Adaptive control charts for monitoring process mean and variance

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ADAPTIVE CONTROL CHARTS FOR
PROCESS MEAN AND VARIANCE

WANG PENG HUI

2006

ADAPTIVE CONTROL CHARTS FOR MONITORING PROCESS MEAN AND VARIANCE

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**SCHOOL OF MECHANICAL AND AEROSPACE ENGINEERING
NANYANG TECHNOLOGICAL UNIVERSITY**

2006

Adaptive Control Charts for Monitoring Process Mean and Variance

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School of Mechanical and Aerospace Engineering

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Doctor of Philosophy

2006

Abstract

Adaptive control charts have been proposed as effective modifications of conventional static control charts. The adaptive features discussed intensively in open literature are variable sampling features, which include the Variable Sampling Interval (VSI), Variable Sample Size (VSS) and Variable Sample Size and Sampling Interval (VSSI) features. The general idea is to vary the time interval to the next sample and/or vary the next sample size depending on the current value of the control statistic, in order to enhance the detection effectiveness of the control charts.

Besides, CUMulative SUM (CUSUM) control charts are widely used in industry to monitor statistic of a quality parameter in a production process, for its advantage in detecting small shifts from the target value. Although the variable sampling features have been developed for some control charts, the study on their integration with the CUSUM statistic(s) for monitoring process mean and variance is insufficient, because of computational difficulties. However, this integration is believed to enhance the control chart performance. As a solution, we consider a single CUSUM statistic, *WLC* (Weighted Loss-function CUSUM), which can monitor two-sided mean shift and an increasing variance shift. Thus, the scheme is named VSSI *WLC*. An algorithm is developed for the optimization design of the VSSI *WLC* scheme. With little adjustment, the algorithm is applicable to the optimization designs of the VSI *WLC* and the VSS *WLC* schemes. In addition, the optimization design of a more complex VSSI 3-CUSUM scheme comprising three CUSUM statistics is also developed. The performance of the proposed and related schemes is compared in terms of steady-state Average Time to Signal (*ATS*). Results show that the VSSI *WLC* and VSI *WLC* schemes are comparable while the VSSI 3-CUSUM scheme is a little inferior: all of them outperform the static and VSSI $\bar{x}$ &*S* charts. The diagnosis studies show that the VSSI and VSI *WLC* schemes are similar in classifying the shift types, both are good at diagnosing joint shift. The VSSI 3-CUSUM scheme is better for diagnosing pure mean and pure variance shifts. Considering performance and simplicity for design and operation, the VSI *WLC* scheme is most recommended.

Variable sampling features are not feasible sometimes, due to extra sampling effort. Therefore, we propose an internal adaptive control chart for monitoring mean and variance with fixed sample size and sampling interval. It is called *AL* (Adaptive Loss-function) chart, which adapts internal parameters by a computer program depending on current value of the control statistic. As a result, its sampling operation is as simple as static control chart's. The formulation and design of the *AL* chart are developed and the performance is compared based on steady-state Average Run Length (*ARL*). From the results, the *AL* chart outperforms other static charts, and is even competitive with the VSSI $\bar{X}$ &*S* charts. The *AL* chart is also found more capable of diagnosing joint shift than others.

Finally is a study for autocorrelated process. Observations are modeled with a common time series model, AR(1). For monitoring mean and variance shifts, two residuals approaches are compared from both statistical analysis and numerical experiments. The approaches can easily extend adaptive control chart uses. Taking the *AL* chart for instance, comparisons under different degrees of autocorrelation show the advantage of the *AL* chart sustains. AR(2) and ARMA(1, 1) models are also discussed.

Computer programs developed in the thesis are available through author's email: wangpenghui@pmail.ntu.edu.sg

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Chapter 1

Introduction

1.1 Background

Monitoring the state of a process through statistics of process variables such as sample mean and sample variance, is an interesting research topic in the field of Quality Control. A process is considered to be in a “state of statistical control” if the variations of the observed sampling results can be attributed to a constant system of chance causes (American Society for Quality Control, 1983). The use of statistical methods to detect the change in a process under statistical control is labeled as Statistical Process Control (SPC). The control chart concept introduced in the 1920’s by Walter Shewhart sets a milestone in SPC methods. It is an online SPC tool for achieving process stability and improving capability of operation around the target with little variability.

For over decades, quality has been established as a key competitive priority in the world of business. As defined by Crosby (1979), quality is conformant to requirements or specifications. Therefore, the objective to detect process change fast never ends. Control chart, as the most widely used and effective tool of SPC, is naturally of interest for advancement. Firstly, without raising the probability of false alarm, the faster a control chart responds to a change the less non-conforming items are produced. Next, a control chart with good diagnosis ability will assist tracking down the assignable causes. Therefore, the performance of a control chart can be generally evaluated from these two aspects. By using an effective control chart to monitor process, organizations can expect an enormous savings due to a dramatic reduction in rework and scrap.

With great facilitation of contemporary technological advances in automated inspection and available computational power, implementation of elaborate monitoring schemes comes to be permissible. Therefore, a group of control charts called adaptive or dynamic charts because of the adaptation of charting parameters during operation have been proposed in recent literature. A development in adaptive control charts is the Variable Sampling Interval (VSI) control chart which allows the time intervals between samples to vary depending on the value of the control statistic. With the similar idea, Variable Sample Size (VSS) control chart and Variable Sample Size and Sampling Interval (VSSI) control chart have also been proposed. Variable sampling schemes are used here to denote all of the control charts with the above three adaptive features. Many papers on variable sampling schemes have shown the advantages of these charts in the improvement of control chart performance compared to the static counterparts.

1.2 Motivation

In SPC, when dealing with a quality characteristic that is a variable, it is usually necessary to monitor both the mean value of the quality characteristic and its variability (Montgomery 2001). However, with a few exceptions, most works in literature are focused on the adaptive schemes for monitoring mean shift. Costa (1999a) studied VSSI $\bar{X}$ & R charts scheme for joint monitoring the process mean and variance. Since it is just based on the traditional $\bar{X}$ & R charts, its detection power may be further improved by integrating the variable sampling features with other control charts such as the Cumulative SUM (CUSUM) type charts which have better performance in detecting small to moderate shifts. Arnold and Reynolds (2001) proposed an approximated scheme of VSSI CUSUM control charts for two-sided mean shifts; they also discussed the complexity in computing statistical properties such as Average Time to Signal (ATS)

of the two-sided VSSI CUSUM scheme. More difficulties will be encountered if variable sampling features are incorporated with the CUSUM scheme for monitoring mean and variance at the same time.

Meanwhile, in the recent development of control charts, there is another trend of charting schemes that use a single chart to monitor process mean and variance simultaneously (called single charts for short). The single chart scheme can eliminate the difficulty of integrating the variable sampling features with the combined CUSUM charts scheme. However, there may be some difficulties in using the single chart. The problem often raised is about the capability of diagnosis in shift type, that is, to correctly identify the type of the signaled shift (due to changes in mean, variance or both). Schemes with separate control charts can perform this by simply observing which control chart has produced the signal. Nevertheless for the single charts, the information for mean and variance shifts is merged; therefore, the interpretation of the signal is not straightforward.

Since the aforementioned variable sampling schemes have to change one or more of the sampling parameters (sample size and/or sampling interval), they encounter some impediments in practical applications. Because their operation needs frequent changes on sampling interval and/or sample size over time, it will increase the difficulty from the operator's angle. From the managerial viewpoint, using different sample sizes and/or sampling intervals causes the lack of measuring instruments and manpower sometimes, and results in the idleness of these resources at other times. Furthermore, automatic sampling and inspection facilities are not as affordable as the computing resources due to the popularity of computers. In view of this, a new kind of adaptive

chart, i.e. the internal adaptive control chart is brought forward to realize adaptation by making full use of computing resources.

The traditional framework for the development of control charts is under the assumption of independence in data. However, many practical processes are inherently autocorrelated, (e.g. process industries). Many studies show that the presence of autocorrelation can cause grave impacts on the performance of the control charts that are designed under independence assumption. Therefore, for the adaptive control charts we need to consider the situations of autocorrelation in order to extend their applications.

1.3 Research Objectives

Motivated by the issues aforementioned, three objectives are set up in this thesis. The first one is to develop CUSUM schemes with variable sampling features for monitoring process mean and variance simultaneously. The second objective is to develop an internal adaptive control chart for joint monitoring purpose. The methods of diagnosis in shift type are also studied. Finally, approaches for applying the adaptive control charts to autocorrelated processes are investigated.

1.4 Scope of the Thesis

All the control charts developed and studied in this thesis are for joint monitoring two-sided mean shift and increasing variance shift. We confine our studies to the increases in variance, which indicates the deterioration of the process quality. The performance of the variable sampling schemes is evaluated in the measure of Average Time to Signal (*ATS*). However, for the fixed sampling schemes, the performance is evaluated in the measure of Average Run Length (*ARL*). The two measures are widely used in the

literature to compare different control charts. There are two modes used to evaluate *ATS* and *ARL*, i.e., the steady-state mode and zero-state mode. Since the steady-state mode is more realistic, it is adopted throughout this thesis. Consequently, the terms of *ATS* and *ARL* always imply the steady-state *ATS* and *ARL*. For the studies in diagnosis of shift type, correct classification rates are taken as the measure.

To describe correlative structures, a general class of models called autoregressive and moving average (ARMA) models (Box and Jenkins 1976) is usually considered. It includes autoregressive (AR) and moving average (MA) models as special cases. This class of models can be used to capture many different types of time series processes, and has been adopted extensively. Among these models, autoregressive process of order one (the AR(1) model) is commonly reported as being most frequently encountered in practice. Therefore, we will focus on the AR(1) case. Moreover, only the positive autocorrelation are treated, because it is much more prevalent than negative autocorrelation in control chart application. The parameters for the AR(1) model are assumed known (estimated from observed data).

1.5 Outline of the Thesis

In the next chapter, a literature review of concept, design and recent development of the control charts is given. Single control charts, adaptive control charts, and some other related areas are introduced with more details. Then, the main body of this thesis is organized as follows.

In Chapter 3, several variable sampling schemes (VSSI *WLC* scheme and VSSI 3-*CUSUM* scheme etc.) are presented. They are both designed by multi-level searching

algorithm in order to minimize Average Time to Signal (ATS) at a specified design point, meanwhile using the in-control ATS_0 and inspection rate R as the constraints. The performance of the VSSI WLC scheme is evaluated in two studies using Design of Experiment (DOE). The diagnosis capability of the proposed control charts is also studied.

Next, Chapter 4 focuses on the formulation and design of a new chart, named as the Adaptive-Loss function (AL) control chart. Comparisons with some fixed sampling control charts and a variable sampling control chart are conducted based on ARL . The diagnosis method for the AL chart is also proposed and studied.

In Chapter 5, two residuals approaches for $AR(1)$ process are compared by both analysis of statistical properties and numerical tests. The rules to select between the two approaches are given for positive $AR(1)$ parameter. Using the selected approach, the performance of the AL chart developed in Chapter 4 is further studied. The $AR(2)$ and $ARMA(1,1)$ models are also discussed briefly.

Finally, Chapter 6 contains the conclusions and summarizes the contributions of this thesis.

Chapter 2

Literature Review

2.1 Introduction to Control Charts

Figure 2-1: A Typical Control Chart

Figure 2-1 shows a typical control chart, which is a graphical display of the statistic of a quality characteristic versus the sample number or time. The chart is characterized by three horizontal lines. *UCL* is the upper control limit, *CL* is the center line, and *LCL* is the lower control limit. A point that plots outside of the control limits is interpreted as that the process is out of control, and corrective action should be taken to eliminate the special causes responsible for the behavior. This kind of control chart is called a Shewhart control chart, as it is based on the principles developed by Dr. W. A. Shewhart (1931).

2.1.1 Variables Control Charts and Attributes Control Charts

Control charts can be classified into two basic categories: charts for variables and charts for attributes. The control charts for variables are used for quality characteristics measured on a numerical scale, such as length, width or temperature. Attribute control charts are for those quality characteristics, which cannot be easily represented numerically. The attribute charts are particularly useful in service and healthcare industries. However, they are generally not as informative as variables charts because there is typically more information in a numerical measurement than in a mere classification of the conformity or nonconformity (Montgomery 2001). The $\bar{X}$ and S (or R) charts, as well as their variations such as S^2 chart and moving range chart, are typical variable control charts and used extensively in manufacturing industry (Bai and Choi 1995, Wu 1994 and 1995). While the $\bar{X}$ chart monitors the shifts in process mean, the S (or R) chart monitors the shifts in variance.

Traditionally, Quality Assurance (QA) engineers have preferred the R chart to the S chart because of the simplicity of calculating sample range from each sample. However, to monitor critical processes or detect small shifts, the sample size needs to be moderately large. For such cases, the R chart loses statistical efficiency and the S chart is preferable. Moreover, the distribution of the statistic S (sample standard deviation) is less dependent on the form of the parent distribution than R (sample range) (Grabov and Ingman 1996).

The two statistics controlled by the $\bar{X}$ & S charts are the sample mean $\bar{x}$ and sample standard deviation s .

$$\begin{aligned}\bar{x} &= \frac{\sum_{i=1}^n x_i}{n} \\ s &= \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}},\end{aligned}\tag{2-1}$$

where x_i is the i th observation in a sample of size n . The control limits of the $\bar{x}$ & S charts are determined by the following formulae (Montgomery 2001):

$$\begin{aligned}LCL_{\bar{x}} &= \mu_0 - k_{\bar{x}} \sigma_0 / \sqrt{n} \\ UCL_{\bar{x}} &= \mu_0 + k_{\bar{x}} \sigma_0 / \sqrt{n} \\ LCL_S &= c_4 \sigma_0 - k_S \sigma_0 \sqrt{1 - c_4^2} \\ UCL_S &= c_4 \sigma_0 + k_S \sigma_0 \sqrt{1 - c_4^2},\end{aligned}\tag{2-2}$$

where μ_0 and σ_0 are the process mean and standard deviation, respectively, under in-control condition; and c_4 is a constant depending on the sample size. In addition, $k_{\bar{x}}$ and k_S are the control limit coefficients. In many applications, $k_{\bar{x}}$ and k_S are set at three. But, they may be adjusted according to the specified probability of a false alarm.

A major disadvantage of any Shewhart control chart is that it only uses the information about the process contained in the last plotted sample point, and ignores any information given by the entire sequence of points. This problem makes the Shewhart control chart relatively insensitive to small or moderate shifts (on the order of 1.5σ or less) in the process. For such cases, the Cumulative-Sum (CUSUM) control chart and the Exponentially Weighted Moving Average (EWMA) control chart are two very effective alternatives (Montgomery 2001).

2.1.2 CUSUM Control Chart and EWMA Control Chart

CUSUM (Cumulative Sum) control chart was first proposed by Page (1954) to incorporate all the information in the sequence of sample points by plotting the cumulative sums of the deviations of the sample values from a target value. The CUSUM chart is very effective in detecting small and moderate shifts. A large number of research works have been devoted to the design and usage of CUSUM chart (Page 1961, Lucas 1976, Gan 1991, Woodall and Adams 1993, Reynolds and Stoumbos 2004b). CUSUM charts for other sample statistics such as sample ranges and sample standard deviations were also discussed (Lowry *et al.* 1995 and Srivastava 1997). Vance (1986) and Hawkins (1992) provided very accurate procedures to calculate the *ARL* of CUSUM chart. However, the CUSUM chart is not as effective as the Shewhart chart in detecting large shifts. Lucas (1982) combined the CUSUM and the Shewhart chart in order to enhance the capability of the CUSUM control chart for detecting large process shifts.

The CUSUM control chart plots the statistic $S_t^+ = \max\{0, z_t - k + S_{t-1}^+\}$ for monitoring its variation in positive direction and $S_t^- = \max\{0, -z_t - k + S_{t-1}^-\}$ in negative direction, where k is the reference value. For a process with variate's in-control distribution $N(\mu_0, \sigma_0^2)$, if it is to monitor its mean by standardized subgroup average, then $z_t = \frac{\bar{x}_t - \mu_0}{\sigma_{\bar{x}}}$

where $\sigma_{\bar{x}} = \frac{\sigma_0}{\sqrt{n}}$. If the controlled statistic $S_t^+(S_t^-)$ exceeds its decision interval,

$h^+(h^-)$, one concludes that the process mean has increased (decreased).

The EWMA (Exponentially Weighted Moving Average) control chart was originally introduced by Roberts (1959). It monitors the exponentially weighted moving average of a process. Lucas and Saccucci (1990) compared the EWMA chart to the CUSUM chart, concluding that there is little practical difference between the two techniques. In this thesis, CUSUM chart will be used and discussed in detail.

2.1.3 Design of Control Charts

The determination of the parameters for control chart is called the design of control chart. Undoubtedly, the charting parameters all have impacts on the performance of the control chart. For control charts using fixed sampling intervals, Average Run Length (*ARL*) serves as the measure of performance. *ARL* is the number of inspected samples required to signal an out-of-control condition. It describes the power (or effectiveness) of the control chart when the process has run out-of-control. The smaller the *ARL*, the earlier a signal will be given, and the more powerful the control chart is. On the other hand, the in-control ARL_0 reflects the false alarm rate when the process is actually in control. The longer the ARL_0 , the less frequently the false alarm takes place. The statistical derivation of *ARL* is obvious if we view the control chart as a sequence of hypothesis tests (Vining 1998). From this perspective, the type I error probability (denoted as α) is the probability for a sample point falling beyond the control limits (indicating an out-of-control condition) when no assignable cause is present at all. And the type II error probability (denoted as β) is the probability for a sample point falling between the control limits when the process is actually out of control. The power, $(1-\beta)$, is the probability of correctly signaling an out-of-control condition. If the observations plotted on the control chart are independent, the average run length of a chart under an in-control process can be calculated by (Montgomery 2001)

$$ARL_0 = \frac{1}{\alpha} \quad (2-3)$$

And the ARL under an out-of-control process is by

$$ARL = \frac{1}{1 - \beta}. \quad (2-4)$$

On the other hand, the average time required to signal an out-of-control state is called Average Time to Signal (ATS). If samples are taken at a fixed interval h , the in-control ATS_0 and out-of-control ATS are directly related to ARL_0 and ARL .

$$ATS_0 = h \cdot ARL_0 \quad \text{and} \quad ATS = h \cdot ARL \quad (2-5)$$

However, for unequal interval situations such as the application of the Variable Sampling Intervals (VSI) control chart, the direct relationship between ATS and ARL is no longer valid. Therefore, ATS has to be used to indicate the performance of a control chart. Similar to ARL_0 and ARL , a larger ATS_0 means fewer false alarms, and a smaller ATS indicates high detection effectiveness.

In the design of a Shewhart control chart, charting parameters normally consist of control limits (UCL and LCL), sample size n and sampling interval h . For other types of control charts, additional parameters may have to be designed, such as the reference parameter k for the CUSUM chart. Generally, moving the control limits further from the center line leads to the decrease of the probability α of type I error or the increase of ARL_0 , i.e., false alarm is reduced. However, at the same time the probability β of type II error will increase, which also results in the increase of ARL , i.e., slow detection of out-of-control cases. On the contrary, if we move the control limits closer to the center line, the consequence will be opposite. In practice, it is customary to base the control

limits on a multiple of the standard deviation. Usually this multiple is three and the limits are called 3-sigma limits. The sample size n and the sampling interval h are also influential parameters on the performance of a control chart. In general, large samples will make it easier to detect process shifts, because it will make the control limits tighter and thus decrease the probability of type II error. Large samples taken at short intervals would, of course, give the best protection against shifts in the process, but this is usually not economically feasible. Current industry practice tends to favor smaller, but more frequent samples (Montgomery 2001). Furthermore, as automatic sensing and measuring technology advances, it is becoming possible to greatly reduce the sampling interval h .

The design algorithm of control chart that only considers statistical criteria can be referred to as statistical design. In addition, there is economic design and hybrid statistical-economic design. However, this study is focused on the statistical design. The statistical design usually includes selecting the control limits, sample size and sampling interval so that the ARL_0 (or ATS_0) of the chart is close to a specified value and the ARL (or ARL) is minimized for detecting certain shifts in process mean and/or variance. This kind of problems in statistical quality control and improvement involve the application of optimization techniques, which were reviewed in a paper by Carlyle *et al.* (2000). In the optimization model, the minimization of ARL (or ATS) is consistent with a management philosophy that encourages doing it right the first time (Lucas 1985). It immediately leads to the reduction of the number of nonconforming units, and therefore directly benefits users. On the other hand, the constraint on ARL_0 (or ATS_0) has been widely used in designing the control charts (Reynolds 1989). Saniga (1989), Kim *et al.* (1992) and Saniga *et al.* (1995) stated that unless constraints are used, the statistical

performance of the economically optimal methods could be poor. Most of the optimization strategies employed in SPC make no attempt to secure a global optimal solution. Instead, they focus on deriving a convenient and systematic procedure for identifying a satisfactory and workable solution that could be adopted in practice (Duncan 1956).

Another performance measure of control charts is the diagnosis capability as mentioned before. Although it is useful, normally it is not used as a critical performance characteristic such as *ATS* or *ARL*.

2.2 Evaluation of Control Chart Performance

As the common measures of control chart performance in statistical design, *ARL* or *ATS* need to be evaluated. In this section, distinctions between the zero-state mode and steady-state mode in the evaluation of *ARL* and *ATS* are first discussed. Then two approaches to obtain *ARL* and *ATS* values are reviewed.

2.2.1 Zero-State Mode vs. Steady-State Mode

In numerous research papers, *ARL* and *ATS* are evaluated under the zero-state mode, in which process shifts are assumed to take place invariably at the beginning of a sampling interval and/or with specified starting values. But in reality, most of the processes start from an in-control state and then shift to an out-of-control state at some random time and with random starting values. As Duncan (1978) mentioned, the assumption behind the zero-state mode “could be a restrictive assumption and research should be done on what will happen when it is removed”. Lucas and Saccucci (1990) gave a definition that a control statistic is in steady state if the process has been in control long enough for the

effect of the starting value to become negligible. The steady-state mode provides a more realistic and accurate *ATS* value compared to that under the zero-state mode. Runger and Pignatiello (1991) and Runger and Montgomery (1993) proposed the algorithms for calculating *ATS* under the steady-state mode in which the process shift is allowed to occur at any time. Costa (1997) replaced the zero-state *ATS* by the steady-state *ATS* for the VSSI $\bar{x}$ chart. Reynolds *et al.* (1990) developed the algorithms to calculate the steady-state *ATS* for CUSUM charts with variable sampling intervals.

2.2.2 Simulation Approach

Simulation has traditionally and frequently been used to analyze the run length distribution of a control chart. To simulate a process, random variates $x_1, x_2, \dots$ are generated from the given distribution. These variates represent the quality characteristics as measured in the field over time, and they are used to simulate the actual working characteristic of the control charts.

Simulation can be used whenever other methods are not feasible. Moreover, due to the simplicity of simulation, it can also be used to check the results obtained by the analytical methods. However, simulation is very time consuming, because 10,000 or more simulated processes are usually required to estimate the result accurately (Champ *et al.* 1990). Simulation approach is broadly used to handle many issues in SQC. To name a few, Gan (1995) and Reynolds and Stoumbos (2004b) used simulation in their papers to obtain the *ARL* of the combined CUSUM charts. Divoky and Taylor (1995) used simulation to study the performance of the trend rules for process mean drifts. Estrop *et al.* (1992) developed a continuous manufacturing process quality model by the approach of computer simulation.

2.2.3 Markov Chain Approach

Some control charts can be modeled by the Markov chain (Champ *et al.* 1990). Through a unified method based on a finite Markov chain approach, the run length distribution and *ARL* of several types of control charts can be evaluated. Especially for the CUSUM and EWMA schemes in which distributional properties of the run length are not easily handled in a closed form. Markov chain approach is one of the most effective approximate methods to study the run length characteristics. An early Markov chain approach to the CUSUM scheme was proposed by Ewan and Kemp (1960). This approach has been used by several researchers, such as Woodall (1983) and Reynolds *et al.* (1990), to compute *ARL* for CUSUM charts. In addition, the method can yield the variance or standard deviation of the run length as a byproduct (Fu *et al.* 2002). Further, successful attempts have been reported for the two-sided CUSUM (Woodall 1984) and multivariate CUSUM (Woodall and Ncube 1985). In these cases, however, the sizes of the transition probability matrices increase exponentially with the dimension of the problem, and so far no efficient way to drastically reduce the matrix size is known (Chao 1999).

From the elementary properties of Markov chains (Cinlar 1975),

$$ARL = \mathbf{B}^T (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{1} \quad (2-6)$$

where $\mathbf{B}^T$ is a row vector of the stationary probabilities. $\mathbf{I}$ is the identity matrix. $\mathbf{Q}$ is the transition probability matrix where the elements associated with the absorbing states have been deleted. Finally, $\mathbf{1}$ is a column vector of units.

ATS can be calculated by

$$ATS = \mathbf{B}^T (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{h} \quad (2-7)$$

where $\mathbf{h}$ is a column vector with elements of sampling intervals corresponding to each state.

The Markov chain approach is brought to wide uses in calculating the *ATS* of variable sampling schemes. The application of equation (2-7) will be illustrated in detail during the design of the VSSI *WLC* scheme in Chapter 3.

