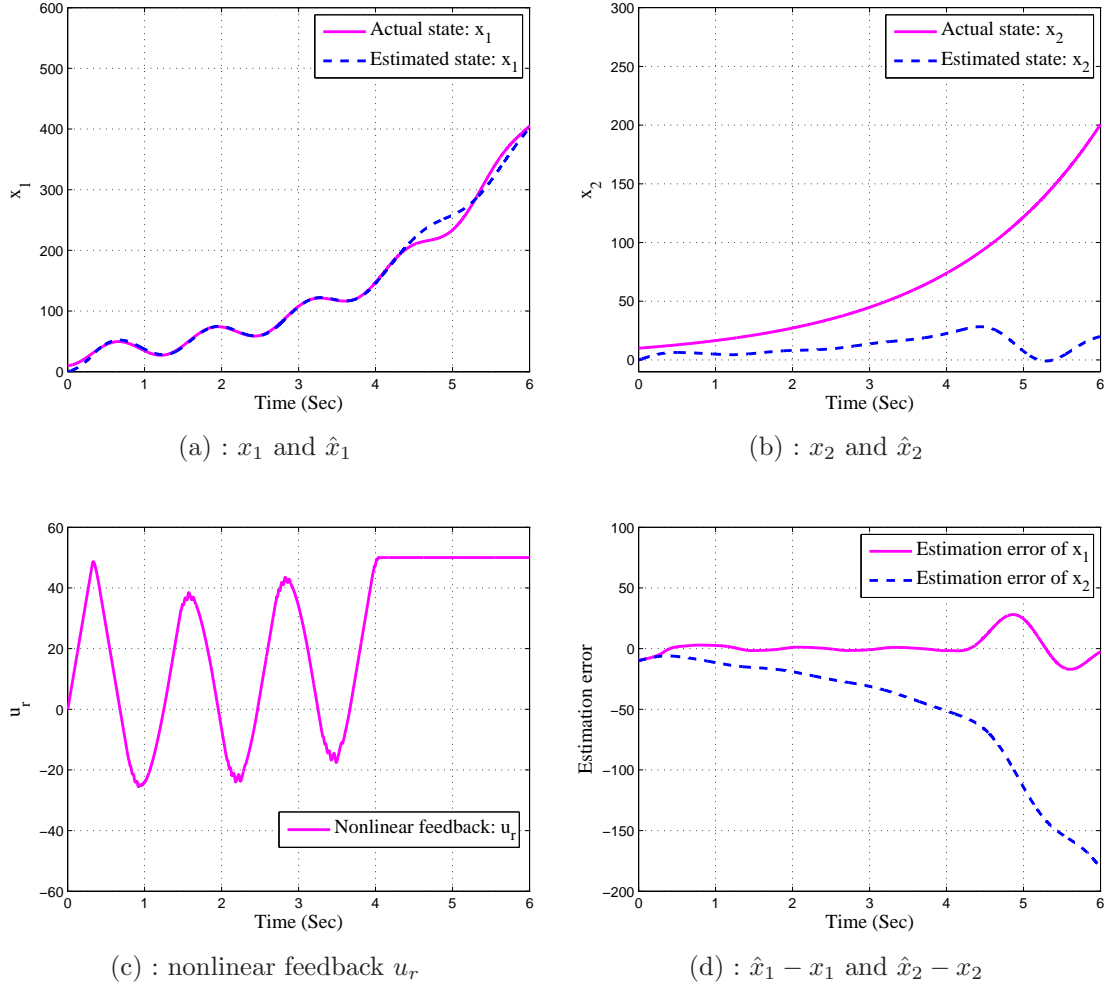


Figure 3.1: The estimation performance with the proposed observer: $k_r = 2.5$

nonlinear feedback should be removed if the reduced order dynamics is divergent on the sliding surface.

3.5 Summary

In this chapter, a hybrid observer has been developed to handle the state estimation of a class of SISO nonlinear systems with the so-called non-matching unknown input. The stability of the developed observer and the design procedure are carefully addressed. The contributions in this chapter can be summarized as:

- (i) A novel hybrid observer that combines a full-order high gain observer with a

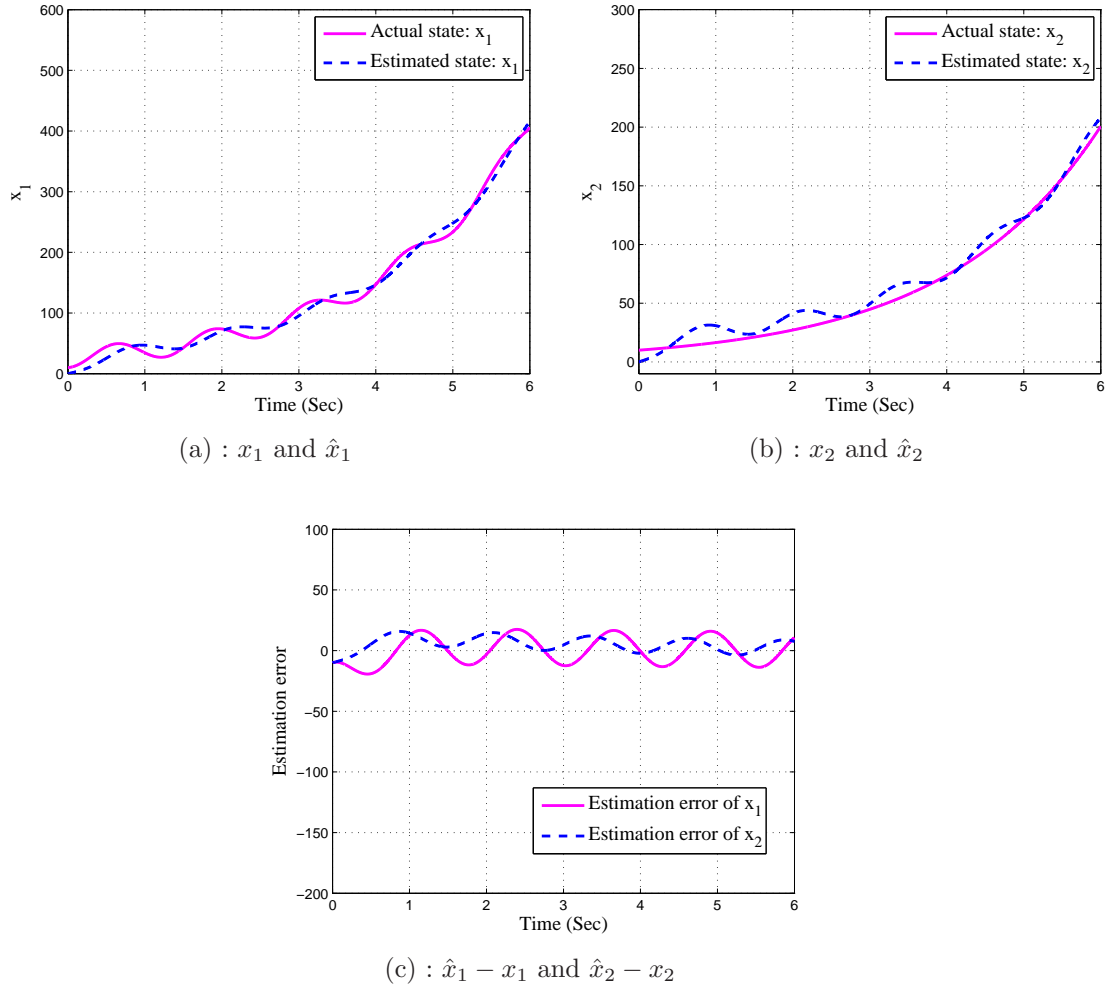


Figure 3.2: The estimation performance with the proposed observer: $k_r = 0$

higher order sliding mode term is proposed, in which the high gain feedback constrains the estimation error to within an invariant bounded set regardless of initial conditions, and the sliding mode term is designed to track the non-matching unknown input.

- (ii) The identifiability of the states and unknown input is carefully addressed. It has been pointed out that the non-matching unknown input can be identified only if the reduced order estimation error dynamics is asymptotically stable, which is only related to the original system structure. In other words, the identifiability can be classified into the following three cases:

- a) In the case when the reduced order dynamics is asymptotically stable,

i.e., Assumption 3.6 is satisfied, then, all system states and unknown input can be exactly identified with the proposed hybrid observer.

- b) In the case when the reduced order dynamics is stable, but not asymptotically stable, i.e., $\lim_{t \rightarrow \infty} \|\hat{\mathbf{x}}_d - \mathbf{x}_d\| \leq \varepsilon_3$ with ε_3 being a positive constant, then, only partial states can be exactly identified on the sliding surface, i.e., $\hat{\mathbf{x}}_s = \mathbf{x}_s$, and the remaining states \mathbf{x}_d and unknown input $d(t)$ failed to be observable. Such conclusion is similar to the results in [62].
- c) In the case when the reduced order dynamics is unstable/divergent, then, it is suggested that the sliding mode feedback should be removed, and the state estimation error can be restrained to within an invariant set. In other words, all system states and the unknown input may not be exactly identified.

Chapter 4

Speed and Parameter Identification in a Series DC Motor

In this chapter, two hybrid observers will be developed and implemented on a series DC motor to identify a non-matching time varying parameter, as well as the unknown speed. Based on the measurable current and input voltage, the unknown speed and non-matching parameter can be exactly estimated without filtering effect. The identified parameter is then used to enhance the speed estimation performance in the presence of external load disturbance. The stability analyses for the proposed observers are given, and the results are experimentally tested.

The chapter is organized as follows: Section 4.1 introduces some existing techniques for solving the state and parameter estimation problems of engineering systems. Section 4.2 presents the mathematical model of a series DC motor, and some background results plus existing problems are also introduced. In section 4.3, a robust hybrid observer is developed to identify the non-matching motor parameter, and a detail stability analysis is given. In section 4.4, the estimated parameter is used in another robust observer to estimate the unknown speed and external load disturbance. In section 4.5, experimental results are presented to illuminate the effectiveness of the proposed observers. Section 4.6 presents detail Monte Carlo studies on the effects of measurement noises in motor resistance and

inductance on the identification of the non-matching motor parameter. Section 4.7 concludes this chapter.

4.1 Introduction

A dc motor in which the field winding is connected in series with the armature coil is referred to as a series dc motor. Due to its special electrical structure, it produces more torque per ampere of current than any other dc motors. And such a motor is widely used in applications that require high starting torque, such as hoists, winches and electric traction applications [106].

In most situations, there are only partial state and parameter information available through the measurement outputs, and these often limit the systems' performance. For such cases, a robust observer with high estimation accuracy is required to recover the unknown states and parameters in real time.

The sliding mode based observer is a well-established and effective candidate to handle the state and parameter estimation problems, due to its robustness, simplicity, and high state estimation accuracy. However, most existing sliding mode observers are designed for systems with direct or indirect (i.e. matched after transformation) matching conditions in the observer sense. In [119, 123], the speed-dependent back-EMF terms of permanent magnet synchronous motor (PMSM) are considered as two matching disturbances, which are used to extract the speed and position information. In [97], an adaptive mechanism appended with sliding mode observer is proposed to identify the unknown back-EMF terms and unknown stator resistance R_s . However, the unknown stator resistance is assumed to be slowly time-changing, and one may treat it as a new unknown state with dynamic equation $\dot{R}_s = 0$. Then, the back-EMF terms can be represented as two matching disturbances. Similarly in [49, 124], the unknown constant parameters or disturbance can be considered as new unknown states, and the identifiability issue can be discussed in the transformed domain. For such cases, the systems can be classified as satisfying the indirect matching condition for the disturbances.

In this chapter, we shall consider a flux-related parameter estimation problem in a series DC motor, i.e., K_T , which is time-varying and non-matching in the observer sense. Then a robust hybrid observer will be developed to asymptotically identify the unknown parameter.

4.2 Preliminaries

4.2.1 Mathematical Model

A series DC motor is configured by connecting the field circuit in series with the armature circuit, and can be modeled as [106, 125]:

$$\begin{aligned} \dot{I}_f &= -\frac{R}{L}I_f - \frac{K_T}{L}I_f\omega + \frac{U}{L} \\ \dot{\omega} &= -\frac{B}{J}\omega + \frac{K_T}{J}I_f^2 - \frac{T_L}{J} \end{aligned} \quad (4.1)$$

where

- I_f : Field/armature current (measurable)
- ω : Electrical angular speed
- U : Voltage input
- R : Resistance
- L : Inductance
- J : Rotor moment of inertia
- B : Viscous-friction coefficient
- T_L : Load torque
- K_T : Field flux-related coefficient

Here, the motor parameter K_T and the load torque T_L are assumed to satisfy the following equations:

$$K_T = K_{T0} + \Delta K_T \quad (4.2)$$

$$T_L = T_{L0} + \Delta T_L \quad (4.3)$$

where K_{T0} and T_{L0} represent the nominal parameter and known load torque, respectively; ΔK_T denotes an unknown time-varying parameter, and ΔT_L denotes the bounded external load disturbance.

Remark 4.1. ΔK_T is modeled as a lumped parameter uncertainty on field flux which may include magnetic saturation effect, imperfect manufacturing effect, etc.

Remark 4.2. The distribution vector $[-I_f\omega/L, I_f^2/J]^T$ of parameter K_T includes the unknown state ω , which implies that K_T is a non-matching parameter in the observer sense.

Remark 4.3. ΔT_L is modeled as a lumped external load disturbance. Furthermore, it can also be treated as the parameter variation of rotor inertia J and viscous-friction coefficient B .

4.2.2 Existing High Gain Observer

In [37], a traditional high gain observer (HGO) is developed for the unknown speed estimation, in which the flux-related coefficient is assumed to be a known constant and there is no external disturbance, i.e., $\Delta K_T = 0$, $\Delta T_L = 0$. Based on the *Lie* transformation, it is given as

$$\begin{aligned}\dot{\hat{I}}_f &= -\frac{R}{L}\hat{I}_f - \frac{K_{T0}}{L}\hat{I}_f\hat{\omega} + \frac{U}{L} - 2\theta e_1 \\ \dot{\hat{\omega}} &= -\frac{B}{J}\hat{\omega} + \frac{K_{T0}}{J}\hat{I}_f^2 - \frac{T_{L0}}{J} + \left[\frac{2\theta R}{K_{T0}\hat{I}_f} + \frac{2\theta\hat{\omega}}{\hat{I}_f} + \frac{\theta^2 L}{K_{T0}\hat{I}_f} \right] e_1\end{aligned}\quad (4.4)$$

where $e_1 = \hat{I}_f - I_f$, \hat{I}_f and $\hat{\omega}$ denote the corresponding estimated current and speed; θ is a positive design parameter for the feedback gain $\mathbf{L} = [2\theta, \theta^2]^T$.

In [125], a linear feedback observer is proposed for speed and load torque estimation in a series DC motor, however, it requires the exact information of the magnetization/saturation curve of the field flux, which means the current dependent parameter K_T is exactly known; furthermore, the unknown load torque is assumed to be constant.

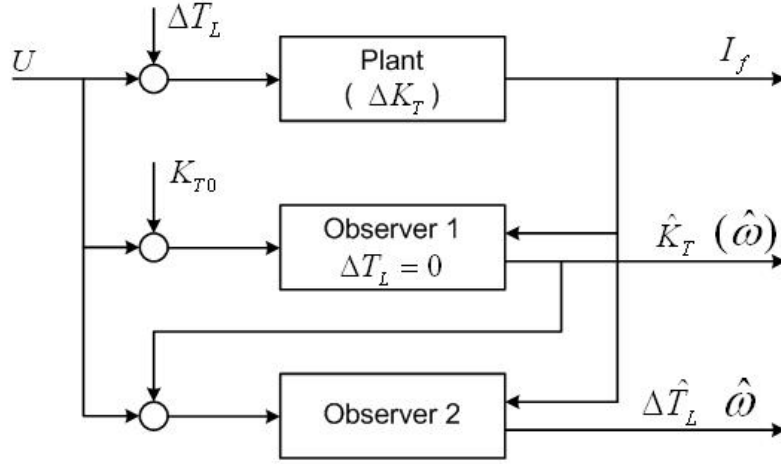


Figure 4.1: System diagram

4.2.3 Problems Formulation

In this chapter, we shall consider the estimation of the unknown parameter K_T and external disturbance ΔT_L in sequence, as well as the unknown speed.

As shown in Figure 4.1, the first observer (observer 1) will identify the unknown parameter K_T with no external disturbance in the initialization stage. From Remark 4.2, the unknown parameter K_T is non-matching in the observer sense and time-varying. The challenging problem is to guarantee that it can be estimated asymptotically.

The second observer (observer 2) is designed based on the identified parameter \hat{K}_T to estimate the unknown speed and external disturbance in the main operation stage. As the identification error of parameter K_T may affect the estimation performance, its effect will be carefully discussed.

4.3 Observer for Non-matching Parameter

In this section, we consider the identification of the non-matching parameter K_T without the external disturbance, i.e., $\Delta T_L = 0$. A hybrid observer that combines high gain observer with robust sliding mode term is proposed, in which the high gain feedback works to speed up the convergence in the beginning and constrains the estimation error to a bounded zone, then the sliding mode term ensures that

the unknown dynamics/disturbance is exactly tracked. The asymptotic stability of the estimation error will be carefully addressed.

4.3.1 Observer Design

For the original system (4.1) with $\Delta T_L = 0$, a hybrid observer can be designed in the form of

$$\begin{aligned}\dot{\hat{I}}_f &= -\frac{R}{L}\hat{I}_f - \frac{K_{T0}}{L}\hat{I}_f\hat{\omega} + \frac{U}{L} - 2\theta e_1 + \rho u_1 \\ \dot{\hat{\omega}} &= -\frac{B}{J}\hat{\omega} + \frac{K_{T0}}{J}\hat{I}_f^2 - \frac{T_L}{J} + \left[\frac{2\theta R}{K_{T0}\hat{I}_f} + \frac{2\theta\hat{\omega}}{\hat{I}_f} + \frac{\theta^2 L}{K_{T0}\hat{I}_f} \right] e_1 - \frac{\hat{I}_f L}{\hat{\omega} J} \rho u_1\end{aligned}\quad (4.5)$$

where θ is the high gain feedback parameter, e_1 denotes the estimation error of current, ρ is a positive tuning parameter to adjust the convergence time, and the robust term u_1 is given by

$$\dot{u}_1 = -\rho_1 \text{sign}(e_1) - \rho_2 \text{sign}(\dot{e}_1) \quad (4.6)$$

with ρ_1 and ρ_2 being two properly chosen sliding gains that satisfy $\rho_1 > \rho_2 > 0$.

Remark 4.4. *The feedback term u_1 in (4.6) is in fact an integral function of a second order sliding mode algorithm (namely twisting algorithm in [53]), which will result in a continuous function.*

Remark 4.5. *Compared with the existing HGO in (4.4), the proposed observer in (4.5) adds nonlinear feedback terms related with u_1 . Such nonlinear feedbacks are used to handle the parameter uncertainty ΔK_T and provide a mechanism to recover the unknown parameter K_T .*

4.3.2 Stability Analysis

Defining estimation error $\mathbf{e} = [e_1, e_2]^T = [\hat{I}_f - I_f, \hat{\omega} - \omega]^T$, then based on (4.1) and (4.5), the dynamics of e_1 can be described as

$$\begin{aligned}\dot{e}_1 &= -\frac{R}{L}e_1 - \frac{K_{T0}}{L}\hat{I}_f\hat{\omega} + \frac{K_T}{L}I_f\omega - 2\theta e_1 + \rho u_1 \\ &= G(\cdot) + \rho u_1\end{aligned}\quad (4.7)$$

where

$$G(\cdot) = -\frac{R}{L}e_1 - \frac{K_{T0}}{L}\hat{I}_f\hat{\omega} + \frac{K_T}{L}I_f\omega - 2\theta e_1$$

Then, by taking derivative on both sides of (4.7), it can be obtained that

$$\ddot{e}_1 = \frac{d}{dt}G(\cdot) - \rho\rho_1\text{sign}(e_1) - \rho\rho_2\text{sign}(\dot{e}_1) \quad (4.8)$$

Considering the second order dynamic system in (4.8), one can always find a positive value ε (independent of ρ_1 and ρ_2) large enough such that $|\frac{d}{dt}G(\cdot)| \leq \varepsilon$. This is true because in real application the unknown parameter K_T is smooth and differentiable and its first order derivative with time is locally bounded, and so are the current and speed.

By defining a Lyapunov function $V_1 = (\dot{e}_1)^2/2 + \rho\rho_1|e_1|$, which is continuously differentiable except on $e_1 = 0$, it can be verified that

$$\begin{aligned}\dot{V}_1 &= \dot{e}_1\ddot{e}_1 + \rho\rho_1\dot{e}_1\text{sign}(e_1) \\ &= \dot{e}_1\frac{d}{dt}G(\cdot) - \rho\rho_1\dot{e}_1\text{sign}(e_1) - \rho\rho_2|\dot{e}_1| + \rho\rho_1\dot{e}_1\text{sign}(e_1) \\ &\leq -(\rho\rho_2 - \varepsilon)|\dot{e}_1| \\ &< 0 \quad (\text{with } \rho_1 > \rho_2 > \varepsilon/\rho)\end{aligned}\quad (4.9)$$

The above inequality is true because the condition $\dot{e} = 0$ cannot hold for any finite interval with $e \neq 0$. In other words, the Lyapunov function V_1 will converge to the origin of the phase plane in finite time, i.e., $\dot{e}_1 = e_1 = 0$. Similar proofs can be found in [53] and [126].

Once the sliding mode occurs, i.e., $\dot{e}_1 = e_1 = 0$, $\hat{I}_f = I_f$, it can be deduced from (4.7) that

$$\rho u_1 = \frac{\hat{I}_f}{L}(K_{T0}\hat{\omega} - K_T\omega) \quad (4.10)$$

The above equation highlights the relationship among the unknown parameter K_T , unknown speed ω and the designed robust term u_1 on the sliding surface, which implies that the parameter K_T can be exactly identified after the estimated speed has converged to its true value, i.e., $\hat{\omega} = \omega$.

Theorem 4.1. *For the system given by (4.1) with unknown parameter K_T but $\Delta T_L = 0$, the proposed observer in (4.5) will ensure that the estimated speed asymptotically converges to its true value on the sliding surface $e_1 = 0$.*

Proof. On the sliding surface $e_1 = 0$, the high gain feedback terms related with θ go to zero, so the remaining dynamics of e_2 can be obtained from (4.1) and (4.5), as

$$\dot{e}_2 = -\frac{B}{J}e_2 + \frac{K_{T0}}{J}I_f^2 - \frac{K_T}{J}I_f^2 - \frac{I_f L}{\hat{\omega} J}\rho u_1 \quad (4.11)$$

By substituting (4.10) into the above equation, we have

$$\begin{aligned} \dot{e}_2 &= -\frac{B}{J}e_2 + \frac{K_{T0}}{J}I_f^2 - \frac{K_T}{J}I_f^2 - \frac{I_f^2}{\hat{\omega} J}(K_{T0}\hat{\omega} - K_T\omega) \\ &= -\frac{B}{J}e_2 - \frac{K_T}{J}I_f^2 + \frac{\omega K_T}{\hat{\omega} J}I_f^2 \\ &= -\frac{B}{J}e_2 - \frac{I_f^2 K_T}{\hat{\omega} J}e_2 \end{aligned} \quad (4.12)$$

Since K_T is a positive motor parameter, $\hat{\omega}$ can be guaranteed to be positive by high gain feedback before sliding mode happens. Therefore, a Lyapunov function can be chosen as $V_2 = (e_2)^2/2$, and the following inequality holds:

$$\dot{V}_2 = -\left(\frac{B}{J} + \frac{I_f^2 K_T}{\hat{\omega} J}\right)|e_2|^2 \leq 0 \quad (4.13)$$

Thus, the remaining estimation error e_2 is asymptotically stable on the sliding

surface $e_1 = 0$. In other words, with the decaying factor $[B/J + I_f^2 K_T / (\hat{\omega} J)]$ and after a finite time, we have $\hat{\omega} = \omega$. ■

After all the estimated states have converged to the true values, i.e., $\hat{I}_f = I_f$, $\hat{\omega} = \omega$. The unknown parameter can be computed from (4.10), as

$$\hat{K}_T = K_{T0} - \frac{L}{\hat{I}_f \hat{\omega}} \rho u_1 \quad (4.14)$$

As the robust term u_1 is designed based on the integral of a second order sliding mode algorithm, which results in a continuous function, the filtering effect is avoided.

Remark 4.6. *From (4.13), it can be seen that the decaying factor for the error e_2 is only dependent on the motor parameters and completely independent of the observer gains. Therefore, the operating procedure of the proposed observer can be described as one in which the high gain feedback works to constrain the estimation error into a bounded zone, where the sliding condition is satisfied, the sliding mode feedback then ensures that the sliding surface is reached in finite time and remains thereafter. Finally, the system structure guarantees asymptotic convergence of the remaining error dynamics.*

4.4 Observer for External Disturbance

With the identified parameter \hat{K}_T , in this section, we shall now consider the estimation of the unknown speed and external load disturbance. The effect of the identification error of K_T on the estimation accuracy will be carefully addressed.

4.4.1 Observer Design and Stability

The system uncertainty can be expressed as

$$d(t) = \frac{K_T I_f}{JL} \Delta T_L \quad (4.15)$$

Then, the original system (4.1) can be rewritten as

$$\begin{aligned}\dot{I}_f &= -\frac{R}{L}I_f - \frac{K_T}{L}I_f\omega + \frac{U}{L} \\ \dot{\omega} &= -\frac{B}{J}\omega + \frac{K_T}{J}I_f^2 - \frac{T_{L0}}{J} - \frac{L}{K_T I_f}d(t)\end{aligned}\quad (4.16)$$

Note that the relative degree between the uncertainty $d(t)$ and the measurable variable I_f is two, so a second order sliding mode algorithm can be used to estimate the unknown speed and reconstruct the uncertainty, as given by

$$\begin{aligned}\dot{\hat{I}}_f &= -\frac{R}{L}\hat{I}_f - \frac{\hat{K}_T}{L}\hat{I}_f\hat{\omega} + \frac{U}{L} - 2\theta e_1 \\ \dot{\hat{\omega}} &= -\frac{B}{J}\hat{\omega} + \frac{\hat{K}_T}{J}\hat{I}_f^2 - \frac{T_{L0}}{J} + \left[\frac{2\theta R}{\hat{K}_T\hat{I}_f} + \frac{2\theta\hat{\omega}}{\hat{I}_f} + \frac{\theta^2 L}{\hat{K}_T\hat{I}_f} \right] e_1 - \frac{L}{\hat{K}_T\hat{I}_f}u_2\end{aligned}\quad (4.17)$$

where \hat{K}_T can be obtained from (4.14), θ is the high gain feedback parameter and u_2 is the robust sliding mode term, given by [69]

$$\begin{aligned}u_2 &= -\rho_3 \text{sign}(z_1 + |e_1|^{1/2} \text{sign}(e_1)) \\ \dot{z}_0 &= v_0 \\ v_0 &= z_1 - 1.5M^{1/2}|z_0 - e_1|^{1/2} \text{sign}(z_0 - e_1) \\ \dot{z}_1 &= -1.1M \text{sign}(z_1 - v_0)\end{aligned}\quad (4.18)$$

Theorem 4.2. *For the given system in (4.16), the proposed observer (4.17)-(4.18) ensures that both the current and speed can be exactly estimated in finite time provided that $\hat{K}_T = K_T$.*

Proof. In order to illuminate the stability of the proposed observer, a nonlinear state transformation will be introduced, which has also been used in [37]. The stability analysis will be discussed in the transformed domain.

Let $\tilde{\mathbf{x}} = [I_f, \omega]^T$, a state transformation function based on *Lie* derivatives can be defined as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \Phi(\tilde{\mathbf{x}}) = \begin{bmatrix} I_f \\ -\frac{R}{L}I_f - \frac{K_T}{L}I_f\omega \end{bmatrix}\quad (4.19)$$

with

$$\frac{\partial \Phi(\tilde{\mathbf{x}})}{\partial \tilde{\mathbf{x}}} = \begin{bmatrix} 1 & 0 \\ -\frac{R}{L} - \frac{K_T}{L}\omega & -\frac{K_T}{L}I_f \end{bmatrix}, \quad \left[\frac{\partial \Phi(\tilde{\mathbf{x}})}{\partial \tilde{\mathbf{x}}} \right]^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{R}{K_T I_f} - \frac{\omega}{I_f} & -\frac{L}{K_T I_f} \end{bmatrix} \quad (4.20)$$

Then the system in (4.16) can be rewritten in the transformed domain, as

$$\begin{aligned} \dot{\mathbf{x}} &= \frac{\partial \Phi(\tilde{\mathbf{x}})}{\partial \tilde{\mathbf{x}}} \dot{\tilde{\mathbf{x}}} \\ &= \frac{\partial \Phi(\tilde{\mathbf{x}})}{\partial \tilde{\mathbf{x}}} \left\{ \begin{bmatrix} -\frac{R}{L}I_f - \frac{K_T}{L}I_f\omega + \frac{U}{L} \\ -\frac{B}{J}\omega + \frac{K_T}{J}I_f^2 - \frac{T_{L0}}{J} - \frac{L}{K_T I_f}d(t) \end{bmatrix} \right\} \\ &= \begin{bmatrix} x_2 \\ f(x_1, x_2) \end{bmatrix} + \begin{bmatrix} \frac{U}{L} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d(t) \end{bmatrix} \end{aligned} \quad (4.21)$$

with the nominal nonlinear function $f(x_1, x_2)$ being given by

$$f(x_1, x_2) = \left(\frac{R}{L} + \frac{K_T}{L}\omega\right)\left(\frac{R}{L}I_f + \frac{K_T}{L}I_f\omega - \frac{U}{L}\right) + \frac{K_T}{L}I_f\left(\frac{B}{J}\omega - \frac{K_T}{J}I_f^2 + \frac{T_{L0}}{J}\right)$$

In other words, the system in (4.16) can be transformed into the following dynamics with a triangular structure:

$$\begin{aligned} \dot{x}_1 &= x_2 + U/L \\ \dot{x}_2 &= f(x_1, x_2) + d(t) \end{aligned} \quad (4.22)$$

Note that $f(x_1, x_2)$ is a nominal nonlinear function of x_1 and x_2 (or, I_f and ω). Together with the system uncertainty $d(t)$, they are locally bounded in real applications.

Similarly, in the case when $\hat{K}_T = K_T$, the proposed observer in (4.17) can be transformed into the same domain via $\hat{\mathbf{x}} = \Phi(\tilde{\mathbf{x}})|_{\tilde{\mathbf{x}}=\hat{\tilde{\mathbf{x}}}}$, with the superscript $\hat{\cdot}$ denoting the corresponding estimated variables. Then, it has

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + U/L - 2\theta e_1 \\ \dot{\hat{x}}_2 &= f(\hat{x}_1, \hat{x}_2) + u_2 - \theta^2 e_1 \end{aligned} \quad (4.23)$$

where $\hat{\mathbf{x}} = [\hat{x}_1, \hat{x}_2]^T = [\hat{I}_f, -\frac{R}{L}\hat{I}_f - \frac{K_T}{L}\hat{I}_f\hat{\omega}]^T$, and $\mathbf{e} = [e_1, e_2]^T = \hat{\mathbf{x}} - \mathbf{x}$.

Therefore, the dynamics of the estimation error \mathbf{e} can be described by

$$\begin{aligned}\dot{e}_1 &= e_2 - 2\theta e_1 \\ \dot{e}_2 &= f(\hat{x}_1, \hat{x}_2) - f(x_1, x_2) - d(t) - \theta^2 e_1 + u_2\end{aligned}\quad (4.24)$$

It has been proven that, with such a triangular dynamic system, the high gain feedback $[-\theta e_1, -\theta^2 e_1]^T$ works to restrain the error \mathbf{e} into an invariant small zone. Details can be found in Corollary 3.1 in Chapter 3.

Now, consider e_1 as the sliding variable, its dynamics can be presented as

$$\begin{aligned}\ddot{e}_1 &= \dot{e}_2 - 2\theta\dot{e}_1 \\ &= f(\hat{x}_1, \hat{x}_2) - f(x_1, x_2) - d(t) - 2\theta e_2 + 3\theta^2 e_1 + u_2\end{aligned}\quad (4.25)$$

For such a second order dynamics of e_1 , the dynamics on the right hand side (without u_2) is locally bounded. Then, the provided second order sliding mode feedback u_2 given by (4.18) ensures that the sliding surface $\dot{e}_1 = e_1 = 0$ will be reached in finite time and remained thereafter, provided that the parameters ρ_3 and M are chosen large enough [69].

Once the sliding surface is reached, it can be deduced from (4.24) that

$$\begin{aligned}e_1 &= \hat{I}_f - I_f = 0 \\ e_2 &= \hat{x}_2 - x_2 = 0 \Rightarrow \hat{\omega} - \omega = 0\end{aligned}\quad (4.26)$$

which means both the current and speed can be exactly estimated by the proposed observer (4.17)-(4.18) in the case when $\hat{K}_T = K_T$. ■

After all the states have converged to the true values, i.e., $\hat{I}_f = I_f$, $\hat{\omega} = \omega$, the system uncertainty $d(t)$ can be reconstructed from the sliding mode term based on the equivalent input concept, as

$$d(t) \approx \{u_2\}_{eq} \quad (4.27)$$

where $\{u_2\}_{eq}$ denotes the equivalent input of u_2 , reconstructed through a low pass filter. Thus, the external disturbance can be obtained by

$$\Delta T_L \approx \frac{JL}{K_T I_f} \{u_2\}_{eq} \quad (4.28)$$

Remark 4.7. *The sliding mode algorithm used in (4.18) comes from [69], but one can choose any other second order sliding mode algorithm to design u_2 .*

Remark 4.8. *The proposed observer in (4.17) combines the high gain observer with sliding mode term, in which the high gain feedback works to speed up the convergence, and will restrain the estimation error bounded to a small zone (as shown in Corollary 3.1 in Chapter 3), then the sliding parameters can be chosen small to reduce the chattering effect on the uncertainty reconstruction.*

4.4.2 Effect of Error in Identified Parameter \hat{K}_T

The above analysis is based on the assumption that the identified parameter \hat{K}_T is exactly its true value. However, there is inevitable identification error in real implementation. Thus, it is necessary to discuss the effect of its identification error on the unknown speed and external disturbance estimations.

Without loss of generality, we assume that $\hat{K}_T = \beta K_T$, where β is a constant value used to indicate the deviation coefficient of the parameter K_T in the identification.

By denoting $\tilde{\omega} = \beta\hat{\omega}$, the proposed observer in (4.17) can be rewritten as

$$\begin{aligned} \dot{\hat{I}}_f &= -\frac{R}{L}\hat{I}_f - \frac{K_T}{L}\hat{I}_f\tilde{\omega} + \frac{U}{L} - 2\theta e_1 \\ \dot{\tilde{\omega}} &= -\frac{B}{J}\tilde{\omega} + \frac{K_T}{J}\hat{I}_f^2 - \frac{T_{L0}}{J} + \left[\frac{2\theta R}{K_T\hat{I}_f} + \frac{2\theta\tilde{\omega}}{\hat{I}_f} + \frac{\theta^2 L}{K_T\hat{I}_f} \right] e_1 - \frac{L}{K_T\hat{I}_f} u_2 + H(\beta) \end{aligned} \quad (4.29)$$

where $H(\beta) = (\beta^2 - 1)\frac{K_T}{J}\hat{I}_f^2 - (\beta - 1)\frac{T_{L0}}{J}$ can be considered as an additional bounded uncertainty.

It is clear that the rewritten observer in (4.29) is similar to the proposed observer in (4.17), and one can readily verified that the sliding surface $\dot{e}_1 = e_1 = 0$

can be reached in finite time and remained thereafter, with the similar procedure in Theorem 4.2.

On the sliding surface, it can be deduced from (4.15), (4.16) and (4.29) that

$$\hat{\omega} = \tilde{\omega}/\beta = \omega/\beta \quad (4.30)$$

$$\Delta T_L = \frac{JL}{K_T I_f} u_2 - [(\beta^2 - 1)K_T I_f^2 - (\beta - 1)T_{L0}] \quad (4.31)$$

Here, the coefficient $1/\beta$ in (4.30) denotes the estimation error of speed, and the residual in the bracket in (4.31) denotes the estimation error of the external load disturbance.

