Reliability analysis of earth slopes accounting for spatial variation

Ji, Jian

2013

Ji, J. (2013). Reliability analysis of earth slopes accounting for spatial variation. Doctoral thesis, Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/52174>

<https://doi.org/10.32657/10356/52174>

**NANYANG
TECHNOLOGICAL
UNIVERSITY**

**RELIABILITY ANALYSIS OF EARTH SLOPES
ACCOUNTING FOR SPATIAL VARIATION**

Ji Jian

SCHOOL OF CIVIL AND ENVIRONMENTAL ENGINEERING

2013

Reliability Analysis of Earth Slopes Accounting For Spatial Variation

Ji Jian

Ji Jian

School of Civil and Environmental Engineering

A thesis submitted to the Nanyang Technological University

in partial fulfilment of the requirement for the degree of

Doctor of Philosophy

2013

Abstract

The stability of earth slopes is an important issue in geotechnical engineering. In the routine stability analysis, soil properties are given deterministic values, based on which a lumped factor of safety is derived as the deterministic solution. One limitation of the deterministic analysis is that the parametric uncertainty which is inherent in geotechnical engineering cannot be taken into account explicitly. In contrast, the uncertainty can be accounted for by adopting probabilistic methods. However, extending deterministic slope stability analysis to probabilistic analysis is often a complicated and challenging task. The objective of this study is to develop some practical approaches of reliability assessment of earth slopes and to investigate the effect of spatial variation on slope reliability.

An efficient approach of first-order reliability method (FORM) is proposed based on a recursive algorithm searching for the design point in the equivalent standard normal variable space (n-space). Based on the recursive algorithm, reliability analysis involving implicit limit state surface can be carried out via finite-difference sensitivity analysis in the n-space (probabilistic sensitivity-based FORM). The probabilistic sensitivity-based FORM is shown to be as efficient as the commonly-used response surface method for FORM (RSM-based FORM), but it needs less evaluation of the limit state surface and hence is more efficient. Application of the probabilistic sensitivity-based FORM to strength-reduction finite element slope reliability analysis is presented via two case studies.

The importance of spatial variation of geotechnical properties is highlighted in this study. Based on the limit equilibrium method (LEM) of slices, two methods, namely, method of autocorrelated slices and method of interpolated autocorrelations are proposed to model the two-dimensional (2-D) spatial variability of soil properties. The

reliability index β is evaluated using a constrained optimization algorithm for FORM in a spreadsheet, where the conventional LEM of slices can be easily integrated with reliability analysis. Based on the two methods, the influence of 2-D spatial variability of soil strength parameters on slope reliability is investigated in detail. Results obtained by the two methods are compared. 2-D spatial variability characterization and a case study involving 2-D spatial variation are presented. A further study of probabilistic finite difference strength-reduction stability analysis involving 2-D spatial variation is also presented, based on the method of interpolated autocorrelations.

Three-dimensional (3-D) slope failure is investigated in this study by probabilistic analysis which accounts for the longitudinal (out-of-plane) spatial variability of shear strength parameters. The 3-D stability model is first analyzed by LEM due to its ease of obtaining an explicit performance function and of transition to probabilistic analysis. Both spatial-averaging approach and spatial-autocorrelation approach are used for the random field analysis. Comparisons between the two methods are made. For a long earth slope, the local 3-D failure with a probabilistic critical width is emphasized. Complementary to the LEM analysis, a probabilistic 3-D strength-reduction finite difference stability analysis is also presented

Finally, an efficient approach for slope system reliability analysis using stand-alone software for deterministic stability analysis is presented. A useful procedure of identifying the local minimum- β slip surfaces is introduced, and a stratified response surface method is proposed to represent the system performance functions (in multiple failure modes). The Kounias-Ditlevsen's bimodal bounds obtained based on multi-mode FORM analyses are compared with system Monte Carlo simulations. The bimodal bounds can be improved using SORM analysis if nonlinearity underlies any of the stratified response surfaces so obtained.

Acknowledgement

This author would like to express his deepest appreciation to his supervisor, Associate Professor Bak Kong Low, for his guidance and encouragement, and for the many invaluable discussions with him. It has been a very memorable and inspiring four-year learning experience. The author would also like to thank the Nanyang Technological University (NTU) of Singapore for granting him the scholarship (2008-2012) for this research program.

Special thanks are due to his associate supervisor, Professor Hongjian Liao of Xi'an Jiaotong University (XJTU) of China, for advising the author's past research work for Master's degree (2005-2008) and co-supervising this research program.

Last, but not least, the author would like to thank his family and friends for their understanding and support.

Table of Contents

Abstract	i
Acknowledgement.....	iii
Table of Contents	iv
List of Tables	xi
List of Figures	xiv
List of Symbols and Abbreviations.....	xxii
Chapter 1 Introduction.....	1
1.1 Research background	1
1.2 Research objectives.....	3
1.3 Outline of the research work	3
Chapter 2 Literature Review	6
2.1 Introduction.....	6
2.2 Limit equilibrium methods for stability analysis of slopes	6
2.2.1 Closed-form solutions	6
2.2.2 Limit equilibrium methods of slices.....	8
2.2.3 Review of techniques used for locating the critical slip surface.....	16
2.2.4 Miscellaneous consideration on slope stability analysis	18
2.3 Strength-reduction technique for stability analysis.....	20
2.3.1 Constitutive model and failure criteria.....	20

2.3.2 Shear strength reduction technique and definition of slope failure	21
2.3.3 Associated flow rule and non-associated flow rule.....	22
2.3.4 A short critical review of the strength-reduction stability analysis	22
2.4 Enhanced limit equilibrium method.....	23
2.5 Review of three-dimensional slope stability analysis methods	24
2.6 Review of methods for characterizing the geotechnical uncertainty	26
2.6.1 Basic statistics quantifying the geotechnical uncertainty	27
2.6.2 Spatial variation and method of analysis.....	31
2.6.3 Random field and variance reduction.....	37
2.7 Review of probabilistic back-analysis in updating geotechnical uncertainties	41
2.8 Review of fundamental probabilistic methods	43
2.8.1 First-order reliability method.....	43
2.8.2 Second-order reliability method	49
2.8.3 System reliability method	52
2.8.4 Monte Carlo simulation and importance sampling technique.....	54
2.8.5 Finite element reliability method.....	56
2.9 Summary.....	59

Chapter 3 Sensitivity-Based Reliability Analysis and Probabilistic

Strength-Reduction Finite Element Analysis.....	61
3.1 Introduction.....	61
3.2 Slope reliability analysis involving implicit performance function	61
3.3 Response surface method for approximating the implicit performance function (Method 1)	63

3.3.1 Polynomial response surface	63
3.3.2 An efficient method of improving the response surface	64
3.3.3 Illustration of the RSM-based FORM using a simple example.....	67
3.4 FORM with probabilistic sensitivity analysis (Method 2)	70
3.4.1 Recursive algorithm for FORM in n-space	70
3.4.2 Reliability and sensitivity analysis in n-space.....	73
3.4.3 Illustration of the probabilistic sensitivity-based FORM using a simple example	76
3.4.4 Application to a slope reliability analysis.....	77
3.4.5 Effect of perturbation values in n-space.....	78
3.5 Comparison between RSM-based and the probabilistic sensitivity-based reliability analyses.....	79
3.6 Probabilistic slope stability analysis by strength-reduction FEM – case studies...	80
3.6.1 Clarence Cannon Dam.....	80
3.6.2 James Bay Dike	90
3.7 Summary.....	97

Chapter 4 Modeling Two-Dimensional Spatial Variation in Slope

Reliability Analysis	99
4.1 Introduction.....	99
4.2 Efficient procedure for slope reliability analysis using FORM.....	100
4.2.1 Systematic search method for critical slip surface.....	100
4.2.2 Slope reliability based on systematic search for the probabilistic critical slip surface.....	101
4.3 Spatial variability of soil properties in homogeneous soils.....	104

4.4 Modeling 2-D spatial variation in slope reliability analysis	105
4.4.1 Method of autocorrelated slices	105
4.4.2 Method of interpolated autocorrelations	114
4.4.3 Using the method of interpolated autocorrelations with independent numerical packages via RSM.....	122
4.4.4 Modeling 2-D spatial variation in strength-reduction FDM stability analysis	123
4.5 Further application to a drained soil slope reliability analysis	129
4.6 Discussion on the use of different 2-D autocorrelation functions	132
4.7 Characterizing the 2-D spatial variation from in-situ tests and case studies	133
4.7.1 Stationarity of random data and effect of trend-removal.....	134
4.7.2 Vertical spatial variability	135
4.7.3 Horizontal spatial variability and effect of sampling plan.....	138
4.7.4 Random measurement errors.....	141
4.7.5 A case study involving 2-D spatial variation – James Bay dike	141
4.8 Summary and conclusions	148

Chapter 5 Three-Dimensional Limit Equilibrium & Finite Difference

Slope Stability Analysis Accounting for Longitudinal Spatial Variation

.....	150
5.1 Introduction.....	150
5.2 Probabilistic limit equilibrium analysis of 3-D slope stability	152
5.2.1 Idealized 3-D model of earth slope failure	152
5.2.2 Deterministic analysis	154

5.2.3 Probabilistic analysis considering the spatial variation.....	155
5.2.4 Comparison of reliability analyses by spatial-averaging and spatial-autocorrelation approaches	162
5.2.5 On the probabilistic critical width of 3-D failure mass.....	167
5.2.6 Discussion on the system reliability of long earth slopes.....	173
5.3 Deterministic 3-D slope stability analysis by strength reduction method	176
5.3.1 Problem description.....	176
5.3.2 Modelling techniques for 3-D stability analysis	176
5.3.3 Undrained analysis	178
5.3.4 Drained analysis.....	187
5.4 Probabilistic treatment of long earth slopes by strength reduction method.....	190
5.4.1 Critical 3-D failure mode with reasonable end-section boundary conditions	190
5.4.2 Critical failure mode in probabilistic 3-D strength reduction analysis	191
5.4.3 Effect of longitudinal autocorrelation distance on probabilistic critical width.....	198
5.5 Summary.....	201

Chapter 6 Stratified Response Surfaces for System Probabilistic

Evaluation of Slopes.....	203
6.1 Introduction.....	203
6.2 System reliability formulation involving multiple failure modes	204
6.3 Observations on local minimum F_s slip surfaces and local minimum β slip surfaces	206
6.4 Using stratified response surfaces to approximate the system performance of	

slopes.....	211
6.5 Case studies.....	214
6.5.1 Case study 1 – A two-layered hypothetical slope	215
6.5.2 Case study 2 – An embankment of cohesive soil resting on undrained soft clay foundation.....	221
6.5.3 Case study 3 – Congress Street cut with probabilistic assessment.....	223
6.5.4 Case study 4 – Example of ACADS 1(c) problem.....	226
6.6 System reliability analysis using the probabilistic sensitivity-based FORM	229
6.7 Summary.....	232
Chapter 7 Concluding Remarks and Recommended Future Studies	233
7.1 Summary and conclusions	233
7.2 Recommendations for future work	236
Table of Contents for the Appendices	238
Appendix A Procedure for first-order reliability method.....	239
Appendix B Procedure for second-order reliability method.....	241
Appendix C An example illustrating the slope reliability analysis using the proposed method of interpolated autocorrelations	249
Appendix D Spreadsheet set-up of the reliability analysis of James Bay dike accounting for 2-D spatial variation	255
Appendix E User-defined FISH subroutine of FLAC for the plane-strain slope stability analysis	257

Appendix F User-defined FISH subroutine of FLAC3D for the	
three-dimensional slope stability analysis.....	261
Appendix G Closed-form solution of undrained slope stability analysis	
with circular slip surfaces	264
References.....	268

List of Tables

Table 2.1 Assumptions, equilibrium conditions, and unknowns in various limit equilibrium methods of slices (after Duncan and Wright 2005).....	14
Table 2.2 Typical values of coefficient of variation for soil strength parameters	30
Table 2.3 One-dimensional autocorrelation functions (after Baecher and Christian 2003)	34
Table 2.4 One-dimensional variance reduction functions (after Vanmarcke 1983)	39
Table 2.5 Closed-form formulas for second-order failure probability approximation (after Chan and Low 2011)	51
Table 3.1 Soil properties of Cannon Dam used for strength-reduction FE stability analysis.....	83
Table 3.2 Comparison between stability analyses of the Cannon Dam	85
Table 3.3 Statistical information of strength parameters of the Cannon Dam (after Hassan and Wolff 1999)	86
Table 3.4 Comparison between reliability analyses of the Cannon Dam.....	89
Table 3.5 Soil properties of James Bay dike used for strength-reduction FE stability analysis.....	91
Table 3.6 Comparison between stability analyses of the James Bay dike	93
Table 3.7 Statistical information of random variables of the James Bay dike (after Christian et al. 1994, El-Ramly 2001, Xu and Low 2006)	94
Table 3.8 Comparison between reliability analyses of the James Bay Dike.....	97

Table 4.1 Expected performance level vs. target reliability indices for slopes (after USACE 1995)	103
Table 4.2 Summary of reliability analyses considering different cases of spatial variation	110
Table 4.3 Soil properties of soft clay foundation and embankment used for strength reduction analysis.....	124
Table 4.4 Results of reliability analysis of a $c - \phi$ slope	132
Table 4.5 Comparison of reliability indices based on two different autocorrelation functions.....	133
Table 4.6 Uncertainties of soil properties of James Bay dike (summarized from the data of Christian et al. 1994).....	143
Table 4.7 Reliability analyses of James Bay dike accounting for 2-D spatial variation .	148
Table 6.1 Uncertain soil properties of case 1	216
Table 6.2 Results of system reliability analysis of a two-layered undrained slope (case 1)	218
Table 6.3 Comparison of system reliability results from different methods (case 1)	221
Table 6.4 Soil properties of case 2	222
Table 6.5 Results of the system reliability analysis of the embankment on soft ground (case 2)	222
Table 6.6 Comparison of system reliability results from different methods (case 2)	223
Table 6.7 Soil properties, the top sand layer has no uncertainty (case 3).....	224

Table 6.8 Results of the system reliability analysis of Congress Street cut (case 3).....	224
Table 6.9 Comparison of system reliability results from different methods (case 3)	225
Table 6.10 Soil properties, Layer 1 is deterministic sand layer (case 4).....	227
Table 6.11 Results of the system reliability analysis of the ACADS example (case 4) ..	228
Table 6.12 Comparison of system reliability results from different methods (case 4) ..	228
Table G.1 Comparison of F_s between the closed-form solution and Bishop's simplified and ordinary methods of slices.....	266

List of Figures

Figure 2.1 Internal forces acting on a slice of failure mass	8
Figure 2.2 Commonly used inter-slice force functions.....	11
Figure 2.3 Grid method for locating the critical circular slip surface.....	16
Figure 2.4 Categories of sources of geotechnical uncertainty (after Christian et al. 1994)	26
Figure 2.5 Components of soil variability	32
Figure 2.6 Correlation between local averages in a 1-D random field	40
Figure 2.7 Updating the statistical distributions by probabilistic back analysis (after Zhang et al. 2010)	43
Figure 2.8 Classical FORM approach in the space of standard normal variables	46
Figure 2.9 Low and Tang's FORM approach based on dispersion ellipsoids in original space.....	47
Figure 2.10 Slope stability analysis by RFEM (after Griffiths et al. 2009): (a) a relatively low spatial autocorrelation distance; (b) a relatively high spatial autocorrelation distance.....	57
Figure 3.1 Schematic interpretation for successive approximation of response surface (after Rajashekhar and Ellingwood 1993)	67
Figure 3.2 Reliability analysis of an example problem by response surface method with iteration for improvement: (a) first iteration; (b) second iteration; and (c) third iteration.....	70
Figure 3.3 Function x_i for obtaining x_i from n_i , coded inside Microsoft EXCEL (after Low and Tang, 2007).....	75
Figure 3.4 Reliability analysis of an example problem by probabilistic sensitivity-based	

FORM.....	76
Figure 3.5 Reliability analysis of a homogeneous $c - \phi$ slope by the recursive algorithm for FORM.....	78
Figure 3.6 Influence of Δn_i of sensitivity analysis on the reliability index β by the recursive algorithm for FORM.....	79
Figure 3.7 Cannon Dam at section of station 12+75.....	82
Figure 3.8 Finite element model of the Cannon Dam.....	84
Figure 3.9 Strength-reduction FE stability analysis of the Cannon Dam.....	85
Figure 3.10 Probabilistic strength-reduction FE analysis of Cannon Dam.....	88
Figure 3.11 Probabilistic critical failure mechanism of the Cannon Dam by strength-reduction FE stability analysis	89
Figure 3.12 Cross-section of a typical James Bay dike.....	90
Figure 3.13 Finite element model of the James Bay dike	92
Figure 3.14 Failure mechanism of the James Bay dike from strength-reduction FE stability analysis.....	92
Figure 3.16 Probabilistic critical failure mechanism of the James Bay dike from strength-reduction FE stability analysis	97
Figure 4.1 Methods of probabilistic slope analysis accounting for spatial variation	100
Figure 4.2 A hypothetical embankment with soft foundation and the soil properties.....	103
Figure 4.3 The deterministic and probabilistic critical slip surfaces are nearly identical when spatial variation is not considered	104
Figure 4.4 Discretizing the 2-D random fields over the slip surface according to slices	106
Figure 4.5 Influence of the number of slices on reliability analysis.....	109

Figure 4.6 Comparison of probabilistic critical slip surfaces with and without spatial variation.....	110
Figure 4.7 Comparison between the reliability analyses by varying the vertical autocorrelation distance δ_y for $\delta_x = 20$ m.....	112
Figure 4.8 Comparison between the reliability analyses by varying the horizontal autocorrelation distance δ_x for $\delta_y = 2$ m.....	113
Figure 4.9 Discretizing the random field using finite control points where variables are spatially correlated.....	115
Figure 4.10 Comparison between reliability analyses using different vertical interpolating frameworks with $\Delta x = 5.0$ m ($\delta_y = 2$ m, $\delta_x = 20$ m).....	120
Figure 4.11 Comparison between reliability analyses using different horizontal interpolating frameworks	121
Figure 4.12 Failure zone indicated by shear strain increment.....	124
Figure 4.13 Failure mass indicated by velocity vectors.....	124
Figure 4.14 Using 32 control points to represent the random field of c_u	125
Figure 4.15 An arbitrary representation of the random field of c_u ($\delta_y = 2$ m, $\delta_x = 20$ m)	125
Figure 4.16 Probabilistic sensitivity-based FORM obtains a β value of 2.572 after four iterations.....	127
Figure 4.17 Profile of the random field of c_u at the design point values	128
Figure 4.18 Failure zone indicated by shear strain increment (random field of c_u is at design point)	129
Figure 4.19 Failure mass indicated by velocity vectors (random field of c_u is at design point).....	129
Figure 4.20 A homogeneous c - ϕ slope.....	130
Figure 4.21 Deterministic critical slip surface and probabilistic critical slip surfaces for a	

homogeneous c - ϕ slope.....	130
Figure 4.22 A measured profile of cone tip resistance q_c within Keswick clay in south Australia (after Jaksa et al. 1999).....	135
Figure 4.23 Sample and fitting autocorrelation function (ACF) based on the residuals of Figure 4.22 (after Jaksa 1999)	137
Figure 4.24 Layout of vertical CPTs in case study 1 (After Jaksa et al. 1999)	139
Figure 4.25 Illustration of the horizontal CPT in case study 2 (After Jaksa et al. 1999)	140
Figure 4.26 Comparison of the CPT data and horizontal spatial variability analyzed from the residuals of the data: (a) case study 1; (b) case study 2	140
Figure 4.27 Soil profile for a cross section (after Christian et al. 1994).....	143
Figure 4.28 Vertical autocovariance function estimated from field vane testing data taken from a typical boring (after Christian et al. 1994)	144
Figure 4.29 Horizontal spatial variability characterization from field vane testing data: (a) sampling plan; (b) autocovariance function (after DeGroot and Baecher 1993)	145
Figure 4.30 Deterministic critical slip surface of James Bay dike	146
Figure 5.1 An idealized model of three-dimensional stability failure.....	153
Figure 5.2 Two ways to compute end-section resisting moment in undrained analysis	154
Figure 5.3 Three-dimensional stability analysis with different width of failure mass.	155
Figure 5.4 Correlation between local averages of c_u over the cylindrical failure surface	157
Figure 5.5 Correlation between point-wise end-section c_u and spatially-averaged c_u over the cylindrical failure surface	159
Figure 5.6 Probabilistic three-dimensional stability analysis by spatial-averaging	

approach.....	161
Figure 5.7 Probabilistic three-dimensional stability analysis by spatial-autocorrelation approach.....	162
Figure 5.8 Reliability indices computed using spatial-autocorrelation and spatial-averaging approaches: (a) random field of c_u follows <i>Normal Distribution</i> ; (b) random field of c_u follows <i>Lognormal Distribution</i>	164
Figure 5.9 Reliability indices as functions of longitudinal autocorrelation distance computed using spatial-averaging approach: (a) random field of c_u follows <i>Normal Distribution</i> ; (b) random field of c_u follows <i>Lognormal Distribution</i>	166
Figure 5.10 Reliability indices as functions of longitudinal autocorrelation distance computed using spatial-autocorrelation approach: (a) random field of c_u follows <i>Normal Distribution</i> ; (b) random field of c_u follows <i>Lognormal Distribution</i>	167
Figure 5.11 Reliability indices computed as functions of width of 3-D failure mass for four cases of δz : (a) by spatial-averaging approach; (b) by spatial-autocorrelation approach.....	169
Figure 5.12 Reliability indices as functions of width of failure mass b computed by spatial-averaging method and spatial-autocorrelation method: (a) longitudinal autocorrelation distance $\delta z = 5$ m; (b) $\delta z = 10$ m; (c) $\delta z = 15$ m; (d) $\delta z = 20$ m	171
Figure 5.13 Refining the discrete points to improve the reliability analysis with spatial-autocorrelations.....	172
Figure 5.14 Relationship between probabilistic critical failure width and autocorrelation distance.....	173
Figure 5.15 Comparison between the probabilities of failure by FORM and by Monte Carlo simulations for the case of δz equal to 20 m.....	173

Figure 5.16 System reliability against the overall length of slope	175
Figure 5.17 A typical 3-D mesh of earth slope for undrained analysis	178
Figure 5.18 A typical 3-D mesh of earth slope for drained analysis.....	178
Figure 5.19 Comparison between the 3-D failure modes with respect to different width of failure mass.....	181
Figure 5.20 Comparison between 3-D strength reduction analysis and limit equilibrium analysis	182
Figure 5.21 Results of undrained analyses for the three cases of slope at: (a) 1H:1V; (b) 1.5H:1V; and (c) 2H:1V	183
Figure 5.22 Plots of end effects versus slope length in undrained analyses for the three cases of slope at: (a) 1H:1V; (b) 1.5H:1V; and (c) 2H:1V.....	185
Figure 5.23 Plots of end effects versus width of failure mass and cross-sectional slope inclination.....	186
Figure 5.24 A drained analysis of slope by strength reduction method	187
Figure 5.25 Plots of end effects versus slope length in drained analyses	189
Figure 5.26 Failure mass was found spreading over the infinitely long slope based on the deterministic 3-D strength-reduction stability analysis	191
Figure 5.27 A typical representation of the longitudinal random field of c_u using the interpolated autocorrelations ($L_{SV} = 100$ m, $\delta z = 5$ m)	193
Figure 5.28 Reliability analysis with $L_{SV} = 60$ m, using thirteen RVs.....	195
Figure 5.29 Reliability analysis with $L_{SV} = 100$ m, using twenty-one RVs.....	196
Figure 5.30 Comparison between the critical- β failure modes obtained using different lengths of probabilistic treatment.....	198
Figure 5.31 Probabilistic critical width of failure mass increases with the increase of longitudinal autocorrelation distance δz	200

Figure 6.1 Single local minimum of F_s (and β) exists in a homogeneous slope	209
Figure 6.2 Local minima of F_s (or β) exist in a multi-layered slope	210
Figure 6.3 A three-layered slope with uncertain soil properties	213
Figure 6.4 A two-layered soil slope with slip surfaces of the multiple failure modes (case 1)	216
Figure 6.5 Procedure of stratified RS approximation using second-order polynomials	219
Figure 6.6 An embankment resting on soft clay foundation (case 2)	222
Figure 6.7 Profile of Congress Street cut (case 3)	224
Figure 6.8 Profile of ACADS 1(c) problem (case 4)	227
Figure 6.9 System reliability analysis based on the probabilistic sensitivity-based FORM.....	231
Figure A.1 Reliability analysis of the plastic collapse of a one-bay frame using the first-order reliability method (FORM)	240
Figure A.2 User-defined function transforming variable from n-space to x-space	240
Figure B.1 Reliability analysis of the plastic collapse of a one-bay frame using the second-order reliability method (SORM)	243
Figure B.2 User-defined function for computing lower triangular matrix based on Cholesky factorization.....	244
Figure B.3 User-defined function for Gram-Schmidt orthogonalization.....	245
Figure B.4 User-defined procedure for determining principal curvatures of limit state surface.....	246
Figure B.5 User-defined function for computing failure probability using SORM formulas. Functions ComplexQuotient and ComplexRoot are required by the Tvedt formula for complex number operations.....	248

Figure C.1 Slope reliability analysis accounting for 2-D spatial variation in a spreadsheet	252
Figure C.2 User-defined VBA function for bilinear interpolation of slice c_u of Figure A.1a using the control point c_u of Figure A.1b	253
Figure C.3 User-defined subroutine for systematic search method	254
Figure D.1 Stability model using the Spencer's method (at the design point).....	255
Figure D.2 Reliability analysis of the James Bay dike considering 2-D spatial variation	256
Figure G.1 A homogeneous slope in undrained stability analysis	264
Figure G.2 Three slip circles to be examined using the closed-form solution	266
Figure G.3 Optimizing the critical F_s based on closed-form solution in a spreadsheet	267

List of Symbols and Abbreviations

The following list of symbols and abbreviations is provided for ease of reference for those symbols and abbreviations that are most frequently used in this thesis. It is therefore not exhaustive. Nonetheless all symbols and abbreviations used in this thesis are defined at their first mention in the main text.

Symbols:

b = width of three-dimensional failure mass of slope over the longitudinal direction

b^* = Probabilistic critical width of failure mass

C = Covariance matrix

c' = Cohesion of soil

c.o.v. = Coefficient of variation

c_u = Undrained shear strength of clay

E = Elastic modulus of soil

$E_{i, i+1}$ = Inter-slice normal force

F_s = Factor of safety by plane-strain stability analysis

F_{3D} = Factor of safety by three-dimensional stability analysis

$g(\mathbf{x})$ = Performance function

$\bar{g}(\mathbf{x})$ = Response surface function

H = Height of slope from the toe to crest

L = Length of three-dimension slope in the longitudinal (out-of-plane) direction

L_{sv} = Length of slope with probabilistic treatment with spatial variation

L = Cholesky matrix (lower triangular)

M_o = Overturning moment

M_R = Resisting moment

M_{ER} = Resisting moment caused by the end-section shearing in three-dimensional slope failure

P_f = Probability of failure

$\mathbf{R}$ = Correlation matrix

$\mathbf{S}$ = Gram-Schmidt orthogonalization matrix

u, U = Standard normal variable

u' = Gram-Schmidt-rotated standard normal variable [$u' = S(u)$]

$V_{i, i+1}$ = Inter-slice shear force

$\mathbf{x}^*, \mathbf{n}^*, \mathbf{u}^*$ = Design points in the original-, equivalent standard normal- and uncorrelated standard normal (reduced) - spaces, respectively

x, X = Random variable

n = Unrotated equivalent standard normal variable [$\Phi(n) = F(x)$]

D = Tangent Depth of slip surface (below ground surface)

$\nabla g(\mathbf{n})$ = Gradient vector of performance function in the n -space

α = Direction cosine vector of performance function at design point

α = Slice-base angle

β = Reliability index

γ = Total unit weight of soil

$\Gamma(T)$ = Variance reduction factor over separation distance T

γ' = Effective unit weight of soil

Δl = Base-length of a slice

θ = Scale of fluctuation

κ = Principal curvature

μ = Mean value

σ = Standard deviation

ν = Poisson's ratio of soil

ϕ = Friction angle of soil

δ = Autocorrelation distance

ρ = Spatial correlation coefficient

$\phi(\cdot)$ = Standard normal pdf

$\Phi(\cdot)$ = Standard normal CDF

Abbreviations:

FDM = Finite difference method

FEM = Finite element method

FORM = First-order reliability method

LEM = Limit equilibrium method

MCS = Monte Carlo simulation

RSM = Response surface method

SORM = Second-order reliability method

SRM = Strength reduction method

Chapter 1 Introduction

1.1 Research background

One aspect that makes geotechnical engineering different from other engineering disciplines is the involvement of soils, the properties of which are highly uncertain in nature. Although the presence of geotechnical uncertainty has long been recognized in the profession, the most widely used method for analysis and design of slope engineering is the lumped factor of safety method. It simplifies the stability analysis by considering uncertain parameters to be deterministic and by accounting for the uncertainties through the use of empirical safety factors. Safety factors are derived based on past experience but do not absolutely guarantee safety or satisfactory performance. Also, they provide no information on how the uncertainties of different parameters influence the safety of the system.

In contrast to the factor-of-safety method (deterministic analysis), probabilistic analysis brings rationality to the consideration of uncertainty in design, provides more information about system behavior, the influence of different uncertain variables on system performance, and the interaction between different system components (Haldar and Mahadevan 2000a). During the past three decades, there has been an increasing demand for probabilistic analysis of the stability of slopes, particularly by applying the results of reliability theory. Reliability theory develops and is more used in the structural, aerospace, and manufacturing industries; and it requires special adaptation to deal with geotechnical engineering. It is well accepted that the probabilistic methods can play a complementary role to the conventional factor-of-safety method.

The application of probabilistic methods to slope stability analysis could be traced to

some pioneer works in the 1970's, such as Lumb (1970, 1974), Matsuo and Kuroda (1974), Yucemem and Tang (1975), Alonso (1976), Tang et al. (1976), and Vanmarcke (1977b, 1980). Some later applications include those by Li and Lumb (1987), Oka and Wu (1990), Chowdhury and Xu (1994), Christian et al. (1994), USACE (1995), Low and Tang (1997b,c), Low (2003) and Low et al (2007), among others. Baecher (1997) concluded that the probabilistic analysis is now on the "threshold of entering mainstream geotechnical practice".

A unique characteristic of a soil property is the spatial variability. Soils are geological materials formed by weathering processes and, save for residual soils, transported by physical means to their present locations. They have been subject to various stresses, pore fluids, and physical and chemical changes. Thus, it is hardly surprising that the physical properties of soils vary from place to place even with homogeneous deposits (Baecher and Christian 2003). However, most past studies carried out probabilistic stability analysis without modeling the spatial variation. This is mainly due to the fact that the quantification of spatial variability requires a large amount of testing data which are not available easily, and the fact that more computational effort is required to model the spatial variation in probabilistic analysis. As a result, characterizing and modeling the spatial variation of soil properties remain challenging tasks in performing probabilistic slope stability analysis.

Although the rationale of probabilistic analysis of earth slopes has been well established and extensive research works have been conducted, the probabilistic analysis of slope stability seems to be still scarce in geotechnical practices. The reluctance of geotechnical practitioner to apply this method is mainly due to the fact that most engineers lack formal training in statistics and probability theory. It is expected that the probabilistic methods will be used more widely to deal with some real engineering problems with the development of computation technology and the increase of investigators that are

familiar with basic probability and reliability theory.

1.2 Research objectives

The global objective of this research work is to overcome some limitations of the application of probabilistic methods to slope stability analysis. The detailed objectives are as follows:

- (1) To develop an efficient procedure for probabilistic strength reduction stability analysis of slopes.
- (2) To investigate the effect of spatial variation of soil properties on the slope reliability. The investigation considers the spatial variation of strength parameters to be two-dimensional.
- (3) To study the three-dimensional (3-D) slope stability and extend the deterministic 3-D analysis to probabilistic analysis. The study aims at revealing the existence of probabilistic critical width of failure mass when out-of-plane spatial variation of soil properties is modeled.
- (4) To develop a new method for the system reliability analysis of slopes.

1.3 Outline of the research work

The thesis contains seven chapters. Chapter 1 presents the research background and the objectives of this research work.

Chapter 2 presents an extensive review of the literature on several subject matters relevant to this research. The review consists of two parts: the first part focuses on various methods used for deterministic slope stability analysis; the second part reviews concepts underlying the probabilistic analysis.

Chapter 3 is dedicated to probabilistic strength reduction slope stability analysis. It is noted that the strength reduction stability analysis is a very attractive alternative to the conventional limit equilibrium methods. Thus it is interesting to extend the strength reduction stability analysis into probabilistic analysis. Nevertheless, the strength reduction stability analysis involves implicit performance functions which make it difficult to carry out probabilistic assessment directly. Monte Carlo simulation is a basic way to do so, but the computational effort can be very intensive. In this regard, this research proposes to use the FORM as an alternative to Monte Carlo simulations for probabilistic strength reduction stability analysis. A new procedure for the reliability analysis incorporating implicit performance function is presented.

Chapter 4 aims at studying the two-dimensional (2-D) spatial variation of soil properties in slope reliability analysis. Two methods, namely the method of autocorrelated slices and the method of interpolated autocorrelations, are proposed for this purpose. Investigation is first carried out based on the limit equilibrium method of slices. First-order reliability method (FORM) is coupled with deterministic slope stability analysis using the constrained optimization approach. Systematic search for the probabilistic critical slip surface is carried out in this study. A case study involving the 2-D spatial variability characterization is presented. Finally, the use of the method of interpolated autocorrelations in a probabilistic strength reduction analysis is illustrated.

In Chapter 5, the three-dimensional (3-D) slope stability is discussed. The discussion begins with an idealized 3-D stability model, for which a closed-form solution of the stability factor is available. The longitudinal (out-of-plane) spatial variation of strength parameters of soils is modeled. Probabilistic analysis based on the idealized 3-D model reveals the existence of probabilistic critical width of failure mass for most long earth

slopes. The 3-D slope stability is further studied using the strength reduction technique. Based on the proposed procedure in Chapter 3, the 3-D strength reduction analysis is extended into probabilistic study.

Chapter 6 aims at developing an efficient method for system reliability analysis of slopes. Since the failure of a slope can occur along any potential slip surface, system reliability analysis is deemed appropriate. However, system reliability analysis of a slope is not an easy task since the potential failure modes are not known and moreover, there are infinite potential failure modes in theory. As a result, a useful procedure of identifying the local minimum- β slip surfaces is introduced, and a stratified response surface method is proposed to represent the system performance functions (in multiple failure modes).

Chapter 7 summarizes the main findings from this research work, with recommendations for future studies.

Chapter 2 Literature Review

2.1 Introduction

An extensive literature review on several subject matters relevant to this research thesis is presented in this chapter. The chapter opens with a review on various methods used for slope stability analysis, including the commonly used limit equilibrium methods and the more recently-developed strength-reduction finite element or finite difference methods. It then presents the concepts of uncertainty in geotechnical engineering, reviews some probabilistic methods available for slope stability assessment, and highlights the important role that reliability approach can play to be complementary to the conventional factor-of-safety methods.

Part I Review of Commonly Used Methods for Slope Stability Analysis

2.2 Limit equilibrium methods for stability analysis of slopes

2.2.1 Closed-form solutions

Closed-form solutions of stability factor F_s exist only when the failure mechanism is very simple and in special conditions. In general, the sliding mass can be treated as a single free-body, with gravity and shear and normal forces acting in equilibrium. Closed-form solutions are available for the following three typical cases:

2.2.1.1 $\phi = 0$ slope with circular slip surface

When a slope consists of $\phi = 0$ soils and the slip surface is circular, the normal force

along the slip circle does not contribute to the shear strength. As a result, the F_s could be obtained easily by taking moment about the centre of rotation. This method was first formulated by Fellenius (Fellenius 1922) in Sweden, and thus is widely known as the Swedish circle method or $\phi = 0$ method. It is best suited for hand calculation for preliminary stability analysis.

2.2.1.2 Infinite slope with plane slip surface

In this case, the slope is assumed to extend infinitely in all directions, and the slip surface is considered parallel to the face of the slope. Equilibrium is considered for a rectangular block bounded by two vertical side planes. Since the slope is infinite, forces on the side planes of the block will exactly balance each other and can be ignored in the stability analysis. Solving the force equilibrium comprising the gravity of the block, the shear and normal forces along the slip surface, the F_s can be accurately obtained.

2.2.1.3 Slope with logarithmic spiral slip surface

By assuming a logarithmic spiral for the slip surface, a statically determinate solution of the F_s could be achieved (Frohlich 1953). No additional assumptions are required in this method. The logarithmic spiral method satisfies moment equilibrium and implicitly satisfies complete static equilibrium, thus it is relatively accurate. Nevertheless, the method seems very complex and difficult to use for hand calculations due to the assumption of logarithmic spiral for slip surface. Some researchers have developed stability charts using the logarithmic spiral method (Wright 1969; Leshchinsky and Volk 1985; Leshchinsky and San 1994).

2.2.2 Limit equilibrium methods of slices

2.2.2.1 A unified formulation of the equilibrium equations

So far, there are more than ten limit equilibrium methods of slices available for slope stability analysis. In particular, the commonly used are those by Bishop (1955), Lowe and Karafiath (1960), Morgenstern and Price (1965), Spencer (1967, 1973), and Janbu (1973). Since most of the existing methods are very similar in their basic formulations with only minor differences in the assumptions on the inter-slice shear forces, it is possible to include them under a unified formulation (Fredlund and Krahn 1977; Espinoza et al. 1994; Cheng and Zhu 2005). In this review, the unified formulation by Cheng and Zhu (2005) that can degenerate into many existing methods of slices is introduced. To begin with, consider a general case of slice with forces acting on it, as shown in Figure 2.1. Three assumptions have been made to solve the equilibrium of the slice: (i) the failure mass is a rigid body; (ii) the base normal force acts at the middle of each slice base; and (iii) Mohr-Coulomb failure criterion is used.

Figure 2.1 Internal forces acting on a slice of failure mass

(a) Force equilibrium

For any slice i , the force equilibrium conditions in horizontal and vertical directions are given by

$$N_i \sin \alpha_i - S_i \cos \alpha_i + HL_i = E_{i,i+1} - E_{i-1,i} \quad (2.1)$$

$$W_i + VL_i - N_i \cos \alpha_i - S_i \sin \alpha_i = V_{i,i+1} - V_{i-1,i} \quad (2.2)$$

Using Mohr-Coulomb failure criterion, the base shear force S_i is related to the base normal force N_i , such that

$$S_i = \frac{N_i \tan \phi_i + c_i l_i}{F} \quad (2.3)$$

The inter-slice shear force $V_{i,i+1}$ is usually assumed to be related to the inter-slice normal force $E_{i,i+1}$ by the relation

$$V_{i,i+1} = \lambda \cdot f(x) \cdot E_{i,i+1} \quad (2.4)$$

where λ is a inter-slice force scaling factor, $f(x)$ is the inter-slice force function that has prescribed values at each slice boundary. Several commonly used types of $f(x)$ will be discussed later.

The boundary conditions to the above three equations are that the inter-slice normal forces for the entry and exit ends will be zero. As a result, considering the force equilibrium, there are a total of $2n+1$ unknowns comprising $n-1$ unknown inter-slice normal forces, n unknown base normal forces, 1 unknown inter-slice force scaling factor λ and 1 unknown factor of safety F to be solved if the failure mass is divided into n slices. However, it is obvious that only $2n$ force equilibrium equations are available so that the problem is over-specified by one

(b) *Moment equilibrium*

The moment equilibrium can be considered in two ways: local moment equilibrium and overall moment equilibrium. The local moment equilibrium is solved separately for each of the n slices. Thus there will be a total of n moment equilibrium equations. At the same time, since taking moment of a single slice requires the knowledge of the location of inter-slice forces, there are $n-1$ unknowns about the location of the $n-1$ inter-slice normal forces. In combination with the force equilibrium, there are $(2n+1)+(n-1) = 3n$ unknowns versus $2n+n = 3n$ equilibrium equations, so that the stability problem can be solved completely.

By taking moment of all the slices about any given point O as shown in Figure 2.1, the overall moment equilibrium is given by

$$\sum_{i=1}^n [W_i(x_{wi} - x_o) + VL_i(x_{Li} - x_o) + HL_i(y_{Li} - y_o) + (N_i \sin \alpha_i - S_i \cos \alpha_i)(y_{bi} - y_o) - (N_i \cos \alpha_i + S_i \sin \alpha_i)(x_{bi} - x_o)] = 0 \quad (2.5)$$

In this situation, there is only one moment equilibrium equation. Since inter-slice force does not contribute to the overall moment, locations of the inter-slice forces are not required. Thus, in combination with the force equilibrium, there are $2n+1$ unknowns versus $2n+1$ equilibrium equations. The stability problem can also be solved completely.

It should be noted that most of the 'rigorous' methods adopt the overall moment equilibrium instead of the local moment equilibrium in their formulations, except for the Janbu rigorous method.

2.2.2.2 Commonly used inter-slice force functions

The inter-slice shear force V is assumed to be related to the inter-slice normal force E by the relation $V = \lambda f(x)E$. Since the slope stability problem is statically indeterminate by nature, there is no theoretical basis to determine $f(x)$ for a general problem and assumptions has to be made. Several types of the commonly used $f(x)$ are plotted in Figure 2.2 and are introduced as follows.

Figure 2.2 Commonly used inter-slice force functions

Type 1: $f(x) = 1.0$. This case is equivalent to the Spencer method and is commonly adopted in many engineering analysis;

Type 2: $f(x) = \sin(x)$. This is a relatively popular alternative to $f(x) = 1.0$. The Sin function is adopted purely because of its simplicity;

Type 3: $f(x)$ = trapezoidal shape. Type 3 $f(x)$ will reduce to type 1 as a special case, but it is seldom adopted in practice;

Type 4: $f(x)$ = error function or the Fredlund–Wilson–Fan force function (Fan et al. 1986) which is in the form of $f(x) = \Psi \exp(-0.5 c^n \eta^n)$, where Ψ , c and n have to be defined by the user. η is a normalized dimensional factor which has a value of

-0.5 entry end and a value of 0.5 at exit end of the slip surface. η varies linearly with the x -coordinates of the slip surface.

2.2.2.3 Reduction to various methods of slices and discussion

The above-mentioned unified formulation can be degenerated into some commonly used methods of slices. Some of them satisfy only the force or the moment equilibrium, while others satisfy both force and moment equilibrium and differ each other by using different types of inter-slice force functions. Some famous methods are summarized in Table 2.1.

The classical *ordinary method of slices* (also known as *Swedish method of slices* or *Fellenius method*) considers only the global moment equilibrium and neglects all the inter-slice forces. Due to the absence of inter-slice forces, the base normal force becomes known. Thus, the factor of safety can be obtained easily without the need of iteration analysis. This method is well known to be conservative, and sometimes the F_s computed from it can be smaller than those from the 'rigorous' methods by 20 – 30 % (Whitman and Bailey 1967; Cheng and Lau 2008), hence the method is seldom adopted in practice. However, this method is usually used for hand calculation due to its simplicity.

The *simplified Bishop method* is one of the most popular LEM of slices. This method satisfies the overall moment equilibrium and vertical force equilibrium but not the horizontal force equilibrium. It is best suitable for circular slip surface analysis, where the centre of the circle is taken as the moment point. The Bishop method has also been used for some non-circular slip surfaces, but as pointed out by Fredlund et al. (1992), the F_s will be dependent on the location of the moment point since there is a net unbalanced horizontal force in the system. Thus the use of the Bishop method to non-circular failure surface is generally not recommended. The simplified Bishop

method is simple for hand calculation and the convergence is always fast. For circular slip surface analysis, the F_s computed by this method can be very close to those by the 'rigorous' methods.

As to the various *force equilibrium methods*, force equilibrium is completely satisfied but the moment equilibrium is not considered. These force equilibrium methods can be used for non-circular slip surface analysis and usually is very fast in computation. But the methods require assumptions about the inclinations of the inter-slice forces, to which the computed F_s could be very sensitive. Duncan and Wright (2005) rightly pointed out that the F_s could vary by 25 % when using different inclinations of the inter-slice forces.

Others satisfying both force and moment equilibrium are called 'rigorous' methods. The *Morgenstern-Price method* is usually based on the inter-slice force function types 1 to 4, though the use of an arbitrary function is possible. If the type 1 inter-slice force function is used, this method will reduce to the Spencer method. The Morgenstern-Price method satisfies both the force and the global moment equilibrium. Since the local moment equilibrium equation is not used in the formulation, the internal forces of an individual slice may not be acceptable. For example, the line of thrust (the point where the inter-slice normal force acts) may fall outside the soil mass when using the Morgenstern-Price method. A more advanced method, the *general limit method of slices* (GLE) is basically similar to the Morgenstern-Price method, except that the line of thrust is determined and is closed at the last slice. The acceptability of the line of thrust for any intermediate slice may still be unacceptable from the GLE analysis. In general, the values of F_s obtained from these 'rigorous' methods are very close to each other (Fredlund and Krahn 1977; Duncan and Wright 1980). The Spencer method is the simplest among these 'rigorous' methods, while Sarma's procedure may be simplest for accommodating the seismic effects.

Table 2.1 Assumptions, equilibrium conditions, and unknowns in various limit equilibrium methods of slices (after Duncan and Wright 2005)

Procedure	Assumptions	Equilibrium equations satisfied		Unknowns solved for	
Ordinary method of slices (also known as Fellenius method)	The slip surface is circular; the forces on the sides of the slices are neglected	1	Σ Moments about center of circle	1	Factor of safety (F)
		1	Total equation	1	Total unknown
Simplified Bishop (Bishop 1955)	The slip surface is circular; the inter-slice forces are horizontal (i.e. no shear force between slices)	1	Σ Moments about center of circle	1	Factor of safety (F)
		n	Σ Forces in the vertical direction	n	Normal force on the base of slices (N)
		$n + 1$	Total equations	$n + 1$	Total unknowns
Force Equilibrium (Lowe and Karafiath (1959); Janbu (1973); Corps of Engineer)	The inclinations of the inter-slice forces are assumed; assumptions vary with procedure	n	Σ Forces in the horizontal direction	1	Factor of safety (F)
		n	Σ Forces in the vertical direction	n	Normal force on the base of slices (N)
		$2n$	Total equations	$n - 1$	Resultant interslice forces (Z)
Spencer (1967)	Inter-slice forces are parallel, (i.e. all have the same inclination). The normal force (N) acts at the center of the base of the slice (typically)	n	Σ Moments about any selected point	1	Factor of safety (F)
		n	Σ Forces in the horizontal direction	1	Interslice force inclination (θ)
		n	Σ Forces in the vertical direction	n	Normal force on the base of slices (N)
		$3n$	Total equations	$n - 1$	Resultant interslice forces (Z)
				$n - 1$	Location of interslice forces (line of thrust)
				$3n$	Total unknowns

Table 2.1 continued

Procedure	Assumptions	Equilibrium equations satisfied	Unknowns solved for
Morgenstern and Price (1965, 1967); Spencer (1973)	Inter-slice shear force is related to inter-slice normal force by $X=\lambda f(x)E$; the normal force (N) acts at the center of the base of the slice (typically)	n Σ Moments about any selected point	1 Factor of safety (F)
		n Σ Forces in the horizontal direction	1 Interslice force scaling factor (λ)
		n Σ Forces in the vertical direction	n Normal force on the base of slices (N)
		$3n$ Total equations	$n-1$ Horizontal interslice forces (E)
			$n-1$ Location of interslice forces (line of thrust)
			$3n$ Total unknowns
Chen and Morgenstern (1983)	The same as M&P, except that inter-slice shear force is related to inter-slice normal force by $X=[\lambda f(x)+f(x)]E$	The same as Morgenstern and Price	The same as Morgenstern and Price
Sarma (1973)	Inter-slice shear force is related to the inter-slice shear strength, S_v , by $X=\lambda f(x)S_v$; S_v depends on shear strength parameters, pore water pressures, and the horizontal component of inter-slice forces; the normal force (N) acts at the center of the base of slices (typically)	n Σ Moments about any selected point	1 Factor of safety (F) or Seismic coefficient (κ)
		n Σ Forces in the horizontal direction	1 Interslice force scaling factor (λ)
		n Σ Forces in the vertical direction	n Normal force on the base of slices (N)
		$3n$ Total equations	$n-1$ Horizontal interslice forces (E)
			$n-1$ Location of interslice forces (line of thrust)
			$3n$ Total unknowns

2.2.3 Review of techniques used for locating the critical slip surface

It is desirable to find the minimum (critical) F_s as well as the location of critical slip surface for the proper design of a slope. In this regard, many techniques for systematic search have been developed. When circular slip surfaces are concerned, the search technique is simple since only three control variables defining a circle are incorporated. The most commonly used approach is the 'grid method', which is shown in Figure 2.3. Each of the pre-defined grid-points is taken as centre of rotation and different radii are considered as circular slip surfaces; when a large number of grid-points are examined, a critical factor of safety as well as the critical slip surface could be found. The method is robust and simple. But the accuracy depends on the spacing between grid-points and the number of radii corresponding to each grid-point. An alternative to the 'grid method' is the approach by varying the entry- and exit- points.

Figure 2.3 Grid method for locating the critical circular slip surface

When non-circular slip surfaces are concerned, the determination of critical slip surface could be much complicated since more control variables are needed to define the slip surface. Baker and Garber (1978) used variational calculus to locate the critical slip

surface but this method is complicated even for a simple slope and is not adopted in practice. Celestino and Duncan (1981) developed a technique of moving one point at a time on a slip surface in one specified direction to find the most critical noncircular surface. Baker (1980) tried to determine the critical slip surface utilizing dynamic programming minimization techniques combined with *Spencer method*. Chen (1992) postulated that using a random trial search would lead to the global minimum factor of safety. Simplex techniques were used by Nguyen (1985) and Chen and Shao (1988). All these methods work well for simple problems. But there are still many limitations associated with them, as addressed by Li and White (1987).

Monte-Carlo-based search techniques have also been proposed to locate the critical slip surface. In this method, large numbers of trial surfaces are generated to ensure a minimum factor of safety. Based on the searching routine, Monte Carlo techniques are divided into two classes, namely random walk and random jump, respectively. Greco (1996) presented an efficient Monte Carlo technique of random walking for locating the critical slip surface. Siegel et al. (1981) attempted to locate the critical slip surface based on random jumping technique. Husein Malkawi et al. (2001a, b) attempted to do Monte Carlo search for the critical slip surface, by coupling both random walking and random jumping procedures.

More recently, some researchers proposed to locate the critical slip surface by *Genetic Algorithm*. For instance, Goh (1999) applied Genetic Algorithm to determine the critical slip surface in multiple-wedge stability analysis. The search strategy was found to be sufficiently robust to handle layered soils with weak, thin layers. Sengupta and Upadhyay (2009) proposed a Genetic Algorithm to locate the critical surface under general conditions with general constraints and demonstrated its distinguishing advantages. Others proposed to use a simulated annealing method to locate the critical slip surface for general slope problems (Cheng 2007).

2.2.4 Miscellaneous consideration on slope stability analysis

2.2.4.1 Local minima of factor of safety

Although many techniques for determining the critical F_s and the critical slip surface have been developed, any single search procedure may be trapped into a local minimum of F_s if multiple minima exist. This is because the objective function of the safety factor F_s is usually non-smooth, non-convex and discontinuous over the solution domain while most optimization methods are usually gradient-type and are applicable only to the continuous function. To overcome this limitation, one is encouraged to perform a number of independent searches by using different starting locations to ensure that all regions of the cross-section are examined and all possible local minima are detected.

2.2.4.2 Pore water pressure

One important factor affecting the slope stability is the change of pore water pressure u . Thus a proper stability analysis should take the u into account. The classical approach uses the pore pressure ratio r_u to define the u , where $r_u = u/\gamma h$. This method is straightforward in concept and makes the computation highly simplified. However, the pore water pressure so defined may diverge from reality. Alternatively, the pore water pressure can be defined by the groundwater table, such that $u = \gamma_w h_w$, where h_w is the height of the water table above the base of the slice. This is the most commonly used method to define the pore pressure, which assumes that pore pressure is hydrostatic. A more reasonable consideration of pore water pressure is by seepage analysis using finite element analysis. With the fast development of computer technology, conducting seepage analysis has been widely accepted in recent years.

2.2.4.2 Tension crack

When a slope near its crest consists of cohesive soils, the computed inter-slice normal forces could be in tension to maintain equilibrium. However, it is desirable to avoid tensile inter-slice normal forces in slope stability analysis for two reasons: first, calculated tensile stresses are controversial and unrealistic since most soils do not have significant tensile strength and thus cannot withstand tension; second, numerical problems could be induced when significant tensile inter-slice forces are encountered. To eliminate the tensile forces from stability analysis, a tension crack is usually used at the crest of slope. The classical approach to estimate the depth of the tension crack is based on Rankine's theory for active earth pressures. For cohesive soils, the depth of vertical crack is estimated to be

$$z_c = \frac{2c}{\gamma} \tan \left(45 + \frac{\phi}{2} \right) \quad (2.6)$$

where c , ϕ , and γ are cohesion, friction angle and unit weight, respectively.

A chart-procedure based on the assumption of parallel inter-slice forces was proposed to estimate the depth of crack by Spencer (1967, 1973). However, this method is still based on Rankine's theory and the depth of crack is not related to the tensile strength of the soil. Baker (1981) presented a variation method for tension cracks. But the approach is only applicable for straight and homogeneous slopes without pore water pressure or external loading.

In reality, tension cracks at the crest of a slope are often an important indicator of the onset of instability. The appearance of tension cracks is likely to decrease stability and therefore increase the tendency of a slope to fail. In stability analysis, it is recommended to consider a tension crack being full, or partly full of water which will exert an extra lateral pressure to the instability of the slope, so that the computed factor

of safety is conservative.

2.3 Strength-reduction technique for stability analysis

For slope stability analysis, an alternative to the limit equilibrium methods of slices is the shear strength reduction technique. Although this technique was proposed as early as 1975 by Zienkiewicz et al. (1975) and has since been used by Naylor (1982) and Matsui and San (1992), among others, its application occurs mostly in the past decade due to the increasing speed of computers. The shear strength reduction technique can be applied to both *finite element analysis* (e.g. Griffiths and Lane 1999) and *finite difference analysis* (e.g. Dawson et al. 1999) of slopes. To better understand the use of strength reduction technique in slope stability analysis, some basic concepts are reviewed below.

2.3.1 Constitutive model and failure criteria

Among a number of commonly used linear and non-linear constitutive models, the elastic-perfectly plastic model with Mohr-Coulomb failure criterion is mostly adopted. It has been shown that the sudden transition of soil behavior from elastic state to plastic state in the elastic-perfectly plastic model helps identify the factor of safety when applying strength-reduction technique (Dawson et al. 1999). Others have used more complicated constitutive models in the strength reduction method. For example, Matsui and San (1992) used the Duncan and Chang's hyperbolic model (Duncan and Chang 1970), and Zhang et al. (2011) used a revised Duncan and Chang's model. However, identifying the failure becomes more difficult. In addition, the factor of safety by strength reduction is found insensitive to the constitutive model used, even though more complicated constitutive models have advantages in predicting the loading-displacement behavior. This is because the stress-dependent stiffness behavior

and hardening effects are excluded when using the strength reduction technique (Plaxis manual; Alkasawneh et al. 2008). Zhang et al. (2011)'s study also revealed that the factors of safety by using Duncan and Chang's model and using the revised model are almost the same.

In this study, the emphasis is on the evaluation of slope stability but not on the load-displacement soil behavior. Thus, the elastic-perfectly plastic model with Mohr-Coulomb failure criterion will be used throughout this study. The model consists of six parameters: unit weight γ , shear strength parameters c and ϕ , Young's modulus E , Poisson's ratio ν , and dilation angle ψ . It is noted that besides the unit weight and the two shear strength parameters which are also used to compute the factor of safety by LEM, additional three parameters are required by the elastic-perfectly plastic model, namely, E , ν and ψ .

2.3.2 Shear strength reduction technique and definition of slope failure

To determine the factor of stability, the shear strength parameters are progressively reduced until an unstable state is detected. A trial reduction factor F^{trial} is simultaneously performed to both cohesion c and friction angle ϕ , such that

$$c_f = c/F^{trial} \quad (2.7)$$

$$\tan \phi_f = \tan \phi / F^{trial} \quad (2.8)$$

where c and ϕ are cohesion and friction angle, respectively, and c_f and ϕ_f are the reduced values that bring the slope to failure. The trial reduction factor F^{trial} which makes the slope unstable is the critical F_s .

One difficulty to the strength-reduction stability analysis lies in the definition of slope failure criteria. So far there is no uniform definition in the literature. Matsui and San

(1992) proposed to use the contour of shear strain as failure indicator; for example, their study of an embankment slope showed that a well-defined failure zone develops from the toe to top surface when the contour of shear strain exceeds 15%. In contrast, Griffiths and Lane (1999) and Dawson et al. (1999) took non-convergence (out-of-balance force) as an indicator for failure in finite element method and finite difference method, respectively. Others considered slopes failed when there are contiguous plastic zones through the toe to the crest (e.g. Zheng et al. 2009).

2.3.3 Associated flow rule and non-associated flow rule

The dilation angle ψ is a parameter that determines the volume change of soils. A negative value of ψ indicates there is decrement of the soil volume, while a positive value indicates an increment. As to the widely used elastic-perfectly plastic constitutive model, dilation angle ψ will be a constant value. When ψ is equal to zero, it implies zero volume change during yielding, i.e. non-associated flow rule; when ψ is equal to the friction angle, it implies that the flow rule is associated. Griffiths and Lane (1999) viewed that the dilation angle is less important in the strength-reduction stability analysis, thus the non-associated flow rule was recommended.