2.2.4 Diagnosis Capability

While the control charts do test the condition of the process, they do not provide much information about the characteristics of the possible shift. But for practitioners, in addition to knowing the process has shifted, they also want to identify the causes and restore the process to normal. To assist their subjective judgments, it is desirable for control chart to be accompanied with some diagnosis capabilities, such as information about the shift type (mean shift or variance shift, or both) and the change point (the time when the shift takes place).

For the joint monitoring of process mean and variance, Guo and Dooley (1992) have presented an example that knowledge on the type of shift (mean shift or variance shift) can narrow down the possible assignable causes to a great extent. Relative to control chart method, it appears that more neural network methods (Guo and Dooley (1992, 1995), Ho and Chang (1999) and Cheng and Chen (2003)) have been developed for the classification purpose with competitive performance. While *ARL* and *ATS* only measure the speed a control chart responds to shifts, other measures are needed for diagnosis of shift type. Correct classification rate on shift type was employed as a performance criterion right after *ARL* in a review paper (Ho and Chang 1999). In the review, a new

combined neural network scheme was proposed and compared with some general SPC charts and other classification techniques.

Change point estimator of CUSUM control chart was proposed by Page (1954). It uses the last time at which the CUSUM statistic that signaled was reset to zero. This estimator is under the assumption that the shift change is of known size $2k$ (k is the reference value). Nishina (1992) proposed an estimator for the change point when an EWMA control chart signals. He compared the CUSUM chart, moving average chart and EWMA chart from the viewpoint of change point estimation, and concluded that the CUSUM chart is recommended among them. Some researchers (Samuel *et al.* (1998 *a*, *b*), Park and Park (2004)) considered the use of maximum likelihood estimators (MLE) for the change point of a step change in the process mean, variance and both respectively. The MLE estimators can be applied to any control chart. Pignatiello and Samuel (2001) compared the performance of the MLE estimator of the change point for the process mean and the built-in change point estimators from CUSUM and EWMA control charts. They concluded the MLE estimator is better for use when the CUSUM or EWMA control chart signaled a change in the process mean.

In this thesis, the scope of diagnosis study is confined to the classification of the shift type.

2.3 Joint Monitoring Scheme

In Statistical Process Control, when dealing with a quality characteristic that is a variable, it is usually necessary to monitor both the mean value of the quality characteristic and its variability (Montgomery 2001). Schemes for this purpose can be

generally classified into two categories: schemes that combine separate control charts for mean and variance, referred as the combined charts; and schemes which employ one statistic for joint monitoring, referred as the single charts. This section reviews some of these schemes. In particular, the combined CUSUM scheme (3-CUSUM), Weighted Loss-function control chart (*WL* chart) and Weighted Loss function CUSUM control Chart (*WL*-CUSUM chart) are introduced with more details, since they are the basis for the new charts proposed and investigated in this thesis.

2.3.1 Combined Charts for Monitoring Mean and Variance

For monitoring individual parameters, some Shewhart type control charts have been mentioned earlier, like $\bar{X}$ chart for process mean and the S (S^2 or R etc.) chart for process variance. In addition, CUSUM chart and EWMA chart serve as the more effective alternatives to Shewhart type charts for small and moderate shifts. Combined charts for monitoring mean and variance can be built based on the separate charts for mean and variance. The combinations of Shewhart charts are commonly used in practice for a long time. They have also been studied in literatures intensively, e.g. Shewhart $\bar{X}$ & R charts (Duncan 1974 and Saniga 1989), Shewhart $\bar{X}$ & S^2 (Reynolds and Ghosh 1981) and many more. The combined CUSUM and Shewhart S^2 by Gan (1989) is another one that follows the line of combined charts. Gan (1995) and Morais and Pacheco (2000) used different approaches to study the performance of combined EWMA schemes. In the paper by Gan (1995), a combined CUSUM scheme was also proposed which, as well as the combined EWMA scheme, were recommended for use from the results of the comparisons. Combined CUSUM scheme and combined EWMA scheme were also studied from the viewpoint of efficient sampling plan recently (Reynolds and Stoumbos 2004 *a, b*).

Combined Triple-CUSUM Scheme (3-CUSUM Scheme)

Gan (1995) discussed a control scheme consisting of two one-sided CUSUM mean charts and two one-sided CUSUM variance charts. As discussed in Section 2.6, only increasing variance shifts are handled in this thesis, therefore a similar 3-CUSUM scheme will be studied. It comprises three cooperative CUSUM charts (two monitoring decreasing and increasing mean shifts and one monitoring increasing variance shift). The statistics to be updated and plotted in the three CUSUM charts are I (for increasing mean shift), D (for decreasing mean shift) and V (for increasing variance shift), respectively.

$$\begin{aligned} I_0 &= 0 \\ I_t &= \max(0, I_{t-1} + z_t - k_I) \end{aligned} \quad (2-8)$$

$$\begin{aligned} D_0 &= 0 \\ D_t &= \min(0, D_{t-1} + z_t + k_D) \end{aligned} \quad (2-9)$$

$$\begin{aligned} V_0 &= 0 \\ V_t &= \max(0, V_{t-1} + \log(s_t^2) - k_V) \end{aligned} \quad (2-10)$$

$$z_t = \frac{\bar{x}_t - \mu_0}{\sigma_0 / \sqrt{n}}, \quad (2-11)$$

where k_I , k_D and k_V are the reference parameters of the three CUSUM charts. $\bar{x}_t$ and s_t^2 are the mean and the variance of the t th sample respectively. The 3-CUSUM scheme will produce an out-of-control signal when one or more of I_t , D_t and V_t fall beyond the respective control limits H_I , H_D and H_V . The 3-CUSUM scheme may be one of the most effective charts for monitoring the two-sided mean shifts and increasing variance shift that can be found in open literature. Since the operation of the 3-CUSUM scheme is quite complicated, an on-line computer may be necessary in actual implementation to facilitate updating and plotting the three statistics. As of the writing, a systematic design procedure for the 3-CUSUM scheme cannot be found in the published literature, not to

mention the 3-*CUSUM* scheme with variable sampling features. In this thesis, a method proposed by Wu and Tian (2005) is employed to design the 3-*CUSUM* scheme for comparative studies.

2.3.2 Single Charts for Monitoring Mean and Variance

If a single chart is able to simultaneously monitor both mean shifts and variance increases, the design and operation can be simplified to a large degree. In view of this, some single control charts have been proposed. These include the Omnibus EWMA chart (Domangue and Patch 1991), the Likelihood Ratio Test (Sullivan and Woodall 1996), the MaxMin EWMA chart (Amin *et al.* 1999), the MaxEWMA chart (Chen *et al.* 2001) and the *B* chart (Grabov and Ingman 1996).

Reynolds and Ghosh (1981) and Cyrus (1997) have developed the *L* chart based on the concept of Loss Function, which is used broadly in industry to measure the quality cost due to poor quality (Taguchi 1986, Spiring and Yeung 1998). The average loss *L* is proportional to the sum of squares of the deviations of the quality characteristic *x* from the target μ_0 (Cyrus 1997).

$$L = \frac{\sum_{i=1}^n (x_i - \mu_0)^2}{n} = \frac{(n-1)s^2 + n(\bar{x} - \mu_0)^2}{n} \quad (2-12)$$

The upper control limit of the *L* chart is determined as follows.

$$UCL_L = \chi_{n,1-\alpha}^2 \frac{\sigma_0^2}{n} \quad (2-13)$$

where, $\chi_{n,1-\alpha}^2$ is the $(1-\alpha)$ percent point of a chi-square distribution with degrees of freedom of *n*. The *L* chart is able to detect the two-sided process mean shifts and increasing variance shift simultaneously. Because, whenever there is a mean shift

(increasing or decreasing) and/or an increasing variance shift, the statistic L becomes larger. And a subsequent sample point will exceed the upper control limit of the L chart shortly. Although control charts based on L are effective to detect large shifts in μ or shifts in σ (Reynolds and Stoumbos 2004 *a, b*), but the overall performance of these charts is not satisfactory. Because it produces considerably larger out-of-control ARL than the $\bar{X}$ & S charts when mean shifts are dominant.

Weighted-Loss-Function Control Chart (WL chart)

The WL chart proposed by Tian (2003) is to mitigate the aforementioned problem suffered by the L chart. The WL chart uses a weighting factor λ ($0 < \lambda < 1$) to adjust the weight of the loss due to the squared mean shift and the weight of the loss attributable to the variance increase in the loss function (equation 2-12). The weighted loss function WL is expressed as follows.

$$WL = \lambda s^2 + (1 - \lambda)(\bar{x} - \mu_0)^2 \quad (2-14)$$

Both statistics L and WL have some similar engineering and managerial interpretation and require the comparable amount of calculation. However, by properly adjusting the weighting factor λ , the WL chart is able to achieve higher overall effectiveness. Consequently, it produces smaller ARL values under most out-of-control situations and always makes the WL chart more effective than the joint $\bar{X}$ & S charts, as well as the conventional L chart, in a large domain of mean and variance shifts. The weighted loss function WL is composed of the term for variance and the term for square of mean shift. The sensitivity of WL to the two terms is different, because the sample variance follows a skewed Chi-square distribution and the sample mean shift follows a normal distribution. As a result, the optimal value of λ is usually close to 0.3 rather than 0.5.

Weighted-Loss-Function CUSUM Chart (WL-CUSUM chart)

By combining the CUSUM scheme with the WL chart, the WL -CUSUM chart was proposed by Wu and Tian (2005). The statistic to be updated and plotted for the WL -CUSUM chart is A_t that is the cumulative sum value of WL up to time t .

$$\begin{aligned} A_0 &= 0 \\ A_t &= \max(0, A_{t-1} + WL_t - k_A), \end{aligned} \quad (2-15)$$

where k_A is the reference parameter. The weighted loss variate WL_t is calculated from the sample mean and sample variance of the t th sample by the WL chart formulae

$$WL_t = \lambda s_t^2 + (1 - \lambda)(\bar{x}_t - \mu_0)^2, \quad (2-16)$$

where $0 \leq \lambda \leq 1$, $\bar{x}_t$ and s_t^2 are the sample mean and sample variance obtained from the t th sample. It has been shown that the WL -CUSUM chart is considerably more powerful than the WL chart, let alone the traditional $\bar{X}$ & S charts.

2.4 Adaptive Control Charts

The conventional control charts are static in the sense that all the charting parameters (the sample size n , the sampling interval h , and the control limits LCL and UCL , etc.) are fixed for the duration of the operation. The advantage of these static control charts is the simplicity for implementation. In order to more effectively utilize the inspection capacity in process control, researchers have investigated the impacts of using adaptive (or dynamic) control schemes. Compared to the static control chart, a control chart is considered adaptive if at least one of the design parameters is allowed to change based on the values of the sample statistics that provide information about the current state of the process. Tagaras (1998) provided a comprehensive survey of the developments of adaptive control charts. The review covers the $\bar{X}$, CUSUM and EWMA adaptive charts that have variable sample size n , sampling interval h , and combinations of them. Also

discussed is the control chart with variable control limit coefficient and Bayesian approach of process control. In this thesis, the adaptive control charts are classified into variable sampling schemes and internal adaptive control charts (those without variable sampling features). The classification is based on whether the adaptation of the parameters of the control chart requires operators' intervention or not. This provides the users with more alternatives to make decisions according to their requirements and resources. However, operators' involvement is not the only consideration in choosing control schemes, because the variable sampling schemes may achieve substantial improvement in detection power in some cases. Therefore, both the variable sampling schemes and internal adaptive schemes are studied in this thesis. It is noted that the studies of new adaptive control charts are not only propelled by the development of the adaptive techniques, but also by the progress in static control charts as those reviewed in Section 2.3.

2.4.1 Variable Sampling Schemes

The adaptation of this type of adaptive schemes is conducted by operators. They usually focus on the changes of parameters for sampling operations, such as sampling interval and sample size. In variable sampling schemes, the region between upper and lower control limits is normally divided into several zones (see Figure 2-2), such that

$$LCL \leq -w_i \leq CL \leq w_i \leq UCL . \quad (2-17)$$

Where, $-w_i$ and w_i are called the lower and upper warning (or threshold) limits. If the sample statistic falls into a certain zone, then the corresponding sampling interval h_i and sample size n_i are used for the next sample. Montgomery (2001) stated that these procedures greatly enhance control chart performance compared to a static control chart when both charts have the equal in-control ATS_0 .

Figure 2-2: Zones of Adaptive Control Charts

Many works have been carried out in this area. Here, three types of variable sampling schemes are briefly summarized, Variable Sampling Interval (VSI), Variable Sample Size (VSS), Variable Sample Size and Sampling Interval (VSSI).

VSI Feature

The VSI $\bar{x}$ control chart was first introduced by Reynolds *et al.* (1988). Then, Reynolds (1989, 1996a), Amin and Miller (1993) continued to contribute to this line of research.

Although different sampling intervals of intermediate lengths could be used, the numerical results from the works by many researchers (Runger and Pignatiello 1991, and Runger and Montgomery 1993) showed that the VSI chart with dual sampling intervals (a short and a long sampling intervals) has gained most of the advantages of the VSI chart and is easy to handle.

Reynolds *et al.* (1990) integrated the VSI feature with the standard CUSUM charts and concluded that the VSI CUSUM chart substantially reduces the *ATS*, especially for

small shifts. Reynolds (1995) also found that the VSI CUSUM charts and the VSI EWMA charts have very similar ATS functions in general. Furthermore, in order to increase the detection speed without resorting to more complex schemes, Reynolds (1996a) added the VSI feature to the $\bar{x}$ charts with runs rules. Baxley (1995) reported a practical application of a variable sampling interval chart.

VSS Feature

Daudin (1992) proposed the DS (double sampling) $\bar{x}$ charts using sample sizes n_1 (first sample) and n_2 (second sample, if needed) at every inspection. The DS $\bar{x}$ chart offers better ARL performance than the static $\bar{x}$ charts. Costa (1994) proposed and examined the properties and performance of the dual sample size VSS $\bar{x}$ charts. He showed that the VSI scheme generally has lower ATS for moderate process shifts, while the VSS scheme has lower ATS for smaller process shifts.

The adaptive sample size policy was extended to the design of the CUSUM charts by Rendtel (1987) and Annadi *et al.* (1995). Rendtel (1987) studied the properties of a VSS CUSUM chart with three possible sample sizes, and concluded that the adaptive chart may reduce out-of-control ARL considerably for small shifts without affecting the in-control ARL_0 .

VSSI Feature

The idea of adapting both the sample size and sampling interval is formed based on the VSI feature and the VSS feature. Previous research suggests the sufficiency of using only two possible sampling intervals to achieve good statistical properties in VSI control charts. There has been no optimality result published for the choice of the

sample size function in the VSS control charts. Considering the simplicity of using two sample sizes, the strategy of dual sample sizes and dual sampling intervals are adopted by most researchers for Variable Sample Size and Sampling Interval (VSSI) control charts. The VSSI feature was applied to $\bar{X}$ charts by Prabhu *et al.* (1994), Costa (1997), and Das *et al.* (1997). It was concluded that the VSSI $\bar{X}$ chart outperforms other charts (e.g. the VSI, VSS and static $\bar{X}$ charts) in terms of *ATS*. Costa (1999a) proposed the VSSI $\bar{X}$ &*R* charts for monitoring both mean and variance shifts. Arnold and Reynolds (2001) presented VSSI CUSUM charts for one-sided and two-sided mean shifts. Wu *et al.* (2005) proposed the Shewhart type VSSI *WL* chart for monitoring two-sided mean shift and increasing variance shift.

2.4.2 Internal Adaptive Schemes

In contrast with the variable sampling schemes, internal adaptive schemes do not need the involvement of operators to make the adaptation. Specially, the sampling parameters (sample size and sampling interval) are fixed, while some other charting parameters (e.g. weighting factor λ in the *WL* chart and control limit) will change over time depending on the process information for a period. Since the entire procedure can be conducted by a software system installed in an on-site computer without any operators' participation, therefore, such adaptive charts avoid a lot of inconvenience in implementation. In the published literature, this type of control charts has been seldom studied. In Tagaras' review (1998), only a few papers for adaptive control limit coefficients chart were referenced. They all belong to Bayesian control charts and are focused on economic rather than statistical performance.

Calabrese (1995) considered a process control with fixed sampling interval and sample size for monitoring the fraction nonconforming using a Bayesian approach. He concluded that the Bayesian scheme results in the reductions in expected cost per time unit. Tagaras (1994) used the Bayesian principle and dynamic programming approach to determine the decision parameters (sampling interval and control limits) for the next sample from the economic point of view. Costa (1999b) extended the study of the $\bar{X}$ chart by including variable action limits. Although this new feature makes the $\bar{X}$ chart more powerful than the static CUSUM scheme in detecting shifts in the process mean, it is still less effective than VSI CUSUM chart. More recently, Capizzi and Masarotto G (2003) proposed a class of adaptive EWMA control charts for monitoring process mean. The new charts can adapt the weight of the past observations using a function of the current error statistic, resulting in more balanced performance than traditional EWMA chart and Shewhart chart.

From the references, one can see that Bayesian approach is a feasible way for changing the control limits. It may also be a sound method for some internal adaptive schemes. Actually, the Bayesian approach has been used widely in economic models of control charts for a long period. Calabrese (1995) and Tagaras (1996) pointed out an important difference between Bayesian control charts and non-Bayesian control charts. The former is by nature dynamic, whereas the latter is either static or dynamic. This difference is concerned with the manner in which past information is weighted. Bayesian charts explicitly take into consideration the length of each sampling interval and the random shock mechanism. However, little work has been found to handle joint shifts of process mean and variance and to evaluate Bayesian control charts from a statistical viewpoint.

In the past, there were some impediments in using Bayesian charts. For instance, the computational requirement of the dynamic programming formulation for the optimization of a Bayesian chart that involves many charting parameters is relatively high. The recent popularization of computer-aided technique has led to great advances in the application of Bayesian methods to complicated problems in various disciplines (Gilks *et al.* 1990).

2.5 Control Charts under Autocorrelation

The fundamental assumption used in the construction of a Shewhart control chart is that the statistics are independent and identically distributed (iid). However, autocorrelation and other systematic time-series effects are often substantial in practice. Autocorrelation in the chemical-process industry has long been recognized. Similar findings in other continuous-process industries have also been cited. Montgomery and Friedman (1989) found that autocorrelation frequently occurs in data used to control processes in computer-integrated manufacturing environment. When autocorrelation is present, traditional control charts for independent data may encounter the serious problems of not detecting special causes that truly exist or of noticing apparent special causes that do not exist. Therefore, if significant autocorrelation exists in the observations from a process, it is not advisable to apply traditional control chart methodology without modification. Two general approaches have been considered in the control chart literature to deal with autocorrelation. In the first approach, traditional control charts are applied but the methods used to estimate process parameters and set the control limits are adjusted in order to account for the autocorrelation (Schmid 1995). It is also called modified approach, which is particularly applicable when the level of autocorrelation is not extremely high (Lu and Reynolds 1999b). In another approach, first developed by

Alwan and Roberts (1988), a time series model is fitted to the process observations and the forecast errors or residuals from this model are monitored by the control charts. The residuals approach is aimed at transforming the autocorrelated observations into independent ones.

Monitoring process mean under autocorrelation has been studied by many authors (Alwan and Radson 1992, Reynolds 1996b, Lu and Reynolds 1999a, Lu and Reynolds 2001, Atienza *et al.* 2002). Besides, a few works for monitoring process variance can be found (MacGregor and Harris 1993, Wieringa 1999). However, work for joint monitoring purpose is apparently less. Lu and Reynolds (1999b) investigated control charts for monitoring the process mean and variance simultaneously when the process was described by an AR(1) plus a random error model. It was mainly focused on the individual observations. For sample size larger than one, assumption of independence in the within sample observations was imposed. However, it is not always the case in reality. This concern was studied by Knoth *et al.* (2001), who assumed that the samples were still independent but the variables within a sample were correlated. This model was proposed based on the fact that the period between two samples is frequently large in comparison with the time needed for taking one sample. Under this model, Knoth *et al.* (2001) and Knoth and Schmid (2002) developed the combined Shewhart charts and combined EWMA scheme in a stationary process AR(1) successively. Their studies show that the residual attempt must be favored if a variance change is present. Furthermore, the combined EWMA scheme excels in detecting small and moderate shifts in mean and variance, while the combined Shewhart scheme only outperforms for large shifts (Knoth and Schmid 2002) in the AR(1) process. The residuals approach adopted in these studies were firstly introduced by Kramer and Schmid (1998). It is

labeled as the revised residuals approach in this thesis since it is different from the traditional residuals approach. Because the comparison of the two residuals approaches has not been found in literature, a study of this issue in the joint monitoring scenario is carried out in Chapter 5.

2.6 Assumptions and Conventions

The following assumptions and conventions have been adopted in this thesis.

- (1) In SPC, it is usually assumed that the quality characteristics follow the normal distribution, and its mean value μ_0 and standard deviation value σ_0 in the in-control status are known or can be estimated. Primarily, several adaptive control charts are proposed for process with independent observations. Therefore, the quality characteristic is assumed to be independent and identically distributed (iid). When mean shift and/or variance shift occurs, the process mean and standard deviation will change to μ and σ respectively.

$$\begin{aligned}\mu &= \mu_0 + \delta_\mu \sigma_0 \\ \sigma &= \delta_\sigma \sigma_0\end{aligned}\tag{2-18}$$

where, δ_μ and δ_σ are the mean shift and standard deviation shift, respectively, in terms of σ_0 . When the process is in control, $\delta_\mu = 0$ and $\delta_\sigma = 1$.

Lastly, control charts are studied under autocorrelation and then the shift model will be redefined accordingly.

- (2) Because control charts are often used to detect deterioration in process quality (Reynolds and Ghosh 1981), only the increasing variance shifts are handled in most of the applications (Reynolds and Stoumbos 2004 *a, b*). This treatment can also

simplify the design procedure and enhance the power for detecting the hazardous increasing variance shifts. As a result, only the increasing variance shift will be handled in this thesis.

- (3) Because all the charts have the same operating characteristics for symmetrical increasing and decreasing mean shifts, only the detection effectiveness against increasing mean shifts will be studied in the comparative studies throughout this thesis.
- (4) For the diagnosis studies, it is assumed that the signal is not a false alarm.
- (5) Since the steady-state mode is more realistic, it is always adopted in this thesis.

2.7 Summary

This chapter reviews the background of this thesis and identifies the research issues. It firstly covers the basics and terms in the control charts, the statistical designs and measures of the control charts. After that, some developments of static control charts for joint monitoring process mean and variance are summarized, especially three advanced charts. On the other hand, the review of literature in adaptive control charts is carried out in two branches newly classified, i.e. variable sampling schemes and internal adaptive schemes. It is hoped to advance the adaptive control charts in both branch. Finally, some researches of control charts in autocorrelation are summed up. As an extension to the control charts under independence assumption, two residuals approaches are worth comparative studies under the joint monitoring scenario.

Chapter 3

Variable Sampling Schemes for Monitoring Process Mean and Variance

Recent research has shown that the variable sampling features (VSS, VSI and VSSI) and CUSUM charts are faster than the traditional static Shewhart charts in detecting process shifts. This chapter proposes several variable sampling schemes, all of which combine the adaptive features and the CUSUM technique for monitoring the process shifts in mean and variance simultaneously. The first scheme is based on the 3-CUSUM scheme, which comprises three individual CUSUM charts for monitoring two one-sided mean shifts and increasing variance shift respectively. The second scheme is based on the *WL*-CUSUM chart, which monitors both mean and variance shifts by manipulating a single CUSUM chart. Since the VSS feature and the VSI feature are just the special cases of the VSSI feature, only the control scheme with VSSI feature is described in detail here.

3.1 Introduction

The Shewhart's $\bar{X}$ & S or $\bar{X}$ & R control charts have been used widely for monitoring process mean and variance in industry (Montgomery 2001). While the $\bar{X}$ & S charts are easy to design and operate, their effectiveness for detecting process shifts in mean and variance is inefficient, especially for small or moderate process shifts. Two advanced techniques, the CUSUM charts and the variable sampling features (VSS, VSI and VSSI), are studied intensively by researchers in an effort to improve the detection effectiveness.

In the combination of the two techniques, the one-sided CUSUM chart with VSSI feature was studied by Arnold and Reynolds (2001). Their work focused on the strategy of dual sample sizes and dual sampling intervals, which is adopted by most studies on control charts with VSSI feature. The new VSSI schemes discussed in this chapter will also use this strategy.

Figure 3-1: A Typical VSSI CUSUM Strategy

$$n_1 \leq n_2, h_1 \geq h_2, 0 \leq w \leq H \quad (3-1)$$

Figure 3-1 is to illustrate the VSSI decision rules of a one-sided CUSUM chart. The warning limit (w) and control limit (H) divide the charting area into three zones. When the process is likely to be in control (charting statistic falls into the central zone), the *relax sample* that employs small sample size n_1 and long sampling interval h_1 will be used for the next sample. Conversely, when the process seems close to an out-of-control condition (charting statistic falls into the warning zone), the *alert sample* that employs large sample size n_2 and short sampling interval h_2 is used. In either case, it is assumed that the last n_i units produced in the sampling interval h_i will be inspected. Finally, if the charting statistic falls into the action zone, a signal will be generated indicating an out-of-control situation. The first sample taken from the process when it just starts up could be chosen at random. It will have little influence on the steady-state value of ATS .