Remark 4.9. *The above analysis results show that with the proposed observer (4.17)-(4.18), the deviation of non-matching motor parameter K_T will directly affect the estimation accuracy of the unknown speed and external disturbance. However, one can predict the estimation error with (4.31)-(4.32) in advance, if the deviation coefficient β is known. This provides the designer with a choice to make trade-off between the simplified compensation of parameter K_T and the estimation accuracy.*

4.5 Experimental Results

4.5.1 Experimental Conditions

As shown in Figure 4.2, the experimental setup consists of a single-pole-pair series dc motor and a dynamometer machine which is used to simulate the load torque on the motor. A 1024 pulse/rev optical encoder is fixed on the rotor to provide the real-time rotor speed for comparison with the estimated value from the proposed observers. A 32-bit fixed point processor, TMS32F2812@150MHz (Texas Instruments DSP), is adopted to execute the observer algorithms, and two 12-bit A/D channels are used to convert the real-time current and input voltage which are measured by sensors. The main motor parameters are summarized in Table 4.1.

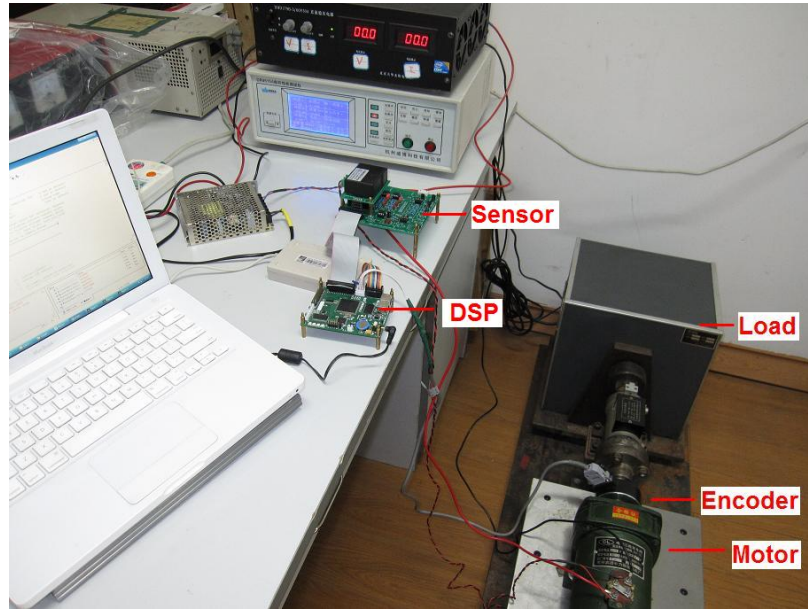


Figure 4.2: Experimental setup

Table 4.1: A series motor parameters

| | | |
|--------------------------------|----------|--|
| Rating current | I | 15 A |
| Rating speed | ω | 115 rad/s |
| Resistance | R | $0.1\ \Omega$ |
| Inductance | L | 1.44 mH |
| Inertia | J | $0.0018\text{ kg} \cdot \text{m}^2$ |
| Viscous friction coefficient | B | $0.0008\text{ N} \cdot \text{m}/\text{rad/s}$ |
| Field flux-related coefficient | K_T | $\text{N} \cdot \text{m}/\text{Wb} \cdot \text{A}$ |
| Load torque | T_L | $\text{N} \cdot \text{m}$ |

For the firmware programming, in order to optimize the execution-speed and improve the mathematical operation accuracy, we use the TI IQmath Library package to seamlessly port the floating-point algorithm into fixed-point code on the device. For example, the multiplication subroutine "IQNmpy(x, y)" takes about 6 execution cycles, the division subroutine "IQNdiv(x, y)" takes about 63 execution cycles, the square root subroutine "IQNsqr(x)" takes about 64 execution cycles, the absolute value subroutine "IQNabs(x)" takes about 2 execution cycles, etc. For more details about the TI IQmath Library package, please refer to: <http://www.ti.com/lit/sw/sprc990/sprc990.pdf>. Meanwhile, some intermedi-

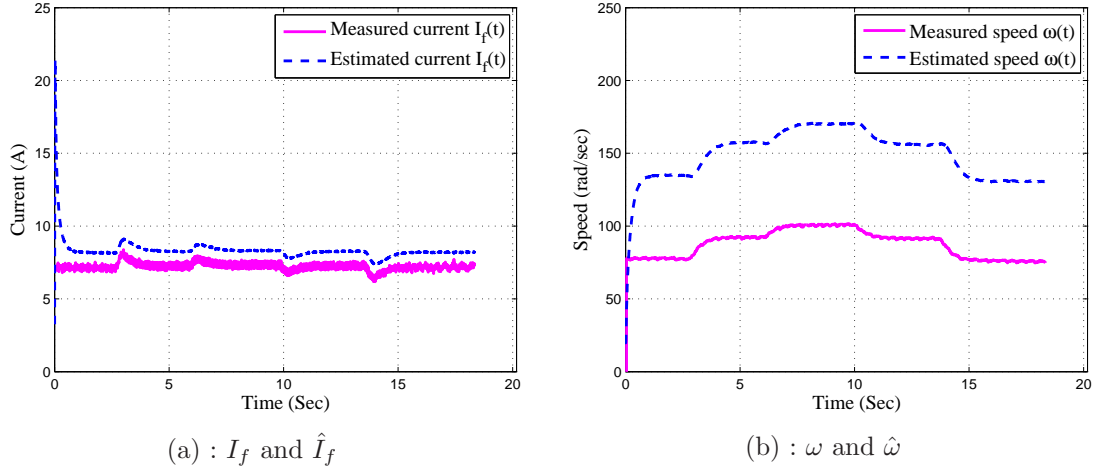


Figure 4.3: Experimental results with the high gain observer, $K_{T0} = 0.003$

ate constants are defined to shorten execution time of the proposed observers, such as $c_1 = T_s/L$, $c_2 = T_s/J$, $c_3 = L/K_{T0}$, $c_4 = L/J$, etc. With T_s being the sampling interval $0.1ms$, as well as the observer execution step.

The following experiments are cataloged in three parts: First, the traditional high gain observer (HGO) given by (4.4) is applied for current and speed estimations. Second, the proposed observer 1 given by (4.5)-(4.6) is performed for current and speed estimations, as well as the identification of the unknown parameter K_T . An indirectly computed K_T is also provided for comparison purpose. Third, the proposed observer 2 given by (4.17)-(4.18) is used to estimate the current and speed, as well as the unknown external disturbance.

4.5.2 State Estimations with High Gain Observer

For comparison purpose, the high gain observer proposed in [37], as given by (4.4), is applied. And the observer parameters are given as: $K_{T0} = 0.003 \text{ N}\cdot\text{m}/\text{Wb}\cdot\text{A}$, $\theta = 2$, $\hat{I}_f(0) = 3 \text{ A}$, $\hat{\omega}(0) = 20 \text{ rad/s}$.

The known load torque is set at $0.24 \text{ N}\cdot\text{m}$ by the dynamometer machine, and the input voltage is provided by a DC power supply which is manually operated to adjust the motor speed. The experimental results are shown in Figure 4.3.

It can be seen that both the estimated current and speed fail to track their

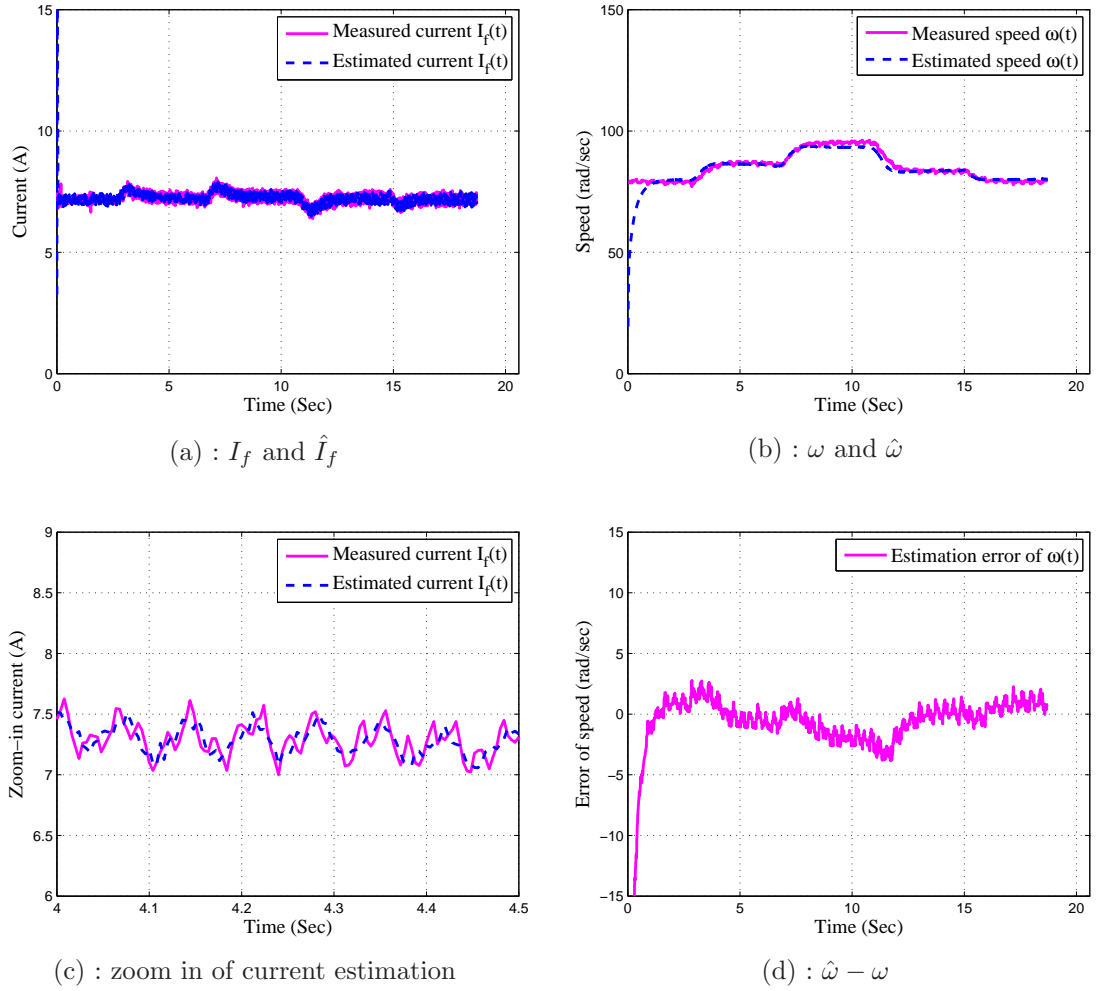


Figure 4.4: Experimental results with the proposed observer 1, $K_{T0} = 0.003$

true values because of the parameter uncertainty ΔK_T . Moreover, there is no mechanism to identify the unknown parameter K_T .

4.5.3 Parameter Identification with Proposed Observer 1

Here, the proposed hybrid observer described in (4.5)-(4.6) is employed, with the additional observer parameters: $\rho_1 = 7$, $\rho_2 = 3$, $\rho = 10000$. The experimental conditions are the same as in the above subsection, and the results are shown in Figure 4.4 and Figure 4.5.

From Figure 4.4, we can see that the estimated current and speed track their corresponding true values. Moreover, as shown in Figure 4.4d, the estimation

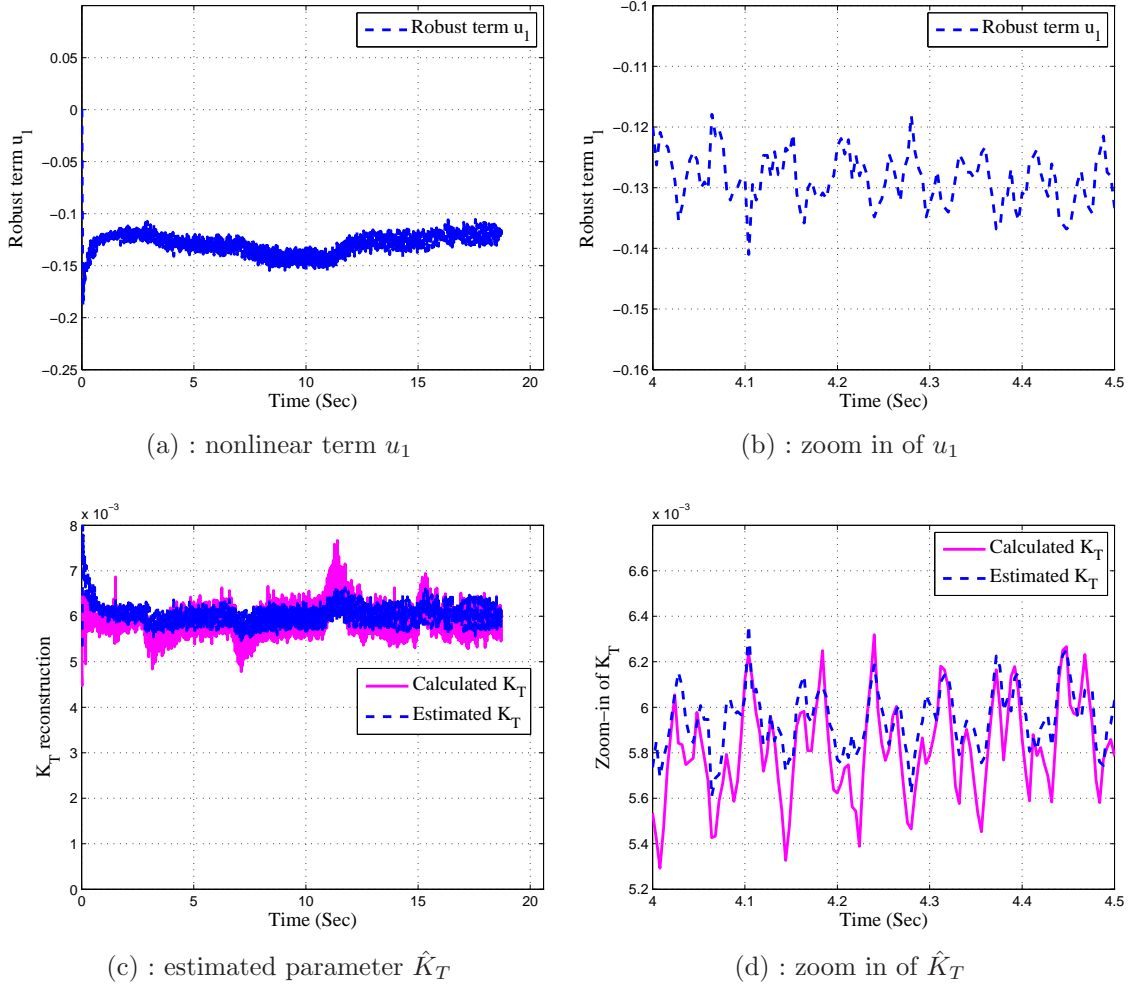


Figure 4.5: Experimental results with the proposed observer 1, $K_{T0} = 0.003$

error of speed is restrained to within $\pm 5 \text{ rad/s}$ after around 2 seconds.

Note that the estimated K_T in Figure 4.5 is a position related function. In order to provide a comparison that the estimated K_T is accurate, an indirectly computed K_T is calculated via

$$K_T \approx (T_L + B\omega)/I_f^2 \quad (4.32)$$

It should be noted that such calculated K_T requires the real-time information of speed and is only suitable for steady-state, i.e., $\dot{\omega} = 0$. It is observed that the estimated K_T tallies well with the computed one, see Figure 4.5c and Figure 4.5d.

In order to demonstrate the robustness of the proposed observer 1 against

the unknown parameter K_T , the same experiment is repeated by setting $K_{T0} = 0.009 \text{ N}\cdot\text{m}/\text{Wb}\cdot\text{A}$. As shown in Figure 4.6, the estimation performance of current and speed remains good.

Comparing the identified parameter K_T in Figure 4.5 and Figure 4.6, they are quite similar, which are position-related functions with an average value of around $0.006 \text{ N}\cdot\text{m}/\text{Wb}\cdot\text{A}$. In other words, these two experiments are conducted under $\pm 50\%$ variation of K_T , and robustness of the proposed observer against the unknown parameter K_T is observed.

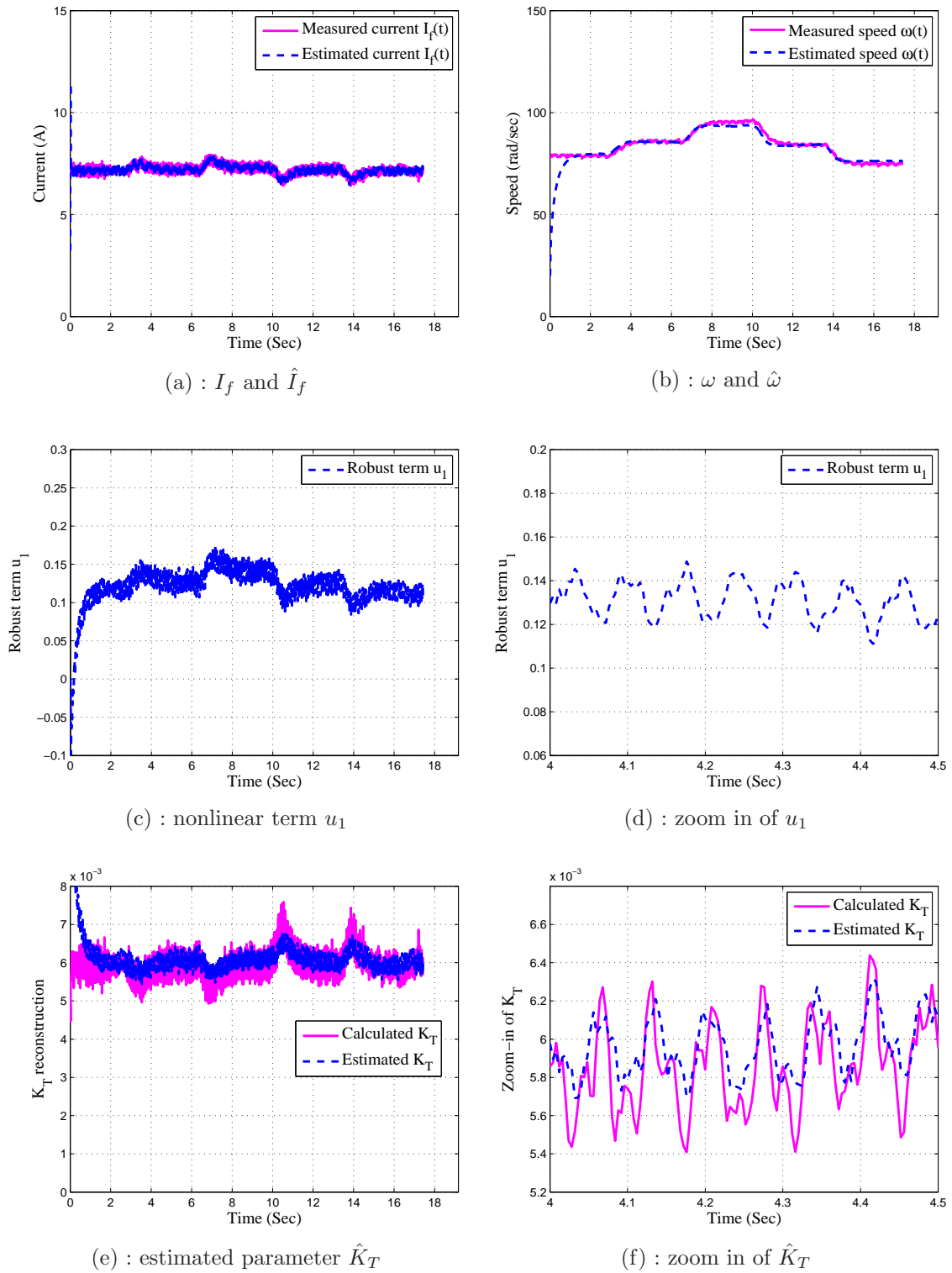
It is worth noting that the big difference between these two experimental results lies with the sliding mode term u_1 , which is designed to track the uncertainties of the systems. From Figure 4.5b and Figure 4.6d, we can see that the sliding mode terms are related to the parameter K_{T0} , current I_f and speed ω , and such relationship guarantees that the unknown parameter K_T can be exactly estimated all the time (after all states have converged to their true values), as given by equation (4.14).

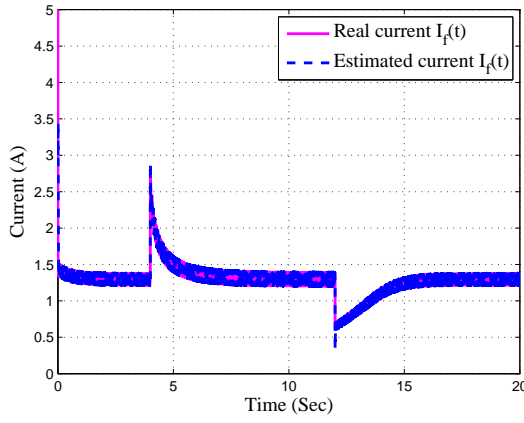
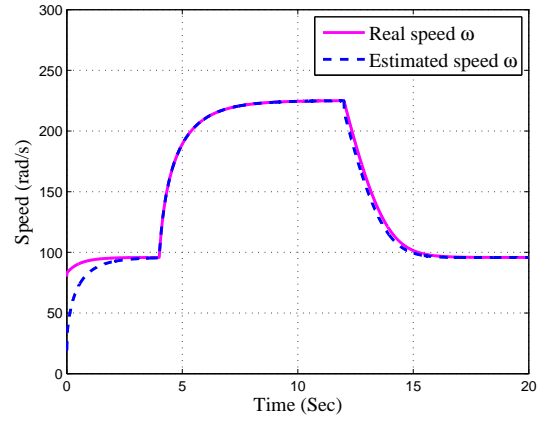
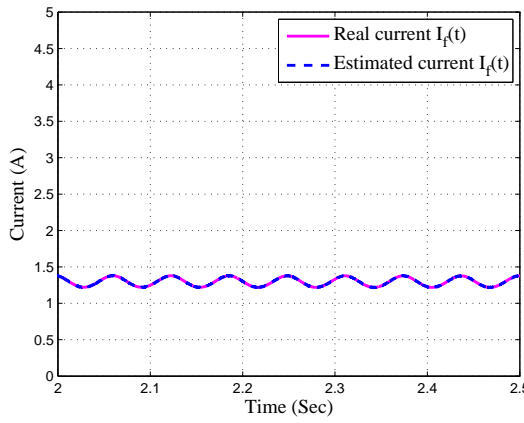
4.5.4 Robustness Study with Proposed Observer 1

It is also of interest to study the estimation performance under high rotor speed and light load condition, but the experimental DC motor we used is not able to operate under this extreme condition because of the limitations of the motor parameters. Thus, we resort to simulation studies to examine the performance of the propose observer under high rotor speed and light load condition.

The real motor parameters are chosen the same as in Table 4.1, except that $K_T = 0.006 + \sin(100t) \text{ N}\cdot\text{m}$, $T_L = 0.01 \text{ N}\cdot\text{m}$, $J = 0.0001 \text{ kg}\cdot\text{m}^2$, $B = 0$. The input voltage U is a periodic step function to adjust the speed. The observer parameter are set as: $\theta = 2$, $\rho = 3000$, $\rho_1 = 7$, $\rho_2 = 3$, $K_{T0} = 0.003 \text{ N}\cdot\text{m}/\text{Wb}\cdot\text{A}$.

The simulation results are shown in Figure 4.7. It can be seen that both the estimated speed $\hat{\omega}$ and parameter \hat{K}_T track their true values well even under large speed variation and light load condition. Meanwhile, a slight estimation mismatch can be observed in Figures 4.7b and 4.7e from 12 *seconds* to 14 *seconds*. This is because the speed estimation error takes a long time to die out for a small

Figure 4.6: Experimental results with the proposed observer 1, $K_{T0} = 0.009$

(a) : I_f and \hat{I}_f (b) : ω and $\hat{\omega}$ 

(c) : zoom-in of current estimation

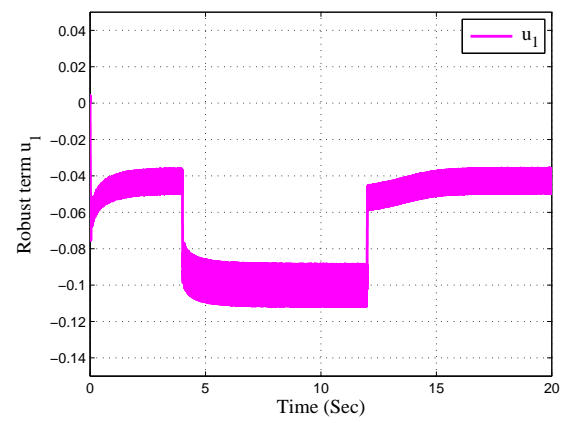
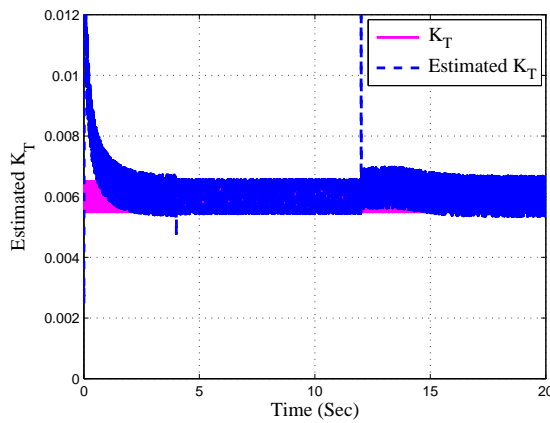
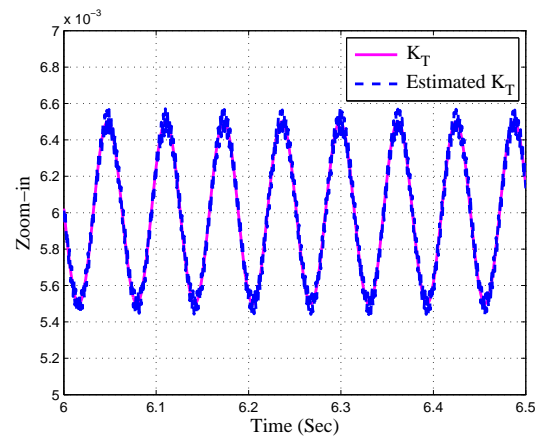
(d) : sliding mode term u_1 (e) : K_T and \hat{K}_T (f) : zoom-in of K_T estimation

Figure 4.7: Simulation results under high speed and light load condition

decaying factor $[B/J + I_f^2 K_T / (\hat{\omega} J)]$ as given in Theorem 4.1.

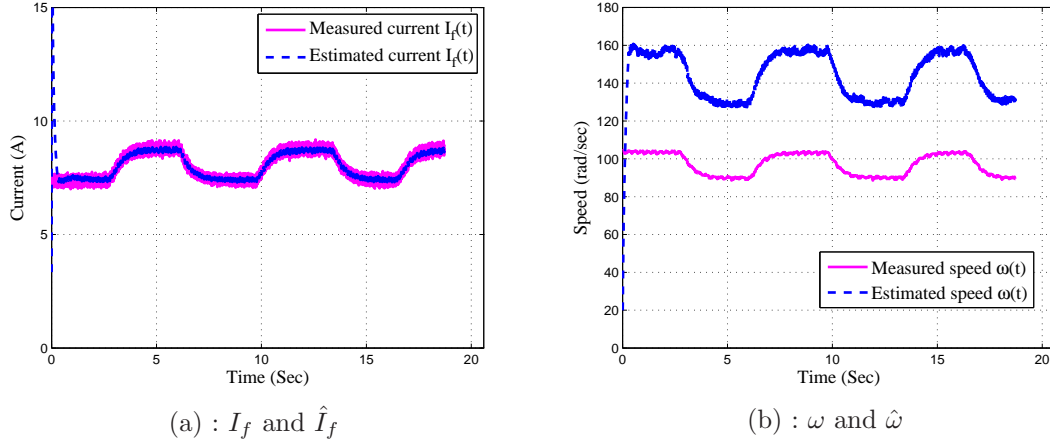


Figure 4.8: Experimental results with the proposed observer 2, $\hat{K}_T = 0.004$

4.5.5 Speed Estimation under External Disturbance

Now, we shall consider the estimation of speed and external disturbance with the proposed observer in (4.17)-(4.18). In order to verify the effect of the identified parameter K_T , we choose $\hat{K}_T = 0.004 \text{ N}\cdot\text{m}/\text{Wb}\cdot\text{A}$, which implies the deviation coefficient β is around $2/3$. Then, the additional observer parameters are given as: $\rho_3 = 0.5$, $M = 1$, $\theta = 2$.

By keeping the input voltage at a fixed value and manually increasing and decreasing the load torque through the dynamometer machine controller, to simulate the external disturbance, we obtained the experimental results as shown in Figure 4.8. It can be seen that the estimated speed is apparently higher than its true value, and it has $\hat{\omega}/\omega \approx 3/2 = 1/\beta$ which exactly tally with the analysis result in (4.31).

With the parameter K_T identified in Figure 4.5c, which is a position-related function, for simplicity, we take its mean as the proposed observer parameter, i.e., $\hat{K}_T = 0.006 \text{ N}\cdot\text{m}/\text{Wb}\cdot\text{A}$. As the deviation coefficient β in Figure 4.5 is $\beta \in [0.006/0.0065, 0.006/0.0055] \approx [0.923, 1.091]$, we can predict that the estimation error of speed will lie within $\hat{\omega}/\omega = 1/\beta \in [0.917, 1.083]$. The experiment is repeated with the same conditions above, and the results are shown in Figure 4.9.

It can be seen that both the estimated current and speed are synchronously varying with their true values, and the tracking errors lie within the above pre-

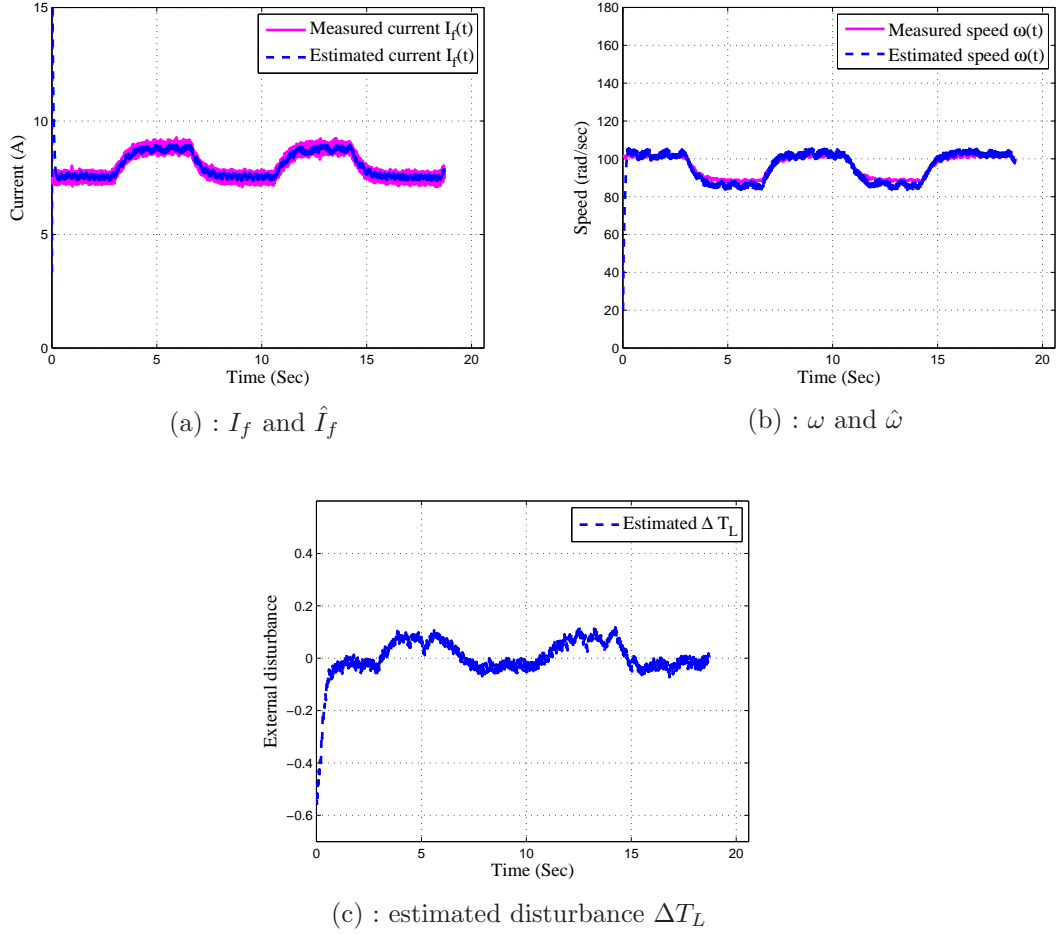


Figure 4.9: Experimental results with the proposed observer 2, $\hat{K}_T = 0.006$

dicted range. Moreover, the unknown external disturbance is successfully reconstructed by a low-pass filter after 1 second.

Small ripples are found on the estimated speed curve. This is normal since the real deviation coefficient β is a varying function with rotor position.

4.6 Simulation Results on the Effects of R and L

In order to illuminate the sensitivity of the estimate \hat{K}_T with respect to the variations of resistance R and inductance L , some Monte carlo simulations will be conducted based on the proposed observer 1.

The proposed observer 1 given by (4.5)-(4.6) is used for the simulation, but R and L are replaced by R_0 and L_0 , respectively. The estimated parameter \hat{K}_T in (4.14) is rewritten as

$$\hat{K}_T = K_{T0} - \frac{L_0}{\hat{I}_f \hat{\omega}} \rho u_1 \quad (4.33)$$

The parameters R_0 and L_0 denote the measured resistance and inductance, which are assumed to be affected by measurement noises (with uniform distribution), i.e., $R_0 = R + \alpha R[2rand(\cdot) - 1]$, $L_0 = L + \gamma L[2rand(\cdot) - 1]$, with α and γ being two tuning scalars, the $rand(\cdot)$ function in Matlab is used to generate pseudorandom values with uniform distribution on the open interval $(0, 1)$. Then, the corresponding expected value and variance of R_0 and L_0 are given as: $\mathbf{E}[R_0] = R$, $\mathbf{VAR}[R_0] = \alpha^2 R^2/3$, and $\mathbf{E}[L_0] = L$, $\mathbf{VAR}[L_0] = \gamma^2 L^2/3$.

The real motor parameters are chosen the same as in Table 4.1, except that $K_T = 0.006 + 0.0005 \sin(100t) \text{ N} \cdot \text{m} / \text{Wb} \cdot \text{A}$, $T_L = 0.2 \text{ N} \cdot \text{m}$. The input voltage U is a periodic step function to adjust the speed. The observer parameters are set as: $\theta = 2$, $\rho = 3000$, $\rho_1 = 7$, $\rho_2 = 3$, $K_{T0} = 0.003 \text{ N} \cdot \text{m} / \text{Wb} \cdot \text{A}$; $\hat{I}_f(0) = 3 \text{ A}$, $\hat{\omega}(0) = 20 \text{ rad/s}$.