2.3.4 A short critical review of the strength-reduction stability analysis

Using the strength reduction method (SRM) in slope stability analysis has some advantages over the traditional limit equilibrium methods (LEM), such as: (i) the failure mechanism of slopes is found automatically from FEM or FDM analysis by progressive reduction of the shear strength; (ii) some assumptions concerning the inter-slice forces in LEM are avoided when using SRM; (iii) it can easily treat many complex conditions and can give information such as stresses, displacements, and deformation of slopes which are not available by the LEM alone.

On the other hand, some disadvantages are mentioned here, including: (i) since the SRM is based on stress-strain analysis, some additional soil parameters which are not required in LEM must be addressed; (ii) convergence of the numerical algorithm may not be attainable in some cases; (iii) where local minima exist, the SRM can only locate the most critical failure mode and may miss other local minima, as illustrated by Cheng and Lau (2008).

It is worth pointing out that some others performed different factoring strategies to soil properties (other than strength parameters) to compute the F_s . Chen and Mizuno (1990) and Swan and Seo (1999) used a gravity/loading-increase method in their finite element stability analyses, where it was found that the conventional strength-reduction method could give rise to a lower F_s value than gravity-increase method. A recent study by Pantelidis and Griffiths (2012) investigated the effect of several commonly-used factoring strategies including both strength-reduction and gravity-increase methods in slope stability analysis.

2.4 Enhanced limit equilibrium method

The enhanced LEM which is also referred to as enhanced limit method involves establishing the stress distribution using finite element analysis at first, and then utilizing the stress field to analyze slope stability (Naylor 1982; Yamagami and Ueta 1988; Zou et al. 1995; Kim and Lee 1997). This approach has been adopted by some numerical packages such as Slope/W (Slope/W menu 2007). The main steps for the use of this method are as follows:

- (i) determine the in-situ stress field using FEA, so that the σ_x , σ_y and τ_{xy} are known within each element;
- (ii) import the stresses information into a conventional limit equilibrium analysis;

- (iii) at mid-point of each slice base, the normal stress and mobilized shear stress can be obtained by interpolation of the imported stress field;
- (iv) compute the available shear strength from computed normal stress in step iii;
- (v) The F_s is defined as the ratio of the total available shear resistance to the total mobilized shear forces along the specified slip surface.

One may notice that the enhanced LEM is somewhat similar to the LEM of slices, except that the stresses are imported by another FEA module. Some advantages of this method are recognized, for example, no assumptions about inter-slice forces, no iterative convergence problems and stresses are much closer to reality, etc. However, the main limitation when using the enhanced LEM is that the effectiveness of this method greatly depends on the accuracy of the stress interpolation from FEA.

2.5 Review of three-dimensional slope stability analysis methods

Baligh and Azzouz (1975) and Azzouz and Baligh (1983) presented a method that extended the concepts of the 2D circular arc shear failure method to 3D slope stability problems. The method was only appropriate for a slope in cohesive soil. For c - ϕ soil slopes, LEM of columns which are extensions of various LEM of slices are developed. For example, Hovland (1977) proposed a general 3D method which is an extension of the 2D ordinary method of slices. Chen and Chameau (1982) extended the Spencer 2D method to 3D stability analysis, where the sliding mass was assumed to be symmetrical and divided into a number of vertical columns. The inter-column forces had the same inclination throughout the mass, and the shear forces were parallel to the base of the column. Hungr (1987) proposed a LEM of columns based on assumptions that the vertical forces of each column could be neglected. It is thus an extension of the simplified Bishop method of slices in 2D. By using the method of columns, Lam and Fredlund (1993) extended the 2D GLE formulation (Fredlund and Krahn 1977) to 3D

formulation. The inter-column force functions of an arbitrary shape to simulate various directions for the inter-column resultant forces were proposed. All the inter-column shear forces acting on the various faces of the column were assumed to be related to their respective normal forces by the inter-column force functions. It was however found that 3D factors of safety determined by the method of columns (Lam and Fredlund 1993) were relatively insensitive to the assumed form of the inter-column force functions. It is noted that most of the above-mentioned 3D formulations assume a plane of symmetry for the sliding mass, which is not the case for complex slopes in reality. To overcome this limitation, other 3D methods incorporating asymmetric sliding were also developed (Yamagami and Jiang 1996; Huang and Tsai 2000; Huang et al. 2002, among others). Cheng and Yip (2007) and Cheng (2008) however pointed out many limitations regarding the use of these asymmetric 3D methods.

Zheng (2009) viewed that the 3D limit equilibrium analysis of slopes could reduce to the solution of a generalized eigenvalue problem in which the largest real eigenvalue is the factor of safety. It was showed that the solution was rigorous and able to accommodate any shape of slip surfaces.

On the other hand, some researchers used finite element strength reduction method to study the 3-D slope stability (e.g. Ugai and Leshchinsky 1995; Griffiths and Marquez 2007; Deng et al. 2007). The 3D strength-reduction stability analysis is simpler in concept and does not involve extra assumptions as compared to the 2D strength-reduction method. Some limitations of the 3D strength-reduction stability analysis, such as the lack of universal convergence criteria, have been examined by Wei et al. (2009)

Part II Review Of Fundamental Concepts for Probabilistic Analysis

2.6 Review of methods for characterizing the geotechnical uncertainty

An important task in the geotechnical engineering is the characterization of soil properties which have uncertain values by nature. The uncertainty associated with a soil property comes from several different sources as noted by Baecher (1987), Christian et al. (1994), and Phoon and Kulhawy (1999a), among others. A typical description of the uncertainty was presented by Christian et al. (1994), as shown in Figure 2.4. A detailed discussion follows.

Figure 2.4 Categories of sources of geotechnical uncertainty (after Christian et al. 1994)

- (i) Data scatter is a combination of inherent or real spatial variation of the soil property and random testing errors. Real spatial variability is caused by natural geologic processes that continually modify the soil mass in situ, resulting in variation of the soil property from point to point. (Spatial variation of soil properties will be further discussed later.) Random testing errors can be attributed to equipment, procedural, operator or random testing effects.

- (ii) Systematic error originates from statistical error in the trend and bias in the measurement procedures. Statistical error in the trend (or mean value) of a soil property is due to insufficient sampling or limited data. A classic example of bias in the measurement procedure is the vane shear testing of undrained shear strength of cohesive soil. The measured strength needs to be corrected using an empirical factor based on the plasticity index of the clay (Bjerrum 1972). Another source of bias, known as model uncertainty, is introduced when field or laboratory measurements are transformed into design values via correlation models.

2.6.1 Basic statistics quantifying the geotechnical uncertainty

To better incorporate the uncertainty of a soil property into geotechnical design and analysis, statistical methods are used to mathematically express such an uncertainty. Suppose a soil property is denoted by a random variable X , and N observations of X are conducted on site or in laboratory. The mean or expected value of X , also known as the first central moment measuring the tendency in the data, could be calculated for the N observations as

$$\text{Mean} = E[X] = \mu_X = \frac{1}{N} \sum_{i=1}^N x_i \quad (2.9)$$

The variance of X , also known as the second central moment measuring the dispersion in the data about the tendency, could be estimated as

$$\text{Variance} = \text{Var}[X] = \sigma_X^2 = \frac{1}{N-1} \sum_{i=1}^N (x_i - \mu_X)^2 \quad (2.10)$$

where μ_X and σ_X are the mean and standard deviation of the data, respectively.

It is noted that the standard deviation does not clearly indicate the degree of dispersion

in X without referring to the mean. Thus it is useful to define another statistic, the coefficient of variation (c.o.v.), which is a non-dimensional term by taking the ratio of the standard deviation and the mean, such that

$$\text{c. o. v. } [X] = \frac{\sigma_X}{\mu_X} \quad (2.11)$$

By this definition, the c.o.v. is zero for a deterministic variable. Otherwise, a smaller value of c.o.v. indicates a smaller amount of uncertainty in the variable, and vice versa. Typical values of c.o.v. pertaining to soil properties are given in Table 2.2.

There are two other useful statistical moments, which are known as the skewness and the kurtosis. They measure the asymmetry and ‘peakedness’ of X , respectively.

$$\text{Skewness} = \gamma[X] = \frac{n}{(n-1)(n-2)} \frac{\sum_{i=1}^N (x_i - \mu_X)^3}{\sigma_X^3} \quad (2.12)$$

$$\text{Kurtosis} = \kappa[X] = \frac{n(n+1)}{(n-1)(n-2)(n-3)} \frac{\sum_{i=1}^N (x_i - \mu_X)^4}{\sigma_X^4} - \frac{3(n-1)^2}{(n-2)(n-3)} \quad (2.13)$$

To carry out probabilistic analysis of geotechnical problems, it is necessary to quantify these basic statistics of geotechnical parameters. However, quantifying the uncertainty of a soil property is generally not an easy task, since it needs a large amount of testing efforts which can be too costly to perform. As a result, reported values of the mean, standard deviation and c.o.v. of soil properties are usually adopted when analyzing slopes or embankments. Table 2.2 shows the ranges of typical values of c.o.v. of soil strength parameters gathered from various sources.

Sometimes there may be correlation between two soil properties X and Y . In this situation, covariance between the two variables could be defined to describe their

correlation, such that

$$Cov[X, Y] = E[(X - E[X])(Y - E[Y])] = \frac{\sum_{i=1}^N (x_i - E[X])(y_i - E[Y])}{N - 1} \quad (2.14)$$

Alternatively, a non-dimensional measure of the degree of correlation between X and Y , the correlation coefficient $\rho_{X,Y}$ is defined by dividing the covariance by the product of the two standard deviations

$$\rho_{X,Y} = \frac{Cov[X, Y]}{\sigma_X \sigma_Y} \quad (2.15)$$

The correlation coefficient may vary from -1 to 1 , with the extremes representing perfect negative and positive correlations, respectively. Pairs of independent random variables have zero correlation coefficient.

Kulhawy and Mayne (1990) summarized the correlation coefficients between commonly used soil properties. However, most of them were developed empirically and are often highly uncertain. In the context of slope design, correlations between several soil properties such as the unit weight γ , the strength parameters c and ϕ are of particular interest. However, correlations reported in the literature are not consistent. Lumb (1970) showed an almost complete lack of correlation between c and $\tan \phi$ for some unsaturated residual soils, and Schultze (1975) showed a similar finding in the study of German Clays. Matsuo and Kuroda (1974) reported strong negative correlation for an unsaturated soil tested in direct shear when sample moisture content was fixed. Wolff (1985), however, reported both negative and positive correlations between c and $\tan \phi$ for the borrow materials of Cannon dam. On the other hand, the cross-correlation between unit weight and shear strength is often not considered. This is due to the little variation of unit weight which is averaged within the soil mass (Alonso 1976).

Table 2.2 Typical values of coefficient of variation for soil strength parameters

Soil property	Range of c.o.v.	Data Sources
ϕ'	0.02-0.05	Lacasse and Nadim (1996)
	0.03-0.22	Kulhawy et al. (1991); Cherubini et al. (1993)
	0.05-0.15	Singh (1971); Hoeg and Murarka (1974); Lumb (1974); Schultze (1975); Phoon and Kulhawy (1999a)
		Schultze (1972); Harr (1977); Kulhawy (1992); Manoliu and Marcu (1993); Meyerhof (1993;1995); Becker (1996b)
	0.05-0.25	
	0.07-0.20	Phoon and Kulhawy (1999b)
	0.15-0.25	Barker et al. (1991)
$\tan \phi'$	0.05-0.15	Lumb (1966, 1974); Schneider (1999)
	0.07-0.15	Schultze (1972); Alonso (1976)
c'	0.20-0.50	Lumb (1966)
	0.25-0.30	Lumb (1974)
	0.30-0.50	Schneider (1999)
c_u	0.05-0.35	Lacasse and Nadim (1996)
	0.10-0.55	Phoon and Kulhawy (1999a; 1999b)
		Alonso (1976); Fredlund and Dahlman (1977); Meyerhof (1993, 1995); Cherubini et al. (1993); Kulhawy (1992); Becker (1996b)
	0.12-0.85	
	0.20-0.40	Schneider (1999)
	0.20-0.50	Lumb (1966, 1974); Singh (1971)

2.6.2 Spatial variation and method of analysis

A unique characteristic of soils is the spatial variability. The spatial variation could be caused by many reasons. For example, the weathering process such as erosion, temperature changes, freezing and thawing and the chemical processes such as oxidation, reduction and carbonation make the soil properties to vary from point to point. Moreover, that soils at a particular site may be residual or transported to their present location makes the soil properties even more spatially variable.

A typical profile of a soil property X measured at location $\mathbf{r}$ (e.g. c_u by CPT test) is shown in Figure 2.5. The measured value of $X(\mathbf{r})$ could be best expressed by two parts: a deterministic trend $t(\mathbf{r})$ that describes the tendency of the value on the whole, and a residual component about the trend $w(\mathbf{r})$, such that

$$X(\mathbf{r}) = t(\mathbf{r}) + w(\mathbf{r}) \quad (2.16)$$

where $\mathbf{r}$ is a position vector specifying the location. In one-dimensional models, $\mathbf{r}$ may be replaced by the horizontal or vertical distance. The trend is characterized deterministically by an equation. The residual component, however, is characterized as a random variable with zero mean and variance given by

$$Var(w) = E[\{X(\mathbf{r}) - t(\mathbf{r})\}^2] \quad (2.17)$$

The division of spatial variation between trend and residuals is an artefact of how to model soil variability. The selection of a certain trend function is a decision on the amount of the spatial variation to be treated as a deterministic function of space, and the amount to be accounted for statistically (Baecher, 1987). The variance of the residual will vary according to the trend function being chosen.

Figure 2.5 Components of soil variability

2.6.2.1 Trend analysis

Trends are often established by using regression analysis, the fitting of well-defined mathematical functions to data points in space. The relationship between the dependent variable (e.g. the soil property) and the independent variable (e.g. distance) can be modelled using either a linear or nonlinear function. Parameters that define the functions are estimated so as to provide a best-fit of the given data, such that the residual is minimised. The method of least square is the most commonly used approach for this purpose, and it can be found in many statistics textbooks (Haldar and Mahadevan 2000a; Baecher and Christian 2003, etc.). Also available is an alternative regression method based on Bayesian statistics.

2.6.2.2 Residual variation

Trend fitting using regression techniques involves several important underlying assumptions pertaining to the residual term. Firstly, the residual is assumed to be a normally distributed random variable with zero mean and constant variance; note this

assumption is usually valid for geotechnical properties. Secondly, the residuals are uncorrelated; this assumption is seldom true. It has been shown that two adjacent soil elements tend to possess similar properties. This characteristic is called *spatial autocorrelation*, because it describes the correlation between a particular property and itself over space. Spatial autocorrelation is a function of separation distance. Correlations are stronger between soil elements that are closer together, and vice versa.

If w_i and w_j represent the residuals at locations i and j , respectively, the spatial relationship between those two sites can be expressed by *autocorrelation function* as follows

$$\rho(\tau) = \frac{E[w_i w_j]}{Var[w]} = \frac{E[w_i w_{i+\tau}]}{Var[w]} \quad (2.18)$$

where τ is the separation distance between locations i and j . By this definition, it is clear that the autocorrelation between soil properties decreases with increment of their separation distance. At zero separation, the autocorrelation is $\rho(\tau = 0) = 1.0$; at extremely large separation, $\rho(\tau = \infty) = 0$. If $\rho(\tau)$ is multiplied by the variance of the residuals, $Var[w]$, the autocovariance function becomes

$$C(\tau) = E[w_i w_{i+\tau}] = Cov[w_i, w_{i+\tau}] \quad (2.19)$$

Several methods of estimating the autocovariance function are available. These include: (1) moment estimation; (2) maximum likelihood estimation; (3) Bayesian estimation. The most widely used one is moment estimation or method of moments, since it is conceptually simple, easy to implement, nonparametric and consistent. However, as DeGroot and Baecher (1993) pointed out, the moment estimator is only asymptotically unbiased and tends to perform poorly when the sample size is small. This deficiency can be overcome with the maximum likelihood estimator, which is asymptotically normal, consistent and has minimum variance. Bayesian estimation of autocovariance has not been widely used in the area of geotechnical engineering, and is less

well-developed as moment estimation. Discussions on the Bayesian approach can be found in Cressie (1993) and Berger (1993). Common autocorrelation functions are shown in Table 2.3.

Table 2.3 One-dimensional autocorrelation functions (after Baecher and Christian 2003)

Model	Equation (Note: δ = autocorrelation distance)
White noise	$\rho(\tau) = \begin{cases} 1 & \text{if } \tau = 0 \\ 0 & \text{otherwise} \end{cases}$
Linear	$\rho(\tau) = \begin{cases} 1 - \tau /\delta & \text{if } \tau \leq \delta \\ 0 & \text{otherwise} \end{cases}$
Exponential	$\rho(\tau) = \exp\left(-\frac{ \tau }{\delta}\right)$
Squared exponential (Gaussian)	$\rho(\tau) = \exp\left(-\frac{\tau^2}{\delta^2}\right)$
Spherical	$\rho(\tau) = 1 - \frac{3\tau}{2\delta} + \frac{\tau^3}{2\delta^3}$
Power	$\rho(\tau) = 1 + \left(\frac{ \tau }{\delta}\right)^{-2\alpha} \quad \alpha > 0$

2.6.2.3 Variograms and geostatistics

In the mining industry, the concept of autocorrelation is particularly important for estimating ore reserves. In geostatistics the variogram is more commonly used to characterize spatial structure, and it is defined as the expected value of the squared difference:

$$2\gamma(\tau) = E[(X_i - X_{i+\tau})^2] = \text{Var}[X_i - X_{i+\tau}] \quad (2.20)$$

where $\gamma(\tau)$ is the semivariogram, X_i and $X_{i+\tau}$ are observed values of the soil property X at locations i and $i + \tau$, respectively.

There is a direct relationship between the variogram and the autocovariance (as well as the autocorrelation) function. If the spatial property X has stationary mean and covariance, the variogram equation can be expressed as

$$2\gamma(\tau) = \text{Var}[X_i] - \text{Var}[X_{i+\tau}] - 2\text{Cov}[X_i, X_{i+\tau}] = 2\{C(0) - C(\tau)\} \quad (2.21)$$

$$\gamma(\tau) = 2\{C(0) - C(\tau)\} \quad (2.22)$$

where $C(\cdot)$ = covariance function. At infinitely large separation (i.e. $\tau \rightarrow \infty$) the semivariogram levels off at a sill value equal to the variance of the spatial variable.

2.6.2.4 Estimating autocorrelation or autocovariance

There are a number of statistical methods for estimating the autocovariance from sample data, such as the method of moments, the maximum likelihood estimates and Bayesian estimate. The method of moments is the one most used. Estimating autocorrelation/autocovariance of soil properties from in-situ testing data can be found in several publications, such as DeGroot and Baecher (1993), Christian et al. (1994), Jaksa et al. (1999), Fenton (1999), Jaksa (2000), Jaksa et al. (2000), Baecher and Christian (2003), Dasaka and Zhang (2012), and Stuedlein et al. (2012).

The following review focuses on the most commonly used method of moments. To estimate the autocovariance or autocorrelation, it is assumed that the statistical moments of a set of data/samples could be used as the corresponding moments of the population being sampled. This requires an adequately large sample size to obtain an unbiased estimate. Consider the measurements $[X_1, \dots, X_n]$ made at equally spaced locations $\mathbf{r} = [r_1, \dots, r_n]$ along a line, as for example in a boring, such that the separation distance is $\tau = r_{i+1} - r_i$. The sample autocovariance of the measurements for any separation $k\tau$ is

$$\text{Cov}(k\tau) = \frac{1}{n-k} \sum_{i=1}^{n-k} [X(r_i) - t(r_i)][X(r_{i+1}) - t(r_{i+1})] \quad (2.23)$$

where $\text{Cov}(k\tau)$ is the sample autocovariance or autocovariance estimator, $n - k$ is the number of data pairs having separation distance $k\tau$, and $t(r_i)$ is the trend removed from the data at location r_i . Often, $t(r_i)$ is simply replaced by the mean of the sample, $\mu_X = (1/n) \sum X(r_i)$. At $k = 0$, $\text{Cov}(k\tau)$ reduces to the sample variance. The corresponding moment estimator of the autocorrelation, $\rho(k\tau)$ is obtained by dividing both sides of Eq. (2.23) by the sample variance, to obtain

$$\rho(k\tau) = \frac{1}{(n-k)\sigma_X^2} \sum_{i=1}^{n-k} [X(r_i) - t(r_i)][X(r_{i+1}) - t(r_{i+1})] \quad (2.24)$$

in which σ_X is the sample standard deviation. Computationally, this simply reduces to taking all data pairs of common separation distance $k\tau$, calculating the correlation coefficient of that set, then plotting the result against $k\tau$.

The above equation is valid for samples separated equally. For a general case, measurements are seldom uniformly spaced, at least in the horizontal plane and seldom lie on a line. For such situations the sample autocovariance can still be used as an estimator, but with some modification. The most common way to accommodate non-uniformly placed measurements is by dividing separation distances into bands, and then taking the averages of the estimators within those bands. This introduces some error into the estimate, but for engineering purposes it is usually of sufficient accuracy.

After obtaining the autocovariance, some existing models (Table 2.3) could be used to best fit the plot of autocovariance against separation distance. The autocorrelation distance could then be estimated.

2.6.3 Random field and variance reduction

2.6.3.1 Spatial averaging and variance reduction in 1-D random fields

Spatial averaging of random fields is among the most important considerations for geotechnical engineering. For example, the failure of a slope depends upon the average strength along the failure surface. Vanmarcke (1977a) pointed out that although the mean of the strength averages (evaluated over intervals of width T) remained constant, their standard deviations would tend to decrease as T increases. When T is large, the fluctuations in the observations tend to cancel out more in the process of spatial averaging, resulting in *variance reduction*. Usually, variance reduction function is used instead of autocorrelation function in order to consider the spatial variation. The variance reduction function is expressed as follows:

$$\text{Var}(X_T) = \Gamma(T)\text{Var}[X] \quad (2.25)$$

where $\Gamma(T)$ = variance reduction factor/function with a value between 0 and 1, $\text{Var}[X]$ = the variance of a random soil property X , and $\text{Var}(X_T)$ = the reduced variance of X over an interval T due to (local) spatial averaging.

An alternative form for the variance function can be defined in terms of local integral of the random process

$$\text{Var}(I_T) = T^2\Gamma(T)\text{Var}[X] \quad (2.26)$$

where $\text{Var}(I_T)$ means the variance of local integral over an interval T . For convenience, the variance reduction function of local integral is introduced, as follows:

$$\Delta(T) = T^2\Gamma(T) \quad (2.27)$$

Vanmarcke (1983) derived two commonly used models of variance reduction function for local averaging, as follows

$$\Gamma(T) = \begin{cases} 1 & \text{if } T \leq \theta \\ \theta/T & \text{if } T \geq \theta \end{cases} \quad (2.28)$$

or

$$\Gamma(T) = \begin{cases} 1 & \text{if } T \leq \frac{\theta}{2} \\ \frac{\theta}{T} \left(1 - \frac{\theta}{4T}\right) & \text{if } T \geq \frac{\theta}{2} \end{cases} \quad (2.29)$$

in which θ = scale of fluctuation.

The variance reduction function can be related to the autocorrelation function mentioned previously via the following equation (Vanmarcke 1983)

$$\Gamma(T) = \frac{2}{T} \int_0^T \left(1 - \frac{\tau}{T}\right) \rho(\tau) d\tau \quad (2.30)$$

Using Eq. (2.30), for example, the most widely used negative exponential autocorrelation function $\rho(\tau) = \exp(-|\tau|/\delta)$ is in correspondence to its variance reduction counterpart in the form of

$$\Gamma(T) = 2 \cdot \left(\frac{\delta}{T}\right)^2 \left(e^{-T/\delta} + \frac{T}{\delta} - 1\right) \quad (2.31)$$

Likewise, other variance reduction functions can be derived from their autocorrelation counterparts, as listed in Table 2.4.

Vanmarcke (1983) also proposed a simple algebraic identity to describe the correlation between two local integrals or local averages over T and T' , such that

$$\rho_{X_T X_{T'}} = \rho_{I_T I_{T'}} = \frac{\text{Cov}[X_T, X_{T'}]}{\sigma_T \sigma_{T'}} = \frac{\sum_{k=0}^3 (-1)^k \Delta(T_k)}{2[\Delta(T)\Delta(T')]^{1/2}} \quad (2.32)$$

where $\Delta(T_k)$ is the variance reduction function over interval T_k , which is defined by Figure 2.6. Eq. (2.32) provides the basis for analyzing stochastic problems involving 1-D random field.

Table 2.4 One-dimensional variance reduction functions (after Vanmarcke 1983)

Description	Autocorrelation function $\rho(\tau)$	Variance reduction function $\Gamma(T)$ (Local averaging)	Variance reduction function $\Delta(T)$ (Local integral)
Triangular	$\begin{cases} 1 - \frac{ \tau }{\delta}, & 0 \leq \tau \leq \delta \\ 0, & \tau \geq \delta \end{cases}$	$\begin{cases} 1 - \frac{T}{3\delta}, & 0 \leq T \leq \delta \\ \frac{\delta}{T} \left(1 - \frac{\delta}{3T}\right), & T \geq \delta \end{cases}$	$\begin{cases} T^2 \left(1 - \frac{T}{3\delta}\right), & 0 \leq T \leq \delta \\ \delta \left(T - \frac{\delta}{3}\right), & T \geq \delta \end{cases}$
Exponential	$e^{- \tau /\delta}, \quad \tau \geq 0$	$2 \left(\frac{\delta}{T}\right)^2 \left(\frac{T}{\delta} - 1 + e^{-T/\delta}\right)$	$2\delta^2 \left(\frac{T}{\delta} - 1 + e^{-T/\delta}\right)$
Squared exponential (Gaussian)	$e^{-(\tau/\delta)^2}, \quad \tau \geq 0$	$\left(\frac{\delta}{T}\right)^2 \left(\sqrt{\pi} \frac{T}{\delta} E\left(\frac{T}{\delta}\right) - 1 + e^{-(T/\delta)^2}\right)$	$\delta^2 \left(\sqrt{\pi} \frac{T}{\delta} E\left(\frac{T}{\delta}\right) - 1 + e^{-(T/\delta)^2}\right)$

Note: $E(\cdot)$ is the “error function” which increase from 0 to 1 as its argument goes from 0 to infinity ∞ . It can be related to Standard Normal cumulative distribution function $\Phi(\cdot)$: $E(x) = 2[\Phi(\sqrt{2}x) - 0.5]$.

Figure 2.6 Correlation between local averages in a 1-D random field

2.6.3.2 Extension to 2-D random field

In a 2-D random field, the local integral I_A and local averaging X_A are considered within a rectangular area $A = T_1 T_2$. This yields a following relationship

$$X_A = \frac{I_A}{A} \quad (2.33)$$

Correspondingly, the variance reduction factors/functions of local integral and local averaging are associated to the point variance by their respective variance functions

$$\text{Var}(X_A) = \Gamma(T_1, T_2) \text{Var}[X] \quad (2.34)$$

$$\text{Var}(I_A) = \Delta(T_1, T_2) \text{Var}[X] \quad (2.35)$$

The two variance reduction functions are related by

$$\Delta(T_1, T_2) = (T_1 T_2)^2 \Gamma(T_1, T_2) \quad (2.36)$$

For a 2-D random field, the relation between autocorrelation function $\rho(\tau_1, \tau_2)$ and variance reduction function $\Gamma(T_1, T_2)$ follows

$$\Gamma(T_1, T_2) = \frac{1}{T_1 T_2} \int_{-T_1}^{+T_1} \int_{-T_2}^{+T_2} \left(1 - \frac{|\tau_1|}{T_1}\right) \left(1 - \frac{|\tau_2|}{T_2}\right) \rho(\tau_1, \tau_2) d\tau_1 d\tau_2 \quad (2.37)$$

Similar to the 1-D case, a simple algebraic identity can be used to describe the correlation between these 2-D local integrals or local averages (Vanmarcke 1983).

2.7 Review of probabilistic back-analysis in updating geotechnical uncertainties

The failure of a slope implies that the factor of safety of the slope at the moment of failure is unity. As such, back analysis of the failed slope is a useful way to improve our knowledge on the geotechnical parameters, such as soil shear strength parameters and pore-water pressure parameters at the moment of slope failure. In general, the back-analysis of slope failure can be categorized into two different types: (1) deterministic back-analysis and (2) probabilistic back-analysis.

In deterministic back-analysis, the slope model and all the geotechnical parameters are believed to be accurate. From deterministic back-analysis, only one geotechnical parameter can be calculated each time (Duncan and Wright 2005). If there are more than one unknown parameters to be back-analyzed, it is essential to select an objective parameter and make assumptions on the others. In this way, one can develop a back-calculated stability charts based on dimensionless parameters from a number of stability analyses (Stauffer and Wright 1984; Duncan and Stark 1992; Jiang and Yamagami 2006). While deterministic back-analysis can provide a useful insight into the conditions in a slope at the time of failure, Duncan and Wright (2006), however, pointed out several limitations when using this method

In probabilistic back-analysis, it is recognized that the slope stability model may not be perfectly accurate and geotechnical parameters are assumed uncertain throughout the back-analysis. Several probabilistic approaches for geotechnical back analysis are available in the literature, including: (1) least-squares method; (2) maximum likelihood method and (3) Bayesian method. For example, Luckman et al. (1987) performed

Bayesian updating for estimating pore pressures using the first-order reliability formulations. Honjo et al. (1994) employed an extended Bayesian method in their back-analysis of an embankment on soft clay; the number of observed data is assumed larger than the number of parameters to be updated. Ledesma et al. (1996) carried out geotechnical back analyses for a tunnel excavation and a large underground cavern based on the maximum likelihood approach. Also, the method of maximum likelihood is applicable only when the number of observed data is larger than the number of parameters to be updated. Gilbert et al. (1998) and Chowdhury et al. (2004) used discrete probability distributions in Bayesian updating of uncertain parameters. In their studies, a continuous random variable was approximated as a discrete variable with sufficient small intervals if accurate reliability evaluation is required. Based on Bayesian updating and system identification approach, Wu et al. (2007) and Wu (2011) presented a case history of probabilistic back-analysis of an embankment built over soft water-treatment sludge. To assure that the sludge would consolidate and gain strength as predicted, a test embankment was built. The observed performance of the test embankment was used to update the material properties used in the prediction for the test embankment. The results by Bayesian updating were then used to design and construct the full-scale embankment. Zhang et al. (2010) proposed two efficient methods based on the Bayesian updating and system identification approach for probabilistic back-analysis of slope failures; the methods are shown to be efficient in updating the statistical distributions of geotechnical parameters such as the shear strength parameters and pore pressure ratio, as shown in Figure 2.7, provided that the uncertainties to be updated follows a multivariate normal distribution. In view of the fact that the number of observed data is often less than the number of geotechnical parameters to be updated, as well as the fact that the prior distribution of uncertain parameters may not necessarily be normally distributed, Zhang et al. (2010) proposed a Markov chain Monte Carlo (MCMC) simulation method also based on Bayes's theorem for the probabilistic back-analysis.

Figure 2.7 Updating the statistical distributions by probabilistic back analysis (after Zhang et al. 2010)

2.8 Review of fundamental probabilistic methods

2.8.1 First-order reliability method

A search in the literature revealed that earlier applications of probabilistic method in the reliability analysis of geotechnical problems include Rosenblueth's point-estimate method (Lind 1983; Nguyen and Chowdhury 1984; Wolff 1985, 1991; Harr 1989; Li 1992)

and the mean-value first-order second-moment (MVFOSM) method (Alonso 1976; Christian et al. 1994; Hassan and Wolff 1999; Wu 2008). MVFOSM is based on a first-order Taylor series approximation of the limit state function linearized at the mean values of the random variables, and it uses only second-moment statistics (means and covariances) of the random variables. The MVFOSM is described in detail in many textbooks, for example Ang and Tang (1984), Haldar and Mahadevan (2000a), and Baecher and Christian (2003). The MVFOSM was originally formulated by Cornell (1969) assuming two statistically-independent normally-distributed random variables R and S such that the limit state function (or performance function) $Z = R - S$ has a normal distribution. An alternative formulation by Rosenbleuth and Esteva (1972) assumed that R and S are statistically-independent lognormal random variables with performance function $Z = \ln R - \ln S$ being normally-distributed. For the general case with performance function $Z = g(x_1, x_2, \dots, x_n)$, the MVFOSM defines the reliability index as (Cornell 1969)

$$\beta = \frac{\mu_Z}{\sigma_Z} \quad (2.38)$$

where μ_Z and σ_Z are the first-order approximations of the mean value and standard deviation of performance function Z , respectively.

$$\mu_Z = g(\mu_1, \mu_2, \dots, \mu_n) \quad (2.39)$$

$$\sigma_Z^2 = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} \text{Cov}(x_i, x_j) \quad (2.40)$$

where μ_i is the mean value of x_i , and $\text{Cov}(x_i, x_j)$ is the covariance of x_i and x_j . The partial derivatives in Eq. (2.40) are evaluated at the mean values.

The MVFOSM is simple to use and gives exact solution when performance function is linear and when all x_i are uncorrelated normal variables. Nonetheless, it suffers from a number of deficiencies (Haldar and Mahadevan 2000a). For example, the probability distributions of the random variables are ignored in MVFOSM. Moreover, there is a lack of invariance in β defined by Eq. (2.38) under different but mechanically equivalent formulations of the performance function (Ditlevsen 1973; USACE 1999).

To overcome the above-mentioned limitations of MVFOSM, Hasofer and Lind (1974) suggested a reliability index being defined as the minimum distance from the origin of the axes in the space of standard normal variables. A matrix formulation of the Hasofer-Lind reliability index for correlated normal random variables as described by Ditlevsen (1981) is

$$\beta = \min_{g(\mathbf{x})=0} \sqrt{(\mathbf{x} - \boldsymbol{\mu})^T \mathbf{C}^{-1} (\mathbf{x} - \boldsymbol{\mu})} \quad (2.41)$$

or equivalently (Low and Tang 1997b)

$$\beta = \min_{g(\mathbf{x})=0} \sqrt{\left[\frac{x_i - \mu_i}{\sigma_i} \right]^T \mathbf{R}^{-1} \left[\frac{x_i - \mu_i}{\sigma_i} \right]} \quad (2.42)$$

where μ_i and σ_i are the mean value and standard deviation of x_i , respectively, $\mathbf{C}$ is the covariance matrix, and $\mathbf{R}$ is the correlation matrix. For correlated non-normal random variables, μ_i and σ_i should be replaced by μ_i^N and σ_i^N – the equivalent normal mean and standard deviation of the non-normal x_i , respectively. The values of μ_i^N and σ_i^N may be estimated based on two-parameter equivalent normal transformation (Rackwitz and Fiessler 1978) or three-parameter equivalent normal transformation (Chen and Lind 1983; Wu and Wirsching 1987). The extension of the Hasofer-Lind index to correlated non-normals is known as the first-order reliability method (FORM).

Most algorithms (e.g., Rackwitz 1976; Liu and Der Kiureghian 1986) for evaluating the Hasofer-Lind reliability index carry out a search for the design point (or checking point or most probable failure point) in the space of standard normal variables (u-space). This standard normal space is obtained from appropriate transformation of the original space of the basic random variables x_i (Rosenblatt 1952; Nataf 1962; Shinozuka 1983; Der Kiureghian and Liu 1986). The design point $\mathbf{u}^*$ is defined as the point of minimum distance from the origin to the transformed performance function $g(\mathbf{u}) = 0$; the minimum distance so computed is the Hasofer-Lind reliability index.

$$\beta = \min \sqrt{\mathbf{u}^{*T} \mathbf{u}^*} \quad (2.43)$$

The classical FORM procedure can be best described by Figure 2.8. In the u-space, the

objective of finding $\mathbf{u}^*$ is equivalent to finding the smallest hyper-sphere tangent to the transformed limit state surface, and the radius of the smallest hyper-sphere is simply the Hasofer-Lind reliability index β .

Figure 2.8 Classical FORM approach in the space of standard normal variables

An alternative interpretation of the Hasofer-Lind reliability index was described in Low and Tang (1997a, 2004, 2007) based on the perspective of an expanding equivalent dispersion ellipsoid in the original space of the basic random variables ($\mathbf{x}$ -space), as shown in Figure 2.9. The procedure aims at finding the smallest hyper-ellipsoid (which has the same aspect ratio as the one-standard-deviation hyper-ellipsoid) that is tangent to the original limit state surface. This hyper-ellipsoid is β times the size of the $1\text{-}\sigma$ hyper-ellipsoid and touches the limit state surface at the design point $\mathbf{x}^*$. The ellipsoid perspective is simpler and more intuitive than the classical approach because it does not involve concepts of transformed space. The Low and Tang's FORM approach based on Microsoft Excel's Solver is demonstrated in Appendix A.

Figure 2.9 Low and Tang's FORM approach based on dispersion ellipsoids in original space

The Hasofer-Lind index and the FORM are more consistent than the MVFOSM because the former method is based on a first-order Taylor series approximation of the limit state function linearized at the design point, and so does not depend on how the limit state function is formulated. However, the MVFOSM appears to be more popular than the FORM in the literature. In this regard, USACE (1999) conceded that “many published analyses of geotechnical problems have not used the Hasofer-Lind method, probably due to its complexity, especially for implicit performance functions such as those in slope stability analysis,” and that “the most common method used in Corps practice is the Taylor series method, based on a Taylor series expansion of the performance function about the expected values.” Low (2003), in his survey of recent papers on geotechnical reliability analysis, also found that the Taylor series mean-value method was more often used than the Hasofer-Lind index.

In the context of FORM probabilistic evaluation, the failure probability P_f is directly inferred from reliability index using the following equation

$$P_f = \Phi(-\beta) \quad (2.44)$$

where $\Phi(.)$ is the cumulative distribution function (CDF) for standard normal distributions.

For most engineering problems, application of this simple first-order theory fulfils all practical needs, as noted by Rackwitz (2001). Two main limitations of the FORM-based probabilistic evaluation are noticeable. First, the curvature of the limit state surface at the design point is always neglected, and this limitation could cause significant error of the estimated failure probability when the limit state surface is highly nonlinear; Second, The FORM is observed inefficient in cases with multiple design points (Dolinski 1983; Der Kiureghian and Dakessian 1998). This is because most optimization algorithms for evaluating Hasofer-Lind index, depending on the initial values, may converge to a local design point. Even if the global design point is found, the Hasofer-Lind index cannot account for any significant contribution to the failure probability from the local design points. A more consistent and selective alternative is the generalized reliability index (Ditlevsen 1979a). The Ditlevsen index is obtained by integrating a weight function over the safe domain in transformed space. The generalized reliability index is consistent and sufficiently selective provided the weight function is rotationally symmetric and positive. However, the evaluation of the generalized reliability index is more tedious than the evaluation of the Hasofer-Lind index because of the difficulty in computing the integral of the weight function.

To overcome the first limitation of FORM-based probabilistic evaluation, the more consistent second-order reliability method (SORM) accounting for the curvature of limit state surface at design point can be used. On the other hand, to overcome the second limitation, the system reliability method accounting for multiple equally-likely Hasofer-Lind indices can be resorted to. The two advanced probabilistic methods are reviewed in the following sections.

2.8.2 Second-order reliability method

The second-order reliability method (SORM) improves the FORM-based probabilistic evaluation by including additional information about the curvature of the limit state surface at design point. Most classical computation procedures of the SORM are developed in the u-space, where the limit state surface at design point is approximated by use of a quadratic expansion. The probability of failure is then estimated from the probability of quadratic forms. A detailed discussion of the SORM can be found in Haldar and Mahadevan (2000a).

The SORM approach was first explored by Fiessler et al. (1979). They used various second-order Taylor expansions to approximate the limit state surface and calculated the failure probability based on the various quadratic approximations. Breitung (1984) derived an asymptotic approximation for multi-normal integrals by applying a quadratic Taylor expansion without mixed terms for the limit state surface at the design point. Breitung's asymptotic estimate is accurate only for large values of β . For problems of low reliability index, Tvedt's (1988, 1990) methods may be used instead. Tvedt's SORM used a parabolic approximation and took full second-order Taylor expansion to approximate the limit state surface at the design point.

In the above approximations, the paraboloid was fitted to the principal curvatures of the limit state surface, defined in terms of the eigenvalues of the matrix of the second-order derivatives of the surface. It is noted that when the number of random variables is large, and when the performance function involves complicated numerical algorithms, computation of the second-order derivatives can be prohibitively costly. To overcome these limitations in the use of SORM, Der Kiureghian et al. (1987) proposed a simpler method based on an approximating paraboloid fitted to the limit state surface at discrete points around the point with minimal distance from the origin. Der Kiureghian and De Stefano (1991) developed an efficient algorithm for computing the principal curvatures in an iterative manner using the gradient of the limit state function. The point-fitted paraboloid is simpler to implement and requires less

computation than the curvature-fitted paraboloid. In particular, Chan and Low (2011) and Chan (2012) implemented the point-fitted paraboloid method in a spreadsheet with simple user-defined VBA subroutines/functions to automatically compute the curvatures. The Chan and Low's SORM approach greatly reduces the computational effort and it largely facilitates the SORM for practical use; one limitation is that the approach is based on explicit performance functions only. The Chan and Low's SORM approach based on Microsoft Excel is demonstrated in Appendix B.

Many others have made various considerations and improvements for SORM. For example, Köylüoglu and Nielsen (1994) suggested a set of simple closed-form approximations for SORM integrals that was free from singularities and remained valid even for negative reliability indices. Cai and Elishakoff (1994) devised an analytical approximate formula, and showed that their proposed method gave accurate results even when the failure surface possessed large curvatures and was relatively close to the origin. Zhao and Ono (1997) introduced the concept of average principal curvature radius and obtained an empirical second-order reliability index, which was shown to be sufficiently accurate in cases with relatively small numbers of random variables and/or relatively small principal curvatures. An improved formulation of the empirical index that gave good approximations even for relatively large numbers of random variables and/or relatively large principal curvatures was obtained later (Zhao and Ono 1999a). For the case of extremely large principal curvatures or the case where a parabolic approximation of the limit state surface might encounter difficulty, Zhao and Ono (1999b) proposed an inverse fast Fourier transformation (IFFT) method for computing the failure probability. Hong (1999) also derived two simple correction factors to be used in conjunction with the Hohenbichler-Rackwitz (1988) formula.

Three different stages of approximations and errors can be identified in SORM (Adhikari 2004): (1) Type-1 error due to approximating a general nonlinear failure surface by a quadratic hyper-surface in the neighborhood of the design point; (2) Type-2 error due to approximating a general PDF by a Gaussian PDF in the neighborhood of the design point; and (3) Type-3 error due to approximations in

evaluating the probability content. Type-1 error is fundamental to SORM and cannot be improved unless higher order derivatives of the failure surface are used (Breitung and Richter 1996). Type-2 error can be avoided provided the basic variables are already Gaussian or transformed to Gaussian using numerical transformations (although equivalent normal transformations may introduce an uncertain amount of error). Extensive work has been done, for example by Tvedt (1983, 1988, 1990), Hohenbichler and Rackwitz (1988), Köylüoglu and Nielsen (1994), Cai and Elishakoff (1994), Zhao and Ono (1999a, b), Polidori et al. (1999), and Hong (1999), to improve the reliability assessment by dealing with Type-3 error.

Several closed-form SORM formulas are listed in Table 2.5. As expected, when the principal curvature κ_i tends to zero (i.e., the LSS is completely plane), all listed SORM equations reduce to the FORM approximation of $\Phi(-\beta)$.

Table 2.5 Closed-form formulas for second-order failure probability approximation (after Chan and Low 2011)

Originator	SORM formula
	$P_f = T_1 + T_2 + T_3$ $T_1 = \Phi(-\beta) \prod_{i=1}^{n-1} (1 + \beta \kappa_i)^{-0.5}$ $T_2 = [\beta \Phi(-\beta) - \phi(-\beta)] \cdot \left\{ \prod_{i=1}^{n-1} (1 + \beta \kappa_i)^{-0.5} - \prod_{i=1}^{n-1} [1 + (1 + \beta) \kappa_i]^{-0.5} \right\}$ $T_3 = (1 + \beta) [\beta \Phi(-\beta) - \phi(-\beta)]$ $\cdot \left\{ \prod_{i=1}^{n-1} (1 + \beta \kappa_i)^{-0.5} - Re \prod_{i=1}^{n-1} [1 + (\sqrt{-1} + \beta) \kappa_i]^{-0.5} \right\}$ <p>where $\phi(\cdot)$ and $\Phi(\cdot)$ are PDF and CDF of the standard normal variable, respectively.</p>
Breitung (1984)	$P_f = \Phi(-\beta) \prod_{i=1}^{n-1} (1 + \beta \kappa_i)^{-0.5}$

Hohenbichler & Rackwitz (1988)	$P_f = \Phi(-\beta) \prod_{i=1}^{n-1} [1 + \kappa_i \psi(\beta)]^{-0.5}$ <p>where $\psi(\beta) = \phi(\beta)/\Phi(-\beta)$</p>
Koyluoglu & Nielsen (1994)	$P_f = \Phi(-\beta) - \Phi(-\beta) \left(\prod_{i=m}^{n-1} [1 - 0.5\kappa_i \xi(\beta)]^{-0.5} \cdot \left\{ 1 - \prod_{i=1}^{m-1} [1 + \kappa_i \psi(\beta)]^{-0.5} \right\} \right)$ $+ \Phi(\beta) \left(\prod_{i=1}^{m-1} [1 + 0.5\kappa_i \psi(\beta)]^{-0.5} \cdot \left\{ 1 - \prod_{i=m}^{n-1} [1 - \kappa_i \xi(\beta)]^{-0.5} \right\} \right)$ <p>where $\psi(\beta) = \phi(\beta)/\Phi(-\beta)$ and $\xi(\beta) = \phi(\beta)/\Phi(\beta)$</p>
Cai & Elishakoff (1994)	$P_f = \Phi(-\beta) - \phi(\beta)(D_1 + D_2 + D_3)$ $D_1 = \sum_{i=1}^{n-1} \lambda_i ; D_2 = -\frac{1}{2}\beta \left(3 \sum_{i=1}^{n-1} \lambda_i^2 + \sum_{i \neq j}^{n-1} \lambda_i \lambda_j \right);$ $D_3 = -\frac{1}{6}(\beta^2 - 1) \left(15 \sum_{i=1}^{n-1} \lambda_i^3 + 9 \sum_{i \neq j}^{n-1} \lambda_i^2 \lambda_j + \sum_{i \neq j \neq k}^{n-1} \lambda_i \lambda_j \lambda_k \right)$ <p>where $\lambda_i = \kappa_i/2$</p>
Hong (1999)	$P_f = C_1 P_0$ $P_0 = \Phi(-\beta) \prod_{i=1}^{n-1} [1 + 2\lambda_i \psi(\beta)]^{-0.5}$ $C_1 = \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{\Phi(-\beta - \{(n-1)\lambda_i/[1 + 2\lambda_i \psi(\beta)]\})}{\Phi(-\beta)} \cdot \exp \left[\frac{(n-1)\lambda_i \psi(\beta)}{1 + 2\lambda_i \psi(\beta)} \right]$
Hong (1999)	$P_f = C_2 P_0$ $P_0 = \Phi(-\beta) \prod_{i=1}^{n-1} [1 + 2\lambda_i \psi(\beta)]^{-0.5}$ $C_2 = 1 - \frac{n-1}{3} + \frac{1}{3} \sum_{i=1}^{n-1} \frac{\Phi(-\beta - \{3\lambda_i/[1 + 2\lambda_i \psi(\beta)]\})}{\Phi(-\beta)} \cdot \exp \left[\frac{3\lambda_i \psi(\beta)}{1 + 2\lambda_i \psi(\beta)} \right]$

2.8.3 System reliability method

Engineering problems often involve multiple failure modes. There may be several potential modes of failure for a system, in which the occurrence of any one of the potential failure modes will result in failure of the system (Ang and Tang 1984). For instance, a semi-gravity retaining wall could fail by sliding, overturning or bearing

capacity failure. A slope may fail along any of several potential sliding surfaces. The calculation of the failure probability of a system by direct simulation method is generally time-consuming. As a result, lower and upper bounds of the failure probability are useful. The review below concerns reliability of series systems.

For positively-correlated failure modes the uni-modal bounds on the system failure probability $P_{f,sys}$ are (Cornell 1967)

$$\max_i P_{f,i} \leq P_{f,sys} \leq 1 - \prod_{i=1}^m (1 - P_{f,i}) \quad (2.45)$$

where $P_{f,i}$ is the failure probability of the i th mode out of m potential failure modes. The values of $P_{f,i}$ can be assessed via a variety of methods, such as FORM, SORM and Monte Carlo simulations. The uni-modal bounds may be too wide for practical use, but can be improved by taking into account the correlation between pairs of failure modes. The resulting bi-modal bounds require the probabilities of joint events. The following bi-modal bounds based on Kounias (1968), Hunter (1976), and Ditlevsen (1979b) are also summarized in Ang and Tang (1984), Madsen et al. (1986), and Haldar and Mahadevan (2000a), such that

$$P_{f,1} + \sum_{i=2}^m \max \left[\left\{ P_{f,1} - \sum_{j=1}^{i-1} P_{f,ij} \right\}, 0 \right] \leq P_{f,sys} \leq \min \left[\left\{ \sum_{i=1}^m P_{f,i} - \sum_{i=2}^m \max_{j<i} P_{f,ij} \right\}, 1 \right] \quad (2.46)$$

where $P_{f,ij}$ is the joint failure probability between the i th and j th modes. Calculating $P_{f,ij}$, however, could be very cumbersome. The computation of the joint failure probability is explained elsewhere such as in Madsen et al. (1986), Hohenbichler and Rackwitz (1983), and Gollwitzer and Rackwitz (1988). In contrast, a weakened version of Eq. (2.46) was proposed by Ditlevsen (1979b), in which the complex problem of estimating system failure probability was reduced to problems of estimating failure probabilities corresponding to various component limit states as well as the correlation coefficients among various component failure modes. A detailed discussion on the Kounias-Ditlevsen's bi-modal bounds theory and its application to system reliability analysis of various slopes will be given in Chapter 6.

2.8.4 Monte Carlo simulation and importance sampling technique

2.8.4.1 Basic concepts of Monte Carlo simulation

The Monte Carlo simulation (MCS) method is a more basic way to perform probabilistic analysis. The method is extremely useful with the fast development of computation technology. In general, MCS method involves first sampling the basic random variables subjected to their statistics and then evaluating the performance function for each set of realizations of the sample data. The number of realizations or MC trials needed for a typical simulation depends on: (1) the desired level of accuracy (e.g. 10% error); (2) the desired confidence interval (e.g. 95%); as well as (3) the unknown probability of failure. The required number of MC trials can potentially be in the order of millions for very small values of P_f , which are not uncommon in engineering problems. This makes the MCS prohibitively costly for many engineering applications.

The basic mathematical definition of failure probability P_f in MCS can be written as

$$P_f = \int_{g(\mathbf{x}) < 0} I_g(\mathbf{x}) f_X(\mathbf{x}) d\mathbf{x} = \frac{1}{N} \sum_{i=1}^N I_g(\mathbf{x}_i) \quad (2.47)$$

where $f_X(\mathbf{x})$ is the joint probability density function of the random variables $\mathbf{X}$, $I_g(\mathbf{x})$ is an indicator function with respect to the performance function $g(\mathbf{x})$ and is defined as

$$I_g(\mathbf{x}) = \begin{cases} 0 & \text{if } g(\mathbf{x}) > 0 \\ 1 & \text{if } g(\mathbf{x}) < 0 \end{cases} \quad (2.48)$$

The efficiency of the MCS can be improved by either reducing the number of trials needed to obtain the same level of accuracy, or increasing the accuracy given the same number of trials. Such enhancement techniques are known as *variance reduction technique* (not to be confused with *variance reduction* in the context of spatial averaging). Many of the variance reduction techniques, including antithetic variables, control variables, stratified sampling, importance sampling and adaptive sampling are

described in detail by Melchers (1999), Haldar and Mahadevan (2000a), Rubinstein and Kroese(2008), among others. In particular, the importance sampling technique is briefly mentioned below.

2.8.4.2 Importance sampling technique

For importance sampling, a new sampling PDF $h_v(\mathbf{v})$ is defined so as to obtain samples in a desired region (e.g. with higher failure probability). This newly defined PDF is known as the sampling density function, and the probability of failure in Eq. (2.47) can be redefined as (Melcher 1989)

$$P_f = \int_{g(\mathbf{x}) < 0} I_g(\mathbf{v}) \frac{f_x(\mathbf{v})}{h_v(\mathbf{v})} h_v(\mathbf{v}) d\mathbf{v} = \frac{1}{N} \sum_{i=1}^N I_g(\mathbf{v}_i) \frac{f_x(\mathbf{v}_i)}{h_v(\mathbf{v}_i)} \quad (2.49)$$

in which the samples $\mathbf{v}$ is now taken from the sampling PDF $h_v(\mathbf{v})$

Several methods have been developed for selecting the appropriate importance sampling density such that the variance of the failure probability estimate can be reduced (e.g., Harbitz 1986; Fujita and Rackwitz 1988; Ibrahim 1991; Melchers 1989, 1999). As opposed to the brute-force MCS, importance sampling performs sampling in the region that contributes most to the failure probability, usually identified as the neighborhood of the design point. Excellent reviews on many of these importance sampling methods can be found in Engelund and Rackwitz (1993), Melchers (1999), Rubinstein and Kroese (2008), among others.

A simple and efficient approach for importance sampling as suggested by Melchers (1989) is briefly mentioned here. The first step is to decide the design point or checking point $\mathbf{x}^*$ which is of maximum likelihood of failure. The design point may be obtained using FORM or other reliability methods. The next step is to choose an appropriate model for the sampling density function $h_v(\mathbf{v})$, and to take samples around the design point. When the original problem is standardized in the u-space, using a multi-normal probability density function with independent components as $h_v(\mathbf{v})$ will greatly simplify the computation, such that

$$h_v(\mathbf{v}) = \phi_n(\mathbf{v}) = \prod_{i=1}^n \phi_i(v_i) \quad (2.50)$$

where $\phi_n(\cdot)$ is of standardized n -dimensional multi-normal distribution (or the joint pdf), and $\phi_i(\cdot)$ is the individual pdf of normal distribution with mean at design point $\mathbf{x}^*$.

Melchers (1989) pointed out that there appears to be little point in applying a MCS if the design point has been located using the reliability methods except to improve the accuracy of results. Thus a direct approach of finding the region of most interest was suggested to implement importance sampling.

In addition to these commonly used variance reduction techniques, there are other advanced simulation techniques, such as the directional simulation method (Bjerager 1988) and subset simulation method (Au and Beck 2001; Au et al. 2007; Au et al. 2010).

The MCS method has been extensively applied for probabilistic slope stability analysis (Kim et al. 1978; Major et al. 1978; Nguyen and Chowdhury 1985; El-Ramly et al. 2002; Cho 2007; Ching et al. 2009, 2010). In addition, MCS are often used as independent and rigorous means of evaluating the probability of failure, for comparison with results obtained using reliability methods (e.g. FORM or SORM).

2.8.5 Finite element reliability method

2.8.5.1 Random finite-element method

The random finite-element method (RFEM) involves generation of a random field of material properties which is mapped onto a finite-element mesh. A conventional finite-element analysis using these randomly generated properties is then performed to compute the response variable, after which the process is repeated many times using Monte Carlo simulation. Prominent pioneer works are largely due to Griffiths and

Fenton (Griffiths and Fenton 2000; 2004; 2009). Paice et al. (1996) studied the effect of random and spatially-correlated soil stiffness on the total settlement under the center of a uniformly-loaded flexible strip footing. Fenton and Vanmarcke (1998) assessed the spatial variability of liquefaction risk. More recent geotechnical applications are summarized here. The bearing capacities of cohesive soil (Griffiths and Fenton 2001) and c- ϕ soil (Fenton and Griffiths 2003) were investigated. The soils were modelled as random media with spatially varying shear strength. The reliability of shallow foundations on spatially-random two-dimensional soil (Fenton and Griffiths 2002) and three-dimensional soil (Fenton and Griffiths 2005) against excessive differential settlement was assessed. Griffiths et al. (2009, 2011) also investigated the failure probability of slopes using the RFEM; it was shown that the spatial variability of soil parameters significantly affects the slope failure mode, as shown in Figure 2.10. Huang et al. (2010a) investigated the consolidation behaviour of soil deposits with spatially varying compressibility and permeability. Hicks and Spencer (2010) used the RFEM to investigate the effect of three-dimensional spatial variation on slope stability.

Figure 2.10 Slope stability analysis by RFEM (after Griffiths et al. 2009): (a) a relatively low spatial autocorrelation distance; (b) a relatively high spatial autocorrelation distance

2.8.5.2 Stochastic finite-element method

Combination of the deterministic finite-element method with a truncated Taylor series reliability method leads to the stochastic finite-element method (SFEM). Prominent pioneer works using the SFEM are summarized below. Contreras (1980) proposed a generic stochastic finite-element method in which stochastic differential and difference

equation theory was applied in structures discretized with the finite-element methodology. Transient structural loads were idealized as stochastic processes. Vanmarcke and Grigoriu (1983) evaluated the second-moment statistics of the deflection of a beam of randomly varying rigidity along its axis, based on a stochastic finite-element formulation in which the elastic characteristics of each finite element are assumed equal to the local average over the element length. Shinozuka and Deodatis (1988) examined the response variability of finite-element systems arising from one-dimensional spatial randomness of the material properties. The problem was analyzed with a first-order Neumann expansion of the stiffness matrix of the system. Spanos and Ghanem (1989) used a Karhunen-Loeve expansion to represent the random material property modelled as a stochastic process.