However, as recommended by most researchers (e.g., Costa 1997), the alert sample should be used as a safeguard to provide additional protection against problems that may occur during the start-up. The VSI scheme uses the same sample sizes for both relax and alert samples, that is, $n_1 = n_2 = n$. And the VSS scheme uses same sampling intervals for both relax and alert samples, that is, $h_1 = h_2 = h$.

However, there is no well-developed algorithm in open literature to design the VSSI CUSUM chart for two-sided mean shifts, let alone the VSSI 3-CUSUM scheme that monitors the two-sided mean shifts and increasing variance shift. It is mainly because of the difficulty in evaluating the *ATS* of the VSSI 3-CUSUM scheme. The interactions among the three CUSUM charts make the design of the VSSI 3-CUSUM scheme much more complicated than the designs of three individual CUSUM charts. It is impossible to express the statistical properties, such as *ATS*, of the VSSI 3-CUSUM scheme simply in terms of the corresponding properties of the three individual CUSUM charts. Arnold and Reynolds (2001) mentioned the use of a two-dimensional Markov chain to evaluate the *ATS* of a VSSI CUSUM scheme for two-sided mean shifts, but they pointed out that considerable computational effort is required. For the VSSI 3-CUSUM scheme, simulation may be the only resort. Furthermore, there are 13 parameters required to be determined in the design of a VSSI 3-CUSUM scheme including k_I , k_D and k_V in equations (2-8) to (2-10), the warning and control limits w_I , w_D , w_V , H_I , H_D and H_V , the sample sizes n_1 and n_2 , and the sampling intervals h_1 and h_2 (note, $k_I = k_D$, $H_I = -H_D$ and $w_I = -w_D$). The implementation of the VSSI 3-CUSUM scheme is also quite complicated, as the three CUSUM charts have to be handled side by side and the three statistics I_t , D_t and V_t have to be updated and plotted for every sample.

Wu *et al.* (2005) proposed a VSSI *WL* chart, which integrates the VSSI feature with the Shewhart type *WL* chart for joint monitoring purpose. As the continuation of the VSSI *WL* chart and the *WL*-CUSUM statistic (Wu and Tian 2005), a VSSI CUSUM chart is developed in this thesis. It is named as the VSSI *WLC* scheme. Like the VSSI 3-*CUSUM* chart, the VSSI *WLC* scheme is able to monitor the two-sided mean shifts, and the increasing variance shift as well. Because, whenever there is a mean shift (increasing or decreasing) and/or an increasing variance shift, the *WLC* statistic becomes larger and a subsequent sample point will exceed the upper control limit of the VSSI *WLC* scheme sooner or later.

Since the *WLC* scheme only consists of a single CUSUM chart, therefore, interaction between charts cannot exist, and the design and operation of the VSSI *WLC* scheme are much simpler than that of the VSSI 3-*CUSUM* scheme. Moreover, the results of a later comparative study show that the VSSI *WLC* scheme (or the even simpler VSI *WLC* scheme) is a little more powerful than the VSSI 3-*CUSUM* scheme from an overall viewpoint. As a result, the VSSI *WLC* and VSI *WLC* schemes are the focuses of this chapter, and their design and operation will be described in detail in the next section.

3.2 VSSI *WLC* Scheme

3.2.1 Operation of the VSSI *WLC* Scheme

The VSSI *WLC* scheme monitors the statistic A_t , which is calculated by (2-15, 2-16).

$$A_0 = 0$$

$$A_t = \max(0, A_{t-1} + WL_t - k_A) \quad (2-15)$$

$$WL_t = \lambda s_t^2 + (1 - \lambda)(\bar{x}_t - \mu_0)^2 \quad (2-16)$$

As a typical VSSI CUSUM chart in Figure 3-1, the VSSI *WLC* scheme has a warning limit w_A and a control limit H_A . The warning and control limits divide the chart into the three zones.

$$\begin{aligned}
 \text{Zone 1 (central zone):} \quad & 0 \leq A_t \leq w_A \\
 \text{Zone 2 (warning zone):} \quad & w_A < A_t \leq H_A \\
 \text{Zone 3 (action zone):} \quad & H_A < A_t
 \end{aligned} \tag{3-2}$$

Zones 1 and 2 are two in-control zones, and zone 3 is an out-of-control one. The type of the next sample depends on the value of A_t in the current sample. The operation procedure of the VSSI *WLC* scheme is outlined below and summarized in flowchart 3-2. It is noted that the charting parameters n_1 , n_2 , h_1 , h_2 , w_A , H_A , k_A , λ are to be designed before the operation. The design will be elaborated later.

- (1) Set A_0 equal to zero and the sample index, f , as two at the beginning of the process.
($(f=1)$ refers to the relax sample and $(f=2)$ to the alert sample)
- (2) In the current sample, take n_f observations at the end of the sampling interval h_f ($f=1$ or 2).
- (3) Calculate the sample mean $\bar{x}_t$ and sample variance s_t^2 by equation (2-1).
- (4) Calculate the weighted loss function WL_t by equation (2-16).
- (5) Update the statistic A_t by equation (2-15) and plot the sample point on the chart of the VSSI *WLC* scheme.
- (6) If A_t is smaller than or equal to H_A , the process is deemed in control. The sample index f for the next sample is equal to one if $A_t \leq w_A$; otherwise, f equals two. Then, go back to step (2).

- (7) If A_t exceeds H_A , the process is thought out of control and a signal is produced. Stop the process immediately, diagnose and remove the assignable cause. Then, resume the process and go back to step (1).

Since only one statistic A_t needs to be updated and plotted on the chart, the operation of the VSSI WLC scheme is obviously simpler than that of the VSSI 3- $CUSUM$ scheme.

Figure 3-2: Operational Flowchart of the VSSI WLC Scheme

3.2.2 Design of the VSSI WLC Scheme

To design a VSSI *WLC* scheme, the following specifications need to be provided:

- R the allowed maximum in-control average inspection rate.
- n_{max} the allowed maximum sample size.
- h_{min} the allowed minimum sampling interval.
- τ the allowed minimum in-control Average Time to Signal.
- d_μ the minimum size of mean shift to be protected against.
- d_σ the minimum size of standard deviation shift to be protected against.

The value of R depends on the available resources for inspection (e.g., manpower and instrument) and the desired out-of-control *ATS* performance. The actual in-control inspection rate r (Arnold and Reynolds 2001) must not exceed R

$$r = \frac{(1 - B_2)n_1 + B_2n_2}{(1 - B_2)h_1 + B_2h_2}. \quad (3-3)$$

where B_2 is the stationary probability for the process being in zone 2 (the warning zone). Under the steady-state mode, the process usually runs in in-control status for a long period and has entered the stationary status before an out-of-control shift takes place at a random time. The values of n_{max} and h_{min} are decided by some practical considerations, such as the number of units that can be inspected in a short time period. The specifications of τ and (d_μ, d_σ) are required by the optimization designs of many control charts. For the optimization design of a VSSI CUSUM chart, the reference parameter k_{csum} is optimized based on the specified (d_μ, d_σ) and the control limit is adjusted so that the resultant in-control ATS_0 is equal or very close to τ (Reynolds and Arnold 2001). The value of τ is decided based on the requirement on the false alarm rate. If handling

false alarms is costly and time consuming, a larger τ should be used to reduce the false alarm frequency. Nevertheless, a large τ will impair the effectiveness of the control chart in the meantime. Ideally, the resultant (or actual) in-control ATS_0 is equal to τ . The specified d_μ and d_σ are the values of δ_μ and δ_σ that should be detected quickly by the control chart, therefore better protected against. The specifications of d_μ and d_σ usually imply that one has little interest in detecting process shifts smaller than the joint d_μ and d_σ . Therefore they represent the minimum sizes of mean shift and standard deviation shift to be protected against. The value of d_μ should be set relatively larger if it is predicted that mean shift be dominant in a process. Conversely, the value of d_σ should be set larger if the variance shift is likely to prevail. The values of the in-control mean μ_0 and standard deviation σ_0 can be set as zero and one, respectively, for the design of the VSSI *WLC* scheme. The resultant charting parameters can be easily modified later based on the actual values of μ_0 and σ_0 that are usually estimated from the data observed during the pilot runs. Based on the specifications, following optimization model for the design of the VSSI *WLC* scheme can be built.

$$\text{Minimize:} \quad U = ATS(d_\mu, d_\sigma) \quad (3-4)$$

$$\text{Subject to:} \quad ATS_0 = \tau \quad (3-5)$$

$$r = R \quad (3-6)$$

$$n_1 \leq n_2 \leq n_{max} \quad (3-7)$$

$$h_1 \geq h_2 \geq h_{min} \quad (3-8)$$

with respect to the eight design variables: $n_1, n_2, h_1, h_2, \lambda, k_A, H_A$ and w_A .

Before the optimization design of the VSSI *WLC* scheme is presented, we need to address the formulations to calculate the performance measures (in-control ATS_0 , r and

out-of-control ATS). The formulae for calculating the cumulative probability function $F(\Gamma)$ of the WL statistic (developed by Wu *et al.* 2005), however, must be introduced first.

Calculation of the Cumulative Probability Function $F(\Gamma)$ of WL (Wu *et al.* 2005)

There are two different cases in which the Weighted Loss Function WL is greater than Γ .

$$WL = \lambda s^2 + (1 - \lambda)(\bar{x} - \mu_0)^2 > \Gamma \quad (3-9)$$

Case 1: The mean shift term, $(1 - \lambda)(\bar{x} - \mu_0)^2$, alone is greater than Γ . That is,

$$\bar{x} < \mu_0 - a \quad \text{or} \quad \bar{x} > \mu_0 + a \quad (3-10)$$

where,

$$a = \sqrt{\frac{\Gamma}{1 - \lambda}} \quad (3-11)$$

For this case, WL is always greater than Γ regardless of the value of s .

Case 2: The mean shift term is smaller than Γ . That is,

$$\mu_0 - a < \bar{x} < \mu_0 + a \quad (3-12)$$

For this case, in order to make WL greater than Γ , we must have

$$s^2 > \frac{\Gamma - (1 - \lambda)(\bar{x} - \mu_0)^2}{\lambda} \quad (3-13)$$

Let

$$Q = \frac{s^2(n-1)}{\sigma^2} = \frac{s^2(n-1)}{(\delta_\sigma \sigma_0)^2} \quad (3-14)$$

Q follows a Chi-square distribution with degrees of freedom of $(n - 1)$. Then, inequality (3-13) is equivalent to

$$Q > b(\bar{x}) \quad (3-15)$$

where, $b(\bar{x})$ is an expression in terms of $\bar{x}$.

$$b(\bar{x}) = \frac{(n-1) \left[\Gamma - (1-\lambda)(\bar{x} - \mu_0)^2 \right]}{\lambda(\delta_\sigma \sigma_0)^2} \quad (3-16)$$

Summarizing the above two cases, the probability that WL is greater than Γ can be calculated by conditioning on the value of sample mean $\bar{x}$.

$$\begin{aligned} \Pr(WL > \Gamma) &= \Pr(\bar{x} < \mu_0 - a) + \Pr(\bar{x} > \mu_0 + a) + \int_{\mu_0 - a}^{\mu_0 + a} p(\bar{x}) f(\bar{x}) d\bar{x} \\ &= P_1 + P_2 + P_3 \end{aligned} \quad (3-17)$$

where, the probabilities P_1 and P_2 can be determined as follows:

$$P_1 = \Pr(\bar{x} < \mu_0 - a) = \Phi \left[\frac{(\mu_0 - a) - (\mu_0 + \delta_\mu \sigma_0)}{\delta_\sigma \sigma_0 / \sqrt{n}} \right] = \Phi \left[\frac{-\sqrt{n}(a + \delta_\mu \sigma_0)}{\delta_\sigma \sigma_0} \right] \quad (3-18)$$

$$P_2 = \Pr(\bar{x} > \mu_0 + a) = 1 - \Phi \left[\frac{(\mu_0 + a) - (\mu_0 + \delta_\mu \sigma_0)}{\delta_\sigma \sigma_0 / \sqrt{n}} \right] = 1 - \Phi \left[\frac{\sqrt{n}(a - \delta_\mu \sigma_0)}{\delta_\sigma \sigma_0} \right] \quad (3-19)$$

The probability P_3 is calculated by integration.

$$P_3 = \int_{\mu_0 - a}^{\mu_0 + a} p(\bar{x}) f(\bar{x}) d\bar{x} \quad (3-20)$$

where, $f(\bar{x})$ is the probability density function of sample mean $\bar{x}$. It follows a normal distribution with mean equal to $(\mu_0 + \delta_\mu \sigma_0)$ and variance equal to $(\delta_\sigma \sigma_0)^2 / n$. $p(\bar{x})$ is the probability that a sample value of WL is greater than Γ when the value of $\bar{x}$ is given. Referring to equation (3-15),

$$p(\bar{x}) = \Pr(Q > b(\bar{x})) = 1 - \chi_{n-1}^2(b(\bar{x})) \quad (3-21)$$

where, $\chi_{n-1}^2(.)$ is the cumulative probability function of a Chi-square distribution with degrees of freedom equal to $(n - 1)$.

Finally, the cumulative probability function $F(\Gamma)$ of WL is calculated by

$$F(\Gamma) = \Pr(WL \leq \Gamma) = 1 - \Pr(WL > \Gamma) = 1 - P_1 - P_2 - P_3 \quad (3-22)$$

All of the following formulae and procedures are firstly derived in this thesis.

Calculation of In-Control ATS_0 and r of VSSI WLC Scheme

Because of the properties of the CUSUM chart, the VSSI WLC scheme can be described by a Markov chain. The values of ATS_0 and ATS are calculated by the Markov chain approach suggested by Brook and Evans (1972). Possible values of A_t are divided into M different transitional states and 1 absorbing state. Transitional states 0 to $(M-1)$ are in control and the absorbing state M is out of control (the value of M is set as 50 in our computation). The width of the interval of an in-control state is

$$d = H_A / M \quad (3-23)$$

The center, O_i , of state i is

$$O_i = i \cdot d \quad i = 0, 1, \dots, M-1 \quad (3-24)$$

The in-control states can be further divided into states in central zone (states 0 to g) and states in warning zone (states $g+1$ to $M-1$), where g is the integer closest to w_A / d .

Let p_{ij} be the transition probability from state i to state j .

$$p_{ij} = \begin{cases} \Pr[O_i + WL - k_A \leq 0.5d] = F[(0.5 - i)d + k_A] & \text{for } j = 0 \\ \Pr[O_j - 0.5d \leq O_i + WL - k_A \leq O_j + 0.5d] \\ = F[(j - i + 0.5)d + k_A] - F[(j - i - 0.5)d + k_A] & \text{for } j > 0 \end{cases} \quad (3-25)$$

where, the cumulative probability function $F(\cdot)$ was just given in equation (3-22). In order to evaluate the in-control ATS_0 and inspection rate r , $F(\cdot)$ must be calculated by using $(\delta_\mu = 0)$ and $(\delta_\sigma = 1)$. Furthermore, sample size n_1 should be used if $i \leq g$ and n_2 is used if $i > g$.

Based on p_{ij} , the in-control transition probability matrix $\mathbf{R}_0$ can be established. It is a $M \times M$ matrix excluding the elements associated with the absorbing (out-of-control) state.

$$\mathbf{R}_0 = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0,M-1} \\ p_{10} & p_{11} & \cdots & p_{1,M-1} \\ \vdots & \vdots & \vdots & \vdots \\ p_{M-1,0} & p_{M-1,1} & \cdots & p_{M-1,M-1} \end{bmatrix} \quad (3-26)$$

ATS_0 is equal to the first element of vector $\mathbf{W}$ given by the following expression.

$$\mathbf{W} = (\mathbf{I} - \mathbf{R}_0)^{-1} \mathbf{H} \quad (3-27)$$

where, $\mathbf{I}$ is an identity matrix and $\mathbf{H} = [\ell_0, \ell_1, \dots, \ell_{M-1}]^T$ is the sampling interval vector.

$$\ell_i = \begin{cases} h_1 & i \leq g \\ h_2 & i > g \end{cases} \quad (3-28)$$

The in-control stationary probability vector $\mathbf{B} = [b_0, b_1, \dots, b_{M-1}]^T$ is obtained by solving the following equation.

$$\mathbf{B} = \mathbf{R}_0^T \mathbf{B} \quad (3-29)$$

subject to

$$\mathbf{1}^T \mathbf{B} = 1 \quad (3-30)$$

where vector $\mathbf{1}$ is a column vector the elements of which are all equal to 1. The stationary probability associated with the alert samples is

$$B_2 = \sum_{i=g+1}^{M-1} b_i \quad (3-31)$$

Finally, the average inspection rate is

$$r = \frac{(1 - B_2)n_1 + B_2n_2}{(1 - B_2)h_1 + B_2h_2} \quad (3-32)$$

Calculation of Out-of-Control *ATS* of VSSI WLC Scheme

The out-of-control transition probability matrix $\mathbf{R}$ for calculating the out-of-control *ATS* can be established like $\mathbf{R}_0$ except that the elements p_{ij} of $\mathbf{R}$ must be evaluated under the out-of-control conditions ($\delta_\mu \neq 0$) and/or ($\delta_\sigma > 1$). It is assumed that the control statistic A_t has reached its stationary distribution at the time when the process shift occurs. It is also assumed that the probability that the process shift takes place in a particular sampling interval is proportional to the product of the stationary probability and the length of that sampling interval, and the random time of process shift has a uniform distribution within the sampling interval (Reynolds *et al.* 1990). Based on these assumptions, the out-of-control *ATS* is calculated by the following equation.

$$ATS = \mathbf{V}^T [(\mathbf{I} - \mathbf{R})^{-1} \mathbf{H} - \mathbf{H} / 2] \quad (3-33)$$

where, $\mathbf{V} = [v_0, v_1, \dots, v_{M-1}]^T$ is the modified stationary probability vector.

$$v_i = \frac{\ell_i b_i}{h_1(1 - B_2) + h_2 B_2} \quad (3-34)$$

Optimization Design of VSSI WLC Scheme

The design of a VSSI *WLC* scheme is to determine the values of eight parameters n_1 , n_2 , h_1 , h_2 , λ , k_A , H_A and w_A . Firstly, h_2 is fixed at h_{min} as recommended by many researchers (Reynolds *et al.* 1988, Prabhu *et al.* 1994). It also substantially reduces the search effort required.

The values of the seven remaining parameters are searched in six levels. At each of the first five levels, the optimal value of a particular parameter (design variable) is sought through a single variable search using the predetermined starting value, step size and search range. From the starting point, the design variable is first increased and decreased by one step in order to find the searching direction leading to the decrease of the objective function. Then the search is conducted along this direction with the given step size until a local optimum is found or the lower or upper bounds of the search range is reached. It is assumed that the objective function is a convex function of each design variable.

The six-level design algorithm of the VSSI *WLC* scheme is outlined below in which the starting value, step size and search range for each parameter are given, and therefore, the design can be easily implemented by a computer program with multiple loops.

Steps of the Optimization Design of the VSSI *WLC* Scheme:

- (0) Enter the specifications of τ , R , n_{max} , h_{min} , d_μ and d_σ . Initialize ATS_{min} as a very large number, say 10^7 (ATS_{min} is used to store the minimum ATS). Set the sampling interval h_2 of the alert sample equal to h_{min} .
- (1) In the top level, the optimal value of the weighting factor λ used in equation (2-14) is searched (starting value: 0.3, step size: 0.01, range: $0 \leq \lambda \leq 1$). From the results of

numerous numerical tests, it is found that the optimal value of λ is always close to 0.3 when the magnitudes of the mean shift and standard deviation shift are comparable. Consequently, 0.3 is used as the starting value of λ .

- (2) In the second level, the optimal value of the sample size n_2 of the alert sample is searched (starting value: n_{max} , step size: 1, range: $2 \leq n_2 \leq n_{max}$).
- (3) In the third level, the optimal value of the sample size n_1 of the relax sample is searched (starting value: n_2 , step size: 1, range: $2 \leq n_1 \leq n_2$). Here, the value of n_2 has been tentatively determined in the second level. The search regions of n_1 and n_2 ensure that neither n_1 nor n_2 be larger than n_{max} (Note, if the VSI *WLC* scheme, instead of the VSSI *WLC* scheme, is to be designed, the only modification is to change the search range of n_1 to $(n_2 \leq n_1 \leq n_2)$. It forces n_1 consistently equal to n_2).
- (4) In the fourth level, the optimal value of the sampling interval h_1 of the relax sample is searched (starting value: h_{min} , step size: Δ_{h1} , range: $h_{min} = h_2 \leq h_1 \leq h_{1max}$). The search regions of h_1 and h_2 ensure that neither h_1 nor h_2 be smaller than h_{min} . It is somewhat difficult to decide Δ_{h1} and h_{1max} directly. However, h_1 is a monotonic increasing function of B_2 (i.e. the stationary probability for the process being in the warning zone, $0 \leq B_2 \leq 1$).

$$h_1(B_2) = \left[\frac{(1 - B_2)n_1 + B_2n_2}{R} - B_2h_2 \right] / (1 - B_2) \quad (3-35)$$

In the designs of some VSSI CUSUM schemes, the effect of B_2 on the chart performance is usually tested within a sub-range of $(0.1 \leq B_2 \leq 0.7)$ with an increment of 0.1 (Reynolds *et al.* 1990). In light of this, the step size Δ_{h1} and the upper bound h_{1max} of h_1 are determined as follows (referring to equation (3-35)),

$$\begin{aligned}\Delta_{h1} &= h_1(0.2) - h_1(0.1) \\ h_{1\max} &= h_1(0.7)\end{aligned}\tag{3-36}$$

(Note, if the VSS *WLC* scheme, instead of the VSSI *WLC* scheme, is to be designed, the only modification is to change the search range of h_1 and h_2 to $h_{\min} \leq h_1 \leq h_{1\max}$ and $h_2 = h_1$. That is h_2 is not fixed to $h_{\min}$ and is forced to be equal to h_1).

- (5) In the fifth level, the optimal value of the reference parameter k_A used in equation (2-15) is searched (starting value: k_{A0} , step size: 0.1, range: $0 \leq k_A \leq \infty$). Since the expected value of the adjusted loss function *WL* is

$$E_{WL}(\delta_\mu, \delta_\sigma) = \lambda(\delta_\sigma \sigma_0)^2 + (1 - \lambda)((\delta_\sigma \sigma_0)^2 / n + (\delta_\mu \sigma_0)^2)\tag{3-37}$$

the starting value k_{A0} is set as

$$k_{A0} = 0.5[E_{WL}(0,1) + E_{WL}(d_\mu, d_\sigma)]\tag{3-38}$$

This temporary setting relates with the common understanding that the reference parameter of the CUSUM chart is usually made equal to half the size of the expected change in the control statistic from the in-control to the out-of-control shift to be protected against.

- (6) In the bottom level, the values of the control and warning limits, H_A and w_A , are adjusted so that the in-control ATS_0 and r are equal (or very close) to their specifications τ and R , respectively. Both ATS_0 and r are functions of H_A and w_A . But, ATS_0 is mainly decided by H_A and r by w_A .

- (6.1) Solve the following equation numerically by adjusting H_A , referring to equation (3-27) (Increasing H_A if $ATS_0 < \tau$ and decreasing H_A if $ATS_0 > \tau$ until the difference between ATS_0 and τ is smaller than a predetermined tolerance).

$$ATS_0(H_A, w_A) = \tau\tag{3-39}$$

- (6.2) Solve the following equation by adjusting w_A , referring to equation (3-32) (Increasing w_A if $r > R$ and decreasing w_A if $r < R$ until the difference between r and R is smaller than a tolerance).

$$r(H_A, w_A) = R \quad (3-40)$$

Steps (6.1) and (6.2) will be carried out iteratively until both ATS_0 and r are equal (or very close) to τ and R at the same time.

- (7) When the values of all the parameters have been tentatively determined, the out-of-control ATS at (d_μ, d_σ) can be calculated, referring to (3-33).
- (8) If the calculated ATS is smaller than the current ATS_{min} , replace the latter by the former, and the current values of the parameters $\lambda, n_1, n_2, h_1, h_2, k_A, H_A$ and w_A are stored as the temporary optimal design.
- (9) At the end of the entire six-level search, the VSSI WLC scheme that results in the minimum ATS at $(\delta_\mu = d_\mu, \delta_\sigma = d_\sigma)$ and satisfies all constraints can be identified. The optimal values of all the charting parameters are also finalized.

Above design algorithm has been implemented in a computer program **VSSI_WLC.C** in C language, by which the design of a typical VSSI WLC scheme can be completed in about one minute of CPU time in a personal computer. The design procedure for the VSI WLC scheme is almost the same except that the third level search for n_1 can be saved ($n_1 \equiv n_2$), and for the VSS WLC scheme just by setting ($h_1 \equiv h_2$) at the fourth level. With some modifications, this algorithm can also be adopted to design other VSS, VSI and VSSI Shewhart or CUSUM schemes for monitoring the process mean and variance. The program is available upon request through email.