The other parameters are given as: simulation cycles $N = 1000$; for each cycle, the duration time $t \in (0, 10)$ in *second*, and simulation step $T_s = 0.5 \text{ ms}$. The major simulation steps are given as:

- 1) Choose the tuning parameters α and γ ;
- 2) Use $rand(\cdot)$ to generate N pairs parameter¹, as

$$(R_0, L_0) = \{(R_0^1, L_0^1), (R_0^2, L_0^2), \dots, (R_0^{1000}, L_0^{1000})\}$$

- 3) Use each (R_0^i, L_0^i) as observer parameters in (4.32) and (4.33), i.e., $R_0 = R_0^i$, $L_0 = L_0^i$; do simulation and record the corresponding estimation error at

¹For $i = 1, \dots, N$; R_0^i and L_0^i are chosen independently.

each step, as

$$\begin{aligned} e_I^i(kT_s) &= \hat{I}_f(kT_s) - I_f(kT_s), \quad k = 1, 2, \dots, n; (n = \frac{10 \text{ sec}}{0.5 \text{ ms}} = 20000) \\ e_S^i(kT_s) &= \hat{\omega}(kT_s) - \omega(kT_s) \\ e_{KT}^i(kT_s) &= \hat{K}_T(kT_s) - K_T(kT_s) \end{aligned}$$

4) Computing the corresponding standard deviation of the estimation error via

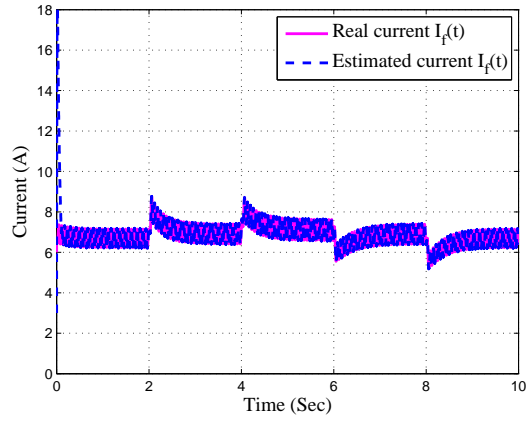
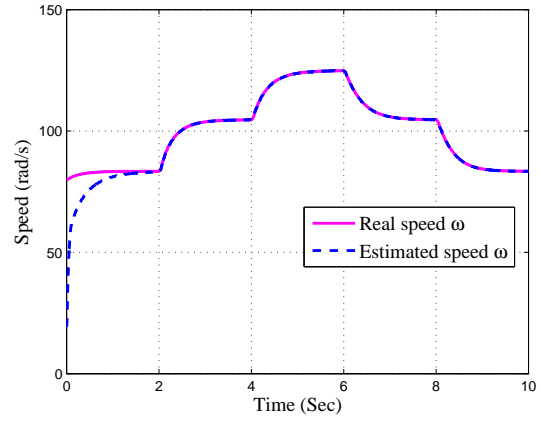
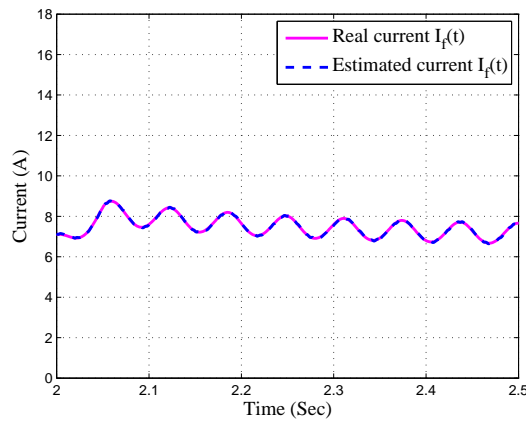
$$\begin{aligned} \sigma_I(kT_s) &= \sqrt{\sum_{i=1}^N (e_I^i(kT_s))^2 / N}, \quad k = 1, 2, \dots, n; \\ \sigma_S(kT_s) &= \sqrt{\sum_{i=1}^N (e_S^i(kT_s))^2 / N} \\ \sigma_{KT}(kT_s) &= \sqrt{\sum_{i=1}^N (e_{KT}^i(kT_s))^2 / N} \end{aligned}$$

5) Draw the figures and repeat all the above steps with different values of α and γ .

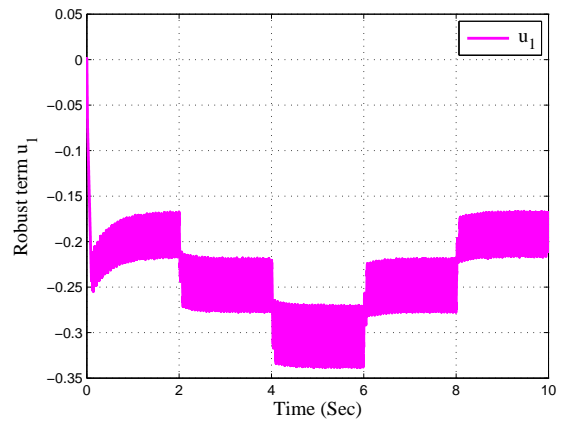
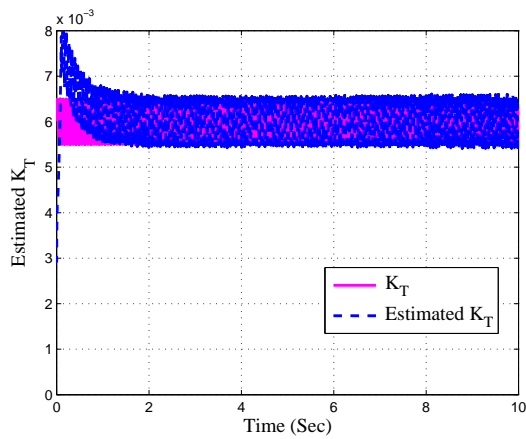
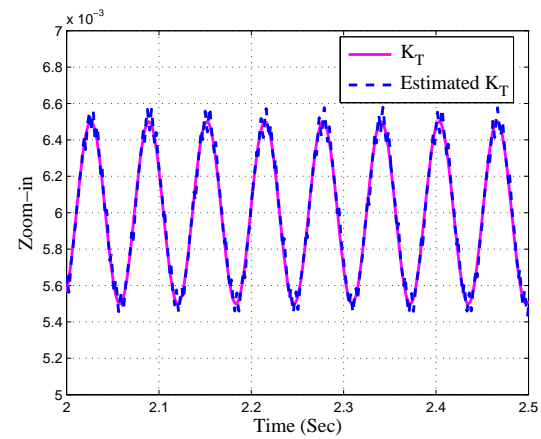
The simulation results are shown in Figure 4.10 - Figure 4.14. It seems the variation of inductance (null expected value) has no obvious effect on the estimation performance of parameter K_T and speed ω , see Figure 4.11 and Figure 4.12, Figure 4.13 and Figure 4.14. On the other hand, the variation of resistance (null expected value) has small effect on the estimation performance, as shown in Figure 4.13 and Figure 4.14. However, all the estimation error will converge into an invariant bounded zone after 2 *seconds*, demonstrating the robustness of estimate \hat{K}_T against R .

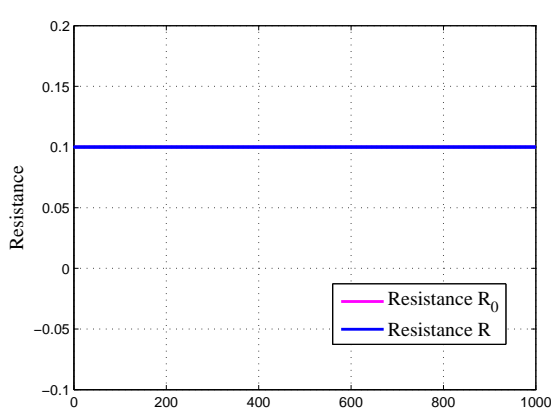
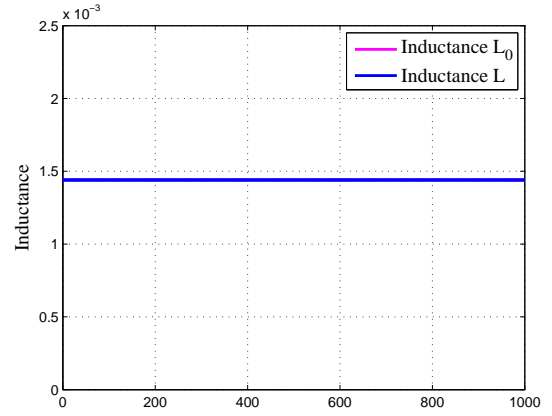
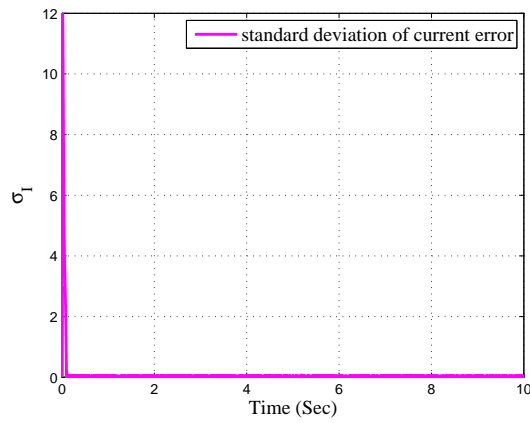
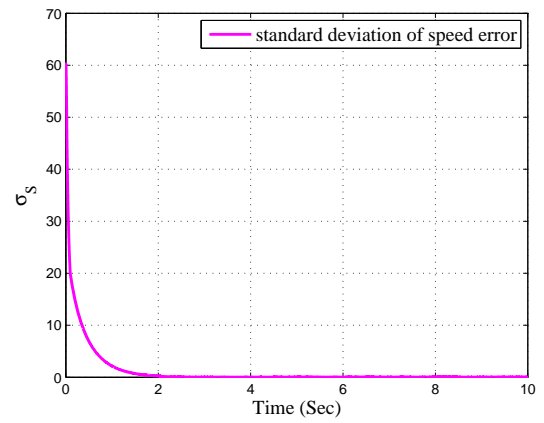
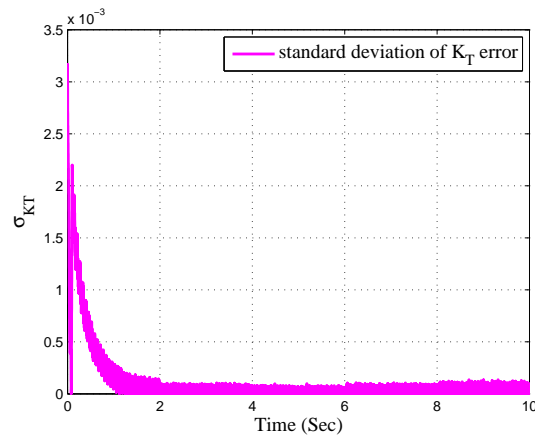
4.7 Summary

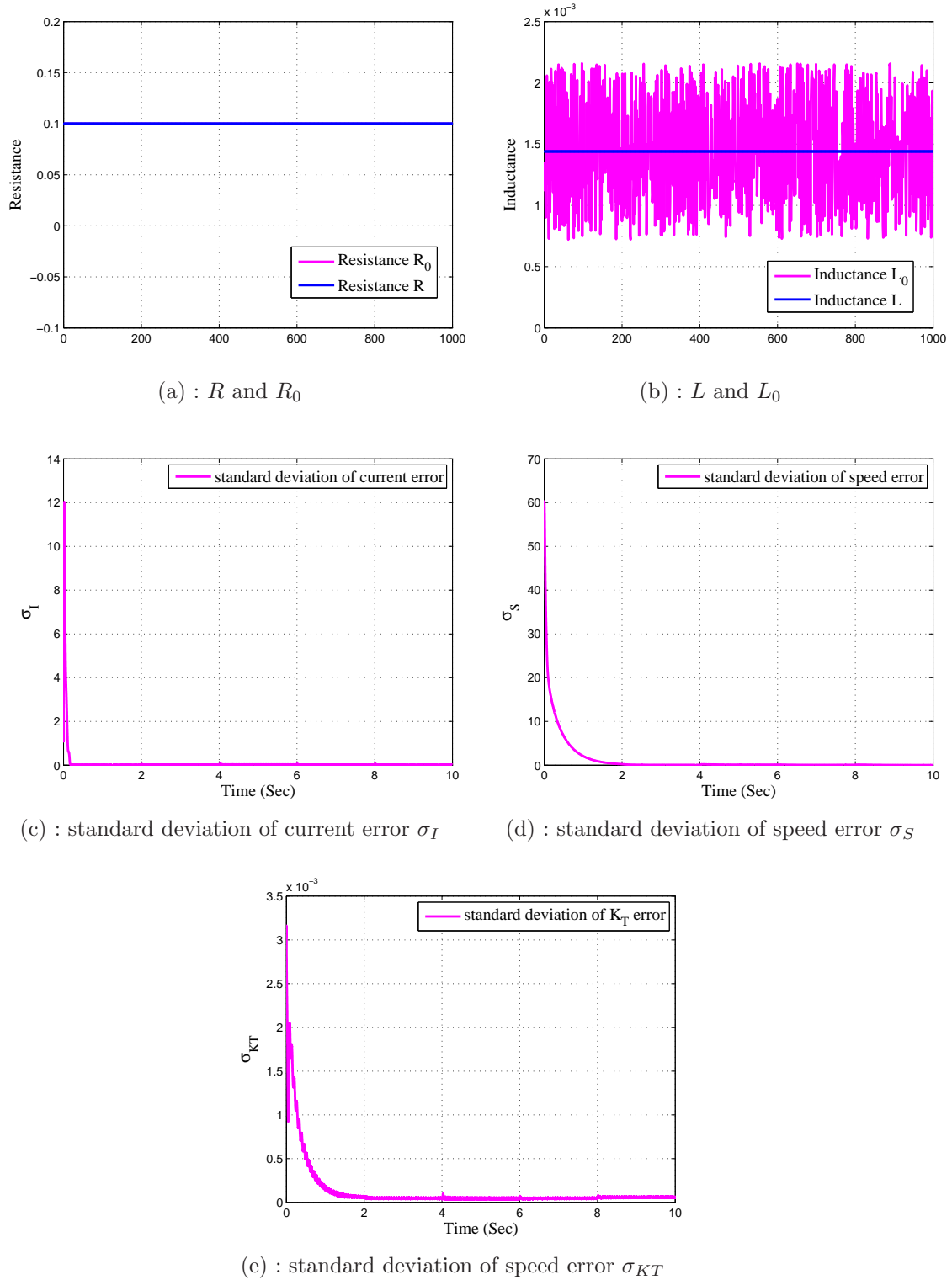
Based on the measurable current and input voltage, a robust hybrid observer has been developed for a series DC motor to identify the non-matching parameter without filtering effect. The asymptotic stability property is theoretically proved and also experimental verified. Then, based on the identified parameter, a second proposed observer is used to handle the unknown external disturbance. The expected estimation performance is demonstrated via experimental results.

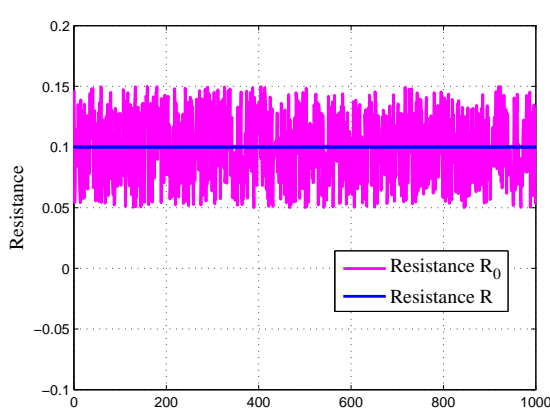
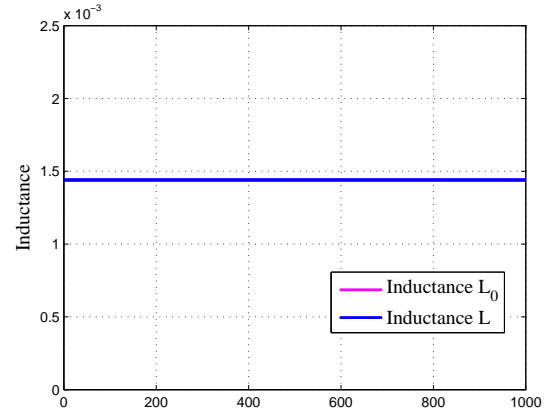
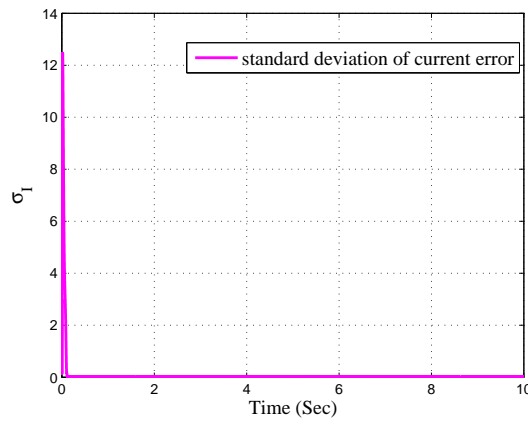
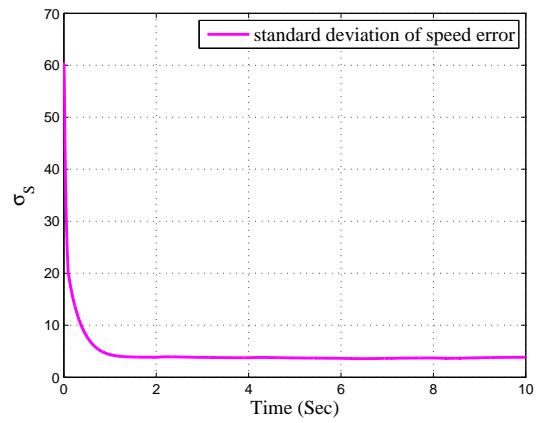
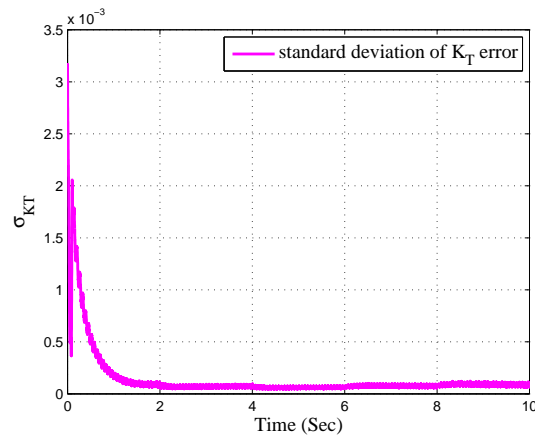
(a) : I_f and \hat{I}_f (b) : ω and $\hat{\omega}$ 

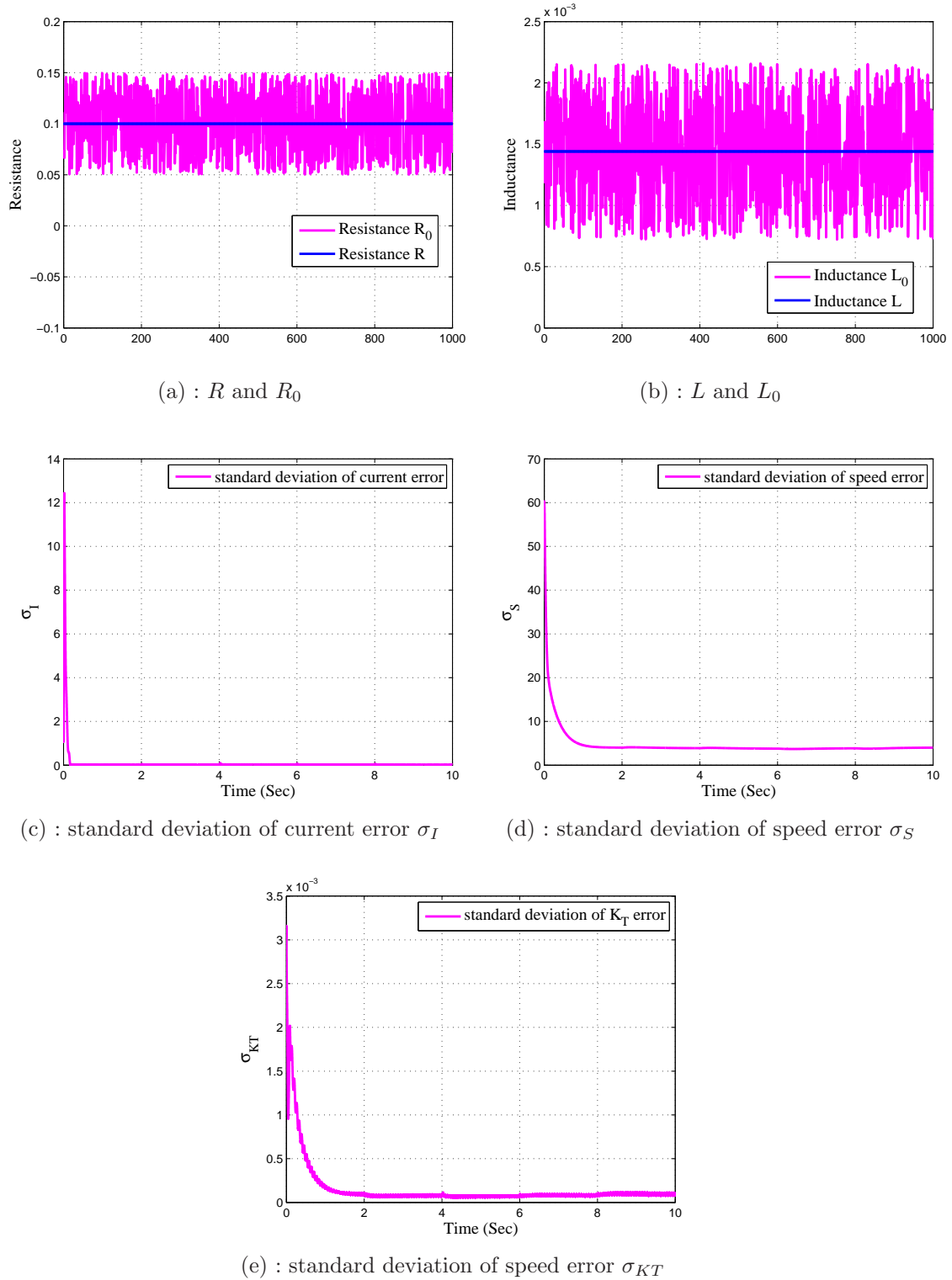
(c) : zoom-in of current estimation

(d) : sliding mode term u_1 (e) : K_T and \hat{K}_T (f) : zoom-in of K_T estimationFigure 4.10: Simulation results with $R_0 = R$, $L_0 = L$, $K_{T0} = 0.003$

(a) : R and R_0 (b) : L and L_0 (c) : standard deviation of current error σ_I (d) : standard deviation of speed error σ_S (e) : standard deviation of speed error σ_{KT} Figure 4.11: Monte carlo simulation results with $\alpha = 0$, $\gamma = 0$

Figure 4.12: Monte carlo simulation results with $\alpha = 0$, $\gamma = 0.5$

(a) : R and R_0 (b) : L and L_0 (c) : standard deviation of current error σ_I (d) : standard deviation of speed error σ_S (e) : standard deviation of speed error σ_{KT} Figure 4.13: Monte carlo simulation results with $\alpha = 0.5$, $\gamma = 0$

Figure 4.14: Monte carlo simulation results with $\alpha = 0.5$, $\gamma = 0.5$

Furthermore, a series of Monte carlo simulations are conducted to illuminate the parameter identification performance against the variations of resistance and inductance.

Chapter 5

A New Perspective on Speed Estimation in a PMSM

In this chapter, a novel perspective on the design of sliding mode observer to handle the speed sensorless estimation of a permanent magnet synchronous motor (PMSM) is presented. The novelty lies with extracting the position related dynamics from the back electromotive forces (EMFs) and modeling them as unknown system states, so that the unknown speed remains as the only system uncertainty that is non-matching in the observer sense. With a one time position calibration signal, the proposed observer ensures that the desirable position information can be exactly recovered without filtering effect.

The chapter is organized as follows: Section 5.1 introduces some existing researches on the speed estimator design of permanent magnet (PM) motors without mechanical sensor. Section 5.2 presents the mathematical model of a surface-mounted permanent magnet synchronous motor (SPMSM), and some existing techniques on sliding mode observer design are introduced. In section 5.3, a novel perspective on the sliding mode observer design for speed and position estimations is developed, and the stability of the estimation error dynamics is carefully addressed. In section 5.4, simulation results are presented to illuminate the estimation performance of the proposed observer. Section 5.5 concludes this chapter.

5.1 Introduction

Permanent magnet synchronous motors (PMSM) have been widely used as servo motors in precision motion control applications due to their high performance and high power density. In many high-performance PMSM drives, the field-oriented or vector control scheme which is based on the real-time information of rotor position/speed, is adopted. However, the presence of physical position sensors presents several problems, such as increasing the cost, shorten the lifetime, degrading the whole system's reliability and so on. Therefore, speed sensorless techniques remains an attractive research topic.

During the past several decades, extensive speed sensorless approaches have been reported and widely performed in many industrial applications, which can be grouped as [127,128]: fundamental model with measured currents and voltages based algebraic operation methods [129–131], back-EMF sensing based methods [132–134], and mathematical model based speed estimators [135–141].

On the other hand, the sliding mode observer has been proven to be one of the effective methods for the speed estimation in electrical drives [143,144], due to its insensitivity to parameter variations and capability of reconstructing the system uncertainties. In [123], based on the measurable currents and input voltages, a sliding mode observer with switching feedbacks has been applied for the speed estimation of a PM synchronous motor. Once the sliding modes happen, the back-EMFs signals can be reconstructed from the corresponding switching terms by low-pass digital filters, from which the unknown speed and position can be obtained. Similar work can be found in [119] where a first-order low pass filter was integrated into the sliding mode observer design. However, for such methods, the estimation performance suffers from the filtering effect, which causes a time delay and requires extra compensation for rotor position estimation. In order to eliminate the filtering effect, an adaptive sliding mode observer using a continuous function to replace the switching form was employed for the sensorless speed control of a PMSM [97], which results in a trade-off between the robustness of the observer and the estimation accuracy of the estimates.

In addition, the back-EMFs based sliding mode observers for speed and po-

sition estimations in electrical motors are mainly based on the one idea that, by considering the rotor position/speed related back-EMFs as matching uncertainties, the desirable position and position information can be extracted from the reconstructed back-EMFs on the sliding surfaces. It should be mentioned that the applications of such methods are strictly limited to the middle or high speed situations, and can hardly be applicable in low speed situations because of the small back-EMFs signals.

In this chapter, we shall propose a novel perspective on position and speed estimations of a surface-mounted permanent magnet synchronous motor (SPMSM), which is to consider the unknown position related dynamics as new system states, the chattering effect on the position estimation is then completely avoided. Furthermore, in order to improve the speed estimation against the filtering effect, a super-twisting sliding algorithm is applied. The observability of the both speed and position will be carefully addressed.

5.2 Preliminaries

5.2.1 Mathematical Model

For a surface-mounted PMSM, the mathematical model in the stator fixed-frame, i.e., the $(\alpha\text{-}\beta)$ frame, can be described as [107, 140]

$$\begin{aligned} \dot{i}_\alpha &= -\frac{R}{L}i_\alpha - \frac{1}{L}e_\alpha + \frac{u_\alpha}{L} \\ \dot{i}_\beta &= -\frac{R}{L}i_\beta - \frac{1}{L}e_\beta + \frac{u_\beta}{L} \end{aligned} \tag{5.1}$$

where

- i_α, i_β : The currents on α -axis and β -axis, respectively
- u_α, u_β : The voltages on α -axis and β -axis, respectively
- e_α, e_β : The back EMFs on α -axis and β -axis, respectively
- R : Stator resistance
- L : Stator inductance

The back electromotive forces can be represented in the form of

$$\begin{aligned} e_\alpha &= -\lambda_f \omega \sin \theta \\ e_\beta &= \lambda_f \omega \cos \theta \end{aligned} \quad (5.2)$$

where λ_f , ω and θ represent the magnetic flux parameter of the PM, the electric angular speed and the rotor angle position, respectively.

In a speed-variable control system with the field-oriented scheme, the rotor position related information, namely the matrix of the Park's transformation [140], is mandatory for the controller design, which is given as

$$\mathbf{T}(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \quad (5.3)$$

Therefore, the objective of the speed sensorless observer design is, based on the measurable currents and input voltages, to provide the estimations of rotor speed ω , as well as the position related functions $\sin \theta$ and $\cos \theta$.

Remark 5.1. *The currents and voltages in the stator fixed frame can be directly computed from the measurable three phase currents and voltages. For ease of analysis, in the following, the currents and voltages in the stator fixed framed are assumed to be known.*

5.2.2 A Conventional Sliding Mode Observer

In general, sliding mode observer design is only based on the dynamic system given by (5.1), to recover the back-EMF signals e_α and e_β . Then, the required information of speed and position can be extracted from the relationship in (5.2). In other words, by considering the EMFs as unknown dynamics in (5.1), a conventional sliding mode observer can be designed in the form of

$$\begin{aligned}\dot{\hat{i}}_\alpha &= -\frac{R}{L}\hat{i}_\alpha + \frac{u_\alpha}{L} + \frac{1}{L}H(\bar{i}_\alpha) \\ \dot{\hat{i}}_\beta &= -\frac{R}{L}\hat{i}_\beta + \frac{u_\beta}{L} - \frac{1}{L}H(\bar{i}_\beta)\end{aligned}\quad (5.4)$$

where $\bar{i}_\alpha = \hat{i}_\alpha - i_\alpha$, and $\bar{i}_\beta = \hat{i}_\beta - i_\beta$ represent the estimation errors of currents, with $\hat{\cdot}$ denoting the corresponding estimate.

$H(\bar{i}_\alpha)$ and $H(\bar{i}_\beta)$ are two properly designed sliding mode terms to ensure that the sliding surfaces $\bar{i}_\alpha = 0$ and $\bar{i}_\beta = 0$ are reached in finite time and remained thereafter. One common choice is given as [123]

$$\begin{aligned}H(\bar{i}_\alpha) &= -\rho_1 \text{sign}(\hat{i}_\alpha - i_\alpha) \\ H(\bar{i}_\beta) &= -\rho_2 \text{sign}(\hat{i}_\beta - i_\beta)\end{aligned}\quad (5.5)$$

with sliding gains chosen large enough, i.e., $\rho_1, \rho_2 > |e_{\alpha,\beta}|$.

Once the sliding modes happen, i.e., $\bar{i}_\alpha = 0$, $\bar{i}_\beta = 0$, based on the equivalent injection input concept of sliding mode [52], the EMFs can be recovered from (5.1) and (5.4), as

$$e_\alpha = -\{H(\bar{i}_\alpha)\}_{eq}; \quad e_\beta = -\{H(\bar{i}_\beta)\}_{eq} \quad (5.6)$$

where $\{\cdot\}_{eq}$ denotes the equivalent injection signal of the corresponding switching term, which can be obtained by a low-pass filter. Therefore, the rotor speed and position can be estimated via

$$\hat{\theta} \approx \tan^{-1} \left(\frac{\{H(\bar{i}_\alpha)\}_{eq}}{\{H(\bar{i}_\beta)\}_{eq}} \right); \quad \hat{\omega} = \dot{\hat{\theta}} \quad (5.7)$$

On the other hand, the desirable rotor speed and the rotor position related functions can also be recovered via

$$\sin \hat{\theta} = \{H(\bar{i}_\alpha)\}_{eq}/(\lambda_f \hat{\omega}), \quad \cos \hat{\theta} = -\{H(\bar{i}_\beta)\}_{eq}/(\lambda_f \hat{\omega}) \quad (5.8)$$

with

$$\hat{\omega} = \sqrt{\{H(\bar{i}_\alpha)\}_{eq}^2 + \{H(\bar{i}_\beta)\}_{eq}^2}/\lambda_f \quad (5.9)$$

Remark 5.2. *With the conventional sliding mode observer in the form of (5.4), the accuracy of speed and position estimations will be degraded by the filtering effect caused during the construction of back-EMF signals (5.6), as well as the computational errors in (5.7) - (5.9), especially at the low-speed situation.*

5.3 Observer Design and Stability

Since it is the information of $\sin \theta$ and $\cos \theta$, but not θ , that is required for the Park's transformation, we define a new system state vector as $\mathbf{x} = [x_1, x_2, x_3, x_4]^T = [i_\alpha, i_\beta, \sin \theta, \cos \theta]^T$. Then, the corresponding system dynamics can be obtained from (5.1) and (5.2), as

$$\begin{aligned} \dot{x}_1 &= -\frac{R}{L}x_1 + \frac{\lambda_f}{L}x_3\omega + \frac{1}{L}u_\alpha \\ \dot{x}_2 &= -\frac{R}{L}x_2 - \frac{\lambda_f}{L}x_4\omega + \frac{1}{L}u_\beta \\ \dot{x}_3 &= x_4\omega \\ \dot{x}_4 &= -x_3\omega \end{aligned} \quad (5.10)$$

with the states x_1 and x_2 being measurable. Note that we have relax the identity condition of $\sin^2 \theta + \cos^2 \theta = 1$.

Further, an implicit condition $\dot{\theta} = \omega$ is applied in the represented dynamic system (5.10), which results in the unknown speed ω being a non-matching in the

observer sense. Then, a sliding mode can be designed in the form of

$$\begin{aligned}
\dot{\hat{x}}_1 &= -\frac{R}{L}\hat{x}_1 + \frac{1}{L}u_\alpha + \frac{\lambda_f}{L}u_1 \\
\dot{\hat{x}}_2 &= -\frac{R}{L}\hat{x}_2 + \frac{1}{L}u_\beta + \frac{\lambda_f}{L}u_2 \\
\dot{\hat{x}}_3 &= -u_2 \\
\dot{\hat{x}}_4 &= -u_1
\end{aligned} \tag{5.11}$$

where $\hat{\mathbf{x}} = [\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4]^T$ denotes the estimate of \mathbf{x} ; u_1 and u_2 are the sliding mode terms, given by

$$\begin{aligned}
u_1 &= -k_1 \text{sign}(\hat{x}_1 - x_1) \\
u_2 &= -k_2 \text{sign}(\hat{x}_2 - x_2)
\end{aligned} \tag{5.12}$$

Theorem 5.1. *For the new dynamic system in (5.10), the proposed observer given by (5.11)-(5.12) ensures that the sliding surfaces $\hat{x}_1 - x_1 = 0$ and $\hat{x}_2 - x_2 = 0$ are reached in finite time and remained thereafter, provided that the sliding gains are chosen to satisfy $k_1, k_2 > |\omega|$.*

Proof. Let the estimation error $\mathbf{e} = [e_1, e_2, e_3, e_4]^T = [\hat{x}_1 - x_1, \hat{x}_2 - x_2, \hat{x}_3 - x_3, \hat{x}_4 - x_4]^T$, the dynamics of e_1 can be obtained from (5.10)-(5.12), as

$$\dot{e}_1 = [-Re_1 - \lambda k_1 \text{sign}(e_1) - \lambda x_3 \omega]/L \tag{5.13}$$

Then, define a Lyapunov function as $V = (e_1)^2/2$, it can be readily deduced that

$$\begin{aligned}
\dot{V} &= e_1 \dot{e}_1 \\
&= [-R(e_1)^2 - \lambda(k_1 - x_3 \omega \text{sign}(e_1))|e_1|]/L \\
&\leq [-R(e_1)^2 - \lambda(k_1 - |\omega|)|e_1|]/L \\
&< 0, \quad (\text{for } V \neq 0)
\end{aligned} \tag{5.14}$$

Therefore, the sliding surface $e_1 = \hat{x}_1 - x_1 = 0$ will be asymptotically reached and remained thereafter. Similarly, the sliding surface $e_2 = \hat{x}_2 - x_2 = 0$ can be

reached by choosing $k_2 > |\omega|$. ■

Theorem 5.2. *For the dynamic system in (5.10) with the stator resistance changing slowly, the proposed observer given by (5.11)-(5.12) ensures that the reduced-order estimation errors of x_3 and x_4 are globally bounded/stable; furthermore, there exists a time constant T_0 such that $\forall t \geq T_0$, it has*

$$\begin{aligned}\hat{x}_3(t) - x_3(t) &= \Delta_1 \\ \hat{x}_4(t) - x_4(t) &= \Delta_2\end{aligned}\tag{5.15}$$

with Δ_1 and Δ_2 being two constant values.