A stochastic finite-element method based on the FORM was described in Der Kiureghian and Ke (1988). The main effort for finite-element implementation was in computing the gradient of response variables in terms of basic load, material property or geometry variables. The response gradient vector can be evaluated through sensitivity analysis using either finite-difference or perturbation methods (Haldar and Mahadevan 2000b). In their book, Haldar and Mahadevan (2000b) present SFEM-based reliability analysis in the context of displacement-based and stress-based deterministic finite-element approaches, considering linear, nonlinear, static and dynamic problems.

A few notable geotechnical applications of SFEM are mentioned here. Baecher and Ingra (1981) applied the stochastic finite-element method in settlement predictions. Ishii and Suzuki (1987) applied the stochastic finite-element method to earth slope stability analysis. Mellah et al. (2000) performed MFOSM-based stochastic finite-element analysis of an embankment with a nonlinear constitutive model. Ghanem et al. (2007) studied an embankment dam problem using a new spectral stochastic finite-element method based on stochastic model reduction for polynomial chaos representation.

2.8.5.3 Comparisons between RFEM and SFEM

Griffiths and Fenton (2009) compared results obtained in a probabilistic settlement analysis of a strip footing between SFEM based on MVFOSM approximations and RFEM based on Monte Carlo simulations. The authors pointed out that the SFEM cannot directly model the influence of spatial variability nor the non-symmetric lognormal distribution. This shortcoming could make the SFEM consistently underestimate the standard deviation of the footing settlement. Thus, the SFEM inevitably leads to un-conservative estimates of the probability of failure. Sudret and Der Kiureghian (2002) compared two methods for finite-element reliability analysis. One is based on direct coupling of finite-element and reliability codes, and the other involves post-processing of results obtained from the spectral stochastic finite-element method (SSFEM). The two procedures are applied to the reliability analysis of the settlement of a foundation on heterogeneous soil. The direct-coupling approach was found to be robust and efficient. The SSFEM approach gave fair results only with high-order polynomial chaos expansions, which lead to prohibitively high computational costs.

2.9 Summary

This Chapter presented an extensive literature review. The first part provided some commonly used methods for slope stability analysis; these methods intend to compute the factor of safety F_s using a combination of soil properties which are deemed to be deterministic, hence all these methods are within the framework of deterministic analysis. The second part reviewed most commonly used probabilistic methods; in the probabilistic analysis the soil properties are taken to be uncertain and a reliability index β or probability of failure P_f is used to evaluate the stability of slopes.

The probabilistic analysis is a more rational approach, since it extends the conventional deterministic analysis by giving them a context and by giving additional information to help engineer interpret the results. However, the probabilistic analysis does not

invalidate the calculations of factors of safety as noted by Christian (2004) and different deterministic and probabilistic approaches are valuable and can contribute in their complementary roles as noted by Low (2010).

Chapter 3 Sensitivity-Based Reliability Analysis and Probabilistic Strength-Reduction Finite Element Analysis

3.1 Introduction

For slope stability analysis, an attractive alternative to the LEM of slices is the *shear strength reduction method (SRM)*, which can be integrated into finite element analysis or finite difference analysis (Zienkiewicz et al. 1975; Naylor 1982; Ugai 1989; Matsui and San 1992; Griffiths and Lane 1999; Dawson et al. 1999). For slopes with embedded structures such as stabilization piles, the SRM is particularly useful. However, for the probabilistic slope stability analysis, the use of SRM is rare. To overcome some limitations that hinder the development of probabilistic SRM stability analysis, a new procedure for FORM based on probabilistic sensitivity analysis is presented in this Chapter.

3.2 Slope reliability analysis involving implicit performance function

The stability of slopes is commonly evaluated using two different types of methods: (i) limit equilibrium methods (LEM), and (ii) strength-reduction method (SRM). The LEM methods have been well documented, and Duncan (1996) presented a comprehensive state-of-the-art review on the methodology. The factor of safety F_s , defined as the ratio of shear strength (or resisting moment) to mobilized shear stress (or overturning moment) of a potential sliding mass, is used as an indicator of its stability. Locations of potential slip surfaces need to be assumed in LEM. The factor of safety is evaluated on all the potential slip surfaces, and a critical slip surface which produces the lowest factor of safety F_s is finally determined. For simple slopes with regular geometry and

$\phi_u = 0$ conditions, a closed-form solution can be derived for F_s . Otherwise, several LEM of slices can be used to compute the F_s . The strength-reduction FEM (e.g., Griffiths and Lane 1999; Zheng and Zhao 2004) or FDM (e.g., Dawson et al. 1999) has proven to be powerful in evaluating slope stability. Some advantages and limitations of the strength-reduction method were summarized in Chapter 2.

However, when the slope stability analyses are extended into probabilistic study, one common limitation of both LEM of slices and SRM-based FEM/FDM is that no closed-form solutions are available. In other words, the performance function of the safety factor is implicit, and it can only be evaluated by numerical simulations. As such, direct use of reliability methods (e.g. FORM/SORM) is not easy since the optimization of reliability index requires an explicit performance function. Although one can use Monte Carlo simulation (MCS) method for probabilistic analysis even when the performance function is implicit (e.g. Griffiths and Fenton 2004), the MCS method may require intensive computational effort especially when the strength reduction method is employed. Thus, using reliability methods for probabilistic slope stability analysis involving implicit performance functions is an attractive alternative (Xu and Low 2006).

Several useful techniques bridging the reliability methods with slope stability analysis involving implicit performance functions have been proposed in the literature. These techniques include response surface method (RSM) and artificial neural networks (ANN). Both methods aim at approximating the actual performance function using closed-form expressions. For example, polynomials are commonly used in RSM, and multilayer functions (also called multilayer perceptrons) are used in ANN. Another technique for reliability analysis incorporating implicit performance functions is the recursive algorithm for FORM. This is less commonly used in the literature.

The following study investigates two techniques for slope reliability when implicit performance functions are encountered, namely, the commonly used RSM and a proposed probabilistic sensitivity-based FORM by recursive algorithm. Comparison between the two techniques is then made. Application of the proposed approach to several case studies selected from the literature will be presented.

3.3 Response surface method for approximating the implicit performance function (Method 1)

3.3.1 Polynomial response surface

Since the introduction of response surface method (RSM) into slope stability analysis by Wong (1985), this approach has been widely used and developed. The basic concept of RSM involves approximating the limit state function or performance function by a simple and explicit polynomial (Bucher and Bourgund 1990; Rajashekhar and Ellingwood 1993; Li 2000; Xu and Low 2006; Li et al. 2011) or by neuron networks (Goh 2003; Elhewy et al. 2006; Goh and Kulhawy 2006; Cho 2009). Reliability analysis is then carried out based on the response surface (RS). It has been suggested that a quadratic polynomial without cross terms is suitable for approximating the implicit performance function associated with slope stability. The expression for this RS model is

$$\bar{g} = \bar{g}(\mathbf{x}) = a + \sum_{i=1}^n b_i x_i + \sum_{i=1}^n c_i x_i^2 \quad (3.1)$$

where $\bar{g}$ = approximate RS function, n = number of basic variables, $\mathbf{x}$ = vector of basic variables ($x_1, \dots, x_n$), e.g., shear strength parameters of soils, and a, b_i, c_i are constants to be determined by numerical experiments.

Eq. (3.1) contains $(2n+1)$ unknown coefficients. To determine the coefficients, $(2n+1)$ sampling points need to be selected, and $(2n+1)$ deterministic stability evaluations be

conducted at these sampling points to obtain the actual performance values g (not the approximate polynomial $\bar{g}$) by numerical method such as LEM of slices or strength reduction FEM/FDM. This will result in $(2n+1)$ equations. Rewrite Eq. (3.1) in matrix form gives

$$\bar{g}(\mathbf{x}) = \mathbf{a}^T \mathbf{x} \quad (3.2)$$

The $(2n+1)$ equations at sampling points can then be expressed by

$$\mathbf{g} = \mathbf{S}\mathbf{a} \quad (3.3)$$

where $\mathbf{a}$ = vector of unknown coefficient, $\mathbf{a} = [a, b_1, \dots, b_n, c_1, \dots, c_n]^T$, $\mathbf{x} = [1, x_1, \dots, x_n, x_1^2, \dots, x_n^2]^T$, $\mathbf{g}$ = vector of the $(2n+1)$ sampling-point performance values g , and $\mathbf{S}$ = sampling matrix with rows denoting the $(2n+1)$ sampling points, i.e., $\mathbf{S} = [\mathbf{x}_1, \dots, \mathbf{x}_{2n+1}]^T$.

By simple matrix operations, the unknown coefficients are obtained as

$$\mathbf{a} = \mathbf{S}^{-1}\mathbf{g} \quad (3.4)$$

Substituting the coefficients into Eq. (3.1) or Eq. (3.2), the approximate RS is obtained.

In the experiments for developing RS, the $(2n+1)$ sampling points are commonly selected around a centre of sampling. For instance, if the mean value point is selected as centre of sampling, $2n$ sampling points will be selected at $(\mu_i \pm h\sigma_i)$ (h is a sampling factor that determines how much the sampling points deviate from the centre point), and one sampling at the centre point.

3.3.2 An efficient method of improving the response surface

Since the RS obtained using the above procedure may not fit the actual performance

function at design point, an iterative procedure is required to progressively improve the approximate RS: (i) construct the first RS $\bar{g}^1(\mathbf{x})$, and perform reliability assessment based on $\bar{g}^1(\mathbf{x})$. This will result in a tentative β value and tentative design point; (ii) construct the improved RS $\bar{g}^2(\mathbf{x})$ based on a new centre point, and find a new tentative β value based on $\bar{g}^2(\mathbf{x})$; and (iii) the iteration stops when the difference between two successive tentative β values is acceptably small and the actual performance function $g(\mathbf{x})$ approaches zero at the final tentative design point.

A simple way for the iterative construction is to use a tentative design point as the centre point around which the next experiment is conducted. This simplified procedure for improvement of RS works well only when the actual performance function is linear. However, for some problems with highly nonlinear performance function, this simplified approach becomes inefficient (e.g. more iterations are incurred), and can even lead to a wrongly converged result.

A more efficient approach is based on linear interpolation to locate a new centre point for the next approximation (Bucher and Bourgund 1990; Rajashekhar and Ellingwood 1993). This approach is briefly introduced as follows.

Consider a two-variable limit state function, the process of successively locating the design point and centre point is shown in Figure 3.1. The first approximation is obtained by conducting sampling in the vicinity of mean value point $\mathbf{x}_{M1}$ as shown in Figure 3.1. Based on the first set of sampling points, a response surface $\bar{g}^1(\mathbf{x})$ is fitted and reliability assessment is then implemented on $\bar{g}^1(\mathbf{x})$. Then a tentative design point of the first iteration, $\mathbf{x}_{D1}$, can be determined. Subsequently, a new centre point needs to be located by linear interpolation in order to do the next computation, such that

$$\mathbf{x}_{M_{i+1}} = \mathbf{x}_{M_i} + (\mathbf{x}_{D_i} - \mathbf{x}_{M_i}) \frac{g(\mathbf{x}_{M_i})}{[g(\mathbf{x}_{M_i}) - g(\mathbf{x}_{D_i})]} \quad (3.5)$$

where $\mathbf{x}_{Mi}$ is the vector denoting i -th experimental center point, $\mathbf{x}_{Di}$ is the vector denoting i -th design point, $g(\mathbf{x}_{Mi})$ means the actual performance function value at point $\mathbf{x}_{Mi}$, and $g(\mathbf{x}_{Di})$ means the actual performance function value at design point $\mathbf{x}_{Di}$.

Re-sampling around the new centre point, a new RS $\bar{g}^2(\mathbf{x})$ is then obtained. The iterative process is repeated until the difference between two successive β values becomes acceptable.

To perform the iterative approximation efficiently, it is interesting to study the effect of the value of sampling factor h . Bucher and Bourgund (1990) used a constant value of $h = 3$ in their numerical examples to achieve a better approximation. Rajashekhar and Ellingwood (1993) suggested that starting the process with $h = 2$ and then repeating once more with $h = 1$ can give a satisfactory response surface around the actual design point. Guan and Melchers' (2001) parametric study provided evidence that h strongly influences the estimated probability of failure and the stability of convergence, depending on the shape of the actual limit state surface. No clear guidance for the selection of h can be given in a single case, but a value between 1 and 3 appears to work in most typical circumstances. In contrast, Wong et al. (2005) proposed that h be chosen according to the coefficients of variation (c.o.v.) of the random variables: low h values for high coefficient of variation and vice versa. On the other hand, it is observed in this study that using a constant h value equal to unity is a simple and efficient method for improving the RS for slope reliability assessment. This may be due to the fact that most slope stability problems show linear or almost linear limit state surface.

Legend: x_1, x_2 = a two-variable space for illustration; $g(\mathbf{x})$ = actual limit state surface; $\bar{g}^1(\mathbf{x})$ = the first approximated response surface; $\bar{g}^2(\mathbf{x})$ = the second approximated response surface; $\bar{g}^3(\mathbf{x})$ = the third approximated response surface; $\mathbf{x}_{M1}$ = center point for the i th approximation; $\mathbf{x}_{Di}$ = design point for the i th approximation

Figure 3.1 Schematic interpretation for successive approximation of response surface (after Rajashekhar and Ellingwood 1993)

3.3.3 Illustration of the RSM-based FORM using a simple example

In this subsection, the response surface method will be illustrated with a problem involving correlated non-normal variables from Ang and Tang (1984), which was solved in Low and Tang (2004, 2007) using the constraint optimization approach. The example problem illustrates a reliability analysis of a steel beam section. The fully plastic flexural capacity is given as YZ , where Y = the yield strength of steel and Z = section modulus of the section. Subjected to a bending moment M at that section, the performance function is defined as $g = YZ - M$. The statistical information of the three

variables Y , Z and M is shown in Figure 3.2. Note that the performance function of this example problem is in closed-form. This may appear not necessary to use a response surface. However, it allows the procedure of the RSM to be explained with clarity, and comparisons to be readily made between the reliability assessments based on the approximate RS and actual performance function. Application of the RSM-based FORM to slope stability problem will be presented later.

Figure 3.2 also shows a detailed procedure of iterative construction of RS in a spreadsheet, where the constraint optimization approach for FORM (Low and Tang, 2007) was used. The Low and Tang (2007)'s FORM approach is illustrated in Appendix A. The first RS was determined by sampling around the mean value point $\mathbf{x}_{M1}$ ($Y=40$, $Z=50$, $M=1000$). Based on the first RS, FORM reliability analysis was then conducted using the constraint optimization approach (not on the closed-form performance function g , but on the obtained RS). This gives a tentative design point and a tentative β value is computed to be 2.6336 (Figure 3.2a).

A new centre point was then computed using Eq. (3.5). Using the new centre point, a second response surface was obtained. Reliability analysis based on the second RS results in a tentative β value of 2.6644 (Figure 3.2b).

Repeating the above procedure, a tentative β value on the third RS was found to be 2.6645 (Figure 3.2c). The difference between the second and the third tentative β values falls below an acceptable threshold. Thus the third response surface was regarded as the final approximation of the actual performance function, and the third tentative β value was taken to be the final one. Note that this value is practically identical to the accurate β based on the actual performance function (2.6646 obtained by Low and Tang, 2004) compared with the roughly-estimated one of 2.6336 based on the first response surface.

(a)

Construction of Response Surface

$\mathbf{x}_1$ vector	Sampling Matrix: $\mathbf{S}_1$	Sampling points with $h=1$	$\mathbf{x}_{Mi}$ Vector	$\mathbf{x}_{Mi+1}$ Vector
1	1	1	1	1
Y	40	45	35	40
Z	50	50	50	50
M	1000	1000	1000	1200
Y^2	1600	2025	1225	1600
Z^2	2500	2500	2500	2500
M^2	1000000	1000000	1000000	1440000
$\mathbf{g}$ vector	1000	1250	750	1100
$\mathbf{a}_1$ vector	a	b_1	b_2	b_3
	-2000	50	40	-1
			c_1	c_2
			5.3E-15	1.4E-14
				1.2E-17

$\mathbf{g}(\mathbf{x}_{Mi}) = 1000$ $\mathbf{g}(\mathbf{x}_{Di}) = 15.871$

$\mathbf{g} = \mathbf{Y}\mathbf{Z} - \mathbf{M}$

"= MMULT($\mathbf{g}$, MINVERSE($\mathbf{S}_1$))", Hold Ctrl and Shift, followed by Enter

Reliability Analysis Based on the Response Surface

Distribution	Variable	Para1	Para2	Para3	Para4	$\mathbf{x}^*$	Correlation matrix $\mathbf{R}$	$\mathbf{n}$
Lognormal	Y	40	5			33.525	1 0.4 0	-1.356
Lognormal	Z	50	2.5			47.549	0.4 1 0	-0.981
ExtValue1	M	1000	200			1578.2	0 0 1	2.206

$\tilde{\mathbf{g}}(\mathbf{x}) = 0.0000$ $\beta = 2.6336$

$\mathbf{g} = \mathbf{a}_1 \mathbf{x}_1$ (based on $\mathbf{x}^*$ values)

$\mathbf{x} = \mathbf{x}_i(\text{Distribution, Para1:Para4, } \mathbf{n})$

(b)

Construction of Response Surface

$\mathbf{x}_2$ vector	Sampling Matrix: $\mathbf{S}_2$	Sampling points with $h=1$	$\mathbf{x}_{Mi}$ Vector	$\mathbf{x}_{Mi+1}$ Vector
1	1	1	1	1
Y	33.4209	38.4209	28.4209	33.4209
Z	47.5092	47.5092	47.5092	47.5092
M	1587.54	1587.54	1587.54	1787.54
Y^2	1116.95	1476.16	807.746	1116.95
Z^2	2257.13	2257.13	2257.13	2257.13
M^2	2520283	2520283	2520283	3195299
$\mathbf{g}$ vector	0.26008	237.806	-237.29	83.8123
$\mathbf{a}_2$ vector	a	b_1	b_2	b_3
	-1587.8	47.5092	33.4209	-1
			c_1	c_2
			0	8.9E-16
				-7E-18

$\mathbf{g}(\mathbf{x}_{Mi}) = 0.26008$ $\mathbf{g}(\mathbf{x}_{Di}) = 0.10268$

$\mathbf{g} = \mathbf{Y}\mathbf{Z} - \mathbf{M}$

"= MMULT($\mathbf{g}$, MINVERSE($\mathbf{S}_2$))", Hold Ctrl and Shift, followed by Enter

Reliability Analysis Based on the Response Surface

Distribution	Variable	Para1	Para2	Para3	Para4	$\mathbf{x}^*$	Correlation matrix $\mathbf{R}$	$\mathbf{n}$
Lognormal	Y	40	5			33.811	1 0.4 0	-1.288
Lognormal	Z	50	2.5			47.772	0.4 1 0	-0.887
ExtValue1	M	1000	200			1615.1	0 0 1	2.297

$\tilde{\mathbf{g}}(\mathbf{x}) = 0.0000$ $\beta = 2.6644$

$\mathbf{g} = \mathbf{a}_2 \mathbf{x}_2$ (based on $\mathbf{x}^*$ values)

$\mathbf{x} = \mathbf{x}_i(\text{Distribution, Para1:Para4, } \mathbf{n})$

(c)

Construction of Response Surface

$\mathbf{x}_3$ vector	Sampling Matrix: $\mathbf{S}_3$			Sampling points with $h=1$				$\mathbf{x}_{Mi}$ Vector	$\mathbf{x}_{Mi+1}$ Vector
1	1	1	1	1	1	1	1	Y	33.4159
Y	34.0659	39.0659	29.0659	34.0659	34.0659	34.0659	34.0659	Z	47.5159
Z	47.9439	47.9439	47.9439	50.4439	45.4439	47.9439	47.9439	M	1587.64
M	1633.14	1633.14	1633.14	1633.14	1633.14	1833.14	1433.14	$g(\mathbf{x}_{Mi})$	0.11069
Y^2	1160.49	1526.15	844.828	1160.49	1160.49	1160.49	1160.49	$g(\mathbf{x}_{Di})$	0.0595
Z^2	2298.61	2298.61	2298.61	2544.58	2065.14	2298.61	2298.61		
M^2	2667150	2667150	2667150	2667150	2667150	3360407	2053894		
$\mathbf{g}$ vector	0.11069 239.83 -239.61 85.2755 -85.054 -199.89 200.111							= YZ- M	
	a	b_1	b_2	b_3	c_1	c_2	c_3		
$\mathbf{a}_3$ vector	-1633.3 47.9439 34.0659 -1 4.4E-15 1.9E-14 -4E-18								
	"= MMULT($\mathbf{g}$,MINVERSE($\mathbf{S}_3$))", Hold Ctrl and Shift, followed by Enter								

Reliability Analysis Based on the Response Surface

Distribution	Variable	Para1	Para2	Para3	Para4	$\mathbf{x}^*$	Correlation matrix $\mathbf{R}$	$\mathbf{n}$
Lognormal	Y	40	5			33.765	1 0.4 0	-1.299
Lognormal	Z	50	2.5			47.746	0.4 1 0	-0.898
ExtValue1	M	1000	200			1612.1	0 0 1	2.290
$\tilde{\mathbf{g}}(\mathbf{x})$	0.0000	$\leftarrow \mathbf{a}_3 \mathbf{x}_3$ (based on $\mathbf{x}^*$ values)				$\mathbf{x} = \mathbf{x_i}(\text{Distribution, Para1:Para4, } \mathbf{n})$		
β	2.6645							

Figure 3.2 Reliability analysis of an example problem by response surface method with iteration for improvement: (a) first iteration; (b) second iteration; and (c) third iteration

3.4 FORM with probabilistic sensitivity analysis (Method 2)

3.4.1 Recursive algorithm for FORM in n-space

It is noted that the RSM-based FORM uses an approximated polynomial as the actual performance function. Thus the efficiency of this method unavoidably depends on the accuracy of the approximated polynomial. For some problems having highly nonlinear performance functions, it may be very hard to approximate them using polynomials.

Thus it is desirable to carry out reliability analysis without having to approximate the actual limit state surface *a priori*. In this circumstance, an alternative Newton-Raphson type recursive algorithm (Rachwitz and Fiessler 1978) can be used to find the design point.

Similar to conventional FORM, the algorithm first linearizes the performance function at each iteration point. Instead of searching for design point by subjecting to an explicit limit state surface, this recursive algorithm uses the derivatives to find the next iteration point, such that (Haldar and Mahadevan 2000a)

$$\mathbf{u}_{k+1} = \frac{1}{|\nabla g(\mathbf{u}_k)|^2} [\nabla g(\mathbf{u}_k)^T \mathbf{u}_k - g(\mathbf{u}_k)] \nabla g(\mathbf{u}_k) \quad (3.6)$$

where $\mathbf{u}_k$ is the k th iteration point in the space of *uncorrelated standard normal variables* (or u-space), $g(\mathbf{u}_k)$ and $\nabla g(\mathbf{u}_k)$ are the performance function and gradient vector of the performance function evaluated at $\mathbf{u}_k$, respectively.

Note that the original recursive algorithm Eq. (3.6) is formulated in the u-space. For some problems involving correlated and non-normal random variables, transformation of random variables from *the original space* (or x-space) or the space of *reduced equivalent normal variables* (or n-space, where $n_i = (x_i - \mu_i^N)/\sigma_i^N$, as detailed by Eq. (2.42)) to u-space is required to use the recursive algorithm. The transformation of variables to u-space may hinder the recursive algorithm for FORM from practical use. To overcome this limitation, the recursive algorithm is reformulated in n-space in this study. It is briefly introduced as follows.

Consider the basic random variables in x-space are subjected to a correlation matrix $\mathbf{R}$. Cholesky decomposition of $\mathbf{R}$ will yield a lower triangular matrix $\mathbf{L}$ and an upper triangular matrix $\mathbf{U} = \mathbf{L}^T$. It is not difficult to see that $\mathbf{u} = \mathbf{L}^{-1}\mathbf{n}$ or $\mathbf{n} = \mathbf{L}\mathbf{u}$. Using the chain rule, one obtains

$$\nabla g(\mathbf{u}_k) = \mathbf{L}^T \nabla g(\mathbf{n}_k) \quad (3.7)$$

Multiplying $\mathbf{L}$ to both sides of Eq. (3.6), and using Eq. (3.7), it is shown that

$$\mathbf{n}_{k+1} = \frac{1}{\nabla g(\mathbf{n}_k)^T \mathbf{R} \nabla g(\mathbf{n}_k)} [\nabla g(\mathbf{n}_k)^T \mathbf{n}_k - g(\mathbf{n}_k)] \mathbf{R} \nabla g(\mathbf{n}_k) \quad (3.8)$$

where $\mathbf{n}_k$ is the k th iteration point in $\mathbf{n}$ -space, $g(\mathbf{n}_k)$ and $\nabla g(\mathbf{n}_k)$ are the performance function and gradient vector of the performance function evaluated at $\mathbf{n}_k$, respectively, and $\mathbf{R}$ is the correlation matrix.

Eq. (3.8) uses the gradient vector of performance function in $\mathbf{n}$ -space which in turn is directly related to its counterpart in $\mathbf{x}$ -space, to find the next iteration point. When the performance function is implicit, the gradient vector has to be evaluated by sensitivity analysis. The sensitivity analysis is usually done by finite difference method using a stand-alone numerical package. Since it is easier to carry out the sensitivity analysis in $\mathbf{x}$ -space than in $\mathbf{u}$ -space, Eq. (3.8) largely facilitates the use of recursive algorithm for FORM in combination with a stand-alone numerical package. As a result, Eq. (3.8) is recommended in this study. A useful procedure of the sensitivity analysis in $\mathbf{n}$ -space and $\mathbf{x}$ -space will be introduced in the next subsection.

In general, the gradient vector is not constant when nonlinear performance function is involved. As a result, iterative evaluation of the Eq. (3.8) is needed to obtain the final iteration point (i.e. the design point $\mathbf{n}^*$). The algorithm is repeated until convergence, satisfying the two criteria: $|\mathbf{n}_{k+1} - \mathbf{n}_k| \leq \epsilon_1$ and $|g(\mathbf{n}_{k+1})| \leq \epsilon_2$; both ϵ_1 and ϵ_2 are small quantities, such as 0.05 as recommended by Haldar and Mahadevan (2000a). For slope reliability performance function, ϵ_2 as used in this study is set to be 0.02.

After obtaining $\mathbf{n}^*$, the reliability index β is computed by (Low and Tang 2007)

$$\beta = \sqrt{\mathbf{n}^{*T} \mathbf{R}^{-1} \mathbf{n}^*} \quad (3.9)$$

3.4.2 Reliability and sensitivity analysis in n-space

A challenging task is to solve the gradient vector of performance function when using the recursive algorithm for FORM. When the performance function is implicit, sensitivity analysis of the performance function against the basic random variables $\mathbf{x}$ is usually used. However, this work can be cumbersome when the basic random variables are correlated non-normals since the sensitivity analysis is usually performed in $\mathbf{x}$ -space while its transformation to $\mathbf{u}$ -space is required (Haldar and Mahadevan 2000a). In contrast, a useful procedure of sensitivity analysis in $\mathbf{n}$ -space to directly implement the recursive algorithm for FORM by Eq. (3.8) is suggested in this study. The detailed discussion follows.

For simplicity, a component of the gradient vector $\nabla g(\mathbf{n}_k)$ is considered, i.e., $\partial g(\mathbf{n}_k)/\partial n_i$, where n_i is the i th component of the reduced vector $\mathbf{n}_k$. Since the analytical differentiation of $g(\mathbf{n}_k)$ is not available, the numerical approach such as finite difference method is used, such that

$$\frac{\partial g(\mathbf{n}_k)}{\partial n_i} = \lim_{\Delta n_i \rightarrow 0} \frac{\Delta g(\mathbf{n}_k)}{\Delta n_i} \quad (3.10)$$

Note that the sensitivity analysis in $\mathbf{n}$ -space needs to be translated into $\mathbf{x}$ -space of original random variables, since the performance function is usually evaluated using the information of original random variables. For random variables in commonly used distributions, the one-to-one translation can be done using a user-defined function x_i as shown in Figure 3.3. The procedure for implementing the recursive algorithm supplemented with sensitivity analysis in $\mathbf{n}$ -space is summarized as follows.

Step 1: First select an initial iteration point $\mathbf{n}_0$ in $\mathbf{n}$ -space, and translate $\mathbf{n}_0$ into $\mathbf{x}_0$. Since $\mathbf{n}_0$ is the vector of reduced equivalent standard normal variables, initial $\mathbf{n}_0$ can

be at its mean value $\mathbf{0}$. Evaluate the performance function $g(\mathbf{n}_0) = g(\mathbf{x}_0)$.

Step 2: Change the value of n_1 (the first component of $\mathbf{n}_0$) to $n_1 + \Delta n_1$, where Δn_1 is a small number. All other components stay at the same value as in step 1. Translate the $(n_1 + \Delta n_1)$ into $(x_1 + \Delta x_1)$ and compute the new value of performance function $g(\mathbf{n}_{0,\text{new}}) = g(\mathbf{x}_{0,\text{new}})$.

Step 3: Compute the change in the value of performance function due to the small change in the value of n_1 , such that $\Delta g(\mathbf{n}_0) = g(\mathbf{n}_{0,\text{new}}) - g(\mathbf{n}_0)$. The approximate derivative of $g(\mathbf{n}_0)$ with respect to n_1 is $\Delta g(\mathbf{n}_0) / \Delta n_1$.

Step 4: Repeat steps 2 to 3 for each component of $\mathbf{n}$ vector. Obtain the gradient vector $\nabla g(\mathbf{n}_0)$.

Step 5: Compute the new $\mathbf{n}$ vector using Eq. (3.8), and compute a tentative β value using Eq. (3.9) where $\mathbf{n}^*$ is taken to be the new $\mathbf{n}$ vector.

Step 6: Repeat steps 1 to 5 until convergence of the $\mathbf{n}$ vector and β value.

Forward difference approach is used in the above-mentioned procedure. Thus $(n+1)m$ times of evaluation of the performance function are required if a problem includes n random variables and the recursive algorithm for FORM is iterated m times.

Type of F(x)	$\mathbf{x} = \mathbf{F}^{-1}[\Phi(\mathbf{n})]$
Normal: mean μ_x , std. dev. σ_x	$x = \mu_x + n\sigma_x$
LogNormal: mean μ_x , std. dev. σ_x	$x = \exp(\lambda + n\zeta)$, $\zeta = \sqrt{\ln(1 + (\sigma_x/\mu_x)^2)}$, $\lambda = \ln \mu_x - 0.5\zeta^2$
Extreme Value (Gumbel): mean μ_x , std. dev. σ_x	$x = u - \frac{\ln[-\ln(\Phi(n))]}{\alpha}$, $\alpha = \frac{1}{\sqrt{6}}\left(\frac{\pi}{\sigma_x}\right)$, $u = \mu_x - \frac{0.5772}{\alpha}$
Exponential: mean μ_x	$x = -\mu_x \ln[1 - \Phi(n)]$
Uniform:	$x = x_{min} + (x_{max} - x_{min}) \cdot \Phi(n)$
Triangular:	$x = a + \sqrt{\Phi(n) \times (m - a) \times (c - a)}$, if $\Phi(n) \leq (m - a)/(c - a)$, $x = c - \sqrt{[1 - \Phi(n)] \times (c - a) \times (c - m)}$, otherwise
Weibull CDF: $1 - \exp[-(x/\lambda)^\alpha]$	$x = \lambda \cdot [-\ln(1 - \Phi(n))]^{1/\alpha}$
Gamma with PDF: $f(x) = \frac{x^{\alpha-1} \exp(-x/\alpha)}{\lambda^\alpha \Gamma(\alpha)}$	Para 1 = α , Para 2 = λ , Mean: $\mu_x = \alpha \times \lambda$, Newton method used to obtain x from n, starting from μ_x
Beta distribution	Para 1 = α , Para 2 = λ , Para 3 = min, Para 4 = max, Mean: $\mu_x = \min + (\max - \min)\alpha/(\alpha + \lambda)$, Newton method used to obtain x from n, starting from μ_x
PERT distribution	Para 1 = min, Para 2 = mode, Para 3 = max, Mean: $\mu_x = (\min + 4 \times \text{mode} + \max)/6$, Newton method used to obtain x from n, starting from μ_x

```

Function x_i(DistributionName, paralist, ni) As Double
    paral = paralist(1): para2 = paralist(2): para3 = paralist(3): para4 = paralist(4)
    With Application.WorksheetFunction
    Select Case UCase(Trim(DistributionName))
    Case "NORMAL": x_i = paral + ni * para2
    Case "LOGNORMAL": lambda = Log(paral) - 0.5 * Log(1 + (para2 / paral) ^ 2)
        zeta = Sqr(Log(1 + (para2 / paral) ^ 2)): x_i = Exp(lambda + zeta * ni)
    Case "EXTVALUE1": alfa = 1.28255 / para2: u = paral - 0.5772 / alfa
        x_i = u - Log(-Log(.NormSDist(ni))) / alfa
    Case "EXPONENTIAL": mean = paral: x_i = -mean * Log(1 - .NormSDist(ni))
    Case "UNIFORM": Min = paral: Max = para2: x_i = Min + (Max - Min) * .NormSDist(ni)
    Case "TRIANGULAR": A = paral: m = para2: C = para3: tem = .NormSDist(ni): maca = (m - A) / (C - A)
        If tem <= maca Then x_i = A + Sqr(tem * (m - A) * (C - A)) Else x_i = C - Sqr((1 - tem) * (C - A) * (C - m))
    Case "WEIBULL": x_i = para2 * (-Log(1 - .NormSDist(ni))) ^ (1 / paral)
    Case "GAMMA": xprev = paral * para2
        For i = 1 To 100
            CDF = .GammaDist(xprev, paral, para2, True): pdf = .GammaDist(xprev, paral, para2, False)
            xnew = xprev - (CDF - .NormSDist(ni)) / pdf
            If Abs((xnew - xprev) / xprev) < 0.000001 Then Exit For
            If xnew <= 0 Then xnew = 0.5 * xprev
            xprev = xnew
        Next i: x_i = xnew
    Case "BETADIST": a1 = paral: a2 = para2: Min = para3: Max = para4: xprev = Min + (Max - Min) * a1 / (a1 + a2)
8: For i = 1 To 100
        CDF = .BetaDist(xprev, a1, a2, Min, Max)
        BetaFunc = Exp(.GammaLn(a1) + .GammaLn(a2) - .GammaLn(a1 + a2))
        pdf = 1 / BetaFunc * (xprev - Min) ^ (a1 - 1) * (Max - xprev) ^ (a2 - 1) / (Max - Min) ^ (a1 + a2 - 1)
        xnew = xprev - (CDF - .NormSDist(ni)) / pdf
        If Abs((xnew - xprev) / xprev) < 0.000001 Then Exit For
        If xnew <= Min Then xnew = 0.5 * (Min + xprev): If xnew >= Max Then xnew = 0.5 * (Max + xprev)
        xprev = xnew
    Next i: x_i = xnew
    Case "PERTDIST": Min = paral: Mode = para2: Max = para3: mean = (Min + 4 * Mode + Max) / 6: xprev = mean
        If Mode = mean Then f = 6 Else f = (2 * Mode - Min - Max) / (Mode - mean)
        a1 = (mean - Min) * f / (Max - Min): a2 = a1 * (Max - mean) / (mean - Min)
        GoTo 8
    End Select: End With
End Function

```

Figure 3.3 Function x_i for obtaining x_i from n_i , coded inside Microsoft EXCEL (after Low and Tang, 2007)

3.4.3 Illustration of the probabilistic sensitivity-based FORM using a simple example

In this subsection, the probabilistic sensitivity-based FORM is also illustrated using the simple example problem as introduced in subsection 3.3.3. The details are shown in Figure 3.4. Initially, the tentative design point $\mathbf{n}_0$ is chosen to be $[0, 0, 0]$. To compute the gradient vector $\nabla g(\mathbf{n}_0)$ or $\Delta g(\mathbf{n}_0)/\Delta n_j$ ($j = 1$ to 3), the forward finite difference sensitivity analysis was performed on each component of $\mathbf{n}_0$. Note that performance function is always evaluated in x -space other than directly in n -space; the one-to-one transformation from n -space to x -space is carried out using the user-defined function x_i in a spreadsheet. Using the proposed 6-step procedure, the first tentative β value was found to be 2.7697. After three iterations, the β value was found converged to 2.6631, which is very close to the accurate value of 2.6646.

Transformation of variables from n -space to x -space										
ProbDist	Variable	Para1	Para2	$\mathbf{x}^*$			Correlation matrix $\mathbf{R}$			$\mathbf{n}$
Lognormal	Y	40	5	39.7	1	0.4	0	0	0.00	
Lognormal	Z	50	2.5	49.9	0.4	1	0	0	0.00	
ExtValue1	M	1000	200	967	0	0	1	0	0.00	

Procedure for Probabilistic Sensitivity-based FORM $\Delta n = 0.5$ $g = YZ - M$										
Step 1	Initial $\mathbf{n}_i$	0	0	0	-2.23	-1.529	1.488	-1.367	-0.938	2.258
	$g(\mathbf{n}_i)$	1015			68.16			-3.785		
Step 2	$g(\mathbf{n}_{i,new})$	1142	1065	916.6	157.5	103.4	-102.7	98.69	36.57	-225.2
Step 3	$\Delta g(\mathbf{n}_i)$	127.3	50.14	-98.33	89.35	35.19	-170.8	102.5	40.36	-221.4
Step 4	$\Delta g(\mathbf{n}_i)/\Delta n_j$	254.6	100.3	-196.7	178.7	70.38	-341.6	205	80.72	-442.8
Step 5	$\mathbf{n}_{i+1}$	-2.23	-1.529	1.488	-1.367	-0.938	2.258	-1.243	-0.852	2.32
	β	2.7697			2.6738			2.6605		
										2.6631

Note: framed cells contain Eqs. (3.8) to (3.10)

Figure 3.4 Reliability analysis of an example problem by probabilistic sensitivity-based FORM

3.4.4 Application to a slope reliability analysis

Low and Tang (2007) showed that a constrained optimization approach for FORM in the n-space can be very efficient when optimization tools such as EXCEL's built-in SOLVER or MATLAB's optimization tool box are available. It is noted that the approach is more used for cases where performance function is explicit. For slope reliability with implicit performance function (e.g. when using Spencer's method of slices), the approach can be used only when the reliability analysis and deterministic stability analysis are integrated in the same platform. The optimization approach for FORM using EXCEL's SOLVER is used to validate the probabilistic sensitivity-based FORM.

To investigate the applicability of the probabilistic sensitivity-based FORM in slope reliability analysis, a homogeneous $1H:1V$ $c - \phi$ slope in Ji et al. (2012) is re-visited. The Spencer's method implemented in a spreadsheet is used to compute the factor of safety F_s . Using the constrained optimization approach by SOLVER, the reliability index β was found to be 1.428. For comparison, Figure 3.5 shows the reliability analysis by the probabilistic sensitivity-based FORM in the n-space. After 3 iterations, the β value has converged to 1.428, which is practically the same as that obtained by the constrained optimization approach by SOLVER. The advantage of the sensitivity-based FORM is that the integration of the reliability methods and deterministic stability model is not required, so that more complicated numerical methods for deterministic stability evaluation (e.g., FEM or FDM) can be used and be extended into probabilistic analysis without much extra effort.

Initial $\mathbf{n}_0$	0.0000	0.0000	Initial $\mathbf{n}_1$	-1.3693	0.1334	Initial $\mathbf{n}_2$	-1.3332	0.2261
$F_s(\mathbf{n}_0)$	1.2256		$F_s(\mathbf{n}_1)$	0.9870		$F_s(\mathbf{n}_2)$	1.0001	
$F_s(\mathbf{n}_{0,new})$	1.3116	1.2620	$F_s(\mathbf{n}_{1,new})$	1.0807	1.0208	$F_s(\mathbf{n}_{2,new})$	1.0946	1.0342
$\Delta g(\mathbf{n}_0)$	0.0859	0.0364	$\Delta g(\mathbf{n}_1)$	0.0938	0.0339	$\Delta g(\mathbf{n}_2)$	0.0945	0.0340
$\Delta g/\Delta n_i$	0.1719	0.0727	$\Delta g/\Delta n_i$	0.1876	0.0677	$\Delta g/\Delta n_i$	0.1890	0.0681
$\mathbf{n}_1$	-1.3693	0.1334	$\mathbf{n}_2$	-1.3332	0.2261	$\mathbf{n}_3$	-1.3343	0.2275
β	1.5100		β	1.4269		β	1.4276	

Note: the factor of safety is computed by Spencer's method in a spreadsheet; the first component of $\mathbf{n}_k$ vector is associated with c and the second component is associated with ϕ ; c and ϕ are correlated with a correlation coefficient equal to -0.5; framed cells contain Eqs. (3.8) to (3.10).

Figure 3.5 Reliability analysis of a homogeneous $c - \phi$ slope by the recursive algorithm for FORM

3.4.5 Effect of perturbation values in n-space

It is noted that a constant perturbation value of Δn_i of 0.5 was used in the above sensitivity analysis for obtaining the gradient vector $\nabla g(\mathbf{n}_k)$. Other perturbation values in the n-space can also be used for the sensitivity analysis. Since the reduced variable n_i has mean value of '0' and standard deviation of '1', a parametric study of the influence of Δn_i (varying from 0.1 to 0.9) on β was carried out. The results are briefly shown in Figure 3.6. For the first iteration, the β value is much sensitive to the perturbation value used. When more iterations were carried out, the β value converges very fast. Thus, it may be concluded that the β value is not really sensitive to the value of Δn_i when an iterative procedure is used to improve the results.

Figure 3.6 Influence of Δn_i of sensitivity analysis on the reliability index β by the recursive algorithm for FORM

3.5 Comparison between RSM-based and the probabilistic sensitivity-based reliability analyses

From the above illustration, it has been shown that both RSM-based FORM and the probabilistic sensitivity-based FORM are able to deal with problems in the absence of explicit performance functions. The concept behind the use of RSM for reliability analysis is simple. The main efforts lie in the estimation of the actual performance function. However, considering computational efficiency and accuracy, the implementation of RSM-based reliability analysis could be impractical when the performance function is highly nonlinear and the number of basic random variables is very large. For example, for a problem involving n basic random variables ($n+1$) experimental evaluation of the performance function are required when using a linear RSM; when using a simple second-order polynomial RSM, $2n+1$ experimental evaluations are required when cross terms are not considered, and $(n+1)(n+2)/2$ experimental evaluations are required when including the cross terms. Note that the number of experimental evaluations increases exponentially with increasing order of polynomial to be used.

In contrast, the probabilistic sensitivity-based FORM does not require approximating the actual performance function, thus it is more suitable to use when no information about the actual performance function (whether linear or nonlinear) is available. Moreover, the number of experimental evaluations of the performance function is reduced significantly especially compared with nonlinear RSM. For example, for a problem involving n basic random variables, only $n+1$ (the same number as used in linear RSM) experimental evaluations of the actual performance function are required for each iteration when forward or backward difference scheme is used for sensitivity analysis, and $2n+1$ experimental evaluations are required for each iteration when central difference scheme is used. The central difference scheme is less commonly used than the forward and backward difference schemes.

3.6 Probabilistic slope stability analysis by strength-reduction FEM – case studies

In this section, two real cases of slope stability problem will be investigated. Both cases are well-documented in the literature. A strength-reduction finite element code, Plaxis, will be used to compute the factor of safety for each deterministic analysis. Using the two techniques for reliability analysis involving implicit performance functions, the finite element analyses of the two case studies are extended into probabilistic assessment.

3.6.1 Clarence Cannon Dam

Problem Background

Clarence Cannon Dam is located in the Salt River in northeastern Missouri and forms Mark Twain Lake. The dam is part of a multi-purpose project which provides flood control, recreation, water supply, fish and wildlife conservation, and hydropower.

Completed in 1983, the dam has a 305 m long earth embankment, a gated concrete spillway section and a concrete powerhouse adjacent to the spillway. The earth embankment volume is approximately 2.74 million m³. The dam crest at elevation 200 m above mean sea level is about 35 m above the floodplain and 42 m above the stream bed.

The embankment section analyzed is located at survey station 12 + 75 and is representative of the dam near its maximum height. The embankment geometry at this location is shown in Figure 3.7. The section includes a compacted clay foundation cutoff trench through an abandoned glacial channel. The cutoff trench and embankment base are constructed of Phase I fill materials and the remainder of the embankment is constructed of Phase II fill materials.

The stability of Cannon Dam has been previously studied by many others. For example, Wolff (1985) investigated four different cases of stability of the dam, such as when the dam is at the end of construction, with partial pool of reservoir, at steady seepage, and with sudden drawdown. Corps' force-equilibrium method was used to compute the factor of safety along a limited number of specified non-circular critical slip surfaces. Hassan and Wolff (1999) analyzed the downstream stability of the dam without considering the pore water pressure, and they employed simplified Bishop method of slices and Spencer method of slices with both circular and non-circular slip surfaces. It is noted that all previous studies found in the literature were limited to limit equilibrium methods of slices. For comparison, the Cannon Dam is re-studied in this study by strength-reduction finite element analysis using the numerical code Plaxis.

Figure 3.7 Cannon Dam at section of station 12+75

Soil properties of the Cannon Dam as reported by Hassan and Wolff (1999) and Slide's manual (<http://www.rocscience.com/products/8/Slide>) are used for the present study. It is noted that only the shear strength parameters of clay fills were reported in past studies. In this study, some missing values of soil properties which were not provided in the past studies are assumed in accordance to their engineering behavior. In addition, elastic modulus E and Poisson's ratio ν are also required in the strength-reduction FE stability analysis (based on elastic-perfectly plastic constitutive model and Mohr-Coulomb failure criterion). In fact, it is shown in the literature that neither E nor ν influences the stability factor (e.g., Griffiths and Lane 1999). Thus the following study assumes nominal values of E and ν which are typical to the relevant geo-materials. The input parameters are detailed in Table 3.1.

Table 3.1 Soil properties of Cannon Dam by strength-reduction FE stability analysis

Material	Young's modulus E (MPa)	Poisson ratio ν	Cohesion c (kPa)	Friction angle ϕ (°)	Unit weight γ (kN/m ³)
Phase II clay fill	15	0.3	143.64	15	19
Phase I clay fill	10	0.3	117.79	8.5	19
Sand filter	30	0.3	0	30	16
Foundation sand	30	0.3	5	18	17.5
Fill	50	0.3	5	35	20
Limestone	20000	0.25	100	35	25

Figure 3.8 shows the finite element model for Cannon Dam. 15-node triangular element is used to mesh the FE model. For the current study, a total of 1828 elements which include 14921 nodes are used. A relatively coarse mesh is applied to the limestone foundation (bottom layer) due to the fact that failure will not take place in this layer. Since the downstream (right side) stability of the dam is of particular interest, as investigated by others (Hassan and Wolff 1999), the mesh has been refined in this side. By trial-and-error, it is found that such a mesh scheme is adequate for the stability analysis.

Figure 3.8 Finite element model of the Cannon Dam

Figure 3.9 shows the displacements and shear strain increments at the state of failure, which is computed using the strength reduction technique of Plaxis. An almost circular failure band is observed, and the failure band tends to cross the Phase I fill, Phase II fill and the foundation sand layers. It is noted that the shear strain increments is larger around the slope toe area, and becomes smaller towards the top surface along the failure band. This implies that the slope failure would first take place around the toe area, and progressively extends to the top surface of the dam.

The critical factor of safety F_s by strength-reduction FE analysis was found to be 2.384. For the cross-validation purpose, the stability of Cannon Dam was also examined using limit equilibrium methods of slices. The results are compared with others in Table 3.2. In general, the factors of safety F_s computed by various methods are in close agreement with each other. For the limit equilibrium methods of slices, when circular slip surface is assumed, generalized limit equilibrium method of slices (GLE) obtains a F_s of 2.471, which is slightly higher than that by simplified Bishop method, 2.462. Since GLE is the most rigorous limit equilibrium method of slices, a non-circular slip surface analysis was also performed using GLE, which yields a lower value of F_s of 2.374 than does GLE with circular slip surface. In comparison, the F_s by strength-reduction FE stability analysis, 2.384, lies between the two values computed by GLE with circular and noncircular slip surfaces. Thus, it may be concluded that the strength-reduction FE stability analysis of Cannon Dam yields a reasonable F_s value compared with various limit equilibrium methods of slices. One advantage of strength-reduction FE stability analysis is that the failure mechanism is found automatically, and does not need

assumptions of the location and shape of slip surfaces.

(a) Failure mass indicated by vector plot of displacements (extreme value: 1.3 m)

(b) An almost circular failure band indicated by contour plot of shear strain increments

Figure 3.9 Strength-reduction FE stability analysis of the Cannon Dam

Table 3.2 Comparison between stability analyses of the Cannon Dam

Method of analyses	Sliding mode	Critical factor of safety F_s
Corps' force-equilibrium method (Wolff 1985)	Non-circular slip surface with 3-segment polyline	2.36
Simplified Bishop method (Hassan and Wolff 1999)	Circular slip surface	2.352
Simplified Bishop method (Hassan and Wolff 1999)	Non-circular slip surface with 3-segment polyline	1.980
Simplified Bishop method in this study	Circular slip surface	2.358
GLE in this study	Circular slip surface	2.471
GLE in this study	Non-circular slip surface	2.374
Strength-reduction FE analysis in this study	Almost circular failure band	2.384

As reported by Wolff (1985) and Hassan and Wolff (1999), strength parameters of the Phase I clay fill and Phase II clay fill were considered as random variables. Based on UU test of record samples, statistical moments for the strength parameters of the two fill layers are summarized in Table 3.3. It is seen that the strength parameters are of very high variability; the coefficient of variation (c.o.v.) for ϕ_1 is as high as 1.0. In addition, cross correlation between cohesion c and friction angle ϕ of each layer is also reported: a positive cross correlation of the strength parameters in Phase I clay fill; but a negative cross correlation in Phase II clay fill. Based on the basic statistical information, a probabilistic investigation of the stability of Cannon Dam can be conducted. For example, Wolff (1985) and Wolff and Harr (1987) used point-estimate method to estimate the probability of failure of the Cannon Dam along a specified non-circular critical slip surface (the deterministic critical one). Hassan and Wolff (1999) carried out a reliability analysis of the Cannon Dam using MVFOSM with search for the critical- β surface; both circular and non-circular slip surface were considered. Bhattacharya et al. (2003) applied a new numerical procedure for locating the critical- β surface in their reliability study of the Cannon Dam.

Table 3.3 Statistical information of strength parameters of the Cannon Dam (after Hassan and Wolff 1999)

Layer description	Strength parameter	Mean value	Standard deviation	Coefficient of variation	Correlation coefficient
Phase II clay fill	c_2 (kPa)	143.64	79	0.55	- 0.55
	ϕ_2 (°)	15	9	0.6	
Phase I clay fill	c_1 (kPa)	117.79	58.89	0.5	0.1
	ϕ_1 (°)	8.5	8.5	1.0	

For the present strength-reduction FE stability analysis of the Cannon Dam, it is also possible to extend into probabilistic analysis using either of the two reliability methods for implicit performance function as introduced previously. Since the probabilistic sensitivity-based FORM requires less runs of FE stability evaluation compared with RSM-based FORM, the former is used for the next probabilistic study. Note that past

studies using point-estimate method and MVFOSM did not mention the statistical distribution of strength parameters but only assumed a lognormal distribution for the computed factor of safety F_s . In contrast, the FORM analysis of this study requires this additional statistical information. Considering the fact that high variability of the strength parameters were reported, Lognormal distribution is recommended in the next probabilistic study, since it helps avoid negative values of strength parameters which is meaningless in practice.

Details of the probabilistic FE evaluation are presented in Figure 3.10, where forward finite difference scheme was used to approximate the gradient vector of performance function. It is shown that the reliability index β converges very fast, although there is still a very small fluctuation of β value after the forth iteration. This is due to the numerical instability of FE analysis when the F_s is close to unity as observed by the author. For convenience, the β value of 2.921 at the fifth iteration is taken to be the final (critical) reliability index. From the final design point, it is shown that the normalized indices n_i for the strength parameters $\{c_2, \phi_2\}$ and $\{c_1, \phi_1\}$ are $\{-0.86, -0.10\}$ and $\{-2.42, -1.46\}$, respectively. Obviously, the n_i 's for $\{c_1, \phi_1\}$ have greater influence on β than those for $\{c_2, \phi_2\}$ do. The implication is that the strength parameters of Phase I clay fill will contribute much more to the stability of the Cannon Dam than does the strength parameters of Phase II clay fill.

Based on the final design point of the fifth iteration, the probabilistic critical failure mechanism of the Cannon Dam can be determined, as shown in Figure 3.11. It is very interesting to compare the probabilistic critical failure mechanism with the deterministic critical failure mechanism of Figure 3.9. Both are directly determined by strength-reduction FE stability analysis without making assumption with respect to the shape and location of slip surface. However, the two failure mechanisms differ very much: an almost circular (critical) failure band was observed from deterministic stability analysis; but a very non-circular (critical) failure band was observed from probabilistic analysis.

Transformation of variables from n -space to x -space

ProbDist	Variable	Para1	Para2	Para3	Para4	$\mathbf{x}^*$	Correlation matrix $\mathbf{R}$	$\mathbf{n}$
Lognormal	c_2	143.6	79.0			80.90	1 -0.55 0 0	-0.86
Lognormal	ϕ_2	15.0	9.0			12.18	-0.55 1 0 0	-0.1
Lognormal	c_1	117.8	58.9			33.67	0 0 1 0.1	-2.42
Lognormal	ϕ_1	8.5	8.5			1.79	0 0 0.1 1	-1.46

Δn_j
0.05
0.2
0.05
0.2

Procedure for Probabilistic Sensitivity-based FORM

$\Delta n_j = \text{varying}, \quad g = F_s - 1$, where F_s is computed by FEM

Step	Initial $\mathbf{n}_i$	$g(\mathbf{n}_i)$	$\Delta g(\mathbf{n}_i)$	$\Delta g(\mathbf{n}_i)/\Delta n_j$	$\mathbf{n}_{i+1}$	β
Step 1	0 0 0 0	1.203	0.023 0.031 0.018 0.042	0.46 0.155 0.36 0.21	-1.303 0.341 -1.325 -0.855	2.045
Step 2	$g(\mathbf{n}_{i,\text{new}})$	1.226 1.234 1.221 1.245	0.005 0.022 0.017 0.044	0.1 0.11 0.34 0.22	-0.233 0.324 -2.133 -1.497	2.561
Step 3	$\Delta g(\mathbf{n}_i)$	0.023 0.031 0.018 0.042	0.008 0.011 0.011 0.028	0.16 0.055 0.22 0.14	-1.184 0.301 -2.136 -1.479	2.785
Step 4	$\Delta g(\mathbf{n}_i)/\Delta n_j$	0.46 0.155 0.36 0.21	0.007 0.017 0.012 0.028	0.14 0.085 0.24 0.14	-0.875 -0.075 -2.385 -1.54	2.934
Step 5	$\mathbf{n}_{i+1}$	-1.303 0.341 -1.325 -0.855	0.004 0.01 0.007 0.015	0.08 0.05 0.14 0.075	-0.86 -0.098 -2.415 -1.457	2.921

Note: A total of 25 FE analysis runs are performed.

Figure 3.10 Probabilistic strength-reduction FE analysis of Cannon Dam

Figure 3.11 Probabilistic critical failure mechanism of the Cannon Dam by strength-reduction FE stability analysis

The results of this probabilistic strength-reduction FE stability analysis of Cannon Dam are compared with others using limit equilibrium methods (LEM), as shown in Table 3.4. Both Hassan and Wolf (1999) and Bhattacharya et al. (2003) used MVFOSM in combination with LEMs of slices. Thus reliability indices β reported by the two studies are in close agreement each other. But it is somewhat strange that the probabilistic critical slip surfaces from the two similar studies differ significantly. In contrast, a relatively higher value of β is found in this study. This is reasonable, since the mean value F_s by FEM is higher than by the other two LEM analyses, while the potential uncertainty of soil properties are the same. Another interesting finding is that even though the β value by FE differs from that by LEM, the probabilistic critical failure band from FE is very similar to the probabilistic critical failure surface from Bhattacharya et al. (2003)'s LEM.

Table 3.4 Comparison between reliability analyses of the Cannon Dam

Method of analyses	Probabilistic critical failure mechanism	Reliability index β
MVFOSM and simplified Bishop method (Hassan and Wolff 1999)	Non-circular surface	2.664
MVFOSM and Spencer method of slices (Bhattacharya et al. 2003)	Non-circular surface	2.674
Probabilistic strength-reduction FE analysis in this study (FORM)	Non-circular band	2.921

3.6.2 James Bay Dike

Problem Background

Construction of approximately 50 km of dikes on soft, sensitive clays was proposed for the James Bay hydroelectric project in northern Quebec, Canada. Figure 3.12 shows the geometry of the embankment and stratigraphy of the foundation soils of a typical James Bay dike (Christian et al. 1994). This embankment was considered to be constructed in a single stage to a height of 12 m with a 56 m wide berm. The foundation of this dike consists of a crust between 1 and 6 m thick, with an average thickness of 4 m. Below the crust is a sensitive marine clay of 8 m thick. A 6.5 m thick lacustrine clay is found below the marine clay. The lacustrine clay is underlain by a very stiff layer of till. Undrained shear strength values of the soils were obtained from field vane tests, and uncertainties of the geotechnical parameters were reported by Soulie et al. (1990) and Christian et al. (1994), among others.