3.3 Design of the VSSI 3-CUSUM Scheme

The design and operation of each of the three CUSUM charts in a VSSI 3-CUSUM scheme is quite similar to that for a VSSI *WLC* scheme. Each CUSUM chart also has three zones divided by the warning and control limits (w_I and H_I of the I chart for increasing mean shift; w_D and H_D of the D chart for decreasing mean shift; w_V and H_V of the V chart for increasing variance shift). A sample taken at the end of a sampling interval is used by all three CUSUM charts. The type of the next sample depends on the values of the statistics I_t , D_t and V_t (equations 2-8 to 2-10) in the current sample. If all of the three statistics fall into their central zones, the relax sample is adopted for the next sample. If one or more of the statistics fall into the action zone, an out-of-control signal will be generated. For any other situations, the process is thought in control, but the alert sample should be chosen.

Due to the interactions among the three CUSUM charts, it is impossible to express the overall *ATS* of the VSSI 3-CUSUM scheme in terms of the *ATS* values of the three individual CUSUM charts. Simulation may have to be used to evaluate the overall *ATS* (Gan 1995) and r . Consequently, it is too time consuming to search a desired solution of a VSSI 3-CUSUM scheme in as many levels as for the VSSI *WLC* scheme. As an alternative, the VSSI 3-CUSUM scheme will use the same sample sizes n_1 , n_2 and sampling intervals h_1 , h_2 as that used by the VSSI *WLC* scheme. It will result in a workable, if not optimal, design of the VSSI 3-CUSUM scheme. Then, among the nine remaining parameters k_I , w_I , H_I , k_D , w_D , H_D , k_V , w_V and H_V ,

$$k_I = k_D, \quad w_I = -w_D, \quad H_I = -H_D \quad (3-41)$$

So, there are still six independent parameters k_I , w_I , H_I , k_V , w_V , H_V to be determined. First, the I chart and V chart are designed separately, in which k_I and k_V are chosen

respectively, in order to minimize the ATS at $\delta_\mu = d_\mu$ and $\delta_\sigma = d_\sigma$. The values of w_I , H_I , w_V and H_V are determined so that the overall in-control ATS_0 and inspection rate r of the VSSI 3-CUSUM scheme will satisfy constraints (3-5) and (3-6). The overall ATS_0 of the VSSI 3-CUSUM scheme can be determined from the $ATS_{0,I}$, $ATS_{0,D}$ and $ATS_{0,V}$ of the three individual CUSUM charts based on an approximate formula by Arnold and Reynolds (2001).

$$\frac{1}{ATS_0} \approx \frac{1}{ATS_{0,I}} + \frac{1}{ATS_{0,D}} + \frac{1}{ATS_{0,V}} \quad (3-42)$$

Normally, in-control ATS_0 for increasing and decreasing mean shift are made equal as in Gan's work (1995), hence

$$ATS_{0,I} = ATS_{0,D} \quad (3-43)$$

Next a parameter η ($0 < \eta < 1$) is introduced to allocate the detection power between chart for mean shift (I and D charts) and chart for variance shift (V chart). It will ensure the satisfaction of (3-42).

$$\begin{aligned} ATS_{0,V} &= ATS_0 / \eta \\ ATS_{0,I} &= ATS_{0,D} = 2ATS_0 / (1 - \eta) \end{aligned} \quad (3-44)$$

Arnold and Reynolds (2001) also used an approximate relationship between the overall stationary probabilities B_2 and its counterparts in the three individual CUSUM charts.

$$B_2 \approx B_{2,I} + B_{2,D} + B_{2,V} \quad (3-45)$$

By making

$$B_2 = \frac{Rh_1 - n_1}{n_2 - n_1 + R(h_1 - h_2)} \quad (3-46)$$

$$\begin{aligned} B_{2,I} &= B_{2,D} = 0.25B_2 \\ B_{2,V} &= 0.5B_2 \end{aligned} \quad (3-47)$$

constraint (3-6) will be approximately satisfied. That is because the value of B_2 determined by equation (3-46) makes the inspection rate r (3-3) of the VSSI 3-CUSUM scheme exactly equal to R . The values of $ATS_{0,I}$, $ATS_{0,D}$, $ATS_{0,V}$, $B_{2,I}$, $B_{2,D}$ and $B_{2,V}$ can be used to obtain a preliminary design for each of the three CUSUM charts. Then, simulation procedure is used to finalize the parameters w_I , H_I , w_V and H_V . It is noted that the design for the VSSI 3-CUSUM scheme is not fully optimized in sense that it adopts the optimal sample sizes n_1 , n_2 and sampling intervals h_1 and h_2 of the VSSI WCL scheme. Due to the intrinsic weak point of the VSSI 3-CUSUM scheme, the computation of ATS of this scheme is very time consuming and, consequently, the full optimization design is almost impossible.

Steps for the Design of the VSSI 3-CUSUM Scheme:

- (1) Carry out the design of a VSSI WLC scheme based on the specifications of n_{max} , h_{min} , τ , R , d_μ and d_σ . The resultant parameters n_1 , n_2 , h_1 and h_2 are adopted by the VSSI 3-CUSUM scheme. Use equation (3-46) to calculate the required B_2 that ensures r being equal to R . Initialize ATS_{min} as a very large number, say 10^7 (ATS_{min} is used to store the minimum ATS).
- (2) In this level, the allocation parameter η in (3-44) is searched (starting value: 0.5, step size: 0.1, range $0 < \eta < 1$).
- (3) The optimization of individual charts and the determination of the overall ATS_0 and r .
 - (3.1) Design the I chart for increasing mean shift. It chooses the value of k_I so that ATS is minimized at $\delta_\mu = d_\mu$ and $\delta_\sigma = d_\sigma$. Meanwhile, the values of H_I^* and w_I^*

(the asterisks attached to H_I^* and w_I^* denote that these two parameters are the tentative ones) are adjusted so that, referring to (3-44).

$$ATS_{0,I} = 2\tau / (1 - \eta) \quad (3-48)$$

$$B_{2,I} = 0.25B_2 \quad (3-49)$$

The procedure is almost the same as that for determining k_A , H_A and w_A of the VSSI *WLC* scheme.

(3.2) For the D chart for decreasing mean shift, simply set $k_D = k_I$, $H_D^* = -H_I^*$ and $w_D^* = -w_I^*$.

(3.3) Design the V chart for increasing variance shift. It includes choosing the value of k_V so that ATS is minimized at $\delta_\mu = d_\mu$ and $\delta_\sigma = d_\sigma$, and adjusting the values of H_V^* and w_V^* so that

$$ATS_{0,V} = \tau / \eta \quad (3-50)$$

$$B_{2,V} = 0.5B_2 \quad (3-51)$$

(3.4) At last, the three CUSUM charts are combined, and the four parameters H_I^* , w_I^* , H_V^* and w_V^* have to be finally adjusted so that the overall ATS_0 and r of the VSSI 3-CUSUM scheme are very close to the specified τ and R , respectively. The final adjustment is carried out with the help of two auxiliary variables a_H and a_w , and always make

$$H_I = a_H H_I^*, \quad w_I = a_w w_I^* \quad (3-52)$$

$$H_V = a_H H_V^*, \quad w_V = a_w w_V^* \quad (3-53)$$

$$H_D = -H_I, \quad w_D = -w_I \quad (3-54)$$

The variable a_H for control limits are primarily used to adjust the overall ATS_0 , while the variable a_w for warning limits are primarily used to adjust the overall

- r . During the final adjustment, the overall ATS_0 and r are evaluated by simulation.
- (4) When the values of all the parameters have been tentatively determined, the overall out-of-control ATS at (d_μ, d_σ) can be simulated.
 - (5) If the simulated ATS is smaller than the current ATS_{min} , replace the latter by the former, and the current values of the parameters $k_I, H_I, w_I, k_D, H_D, w_D, k_V, H_V$ and w_V are stored as the temporary optimal design.
 - (6) At the end of the search, the VSSI 3-*CUSUM* scheme that results in the minimum ATS at $(\delta_\mu = d_\mu, \delta_\sigma = d_\sigma)$ and satisfies all constraints can be identified. The optimal values of all the charting parameters are also finalized.

Since simulation has to be employed during the final adjustment, the design of the VSSI 3-*CUSUM* scheme is more time consuming even when it uses the resultant parameters n_1, n_2, h_1 and h_2 from the VSSI *WLC* chart. Normally it takes about 200 seconds for one optimization design.

3.4 Comparative Studies

Two comparative studies are conducted to evaluate the performance of the VSSI *WLC* scheme, the VSSI 3-*CUSUM* scheme and other charts. The first study is to compare the performance of the VSSI *WLC* scheme with non-*WLC* based schemes. Secondly, several *WLC* based charts with or without variable sampling features are compared. In each study, the comparison is first conducted for a typical case according to the norm of SPC practice. Then, a 2^4 factorial experiment (on n_{max}, h_{min}, τ and R) and a 2^2 factorial experiment (on design point d_μ and d_σ) are carried out to further compare the charts.

3.4.1 Study one

The effectiveness of the following four control charts is compared in this study.

- (1) Static $\bar{X}$ & S charts
- (2) VSSI $\bar{X}$ & S charts
- (3) VSSI 3- $CUSUM$ scheme
- (4) VSSI WLC scheme

The design procedures for the first three charts are similar to that for the VSSI WLC scheme. Namely, all the parameters of these three charts are optimized in order to minimize the out-of-control ATS at $(\delta_\mu = d_\mu, \delta_\sigma = d_\sigma)$.

The static $\bar{X}$ & S charts use fixed sample size n and sampling interval h . But both n and h will be optimized. Another parameter to be optimized is the allocation factor η ($0 \leq \eta \leq 1$) used to distribute the overall type I error α between the $\bar{X}$ chart and the S chart.

$$\begin{aligned}\alpha_s &= \eta\alpha \\ \alpha_{\bar{x}} &= (\alpha - \alpha_s) / (1 - \alpha_s)\end{aligned}\tag{3-55}$$

The VSSI $\bar{X}$ & S charts use the statistic $z_t (= \sqrt{n}(\bar{x} - \mu_0) / \sigma_0)$ to monitor the mean shifts and $q (= s / \sigma_0)$ for variance shift (Costa 1999a). The variable z_t is independent of the sample size when process is in control, whereas q is not. Therefore, the design of the VSSI $\bar{X}$ & S charts requires the determination of 13 parameters, i.e., the allocation factor η , the sample sizes and sampling intervals n_1, n_2, h_1, h_2 ; the lower and upper warning and control limits $w_{Z,lower}, w_{Z,upper}, LCL_Z$ and UCL_Z of the $\bar{X}$ chart; the warning and control limits w_{q1}, UCL_{q1} (for the relax sample) and w_{q2}, UCL_{q2} (for the alert sample) of the S chart. The lower side warning and control limits of the $\bar{X}$ chart are opposite to the upper side ones (i.e. $w_{Z,lower} = -w_{Z,upper}, LCL_Z = -UCL_Z$). The whole design procedure is similar to that for the VSSI WLC scheme.

Study 1.1: Comparison under the general conditions

The comparison is first conducted under the following general conditions:

$$n_{max} = 15, h_{min} = 0.2, \tau = 400, R = 5, d_{\mu} = 0.8, d_{\sigma} = 1.4 \quad (3-56)$$

where, h_{min} and τ are expressed in terms of the time unit to be used, and R indicates the average number of items to be inspected in a time unit. The specified set of process shifts ($d_{\mu} = 0.8, d_{\sigma} = 1.4$) is a reasonable compromise between mean and standard deviation shifts, and also between small and large shifts.

The chart parameters for the four control schemes are worked out and listed below. Also displayed is the inspection rate r produced by each scheme.

$$\begin{aligned} \text{Static } \bar{X}\&S \text{ charts: } n = 15, h = 3, UCL_Z = 2.7906, LCL_Z = -2.7906, UCL_q = 1.5524, \\ r &= 5.000 \end{aligned}$$

$$\begin{aligned} \text{VSSI } \bar{X}\&S \text{ charts: } n_1 = 5, n_2 = 14, h_1 = 1.65, h_2 = 0.20, w_{Z,upper} = 1.6184, UCL_Z = \\ &3.0379, w_{Z,lower} = -1.6184, LCL_Z = -3.0379, w_{q1} = 1.3823, \\ &UCL_{q1} = 2.1461, w_{q2} = 1.2280, UCL_{q2} = 1.6284, r = 5.0000 \end{aligned}$$

$$\begin{aligned} \text{VSSI 3-}CUSUM: n_1 = 6, n_2 = 7, h_1 = 1.67, h_2 = 0.20, w_I = 0.4912, H_I = 2.7273, \\ k_I = 1.1000, w_V = 0.4923, H_V = 1.7576, k_V = 0.2365, r = 4.9127 \end{aligned}$$

$$\begin{aligned} \text{VSSI } WLC \text{ scheme: } n_1 = 6, n_2 = 7, h_1 = 1.67, h_2 = 0.20, \lambda = 0.29, w_A = 0.0095, H_A = \\ 0.9981, k_A = 0.5831, r = 5.0252 \end{aligned}$$

(3-57)

It is noted that constraints (3-6), (3-7) and (3-8) pertaining to R , n_{max} and h_{min} are satisfied by all of the charts.

The parameters of above charts have been designed in order to minimize the out-of-control ATS at the design point ($d_{\mu} = 0.8, d_{\sigma} = 1.4$). However, in most of the

applications, it is hoped that the control charts be effective over a large domain of the process shifts. The values of ATS of the four charts in the shift domain of $(0 < \delta_\mu \leq 2, 1 < \delta_\sigma \leq 2)$ are displayed in Table 3-1. There are totally 65 out-of-control cells (combinations of the discrete values of δ_μ and δ_σ) and one in-control cell ($\delta_\mu = 0$ and $\delta_\sigma = 1$). Each cell contains the values of the ATS of the four charts (from the top to bottom: static $\bar{X}$ & S charts, VSSI $\bar{X}$ & S charts, VSSI 3- $CUSUM$ scheme, VSSI WLC scheme) for a particular combination of the values of δ_μ and δ_σ . There are several interesting findings listed below:

Table 3-1: ATS Values for the Four Charts in Study One

($n_{max} = 15, h_{min} = 0.2, R = 5, \tau = 400, d_\mu = 0.8, d_\sigma = 1.4$)

δ_σ	chart	δ_μ										
		0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
1.0	$\bar{X}S$ 1	400.00	122.03	25.96	7.82	3.33	1.99	1.60	1.51	1.50	1.50	1.50
	$\bar{X}S$ 2	400.00	173.84	33.27	6.70	2.48	1.53	1.16	0.99	0.89	0.85	0.82
	3- C	400.46	113.50	14.78	4.13	2.10	1.43	1.13	0.98	0.90	0.86	0.84
	WLC	396.55	157.46	30.26	7.02	2.77	1.62	1.17	0.97	0.88	0.83	0.81
1.2	$\bar{X}S$ 1	39.83	28.80	13.80	6.32	3.32	2.13	1.68	1.54	1.51	1.50	1.50
	$\bar{X}S$ 2	33.02	23.24	9.90	4.02	2.15	1.49	1.19	1.02	0.92	0.87	0.84
	3- C	14.11	11.33	6.06	3.15	1.99	1.43	1.17	1.02	0.93	0.88	0.84
	WLC	15.22	11.08	5.77	3.13	1.97	1.42	1.13	0.97	0.89	0.84	0.82
1.4	$\bar{X}S$ 1	9.22	8.32	6.25	4.23	2.86	2.10	1.72	1.57	1.52	1.50	1.50
	$\bar{X}S$ 2	5.29	4.69	3.43	2.36	1.72	1.37	1.16	1.03	0.94	0.89	0.85
	3- C	3.69	3.44	2.87	2.18	1.69	1.37	1.18	1.02	0.95	0.89	0.85
	WLC	3.47	3.17	2.55	1.96	1.54	1.26	1.08	0.96	0.89	0.85	0.82
1.6	$\bar{X}S$ 1	3.94	3.77	3.33	2.78	2.28	1.92	1.69	1.57	1.52	1.51	1.50
	$\bar{X}S$ 2	2.17	2.08	1.87	1.62	1.40	1.23	1.10	1.01	0.94	0.89	0.86
	3- C	2.12	2.05	1.87	1.63	1.44	1.26	1.12	1.02	0.95	0.90	0.86
	WLC	1.89	1.83	1.66	1.47	1.28	1.14	1.02	0.94	0.89	0.85	0.83
1.8	$\bar{X}S$ 1	2.43	2.38	2.25	2.08	1.89	1.74	1.63	1.56	1.52	1.51	1.50
	$\bar{X}S$ 2	1.47	1.45	1.39	1.30	1.21	1.12	1.04	0.98	0.93	0.89	0.86
	3- C	1.54	1.51	1.44	1.34	1.25	1.16	1.08	1.00	0.95	0.90	0.87
	WLC	1.38	1.36	1.30	1.21	1.12	1.04	0.97	0.92	0.88	0.85	0.83
2.0	$\bar{X}S$ 1	1.88	1.86	1.82	1.76	1.69	1.62	1.57	1.54	1.52	1.51	1.50
	$\bar{X}S$ 2	1.21	1.20	1.17	1.13	1.09	1.04	0.99	0.95	0.92	0.89	0.86
	3- C	1.27	1.27	1.25	1.19	1.13	1.07	1.01	0.98	0.93	0.90	0.87
	WLC	1.15	1.14	1.11	1.07	1.02	0.98	0.93	0.90	0.87	0.85	0.83

- (1) Firstly, all of the charts generate identical value of ATS_0 (almost equal to $\tau = 400$) when the process is in control ($\delta_\mu = 0$, $\delta_\sigma = 1$). This ensures that the constraint of ($ATS_0 = \tau$) is satisfied and the comparison between the charts is fair.
- (2) The VSSI *WLC* scheme obviously excels over other three charts in the whole shift domain except in a few cells where variance shift does not exist. Specifically, among the 65 out-of-control cells, the VSSI *WLC* is less effective than the static $\bar{X}$ &*S* charts, the VSSI $\bar{X}$ &*S* charts and the VSSI 3-*CUSUM* scheme only in two, four and seven cells, respectively. The reason that a single VSSI *WLC* scheme outperforms the VSSI 3-*CUSUM* scheme may be that the statistic *WL* (equation 2-14) used by the *WLC* scheme integrates the sample mean $\bar{x}$ and sample standard deviation *s* which jointly push *WL* up more quickly when mean and variance shifts take place simultaneously.
- (3) The VSSI 3-*CUSUM* scheme is the second most powerful chart from an overall viewpoint. It outperforms the static $\bar{X}$ &*S* charts in the whole shift domain. The VSSI 3-*CUSUM* scheme is more effective than the VSSI $\bar{X}$ &*S* charts when the process shifts δ_μ and/or δ_σ are small or moderate. However, it is inferior to the latter when δ_μ and/or δ_σ are large.
- (4) The adaptive feature also makes the VSSI $\bar{X}$ &*S* charts more powerful than its static counterpart except in two cells where δ_μ is small and δ_σ is equal to one.

With these findings, a general quantification of the superiority of the VSSI *WLC* chart is desirable. The Mean Relative Average Time to Signal (*MRATS*) with respect to the VSSI *WLC* approach can serve as a general quantitative measurement of the effectiveness of a chart compared to the VSSI *WLC* scheme:

$$MRATS = \frac{\sum_{j=1}^m \frac{ATS(\delta_{\mu,j}, \delta_{\sigma,j})}{ATS_{VSSI_WLC}(\delta_{\mu,j}, \delta_{\sigma,j})}}{m} \quad (3-58)$$

where, $\delta_{\mu,j}$ and $\delta_{\sigma,j}$ are the values of the mean and standard deviation shifts in the j th cell and m ($= 65$) is the total number of the out-of-control cells in Table 3-1. $ATS(\delta_{\mu,j}, \delta_{\sigma,j})$ is the value of the out-of-control ATS produced by a chart at $(\delta_{\mu,j}, \delta_{\sigma,j})$ and $ATS_{VSSI_WLC}(\delta_{\mu,j}, \delta_{\sigma,j})$ is the value generated by the VSSI WLC scheme for the same shifts of $\delta_{\mu,j}$ and $\delta_{\sigma,j}$. When the process shift is large, the difference of the ATS values between the charts becomes small. However, the impact of an out-of-control case cannot be judged only by the difference of ATS values, but also by the difference of the average number of defectives produced in a sampling interval. When a severe process shift occurs, much more defectives (or much higher loss function value) are produced in each sampling interval. Therefore, even a minor difference in ATS values may result in a substantial difference in costs. In view of this, the ratio of the ATS values of two charts in a cell is a more appropriate measure of the relative effectiveness than the difference of the ATS values.

Obviously, if the value of $MRATS$ (equation 3-58) of a particular chart is larger than one, this chart is generally less effective than the VSSI WLC scheme in the process shift domain ($0 < \delta_{\mu} \leq 2$, $1 < \delta_{\sigma} \leq 2$), and *vice versa*. The values of the $MRATS$ for the four charts are calculated from the data in Table 3-1 and listed in the row of RUN 0 in Table 3-2. These $MRATS$ values indicate that the VSSI WLC scheme is the most effective chart for the given design specifications (equation 3-56). It is not only more powerful than the static $\bar{X}$ & S charts by 75.84% on average, but is also more effective than the VSSI $\bar{X}$ & S charts and the VSSI 3- $CUSUM$ scheme, by 12.31% and 4.28% respectively.

Study 1.2: A 2^4 experiment pertaining to n_{max} , h_{min} , τ and R

Next, a 2^4 factorial experiment (Montgomery 2001) is employed to study the effects of the first four factors (specifications) n_{max} , h_{min} , τ and R on the relative effectiveness of the four charts (Other two factors d_μ and d_σ are still fixed at 0.8 and 1.2, respectively, for the time being). Each of the four factors varies at two levels, resulting in 16 runs (i.e., 16 combinations of the values of the four factors). The low and high levels for each factor are given below:

n_{max}	10	20	
h_{min}	0.1	0.3	
τ	200	600	
R	2	8	(3-59)

The ATS values of the four charts under each of these 16 runs disclose the similar operating characteristics as that shown in Table 3-1 for RUN 0 (it is the run at the center of the 2^4 factorial experiment). The $MRATS$ values of the charts in these 16 runs are also calculated and enumerated in Table 3-2. It can be seen that the VSSI WLC scheme is the most effective chart except for four runs where the VSSI 3- $CUSUM$ scheme performs a little better.

Finally, the grand average $\overline{MRATS}$, the average of the $MRATS$ values for a chart over the 17 runs, is calculated and listed at the bottom of Table 3-2. The values of $\overline{MRATS}$ indicate that, taking all different combinations of the values of n_{max} , h_{min} , τ , R , δ_μ and δ_σ into consideration (d_μ and d_σ are fixed), the VSSI WLC scheme is more effective than

the static $\bar{X}$ & S charts, the VSSI $\bar{X}$ & S charts and the VSSI 3- $CUSUM$ scheme by 80.07, 16.97 and 2.74 percent, respectively.

Table 3-2: $MRATS$ Values for the Four Charts in Study One

RUN	τ	R	n_{max}	h_{min}	Static $\overline{XS}$	VSSI $\overline{XS}$	VSSI 3-C
0	400	5	15	0.2	1.7584	1.1231	1.0428
1	200	2	10	0.1	1.6470	1.1798	0.9906
2				0.3	1.5216	1.1168	1.0029
3			20	0.1	1.7291	1.1530	0.9898
4				0.3	1.6984	1.1497	0.9935
5		8	10	0.1	1.7517	1.2176	1.0518
6				0.3	1.4310	1.1293	1.0450
7			20	0.1	1.9495	1.1344	1.0611
8				0.3	1.6226	1.0791	1.0163
9	600	2	10	0.1	1.8946	1.2043	1.0251
10				0.3	1.7324	1.2248	1.0350
11			20	0.1	2.2255	1.2541	1.0174
12				0.3	2.0064	1.1680	1.0241
13		8	10	0.1	1.9464	1.2021	1.0186
14				0.3	1.7049	1.2986	1.0864
15			20	0.1	2.2436	1.1774	1.0676
16				0.3	1.7488	1.0728	0.9976
$\overline{MRATS}$					1.8007	1.1697	1.0274

If the ease of design and operation is also taken into consideration, the VSSI *WLC* scheme is more likely to be a better choice than the VSSI 3-*CUSUM* scheme. The operation of the VSSI *WLC* scheme is even simpler than the operation of the VSSI $\bar{X}$ & S charts for the latter one has to plot two statistics z and q in two charts. Moreover, during the operation, operator has to judge which warning and control limit of the S chart should be used for relax sample or alert sample.

The effects of the four factors n_{max} , h_{min} , τ and R on the performance (the values of *MRATS*) of the control charts are also studied. The effects of all four factors are very insignificant for the VSSI $\bar{X}$ & S charts and the VSSI 3-*CUSUM* scheme. It implies that the superiority of the VSSI *WLC* scheme over these two schemes is quite consistent and not affected significantly by the changes of n_{max} , h_{min} , τ and R . The effect of R on the static $\bar{X}$ & S charts is also insignificant. But, the effects of n_{max} , h_{min} and τ on the *MRATS* value of the static $\bar{X}$ & S charts are 0.1993, -0.2401 and 0.2689, respectively, and considered relatively high. It means that the superiority of the VSSI *WLC* scheme over the static $\bar{X}$ & S charts is more notable when n_{max} and τ are larger and/or h_{min} is smaller.