Proof. According to Theorem 5.1, the proposed observer ensure that the sliding surfaces are reached in a finite time, denoted as T_0 , and remained thereafter. In other words, $\forall t \geq T_0$, it has $\hat{x}_1 = x_1$, and $\hat{x}_2 = x_2$. Then, it can be deduced from (5.10) and (5.11) that

$$\begin{aligned}x_3\omega &= u_1 = -k_1\text{sign}(\hat{x}_1 - x_1) \\ -x_4\omega &= u_2 = -k_2\text{sign}(\hat{x}_2 - x_2)\end{aligned}\tag{5.16}$$

By substituting (5.16) into the dynamics of $e_3 = \hat{x}_3 - x_3$ and $e_4 = \hat{x}_4 - x_4$, it can be readily obtained that

$$\begin{aligned}\dot{e}_3 &= -u_2 - x_4\omega = 0 \\ \dot{e}_4 &= -u_1 + x_3\omega = 0\end{aligned}\tag{5.17}$$

The above two equations imply that, after the sliding surfaces are reached, i.e., $t \geq T_0$, the estimation errors of x_3 and x_4 are constrained into two constants. In other words, the two equalities in (5.15) are true. ■

Corollary 5.1. *On the sliding modes surfaces, the unknown rotor position related states x_3 and x_4 can be exactly estimated by the proposed observer in (5.11), provided that there exist two time instants $T_1 \geq T_0$ and $T_2 \geq T_0$ such that $x_3(t = T_1)$ and $x_4(t = T_2)$ are measurable.*

Proof. According to Theorem 5.2, Δ_1 and Δ_2 are time-independent constants, then they can be exactly estimated at the time instants T_1 and T_2 , respectively; i.e., $\Delta_1 = (\hat{x}_3 - x_3)|_{t=T_1}$ and $\Delta_2 = (\hat{x}_4 - x_4)|_{t=T_2}$. Therefore, $\forall t \geq \max\{T_1, T_2\}$, the states x_3 and x_4 can be exactly estimated by \hat{x}_3 and \hat{x}_4 based on the calibration of Δ_1 and Δ_2 , respectively. ■

Remark 5.3. *In real applications, the calibration time instants can be chosen while the phase back-EMFs are crossing zeros. For ease of analysis, we assume that $e_\alpha|_{t=T_1} = -\lambda_f \omega x_3|_{t=T_1} = 0$, and $e_\beta|_{t=T_2} = \lambda_f \omega x_4|_{t=T_2} = 0$. In other words, together with the back-EMF sensing techniques [132–134], the proposed observer given by (5.11)–(5.12) provides a novel and robust estimations of the rotor position related signals $\sin \theta$ and $\cos \theta$ for the Park's transformation (5.3), without filtering effect.*

Remark 5.4. *Form the observer design point of view, the proposed sliding mode observer given by (5.11) ensures that only partial states of the system (5.10) can be exactly identified on the sliding surfaces (see Theorem 5.1), and the remaining states and unknown speed failed to be observable (see Theorem 5.2). Compared with the conclusions in Chapter 3, this can be considered as a practical example of case 2, in which the reduced order dynamics is stable, but not asymptotically stable.*

For the unknown rotor speed ω estimation, it can be extracted from the identified rotor position related states x_3 and x_4 after $t \geq \max\{T_1, T_2\}$, as

$$\hat{\omega} = \tan^{-1} \left(\frac{\hat{x}_3}{\hat{x}_4} \right) \Big|_{t \geq \max\{T_1, T_2\}} \quad (5.18)$$

Besides, it can also be reconstructed from the sliding mode terms once the sliding surfaces are reached, i.e.,

$$\hat{\omega} = \sqrt{(\{u_1\}_{eq})^2 + (\{u_2\}_{eq})^2} \Big|_{t \geq T_0} \quad (5.19)$$

with $(\{u_1\}_{eq})$ and $(\{u_2\}_{eq})$ being the equivalent injection signals of u_1 and u_2 , respectively, which can be obtained via low-pass filters.

Note that the formula in (5.18) involves the division operator which may cause computational error when the denominator is close to zero. For the other formula in (5.19), the accuracy will be affected by the filtering effect because of the switching mechanism in u_1 and u_2 .

Remark 5.5. *In order to improve the speed estimation performance, the sliding mode terms u_1 and u_2 in the proposed observer given by (5.11) can be replaced with the super-twisting algorithm, which are given in the form of [53]*

$$\begin{aligned} u_1 &= -k_{11}|\hat{x}_1 - x_1|^{1/2} \text{sign}(\hat{x}_1 - x_1) + v_1 \\ \dot{v}_1 &= -k_{12} \text{sign}(\hat{x}_1 - x_1) \end{aligned} \quad (5.20)$$

and

$$\begin{aligned} u_2 &= -k_{21}|\hat{x}_2 - x_2|^{1/2} \text{sign}(\hat{x}_2 - x_2) + v_1 \\ \dot{v}_2 &= -k_{22} \text{sign}(\hat{x}_2 - x_2) \end{aligned} \quad (5.21)$$

with k_{11} , k_{12} , k_{13} and k_{14} being the properly chosen positive sliding gains.

It has been proven by geometrical methods [53], or by means of the Homogeneity properties of the algorithm [71], that the super-twisting algorithm given by (5.20) and (5.21) will ensure the sliding surfaces $\dot{e}_1 = e_1 = 0$ and $\dot{e}_2 = e_2 = 0$ are reached in a finite time and remained thereafter. Then, the speed estimation formula in (5.19) can be updated as

$$\hat{\omega} = \sqrt{(u_1)^2 + (u_2)^2} |_{t \geq T_0} \quad (5.22)$$

Thus, the filtering effect on the speed estimation can be completely avoided.

5.4 Simulation Results

For the simulation purpose, the motor parameters in (5.10) are chosen as [107]: $R = 0.3 \text{ ohm}$, $L = 3.366 \text{ mH}$, $\lambda_f = 0.0776 \text{ Wb}$, and the real rotor speed is assumed as $\omega = 3 \times (1 + 0.6 \sin t)$, with the initial values of the motor as: $I_\alpha(0) =$

$$I_\beta(0) = 10 \text{ A}, \theta(0) = 0.2\pi.$$

The following simulations are cataloged in four parts: First, the conventional sliding mode observer given by (5.4)-(5.5) is applied to estimate the position related functions and speed via (5.8) and (5.9), respectively. Second, the proposed sliding mode observer given by (5.11)-(5.12) is performed for the position related functions and speed estimations. Third, based on Remark 5.5, the proposed observer (5.11) appended with (5.20)-(5.21) is used for speed estimation, as well as the position related functions. Last, in order to study the robustness of proposed observers, similar simulations are repeated with 30 dB current measurement noise.

5.4.1 With Conventional Sliding Mode Observer

For the purpose of comparison, the conventional sliding mode observer given by (5.4)-(5.5) is applied. The observer parameters are given as: $\rho_1 = \rho_2 = 0.7$, $\hat{I}_\alpha(0) = \hat{I}_\beta(0) = 0$, $\hat{\omega}(0) = 0$, $\hat{\theta}(0) = 0$; the simulation step is set to 0.5 ms.

The simulation results are shown in Figure 5.1, in which two low-pass filters are used for the reconstruction of back-EMFs, so that the unknown speed can be recovered. However, it causes an undesirable time delay for the speed estimation, as shown in Figure 5.1e. Similar issues are also involved in the reconstruction of the rotor position related functions, see Figure 5.1c and Figure 5.1d. Note that there is no calibration signal required for the conventional sliding mode observer.

5.4.2 With Proposed Sliding Mode Observer 1

Here, the proposed observer described by (5.11)-(5.12) is employed, with observer parameters $k_1 = k_2 = 7$. The other simulation parameters are chosen the same as in the above subsection.

According to Corollary 5.1, we assume that, at the time instant $T_1 = T_2 = 6 \text{ seconds}$, the back-EMFs signals can be exactly measured for calibration purpose. The simulation results are shown in Figure 5.2. It can be found in Figure 5.2c and Figure 5.2d that the desirable rotor position related signals can be exactly estimated without filtering effect after the calibration. However, there still exists

small ripples on the recovered speed signal (via the formula of (5.19)) because of the requirement of low-pass filters, as shown in Figure 5.2e.

5.4.3 With Proposed Sliding Mode Observer 2

Now, we shall consider the estimation performance with the proposed observer of (5.11), appended with the sliding mode terms given by (5.20) and (5.21). The simulation parameters are chosen the same as above, except that the sliding gains are given as $k_{11} = k_{21} = 1$, $k_{12} = k_{22} = 20$.

The improved performance can be clearly seen in Figure 5.3, and the visible ripples disappear in speed estimation, as shown in Figure 5.3e.

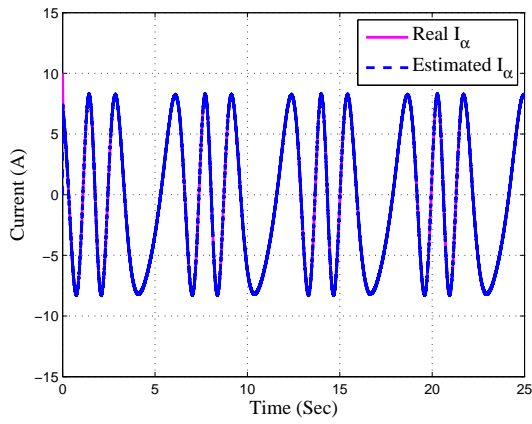
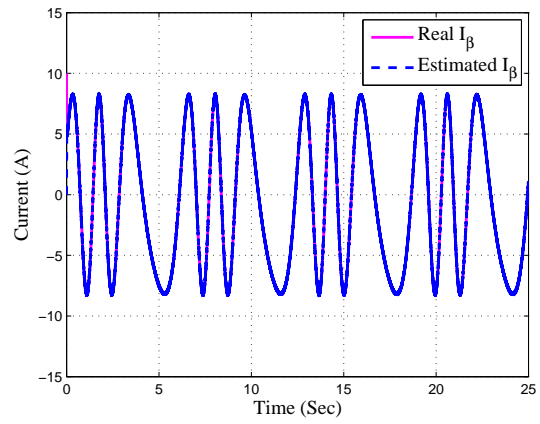
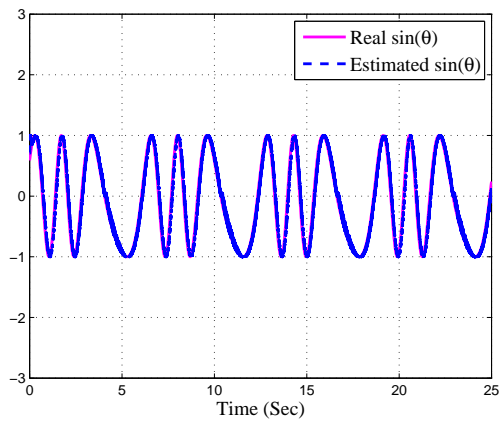
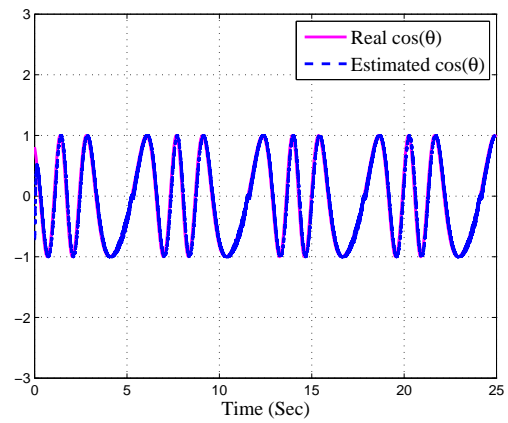
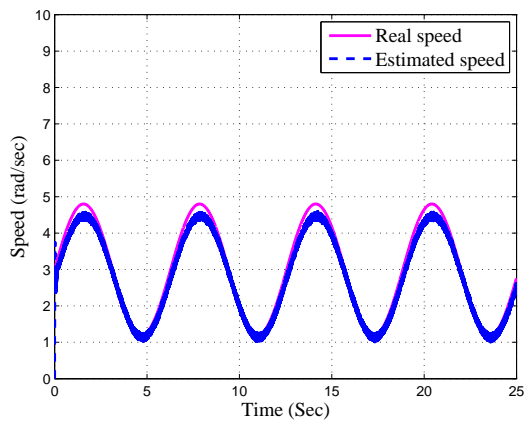
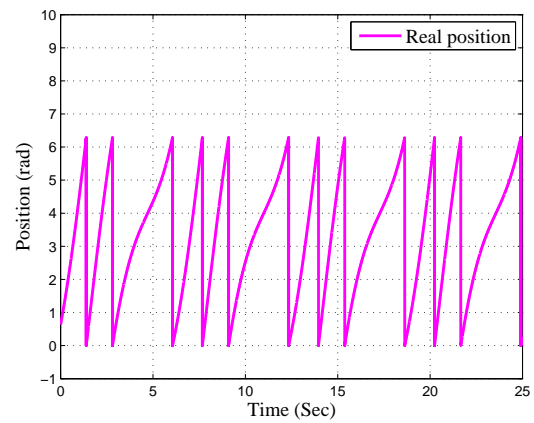
5.4.4 Robustness Study with Measurement Noise

In order to study the estimation performance in case of measurement noise, the above simulations are re-conducted by adding $30dB$ white-noise on the measurement signals, with functions $awgn(I_\alpha, 30, '')$ and $awgn(I_\beta, 30, '')$ in Matlab, see Figure 5.4a, Figure 5.5a and Figure 5.6a.

The simulation parameters are chosen the same as above, except that a simple digital filter is applied on the currents to handle the measurement noise, i.e., $I_\alpha(k) = \sum_{i=0}^9 I_\alpha(k-i)$ and $I_\beta(k) = \sum_{i=0}^9 I_\beta(k-i)$. Then, the simulation results are shown in Figure 5.4, Figure 5.5 and Figure 5.6. Clearly, the proposed observer 2 provides a better estimation performance on the speed and rotor position related functions than both the conventional one and the proposed observer 1.

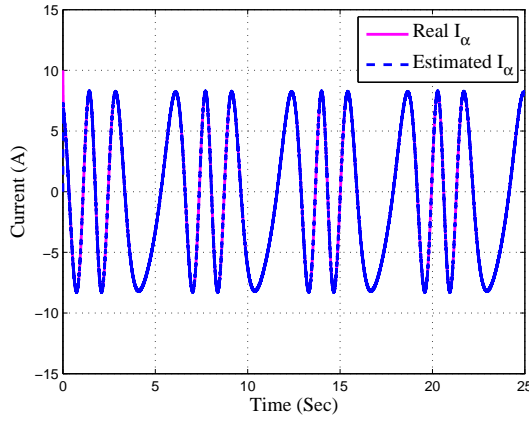
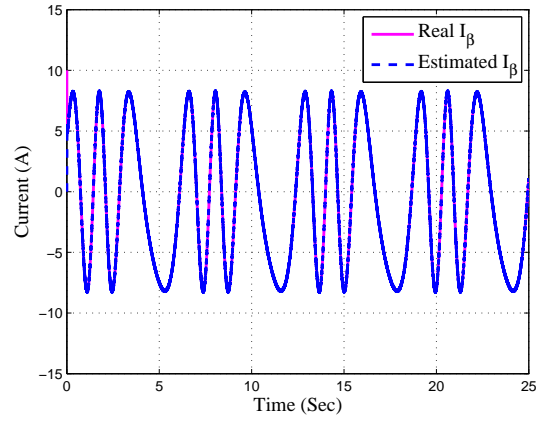
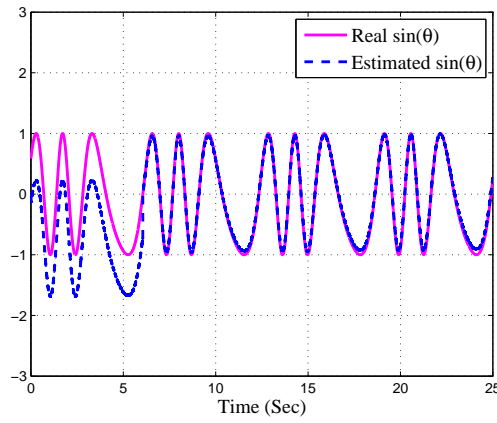
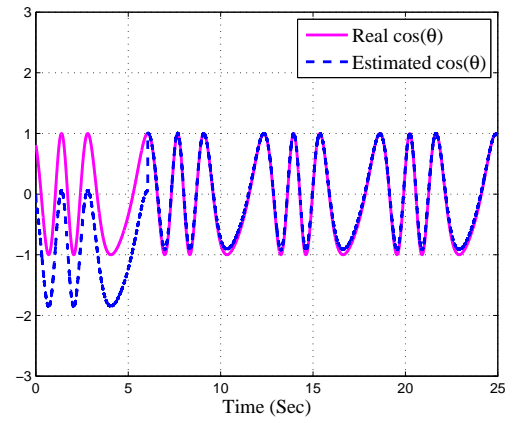
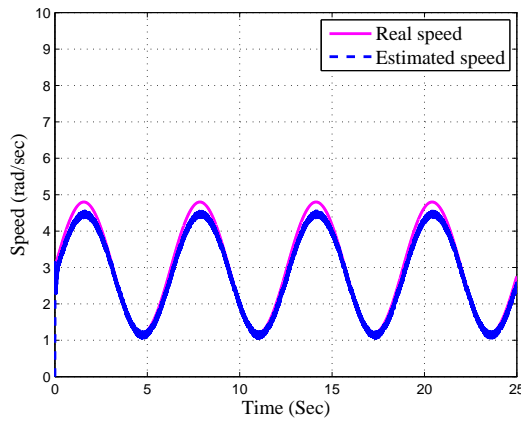
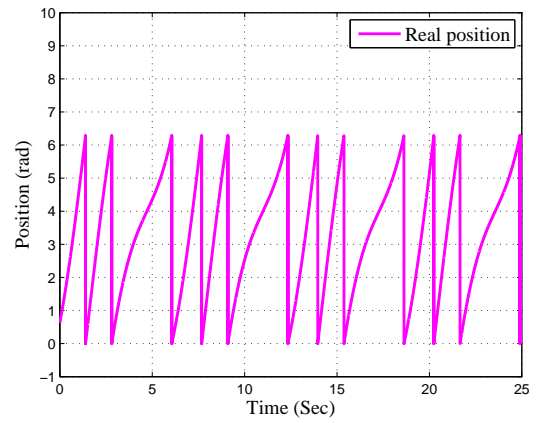
5.5 Summary

Based on the idea that the chattering/filtering phenomenon of the sliding mode observer affects only the reconstructed uncertainties, but not the unknown system states, a novel perspective on the estimation of rotor position related signals in a surface-mounted PMSM is presented. With a one time calibration of the estimated position information, the real-time position related Park's transformation matrix can be exactly identified without filtering effect. Moreover, the rotor speed

(a) : I_α and \hat{I}_α (b) : I_β and \hat{I}_β (c) : $\sin \theta$ and its estimate(d) : $\cos \theta$ and its estimate(e) : ω and $\hat{\omega}$ 

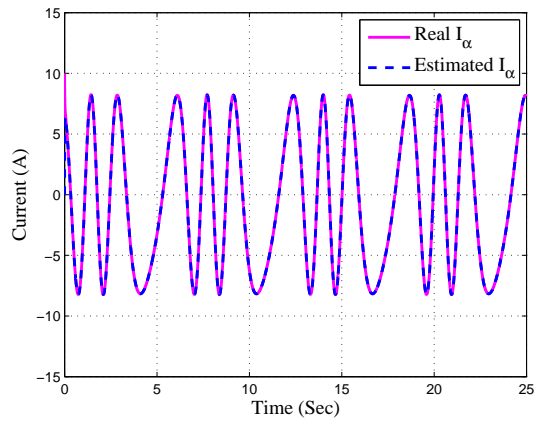
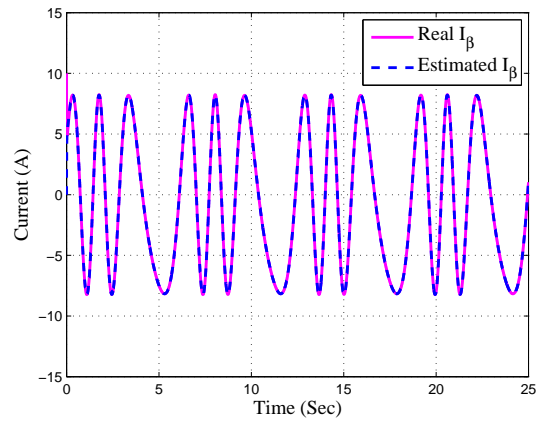
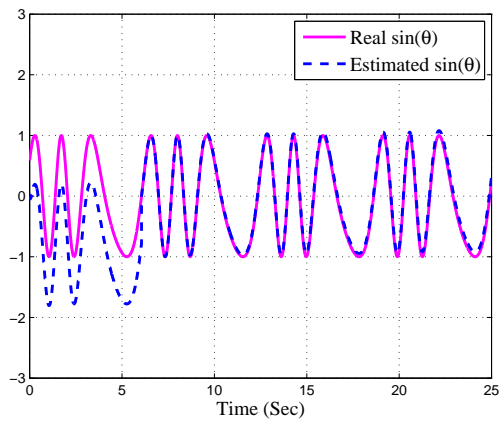
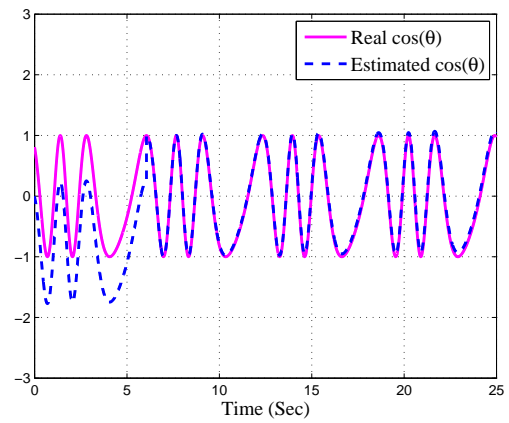
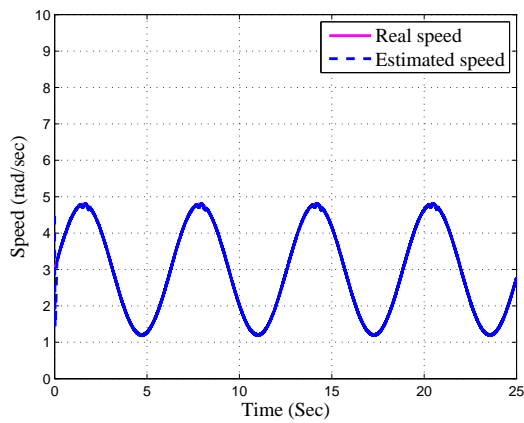
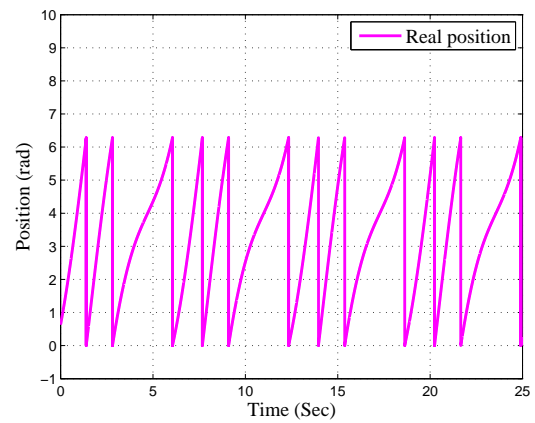
(f) : real position signal

Figure 5.1: The estimation performance with the conventional SMO

(a) : I_α and \hat{I}_α (b) : I_β and \hat{I}_β (c) : $\sin \theta$ and its estimate(d) : $\cos \theta$ and its estimate(e) : ω and $\hat{\omega}$ 

(f) : real position signal

Figure 5.2: The estimation performance with the proposed SMO1

(a) : I_α and \hat{I}_α (b) : I_β and \hat{I}_β (c) : $\sin \theta$ and its estimate(d) : $\cos \theta$ and its estimate(e) : ω and $\hat{\omega}$ 

(f) : real position signal

Figure 5.3: The estimation performance with the proposed SMO2

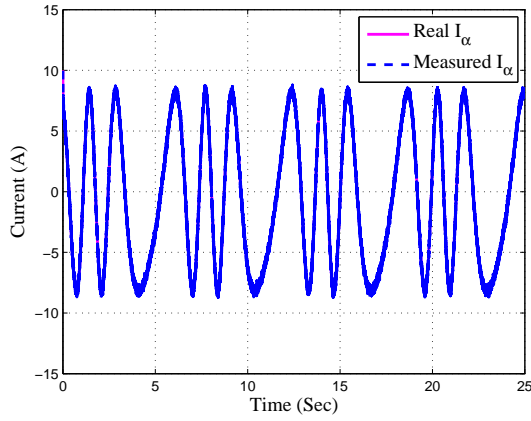
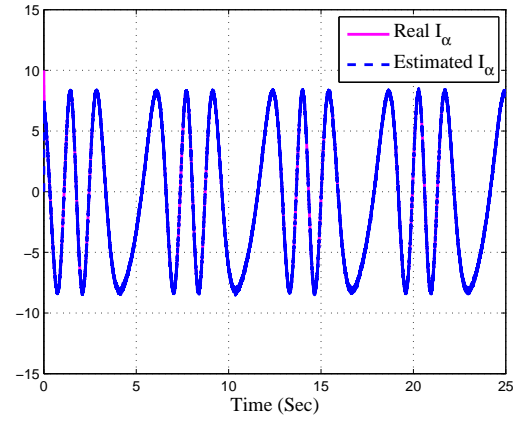
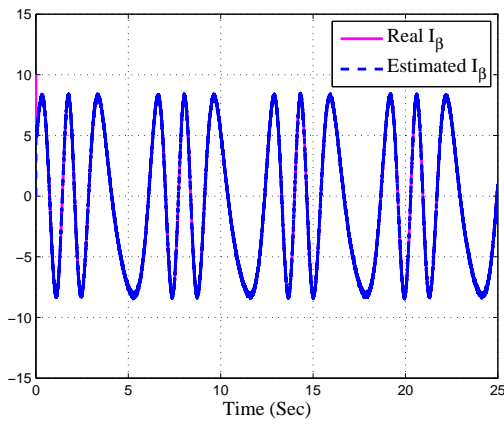
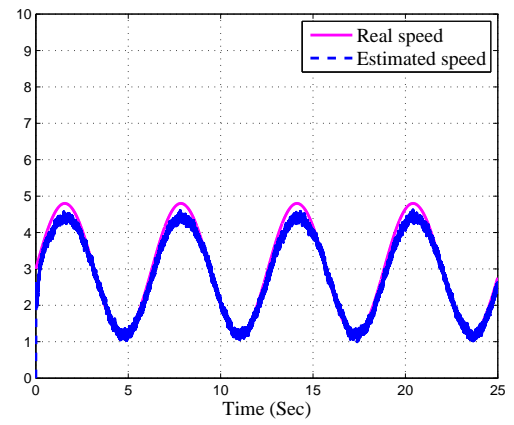
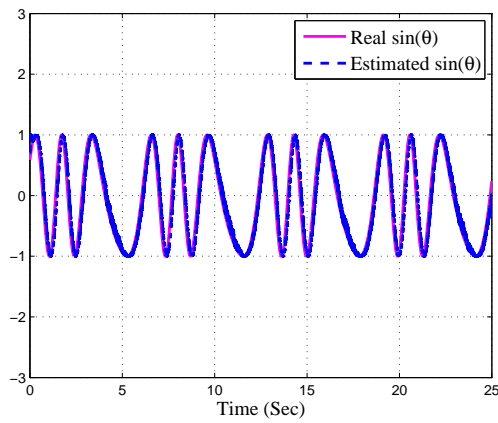
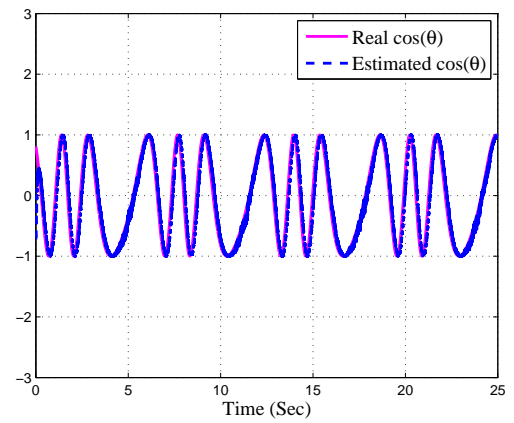
(a) : real I_α and measured \hat{I}_α (b) : real I_α and estimated \hat{I}_α (c) : real I_β and estimated \hat{I}_β (d) : ω and $\hat{\omega}$ (e) : $\sin \theta$ and its estimate(f) : $\cos \theta$ and its estimate

Figure 5.4: The conventional SMO results with measurement noise

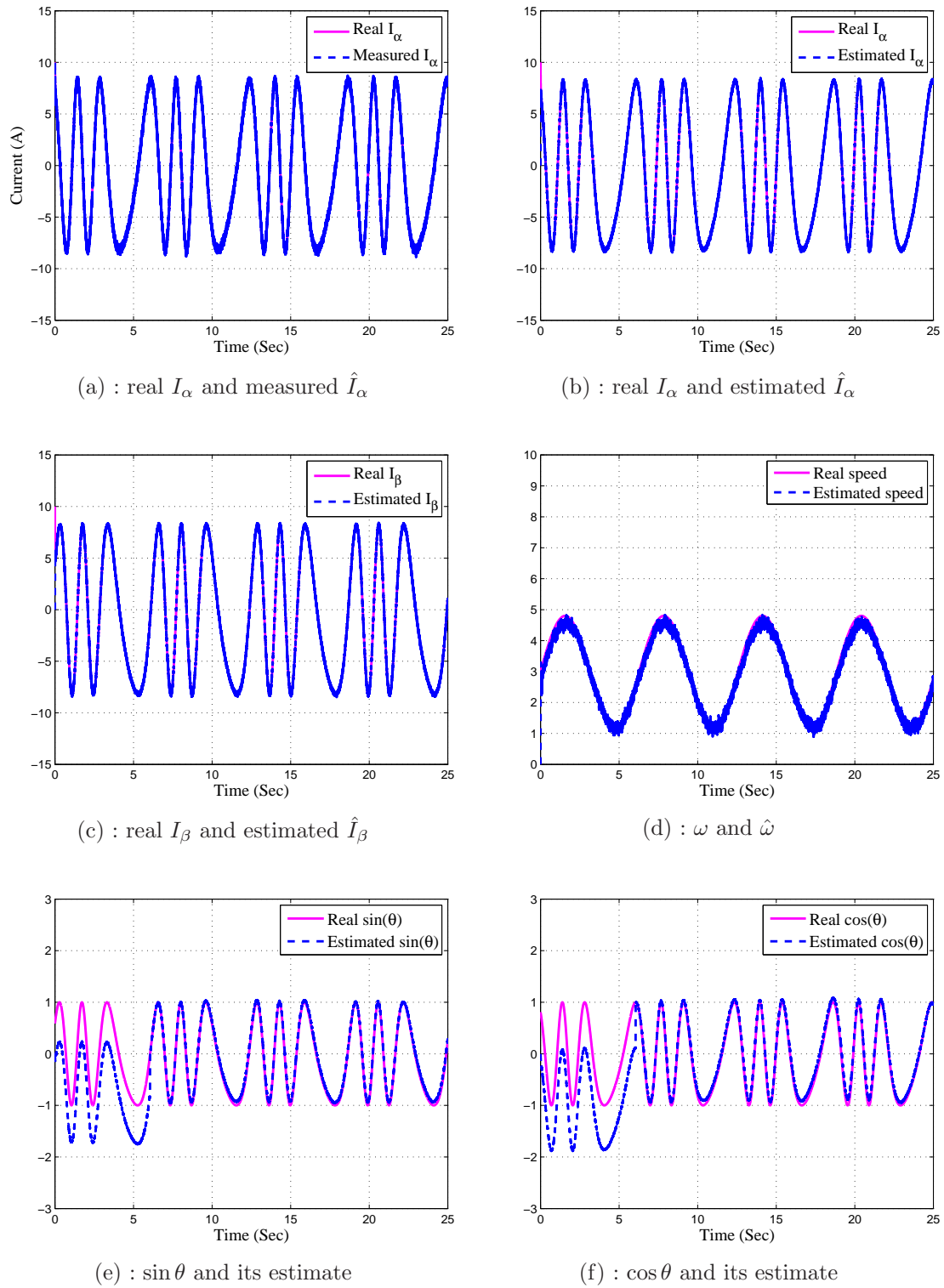


Figure 5.5: The proposed SMO1 results with measurement noise

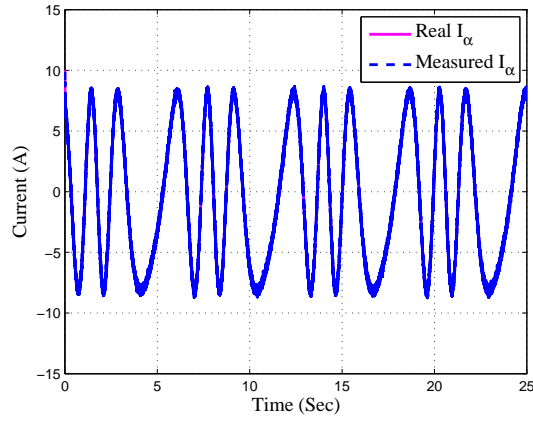
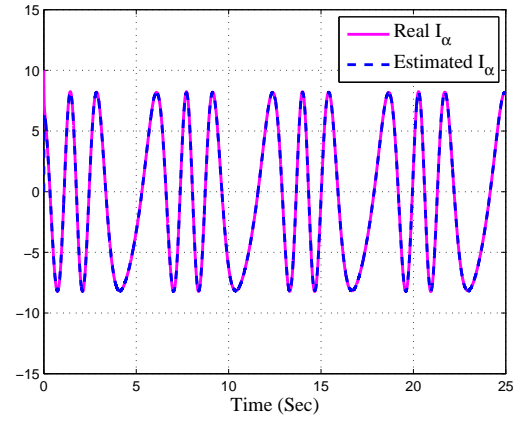
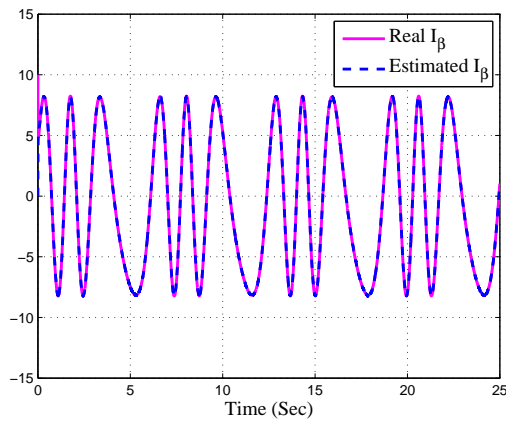
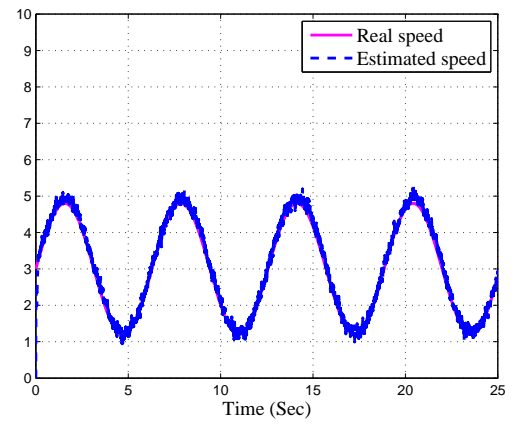
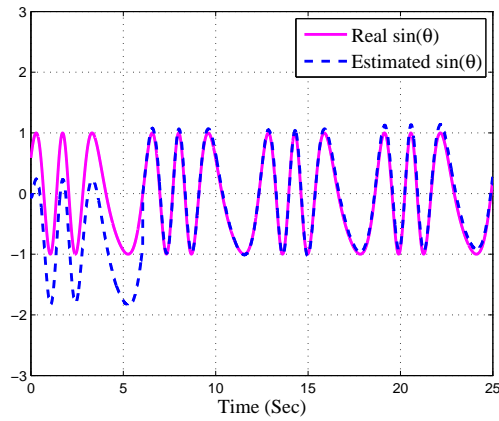
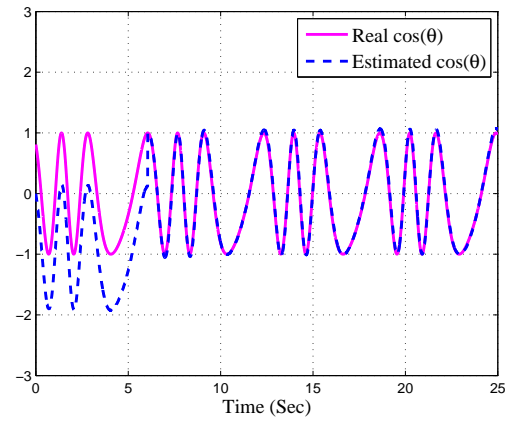
(a) : real I_α and measured \hat{I}_α (b) : real I_α and estimated \hat{I}_α (c) : real I_β and estimated \hat{I}_β (d) : ω and $\hat{\omega}$ (e) : $\sin \theta$ and its estimate(f) : $\cos \theta$ and its estimate

Figure 5.6: The proposed SMO2 results with measurement noise

can be estimated without filtering effect with the application of the super-twisting algorithm.

Chapter 6

State and Unknown Input Estimations in MIMO Systems

In the 3 preceding chapters, we have studied the design of hybrid nonlinear observers that combine full-order high gain feedback with higher-order sliding mode feedback, and demonstrated their different characteristics with practical examples of DC and PM motors, respectively. We showed that the stability of the reduced-order dynamics is completely independent of the observer gains, and is only related to the original system structure.

In this chapter, we shall look into the state estimation of a class of multi-input-multi-output (MIMO) nonlinear systems where the number of output measurements is more than the number of unknown inputs. Then, with less restrictive system structure assumptions, we are able to ensure asymptotic stability of the reduced order dynamics. For the systems under consideration, the unknown inputs still enter the systems via unknown-state dependent matrices. From the observer design point of view, such unknown inputs are termed as non-matching in the observer sense and the systems may not satisfy the strict involutive condition. For this class of systems, we shall show that the identifiability of both the unknown inputs and states can be guaranteed. A numerical example is given to illuminate the effectiveness of the proposed observer.