Figure 3.12 Cross-section of a typical James Bay dike

Strength-reduction FE stability analysis based on mean values

The stability of the James Bay dike has been extensively investigated by many researchers (Christian et al. 1994, El-Ramly et al. 2002, Xu and Low 2006, among others). This study re-examines the stability problem by strength-reduction finite element analysis, in order to provide basis for the probabilistic FE analysis based on

the proposed approach. Table 3.5 lists the soil properties used in strength-reduction finite element stability analysis. While the strength parameters are adapted from data as reported by Christian et al. (1994), nominal values for elastic modulus E and Poisson's ratio ν are assumed in accordance with their engineering behavior.

Table 3.5 Soil properties of James Bay dike used for strength-reduction FE stability analysis

Layer description	Young's modulus E (MPa)	Poisson ratio ν	Cohesion c (kPa)	Friction angle ϕ (°)	Unit weight γ (kN/m ³)
Embankment fills	30	0.3	0	30	20
Crust clay	15	0.495	47	0	19
Marine clay	10	0.495	34.5	0	19
Lacustrine clay	8	0.495	31.2	0	20.5

Figure 3.13 shows the finite element model of the James Bay dike. 15-node triangular element is used to mesh the FE model. For the current study, a total of 3042 elements consisting of 24785 nodes are used. Also by trial-and-error, it is found that such a mesh scheme is satisfactory for the stability analysis of the dike.

Figure 3.14 shows the displacements and shear strain increments at the state of failure. A noncircular failure band is observed, and it tends to touch the firm base. The critical factor of safety F_s by strength-reduction FE analysis was found to be 1.238. For cross-validation purpose, the stability of the James Bay dike was also examined using limit equilibrium methods of slices. The results of this study are compared with others in Table 3.6. In general, the factors of safety F_s computed by various methods are in close agreement with each other. For the limit equilibrium methods of slices, when circular slip surface is assumed, generalized limit equilibrium method of slices (using 50 slices) obtains a F_s of 1.473, which is slightly higher than that by the simplified Bishop method analyses performed by others. Since GLE is the most rigorous limit equilibrium method of slices, a non-circular slip surface analysis is also performed using GLE, which yields a much lower value of F_s of 1.150. In comparison, the F_s by strength-reduction FE stability analysis, 1.238, lies between the two values computed

by GLE with circular and noncircular slip surfaces, and is closer to the later. In particular, the failure mechanism developing naturally from the FE analysis exhibits a non-circular band, which is very similar to the non-circular critical slip surface by LEM analysis.

Figure 3.13 Finite element model of the James Bay dike

(a) Failure mass indicated by vector plot of displacements (extreme value: 0.51 m)

(b) Non-circular failure band indicated by contour plot of shear strain increments

Figure 3.14 Failure mechanism of the James Bay dike from strength-reduction FE stability analysis

Table 3.6 Comparison between stability analyses of the James Bay dike

Method of analyses	Sliding mode	Critical factor of safety F_s
Simplified Bishop method by Christian et al. (1994)	Circular slip surface	1.453
Simplified Bishop method by El-Ramly et al. (2002)	Circular slip surface	1.46
Strength-reduction FE analysis by Xu and Low (2006)	Non-circular failure band	1.247
GLE in this study	Circular slip surface	1.473
	Non-circular slip surface	1.150
Strength-reduction FE analysis in this study	Non-circular failure band	1.238

Probabilistic strength-reduction FE stability analysis

Christian et al. (1994) and El-Ramly et al. (2002) carried out probabilistic analyses of this example using the MVFOSM method and Monte Carlo simulation, respectively. Both used limit equilibrium methods of slices in their calculation of stability factor F_s . In contrast, Xu and Low (2006) conducted probabilistic finite element analysis for the James Bay Diike, using the response surface method. An alternative probabilistic finite element analysis for the James Bay dike can be made by use of the proposed sensitivity-based FORM approach. This is presented as follows.

According to Christian et al. (1994), the uncertainty involved in the James Bay dike is the result of many factors. Two sources of uncertainty were considered most important in the stability evaluation: (i) uncertainty in soil profile and fill properties, this includes uncertainty about the thickness of the crust and the depth to the till as well as the unit weight and friction angle of the fill; (ii) uncertainty in shear strength of the foundation clay, including the c_u of marine and lacustrine clays based on the field vane test. It is noted that the first type uncertainty inevitably induces changes of the thickness of the crust as well as changes of the depth to the till. Such geometry changes require FEM re-mesh during the FEM analysis. To eliminate the possible effect of such FEM re-mesh on the computed F_s , the FEM model uses a very fine mesh with 24785 nodes and each

re-meshing uses the same number of nodes. Since the change of slope geometry is very small, the nodal density is almost the same during re-mesh. As a result, the FEM re-mesh has little influence on the F_s so computed. Table 3.7 summarizes the six random variables to be considered in the probabilistic stability analysis. All the variables are assumed to be normally distributed. The standard deviations as shown here are the end result of a detailed statistical analysis (El-Ramly 2001), which includes the removal of random measurement error, estimation of statistical error, accounting for bias due to Bjerrum's vane correction factor, and variance reduction due to spatial averaging.

Table 3.7 Statistical information of random variables of the James Bay dike (after Christian et al. 1994, El-Ramly 2001, Xu and Low 2006)

Random variables	Mean value	Standard deviation	Coefficient of variation
Unit weight of embankment fill, γ_{fill} (kN/m ³)	20.0	1.10	0.055
Friction angle of embankment fill, ϕ_{fill} (°)	30.0	1.79	0.060
Thickness of clay crust, t_{cr} (m)	4.0	0.48	0.120
Strength of marine clay, c_{uM} (kPa)	34.5	3.95	0.114
Strength of lacustrine, c_{uL} (kPa)	31.2	6.31	0.202
Depth of till layer, D_{till} (m)	18.5	1.00	0.054

Details of the probabilistic FE stability analysis are presented in Figure 3.15, where forward finite difference scheme was used to approximate the gradient vector of performance function. It is shown that the reliability index β converges very fast, and the β value of 1.361 after four iterations is taken to be the final (critical) reliability index. Based on the final design point of the fourth iteration, the probabilistic critical failure mechanism of the James Bay dike can be determined, as shown in Figure 3.16. It is found that the probabilistic failure so obtained and the previously obtained deterministic critical failure (Figure 3.14) are of little difference – the failure bands from the two analyses are almost identical. This finding is somewhat different from the other case study of Cannon Dam.

Table 3.8 compares the results of this probabilistic study of James Bay dike with others.

When the GLE with circular slip surface is used in probabilistic sensitivity-based FORM analysis, the β value is found to be 2.951. Christian et al. (1994) used Mean Value FOSM in combination with Bishop method of circular slip surface, yielding a β value of 2.66. In general, the two reliability indices based on LEMs with circular slip surfaces are close to each other. El-Ramly et al. (2002) performed a Monte Carlo simulation fixed on the deterministic critical circular slip surface, and found that the failure probability P_f is around 0.24 %. This value of P_f is slightly higher than that from the FORM prediction in this study, 0.16 %. Note that the relatively close agreement between the above-mentioned studies is due to the fact that circular slip surfaces were assumed in the calculation of F_s and β values.

In contrast, the reliability index from probabilistic sensitivity-based strength-reduction FE analyses was found to be 1.361 in this study. This value is quite close to the value of 1.49 obtained by RSM-based probabilistic FE analysis (Xu and Low 2006). Obviously, both probabilistic FE analyses yield β values significantly smaller than those by LEMs with circular slip surfaces. For example, the percentage difference between the reliability indices by probabilistic sensitivity-based FE analysis and GLE analysis with circular slip surface is almost 73.7%. On the other hand, the reliability indices from probabilistic FE analyses were slightly higher than that by GLE with noncircular slip surface. For example, the percentage difference between the reliability indices by probabilistic sensitivity-based FE analysis and GLE analysis with non-circular slip surface is about 34.5%. This is reasonable, since the critical F_s by FE analyses were previously found significantly smaller than those by LEMs with circular slip surfaces, whereas slightly higher than that by LEM with noncircular slip surface. In general, smaller factor of safety leads to smaller reliability index, and hence higher failure probability. Note that in the two different studies of probabilistic FE analyses, the β value of 1.361 found by this study is valid for the F_s of 1.238, while the β value of 1.49 found by Xu and Low (2006) is valid for the F_s of 1.247. The advantage of this study is that knowledge of the implicit performance function associated with FE analysis is not required, and relatively fewer runs of FE stability calculations are needed in comparison with the RSM-based probabilistic FE analysis.

Transformation of variables from n -space to x -space

Variables	ProbDist	Mean	SD	$\mathbf{x}^*$	Correlation matrix $\mathbf{R}$						$\mathbf{n}$
γ_{fill}	Normal	20	1.1	20.57	1	0	0	0	0	0	0.52
Φ_{fill}	Normal	30	1.79	29.86	0	1	0	0	0	0	-0.08
t_{cr}	Normal	4	0.48	4.13	0	0	1	0	0	0	0.27
c_{uM}	Normal	34.5	3.95	34.26	0	0	0	1	0	0	-0.06
c_{uL}	Normal	31.2	6.31	23.50	0	0	0	0	1	0	-1.22
D_{till}	Normal	18.5	1	18.59	0	0	0	0	0	1	0.09

Procedure for Probabilistic Sensitivity-based FORM

$\Delta n = 0.5$

$g = F_s - 1$, where F_s is computed by FEM

Initial $\mathbf{n}_i$	0	0	0	0	0	0	0.55	-0.09	-0.05	-0.21	-1.3	0.27	0.34	0.08	-0.03	-0.14	-1.26	0.23	0.64	-0.07	-0.03	-0.21	-1.11	0.41
$g(\mathbf{n}_i)$	0.238						-0.019						0.018						0.016					
$g(\mathbf{n}_{i,\text{new}})$	0.207	0.243	0.241	0.250	0.311	0.223	-0.044	-0.025	-0.017	-0.009	0.074	-0.036	-0.028	0.023	0.020	0.033	0.097	-0.011	-0.018	0.021	-0.002	0.020	0.096	0.010
$\Delta g(\mathbf{n}_i)$	-0.031	0.005	0	0.01	0.07	-0.01	-0.03	-0.01	0	0.01	0.09	-0.02	-0.05	0	0	0.01	0.08	-0.03	-0.03	0	-0.02	0	0.08	-0.01
$\Delta g(\mathbf{n}_i)/\Delta n_j$	-0.062	0.01	0.01	0.02	0.15	-0.03	-0.05	-0.01	0	0.02	0.19	-0.03	-0.09	0.01	0	0.03	0.16	-0.06	-0.07	0.01	-0.04	0.01	0.16	-0.01
$\mathbf{n}_{i+1}$	0.55117	-0.09	-0.05	-0.21	-1.3	0.27	0.34	0.08	-0.03	-0.14	-1.26	0.23	0.64	-0.07	-0.03	-0.21	-1.11	0.41	0.52	-0.08	0.27	-0.06	-1.22	0.09
β	1.455						1.332						1.360						1.361					

Figure 3.15 Probabilistic strength-reduction FE analysis of James Bay Dike

Figure 3.16 Probabilistic critical failure mechanism of the James Bay dike from strength-reduction FE stability analysis

Table 3.8 Comparison between reliability analyses of the James Bay Dike

Method of analyses	Probabilistic critical failure mechanism	Reliability index β	Failure Probability P_f
MFOSM and simplified Bishop method (Christian et al. 1994)	Circular surface	2.66	/
Monte Carlo simulation and Bishop method of slices (El-Ramly et al. 2002)	Circular surface	/	0.24 %
Sensitivity-based FORM using GLE in this study	Circular surface	2.951	0.16 %
	Non-circular surface	0.961	16.83 %
RSM-based probabilistic FE analysis (Xu and Low 2006)	Non-circular band	1.49	6.81 %
Sensitivity-based Probabilistic FE analysis in this study	Non-circular band	1.361	8.68 %

3.7 Summary

Strength-reduction finite element analyses of slopes were conducted in this Chapter. Two techniques for reliability analysis involving implicit performance functions were introduced in this Chapter, namely response surface method and probabilistic sensitivity analysis. Formulation of the probabilistic sensitivity-based FORM in the n -space is one of the original contributions of this thesis study. While both enable the strength-reduction finite element stability analysis of slopes by stand-alone numerical codes (e.g. Plaxis) to be extended into probabilistic assessment, the newly developed

FORM with probabilistic sensitivity analysis is more efficient than the RSM-based FORM. The proposed approach was applied to two case studies, the Clarence Cannon Dam and a James Bay dike.

For the Clarence Cannon Dam, strength-reduction FE analysis showed that there is an almost circular failure band if deterministic (mean) values of the strength parameters were used. Moreover, the F_s by FE analysis was consistent with the one by LEM with circular slip surface. When probabilistic assessment was conducted, it was shown that the probabilistic critical failure band could be very non-circular. This is very different from deterministic FE analysis.

For the James Bay dike, the deterministic strength-reduction FE analysis revealed a non-circular failure band. In addition, the F_s by FE analysis was found more consistent with the one by LEM with non-circular slip surface. When probabilistic assessment was conducted, the probabilistic critical failure band was also found to be non-circular and almost the same as the deterministic critical failure band.

Chapter 4 Modeling Two-Dimensional Spatial Variation in Slope Reliability Analysis

4.1 Introduction

For geotechnical materials, an essential characteristic is the spatial variation. Past studies dealing with spatial variation in slope stability are often limited to one-dimensional (1-D) consideration, i.e., either along the critical failure slip surface or solely along vertical direction or horizontal direction (Low and Tang (1997b), Low (2003), Low et al. (2007), Cho (2007, 2010), and Wang et al. (2011), among others). Some others studied the influence of spatial variation on the slope reliability by using the random finite-element method (RFEM) (Griffiths and Fenton 2004; Griffiths et al. 2009; Hicks and Spencer 2010; and Schweiger and Peschl 2005), in which 2-D spatial variation was taken into account. It was noted that the number of spatially correlated random variables assigned to elements is commonly very large in the advanced RFEM so that only Monte Carlo simulation can be employed. The disadvantage of Monte-Carlo-based RFEM lies in the intensive computation.

In contrast, this study presents an alternative approach to consider the 2-D spatial variation in slope reliability analysis. Figure 4.1 shows the two methods to be proposed in the following sections, namely the method of autocorrelated slices and the method of interpolated autocorrelations, in the context of some existing methods of probabilistic slope analysis involving 2-D spatial variation. It will be shown in subsection 4.4.3 that the proposed FORM-based interpolated autocorrelations technique (which is developed by means of limit equilibrium method of slices in this study) can be coupled with stand-alone finite element packages, and is an attractively efficient alternative to the recently introduced MC-based probabilistic finite element analysis of slopes.

Figure 4.1 Methods of probabilistic slope analysis accounting for spatial variation

4.2 Efficient procedure for slope reliability analysis using FORM

4.2.1 Systematic search method for critical slip surface

Low (2003) conducted slope stability analysis using a reformulated Spencer's method in the Microsoft Excel spreadsheet platform. The optimization routine Solver that resides in the spreadsheet was invoked to search for the critical noncircular slip surface and the corresponding critical factor of safety F_s (here referred to as direct Solver search). The details will not be repeated here. Instead, this study uses a systematic search based on grid method to locate the critical slip surface. Excel's Solver is used only to solve the F_s corresponding to a specific slip surface. The systematic search for the critical F_s slip surface is carried out by a short and simple user-defined Visual Basic for Applications (VBA) subroutine. It is more robust than the direct Solver search approach in cases where local minima exist. Many commercial programs with systematic search for the critical slip surface are available for slope stability analysis. The advantage of the present approach is in the relative ease of transiting from a deterministic analysis to a probabilistic analysis in a ubiquitous platform. If desired, one can implement the procedures in the MATLAB platform.

4.2.2 Slope reliability based on systematic search for the probabilistic critical slip surface

The merits of probabilistic analysis of slopes instead of the conventional deterministic factor of safety approach have been widely acknowledged and accepted by many researchers. Monte Carlo simulation is a basic technique to probabilistically evaluate the slope stability. On the other hand, a more efficient alternative is the Hasofer-Lind (1974) second-moment reliability index β for correlated normals and first-order reliability method (FORM) for correlated non-normals, which reflect the uncertainty of the parameters as well as their correlation structure. From the geometric point of view, the FORM evaluates the minimum distance from the origin of the rotated and dimensionless reduced coordinate system to the limit state surface (Haldar and Mahadevan 2000a), or the shortest distance measured in units of directional standard deviations from the safe mean-value point to the most probable failure combination of parameters (“the design point”) on the limit state surface (LSS). This requires a definition of the LSS prior to conducting reliability analysis. In slope reliability, the LSS can be expressed as:

$$g(\mathbf{x}) = F_s - 1 \quad (4.1)$$

where F_s is the stability factor, which can be evaluated by LEM or FEM/FDM. The value of $g(\mathbf{x})$ is also called margin of safety, which means safety when $g(\mathbf{x}) > 0$; and failure otherwise.

Extension of the deterministic slope stability analysis into probabilistic analysis was illustrated in Low and Tang (1997b). One may note that for any predefined slip surface there exists a factor of safety F_s which is a function of soil properties. If uncertainty of soil properties is considered, i.e., the soil properties are treated as random variables, one can always find a combination of these random variables which satisfies the condition $F_s = 1$ and results in a minimum value of β . The concept of β and its computation are described in section 2.8.1 in Chapter 2. Bearing this in mind, one can easily perform slope reliability analysis with respect to a particular slip surface by

using optimization tools such as Excel Solver or MATLAB optimization toolbox. When a large number of slip surfaces are investigated, the one with the smallest reliability index β may be taken as the probabilistic critical slip surface. In this chapter, a systematic search method (SSM) is used to locate the probabilistic critical slip surface as well as the reliability index β . The SSM is devised based on the well-known grid method which is commonly used for locating the critical F_s slip surface. Such a systematic search method can best be interpreted by the following steps:

- (1) At first, a grid range is identified to define many possible centers of slip circles;
- (2) At each of these centers, a number of slip circles are defined by varying the radius;
- (3) On each of these slip circles, slope reliability is investigated with a search for the reliability index β . Among the many predefined slip circles, the one with minimum β value would be regarded as the probabilistic critical slip surface.

It is worth pointing out that the probabilistic critical slip surface may not coincide with the deterministic critical slip surface, as noted by Li and Lumb (1987), Hassan and Wolff (1999), Bhattacharya et al. (2003), and Xue and Gavin (2007).

Consider an embankment underlain by soft clay foundation as shown in Figure 4.2. The mean values of soil properties are also shown. The F_s based on the mean values was found to be 1.462 by the Spencer's method. For the probabilistic analysis, the undrained shear strength c_u of the foundation is taken to be uncertain parameter, and other parameters are assumed to have deterministic values. The uncertain parameter c_u is assumed to be normally distributed (Gaussian). According to Table 2.2, typical values of c.o.v. of c_u range between 0.1 and 0.55, although some others reported c.o.v. as high as 0.85. This study uses an intermediate value of 0.25 for the c.o.v. of c_u . Referring to the computed F_s , the safety condition is deemed satisfied. However, when taking c_u as a single random variable to account for its uncertainty (without spatial variation), the reliability index was found to be 1.320, which gives a probability of failure of 9.35%, which is unsatisfactory according to Table 4.1.

Figure 4.3 compares the probabilistic critical slip surface corresponding to β of 1.320 and the deterministic critical slip surface having minimum F_s . The two slip surfaces are practically identical for the case in hand. This can be anticipated, since there is only one random variable c_u in this case, and the performance function has a linear relationship with c_u .

Table 4.1 Expected performance level vs. target reliability indices for slopes (after USACE 1995)

Expected performance level	Reliability index β	Probability of failure P_f
High	5.0	0.00003%
Good	4.0	0.003%
Above average	3.0	0.1%
Below average	2.5	0.6%
Poor	2.0	2.3%
Unsatisfactory	1.5	7.0%
Hazardous	1.0	16%

Figure 4.2 A hypothetical embankment with soft foundation and the soil properties

Figure 4.3 The deterministic and probabilistic critical slip surfaces are nearly identical when spatial variation is not considered

4.3 Spatial variability of soil properties in homogeneous soils

The reliability assessment described in section 4.2 is based on simple basic statistical information of soil properties. In practice, soils are geological materials formed by physical weathering processes (erosion, temperature changes, freezing and thawing) and chemical weathering processes. Soils at a particular site may be residual or transported by physical means to their present location. Thus, the physical properties of soils vary from point to point even within a 'homogeneous' deposit. This phenomenon makes it necessary to model soils as random fields in geotechnical reliability analysis. The random field of a soil property can be mathematically described using some commonly-used spatial correlation models.

A well-known approach for modeling spatial variation is the spatial-averaging method proposed by Vanmarcke (1977a, 1977 b, 1983). The spatial-averaging method considers that the variance of the random field should be reduced as compared to its original or point-wise value. The amount of variance reduction depends on the extent of random field of interest.

Another method dealing with the spatial variation is the spatial autocorrelation

method. It considers the soil properties at a particular point as random variables, and uses autocorrelation function to model the spatial correlation of the point-wise soil properties. However, the point to point variation of a random field is very difficult to obtain in practice and is often of no interest. Therefore, several methods have been proposed for the discretization of random fields into finite random variables. These can be classified into three main approaches (Haldar and Mahadevan 2000b): (1) discretizing the random fields into finite elements and defining random variables at one or several selected locations in each element. This is referred to as the midpoint method or nodal point method; (2) defining a random variable for each element using the spatial average of the random field over the element, and computing the correlation between the spatial averages using the random field theory as proposed by Vanmarcke (1983); and (3) using some form of series expansion to break a random field into a set of random variables. After the discretization of random fields into finite random variables, spatial correlation between these variables can be modeled. The details of modeling spatial variation are discussed next.

4.4 Modeling 2-D spatial variation in slope reliability analysis

4.4.1 Method of autocorrelated slices

In this study slope reliability analysis is first performed using limit equilibrium methods of slices. The failure mass is discretized into finite vertical slices. Thus, it is convenient to take the strength parameter at the midpoint along the base of each slice as a random variable (Figure 4.4), and model spatial variation between these random variables based on the random fields theory. This will be referred to as the *method of autocorrelated slices*, in contrast to the *method of interpolated autocorrelations* of the next sub-section. Although the latter method (focus of this study) is more versatile and can be applied to probabilistic finite-element/finite-difference analysis of slopes (described later), the former is discussed in this sub-section because it provides an interesting comparison with and a logical progression to the latter.

Figure 4.4 Discretizing the 2-D random fields over the slip surface according to slices

Several published works have applied the method of autocorrelated slices directly or indirectly. Xu (2006) studied the influence of spatial variation on slope reliability by treating the strength parameters at the mid-point along the base of slices as correlated random variables, and made comparisons with the variance reduction method. Low et al. (2007) studied the effect of horizontal spatial variability of soil properties on a Norway slope, also using the method of autocorrelated slices. Cho (2007) used the so-called mid-point discretization of random field in modeling spatial variation and carried out Monte Carlo simulation.

The correlation between spatial quantities may be described by a certain model that decays over the separation distance between them. Such a model is usually referred to as an autocorrelation function. In general, a decaying model that best fits the autocovariance plotting based on statistical data may be selected as the autocorrelation function. However, the determination of a proper decaying model is not an easy task since it needs a large amount of in-situ testing data. Therefore, some simple but efficient decaying models are usually assumed for autocorrelation functions. For example, the simple negative exponential and square-root exponential functions as shown by Eqs. (4.2) and (4.3), respectively, are usually used for 2-D autocorrelation functions.

$$\rho_{ij} = \exp \left[- \left(\frac{|x_i - x_j|}{\delta_x} + \frac{|y_i - y_j|}{\delta_y} \right) \right] \quad (4.2)$$

$$\rho_{ij} = \exp \left(- \sqrt{\left(\frac{x_i - x_j}{\delta_x} \right)^2 + \left(\frac{y_i - y_j}{\delta_y} \right)^2} \right) \quad (4.3)$$

where (x_i, y_i) denotes the position of a random variable; δ_x and δ_y are the horizontal and vertical autocorrelation distances, respectively.

Li and Lumb (1987) applied four different types of autocorrelation functions in a slope reliability analysis, including simple negative exponential, square exponential, second-order autoregressive and cosine exponential functions. They concluded that the reliability results were not sensitive to the form of autocorrelation function, provided that the scale of fluctuation, which is two times the autocorrelation distance, is small compared with the slip surface, and suggested using the negative exponential autocorrelation function due to its simplicity. For convenience, the simple negative exponential autocorrelation function in Eq. (4.2) is also used in this study, unless stated otherwise. However, the square-root function in Eq. (4.3) is more consistent since it naturally reduces to the 1-D case when δ_x is equal to δ_y . A comparison between the reliability indices computed by using the two different autocorrelation functions will be presented in section 4.6

The analysis of the two-layer embankment of Figure 4.2 will now be extended to reflect spatial variability of c_u . The spatial variability of the undrained shear strength of the soft foundation is considered both vertically and horizontally. Prior to conducting numerical analysis, it is important to characterize the spatial variability. Methods for spatial variation characterization were briefly mentioned in section 2.6 of Chapter 2. DeGroot (1993) reported that the autocorrelation distance for in situ soils typically lies between 0.5 to 3 m in the vertical direction, and 15 to 30 m in the horizontal direction. Accordingly, δ_y is assumed to be 2 m and δ_x to be 20 m for the purpose of parametric study. A case study based on realistic spatial variability obtained from statistical

analysis of in-situ CPT data will be presented in section 4.7.

In principle, a proper slope reliability analysis considering spatial variation should involve finding the probabilistic critical slip surface. In Monte-Carlo-based random finite element method, the probabilistic critical slip surface can be “sought out” automatically and simultaneously. However, when FORM combined with limit equilibrium methods is employed, previous studies involving determination of probabilistic critical slip surface were commonly performed without considering spatial variation. In this study, as part of slope reliability analysis involving 2-D spatial variation, search for such a “real” probabilistic critical slip surface was realized by using the systematic search method.

In the search process, all trial slip circles have the same number of slices, so that the same number of random variables and the same size of correlation matrix can continue to be used. The study was done by first investigating how many slices should be used in the reliability analysis. Suppose the number of slices with base in the random field of the foundation soil is n . There will be a total of n spatially correlated random variables and an $n \times n$ spatial correlation matrix for all trial slip circles. The correlation matrix $\mathbf{R}$ was constructed using Eq. (4.2) with the geometrical information of the slices. The influence of n on reliability index β has been plotted in Figure 4.5, where 2-D spatial variation has been considered. It is shown that β for the two-layer embankment is not sensitive to the number of slices n when more than 24 slices are used.

For convenience, the following study will use 24 slices for the random field. The β value was then found to be 2.256 when normal distribution is assumed for all the random variables, corresponding to a probability of failure of 1.20% by Eq. (2.44). Note that the probabilistic critical slip surface obtained by considering 2-D spatial variation differs significantly from both the deterministic and the traditional probabilistic critical slip surfaces (i.e., without consideration of spatial variation), as shown in Figure 4.6. For comparison, Monte Carlo simulation with Latin Hypercube sampling using @RISK Version 5.5 (<http://www.palisade.com/risk>) was also carried out. Unlike FORM analysis,

Monte Carlo simulation performs numerous realizations (trials) for the random variables subject to their correlation matrix $\mathbf{R}$. A deterministic slope stability analysis is implemented for each realization. The probability of failure is directly inferred by dividing the number of failures by the number of total realizations. Based on this, the probability of failure by MC simulation was found to be 1.24% with 200,000 iterations, versus FORM's 1.20%.

Figure 4.5 Influence of the number of slices on reliability analysis

It is interesting to study the effect of spatial variation on the slope reliability. Details are listed in Table 4.2, where a 1000 m autocorrelation distance implies perfect correlation of the soil properties, i.e., no spatial variation. Based on the methodologies described in this study, it was observed that ignoring spatial variation will overestimate the probability of failure. This can also be anticipated from the concept of variance reduction (Vanmarcke 1977b): when the spatial variability of strength parameters is modeled, their uncertainties are averaged over the slip surface, hence the zonal coefficients of variation are reduced while the mean values remain constant. As a result, the probability of failure decreases.

Table 4.2 also shows that cases considering only 1-D spatial variation (either in vertical or horizontal direction) fall in between the previous two extreme cases. However, the

influence of vertical spatial variation on β and P_f is much more significant than the influence of horizontal spatial variation. This is investigated further below.

- Probabilistic critical slip surface considering spatial variation
- · - · - Probabilistic critical slip surface without considering spatial variation
- Deterministic critical slip surface

Note: Probabilistic critical surface without considering spatial variation was found almost identical to the deterministic critical slip surface

Figure 4.6 Comparison of probabilistic critical slip surfaces with and without spatial variation

Table 4.2 Summary of reliability analyses considering different cases of spatial variation

Description of autocorrelation distance	Reliability index β	Probability of failure P_f
$\delta_y = 2$ m, $\delta_x = 20$ m	2.256	1.20%
$\delta_y = 2$ m, $\delta_x = 1000^a$ m	1.997	2.29%
$\delta_y = 1000^b$ m, $\delta_x = 20$ m	1.711	4.35%
$\delta_y = 1000^c$ m, $\delta_x = 1000^c$ m	1.487	6.85%

^aCorresponding to 1-D spatial variation in vertical direction

^bCorresponding to 1-D spatial variation in horizontal direction

^cCorresponding to no spatial variation (perfect correlation)

It is clear that the autocorrelation distance δ affects β to some extent. Based on this, a

series of slope reliability analyses have been carried out by varying δ values of the soft foundation. Consider first the influence of vertical autocorrelation distance δ_y on the slope reliability by assuming a constant horizontal autocorrelation distance δ_x of 20 m. Several values of δ_y from 0.5 to 5 m have been investigated successively. It was found that the β value decreases with increase in δ_y value, as shown in Figure 4.7.

In reliability analysis, it is very important to find the design point, by which the β value is calculated. Thus, investigations on how the random variables behave at the design point were also conducted. Figure 4.7 clearly shows that the profiles of design point vary significantly with increase in δ_y value. Furthermore, fluctuation in design point values tends to smooth out when δ_y is very large, e.g., 1000 m. This is because the strong correlations among the random c_u values allow only very slight deviations between slices. At the same time, reliability index β decreases in tandem with increase in δ_y value, as expected.

Next consider the influence of horizontal autocorrelation distance δ_x on the slope reliability by assuming a constant vertical autocorrelation distance δ_y of 2 m. Several values of δ_x ranging from 10 to 40 m were investigated. Comparisons between these reliability analyses are plotted in Figure 4.8.

Increasing δ_x has the same effect of reducing β , albeit not as much as with increasing δ_y . This can be observed from Figure 4.7 and Figure 4.8. The profiles of design point do not vary very much even for big difference in δ_x . Therefore, one may conclude that although both vertical and horizontal spatial variation can affect the reliability analysis, the influence of the vertical spatial variation on β and P_f is much more significant than the influence of the horizontal spatial variation, when the horizontal autocorrelation distance δ_x is an order of magnitude bigger than the vertical autocorrelation distance δ_y . The nature of the sedimentation process which give rise to clay deposits is such that δ_x is logically greater than δ_y , which means that soil properties (e.g. c_u) are more strongly correlated in the horizontal direction than in the vertical direction.

Figure 4.7 Comparison between the reliability analyses by varying the vertical autocorrelation distance δ_y for $\delta_x = 20$ m

Figure 4.8 Comparison between the reliability analyses by varying the horizontal autocorrelation distance δ_x for $\delta_y = 2 \text{ m}$

4.4.2 Method of interpolated autocorrelations in LEM

Although the method of autocorrelated slices is straightforward and able to deal with problems involving spatial variation, the disadvantage is that slope reliability analysis based on this method is strictly confined to limit equilibrium framework. Therefore, an alternative method by interpolating the soil properties is explored in this section to efficiently handle slope reliability analysis that incorporates spatial variation. Low and Tang (1997b) first used the method of interpolated autocorrelations to model the 1-D spatial variability of undrained shear strength in a soft foundation supporting an embankment. The present study extends the interpolation technique to model 2-D spatial variation in reliability analysis. The basic idea is to discretize the random fields using a mesh of finite control points (Figure 4.9), where the soil properties are taken to be random variables and spatially correlated. Soil properties at any other location X within the random field are then interpolated with values at the nearest control points surrounding it. In this study, bilinear interpolation based on 4 nodal points (in this context, the nodal point denotes the control point used for interpolation of random variables, which should not be confused with the concept in finite element method) is used in the method of interpolated autocorrelations. By this procedure, if one uses a 5×6 mesh to discretize the random field of c_u for the case in hand, for example, there will be 30 random variables in total to be considered. The correlation matrix $\mathbf{R}$ will be of size 30×30 , which is obtained using Eq. (4.2) based on the coordinates of the control points and the autocorrelation distances.

Searching for the probabilistic critical slip surface considering spatial variation is simpler when using the method of interpolated autocorrelations, because the locations of the control points are always fixed which means that the structure of the correlation matrix are invariable regardless of the position of the slip surface. Nevertheless, in order to have results that are comparable, the method of interpolated autocorrelations is first investigated based on the probabilistic critical slip surface obtained earlier in the method of autocorrelated slices.

The accuracy of the method of interpolated autocorrelations depends on the spacings of control points in both directions. It is anticipated that the finer the random field discretization is, the more accurate the reliability analysis will be in accounting for spatial variation. However, computational efficiency may suffer if a very finely-divided random field mesh is applied, since there will be a large number of random variables. Hence, the objective is to find a suitable spacing of control points that can balance accuracy with efficiency.

Figure 4.9 Discretizing the random field using finite control points where variables are spatially correlated

It was concluded in subsection 4.4.1 that spatial variation in the vertical direction affects slope reliability much more than that in the horizontal direction. Thus it is expected that the refinement of the random field mesh in vertical direction may influence the slope reliability much more than the refinement in horizontal direction. To simplify the analysis of the method of interpolated autocorrelations of this section, the study of the 2-D mesh refinement of the random field in the vertical direction is given higher priority than that in the horizontal direction. This is discussed in detail as follows.

In general, random variables at control points influence the profile of design point. To represent the curved shapes of design point profiles as shown in Figure 4.7 and Figure 4.8, at least three control points are required in the horizontal direction. In fact, there are so many potential slip surfaces to consider that it can be very difficult to decide

which three points should be selected. Nevertheless, it can be expected that the potential slip surface should cover the slope (from crest to toe). Hence, the discretization in the horizontal direction can be determined in such a way that the horizontal nodal spacing of control points Δx is equal to half of the horizontal distance between slope toe and crest. A short discussion on the horizontal refinement will be provided in the later part of this section. Consider the ratio of vertical spacing of control points to vertical autocorrelation distance

$$\eta_y = \frac{\Delta y}{\delta_y} \quad (4.4)$$

where Δy = vertical spacing of control points of random variables; δ_y = autocorrelation distance in the vertical direction.

The results are shown in Figure 4.10, where the horizontal nodal spacing of control points is 5 m. Furthermore, the method of autocorrelated slices of subsection 4.4.1 is used as the benchmark for exploring the method of interpolated autocorrelations, since the former considers all the slice-base mid-point variables to be directly autocorrelated without interpolation. The reliability indices obtained by methods of interpolated autocorrelations differ very little from those obtained by the method of autocorrelated slices, e.g., less than 3% for the case in hand when a suitable discretization scheme is selected (e.g., $\eta_y = 1.0$, $\Delta x = 5$ m). The case with Δy equal to 2 times the autocorrelation distance δ_y (i.e. $\eta_y = 2.0$) yields a reliability index value which is about 96% ($=2.158/2.256$) of the accurate value. Further investigations show that significant error could occur when Δy exceeds 2 times the vertical autocorrelation distance. This finding is consistent with previous studies, e.g. Vanmarcke (1977b). It is interesting to note the tradeoff between small Δy (which results in good accuracy but bigger number of random variables) and large Δy (which results in smaller number of random variables but less accurate reliability analysis).

Next, the horizontal refinement was examined based on a relatively accurate vertical discretization of η_y equal to 1. It has been shown in Figure 4.7, Figure 4.8 and Figure 4.10 that the profiles of design point values tend to be curved below the slope. It is

therefore reasonable to have some horizontal control points below the slope (between the crest and toe) since the purpose of using control points is to represent the curved profiles of design point values. Parametric studies were conducted by varying the value of Δx while fixing a horizontal control point below the mid-slope. The influences of Δx on both the β value and the profile of design point values are shown in Figure 4.11. As expected, neither the β values nor the profiles of design point values are sensitive to Δx . This is consistent with the results shown in Figure 4.8. Based on parametric studies, η_y is recommended to be 1 to 1.5, and Δx to be half of the horizontal distance between slope crest and toe.

It is worth pointing out that β values from the two specific interpolation schemes $\eta_y = 1.5$ and $\Delta x = 10$ m do not follow a monotonic trend, as shown in Figure 4.10 and Figure 4.11. In fact, it is clear that the more dense the discretization (control) points, the more accurate the representation of the random field. However, a denser discretization scheme will lead to too many random variables so that FORM-based reliability analysis is difficult to implement. So, the purpose of the method of interpolated autocorrelations is to use as few discretization points as possible to represent the random field. When the discretization is too coarse, such as $\eta_y = 1.5$ and $\Delta x = 10$ m or even coarser, the representation by the method of interpolated autocorrelations may be significantly distorted. The consequence is that the reliability index β so obtained may deviate too much from the accurate value. Such a deviation may not follow a monotonic trend. On the other hand, we can easily find that when the discretization is dense enough, the β value tends to be insensitive. For instance, for the three case of $\eta_y = 0.7, 1.0$ and 1.5 , the β values differ by a percentage error not exceeding 2%.

In addition, a reliability analysis by the method of interpolated autocorrelations with search for probabilistic critical slip surface was conducted based on the discretization scheme $\eta_y = 1.0$ and $\Delta x = 5$ m. The β value was also found to be 2.242, and the probabilistic critical slip surface found by the method of interpolated autocorrelations is exactly the same as that by the method of autocorrelated slices.

In conclusion, the method of interpolated autocorrelations is comparable with the relatively accurate method of autocorrelated slices in slope reliability analysis that accounts for 2-D spatial variation. Several advantages of the method of interpolated autocorrelations are: (1) one is not confined to just limit equilibrium methods of slices in performing slope reliability analysis; more robust methods (e.g., strength-reduction FEM or FDM) may be used as well; (2) since the random field is reduced to a small number of random variables at control points, FORM can be easily performed in combination with either LEM or strength-reduction FEM/FDM, as an alternative to Monte Carlo simulations; (3) once the discretization scheme has been decided, the structure of spatial correlation matrix and random variables do not change during the search for critical slip surface, in cases where limit equilibrium method of slices is used.

An interesting observation from Figure 4.7 to Figure 4.11 is that the profiles for design point c_u values appears to be symmetric about $x = 5$ m, i.e., the mid-slope line. For the example slope, the symmetry of the design point profile of c_u is discussed below:

- (1) The design point c_u values shown in Figure 4.7 to Figure 4.11 are along the slip surfaces in clay foundation. In deterministic stability analysis, it is found that the critical slip surface in clay foundation is always symmetric about mid-slope line (in vertical). For probabilistic analysis, it is also found that the probabilistic critical slip surface in clay foundation is symmetric about mid-slope line. The symmetry of slip surface gives rise to symmetry of the spatial correlation. On the other hand, the random field of c_u is considered to be 'homogeneous'; in other words, any point-wise c_u within the random field has the same statistical characteristics. As a result, the sensitivity of two random c_u symmetric about mid-slope line tends to be the same. In reliability analysis, the outcome is the symmetry of design point c_u profile.
- (2) In this probabilistic analysis accounting for spatial variability of c_u , the two different methods are proposed to treat the spatial correlation (based on the same information of autocorrelation distance which is an intrinsic material property), and both results show that the design point c_u values are symmetric about $x = 5$ m, i.e., the mid-slope line. Thus it may not be just an artifact of the

way the spatial correlation is treated.

The practical implication of these results is that for the given slope profile and soil properties, when the random field of c_u exhibits the design point profile, the slope failure is most likely to take place.

Figure 4.10 Comparison between reliability analyses using different vertical interpolating frameworks with $\Delta x = 5.0\text{m}$ ($\delta_y = 2\text{m}$, $\delta_x = 20\text{m}$)

Figure 4.11 Comparison between reliability analyses using different horizontal interpolating frameworks

4.4.3 Using the method of interpolated autocorrelations with independent numerical packages via RSM

Section 4.4.2 has shown the success of application of the method of interpolated autocorrelations to slope reliability analysis accounting for 2-D spatial variation, which is based on coupling FORM and stability model in the same platform. Alternatively, the method of interpolated autocorrelations can also be used in combination with independent numerical packages, and slope reliability analysis can be realized by use of *response surface method* (RSM) or *neural networks*. The basic concept of RSM and a RSM-based reliability analysis were introduced in Chapter 3. The following study presents a RSM-based slope reliability analysis with 2-D spatial variation being modeled using the method of interpolated autocorrelations. For simplicity, the RSM is considered to be a second-order polynomial without cross terms.

To use RSM for reliability analysis, one needs to first select a number of sampling points (e.g. $\mu_i \pm \sigma_i$) and to conduct a series of deterministic slope stability evaluation on these sampling points. An approximate performance function (or response surface) is then determined and reliability analysis is carried out.

Consider again the example embankment with soft clay foundation (Figure 4.2). To investigate the applicability of using interpolated autocorrelations with RSM, two discretization schemes for the random field of c_u , case 1 ($\Delta x = 5$ m, $\eta_y = 1.0$) and case 2 ($\Delta x = 5$ m, $\eta_y = 1.5$) will be used for discussion. In both cases 24 discretization points are employed, hence generating 24 spatially correlation c_u values. To approximate the performance function comprising these 24 random variables, a second-order polynomial without cross terms is used, such that $g(\mathbf{c}_u) = F_s(\mathbf{c}_u) - 1 = a + (b_1 \mathbf{c}_{u1} + \dots + b_{24} \mathbf{c}_{u24}) + (c_1 c_{u1}^2 + \dots + c_{24} c_{u24}^2)$. There are a total of 49 unknown coefficients in the polynomial, so 49 sampling points are needed to determine them. The 49 sampling points (each comprising 24 random c_u variables) are selected in a way that each random variable is disturbed twice while keeping others unchanged, e.g., $\mu_i \pm \sigma_i$ approach. At each of the 49 sampling points, a stability analysis can be run using an independent

numerical package. In this study, the stability is solved using a spreadsheet. As a result, the polynomial can be easily obtained.

The polynomial so obtained may not be the true performance function. So an iterative procedure is carried out. An illustrative example of the iterative procedure of RSM was given in Subsection 3.3. For the present study, reliability indices of the two discretization schemes based on RSM after second iteration were computed to be 2.245 and 2.249, respectively, which are very close to the direct reliability analyses (shown in Figure 4.10).

4.4.4 Modeling 2-D spatial variation in strength-reduction FDM stability analysis

The proposed interpolated autocorrelation approach is also the basis to develop the probabilistic strength-reduction stability analysis involving 2-D spatial variation. One implementation requirement must however be met: the strength reduction program (in either FEM or FDM) must be able to interpolate the shear strength parameters of all the elements from each input of 2-D grid of shear strength values. The following study uses the numerical package of FDM FLAC version 7.0, which enables the strength reduction technique to be done automatically. In particular, a FISH (which is a programming language for FLAC) subroutine was written to implement the 2-D interpolation of random variables.

The elastic-perfectly plastic constitutive model with Mohr-Coulomb failure criterion is used for strength-reduction analysis of the example slope, where reduction is simultaneously applied to c and $\tan \phi$. Soil properties used for the strength-reduction stability analysis are listed in Table 4.3.

Figure 4.12 and Figure 4.13 show the results of deterministic stability analysis of the embankment with soft clay foundation. The computed factor of safety is 1.514 by FLAC version 7.0. This value is slightly higher than the F_s of 1.462 as computed by Spencer's method with circular slip surface by 3.6 %.

Table 4.3 Soil properties of soft clay foundation and embankment used for strength reduction analysis

Soil Layer	Cohesion c (kPa)	Friction angle ϕ ($^{\circ}$)	Young's m. E (MPa)	Poisson ratio ν	Remarks
Embankment	5	30	50	0.3	Non-associated
Clay foundation	25	0	10	0.495	flow rules

Figure 4.12 Failure zone indicated by shear strain increment

Figure 4.13 Failure mass indicated by velocity vectors

The deterministic strength-reduction stability analysis is extended into probabilistic study with the probabilistic sensitivity-based FORM proposed in Chapter 3. In particular, spatial variation of the undrained shear strength c_u of the foundation clay is modelled using the method of interpolated autocorrelations.

Considering that the failure zone develops mainly near the slope, the following study uses a relatively dense random field discretization scheme near the slope and a relatively coarse discretization scheme far away from the slope, in order to reduce the number of random variables involved in FORM-based probabilistic analysis. As a result, 32 control points are used to represent the random field of c_u for the foundation. The arrangement of the 32 control points is shown in Figure 4.14.

The random field c_u value at each grid-point (for FLAC FDM discretization) can be interpolated from the correlated random c_u at the 32 control points. An arbitrary representation of the random field c_u is shown in Figure 4.15, in which the c_u values at the 32 control points are generated randomly subjected to their correlation matrix.

Figure 4.14 Using 32 control points to represent the random field of c_u

Figure 4.15 An arbitrary representation of the random field of c_u ($\delta_y = 2$ m, $\delta_x = 20$ m)

Figure 4.16 displays the detailed reliability analysis using the sensitivity-based FORM in n-space. For each iteration, computation of the gradient vector of performance function (by sensitivity analysis) requires (32+1) runs of strength-reduction stability evaluation, which can be carried out automatically by a user-defined FISH subroutine in FLAC version 7.0. Details of the subroutine can be found in Appendix E. The β value converges after four iterations, and the ultimate β is about 2.572. Note that this value is valid for the deterministic F_s of 1.514 as obtained by FLAC's strength reduction analysis, while the previous β value of 2.242 is valid for the deterministic F_s of 1.462 as obtained by LEM analysis.

Using the design point values of the 32 random c_u , representation of the random field that will most probably yield the slope failure is shown in Figure 4.17. Compared with the arbitrary representation as shown in Figure 4.15, the random field of c_u at design point tends to be smoothed out. It is interesting to see that there is a sensitive zone of c_u (middle area of the contour plot) which implies that slope failure will probably start from this area, and progressively extend to other areas. Thus, geotechnical practitioner may use the proposed reliability analysis procedure to foresee the potential sensitive zone when a random field of soil property is involved, so that economic decision of reinforcement can be made.

Initial $\mathbf{n}_0$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
$F_s(\mathbf{n}_0)$	1.514																															
$F_s(\mathbf{n}_{0,new})$	1.518	1.518	1.514	1.514	1.522	1.525	1.522	1.518	1.518	1.522	1.529	1.529	1.518	1.518	1.522	1.533	1.518	1.522	1.529	1.525	1.522	1.525	1.522	1.518	1.518	1.518	1.518	1.514	1.514	1.514	1.514	1.514
$\Delta g(\mathbf{n}_0)$	0.00	0.00	0.00	0.00	0.01	0.01	0.01	0.00	0.00	0.01	0.02	0.02	0.00	0.00	0.01	0.02	0.00	0.01	0.02	0.01	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
$\Delta g/\Delta n_i$	0.01	0.01	0.00	0.00	0.02	0.02	0.02	0.01	0.01	0.02	0.03	0.03	0.01	0.01	0.02	0.04	0.01	0.02	0.03	0.02	0.02	0.02	0.02	0.01	0.01	0.01	0.01	0.00	0.00	0.00	0.00	
$\mathbf{n}_1$	-0.59	-0.83	-0.86	-0.71	-0.86	-1.25	-1.37	-1.16	-0.92	-1.37	-1.59	-1.39	-0.94	-1.39	-1.64	-1.46	-0.93	-1.38	-1.60	-1.36	-0.88	-1.27	-1.41	-1.15	-0.73	-1.05	-1.15	-0.91	-0.44	-0.63	-0.70	-0.55
β	2.360																															
Initial $\mathbf{n}_1$	-0.59	-0.83	-0.86	-0.71	-0.86	-1.25	-1.37	-1.16	-0.92	-1.37	-1.59	-1.39	-0.94	-1.39	-1.64	-1.46	-0.93	-1.38	-1.60	-1.36	-0.88	-1.27	-1.41	-1.15	-0.73	-1.05	-1.15	-0.91	-0.44	-0.63	-0.70	-0.55
$F_s(\mathbf{n}_1)$	1.057																															
$F_s(\mathbf{n}_{1,new})$	1.057	1.057	1.057	1.057	1.065	1.068	1.065	1.057	1.057	1.065	1.072	1.068	1.057	1.057	1.065	1.072	1.057	1.061	1.068	1.065	1.061	1.068	1.065	1.057	1.061	1.061	1.057	1.057	1.057	1.057	1.057	
$\Delta g(\mathbf{n}_1)$	0.00	0.00	0.00	0.00	0.01	0.01	0.01	0.00	0.00	0.01	0.02	0.01	0.00	0.00	0.01	0.02	0.00	0.00	0.01	0.01	0.00	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
$\Delta g/\Delta n_i$	0.00	0.00	0.00	0.00	0.02	0.02	0.02	0.00	0.00	0.02	0.03	0.02	0.00	0.00	0.02	0.03	0.00	0.01	0.02	0.02	0.01	0.02	0.02	0.00	0.01	0.01	0.00	0.00	0.00	0.00	0.00	
$\mathbf{n}_2$	-0.50	-0.88	-1.04	-0.76	-0.83	-1.45	-1.71	-1.26	-0.84	-1.57	-1.97	-1.54	-0.82	-1.54	-1.98	-1.61	-0.82	-1.52	-1.90	-1.47	-0.81	-1.44	-1.66	-1.21	-0.70	-1.19	-1.32	-0.95	-0.42	-0.72	-0.80	-0.58
β	2.679																															
Initial $\mathbf{n}_2$	-0.50	-0.88	-1.04	-0.76	-0.83	-1.45	-1.71	-1.26	-0.84	-1.57	-1.97	-1.54	-0.82	-1.54	-1.98	-1.61	-0.82	-1.52	-1.90	-1.47	-0.81	-1.44	-1.66	-1.21	-0.70	-1.19	-1.32	-0.95	-0.42	-0.72	-0.80	-0.58
$F_s(\mathbf{n}_2)$	0.986																															
$F_s(\mathbf{n}_{2,new})$	0.986	0.986	0.986	0.986	0.994	1.002	0.994	0.986	0.990	0.994	1.006	0.994	0.986	0.986	0.998	1.002	0.990	0.994	1.002	0.994	0.990	0.998	0.994	0.986	0.990	0.990	0.986	0.986	0.986	0.986	0.986	
$\Delta g(\mathbf{n}_2)$	0.00	0.00	0.00	0.00	0.01	0.02	0.01	0.00	0.00	0.01	0.02	0.01	0.00	0.00	0.01	0.02	0.00	0.01	0.02	0.01	0.00	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
$\Delta g/\Delta n_i$	0.00	0.00	0.00	0.00	0.02	0.03	0.02	0.00	0.01	0.02	0.04	0.02	0.00	0.00	0.02	0.03	0.01	0.02	0.03	0.02	0.01	0.02	0.02	0.00	0.01	0.01	0.00	0.00	0.00	0.00	0.00	
$\mathbf{n}_3$	-0.54	-0.89	-1.00	-0.64	-0.90	-1.47	-1.66	-1.06	-0.95	-1.59	-1.93	-1.29	-0.93	-1.57	-1.96	-1.38	-0.94	-1.56	-1.88	-1.27	-0.87	-1.43	-1.62	-1.05	-0.74	-1.17	-1.28	-0.82	-0.45	-0.71	-0.78	-0.50
β	2.590																															
Initial $\mathbf{n}_3$	-0.54	-0.89	-1.00	-0.64	-0.90	-1.47	-1.66	-1.06	-0.95	-1.59	-1.93	-1.29	-0.93	-1.57	-1.96	-1.38	-0.94	-1.56	-1.88	-1.27	-0.87	-1.43	-1.62	-1.05	-0.74	-1.17	-1.28	-0.82	-0.45	-0.71	-0.78	-0.50
$F_s(\mathbf{n}_3)$	1.002																															
$F_s(\mathbf{n}_{3,new})$	1.002	1.002	1.002	1.002	1.010	1.014	1.006	1.002	1.002	1.010	1.025	1.006	1.002	1.002	1.018	1.010	1.006	1.010	1.018	1.006	1.006	1.014	1.006	1.002	1.002	1.002	1.002	1.002	1.002	1.002	1.002	
$\Delta g(\mathbf{n}_3)$	0.00	0.00	0.00	0.00	0.01	0.01	0.00	0.00	0.00	0.01	0.02	0.00	0.00	0.00	0.02	0.01	0.00	0.01	0.02	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
$\Delta g/\Delta n_i$	0.00	0.00	0.00	0.00	0.02	0.02	0.01	0.00	0.00	0.02	0.05	0.01	0.00	0.00	0.03	0.02	0.01	0.02	0.03	0.01	0.01	0.02	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
$\mathbf{n}_4$	-0.50	-0.87	-1.04	-0.54	-0.83	-1.44	-1.71	-0.89	-0.86	-1.59	-2.05	-1.09	-0.85	-1.58	-2.09	-1.14	-0.86	-1.56	-1.96	-1.05	-0.78	-1.39	-1.63	-0.86	-0.61	-1.09	-1.27	-0.67	-0.37	-0.66	-0.77	-0.41
β	2.572																															

Note: $F_s(\mathbf{n})$ is the factor of safety computed using FLAC version 7.0, based on the 32 spatially correlated random variables of c_u ; $g = F_s - 1$; framed cells contain equations.

Figure 4.16 Probabilistic sensitivity-based FORM obtains a β value of 2.572 after four iterations

Figure 4.17 Profile of the random field of c_u at the design point values

It is also interesting to investigate the slope failure when the random field of c_u is at the design point, since such a failure will indicate the *probabilistic* critical state of slope failure. The results are presented in Figure 4.18 and Figure 4.19. As expected, the factor of safety F_s based on the design point of random field was found to be 0.998 (F_s should theoretically be 1.0 if the design point is accurate), which again validates the convergence of the previous reliability analysis. Along the failure zone of the slope, larger shear strain increment was found at the middle part, which is consistent with the sensitive zone of c_u as shown in Figure 4.17.

From the deterministic stability analysis, the failure mass tends to touch the bottom of the soft foundation (Figure 4.13). In contrast, Figure 4.19 of probabilistic random field analysis vividly shows that the failure mass may not develop so deeply as to the bottom when spatial variation is taken into consideration. This observation from probabilistic strength-reduction stability analysis is also consistent with the previous probabilistic limit equilibrium analysis.

Figure 4.18 Failure zone indicated by shear strain increment (random field of c_u is at design point)

Figure 4.19 Failure mass indicated by velocity vectors (random field of c_u is at design point)

4.5 Further application to a drained soil slope reliability analysis

Methods of modeling the 2-D spatial variation of soil properties are further applied to drained slope reliability analysis. Note that reliability analysis that deals with spatial variability of friction angle ϕ' is not common in the literature. In contrast, this study considers spatial variation to both cohesion c' and friction angle ϕ' . Griffiths et al. (2009) investigated the spatial variation of a c - ϕ slope using random finite element method based on Monte Carlo simulation, by assuming the same autocorrelation distance for both c' and $\tan \phi'$ of a homogeneous soil layer. This assumption is adopted in this study.

Consider the homogeneous slope shown in Figure 4.20. The coefficients of variation are 0.3 for c' and 0.1 for ϕ' , respectively. Both random variables are assumed to be normally distributed. In addition, cross correlation between c' and ϕ' is modeled by assuming a cross correlation coefficient $\rho_{c-\phi} = -0.5$, based on past findings that show

negative cross correlation between c' and ϕ' in natural soils (Matsuo and Kuroda 1974).

Figure 4.20 A homogeneous c - ϕ slope

Spencer's method with 24 slices was used in the stability analysis. The critical factor of safety F_s of deterministic analysis was found to be 1.226 and probabilistic analysis gave a reliability index of 1.428. The deterministic critical slip surface and reliability-based probabilistic critical slip surface are shown in Figure 4.21.

Figure 4.21 Deterministic critical slip surface and probabilistic critical slip surfaces for a homogeneous c - ϕ slope

When 2-D spatial variation is modeled for this c - ϕ slope, the random fields of c' and ϕ'

can share the same discretization scheme, since equal autocorrelation distance for both c' and ϕ' has been assumed. In this example, the autocorrelation distances are: $\delta_{c_y} = \delta_{\phi_y} = \delta_y = 3 \text{ m}$, $\delta_{c_x} = \delta_{\phi_x} = \delta_x = 30 \text{ m}$.

At each control point discretizing the random field, the c' and ϕ' are spatially correlated with those at other points, such that

$$\rho_{c_{ij}} = \rho_{\phi_{ij}} = \rho_{ij} \quad (4.5)$$

where $\rho_{c_{ij}}$ = autocorrelation coefficient between c' at locations i and j , $\rho_{\phi_{ij}}$ = autocorrelation coefficient between ϕ' at locations i and j . ρ_{ij} can be computed based on Eq. (4.2).