Study 1.3: A 2^2 experiment pertaining to d_μ and d_σ

Finally, it is desired to conduct a 2^6 factorial experiment to study the effects of the six factors n_{max} , h_{min} , τ , R , d_μ and d_σ all together. However, since it has been observed in the preceding 2^4 experiment that the *MRATS* values of the VSSI $\bar{X}$ & S charts and the VSSI 3-*CUSUM* scheme are insensitive to n_{max} , h_{min} , τ and R , we may simplify the overall 2^6 experiment to a 2^2 experiment focusing on the last two specifications d_μ and d_σ (The first four factors n_{max} , h_{min} , τ and R are fixed at their nominal values as in (3-56)). The comparison with the static $\bar{X}$ & S charts is not a main concern, as this scheme is

obviously inferior to all other schemes. Each of d_μ and d_σ varies at two levels, resulting in four runs (Table 3-3).

$$\begin{array}{lll} d_\mu & 0.4 & 1.2 \\ d_\sigma & 1.2 & 1.6 \end{array} \quad (3-60)$$

Table 3-3: Factorial Experiment on d_μ and d_σ in Study One

$$(n_{max} = 15, h_{min} = 0.2, R = 5, \tau = 400)$$

RUN	d_μ	d_σ	$MRATS$		
			Static $\bar{X}\bar{S}$	VSSI $\bar{X}\bar{S}$	VSSI 3- C
0	0.8	1.4	1.7584	1.1231	1.0428
1	0.4	1.2	1.5930	2.0481	1.1027
2		1.6	1.8339	1.1105	1.0597
3	1.2	1.2	1.7360	1.1324	1.0752
4		1.6	1.6967	1.1345	1.0388
$\overline{MRATS}$			1.7236	1.3097	1.0639

In Table 3-3, RUN 1 ($d_\mu = 0.4$ and $d_\sigma = 1.2$) is used when the charts are designed to detect joint mean shift δ_μ and standard deviation shift δ_σ of small sizes; RUN 2 ($d_\mu = 0.4$ and $d_\sigma = 1.6$) for detecting prevailing δ_σ ; RUN 3 ($d_\mu = 1.2$ and $d_\sigma = 1.2$) for detecting dominant δ_μ ; Finally, RUN 4 ($d_\mu = 1.2$ and $d_\sigma = 1.6$) for detecting joint δ_μ and δ_σ of large magnitudes.

The *MRATS* values of the charts in above four runs and RUN 0 (i.e., the run at the center of this 2^2 factorial experiment) are enumerated in Table 3-3. It can be seen that the VSSI *WLC* scheme is always the most effective chart in all of the five runs. The grand average $\overline{MRATS}$ of a chart over the five runs is also calculated and listed at the bottom of Table 3-3. The values of $\overline{MRATS}$ indicate that, in a most general and overall

sense and in a broad process shift domain, the VSSI *WLC* scheme is more effective than the static $\bar{x}$ &*S* charts, the VSSI $\bar{x}$ &*S* charts and the VSSI 3-*CUSUM* scheme by 72.36, 30.97 and 6.39 percent, respectively.

The effects of d_μ and d_σ on the values of *MRATS* of the VSSI $\bar{x}$ &*S* charts are -0.4458 and -0.4677, respectively, that are very significant. It means that the VSSI *WLC* scheme is much more effective than the VSSI $\bar{x}$ &*S* charts when used to detect small mean and variance shifts. It seems quite natural as the *CUSUM* type charts are significantly superior to the Shewhart type charts for detecting small process shifts.

The effectiveness of the VSSI 3-*CUSUM* scheme relative to the VSSI *WLC* scheme does not vary substantially along with the changes of d_μ and d_σ .

3.4.2 Study two

Next, the effectiveness of four *WLC* schemes, i.e., the static, VSS, VSI and VSSI *WLC* schemes are compared. The VSS *WLC* scheme uses the same sampling interval for both relax and alert samples, that is, $h_1 = h_2 = h$. For the VSI *WLC* scheme, $n_1 = n_2 = n$. For the static *WLC* scheme, fixed sample size n and fixed sampling interval h are always used, but they are adjusted to minimize *ATS* at $(\delta_\mu = d_\mu, \delta_\sigma = d_\sigma)$. The parameters of these three *WLC* schemes are designed by the procedures similar to that for the VSSI *WLC* scheme.

Study 2.1: Comparison under the general conditions

As in study one, the comparison is first conducted under the general conditions:

$$n_{max} = 15, h_{min} = 0.2, \tau = 400, R = 5, d_\mu = 0.8, d_\sigma = 1.4 \quad (3-61)$$

The charting parameters, as well as the inspection rate r , of the four *WLC* schemes are worked out and listed below.

Static *WLC* scheme: $n = 14, h = 2.80, H_A = 0.2267, k_A = 0.5420, \lambda = 0.28,$
 $r = 5.0000$

VSS *WLC* scheme: $n_1 = 10, n_2 = 15, h = 2.10, w_A = 0.0828, H_A = 0.4723,$
 $k_A = 0.5420, \lambda = 0.31, r = 5.0453$

VSI *WLC* scheme: $n = 6, h_1 = 1.59, h_2 = 0.20, w_A = 0.0087, H_A = 1.1003,$
 $k_A = 0.5831, \lambda = 0.28, r = 5.0262$

VSSI *WLC* scheme: $n_1 = 6, n_2 = 7, h_1 = 1.67, h_2 = 0.20, w_A = 0.0095,$
 $H_A = 0.9981, k_A = 0.5831, \lambda = 0.29, r = 5.0252$

(3-62)

Again, constraints (3-6), (3-7) and (3-8) pertaining to R, n_{max} and h_{min} are fully satisfied by all of the *WLC* schemes.

Each cell in Table 3-4 contains the *ATS* values of the four schemes (from the top to bottom: static *WLC* scheme, VSS *WLC* scheme, VSI *WLC* scheme, VSSI *WLC* scheme) for a particular combination of δ_μ and δ_σ within the shift domain of $(0 < \delta_\mu \leq 2, 1 < \delta_\sigma \leq 2)$.

From Table 3-4, it can be found that,

- (1) All of the charts generate identical ATS_0 value (almost equal to τ) when the process is in control.

Table 3-4: *ATS* Values for the Four *WLC* Schemes in Study Two

$$(n_{max} = 15, h_{min} = 0.2, R = 5, \tau = 400, d_{\mu} = 0.8, d_{\sigma} = 1.4)$$

δ_{σ}	chart	δ_{μ}										
		0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
1.0	Static	397.66	135.99	30.53	9.33	4.00	2.28	1.65	1.45	1.41	1.40	1.40
	VSS	396.42	160.24	36.78	10.38	4.37	2.45	1.65	1.27	1.12	1.06	1.05
	VSI	396.09	154.00	29.34	7.05	2.80	1.62	1.16	0.95	0.85	0.80	0.78
	VSSI	396.55	157.46	30.26	7.02	2.77	1.62	1.17	0.97	0.88	0.83	0.81
1.2	Static	27.10	19.08	9.61	5.00	2.96	2.04	1.62	1.46	1.41	1.40	1.40
	VSS	24.06	17.54	9.23	4.98	3.02	2.05	1.54	1.26	1.13	1.07	1.06
	VSI	15.03	11.08	5.88	3.20	2.00	1.42	1.11	0.95	0.86	0.81	0.78
	VSSI	15.22	11.08	5.77	3.13	1.97	1.42	1.13	0.97	0.89	0.84	0.82
1.4	Static	6.90	6.09	4.52	3.19	2.33	1.84	1.58	1.46	1.42	1.40	1.40
	VSS	6.04	5.43	4.20	3.07	2.27	1.76	1.43	1.24	1.13	1.08	1.06
	VSI	3.59	3.28	2.62	2.00	1.55	1.25	1.06	0.94	0.86	0.82	0.79
	VSSI	3.47	3.17	2.55	1.96	1.54	1.26	1.08	0.96	0.89	0.85	0.82
1.6	Static	3.36	3.18	2.77	2.31	1.94	1.68	1.53	1.45	1.42	1.41	1.40
	VSS	3.07	2.93	2.58	2.18	1.82	1.54	1.34	1.20	1.13	1.08	1.06
	VSI	1.94	1.87	1.69	1.48	1.28	1.12	1.00	0.92	0.86	0.82	0.79
	VSSI	1.89	1.83	1.66	1.47	1.28	1.14	1.02	0.94	0.89	0.85	0.83
1.8	Static	2.23	2.17	2.03	1.86	1.69	1.57	1.49	1.44	1.42	1.41	1.40
	VSS	2.06	2.01	1.88	1.71	1.53	1.38	1.26	1.17	1.12	1.08	1.06
	VSI	1.39	1.37	1.30	1.21	1.11	1.03	0.95	0.89	0.85	0.82	0.80
	VSSI	1.38	1.36	1.30	1.21	1.12	1.04	0.97	0.92	0.88	0.85	0.83
2.0	Static	1.77	1.75	1.70	1.63	1.56	1.50	1.45	1.43	1.41	1.41	1.40
	VSS	1.59	1.57	1.52	1.44	1.35	1.26	1.19	1.14	1.10	1.08	1.06
	VSI	1.14	1.13	1.10	1.06	1.01	0.95	0.91	0.87	0.84	0.81	0.80
	VSSI	1.15	1.14	1.11	1.07	1.02	0.98	0.93	0.90	0.87	0.85	0.83

- (2) The VSSI *WLC* scheme significantly excels the static *WLC* scheme in the whole shift domain except in two cells where δ_{μ} is very small and δ_{σ} is equal to 1. It also outperforms the VSS *WLC* scheme overwhelmingly and substantially. However, even though the *ATS* value of the VSSI *WLC* scheme is smaller than that of the VSI *WLC* scheme at the design point ($d_{\mu} = 0.8, d_{\sigma} = 1.4$), the differences of the *ATS* values of these two schemes in all of the cells are very negligible. From a practical viewpoint, it can be concluded that the effectiveness of the VSI and VSSI *WLC* schemes is nearly identical in the whole process shift domain.

- (3) Since the VSI *WLC* scheme is in fact as powerful as the VSSI *WLC* scheme, the comparison between the VSI *WLC* scheme and the static *WLC* scheme (or VSS *WLC* scheme) is similar to that between the VSSI *WLC* scheme and the static *WLC* scheme (or VSS *WLC* scheme).
- (4) The VSS *WLC* scheme is considerably more powerful than its static counterpart except in a few cells where δ_μ and/or δ_σ are small.

Study 2.2: A 2^4 experiment pertaining to n_{max} , h_{min} , τ and R

Next, a 2^4 factorial experiment pertaining to n_{max} , h_{min} , τ and R is conducted under the same conditions as in study one. The values of *MRATS* of 17 runs (including RUN 0) and the values of the grand average $\overline{MRATS}$ are all calculated and enumerated in Table 3-5. From the values of $\overline{MRATS}$, it can be seen that the VSSI *WLC* scheme is more effective than its static and VSS counterparts by 51.62 and 37.71 percent, respectively. It is the improvement attributable to the application of the adaptive features to the control charts. It is also found that the VSI *WLC* scheme possesses nearly all of the advantages of its VSSI counterpart. Furthermore, while the performance comparison of the static and VSS *WLC* schemes are more or less affected by the values of n_{max} , h_{min} , τ and R , the relative effectiveness (the *MRATS* value) of the VSI *WLC* scheme is insensitive to the changes of these four factors.

Table 3-5: *MRATS* Values for the Four *WLC* Schemes in Study Two

RUN	τ	R	n_{max}	h_{min}	Static WLC	VSS WLC	VSI WLC
0	400	5	15	0.2	1.5715	1.3840	0.9863
1	200	2	10	0.1	1.4681	1.3117	1.0354
2				0.3	1.3553	1.2200	0.9760
3			20	0.1	1.4286	1.3994	1.0175
4				0.3	1.4001	1.3710	1.0796
5		8	10	0.1	1.5235	1.3702	1.0988
6				0.3	1.2364	1.1242	0.9643
7			20	0.1	1.6041	1.4341	0.9817
8				0.3	1.3451	1.2126	0.9949
9	600	2	10	0.1	1.6835	1.5265	1.0409
10				0.3	1.5302	1.3946	0.9918
11			20	0.1	1.8048	1.6140	1.0295
12				0.3	1.6333	1.4669	1.0284
13		8	10	0.1	1.5260	1.3983	1.0219
14				0.3	1.3399	1.2256	1.0253
15			20	0.1	1.8636	1.6520	0.9909
16				0.3	1.4612	1.3062	1.0314
$\overline{MRATS}$					1.5162	1.3771	1.0173

Study 2.3: A 2^2 experiment pertaining to d_μ and d_σ

As in study one, a 2^2 experiment focusing on the last two specifications d_μ and d_σ under the same conditions is also conducted. The results are shown in Table 3-6. The values of the grand average $\overline{MRATS}$ of the static, VSS and VSI *WLC* schemes produced in this 2^2

experiment are roughly close to that obtained in the preceding 2^4 experiments. Therefore, from a most general and overall viewpoint, it can be concluded that the VSSI *WLC* scheme is more effective than the static and VSS *WLC* schemes by 50.94 and 37.07 percent, respectively, and the VSSI and VSI *WLC* schemes are almost equal in performance for all different combinations of n_{max} , h_{min} , τ , R , d_μ and d_σ . Since the VSI *WLC* scheme is relatively easier to implement, it should be the most desirable scheme for detecting both mean and variance shifts.

Table 3-6: Factorial Experiment on d_μ and d_σ in Study Two

$$(n_{max} = 15, h_{min} = 0.2, R = 5, \tau = 400)$$

RUN	d_μ	d_σ	<i>MRATS</i>		
			Static <i>WLC</i>	VSS <i>WLC</i>	VSI <i>WLC</i>
0	0.8	1.4	1.5715	1.3840	0.9863
1	0.4	1.2	1.3948	1.2651	1.0035
2		1.6	1.5305	1.3834	1.0262
3	1.2	1.2	1.6320	1.5018	0.9978
4		1.6	1.4181	1.3193	0.9757
<i>MRATS</i>			1.5094	1.3707	0.9979

Since the factorial experiments in the above two studies are all carried out under the same conditions and the values of *MRATS* of all charts are calculated by using the VSSI *WLC* scheme as the benchmark, we can rank the seven charts all together according to the values of $\overline{MRATS}$ in Tables 3-3 and 3-6. It can be seen that the VSSI and VSI *WLC* schemes share the first position, followed by the VSSI 3-*CUSUM* scheme, and then the VSSI $\bar{X}$ &*S* charts, the VSS *WLC* scheme and the static *WLC* scheme. The static $\bar{X}$ &*S* charts rank the last. The VSI *WLC* scheme is not only substantially more effective than the static and VSSI $\bar{X}$ &*S* charts, but also outperforms the VSSI 3-*CUSUM* scheme to some degree. If the ease of operation is also taken into consideration, the VSI *WLC* scheme is undoubtedly the most promising control scheme for detecting process shifts

in mean and variance. During the operation of a VSI *WLC* scheme, only the sample interval has to be changed and the sample size is a constant. As a result, its operation is even simpler than that of the VSSI $\bar{X}$ & S charts. While the VSI *WLC* scheme is based on only seven parameters (n , h_1 , h_2 , w_A , H_A , k_A and λ) during the operation and the plot of a single statistic WL on a single chart, the VSSI $\bar{X}$ & S charts are based on eleven parameters (n_1 , n_2 , h_1 , h_2 , w_z , LCL_z , UCL_z , w_{q1} , UCL_{q1} , w_{q2} and UCL_{q2}) and the plots of two statistics z and q on two charts, and the VSSI 3-*CUSUM* scheme is based on thirteen parameters (n_1 , n_2 , h_1 , h_2 , w_L , H_L , k_L , w_D , H_D , k_D , w_V , H_V and k_V) and the plots of three statistics on three charts.

The design of a typical VSI *WLC* scheme requires only 20 seconds of CPU time in a personal computer (Pentium IV 2.4 GHz). This value is equal to 60 seconds for the design of a typical VSSI *WLC* scheme. For the design of a corresponding VSSI 3-*CUSUM* scheme, 200 seconds of CPU time are required even when it uses the sample sizes n_1 , n_2 and sampling intervals h_1 , h_2 designed for the VSSI *WLC* scheme. In fact, without the design procedure proposed for the adaptive *WLC* scheme, it is very difficult to decide the reasonable values for the sample sizes and sampling intervals of a VSSI 3-*CUSUM* scheme.

3.5 Diagnosis of the Process Shift

It is also useful to diagnose the process shifts for an out-of-control case, that is, to identify the type of the shift (mean shift or variance shift, or both), to estimate the magnitudes of δ_μ and δ_σ , and to locate the change point (the time when the shift takes place). The results of a trustworthy diagnosis make it easier for the users to identify and eliminate the assignable cause that produces the out-of-control case (Sullivan and

Woodall 1996, Reynolds and Stoumbos 2001).

Control schemes comprising several charts are usually straightforward at diagnosis, because the chart that signals will provide identification of the type of shift. On the other hand, the VSI *WLC* and VSSI *WLC* schemes can make use of the self-contained CUSUM estimator of the change point to diagnose the shift type. Usually, when the process is in control, only a few isolated sample points carry a A_t value larger than zero. However, when the process turns to out of control, most of the sample points result in ($A_t > 0$). The pattern of the sample points of a CUSUM chart often pinpoints the change point quite accurately (Nishina 1992). The magnitudes of δ_μ and δ_σ can be estimated from the $\bar{x}$ and s^2 values of the samples taken after the estimated change point. For a VSSI *WLC* scheme, the process shifts in mean and standard deviation can be estimated by the equations follow (Montgomery 2001).

$$\hat{\delta}_\mu = \left(\frac{n_1 \sum_{j=1}^{m_1} \bar{x}_{1j} + n_2 \sum_{j=1}^{m_2} \bar{x}_{2j}}{m_1 n_1 + m_2 n_2} - \mu_0 \right) / \sigma_0 \quad (3-63)$$

$$\hat{\delta}_\sigma = \left(\sqrt{\frac{\frac{(n_1 - 1) \sum_{j=1}^{m_1} s_{1j}^2}{m_1} + \frac{(n_2 - 1) \sum_{j=1}^{m_2} s_{2j}^2}{m_2}}{n_1 + n_2 - 2}} \right) / \sigma_0 \quad (3-64)$$

where, m_1 and m_2 are the numbers of the relax and alert samples taken after the estimated change point, and $\bar{x}_{ij}$ and s_{ij}^2 are the sample mean and sample variance of the j th sample of size n_i after the change point. The sample values $\bar{x}_{ij}$ and s_{ij}^2 are available during the calculation of A_t at each sample point.

The estimated $\hat{\delta}_\mu$ and $\hat{\delta}_\sigma$ are, used to identify the type of the shift.

- (1) If $(\hat{\delta}_\mu \geq b_\mu)$ and $(\hat{\delta}_\sigma < b_\sigma)$, an increasing shift in process mean is indicated.
- (2) If $(\hat{\delta}_\mu \leq -b_\mu)$ and $(\hat{\delta}_\sigma < b_\sigma)$, a decreasing shift in process mean is indicated.
- (3) If $(-b_\mu < \hat{\delta}_\mu < b_\mu)$ and $(\hat{\delta}_\sigma \geq b_\sigma)$, an increasing shift in variance is indicated.
- (4) If $(\hat{\delta}_\mu \geq b_\mu)$ and $(\hat{\delta}_\sigma \geq b_\sigma)$, a joint increasing mean and variance shift is indicated.
- (5) If $(\hat{\delta}_\mu \leq -b_\mu)$ and $(\hat{\delta}_\sigma \geq b_\sigma)$, a joint decreasing mean and increasing variance shift is indicated.
- (6) Finally, if $(-b_\mu < \hat{\delta}_\mu < b_\mu)$ and $(\hat{\delta}_\sigma < b_\sigma)$, no diagnosis is indicated.

Here b_u and b_σ are set at 0.2 and 1.2 respectively. Usually, values of b_u and b_σ are decided by the QA engineer based on their perception of a noticeable mean shift and a standard deviation shift.

Besides the VSI *WLC* and VSSI *WLC* schemes, the static $\bar{X}$ &*S* charts and the VSSI 3-*CUSUM* scheme are also studied in an illustrative comparison for diagnosis capability. The diagnosis rules used for the static $\bar{X}$ &*S* charts and the VSSI 3-*CUSUM* scheme here are also common for other combined schemes.

Diagnosis rules for the static $\bar{X}$ &*S* charts at the time of chart(s) signaling,

- (1) If the upper control limit of the $\bar{X}$ chart (UCL_Z) is exceeded alone, increasing mean shift is indicated.
- (2) If the lower control limit of the $\bar{X}$ chart (LCL_Z) is exceeded alone, decreasing mean shift is indicated.

- (3) If the upper control limit of the S chart (UCL_q) is exceeded alone, increasing variance shift is indicated.
- (4) If the UCL_Z and the UCL_q are exceeded together, joint increasing mean shift and increasing variance shift is indicated.
- (5) If the LCL_Z and the UCL_q are exceeded together, joint decreasing mean shift and increasing variance shift is indicated.

Diagnosis rules for the VSSI 3-CUSUM scheme at the time of chart(s) signaling,

- (1) If the control limit of the I chart (H_I) is exceeded alone, increasing mean shift is indicated.
- (2) If the control limit of the D chart (H_D) is exceeded alone, decreasing mean shift is indicated.
- (3) If the control limit of the V chart (H_V) is exceeded alone, increasing variance shift is indicated.
- (4) If the H_I and the H_V are exceeded together, joint increasing mean shift and increasing variance shift is indicated.
- (5) If the H_D and the H_V are exceeded together, joint decreasing mean shift and increasing variance shift is indicated.

Using above diagnosis rules, a simulation study is conducted under the general condition ($n_{max} = 15$, $h_{min} = 0.2$, $\tau = 400$, $R = 5$, $d_\mu = 0.8$, $d_\sigma = 1.4$). For each of the process shift combinations (δ_μ , δ_σ) as shown in Table (3-7), a series of 10,000 simulation runs are conducted. The correct classification rates together with the ATS values produced by the static $\bar{X}$ & S charts, the VSSI 3-CUSUM scheme, the VSI WLC scheme and the VSSI WLC scheme are listed in Table 3-7.

Table 3-7: Rates of Correct Diagnoses of Four Charts

	δ_μ	δ_σ	ATS				Correct classification rate %			
			Static $\bar{X}\&S$	VSSI 3-C	VSI WLC	VSSI WLC	Static $\bar{X}\&S$	VSSI 3-C	VSI WLC	VSSI WLC
	0.0	1.0	400.25	398.53	413.03	414.26	—	—	—	—
δ_μ shift	0.5	1.0	13.65	6.99	13.87	14.21	98.87	98.53	69.36	64.21
	1.0	1.0	1.96	1.44	1.64	1.62	99.71	99.85	81.15	80.56
	1.5	1.0	1.50	0.94	0.89	0.93	99.82	99.95	80.47	79.25
δ_σ shift	0.0	1.2	39.75	14.12	15.24	15.39	72.32	88.62	41.38	38.92
	0.0	1.4	9.31	3.77	3.62	3.51	83.27	90.10	40.28	40.24
	0.0	1.7	2.98	1.73	1.62	1.60	85.11	86.52	28.76	28.16
Joint Shift	0.5	1.2	9.35	4.36	4.34	4.26	4.79	3.48	67.50	68.67
	1.0	1.4	2.09	1.38	1.26	1.25	22.81	9.98	74.15	74.08
	1.5	1.7	1.54	0.98	0.88	0.91	62.03	18.35	83.18	83.83

Generally, the VSI and VSSI *WLC* schemes have similar performance. For diagnosing joint mean and variance shifts, the VSI and VSSI *WLC* schemes are always much more capable than the $\bar{X}\&S$ charts and the VSSI 3-*CUSUM* scheme. On the other hand, the $\bar{X}\&S$ charts and the VSSI 3-*CUSUM* scheme are favored for mean shift and variance shift.

3.6 Design Table for the VSI *WLC* Scheme

As aforementioned, the VSI *WLC* scheme is as efficient as the VSSI *WLC* scheme and easier to implement, hence we focus on the design of the VSI *WLC* scheme here. In order to facilitate the designs of the VSI *WLC* schemes, design tables (Table 3-8, 3-9, 3-10) are provided. For brevity of exposition, we just use the general shifts combination ($d_\mu = 0.8$, $d_\sigma = 1.4$) to carry out the designs.

The designs are presented according to the specified values of the four input parameters, each at three levels.

τ	200	400	600
R	2	5	8
h_{min}	0.1	0.2	0.3
n_{max}	10	15	20

(3-65)

However, practitioners can design a computer program following the algorithm presented in this chapter for other cases. In the design table, for each set of the values of τ , R , h_{min} and n_{max} , the values of n , h_1 , h_2 , λ , w_A' , H_A' and k_A' are provided. The primes attached to w_A' , H_A' and k_A' denote that these parameters are calculated under the standard condition (i.e., $\mu_0 = 0$, $\sigma_0 = 1$). For a general condition ($\mu_0 \neq 0$ and/or $\sigma_0 \neq 1$), these parameters are obtained by a simple modification.

$$w_A = \sigma_0^2 w_A' \quad (3-66)$$

$$H_A = \sigma_0^2 H_A' \quad (3-67)$$

$$k_A = \sigma_0^2 k_A' \quad (3-68)$$

It is noted that the adjusted loss function WL (equation 2-11) can be expressed as

$$WL = \sigma_0^2 w\ell \quad (3-69)$$

where,

$$w\ell = \lambda \left(\frac{s}{\sigma_0} \right)^2 + (1 - \lambda) \left(\frac{\bar{x} - \mu_0}{\sigma_0} \right)^2 \quad (3-70)$$

As a special case of WL , $w\ell$ is the weighted loss function under the standard condition ($\mu_0 = 0$ and $\sigma_0 = 1$). It can be shown that a general-condition VSI WLC scheme with w_A , H_A and k_A will produce the same ATS value as a standard-condition VSI WLC scheme with w_A' , H_A' and k_A' against the same shift combination. For detailed verification, refer to Appendix A.