The organization of this chapter is as follows: Section 6.1 introduces some existing techniques on the estimation problem of MIMO systems. In section 6.2,

the system description is presented, as well as some nonlinear transformation results. In section 6.3, a robust hybrid observer which consists of high order sliding mode terms and a reduced-order high gain feedback is proposed, and the observability of the unknown inputs and states is carefully addressed. In section 6.4, a six-order numerical example is used to demonstrate the proposed observer design. Section 6.5 concludes this chapter.

6.1 Introduction

In general, sliding mode observer design for state and unknown input estimations of uncertain MIMO nonlinear systems is often based on the requirement of involutive condition [105], with which the original uncertain system can be decomposed into two subsystems: the first one, which is constructed via *Lie* derivatives of the system outputs, has a differential structure with the unknown inputs appearing in the corresponding last dynamic equation, and the other subsystem has a nominal dynamics. However, the observability of the unknown inputs can be ensured only when the nominal subsystem is self asymptotically stable on the sliding surfaces, which is difficult to verify [78]. It also has been mentioned in [62] that the remaining system states will fail to be identified if the reduced-order nominal subsystem is unstable.

In addition, another critical obstacle for unknown input estimations of a MIMO system lies with that the involutive condition only guarantees the existence of the nominal subsystem, but no explicit formulation has been proposed on how to construct it. Therefore, the stability of the nominal subsystem is difficult to check, and the identifiability of the unknown inputs and states of the original system cannot be guaranteed.

In this chapter, we shall consider a class of uncertain MIMO nonlinear systems, in which the unknown inputs are non-matching in the observer sense, as defined in Definition 3.1. With the assumption that the number of measurement outputs is more than the number of unknown inputs, a hybrid observer that combines higher order sliding mode feedbacks with a reduced-order high gain feedback is proposed,

in which the high order sliding mode feedbacks work to ensure the corresponding sliding mode surfaces are reached individually and remained thereafter, and the high gain feedback designed based on an extra output guarantees the asymptotic stability of the remaining estimation dynamics on the sliding surfaces. The identifiability of the unknown inputs and states can be guaranteed based on the idea that the unknown inputs can be replaced by some nominal dynamics when the corresponding sliding mode happens.

6.2 Preliminaries

6.2.1 System Dynamics

Consider the class of locally stable multi-input-multi-output systems that is described as

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\varphi(t) \\ \mathbf{y} &= \mathbf{h}(\mathbf{x})\end{aligned}\tag{6.1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^{m+1}$, $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^n$, $\mathbf{h}(\mathbf{x}) = [h_1(\mathbf{x}), \dots, h_{m+1}(\mathbf{x})]^T \in \mathbb{R}^{m+1}$, $\mathbf{G}(\mathbf{x}) = [\mathbf{g}_1(\mathbf{x}), \dots, \mathbf{g}_m(\mathbf{x})] \in \mathbb{R}^{n \times m}$, $\varphi(t) = [\varphi_1(t), \dots, \varphi_m(t)]^T \in \mathbb{R}^m$, and $\mathbf{g}_i(\mathbf{x}) \in \mathbb{R}^n$, $i = 1, \dots, m < n$ are smooth vector and matrix functions defined on an open set $\Omega \subset \mathbb{R}^n$. $\varphi(t)$ is the unknown input vector, which is required to be identified, together with the state vector \mathbf{x} .

Assumption 6.1. *The first m outputs have a vector relative degree $\{r_1, \dots, r_m\}$ corresponding to $\mathbf{G}(\mathbf{x})$ at each point $\mathbf{x} \in \Omega$, i.e.,*

$$\begin{aligned}L_{g_j} L_f^k h_i(\mathbf{x}) &= 0, \quad \forall j = 1, \dots, m, \quad \forall k < r_i - 1, \quad \forall i = 1, \dots, m \\ L_{g_j} L_f^{r_i-1} h_i(\mathbf{x}) &\neq 0 \quad \text{for at least one } 1 \leq j \leq m\end{aligned}$$

where Lie derivative is defined as $L_f h_i(\mathbf{x}) = [\partial h_i(\mathbf{x}) / \partial \mathbf{x}] \mathbf{f}$. Furthermore, the fol-

lowing $m \times m$ matrix

$$\mathbf{E}(\mathbf{x}) = \begin{pmatrix} L_{g_1} L_f^{r_1-1} h_1(\mathbf{x}) & L_{g_2} L_f^{r_1-1} h_1(\mathbf{x}) & \dots & L_{g_m} L_f^{r_1-1} h_1(\mathbf{x}) \\ L_{g_1} L_f^{r_2-1} h_2(\mathbf{x}) & L_{g_2} L_f^{r_2-1} h_2(\mathbf{x}) & \dots & L_{g_m} L_f^{r_2-1} h_2(\mathbf{x}) \\ \vdots & \vdots & \vdots & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(\mathbf{x}) & L_{g_2} L_f^{r_m-1} h_m(\mathbf{x}) & \dots & L_{g_m} L_f^{r_m-1} h_m(\mathbf{x}) \end{pmatrix}$$

is non-singular at each point $\mathbf{x} \in \Omega$.

Assumption 6.2. The extra output $h_{m+1}(\mathbf{x})$ is said to have a relative degree r_{m+1} with respect to the unknown inputs at each point $\mathbf{x} \in \Omega$. i.e.,

$$\begin{aligned} L_{g_i} L_f^k h_{m+1}(\mathbf{x}) &= 0, \quad \forall j = 1, \dots, m, \quad \forall k < r_{m+1} - 1, \\ L_{g_i} L_f^{r_{m+1}-1} h_{m+1}(\mathbf{x}) &\neq 0 \quad \text{for at least one } 1 \leq j \leq m \end{aligned}$$

with the total relative degree $r_{to} = \sum_{i=1}^{m+1} r_i = r_{sm} + r_{m+1} \leq n$, where $r_{sm} = \sum_{i=1}^m r_i$.

Note that Assumption 6.1 on the definition of vector relative degree is firstly proposed in [105] for a square system, and similar definitions can also be found in [62, 78]. The non-singular matrix $\mathbf{E}(\mathbf{x})$ is required to ensure that all the unknown inputs can be reconstructed individually after all the states have converged to their true values. Assumption 6.2 is a new definition of the relative degree between the extra output $h_{m+1}(\mathbf{x})$ and the m unknown inputs $\varphi_i(t)$, $i = 1, \dots, m$.

6.2.2 Coordinate Transformation

Based on the system given by (6.1), a new basis chosen based on *Lie* derivatives is given as follows:

$$\begin{aligned}
 [\xi^T, \eta^T]^T &\in \mathbb{R}^n : \quad i = 1, \dots, m \\
 \xi &= \begin{pmatrix} \xi^1 \\ \xi^2 \\ \vdots \\ \xi^m \end{pmatrix} \in \mathbb{R}^{r_{sm}}, \quad \xi^i = \begin{pmatrix} \xi_1^i \\ \xi_2^i \\ \vdots \\ \xi_{r_i}^i \end{pmatrix} = \begin{pmatrix} \phi_1^i(\mathbf{x}) \\ \phi_2^i(\mathbf{x}) \\ \vdots \\ \phi_{r_i}^i(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} h_i(\mathbf{x}) \\ L_{\mathbf{f}} h_i(\mathbf{x}) \\ \vdots \\ L_{\mathbf{f}}^{r_i-1} h_i(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^{r_i} \\
 \eta &= \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-r_{sm}} \end{pmatrix} = \begin{pmatrix} \phi_{r_{sm}+1}(\mathbf{x}) \\ \phi_{r_{sm}+2}(\mathbf{x}) \\ \vdots \\ \phi_n(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} h_{m+1}(\mathbf{x}) \\ L_{\mathbf{f}} h_{m+1}(\mathbf{x}) \\ \vdots \\ L_{\mathbf{f}}^{n-r_{sm}-1} h_{m+1}(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^{n-r_{sm}}
 \end{aligned} \tag{6.2}$$

Assumption 6.3. *The mapping function $\Phi(\mathbf{x})$ defined below is a local diffeomorphism $\forall \mathbf{x} \in \Omega$, which means $\mathbf{x} = \Phi^{-1}(\xi, \eta)$,*

$$\begin{aligned}
 \Phi(\mathbf{x}) &= [\phi_1^1(\mathbf{x}), \dots, \phi_{r_1}^1(\mathbf{x}), \phi_1^2(\mathbf{x}), \dots, \phi_{r_2}^2(\mathbf{x}), \dots, \\
 &\quad \phi_1^m(\mathbf{x}), \dots, \phi_{r_m}^m(\mathbf{x}), \phi_{r_{sm}+1}(\mathbf{x}), \dots, \phi_n(\mathbf{x})]^T \in \mathbb{R}^n
 \end{aligned}$$

$$i.e., \quad \Phi(\xi, \eta) = [\xi_1^1, \dots, \xi_{r_1}^1, \xi_1^2, \dots, \xi_{r_2}^2, \dots, \xi_1^m, \dots, \xi_{r_m}^m, \eta_1, \dots, \eta_{n-r_{sm}}]^T \in \mathbb{R}^n$$

It can be seen that the mapping function $\Phi(\xi, \eta)$ is strictly defined based on the *Lie* derivatives of the measurement outputs. Assumption 6.3 implies that the state \mathbf{x} can be recovered by means of inverse mapping, so the observer design of the original system (6.1) can be discussed in the new transformed domain.

Suppose that the original system given by (6.1) satisfies Assumptions 6.1-6.3, then it can be represented in the form of

$$\begin{aligned}
 \dot{\xi}^i &= \Lambda_i \xi^i + \Psi^i(\xi, \eta) + \lambda^i(\xi, \eta, \varphi(t)), \quad \forall i = 1, \dots, m \\
 \dot{\eta} &= \mathbf{A}\eta + \alpha(\xi, \eta) + \mathbf{P}(\xi, \eta)\varphi(t)
 \end{aligned} \tag{6.3}$$

where

$$\Lambda_i = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in \mathfrak{R}^{r_i \times r_i}$$

$$\Psi^i(\xi, \eta) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ L_{\mathbf{f}}^{r_i} h_i(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ L_{\mathbf{f}}^{r_i} h_i(\Phi^{-1}(\xi, \eta)) \end{pmatrix} \in \mathfrak{R}^{r_i}$$

$$\lambda^i(\xi, \eta, \varphi(t)) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{j=1}^m L_{\mathbf{g}_j} L_{\mathbf{f}}^{r_i-1} h_i(\mathbf{x}) \varphi_j(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{j=1}^m L_{\mathbf{g}_j} L_{\mathbf{f}}^{r_i-1} h_i(\Phi^{-1}(\xi, \eta)) \varphi_j(t) \end{pmatrix}$$

and

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in \mathfrak{R}^{(n-r_{sm}) \times (n-r_{sm})}, \quad \alpha(\xi, \eta) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ L_{\mathbf{f}}^{n-r_{sm}} h_{m+1}(\Phi^{-1}(\xi, \eta)) \end{pmatrix}$$

$$\mathbf{P}(\xi, \eta) = [\mathbf{p}^1(\xi, \eta), \mathbf{p}^2(\xi, \eta), \dots, \mathbf{p}^m(\xi, \eta)] \in \mathfrak{R}^{(n-r_{sm}) \times m}$$

$$\mathbf{p}^i(\xi, \eta) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ L_{\mathbf{g}_i} L_{\mathbf{f}}^{r_{m+1}-1} h_{m+1}(\mathbf{x}) \\ \vdots \\ L_{\mathbf{g}_i} L_{\mathbf{f}}^{(n-r_{sm}+1)} h_{m+1}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ L_{\mathbf{g}_i} L_{\mathbf{f}}^{r_{m+1}-1} h_{m+1}(\Phi^{-1}(\xi, \eta)) \\ \vdots \\ L_{\mathbf{g}_i} L_{\mathbf{f}}^{(n-r_{sm}+1)} h_{m+1}(\Phi^{-1}(\xi, \eta)) \end{pmatrix} \in \mathfrak{R}^{n-r_{sm}}$$

It can be seen that the original system given in (6.1) has been decomposed into two subsystems that are both described in the I/O form. As shown in (6.2), for the ξ subsystem, the dynamics of each ξ^i has a differentiator structure with unknown inputs appearing in the corresponding last dynamic equation, respectively. For the η subsystem, it can be considered as a non-minimum phase dynamics, in which the distribution matrix $\mathbf{P}(\xi, \eta)$ includes unknown states, i.e., the unknown input vector $\varphi(t)$ is non-matching in the observer sense.

6.2.3 Problem Formulation and Difficulties

The objective of this chapter is to design a robust observer for the estimation of ξ and η in the transformed dynamic system (6.3), thereby recovering the state \mathbf{x} and the unknown input $\varphi(t)$ in the original system (6.1). The observer design in the remaining part of this chapter will be discussed in the context of the transformed domain, and we address the asymptotic convergence of the estimates $\hat{\xi}$, $\hat{\eta}$ and $\hat{\varphi}(t)$ to their true values, i.e.,

$$\lim_{t \rightarrow \infty} \|\hat{\xi} - \xi\| = 0 \quad (6.4)$$

$$\lim_{t \rightarrow \infty} \|\hat{\eta} - \eta\| = 0 \quad (6.5)$$

$$\lim_{t \rightarrow \infty} \|\hat{\varphi}(t) - \varphi(t)\| = 0 \quad (6.6)$$

It should be mentioned that the unknown input observer design for MIMO nonlinear systems is often based on the involutive condition of the distribution matrix $\mathbf{G}(\mathbf{x})$. As a result, the original uncertain system can be decomposed into two subsystems, of which only one is affected by the unknown inputs and it has the same structure of ξ in (6.3). The other subsystem's dynamics is assumed to be nominal and self stable which is difficult to check, because the involutive condition only guarantees the existence of such a nominal subsystem, but no explicit formulation is available on how to construct it [62, 105].

In order to provide an explicit state transformation formulation and to guarantee the identifiability of the unknown input, we consider a class of nonlinear uncertain systems that can be transformed into the I/O form as in (6.3). Unlike

the nominal dynamic subsystem in [62, 78], the η subsystem in (6.3) is obtained by the *Lie* derivatives of an extra measurement output, i.e., $\eta_1 = h_{m+1}(\mathbf{x})$, and the relative degree between the extra output and unknown inputs is assumed to be r_{m+1} , as defined in Assumption 2. To obtain the asymptotic stability of the observer, the following structure assumptions are required.

Assumption 6.4. *For the ξ subsystem, the components of vector $\Psi^i(\xi, \eta)$ are assumed to be Lipschitz functions with respect to η , and each component includes only the upper r_{m+1} components of vector η , which is more general than the triangular structure, i.e.,*

$$L_f^{r_i} h_i(\Phi^{-1}(\xi, \eta)) = \psi_i(\xi, \eta_1, \dots, \eta_{r_{m+1}}), \quad i = 1, \dots, m \quad (6.7)$$

Assumption 6.5. *The Lie derivative function $L_f^{n-r_{sm}} h_{m+1}(\Phi^{-1}(\xi, \eta))$ in the vector of $\alpha(\xi, \eta)$ is a Lipschitz function with respect to their arguments, i.e.,*

$$\left| L_f^{n-r_{sm}} h_{m+1}(\Phi^{-1}(\hat{\xi}, \hat{\eta})) - L_f^{n-r_{sm}} h_{m+1}(\Phi^{-1}(\xi, \eta)) \right| \leq l_\alpha (\|\hat{\xi} - \xi\| + \|\hat{\eta} - \eta\|) \quad (6.8)$$

with l_α being a Lipschitz constant.

Assumption 6.6. *Consider the matrix $\mathbf{P}(\xi, \eta) \mathbf{E}^{-1}(\Phi^{-1}(\hat{\xi}, \hat{\eta})) \triangleq \bar{\mathbf{P}}(\xi, \eta)$, each column vector has the triangular structure with respect to η , and their moduli are locally bounded on \mathfrak{R} ; furthermore, their components are all Lipschitz functions. In other words, let $\bar{\mathbf{P}}(\xi, \eta) = [\bar{\mathbf{p}}^1, \dots, \bar{\mathbf{p}}^m] \in \mathfrak{R}^{(n-r_{sm}) \times m}$, it has*

$$\begin{aligned} \bar{\mathbf{p}}^i(\xi, \eta) &= [0, \dots, 0, \bar{p}_{r_{m+1}}^i(\xi, \eta_1, \dots, \eta_{r_{m+1}}), \dots, \\ &\quad \bar{p}_{r_{m+1}}^i(\xi, \eta_1, \dots, \eta_{n-r_{sm}})]^T \in \mathfrak{R}^{n-r_{sm}}, \quad i = 1, \dots, m \end{aligned} \quad (6.9)$$

with $\bar{p}_{r_{m+1}}^i(\xi, \eta_1, \dots, \eta_k)$, $\forall i = 1, \dots, m, \forall k = r_{m+1}, \dots, n - r_{sm}$, being Lipschitz functions.

Assumption 6.4 means that the ξ subsystem is only related to the first r_{m+1} states of the η subsystem, with r_{m+1} being the relative degree between the unknown inputs and the measurable output in the η subsystem. In the case when

the relative degree is full order, i.e, $r_{m+1} = n - r_{sm}$, then the new function given in (6.7) can be rewritten as $\psi_i(\xi, \eta)$ which has the similar structure as in [78].

Assumptions 6.5 and 6.6 can be conservative but they are required to ensure the uniform observability of the η subsystem. It has been proven in [35] that a nominal nonlinear system with triangular structure is uniformly observer for any known input. Together with Assumptions 6.4-6.6, they allow us to guarantee the uniform observability of the η subsystem if the unknown input $\varphi(t)$ can be replaced by some nominal dynamics. Besides, in Assumption 6.6, the first $(r_{m+1} - 1)$ components of each column vectors $\bar{\mathbf{p}}^i(\cdot)$ are zeros, because the relative degree between the extra measurement output $\eta_1 = h_{m+1}(\mathbf{x})$ and the unknown input vector $\varphi(t)$ is assumed to be r_{m+1} in Assumption 6.2.

6.3 Robust Hybrid Observer Design

In this section, a robust nonlinear observer that combines the high order sliding mode feedbacks with a reduced-order high gain feedback is proposed for handling the state estimation of the transformed dynamic system given by (6.3), and the unknown input will be reconstructed after all the states have converged to their true values.

For the ξ subsystem, it can be considered as m dynamics of $\xi^i, i = 1, \dots, m$, with each of them processing the differentiator structure and the unknown input appearing only in the corresponding last dynamic equation. Then, the quasi-continuous sliding mode observer with the higher-order sliding mode differentiator can be used to handle these m dynamics, and the corresponding m sliding surfaces will be individually reached in finite time.

Once all the sliding surfaces are reached, the unknown inputs in the η subsystem can be completely replaced with some nominal dynamics and hence the uniform observability property of the η subsystem can be ensured. Then, the reduced-order high gain feedback which is designed based on the extra measurement output will guarantee the asymptotic convergence for the remaining estimation error on the sliding surfaces.

6.3.1 Sliding Mode Feedbacks Design

By defining the m sliding variables $\sigma_i = \hat{\xi}_1^i - \xi_1^i = \hat{\xi}_1^i - h_i(\mathbf{x})$, $i = 1, \dots, m$, the derivatives of σ_i can be estimated in finite time by the higher-order sliding mode differentiator [69], which can be written in the form of

$$\begin{aligned}
\dot{z}_0^i &= v_0^i, \\
v_0^i &= -\lambda_0^i |z_0^i - \sigma_i|^{(r_i-1)/r_i} \text{sign}(z_0^i - \sigma_i) + z_1^i \\
\dot{z}_1^i &= v_1^i, \\
v_1^i &= -\lambda_1^i |z_1^i - v_0^i|^{(r_i-2)/(r_i-1)} \text{sign}(z_1^i - v_0^i) + z_2^i \\
&\vdots \\
\dot{z}_{r_i-2}^i &= v_{r_i-2}^i, \\
v_{r_i-2}^i &= -\lambda_{r_i-2}^i |z_{r_i-2}^i - v_{r_i-3}^i|^{1/2} \text{sign}(z_{r_i-2}^i - v_{r_i-3}^i) + z_{r_i-1}^i \\
\dot{z}_{r_i-1}^i &= -\lambda_{r_i-1}^i \text{sign}(z_{r_i-1}^i - v_{r_i-2}^i)
\end{aligned} \tag{6.10}$$

where the positive parameters λ_k^i , $k = 0, \dots, r_i - 1$, $i = 1, \dots, m$ are properly chosen. Then, the following equalities are true after a finite time transient process,

$$z_k^i - \sigma_i^{(k)} = 0, \quad k = 0, \dots, r_i - 1, \quad i = 1, \dots, m \tag{6.11}$$

where $\sigma_i^{(k)}$ denotes the k th-order derivative of σ_i .

Based on (6.10), the m sliding mode feedbacks can be designed based on the quasi-continuous high order sliding mode technique [71], as given in the form of

$$u_{r_i} = -\rho_i \Gamma_{k,r_i}(z_0^i, z_1^i, \dots, z_{r_i-1}^i), \quad i = 1, \dots, m \tag{6.12}$$

where, for $k = 0, 1, \dots, r_i - 1$, Γ_{k,r_i} are computed via

$$\begin{aligned}
\gamma_{0,r_i} &= z_0^i, \quad N_{0,r_i} = |z_0^i|, \quad \Gamma_{0,r_i} = \gamma_{0,r_i}/N_{0,r_i} = \text{sign}(z_0^i) \\
\gamma_{k,r_i} &= z_k^i + \beta_k^i N_{k-1,r_i}^{(r_i-k)/(r_i-k+1)} \Gamma_{k-1,r_i}, \\
N_{k,r_i} &= |z_k^i| + \beta_k^i N_{k-1,r_i}^{(r_i-k)/(r_i-k+1)}, \quad \Gamma_{k,r_i} = \gamma_{k,r_i}/N_{k,r_i}
\end{aligned} \tag{6.13}$$

Here, the positive tuning parameters β_k^i , $k = 1, \dots, r_i - 1$, $i = 1, \dots, m$ are

chosen in advanced, and the sliding gains ρ_i should be chosen large enough.

Now, considering the m dynamic systems of ξ^i in the ξ subsystem (6.3), the corresponding high order sliding mode observer can be designed as

$$\dot{\hat{\xi}}^i = \Lambda_i \hat{\xi}^i + \hat{\Psi}^i(\hat{\xi}, \hat{\eta}) + \hat{\lambda}^i(u_{r_i}), \quad i = 1, \dots, m \quad (6.14)$$

where $\hat{\xi} = [\hat{\xi}^1, \hat{\xi}^2, \dots, \hat{\xi}^m]^T$ and $\hat{\eta}$ are the estimated vectors of ξ and η respectively. Furthermore,

$$\hat{\xi}^i = \begin{pmatrix} \hat{\xi}_1^i \\ \hat{\xi}_2^i \\ \vdots \\ \hat{\xi}_{r_i}^i \end{pmatrix}, \quad \hat{\Psi}^i(\hat{\xi}, \hat{\eta}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hat{\psi}_i(\hat{\xi}, \hat{\eta}) \end{pmatrix}, \quad \hat{\lambda}^i(u_{r_i}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ u_{r_i} \end{pmatrix}$$

Here, the function $\psi_i(\cdot)$ is exactly the same as $L_{\mathbf{f}}^{r_i} h_i(\cdot)$, as assumed in Assumption 6.4.

It has been proven in [69] that such sliding mode observer in (6.14) ensures the sliding surfaces are reached in finite time and remained thereafter, provided that the sliding parameters are properly chosen. In other words, it has

$$\sigma_i^{(r_i-1)} = \dots = \sigma_i = 0, \quad i = 1, \dots, m \quad (6.15)$$

which implies the state ξ can be exactly estimated after a finite time transient: $\hat{\xi}^i = \xi^i$, $i = 1, \dots, m$, i.e., $\hat{\xi} = \xi$. Then, based on the equivalent injection input

concept and together with Assumption 6.4, the following equalities hold:

$$\begin{aligned}
u_{r_1} + \hat{\psi}_1(\xi, \hat{\eta}_1, \dots, \hat{\eta}_{r_{m+1}}) - \psi_1(\xi, \eta_1, \dots, \eta_{r_{m+1}}) &= \sum_{j=1}^m L_{\mathbf{g}_j} L_{\mathbf{f}}^{r_1-1} h_1(\Phi^{-1}(\xi, \eta)) \varphi_j(t) \\
u_{r_2} + \hat{\psi}_2(\xi, \hat{\eta}_1, \dots, \hat{\eta}_{r_{m+1}}) - \psi_2(\xi, \eta_1, \dots, \eta_{r_{m+1}}) &= \sum_{j=1}^m L_{\mathbf{g}_j} L_{\mathbf{f}}^{r_2-1} h_2(\Phi^{-1}(\xi, \eta)) \varphi_j(t) \\
&\vdots \\
u_{r_m} + \hat{\psi}_m(\xi, \hat{\eta}_1, \dots, \hat{\eta}_{r_{m+1}}) - \psi_m(\xi, \eta_1, \dots, \eta_{r_{m+1}}) &= \sum_{j=1}^m L_{\mathbf{g}_j} L_{\mathbf{f}}^{r_m-1} h_m(\Phi^{-1}(\xi, \eta)) \varphi_j(t)
\end{aligned} \tag{6.16}$$

With the definition of matrix $\mathbf{E}(\Phi^{-1}(\xi, \eta))$ in Assumption 6.1, the above equalities can be represented in a vector/matrix form as

$$\mathbf{u} + \Delta(\hat{\eta}, \eta) = \mathbf{E}(\Phi^{-1}(\xi, \eta)) \varphi(t) \triangleq \mathbf{E}(\xi, \eta) \varphi(t) \tag{6.17}$$

where

$$\mathbf{u} = \begin{pmatrix} u_{r_1} \\ u_{r_2} \\ \vdots \\ u_{r_m} \end{pmatrix} \in \mathbb{R}^m, \quad \Delta(\hat{\eta}, \eta) = \begin{pmatrix} \hat{\psi}_1(\xi, \hat{\eta}_1, \dots, \hat{\eta}_{r_{m+1}}) - \psi_1(\xi, \eta_1, \dots, \eta_{r_{m+1}}) \\ \hat{\psi}_2(\xi, \hat{\eta}_1, \dots, \hat{\eta}_{r_{m+1}}) - \psi_2(\xi, \eta_1, \dots, \eta_{r_{m+1}}) \\ \vdots \\ \hat{\psi}_m(\xi, \hat{\eta}_1, \dots, \hat{\eta}_{r_{m+1}}) - \psi_m(\xi, \eta_1, \dots, \eta_{r_{m+1}}) \end{pmatrix}$$

Furthermore, as the matrix $\mathbf{E}(\cdot)$ is assumed to be non-singular, the unknown input vector $\varphi(t)$ in (6.17) can be obtained as

$$\varphi(t) = \mathbf{E}^{-1}(\xi, \eta) [\mathbf{u} + \Delta(\hat{\eta}, \eta)] \tag{6.18}$$

The above equation (6.18) shows the relationship between the unknown input vector $\varphi(t)$ and the sliding mode terms related vector \mathbf{u} , which implies the unknown input vector can be recovered from the vector \mathbf{u} if all states can converge to their true values, i.e., $\Delta(\hat{\eta}, \eta) = 0$.

In the following section, we shall consider a reduced-order high gain feedback design to ensure that the remaining states in the η subsystem can be asymptoti-

cally estimated.

6.3.2 High Gain Feedback Design on the Sliding Surfaces

On the sliding surfaces given by (6.15), the η subsystem can be reduced into the following form by substituting equation (6.18) into the dynamics in (6.3), as

$$\dot{\eta} = \mathbf{A}\eta + \alpha(\xi, \eta) + \mathbf{P}(\xi, \eta)\mathbf{E}^{-1}(\xi, \eta)[\mathbf{u} + \Delta(\hat{\eta}, \eta)] \quad (6.19)$$

Clearly, the unknown input vector in the η subsystem is completely cancelled while all the sliding modes occur. Together with the triangular structure assumptions in Assumption 6.4 and Assumption 6.6, the uniform observability of the η subsystem is guaranteed.

Then, an $(n - r_{sm})$ th-order high gain feedback can be designed for the remaining state estimation of the η subsystem, as given by

$$\dot{\hat{\eta}} = \mathbf{A}\hat{\eta} + \alpha(\hat{\xi}, \hat{\eta}) + \mathbf{P}(\hat{\xi}, \hat{\eta})\mathbf{E}^{-1}(\hat{\xi}, \hat{\eta})\mathbf{u} + \mathbf{L}[h_{m+1}(\mathbf{x}) - \mathbf{C}\hat{\eta}] \quad (6.20)$$

where $h_{m+1}(\mathbf{x})$ denotes the extra measurement output of the original system (6.1), the constant row vector \mathbf{C} is defined as $\mathbf{C} = [1, 0, \dots, 0] \in \Re^{n-r_{sm}}$, and the feedback gain \mathbf{L} is designed based on the high gain theory [36], given as

$$\mathbf{L} = \mathbf{S}_\theta^{-1}\mathbf{C}^T = [\theta C_{n-r_{sm}}^1, \theta^2 C_{n-r_{sm}}^2, \dots, \theta^{n-r_{sm}} C_{n-r_{sm}}^{n-r_{sm}}]^T \in \Re^{n-r_{sm}} \quad (6.21)$$

with \mathbf{S}_θ being a positive symmetric matrix function of parameter θ , as

$$\mathbf{S}_\theta(i, j) = \frac{(-1)^{i+j} C_{i+j-2}^{j-1}}{\theta^{i+j-1}}, \quad 1 \leq i, j \leq n - r_{sm}, \quad C_m^k = \frac{m!}{(m-k)!k!}$$

By defining the remaining estimation error $\mathbf{e}_\eta = \hat{\eta} - \eta = [e_1^\eta, \dots, e_{n-r_{sm}}^\eta]^T \in \Re^{n-r_{sm}}$, the following Theorem guarantees that the estimation error \mathbf{e}_η is asymptotically stable on the sliding surfaces.

Theorem 6.1. *On the sliding surfaces (6.15), the proposed high gain observer given by (6.20)-(6.21) ensures that the remaining state η can be asymptotically*

identified, provided that Assumptions 6.1-6.6 are satisfied and the high gain feedback parameter θ is chosen large enough, i.e., $\theta > \max\{\theta_0, 1\}$, with θ_0 being given by

$$\theta_0 = \sigma(\mathbf{S}_1)[2l_\alpha + 2(n - r_{to} + 1)l_p\bar{\rho} + 2ml_\psi b_p] \quad (6.22)$$

where \mathbf{S}_1 is equal to the matrix \mathbf{S}_θ by setting $\theta = 1$, $\sigma(\mathbf{S}_1)$ denotes the condition number of \mathbf{S}_1 ; l_α , l_p , and l_ψ are some positive Lipschitz constants; $\bar{\rho}$ and b_p are positive constants related to \mathbf{u} and $\bar{\mathbf{P}}(\cdot)$, respectively; and r_{to} is the total relative degree.

Proof. On the sliding surfaces, i.e., $\hat{\xi} = \xi$, the η subsystem in (6.3) can be rewritten in the form of (6.19), then the dynamics of the estimation error \mathbf{e}_η can be obtained from (6.19) and (6.20), as

$$\begin{aligned} \dot{\mathbf{e}}_\eta &= (\mathbf{A} - \mathbf{S}_\theta^{-1}\mathbf{C}^T\mathbf{C})\mathbf{e}_\eta + \alpha(\xi, \hat{\eta}) - \alpha(\xi, \eta) - \bar{\mathbf{P}}(\xi, \eta)\Delta(\hat{\eta}, \eta) \\ &\quad + [\bar{\mathbf{P}}(\xi, \hat{\eta}) - \bar{\mathbf{P}}(\xi, \eta)]\mathbf{u} \end{aligned} \quad (6.23)$$

with $\bar{\mathbf{P}}(\cdot) = \mathbf{P}(\cdot)\mathbf{E}^{-1}(\cdot)$ being defined in Assumption 6.6, \mathbf{u} and $\Delta(\hat{\eta}, \eta)$ being defined in (6.17). For ease of analysis, the following equalities are introduced [36]:

$$\begin{aligned} \Delta_\theta &= \text{diag}\{1, \frac{1}{\theta}, \dots, \frac{1}{\theta^{n-r_{sm}-1}}\}, & \mathbf{S}_\theta &= \frac{1}{\theta}\Delta_\theta\mathbf{S}_1\Delta_\theta, \\ \Delta_\theta\mathbf{A}\Delta_\theta^{-1} &= \theta\mathbf{A}, & \mathbf{C}\Delta_\theta &= \mathbf{C}\Delta_\theta^{-1} = \mathbf{C}, \\ \theta\mathbf{S}_\theta + \mathbf{A}^T\mathbf{S}_\theta + \mathbf{S}_\theta\mathbf{A} - \mathbf{C}^T\mathbf{C} &= 0 \end{aligned}$$

By setting $\mathbf{e}_d = \Delta_\theta\mathbf{e}_\eta$, and define a Lyapunov function as $V = \mathbf{e}_d^T\mathbf{S}_1\mathbf{e}_d$. Then, it can be deduced from (6.23) that

$$\begin{aligned} \dot{V} &= 2\mathbf{e}_d^T\mathbf{S}_1\Delta_\theta\dot{\mathbf{e}}_\eta \\ &= 2\mathbf{e}_d^T\mathbf{S}_1\Delta_\theta(\mathbf{A} - \mathbf{S}_\theta^{-1}\mathbf{C}^T\mathbf{C})\mathbf{e}_\eta + 2\mathbf{e}_d^T\mathbf{S}_1\Delta_\theta[\alpha(\xi, \hat{\eta}) - \alpha(\xi, \eta)] \\ &\quad - 2\mathbf{e}_d^T\mathbf{S}_1\Delta_\theta\bar{\mathbf{P}}(\xi, \eta)\Delta(\hat{\eta}, \eta) + 2\mathbf{e}_d^T\mathbf{S}_1\Delta_\theta[\bar{\mathbf{P}}(\xi, \hat{\eta}) - \bar{\mathbf{P}}(\xi, \eta)]\mathbf{u} \\ &= 2\theta\mathbf{e}_d^T\mathbf{S}_1\mathbf{A}\mathbf{e}_d - 2\theta\|\mathbf{C}\mathbf{e}_d\|^2 + 2\mathbf{e}_d^T\mathbf{S}_1\Delta_\theta[\alpha(\xi, \hat{\eta}) - \alpha(\xi, \eta)] \\ &\quad - 2\mathbf{e}_d^T\mathbf{S}_1\Delta_\theta\bar{\mathbf{P}}(\xi, \eta)\Delta(\hat{\eta}, \eta) + 2\mathbf{e}_d^T\mathbf{S}_1\Delta_\theta[\bar{\mathbf{P}}(\xi, \hat{\eta}) - \bar{\mathbf{P}}(\xi, \eta)]\mathbf{u} \end{aligned} \quad (6.24)$$

Furthermore, with the equalities listed above, it has

$$2\mathbf{e}_d^T \mathbf{S}_1 \mathbf{A} \mathbf{e}_d = \mathbf{e}_d^T (\mathbf{S}_1 \mathbf{A} + \mathbf{A}^T \mathbf{S}_1) \mathbf{e}_d = -\mathbf{e}_d^T \mathbf{S}_1 \mathbf{e}_d + \|\mathbf{C} \mathbf{e}_d\|^2 \quad (6.25)$$

Based on Assumption 6.5, $\alpha(\cdot)$ is a Lipschitz function vector, and for any $\theta > 1$, we have

$$\|\Delta_\theta[\alpha(\xi, \hat{\eta}) - \alpha(\xi, \eta)]\| \leq \frac{1}{\theta^{n-r_{sm}-1}} l_\alpha \|\hat{\eta} - \eta\| \leq l_\alpha \|\Delta_\theta \mathbf{e}_\eta\| = l_\alpha \|\mathbf{e}_d\| \quad (6.26)$$

where l_α is the Lipschitz constant of the function $L_{\mathbf{f}}^{n-r_{sm}} h_{m+1}(\cdot)$ defined in Assumption 6.5.