Regarding the cross correlation between c and ϕ in a discretized random field, a simple model is assumed

$$\rho_{c_i-\phi_j} = \rho_{c-\phi} \times \rho_{ij} \quad (4.6)$$

where $\rho_{c-\phi}$ = cross correlation coefficient between c - ϕ without spatial variability, and $\rho_{c-\phi} = -0.5$ in this case; ρ_{ij} = autocorrelation coefficient of either c' or ϕ' . This implies that the c - ϕ cross correlation coefficient decays likewise over space, which is better than the assumption of constant cross correlation coefficient of c - ϕ in previous studies, e.g., Soubra and Youssef Abdel Massih (2010), because Eq. (4.6) can help avoid non-positive definite correlation matrix (which is not allowed statistically). The need for positive definite correlation matrix in statistics are discussed by Ditelevsen (1981), Low and Tang (2004), for example.

Based on Eqs. (4.2), (4.5) and (4.6), the correlation matrix of random variables at control points is readily constructed. Subsequently, FORM-based reliability analysis involving 2-D spatial variation of this c - ϕ slope was conducted using both the method of autocorrelated slices and the method of interpolated autocorrelations. The results are listed in Table 4.4.

Table 4.4 Results of reliability analysis of a c - ϕ slope

Description of autocorrelation distance	Method of autocorrelated slices		Method of interpolated autocorrelations	
	β	P_f	β	P_f
$\delta_y = 3$ m, $\delta_x = 30$ m	2.245	1.24 %	2.232	1.28 %
$\delta_y = 3$ m, $\delta_x = 1000$ m	2.159	1.54 %	2.118	1.71 %
$\delta_y = 1000$ m, $\delta_x = 30$ m	1.533	6.26 %	1.526	6.35 %
$\delta_y = 1000$ m, $\delta_x = 1000$ m	1.434	7.58 %	1.457	7.26 %

Good consistency was also observed between the two methods for this simple c - ϕ slope. Similar to the undrained analysis of soft foundation below embankment, reliability analysis that considers 2-D spatial variation tends to give bigger β value than no spatial variation, or merely 1-D spatial variation in the horizontal or vertical direction. If only 1-D spatial variation is considered, the contribution of spatial variation in vertical direction is more significant than the contribution in horizontal direction. Comparison of various critical slip surfaces are also plotted in Figure 4.21. The probabilistic critical slip surfaces are practically identical for this c - ϕ slope whether or not the spatial variation is considered. This is somewhat different from the undrained analysis (Figure 4.6). In fact, the difference between such surfaces cannot be anticipated, and can only be determined case by case. In cases where soil's spatial variability is very large, β can be very sensitive to the locations of slip surfaces (or averaging length or field). This will result in a significant difference between the probabilistic critical slip surface without modeling spatial variation and that which accounts for spatial variation. Otherwise, there is not much difference between the two probabilistic critical slip surfaces.

4.6 Discussion on the use of different 2-D autocorrelation functions

It is noted that the simple negative exponential autocorrelation function Eq.(4.2) was used in this study. The simple function is more suitable for 2-D spatial variation with δ_x being an order of magnitude bigger than δ_y . One disadvantage of the simple function is that there is no rotational symmetry. Suppose a 2-D random field is isotropic, such that

the vertical autocorrelation distance is equal to the horizontal one. In this case, the 2-D random field reduces to a 1-D random field (isotropic). However, the simple negative exponential function of 2-D cannot reduce to the form of 1-D. In contrast, the square-root exponential function Eq. (4.3) will have rotational symmetry.

Based on the example two-layer embankment, comparisons between the reliability indices computed based on the simple negative exponential and square-root exponential autocorrelation functions are presented in Table 4.5. It is found that the difference between reliability indices computed by using the two different autocorrelation functions is as large as 10.6% (with reference to square-root exponential function) when the 2-D spatial variation reduces to 1-D. With increase of the ratio of δ_x to δ_y , such a difference decreases.

Table 4.5 Comparison of reliability indices based on two different autocorrelation functions

Spatial variability	β by simple exponential	β by square-root exponential	Difference in β values (%)
$\delta_y = 2 \text{ m}, \delta_x = 2 \text{ m}$	3.847	3.477	10.6
$\delta_y = 2 \text{ m}, \delta_x = 10 \text{ m}$	2.500	2.270	10.1
$\delta_y = 2 \text{ m}, \delta_x = 20 \text{ m}$	2.256	2.105	7.1
$\delta_y = 2 \text{ m}, \delta_x = 50 \text{ m}$	2.100	2.023	3.8
$\delta_y = 2 \text{ m}, \delta_x = 1000 \text{ m}$	1.997	1.993	0.2

4.7 Characterizing the 2-D spatial variation from in-situ tests and case studies

In recent years, geotechnical reliability analysis involving spatial variation has been an active topic in the literature. While most studies aimed at numerical analysis using reported or reasonably-assumed spatial variation characteristics, only a few studies were carried out in characterizing soil spatial variability from in-situ testing data. Some

important issues in spatial variation characterization are mentioned below.

4.7.1 Stationarity of random data and effect of trend-removal

A set of random data is stationary if it satisfies the following four conditions (Jaksa et al. 1999): (i) the mean is constant with distance, i.e. no trend exists in the data; (ii) the variance is constant with distance; (iii) there are no seasonal variations of the data, implying that the data are time-independent (iv) there are no irregular fluctuations. With the assumption of stationarity of a random field, the spatial variation characterization could be greatly simplified since a uniform probabilistic law that is independent of the location and time is applicable to the whole random field.

In geotechnical engineering, the data obtained by in-situ tests such as FVT and CPT/SPT are often used for the characterization of spatial variability of shear strengths of soils. However, the in-situ testing data commonly vary with depth if the sounding is taken vertically. Thus, there is often a lack of stationarity in the CPT data. For example, Figure 4.22 shows a measured cone tip resistance value q_c , by CPT test. The measured data show obvious variation with depth and there is a deterministic trend for the variation of mean value. To study the spatial variation, the non-stationary CPT data should first be transformed to be stationary. This can be achieved by removing a low-order polynomial trend from within the depth-dependent CPT data. Usually, the low-order polynomial trend to be removed is no more than second-order and can be found by using ordinary least squares technique (OLS). After the trend removal, the CPT data could be regarded to be stationary. Jaksa (1999, 2000) suggested that the transformed CPT data should be tested against stationarity and Jaksa et al. (1997a, b) found that Kendall's τ test (Daniel 1990) is a useful tool for assessing whether data are stationary. The technique is recently used by Stuedlein et al. (2012) to test the stationarity of their linearly-detrended CPT data. On the other hand, Fenton (2000) questioned the necessity of stationarity test using Kendall's τ test method due to the fact that stationarity can be strictly verified only if data is taken over an infinite extent which never occurs in practice. Thus Fenton (2000) suggested that the trend-removal

CPT data could be simply assumed stationary without any test, considering the fact that the assumption of stationarity is always conservative.

Figure 4.22 A measured profile of cone tip resistance q_c within Keswick clay in south Australia (after Jaksa et al. 1999)

4.7.2 Vertical spatial variability

The characterization of vertical spatial variability of soil strength parameters is commonly carried out based on CPT/SPT data (Jaksa et al. 1997a, 1999; Fenton 1999). The characterization procedures by different studies are similar. As described in Jaksa et al. (1997a, 1999), the main steps are summarized below:

- Step 1: Carry out in-situ tests (CPT/SPT) based on a suitable sampling plan to gather sufficient data for quantifying vertical spatial variability. A sampling plan is a program of action for collecting data from a sampled population.
- Step 2: Plot CPT/SPT data from a typical vertical sounding against depth below the ground surface. In many practical instances, plots of in-situ test measurement, e.g. CPT q_c or SPT N , with depth have been used to examine vertical spatial variability directly. Otherwise, design soil properties such as c_u or ϕ , which are correlated from in-situ test measurement via established transformation models may be plotted instead.
- Step 3: Remove deterministic trend from CPT/SPT data to make data set stationary. The trend can usually be described by some linear or quadratic function. Ordinary least-squares method is used to determine the most appropriate trend function, which is then subtracted from the original data.
- Step 4: Evaluate the sample autocorrelation (or autocovariance) function for the residuals off the trend. The most common method of estimating autocorrelation (or autocovariance) functions is the method of moments.
- Step 5: Obtain the theoretical best fit for the sample autocorrelation function by using regression analysis based on a least-squares error approach. This can be done by fitting one of a number of existing models to the sample autocorrelation function, followed by determining the autocorrelation distance (or scale of fluctuation) based on the best-fit model.

For example, based on the above steps, the measured CPT data as shown in Figure 4.22 were first removed of a deterministic trend of a quadratic polynomial (Jaksa et al. 1997a, 1999). For the residuals after trend-removal of the CPT data, the sample autocovariance or autocorrelation function is plotted in Figure 4.23, using the method of moments [Eqs. (2.23) and (2.24)]. The two models as used by Jaksa et al. (1999) are the negative exponential and Gaussian (exponential quadratic) forms, respectively:

$$\text{Model 1: } \rho = \exp(-|y|/\delta_y)$$

$$\text{Model 2: } \rho = \exp(-y^2/\delta_y^2)$$

The above two models yield vertical autocorrelation distance δ_y , of 0.091 and 0.097 m, respectively.

The vertical autocorrelation distance of Keswick clay found by Jaksa et al. (1997a) seems very small as compared with some reported values typically ranging from 1 to 5 m by others (e.g., Fenton 1999). This could be caused by two reasons: (1) the soil properties at different sites where different tests are conducted are different in nature; (2) the procedure of exploration including the sample spacing and the domain size may have influences on the outcome of characterization of spatial variation (DeGroot and Baecher 1993; Jaksa et al. 1997a, 1999; Fenton 2000). The second reason is known as fractal behavior (Jaksa et al. 1999) and it is explained in the following study of horizontal spatial variability by different sampling plans.

Figure 4.23 Sample and fitting autocorrelation function (ACF) based on the residuals of Figure 4.22 (after Jaksa 1999)

4.7.3 Horizontal spatial variability and effect of sampling plan

While many studies discussed the vertical spatial variation from CPT/SPT testing data, only a few carried out the work of characterizing horizontal spatial variation, due to the much more testing effort and cost. Jaksa et al. (1997a, 1999) presented two case studies involving CPTs performed in a relatively homogeneous OC clay – the Keswick clay in south Australia. The first case study involved 223 CPTs performed in a 50 X 50 m area, each to a typical depth of 5 m. The CPTs were arranged in a grid layout, with adjacent CPTs being spaced at 0.5, 1, and 5 m horizontal intervals. The grid layout of the first case study is shown in Figure 4.24. In particular, the measured CPT data from soundings CD1 to CD50 horizontally spaced at 0.5 m are discussed here.

In the second case study, a single horizontal CPT being able to obtain a large quantity of closely-spaced horizontal data was performed in a suitable embankment at Keswick which was known to consist of Keswick clay. An electrical cone penetrometer was designed and it was driven laterally into the embankment to a total lateral extent of 7.62 m (Jaksa 1997a). Figure 4.25 illustrates the experiment conducted in the second case study.

In general, the in-situ soils in the two case studies could be considered as the same material, but the spatial variability is tested from two different sampling plans. The results obtained from the two case studies are compared in Figure 4.26. The horizontal autocorrelation distance δ_x is found between 1 to 2 m from the first case study, while the δ_x found from the second case study is around 150 mm. The comparison shows that the horizontal spatial variability of a soil could be very different from different testing or sampling plans. The contradictory results are due to the fact of nested structures within the Keswick clay. Nested structures are sources of variability which could be influenced by the scale of observation (Vanmarcke 1978), and Jaksa (1999, 2000) and Fenton (1999, 2000) called such a phenomenon as ‘fractal behavior’ of soil spatial variation. The recent publication by Dasaka and Zhang (2012) also showed that the sampling plan could affect the autocorrelation distance characterized from SPT

data, coming to a conclusion that the scale of fluctuation increases with the size of the sampling domain.

Thus, to avoid making misleading results of spatial variation characterization, the in-situ tests should be conducted with a number of borings located at separation distances less than the autocorrelation distance (DeGroot and Baecher 1993). However, it is not possible to know the required minimum sampling interval of any material prior to testing. Jaksa et al. (1997b) suggested that in-situ tests should begin sampling with a relatively coarse grid, and then to infill with a repeatedly finer grid, until the sample spacing no longer influences the resultant sample ACF and experimental semivariogram.

Figure 4.24 Layout of vertical CPTs in case study 1 (After Jaksa et al. 1999)

Figure 4.25 Illustration of the horizontal CPT in case study 2 (After Jaksa et al. 1999)

Figure 4.26 Comparison of the CPT data and horizontal spatial variability analyzed from the residuals of the data: (a) case study 1; (b) case study 2

4.7.4 Random measurement errors

The random measurement errors refer to the source of spatial variability which was caused by taking CPTs/SPTs at different soundings and combining the CPT data into a single population. In general, the measurement errors affects estimates of variability at small scales but not significantly affects the large-scale estimates (Jaksa et al. 1997b, Fenton 2000). In order to better assess spatial variability, it is crucial that the same method is used in measuring the variability of soil parameters. Some factors (e.g., sampling interval and penetration speed, etc.) which may influence the CPT reading need to be taken into account when the CPT data are used for spatial variability characterization. The horizontal CPT and vertical CPT are also unlikely to give comparable q_c values as the stress states which control the cone resistance are different. Vertical CPT conducted at very close intervals would be severely affected by previous soundings due to soil disturbance and the cavities left by previous soundings. As a result, the CPT data should be used with great caution due to the complicated nature of CPT testing.

4.7.5 A case study involving 2-D spatial variation – James Bay dike

The uncertainty of soil properties in the James Bay dike has been well-documented by Christian et al. (1994). The case study of James Bay dike presented in Chapter 3 implicitly considered spatial variation by using reduced values for standard deviations prior to carrying out the analysis. Using reduced standard deviation to account for spatial variation was also adopted by many others, such as Christian et al. (1994), El-Ramly et al. (2002). In this section, the James Bay dike is re-visited and studied using spatial-autocorrelation method to model the spatial variability of the clay foundation. The spatial variation is explicitly modeled in tandem with reliability analysis.

4.7.5.1 Basic soil parameter uncertainty analysis

Figure 4.27 shows a simplified soil profile of the foundation layers, which is composed of 4 m of crust underlain by 8 m of highly sensitive marine clay underlain in turn by 6.5 m of lacustrine clay. The total thickness of the foundation layers is 18.5 m. The undrained shear strengths of foundation clays were obtained from field vane tests. For the single stage construction of the embankment with height of 12 m, uncertainties were observed in several aspects, including uncertainties associated with the soil profile and embankment fill and uncertainties of the undrained shear strengths of foundation clays based on field vane data. Detailed analyses of the above-mentioned uncertainties were given in Christian et al. (1994). In their original study, soil profile uncertainties were considered to the depth of crust and depth to till. The two sources of uncertainties were found to be of marginal influence on the ultimate reliability index as reported in their study. On the other hand, these two sources of uncertainties tend to complicate the reliability analysis since the geometry of the dike has to vary in the analysis. Moreover, knowledge of the statistical distribution of soil profile uncertainties is scarce. As a result, the soil profile uncertainties will not be modeled in this study, and only the uncertainties associated with soil properties will be considered. They are summarized in Table 4.6. The variances of uncertain parameters are divided into two types: spatial variability and systematic errors. According to Christian et al. (1994), these two types of uncertainty contribute to the reliability analysis very differently. The uncertainty caused by systematic error propagates almost unchanged throughout the reliability analysis; in contrast, the uncertainty due to spatial variability tends to cancel out with increase of the volume of soil. Thus they should be handled in different ways, which will be illustrated in the following reliability analysis.

Figure 4.27 Soil profile for a cross section (after Christian et al. 1994)

Table 4.6 Uncertainties of soil properties of James Bay dike (summarized from the data of Christian et al. 1994)

Variable	Mean	Variance	
		Spatial	Systematic
Fill density γ'_{fill}	20 kN/m ³	1	1
Fill friction ϕ_{fill}	30°	1	3
Marine clay c_{uM}	34.5 kPa	39.8	7.6
Lacustrine clay c_{uL}	31.2 kPa	72	24.9

4.7.5.2 Characteristics of spatial variability

From the field vane data tested in James Bay dike area, it is possible to characterize the spatial variability of the foundation clays. The commonly used approach of spatial variability characterization is the autocovariance function. Since the autocovariance

functions for soil properties are different in the vertical and horizontal directions, the vertical autocorrelations should be estimated using testing data from the same boring, and the horizontal autocorrelations using data from different borings at the same elevation. Detailed discussions on the spatial variability characterization for James Bay dike were presented in DeGroot and Baecher (1993) and Christian et al. (1994).

Figure 4.28 shows the autocovariance function fitted to the observed testing data in the Marine clay in vertical direction; the vertical autocorrelation distance is in the range of 1 to 2 m as observed from this plot. Figure 4.29 shows the autocovariance function fitted to the observed testing data in the Marine clay in horizontal direction; the horizontal autocorrelation distance is about 30 m from the DeGroot and Baecher (1993)'s analysis. There is no description on spatial variability characterization in the Lacustrine clay. However, it is reasonable to assume that the Lacustrine clay may show similar spatial variability to the Marine clay since the two clay layers are in the same site and experience the same geological processes. Also not available is the spatial variability of the embankment fill properties. Considering the fact that the embankment fill is probably man-made materials which could be very homogeneous, this study will take the embankment fill as perfect correlation materials, i.e., spatial variability will not be considered for the embankment fill. Similar considerations are also found in other studies, such as El-Ramly et al. (2002).

Figure 4.28 Vertical autocovariance function estimated from field vane testing data taken from a typical boring (after Christian et al. 1994)

Figure 4.29 Horizontal spatial variability characterization from field vane testing data: (a) sampling plan; (b) autocovariance function (after DeGroot and Baecher 1993)

4.7.5.3 First-order reliability analysis

The stability analysis of James Bay dike is carried out using the Spencer's method with half-sine inter-slice force function, which is equivalent to the Morgenstern-Price method. The stability model is set up in a spreadsheet so that the first-order reliability analysis can be directly coupled with the stability model. The spreadsheet set-up of the stability and reliability analyses of James Bay dike are given in Appendix D. The factor of safety based on mean values of soil parameters is found to be 1.473. This value is close to those reported by others, e.g., 1.453 by Christian et al. (1994) and 1.46 by

El-Ramly et al. (2002). The slight difference may be due to the use of different methods and to the modeling errors among different investigations. The critical circular slip surface is found tangent to the bottom of Lacustrine clay (Figure 4.30). This is also consistent with past studies. Thus, the deterministic stability model set-up in this study is sufficiently accurate, so that it may provide a good basis for the following reliability analysis.

Deterministic stability analysis: $F_s = 1.473$

Figure 4.30 Deterministic critical slip surface of James Bay dike

Based on the above uncertainty analyses of soil parameters, spatial variability should be considered to the undrained shear strengths of Marine clay and Lacustrine clay. Since these two clays are different types of materials, there is no cross-correlation between the undrained strengths of the two clays, i.e., spatial variability is only considered within the same clay layer. Considering the fact that the vertical autocorrelation distance is between 1 to 2 m, an intermediate value of 1.5 m is used in this study. Accordingly, the maximum spacing between mid-points of slice-bases is refined to be less than 1.5 m.

The method of autocorrelated slices is used to model the 2-D spatial variation in James Bay dike since both stability model and reliability analysis are set up in the same spreadsheet. For the embankment fill, since perfect spatial-autocorrelation is considered, two random variables separately accounting for the spatial uncertainty and

systematic uncertainty are used for each uncertain parameter. The spatial uncertainty is inherent of geotechnical materials and is related to the location of soils, while the systematic uncertainty is caused by bias and errors in sampling and measurement processes. Thus the two uncertainties are independent to each other. For the Marine and Lacustrine clays, discretization of the random fields has to be carried out to model their spatial variability. According to the discretization of slip surface (i.e., number of autocorrelated slices), 12 random variables are used to model the spatial uncertainty of Marine clay and 16 random variables are used to model the spatial uncertainty of Lacustrine clay. Also, the spatial uncertainties are independent on the systematic uncertainties.

Based on the above treatment of spatial and systematic uncertainties, there are a total of 34 random variables to be considered in the reliability analysis of James Bay dike. Using square-root exponential autocorrelation function, the correlation matrix which accounts for spatial autocorrelations is obtained. The spreadsheet set-up for FORM based on the correlation matrix is shown in Appendix D. The results of reliability analyses with respect to several cases of spatial variation are shown in Table 4.7.

For the case of $\delta_x = 30$ m and $\delta_y = 1.5$ m, as obtained from the statistical analyses of field vane data, the reliability index is 2.913 corresponding to a failure probability of 0.19%. Note that this value accounts for the real 2-D effect of spatial variability. In contrast, some others studying the James Bay dike did not properly consider the 2-D effect of spatial variability. For example, El-Ramly et al. (2002) assumed that the clays have an isotropic autocorrelation distance of 30 m and reported their value of failure probability to be 0.24% based on Monte Carlo simulation, and Christian et al. (1994) reported their value of reliability index to be 2.66 based on FOSM analysis where spatial variability is considered against the length of slip surface (1-D case). Since different studies yield slightly different factors of safety, which in turn lead to different reliability indices or failure probabilities, it is difficult to make comparison between the reliability analyses by different studies. Hence, a further investigation of the James Bay dike assuming isotropic spatial variability (1-D) with spatial autocorrelation distance

equal to 30 m is also conducted in this study. The reliability index is then found to be 2.552 yielding a failure probability of 0.54%. This 1-D consideration overestimates the failure probability by almost 300%, in comparison with the real 2-D analysis.

Table 4.7 Reliability analyses of James Bay dike accounting for 2-D spatial variation

Autocorrelation distances (m)	Reliability index β	Failure Probability (%)	F_s based on mean values	Method of analysis
$\delta_x = 30, \delta_y = 30$	2.66	0.39	1.453	FOSM by Christian et al. (1994)
$\delta_x = 30, \delta_y = 30$	/	0.24	1.46	MC by El-Ramly et al. (2002)
$\delta_x = 30, \delta_y = 30$	2.552	0.54	1.473	FORM by this study
$\delta_x = 30, \delta_y = 1.5$	2.913	0.18	1.473	FORM by this study

4.8 Summary and conclusions

Slope reliability analysis incorporating 2-D spatial variability was carried out with search for the probabilistic critical slip surface. It was found that the probabilistic critical slip surface may not necessarily coincide with the deterministic critical slip surface, especially when spatial variation is modeled.

Influence of 2-D spatial variability of soil properties on the slope reliability was investigated. Two methods, namely the method of autocorrelated slices and the method of interpolated autocorrelations, were presented. The first method is directly related to the limit equilibrium method of slices which is widely used in slope stability

analysis. Uncertain soil properties at the mid-points of the bases of slices are modeled as spatially correlated random variables. Based on this method, good agreement was obtained between FORM results and Monte Carlo simulations using @RISK for slope reliability with 2-D spatial variation. This study shows that the probability of failure may be considerably overestimated when spatial variation of shear strength is ignored, that is, a slope may be much safer than it is predicted without consideration of spatial variation. Furthermore, it was shown that the influence of the vertical spatial variation on β and P_f is much more significant than the influence of the horizontal spatial variation, when the horizontal autocorrelation distance δ_x is an order of magnitude bigger than the vertical autocorrelation distance δ_y .

The second method (interpolated autocorrelations) for slope reliability analysis involving spatial variation was shown to be as good as the method of autocorrelated slices. In addition, it possesses some remarkable advantages: (1) once the discretization scheme has been decided, the structure of spatial correlation matrix and number of random variables do not change during the search for the critical slip surface (in cases where limit equilibrium method of slices is used for slope reliability analysis), so that search for the probabilistic critical slip surface accounting for spatial variation can be done efficiently; (2) one is not confined to just limit equilibrium methods of slices in performing slope reliability analysis; a FORM-based strength-reduction FEM or FDM analysis of slopes may be used as well.

Finally, two case studies which involve 2-D spatial variability characterization from in-situ testing data were reviewed. Based on the reported 2-D spatial variability parameters, this study conducted a new reliability analysis of the well-know James Bay dike.

Chapter 5 Three-Dimensional Limit Equilibrium & Finite Difference Slope Stability Analysis Accounting for Longitudinal Spatial Variation

5.1 Introduction

The earth slope stability has been extensively studied by plane strain analysis or two-dimensional (2-D) analysis. A more rational study should be carried out using three-dimensional (3-D) analysis, since slope failures are 3-D in reality. However, compared to the prevailing 2-D studies, the 3-D stability analysis is less commonly conducted. The main causes for the reluctance of using 3-D analysis are probably due to: first, the 2-D modeling method can significantly simplify the analysis; and second, most studies show that 3-D modeling of slope stability tends to give rise to a higher factor of safety, implying that the 2-D analysis is generally a conservative method.

The 3-D limit equilibrium method for slope stability analysis develops mainly by extending the existing 2-D limit equilibrium methods. For example, Baligh & Azzouz (1975) and Azzouz & Baligh (1983) studied a cohesive soil slope using 3-D analysis that is developed from the 2-D circular arc method. They found that the 3-D effect can lead to as much as 40% increase in the factor of safety. Many others used the concept of limit equilibrium methods of columns in dealing with 3-D analysis. However, various assumptions about both the inter-column forces and the locations and shapes of sliding surface are required (Hovland 1977; Chen and Chameau 1983; Hungr 1987; Leshchinsky and Huang 1992; Lam and Fredlund 1993; Huang and Tsai 2000; Huang et al. 2002; Cheng and Yip 2007; among others).

On the other hand, some researchers performed 3-D stability analyses using strength-reduction finite element method and made comparisons with LEM analysis (e.g. Ugai and Leshchinsky 1995; Deng et al. 2007; Griffiths and Marquez 2007; Li et al. 2009, 2010). The 3-D strength-reduction stability analysis is simpler in concept and does not involve extra assumptions as compared to the 2-D strength-reduction method. Some limitations of the 3-D strength-reduction stability analysis, such as the lack of universal convergence criteria, were pointed out by Wei et al. (2009)

While most 3-D stability analyses were conducted in a deterministic way, some others used probabilistic methods to study the 3-D slope failures (Vanmarcke 1977b, 1980, 2011; Yüçemen and Al-Homoud 1990; Griffiths et al. 2009; Hicks and Spencer 2010). The initial work by Vanmarcke reveals that there would be a probabilistic critical width of failure mass for long earth slopes due to the longitudinal (out-of-plane) spatial variability of soil properties. The probabilistic critical failure width may be very small compared to the total length of the long slope. On the other hand, since the probabilistic critical failure width may occur randomly along the overall length, a system failure should be examined. The system reliability analysis showed that the stability of long earth slope depends on the total length.

A more straightforward way for system failure analysis is the Monte Carlo simulations. For example, Griffiths et al. (2009) carried out probabilistic 3-D slope stability analysis using the random finite element method (RFEM). The spatial variation of strength parameters is considered in the out-of-plane direction. It was found that the longer the slope in the out-of-plane direction, the more likely it is that the 2-D analysis will underestimate the probability of failure. Hicks and Spencer (2010) carried out probabilistic analysis of a long slope cut by 3-D finite element modeling, which also showed that the length of slope affects the performance of 3-D stability. However, the probabilistic strength-reduction stability analyses by MC simulations all yields system

failure probability and fails to study the local failure with critical failure width, as rightly pointed out by Vanmarcke (2011).

This study investigates the probabilistic 3-D slope stability by both LEM and strength-reduction method (using a finite difference numerical package FLAC 3D). The probabilistic analysis accounts for longitudinal spatial variability of shear strength parameters. The first-order reliability method (FORM) is the basis for the probabilistic analysis which aims at investigating the local-failure but not the system-failure.

5.2 Probabilistic limit equilibrium analysis of 3-D slope stability

5.2.1 Idealized 3-D model of earth slope failure

A pioneer work of probabilistic limit equilibrium analysis of 3-D slope stability using an idealized 3-D failure model can be traced to Vanmarcke (1977b, 1980). The model assumes that the 3-D failure surface is cylindrical and extends along the longitudinal axis over a finite width b . The cylindrical failure mass has vertical cut-offs at two ends. Moment equilibrium of the cylindrical failure mass about an axial line is the basis to derive the factor of safety for 3-D failure, F_{3D} . The idealized 3-D model for limit equilibrium analysis is illustrated by Figure 5.1.

For a 3-D failure mass with width equal to b , the factor of safety F_{3D} , defined as the ratio of the resisting moment to the overturning moment, is written as

$$F_{3D} = \frac{\sum M_R}{\sum M_O} = \frac{M_R b + 2M_{ER}}{M_O b} \quad (5.1)$$

where M_R and M_O are the resisting and overturning moments by plane strain analysis (moment per unit longitudinal length), respectively, M_{ER} is the resisting moment resulted from shear of the end section.

Figure 5.1 An idealized model of three-dimensional stability failure

The probabilistic analysis of 3-D slopes can be easily done when the performance function is explicitly known, i.e., a closed-form solution of the factor of safety, F_{3D} is available. Clearly, only limit equilibrium analysis on some specific slope models can attain such a closed-form solution. For example, when the slope is homogeneous and in undrained analysis, closed-form solutions of M_R and M_O are available. The details are presented in the Appendix G. Since only c_u contributes to the resisting moment, the end-section resisting moment M_{ER} can be obtained by integrating the resisting moment of dA or $dx dy$ over the area of end section, as shown in Figure 5.2.

$$M_{ER} = \iint_A c_u \sqrt{(x - x_c)^2 + (y - y_c)^2} dx dy \quad (5.2)$$

Also, the same value of M_{ER} can be obtained by integrating the resisting moment of dr . However, both the integration function and boundary condition are more complicated by doing so. Thus, Eq. (5.2) is recommended for the calculation of M_{ER} in this study.

Figure 5.2 Two ways to compute end-section resisting moment in undrained analysis

5.2.2 Deterministic analysis

The above-mentioned 3-D model (Figure 5.1) is applied to a hypothetical earth slope. Suppose the slope is inclined at 1.5H:1V, the height of the slope H is 5 m and the depth of foundation D is 2 m. The undrained strength c_u is 25 kPa. A plane strain stability analysis with search for the critical slip circle obtains a factor of safety F_s of 1.596. Based on the cylindrical failure mass as extended axially from the plane strain critical slip circle, 3-D stability analyses are performed and the results are shown in Figure 5.3.

Clearly, the F_{3D} which considers the end-section effects is consistently larger than the F_s by plane strain analysis. The difference between F_{3D} and F_s can be very significant when the width of failure mass b is small. For example, F_{3D} could be about 1.5 times of its plane strain counterpart F_s when the width b is at $2H$. The difference rapidly and consistently reduces to a negligible value with increase of the width b . This finding, although based on an idealized 3-D model with undrained analysis, is highly consistent with past studies using more sophisticated methods of 3-D analysis.

From the observation of deterministic analysis, it seems that the larger the width of failure mass, the smaller the factor of safety by 3-D analysis. It implicitly indicates that a 3-D slope failure could take place with sliding mass as wide as possible. However, a 3-D slope failure as observed in a real world is only of a certain extent, which seems to conflict with the above deterministic analysis. Regarding the finite-width 3-D failures as observed in reality, Duncan and Wright (2005) wrote “many 3-D failures occur because soil properties and pore-water pressures vary along the length of the slope (i.e., subsurface conditions and soil properties are not uniform)” and “3-D slope failures may also occur due to the 3-D geometry of the slope”. The first scenario caused by the variation of soil properties is investigated in detail in the next subsection.

Figure 5.3 Three-dimensional stability analysis with different width of failure mass

5.2.3 Probabilistic analysis considering the spatial variation

Consider the undrained strength c_u as the main source of uncertainty in this example of a homogeneous slope. A typical value of coefficient of variation of 0.3 is assumed for c_u , and also, the c_u conforms to normal distribution. Moreover, c_u is taken as a random field over the longitudinal direction, which could be expressed using a longitudinal autocorrelation distance δ_z , and an exponential autocorrelation model. Other soil

properties such as unit weight are treated as deterministic parameters.

5.2.3.1 FORM analysis based on spatial-averaging of the random field

The spatial-averaging approach is based on variance reduction of random field over a local interval or area as proposed by Vanmarcke (1983). For the present study, the shear strength c_u is taken to be a random field. For the example 3-D slope, the uncertainty of c_u acting on the cylindrical failure surface and the uncertainty of c_u acting on the two end sections should be treated separately when variance reduction method is used. For a given interval T over the longitudinal direction, the standard deviation of c_u on the cylindrical failure surface considering longitudinal spatial variation is simply obtained by applying the reduction factor $\sqrt{\Gamma(T)}$ to the point-wise or original standard deviation. In contrast, there is no variance reduction (by $\sqrt{\Gamma(T)}$) to c_u on the two end sections which are vertical cutoffs (on a vertical cutoff, $T = 0$ and $\Gamma(T) = 1$). As a result, two different types of correlation are involved: correlation between spatially averaged random variables and correlation between point-wise random variable and spatially averaged variable; the latter is a special case of the former.

Correlation between spatially averaged c_u over the cylindrical failure surface

In the spatial-averaging approach, the longitudinal variation of c_u on the cylindrical failure surface is modeled by a number of local averages. If the averaging interval T is constant and uniform, the correlation coefficient between any two local averages is (Vanmarcke, 1983)

$$\rho[c_{u_T}, c_{u_T}] = \frac{\sum_{k=0}^3 (-1)^k \Delta(T_k)}{2\Delta(T)} \quad (5.3)$$

where c_{u_T} is a spatially averaged c_u over interval T , the interval T_k is as defined in Figure 5.4. The details are referred to Chapter 2, section 2.6.3.

General case:

$$\rho[c_{u_T}, c_{u_{T'}}] = \frac{\sum_{k=0}^3 (-1)^k \Delta(T_k)}{2[\Delta(T)\Delta(T')]^{1/2}}$$

Special case with uniform intervals, i.e., $T = T'$:

$$\rho[c_{u_T}, c_{u_{T'}}] = \frac{\sum_{k=0}^3 (-1)^k \Delta(T_k)}{2\Delta(T)}$$

where, $\Delta(T) = T^2 \Gamma(T)$

Figure 5.4 Correlation between local averages of c_u over the cylindrical failure surface

Correlation between end-section c_u and cylindrical failure surface c_u

The covariance between the point-wise end-section $c_{u_{\text{end}}}$ and the spatially averaged c_{u_T} on the cylindrical failure surface could be computed by (Vanmarcke 1983)

$$\text{Cov}[c_{u_{\text{end}}}, c_{u_T}] = \frac{\sigma^2}{2T} [\dot{\Delta}(T_2) - \dot{\Delta}(T_1)] \quad (5.4)$$

where $\text{Cov}[c_{u_{\text{end}}}, c_{u_T}]$ is the covariance of the point-wise end-section $c_{u_{\text{end}}}$ and the spatially averaged c_{u_T} (local average), the intervals T_1 and T_2 are as defined in Figure 5.5.

From statistical mathematics, the correlation coefficient is given by

$$\rho[c_{u_{\text{end}}}, c_{u_T}] = \frac{\text{Cov}[c_{u_{\text{end}}}, c_{u_T}]}{\sigma[c_{u_{\text{end}}}] \cdot \sigma[c_{u_T}]} = \frac{[\dot{\Delta}(T_2) - \dot{\Delta}(T_1)]}{2T\sqrt{\Gamma(T)}} \quad (5.5)$$

where $\sigma[c_{u_{\text{end}}}]$ is the standard deviation of the end-section $c_{u_{\text{end}}}$, and $\sigma[c_{u_{\text{end}}}] = \sigma$; $\sigma[c_{u_T}]$ is the standard deviation of the spatially averaged c_{u_T} , and $\sigma[c_{u_T}] = \sigma\sqrt{\Gamma(T)}$; $\dot{\Delta}(\cdot)$ is the first derivative of the variance function $\Delta(\cdot)$, such that

$$\Delta(T) = T^2\Gamma(T) \quad (5.6)$$

$$\dot{\Delta}(T_1) = \left. \frac{d[\Delta(T)]}{dT} \right|_{T=T_1} \quad (5.7)$$

The widely used variance reduction function based on exponential autocorrelation model is used, such that

$$\rho(T) = e^{-T/\delta_z} \quad (5.8)$$

$$\Gamma(T) = 2 \cdot \left(\frac{\delta_z}{T}\right)^2 \left(e^{-T/\delta_z} + \frac{T}{\delta_z} - 1\right) \quad (5.9)$$

Based on this model, Eqs. (5.3) and (5.5) are finally shown to be

$$\rho[c_{u_T}, c_{u_{T'}}] = \frac{1 - e^{-b/\delta_z}}{\sqrt{2(e^{-b/\delta_z} + b/\delta_z - 1)}} \quad (5.10)$$

$$\rho[c_{u_{\text{end}}}, c_{u_T}] = \frac{\delta_z(e^{-T_1/\delta_z} - e^{-T_2/\delta_z})}{T\sqrt{\Gamma(T)}} \quad (5.11)$$

The correlation between the two end-section c_u values is simply obtained by

$$\rho[c_{u_{\text{end}1}}, c_{u_{\text{end}2}}] = e^{-b/\delta_z} \quad (5.12)$$

From the above random field analysis, a typical spreadsheet set-up for the 3-D slope reliability analysis based on spatial-averaging approach is shown in Figure 5.6, where the width of 3-D failure mass b is taken to be 30 m, and longitudinal autocorrelation distance δ_z being 5 m. Parametric studies on both varied b and δ_z values shall be conducted in the following sections.

General case:

$$\rho[c_{u_{\text{end}}}, c_{u_T}] = \frac{[\dot{\Delta}(T_2) - \dot{\Delta}(T_1)]}{2T\sqrt{\Gamma(T)}}$$

Special case with uniform intervals, i.e., $T_1 = 0$, $T_2 = T$:

$$\rho[c_{u_{\text{end}}}, c_{u_T}] = \frac{\dot{\Delta}(T)}{2T\sqrt{\Gamma(T)}}$$

where, $\Delta(T) = T^2\Gamma(T)$ and $\dot{\Delta}(T_1) = \left. \frac{d[\Delta(T)]}{dT} \right|_{T=T_1}$

Figure 5.5 Correlation between point-wise end-section c_u and spatially-averaged c_u over the cylindrical failure surface

5.2.3.2 FORM analysis based on spatial-autocorrelation of the random field

In the study of longitudinal spatial variation in probabilistic 3-D slope stability analysis, an alternative to the spatial-averaging approach is the spatial-autocorrelation approach, which directly uses the original variances of random variables and autocorrelation function. The principle of spatial-autocorrelation approach is to use finite control points or discrete points to discretize the random field. c_{ui} 's at these discrete points are treated as spatially correlated random variables subjected to the correlation function as shown in Eq. (5.13). c_u values at other locations are linearly interpolated from these spatially correlated random variables at discrete points. Also, the spatial-autocorrelation approach is the basis of the method of interpolated autocorrelations as used in Chapter 4 for 2-D random field discretization.

$$\rho[c_{ui}, c_{uj}] = e^{-|z_i - z_j|/\delta_z} \quad (5.13)$$

where z_i and z_j are the z -coordinates of the control points or discrete points i and j , respectively.

A typical spreadsheet set-up for the 3-D slope reliability analysis based on spatial-autocorrelation approach is shown in Figure 5.7, where model parameters are the same as those in Figure 5.6.

Figure 5.6 Probabilistic three-dimensional stability analysis by spatial-averaging approach

Figure 5.7 Probabilistic three-dimensional stability analysis by spatial-autocorrelation approach

5.2.4 Comparison of reliability analyses by spatial-averaging and spatial-autocorrelation approaches

In Figure 5.8, reliability indices β were computed as a function of the normalized spatial-averaging interval T/δ_z or normalized separation distance s/δ_z between adjacent spatial-autocorrelations. Note that the x -axis has been plotted in log-scale. A small value of T/δ_z and/or s/δ_z means a large number of spatially averaged c_{uT} (local

averages) and/or discrete points, and vice versa. It can be seen that the reliability index β obtained based on the spatial-autocorrelation method is consistent with that obtained based on the spatial-averaging method, at small values of T/δ_z and/or s/δ_z , for both *Normal* random field of c_u (as shown in Figure 5.8a) and *Lognormal* random field of c_u (as shown in Figure 5.8b). As the interval of local averages T/δ_z or separation distance s/δ_z reduces to an infinitely small value, the two curves obtained by spatial-averaging approach and by spatial-autocorrelation approach meet each other. This is logical, because an infinitely small interval is equivalent to a discrete point, so that the correlation between local averages and autocorrelation between discrete-point random variables are the same. Moreover, the variance reduction factor to be applied to each local average will be '1' in this circumstance. Figure 5.8a shows that in the case of normally distributed random field the spatial-averaging approach yields accurate β value while spatial-autocorrelation approach tends to underestimate β ; Figure 5.8b, on the other hand, shows that in the case of lognormally distributed random field, the spatial-averaging approach tends to overestimate the β , while the spatial-autocorrelation approach still underestimate the β . In general, the opposite trends are caused by the underlying difference between the two approaches. The variance of the random variables in the spatial-autocorrelation approach is equal to the point-wise variance of the random field, whereas in the spatial-averaging method the variance is reduced through a variance function. Naturally the higher variance of the random variables produces a lower value of the reliability index. In either case, it may be concluded that the spatial-autocorrelation approach always yields a lower estimate of β than the spatial-averaging approach.

Figure 5.8 Reliability indices computed using spatial-autocorrelation and spatial-averaging approaches: (a) random field of c_u follows *Normal Distribution*; (b) random field of c_u follows *Lognormal Distribution*

Figure 5.9a and Figure 5.9b show the relationships between reliability index β and longitudinal autocorrelation distance δ_z for the spatial-averaging method with five different schemes of averaging. The results obtained based on 1, 5, 9, 13 and 41 local averages are shown. The curve corresponding to 41 local averages is taken as the benchmark for comparison. For the case of *Normal* random field of c_u , the accuracy of

the spatial averaging method is totally independent of the autocorrelation distance δ_z and the number of local averages used; for the case of *LogNormal* random field of c_u , the accuracy of the spatial averaging method is slightly affected by both.

Figure 5.10a and Figure 5.10b show the relationships between reliability index β and longitudinal autocorrelation distance δ_z for the spatial-autocorrelation method with five different schemes of discretization. The results obtained based on 2, 6, 10, 14 and 42 local averages are shown. The curve corresponding to 42 discrete points is taken as the benchmark for comparison. Compared with the spatial-averaging method, the accuracy of spatial-autocorrelation method is more dependent on the number of discrete points used, whether c_u is modeled by *Normal* or *LogNormal* random field. On the other hand, the number of discrete points to be used becomes less important when it exceeds a certain minimum. For example, using 6 discrete points can achieve practically the same accuracy as using 42 discrete points. The error is below 5 per cent. For the case in hand, using 6 discrete points means the uniform separation distance s is $30/6 = 5$ m, which is equal to the smallest δ_z among the four values investigated. Based on the above observations, the separation distance between discrete points should be less than the autocorrelation distance to achieve relatively good accuracy. This observation is consistent with the findings by Low (2001) and Ji et al. (2012).

(a)

(b)

Figure 5.9 Reliability indices as functions of longitudinal autocorrelation distance computed using spatial-averaging approach: (a) random field of c_u follows *Normal Distribution*; (b) random field of c_u follows *Lognormal Distribution*

Figure 5.10 Reliability indices as functions of longitudinal autocorrelation distance computed using spatial-autocorrelation approach: (a) random field of c_u follows Normal Distribution; (b) random field of c_u follows Lognormal Distribution

5.2.5 On the probabilistic critical width of 3-D failure mass

The discussions presented above are solely based on a 3-D failure mass with width b equal to 30 m. It is interesting to perform more extensive parameter studies by varying

the width b . For simplicity and to exclude potential inaccuracy caused by random field discretization, the following discussions are based on the *Normal* random field of c_u (corresponding to which the reliability analysis is accurate when spatial-averaging approach is used, see Figure 5.9a).

The reliability indices obtained by both spatial-averaging approach and spatial-autocorrelation approach are shown in Figure 5.11. First consider the case of perfect autocorrelation, i.e., the probabilistic analysis without modeling the longitudinal spatial variation. In this case, there is no reduction to the point-wise variance if the spatial-averaging approach is used. The reliability index was found to rapidly and consistently reduce to a constant value when the width of failure mass increases infinitely. The physical meaning is that the larger the width of failure mass, the smaller the reliability index and thus the bigger the probability of failure. In other words, the probabilistic 3-D stability analysis without modeling spatial variation also indicates that the wider the sliding mass, the more likely the slope will fail. This is somewhat similar to the previous finding by deterministic analysis as indicated by Figure 5.3.

When the autocorrelation distance is modeled as 5 m (which is a very typical value for geotechnical materials), a minimum value of the reliability index was found when the normalized width of failure mass equals to 1.94 (or $b = 9.68$ m). The width of failure mass that yields minimum reliability index is termed the *probabilistic critical width* b^* . In this case, the implication is that there exists a certain width at which the 3-D slope will most probably fail. This finding is very different from those of the deterministic analysis and the probabilistic analysis without spatial variation, and in particular, this finding is more consistent with real 3-D slope failure. Another three typical cases of autocorrelation distances δ_z at 10, 15 and 20 m, respectively, are investigated. The results are also shown in Figure 5.11. It is clear that using a larger value of δ_z in

probabilistic 3-D stability analysis consistently yields a lower value of reliability index or a higher probability of failure.

Figure 5.11 Reliability indices computed as functions of width of 3-D failure mass for four cases of δ_z : (a) by spatial-averaging approach; (b) by spatial-autocorrelation approach

Figure 5.12 compares the two approaches for individual longitudinal autocorrelation distance. All the four cases show that the results by the two approaches agree with each other very well. In Figure 5.12a, for the case where the autocorrelation distance is

5 m (a relatively small value), there is some discrepancy between the results by the two different approaches. In general, the reliability index by the spatial-autocorrelation approach is slightly lower than that by the spatial-averaging approach. However, with increasing the autocorrelation distance, the agreement between the two approaches becomes better. This is because the relative density of discrete points (which is measured as the ratio of autocorrelation distance to the spacing between discrete points) becomes larger. In general, the finer the discretization scheme, the more accurate the estimate by the spatial-autocorrelation approach. Based on this conclusion, the slight difference as shown in Figure 5.12 for longitudinal autocorrelation distance at 5 m (coarse) could be reduced even further by increasing the number of discrete points or decreasing the spacing between them. For example, Figure 5.13 shows a further study by the spatial-autocorrelation approach, where the spacing is reduced from 5 m to 3 m (refined). Obviously, using a refined scheme of discrete points indeed yields a curve that is closer to the one by spatial-averaging approach. On the other hand, more discrete points have to be used and thus more computational efforts are incurred in a refined scheme. In fact, the improvement of accuracy becomes minor when the spacing between discrete points falls below a certain value.

(b)

(c)

(d)

Figure 5.12 Reliability indices as functions of width of failure mass b computed by spatial-averaging method and spatial-autocorrelation method: (a) longitudinal autocorrelation distance $\delta_z = 5$ m; (b) $\delta_z = 10$ m; (c) $\delta_z = 15$ m; (d) $\delta_z = 20$ m

Figure 5.13 Refining the discrete points to improve the reliability analysis with spatial-autocorrelations

Moreover, it is found that the longitudinal autocorrelation distance δ_z significantly influences the computed probabilistic critical width of failure mass b^* , as shown in Figure 5.14. The b^* almost proportionally increases with δ_z for the four cases analyzed.

It is noted that the above analyses based on FORM use the reliability index β for probabilistic assessments. The probability of failure based on FORM could be estimated using the well-established formula $P_f \approx \Phi(-\beta)$. It is desirable to compare the estimated P_f with that by Monte Carlo (MC) simulations to verify the results of FORM. For the example 3-D slope, such a comparison is feasible since the closed-form solution of the problem makes it easy to carry out the MC simulations. The comparison is shown in Figure 5.15, where δ_z is equal to 20 m. Obviously, the P_f estimated by FORM agrees very well with that by MC simulations with Latin Hypercube sampling technique (a commercial software @RISK was used for the MC simulations and the simulation was repeated five times, each with 500,000 iterations). Further investigation showed that the consistency between FORM and MC simulations holds for other cases of δ_z .

Figure 5.14 Relationship between probabilistic critical failure width and autocorrelation distance

Figure 5.15 Comparison between the probabilities of failure by FORM and by Monte Carlo simulations for the case of δ_z equal to 20 m

5.2.6 Discussion on the system reliability of long earth slopes

From the above analyses, it was shown that a probabilistic critical width of failure mass b^* exists for a long earth slope due to the longitudinal spatial variation of soil properties. The risk analysis with respect to a particular b or b^* is in fact a local-failure-risk analysis.

Another aspect of risk analysis for a long earth slope is the ‘system reliability’, which considers that the event of 3-D slope failure with width of b^* can occur randomly along the longitudinal axis. Each local-failure with width of b^* is a component of the system failure. Obviously, when the overall length of a 3-D slope B is larger than the probabilistic critical width of failure mass b^* , the chance of one or more failures somewhere along the slope axis and hence the probability of ‘system failure’ steadily increases with the overall length B (Vanmarcke 1980, 2011). Denoting the slope reliability of a local-failure as β_{b^*} or $P_f(b^*)$, and the reliability of the ‘system failure’ as β_B or $P_f(B)$, respectively, Vanmarcke (2011) showed that

$$P_s(B) = \begin{cases} [1 - P_f(b^*)] \exp\{-(B - b^*)v(\beta_{b^*})\}, & B \geq b^* \\ 1 - P_f(b^*), & B \leq b^* \end{cases} \quad (5.14)$$

where $P_s(B)$ is the survival probability, and $P_s(B) = 1 - P_f(B)$; $v(\beta_{b^*})$ is the rate of decay of ‘system reliability’ per unit length along the slope axis, and it was shown to be

$$v(\beta_{b^*}) = \frac{\sqrt{1 - \rho(b^*)}}{\pi\sqrt{2}b^*\Gamma(b^*)} \exp(-\beta_{b^*}^2/2) \quad (5.15)$$

where $\rho(b^*)$ and $\Gamma(b^*)$ are the autocorrelation function and variance reduction function evaluated at separation distance b^* , respectively.

For the example 3-D slope with exponential autocorrelation function and autocorrelation distance of 5 m, the system reliability could be estimated using the above two equations, and it is plotted against the overall length B , as shown in Figure 5.16. That the system failure probability $P_f(B)$ almost linearly increases with the overall length B is only valid when the $P_f(B)$ is well below 1 (Vanmarcke 2011). Otherwise, Monte Carlo simulations have to be carried out for the determination of system failure probability.

Figure 5.16 System reliability against the overall length of slope

From the above studies, it is concluded that the longitudinal spatial variation of undrained strength can be modeled using either spatial-averaging approach proposed by Vanmarcke (1977b, 1980) or the spatial-autocorrelation approach with linear interpolations for the random field proposed in this study. When the two methods are used under the same conditions, the results by the two methods are highly consistent. When using the spatial-averaging method to study the reliability of the 3-D slope stability, it is important to correctly compute the correlation between the point-wise end-section c_u and the spatially averaged c_u on the cylindrical slip surface. Usually, a thorough understanding of the random fields theory is required to correctly use this method when the model of analysis is complicated. In contrast, the method of spatial-autocorrelation is more straightforward in concept and simpler to use. Moreover, the method is easier to accommodate the modeling of spatial variation by using other sophisticated procedures such as strength reduction finite element or finite difference method for slope stability analysis. This will be investigated in the subsequent sections.

5.3 Deterministic 3-D slope stability analysis by strength reduction method

5.3.1 Problem description

So far, the 3-D slope stability analyses were discussed with limit equilibrium analysis (based on an idealized 3-D stability model). As a complement, the following study will examine the problem using strength reduction finite difference method (SRFDM or SRM for short). Consider again the 3-D homogeneous earth slope with geometry shown in Figure 5.1 and Figure 5.2. The height of slope H is 5 m with foundation depth D of 2 m. For SRM stability analysis, the elastic-perfectly plastic constitutive model with Mohr-Coulomb failure criterion is used. As such, elastic modulus E and Poisson's ratio ν are required and they are assumed to be 8 MPa and 0.3 (drained analysis) or 0.495 (undrained analysis), respectively, which are typical for clay soil.

In this study, the well-known commercial software FLAC 3D version 3.1 is adopted for the SRM factor of safety analysis. To determine the critical factor of safety F_s , FLAC 3D uses the non-convergence failure criterion (unique to factor of safety analysis, not to be confused with the Mohr-Coulomb failure criterion). For each step of strength reduction, the FLAC 3D computes the total out-of-balanced force; at a calculation step where the computed total out-of-balanced force cannot converge to a prescribed small limit, e.g., 1×10^{-5} by default, the reduction factor is taken to be the critical F_s .

5.3.2 Modelling techniques for 3-D stability analysis

It is noted that FLAC 3D version 3.1 as used by the author requires much computational effort especially when the width of 3-D failure mass is large (since more FDM grid points have to be used). Thus, it is desirable to use a relatively coarse mesh of grid. For the problem in hand, trial-and-error has been conducted to find a suitable

mesh size that achieves a balance between accuracy and computational effort. Figure 5.17 shows a typical mesh of grid for undrained analysis. It is recommended to refine the mesh in the lower part (slope foundation) where failure zones are most likely to develop. In contrast, the mesh for drained analysis is refined in the upper part, as shown in Figure 5.18. In short, the mesh should be more refined in those areas where failure zones tend to develop. In this way, the numerical analysis can be more efficient.

The boundary conditions need to be applied appropriately in 3-D stability analysis. Past studies showed that the F_s by 3-D analysis may be very sensitive to the end-section boundary conditions applied (Chugh 2003). Griffiths and Marquez (2007) considered the displacement boundaries at bottom and both end-sections fixed (fixed end-section), and other displacement boundaries with roller. The fixed end-section boundary conditions are widely adopted in performing 3-D stability evaluation (by deterministic analysis). One question arising from the use of such boundary conditions is that the 3-D failure is not brought about naturally, but is probably due to the fixed end-section boundary conditions applied. However, in deterministic 3-D analysis the fixed end-section boundary conditions have to be imposed on end sections otherwise it is difficult to model the 3-D failure, especially when there is no weak zones which itself can produce a 3-D failure. As a start, the widely-used fixed end-section boundary conditions are also adopted in the following deterministic 3-D stability analysis unless stated otherwise. The inconsistency caused by fixed end-section boundary conditions can be avoided by probabilistic analysis, which will be discussed in a subsection 5.4.1.

For the deterministic stability analysis of homogeneous soil slopes with symmetric geometries, symmetric analysis could be used to reduce the computational effort. The displacement in longitudinal direction is zero on the plane of symmetry. In this study, symmetric analysis will also be used where appropriate.

Figure 5.17 A typical 3-D mesh of earth slope for undrained analysis

Figure 5.18 A typical 3-D mesh of earth slope for drained analysis

5.3.3 Undrained analysis

One important factor influencing the 3-D slope stability is the end-section effects (or end effects), which are caused by the side shear resistances. Due to the end effects, the factor of safety by 3-D analysis can be very different from that by 2-D (plane strain), and the former is generally found to be larger than the latter. Past studies have shown that the end effects significantly depend on the width of failure mass b . In general, it diminishes with the increase of b value considered. For the example slopes, the end

effects are first examined in undrained analysis. Further studies incorporating drained analysis shall be discussed in the subsection 5.3.4.

To investigate the end effects in 3-D SRM stability analysis, a series of undrained analyses by varying the width of failure mass were carried out. The failure modes with respect to different widths are presented in Figure 5.19, where slope height $H = 5$ m and undrained strength $c_u = 25$ kPa. As expected, the 3-D failure modes are different from the idealized model with vertical cut-off at end section as used in the previous LEM analysis. Concerning the factor of safety F_{3D} , the SRM analyses also show that it decreases and converges to its plane strain counterpart. Figure 5.20 compares the 3-D SRM analysis with the LEM analysis. It is shown that the F_{3D} by SRM is generally lower than that by the previous LEM where an idealized 3-D model of failure mass has been used. The difference is mainly due to the assumptions of vertical cut-offs for the two end sections used in LEM. One may also note that the difference between SRM and LEM analyses is large when the width of failure mass b is small. This is because when b is small, the contribution of end effects to the 3-D stability is large, so that the error (i.e. the difference between the two analyses) caused by the assumptions of end section becomes large.

*Chapter 5 Three-dimensional Limit Equilibrium & Finite Difference Slope Stability Analysis
Accounting for Longitudinal Spatial Variation*

(a) Slope failure in 3-D analysis with normalized width b/H equal to 2

(b) Slope failure in 3-D analysis with normalized width b/H equal to 4

(c) Slope failure in 3-D analysis with normalized width b/H equal to 6

*Chapter 5 Three-dimensional Limit Equilibrium & Finite Difference Slope Stability Analysis
Accounting for Longitudinal Spatial Variation*

(d) Slope failure in 3-D analysis with normalized width b/H equal to 10

(e) Slope failure in 3-D analysis with normalized width b/H equal to 20

Note: in-plane slope height $H = 5$ m; undrained strength $c_u = 25$ kPa; Young's modulus $E_u = 8$ MPa

Figure 5.19 Comparison between the 3-D failure modes with respect to different width of failure mass

Figure 5.20 Comparison between 3-D strength reduction analysis and limit equilibrium analysis

Since the width of failure mass b may not be the only parameter that influences the performance of 3-D slope stability, further investigation was performed by extensive parametric studies. Figure 5.21 shows the results of parametric studies on the long earth slope with three different inclinations, each of which has been evaluated with three normalized undrained shear strengths $c_u/\gamma H$ at 0.2 ($c_u = 19$ kPa), 0.26 ($c_u = 25$ kPa) and 0.3 ($c_u = 28.5$ kPa), respectively. It was found that for each case of the undrained analyses, the factor of safety F_{3D} decreases and converges to a certain value with increasing the b value. The converged value of F_{3D} is practically the same as the factor of safety by 2-D (plane strain). This finding is similar to the 3-D analysis by many others using finite element strength reduction method (e.g. Griffiths and Marquez 2007).