With the help of the design tables, the design of a VSI *WLC* scheme is straightforward.

- (1) Specify τ , R , h_{min} , n_{max} . Each of them can be chosen from three different levels.
- (2) Find the sample sizes, sampling intervals and weighting factor n , h_1 , h_2 , λ and the standard-condition parameters w_A' , H_A' and k_A' from the design table.
- (3) Calculate the parameters w_A , H_A and k_A under the general condition by equations (3-66) to (3-68).

Table 3-8: Design Table of the VSI *WLC* Scheme ($\tau = 200$, $d_\mu = 0.8$, $d_\sigma = 1.4$)

R	h_{min}	n_{max}	n	h_1	h_2	λ	k_A'	w_A'	H_A'
2	0.1	10	4	2.7389	0.1	0.29	0.7160	0.0097	1.1026
		15	4	3.0028	0.1	0.30	0.6831	0.0020	1.2104
		20	4	2.7389	0.1	0.32	0.7543	0.0020	1.1007
	0.2	10	4	2.7000	0.2	0.29	0.7160	0.0096	1.1036
		15	4	2.9500	0.2	0.30	0.6831	0.0020	1.2117
		20	4	2.7000	0.2	0.32	0.7543	0.0020	1.1015
	0.3	10	4	2.6611	0.3	0.29	0.7160	0.0097	1.1045
		15	4	2.8972	0.3	0.30	0.6831	0.0020	1.2130
		20	5	3.3556	0.3	0.30	0.6543	0.0020	0.8124
5	0.1	10	4	1.0722	0.1	0.29	0.7160	0.0152	1.5193
		15	4	1.1694	0.1	0.30	0.6831	0.0138	1.6640
		20	5	1.3500	0.1	0.30	0.6543	0.0096	1.1354
	0.2	10	6	1.5889	0.2	0.31	0.6160	0.0066	0.9087
		15	5	1.3111	0.2	0.32	0.6831	0.0020	1.1262
		20	5	1.3111	0.2	0.30	0.6543	0.0075	1.1427
	0.3	10	6	1.5500	0.3	0.31	0.6160	0.0020	0.9078
		15	6	1.4250	0.3	0.32	0.6831	0.0020	0.7911
		20	7	1.8278	0.3	0.29	0.5543	0.0065	0.7468
8	0.1	10	6	1.0028	0.1	0.31	0.6160	0.0092	1.0452
		15	5	0.8292	0.1	0.32	0.6831	0.0020	1.2945
		20	5	0.8292	0.1	0.30	0.6543	0.0115	1.3070
	0.2	10	7	1.0438	0.2	0.30	0.6160	0.0020	0.7542
		15	8	1.2000	0.2	0.30	0.5831	0.0020	0.6432
		20	7	1.1375	0.2	0.29	0.5543	0.0077	0.8685
	0.3	10	7	1.0187	0.3	0.30	0.6160	0.0020	0.7536
		15	9	1.3312	0.3	0.32	0.5831	0.0020	0.5621
		20	8	1.1750	0.3	0.29	0.5543	0.0293	0.6717

Table 3-9: Design Table of the VSI *WLC* Scheme ($\tau = 400$, $d_\mu = 0.8$, $d_\sigma = 1.4$)

R	h_{min}	n_{max}	n	h_1	h_2	λ	k_A'	w_A'	H_A'
2	0.1	10	4	2.7389	0.1	0.29	0.7160	0.0141	1.4154
		15	4	3.0028	0.1	0.30	0.6831	0.0148	1.5472
		20	4	2.7389	0.1	0.32	0.7543	0.0124	1.4020
	0.2	10	4	2.7000	0.2	0.30	0.7160	0.0331	1.4422
		15	4	2.9500	0.2	0.30	0.6831	0.0149	1.5446
		20	4	2.9500	0.2	0.29	0.6543	0.0485	1.6174
	0.3	10	4	2.6611	0.3	0.30	0.7160	0.0170	1.4449
		15	4	2.8972	0.3	0.30	0.6831	0.0140	1.5481
		20	5	3.3556	0.3	0.30	0.6543	0.0069	1.0584
5	0.1	10	4	1.0722	0.1	0.31	0.7160	0.0574	1.9136
		15	4	1.1694	0.1	0.30	0.6831	0.0195	2.0031
		20	5	1.3500	0.1	0.30	0.6543	0.0131	1.3872
	0.2	10	7	1.7000	0.2	0.30	0.6160	0.0046	0.8036
		15	6	1.5889	0.2	0.28	0.5831	0.0087	1.1003
		20	5	1.3111	0.2	0.30	0.6543	0.0135	1.3841
	0.3	10	7	1.6750	0.3	0.30	0.6160	0.0029	0.8036
		15	8	1.9250	0.3	0.30	0.5831	0.0029	0.6853
		20	7	1.6750	0.3	0.33	0.6543	0.0020	0.8143
8	0.1	10	6	1.0028	0.1	0.31	0.6160	0.0090	1.2537
		15	5	0.8292	0.1	0.32	0.6831	0.0020	1.5392
		20	5	0.8292	0.1	0.30	0.6543	0.0133	1.5545
	0.2	10	7	1.0438	0.2	0.30	0.6160	0.0046	0.9137
		15	7	1.1375	0.2	0.31	0.5831	0.0020	1.0373
		20	7	1.1375	0.2	0.29	0.5543	0.0101	1.0493
	0.3	10	7	1.0187	0.3	0.30	0.6160	0.0020	0.9100
		15	8	1.1750	0.3	0.30	0.5831	0.0029	0.7811
		20	9	1.3312	0.3	0.29	0.5543	0.0020	0.6557

Table 3-10: Design Table of the VSI *WLC* Scheme ($\tau = 600$, $d_\mu = 0.8$, $d_\sigma = 1.4$)

R	h_{min}	n_{max}	n	h_1	h_2	λ	k_A'	w_A'	H_A'
2	0.1	10	4	2.7389	0.1	0.30	0.7160	0.0170	1.6288
		15	4	3.0028	0.1	0.30	0.6831	0.0169	1.7500
		20	4	2.7389	0.1	0.32	0.7543	0.0148	1.5896
	0.2	10	4	2.7000	0.2	0.30	0.7160	0.0170	1.6294
		15	4	2.9500	0.2	0.30	0.6831	0.0169	1.7454
		20	5	3.3944	0.2	0.30	0.6543	0.0100	1.2001
	0.3	10	4	2.6611	0.3	0.30	0.7160	0.0170	1.6300
		15	4	2.8972	0.3	0.30	0.6831	0.0169	1.7461
		20	5	3.3556	0.3	0.30	0.6543	0.0100	1.2004
5	0.1	10	6	1.6278	0.1	0.31	0.6160	0.0098	1.2354
		15	4	1.1694	0.1	0.31	0.6831	0.0227	2.2712
		20	5	1.3500	0.1	0.30	0.6543	0.0148	1.5320
	0.2	10	6	1.5889	0.2	0.31	0.6160	0.0112	1.2348
		15	5	1.3111	0.2	0.32	0.6831	0.0020	1.5163
		20	6	1.5889	0.2	0.34	0.6543	0.0020	1.2414
	0.3	10	7	1.6750	0.3	0.30	0.6160	0.0046	0.8973
		15	7	1.8278	0.3	0.31	0.5831	0.0020	1.0207
		20	7	1.8278	0.3	0.29	0.5543	0.0095	1.0323
8	0.1	10	7	1.0687	0.1	0.30	0.6160	0.0062	1.0078
		15	5	0.8292	0.1	0.32	0.6831	0.0020	1.6830
		20	5	0.8292	0.1	0.30	0.6543	0.0155	1.7023
	0.2	10	7	1.0438	0.2	0.30	0.6160	0.0046	1.0080
		15	8	1.2000	0.2	0.30	0.5831	0.0062	0.8656
		20	8	1.3111	0.2	0.31	0.5543	0.0020	0.9849
	0.3	10	8	1.1750	0.3	0.32	0.6160	0.0020	0.8484
		15	9	1.3312	0.3	0.31	0.5831	0.0020	0.7247
		20	10	1.4875	0.3	0.31	0.5543	0.0020	0.6606

3.7 Example

In this section, an example is used to demonstrate the application of the VSI *WLC* scheme. A manufacturing factory produces a type of bearing for automobile. The inside diameter x of the bearing is a key dimension. Its nominal value and tolerance are specified as $25 \pm 0.08 \text{ mm}$. From the pilot runs, it is found that the probability distribution of x can be very well approximated by a normal one. The process mean μ_0 can be easily adjusted to the nominal value of 25 mm . The standard deviation σ_0 is estimated as 0.015 mm from the historic data. A VSI *WLC* scheme is to be built to monitor both mean and variance shifts of x . The design is carried out as follows:

- (1) The design specifications are decided as $n_{\max} = 15$, $h_{\min} = 0.2$ hour, $\tau = 400$ hours and $R = 5$ / hour.
- (2) From Table 3-9, the values of the charting parameters can be found. The sample sizes and sampling intervals are $n = 6$, $h_1 = 1.59$, $h_2 = 0.2$. The weighting factor is $\lambda = 0.28$, and the standard-condition parameters are $w_A' = 0.0087$, $H_A' = 1.1003$, $k_A' = 0.5831$.
- (3) The general-condition values of the parameters w_A , H_A and k_A are calculated by equations (3-66) to (3-68). The final values are

$$w_A = 1.9575 \times 10^{-6} \text{ mm}, H_A = 2.4757 \times 10^{-4} \text{ mm}, k_A = 1.3120 \times 10^{-4} \text{ mm}$$

The computer program can work out the same design in 10 seconds of CPU time in a Pentium IV (2.4 GHz) personal computer. The values of *ATS* versus the process shifts in the domain of $(0 < \delta_\mu \leq 2, 1 < \delta_\sigma \leq 2)$ for this VSI *WLC* scheme are enumerated in Table 3-4.

The control chart of the VSI *WLC* scheme is drawn in Figure 3-3. It also showcases a simulation run of the process. The real process stays in control until $t = 82$ when the following out of control case takes place:

$$\delta_\mu = 0.4, \delta_\sigma = 1.2 .$$

The problem is signaled by the VSI *WLC* scheme at *Sample No.* 91 ($t = 87.7$). The pattern of the sample points in Figure 3-3 suggests that something might go wrong around the *Sample No.* 70 (i.e. $t = 82.11$), because all the subsequent sampling points (one relax samples and 20 alert samples) result in an A_t value greater than zero. Using equations (3-63) and (3-64), the values of the mean shift and standard deviation shift can be estimated as

$$\hat{\delta}_\mu = 0.4026, \hat{\delta}_\sigma = 1.2239 .$$

These estimated values are quite close to the actual ones. Since both $\hat{\delta}_\mu$ and $\hat{\delta}_\sigma$ are significant, the message is that both increasing mean shift and increasing variance shift exist.

Figure 3-3: An Illustrative Example of the VSI *WLC* Scheme

Figure 3-3: An Illustrative Example of the VSI WLC Scheme

3.8 Summary and Discussion

This chapter proposes several adaptive *WLC* schemes that have higher effectiveness for detecting the mean shift and increasing variance shift, without increasing the false alarm rate and the inspection rate. We also developed the design algorithm for the VSSI 3-*CUSUM* scheme.

Broad and comprehensive performance studies have been conducted to compare the performance characteristics of the VSSI *WLC* scheme with many other control charts (schemes) for detecting both mean and variance shifts. The degree of difficulty in designing and operating the charts is also discussed. The results of these studies provide useful guidelines for the practitioners to decide a most suitable control scheme for their applications.

Generally speaking, the VSI *WLC* scheme is particularly recommended, because it enjoys almost all of the performance advantages of the VSSI *WLC* scheme and is more, or substantially more, powerful than all other control schemes for detecting shifts in mean and variance. Unlike some other CUSUM and adaptive charts that improve the detection power only for small and moderate process shifts, the VSI *WLC* scheme outperforms other charts almost in the whole process shift domain. Moreover, the superiority of the VSI *WLC* scheme over many other charts is not obviously affected by the changes of the specified parameters n_{max} , h_{min} , τ , R , d_μ and d_σ .

More importantly, the VSI *WLC* scheme is much simpler to design and implement than many other adaptive CUSUM schemes. The simplicity in design is mainly attributable to the fact that a single CUSUM chart used in the VSI *WLC* scheme cannot interact with

any other charts. As a result, the design can be carried out in a well-formulated procedure. A program in C language has been developed for the design of the VSI *WLC* scheme, which can be finished in about 20 seconds of CPU time in a personal computer (Pentium IV 2.4 GHz). It is available upon request by author's email; moreover practitioners can develop their own computer program by following the guidelines provided in this chapter. Since only the sampling interval has to be changed depending on the current process status and only one statistic has to be maintained and plotted, the operation of the VSI *WLC* scheme is much easier than that of the VSSI 3-*CUSUM* scheme. In view of the operational simplicity, the VSI *WLC* scheme may be even more tractable than the VSSI $\bar{X}$ &*S* charts.

The design algorithm for the VSSI *WLC* scheme can be extended, with some modification, to the optimization designs of many other adaptive Shewhart and CUSUM charts for monitoring process shifts in mean and variance.

Finally, the diagnosis capability of the VSI *WLC* and VSSI *WLC* schemes has been investigated and the guidelines for diagnosis are provided. It has been shown that the VSI and VSSI *WLC* schemes are considerably more capable for diagnosing the out-of-control cases than the $\bar{X}$ &*S* charts and the VSSI 3-*CUSUM* scheme when the process mean and variance shift simultaneously.

Chapter 4

An Internal Adaptive Chart for Monitoring Process Mean and Variance

In this chapter an internal adaptive scheme named Adaptive-Loss-Function (*AL*) control chart is proposed for monitoring the process shifts in mean and variance simultaneously. The weights of the losses due to mean shift and variance shift in the loss function are adjusted dynamically through a weighting factor a depending on the estimation of current process status.

4.1 Introduction

As shown in Chapter 3, the adaptive features (VSS, or VSI, or VSSI) can greatly enhance control chart performance in terms of the detection effectiveness. However, their implementation is confronted with some difficulties from operational or managerial perspectives. Firstly, the frequent change of sample sizes and/or sampling intervals may encounter resistance from the operators. Secondly, using different sample sizes and/or sampling intervals causes the lack of the measuring instruments and manpower sometimes, and results in the idleness of these resources at other times. Moreover, the changes of sampling intervals may violate rational subgrouping principles.

With the advances in information technology, on-line measurement and distributed computing systems are becoming the norm in today's SPC applications (Woodall and Montgomery 1999). However, computers have been used only to calculate some very

simple statistics (e.g., sample mean and sample range) in many computer-aided SPC systems. As computer cannot take samples and make observations by its own, it is not able to overcome the difficulty in the implementation of the aforementioned adaptive charts.

Even automatic measuring systems cannot overcome these barriers. When using VSS, VSI or VSSI scheme manually, the amount of inspection made by a human operator will be adjusted according to different circumstances (i.e. the process is likely to be in control or tends to be out of control). However, when an automatic measuring system is in use, it must always be made full use and run at the highest possible and constant pace. It makes no sense to decrease the inspection rate of an automatic system even when the process is in control.

Nevertheless, the adaptation of some charting parameters can be conducted on-line by a software system and, thus, does not require the intervention of human operators. Since the parameter adaptation is carried out by a computer, such adaptive charts are referred to as the *internal adaptive charts*. Like the conventional static charts, an internal adaptive chart uses fixed sample size and fixed sampling interval, and leaves all of the complicated procedure of parameter adaptation to a computer. The internal adaptive charts are appealing as long as an on-site computer is available and a computer program is developed and installed in the computer. The fixed sample size of an internal adaptive chart is decided based on the available resources and its fixed sampling interval is decided according to the rational subgrouping. Manpower and other resources can be utilized at a constant rate.

This chapter proposes an internal adaptive control chart based on the *WL* chart. As mentioned in Chapter 2, the weighting factor λ is determined by an off-line optimization design so that the *WL* chart can allocate its power for detecting mean shifts and variance increase shift properly and achieve the highest detection effectiveness for some specified process shifts. However, since the *WL* chart uses a fixed value of λ for the entire duration of operation, it may not always have optimal performance for a variety of situations. This becomes a barrier to the implementation of the *WL* chart as it is often impossible to predict the actual sizes of mean shifts and variance shifts of an individual process (Reynolds and Stoumbos 2004b).

In the new chart, the weighting factor λ is treated as an adaptable parameter for the on-line adjustment between the weights for s^2 and the weight for $(\bar{x} - \mu_0)^2$ in the loss function. The corresponding Adaptive Loss Function chart (*AL* chart) is based on the following statistic *AL*

$$AL = as^2 + (1 - a)(\bar{x} - \mu_0)^2. \quad (4-1)$$

Unlike λ in *WL*, the weighting factor a ($0 \leq a \leq 1$) in *AL* is adaptable. A procedure similar to the Bayesian method is developed to estimate the current process status (or decide which of δ_μ and δ_σ is more severe). In general, a small a is adopted when mean shift is dominant and a large a is used when variance shift prevails. Further studies will show that a is either 0 or 1 if it changes from its initial value. By this way, the increase of the statistic *AL* is accelerated when the process is out of control so that the *AL* chart is more sensitive to current process shifts. The *AL* chart is an internal adaptive chart since both the estimation of the current process status and the adjustment of parameter a are carried out by a computer program. Operators only have to take a sample of fixed size at the end of each fixed sampling interval.

4.2 The Formulation of the AL Chart

In a variable sampling scheme, the warning and control limits divide the chart into central, warning and action zones in order to differentiate the magnitudes of shifts. Similarly, in an *AL* chart, six representative *status points* $P_i(\delta_{\mu,i}, \delta_{\sigma,i})$ are located in the half space ($\delta_{\sigma} \geq 1$) of the process shift domain (Figure 4-1) in order to distinguish the types of process shifts. Each status point P_i is characterized by a pair of $\delta_{\mu,i}$ (mean shift) and $\delta_{\sigma,i}$ (standard deviation shift).

$P_0(\delta_{\mu,0} = 0.0, \delta_{\sigma,0} = 1.0)$	represents an in-control status;
$P_1(\delta_{\mu,1} > 0.0, \delta_{\sigma,1} = 1.0)$	represents an increasing mean shift;
$P_2(\delta_{\mu,2} > 0.0, \delta_{\sigma,2} > 1.0)$	represents a joint increasing mean and increasing standard deviation shifts;
$P_3(\delta_{\mu,3} = 0.0, \delta_{\sigma,3} > 1.0)$	represents an increasing standard deviation shift;
$P_4(\delta_{\mu,4} < 0.0, \delta_{\sigma,4} > 1.0)$	represents a joint decreasing mean and increasing standard deviation shifts;
$P_5(\delta_{\mu,5} < 0.0, \delta_{\sigma,5} = 1.0)$	represents a decreasing mean shift.

Figure 4-1: The Six Status Points of the *AL* Chart

For the determination of $\delta_{\mu,i}$ and $\delta_{\sigma,i}$ ($i = 1, 2, \dots, 5$), the following three rules are used.

R1: The pair of shifts, $\delta_{\mu,i}$ and $\delta_{\sigma,i}$, of each out-of-control status point P_i produce the same value of loss function, that is, $(\delta_{\sigma,i}^2 + \delta_{\mu,i}^2)\sigma_0^2$ is identical. It ensures that each out-of-control status point has the same impact on the product quality.

R2: For status point P_2 (representing the joint increasing mean and increasing standard deviation shifts), its mean shift component ($\delta_{\mu,2}$) and standard deviation shift component ($\delta_{\sigma,2}$) are proportional using the mean shift $\delta_{\mu,1}$ of P_1 and the standard deviation shift $\delta_{\sigma,3}$ of P_3 as the benchmarks, that is,

$$\frac{\delta_{\mu,2}}{\delta_{\sigma,2} - 1} = \frac{\delta_{\mu,1}}{\delta_{\sigma,3} - 1} \quad (4-2)$$

It makes P_2 a sound representative of the joint increasing mean and increasing standard deviation shifts.

R3: Points P_1 and P_5 are symmetrical about the vertical line ($\delta_{\mu} = 0$); so are P_2 and P_4 .

The locations of all the status points are uniquely determined by a single parameter, D_{μ} , that is the specified value of the mean shift $\delta_{\mu,1}$ of point P_1 . The values of $\delta_{\mu,i}$ and $\delta_{\sigma,i}$ of other points are determined according to D_{μ} and the three rules presented above.

$$\delta_{\mu,1} = D_{\mu}, \quad \delta_{\sigma,1} = 1$$

$$\delta_{\sigma,2} = \frac{2D_{\mu}^2 + \sqrt{4D_{\mu}^4 - 4[(\delta_{\sigma,3} - 1)^2 + D_{\mu}^2] [D_{\mu}^2 - \delta_{\sigma,3}^2(\delta_{\sigma,3} - 1)^2]}}{2[(\delta_{\sigma,3} - 1)^2 + D_{\mu}^2]} \quad (\text{using R1 and R2})$$

$$\delta_{\mu,2} = \frac{D_{\mu}(\delta_{\sigma,2} - 1)}{\delta_{\sigma,3} - 1}$$

$$\delta_{\mu,3} = 0, \quad \delta_{\sigma,3} = \sqrt{1 + D_{\mu}^2} \quad (\text{using R1})$$

$$\delta_{\mu,4} = -\delta_{\mu,2}, \quad \delta_{\sigma,4} = \delta_{\sigma,2} \quad (\text{using R3})$$

$$\delta_{\mu,5} = -D_{\mu}, \quad \delta_{\sigma,5} = 1 \quad (\text{using R3})$$

(4-3)

Where, the mean shift component ($\delta_{\mu,2}$) and standard deviation shift component ($\delta_{\sigma,2}$) of P_2 are determined by solving the following three simultaneous equations based on R1 and R2.

$$\delta_{\sigma,2}^2 + \delta_{\mu,2}^2 = 1 + D_{\mu}^2$$

$$\delta_{\sigma,3}^2 = 1 + D_{\mu}^2$$

$$\frac{\delta_{\mu,2}}{\delta_{\sigma,2} - 1} = \frac{D_{\mu}}{\delta_{\sigma,3} - 1} \quad (4-4)$$

Each status point is also associated with a specific value a_i for the weighting factor a in equation (4-1). The value a_0 for the in-control point P_0 is set equal to the optimal value of the weighting factor λ of the corresponding WL chart. The values a_i ($i = 1, 2, \dots, 5$) for the five out-of-control points P_i are determined by the two terms in the addition of the expected value $E(AL)$ of AL at $(\delta_{\mu,i}, \delta_{\sigma,i})$. At status point P_i , according to equation (4-1),

$$E(AL) = a_i (\delta_{\sigma,i} \sigma_0)^2 + (1 - a_i) \left[\frac{(\delta_{\sigma,i} \sigma_0)^2}{n} + (\delta_{\mu,i} \sigma_0)^2 \right]$$

$$\sigma_0^2 \left[\left(1 - \frac{1}{n} \right) \delta_{\sigma,i}^2 - \delta_{\mu,i}^2 \right] > 0, \text{ set } a_i = 1; \text{ otherwise, } a_i = 0. \quad (4-5)$$

Let p_i be the probability that the current process status is equal to that represented by point P_i (i.e. $\delta_\mu = \delta_{\mu,i}$, $\delta_\sigma = \delta_{\sigma,i}$). If p_k is the greatest of all p_i ($i = 0, 1, \dots, 5$), P_k is called the *key point* and the associated a_k is assigned to a . Then AL in equation (4-1) is

$$AL = a_k s^2 + (1 - a_k)(\bar{x} - \mu_0)^2 \quad (4-6)$$

This procedure embodies the internal adaptive feature of the AL chart. The value of a depends on which status point is currently the key point (or depends on the likely status of the current process). It leads to an increase of the value of a to 1 when variance shift is dominant and a decrease of the value of a to 0 when mean shift is prevailing.