Similarly, Assumption 6.6 assumes that all the column vectors $\bar{\mathbf{p}}^i(\cdot)$ of the matrix $\bar{\mathbf{P}}(\cdot)$, $i = 1, \dots, m$, are Lipschitz functions with the triangular structure in (6.9). Then for any $\theta > 1$, it can be obtained that

$$\begin{aligned} \|\Delta_\theta[\bar{\mathbf{P}}(\xi, \hat{\eta}) - \bar{\mathbf{P}}(\xi, \eta)]\mathbf{u}\| &\leq \sum_{i=1}^m \|\Delta_\theta[\bar{\mathbf{p}}^i(\hat{\eta}) - \bar{\mathbf{p}}^i(\eta)]u_{r_i}\| \\ &\leq \sum_{i=1}^m \rho_i \sum_{k=r_{m+1}}^{n-r_{sm}} \frac{1}{\theta^{k-1}} l_{\mathbf{p}_{(k,i)}} \|\bar{\mathbf{e}}_k\| \\ &\leq \sum_{i=1}^m \rho_i (n - r_{sm} - r_{m+1} + 1) l_{\mathbf{p}_i} \|\Delta_\theta \mathbf{e}_\eta\| \quad (6.27) \\ &\leq (n - r_{sm} - r_{m+1} + 1) l_{\mathbf{p}} \bar{\rho} \|\mathbf{e}_d\| \\ &= (n - r_{to} + 1) l_{\mathbf{p}} \bar{\rho} \|\mathbf{e}_d\| \end{aligned}$$

where $\bar{\mathbf{e}}_k \triangleq [e_1^\eta, \dots, e_k^\eta, 0, \dots, 0]^T \in \mathbb{R}^{n-r_{sm}}$, $k = r_{m+1}, \dots, n - r_{sm}$, are partial estimation errors of \mathbf{e}_η ; the constant $\bar{\rho} = \sum_{i=1}^m \rho_i$ denotes the sum of sliding gains of \mathbf{u} ; $l_{\mathbf{p}_{(k,i)}}$ denotes the Lipschitz constant of the k -th row and i -th column component function of matrix $\bar{\mathbf{P}}(\cdot)$, $l_{\mathbf{p}_i}$ means the maximum Lipschitz constant number in the i -th column, and $l_{\mathbf{p}} = \max\{l_{\mathbf{p}_1}, l_{\mathbf{p}_2}, \dots, l_{\mathbf{p}_m}\}$; the constant number of $(n - r_{to} + 1)$ comes up because the relative degree of the unknown input is assumed to be r_{m+1} for the η subsystem in Assumption 6.2, and r_{to} denotes the total relative degree of the original system in (6.1), as defined in Assumption 6.2, i.e., $r_{to} = r_{sm} + r_{m+1}$.

According to Assumption 6.4, the functions $\psi_i(\cdot)$, $i = 1, \dots, m$, are Lipschitz functions with respect to η , and only include the upper r_{m+1} dynamics of η ,

i.e., $\{\eta_1, \eta_2, \dots, \eta_{r_{m+1}}\}$. Therefore, on the sliding surfaces, i.e., $\hat{\xi} = \xi$, it can be deduced that

$$\left| \hat{\psi}_i(\xi, \hat{\eta}_1, \dots, \hat{\eta}_{r_{m+1}}) - \psi_i(\xi, \eta_1, \dots, \eta_{r_{m+1}}) \right| \leq l_{\psi_i} \|\bar{\mathbf{e}}_{r_{m+1}}\|, \quad i = 1, \dots, m \quad (6.28)$$

where $\bar{\mathbf{e}}_{r_{m+1}} = [e_1^\eta, \dots, e_{r_{m+1}}^\eta, 0, \dots, 0]^T \in \mathbb{R}^{n-r_{sm}}$ is a partial estimation error of \mathbf{e}_η , and l_{ψ_i} denotes the Lipschitz constant of the function $\psi_i(\cdot)$.

Considering the moduli of the vectors $\bar{\mathbf{p}}^i(\xi, \eta)$ are locally bounded as assumed in Assumption 6.6, and they have the triangular structures as shown in (6.9), we have

$$\|\Delta_\theta \bar{\mathbf{p}}^i(\xi, \eta)\| \leq \frac{1}{\theta^{r_{m+1}-1}} b_{\mathbf{p}_i}, \quad i = 1, \dots, m \quad (6.29)$$

where $b_{\mathbf{p}_i}$ denotes the upper bound of the moduli of vector $\bar{\mathbf{p}}^i(\xi, \eta)$.

Therefore, from (6.28) and (6.29), it can be deduced that

$$\begin{aligned} \|\Delta_\theta \bar{\mathbf{P}}(\xi, \eta) \Delta(\hat{\eta}, \eta)\| &\leq \sum_{i=1}^m \|\Delta_\theta \bar{\mathbf{p}}^i(\xi, \eta) [\hat{\psi}_i(\xi, \hat{\eta}_1, \dots, \hat{\eta}_{r_{m+1}}) - \psi_i(\xi, \eta_1, \dots, \eta_{r_{m+1}})]\| \\ &\leq \sum_{i=1}^m \frac{1}{\theta^{r_{m+1}-1}} b_{\mathbf{p}_i} l_{\psi_i} \|\bar{\mathbf{e}}_{r_{m+1}}\| \\ &\leq \sum_{i=1}^m l_{\psi_i} b_{\mathbf{p}_i} \|\Delta_\theta \mathbf{e}_\eta\| \\ &\leq m l_\psi b_{\mathbf{p}} \|\mathbf{e}_d\| \end{aligned} \quad (6.30)$$

where $b_{\mathbf{p}} = \max\{b_{\mathbf{p}_1}, \dots, b_{\mathbf{p}_m}\}$, and $l_\psi = \max\{l_{\psi_1}, \dots, l_{\psi_m}\}$.

Finally, substituting (6.25), (6.26), (6.27) and (6.30) into (6.24), it can be obtained that

$$\begin{aligned} \dot{V} &\leq -\theta \mathbf{e}_d^T \mathbf{S}_1 \mathbf{e}_d - \theta \|\mathbf{C} \mathbf{e}_d\|^2 + 2l_\alpha \|\mathbf{e}_d^T \mathbf{S}_1\| \|\mathbf{e}_d\| + 2(n - r_{to} + 1) l_{\mathbf{p}} \bar{\rho} \|\mathbf{e}_d^T \mathbf{S}_1\| \|\mathbf{e}_d\| \\ &\quad + 2m l_\psi b_{\mathbf{p}} \|\mathbf{e}_d^T \mathbf{S}_1\| \|\mathbf{e}_d\| \\ &\leq -\theta V + \sigma(\mathbf{S}_1) [2l_\alpha + 2(n - r_{to} + 1) l_{\mathbf{p}} \bar{\rho} + 2m l_\psi b_{\mathbf{p}}] V \\ &= -(\theta - \theta_0) V \end{aligned} \quad (6.31)$$

with θ_0 being defined in (6.22).

Therefore, it is clear that by choosing the high gain feedback parameter $\theta > \theta_0$,

we have $\dot{V} < 0$, which implies the estimation error dynamics of \mathbf{e}_η is asymptotically stable. In other words, it has $\hat{\eta} = \eta$. ■

6.3.3 Unknown Input Reconstruction

As discussed in the above two subsections, the proposed hybrid observer consists of two parts: the high order sliding mode observer given by (6.14) which is used to handle the ξ subsystem, and the $(n - r_{sm})$ th order high gain observer given by (6.20) that is used to handle the η subsystem. In essence, the operating procedure of the proposed observer can be described as one in which the high order sliding mode observer works individually to ensure the corresponding sliding surfaces are reached in finite time and remained thereafter, then the reduced-order high gain observer guarantees the asymptotic stability of the remaining estimation error dynamics on the sliding surfaces.

Thereafter, after all states have converged to their true values, i.e., $\hat{\xi} = \xi$, $\hat{\eta} = \eta$, it can be obtained from (6.17) and (6.18) that

$$\Delta(\hat{\eta}, \eta) = 0, \quad \varphi(t) = \mathbf{E}^{-1}(\hat{\xi}, \hat{\eta})\mathbf{u} \quad (6.32)$$

which means the unknown input vector can be successfully reconstructed from the sliding mode terms. For the original system given by (6.1), all the state \mathbf{x} can be estimated by the inverse mapping, i.e., $\hat{\mathbf{x}} = \Phi^{-1}(\hat{\xi}, \hat{\eta}) = \Phi^{-1}(\xi, \eta) = \mathbf{x}$.

Remark 6.1. Note that the components of vector \mathbf{u} , i.e., u_{r_i} , $i = 1, \dots, m$, defined in (6.12) are switching functions, thus proper low-pass filters are required for the reconstruction of the unknown input vector $\varphi(t)$ via equation (6.32).

6.4 Numerical Simulations

In this section, the proposed observer is demonstrated for a six-order numerical example, which is described as

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} -4x_1 + x_2x_5 \\ x_4 \\ x_5 \\ -4x_4 - x_2^2 + 4 \\ x_6 + \cos x_5 \\ -3x_6 - 3x_5 - x_3 \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ x_1 & 0.4x_4 \\ 1 & 0.5x_6 \end{pmatrix}}_{\mathbf{G}(\mathbf{x})} \underbrace{\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}}_{\varphi(t)} \quad (6.33)$$

$$\mathbf{y} = [h_1(\mathbf{x}), h_2(\mathbf{x}), h_3(\mathbf{x})]^T = [x_1, x_2, x_3]^T \quad (6.34)$$

where $\mathbf{x} = [x_1, x_2, x_3, x_4, x_5, x_6]^T \in \mathbb{R}^6$ is the system state, $\mathbf{y} \in \mathbb{R}^3$ is the system measurement output, $\varphi(t) = [\varphi_1(t), \varphi_2(t)]^T \in \mathbb{R}^2$ is the system unknown input which needs to be identified, and $\mathbf{G}(\mathbf{x}) = [\mathbf{g}_1(\mathbf{x}), \mathbf{g}_2(\mathbf{x})] \in \mathbb{R}^{6 \times 2}$, $\mathbf{g}_1(\mathbf{x}), \mathbf{g}_2(\mathbf{x}) \in \mathbb{R}^6$ are the distribution matrix and vectors. Note that the distribution matrix $\mathbf{G}(\mathbf{x})$ includes the unknown states x_4 and x_6 , which means the unknown input $\varphi(t)$ is non-matching in the observer sense.

By direct computation, it follows that

$$\begin{aligned} L_{\mathbf{g}_1} h_1(\mathbf{x}) &= 1, \\ L_{\mathbf{g}_1} h_2(\mathbf{x}) &= L_{\mathbf{g}_2} h_2(\mathbf{x}) = 0, & L_{\mathbf{g}_2} L_{\mathbf{f}} h_2(\mathbf{x}) &= 1, \\ L_{\mathbf{g}_1} h_3(\mathbf{x}) &= L_{\mathbf{g}_2} h_3(\mathbf{x}) = 0, & L_{\mathbf{g}_1} L_{\mathbf{f}} h_3(\mathbf{x}) &= x_1, \end{aligned} \quad (6.35)$$

It implies that the first two outputs in the system (6.33)-(6.34) have a vector relative degree $\{1, 2\}$, and the extra output $h_3(\mathbf{x})$ has relative degree 2 with respect to the unknown input $\varphi(t)$. Therefore, according to Assumption 6.1, the matrix

$\mathbf{E}(\mathbf{x})$ can be defined as

$$\mathbf{E}(\mathbf{x}) = \begin{pmatrix} L_{\mathbf{g}_1} h_1(\mathbf{x}) & L_{\mathbf{g}_2} h_1(\mathbf{x}) \\ L_{\mathbf{g}_1} L_{\mathbf{f}} h_2(\mathbf{x}) & L_{\mathbf{g}_2} L_{\mathbf{f}} h_2(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (6.36)$$

which is nonsingular.

Then a coordinate transformation can be chosen as $(\xi, \eta) : \xi_1^1 = x_1, \xi_1^2 = x_2, \xi_2^2 = x_4, \eta_1 = x_3, \eta_2 = x_5, \eta_3 = x_6 + \cos x_5$. The system (6.33)-(6.34) in the new basis can be described by

$$\begin{aligned} \dot{\xi}_1^1 &= -4\xi_1^1 + \xi_1^2 \eta_2 + \varphi_1(t) \\ \dot{\xi}_1^2 &= \xi_2^2 \\ \dot{\xi}_2^2 &= -4\xi_2^2 - (\xi_1^2)^2 + 4 + \varphi_2(t) \\ \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \eta_3 + \xi_1^1 \varphi_1(t) + 0.4\xi_2^2 \varphi_2(t) \\ \dot{\eta}_3 &= -3(\eta_3 - \cos \eta_2) - 3\eta_2 - \eta_1 - \eta_3 \sin \eta_2 + [1 - \xi_1^1 \sin \eta_2] \varphi_1(t) \\ &\quad + [0.5(\eta_3 - \cos \eta_2) - 0.4\xi_2^2 \sin \eta_2] \varphi_2(t) \end{aligned} \quad (6.37)$$

It can be verified that the mapping function $\Phi(\mathbf{x}) = [\xi_1^1, \xi_1^2, \xi_2^2, \eta_1, \eta_2, \eta_3]^T$ is diffeomorphism $\forall \mathbf{x}$, and Assumptions 6.1-6.6 are all satisfied. Then, by setting the corresponding estimated dynamics as $\{\hat{\xi}_1^1, \hat{\xi}_1^2, \hat{\xi}_2^2, \hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3\}$, a robust observer can be designed in the form of

$$\begin{aligned} \dot{\hat{\xi}}_1^1 &= -4\hat{\xi}_1^1 + \hat{\xi}_1^2 \hat{\eta}_2 + u_1 \\ \dot{\hat{\xi}}_1^2 &= \hat{\xi}_2^2 \\ \dot{\hat{\xi}}_2^2 &= -4\hat{\xi}_2^2 - (\hat{\xi}_1^2)^2 + 4 + u_2 \\ \dot{\hat{\eta}}_1 &= \hat{\eta}_2 - 3\theta[\hat{\eta}_1 - h_3(\mathbf{x})] \\ \dot{\hat{\eta}}_2 &= \hat{\eta}_3 + \hat{\xi}_1^1 u_1 + 0.4\hat{\xi}_2^2 u_2 - 3\theta^2[\hat{\eta}_1 - h_3(\mathbf{x})] \\ \dot{\hat{\eta}}_3 &= -3(\hat{\eta}_3 - \cos \hat{\eta}_2) - 3\hat{\eta}_2 - \hat{\eta}_1 - \hat{\eta}_3 \sin \hat{\eta}_2 + [1 - \hat{\xi}_1^1 \sin \hat{\eta}_2] u_1 \\ &\quad - \theta^3[\hat{\eta}_1 - h_3(\mathbf{x})] + [0.5(\hat{\eta}_3 - \cos \hat{\eta}_2) - 0.4\hat{\xi}_2^2 \sin \hat{\eta}_2] u_2 \end{aligned} \quad (6.38)$$

where θ is the high gain feedback parameter, u_1 and u_2 are the first-order and second-order sliding mode terms, given by [69], [71]

$$\begin{aligned}
u_1 &= -\rho_1 \text{sign}(\hat{\xi}_1^1 - h_1(\mathbf{x})) \\
u_2 &= -\rho_2(z_1 + |z_0|^{1/2} \text{sign}(z_0)) / (|z_1| + |z_0|^{1/2}) \\
\dot{z}_0 &= v_0 \\
v_0 &= -1.5M^{1/2}|z_0 - [\hat{\xi}_1^2 - h_2(\mathbf{x})]|^{1/2} \text{sign}[z_0 - (\hat{\xi}_1^2 - h_2(\mathbf{x}))] + z_1 \\
\dot{z}_1 &= -1.1M \text{sign}(z_1 - v_0)
\end{aligned} \tag{6.39}$$

with M being a tuning parameter.

For the simulation, we choose $\varphi_1(t) = 2 \sin(5t)$ and $\varphi_2(t) = 2 \sin(2t) + \cos t$. Then, the parameters for observer (6.38)-(6.39) are chosen as: $\rho_1 = 3, \rho_2 = 5, \theta = 2, M = 20$. The simulation is performed with the initial values $\mathbf{x}_0 = [6, 6, 6, 6, 6, 6]^T$, $\hat{\mathbf{x}}_0 = [2, 2, 2, 0, 0, 0]^T$, $z_0 = z_1 = v_0 = 1$.

As the mapping function $\Phi(\mathbf{x})$ is diffeomorphism $\forall \mathbf{x}$, the estimated states for the original system in (6.33) can be obtained by inverse mapping, as: $\hat{x}_1 = \hat{\xi}_1^1$, $\hat{x}_2 = \hat{\xi}_1^2$, $\hat{x}_3 = \hat{\eta}_1$, $\hat{x}_4 = \hat{\xi}_2^2$, $\hat{x}_5 = \hat{\eta}_2$, $\hat{x}_6 = \hat{\eta}_3 - \cos(\hat{\eta}_2)$. Then the simulation results are shown in Figure 6.1.

From Figure 6.1d, it shows that all the estimated states converge to their true values after 5s, then, the unknown inputs $\varphi_1(t)$ and $\varphi_2(t)$ can be reconstructed from the sliding mode terms, i.e., $\varphi_1(t) = u_1$, $\varphi_2(t) = u_2$, as shown in Figure 6.1e and Figure 6.1f.

6.5 Summary

In this chapter, a robust observer has been developed for a class of uncertain MIMO nonlinear systems in which the unknown inputs enter the system dynamics via the unknown-state dependent distribution matrix. If the number of measurement outputs is more than the number of unknown inputs, and with proper structure assumptions, the uniform observability of the states and unknown inputs can be guaranteed without the requirement of the involutive property.

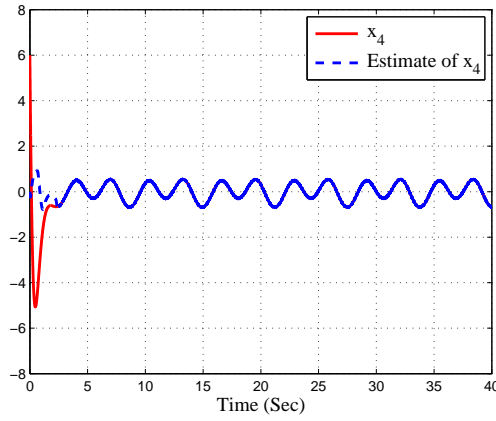
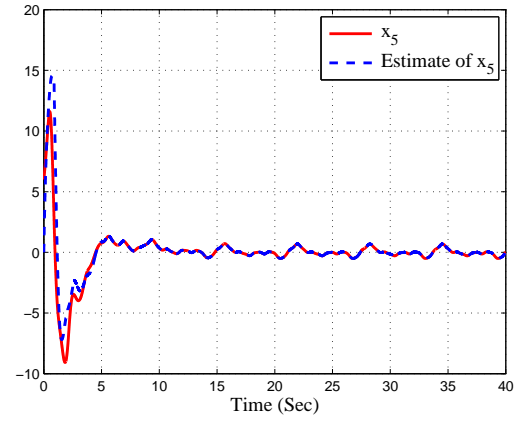
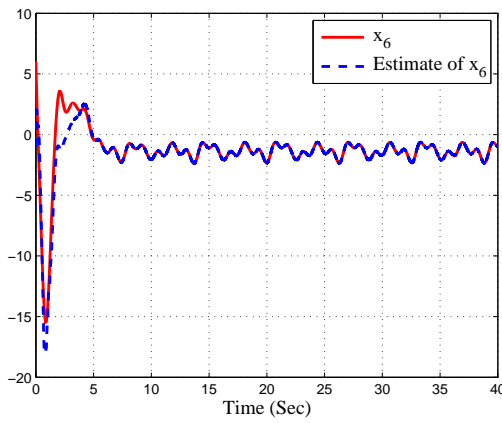
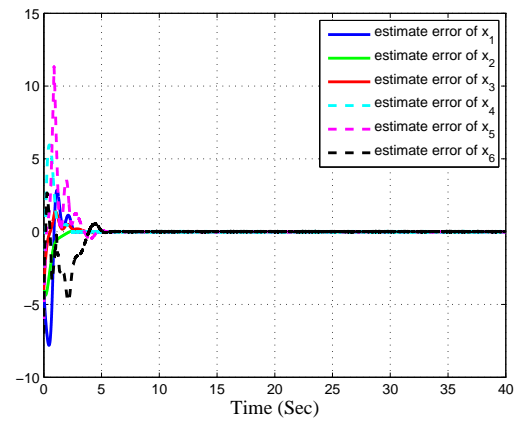
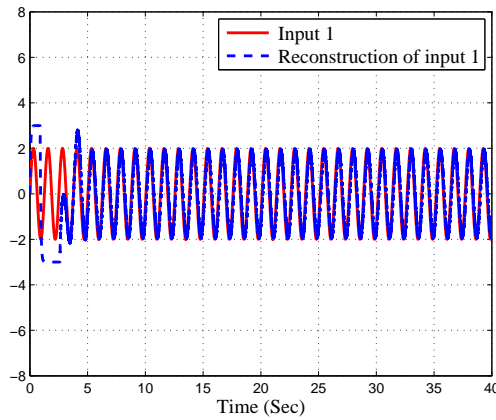
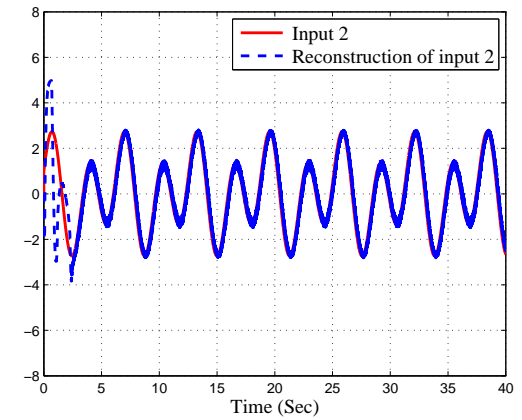
(a) : x_4 and \hat{x}_4 (b) : x_5 and \hat{x}_5 (c) : x_6 and \hat{x}_6 (d) : x_5 and \hat{x}_5 (e) : $\varphi_1(t)$ and $\hat{\varphi}_1(t)$ (f) : $\varphi_2(t)$ and $\hat{\varphi}_2(t)$

Figure 6.1: The estimation performance with the proposed hybrid observer

Chapter 7

A Recursive Sliding Mode Observer for Input Identification

In chapter 6, we have studied the hybrid nonlinear observer design for state and non-matching unknown input estimation of a class of nonlinear MIMO uncertain systems, in which the measurement outputs are noiseless (e.g. the uncertainties are not included in the system measurements). However, in some particular systems with a direct feedthrough path, the unknown inputs would appear not only in the state dynamics, but also in the measurement outputs. For such systems, the difficulty of the state and unknown input observer designs lies with how to separate the "clean" state dynamics contaminated by the unknown inputs from the measurement outputs.

In [84], a high-order sliding mode observer is designed for state estimation of a class of linear uncertain systems, in which not all the outputs of the system contain information affected by the unknown inputs, and strong observability or strong detectability assumption is required.

In this chapter, we shall extend the results in Chapter 6 to study the state estimation and unknown input identification problems of a class of MIMO nonlinear systems, in which the unknown inputs are non-matching in the observer sense and that they appear in both the dynamics of states and the measurement outputs. A hybrid observer that combines higher order sliding mode observers with a high gain observer is proposed, but we develop a new feature in which

the sliding mode observers are designed with recursive structures to ensure that the sliding mode surfaces are reached sequentially, and that the valuable signals in the measurement outputs are gradually extracted by cancelling the unknown inputs in sequence. Then, the high gain feedback works to guarantee that the unknown inputs and the states can be identified asymptotically.

The chapter is organized as follows: Section 7.1 presents the system dynamics and problem formulation. In section 7.2, a robust observer that combines a novel recursive high order sliding mode algorithm with high gain feedback is proposed; and a modified recursive sliding mode algorithm is also developed to improve the estimation performance. In section 7.3, a six-order numerical example is used to demonstrate the effectiveness of the proposed observer design. Section 7.4 concludes this chapter.

7.1 Preliminaries

7.1.1 System Dynamics

In this chapter, we shall consider a class of stable uncertain systems with m inputs and $m + 1$ outputs, for which the dynamics can be described in the form of

$$\begin{aligned}
 \dot{x}_1^i &= x_2^i \\
 &\vdots \\
 \dot{x}_{r_i-1}^i &= x_{r_i}^i \\
 \dot{x}_{r_i}^i &= f_i(\mathbf{x}_d^1, \dots, \mathbf{x}_d^i) + b_i x_1^{i+1} + \varphi_i(t), & i = 1, \dots, m \\
 \dot{\mathbf{x}}_s &= \mathbf{A}\mathbf{x}_s + \alpha(\mathbf{x}_d, \mathbf{x}_s) + \mathbf{P}(\mathbf{x}_d, \mathbf{x}_s)\varphi(t) \\
 \mathbf{y} &= \mathbf{h}(\mathbf{x}) + \mathbf{D}\varphi(t)
 \end{aligned} \tag{7.1}$$

where ¹

$$\mathbf{x} = [(\mathbf{x}_d)^T, (\mathbf{x}_s)^T]^T, \quad \mathbf{x}_d = \begin{pmatrix} \mathbf{x}_d^1 \\ \mathbf{x}_d^2 \\ \vdots \\ \mathbf{x}_d^m \end{pmatrix} \in \mathbb{R}^r, \quad \mathbf{x}_d^i = \begin{pmatrix} x_1^i \\ x_2^i \\ \vdots \\ x_{r_i}^i \end{pmatrix} \in \mathbb{R}^{r_i}$$

$$x_1^{m+1} \triangleq x_1^s, \quad \mathbf{x}_s = \begin{pmatrix} x_1^s \\ x_2^s \\ \vdots \\ x_{n-r}^s \end{pmatrix} \in \mathbb{R}^{n-r}, \quad \varphi(t) = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \vdots \\ \varphi_m(t) \end{pmatrix} \in \mathbb{R}^m$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{(n-r) \times (n-r)}, \quad \mathbf{h}(\mathbf{x}) = \begin{pmatrix} x_1^1 \\ \vdots \\ x_1^m \\ x_1^s \end{pmatrix} \in \mathbb{R}^{m+1}$$

with $r = \sum_{i=1}^m r_i$; $\mathbf{x} \in \mathbb{R}^n$ denotes the system state vector, $\varphi(t) \in \mathbb{R}^m$ the unknown input vector, $\mathbf{y} = [y_1, \dots, y_{m+1}]^T \in \mathbb{R}^{m+1}$ the system measurement output; b_i , $i = 1, \dots, m$ are known constant scalars, $\alpha(\cdot)$ is a known smooth vector on \mathbb{R}^{n-r} , $\mathbf{D} \in \mathbb{R}^{(m+1) \times m}$ a constant distribution matrix of unknown inputs of the system measurements, and $\mathbf{P}(\mathbf{x}_d, \mathbf{x}_s) = [\mathbf{p}^1, \mathbf{p}^2, \dots, \mathbf{p}^m] \in \mathbb{R}^{(n-r) \times m}$ the distribution matrix of unknown inputs of the dynamics of \mathbf{x}_s , with its column vector component $\mathbf{p}^i \in \mathbb{R}^{n-r}$, $i = 1, \dots, m$.

Note that the system given by (7.1) has been decomposed into two subsystems. For the \mathbf{x}_d subsystem, the dynamics of each \mathbf{x}_d^i is represented in the I/O form with differentiator structure, and the unknown inputs appear in the corresponding last dynamic equation, respectively. For the \mathbf{x}_s subsystem, it can be considered as a non-minimum phase dynamics with the distribution matrix $\mathbf{P}(\mathbf{x}_d, \mathbf{x}_s)$ including unknown states.

¹Through out this chapter, the variables x_1^{m+1} and x_1^s denote the same variable for notational simplicity.

For the measurement output \mathbf{y} , it is assumed to be directly affected by the unknown input $\varphi(t)$ via a constant matrix \mathbf{D} . This happens when the system model has a direct feedthrough, otherwise, \mathbf{D} is a zero matrix.

7.1.2 Problem Formulation

The problem considered in this chapter is to design a robust observer for the system (7.1) with only the measurement \mathbf{y} available, to guarantee that the estimates $\hat{\mathbf{x}}(t)$ and $\hat{\varphi}(t)$ will asymptotically converge to their true values, i.e.,

$$\lim_{t \rightarrow \infty} \|\hat{\mathbf{x}} - \mathbf{x}\| = 0 \quad (7.2)$$

$$\lim_{t \rightarrow \infty} \|\hat{\varphi}(t) - \varphi(t)\| = 0 \quad (7.3)$$

Consider the subsystem of \mathbf{x}_s , the unknown input $\varphi(t)$ is said to not satisfy the matching condition in the observer sense, because the distribution matrix $\mathbf{P}(\mathbf{x}_d, \mathbf{x}_s)$ includes the unknown states of \mathbf{x} . To the best of our knowledge, there are few effective existing works that can handle such non-matching unknown inputs. Although the disturbance considered in [124] is non-matching, it is assumed to be an unknown constant. In [78], a higher order sliding mode observer is proposed to estimate the unknown states and inputs of a MIMO nonlinear system, which can be decomposed into two subsystems based on the involutive condition. Of these two subsystems, only one is affected by the unknown inputs with the similar structure of \mathbf{x}_d in (7.1), and the other subsystem's so-called internal dynamics is assumed to be observable and self stable which is difficult to check. Furthermore, as mentioned in [62], the stability of the internal dynamics plays an important role on the identifiability of the unknown inputs, and it may also affect the estimation performance of the remaining states. One illustrative numerical example is given as

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1(t) \\ \dot{x}_2 &= x_2 \end{aligned} \quad (\text{with } y = x_1) \quad (7.4)$$

It can be seen that the unknown input $\varphi_1(t)$ in (7.4) is difficult to be identified

since the internal dynamics of x_2 is unstable/divergent. So, it is necessary to further explore the stability of the internal dynamics. The involutive condition used in [62, 78, 105] only guarantees the existence of the internal dynamics subsystem that is independent of uncertainties, but no explicit methodology is available on how to construct such internal dynamics from the original nonlinear system, and the corresponding stability is difficult to be verified.

For the class of nonlinear uncertain systems that is expressed in the form as shown in (7.1), if we compare the \mathbf{x}_s subsystem with the internal dynamical subsystem mentioned in [62, 78], the existence of the distribution matrix $\mathbf{P}(\mathbf{x}_d, \mathbf{x}_s)$ means that the involutive condition is not satisfied. Moreover, a feedthrough path with matrix \mathbf{D} is also considered since it often appears in a general control system. The existence of non-matching unknown inputs in both the \mathbf{x}_s subsystem and the measurement outputs \mathbf{y} remains a challenging problem for the observer design.

In order to guarantee the identifiability of the unknown inputs, in this paper, the number of the measurable outputs is assumed to be one more than the number of the unknown inputs. Such assumption ensures that at least one clean output signal can be extracted in the initial stages, and the residual measurable output (i.e., related to x_1^s) will be used to handle the observability of the \mathbf{x}_s subsystem. In addition, the following additional structural assumptions are required.

Assumption 7.1. *The system in (7.1) satisfies, $i = 1, \dots, m$*

$$\begin{aligned}\alpha(\mathbf{x}_d, \mathbf{x}_s) &= [\alpha_1(\mathbf{x}_d, x_1^s), \alpha_2(\mathbf{x}_d, x_1^s, x_2^s), \dots, \alpha_{n-r}(\mathbf{x}_d, x_1^s, x_2^s, \dots, x_{n-r}^s)]^T \\ \mathbf{p}^i(\mathbf{x}_d, \mathbf{x}_s) &= [p_{1i}(\mathbf{x}_d, x_1^s), p_{2i}(\mathbf{x}_d, x_1^s, x_2^s), \dots, p_{(n-r)i}(\mathbf{x}_d, x_1^s, x_2^s, \dots, x_{n-r}^s)]^T\end{aligned}\quad (7.5)$$

Assumption 7.2. *The functions $\alpha_k(\mathbf{x}_d, x_1^s, \dots, x_k^s)$, $p_{ki}(\mathbf{x}_d, x_1^s, \dots, x_k^s)$; $i = 1, \dots, m$, $k = 1, \dots, n - r$ are Lipschitz functions with respect to \mathbf{x} .*

Assumption 7.3. *The distribution matrix $\mathbf{P}(\mathbf{x}_d, \mathbf{x}_s)$ with functions $p_{ki}(\cdot)$; $k = 1, \dots, n - r$, $i = 1, \dots, m$ are bounded with respect to their arguments.*

Assumption 7.4. *The feedthrough matrix \mathbf{D} is a lower triangular matrix in the*

form of

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ d_{11} & 0 & 0 & \dots & 0 \\ d_{21} & d_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{m1} & d_{m2} & d_{m3} & \dots & d_{mm} \end{pmatrix} \in \Re^{(m+1) \times m} \quad (7.6)$$

Assumption 7.5. *The coefficients $(1 - d_{kk}b_k)$, $k = 1, \dots, m$, are assumed to be non-zeros.*

In Assumption 7.1, the \mathbf{x}_s subsystem given in (7.1) is assumed to satisfy the triangular structure, and it has been proven in [33] that such structure guarantees the uniform observability of the state \mathbf{x}_s for any known input. This allows us to design appropriate robust terms to approach the unknown inputs and thereby ensuring the uniform observability of the \mathbf{x}_s subsystem. Assumption 7.2 and Assumption 7.3 are required in order to achieve asymptotic stability of the estimation of the state \mathbf{x}_s .

For the dynamics of each \mathbf{x}_d^i , $i = 1, \dots, m$, in (7.1), it can be noticed that the relative degree between x_1^i and the corresponding unknown input $\varphi_i(t)$ is r_i . Furthermore, the structural connection between the dynamics \mathbf{x}_d^i and \mathbf{x}_d^{i+1} is made through a lower triangular form, i.e., $f_i(\mathbf{x}_d^1, \dots, \mathbf{x}_d^i) + b_i x_1^{i+1}$. This structure can be conservative, but together with Assumption 7.4 and Assumption 7.5, they are required to ensure that the unknown inputs on the measurement outputs can be cancelled in sequence and thereby the identifiability of all states and unknown inputs can be guaranteed.

7.2 Robust Nonlinear Observer

In this section, we shall consider the observer design for the nonlinear uncertain system given in (7.1), and the proposed observer can be decomposed into higher order sliding mode feedbacks and a high gain feedback, which are used to handle the two subsystems of (7.1), respectively.

For the \mathbf{x}_d subsystem, the m dynamics of $\mathbf{x}_d^i, i = 1, \dots, m$, have the differentiator structure, and the uncertainties appear in the corresponding highest dimension dynamics. Then, the quasi-continuous sliding mode observer with higher-order sliding mode differentiator can be used to handle such subsystems. However, unlike the works in [62, 78] and Chapter 6, as the measurable outputs are also distorted by the uncertainties, a hierarchical observer design method is proposed to ensure that the sliding surfaces are reached sequentially.

Once all the sliding surfaces are reached and remained thereafter, based on the equivalent control concept, the unknown inputs in the \mathbf{x}_s subsystem can be replaced by some nominal terms. Therefore, the uniform observability of \mathbf{x}_s can be guaranteed. Then a high gain feedback can be designed to ensure the asymptotic stability of the remaining estimation error dynamics on the sliding surfaces.