Figure 5.21 Results of undrained analyses for the three cases of slope at: (a) 1H:1V; (b) 1.5H:1V; and (c) 2H:1V

To better describe the end effects in 3-D stability analysis, a *factor of end effects*, f_e , may be defined as

$$f_e = \frac{F_{3D}}{F_s} \quad (5.16)$$

where F_{3D} is the factor of safety by 3-D analysis, F_s is the factor of safety by plane strain analysis (2-D). A large value of f_e means that the 3-D stability analysis is very different from the 2-D one, and vice versa. In particular, when f_e is equal to unity, there is no difference between 3-D and 2-D stability analyses. This extreme case can only occur when the width b as modeled in 3-D analysis is infinitely large.

Plots showing the end effects in the undrained analyses are presented in Figure 5.22. It is very interesting to see that for each of the three slopes, the factor of end effects f_e seems only dependent on the width of failure mass b analyzed, and not on the undrained shear strength c_u at all. In general, the end effects are very significant when the slope is narrow in longitudinal direction and it decreases rapidly with increasing b/H . In particular, the end effects in 3-D analysis can cause more than 40% increase of the factor of safety compared to the F_s by 2-D analysis. This finding by 3-D strength reduction analysis is consistent with past finding by 3-D limit equilibrium analysis.

When the normalized width b/H exceeds a certain value, the end effects will not be so important for 3-D stability analysis. For instance, when b/H is larger than 10, the f_e is less than 1.05, which is very close to 1. In this circumstance, 2-D stability analysis can be used as a substitute for the more complicated 3-D analysis without causing much error, e.g. within 5% in error.

Figure 5.22 Plots of end effects versus slope length in undrained analyses for the three cases of slope at: (a) 1H:1V; (b) 1.5H:1V; and (c) 2H:1V

On the other hand, the end effects show some difference for slopes with different cross-sectional inclinations, as shown in Figure 5.23 for the case with undrained strength $c_u/\gamma H$ at 0.26. On the whole, the steeper the slope, the stronger the end effects.

Based on the above findings, it was concluded that the end effects have no relationship with the undrained shear strength c_u when the long earth slope is homogeneous. In addition, it was also concluded that end effects can severely affect the 3-D stability analysis when the failure mass is relatively narrow in longitudinal direction and the cross-sectional inclination of slope is relatively large. This is somewhat consistent with the work by Lefebvre and Duncan (1973) concluding that neglecting the end effects can give rise to very inaccurate factor of safety especially in narrow slopes with slope surface angles steeper than 20 degrees.

Figure 5.23 Plots of end effects versus width of failure mass and cross-sectional slope inclination

5.3.4 Drained analysis

Consider the 5 m height (H) slope with cross-sectional inclination at 1.5H:1V. Investigation by drained analysis is performed on the 3-D slope model (Figure 5.18), and a typical plot of the failure mass is shown in Figure 5.24.

Note: Slope height $H = 5$ m, width of failure mass $b = 15$ m;
Soil properties $E = 8$ MPa, $\nu = 0.3$, $c = 5$ kPa, $\phi = 30^\circ$.

Figure 5.24 A drained analysis of slope by strength reduction method

In comparison with undrained analysis that only involves a single strength parameter c_u , the drained analysis can be more complicated since parametric studies should be conducted on both cohesion c and friction angle ϕ . In this study, shear strength combinations of cohesion c valued at 0 kPa, 5 kPa and 10 kPa and friction angle ϕ valued at 20° , 25° and 30° are investigated. For each strength combination, end effects evaluated at different b values are shown in Figure 5.25. Similar to the results of undrained analysis, the end effects in drained analysis are more obvious when the width of failure mass in normalized value b/H is relatively small. Also, the threshold value of b/H is around 10, which has also been observed in undrained analysis. However, unlike the undrained analysis where end effects were found independent of

c_u , the end effects in drained analysis seem to be influenced by both cohesion c and friction angle ϕ . The results from drained case studies are summarized below:

- (a) In the case where cohesionless soils are involved ($c = 0$), the influence of end effects were found very minimal. For example, the maximum f_e is only around 1.03. So, for cohesionless soils, 3-D stability analysis does not have advantages over the 2-D analysis regardless of the width of failure mass.
- (b) In the other two cases where cohesive soils are involved, the end effects were found slightly sensitive to the shear strength parameters when b/H is relatively small (e.g. less than 10). With respect to a constant value of cohesion c , a smaller value of friction angle ϕ yields a larger value of the factor of end effects f_e . In contrast, with respect to a constant value of friction angle ϕ , a smaller value of cohesion c may yield a smaller value of the factor of end effects f_e .
- (c) On the other hand, in any of the drained analyses, it is found that the end effects are more dependent on the width of failure mass than on the shear strength parameters.

Note: in-plane slope height $H = 5$ m; slope at $1.5H:1V$

Figure 5.25 Plots of end effects versus slope length in drained analyses

5.4 Probabilistic treatment of long earth slopes by strength reduction method

5.4.1 Critical 3-D failure mode with reasonable end-section boundary conditions

For 3-D FEM/FDM stability analysis, the boundary conditions should be applied appropriately, since 3-D failure could be very sensitive to them. Conventional deterministic FEM/FDM stability analysis adopts fixed end-section boundary conditions (displacements are fixed in all three dimensions at the two end sections) to model the 3-D failure mode. The fixed end-section boundary conditions, however, may cause artificial 3-D failure. As such, the smooth end-section boundary conditions (i.e., zero displacement in the longitudinal/out-of-plane direction and free moving otherwise) are particularly appropriate when studying the 3-D failure mode of a long earth slope. The smooth end-section boundary conditions will be used in the following study unless stated otherwise.

Consider a 5 m high slope with length of 50 m, as shown in Figure 5.26. First consider deterministic 3-D strength-reduction FDM analysis with a homogeneous shear strength c_u . The elastic modulus E and Poisson's ratio ν are set to be 8 MPa and 0.495, respectively. With smooth end-section boundary conditions, there is no 3-D failure mode (usually with a finite failure width) produced in the deterministic analysis. The F_{3D} by undrained analysis was found to be 1.58, which is the same as the plane strain F_s . In other words, a long earth slope shall take overall failure based on the deterministic analysis.

In reality, slope failure over the whole length is almost impossible and only a small extent of a long earth slope may fail. Thus, the above deterministic 3-D stability analysis of infinitely long earth slope cannot explain a real slope failure. To overcome

the shortcomings of deterministic 3-D stability analysis, a probabilistic analysis involving spatial variation of undrained strength c_u is considered next.

Figure 5.26 Failure mass was found spreading over the infinitely long slope based on the deterministic 3-D strength-reduction stability analysis

5.4.2 Critical failure mode in probabilistic 3-D strength reduction analysis

In the previous deterministic strength-reduction stability analysis, numerical simulations were performed on the right-side of the 3-D model, due to the symmetries of both the slope geometry and soil properties over the longitudinal direction. However, such symmetries are not applicable for probabilistic analysis when spatial variation is involved. This is because the soil properties are taken to be random fields so that the 3-D model contains heterogeneous materials which are asymmetric in nature. In their random finite element analysis of a 3-D slope, Griffiths et al. (2009) also found that the symmetric analysis is incompatible with probabilistic analysis involving spatial variation of soil properties.

For the infinitely long earth slope, the c_u over the whole length should be considered as a random field by default. However, for numerical simulations, only a small extent of the infinitely long earth slope that covers the potential failure mass needs to be modelled as a random field. This is because c_u far away from the potential failure mass does not contribute to the 3-D slope failure. Such a small extent with probabilistic treatment is defined as L_{SV} . For probabilistic strength reduction analysis, L_{SV} has to be determined before performing the numerical simulation. A preliminary estimate of L_{SV} could be made based on the previous findings by LEM analysis. For example, when the longitudinal autocorrelation distance δ_z was less than 20 m, the critical width of failure mass b^* was less than 30 m (Figure 5.14). Thus, the following probabilistic strength reduction analyses are conducted on two different L_{SV} , 60 m and 100 m, respectively. Both values of L_{SV} are sufficiently large to cover the potential b^* . The properties of random field of c_u are the same as in the LEM analysis. Thus, for the L_{SV} of 60 m, a total of 13 control points equally spaced at 5 m (equal to δ_z) were used to discretize the random field over L_{SV} ; for the L_{SV} of 100 m, a total of 21 control points equally spaced at 5 m were used to discretize the random field over L_{SV} . Considering the control-point c_u values as spatially correlated random variables, a typical representation of the random field of c_u is shown in Figure 5.27. Note that smooth end-section boundary conditions are used, so that artificial 3-D failure caused by fixed end-section boundary conditions (commonly performed in deterministic 3-D analysis) is avoided.

Based on this simplified treatment of the longitudinal random field of c_u , probabilistic 3-D stability evaluation can be conducted using any existing method of probabilistic analysis, such as MC simulations or FORM and its extensions. Since the 3-D stability is computed using strength-reduction finite-difference method which requires much computation time even for a single iteration, MC simulations are not feasible in this circumstance. On the other hand, since the method involves implicit performance

functions, probabilistic analyses based on the sensitivity-based FORM (as proposed in Chapter 3) are used.

Figure 5.27 A typical representation of the longitudinal random field of c_u using the interpolated autocorrelations ($L_{SV} = 100$ m, $\delta_z = 5$ m)

Figure 5.28 and Figure 5.29 present the results of reliability analyses using the two different schemes of L_{SV} . The reliability index β value converges to 2.236 for $L_{SV} = 60$ m, and to 2.230 for $L_{SV} = 100$ m. The β values obtained by using the two different probabilistically treated lengths L_{SV} are almost the same. This implies that the reliability analysis is insensitive to the L_{SV} to be used, provided that the L_{SV} is large enough to cover the potential failure mass b^* .

A vivid illustration is to present the probabilistic critical (critical- β) width based on design point corresponding to the converged β value, as shown in Figure 5.30. For the problem in hand, a specific value of probabilistic critical width of failure mass b^* is not available due to the SRM itself. As a result, one can only estimate the probabilistic critical width b^* from the contour plot. For example, the b^* is roughly estimated to be 20 m for the problem studied. This value is about four times of the autocorrelation distance δ_z of 5 m. Also shown in Figure 5.30 is that the critical- β widths in both cases of L_{SV} are almost the same. This is consistent with the above-mentioned insensitivity of

β value to L_{SV} .

In conclusion, for an infinitely long earth slope, the longitudinal spatial variation of soil properties is a very important factor affecting local failure. Without modelling the longitudinal spatial variation, a long earth slope shows overall failure over its entire length. This is inconsistent with real cases. However, probabilistic analysis accounting for longitudinal spatial variation can reasonably explain the existence of critical width of failure mass (local failure).

Initial $\mathbf{n}_0$	-0.50	-0.50	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$F_s(\mathbf{n}_0)$	1.522											
$F_s(\mathbf{n}_{0,new})$	1.522	1.525	1.529	1.533	1.541	1.545	1.549	1.545	1.541	1.533	1.529	1.525
$\Delta g(\mathbf{n}_0)$	0.00	0.00	0.01	0.01	0.02	0.02	0.03	0.02	0.02	0.01	0.01	0.00
$\Delta g/\Delta n_i$	0.00	0.01	0.01	0.02	0.04	0.05	0.05	0.05	0.04	0.02	0.01	0.00
$\mathbf{n}_1$	-0.15	-0.39	-0.75	-1.18	-1.72	-2.10	-2.29	-2.10	-1.72	-1.18	-0.75	-0.39
β	3.453											

Initial $\mathbf{n}_1$	-0.15	-0.39	-0.75	-1.18	-1.72	-2.10	-2.29	-2.10	-1.72	-1.18	-0.75	-0.39
$F_s(\mathbf{n}_1)$	0.673											
$F_s(\mathbf{n}_{1,new})$	0.673	0.673	0.673	0.673	0.685	0.731	0.757	0.731	0.685	0.673	0.673	0.673
$\Delta g(\mathbf{n}_1)$	0.00	0.00	0.00	0.00	0.01	0.06	0.08	0.06	0.01	0.00	0.00	0.00
$\Delta g/\Delta n_i$	0.00	0.00	0.00	0.00	0.02	0.12	0.17	0.12	0.02	0.00	0.00	0.00
$\mathbf{n}_2$	-0.01	-0.03	-0.08	-0.23	-0.63	-1.34	-1.71	-1.34	-0.63	-0.23	-0.08	-0.03
β	2.028											

Initial $\mathbf{n}_2$	-0.01	-0.03	-0.08	-0.23	-0.63	-1.34	-1.71	-1.34	-0.63	-0.23	-0.08	-0.03
$F_s(\mathbf{n}_2)$	1.080											
$F_s(\mathbf{n}_{2,new})$	1.080	1.080	1.080	1.080	1.092	1.143	1.168	1.143	1.092	1.080	1.080	1.080
$\Delta g(\mathbf{n}_2)$	0.00	0.00	0.00	0.00	0.01	0.06	0.09	0.06	0.01	0.00	0.00	0.00
$\Delta g/\Delta n_i$	0.00	0.00	0.00	0.00	0.02	0.13	0.18	0.13	0.02	0.00	0.00	0.00
$\mathbf{n}_3$	-0.01	-0.03	-0.09	-0.26	-0.70	-1.50	-1.90	-1.50	-0.70	-0.26	-0.09	-0.03
β	2.271											

Initial $\mathbf{n}_3$	-0.01	-0.03	-0.09	-0.26	-0.70	-1.50	-1.90	-1.50	-0.70	-0.26	-0.09	-0.03
$F_s(\mathbf{n}_3)$	0.986											
$F_s(\mathbf{n}_{3,new})$	0.986	0.986	0.986	0.986	0.998	1.051	1.084	1.051	0.998	0.986	0.986	0.986
$\Delta g(\mathbf{n}_3)$	0.00	0.00	0.00	0.00	0.01	0.06	0.10	0.06	0.01	0.00	0.00	0.00
$\Delta g/\Delta n_i$	0.00	0.00	0.00	0.00	0.02	0.13	0.20	0.13	0.02	0.00	0.00	0.00
$\mathbf{n}_4$	-0.01	-0.03	-0.09	-0.25	-0.67	-1.46	-1.89	-1.46	-0.67	-0.25	-0.09	-0.03
β	2.231											

Initial $\mathbf{n}_4$	-0.01	-0.03	-0.09	-0.25	-0.67	-1.46	-1.89	-1.46	-0.67	-0.25	-0.09	-0.03
$F_s(\mathbf{n}_4)$	1.002											
$F_s(\mathbf{n}_{4,new})$	1.002	1.002	1.002	1.002	1.010	1.065	1.100	1.065	1.010	1.002	1.002	1.002
$\Delta g(\mathbf{n}_4)$	0.00	0.00	0.00	0.00	0.01	0.06	0.10	0.06	0.01	0.00	0.00	0.00
$\Delta g/\Delta n_i$	0.00	0.00	0.00	0.00	0.02	0.13	0.20	0.13	0.02	0.00	0.00	0.00
$\mathbf{n}_5$	-0.01	-0.03	-0.08	-0.23	-0.63	-1.45	-1.92	-1.45	-0.63	-0.23	-0.08	-0.03
β	2.236											

Figure 5.28 Reliability analysis with $L_{SV} = 60$ m, using thirteen RVs

Initial $\mathbf{n}_0$	0.00	0.00	0.00	0.00	0.00	-0.50	-0.50	-0.50	-0.50	-0.50	-0.50	-0.50	-0.50	-0.50	-0.50	0.00	0.00	0.00	0.00	0.00
$F_s(\mathbf{n}_0)$	1.397																			
$F_s(\mathbf{n}_{0,new})$	1.397	1.397	1.397	1.397	1.397	1.400	1.410	1.418	1.436	1.420	1.426	1.420	1.436	1.418	1.410	1.400	1.397	1.397	1.397	1.397
$\Delta \mathbf{g}(\mathbf{n}_0)$	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.04	0.02	0.03	0.02	0.04	0.02	0.01	0.00	0.00	0.00	0.00	0.00
$\Delta \mathbf{g}/\Delta \mathbf{n}_i$	0.00	0.00	0.00	0.00	0.00	0.01	0.03	0.04	0.08	0.05	0.06	0.05	0.08	0.04	0.03	0.01	0.00	0.00	0.00	0.00
$\mathbf{n}_1$	0.00	-0.01	-0.02	-0.05	-0.13	-0.34	-0.75	-1.19	-1.64	-1.48	-1.53	-1.48	-1.64	-1.19	-0.75	-0.34	-0.13	-0.05	-0.02	-0.01
β	2.846																			
Initial $\mathbf{n}_1$	0.00	-0.01	-0.02	-0.05	-0.13	-0.34	-0.75	-1.19	-1.64	-1.48	-1.53	-1.48	-1.64	-1.19	-0.75	-0.34	-0.13	-0.05	-0.02	-0.01
$F_s(\mathbf{n}_1)$	0.939																			
$F_s(\mathbf{n}_{1,new})$	0.939	0.939	0.939	0.939	0.939	0.939	0.939	0.947	0.967	0.982	0.986	0.982	0.967	0.947	0.939	0.939	0.939	0.939	0.939	0.939
$\Delta \mathbf{g}(\mathbf{n}_1)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.03	0.04	0.05	0.04	0.03	0.01	0.00	0.00	0.00	0.00	0.00	0.00
$\Delta \mathbf{g}/\Delta \mathbf{n}_i$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.06	0.09	0.09	0.09	0.06	0.02	0.00	0.00	0.00	0.00	0.00	0.00
$\mathbf{n}_2$	0.00	0.00	0.00	-0.01	-0.03	-0.07	-0.20	-0.53	-1.08	-1.53	-1.68	-1.53	-1.08	-0.53	-0.20	-0.07	-0.03	-0.01	0.00	0.00
β	2.327																			
Initial $\mathbf{n}_2$	0.00	0.00	0.00	-0.01	-0.03	-0.07	-0.20	-0.53	-1.08	-1.53	-1.68	-1.53	-1.08	-0.53	-0.20	-0.07	-0.03	-0.01	0.00	0.00
$F_s(\mathbf{n}_2)$	0.994																			
$F_s(\mathbf{n}_{2,new})$	0.994	0.994	0.994	0.994	0.994	0.994	0.994	0.994	1.015	1.049	1.067	1.049	1.015	0.994	0.994	0.994	0.994	0.994	0.994	0.994
$\Delta \mathbf{g}(\mathbf{n}_2)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.05	0.07	0.05	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\Delta \mathbf{g}/\Delta \mathbf{n}_i$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.04	0.11	0.15	0.11	0.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\mathbf{n}_3$	0.00	0.00	0.00	-0.01	-0.02	-0.04	-0.11	-0.31	-0.83	-1.50	-1.83	-1.50	-0.83	-0.31	-0.11	-0.04	-0.02	-0.01	0.00	0.00
β	2.262																			
Initial $\mathbf{n}_3$	0.00	0.00	0.00	-0.01	-0.02	-0.04	-0.11	-0.31	-0.83	-1.50	-1.83	-1.50	-0.83	-0.31	-0.11	-0.04	-0.02	-0.01	0.00	0.00
$F_s(\mathbf{n}_3)$	0.994																			
$F_s(\mathbf{n}_{3,new})$	0.994	0.994	0.994	0.994	0.994	0.994	0.994	0.994	1.006	1.053	1.084	1.053	1.006	0.994	0.994	0.994	0.994	0.994	0.994	0.994
$\Delta \mathbf{g}(\mathbf{n}_3)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.06	0.09	0.06	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\Delta \mathbf{g}/\Delta \mathbf{n}_i$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.12	0.18	0.12	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\mathbf{n}_4$	0.00	0.00	0.00	0.00	-0.01	-0.03	-0.09	-0.25	-0.68	-1.46	-1.89	-1.46	-0.68	-0.25	-0.09	-0.03	-0.01	0.00	0.00	0.00
β	2.230																			

Figure 5.29 Reliability analysis with $L_{SV} = 100$ m, using twenty-one RVs

(a) Design point profile and probabilistic critical failure width based on $L_{SV} = 60 \text{ m}$

(b) Design point profile and probabilistic critical failure width based on $L_{SV} = 100$ m

Figure 5.30 Comparison between the critical- β failure modes obtained using different lengths of probabilistic treatment

5.4.3 Effect of longitudinal autocorrelation distance on probabilistic critical width

In the previous LEM analysis (on an idealized 3-D slope model), it was shown that the probabilistic critical width of failure mass b^* is related to longitudinal spatial autocorrelation distance δ_z . The finding is validated here by using the probabilistic SRM analysis. In addition to the probabilistic SRM stability analysis using 5 m of δ_z ,

further investigations using two more δ_z values of 10 and 15 m, respectively, are conducted. The results are compared in Figure 5.31. It is obvious that b^* increases with increase of δ_z used. For the case in hand, estimated values for critical- β width with respect to δ_z of 5, 10 and 15 m are 20, 25, and 30 m, respectively.

(a) Design point profile and probabilistic critical width of failure mass when $\delta_z = 5$ m

(b) Design point profile and probabilistic critical width of failure mass when $\delta_z = 10$ m

(c) Design point profile and probabilistic critical width of failure mass when $\delta_z = 15$ m

Figure 5.31 Probabilistic critical width of failure mass increases with the increase of longitudinal autocorrelation distance δ_z

5.5 Summary

This chapter investigated the three-dimensional slope stability by both deterministic analysis and probabilistic analysis.

The investigation was first carried out using an idealized three-dimensional (3-D) stability model so that explicit performance function of the 3-D failure could be attained by limit equilibrium analysis. The deterministic analysis showed that the factor of safety by 3-D analysis, F_{3D} , could be larger than its plane strain counterpart by more than 40 % when the width of failure mass b is sufficiently small. The F_{3D} consistently decreased with increase in b . The 3-D stability analysis thus implied an infinite deterministic critical width b^* along the longitudinal direction. This conclusion seems inconsistent with the observation from real cases. A more consistent investigation was further conducted by probabilistic analysis which incorporates the longitudinal spatial variation of the strength parameters. The spatial-averaging approach proposed by Vanmarcke was first employed to model the longitudinal spatial variation. It was found that there is a most probable critical width of failure mass b^* which is very different from the deterministic analysis. In addition, the magnitude of b^* was found to be dependent on the longitudinal spatial autocorrelation distance to some extent. As an alternative to the spatial-averaging approach, the spatial-autocorrelation approach with linear interpolation for the random field was proposed to deal with the longitudinal spatial variation. It was found that the two methods agree well in dealing with the spatial variation when they were used under the same conditions.

The 3-D stability analysis was then investigated by strength reduction method. The deterministic strength reduction stability analyses were discussed in both undrained and drained conditions. In undrained condition, parametric studies showed that the

contribution of end effects to the F_{3D} is affected by the width of failure mass and the cross-sectional slope inclination, but is not influenced by the undrained strength c_u at all. In drained conditions, parametric studies showed that the contribution of end effects will also be affected by the shear strength parameters c and ϕ . Finally, a probabilistic strength reduction finite difference stability analysis which incorporates the longitudinal spatial variation of undrained strength was demonstrated, using the method of interpolated autocorrelations. Similar to the limit equilibrium analysis, the probabilistic strength reduction analysis also revealed that there is a most probable critical width of failure mass b^* which is dependent on the longitudinal autocorrelation distance. Using probabilistic sensitivity-based FORM for the 3-D strength reduction slope reliability analysis is one of the original contributions of this thesis.

Chapter 6 Stratified Response Surfaces for System Probabilistic Evaluation of Slopes

6.1 Introduction

Past researches on reliability of slopes typically aimed at locating the critical failure surface corresponding to either the minimum F_s or minimum β value (Li and Lumb 1987; Hassan and Wolff 1999; Low 2003; Bhattacharya et al. 2003; and Xue and Gavin 2007). In contrast, Oka and Wu (1990) investigated the system reliability of a slope and found that large difference may occur between the failure probability of the system and that of the critical F_s surface. Chowdhury and Xu (1995) studied slope system reliability on a number of preselected slip surfaces which were taken to be important elements or modes. Low et al. (2011) studied the system reliability of a slope by considering several slip surfaces corresponding to the local minima of reliability indices. Based on the critical β surfaces, system reliability assessment was conducted using values of β indices and design points obtained from FORM. It was concluded in Low et al. (2011) that among the many potential failure surfaces or modes in a slope system, only those corresponding to local minima of β have real effect on the system reliability.

On the other hand, system reliability can be attained in a straightforward way by conducting MC simulations for probabilistic slope analysis, either using strength reduction technique (Griffiths and Fenton 2004; Huang et al. 2010) or limit equilibrium method of slices (Ching et al. 2009; Slide, <http://www.rocscience.com/products/8/Slide>). The disadvantage is that the computational effort can be extremely intensive. In this regard, Ching et al. (2009) proposed to use the importance sampling technique for

slope system reliability analysis.

This study proposes to use a 'stratified response surface method' for the system reliability analysis of slopes. FORM/SORM is the basis of the proposed approach. Unlike past studies that involved integration of the reliability analysis with the conventional deterministic analysis in the same platform, a group of stratified response surfaces are constructed for the most probable failure modes, and the reliability analysis can be extended easily to complex cases with implicit performance functions that can only be evaluated using numerical packages. The proposed method will be demonstrated using several case studies from the literature.

6.2 System reliability formulation involving multiple failure modes

In most engineering problems, the system can fail in more than one failure mode. For instance, a semi-gravity retaining wall could fail by sliding, overturning or bearing capacity failure. A slope may fail along any of several potential sliding surfaces. For this reason, reliability-bounds theories are available to handle problems with multiple failure modes. Suppose there are m potential failure modes to be considered, the bi-modal bounds for system reliability are expressed by (e.g., Ang and Tang 1984; Haldar and Mahadevan 2000a)

$$P_{F_1} + \sum_{i=2}^m \max \left[\left\{ P_{F_i} - \sum_{j=1}^{i-1} P(E_i E_j) \right\}; 0 \right] \leq P_F \leq \min \left[\left\{ \sum_{i=1}^m P_{F_i} - \sum_{i=2}^m \max_{j < i} P(E_i E_j) \right\}; 1 \right] \quad (6.1)$$

where P_{F_1} = the largest failure probability among the potential failure modes, P_{F_i} = the failure probability of the i -th failure mode, E_i = the event of failure for the i -th mode, $P(E_i E_j)$ = the joint probability of events E_i and E_j , and P_F = the system failure probability.

The joint probability involved in the above equation is usually not equal to zero, particularly when the failure modes are correlated. To solve such a problem, a weakened version of the above bi-modal bounds has been proposed by Kounias (1968),

Hunter (1976), and Ditlevsen (1979) for Gaussian random variables. In brief, the joint probability $P(E_i E_j)$ can be reduced to

$$\max[P(A), P(B)] \leq P(E_i E_j) \leq P(A) + P(B) \quad (6.2)$$

on condition that the two events are positively correlated, or otherwise,

$$0 \leq P(E_i E_j) \leq \min[P(A), P(B)] \quad (6.3)$$

where the terms $P(A)$ and $P(B)$ are defined by

$$P(A) = \Phi(-\beta_i) \Phi\left(-\frac{\beta_j - \rho\beta_i}{\sqrt{1-\rho^2}}\right) \quad (6.4)$$

$$P(B) = \Phi(-\beta_j) \Phi\left(-\frac{\beta_i - \rho\beta_j}{\sqrt{1-\rho^2}}\right) \quad (6.5)$$

where β_i = reliability index corresponding to failure mode i , ρ = correlation coefficient between modes i and j .

In Eqs. (6.4) and (6.5), the reliability index β can be computed using FORM or back-calculated from the SORM, and the correlation coefficient ρ between failure modes can be estimated from the direction cosines of the design point in the u-space as follows (e.g. Ang and Tang 1984):

$$\rho = \sum_{k=1}^n \alpha_{ik}^* \alpha_{jk}^* = \mathbf{\alpha}_i^{*T} \mathbf{\alpha}_j^* = \frac{\mathbf{u}_i^{*T} \mathbf{u}_j^*}{\beta_i \beta_j} \quad (6.6)$$

where α_{ik}^* = the k -th direction cosine of failure mode i or performance function g_i at the design/failure point in the u-space, $\mathbf{u}_i^*$ = design point corresponding to failure mode i .

Alternatively, when the Low and Tang 2007 FORM procedure is used, one can obtain the correlation coefficient ρ in terms of the $\mathbf{n}^*$ vectors using Cholesky decomposition

of the correlation matrix $\mathbf{R}$, as follows (Low et al. 2011):

$$\begin{aligned}\rho &= \frac{\mathbf{u}_i^{*T} \mathbf{u}_j^*}{\beta_i \beta_j} \\ &= \frac{(\mathbf{L}^{-1} \mathbf{n}_i^*)^T (\mathbf{L}^{-1} \mathbf{n}_j^*)}{\beta_i \beta_j} = \frac{\mathbf{n}_i^{*T} (\mathbf{L}^T)^{-1} \mathbf{L}^{-1} \mathbf{n}_j^*}{\beta_i \beta_j} \\ &= \frac{1}{\beta_i \beta_j} \mathbf{n}_i^{*T} \mathbf{R}^{-1} \mathbf{n}_j^*\end{aligned}\tag{6.7}$$

in which the correlation matrix $\mathbf{R} = \mathbf{L}\mathbf{U} = \mathbf{L}\mathbf{L}^T$, and $\mathbf{L}$ and $\mathbf{U}$ are the lower and the upper triangular matrices by Cholesky decomposition, respectively.

6.3 Observations on local minimum F_s slip surfaces and local minimum β slip surfaces

In deterministic slope stability analysis, it is well-known that there may exist local minima of factors of safety F_s , each of which is associated with a local minimum F_s slip surface. A simple approach to locate the local minimum F_s slip surfaces is the variation of depth of slip surface, via two steps: (1) Select a specific horizontal line with y-coordinate at D , and run stability analyses with all slip surfaces tangent to this line, so that a *critical- D F_s* as well as a *deterministic-critical- D slip surface* is determined; (2) vary the tangent line D , and find a series of *critical- D F_s* as well as *deterministic-critical- D slip surfaces*. By plotting the *critical- D F_s* versus the tangent line D , local minimum F_s slip surfaces can be found finally.

Similarly, in slope reliability analysis, one can locate the local minimum β slip surfaces by plotting *critical- D β* versus the tangent line D .

Figure 6.1 shows the study of a simple homogeneous slope, illustrating both the local minimum F_s slip surfaces and local minimum β slip surfaces. The F_s was computed by

a closed-form solution since only undrained analysis and circular slip surfaces were incorporated. The details of the closed-form solution formulation are presented in Appendix G. The reliability index β was computed using the FORM algorithm of Low and Tang (2007). Clearly, in a single-layered homogeneous slope, there is only a single local minimum of F_s (Figure 6.1b) and a single local minimum of β (Figure 6.1c). In addition, the local minimum F_s slip circle by deterministic analysis is found to be identical to the local minimum β slip circle by reliability analysis. In other words, for the example homogeneous slope the deterministic critical slip surface and the probabilistic critical slip surface are identical.

This phenomenon is further investigated by extending the study of the single-layered slope to a multi-layered slope. Consider a two-layered slope which will be studied in detail in subsection 6.5.1. A spreadsheet model that integrates the factor of safety calculation and reliability analysis was built as shown in Figure 6.2a. The results of deterministic analysis and reliability analysis are shown in Figure 6.2b&c, respectively. It is observed that two sets of local minima of F_s and local minima of β exist. Moreover, further investigation showed that for each local minimum tangent line D , the local minimum F_s slip surface is identical to the local minimum β slip surface.

Another observation is that the local minimum F_s slip surfaces could be identified by strength-reduction of one soil layer at a time (single-layer-strength-reduction). Consider the two-layered slope, for example. When using a reduced strength of clay layer 1 (upper layer) in combination with the original strength of clay layer 2 (lower layer) to run a stability evaluation, the critical slip surface so obtained will be very close to the slip surface S1 with a local minimum of F_s (as shown in Figure 6.2). Furthermore, when using a reduced strength of clay layer 2 in combination with the original strength of clay layer 1 to run a stability evaluation, the critical slip surface so obtained will be close to the slip surface S2 with a local minimum of F_s .

This observation is very useful for slope reliability analysis. One can first locate the local minimum F_s slip surfaces using the single-layer-strength-reduction technique by conventional deterministic stability analysis, and then treat them as local minimum β slip surfaces. This observation is useful for performing system reliability analysis of multi-layered slopes, since slope system reliability analysis could be conducted only on the several local minimum β slip surfaces so obtained. This is because the system reliability of a given slope is mainly governed by the local minimum β failure surfaces (modes), as suggested by Low et al. (2011). Furthermore, this observation helps in developing the stratified response surfaces for dealing with implicit performance functions. The detailed description will be given in subsection 6.4.

- (a) An undrained slope with $c_u = 50$ kPa, and c.o.v. = 0.3

(Units:m,kN/m,kN.m/m,kN/m²,kN/m³)

Stability Model								
xc	yc	R	H	D	Dmax	NumCu	cotΩ	
10.00	19.07	29.07	10	10.00	10	1	2	
θ	θ _h	M _R	M _O	F _s				
0.86	0.40	56000	56000	1.000				
γ	18.5							
c _u	31.426	s _x	s _y	xmin	xmax			
		0	0	-11.94	37.62			
c _{u_Aver}	31.426			ymin	ymax			
				0	10			
Reliability analysis								
ProbDist	Variable	μ	σ	x*	CorrMat	n		
Lognormal	c _{u1}	50	15	31.43	1	-1.435		
PerFun	0.000							
Beta	1.4352

- (b) Plotting of critical- $D F_s$ vs. tangent line D

- (c) Plotting of critical- $D \beta$ vs. tangent line D

Figure 6.1 Single local minimum of F_s (and β) exists in a homogeneous slope

(a) Spreadsheet model analyzing a two-layered slope

Units:m,kN/m,kN.m/m,kN/m²,kN/m³)

Stability Model

xc	yc	R	H	D	Dmax	NumCu	cotΩ
15.88	35.41	39.41	24	4.00	10	1	1.33

θ	θ _h	M _R	M _O	θ1	θ2	F _s
0.45	0.82	198717	198717	0.55	1.18	1.000

γ	19	s _x	s _y	xmin	xmax
c _u	73.91	0	0	-1.42	53.60
c _u Aver	73.91	ymin ymax			
		0	24		

Reliability analysis

Distributi	Variab	Mean	SD	x*	CorrMat	n
Lognorma	c _{u1}	120	36	78.35	1 0	-1.31
Lognorma	c _{u2}	160	48	71.85	0 1	-2.58
PerFun	0.0000					
Beta	2.8919

(b) Plotting of critical-D F_s vs. tangent line D

(c) Plotting of critical-D β vs. tangent line D

Figure 6.2 Local minima of F_s (or β) exist in a multi-layered slope

6.4 Using stratified response surfaces to approximate the system performance of slopes

A slope failure can take place along any of a large number of slip surfaces, and each slip surface represents a potential failure mode. Thus a slope failure should be considered as a system failure problem. That the number of potential failure modes is unknown and the performance function for each failure mode is implicit makes the slope system reliability analysis particularly cumbersome. Response surface method (RSM) has been used for slope reliability analysis with implicit performance functions. The details were discussed in Chapter 3. However, past studies using RSM typically aimed at approximating the performance of slope failure using a single polynomial-based response surface. It is understandable that a single response surface can only represent one performance function at a time; multiple response surfaces are needed for problems with multiple failure modes. Thus, when RSM is used for system reliability analysis of slopes, a better way is to use a group of response surfaces to represent the potentially multiple slope failure modes. This idea is not commonly used in the literature, except in Zhang et al. (2011) and Ji and Low (2012), to the best knowledge of the author. In Zhang et al. (2011), several representative slip surfaces were selected from a large number of candidate circular slip surfaces. The performance function of each representative slip surface was approximated by a second-order polynomial and slope system reliability was then evaluated based on these polynomials. Nevertheless, their procedure of identifying the representative slip surfaces is not an easy task, since correlation coefficients among all the candidate (failure modes) slip surfaces need to be computed to single out the representative surfaces.

In contrast, the ‘stratified response surface method’ proposed by Ji and Low (2012) for system reliability analysis of slopes constructs multiple response surfaces according to

the stratification of a slope. Prior to constructing the stratified response surfaces, one needs to determine how many are required. In this regard, the above-mentioned single-layer-strength-reduction technique can be used, as follows: (1) lower the soil strengths of one of the layers by, say one to three standard deviations below their mean values while keeping the soil strengths of other layers at their mean values, and use the reduced soil strengths to run a stability analysis; this will lead to a critical- F_s surface for the layer with reduced strengths; (2) repeat the above operation for every layer that has uncertain soil properties, and record all the critical- F_s surfaces; (3) each of the critical- F_s surfaces so obtained can be regarded as a local minimum F_s slip surface as well as a local minimum β slip surface, which in turn corresponds to the most probable failure mode for that layer.

For each of the so-obtained local minimum β slip surfaces, a polynomial response surface can be constructed to represent the failure mode. Thus, a group of stratified response surfaces are finally constructed in accordance with all the local minimum β slip surfaces. Note that a similar procedure was also used by Hassan and Wolff (1999) to locate the critical β slip surface, whereas this paper further extends the procedure to system reliability analysis since each so-obtained critical- F_s slip surface can be regarded as a local minimum F_s slip surface as well as a local minimum β surface, as illustrated in section 6.3.

For simple slopes with regularly layered strata, the procedure is very efficient for obtaining the local minimum β slip surfaces, and in turn the stratified response surfaces; for some slopes with complex stratifications such as in Figure 6.3, the so obtained local minimum β slip surface may not be accurate, so that an optimization of the local minimum β slip surface has to be implemented in the development of stratified response surface. The details of stratified response surface method with optimization of local minimum β slip surfaces are explained as follows.

Figure 6.3 A three-layered slope with uncertain soil properties

Again, consider the slope profile in Figure 6.3. Suppose the cohesions c_i and friction angles ϕ_i of each of the three soil layers are random variables, each with a mean value and a standard deviation. To investigate the reliability of the slope, three failure modes can be considered: (a) slope failure with slip surface contained entirely in layer 1 (found by strength reduction to c_1 and ϕ_1) and involving only the shear strength uncertainty of soil layer 1; (b) slope failure with slip surface traversing layers 1 and 2 (found by strength reduction to c_2 and ϕ_2) and involving the shear strength uncertainties of both soil layers 1 and 2; and (c) slope failure with slip surface extending into layer 3 (found by strength reduction to c_3 and ϕ_3) and involving the shear strength uncertainties of all three soil layers. The three performance functions for system reliability analysis of this slope can be described as follows:

$$\left. \begin{aligned} g_1 &= g_1(c_1, \phi_1) \\ g_2 &= g_2(c_1, \phi_1, c_2, \phi_2) \\ g_3 &= g_3(c_1, \phi_1, c_2, \phi_2, c_3, \phi_3) \end{aligned} \right\} \quad (6.8)$$

where g_i ($i = 1$ to 3) represents the performance function for the event where the failure slip surface extends into the i -th layer, c_i and ϕ_i denote the soil shear strength properties in the i -th layer. These three performance functions constitute a series system. Failure of any one performance function can give rise to the failure of the whole system.

When constructing a response surface to approximate the individual performance function, only those parameters underlying the performance function are treated as random variables. For instance, when constructing response surface $\bar{g}_2$ and conducting reliability analysis on $\bar{g}_2$, c_1 , ϕ_1 and c_2 , ϕ_2 are the basic random variables and other parameters are assigned their mean values.

Each of the stratified response surfaces is progressively refined by conducting a search for the local minimum β slip surface. The candidate slip surfaces must be properly constrained so as to be consistent with the physical meaning of the response surface function. For example, since the performance function g_2 is a failure mode involving the local minimum β slip surface extending into the middle layer (layer 2), the candidate slip surfaces (based on which a search of the critical one is required for each response surface experiment) should be confined to those traversing layers 1 and 2 only when one develops the response surface $\bar{g}_2$ for approximating g_2 . The rest of the stratified response surfaces with search for the associated local minimum β slip surfaces are developed likewise.

6.5 Case studies

In this section, four cases from the literature are analyzed by using the proposed method. Slide V6.0 (<http://www.roscience.com/products/Slide.asp>), a 2D limit equilibrium slope stability program, is used for deterministic stability evaluations in the experiments for developing stratified response surfaces. Slide V6.0 can also be used to carry out probabilistic slope stability analysis (for system/overall probabilistic assessment) based on the direct Monte Carlo simulation method. This is used for verification purposes in the case studies. Since MC-based overall probabilistic assessment by Slide is more reliable for circular slip surfaces, the case studies by the

proposed method will also be conducted on circular slip surfaces, so that consistent comparisons can be made. For case 1, Ordinary method of slices is used to compute the factor of safety, since only undrained soils are involved. For the other cases, Spencer's method is used when c - ϕ soils are involved. Although limit equilibrium methods of slices with circular slip surfaces are used in developing the stratified response surfaces, other numerical methods such as LEM with noncircular slip surfaces or finite element or finite difference method can also be used.

6.5.1 Case study 1 – A two-layered hypothetical slope

An example slope from Ching et al. (2009) consists of two layers of soils (Figure 6.4), with undrained strength parameters as shown in Table 6.1. Uncertainty in the undrained shear strength is considered for both layers, with lognormal distribution and c.o.v. of 0.3. In Ching et al. (2009), Monte Carlo simulations were performed to obtain the system probability of failure, and importance sampling technique was used to reduce the computational effort. In contrast, Low et al. (2011) analyzed this same case using FORM with multiple-mode system reliability analysis. The reliability analysis was coupled with limit equilibrium method and a search for reliability-based critical slip surfaces was conducted. It was shown that multiple-mode system reliability analysis based on FORM yielded failure probability bounds which were consistent with MC simulation results.

Figure 6.4 A two-layered soil slope with slip surfaces of the multiple failure modes (case 1)

Table 6.1 Uncertain soil properties of case 1

Slope layers	Probability distribution	Undrained strength c_u	
		Mean (kPa)	c.o.v.
Clay 1	Lognormal	120	0.3
Clay 2	Lognormal	160	0.3

In this study, the proposed stratified response surface method is applied to the case of Figure 6.4. According to the profile of the slope, two possible failure modes should be considered, namely, failure in the upper layer due to the uncertainty of clay layer 1 and failure extending into the lower layer and involving the uncertainties of both clay layers 1 and 2. Two slip surfaces concerning the two possible failure modes can be readily obtained using the suggested procedure of single-layer-strength-reduction, and they are shown in Figure 6.4. The critical factor of safety for this case was found to be 1.997, which can be easily obtained by Slide V6.0 using its “slope search” option. To study the system failure probability, two separate stratified response surfaces are developed and the system reliability analyses are summarized below.

The stratified response surfaces are obtained progressively, and they both converge on the second iteration for the case in hand. Therefore, a total of sixteen sampling points were selected and Slide V6.0 was used to evaluate the factor of safety at the sixteen sampling points. Direct Monte Carlo simulations with Latin Hypercube sampling technique using @RISK 5.0 (<http://www.palisade.com/risk>) were carried out to validate the accuracy of FORM analyses on the stratified response surfaces. It was found that P_{f1} and P_{f2} (failure probabilities corresponding to mode 1 and 2) are about 0.26 % and 0.14 %, respectively. Both of these values are comparable with those of FORM analyses in Table 6.2. MC simulations were also carried out using Slide V6.0, for comparison with the results obtained by Stratified RSM. It is noted that without considering this two-layered slope as a system failure, a lower value of failure probability is predicted, which significantly underestimates the real failure probability. A much higher failure probability of about 0.405 % is obtained by the Slide's system/overall MC simulation (which searches for the critical F_s surface for every trial in the MC simulation). It can be seen from Table 6.3 that the system reliability bounds obtained by the proposed Stratified RSM are consistent with previous studies and also with the Slide's overall MC simulation based on Slide.

The Slide's system/overall MC simulations can be very time-consuming if one uses the 'default' MC options for a problem with small failure probability (which requires a large number of stability evaluations/trials). A more efficient technique is described here. For example, if knowledge of the slip surfaces of local minima of β values is available, one can specify these local minimum β slip surfaces in Slide so that a new search for the critical slip surface need not be carried out for every MC trial. This implies that only several stability evaluations corresponding to these specific local minimum β slip surfaces are done for each trial in performing the system/overall MC simulation. For comparison, the two local minimum β slip surfaces given in Low et al. (2011) is studied here using Slide's system/overall MC simulation. Using the above

technique, three MC simulations each with 20,000 trials taking less than one minute yield $P_f = 0.42\% \pm$, which is practically the same result as that with the 'default' settings for MC simulation ($P_f = 0.405\% \pm$) which took several hours. It may seem somewhat paradoxical that the Slide's 'default' settings for overall MC simulation which search for the most critical slip surface among a large number of predefined slip surfaces obtain a slightly lower failure probability than does the suggested technique which only searches between the two local minimum β slip surfaces. Such a slight inconsistency could be due to two possibilities: first, the large number of predefined slip surfaces used in the 'default' settings may not contain the two local minimum β slip surfaces; and second, the results of MC simulations themselves involve fluctuation which could give rise to slightly different values of P_f between any two simulations.

Table 6.2 Results of system reliability analysis of a two-layered undrained slope (case 1)

Slip surface extending to	Stratified RS	FORM index β	Failure probability	Design point values	Direction cosines	System reliability
Clay layer 1	$\bar{g}_1$	$\beta_1 = 2.8003$	$P_{f1} = 0.26\%$	$c_{u1}^* = 50.52$	$\alpha_{cu1} = -1$	$\rho_{\bar{g}_1, \bar{g}_2} = 0.478$
Clay layer 2	$\bar{g}_2$	$\beta_2 = 2.9441$	$P_{f2} = 0.16\%$	$c_{u1}^* = 76.07$ $c_{u2}^* = 71.72$	$\alpha_{cu1} = -0.478$ $\alpha_{cu2} = -0.879$	
						$P_f = \{0.399\%, 0.408\%\}$

Note:

- (1) The two stratified RS are obtained as follows:

$$\bar{g}_1 = g_1(c_{u1}) = [-1.00556, 0.019884, -3.9E-7] [1, c_{u1}, c_{u1}^2]^T$$

$$\bar{g}_2 = g_2(c_{u1}, c_{u2}) = [-1.01732, 0.00452, 0.009482, 7.64E-7, -2.6E-6] [1, c_{u1}, c_{u2}, c_{u1}^2, c_{u2}^2]^T$$

- (2) Each of the stratified RS is approximated using a second-order polynomial without cross-term; refer to Figure 6.5 for the details of the polynomial approximation; the procedure is described in Section 3.3.3

(a) RSM for approximating the first polynomial $\bar{g}_1$

Figure 6.5 Procedure of stratified RS approximation using second-order polynomials

(b) RSM for approximating the second polynomial $\bar{g}_2$

Figure 6.5 continued

Table 6.3 Comparison of system reliability results from different methods (case 1)

Method of analysis	Failure probability P_f
FORM system reliability bounds based on Stratified RSM ^a	{0.399 %, 0.408 %}
MCS based on Stratified RSM with 50000 samplings ^b	0.390 % $\pm$
Slide V6.0 with 20000 samplings ^c	0.405 % $\pm$
Results from Ching et al. (2009) using importance sampling (1000 samplings)	$\mu_{P_f} = 0.41\%$, $\sigma_{P_f} = 0.027\%$
MCS by Ching et al. (2009) with 10000 samplings	$\mu_{P_f} = 0.44\%$, $\sigma_{P_f} = 0.067\%$
Results from Low et al. (2011)'s reliability bounds ^d	{0.432 %, 0.441 %}

a: Stratified RSM = Stratified Response Surface Method;

b&c: MC simulations were done three times; the P_f as shown is the average of the three slightly different P_f ; this approach of MC simulations is also applied to the other case studies;

d: Low et al. (2011)'s coupled β and LEM analysis obtained $\beta_1 = 2.795$ and $\beta_2 = 2.893$.

6.5.2 Case study 2 – An embankment of cohesive soil resting on undrained soft clay foundation

This example (Figure 6.6) originates from Chowdhury and Xu (1995). In their work, the system reliability was analyzed by identifying several slip surfaces as important failure modes. In this study, the example is re-examined by the stratified response surface method. The critical factor of safety based on mean values of soil properties was found to be 1.176. All the soil strength parameters are treated as normally distributed random variables, as listed in Table 6.4. It is anticipated that two modes of failure can occur, namely, failure in the embankment only and failure traversing both embankment and soft clay foundation. Hence, two stratified response surfaces are developed according to the soil profiles and system reliability assessment is then carried out as shown in Table 6.5.

Direct MC simulations using @RISK were also performed on the two stratified response surfaces. P_{f1} and P_{f2} from direct MC simulations are about 19.71 % and 21.25 %, respectively, which compare with those of FORM estimate as shown in Table 6.5. The results of this study are compared with others as shown in Table 6.6.

Figure 6.6 An embankment resting on soft clay foundation (case 2)

Table 6.4 Soil properties of case 2

Slope profile	Cohesion c		Friction angle ϕ	
	Mean (kPa)	c.o.v.	Mean ($^{\circ}$)	c.o.v.
Embankment	10	0.20	12	0.25
Clay foundation	40	0.20	0	0

Table 6.5 Results of the system reliability analysis of the embankment on soft ground (case 2)

Slip surface extending to	Stratified RS	FORM index β	Failure probability	Design point values	Direction cosines	System reliability
Embankment	$\bar{g}_1$	$\beta_1 = 0.8471$	$P_{f1} = 19.85\%$	$c_m^* = 9.07$ $\phi_m^* = 9.88$	$\alpha_{c_m} = -0.55$, $\alpha_{\phi_m} = -0.835$	$\rho_{\bar{g}_1, \bar{g}_2} = 0.0854$
Soft clay foundation	$\bar{g}_2$	$\beta_2 = 0.8067$	$P_{f2} = 20.99\%$	$c_m^* = 9.93$ $\phi_m^* = 11.83$ $c_u^* = 33.57$	$\alpha_{c_m} = -0.046$, $\alpha_{\phi_m} = -0.072$, $\alpha_{c_u} = -0.996$	

Note: $\bar{g}_1 = g_1(c_m, \phi_m) = [-1.01948, 0.055507, 0.052744, -1.3E-4, 5.56E-5] [1, c_m, \phi_m, c_m^2, \phi_m^2]^T$

$\bar{g}_2 = g_2(c_m, \phi_m, c_u) = [-1.05376, 0.005, 0.009102, 0.027924, 0, -1.7E-4, -1.6E-5] [1, c_m, \phi_m, c_u, c_m^2, \phi_m^2, c_u^2]^T$

Table 6.6 Comparison of system reliability results from different methods (case 2)

Method of analysis	P_f
FORM system reliability bounds based on Stratified RSM	{31.7 %, 36.3 %}
MCS based on Stratified RSM with 50000 samplings	35.75 % $\pm$
Slide V6.0 with 2000 samplings	36.1 % $\pm$
Study by Chowdhury and Xu (1995)	{22.7%, 44.6%}

6.5.3 Case study 3 – Congress Street cut with probabilistic assessment

The Congress Street cut failed during excavation for a length of about 60 m. Many studies have been performed on this well-known failure case in the literature (Ireland 1954; Tang et al. 1976; Oka and Wu 1990; Chowdhury and Xu 1995; Ching et al. 2009). The approximate dimensions of the cut at the time of failure are shown in Figure 6.7. Uncertainties of the undrained shear strength parameters are considered in the clay layers, while the properties of the top sand layer are taken to be deterministic. Table 6.7 presents the soil properties as used by Chowdhury and Xu (1995). All the random variables are assumed to be normally distributed. Since failure can occur with the slip surface extending to each of the clay layers, there are in total three possible critical failure modes to be examined. The details of reliability assessment based on stratified response surfaces are summarized in Table 6.8.

Figure 6.7 Profile of Congress Street cut (case 3)

Table 6.7 Soil properties, the top sand layer has no uncertainty (case 3)

Soil description	Probability distribution	Undrained strength c_u	
		Mean (kPa)	c.o.v.
Clay 1	Normal	55	0.37
Clay 2	Normal	43	0.19
Clay 3	Normal	56	0.24

Table 6.8 Results of the system reliability analysis of Congress Street cut (case 3)

Slip surface extending to	Stratified RS	FORM index β	Failure probability	Design point values	Direction cosines	System reliability
Clay layer 1	$\bar{g}_1$	$\beta_1 = 1.6658$	$P_{f1} = 4.79\%$	$c_{u1}^* = 21.02$	$\alpha_{cu1} = -1$	$\rho_{\bar{g}_1, \bar{g}_2} = 0.512,$
Clay layer 2	$\bar{g}_2$	$\beta_2 = 0.6978$	$P_{f2} = 24.27\%$	$c_{u1}^* = 47.71$ $c_{u2}^* = 38.09$	$\alpha_{cu1} = -0.512,$ $\alpha_{cu2} = -0.859$	$\rho_{\bar{g}_1, \bar{g}_3} = 0.323,$ $\rho_{\bar{g}_3, \bar{g}_2} = 0.382$
Clay layer 3	$\bar{g}_3$	$\beta_3 = 0.7328$	$P_{f3} = 23.18\%$	$c_{u1}^* = 50.18$ $c_{u2}^* = 41.49$ $c_{u3}^* = 47.18$	$\alpha_{cu1} = -0.323,$ $\alpha_{cu2} = -0.252,$ $\alpha_{cu3} = -0.912$	$P_f = \{31.22\%, 41.84\%\}$

Note: $\bar{g}_1 = g_1(c_{u1}) = [-0.99248, 0.049366, -0.0001] [1, c_{u1}, c_{u1}^2]^T$

$\bar{g}_2 = g_2(c_{u1}, c_{u2}) = [-0.95641, 0.004943, 0.017859, -1.2E-6, 2.97E-5] [1, c_{u1}, c_{u2}, c_{u1}^2, c_{u2}^2]^T$

$\bar{g}_3 = g_3(c_{u1}, c_{u2}, c_{u3}) = [-1.14408, 0.003185, 0.009424, 0.015832, -2.4E-6, -4.5E-5, -3.2E-5] [1, c_{u1}, c_{u2}, c_{u3}, c_{u1}^2, c_{u2}^2, c_{u3}^2]^T$

It was noted, for example in Ang and Tang (1984), that the bimodal bounds on failure probability of systems with multiple failure modes will depend on the ordering of the individual failure modes. It was suggested, for example in Haldar and Mahadevan (2000a), that ordering the failure modes in decreasing probabilities of failure (i.e., increasing β values) will lead to closer bounds. This has been done for the present case, yielding $P_f = \{31.22 \%, 41.84 \%\}$ as shown in Table 6.8. Table 6.9 presents comparisons between this study and the studies by others. In general, the system reliability bounds obtained by the proposed approach compare well with the Monte Carlo simulations and the bounds of Chowdhury and Xu (1995) which studied the failure modes 2 and 3 only.

Table 6.9 Comparison of system reliability results from different methods (case 3)

Method of analysis	P_f
FORM system reliability bounds based on Stratified RSM	$\{31.22 \%, 41.84 \%\}$
MCS based on Stratified RSM with 50000 samplings ^a	$39.11 \% \pm$
Slide V6.0 with 10000 samplings	$36.76 \% \pm$
Study by Chowdhury and Xu (1995)	$\{27.4\%, 44.7\%\}$

a: Direct MC simulations for the stratified response surfaces approximately yielded $P_{f1} = 4.79 \%$, $P_{f2} = 24.14 \%$, and $P_{f3} = 24.03 \%$.

It is observed that the P_{f1} is quite small compared with P_{f2} and P_{f3} . Therefore, it is interesting to investigate the influence of P_{f1} on the system reliability. The investigation was done by ignoring P_{f1} and the recomputed bounds were found to be $P_f = \{31.22 \%, 39.25 \%\}$. This only differs slightly from the original bounds obtained by considering all three failure modes, as shown in Table 6.8. The new system reliability bounds (without considering P_{f1}) also compare well with Slide V6.0's overall/system MC simulation. This shows that among the critical failure modes the one with relatively small chance of occurrence has minor influence on the overall system reliability.

6.5.4 Case study 4 – Example of ACADS 1(c) problem

In 1988 a set of five basic slope stability problems, together with five variants, was distributed both in the Australian Geomechanics profession and overseas as part of a survey sponsored by ACADS (Giam and Donald 1989). The present study uses the ACADS 1(c) problem as shown in Figure 6.8 for illustration. Past treatments of this problem have been limited to deterministic stability analysis. The present study extends the problem into a probabilistic analysis with an assumption of coefficient of variation of 0.3 for the shear strength parameters of layers 2 and 3 (Table 6.10), and normal distribution is assumed for the random variables.

Slide V6.0 is used to perform the stability analysis with circular slip surfaces. The critical F_s was found to be 1.406 at the mean value point, and the deterministic critical slip surface (S) traverses all the three layers of sand/soils. Unlike the preceding example slopes, which are layered regularly in the horizontal direction, the stratification of ACADS 1(c) is rather complicated so that experience is needed to recognize the possible failure modes in order to efficiently implement the system reliability assessment. The suggested procedure for identifying the possible critical failure modes was applied first. It was found that critical slip surface tends to traverse the sand layer and soil layer 3 (e.g. S1) when the uncertain soil strength properties in layer 3 are lowered to, say three standard deviation below their mean values; while the critical slip surface tends to traverse the sand layer and soil layers 2 and 3 (e.g. S2) when the uncertain soil strength properties in layer 2 are lowered. Therefore, the subsequent analysis considers the above two slip surfaces as the two most critical failure modes, which are represented by the performance functions $g_1(c_3, \phi_3)$ and $g_2(c_3, \phi_3, c_2, \phi_2)$, respectively. The failure mode with slip surface traversing the sand layer and soil layer 2 only is less critical and hence excluded from the following analysis.