Since production processes often operate in the in-control status and produce acceptable product for relatively long periods of time (Montgomery 2001), the initial value $p_0(0)$ of the probability p_0 of the in-control point P_0 is set to $(1 - \varepsilon)$ (say, $\omega = 0.000001$) at the commence of a process. The initial values of $p_i(0)$ for other points are very small.

$$p_i(0) = \omega / 5 \quad i = 1, 2, \dots, 5. \quad (4-7)$$

The probabilities p_i are subsequently updated by a procedure similar to the Bayesian method based on the values of the sample mean $\bar{x}$ and sample standard deviation s in each forthcoming sample.

$$p_i(t) = c \left(f_{\bar{x},i}(\bar{x}) f_{s,i}(s) \right)^w p_i(t-1) \quad i = 0, 1, \dots, 5 \quad (4-8)$$

where, $p_i(t)$ and $p_i(t-1)$ are the values of p_i at the t th sample and $(t-1)$ th sample, respectively. The constant c is to make the sum of the values of $p_i(t)$ always equal to one.

$$c = \frac{1}{\sum_{i=0}^5 \left(f_{\bar{x},i}(\bar{x}) f_{s,i}(s) \right)^w p_i(t-1)} \quad (4-9)$$

And $f_{\bar{x},i}(\bar{x})$ and $f_{s,i}(s)$ are the probability density functions of the sample mean $\bar{x}$ and the sample standard deviation s at point P_i .

$$f_{\bar{x},i}(\bar{x}) = \frac{\sqrt{n}}{\sqrt{2\pi}\delta_{\sigma,i}\sigma_0} \exp\left[-0.5\left(\frac{\bar{x} - (\mu_0 + \delta_{\mu,i}\sigma_0)}{\delta_{\sigma,i}\sigma_0 / \sqrt{n}}\right)^2\right] \quad (4-10)$$

$$f_{s,i}(s) = \frac{2s(n-1)f_{n-1}\left(\frac{s^2(n-1)}{(\delta_{\sigma,i}\sigma_0)^2}\right)}{(\delta_{\sigma,i}\sigma_0)^2} \quad (4-11)$$

In equation (4-11), $f_{n-1}(\cdot)$ is the probability density function of a Chi-square distribution with $(n - 1)$ degrees of freedom.

$$f_{n-1}(y) = \frac{y^{0.5(n-1)} e^{-0.5y}}{2^{0.5(n-1)} \Gamma(0.5(n-1))} \quad (4-12)$$

The power ψ in equation (4-8) is used to accelerate the increase of the probability p_k of the key point P_k when an out-of-control case occurs. The value of ψ is determined by an optimization search during the design of the AL chart. In comparison with the typical Bayesian approach using certain prior probability function, the updating procedure (4-8) uses the posterior probability of each shift at the previous sampling as the prior probability for the next sampling. This is reasonable when there are no changes to the process. However if a shift happens during a sampling interval, the prior probability of the shift happening will also changes accordingly. In this case, the posterior probability will update slowly due to the outdated prior probability if typical approach is used, i.e. $\psi = 1$. Intensive numerical tests show that by using an optimal ψ value, the probability p_i corresponding to the current process shift type will be faster. As a result, the out-of-control case is likely to be signaled earlier.

The current values of the probabilities p_i of status points P_i provide the information for the answers to the following three questions.

(1) Is the process probably remaining in control or going out of control?

The probability p_0 is often much greater than any other p_i as long as the process remains in control and the weighting factor a in equation (4-1) continuously takes the value of a_0 .

(2) What is the type of the process shift?

Eventually some assignable causes will occur resulting in a process shift. If the process mean shift and standard deviation shift are close to $\delta_{\mu,k}$ and $\delta_{\sigma,k}$ that are associated with status point P_k , then the values of $f_{\bar{x},k}(\bar{x})$ and $f_{s,k}(s)$ in equation (4-8) are likely to increase, so is the probability p_k . At some time, the probability p_k will become larger than any other p_i (including p_0). At this point the weighting factor a will take the value of a_k to update the current value of AL in equation (4-1). For example, if an out-of-control case with prevailing increasing mean shift takes place, the process status $(\delta_{\mu}, \delta_{\sigma})$ is closer to status point P_1 than to any other points. The probability p_1 will increase quickly and exceeds any other p_i . Consequently, the weighting factor a is made equal to a_1 that allocates all weight to the loss term due to increasing mean shift, and AL is likely to get a large value. On the other hand, if a prevailing increasing variance shift occurs, the process status $(\delta_{\mu}, \delta_{\sigma})$ is closest to status point P_3 . The weighting factor a is made equal to a_3 giving all weight to the loss term due to increasing variance shift. It again makes AL greater. It is noted that the event in which the value of p_k of an out-of-control point is the greatest is similar to the event in which a sample point falls into the warning zone in a variable sampling scheme. Therefore, the AL chart will not produce an out-of-

control signal just because of p_k being the greatest; otherwise the false alarm is too frequent. The AL chart produces a signal only when the statistic AL is greater than the control limit.

(3) Is the mean shift an increasing one or a decreasing one?

If p_1 or p_2 is the greatest, the corresponding mean shift is likely to be an increasing one. Conversely, if p_4 or p_5 is the greatest, the corresponding mean shift is likely to be a decreasing one. With this information available, equation (4-6) for calculating the statistic AL can be modified as follows in order to further enhance the power of the AL chart.

$$AL = a_k s^2 + (1 - a_k) \Delta \bar{x}^2 \quad (4-13)$$

where, the mean shift $\Delta \bar{x}$ is calculated differently for the following three different cases.

Case 1: the value of p_0 or p_3 is the greatest (mean shift is unlikely)

$$\Delta \bar{x} = \bar{x} - \mu_0 \quad (4-14)$$

Case 2: the value of p_1 or p_2 is the greatest (increasing mean shift is likely)

$$\Delta \bar{x} = \max(0, \bar{x} - \mu_0) \quad (4-15)$$

Meanwhile the control limit UCL_{AL} of the AL chart is reduced to ξUCL_{AL} ($0 < \xi < 1$).

The determination of ξ will be discussed shortly.

Case 3: the value of p_4 or p_5 is the greatest (decreasing mean shift is likely)

$$\Delta \bar{x} = \min(0, \bar{x} - \mu_0) \quad (4-16)$$

Meanwhile the control limit of the AL chart is reduced by the same factor ξ as in case 2.

In case 2, equation (4-15) implies preventing the detection of a decreasing mean shift, because a decreasing mean shift (e.g. $\bar{x} < \mu_0$) is quite unlikely when p_1 or p_2 is the greatest. Since this action will reduce the probability of type I error, the control limit of the AL chart is allowed to be tightened by a factor of ζ without increasing the false alarm rate. As a result, the power for detecting a likely increasing mean shift is enhanced. It is similar to the one-sided $\bar{X}$ chart (for increasing mean shifts) that is used to replace the two-sided $\bar{X}$ chart when a decreasing mean shift is nearly impossible in a process. For a given type I error probability, the one-sided $\bar{X}$ chart has a tighter upper control limit than the two-sided $\bar{X}$ chart and the former is more effective than the latter for detecting increasing mean shifts. The actions taken in case 3 can be explained similarly.

The procedure to run an AL chart is as easy as that for a static $\bar{X}$ & S chart. The operators simply enter a fixed number (n) of observations x_i into the computer for each fixed sampling interval (h). All other tasks are left to an operational computer program. The pseudo code of the program (that also interprets the operation of the AL chart) is listed below:

- 1 Accept the n observations x_i for the current sample from the operators.
- 2 Calculate the sample mean and sample standard deviation.

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n}, \quad s = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}} \quad (4-17)$$

- 3 Calculate $f_{\bar{x},i}(\bar{x})$ ($i = 0, 1, \dots, 5$).

$$f_{\bar{x},i}(\bar{x}) = \frac{\sqrt{n}}{\sqrt{2\pi}\delta_{\sigma,i}\sigma_0} \exp\left[-0.5\left(\frac{\bar{x} - (\mu_0 + \delta_{\mu,i}\sigma_0)}{\delta_{\sigma,i}\sigma_0 / \sqrt{n}}\right)^2\right] \quad (4-18)$$

4 Calculate $f_{s,i}(s)$ ($i = 0, 1, \dots, 5$).

$$y = \frac{s^2(n-1)}{(\delta_{\sigma,i}\sigma_0)^2}$$

$$f_{n-1}(y) = \frac{y^{0.5(n-3)} e^{-0.5y}}{2^{0.5(n-1)} \Gamma(0.5(n-1))}$$

$$f_{s,i}(s) = \frac{2s(n-1)f_{n-1}(y)}{(\delta_{\sigma,i}\sigma_0)^2} \quad (4-19)$$

5 Calculate the constant c .

$$c = \frac{1}{\sum_{i=0}^5 (f_{\bar{x},i}(\bar{x})f_{s,i}(s))^w p_i(t-1)} \quad (4-20)$$

6 Update $p_i(t)$ ($i = 0, 1, \dots, 5$).

$$p_i(t) = c(f_{\bar{x},i}(\bar{x})f_{s,i}(s))^w p_i(t-1) \quad (4-21)$$

7 Identify the key point P_k that has the highest value $p_k(t)$ among all six probabilities $p_i(t)$.

8 Make the parameter a equal to a_k and then calculate AL .

$$AL = a_k s^2 + (1 - a_k) \Delta \bar{x}^2 \quad (4-22)$$

$$\Delta \bar{x} = \begin{cases} \bar{x} - \mu_0 & \text{if } k = 0, \text{ or } 3 \\ \max(0, \bar{x} - \mu_0) & \text{if } k = 1, \text{ or } 2 \\ \min(0, \bar{x} - \mu_0) & \text{if } k = 4, \text{ or } 5 \end{cases} \quad (4-23)$$

9 Determine the current control limit ucl_{AL} from the nominal limit UCL_{AL} .

$$ucl_{AL} = \begin{cases} UCL_{AL} & \text{if } k = 0, \text{ or } 3 \\ \xi UCL_{AL} & \text{otherwise} \end{cases} \quad (4-24)$$

- 10 If the calculated AL is smaller than or equal to ucl_{AL} , go back to step (1) (process is in control).
- 11 Otherwise (i.e. $AL > ucl_{AL}$), print out an alarming message (e.g. “process is out of control”) and stop the current process. Also printed are the $p_i(t)$ values that will help the operators to diagnose the assignable cause.

The above eleven steps seem complex for a human operator. However, only the first step is carried out by the operator, the others can be easily implemented by a computer program which can be coded in any computer programming language.

4.3 Design of the AL chart

To design an AL chart, three parameters need to be specified:

- n sample size.
- τ the allowed minimum in-control Average Run Length (ARL_0).
- D_μ the minimum mean shift size to be protected against, also as the mean shift value of status point P_1 (see equation (4-3)).

The value of n is usually decided based on the available inspection resources (e.g. manpower and instrument) and the desired out-of-control ARL performance. The specifications of τ and D_μ are required by the optimization designs of many control charts. For example for the optimization design of a CUSUM chart, the reference parameter k_{cushum} is optimized based on the specified D_μ and the control limit is adjusted so that the in-control ARL_0 is equal to τ . The value of τ is decided based on the requirement on the false alarm rate. If handling false alarms is costly and time

consuming, a larger τ should be used to reduce the false alarm frequency. Nevertheless a large τ will impair the effectiveness of the control chart in the meantime. Ideally the resultant (or actual) in-control ARL_0 is equal or very close to τ . The value of D_μ depends on the mean shift that should be detected quickly by the AL chart. The value of D_μ should be set relatively larger if a large mean shift in a process is predicted, and *vice versa*. As for the design in last chapter, the values of the in-control mean μ_0 and standard deviation σ_0 can be set as zero and one in the design of the AL chart. The resultant charting parameters can be easily modified later based on the actual values of μ_0 and σ_0 that are usually estimated from the data observed during the pilot runs.

The design of an AL chart will determine the values of the mean shift $\delta_{\mu,i}$, the standard deviation shift $\delta_{\sigma,i}$ and the weighting factor a_i that are associated with the six status points P_i ; the power ψ in equation (4-8) for accelerating the increase of p_k ; the control limit UCL_{AL} and the reducing factor ζ . The design steps are outlined below:

- (1) Determine the values of the process shifts $\delta_{\mu,i}$ and $\delta_{\sigma,i}$ ($i = 1, 2, \dots, 5$) by equation (4-3), based on the specified value of D_μ .
- (2) Carry out an optimization design to determine the optimal value of the weighting factor λ for a WL chart (Tian 2003) under the same specifications. Set the weighting factor a_0 equal to λ (a_0 is associated with the in-control status point P_0).
- (3) Determine the weighting factor a_i that is associated with the out-of-control points P_i ($i = 1, 2, \dots, 5$) based on the judgment method (given in equation (4-5)).
- (4) Determine the power ψ , the control limit UCL_{AL} and the reducing factor ζ by using the following optimization model:

$$\text{Minimize: } U = \left(\sum_{i=1}^5 ARL(\delta_{\mu,i}, \delta_{\sigma,i}) \right) / 5 \quad (4-25)$$

$$\text{Subject to: } ARL_0 = \tau \quad (4-26)$$

With respect to the design variables: ψ, UCL_{AL}, ξ

The objective function U is the average value of the ARL s that are evaluated at the five out-of-control status points rather than the ARL value at a single point. In other words, the objective function takes into consideration of the effectiveness of the AL chart for detecting different combinations of mean and variance shifts. However, if historical records indicate that a particular process shift pair $(\delta_{\mu,i}, \delta_{\sigma,i})$ happens more frequently than others, a relatively larger weight ($> 1/5$) can be assigned to the corresponding $ARL(\delta_{\mu,i}, \delta_{\sigma,i})$ in the objective function.

The search for reducing factor ξ in (4-24) and power ψ in (4-8) is conducted in an iterative manner in rules (4.1) and (4.2) below.

Search algorithm of reducing factor ξ and power ψ :

(4.1) Search of reducing factor ξ (starting value: 0.6, step size: 0.05, range: $0.1 \leq \xi \leq 1$) with the objective function to minimize U in (4-25). For every given value of ξ , UCL_{AL} has to be adjusted so that ARL_0 is roughly equal to τ .

(4.2) Search of power ψ (starting value: 2.5, step size: 0.1, range: $0.2 \leq \psi \leq 5$) with the objective function to minimize U . For every given value of ψ , UCL_{AL} has to be adjusted so that ARL_0 is roughly equal to τ .

During the searching of one variable the other variable is fixed in a temporarily determined value. The starting values of ξ and ψ in the first iteration are determined by best guess. The starting values for the subsequent iterations take the optimal values of the previous iteration. In both (4.1) and (4.2), the optimal value of a particular parameter (design variable) is sought through a single variable search. From a starting point, the design variable is first increased and decreased by one step in order to find the searching direction leading to the decrease of the objective function. Then the search is conducted along this direction with a given step size until a local optimum is found or the lower or upper bounds of the search range is reached. It is assumed that the objective function is a convex function of each design variable.

The iterative search will finish as soon as the results of ξ and ψ from (4.1) and those from (4.2) coincide. In the end, all charting parameters are finalized.

Both the in-control ARL_0 and the out-of-control ARL are evaluated by simulation. The simulation program is developed using the operational pseudo code that is described in the last section. Whenever the AL chart signals (i.e. $AL > ucl_{AL}$ at step (11)), a sample value of the run length RL is obtained. If the simulation is repeated N times ($N = 10,000$ in this study), the average RL is an estimate of ARL_0 or ARL . In the simulation program, the observations of x_i are generated from a normal distribution, $N(\mu_0, \sigma_0^2)$ for an in-control case, and another normal distribution, $N(\mu_0 + \delta_{\mu,i}\sigma_0, \delta_{\sigma,i}^2\sigma_0^2)$, for an out-of-control case. For an out-of-control case, process shifts may be assumed to occur at the beginning of the process (zero-state mode) or after a relatively long time when the process has reached a stationary status (steady-state mode). The steady-state mode is

adopted in this study since it is more realistic. Under the steady-state mode, in each simulation iteration, the process stays in the in-control status for the first g samples (say, $g = 50$, it is long enough for the process to reach the stationary status) and then turns to the out-of-control status. Any signal that is produced in the first g samples is treated as a false alarm.

A computer program *AL_CHART.C* in C language has been developed to design an *AL* chart for a given set of specifications, and it is available upon request by author's email.

4.4 Comparative Studies

This section compares the performance of five control charts (the static $\bar{X}$ & S charts, the *WL* chart, the VSSI $\bar{X}$ & S charts, the 3-*CUSUM* scheme, and the *AL* chart) in terms of the detection effectiveness. Because the different operational properties between the variable sampling schemes and the fixed sampling schemes, it is unfair to the latter if the detection effectiveness of the two types of schemes is compared. Therefore, we primarily consider the fixed sampling schemes here. An exception is the VSSI $\bar{X}$ & S charts, which are used as a representative of the variable sampling schemes and serve as just a reference. The parameters of all the charts are determined through the optimization design in which the objective function to be minimized is the average value of *ARL*s evaluated at the five out-of-control status points calculated by equation (4-25) and the constraint function makes the resultant in-control ARL_0 equal to the specified τ in equation (4-26). A fixed sample size n and a fixed sampling interval h are used by each chart except the VSSI $\bar{X}$ & S charts. Moreover steady-state mode is adopted for all the charts.

(1) The static $\bar{X}$ & S charts

As discussed in Study one of Section 3.4, the design of the static $\bar{X}$ & S charts optimizes an allocation factor η ($0 \leq \eta \leq 1$) that distributes the overall type I error α (or the power) between the $\bar{X}$ chart and the S chart.

$$\begin{aligned}\alpha_s &= \eta\alpha \\ \alpha_{\bar{x}} &= (\alpha - \alpha_s)/(1 - \alpha_s)\end{aligned}\tag{4-27}$$

(2) The WL chart

The design of the WL chart optimizes the weighting factor λ in equation (2-14) in order to minimize U in equation (4-25).

(3) The VSSI $\bar{X}$ & S charts

As mentioned in Study one of Section 3.4, the optimization design of the VSSI $\bar{X}$ & S charts has to determine the optimal values of 13 parameters in order to minimize U in equation (4-25). They are the allocation factor η in equation (4-27); the sample sizes and sampling intervals n_1, n_2, h_1, h_2 ; the lower and upper warning and control limits $w_{Z,lower}, w_{Z,upper}, LCL_Z$ and UCL_Z of the $\bar{X}$ chart; the warning and control limits w_{q1}, UCL_{q1} (for the sample using a small sample size n_1 and a large sampling interval h_1) and w_{q2}, UCL_{q2} (for the sample using a large n_2 and a small h_2) of the S chart. The lower side warning and control limits of the $\bar{X}$ chart are symmetrical to the upper side ones (i.e. $w_{Z,lower} = -w_{Z,upper}, LCL_Z = -UCL_Z$). The maximum sample size n_{max} of the VSSI $\bar{X}$ & S charts is limited to $3n$ and the minimum sampling interval h_{min} is constrained to $0.3h$, where h is the sampling interval used by other charts. The values of n_{max} and h_{min} are decided by some practical considerations such as the number of units that can be inspected in a short time period. Since the VSSI $\bar{X}$ & S charts use variable sampling

intervals, their performance should be measured by *ATS* (Average Time to Signal) rather than *ARL*. However, in this study, the VSSI $\bar{X}$ &*S* charts will take h (the fixed sampling interval that is used by the four static charts) as the time unit, thus the *ATS* measure is equivalent to the *ARL* measure. Finally, in addition to the constraint ($ARL_0 = \tau$), the inspection rate of the VSSI $\bar{X}$ &*S* charts is set equal to the constant inspection rate (n / h) for the four static charts (Reynolds and Stoumbos 2004b). It ensures a fair basis for the comparison between the VSSI $\bar{X}$ &*S* charts and the static charts.

(4) The 3-*CUSUM* scheme

The design of the 3-*CUSUM* scheme follows the optimization procedure in Wu and Tian (2005). In the design of a 3-*CUSUM* scheme, an allocation factor similar to η in equation (3-44) is optimized in order to achieve the best allocation of the power between the *CUSUM* charts for detecting mean shifts and the *CUSUM* chart for detecting variance increases. In addition, the optimal values of the reference parameters k_I , k_D , k_V and the control limits H_I , H_D , H_V of the three *CUSUM* charts have to be searched ($k_I = k_D$, $H_I = -H_D$).

(5) The internal adaptive *AL* chart that is proposed in this chapter

Without the loss of generality, we set $\mu_0 = 0$ and $\sigma_0 = 1$ for the comparative studies. As in Chapter 3, three studies are conducted to evaluate the performance of the *AL* chart. The comparison is started with a general case ($n = 5$, $\tau = 400$, $D_\mu = 1$) according to the norm of SPC practice (study one). Then, a 2^2 factorial experiment on parameters n and τ are carried out with D_μ fixed at 1 (study two). Finally, experiments on different values of D_μ are followed to further compare the charts (study three).

4.4.1 Study one

The comparison is first conducted under the following general conditions:

$$n = 5, \tau = 400, D_\mu = 1 \quad (4-28)$$

The specification of $(D_\mu = 1)$ is commonly used by the optimization designs of many other charts, e.g. the CUSUM chart (Montgomery 2001). As a result, we have the following status points (Figure 4-1):

$$\begin{aligned} &P_0(\delta_{\mu,0} = 0.0, \delta_{\sigma,0} = 1.0), \\ &P_1(\delta_{\mu,1} = 1.0, \delta_{\sigma,1} = 1.0), P_2(\delta_{\mu,2} = 0.6357, \delta_{\sigma,2} = 1.2633) \\ &P_3(\delta_{\mu,3} = 0.0, \delta_{\sigma,3} = 1.4142), P_4(\delta_{\mu,4} = -0.6357, \delta_{\sigma,4} = 1.2633), \\ &P_5(\delta_{\mu,5} = -1.0, \delta_{\sigma,5} = 1.0) \end{aligned} \quad (4-29)$$

Based on the specifications (4-28), the optimal values of the charting parameters for the individual control charts are determined and listed below:

$$\begin{aligned} \text{Static } \bar{X} \&S \text{ charts: } &LCL_{\bar{X}} = -3.1295, UCL_{\bar{X}} = 3.1295, UCL_S = 2.1854 \\ WL \text{ chart: } &\lambda = 0.3, UCL_{WL} = 1.7422 \\ VSSI \bar{X} \&S \text{ charts: } &n_1 = 3, n_2 = 15, h_1 = 1.2750, h_2 = 0.3, w_{Z,upper} = 1.6184, \\ &UCL_Z = 3.2047, w_{Z,lower} = -1.6184, LCL_Z = -3.2047, \\ &w_{q1} = 1.4995, UCL_{q1} = 2.5705, w_{q2} = 1.2203, UCL_{q2} = 1.5868 \\ 3-CUSUM \text{ scheme: } &k_I = 0.45, H_I = 1.2868, k_D = 0.45, H_D = -1.2868, k_V = 0.1966, \\ &H_V = 2.5986 \\ AL \text{ chart: } &a_0 = 0.3, a_1 = 0.0, a_2 = 1.0, a_3 = 1.0, a_4 = 1.0, a_5 = 0.0, \xi = 0.3, \\ &\psi = 2.5, UCL_{AL} = 2.0798 \end{aligned} \quad (4-30)$$

The values of ARL of the five charts are calculated in a shift domain of ($0 \leq \delta_\mu \leq 2$, $1 \leq \delta_\sigma \leq 2$) and displayed in Table 4-1 where there are a total of 65 out-of-control cells (combinations of the discrete values of δ_μ and δ_σ) and one in-control cell ($\delta_\mu = 0$ and $\delta_\sigma = 1$). Each cell in this table contains the values of the ARL of the five charts for a particular combination of δ_μ and δ_σ . There are several interesting findings.