7.2.1 Higher-order Sliding Mode Observer

Let $\hat{\mathbf{x}}_d$ and $\hat{\mathbf{x}}_s$ denote the estimated vectors of \mathbf{x}_d and \mathbf{x}_s , respectively. Based on the structure of the measurable outputs $\mathbf{y} = [y_1, \dots, y_{m+1}]^T$ in (7.1) with Assumption 7.4 and Assumption 7.5 being satisfied, A higher order sliding mode observer for $\mathbf{x}_d^i, i = 1, \dots, m$ can be designed in the form of

$$\begin{aligned} \dot{\hat{x}}_1^i &= \hat{x}_2^i \\ &\vdots \\ \dot{\hat{x}}_{r_i-1}^i &= \hat{x}_{r_i}^i \\ \dot{\hat{x}}_{r_i}^i &= f_i(\hat{\mathbf{x}}_d^1, \dots, \hat{\mathbf{x}}_d^i) + b_i \hat{x}_1^{i+1} + u_i \quad (\hat{x}_1^{m+1} \triangleq \hat{x}_1^s) \end{aligned} \quad (7.7)$$

where $u_i, i = 1, \dots, m$ are the robust sliding mode terms, designed to deal with the corresponding unknown inputs $\varphi_i(t)$. In order to design the sliding mode terms, the m sliding variables are chosen based on the following algorithm:

$$\begin{aligned} \sigma_1 &= \hat{x}_1^1 - y_1 \\ \sigma_2 &= [\hat{x}_1^2 + d_{11}\{u_1\}_{eq} - y_2]/[1 - d_{11}b_1] \\ &\vdots \\ \sigma_m &= [\hat{x}_1^m + \sum_{i=1}^{m-1} d_{(m-1)i}\{u_i\}_{eq} - y_m]/[1 - d_{(m-1)(m-1)}b_{m-1}] \end{aligned} \quad (7.8)$$

Here, $\{u_i\}_{eq}$, $i = 1, \dots, m$ are the equivalent signals of the corresponding sliding mode terms u_i ; d_{ki} , $k = 1, \dots, m - 1$ are the elements of matrix \mathbf{D} defined in Assumption 7.4. So, the derivatives of σ_i , $i = 1, \dots, m$ can be estimated via [69]

$$\begin{aligned}
\dot{z}_0^i &= v_0^i, \\
v_0^i &= -\lambda_0^i |z_0^i - \sigma_i|^{(r_i-1)/r_i} \text{sign}(z_0^i - \sigma_i) + z_1^i \\
\dot{z}_1^i &= v_1^i, \\
v_1^i &= -\lambda_1^i |z_1^i - v_0^i|^{(r_i-2)/(r_i-1)} \text{sign}(z_1^i - v_0^i) + z_2^i \\
&\vdots \\
\dot{z}_{r_i-2}^i &= v_{r_i-2}^i, \\
v_{r_i-2}^i &= -\lambda_{r_i-2}^i |z_{r_i-2}^i - v_{r_i-3}^i|^{1/2} \text{sign}(z_{r_i-2}^i - v_{r_i-3}^i) + z_{r_i-1}^i \\
\dot{z}_{r_i-1}^i &= -\lambda_{r_i-1}^i \text{sign}(z_{r_i-1}^i - v_{r_i-2}^i)
\end{aligned} \tag{7.9}$$

where the positive parameters λ_k^i , $k = 0, \dots, r_i - 1$, $i = 1, \dots, m$ are properly chosen. Then, the following equalities are true after a finite time transient process,

$$z_k^i - \sigma_i^{(k)} = 0, \quad k = 0, \dots, r_i - 1, \quad i = 1, \dots, m \tag{7.10}$$

with $\sigma_i^{(k)}$ denoting the k th-order derivative of σ_i for the dynamics of \mathbf{x}^i . Thus, the quasi-continuous higher-order sliding mode terms can be designed as [71]

$$u_i = -\rho_i \Gamma_{k,r_i}(z_0^i, z_1^i, \dots, z_{r_i-1}^i), \quad i = 1, \dots, m \tag{7.11}$$

where, for $k = 0, 1, \dots, r_i - 1$, Γ_{k,r_i} are computed via

$$\begin{aligned}
\gamma_{0,r_i} &= z_0^i, & N_{0,r_i} &= |z_0^i|, & \Gamma_{0,r_i} &= \gamma_{0,r_i}/N_{0,r_i} = \text{sign}(z_0^i) \\
\gamma_{k,r_i} &= z_k^i + \beta_k^i N_{k-1,r_i}^{(r_i-k)/(r_i-k+1)} \Gamma_{k-1,r_i}, \\
N_{k,r_i} &= |z_k^i| + \beta_k^i N_{k-1,r_i}^{(r_i-k)/(r_i-k+1)}, & \Gamma_{k,r_i} &= \gamma_{k,r_i}/N_{k,r_i}
\end{aligned} \tag{7.12}$$

Here, the positive tuning parameters β_k^i , $k = 1, \dots, r_i - 1$, $i = 1, \dots, m$ are chosen in advanced, and the sliding gains ρ_i should be chosen large enough to cover the upper bounds of the unknown inputs.

Theorem 7.1. *Suppose that the dynamic subsystem of \mathbf{x}_d given by (7.1) satisfies*

Assumptions 4 and 5, the proposed higher order sliding mode observer (7.7)-(7.12) ensures the m sliding surfaces $\sigma_i^{(r_i-1)} = \dots = \sigma_i = 0$, $i = 1, \dots, m$ are reached in finite time and remained thereafter. Furthermore, once all the sliding surfaces are reached, the variables \mathbf{x}_d^i can be exactly estimated by $\hat{\mathbf{x}}_d^i$, i.e., $\hat{\mathbf{x}}_d = \mathbf{x}_d$. Furthermore, the unknown inputs can be represented as

$$\varphi(t) = \mathbf{u} + \Delta(\hat{x}_1^s, x_1^s) \quad (7.13)$$

where

$$\mathbf{u} = \begin{pmatrix} \{u_1\}_{eq} \\ \{u_2\}_{eq} \\ \vdots \\ \{u_{m-1}\}_{eq} \\ \{u_m\}_{eq} \end{pmatrix} \in \Re^m, \quad \Delta(\hat{\eta}_1, \eta_1) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_m[\hat{x}_1^s - x_1^s] \end{pmatrix} \in \Re^m$$

Proof. Consider the observer structure given by (7.7)-(7.12), the sliding variables and the sliding mode terms are coupled with each other. In other words, the sliding mode term u_k defined based on σ_k is also related to the previous robust term u_{k-1} . i.e., $u_k \leftarrow \sigma_k/x_1^k \leftarrow u_{k-1}$. The sliding variables are chosen in such a recursive structure of (7.8), in order to cancel the unknown inputs/disturbances in the measurable outputs sequentially, and to obtain a clean relationship between sliding variable σ_k and its corresponding state estimation error $\hat{x}_1^k - x_1^k$.

With Assumption 7.4, the $(m+1)$ measurement outputs of \mathbf{y} in (7.1) can be rewritten as

$$\begin{aligned} y_1 &= x_1^1 \\ y_2 &= x_1^2 + d_{11}\varphi_1(t) \\ &\vdots \\ y_m &= x_1^m + \sum_{i=1}^{m-1} d_{(m-1)i}\varphi_i(t) \\ y_{m+1} &= x_1^s + \sum_{i=1}^m d_{mi}\varphi_i(t) \end{aligned} \quad (7.14)$$

Then, the m sliding variables in (7.8) can be expressed as

$$\begin{aligned}
 \sigma_1 &= \hat{x}_1^1 - x_1^1 \\
 \sigma_2 &= \{\hat{x}_1^2 - x_1^2 + d_{11}[\{u_1\}_{eq} - \varphi_1(t)]\}/[1 - d_{11}b_1] \\
 &\vdots \\
 \sigma_m &= \{\hat{x}_1^m - x_1^m + \sum_{i=1}^{m-1} d_{(m-1)i}[\{u_i\}_{eq} - \varphi_i(t)]\}/[1 - d_{(m-1)(m-1)}b_{m-1}]
 \end{aligned} \tag{7.15}$$

Clearly, the first sliding variable σ_1 is clean with respect to the unknown inputs, and its dynamic can be obtained from (7.1) and (7.7), as

$$\sigma_1^{(r_1)} = f_1(\hat{\mathbf{x}}_d^1) - f_1(\mathbf{x}_d^1) + b_1[\hat{x}_1^2 - x_1^2] - \varphi_1(t) + u_1 \tag{7.16}$$

where u_1 is designed based on quasi-continuous sliding mode (7.11)-(7.12). As proven in [71], by properly choosing sliding gain and tuning parameters, the sliding surfaces $\sigma_1^{(r_1-1)} = \dots = \sigma_1 = 0$ will be reached in finite time and remained thereafter, which implies $\hat{x}_k^1 = x_k^1$, $k = 1, \dots, r_1$, i.e., $\hat{\mathbf{x}}_d^1 = \mathbf{x}_d^1$. Thus, based on the equivalent injection concept, it can be obtained from (7.16) that

$$\{u_1\}_{eq} - \varphi_1(t) = -b_1[\hat{x}_1^2 - x_1^2] \tag{7.17}$$

where $\{u_1\}_{eq}$ denotes the equivalent signal of u_1 . By substituting the above equation into the sliding variable σ_2 in (7.15), then it has

$$\sigma_2 = \hat{x}_1^2 - x_1^2 \tag{7.18}$$

It means that the sliding mode of σ_1 will result in the next sliding variable σ_2 being without any uncertainty. And the corresponding dynamics can be written as

$$\begin{aligned}
 \sigma_2^{(r_2)} &= (\hat{x}_1^2)^{(r_2)} - (x_1^2)^{(r_2)} \\
 &= f_2(\mathbf{x}_d^1, \hat{\mathbf{x}}_d^2) - f_2(\mathbf{x}_d^1, \mathbf{x}_d^2) + b_2[\hat{x}_1^3 - x_1^3] - \varphi_2(t) + u_2
 \end{aligned} \tag{7.19}$$

Similarly, the sliding mode u_2 given in (7.11)-(7.12) ensures the sliding surface $\sigma_2^{(r_2-1)} = \dots = \sigma_2 = 0$ will be reached in finite time and remained thereafter,

which implies $\hat{x}_k^2 = x_k^2$, $k = 1, \dots, r_2$, i.e., $\hat{\mathbf{x}}_d^2 = \mathbf{x}_d^2$. Thus, it can be deduced that

$$\begin{aligned} \{u_1\}_{eq} - \varphi_1(t) &= 0 \\ \{u_2\}_{eq} - \varphi_2(t) &= -b_2[\hat{x}_1^3 - x_1^3] \end{aligned} \quad (7.20)$$

Then, the sliding variable σ_3 in (7.15), when the sliding mode of σ_1 and σ_2 occur, can be simplified into

$$\sigma_3 = \hat{x}_1^3 - x_1^3 \quad (7.21)$$

Compared (7.17) with (7.20), we can extend the above obtained results into more general cases. Then, in order to complete the proof, the following Lemma 7.1 is introduced.

Lemma 7.1. *Suppose the following two propositions are true for a given k such that $2 \leq k \leq m - 1$,*

$$\begin{aligned} 1) \quad & \hat{\mathbf{x}}_d^i = \mathbf{x}_d^i, \quad \sigma_i^{(r_i-1)} = \dots = \sigma_i = 0; \quad i = 1, \dots, k; \\ 2) \quad & \{u_i\}_{eq} - \varphi_i(t) = 0, \quad i = 1, \dots, k-1; \\ & \{u_k\}_{eq} - \varphi_k(t) = -b_k[\hat{x}_1^{k+1} - x_1^{k+1}] \end{aligned}$$

then it will be also true for $k+1$.

Proof. First, based on the previous analysis and (7.20), the above two propositions are true for $k = 2$. So, we only need to prove these propositions still hold for $k' = k + 1$.

By substituting 2) into the sliding variable σ_{k+1} in (7.15), it can be obtained that

$$\sigma_{k'} = \sigma_{k+1} = \hat{x}_1^{k+1} - x_1^{k+1} \quad (7.22)$$

From (7.1) and (7.7) the dynamic of σ_{k+1} can be described as

$$\begin{aligned} \sigma_{k+1}^{(r_{k+1})} &= (\hat{x}_1^{k+1} - x_1^{k+1})^{(r_{k+1})} \\ &= f_{k+1}(\mathbf{x}_d^1, \dots, \mathbf{x}_d^k, \hat{\mathbf{x}}_d^{k+1}) - f_{k+1}(\mathbf{x}_d^1, \dots, \mathbf{x}_d^k, \mathbf{x}_d^{k+1}) \\ &\quad + b_{k+1}[\hat{x}_1^{k+2} - x_1^{k+2}] - \varphi_{k+1}(t) + u_{k+1} \end{aligned} \quad (7.23)$$

Consider this r_{k+1} order differentiator dynamic of σ_{k+1} , the quasi-continuous sliding mode controller u_{k+1} defined in (7.11)-(7.12) can ensure a new sliding surface $\sigma_{k+1}^{(r_{k+1}-1)} = \dots = \sigma_{k+1} = 0$ is reached in finite time and remained thereafter. Together with (7.21), it has $\hat{x}_j^{k+1} = x_j^{k+1}, j = 1, \dots, r_{k+1}$. i.e., $\hat{\mathbf{x}}_d^{k+1} = \mathbf{x}_d^{k+1}$. So, we can say that the statement 1) is true for $i = k + 1$.

Now, based on the equivalent injection concept, and substitute $\hat{\mathbf{x}}_d^{k+1} = \mathbf{x}_d^{k+1}$ into statement 2), it can be obtained that

$$\begin{aligned} \{u_i\}_{eq} - \varphi_i(t) &= 0, & i &= 1, \dots, k \\ \{u_{k+1}\}_{eq} - \varphi_{k+1}(t) &= -b_{k+1}[\hat{x}_1^{k+2} - x_1^{k+2}] \end{aligned} \quad (7.24)$$

Thus, these two statements are also true for $k' = k + 1$. ■

Now, by substituting $k = m$ into *Lemma 1*, the following equalities hold:

$$\begin{aligned} \hat{\mathbf{x}}_d^i &= \mathbf{x}_d^i, \quad \sigma_i^{(r_i-1)} = \dots = \sigma_i = 0; & i &= 1, \dots, m; \\ \{u_i\}_{eq} - \varphi_i(t) &= 0, & i &= 1, \dots, m-1; \\ \{u_m\}_{eq} - \varphi_m(t) &= -b_m[\hat{x}_1^{m+1} - x_1^{m+1}] = -b_m[\hat{x}_1^s - x_1^s] \end{aligned}$$

By defining \mathbf{u} and $\Delta(\hat{x}_1^s, x_1^s)$ as in Theorem 7.1. We can obtain the equation in (7.13). ■

For the sliding variables given in (7.8), they are chosen in a recursive way to sequentially cancel the unknown inputs in the system measurement outputs. Due to the structure of the proposed observer in (7.7)-(7.12), the sliding surfaces will be approached in a fixed sequential order, and a properly designed low-pass filter is required to obtain the equivalent signal of the corresponding sliding mode terms.

Remark 7.1. *For such a recursive sliding mode observer structure given in (7.6)-(7.12), the implementation error of the equivalent signal will be passed down to the remaining dynamics, so the low-pass filter will affect not just the estimation performance of the states, but also the performance of unknown inputs identification.*

7.2.2 Reduced-order Dynamics on the Sliding Surfaces

Once the sliding mode occurs, the sliding variables will reach the sliding surfaces and remain on them thereafter, i.e., $\hat{\mathbf{x}}_d = \mathbf{x}_d$. Then according to Theorem 7.1, the unknown inputs can be rewritten as

$$\varphi(t) = \mathbf{u} + \Delta(\hat{x}_1^s, x_1^s) \quad (7.25)$$

Therefore, the dynamics of subsystem \mathbf{x}_s in (7.1) can be described as

$$\dot{\mathbf{x}}_s = \mathbf{A}\mathbf{x}_s + \alpha(\mathbf{x}_d, \mathbf{x}_s) + \mathbf{P}(\mathbf{x}_d, \mathbf{x}_s)[\mathbf{u} + \Delta(\hat{x}_1^s, x_1^s)] \quad (7.26)$$

Note that the unknown inputs completely disappear in the above dynamics, and this allows us to design a corresponding observer to ensure the uniform observability of \mathbf{x}_s on the sliding surfaces.

Hence, a high gain feedback observer for the \mathbf{x}_s subsystem can be designed in the form of

$$\dot{\hat{\mathbf{x}}}_s = \mathbf{A}\hat{\mathbf{x}}_s + \alpha(\hat{\mathbf{x}}_d, \hat{\mathbf{x}}_s) + \mathbf{P}(\hat{\mathbf{x}}_d, \hat{\mathbf{x}}_s)\mathbf{u} - \mathbf{L}\sigma_{m+1} \quad (7.27)$$

where $\mathbf{L}\sigma_{m+1}$ denotes the high gain feedback term, with σ_{m+1} being defined as

$$\sigma_{m+1} = [\hat{x}_1^s + \sum_{i=1}^m d_{mi}\{u_i\}_{eq} - y_{m+1}] / [1 - d_{mm}b_m] \quad (7.28)$$

and the constant feedback gain \mathbf{L} is chosen based on HGO [36], such that

$$\begin{aligned} \mathbf{L} &= \mathbf{S}_\theta^{-1}\mathbf{C}^T = [\theta C_{n-r}^1, \theta^2 C_{n-r}^2, \dots, \theta^{n-r} C_{n-r}^{n-r}]^T \\ \mathbf{C} &= [1, 0, \dots, 0] \in \Re^{n-r} \end{aligned} \quad (7.29)$$

Here, \mathbf{S}_θ is a positive symmetric matrix function of parameter θ , and its $(i, j)^{th}$ element is given by

$$\mathbf{S}_\theta(i, j) = \frac{(-1)^{i+j} C_{i+j-2}^{j-1}}{\theta^{i+j-1}}, \quad 1 \leq i, j \leq n-r, \quad C_m^k = \frac{m!}{(m-k)!k!}$$

Define the estimation error $\mathbf{e}_s = \hat{\mathbf{x}}_s - \mathbf{x}_s = [e_1^s, \dots, e_{n-r}^s]^T \in \Re^{n-r}$, the

following Theorem 7.2 guarantees that the estimation error \mathbf{e}_s is asymptotically stable on the sliding surfaces.

Theorem 7.2. *On the sliding surfaces, i.e., $\hat{\mathbf{x}}_d = \mathbf{x}_d$, and with the \mathbf{x}_s subsystem given in (7.1) satisfying Assumptions 1, 2 and 3, the proposed high gain observer (7.27)-(7.29) ensures that the estimation error \mathbf{e}_s is asymptotically stable, provided that the high gain parameter $\theta > \max\{\theta_0, 1\}$, with*

$$\theta_0 = 2\sigma(\mathbf{S}_1)[(n-r)l_\alpha + (n-r)\bar{\rho}l_p + \bar{P}b_m] \quad (7.30)$$

where \mathbf{S}_1 is equal to the matrix \mathbf{S}_θ by setting $\theta = 1$, $\sigma(\mathbf{S}_1)$ denotes the condition number of \mathbf{S}_1 ; l_α and l_p are the corresponding Lipschitz constants; $\bar{\rho} = \sum_{i=1}^m \rho_i$, and \bar{P} denotes the upper bound of the modulus of distribution matrix $\mathbf{P}(\mathbf{x}_d, \mathbf{x}_s)$.

Proof. On the sliding surfaces, i.e., $\hat{\mathbf{x}}_d = \mathbf{x}_d$, the feedback σ_{m+1} given in (7.28) can be simplified into the following equation with Theorem 7.1, as

$$\begin{aligned} \sigma_{m+1} &= [\hat{x}_1^s + \sum_{i=1}^m d_{mi}\{u_i\}_{eq} - (x_1^s + \sum_{i=1}^m d_{mi}\varphi_i(t))]/[1 - d_{mm}b_m] \\ &= [\hat{x}_1^s - x_1^s + d_{mm}(\{u_m\}_{eq} - \varphi_m(t))]/[1 - d_{mm}b_m] \\ &= \hat{x}_1^s - x_1^s \\ &= \mathbf{C}\mathbf{e}_s \end{aligned} \quad (7.31)$$

Since the dynamics of the \mathbf{x}_s subsystem on the sliding surfaces can be rewritten as in (7.26), the dynamics of the estimation error \mathbf{e}_s can be obtained from (7.26) and (7.27), as

$$\begin{aligned} \dot{\mathbf{e}}_s &= (\mathbf{A} - \mathbf{S}_\theta^{-1}\mathbf{C}^T\mathbf{C})\mathbf{e}_s + \alpha(\mathbf{x}_d, \hat{\mathbf{x}}_s) - \alpha(\mathbf{x}_d, \mathbf{x}_s) + [\mathbf{P}(\mathbf{x}_d, \hat{\mathbf{x}}_s) - \mathbf{P}(\mathbf{x}_d, \mathbf{x}_s)]\mathbf{u} \\ &\quad - \mathbf{P}(\mathbf{x}_d, \mathbf{x}_s)\Delta(\hat{x}_1^s, x_1^s) \end{aligned} \quad (7.32)$$

In order to prove the asymptotic stability of the above dynamics and for ease of

analysis, the following equalities are introduced:

$$\begin{aligned}\Delta_\theta &= \text{diag}(1, \frac{1}{\theta}, \dots, \frac{1}{\theta^{n-1}}) & \mathbf{S}_\theta &= \frac{1}{\theta} \Delta_\theta \mathbf{S}_1 \Delta_\theta \\ \Delta_\theta \mathbf{A} \Delta_\theta^{-1} &= \theta \mathbf{A} & \mathbf{C} \Delta_\theta &= \mathbf{C} \Delta_\theta^{-1} = \mathbf{C} \\ \theta \mathbf{S}_\theta + \mathbf{A}^T \mathbf{S}_\theta + \mathbf{S}_\theta \mathbf{A} - \mathbf{C}^T \mathbf{C} &= 0\end{aligned}$$

By setting $\xi = \Delta_\theta \mathbf{e}_s$, and defining a Lyapunov function as $V = \xi^T \mathbf{S}_1 \xi$, it can be deduced from (7.32) that

$$\begin{aligned}\dot{V} &= 2\xi^T \mathbf{S}_1 \Delta_\theta \dot{\mathbf{e}}_s \\ &= 2\xi^T \mathbf{S}_1 \Delta_\theta (\mathbf{A} - \mathbf{S}_\theta^{-1} \mathbf{C}^T \mathbf{C}) \mathbf{e}_s + 2\xi^T \mathbf{S}_1 \Delta_\theta [\alpha(\mathbf{x}_d, \hat{\mathbf{x}}_s) - \alpha(\mathbf{x}_d, \mathbf{x}_s)] \\ &\quad - 2\xi^T \mathbf{S}_1 \Delta_\theta \mathbf{P}(\mathbf{x}_d, \mathbf{x}_s) \Delta(\hat{x}_1^s, x_1^s) + 2\xi^T \mathbf{S}_1 \Delta_\theta [\mathbf{P}(\mathbf{x}_d, \hat{\mathbf{x}}_s) - \mathbf{P}(\mathbf{x}_d, \mathbf{x}_s)] \mathbf{u} \\ &= 2\theta \xi^T \mathbf{S}_1 \mathbf{A} \xi - 2\theta \|\mathbf{C} \xi\|^2 + 2\xi^T \mathbf{S}_1 \Delta_\theta [\alpha(\mathbf{x}_d, \hat{\mathbf{x}}_s) - \alpha(\mathbf{x}_d, \mathbf{x}_s)] \\ &\quad - 2\xi^T \mathbf{S}_1 \Delta_\theta \mathbf{P}(\mathbf{x}_d, \mathbf{x}_s) \Delta(\hat{x}_1^s, x_1^s) + 2\xi^T \mathbf{S}_1 \Delta_\theta [\mathbf{P}(\mathbf{x}_d, \hat{\mathbf{x}}_s) - \mathbf{P}(\mathbf{x}_d, \mathbf{x}_s)] \mathbf{u}\end{aligned}\tag{7.33}$$

With the equalities listed above, we have

$$2\xi^T \mathbf{S}_1 \mathbf{A} \xi = -\xi^T \mathbf{S}_1 \xi + \|\mathbf{C} \xi\|^2\tag{7.34}$$

Based on Assumption 7.1 and Assumption 7.2, the vector $\alpha(\cdot)$ is a Lipschitz function with respect to \mathbf{x} , and for any $\theta > 1$, it has

$$\begin{aligned}\|\Delta_\theta [\alpha(\mathbf{x}_d, \hat{\mathbf{x}}_s) - \alpha(\mathbf{x}_d, \mathbf{x}_s)]\| &\leq \sum_{k=1}^{n-r} \frac{1}{\theta^{k-1}} |\alpha_k(\mathbf{x}_d, \hat{x}_1^s, \dots, \hat{x}_k^s) - \alpha_k(\mathbf{x}_d, x_1^s, \dots, x_k^s)| \\ &\leq \sum_{k=1}^{n-r} \frac{1}{\theta^{k-1}} l_{\alpha k} \|\mathbf{e}_k^s\| \\ &\leq \sum_{k=1}^{n-r} l_{\alpha k} \|\Delta_\theta \mathbf{e}_s\| \\ &\leq (n-r) l_\alpha \|\xi\|\end{aligned}\tag{7.35}$$

where $\mathbf{e}_k^s = [e_1^s, \dots, e_k^s, 0, \dots, 0]^T \in \mathfrak{R}^{n-r}$ denotes the partial estimation error of \mathbf{e}_s ; and $l_\alpha = \sup_k |l_{\alpha k}|$, with $l_{\alpha k}$ being the Lipschitz constant of function $\alpha_k(\cdot)$.

Similarly, it can be obtained that

$$\begin{aligned}
\|\Delta_\theta[\mathbf{P}(\mathbf{x}_d, \hat{\mathbf{x}}_s) - \mathbf{P}(\mathbf{x}_d, \mathbf{x}_s)]\mathbf{u}\| &\leq \sum_{i=1}^m \|\Delta_\theta[\mathbf{p}^i(\mathbf{x}_d, \hat{\mathbf{x}}_s) - \mathbf{p}^i(\mathbf{x}_d, \mathbf{x}_s)]u_i\| \\
&\leq \sum_{i=1}^m \sum_{k=1}^{n-r} \frac{|u_i|}{\theta^{k-1}} |p_{ki}(\mathbf{x}_d, \hat{x}_1^s, \dots, \hat{x}_k^s) - p_{ki}(\mathbf{x}_d, x_1^s, \dots, x_k^s)| \\
&\leq \sum_{i=1}^m (n-r)\rho_i l_{\mathbf{p}i} \|\xi\| \\
&\leq (n-r)\bar{\rho} l_{\mathbf{p}} \|\xi\|
\end{aligned} \tag{7.36}$$

where $\bar{\rho} = \sum_{i=1}^m \rho_i$, with ρ_i being the sliding gain of u_i in (7.11); $l_{\mathbf{p}i}$ denotes the maximum Lipschitz constant number in the i th-column vector $\mathbf{p}^i(\cdot)$, and $l_{\mathbf{p}} = \sup_i |l_{\mathbf{p}i}|$.

Considering the bounded distribution matrix $\mathbf{P}(\mathbf{x}_d, \mathbf{x}_s)$ in Assumption 7.3, and together with the structure of $\Delta(\hat{x}_1^s, x_1^s)$ in (7.13), it has

$$\|\Delta_\theta \mathbf{P}(\mathbf{x}_d, \mathbf{x}_s) \Delta(\hat{x}_1^s, x_1^s)\| \leq \bar{P} b_m \|\xi\| \tag{7.37}$$

where \bar{P} denotes the upper bound of the modulus of the distribution matrix $\mathbf{P}(\mathbf{x}_d, \mathbf{x}_s)$, and b_m is the system coefficient in (7.1).

Therefore, substituting (7.34)-(7.34) into (7.33), we can obtain that

$$\begin{aligned}
\dot{V} &\leq -\theta \xi^T \mathbf{S}_1 \xi - \theta \|\mathbf{C}\xi\|^2 + 2\|\xi^T \mathbf{S}_1\| \|\xi\| [(n-r)l_\alpha + (n-r)\bar{\rho} l_{\mathbf{p}} + \bar{P} b_m] \\
&\leq -\theta V + 2\sigma(\mathbf{S}_1) [(n-r)l_\alpha + (n-r)\bar{\rho} l_{\mathbf{p}} + \bar{P} b_m] V \\
&= -(\theta - \theta_0) V
\end{aligned} \tag{7.38}$$

with θ_0 being defined as $\theta_0 = 2\sigma(\mathbf{S}_1) [(n-r)l_\alpha + (n-r)\bar{\rho} l_{\mathbf{p}} + \bar{P} b_m]$.

Now, it can be concluded that the estimation error \mathbf{e}_s and ξ will asymptotically converge to zero by choosing $\theta > \max\{\theta_0, 1\}$. In other words, the remaining state \mathbf{x}_s can be exactly estimated by the proposed high gain observer in (7.27)-(7.29) on the sliding mode surfaces, i.e., $\hat{\mathbf{x}}_s = \mathbf{x}_s$. ■

7.2.3 Unknown Inputs Reconstruction

Based on the analysis in the above subsections, the proposed observer to handle the nonlinear uncertain system (7.1) consists of two parts: the higher order sliding mode observers given in (7.7)-(7.12) and a high gain observer given in (7.27)-(7.29). Then, the operating procedure of the proposed observer can be described as one in which the sliding mode observers work to ensure the corresponding sliding mode surfaces are reached sequentially, then the high gain observer guarantees the asymptotic stability of the remaining estimation error dynamics.

After all the states have converged to their true values, i.e., $\hat{\mathbf{x}}_d = \mathbf{x}_d$, $\hat{\mathbf{x}}_s = \mathbf{x}_s$, it can be obtained from (7.13) that

$$\Delta(\hat{x}_1^s, x_1^s) = 0, \quad \varphi(t) = \mathbf{u} \quad (7.39)$$

which means the unknown inputs can be reconstructed from their corresponding sliding mode terms, i.e., $\varphi_i(t) = \{u_i\}_{eq}$, $i = 1, \dots, m$.

7.2.4 Modification of Sliding Mode Observer

As claimed in Remark 7.1, the proposed recursive sliding mode observers in (7.7)-(7.12) are sensitive to the low pass filters' parameters which are required to obtain the equivalent injection signals of the sliding mode terms. As a result, the estimation accuracies of the states and the unknown inputs will be affected. So it is necessary to consider how to remove or attenuate the low pass filters' effect.

As mentioned in [78, 82], the unknown inputs may be recovered from the corresponding equations instead of reconstructing from the additional low pass filters. This is reasonable since the sliding mode differentiator can be treated as a robust nonlinear filter. In this section, we shall consider the higher order sliding mode differentiator design and the most challenging task lies with the definition of intermediate variables which are used to handle the unknown inputs' effect on the measurement outputs.

Assumption 7.6. *The unknown input vector $\varphi(t)$ is locally smooth.*

In keeping with the previous sections, the robust terms \bar{u}_i are chosen based on the corresponding equations, as

$$\bar{u}_i = \dot{\hat{x}}_{r_i}^i - f_i(\hat{\mathbf{x}}_d^1, \dots, \hat{\mathbf{x}}_d^i) - b_i \hat{x}_1^{i+1}, \quad i = 1, \dots, m \quad (7.40)$$

where $\dot{\hat{x}}_{r_i}^i$ indicates the estimated value of $\dot{x}_{r_i}^i$, and \hat{x}_1^{m+1} is used to denote \hat{x}_1^s for notational simplicity.

Then, based on the system measurement outputs \mathbf{y} in (7.1), we define new variables \bar{y}_i , $i = 1, \dots, m$ as

$$\begin{aligned} \bar{y}_1 &= y_1 \\ \bar{y}_2 &= [y_2 - d_{11}\bar{u}_1 - d_{11}b_1\hat{x}_1^2]/[1 - d_{11}b_1] \\ &\vdots \\ \bar{y}_m &= [y_m - \sum_{i=1}^{m-1} d_{(m-1)i}\bar{u}_i - d_{(m-1)(m-1)}b_{m-1}\hat{x}_1^m]/[1 - d_{(m-1)(m-1)}b_{m-1}] \end{aligned} \quad (7.41)$$

Note that only the first m outputs of \mathbf{y} are used to construct these new variables \bar{y}_i , which are defined to ensure the clean signals x_1^i can be approached sequentially, i.e., $\bar{y}_i \rightarrow x_1^i$, $i = 1, \dots, m$. Then, the $(r_i + 1)$ -th order sliding mode differentiator [69] for the derivatives $\bar{y}_i^{(k)}$, $i = 1, \dots, m$, $k = 1, \dots, r_i$ can be expressed in the form of

$$\begin{aligned} \dot{\hat{x}}_1^i &= \bar{v}_0^i, \\ \bar{v}_0^i &= -\bar{\lambda}_0^i |\hat{x}_1^i - \bar{y}_1|^{(r_i-1)/r_i} \text{sign}(\hat{x}_1^i - \bar{y}_1) + \hat{x}_2^i \\ \dot{\hat{x}}_2^i &= \bar{v}_1^i, \\ \bar{v}_1^i &= -\bar{\lambda}_1^i |\hat{x}_2^i - \bar{v}_0^i|^{(r_i-2)/(r_i-1)} \text{sign}(\hat{x}_2^i - \bar{v}_0^i) + \hat{x}_3^i \\ &\vdots \\ \dot{\hat{x}}_{r_i}^i &= \bar{v}_{r_i-1}^i, \\ \bar{v}_{r_i-1}^i &= -\bar{\lambda}_{r_i-1}^i |\hat{x}_{r_i}^i - \bar{v}_{r_i-2}^i|^{1/2} \text{sign}(\hat{x}_{r_i}^i - \bar{v}_{r_i-2}^i) + \dot{\hat{x}}_{r_i}^i \\ \ddot{\hat{x}}_{r_i}^i &= -\bar{\lambda}_{r_i}^i \text{sign}(\hat{x}_{r_i}^i - \bar{v}_{r_i-1}^i) \end{aligned} \quad (7.42)$$

where the positive parameters $\bar{\lambda}_k^i$, $k = 0, \dots, r_i$, $i = 1, \dots, m$ are properly chosen. Then the following equalities are true after a finite time transient process,

$$\hat{x}_1^i = \bar{y}_i, \quad \dots, \quad \hat{x}_{r_i}^i = \bar{y}_i^{(r_i-1)}, \quad \dot{\hat{x}}_{r_i}^i = \bar{y}_i^{(r_i)}, \quad i = 1, \dots, m \quad (7.43)$$

Theorem 7.3. *Suppose that the dynamic subsystem of \mathbf{x}_d given by (7.1) satisfies Assumptions 7.4 - 7.6, the higher-order sliding mode differentiators given by (7.41)-(7.42) ensures that, after a finite time transient, the variables x_k^i , $i = 1, \dots, m$, $k = 1, \dots, r_i$ can be exactly estimated by \hat{x}_k^i , i.e., $\hat{\mathbf{x}}_d = \mathbf{x}_d$. Furthermore, once the variable \mathbf{x}_d has been estimated by $\hat{\mathbf{x}}_d$, the unknown inputs can be re-presented as*

$$\varphi(t) = \mathbf{u} + \Delta(\hat{x}_1^s, x_1^s) \quad (7.44)$$

where

$$\mathbf{u} = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_m \end{pmatrix} \in \mathbb{R}^m, \quad \Delta(\hat{x}_1^s, x_1^s) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_m[\hat{x}_1^s - x_1^s] \end{pmatrix} \in \mathbb{R}^m$$

Proof. Consider the equations in (7.43) and the differentiator structures of the \mathbf{x}_d subsystem given in (7.1), it is clear that this theorem holds if and only if $\bar{y}_i = x_1^i$, $i = 1, \dots, m$ can be reached and remained thereafter.

By substituting (7.14) into the definition of \bar{y}_i given by (7.41), it can be deduced that

$$\begin{aligned} \bar{y}_1 &= x_1^1 \\ \bar{y}_2 &= \{x_1^2 + d_{11}[\varphi_1(t) - \bar{u}_1] - d_{11}b_1\hat{x}_1^2\}/[1 - d_{11}b_1] \\ &\vdots \\ \bar{y}_m &= \{x_1^m + \sum_{i=1}^{m-1} d_{(m-1)i}[\varphi_i(t) - \bar{u}_i] - d_{(m-1)(m-1)}b_{m-1}\hat{x}_1^m\} \\ &\quad / [1 - d_{(m-1)(m-1)}b_{m-1}] \end{aligned} \quad (7.45)$$

Clearly, \bar{y}_1 is exactly equal to x_1^1 in the initial states, and with the proposed higher-order sliding mode differentiator in (7.42), the corresponding result in (7.43) can be described as

$$\hat{x}_1^1 = \bar{y}_1 = x_1^1, \quad \dots, \quad \hat{x}_{r_1}^1 = \bar{y}_1^{(r_1-1)} = x_{r_1}^1, \quad \dot{\hat{x}}_{r_1}^1 = \bar{y}_1^{(r_1)} = \dot{x}_{r_1}^1 \quad (7.46)$$

which implies $\hat{\mathbf{x}}_d^1 = \mathbf{x}_d^1$. Together with the dynamics of $x_{r_1}^1$ in (7.1), the robust

term \bar{u}_1 defined in (7.40) can be simplified into

$$\bar{u}_1 = \dot{x}_{r_1}^1 - f_1(\mathbf{x}_d^1) - b_1 \hat{x}_1^2 = b_1(x_1^2 - \hat{x}_1^2) + \varphi_1(t) \quad (7.47)$$

Then, with (7.45), it can be deduced that

$$\bar{y}_2 = x_1^2 \quad (7.48)$$

Now, it can be seen that \bar{y}_2 is exactly equal to ξ_1^2 once the higher order sliding mode differentiator reaches its sliding surface, i.e., $\hat{\mathbf{x}}_d^1 = \mathbf{x}_d^1$. After that, the differentiator of \bar{y}_2 in (7.42) kicks in and the following equations will hold after a finite time:

$$\hat{x}_1^2 = \bar{y}_2 = x_1^2, \quad \dots, \quad \hat{x}_{r_2}^2 = \bar{y}_2^{(r_2-1)} = x_{r_2}^2, \quad \dot{\hat{x}}_{r_2}^2 = \bar{y}_2^{(r_2)} = \dot{x}_{r_2}^2 \quad (7.49)$$

which means $\hat{\mathbf{x}}_d^2 = \mathbf{x}_d^2$. Thus, the robust terms \bar{u}_1 and \bar{u}_2 can be rewritten as

$$\begin{aligned} \bar{u}_1 &= \varphi_1(t) \\ \bar{u}_2 &= b_2(x_1^3 - \hat{x}_1^3) + \varphi_2(t) \end{aligned} \quad (7.50)$$

In order to complete the proof, we need to introduce the following Lemma 7.2.