Next, stratified response surfaces $\bar{g}_1$ and $\bar{g}_2$ are constructed using second-order polynomials without cross-terms, in tandem with refinement of the critical slip surfaces S1 and S2, respectively. It was found that the refined critical slip surface for failure mode-1 denoted as S1' (critical- β surface) in Figure 6.8 differs slightly from S1, while the refined critical slip surface for failure mode-2 denoted as S2' in Figure 6.8 differs significantly from S2. Results of the reliability analysis are summarized in Table 6.11.

Figure 6.8 Profile of ACADS 1(c) problem (case 4)

Table 6.10 Soil properties, Layer 1 is deterministic sand layer (case 4)

Slope profile	Cohesion		Friction angle	
	Mean (kPa)	c.o.v.	Mean (°)	c.o.v.
Layer 1	0	NA	38	NA
Layer 2	5.3	0.3	23	0.2
Layer 3	7.2	0.3	20	0.2

Table 6.11 Results of the system reliability analysis of the ACADS example (case 4)

Slip surface traversing	Stratified RS	FORM index β	Failure probability	Design point values	Direction cosines	System reliability
Sand layer & Soil layer 3	$\bar{g}_1$	$\beta_1 = 2.4639$	$P_{f1} = 0.69 \%$	$c_3^* = 4.18$ $\phi_3^* = 11.89$	$\alpha_{c_3} = -0.568,$ $\alpha_{\phi_3} = -0.823$	$\rho_{\bar{g}_1, \bar{g}_2} = 0.94$
Sand layer, Soil layer 3 & soil layer 2	$\bar{g}_2$	$\beta_2 = 2.3564$	$P_{f2} = 0.92 \%$	$c_3^* = 5.49$ $\phi_3^* = 11.43$ $c_2^* = 5.13$ $\phi_2^* = 20.38$	$\alpha_{c_3} = -0.336,$ $\alpha_{\phi_3} = -0.909,$ $\alpha_{c_2} = -0.046,$ $\alpha_{\phi_2} = -0.241$	$P_f = \{1.08 \%,$ $1.30 \%\}$

Note: $\bar{g}_1 = g_1(c_3, \phi_3) = [-0.91737, 0.067139, 0.058662, -4.3E-4, -3.8E-4] [1, c_3, \phi_3, c_3^2, \phi_3^2]^T$
 $\bar{g}_2 = g_2(c_3, \phi_3, c_2, \phi_2) = [-1.32797, 0.043427, 0.062433, 0.00566, 0.037257, -0.00118, -7.8E-4, -2.0E-18,$
 $-7.0E-4] [1, c_3, \phi_3, c_2, \phi_2, c_3^2, \phi_3^2, c_2^2, \phi_2^2]^T$

Table 6.12 Comparison of system reliability results from different methods (case 4)

Method of analysis	P_f
FORM system reliability bounds based on Stratified RSM	{1.08 %, 1.30 %}
SORM system reliability bounds based on Stratified RSM	{1.35 %, 1.53 %}
MCS based on Stratified RSM with 50000 samplings	1.34 % $\pm$
Slide V6.0 with 20000 samplings	1.40 % $\pm$

As shown in Table 6.12, direct Monte Carlo simulations on the Stratified RSM gives a system failure probability of about 1.34 %, which differs somewhat from the FORM-based system reliability bounds. The underlying reason for this inconsistency can be explained by investigating the individual stratified response surfaces. Direct MC simulations on the response surfaces show that $P_{f1} = 0.69 \%$ and $P_{f2} = 1.26 \%$. It is found that $\bar{g}_1$ is quite linear because reliability analysis using FORM yields practically the same failure probability as MC simulations. However, the approximated $\bar{g}_2$ appears to be nonlinear since FORM only gives a P_{f2} of 0.92 % (Table 6.11). Therefore, the inaccuracy of the FORM-based system reliability bounds is due to underestimation of the failure probability corresponding to failure mode 2.

A more rational system reliability analysis for such a problem can be carried out by considering second-order reliability method (SORM) for the nonlinear performance function. For the second response surface $\bar{g}_2$, a SORM analysis (refer to Appendix B for the details of SORM) yielded a probability of failure $P_{f2,SORM}$ equal to 1.25 %, which significantly improved the estimate by FORM (0.92 %). The equivalent FORM reliability index, $\beta_{2,SORM} = 2.2414$, was back-calculated from $P_{f2,SORM}$ using the formula $\beta_{2,SORM} = -\Phi^{-1}(P_{f2,SORM})$. Assuming that the direction cosine vector remains unchanged when equivalent FORM reliability index is used, as suggested by Low et al. (2011), using $\beta_{2,SORM}$ in combination with the original β_1 obtained a refined system reliability bounds as $P_{f,SORM} = \{1.35 \%, 1.53 \%\}$, which seems more consistent with the MC simulation results of 1.34% based on the Stratified RSM.

6.6 System reliability analysis using the probabilistic sensitivity-based FORM

As an alternative to the Stratified RSM, the probabilistic sensitivity-based FORM as proposed in Chapter 3 can also be used for system reliability analysis of multi-layered earth slopes. Both methods aim at solving the slope system reliability involving implicit performance functions, but they differ in the way of locating the multi-mode design points and reliability indices. The pros and cons of the RSM and probabilistic sensitivity-based FORM have been discussed in Chapter 3. Consider again the two-layered slope in case study 1. A system reliability analysis based on the results of probabilistic sensitivity-based FORM for the two failure modes is presented in Figure 6.9.

For failure mode-1, the initial search point in the n-space is (-3.0, 0). The physical meaning of this initial search point is that the value of first component (i.e., c_{u1}) is reduced by three standard deviations, while the value of the other component remains at its mean value. Using such an initial search point is equivalent to the

single-layer-strength-reduction technique as discussed in section 6.3. Moreover, the initial search point converges to the design point of failure mode-1, and yields the first minimum β value of 2.7971, compared to 2.8003 obtained by the Stratified RSM and 2.795 obtained by Low et al. (2011).

For failure mode-2, the initial search point in the n-space is (0, -3.0), which is equivalent to the strength-reduction of c_{u2} of soil layer 2. This initial search point ensures that the probabilistic sensitivity-based FORM is carried out with respect to failure mode-2 only (i.e., the failure surface always traverses soil layer 2). After four iterations, the initial search point converges to the design point of failure mode-2, and yields the second minimum β value of 2.9226, compared to 2.9441 obtained by the stratified RSM and 2.893 obtained by Low et al. (2011).

The Kounias-Ditlevsen system reliability bounds, based on the design points and reliability indices corresponding to the two failure modes, are {0.414%, 0.422%}. Note that the bounds are also in agreement with those reported in Table 6.3.

It may be noted that the system reliability bounds based on results of probabilistic sensitivity-based multi-mode FORM analyses are only reliable when the actual multi-mode limit state surfaces are linear. Otherwise, SORM analysis should be conducted to obtain more accurate bounds. One limitation of the probabilistic sensitivity-based FORM is that it cannot be extended to SORM analysis directly, since the performance functions are always not available. However, such a limitation can be easily overcome by using polynomials to approximate the actual performance functions at the design points obtained from the probabilistic sensitivity-based FORM analyses.

Transformation of variables from n-space to x-space

ProbDist	Variable	Para1	Para2	$\mathbf{x}^*$	Correlation matrix $\mathbf{R}$	$\mathbf{n}$
Lognorm	cu1	120	36	114.9	1 0	0.00
Lognorm	cu2	160	48	63.52	0 1	-3.00

Probabilistic Sensitivity-Based FORM Analysis of Failure Mode-1

$$\Delta n = -0.5 \quad g = F_s - 1$$

Step 1	Initial $\mathbf{n}_i$	-3.00	0.00	-2.78	0.00	-2.80	0.00
	$g(\mathbf{n}_i)$	-0.058		0.006		0.000	
Step 2	$g(\mathbf{n}_{i,new})$	-0.187	-0.058	-0.131	0.006	-0.137	0.000
Step 3	$\Delta g(\mathbf{n}_i)$	-0.129	0	-0.137	0	-0.137	0
Step 4	$\Delta g(\mathbf{n}_i)/\Delta n_j$	0.258	0	0.274	0	0.274	0
Step 5	$\mathbf{n}_{i+1}$	-2.7752	0	-2.797	0	-2.797	0
	β	2.7752		2.7971		2.7971	

Probabilistic Sensitivity-Based FORM Analysis of Failure mode-2

$$\Delta n = -0.1 \quad g = F_s - 1$$

Step 1	Initial $\mathbf{n}_i$	0.00	-3.00	-1.72	-2.09	-1.09	-2.66	-1.36	-2.59
	$g(\mathbf{n}_i)$	0.087		0.068		0.013		0.000	
Step 2	$g(\mathbf{n}_{i,new})$	0.073	0.070	0.059	0.046	0.003	-0.006	-0.010	-0.019
Step 3	$\Delta g(\mathbf{n}_i)$	-0.014	-0.017	-0.009	-0.022	-0.01	-0.019	-0.010	-0.019
Step 4	$\Delta g(\mathbf{n}_i)/\Delta n_j$	0.14	0.17	0.09	0.22	0.1	0.19	0.1	0.19
Step 5	$\mathbf{n}_{i+1}$	-1.7233	-2.093	-1.089	-2.6613	-1.361	-2.586	-1.361	-2.586
	β	2.7108		2.8754		2.9226		2.9226	

System Reliability Based on the Two-Mode FORM Results

Failure mode-1		Failure mode-2		CorrCoeff		BetaVector	LowerBound	UpperBound
n^*	β	n^*	β					
-2.80	2.797	-1.36	2.923	1	0.466	2.797	0.414%	0.422%
0.00		-2.59		0.466	1	2.923		

Figure 6.9 System reliability analysis based on the probabilistic sensitivity-based FORM

6.7 Summary

This chapter confines itself to slope system reliability analysis using a stratified response surface method, which is one of the original contributions of this thesis study. The method is straightforward in concept and is very suitable for coupling reliability methods with stand-alone numerical packages that only deal with deterministic stability analysis. The proposed method was applied to several slope stability problems reported in the literature. Results obtained by the proposed approach were compared with those obtained using Monte Carlo simulations and with the results of previous studies. It was observed that, failure probability from system reliability analysis can differ significantly from single-mode reliability analysis, especially when multi-layered slopes are considered. For all the four cases studied, the system failure probabilities by MC simulations are within the system reliability bounds obtained by Stratified RSM. The comparative studies have shown that the proposed approach is practical, efficient and promising for system reliability assessment of slopes.

It was shown in this chapter that the stratified response surfaces are almost linear when the slopes are horizontally layered, so that system reliability bounds based on FORM are adequately accurate. In contrast, the stratified response surfaces tend to be nonlinear for slopes with complex stratification. In such cases, SORM can be used to improve the system reliability assessment.

Although the analyses of slope stability in this chapter are performed by limit equilibrium methods of slices, the proposed stratified response surface method for system reliability analysis of slopes can also be combined with strength-reduction finite element or finite difference analysis of stability of slopes.

Chapter 7 Concluding Remarks and Recommended Future Studies

7.1 Summary and conclusions

A summary of the findings of this study and of the conclusions drawn is as follows:

Two techniques for reliability analysis involving implicit performance functions, namely response surface method (RSM) and probabilistic sensitivity analysis, were introduced in this study. A new formula of recursive algorithm for locating the design point in n -space was derived and procedures for probabilistic sensitivity-based FORM using the new formula were given. It was shown that while both render the strength-reduction finite element stability analysis of slopes by stand-alone numerical codes (e.g. Plaxis) extendable to probabilistic assessment, the proposed probabilistic sensitivity-based FORM is more efficient than the RSM-based FORM. The proposed approach was then applied to two case studies, the Clarence Cannon Dam and the James Bay dike.

Influence of two-dimensional (2-D) spatial variability of soil strength parameters on slope reliability was investigated. Two methods, namely the method of autocorrelated slices and the method of interpolated autocorrelations, were presented to model the 2-D spatial variation. The first method was directly related to the limit equilibrium method of slices; soil strength parameters at the mid-point of each slice-base were treated as spatially-correlated random variables. Based on this method, reliability analyses of a hypothetical embankment slope with soft clay foundation and of a hypothetical c - ϕ slope were carried out, with automatic search for the probabilistic

critical slip surface. It was found that the probabilistic critical slip surface may not necessarily coincide with the deterministic critical slip surface when 2-D spatial variation was modeled. At the same time, it was found that the probability of failure may be considerably overestimated when spatial variation was ignored in the reliability analysis (equivalent to 'perfect autocorrelation'). Thus, it was concluded in this study that a reliability analysis that does not account for spatial variability results in a more conservative assessment of the stability of slopes. Furthermore, it was shown that the influence of the vertical spatial variation on β and on P_f was much more significant than the influence of the horizontal spatial variation. This is because the horizontal autocorrelation distance δ_x is usually an order of magnitude greater than the vertical autocorrelation distance δ_y . The proposed second method (interpolated autocorrelations) for slope reliability analysis was shown to be as good as the method of autocorrelated slices in dealing with 2-D spatial variation. Some merits of the proposed method were pointed out. It was shown that modeling 2-D spatial variation in a FORM-based strength-reduction slope stability analysis could be carried out efficiently when using the proposed method of interpolated autocorrelations.

The study further investigated three-dimensional (3-D) slope stability by both deterministic analysis and probabilistic analysis. The investigation was first carried out using an idealized 3-D stability model so that explicit performance function of the 3-D failure could be derived by limit equilibrium analysis. The deterministic analysis showed that the factor of safety by 3-D analysis, F_{3D} , could be larger than its plane strain counterpart by more than 40 % when the width of failure mass b was sufficiently small. The value of F_{3D} decreases as b increases. The 3-D stability analysis thus implies an infinite deterministic critical width b^* along the longitudinal (i.e., out-of-plane) direction. This conclusion seems inconsistent with the observation from real cases. A more consistent investigation was further conducted by probabilistic analysis which incorporates the longitudinal spatial variation of the soil strength parameters. The

spatial-averaging approach proposed by Vanmarcke (1983) was first employed to model the longitudinal spatial variation. It was found that there is a probabilistic critical width of failure mass b^* which is very different from the deterministic analysis. Further study showed that b^* is dependent on the longitudinal spatial autocorrelation distance. As an alternative to the spatial-averaging approach, the spatial-autocorrelation approach with linear interpolation for the random field was proposed to deal with the longitudinal spatial variation. It was found that the two methods agree fairly well in dealing with the spatial variation when they were used under the same conditions.

The 3-D stability analysis was then investigated by strength reduction method. The deterministic strength reduction stability analyses were discussed in both undrained and drained conditions. In undrained condition, parametric studies showed that the contribution of end effects to the F_{3D} is affected by the width of failure mass and the cross-sectional slope inclination, but is not influenced by the undrained strength c_u at all. In drained conditions, parameter studies showed that the contribution of end effects will also be affected by the shear strength parameters c and ϕ . A probabilistic strength reduction 3-D stability analysis which incorporates the longitudinal spatial variation of undrained strength was also demonstrated, using the method of spatial-autocorrelation approach (interpolated autocorrelations). Similar to the probabilistic limit equilibrium analysis, the probabilistic strength reduction analysis also revealed that there is a probabilistic critical width of failure mass b^* which is dependent on the longitudinal autocorrelation distance used.

Finally, slope system reliability analysis involving multiple failure modes was studied, using the Kounias-Ditlevsen bimodal bounds theory. A simple procedure for approximating the slip surfaces corresponding to potential failure modes was suggested. When using stand-alone numerical packages for slope reliability analysis,

the stratified response surface method (which is one of the original contributions of this thesis) was proposed for practical use. The proposed method was applied to several slope stability problems reported in the literature. It was observed that, failure probability from system reliability analysis may differ significantly from single-mode reliability analysis, especially when multi-layered slopes were considered. For all the four cases studied, the system failure probabilities by MC simulations were within the system reliability bounds obtained by Stratified RSM. Moreover, the so-obtained stratified response surfaces were almost linear when the slopes were horizontally layered, so that system reliability bounds based on FORM were adequately accurate. In contrast, the stratified response surfaces could be nonlinear for slopes with complex stratification. In such cases, it was shown that SORM could be used to improve the system reliability assessment. The comparative studies showed that the proposed approach is practical, efficient and promising for system reliability assessment of slopes.

7.2 Recommendations for future work

Some suggestions for future research work are given below:

- (i) In the development of the probabilistic sensitivity-based FORM in the n -space, the Newton-Raphson type recursive algorithm was used for its simplicity and least computation requirement. Nevertheless, this algorithm may fail to converge in some situations, as noted by Haldar and Mahadevan (2000a). Thus, further investigation could be done by using more advanced optimization algorithms.
- (ii) The artificial neural network approach may be used as an alternative to the polynomial-based response surface method for cases with implicit performance functions.

- (iii) Bilinear interpolation scheme was used for the method of interpolated autocorrelations in dealing with 2-D spatial variation. Other interpolation schemes could be explored to improve the reliability analysis.
- (iv) In the reliability analysis of 3-D slope, only longitudinal spatial variation was considered. A more realistic study may be conducted by considering spatial variation in 3-D.
- (v) Also in the reliability analysis of 3-D slope, only local failure which shows a probabilistic critical width of failure mass was examined in detail. System failure can be considered in future studies.
- (vi) For the plane strain system reliability analysis of slopes (Chapter 6), failure was assumed to occur along circular slip surfaces. More advanced slope system reliability analysis may be carried out using LEM with non-circular slip surfaces or using strength-reduction FEM or FDM.
- (vii) This study focused mainly on developing and applying reliability methods to analyze slope stability, by assuming reasonable values or directly using previously reported values for uncertain soil properties. More research on the estimation of geotechnical uncertainties may be done in the future, either by statistical analysis of in-situ testing data (DeGroot and Baecher 1993; Jaksa et al. 2000; Dasaka and Zhang 2012) or using methods for probabilistic back analysis of geotechnical uncertainties based on observed performance (Gilbert et al. 1998; Wu et al. 2007; Zhang et al. 2010; Wu 2011).

Table of Contents for the Appendices

Appendix A Procedure for first-order reliability method	239
Appendix B Procedure for second-order reliability method.....	241
Appendix C An example illustrating the slope reliability analysis using the proposed method of interpolated autocorrelations	249
Appendix D Spreadsheet set-up of the reliability analysis of James Bay dike accounting for 2-D spatial variation.....	255
Appendix E User-defined FISH subroutine of FLAC for the plane-strain slope stability analysis	257
Appendix F User-defined FISH subroutine of FLAC3D for the three-dimensional slope stability analysis	261
Appendix G Closed-form solution of undrained slope stability analysis with circular slip surfaces.....	264

Appendix A

Procedure for first-order reliability method

The following procedure describes an efficient algorithm for the computation of the Hasofer-Lind reliability index. The procedure was originally proposed by Low and Tang (1997a, 2007). A commonly-cited example pertains to one plastic collapse failure mechanism of a one-bay frame (Der Kiureghian et al. 1987; Tvedt 1990; Hong 1999; Zhao and Ono 1999):

$$g(x) = x_1 + 2x_2 + 2x_3 + x_4 - 5x_5 - 5x_6 \quad (\text{A.1})$$

where the x_i 's are statistically independent lognormal random variables with means $\mu_1 = \dots = \mu_4 = 120$, $\mu_5 = 50$, $\mu_6 = 40$, and standard deviations $\sigma_1 = \dots = \sigma_4 = 12$, $\sigma_5 = 15$ and $\sigma_6 = 12$. A first-order reliability analysis of the example based on Low and Tang algorithm (2007) is carried out in a spreadsheet of Microsoft Excel as shown in Figure A.1. The quadratic form in Eq. (3.9), i.e., $\beta = \sqrt{\mathbf{n}^{*T} \mathbf{R}^{-1} \mathbf{n}^*}$, is used to compute the reliability index β . For the present case the correlation matrix $\mathbf{R}$ is a 6×6 identity matrix. The equivalent standard normal vector $\mathbf{n}$ is varied as six dimensionless numerical values; the corresponding x values are calculated from $\mathbf{n}$ based on a transformation function $x = F^{-1}[\Phi(n)]$; a user-defined VBA function x_i as shown in Figure A.2 can be used to automatically compute the x values (in column $\mathbf{x}^*$) in the spreadsheet. The β index is obtained using Solver, by minimizing the quadratic form subject to $g(x) = 0$. The first-order reliability index so obtained is 2.3482; the inferred failure probability is 0.94% based on $\Phi(-\beta)$. The design point in the original space is $\mathbf{x}^* = \{117.268, 115.242, 115.242, 117.268, 83.650, 55.451\}$.

Appendix B

Procedure for second-order reliability method

The procedure described below is based on the point-fitted paraboloid method of Der Kiureghian et al. (1987). The example problem (Figure A.1) as discussed in Appendix A is re-visited with second-order reliability analysis, which is carried out in a Microsoft Excel spreadsheet as shown in Figure B.1. The first-order reliability index is 2.348; the inferred failure probability is 0.0094 based on $\Phi(-\beta)$. The exact failure probability is 0.0121 (Der Kiureghian et al. 1987); the relative percentage error of the FORM estimate is therefore about -22%.

The design point in the original space and in the reduced space are $\mathbf{x}^* = \{117.268, 115.242, 115.242, 117.268, 83.650, 55.451\}$ and $\mathbf{n}^* = \{-0.181, -0.356, -0.356, -0.181, 1.900, 1.259\}$, respectively. The design point in the standard normal space $\mathbf{u}^*$ is equal to $\mathbf{n}^*$ because the random variables $\mathbf{x}$ are statistically independent. The direction cosines $\boldsymbol{\alpha}$ at the design point $\mathbf{u}^*$ are $\boldsymbol{\alpha} = \mathbf{u}^*/\beta$. If $\mathbf{x}$ are correlated, $\mathbf{u}^*$ can be obtained from $\mathbf{n}^*$ by Cholesky factorization $\mathbf{u}^* = \mathbf{L}^{-1}\mathbf{n}^*$ where $\mathbf{L}$ is the lower Cholesky matrix computed based on the correlation matrix $\mathbf{R}$. The user-defined matrix function *Cholesky* (Figure B.2) can be used to obtain the lower Cholesky matrix. The VBA code in Figure B.2 is adapted from an opensource Microsoft Excel add-in, the *Matrix.xla* v 2.3 (<http://digilander.libero.it/foxes>).

The second-order approximation is carried out in a rotated standard normal space $\mathbf{u}'$ in which the u'_n axis (corresponding to the last component of vector $\mathbf{u}'$) coincides with $\mathbf{u}^*$ based on Gram-Schmidt orthogonalization. The Gram-Schmidt rotation matrix $\mathbf{S}$ is computed based on direction cosines $\boldsymbol{\alpha}$, which are related to $\mathbf{u}^*$ by $\boldsymbol{\alpha} = \mathbf{u}^*/\beta$. The user-defined matrix function *GramSchmidt* (Figure B.3, adapted from Chan and Low

(2011)) can be used to obtain the Gram-Schmidt orthogonalization matrix. For $n = 6$ random variables, $2(n - 1) = 10$ points are needed to construct the point-fitted paraboloid. According to Der Kiureghian et al. (1987) the points to be selected are as follows:

$$\begin{array}{cccccccccc}
 -k\beta & +k\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & -k\beta & +k\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & -k\beta & +k\beta & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & -k\beta & +k\beta & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k\beta & +k\beta \\
 u'_{n,1-} & u'_{n,1+} & u'_{n,2-} & u'_{n,2+} & u'_{n,3-} & u'_{n,3+} & u'_{n,4-} & u'_{n,4+} & u'_{n,5-} & u'_{n,5+}
 \end{array} \quad (B.1)$$

The coefficient k is equal to 1 since $\beta < 3$. The unknown coordinates $u'_{n,i\pm}$ (for $i=1, 2, \dots, 5$) can be determined using the secant-rule method (Der Kiureghian et al. 1987) or by an user-defined VBA subroutine which calls Microsoft Excel's build-in *GoalSeek* program to automatically determine the unknowns (Chan and Low 2011).

Based on Chan and Low (2011)'s approach, a spreadsheet set-up extending the first-order reliability analysis to the second-order reliability analysis of the example problem is presented in Figure B.1. The column vector $\mathbf{n}$ (boxed) is back-calculated from $\mathbf{u}'$ by $\mathbf{n} = \mathbf{L}\mathbf{u} = \mathbf{L}\mathbf{S}^{-1}\mathbf{u}'$. The cell labeled as $g(\mathbf{x})$ contains the performance function and depends on the $\mathbf{x}$ values obtained from $\mathbf{n}$ via the user-defined VBA function x_i (Figure A.2). The program *CurvatureSORM* (Figure B.4), when run, automatically determines the unknown coordinates $u'_{n,i\pm}$ by calling Microsoft Excel's Goal Seek program and displays the computed principal curvatures in column κ . The user-defined function *ProbFailSORM* (Figure B.5) allows the user to compute second-order failure probabilities based on the principal curvatures obtained above and the reliability index β . Several commonly-used formulas computing the SORM failure probability are coded into the function. Obviously, most of the SORM formula obtained failure probability almost the same as the exact solution.

Figure B.1 Reliability analysis of the plastic collapse of a one-bay frame using the second-order reliability method (SORM)

```

Function Cholesky(Mat As Range)
Dim a, L() As Double, s As Double
a = Mat
n = Mat.Rows.Count
m = Mat.Columns.Count
If n <> m Then
    Cholesky = "?"
    Exit Function
End If

ReDim L(1 To n, 1 To n)
For j = 1 To n
    s = 0
    For k = 1 To j - 1
        s = s + L(j, k) ^ 2
    Next k
    L(j, j) = a(j, j) - s
    If L(j, j) <= 0 Then Exit For
    L(j, j) = Sqr(L(j, j))

    For i = j + 1 To n
        s = 0
        For k = 1 To j - 1
            s = s + L(i, k) * L(j, k)
        Next k
        L(i, j) = (a(i, j) - s) / L(j, j)
    Next i
Next j
Cholesky = L
End Function

```

Figure B.2 User-defined function for computing lower triangular matrix based on Cholesky factorization

```

Function GramSchmidt(DirectionCosines)
Dim dircos() As Double: Dim Tem() As Double
Dim Bef() As Double: Dim Aft() As Double
VarNum = DirectionCosines.Rows.Count
ReDim dircos(1 To VarNum) As Double: ReDim Tem(1 To VarNum) As Double
ReDim Bef(1 To VarNum, 1 To VarNum) As Double
ReDim Aft(1 To VarNum, 1 To VarNum) As Double
For i = 1 To VarNum
dircos(i) = DirectionCosines(i)
Next i
For i = 1 To VarNum
For j = 1 To VarNum
Bef(i, j) = 0
If i = j Then Bef(i, j) = 1: If i = VarNum Then Bef(i, j) = dircos(j)
Next j
Next i
For i = 1 To VarNum
x = VarNum - i + 1
If x = VarNum Then
For j = 1 To VarNum
Aft(x, j) = dircos(j)
Next j
End If
If x < VarNum Then
SumSq = 0
For k = 1 To VarNum
Tem(k) = 0
For L = 1 To VarNum - x
Numer = Aft(x + L, x)
For m = 1 To VarNum
Tem(m) = Tem(m) + Numer * Aft(x + L, m)
Next m
Next L
Aft(x, k) = Bef(x, k) - Tem(k): SumSq = SumSq + Aft(x, k) ^ 2
Next k
For j = 1 To VarNum
Aft(x, j) = Aft(x, j) / Sqr(SumSq)
Next j
End If
Next i
GramSchmidt = Aft
End Function

```

Figure B.3 User-defined function for Gram-Schmidt orthogonalization


```

Sub CurvaturesKappa()
'CrMatrix, beta, u_prime, PerFunc and Curvatures are pre-defined
'cells\blocks in spreadsheet.
'Also, calculation maximum change changed from the default 0.001 to 0.000001
'in Excel Options\Formulas

n = Range("CrMatrix").Rows.Count
beta = Range("beta").Value
If beta < 3 Then k = 1 Else k = 3 / beta
For i = 1 To (n - 1)
    Range("u_prime").Value = 0
    Range("u_prime").Cells(i).Value = -k * beta
    Range("u_prime").Cells(n).Value = beta
    Range("PerFunc").GoalSeek Goal:=0, ChangingCell:=Range("u_prime").Cells(n)
    u_prime_n1 = Range("u_prime").Cells(n).Value
    Range("u_prime").Cells(i).Value = k * beta
    Range("u_prime").Cells(n).Value = beta
    Range("PerFunc").GoalSeek Goal:=0, ChangingCell:=Range("u_prime").Cells(n)
    u_prime_n2 = Range("u_prime").Cells(n).Value
    kappa1 = 2 * (u_prime_n1 - beta) / (k * beta) ^ 2
    kappa2 = 2 * (u_prime_n2 - beta) / (k * beta) ^ 2
    tem1 = 1 / Sqr(1 + beta * kappa1): tem2 = 1 / Sqr(1 + beta * kappa2)
    Range("Curvatures").Cells(i).Value = 1 / beta * (4 / (tem1 + tem2) ^ 2 - 1)
Next i
End Sub

```

Figure B.4 User-defined procedure for determining principal curvatures of limit state surface

```

Function ProbFailSORM(FormulaName, beta, curvatures)
Dim kappa() As Double, temp3re(1 To 2) As Double, temp3im(1 To 2) As Double
CurNum = WorksheetFunction.CountA(curvatures): CellNum = curvatures.Count: k = 1
ReDim kappa(1 To CurNum) As Double
For i = 1 To CellNum
If curvatures(i) <> "" Then kappa(k) = curvatures(i): k = k + 1
Next i
With Application.WorksheetFunction
Select Case UCase(Trim(FormulaName)) 'trim leading/trailing spaces & convert to uppercase
Case "TVEDT": temp1 = 1: temp2 = 1: temp3re(1) = 1: temp3im(1) = 0
For i = 1 To CurNum
temp1 = temp1 / Sqr(1 + beta * kappa(i)): temp2 = temp2 / Sqr(1 + (beta + 1) * kappa(i))
temp3re(2) = ComplexQuotient(temp3re(1), temp3im(1), 1 + beta * kappa(i), kappa(i), 1)
temp3im(2) = ComplexQuotient(temp3re(1), temp3im(1), 1 + beta * kappa(i), kappa(i), 2)
temp3re(1) = temp3re(2): temp3im(1) = temp3im(2): temp3 = ComplexRoot(temp3re(1), temp3im(1), 1)
Next i
T1 = .NormSDist(-beta) * temp1: T2 = (beta * .NormSDist(-beta) - .NormDist(-beta, 0, 1, False)) *
(temp1 - temp2)
T3 = (beta + 1) * (beta * .NormSDist(-beta) - .NormDist(-beta, 0, 1, False)) * (temp1 - temp3)
ProbFailSORM = T1 + T2 + T3
Case "BREITUNG": temp = 1
For i = 1 To CurNum
temp = temp / Sqr(1 + beta * kappa(i))
Next i
ProbFailSORM = .NormSDist(-beta) * temp
Case "HOHENBICHLER_RACKWITZ": temp = 1
For i = 1 To CurNum
temp = temp / Sqr(1 + kappa(i) * .NormDist(beta, 0, 1, False) / .NormSDist(-beta))
Next i
ProbFailSORM = .NormSDist(-beta) * temp
Case "KOYLUOGLU_NIELSEN": temp1 = 1: temp2 = 1: temp3 = 1: temp4 = 1
For i = 1 To CurNum
If kappa(i) > 0 Then
temp1 = temp1 / Sqr(1 + kappa(i) * .NormDist(beta, 0, 1, False) / .NormSDist(-beta))
temp2 = temp2 / Sqr(1 + 0.5 * kappa(i) * .NormDist(beta, 0, 1, False) / .NormSDist(-beta))
ElseIf kappa(i) < 0 Then
temp3 = temp3 / Sqr(1 - kappa(i) * .NormDist(beta, 0, 1, False) / .NormSDist(beta))
temp4 = temp4 / Sqr(1 - 0.5 * kappa(i) * .NormDist(beta, 0, 1, False) / .NormSDist(beta))
End If
Next i
ProbFailSORM = .NormSDist(-beta) - .NormSDist(-beta) * temp4 * (1 - temp1) + .NormSDist(beta) *
temp2 * (1 - temp3)
Case "CAI_ELISHAKOFF": temp1 = 0: temp2 = 0: temp3 = 0
For i = 1 To CurNum
temp1 = temp1 + kappa(i): temp2 = temp2 + kappa(i) ^ 2: temp3 = temp3 + kappa(i) ^ 3
Next i
D1 = temp1 / 2: D2 = -0.5 * beta * (3 * temp2 / 4 + (temp1 ^ 2 - temp2) / 4)
D3 = (beta ^ 2 - 1) / 6 * (15 * temp3 / 8 + 9 * (temp1 * temp2 - temp3) / 8 + (temp1 ^ 3 + 2 * temp3 - 3 *
temp1 * temp2) / 8)
ProbFailSORM = .NormSDist(-beta) - .NormDist(beta, 0, 1, False) * (D1 + D2 + D3)
Case "HONG(P3)": temp = 1: m = .NormDist(beta, 0, 1, False) / .NormSDist(-beta): C1 = 0
For i = 1 To CurNum
temp = temp / Sqr(1 + kappa(i) * m): lamda = kappa(i) / 2
C1 = C1 + (1 / CurNum) * (.NormSDist(-beta) - CurNum * lamda / (1 + 2 * m * lamda))
/ .NormSDist(-beta)) * Exp(CurNum * m * lamda / (1 + 2 * m * lamda))

```

```

Next i
ProbFailSORM = .NormSDist(-beta) * temp * C1
Case "HONG(P4)": temp = 1: m = .NormDist(beta, 0, 1, False) / .NormSDist(-beta): C2 = 1 - CurNum /
3
For i = 1 To CurNum
temp = temp / Sqr(1 + kappa(i) * m): lamda = kappa(i) / 2
C2 = C2 + (1 / 3) * (.NormSDist(-beta - 3 * lamda / (1 + 2 * m * lamda)) / .NormSDist(-beta)) * Exp(3 * m
* lamda / (1 + 2 * m * lamda))
Next i
ProbFailSORM = .NormSDist(-beta) * temp * C2
End Select
End With
End Function

Function ComplexQuotient(Re1, Im1, Re2, Im2, code)
If code = 1 Then ComplexQuotient = (Re1 * Re2 + Im1 * Im2) / (Re2 ^ 2 + Im2 ^ 2)
If code = 2 Then ComplexQuotient = (Im1 * Re2 - Re1 * Im2) / (Re2 ^ 2 + Im2 ^ 2)
End Function

Function ComplexRoot(Re, Im, code)
R = Sqr(Re ^ 2 + Im ^ 2)
If code = 1 Then ComplexRoot = Sqr((R + Re) / 2): If code = 2 Then ComplexRoot = Im / Sqr(2 * (R +
Re))
End Function

```

Figure B.5 User-defined function for computing failure probability using SORM formulas. Functions ComplexQuotient and ComplexRoot are required by the Tvedt formula for complex number operations

Appendix C

An example illustrating the slope reliability analysis using the proposed method of interpolated autocorrelations

Figure C.1 shows a spreadsheet set-up for both the deterministic stability analysis and reliability analysis of the example embankment on soft clay (Figure 4.2) using the method of interpolated autocorrelations. Most of the VBA subroutines and Macros used were presented in (Low 2003; Low and Tang 1997) and can be found at <http://alum.mit.edu/www/bklow>. A new VBA pertaining to the method of interpolated autocorrelations is provided in Figure C.2. Interested readers can carry out the proposed slope reliability analysis by the following steps.

- Step 1: Set up the analytical model for slope stability analysis using methods of slices, e.g. the Spencer's method used in this study. Figure C.1a uses 32 slices.
- Step 2: Determine the mesh of discretization of the random field, and establish the vector/matrix of the random variables. For example, when the undrained shear strength c_u of the soft ground is taken to be a random field (as illustrated in Figure C.1), a **c_u -matrix** can be built with x- and y- coordinates denoting the position of the control points and with entries denoting the basic variables at control points; Figure C.1b uses a grid of 4×6 control points, i.e., 24 random c_u values for use in interpolation.
- Step 3: For all the random variables of **c_u -matrix**, re-arrange them in a single vector **c_u^*** and build the analytical model for reliability analysis; use the vectors of x- and y- coordinates and autocorrelation function Eq. (4.2) to obtain the 24×24 autocorrelation matrix **R**, Figure C.1c.
- Step 4: The VBA function code of Figure C.2 is invoked in the column labelled “**c**” in Figure C.1a, from slice #5 to slice #32 which have bases in the foundation clay. It

uses the 4×6 c_u control points of Figure C.1b to interpolate for the c_u values at the mid-points of the bases of slices #5-#32. The inputs required for this VBA function are the mid-point (x_{bmid} , y_{bmid}) of the base of each slice, the thickness of the clay (d_{max}), the 4×6 matrix of c_u control points (c_{um} , which is the **c_u Matrix** of Figure C.1b), and the 6 x-coordinate values (x_{cum}) and 4 y-coordinate values (y_{cum}) of Figure C.1b which define the grid of c_u control points. The factor of safety F_s depends on the c_u values at the bases of slices #5-#32, and hence is a function of the 4×6 matrix of c_u control points.

Step 5: Solver residing in Excel is invoked to optimize the β value (cell V19 of Figure C.1c), by changing the 24 control point c_u values (cells AA22-AA45), subject to the constraint that $F_s=1$. With a systematic search method, a reliability index of about 2.242 is obtained (that of $\eta_y = 1.0$ in Figure 4.10). Figure C.3 presents a user-defined subroutine in Excel-VBA which is created to implement the systematic search method.

(a) Spreadsheet set-up of Spencer's method for slope stability analysis

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
1									SPENCER method		λ' F_s DumEq							
2											0.090 1.000 1.000		ΣM $\Sigma Forces$		Varying λ			
3											0.00 0.00		Array formulas		TRUE			
4													Varying side-force angle λ					
5																		
6																		
7																		
8																		
9	slope angle																	
10	Ω	H	h_c	γ_w	P_w	M_w	r_u		x_c	y_c	R	x_{min}	x_{max}	dx
11	26.57	5	0.00	10	0	0	0		5.00	6.62	10.97	-3.75	15.85	0.625				
12									Center of Rotation		4.4	xcrest						
13	Units: m, kN/m ³ , kN/m ² , kN, or other consistent set of units.												10		$\lambda = \lambda' \sin \left[\frac{(x_i - x_0)}{(x_n - x_0)} \pi \right]$			
14									framed cells contain equations									
15	#	x	y_{bot}	y_{top}	γ_{ave}	c	ϕ	W	α_{ra}	u	I	P	T	E	λ	σ'	L_x	L_y
16	0	15.85	5.00	5.00										0	0.000			
17	1	15.32	2.91	5.00	19.00	5.00	30	10.4	1.324	0.00	2.15	0.00	10.77	-2.6	0.008	0.00	10.59	2.66
18	2	14.80	1.69	5.00	19.00	5.00	30	27.0	1.165	0.00	1.33	22.34	19.56	10.2	0.015	16.77	10.06	4.32
19	3	14.27	0.76	5.00	19.00	5.00	30	37.7	1.056	0.00	1.07	32.77	24.25	26.8	0.023	30.72	9.54	5.39
20	4	13.75	0.00	5.00	19.00	5.00	30	46.1	0.965	0.00	0.92	39.87	27.63	43.8	0.030	43.20	9.01	6.24
21	5	13.12	-0.75	5.00	19.00	19.67	0	63.8	0.879	0.00	0.98	73.55	19.25	88.1	0.038	75.14	8.44	6.99
22	6	12.50	-1.39	5.00	19.00	18.31	0	72.1	0.793	0.00	0.89	82.07	16.31	135.2	0.046	92.14	7.81	7.69
23	7	11.87	-1.93	5.00	19.00	17.07	0	79.1	0.715	0.00	0.83	87.75	14.12	182.1	0.054	106.05	7.19	8.28
24	8	11.25	-2.40	5.00	19.00	16.14	0	85.1	0.642	0.00	0.78	91.78	12.59	226.9	0.061	117.66	6.56	8.78
25	9	10.62	-2.80	5.00	19.00	15.64	0	90.2	0.572	0.00	0.74	94.78	11.62	268.5	0.067	127.52	5.94	9.22
26	10	10.00	-3.15	5.00	19.00	15.15	0	94.7	0.506	0.00	0.71	97.32	10.82	306.1	0.073	136.25	5.31	9.59
27	11	9.37	-3.44	4.69	19.00	14.75	0	96.6	0.442	0.00	0.69	97.56	10.20	338.6	0.078	141.15	4.69	9.91
28	12	8.75	-3.69	4.37	19.00	14.44	0	96.1	0.379	0.00	0.67	95.76	9.72	365.1	0.082	142.34	4.06	10.18
29	13	8.12	-3.90	4.06	19.00	14.15	0	95.1	0.319	0.00	0.66	93.95	9.31	385.7	0.085	142.78	3.44	10.41
30	14	7.50	-4.06	3.75	19.00	13.88	0	93.6	0.259	0.00	0.65	92.12	8.98	400.6	0.088	142.49	2.81	10.60
31	15	6.87	-4.19	3.44	19.00	14.17	0	91.7	0.201	0.00	0.64	90.19	9.04	409.8	0.090	141.44	2.19	10.74
32	16	6.25	-4.28	3.12	19.00	14.43	0	89.2	0.143	0.00	0.63	88.21	9.11	413.3	0.090	139.73	1.56	10.85
33	17	5.62	-4.33	2.81	19.00	14.57	0	86.4	0.086	0.00	0.63	86.14	9.14	411.6	0.090	137.35	0.94	10.92
34	18	5.00	-4.35	2.50	19.00	14.59	0	83.1	0.028	0.00	0.63	83.91	9.12	404.8	0.089	134.22	0.31	10.96
35	19	4.37	-4.33	2.19	19.00	14.59	0	79.4	-0.028	0.00	0.63	81.47	9.12	393.4	0.087	130.32	-0.31	10.96
36	20	3.75	-4.28	1.88	19.00	14.57	0	75.2	-0.086	0.00	0.63	78.77	9.14	377.6	0.084	125.60	-0.94	10.92
37	21	3.12	-4.19	1.56	19.00	14.43	0	70.7	-0.143	0.00	0.63	75.75	9.11	357.8	0.081	119.98	-1.56	10.85
38	22	2.50	-4.06	1.25	19.00	14.17	0	65.7	-0.201	0.00	0.64	72.32	9.04	334.5	0.076	113.42	-2.19	10.74
39	23	1.87	-3.90	0.94	19.00	13.88	0	60.2	-0.259	0.00	0.65	68.45	8.97	308.3	0.071	105.87	-2.81	10.60
40	24	1.25	-3.69	0.63	19.00	14.15	0	54.3	-0.319	0.00	0.66	64.19	9.31	279.3	0.065	97.56	-3.44	10.41
41	25	0.62	-3.44	0.31	19.00	14.44	0	47.9	-0.379	0.00	0.67	59.39	9.72	248.3	0.058	88.28	-4.06	10.18
42	26	0.00	-3.15	0.00	19.00	14.75	0	41.0	-0.442	0.00	0.69	53.93	10.19	216.0	0.051	78.03	-4.69	9.91
43	27	-0.62	-2.80	0.00	19.00	15.15	0	35.3	-0.506	0.00	0.71	49.92	10.82	182.3	0.043	69.88	-5.31	9.59
44	28	-1.25	-2.40	0.00	19.00	15.63	0	30.9	-0.572	0.00	0.74	47.44	11.62	146.9	0.035	63.83	-5.94	9.22
45	29	-1.87	-1.93	0.00	19.00	16.13	0	25.7	-0.642	0.00	0.78	44.25	12.58	110.3	0.027	56.74	-6.56	8.78
46	30	-2.50	-1.39	0.00	19.00	17.06	0	19.7	-0.715	0.00	0.83	40.52	14.12	73.1	0.018	48.96	-7.19	8.28
47	31	-3.12	-0.75	0.00	19.00	18.30	0	12.7	-0.793	0.00	0.89	36.10	16.30	35.9	0.009	40.52	-7.81	7.69
48	32	-3.75	0.00	0.00	19.00	19.66	0	4.5	-0.879	0.00	0.98	30.73	19.25	0.0	0.000	31.39	-8.44	6.99

(b) c_u -matrix containing the basic random variables at control points

	AB	AC	AD	AE	AF	AG	AH	AI	AJ	AK	AL	AM	AN	AO	AP	AQ
6	c _u -Matrix															
7	y-coordinate	0	20.5	19.9	19.9	19.9	20.5	21.5	<div>The c_u-matrix which connects with cells AA22-AA45 of Fig. A.1c, is used to interpolate cells F17-F48 of Fig. A.1a</div>							
8		-2	17.2	15.8	15.6	15.8	17.3	19.0								
9		-4	16.4	14.2	13.4	14.2	16.4	18.3								
10		-6	21.5	20.6	20.2	20.6	21.5	22.3								
11																
12																
13																

```

Function cu_mid(xbmid, ybmid, dmax, cum, x_cum, y_cum)
'cum: cu_matrix
'x_cum: vector of x-coordinates for cu_matrix
'y_cum: vector of y-coordinates for cu_matrix
If Abs(ybmid) > dmax Then
    cu_mid = 300
    Exit Function
End If
For i = 2 To x_cum.Count
    If xbmid <= x_cum(i) Then
        For j = 2 To y_cum.Count
            If ybmid >= y_cum(j) Then
                m = ((ybmid - y_cum(j - 1)) / (y_cum(j) - y_cum(j - 1))) * (cum(j, i - 1) - cum(j - 1, i - 1)) + cum(j - 1, i - 1)
                n = ((ybmid - y_cum(j - 1)) / (y_cum(j) - y_cum(j - 1))) * (cum(j, i) - cum(j - 1, i)) + cum(j - 1, i)
                cu_mid = (xbmid - x_cum(i - 1)) / (x_cum(i) - x_cum(i - 1)) * (n - m) + m
            End If
        Next j
    End If
Next i
End Function

```

Figure C.2 User-defined VBA function for bilinear interpolation of slice c_u of Figure A.1a using the control point c_u of Figure A.1b


```

'A user-defined subroutine for Systematic Search Method (SSM)
'The search is based on grid method
'The SSM aims at finding the minimum reliability index as well as the probabilistic critical slip
circle

Sub SSM()
Application.ScreenUpdating = False
'parameters for grid discretization
n_xc = 5: n_yc = 5
'number of radius with respect to each circle centre
n_R = 10
'definition of grid range
xc_min = 4.5: xc_max = 5.5: yc_min = 6: yc_max = 8
'definition of radius range
T_min = -6: T_max = -3
dT = (T_max - T_min) / n_R
'defining an initial value of reliability index
RI = 10
For i = 0 To n_xc
    xc = xc_min + i * (xc_max - xc_min) / n_xc
    For j = 0 To n_yc
        yc = yc_min + j * (yc_max - yc_min) / n_yc
        For k = 0 To n_R
            R = yc + Abs(T_min + k * dT)
            Range("xc").Value = xc
            Range("yc").Value = yc
            Range("Radius").Value = R
            Range("F").Value = 1
            Range("lamda").Value = 0
            Range("AA22:AA45").Value = Range("AB22:AB45").Value
            'solver need to be set up before running this sub
            Solversolve True

            If Range("Beta").Value <= RI Then
                RI = Range("Beta").Value
                xc_target = xc
                yc_target = yc
                R_target = R
            End If
        Next k
    Next j
Next i

Range("xc").Value = xc_target
Range("yc").Value = yc_target
Range("Radius").Value = R_target
Application.ScreenUpdating = True
Solversolve True
End Sub

```

Figure C.3 User-defined subroutine for systematic search method

**Spreadsheet set-up of the reliability analysis of James Bay dike
accounting for 2-D spatial variation**

#	x	y _{bot}	y _{top}	γ _{ave}	c	φ	W	α _{ra}	u	l	P	T	E	λ	σ'	L _x	L _y
0	127.82	12.00	12.00										0	0.000			
1	125.09	9.70	12.00	21.41	0.00	29.39	67.1	0.699	0.00	3.57	59.46	33.49	12.7	0.002	16.67	80.45	95.64
2	122.36	7.54	12.00	21.41	0.00	29.39	197.5	0.671	0.00	3.49	174.14	98.07	44.2	0.004	49.96	77.72	97.87
3	119.63	5.49	12.00	21.41	0.00	29.39	320.6	0.644	0.00	3.41	281.52	158.54	86.3	0.005	82.50	74.99	99.98
4	116.90	3.55	12.00	21.41	0.00	29.39	437.0	0.617	0.00	3.35	382.39	215.34	131.7	0.007	114.29	72.27	101.97
5	114.17	1.73	12.00	21.41	0.00	29.39	546.9	0.590	0.00	3.29	477.42	268.85	173.9	0.009	145.33	69.54	103.85
6	111.44	0.00	12.00	21.41	0.00	29.39	650.7	0.564	0.00	3.23	567.17	319.40	207.1	0.011	175.60	66.81	105.63
7	109.69	-1.06	12.00	21.30	47.00	0	467.5	0.543	0.00	2.05	485.44	96.15	375.5	0.012	237.29	64.57	107.02
8	107.94	-2.07	12.00	21.13	47.00	0	502.1	0.527	0.00	2.03	522.23	95.23	555.6	0.013	257.75	62.81	108.06
9	106.19	-3.06	12.00	20.98	47.00	0	535.3	0.510	0.00	2.01	557.01	94.35	745.4	0.014	277.46	61.06	109.06
10	104.43	-4.00	12.00	20.86	47.00	0	567.3	0.494	0.00	1.99	589.91	93.53	943.0	0.015	296.44	59.31	110.02
11	101.33	-5.59	12.00	20.72	31.89	0	1081.7	0.472	0.00	3.49	1148.07	111.33	1366.4	0.017	328.86	56.88	111.29
36	-6.22	-7.07	0.00	19.00	31.48	0	458.3	-0.417	0.00	3.40	555.03	107.03	1162.4	0.009	163.22	-50.66	114.25
37	-9.33	-5.59	0.00	19.00	31.61	0	373.8	-0.445	0.00	3.44	470.79	108.84	861.6	0.007	136.71	-53.77	112.82
38	-12.43	-4.00	0.00	19.00	31.89	0	283.2	-0.472	0.00	3.49	378.28	111.33	590.3	0.005	108.36	-56.88	111.29
39	-14.19	-3.06	0.00	19.00	47.00	0	117.4	-0.494	0.00	1.99	185.25	93.53	420.1	0.003	93.09	-59.31	110.02
40	-15.94	-2.07	0.00	19.00	47.00	0	85.4	-0.510	0.00	2.01	151.65	94.35	263.7	0.002	75.54	-61.06	109.06
41	-17.69	-1.06	0.00	19.00	47.00	0	52.1	-0.527	0.00	2.03	116.17	95.23	123.0	0.001	57.34	-62.81	108.06
42	-19.44	0.00	0.00	19.00	47.00	0	17.6	-0.543	0.00	2.05	78.71	96.15	0.0	0.000	38.48	-64.57	107.02

Figure D.1 Stability model using the Spencer's method (at the design point)

Beta	2.9128
------	--------

Figure D.2 Reliability analysis of the James Bay dike considering 2-D spatial variation

Appendix E

User-defined FISH subroutine of FLAC for the plane-strain slope stability analysis

This appendix presents a user-defined FISH subroutine; it is to be read in conjunction with the content of section 4.4.4. The subroutine performs a plane-strain stability analysis of a two-layer earth slope. The undrained shear strength c_u of the foundation layer is taken to be a random field.

```
;FISH subroutine created in FLAC version 7.0
```

```
;------
```

```
;generate slope geometry and mesh-----
```

```
grid 100,50  
gen -20.0,-6.0 -20.0,0.0 0.0,0.0 0.0,-6.0 i 1 30 j 1 25  
model mohr i 1 29 j 1 24  
gen 0,-6.0 0.0,0.0 10.0,0.0 10.0,-6.0 i 30 50 j 1 25  
model mohr i 30 49 j 1 24  
gen 10,-6 10,0 30,0 30,-6 i 50 80 j 1 25  
model mohr i 50 79 j 1 24  
gen 0,0 0,5 10,5 10,0 i 30 50 j 25 35  
model mohr i 30 49 j 25 34  
gen 10,0 10,5 30,5 30,0 i 50 80 j 25 35  
model mohr i 50 79 j 1 34  
gen line 0,0 10,5  
model null region 40,32
```

```
;define soil properties-----
```

```
prop density=1900.0 bulk=3.33E8 shear=3.34E6 cohesion=25000.0 &  
friction=0.0 tension=1E10 dila=0 j 1 25  
prop density=1900.0 bulk=4.15E7 shear=1.92E7 cohesion=5000.0 &  
friction=30.0 tension=1E10 dila=20 j 25 35
```

```
;apply boundary conditions -----
```

```
fix x y i 1 80 j 1  
fix x i 80 j 1 35
```

```

fix x i 1 j 1 25
set gravity=10.0

;define information of the control points -----
def control_points
  array cu_M(100,6) x_coord(6) y_coord(4)
  array cu_M_ASCII(100) x_coord_ASCII(6) y_coord_ASCII(4)
  ; Read the information of control points
  x_coord_input = 'x_coord.txt'
  y_coord_input = 'y_coord.txt'
  cu_M_input = 'cu_M.txt'

  status1 = open(x_coord_input, 0, 1)
  status1 = read(x_coord_ASCII,6)
  status1 = close
  status2 = open(y_coord_input, 0, 1)
  status2 = read(y_coord_ASCII,4)
  status2 = close
  status3 = open(cu_M_input, 0, 1)
  status3 = read(cu_M_ASCII,100)
  status3 = close

  loop i(1,4)
    y_coord(i)=parse(y_coord_ASCII(i),1)
  end_loop
  loop i(1,100)
    loop j(1,6)
      x_coord(j)=parse(x_coord_ASCII(j),1)
      cu_M(i,j)=parse(cu_M_ASCII(i),j)*1000
    end_loop
  end_loop
end

;interpolation of zone parameter -----
def parameter_interpol
  loop i (1,izones)
    loop j (1,jzones)
      yc = (y(i,j)+y(i+1,j)+y(i,j+1)+y(i+1,j+1))/4.0
      xc = (x(i,j)+x(i+1,j)+x(i,j+1)+x(i+1,j+1))/4.0

  if yc<=y_coord(1) then
    if yc>=y_coord(4) then
      if xc<=x_coord(6) then

```

```

    if xc>=x_coord(1) then

;interpolation
loop kk(2,6)
  k=kk-1
  If xc <= x_coord(kk) Then
    loop tt(2,4)
      t=tt-1
      cu_M_tt=tt + 4 * (numb_fos_run - 1)
      cu_M_t=cu_M_tt - 1
      If yc >= y_coord(tt) Then

cu_1_1 = ((yc - y_coord(t)) / (y_coord(tt) - y_coord(t)))
cu_1 = cu_M(cu_M_t,k)+cu_1_1 * (cu_M(cu_M_tt, k) - cu_M(cu_M_t, k))
cu_2_1 = ((yc - y_coord(t)) / (y_coord(tt) - y_coord(t)))
cu_2 = cu_M(cu_M_t,kk)+cu_2_1 * (cu_M(cu_M_tt, kk) - cu_M(cu_M_t, kk))
;cu_1 is the interpolation value of left column, cu_2 is the left interpolation-----

cu_interpol_1 = (xc - x_coord(k)) / (x_coord(kk) - x_coord(k))
cu_interpol = cu_interpol_1 * (cu_2 - cu_1) + cu_1
;cu_interpol is interpolated from values of m and n-----

cohesion(i,j)=cu_interpol

    else
    end_loop
  end_if

    else
    end_loop
  end_if

end_if
end_if
end_if
end_if

end_loop
end_loop
end

;a single run of stability analysis -----
def fos_run

```

```
array fos_vector (25)

loop numb_fos_run (1,25)
command
;call interpolation -----
control_points
parameter_interpol
;solve the factor of safety -----
solve fos ;associated
end_command
fos_vector(numb_fos_run)=fos
end_loop
end
```

Appendix F

User-defined FISH subroutine of FLAC3D for the three-dimensional slope stability analysis

This appendix presents a user-defined FISH subroutine; it is to be read in conjunction with the content of section 5.4.2. The subroutine performs a 3-D stability analysis of a homogeneous slope. The undrained shear strength c_u is taken to be a random field in the longitudinal direction.

```
;FISH subroutine created in FLAC-3D version 3.1
```

```
;------
```

```
;generate the geometries -----
```

```
gen zone uwedge p0 7.5 0 50 p1 add 0 5 0 p2 add 0 0 -50 p3 add -7.5 0 0 &  
size 5 30 10 ratio 1.05 0.95238 1 ;second component of size is z direction  
gen zone brick p0 0 -2 0 p1 add 7.5 0 0 p2 add 0 2 0 p3 add 0 0 50 &  
size 10 4 30 ratio 1 1 1.05
```

```
gen zone brick p0 7.5 0 0 p1 add 10 0 0 p2 add 0 5 0 p3 add 0 0 50 &  
size 10 5 30 ratio 1.05 1.05 1.05  
gen zone brick p0 7.5 -2 0 p1 add 10 0 0 p2 add 0 2 0 p3 add 0 0 50 &  
size 10 4 30 ratio 1.05 1 1.05
```

```
gen zone brick p0 -6 -2 0 p1 add 6 0 0 p2 add 0 2 0 p3 add 0 0 50 &  
size 8 4 30 ratio 0.95 1 1.05
```

```
gen merge 0.15  
gen zone reflect dip 180
```

```
;boundary conditions -----
```

```
fix x range y -2.1 -1.9  
fix y range y -2.1 -1.9
```

```
fix x range x -6.1 -5.9  
fix x range x 16.9 18.1
```



```

;fix x range z -0.1 0.1
;fix y range z -0.1 0.1

fix z range z -0.1 0.1
fix x range z 49.9 50.1
fix y range z 49.9 50.1
fix z range z 49.9 50.1

;initial conditions -----
initial density 1900
set g 0 -10 0
ini syy -6.25e4 grad 0 2.5e3 0
ini sxx -6.25e4 grad 0 0 0
ini szz -6.25e4 grad 0 0 0

;assign zones parameters -----
mod mo
pro bulk 2.67e8 shear 2.68e6 fric 0 coh 25e3 range y 0 5
pro bulk 2.67e8 shear 2.68e6 fric 0 coh 25e3 range y -2 0

;define the control points -----
def control_points
  array cu_M(11) z_coord(11) cu_M_ASCII(11)
  ; Read the information of control points
  cu_M_input = 'cu_M.txt'

  status = open(cu_M_input, 0, 1)
  status = read(cu_M_ASCII,11)
  status = close

  loop i(1,11)
    z_coord(i)=parse(cu_M_ASCII(i),1)
    cu_M(i)=parse(cu_M_ASCII(i),2)*1000
  end_loop
end

;call control points -----
control_points

;interpolation of zone parameter -----
def parameter_interpol
pnt= zone_head

```

```

loop while pnt # null
z_width= z_zcen(pnt)

if z_width<=z_coord(11) then
  if z_width>=z_coord(1) then

;interpolation
loop kk(2,11)
  k=kk-1
  If z_width <= z_coord(kk) Then
    cu_interpol =
cu_M(k)+(cu_M(kk)-cu_M(k))*(z_width-z_coord(k))/(z_coord(kk)-z_coord(k))
    ;cu_interpol is interpolated from values of m and n

z_prop(pnt,'cohesion')=cu_interpol

    else
    end_loop
  end_if

end_if
end_if

pnt=z_next(pnt)
end_loop
end

;call interpolation -----
parameter_interpol
plo blo pro coh

;calculate the factor of safety -----
solve fos

```

Appendix G

Closed-form solution of undrained slope stability analysis with circular slip surfaces

The stability model presented below extends the work of Low and Tang (1997b) to build a closed-form solution of factor of safety F_s . This study simplifies the previous work by conducting stability analysis in undrained condition only, so that all the equations are exactly derived. It will be shown that the search for the critical slip surface in various circumstances can be automatically done using a spreadsheet's optimization tool.