Lemma 7.2. *Suppose the following proposition is true for a given integer k such that $2 \leq k \leq m-1$,*

$$\begin{aligned} 1) \quad \hat{\mathbf{x}}_d^i &= \mathbf{x}_d^i, \quad \bar{y}_i = x_1^i; & i &= 1, \dots, k; \\ 2) \quad \bar{u}_i - \varphi_i(t) &= 0, & i &= 1, \dots, k-1; \\ \bar{u}_k - \varphi_k(t) &= -b_k[\hat{x}_1^{k+1} - x_1^{k+1}] \end{aligned}$$

then it will be also true for $k+1$.

Proof. First, based on the previous analysis, the above two propositions are true for $k=2$. Next, we shall prove these propositions still hold for $k'=k+1$.

By substituting 2) into \bar{y}_{k+1} given in (7.45), it can be obtained that

$$\begin{aligned}\bar{y}_{k+1} &= \{x_1^{k+1} + \sum_{i=1}^k d_{ki}[\varphi_i(t) - \bar{u}_i] - d_{kk}b_k\hat{x}_1^{k+1}\}/[1 - d_{kk}b_k] \\ &= \{x_1^{k+1} + d_{kk}[\varphi_k(t) - \bar{u}_k] - d_{kk}b_k\hat{x}_1^{k+1}\}/[1 - d_{kk}b_k] \\ &= x_1^{k+1}\end{aligned}\quad (7.51)$$

Then, with the higher-order sliding mode differentiator of \bar{y}_{k+1} in (7.42), the following equations will be true in finite time:

$$\hat{x}_1^{k+1} = x_1^{k+1}, \quad \dots, \quad \hat{x}_{r_{k+1}}^{k+1} = x_{r_{k+1}}^{k+1}, \quad \dot{\hat{x}}_{r_{k+1}}^{k+1} = \dot{x}_{r_{k+1}}^{k+1}, \quad (\text{i.e., } \hat{\mathbf{x}}_d^{k+1} = \mathbf{x}_d^{k+1}) \quad (7.52)$$

With the dynamic equation of $x_{r_{k+1}}^{k+1}$ in (7.1) and the definition of \bar{u}_{k+1} in (7.40), it can be deduced that

$$\begin{aligned}\bar{u}_k &= \varphi_k(t) \\ \bar{u}_{k+1} &= \dot{x}_{r_{k+1}}^{k+1} - f_{k+1}(\mathbf{x}_d^1, \dots, \mathbf{x}_d^{k+1}) - b_{k+1}\hat{x}_1^{k+2} \\ &= -b_{k+1}[\hat{x}_1^{k+2} - x_1^{k+2}] + \varphi_{k+1}(t)\end{aligned}\quad (7.53)$$

Thus, these two statements are also true for $k' = k + 1$. ■

Now, by substituting $k = m$ into Lemma 7.2, the following equalities hold:

$$\begin{aligned}\hat{\mathbf{x}}_d^i &= \mathbf{x}_d^i, \quad \bar{y}_i = x_1^i, & i &= 1, \dots, m; \\ \bar{u}_i &= \varphi_i(t), & i &= 1, \dots, m-1; \\ \bar{u}_m - \varphi_m(t) &= -b_m[\hat{x}_1^{m+1} - x_1^{m+1}] = -b_m[\hat{x}_1^s - x_1^s]\end{aligned}$$

By defining \mathbf{u} and $\Delta(\hat{x}_1^s, x_1^s)$ as in Theorem 7.3, we can obtain equation (7.44). ■

Clearly, the result in Theorem 7.3 is similar to Theorem 7.1, so the previous results on the remaining dynamics and the reconstruction of the unknown inputs can be directly applied. Moreover, as the new robust terms \bar{u}_i are obtained by solving the corresponding equations, the additional low pass filters are successfully removed, and the unknown inputs can be directly obtained after all the states have converged to the true values.

It has been proven in [69] that the sliding accuracy can be improved with the order of sliding mode algorithm. Thus, this provides a potential to increase the estimation performance by artificially increase the order of the proposed higher order sliding mode differentiator in (7.42).

Remark 7.2. *In this subsection, the higher order sliding mode differentiator algorithm [69] is used to attenuate the chattering effect, as well as to avoid the requirement of additional low pass filters. Moreover, in the case when the relative degree of any dynamic subsystem in (7.1) is one or two, i.e., $r_i = 1$ or $r_i = 2$, $i \in \{1, \dots, m\}$, the super-twisting algorithm in [53] or the modified super-twisting algorithm in [86] can be applied to avoid the requirement of low pass filters, with slight adjustment of u_i given in (7.11).*

Remark 7.3. *For a given nonlinear uncertain system that can be transformed into the system given in (7.1) through a nonlinear transformation (such as the Lie derivatives), and all the mentioned Assumptions are satisfied in the transformed domain, then an observer design for the original system can be performed by the means of inverse transformation.*

7.3 Numerical Simulations

In this section, a six-order numerical example is used to demonstrate effectiveness of the the proposed observer. Consider

$$\dot{\mathbf{x}} = \begin{pmatrix} -4(x_1)^2 + 0.2x_2 + 10 \\ x_4 \\ x_5 \\ -4x_2 + 0.2x_3 + 10 \\ x_6 \\ -3x_6 - x_3x_5 - x_4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0.4 \\ 1 & 0.5x_4 \end{pmatrix} \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} \quad (7.54)$$

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0.3 & 0 \\ 0.5 & 0.1 \end{pmatrix} \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} \quad (7.55)$$

where $\mathbf{x} = [x_1, x_2, x_3, x_4, x_5, x_6]^T \in \mathfrak{R}^6$ is the system state, $\mathbf{y} = [y_1, y_2, y_3]^T \in \mathfrak{R}^3$ is the system output, $\varphi(t) = [\varphi_1(t), \varphi_2(t)]^T$ is the system unknown input which needs to be reconstructed.

Then, under the coordinate transformation: $x_1^1 = x_1$, $x_1^2 = x_2$, $x_1^2 = x_4$, $x_1^s = x_3$, $x_2^s = x_5$, $x_3^s = x_6$, and $\mathbf{x} = [(\mathbf{x}_d)^T, (\mathbf{x}_s)^T]^T$, $\mathbf{x}_d = [x_d^1, x_1^2, x_2^2]^T \in \mathfrak{R}^3$, $\mathbf{x}_s = [x_1^s, x_2^s, x_3^s]^T \in \mathfrak{R}^3$. The original numerical example can be rewritten in the form of (7.1), as

$$\begin{aligned} \dot{x}_1^1 &= -4(x_1^1)^2 + 0.2x_1^2 + 10 + \varphi_1(t) \\ \dot{x}_1^2 &= x_2^2 \\ \dot{x}_2^2 &= -4x_1^2 + 0.2x_1^s + 10 + \varphi_2(t) \\ \dot{x}_1^s &= x_2^s \\ \dot{x}_2^s &= x_3^s + \varphi_1(t) + 0.4\varphi_2(t) \\ \dot{x}_3^s &= -3x_3^s - x_1^s x_2^s - x_2^2 + \varphi_1(t) + 0.5x_2^2 \varphi_2(t) \end{aligned} \quad (7.56)$$

with the measurable outputs as

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1^1 \\ x_1^2 + 0.3\varphi_1(t) \\ x_1^s + 0.5\varphi_1(t) + 0.1\varphi_2(t) \end{pmatrix}, \quad \text{with } \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0.3 & 0 \\ 0.5 & 0.1 \end{pmatrix} \quad (7.57)$$

It is clear that the transformed nonlinear system has the unknown inputs appearing not only just in the state dynamics, but also in the measurement outputs, and that Assumptions 7.1-7.5 are satisfied. Then, the proposed observer can be used to estimate the states and identify the unknown inputs.

7.3.1 Proposed Robust Observer

By setting the corresponding estimated variables as $\{\hat{x}_1^1, \hat{x}_1^2, \hat{x}_2^2, \hat{x}_1^s, \hat{x}_2^s, \hat{x}_3^s\}$, a robust observer can be designed in the form of

$$\begin{aligned}
 \dot{\hat{x}}_1^1 &= -4(\hat{x}_1^1)^2 + 0.2\hat{x}_1^2 + 10 + u_1 \\
 \dot{\hat{x}}_1^2 &= \hat{x}_2^2 \\
 \dot{\hat{x}}_2^2 &= -4\hat{x}_1^2 + 0.2\hat{x}_1^s + 10 + u_2 \\
 \dot{\hat{x}}_1^s &= \hat{x}_2^s - 3\theta\sigma_3 \\
 \dot{\hat{x}}_2^s &= \hat{x}_3^s + u_1 + 0.4u_2 - 3\theta^2\sigma_3 \\
 \dot{\hat{x}}_3^s &= -3\hat{x}_3^s - \hat{x}_1^s\hat{x}_2^s - \hat{x}_2^2 + u_1 + 0.5\hat{x}_2^2u_2 - \theta^3\sigma_3
 \end{aligned} \tag{7.58}$$

with

$$\begin{aligned}
 \sigma_1 &= \hat{x}_1^1 - y_1 \\
 \sigma_2 &= [\hat{x}_1^2 + 0.3\{u_1\}_{eq} - y_2]/0.94 \\
 \sigma_3 &= [\hat{x}_1^s + 0.5\{u_1\}_{eq} + 0.1\{u_2\}_{eq} - y_3]/0.98
 \end{aligned} \tag{7.59}$$

where θ is the high gain feedback parameter, u_1 and u_2 are the first-order and second-order sliding mode terms. Based on (7.11) and (7.12), we have

$$\begin{aligned}
 u_1 &= -\rho_1 \text{sign}(\sigma_1) \\
 u_2 &= -\rho_2(z_1 + |z_0|^{1/2} \text{sign}(z_0))/(|z_1| + |z_0|^{1/2}) \\
 \dot{z}_0 &= v_0 \\
 v_0 &= -1.5M^{1/2}|z_0 - \sigma_2|^{1/2} \text{sign}(z_0 - \sigma_2) + z_1 \\
 \dot{z}_1 &= -1.1M \text{sign}(z_1 - v_0)
 \end{aligned} \tag{7.60}$$

with M being a turning parameter, $\{u_1\}_{eq}$ and $\{u_2\}_{eq}$ are the equivalent signals of u_1 and u_2 , respectively, which are obtained through low-pass filters.

In order to demonstrate the performance of the proposed approach, an observer without any equivalent signal compensation on the measurable outputs, i.e., $\{u_1\}_{eq} = \{u_2\}_{eq} = 0$, is also applied.

For simulation purposes, we choose $\varphi_1(t) = 2 \sin(t)$ and $\varphi_2(t) = 2 \sin(3t) + 1 \cos t$. Then, the parameters for observer (7.58)-(7.60) are chosen as: $\rho_1 = 3, \rho_2 = 4, \theta = 4, M = 65$. The simulation is performed with the initial values

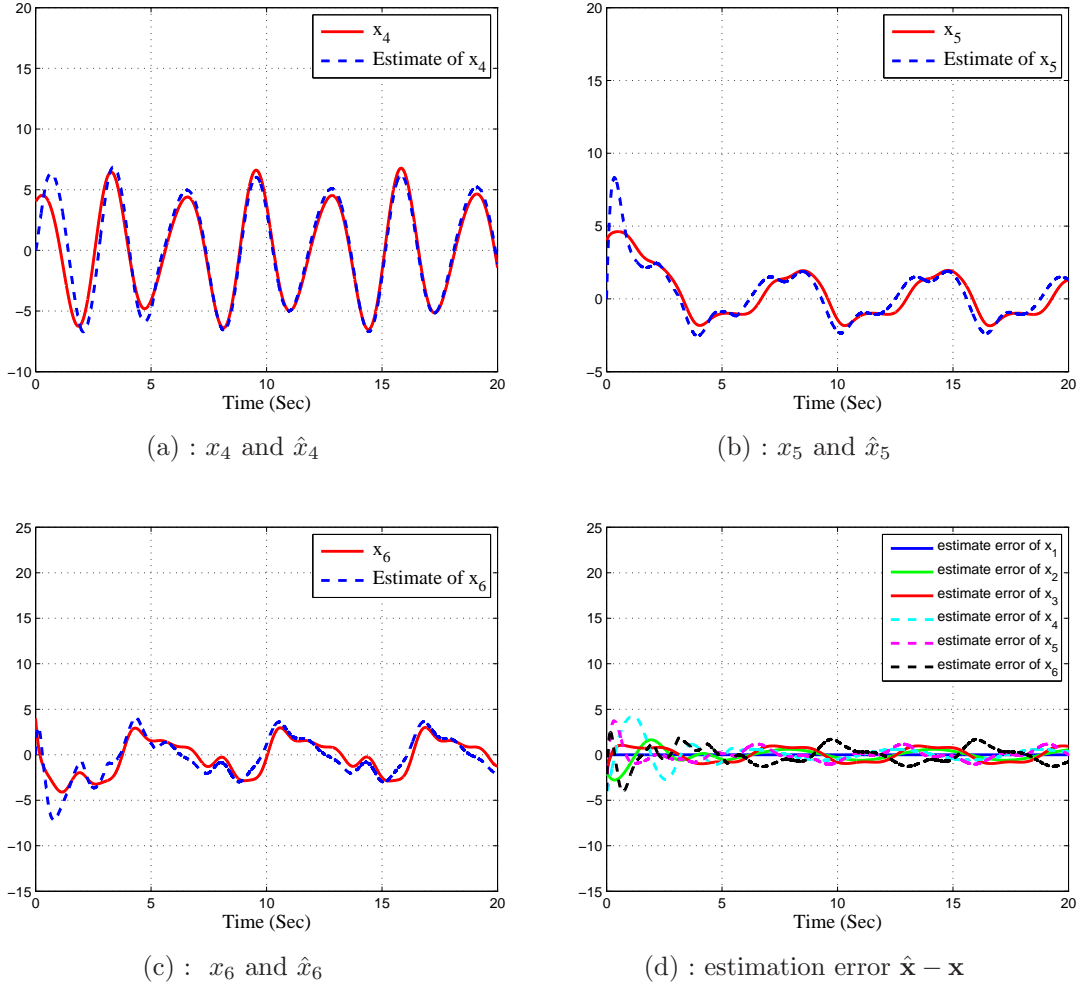


Figure 7.1: The estimation performance without compensation

$\mathbf{x}_0 = [2, 2, 2, 4, 4, 4]^T$, $\hat{\mathbf{x}}_0 = [0, 0, 0, 0, 0, 0]^T$, $z_0 = z_1 = v_0 = 1$. The simulation step is set to 0.5 ms.

First, the observer without compensation on outputs is employed, and the performance is shown in Figure 7.1. It is clear that the estimated states fail to track the true values due to the uncertainties in the system outputs.

Then, the proposed observer described by (7.58)-(7.60) is applied, and the improved estimation performance can be clearly seen in Figure 7.2. And the unknown inputs are successfully reconstructed after about 7s, as shown in Figure 7.2e and Figure 7.2e.

However, there exist small ripples in the estimated states and the reconstructed

unknown inputs, see Figure 7.2c and Figure 7.2f. This is mainly caused by the low pass filters, which are necessary for the reconstruction of the equivalent signals of u_1 and u_2 .

7.3.2 Modification of Proposed Observer

In order to eliminate the additional low pass filter's effect and to improve the estimation performance, we suggest to use the higher order sliding mode differentiator to replace the previous quasi-sliding mode observer, since the unknown inputs are assumed to be smooth, i.e., Assumption 7.6 is satisfied.

According to (7.40) and (7.41), we define \bar{u}_i and \bar{y}_i , $i = 1, 2$ as follows:

$$\begin{aligned}\bar{u}_1 &= z_1^1 + 4(\hat{x}_1^1)^2 - 0.2\hat{x}_1^2 - 10 \\ \bar{u}_2 &= z_1^2 + 4\hat{x}_1^2 - 0.2\hat{x}_1^s - 10 \\ \bar{y}_1 &= y_1 \\ \bar{y}_2 &= [y_2 - 0.3\bar{u}_1 - 0.06\hat{x}_1^2]/0.94\end{aligned}\tag{7.61}$$

Here, z_1^1 and z_2^2 are used to indicate the estimated values of \hat{x}_1^1 and \hat{x}_2^2 respectively. Then, a third order and a forth order sliding mode differentiators are proposed for x_1^1 and x_1^2 , given as

$$\begin{aligned}\dot{\hat{x}}_1^1 &= v_0^1 \\ v_0^1 &= -3L^{1/3}|\hat{x}_1^1 - \bar{y}_1|^{2/3}\text{sign}(\hat{x}_1^1 - \bar{y}_1) + z_1^1 \\ \dot{z}_1^1 &= v_1^1 \\ v_1^1 &= -1.5L^{1/2}|z_1^1 - v_0^1|^{1/2}\text{sign}(z_1^1 - v_0^1) + z_2^1 \\ \dot{z}_2^1 &= -1.1L\text{sign}(z_2^1 - v_1^1)\end{aligned}\tag{7.62}$$

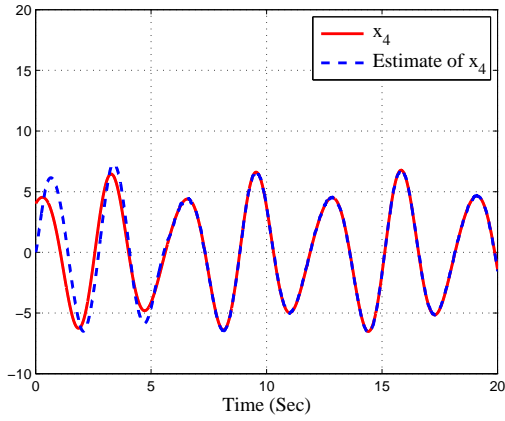
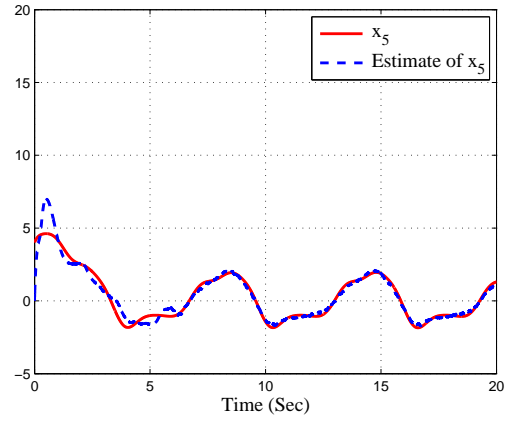
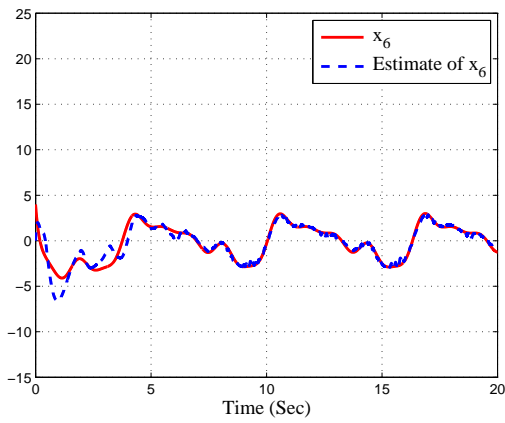
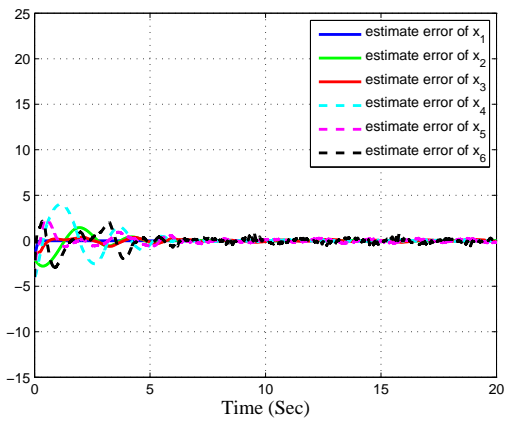
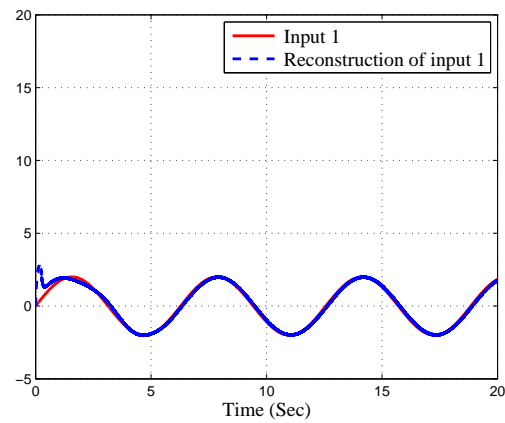
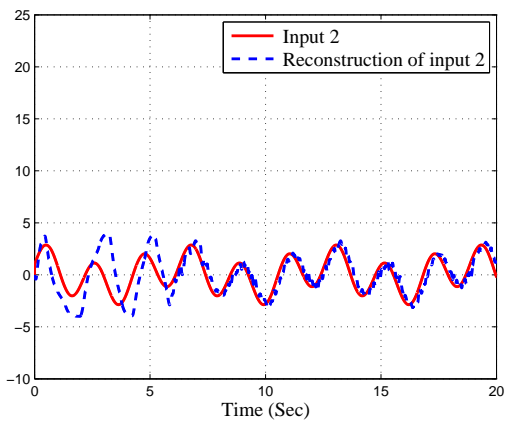
(a) : x_4 and \hat{x}_4 (b) : x_5 and \hat{x}_5 (c) : x_6 and \hat{x}_6 (d) : estimation error $\hat{\mathbf{x}} - \mathbf{x}$ (e) : $\varphi_1(t)$ and $\hat{\varphi}_1(t)$ (f) : $\varphi_2(t)$ and $\hat{\varphi}_2(t)$

Figure 7.2: The estimation performance with the proposed observer

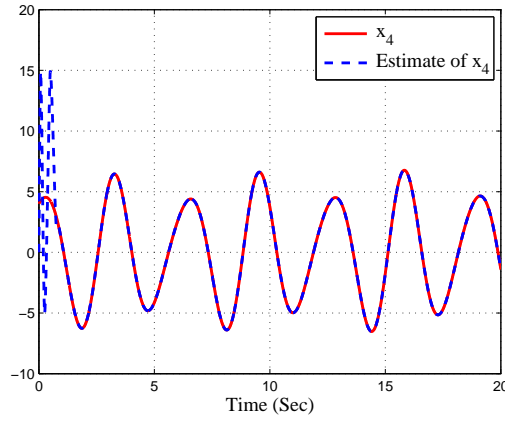
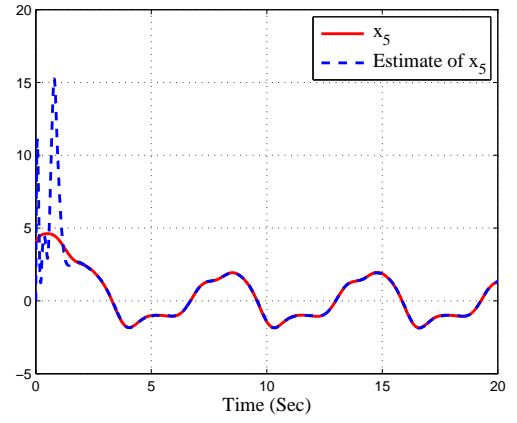
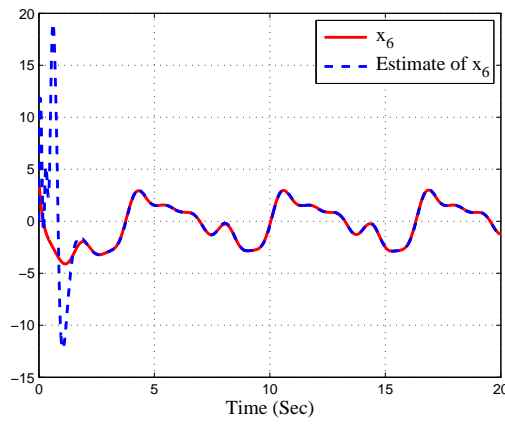
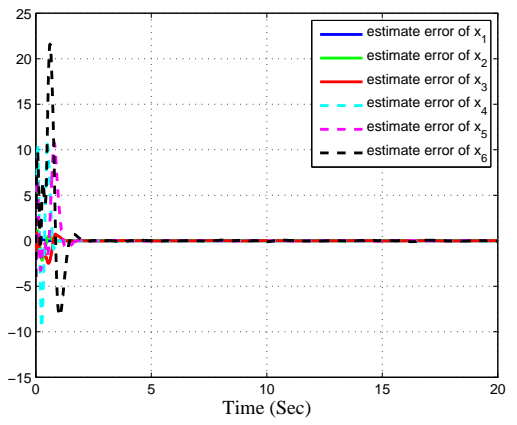
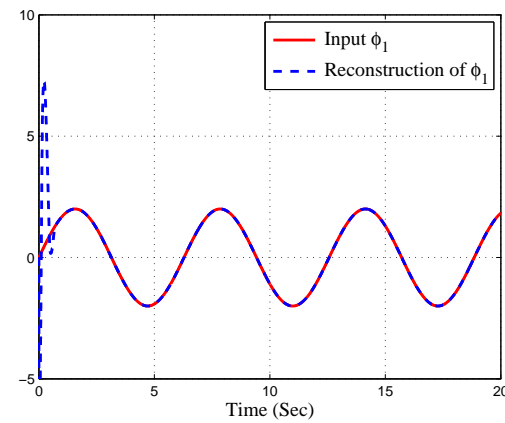
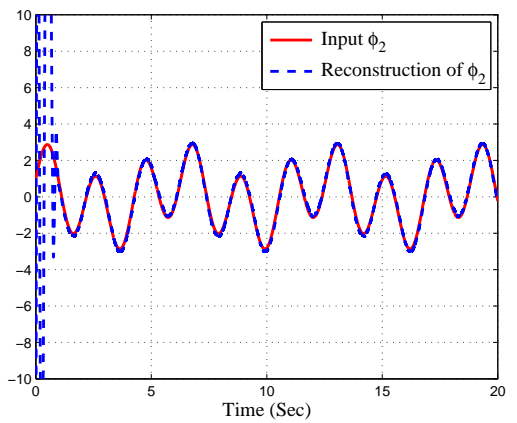
(a) : x_4 and \hat{x}_4 (b) : x_5 and \hat{x}_5 (c) : x_6 and \hat{x}_6 (d) : estimation error $\hat{\mathbf{x}} - \mathbf{x}$ (e) : $\varphi_1(t)$ and $\hat{\varphi}_1(t)$ (f) : $\varphi_2(t)$ and $\hat{\varphi}_2(t)$

Figure 7.3: The estimation performance with the modified observer

and

$$\begin{aligned}
\dot{\hat{x}}_1^2 &= v_0^2 \\
v_0^2 &= -5K^{1/4}|\hat{x}_1^2 - \bar{y}_2|^{3/4}\text{sign}(\hat{x}_1^2 - \bar{y}_2) + \hat{x}_2^2 \\
\dot{\hat{x}}_2^2 &= v_1^2 \\
v_1^2 &= -3K^{1/3}|\hat{x}_2^2 - v_0^2|^{2/3}\text{sign}(\hat{x}_2^2 - v_0^2) + z_2^2 \\
\dot{z}_2^2 &= v_2^2 \\
v_2^2 &= -1.5K^{1/2}|z_2^2 - v_1^2|^{1/2}\text{sign}(z_2^2 - v_1^2) + z_3^2 \\
\dot{z}_3^2 &= -1.1K\text{sign}(z_3^2 - v_2^2)
\end{aligned} \tag{7.63}$$

with L and M being the tuning parameters.

Then, by defining σ_3 similar to (7.59), as

$$\sigma_3 = [\hat{x}_1^s + 0.5\bar{u}_1 + 0.1\bar{u}_2 - y_3]/0.98 \tag{7.64}$$

Similar to (7.58), a high gain observer can be designed for the remaining dynamics of \mathbf{x}_s , as

$$\begin{aligned}
\dot{\hat{x}}_1^s &= \hat{x}_2^s - 3\theta\sigma_3 \\
\dot{\hat{x}}_2^s &= \hat{x}_3^s + \bar{u}_1 + 0.4\bar{u}_2 - 3\theta^2\sigma_3 \\
\dot{\hat{x}}_3^s &= -3\hat{x}_3^s - \hat{x}_1^s\hat{x}_2^s - \hat{x}_2^2 + \bar{u}_1 + 0.5\hat{x}_2^2\bar{u}_2 - \theta^3\sigma_3
\end{aligned} \tag{7.65}$$

We choose the same simulation parameters as before except for the tuning parameters which are set as: $L = 45$, $K = 200$. And the simulations results are shown in Figure 7.3.

Compared with the previous results in Figure 7.2, it can be seen that the estimation performance is greatly improved. Moreover, the unknown inputs can be reconstructed with a better accuracy, as seen in Figure 7.3e and Figure 7.3f, and the visible ripples have been significantly attenuated.

7.4 Summary

In this chapter, a robust observer based on a recursive higher order sliding mode algorithm has been developed for a class of MIMO nonlinear systems with unknown inputs in states and measurable outputs. With more output measurements

than the number of unknown inputs, plus proper assumptions on the system structure, the restrictive involutive condition is not required, and the states and unknown inputs can be identified asymptotically. Furthermore, a modified observer without low pass filtering is also presented to improve the estimation performance if the unknown inputs are smooth.

The contribution of in this chapter can be summarized as:

- (i) A robust hybrid observer is proposed to handle a class of MIMO nonlinear systems with the unknown inputs appearing in both the states and outputs.
- (ii) A novel HSMC based recursive algorithm is developed to track and recover the unknown inputs.
- (iii) The restrictive involutive condition is not imposed because of an unknown-state dependent distribution matrix of the unknown inputs, and a high gain feedback is proposed to ensure identifiability of the unknown inputs.

Chapter 8

Conclusion and Future Work

8.1 Conclusion

In this thesis, we have investigated the state and unknown input estimations of nonlinear uncertain systems, in which the unknown input enters the systems through an unknown-state-dependent distribution vector, i.e., we considered the problem of non-matching disturbance in the observer sense. The identifiability of the unknown input and system states is carefully addressed based on the idea that *the unknown input can be replaced by some nominal dynamics while the corresponding sliding mode surface is reached*. The contribution of the thesis can be summarized as follows:

- (i) A hybrid observer which integrates a full-order high gain feedback with a higher-order sliding mode term is developed for a class of SISO uncertain systems. With the high gain feedback, the state estimation error will converge into an invariant set regardless of the initial conditions, in which the sliding condition is satisfied thereby ensuring the sliding surface is reached. However, the identifiability of the unknown input as well as system states is strictly related to the stability of the reduced-order dynamic system structure, which can be classified into three categories:
 - The reduced-order dynamics is asymptotically stable on the sliding surface. Then, all states and unknown input can be asymptotically

identified with the proposed hybrid observer.

- The reduced-order dynamics is stable, but not asymptotically stable. Then, only partial states can be exactly identified.
- The reduced-order dynamics is divergent. Then, the states estimation error will fall into an invariant set instead of converging to zeros, and the unknown input fails to be identified.

- (ii) To verify the effectiveness of the proposed hybrid observer design approach, it was implemented on a series DC motor for a non-matching time-varying parameter identification. From the experimental results, the non-matching motor parameter can be successfully identified based on the measurable current and input voltage, as well as the unknown rotor speed. The identified parameter is then used to enhance the speed estimation performance in the presence of external disturbance. Monte carlo simulations are also conducted to illuminate the identification accuracy with respect to measurement noises of motor resistance and inductance.
- (iii) Based on the idea that the sliding mode chattering affects only the accuracy of the reconstructed uncertainties, but not the system states, a novel perspective on the sliding mode observer design for speed and position estimations of a surface-mounted PMSM is presented. With a one time calibration of the position estimation, which can be conducted by sensing the zero-crossing of the back-EMFs, the desirable speed and rotor information can be exactly estimated without filtering effect.
- (iv) To handle the estimation problems of a class of MIMO nonlinear systems with non-matching inputs, a hybrid observer that combines multiple higher order sliding mode feedbacks with a reduced-order high gain feedback is proposed. With proper system structure assumptions and that the number of measurement outputs is assumed to be one more than the number of unknown inputs, then the unknown inputs, as well as the full-order system state, can be asymptotically estimated without the requirement of the restrictive involutive condition.

- (v) A more general uncertain MIMO nonlinear system in which the unknown inputs appear in both the state dynamics and the measurement outputs is also studied. A recursive sliding mode observer integrated with a reduced-order high gain feedback is developed, in which a novel recursive sliding observer ensures that the sliding surfaces are reached sequentially, meanwhile, the valuable signals in the measurement outputs can be gradually extracted by cancelling the unknown inputs in sequence. The reduced-order high gain feedback designed based on the extra measurement output will work to guarantee that both the unknown inputs and states can be identified asymptotically.

8.2 Future Work

Several interesting research issues that could serve as future research directions are as follows:

- (i) Note that the proposed hybrid observers in this thesis are developed and analyzed in the continuous-time domain. It would be interesting and useful to extend these works into the discrete-time domain for ease of implementation in real systems, for example by discretization via *Taylor* series expansion. Then, the relationship between stability, estimation error, and sampling interval need to be carefully addressed.

It should be mentioned that the stability and sliding conditions of discrete-time sliding mode (DSM) are quite different from its continuous counterpart, and remained a challenging problem.

- (ii) The proposed observer in Chapter 5 for speed and position related dynamics estimations of a surface-mounted PMSM is proved to be robust against the chattering/filtering effect. It would be necessary and significant to take motor parameter variations into consideration, especially the variation in winding resistance due to temperature changing. For such case, an additional dynamic equation $\dot{R}_s = 0$ can be used, with R_s denoting the winding resistance.

Moreover, experimental verification of speed sensorless control should be conducted based on the proposed observer, to demonstrate the effectiveness of the proposed approach.

- (iii) In most practical systems, the states as well as system uncertainties are restrained by some particular conditions (e.g. $(\sin \theta)^2 + (\cos \theta)^2 = 1$ in Chapter 5). It would be significant to consider the sliding mode observer design for these particular systems with the required conditions, since the intrinsic property of sliding mode approaches is to constrain the system trajectory towards a predefined manifold and staying on it thereafter.

In fact, the restrained condition, $(\sin \theta)^2 + (\cos \theta)^2 = 1$, has been partially included in the modeling of the dynamic systems of (5.10) by taking differential operator. As a result, a one time calibration signal that works as initial conditions is required to recover the real-time system states.

- (iv) The proposed hybrid observers in Chapter 6 and Chapter 7 are developed for state and unknown input estimations of a class of uncertain MIMO nonlinear systems. It would be significant to extend and apply such results into some practical engineering systems, such as for observer-based fault detection or system health diagnosis purposes.
- (v) Although the hybrid sliding observer design in this thesis was limited to a combination of sliding mode techniques with high gain feedback for state and non-matching unknown input estimations, it would be significant to extend such design methodology to other research methods, such as by integrating the sliding mode observer with adaptive techniques to handle parameter variations [97] or to obtain variable sliding gains for chattering attenuation [147].

In fact, the integration of sliding mode techniques with some soft-computing (SC) approaches [98] to achieve higher performance has attracted increasing attention in recent years, such as neural networks (NNs), fuzzy logic (FL), and so on.

Author's Publications

1. Y. Zhou, Y. C. Soh, and J. X. Shen, "Robust hybrid observer for speed sensorless estimation of a DC motor," in *The 9th IEEE International Conference on Control and Automation (ICCA)*, Santiago, Chile, pp. 784-789, Dec. 19-21, 2011.
2. Y. Zhou, Y. C. Soh, and J. X. Shen, "Speed estimation and nonmatched time-varying parameter identification for a DC motor with hybrid sliding-mode observer," *IEEE Transactions on Industrial Electronics*, 2012.
3. Y. Zhou, Y. C. Soh, and J. X. Shen, "High gain observer with higher order sliding mode for state and unknown disturbance estimations," *International Journal of Robust and Nonlinear Control*, 2013.
4. Y. Zhou, Y. C. Soh, and J. X. Shen, "Recursive sliding mode observer for robust states estimation and unknown inputs identification of a class of nonlinear systems," Submission with revision.
5. Y. Zhou, Y. C. Soh, and J. X. Shen, "Robust hybrid observer for state and unknown input estimation in nonlinear systems," Submission with revision.

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