Figure G.1 shows an example slope, the toe of which is defined as the origin of the x - y coordinate system. The coordinate values of x_c and y_c represent the center of an arbitrary slip surface of circle, of radius R equal to $(y_c + D)$, where D is an arbitrary "tangent depth".

Figure G.1 A homogeneous slope in undrained stability analysis

The overturning moment M_o derives from the weight of block ABCDA, as follows.

$$M_o = \frac{1}{6} rH \left(3R^2 - 3y_c^2 - 3x_c^2 + 3y_c H - H^2 \cot^2 \Omega + 3x_c H \cot \Omega - H^2 \right) \quad (G.1)$$

The resisting moment M_R derives from the shear strength c_u along the arc CDEF. For conventional stability analysis that ignores the spatial variation, M_R is shown to be

$$M_R = (2\theta + \theta_h) R^2 c_u \quad (G.2)$$

Thus, the factor of safety with respect to arbitrary slip circle (x_c, y_c, R) is

$$F_s = \frac{M_R}{M_o} = \frac{6c_u R^2 (2\theta + \theta_h)}{rH \left(3R^2 - 3x_c^2 - 3y_c^2 + 3y_c H - H^2 \cot^2 \Omega + 3x_c H \cot \Omega - H^2 \right)} \quad (G.3)$$

However, Eq. (G.3) as a general closed-form solution of F_s is valid only for cases with deep slip circles (i.e. with tangent depth $D > 0$), as illustrated by Low and Tang (1997b). When a slip circle intersects the slope (e.g. a surface failure S_1 or S_2 as shown in Figure G.2), Eq. (G.3) has to be corrected in this study by substituting $(x_c - s_x)$, $(y_c - s_y)$ and $(y_{max} - y_{min})$ for the x_c , y_c and H , respectively. Note that y_{max} is the y -coordinate of entry point of the slip surface, y_{min} is the y -coordinate of exit point, s_x and s_y are indicators of surface failure and are defined as

$$s_x = s_y \cot \Omega \quad (G.4)$$

$$s_y = \frac{(x_c \cot \Omega + y_c) - \sqrt{(x_c \cot \Omega + y_c)^2 - (1 + \cot^2 \Omega)(x_c^2 + y_c^2 - R^2)}}{1 + \cot^2 \Omega} \quad (G.5)$$

The physical meanings of s_x and s_y are the x - and y -coordinates of the exit point when the exit point lies on the slope. Otherwise, both s_x and s_y equal to zero or there is no need to correct the Eq. (G.3).

Figure G.2 Three slip circles to be examined using the closed-form solution

For validation purpose, the factor of safety F_s using the above-mentioned equations is computed on three arbitrary slip circles as shown in Figure G.2. Bishop's simplified method of slices or ordinary method of slices is used for comparison. The results are shown in Table G.1

Table G.1 Comparison of F_s between the closed-form solution and Bishop's simplified and ordinary methods of slices

Slip No.	circle	Closed-form solution	Bishop's method			
			25 slices	50 slices	25 slices	50 slices
S		1.592	1.601	1.595	1.601	1.595
S ₁		2.254	2.259	2.256	2.259	2.256
S ₂		3.867	3.876	3.869	3.876	3.869

Since the example slope was examined in undrained condition, the Bishop's simplified method of slices and ordinary method of slices obtained the same result. It was found that the closed-form solution of F_s obtains practically the same results as compared to

Bishop's simplified method and ordinary method of slices on all the three slip circles, which in turn implies that the Eqs. (G.1) to (G.5) developed in this study are suitable for undrained stability analysis of slopes concerning various circular slip surfaces. It was also found that the more slices used, the closer the F_s between LEM of slices and the closed-form solution. Thus, although the Bishop's simplified method and ordinary method of slices are adequately accurate for undrained stability analysis, the number of slices used in the two methods affects to some extent the F_s . This minor error caused by the number of slices is easy to understand: the LEM of slices take the base-line of each of the slices to be a straight line other than an arc-segment, while this source of error is avoided in the closed-form solution.

With the closed-form solution of F_s , search for the critical factor of safety and the corresponding critical slip surface can be easily carried out, for example using the spreadsheet Solver. For the case in hand, the critical F_s was found to be 1.591, and the critical slip circle is $(x_c, y_c, R) = (10.00, 19.07, 29.07)$, as shown in Figure G.3.

(Units:m,kN/m,m,kN.m/m,kN/m ² ,kN/m ³)									
	xc	yc	R	H	D	Dmax	NumCu	cotΩ	
	10.00	19.07	29.07	10	10.00	10	1	2	
	θ	θ _h	M _R	M _O	F _s	s _x	s _y	xmin	xmax
	0.86	0.40	89099.70	56000.66	1.591	0	0	-11.9405	37.61837
	γ	18.5						ymin	ymax
	c _u	50						0	10

Figure G.3 Optimizing the critical F_s based on closed-form solution in a spreadsheet

It is worth pointing out that the critical slip circle is not sensitive to the value of undrained shear strength c_u for the case in hand. For example, one will get the same critical slip circle (with different F_s) if the c_u value as shown in Figure G.3 is changed to 100 kPa. This can easily be explained using the closed-form solution of Eq.(G.3).

References

- Adhikari, S. (2004). "Reliability analysis using parabolic failure surface approximation", Journal of Engineering Mechanics-ASCE, Vol. 130, No. 12, pp. 1407-1427.
- Alkasawneh, W., Husein Malkawi, A. I., Nusairat, J. H. and Albataineh, N. (2008). "A comparative study of various commercially available programs in slope stability analysis", Computers and Geotechnics, Vol. 35, No. 3, pp. 428-435.
- Alonso, E. (1976). "Risk analysis of slopes and its application to slopes in Canadian sensitive clays", Geotechnique, Vol. 26, No. 3, pp. 453-472.
- Ang, A. H.-S. and Tang, W. H. (1984). Probability concepts in engineering design, Vol. 2: Decision, Risk and Reliability, Wiley, New York.
- Au, S. K. and Beck, J. L. (2001). "Estimation of small failure probabilities in high dimensions by subset simulation", Probabilistic Engineering Mechanics, Vol. 16, No. 4, pp. 263-277.
- Au, S. K., Cao, Z. J. and Wang, Y. (2010). "Implementing advanced Monte Carlo simulation under spreadsheet environment", Structural Safety, Vol. 32, No. 5, pp. 281-292.
- Au, S. K., Ching, J. and Beck, J. L. (2007). "Application of subset simulation methods to reliability benchmark problems", Structural Safety, Vol. 29, No. 3, pp. 183-193.
- Auvinet, G. and González, J. L. (2000). "Three-dimensional reliability analysis of earth slopes", Computers and Geotechnics, Vol. 26, No. 3-4, pp. 247-261.
- Azzouz, A. S. and Baligh, M. M. (1983). "Loaded areas on cohesive slopes", Journal of Geotechnical Engineering-ASCE, Vol. 109, No. 5, pp. 724-729.
- Baecher, G. B. (1987). "Statistical analysis of geotechnical data", USACE Waterways Experiment Station, Vicksburg, Mississippi.
- Baecher, G. B. and Christian, J. T. (2003). Reliability and statistics in geotechnical engineering, Wiley, Chichester, U. K.
- Baecher, G. B. and Ingra, T. S. (1981). "Stochastic FEM in settlement predictions", Journal of the Geotechnical Engineering Division - ASCE, Vol. 107, No. 4, pp. 449-463.
- Baker, R. (1980). "Determination of the critical slip surface in slope stability

- computations", International Journal for Numerical & Analytical Methods in Geomechanics, Vol. 4, No. 4, pp. 333-359.
- Baker, R. (1981). "Tensile strength, tension cracks, and stability of slopes", Soils & Foundations, Vol. 21, No. 2, pp. 1-17.
- Baker, R. and Garber, M. (1978). "Theoretical analysis of the stability of slopes", Geotechnique, Vol. 28, No. 4, pp. 395-411.
- Baligh, M. M. and Azzouz, A. S. (1975). "End effects on stability of cohesive slopes", Journal of Geotechnical Engineering Division, ASCE, Vol. 101, No. 11, pp. 1105-1117.
- Barker, R. M., Duncan, J. M., Rojiani, K. B., Ooi, P. S., Tan, C. K. and Kim, S. G. (1991). "Manuals for the design of bridge foundations", Washington.
- Becker (1996). "Eighteenth Canadian Geotechnical Colloquium: Limit states design for foundations. Part 2. Development for the National Building Code of Canada", Canadian Geotechnical Journal, Vol. 33, pp. 984-1007.
- Berger, J. O. (1993). Statistical decision theory and Bayesian analysis, Springer-Verlag, New York.
- Bhattacharya, G., Jana, D., Ojha, S. and Chakraborty, S. (2003). "Direct search for minimum reliability index of earth slopes", Computers and Geotechnics, Vol. 30, No. 6, pp. 455-462.
- Bishop, A. W. (1955). "The use of slip circle in the stability analysis of slopes", Geotechnique, Vol. 5, No. 1, pp. 7-17.
- Bjerager, P. (1988). "Probability integration by directional simulation", Journal of Engineering Mechanics - ASCE, Vol. 114, No. 8, pp. 1285-1302.
- Bjerrum, L. (1972). "Embankments on soft ground", Proceedings of ASCE Special Conference on Performance of Earth and Earth-supported Structures, Purdue University, West Lafayette, Indiana, Vol. Vol. 2, pp. 1-54.
- Breitung, K. (1984). "Asymptotic approximations for multinormal integrals", Journal of Engineering Mechanics, ASCE, Vol. 110, No. 3, pp. 357-366.
- Breitung, K. (1989). "Asymptotic approximations for probability integrals", Probabilistic Engineering Mechanics, Vol. 4, No. 4, pp. 187-190.
- Bucher, C. G. and Bourgund, U. (1990). "A fast and efficient response surface approach

- for structural reliability problems", Structural Safety, Vol. 7, No. 1, pp. 57-66.
- Cai, G. Q. and Elishakoff, I. (1994). "Refined second-order reliability analysis", Structural Safety, Vol. 14, No. 4, pp. 267-276.
- Celestino, T. B. and Duncan, J. M. (1981). "Simplified search for noncircular slip surfaces", Proceedings of the International Conference on Soil Mechanics and Foundation Engineering, Vol. 3, pp. 25-32.
- Chan, C. L. (2012). "Reliability assessment of laterally loaded single piles", Ph.D. thesis, Nanyang Technological University, Singapore.
- Chan, C. L. and Low, B. K. (2011). "Practical second-order reliability analysis applied to foundation engineering", International Journal for Numerical and Analytical Methods in Geomechanics, DOI: 10.1002/nag.1057.
- Chen, R. H. and Chameau, J. L. (1983). "Three-dimensional limit equilibrium analysis of slopes", Geotechnique, Vol. 33, No. 1, pp. 31-40.
- Chen, W. F. and Mizuno, E. (1990). Nonlinear analysis in soil mechanics: Theory and implementation, Elsevier, Amsterdam.
- Chen, X. and Lind, N. C. (1983). "Fast probability integration by three-parameter normal tail approximation", Structural Safety, Vol. 1, No. 4, pp. 269-276.
- Chen, Z.-Y. (1992). "Random trials used in determining global minimum factors of safety of slopes", Canadian Geotechnical Journal, Vol. 29, pp. 225-233.
- Chen, Z.-Y. and Morgenstern, N. R. (1983). "Extensions to the generalized method of slices for stability analysis", Canadian Geotechnical Journal, Vol. 20, No. 1, pp. 104-119.
- Chen, Z.-Y. and Shao, C. M. (1988). "Evaluation of minimum factor of safety in slope stability analysis", Canadian Geotechnical Journal, Vol. 25, pp. 735-748.
- Cheng, Y. M. (2007). "Global optimization analysis of slope stability by simulated annealing with dynamic bounds and Dirac function", Engineering Optimization, Vol. 39, No. 1, pp. 17-32.
- Cheng, Y. M. and Lau, C. K. (2008). Slope stability analysis and stabilization - new methods and insight, Routledge, New York, NY, USA.
- Cheng, Y. M. and Yip, C. J. (2007). "Three-dimensional asymmetrical slope stability analysis extension of Bishop's, Janbu's, and Morgenstern-Price's techniques",

- Journal of Geotechnical and Geoenvironmental Engineering, Vol. 133, No. 12, pp. 1544-1555.
- Cheng, Y. M. and Zhu, L. J. (2005). "Unified formulation for two dimensional slope stability analysis and limitations in factor of safety determination", Soils & Foundations, Vol. 44, No. 6, pp. 121-128.
- Cherubini, C., Giasi, C. I. and Rethati, L. (1993). "The coefficients of variation of some geotechnical parameters", Canadian Geotechnical Journal, Vol. 44, No. 7, pp. 753-764.
- Ching, J., Phoon, K. K. and Hu, Y. G. (2009). "Efficient evaluation of reliability for slopes with circular slip surfaces using importance sampling", Journal of Geotechnical and Geoenvironmental Engineering, ASCE, Vol. 135, No. 6, pp. 768-777.
- Ching, J., Phoon, K. K. and Hu, Y. G. (2010). "Observations on limit equilibrium-based slope reliability problems with inclined weak seams", Journal of Engineering Mechanics, Vol. 136, No. 10, pp. 1220-1233.
- Cho, S. E. (2007). "Effects of spatial variability of soil properties on slope stability", Engineering Geology, Vol. 92, pp. 97-109.
- Cho, S. E. (2009). "Probabilistic stability analyses of slopes using the ANN-based response surface", Computers and Geotechnics, Vol. 36, pp. 787-797.
- Cho, S. E. (2010). "Probabilistic assessment of slope stability that considers the spatial variability of soil properties", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 136, No. 7, pp. 975-984.
- Chowdhury, R., Zhang, S. and Flentje, P. (2004). "Reliability updating and geotechnical back-analysis", Proceedings of the Advances in geotechnical engineering: the Skempton conference, London: Thomas Telford, pp. 815 - 821.
- Chowdhury, R. N. and Xu, D. W. (1995). "Geotechnical system reliability of slopes", Reliability Engineering and System Safety, Vol. 47, pp. 141-151.
- Christian, J. T. (2004). "Geotechnical engineering reliability: How well do we know what we are doing?", Journal of Geotechnical and Geoenvironmental Engineering - ASCE, Vol. 130, pp. 985-1004.
- Christian, J. T., Ladd, C. C. and Baecher, G. B. (1994). "Reliability applied to slope

- stability analysis", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 120, No. 12, pp. 27.
- Chugh, A. K. (2003). "On the boundary conditions in slope stability analysis", International Journal for Numerical and Analytical Methods in Geomechanics, Vol. 27, No. 11, pp. 905-926.
- Contreras, H. (1980). "The stochastic finite-element method", Computers & Structures, Vol. 12, No. 3, pp. 341-348.
- Cornell, C. A. (1967). "Bounds on reliability of structural systems", Journal of the Structural Division, ASCE, Vol. 93, No. ST1, pp. 171-200.
- Cornell, C. A. (1969). "A probability-based structural code", J. Amer. Conc. Inst., Vol. 66, No. 12, pp. 974-985.
- Cressie, N. A. C. (1993). Statistics for spatial data, Wiley, New York.
- Dasaka, S. M. and Zhang, L. M. (2012). "Spatial variability of in situ weathered soil", Geotechnique, Vol. 62, No. 5, pp. 375-384.
- Dawson, E. M., Roth, W. H. and Drescher, A. (1999). "Slope stability analysis by strength reduction", Geotechnique, Vol. 49, No. 6, pp. 835-840.
- DeGroot, D. J. and Baecher, G. B. (1993). "Estimating autocovariance of in-situ soil properties", Journal of Geotechnical Engineering - ASCE, Vol. 119, No. 1, pp. 147-166.
- Deng, J. H., Tham, L. G., Lee, C. F. and Yang, Z. Y. (2007). "Three-dimensional stability evaluation of a preexisting landslide with multiple sliding directions by the strength-reduction technique", Canadian Geotechnical Journal, Vol. 44, No. 3, pp. 343-354.
- Der Kiureghian, A. and Dakessian, T. (1998). "Multiple design points in first and second-order reliability", Structural Safety, Vol. 20, No. 1, pp. 37-49.
- Der Kiureghian, A. and De Stefano, M. (1991). "Efficient algorithm for second-order reliability analysis", Journal of Engineering Mechanics, Vol. 117, No. 12, pp. 2904-2923.
- Der Kiureghian, A. and Ke, J.-B. (1988). "The stochastic finite element method in structural reliability", Probabilistic Engineering Mechanics, Vol. 3, No. 2, pp. 83-91.

- Der Kiureghian, A., Lin, H. Z. and Hwang, S. J. (1987). "Second-order reliability approximations", Journal of Engineering Mechanics, ASCE, Vol. 113, No. 8, pp. 1208-1225.
- Der Kiureghian, A. and Liu, P. L. (1986). "Structural reliability under incomplete probability information", Journal of Engineering Mechanics, ASCE, Vol. 112, No. 1, pp. 85-104.
- Ditlevsen, O. (1973). "Structural reliability and the invariance problem", Research Report No. 22, Solid Mechanics Division, University of Waterloo, Waterloo, Canada.
- Ditlevsen, O. (1979a). "Generalized second moment reliability index", Journal of Structural Mechanics, Vol. 7, No. 4, pp. 435-451.
- Ditlevsen, O. (1979b). "Narrow reliability bounds for structural systems", Journal of Structural Mechanics, Vol. 7, No. 4, pp. 453-472.
- Ditlevsen, O. (1981). Uncertainty modelling: With applications to multidimensional civil engineering systems, McGraw-Hill Book Co., Inc., New York.
- Ditlevsen, O. (1985). "Comments on "first-order second-moment approximation in reliability of structural systems: critical review and alternative approach", by Krzysztof Dolinski", Structural Safety, Vol. 3, No. 1, pp. 59-61.
- Dolinski, K. (1983). "First-order second-moment approximation in reliability of structural systems - critical review and alternative approach", Structural Safety, Vol. 1, No. 3, pp. 211-231.
- Duncan, J. M. (1996). "State of the art: Limit equilibrium and finite-element analysis of slopes", Journal of Geotechnical Engineering, Vol. 122, No. 6, pp. 577-596.
- Duncan, J. M. and Chang, C. Y. (1970). "Nonlinear analysis of stress and strain in soils", Journal of the soil mechanics and foundations division, ASCE, Vol. 96, No. 5, pp. 1629-1653.
- Duncan, J. M. and Stark, T. D. (1992). "Soil strengths from back analysis of slope failures, *Stability and performance of slopes and embankments: II*", Geotechnical Special Publication 31, University of California, Berkeley, CA, pp. 890-904.
- Duncan, J. M. and Wright, S. G. (1980). "The accuracy of equilibrium methods of slope stability analysis", Engineering Geology, Vol. 16, No. 1-2, pp. 5-17.

- Duncan, J. M. and Wright, S. G. (2005). Soil strength and slope stability, John Wiley & Sons, Inc., Hoboken, New Jersey.
- El-Ramly, H. (2001). "Probabilistic analyses of landslide hazards and risks: Bridging theory and practice", Ph.D. thesis, University of Alberta, Alberta, Canada.
- El-Ramly, H., Morgenstern, N. R. and Cruden, D. M. (2002). "Probabilistic slope stability analysis for practice", Canadian Geotechnical Journal, Vol. 39, No. 3, pp. 665-683.
- Elhewy, A. H., Mesbahi, E. and Pu, Y. (2006). "Reliability analysis of structures using neural network method", Probabilistic Engineering Mechanics, Vol. 21, pp. 44-53.
- Engelund, S. and Rackwitz, R. (1993). "A benchmark study on importance sampling techniques in structural reliability", Structural Safety, Vol. 12, No. 4, pp. 255-276.
- Espinoza, R. D., Bourdeau, P. L. and Muhunthan, B. (1994). "Unified formulation for analysis of slopes with general slip surface", Journal of Geotechnical Engineering - ASCE, Vol. 120, No. 7, pp. 1185-1204.
- Fan, K., Fredlund, D. G. and Wilson, G. W. (1986). "An interslice force function for limit equilibrium slope stability analysis", Canadian Geotechnical Journal, Vol. 23, No. 3, pp. 287-296.
- Fellenius, W. (1922). Statens Järnvägars Geotekniska Kommission, Stockholm, Sweden.
- Fenton, G. A. (1999). "Random field modeling of CPT data", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 125, No. 6, pp. 486-498.
- Fenton, G. A. (2000). "Random field modeling of CPT data - Discussion", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 126, No. 12, pp. 1212-1214.
- Fenton, G. A. and Griffiths, D. V. (2002). "Probabilistic foundation settlement on spatially random soil", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 128, No. 5, pp. 381-390.
- Fenton, G. A. and Griffiths, D. V. (2003). "Bearing-capacity prediction of spatially random c-phi soils", Canadian Geotechnical Journal, Vol. 40, No. 1, pp. 54-65.
- Fenton, G. A. and Griffiths, D. V. (2005). "Three-dimensional probabilistic foundation

- settlement", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 131, No. 2, pp. 232-239.
- Fenton, G. A. and Vanmarcke, E. H. (1998). "Spatial variation in liquefaction risk", Geotechnique, Vol. 48, No. 6, pp. 819-831.
- Fiessler, B., Neumann, H. J. and Rackwitz, R. (1979). "Quadratic limit states in structural reliability", Journal of Engineering Mechanics, ASCE, Vol. 105, No. 4, pp. 661-676.
- Fredlund, D. G. and Krahn, J. (1977). "Comparison of slope stability methods of analysis", Canadian Geotechnical Journal, Vol. 14, No. 3, pp. 429-439.
- Fredlund, D. G., Zhang, Z. M. and Lam, L. (1992). "Effect of the axis of moment equilibrium in slope stability analysis", Canadian Geotechnical Journal, Vol. 29, No. 3, pp. 456-465.
- Frohlich, O. K. (1953). "The factor of safety with respect to sliding of a mass of soil along the arc of a logarithmic spiral", Proceedings of the 3rd International Conference on Soil Mechanics and Foundation Engineering, Switzerland, Vol. 2, pp. 230-233.
- Fujita, M. and Rackwitz, R. (1988). "Updating first-and second-order reliability estimates by importance sampling", Structural Engineering and Earthquake Engineering, Japan Society of Civil Engineers, Vol. 9, No. 1, pp. 53-59.
- Ghanem, R., Saad, G. and Doostan, A. (2007). "Efficient solution of stochastic systems: Application to the embankment dam problem", Structural Safety, Vol. 29, No. 3, pp. 238-251.
- Giam, P. S. K. and Donald, I. B. (1989). "Example problems for testing soil slope stability programs", Monash University.
- Gilbert, R. B., Wright, S. G., and Liedtke, E. (1998). "Uncertainty in back-analysis of slopes: Kettleman Hills case history", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 124, No. 12, pp. 1167-1176.
- Goh, A. T. C. (1999). "Genetic algorithm search for critical slip surface in multiple-wedge stability analysis", Canadian Geotechnical Journal, Vol. 36, pp. 382-391.
- Goh, A. T. C. and Kulhawy, F. H. (2006). "Reliability assessment of geotechnical

- serviceability state using neural networks", GeoCongress, pp. 1-4.
- Goh, A. T. C. K. F. H. (2003). "Neural network approach to model the limit state surface for reliability analysis", Canadian Geotechnical Journal, Vol. 40, pp. 1235-1244.
- Gollwitzer, S. and Rackwitz, R. (1988). "An efficient numerical solution to the multinormal integral", Probabilistic Engineering Mechanics, Vol. 3, No. 2, pp. 98-101.
- Greco, V. R. (1996). "Efficient Monte Carlo technique for locating critical slip surface", Journal of Geotechnical Engineering, ASCE, Vol. 122, No. 7, pp. 517-525.
- Griffiths, D. V. and Fenton, G. A. (2000). "Influence of soil strength spatial variability on the stability of an undrained clay slope by finite elements", Proceedings of Sessions of Geo-Denver 2000 - Slope Stability 2000, GSP 101.
- Griffiths, D. V. and Fenton, G. A. (2001). "Bearing capacity of spatially random soil: The undrained clay Prandtl problem revisited", Geotechnique, Vol. 51, No. 4, pp. 351-359.
- Griffiths, D. V. and Fenton, G. A. (2004). "Probabilistic slope stability analysis by finite elements", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 130, No. 5, pp. 507-518.
- Griffiths, D. V. and Fenton, G. A. (2009). "Probabilistic Settlement Analysis by Stochastic and Random Finite-Element Methods", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 135, No. 11, pp. 1629-1637.
- Griffiths, D. V., Fenton, G. A. and Denavit, M. D. (2007). "Traditional and advanced probabilistic slope stability analysis".
- Griffiths, D. V., Huang, J. and Fenton, G. A. (2009). "Influence of spatial variability on slope reliability using 2-D random fields", Journal of Geotechnical and Geoenvironmental Engineering, pp. 1367-1378.
- Griffiths, D. V., Huang, J. and Fenton, G. A. (2011). "Probabilistic infinite slope analysis", Computers and Geotechnics, Vol. 38, No. 4, pp. 577-584.
- Griffiths, D. V., Huang, J. S. and Fenton, G. A. (2009). "On the reliability of earth slopes in three dimensions", Proceedings of the Royal Society a-Mathematical Physical and Engineering Sciences, Vol. 465, No. 2110, pp. 3145-3164.

- Griffiths, D. V. and Lane, P. A. (1999). "Slope stability analysis by finite elements", Géotechnique, Vol. 49, No. 3, pp. 387-403.
- Griffiths, D. V. and Marquez, R. M. (2007). "Three-dimensional slope stability analysis by elasto-plastic finite elements", Géotechnique, Vol. 57, No. 6, pp. 537-546.
- Guan, X. L. and Melchers, R. E. (2001). "Effect of response surface parameter variation on structural reliability estimates", Structural Safety, Vol. 23, pp. 429-444.
- Haldar, A. and Mahadevan, S. (2000a). Probability, reliability and statistical methods in engineering design, Wiley, New York.
- Haldar, A. and Mahadevan, S. (2000b). Reliability assessment using stochastic finite element analysis, Wiley, New York.
- Harbitz, A. (1986). "An efficient sampling method for probability of failure calculation", Structural Safety, Vol. 3, No. 2, pp. 109-115.
- Harr, M. E. (1977). Mechanics of particulate media: A probabilistic approach, McGraw-Hill, New York.
- Harr, M. E. (1989). "Probabilistic estimates for multivariate analysis", Applied Mathematical Modelling, Vol. 13, No. 5, pp. 313-318.
- Hasofer, A. M. and Lind, N. C. (1974). "Exact and invariant second moment code format", Journal of the Engineering Mechanics, ASCE, Vol. 100, No. EM1, pp. 111-121.
- Hassan, A. M. and Wolff, T. F. (1999). "Search algorithm for minimum reliability index of earth slopes", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 125, No. 4, pp. 301-308.
- Hicks, M. A. and Spencer, W. A. (2010). "Influence of heterogeneity on the reliability and failure of a long 3D slope", Computers and Geotechnics, Vol. 37, No. 7-8, pp. 948-955.
- Hoeg, K. and Murarka, R. P. (1974). "Probabilistic analysis of a retaining wall", Journal of the Geotechnical Engineering Division, ASCE, Vol. 100, No. GT3, pp. 349-370.
- Hohenbichler, M. and Rackwitz, R. (1983). "First-order concepts in system reliability", Structural Safety, Vol. 1, No. 3, pp. 177-188.
- Hohenbichler, M. and Rackwitz, R. (1988). "Improvement of second-order reliability

- estimates by importance sampling", Journal of Engineering Mechanics-ASCE, Vol. 114, No. 12, pp. 2195-2199.
- Hong, H. P. (1999). "Simple approximations for improving second-order reliability estimates", Journal of Engineering Mechanics-ASCE, Vol. 125, No. 5, pp. 592-595.
- Honjo, Y., Wentsung, L. and Guha, S. (1994). "Inverse analysis of an embankment on soft clay by extended bayesian method", International Journal for Numerical and Analytical Methods in Geomechanics, Vol. 18, No. 10, pp. 709-734.
- Hovland, H. J. (1977). "Three-dimensional slope stability analysis method", Journal of Geotechnical Engineering Division, ASCE, Vol. 103, No. GT9, pp. 971-986.
- Huang, C. C. and Tsai, C. C. (2000). "New method for 3D and asymmetrical slope stability analysis", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 126, No. 10, pp. 917-927.
- Huang, C. C., Tsai, C. C. and Chen, Y. H. (2002). "Generalized method for three-dimensional slope stability analysis", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 128, No. 10, pp. 836-848.
- Huang, J., Griffiths, D. V. and Fenton, G. A. (2010b). "System reliability of slopes by RFEM", Soils & Foundations, Vol. 50, No. 3, pp. 345-355.
- Huang, J. S., Griffiths, D. V. and Fenton, G. A. (2010a). "Probabilistic analysis of coupled soil consolidation", Journal of Geotechnical and Geoenvironmental Engineering - ASCE, Vol. 136, No. 3, pp. 417-430.
- Hungr, O. (1987). "An extension of Bishop's simplified method of slope stability analysis to three dimensions", Geotechnique, Vol. 37, No. 1, pp. 113-117.
- Hunter, D. (1976). "An upper bound for the probability of a union", Journal of Applied Probability, Vol. 13, No. 3, pp. 597-603.
- Husein Malkawi, A. I., Hassen, W. F. and Sarma, S. k. (2001a). "An efficient search method for finding critical circular slip surface using Monte Carlo techniques", Canadian Geotechnical Journal.
- Husein Malkawi, A. I., Hassen, W. F. and Sarma, S. k. (2001b). "Global search method for locating general slip surface using Monte Carlo techniques", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 127, No. 8, pp.

688-698.

- Ibrahim, Y. (1991). "Observations on applications of importance sampling in structural reliability analysis", Structural Safety, Vol. 9, No. 4, pp. 269-281.
- Ishii, K. and Suzuki, M. (1987). "Stochastic finite-element method for slope stability analysis", Structural Safety, Vol. 4, No. 2, pp. 111-129.
- Jaksa, M. B. (2000). "Random field modeling of CPT data - Discussion", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 126, No. 12, pp. 1212-1214.
- Jaksa, M. B., Brooker, P. I. and Kaggwa, W. S. (1997a). "Modelling the spatial variability of the undrained shear strength of clay soils using geostatistics", Geostatistics Wollongong 96 - Proceedings of the Fifth International Geostatistics Congress, Wollongong, Australia, September 1996, pp. 1284-1295.
- Jaksa, M. B., Brooker, P. I. and Kaggwa, W. S. (1997b). "Inaccuracies associated with estimating random measurement errors", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 123, No. 5, pp. 393-401.
- Jaksa, M. B., Kaggwa, W. S. and Brooker, P. I. (1999). Experimental evaluation of the scale of fluctuation of a stiff clay.
- Janbu, N. (1973). "Slope stability computations", Embankment-Dam Eng., pp. 47-86.
- Ji, J., Liao, H. J. and Low, B. K. (2012). "Modeling 2-D spatial variation in slope reliability analysis using interpolated autocorrelations", Computers and Geotechnics, Vol. 40, pp. 135-146.
- Ji, J. and Low, B. K. (2012) "Stratified response surfaces for system probabilistic evaluation of slopes", Journal of Geotechnical and Geoenvironmental Engineering, ASCE, Vol. 138, No. 11, pp. 1398-1406.
- Kim, H.-S., Major, G. and Ross-Brown, D. M. (1978). "Applications of Monte Carlo techniques to slope stability analysis", Proceedings of the 19th US Symposium on Rock Mechanics, Nevada, USA, pp. 28-39.
- Kim, J. Y. and Lee, S. R. (1997). "An improved search strategy for the critical slip surface using finite element stress fields", Computers and Geotechnics, Vol. 21, No. 4, pp. 295-313.
- Kounias, E. G. (1968). "Bounds for the probability of a union, with applications", The

- Annals of Mathematical Statistics, Institute of Mathematical Statistics, Vol. 39, No. 6, pp. 2154-2158.
- Koyluoglu, H. U. and Nielsen, S. R. K. (1994). "New approximations for SORM integrals", Structural Safety, Vol. 13, No. 4, pp. 235-246.
- Kulhawy, F. H. (1992). "On evaluation of static soil properties", Proceedings of ASCE Specialty Symposim on Stability and Performance of Slopes and Embankments, New York, pp. 95-115.
- Kulhawy, F. H. and Mayne, P. W. (1990). "Manual on estimating soil properties for foundation design", Cornell Univ., Ithaca, NY (USA). .
- Kulhawy, F. H., Roth, M. S. and Grigoriu, M. D. (1991). "Some statistical evaluation of geotechnical properties", Proceedings of 6th International Conference on Applications of Statistics and Probability in Civil Engineering, Mexico City, pp. 705-712.
- Lacasse, S. and Nadim, F. (1996). "Uncertainties in characterizing soil properties", Proceedings of Uncertainty in the Geologic Environment Madison, Wisconsin, pp. 49-75.
- Lam, L. and Fredlund, D. G. (1993). "A general limit equilibrium-model for three-dimensional slope stability analysis", Canadian Geotechnical Journal, Vol. 30, No. 6, pp. 905-919.
- Ledesma, A., Gens, A. and Alonso, E. E. (1996). "Parameter and variance estimation in geotechnical backanalysis using prior information", International Journal for Numerical and Analytical Methods in Geomechanics, Vol. 20, No. 2, pp. 119-141.
- Lefebvre, G., Duncan, J. M. and Wilson, E. L. (1973). "Three-dimensional finite element analyses of dams", Journal of the Soil Mechanics and Foundations Division, Vol. 99, No. 7, pp. 495-507.
- Leshchinsky, D. and Huang, C. C. (1992). "Generalized three-dimensional slope stability analysis", Journal of Geotechnical Engineering-ASCE, Vol. 118, No. 11, pp. 1748-1764.
- Leshchinsky, D. and San, K. C. (1994). "Pseudostatic seismic stability of slopes - design charts", Journal of Geotechnical Engineering - ASCE, Vol. 120, No. 9, pp.

1514-1532.

- Leshchinsky, D. and Volk, J. C. (1985). "Stability charts for geotextile-reinforced walls", Transportation Research Board, National Research Council, Washington DC.
- Li, D., Chen, Y., Lu, W. and Zhou, C. (2011). "Stochastic response surface method for reliability analysis of rock slopes involving correlated non-normal variables", Computers and Geotechnics, Vol. 38, No. 1, pp. 58-68.
- Li, G. J. (2000). "Soft clay consolidation under reclamation fill and reliability analysis", Ph.D. Thesis, Nanyang Technological University, Singapore.
- Li, K. S. (1992). "Some common mistakes in probabilistic analysis of slopes", Proceedings of 6th International Symposium on landslides, Christchurch, New Zealand, pp. 475-480.
- Li, K. S. and White, W. (1987). "Rapid evaluation of the critical slip surface in slope stability problems", International Journal for Numerical & Analytical Methods in Geomechanics, Vol. 11, No. 5, pp. 449-473.
- Lind, N. C. (1983). "Modelling uncertainty in discrete dynamical systems", Applied Mathematical Modelling, Vol. 7, No. 3, pp. 146-152.
- Liu, P. L. and Der Kiureghian, A. (1986). "Optimization algorithms for structural reliability analysis", Report UCB/SESM-86/09, Dept. of Civil Engineering, University of California, Berkeley.
- Low, B. K. (2001). "Probabilistic slope analysis involving generalized slip surface in stochastic soil medium", Proceedings of the 14th SEAGC, A.A. Balkema: Hong Kong, pp. 825-830.
- Low, B. K. (2003). "Practical probabilistic slope stability analysis", Proc. Soil and Rock America, 12th Panamerican Conf. on Soil Mechanics and Geotechnical Engineering and 39th U.S. Rock Mechanics Symp., MIT, Cambridge, Massachusetts, Vol. 2, pp. 2777-2784.
- Low, B. K. (2008). "Practical reliability approach using spreadsheet", Reliability-based design in geotechnical engineering: Computations and applications, Eds. Phoon, K. K. London, Taylor & Francis, pp. 134-168.
- Low, B. K. (2010). "Slope reliability analysis: some insights and guidance for practitioners", Proceedings of The 17th Southeast Asian Geotechnical

- Conference, Taipei, Taiwan, pp. 231-234.
- Low, B. K., Gilbert, R. B. and Wright, S. G. (1998). "Slope reliability analysis using generalized method of slices", Journal of Geotechnical and Geoenvironmental Engineering, ASCE, Vol. 124, pp. 350-362.
- Low, B. K., Lacasse, S. and Nadim, F. (2007). "Slope reliability analysis accounting for spatial variation", Georisk, Vol. 1, No. 4, pp. 177-189.
- Low, B. K. and Tang, W. H. (1997a). "Efficient reliability evaluation using spreadsheet", Journal of Engineering Mechanics, ASCE, Vol. 123, No. 7, pp. 749-752.
- Low, B. K. and Tang, W. H. (1997b). "Reliability analysis of reinforced embankments on soft ground", Canadian Geotechnical Journal, Vol. 34, pp. 672-685.
- Low, B. K. and Tang, W. H. (1997c). "Probabilistic slope analysis using Janbu's generalized procedure of slices", Computers and Geotechnics, Vol. 21, No. 2, pp. 121-142.
- Low, B. K. and Tang, W. H. (2004). "Reliability analysis using object-oriented constrained optimization", Structural Safety, Vol. 26, pp. 69-89.
- Low, B. K. and Tang, W. H. (2007). "Efficient spreadsheet algorithm for first-order reliability method", Journal of Engineering Mechanics, Vol. 133, No. 12, pp. 1378-1387.
- Low, B. K., Zhang, J. and Tang, W. H. (2011). "Efficient system reliability analysis illustrated for a retaining wall and a soil slope", Computers and Geotechnics, Vol. 38, No. 2, pp. 196-204.
- Lowe, J. and Karafiath, L. (1959). "Stability of earth dams upon drawdown", Proceedings of the First PanAmerican Conference on Soil Mechanics and Foundation Engineering, Mexico City.
- Luckman, P. G., Der Kiureghian, A. and Sitar, N. (1987). "Use of stochastic stability analysis for Bayesian back calculation of pore pressures acting in a cut at failure", Proceedings of 5th Int Conf on Application of Statistics and Probability in Soil and Struct Engr, Vancouver. Ontario: University of Waterloo Press, pp. 922-929.
- Lumb, P. (1966). "Variability of natural soils", Canadian Geotechnical Journal, Vol. 3, pp. 74-97.

- Lumb, P. (1970). "Safety factors and the probability distribution of soil strength", Canadian Geotechnical Journal, Vol. 7, No. 3, pp. 225-242.
- Lumb, P. (1974). "Application of statistics in soil mechanics", Soil mechanics: New horizons, pp. 44-111.
- Madsen, H. O., Krenk, S. and Lind, N. C. (1986). Methods of structural safety, Prentice Hall, Englewood Cliffs, New Jersey.
- Major, G., Ross-Brown, D. M. and Kim, H.-S. (1978). "A general probabilistic analysis for three-dimensional wedge failures", Proceedings of the 19th US Symposium on Rock Mechanics, Nevada, USA, 45-56.
- Manoliu, I. and Marcu, A. (1993). "25 years of utilization of the limit state concept in the Romanian Code for geotechnical design", Proceedings of International Symposium on Limit State Design in Geotechnical Engineering, Copenhagen, pp. 533-542.
- Matsui, T. and San, K.-C. (1992). "Finite element slope stability analysis by shear strength reduction technique", Soils and Foundations, Vol. 32, No. 1, pp. 59-70.
- Matsuo, M. and Kuroda, K. (1974). "Probabilistic approach to design of embankments", Soils & Foundations, Vol. 14, No. 2, pp. 1-17.
- Melchers, R. E. (1989). "Importance sampling in structural systems", Structural Safety, Vol. 6, No. 1, pp. 3-10.
- Melchers, R. E. (1999). Structural reliability analysis and prediction, JOHN WILEY & SONS, New York.
- Meyerhof, G. G. (1993). "Development of geotechnical limit state design", Proceedings of International Symposium on Limit State Design in Geotechnical Engineering, Copenhagen, pp. 1-12.
- Meyerhof, G. G. (1995). "Development of geotechnical limit state design", Canadian Geotechnical Journal, Vol. 32, pp. 128-136.
- Morgenstern, N. R. and Price, V. E. (1965). "The analysis of the stability of general slip surfaces", Geotechnique, Vol. 15, No. 1, pp. 79-93.
- Morgenstern, N. R. and Price, V. E. (1967). "A numerical method for solving the equations of stability of general slip surfaces", Computer Journal, Vol. 9, No. 4,

pp. 388-393.

- Nataf, A. (1962). "Determination des distributions dont les marges sont donnees", Comptes Rendus de l'Academie des Sciences, Vol. 225, pp. 42-43.
- Naylor, D. J. (1982). "Finite elements and slope stability", Numerical Methods in Geomechanics, Vol. 92, pp. 229-244.
- Nguyen, V. U. (1985). "Determination of critical slope failure surfaces", Journal of Geotechnical Engineering, ASCE, Vol. 111, No. 2, pp. 238-250.
- Nguyen, V. U. and Chowdhury, R. N. (1984). "Probabilistic study of spoil pile stability in strip coal mines-two techniques compared", International Journal of Rock Mechanics, Mining Science and Geomechanics, Vol. 21, No. 6, pp. 303-312.
- Nguyen, V. U. and Chowdhury, R. N. (1985). "Simulation for risk analysis with correlated variables", Geotechnique, Vol. 35, No. 1, pp. 47-58.
- Oka, Y. and Wu, T. H. (1990). "System reliability of slope stability", Journal of Geotechnical Engineering, ASCE, Vol. 116, No. 8, pp. 1185-1189.
- Paice, G. M., Griffiths, D. V. and Fenton, G. A. (1996). "Finite element modeling of settlements on spatially random soil", Journal of Geotechnical Engineering-ASCE, Vol. 122, No. 9, pp. 777-779.
- Pantelidis, L. and Griffiths, D. V. (2012). "Stability assessment of slopes using different factoring strategies", Journal of Geotechnical and Geoenvironmental Engineering, ASCE, DOI: 10.1061/(ASCE)GT.1943-5606.0000678.
- Phoon, K. K. and Kulhawy, F. H. (1999a). "Characterization of geotechnical variability", Canadian Geotechnical Journal, Vol. 36, No. 4, pp. 612-624.
- Phoon, K. K. and Kulhawy, F. H. (1999b). "Evaluation of geotechnical property variability", Canadian Geotechnical Journal, Vol. 36, No. 4, pp. 625-639.
- Polidori, D. C., Beck, J. L. and Papadimitriou, C. (1999). "New approximations for reliability integrals", Journal of Engineering Mechanics-ASCE, Vol. 125, No. 4, pp. 466-475.
- Rackwitz, R. (2001). "Reliability analysis: A review and some perspectives", Structural Safety, Vol. 23, No. 4, pp. 365-395.
- Rackwitz, R. and Fiessler, B. (1976). "Note on discrete safety checking when using non-normal stochastic models for basic variables", MIT, Cambridge,

Massachusetts.

- Rackwitz, R. and Fiessler, B. (1978). "Structural reliability under combined random load sequences", Computers and Structures, Vol. 9, pp. 484-494.
- Rajashekhar, M. R. and Ellingwood, B. R. (1993). "A new look at the response surface approach for reliability analysis", Structural Safety, Vol. 12, No. 205-220.
- Rosenblatt, M. (1952). "Remarks on a multivariate transformation", Annals of Mathematical Statistics, Vol. 23, No. 3, pp. 470-472.
- Rosenblueth, E. and Esteva, L. (1972). Reliability basis for some Mexican codes. Probabilistic design of Reinforced Concrete Building Detroit, Mich., ACI Public. SP-31, ACI: 1-42.
- Rubinstein, R. Y. and Kroese, D. P. (2008). Simulation and the Monte Carlo method, Wiley, Hoboken, New Jersey.
- Sarma, S. K. (1973). "Stability analysis of embankments and slopes", Geotechnique, Vol. 23, No. 3, pp. 423-433.
- Schneider, H. R. (1999). "Definition and determination of characteristic soil properties", Proceedings of 12th International Conference on Soil Mechanics and Geotechnical Engineering, Hamburg, pp. 2271-2274.
- Schultze, E. (1975). "Some aspects concerning the application of statistics and probability to foundation structures", Proceedings of the 2nd International Conference on Applications of Statistics and Probability in Soil and Structural Engineering, Aachen, pp. 457-494.
- Schweiger, H. F. and Peschl, G. M. (2005). "Reliability analysis in geotechnics with the random set finite element method", Computers and Geotechnics, Vol. 32, No. 6, pp. 422-435.
- Sengupta, A. and Upadhyay, A. (2009). "Locating the critical failure surface in a slope stability analysis by genetic algorithm", Applied Soft Computing, Vol. 9, No. 1, pp. 387-392.
- Shinozuka, M. (1983). "Basic analysis of structural safety", Journal of Structural Engineering, ASCE, Vol. 109, No. 3, pp. 721-740.
- Shinozuka, M. and Deodatis, G. (1988). "Response variability of stochastic finite-element systems", Journal of Engineering Mechanics - ASCE, Vol. 114,

No. 3, pp. 499-519.

- Siegel, R. A., Lovell, C. W. and Kovacs, W. D. (1981). "Random surface generation in stability analysis", Journal of the Soil Mechanics and Foundations Division, Vol. 107, No. 7, pp. 996-1002.
- Singh, A. (1971). "How reliable is the factor of safety in foundation engineering?", Proceedings of 1st International Conference on Applications of Statistics and Probability in Soil and Structural Engineering, Hong Kong, pp. 389-424.
- Soubra, A. H. and Youssef Abdel Massih, D. S. (2010). "Probabilistic analysis and design at the ultimate limit state of oblique loaded strip footings", Geotechnique, Vol. 60, No. 4, pp. 275-285.
- Spanos, P. D. and Ghanem, R. (1989). "Stochastic finite-element expansion for random-media", Journal of Engineering Mechanics-ASCE, Vol. 115, No. 5, pp. 1035-1053.
- Spencer, E. (1967). "A method of analysis of the stability of embankments assuming parallel inter-slice forces", Géotechnique, Vol. 17, No. 1, pp. 11-26.
- Spencer, E. (1973). "Thrust line criterion in embankment stability analysis", Geotechnique, Vol. 23, No. 1, pp. 85-100.
- Stauffer, P. A. and Wright, S. G. (1984). "An examination of earth slope failures in Texas", Research Report 353-3F, Center for Transportation Research, University of Texas, Austin, TX.
- Stuedlein, A., Kramer, S., Arduino, P. and Holtz, R. (2012) "Geotechnical characterization and random field modeling of desiccated clay", Journal of Geotechnical and Geoenvironmental Engineering, ASCE, DOI: 10.1061/(ASCE)GT.1943-5606.0000723.
- Sudret, B. and Kiureghian, A. D. (2002). "Comparison of finite element reliability methods", Probabilistic Engineering Mechanics, Vol. 17, No. 4, pp. 337-348.
- Swan, C. C. and Seo, Y. K. (1999). "Limit state analysis of earthen slopes using dual continuum/FEM approaches", International Journal for Numerical and Analytical Methods in Geomechanics, Vol. 23, No. 12, pp. 1359-1371.
- Tvedt, L. (1988). "Second-order reliability by an exact integral", Proceedings of 2nd IFIP Working Conference on Reliability and Optimization on Structural Systems,

- Berlin, Germany, pp. 377-384.
- Tvedt, L. (1990). "Distribution of quadratic forms in normal space: Application to structural reliability", Journal of Engineering Mechanics, ASCE, Vol. 116, No. 6, pp. 1183-1197.
- U.S. Army Corps of Engineers (1999). Risk-based analysis in geotechnical engineering for support of planning studies, **Appendix A:** A11-A23, <http://www.usace.army.mil/publications/eng-tech-ltrs/etl1110-2-556/toc.html>.
- Ugai, K. (1989). "A method of calculation of total factor of safety of slopes by elasto-plastic FEM", Soils & Foundations, Vol. 29, No. 2, pp. 190-195.
- Ugai, K. and Leshchinsky, D. (1995). "Three-dimensional limit equilibrium and finite element analyses: A comparison of results", Soils and Foundations, Vol. 35, No. 4, pp. 1-7.
- Vanmarcke, E. (2011). "Risk of limit-equilibrium failure of long earth slopes: How it depends on length", GeoRisk 2011, ASCE, pp. 1-24.
- Vanmarcke, E. and Grigoriu, M. (1983). "Stochastic finite-element analysis of simple beams", Journal of Engineering Mechanics - ASCE, Vol. 109, No. 5, pp. 1203-1214.
- Vanmarcke, E. H. (1977a). "Probabilistic modeling of soil profiles", Journal of the Geotechnical Engineering Division, Vol. 103, No. 11, pp. 1227-1246.
- Vanmarcke, E. H. (1977b). "Reliability of earth slopes", Journal of the Geotechnical Engineering Division, Vol. 103, No. GT11, pp. 1247-1265.
- Vanmarcke, E. H. (1980). "Probabilistic stability analysis of earth slopes", Engineering Geology, Vol. 16, No. 1, pp. 29-50.
- Vanmarcke, E. H. (1983). Random fields analysis and synthesis, MIT Press, Cambridge.
- Wang, Y., Cao, Z. J. and Au, S. K. (2011). "Practical reliability analysis of slope stability by advanced Monte Carlo simulations in a spreadsheet", Canadian Geotechnical Journal, Vol. 48, No. 1, pp. 162-172.
- Wei, W. B., Cheng, Y. M. and Li, L. (2009). "Three-dimensional slope failure analysis by the strength reduction and limit equilibrium methods", Computers and Geotechnics, Vol. 36, pp. 70-80.

- Whitman, R. V. and Bailey, W. A. (1967). "Use of computers for slope stability analyses", Journal of the soil mechanics and foundations division, ASCE, Vol. 93, No. 4, pp. 475-498.
- Wolff, T. F. (1985). "Analysis and design of embankment dam slopes: A probabilistic approach", Ph.D., Purdue University.
- Wolff, T. F. (1991). "Embankment reliability versus factor of safety - Before and after slide repair", International Journal for Numerical and Analytical Methods in Geomechanics, Vol. 15, No. 1, pp. 41-50.
- Wolff, T. F. and Harr, M. E. (1987). "Slope design for earth dams", Reliability and Risk Analysis in Civil Engineering 2, Proceedings of the 5th International Conference on Applications of Statistics and Probability in Soil and Structural Engineering, Vancouver, BC, Canada, pp. 725-732.
- Wong, F. S. (1985). "Slope reliability and response surface method", Journal of Geotechnical Engineering - ASCE, Vol. 111, No. 1, pp. 32-53.
- Wong, S. M., Hobbs, R. E. and Onof, C. (2005). "An adaptive response surface method for reliability analysis of structures with multiple loading sequences", Structural Safety, Vol. 27, No. 4, pp. 287-308.
- Wright, S. G. (1969). "A study of slope stability and the undrained shear strength of clay shales", Ph.D. dissertation, University of California, Berkeley, CA.
- Wu, T. H. (2008). "Reliability analysis of slopes", Reliability-based design in geotechnical engineering: Computations and applications, Eds. Phoon, K. K. London, Taylor & Francis, pp. 413-447.
- Wu, T. H. (2011). "2008 Peck Lecture: The Observational Method: Case History and Models", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 137, No. 10, pp. 862-873.
- Wu, T. H., Tang, W. H., Sangrey, D. A. and Baecher, G. B. (1989). "Reliability of offshore foundations - State of the art", Journal of Geotechnical Engineering - ASCE, Vol. 115, No. 2, pp. 157-178.
- Wu, T. H., Zhou, S. Z. and Gale, S. M. (2007). "Embankment on sludge: predicted and observed performances", Canadian Geotechnical Journal, Vol. 44, No. 5, pp. 545-563.

- Wu, Y. T. and Wirsching, P. H. (1987). "New algorithm for structural reliability estimation", Journal of Engineering Mechanics, ASCE, Vol. 113, No. 9, pp. 1319-1336.
- Xu, B. (2006). "Behaviour of reinforced embankment on soft ground and reliability analysis", Ph.D. thesis, Nanyang Technological University, Singapore.
- Xu, B. and Low, B. K. (2006). "Probabilistic stability analyses of embankments based on finite-element method", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 132, No. 11, pp. 1444-1454.
- Xue, J. F. and Gavin, K. (2007). "Simultaneous determination of critical slip surface and reliability index for slopes", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 133, No. 7, pp. 878-886.
- Yamagami, T. and Ueta, Y. (1988). "Search for critical slip line in finite element stress fields by dynamic programming", Proceedings of the 6th international conference on numerical methods in geomechanics, Rotterdam: A.A. Balkema.
- Yüccemen, M. S. and Al-Homoud, A. S. (1990). "Probabilistic three-dimensional stability analysis of slopes", Structural Safety, Vol. 9, No. 1, pp. 1-20.
- Zhang, J., Tang, W. H. and Zhang, L. M. (2010). "Efficient probabilistic back-analysis of slope stability model parameters", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 136, No. 1, pp. 99-109.
- Zhang, J., Zhang, L. M. and Tang, W. H. (2011). "New methods for system reliability analysis of soil slopes", Canadian Geotechnical Journal, Vol. 48, No. 7, pp. 1138-1148.
- Zhang, K., Li, W. and Shi, J. (2011). "FEM stability analysis on soil slope with different constitutive models", Geotechnical Special Publication No. 216, ASCE, pp. 26-33.
- Zhang, L. L., Zhang, J., Zhang, L. M. and Tang, W. H. (2010). "Back analysis of slope failure with Markov chain Monte Carlo simulation", Computers and Geotechnics, Vol. 37, No. 7-8, pp. 905-912.
- Zhao, Y. G. and Ono, T. (1997). "An empirical reliability index based on SORM", Proceedings of 7th International Conference on Structural Safety and Reliability, Kyoto, Japan.

- Zhao, Y. G. and Ono, T. (1999a). "New approximations for SORM: Part 1", Journal of Engineering Mechanics, Vol. 125, No. 1, pp. 79-85.
- Zhao, Y. G. and Ono, T. (1999b). "New approximations for SORM: Part 2", Journal of Engineering Mechanics, Vol. 125, No. 1, pp. 86-93.
- Zhao, Y. G. and Ono, T. (2001). "Moment methods for structural reliability", Structural Safety, Vol. 23, No. 1, pp. 47-75.
- Zheng, H. (2009). "Eigenvalue Problem from the Stability Analysis of Slopes", Journal of Geotechnical and Geoenvironmental Engineering, Vol. 135, No. 5, pp. 647-656.
- Zheng, H., Sun, G. and Liu, D. (2009). "A practical procedure for searching critical slip surfaces of slopes based on the strength reduction technique", Computers and Geotechnics, Vol. 36, pp. 1-5.
- Zheng, Y. R. and Zhao, S. Y. (2004). "Application of strength reduction FEM in soil and rock slope", Chinese Journal of Rock Mechanics and Engineering, Vol. 19 No. 34.
- Zienkiewicz, O. C., Humpheson, C. and Lewis, R. W. (1975). "Associated and non-associated viscoplasticity and plasticity in soil mechanics", Géotechnique, Vol. 25, No. 4, pp. 671-689.
- Zou, J. Z., Williams, D. J. and Xiong, W. L. (1995). "Search for critical slip surfaces based on finite-element method", Canadian Geotechnical Journal, Vol. 32, No. 2, pp. 233-246.