Table 4-1: ARL Values for Five Charts ($n = 5$, $\tau = 400$, $D_\mu = 1$)

δ_σ	Chart	δ_μ										
		0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
1.0	Static $\bar{X}S$	400.01	218.45	74.20	26.57	11.02	5.36	3.05	2.00	1.49	1.23	1.10
	WL	399.66	213.25	73.42	27.22	11.61	5.73	3.26	2.12	1.56	1.27	1.12
	VSSI $\bar{X}S$	400.92	215.14	43.06	8.41	3.49	2.41	1.96	1.71	1.55	1.44	1.36
	3-C	401.75	116.33	21.80	7.72	4.31	2.97	2.29	1.91	1.63	1.41	1.26
	AL	401.28	100.98	20.01	7.60	4.25	2.86	2.16	1.76	1.50	1.31	1.18
1.2	Static $\bar{X}S$	52.45	41.59	24.16	12.89	7.11	4.24	2.77	1.98	1.54	1.29	1.15
	WL	47.17	35.89	20.03	10.79	6.16	3.82	2.58	1.90	1.51	1.28	1.14
	VSSI $\bar{X}S$	30.74	23.46	11.21	4.99	2.98	2.27	1.93	1.71	1.57	1.47	1.39
	3-C	25.26	20.55	11.50	6.37	4.09	2.97	2.30	1.92	1.65	1.46	1.29
	AL	25.08	17.77	9.36	5.50	3.68	2.69	2.11	1.75	1.48	1.32	1.21
1.4	Static $\bar{X}S$	14.46	13.08	10.04	7.05	4.84	3.39	2.48	1.91	1.56	1.33	1.19
	WL	13.01	11.47	8.42	5.78	4.00	2.88	2.18	1.73	1.45	1.27	1.16
	VSSI $\bar{X}S$	5.56	5.10	4.06	3.08	2.45	2.08	1.85	1.69	1.57	1.48	1.41
	3-C	8.43	7.81	6.27	4.84	3.64	2.83	2.33	1.94	1.66	1.47	1.32
	AL	7.64	6.95	5.40	4.06	3.08	2.46	1.99	1.69	1.48	1.32	1.23
1.6	Static $\bar{X}S$	6.32	6.02	5.24	4.30	3.42	2.71	2.18	1.80	1.53	1.34	1.22
	WL	5.79	5.44	4.60	3.68	2.91	2.32	1.90	1.60	1.40	1.26	1.16
	VSSI $\bar{X}S$	2.78	2.71	2.52	2.30	2.08	1.90	1.76	1.64	1.55	1.48	1.42
	3-C	4.91	4.70	4.30	3.70	3.15	2.64	2.23	1.92	1.67	1.51	1.36
	AL	4.29	4.15	3.64	3.08	2.55	2.15	1.85	1.62	1.45	1.31	1.22
1.8	Static $\bar{X}S$	3.67	3.58	3.31	2.95	2.56	2.20	1.90	1.66	1.47	1.33	1.22
	WL	3.43	3.31	3.02	2.64	2.27	1.95	1.69	1.50	1.35	1.24	1.16
	VSSI $\bar{X}S$	2.12	2.09	2.04	1.95	1.86	1.76	1.67	1.59	1.52	1.46	1.41
	3-C	3.57	3.52	3.30	3.03	2.70	2.38	2.12	1.86	1.66	1.51	1.38
	AL	3.00	2.90	2.70	2.44	2.19	1.91	1.72	1.55	1.40	1.29	1.21
2.0	Static $\bar{X}S$	2.55	2.51	2.40	2.23	2.04	1.85	1.68	1.52	1.39	1.29	1.21
	WL	2.41	2.37	2.24	2.07	1.88	1.70	1.54	1.40	1.30	1.21	1.15
	VSSI $\bar{X}S$	1.84	1.83	1.81	1.76	1.71	1.66	1.60	1.54	1.49	1.45	1.41
	3-C	2.87	2.85	2.73	2.60	2.38	2.19	1.99	1.79	1.66	1.51	1.40
	AL	2.31	2.27	2.19	2.03	1.88	1.74	1.59	1.45	1.35	1.27	1.21

- (1) The five charts generate nearly identical values of ARL_0 that are very close to τ (= 400) when the process is in control ($\delta_\mu = 0$, $\delta_\sigma = 1$). This suggests that the charts have very similar false alarm rates which provide a common ground for the comparison.
- (2) The AL chart considerably outperforms the static $\bar{X}$ & S charts and the WL chart in most cases except for several cells at the right hand side of Table 4-1 (for large δ_μ) where the ARL of the AL chart is slightly larger than that of the other two charts. The superiority of the AL chart is more notable for mean shifts of small to moderate sizes. For example, when $\delta_\mu = 0.4$ and $\delta_\sigma = 1.0$, the ratio of the ARL values between the static $\bar{X}$ & S charts and the AL chart is as high as 3.71; and the ARL ratio between the WL chart and the AL chart is 3.67.
- (3) While the AL chart is less effective than the VSSI $\bar{X}$ & S charts for detecting joint mean and variance shifts of moderate magnitudes (in the central area of Table 4-1), the former outperforms the latter for detecting small or large process shifts. Their overall detection effectiveness in the whole process shift domain seems comparable.
- (4) Finally, the AL chart is more effective than the 3- $CUSUM$ scheme in every cell in Table 4-1 (i.e. for every shift combination of δ_μ and δ_σ). The superiority is relatively more substantial for joint mean and variance shifts than for mean shifts or for variance shifts.

Similar to *MRATS* (3-58) in Chapter 3, the average ratio *MRARL*, of the *ARL* values is used as a general quantitative measurement of the relative effectiveness of a chart compared to the *AL* chart.

$$MRARL = \frac{\sum_{j=1}^m \frac{ARL(\delta_{\mu,j}, \delta_{\sigma,j})}{ARL_{AL}(\delta_{\mu,j}, \delta_{\sigma,j})}}{m} \quad (4-31)$$

where m is the number of cells in the region of interest; $ARL(\delta_{\mu,j}, \delta_{\sigma,j})$ is the value of the out-of-control *ARL* that is produced by a chart at $(\delta_{\mu,j}, \delta_{\sigma,j})$ in the j th cell and $ARL_{AL}(\delta_{\mu,j}, \delta_{\sigma,j})$ is the value that is generated by the *AL* chart at the same shift point $(\delta_{\mu,j}, \delta_{\sigma,j})$. Since there are less input factors here (only n and τ) than in Chapter 3, the following breakdown regions of shifts are considered.

Small shift region ($0.0 \leq \delta_{\mu} \leq 1.0$, $1.0 \leq \delta_{\sigma} \leq 1.6$, excluding $\delta_{\mu} = 0$ & $\delta_{\sigma} = 1$)

- (1) $MRARL(0.0 < \delta_{\mu} \leq 1.0, \delta_{\sigma} = 1.0)$: mean shift
- (2) $MRARL(\delta_{\mu} = 0.0, 1.0 < \delta_{\sigma} \leq 1.6)$: standard deviation shift
- (3) $MRARL(0.0 \leq \delta_{\mu} \leq 1.0, 1.0 \leq \delta_{\sigma} \leq 1.6, \text{excluding } \delta_{\mu} = 0 \text{ \& } \delta_{\sigma} = 1)$: all shifts

Whole shift region ($0.0 \leq \delta_{\mu} \leq 2.0$, $1.0 \leq \delta_{\sigma} \leq 2.0$, excluding $\delta_{\mu} = 0$ & $\delta_{\sigma} = 1$)

- (4) $MRARL(0.0 < \delta_{\mu} \leq 2.0, \delta_{\sigma} = 1.0)$: mean shift
- (5) $MRARL(\delta_{\mu} = 0.0, 1.0 < \delta_{\sigma} \leq 2.0)$: standard deviation shift
- (6) $MRARL(0.0 \leq \delta_{\mu} \leq 2.0, 1.0 \leq \delta_{\sigma} \leq 2.0, \text{excluding } \delta_{\mu} = 0 \text{ \& } \delta_{\sigma} = 1)$: all shifts.

The *MRARL* values of the static $\bar{X}$ &*S* charts, the *WL* chart, the *VSSI* $\bar{X}$ &*S* charts and the 3-*CUSUM* scheme are calculated for the above six shift regions using the data in Table 4-1. The results in this study are listed in the first row (RUN(0)) in Table 4-2.

Table 4-2: *MRARL* Values for Four Charts ($D_\mu = 1$)

Chart	Small shift ($0 \leq \delta_\mu \leq 1, 1 \leq \delta_\sigma \leq 1.6$, excluding $\delta_\mu = 0$ & $\delta_\sigma = 1$)			Whole shift ($0 \leq \delta_\mu \leq 2, 1 \leq \delta_\sigma \leq 2$ excluding $\delta_\mu = 0$ & $\delta_\sigma = 1$)		
	δ_μ (1)	δ_σ (2)	All shift (3)	δ_μ (4)	δ_σ (5)	All shift (6)
RUN(0): $n = 5, \tau = 400$						
Static $\bar{X}$ & S	2.7680	1.8189	1.9732	1.9241	1.5571	1.4024
WL	2.8196	1.6445	1.7976	1.9766	1.4240	1.2991
VSSI $\bar{X}$ & S	1.4108	0.8666	0.9748	1.2220	0.8207	0.9874
3- C	1.0624	1.0848	1.1355	1.0683	1.1376	1.1585
RUN(1): $n = 3, \tau = 200$						
Static $\bar{X}$ & S	2.4808	1.4515	1.7185	1.8438	1.3224	1.3083
WL	2.4555	1.3535	1.6102	1.8431	1.2389	1.2352
VSSI $\bar{X}$ & S	1.3426	0.8567	0.9959	1.2555	0.8504	1.0787
3- C	0.9348	1.2438	1.1067	0.9779	1.2639	1.1184
RUN(2): $n = 3, \tau = 600$						
Static $\bar{X}$ & S	4.0196	1.8374	2.3958	2.7478	1.6000	1.6183
WL	4.0417	1.6506	2.2226	2.7901	1.4434	1.5067
VSSI $\bar{X}$ & S	1.5830	0.8799	1.0522	1.3712	0.8399	1.0964
3- C	1.0041	1.2618	1.1658	1.0396	1.2795	1.1753
RUN(3): $n = 7, \tau = 200$						
Static $\bar{X}$ & S	1.9216	1.4856	1.5456	1.4706	1.3088	1.2134
WL	2.0143	1.3489	1.4192	1.5381	1.2075	1.1436
VSSI $\bar{X}$ & S	1.2464	0.8511	0.9406	1.1335	0.8258	0.9505
3- C	0.9811	1.1001	1.0976	0.9930	1.1233	1.0894
RUN(4): $n = 7, \tau = 600$						
Static $\bar{X}$ & S	2.8130	1.9620	2.0408	1.9259	1.6293	1.4180
WL	2.9649	1.7642	1.8415	2.0333	1.4860	1.3103
VSSI $\bar{X}$ & S	1.2113	0.8533	0.9223	1.1592	0.8401	1.0159
3- C	1.1747	1.0714	1.1663	1.0961	1.1090	1.1368

It is found from RUN(0) in Table 4-2, that all of the *MRARL* values of the static $\bar{X}$ & S charts, the WL chart and the 3-*CUSUM* scheme are greater than one. It indicates that the

AL chart outperforms the three charts in all six regions. In the whole shift region, on average, the *AL* chart is 92.41% more effective than the static $\bar{X}$ &*S* charts in detecting mean shifts, 55.71% more effective in detecting standard deviation shifts and 40.24% more powerful in all shifts. The superiority of the *AL* chart is even more prominent in the small shift region where it outperforms the static $\bar{X}$ &*S* charts by 176.80% in detecting mean shifts, by 81.89% in detecting standard deviation shifts and by 97.32% in all shifts. The superiority of the *AL* chart over the *WL* chart is also substantial. In the whole shift region, the former is 97.66%, 42.40% and 29.91% more effective than the latter in signaling mean shifts, standard deviation shifts and all shifts, respectively. In the small shift region, these values are 181.96%, 64.45% and 79.76%, respectively. The superiority of the *AL* chart over the 3-*CUSUM* scheme is also considerable in view of all shifts, namely, 13.55% in the small shift region and 15.85% in the whole shift region. The VSSI $\bar{X}$ &*S* charts is the only scheme that has comparable effectiveness to the *AL* chart. While the VSSI $\bar{X}$ &*S* charts are more powerful for detecting standard deviation shifts, the latter is superior to the former in detecting mean shifts. The overall detection effectiveness of the two schemes is close.

It is also found that, even though the Shewhart-type control charts are usually considered less effective than the CUSUM-type charts, the Shewhart-type control charts with adaptive features (i.e. the *AL* chart and the VSSI $\bar{X}$ &*S* charts) generally outperform the static CUSUM-type charts (e.g. the 3-*CUSUM* scheme.)

4.4.2 Study two

Next, a 2^2 factorial experiment (Montgomery 2001) is employed to study the effects of the two factors n and τ on the relative effectiveness of the five charts (the other factors

D_μ is still fixed at 1.0 for the time being). Each of the two factors varies at two levels, resulting in 4 runs (i.e., 4 combinations of the values of the two factors). The low and high levels for each factor are decided below:

$$\begin{array}{lll} n & 3 & 7 \\ \tau & 200 & 600 \end{array} \quad (4-32)$$

The values of ARL of the five charts under the four additional runs disclose similar operating characteristics as shown in Table 4-1 for RUN 0 (it is the run at the center of the 2^2 factorial experiments). The $MRARL$ values (or the relative effectiveness) of the static $\bar{X}$ & S charts, the WL chart, the VSSI $\bar{X}$ & S charts and the 3- $CUSUM$ scheme are also calculated for different process shift regions and presented in Table 4-2. The $MRARL$ values are quite similar to that for RUN (0) with only two exceptions as follows.

- (1) The general effectiveness for all shifts in the whole shift region (column (6) in Table 4-2) of the AL chart is higher than that of the VSSI $\bar{X}$ & S charts in RUNs (1), (2) and (4). Especially, the former is more powerful than the latter by almost 10% in RUN (2). This again suggests that the overall detection effectiveness of the AL chart and the VSSI $\bar{X}$ & S charts are similar.
- (2) When the value of τ is small (= 200) as in RUNs (1) and (3), the 3- $CUSUM$ scheme slightly outperforms the AL chart for detecting mean shifts.

Finally, the grand average $\overline{MRARL}$, the average of the $MRARL$ in column (6) of Table 4-2 for a chart over the 5 runs, is calculated and listed at the bottom of Table 4-3. The

values of $\overline{MRARL}$ indicate that, from an overall viewpoint (covering all different combinations of the values of n , τ , δ_μ and δ_σ), the AL chart is more effective than the static $\bar{X}\&S$, the WL chart, VSSI $\bar{X}\&S$ and the 3- $CUSUM$ scheme by 39.21, 29.90, 2.58 and 13.57 percent, respectively.

Table 4-3: $MRARL(6)$ Values for Four Charts ($D_\mu = 1$)

RUN	n	τ	Static $\bar{X}\&S$	WL	VSSI $\bar{X}\&S$	3- C
0	5	400	1.4024	1.2991	0.9874	1.1585
1	3	200	1.3083	1.2352	1.0787	1.1184
2		600	1.6183	1.5067	1.0964	1.1753
3	7	200	1.2134	1.1436	0.9505	1.0894
4		600	1.4180	1.3103	1.0159	1.1368
$\overline{MRARL}$			1.3921	1.2990	1.0258	1.1357

The effects of the two factors n and τ on the performance (the values of $MRARL(6)$) are also studied. In a series of two-way ANOVA tests, the smallest p -value (the smallest level of significance that would lead to rejection of null hypothesis) found among eight p -values (of n effect and τ effect in each scheme) is 0.058 from effect τ on the 3- $CUSUM$ scheme. Hence the effects of both factors are very insignificant for the four schemes. It implies that the superiority of the AL chart over these four schemes is quite consistent and not affected significantly by the changes of n and τ . The effects of the factors n and τ on the response $MRARL(6)$ for all four schemes are found to be negative and positive respectively. In other words, the superiority of the AL chart over the other charts is relatively more notable when n is small and/or τ is large. Comparing RUN (1)

to RUN (2) and RUN (3) to RUN (4), it is easily noted that the superiority of the *AL* chart is more evident when τ is large, that is, when the false alarm rate is required to be low. It may be because that *ARL* becomes large along with the increase of τ . It gives the updating mechanism (equation 4-8) more time and chance to identify the most suitable value for the weighting factor a , so that the *AL* chart will work more effectively.

4.4.3 Study Three

In this study, the effectiveness of the five charts is compared under four different values of D_μ (0.6, 0.8, 1.2, 1.4). Here, n and τ are fixed at 5 and 400 respectively as in study one. The overall ratios in the whole shift region, i.e. $MRARL(6)$, of the four runs together with the previous run ($D_\mu = 1.0$) are listed in Table 4-4.

It is found that the effect of D_μ is more substantial for the VSSI $\bar{X}\&S$ charts and the 3-*CUSUM* scheme. We observe that the performance of the VSSI $\bar{X}\&S$ charts is close to that of the *AL* chart when D_μ is one. With the decrease of D_μ , the VSSI $\bar{X}\&S$ charts perform rather poorly. On the other direction, it gradually outperforms the *AL* chart when D_μ is larger than one. However, the specification of $D_\mu > 1$ is less common in practice. Compared with the 3-*CUSUM* scheme, we can also easily observe the trend of increasing advantage for the *AL* chart with the decrease of D_μ specification. Finally, the bottom row of Table 4-4 shows that, from a most general and overall viewpoint, the *AL* chart is more effective than the static $\bar{X}\&S$ charts, the *WL* chart, the VSSI $\bar{X}\&S$ charts and the 3-*CUSUM* scheme by 36.28, 26.34, 26.78, and 14.31 percent, respectively.

Table 4-4: Four Additional Experiments on D_μ ($n = 5$, $\tau = 400$)

RUN	D_μ	$MRARL(6)$			
		Static $\bar{X}\&S$	WL	VSSI $\bar{X}\&S$	3- C
1	0.6	1.3065	1.2166	2.2616	1.1776
2	0.8	1.4140	1.3123	1.3304	1.2060
0	1.0	1.4024	1.2991	0.9874	1.1585
3	1.2	1.3596	1.2583	0.9105	1.0923
4	1.4	1.3315	1.2307	0.8492	1.0810
$\overline{MRARL(6)}$		1.3628	1.2634	1.2678	1.1431

The results of the comparative studies show that, under general conditions, the AL chart is not only considerably more effective than the static $\bar{X}\&S$ charts, but also outperforms the 3- $CUSUM$ scheme under most conditions. Finally, while the AL chart is not completely better than the VSSI $\bar{X}\&S$ charts in terms of the overall detection effectiveness, the former is much easier for SPC operators to implement. As noted earlier, the implementation of the variable sampling schemes (including VSSI schemes) are confronted with some practical problems, such as the allocation of inspection resources and extra workload for operators etc. In contrast, the user of the AL chart has no such worries as long as an on-site computer is available, which should not be a problem for today's environment.

4.5 Diagnosis of the Process Shift

The AL chart comprises only one chart. However, it can make use of the probabilities p_i pertaining to the five out-of-control status points to diagnose the types of shifts. At the

time of signaling, p_k (the largest probability of p_i 's) indicates the process is probably shifted to process P_k ($\delta_{\mu,k}$, $\delta_{\sigma,k}$). Specifically,

- 1 If p_1 is greatest, increasing mean shift is indicated.
- 2 If p_2 is greatest, joint increasing mean shift and increasing variance shift is indicated.
- 3 If p_3 is greatest, increasing variance shift is indicated.
- 4 If p_4 is greatest, joint decreasing mean shift and increasing variance shift is indicated.
- 5 If p_5 is greatest, decreasing mean shift is indicated.
- 6 If p_0 is greatest, use the second largest p_i value to diagnose.

The diagnosis capability of the AL chart has been compared with that of the static $\bar{X}\&S$ charts and the 3- $CUSUM$ scheme. The diagnosis rules used for the static $\bar{X}\&S$ charts were introduced in Section 3.5, and the rules for the 3- $CUSUM$ scheme are exactly the same as those for the VSSI 3- $CUSUM$ scheme in Section 3.5.

Table 4-5: Rates of Correct Diagnoses of Three Charts

	δ_{μ}	δ_{σ}	ATS			Correct classification rate %		
			Static $\bar{X}\&S$	3-C	AL	Static $\bar{X}\&S$	3-C	AL
	0.0	1.0	392.85	403.34	410.17	—	—	—
δ_{μ} shift	0.5	1.0	43.65	12.00	11.55	96.76	98.60	60.75
	1.0	1.0	5.46	2.95	2.84	99.59	99.68	88.15
	1.5	1.0	1.69	1.75	1.61	99.79	99.88	89.37
δ_{σ} shift	0.0	1.2	51.92	25.66	24.99	52.41	81.37	57.41
	0.0	1.4	14.55	8.34	7.65	63.30	82.70	64.67
	0.0	1.7	4.63	4.11	3.48	69.16	80.20	63.47
Joint Shift	0.5	1.2	17.62	8.47	7.03	0.83	2.33	61.06
	1.0	1.4	3.38	2.85	2.44	4.06	5.72	54.13
	1.5	1.7	1.61	1.79	1.50	13.43	10.54	66.23

Using these diagnosis rules, a simulation study is conducted under the general condition ($n = 5$, $\tau = 400$, $D_{\mu} = 1$). For each of the process shift combinations (δ_{μ} , δ_{σ}) in Table 4-5, a series of 10,000 simulation runs are conducted. The correct classification rates

together with the ARL values made by the static $\bar{X}$ & S charts, the 3- $CUSUM$ scheme and the AL scheme are listed.

It is found that the combined charts (static $\bar{X}$ & S charts and the 3- $CUSUM$ scheme) are good at identifying mean shifts, but they perform very poorly for joint shift. In contrast, the AL chart is less capable in diagnosing mean shift, but it has obviously higher correct classification rates for joint mean and variance shift. For the variance shift, the AL chart has similar diagnosis capability as the static $\bar{X}$ & S charts and both of them are inferior to the 3- $CUSUM$ scheme. In conclusion, the 3- $CUSUM$ scheme is favored for pure mean or variance shift, while the AL chart is favored for joint shift. In another work (Wang and Wu 2004), it has been found that if only the pure mean or variance shifts are to be classified, the AL chart is generally to be favored in diagnosis performance relative to the $\bar{X}$ & R charts, combined EWMA scheme and a Combined-Neural Network method (Ho and Chang (1999)).

Finally, if the change point problem is concerned, the joint maximum likelihood estimator in Park and Park (2004) can be applied to the AL chart.

4.6 Design Table for the AL chart

In order to facilitate the design of the AL charts (i.e., the determination of UCL_{AL} , ξ , ψ , a_0), a design table (Table 4-6) is provided for nine usual in-control Average Run Lengths (200, 300, ..., 1000) and eight sample sizes (3, 4, 5, 6, 8, 12, 18, 25). The mean shift value of design is set at one ($D_\mu = 1$), all other charting parameters (out-of-control status points P_i and weighting factors a_i) are calculated from equations (4-3 and 4-5). In the design table, for each pair of the values of τ and n , a set of parameters $UCL_{standard}$

(standard upper control limit), ζ (reducing factor), ψ (acceleration power) and a_0 (weight at P_0) can be found. $UCL_{standard}$ is determined under the standard condition ($\mu_0 = 0$, $\sigma_0 = 1$). For a general condition ($\mu_0 \neq 0$ and/or $\sigma_0 \neq 1$), UCL_{AL} is obtained by a simple modification (see Appendix A for details).

$$UCL_{AL} = \sigma_0^2 UCL_{standard} \quad (4-33)$$

A general-condition AL chart with an upper control limit UCL_{AL} produces the same ARL as a standard-condition AL chart with $UCL_{standard}$ for given values of δ_μ and δ_σ .

With the help of the design table (4-6), the design of an AL chart becomes very simple.

- 1 Specify μ_0 , σ_0 , n and τ . ($D_\mu = 1$)
- 2 From the design table, find a pair of n and τ which are closest to the specified values, and then select the corresponding $UCL_{standard}$, ζ , ψ and a_0 .
- 3 Calculate the general-condition UCL_{AL} by using equation (4-33).
- 4 All other charting parameters can be found by respective equations.

(1) Status points $P_i(\delta_{\mu,i}, \delta_{\sigma,i})$

$P_i(\delta_{\mu,i}, \delta_{\sigma,i})$ ($i = 1, 2, \dots, 5$) by equation (4-3), while $P_0(0, 1)$ are fixed.

$$\delta_{\mu,1} = D_\mu, \quad \delta_{\sigma,1} = 1$$

$$\delta_{\sigma,2} = \frac{2D_\mu^2 + \sqrt{4D_\mu^4 - 4[(\delta_{\sigma,3} - 1)^2 + D_\mu^2] [D_\mu^2 - \delta_{\sigma,3}^2(\delta_{\sigma,3} - 1)^2]}}{2[(\delta_{\sigma,3} - 1)^2 + D_\mu^2]}$$

$$\delta_{\mu,2} = \frac{D_\mu(\delta_{\sigma,2} - 1)}{\delta_{\sigma,3} - 1}$$

$$\delta_{\mu,3} = 0, \quad \delta_{\sigma,3} = \sqrt{1 + D_\mu^2}$$

$$\delta_{\mu,4} = -\delta_{\mu,2}, \quad \delta_{\sigma,4} = \delta_{\sigma,2}$$

$$\delta_{\mu,5} = -D_{\mu}, \quad \delta_{\sigma,5} = 1 \quad (4-3)$$

(2) Other weighting factors.

a_i ($i = 1, 2, \dots, 5$) are obtained by maximizing $E(AL)$ at P_i with equation (4-5).

Table 4-6: Design Table of the *AL* Charts ($D_\mu = 1$)

Table 4-6: Design Table of the AL chart ($D_\mu = 1$)

τ	n	$UCL_{standard}$	ξ	ψ	a_0	τ	n	$UCL_{standard}$	ξ	ψ	a_0	τ	n	$UCL_{standard}$	ξ	ψ	a_0
200	3	2.9674	0.30	3.0	0.26	500	3	3.1382	0.35	2.4	0.28	800	3	3.4626	0.20	2.2	0.28
	4	2.3124	0.40	3.0	0.30		4	2.5642	0.20	2.4	0.30		4	3.0521	0.30	2.3	0.30
	5	1.9413	0.50	3.0	0.31		5	2.1422	0.25	2.4	0.31		5	2.1638	0.40	2.2	0.31
	6	1.4601	0.40	2.7	0.30		6	1.7660	0.45	2.4	0.31		6	1.9470	0.25	2.2	0.32
	8	1.2710	0.55	2.9	0.31		8	1.4632	0.40	2.4	0.32		8	1.5254	0.40	2.2	0.32
	12	0.9584	0.50	3.0	0.30		12	1.1806	0.50	2.5	0.31		12	1.1337	0.40	2.2	0.32
	18	0.7347	0.70	2.9	0.30		18	0.8431	0.70	2.6	0.31		18	0.8680	0.70	2.3	0.31
	25	0.6202	0.65	3.1	0.29		25	0.6856	0.70	2.5	0.30		25	0.7239	0.60	2.5	0.30
300	3	3.4031	0.10	2.7	0.27	600	3	3.5857	0.25	2.4	0.28	900	3	3.5909	0.45	2.3	0.28
	4	2.4787	0.30	2.7	0.30		4	2.5949	0.15	2.3	0.30		4	2.6898	0.40	2.2	0.30
	5	1.9565	0.45	2.7	0.30		5	2.0820	0.35	2.3	0.31		5	2.3632	0.30	2.2	0.32
	6	1.7428	0.50	2.7	0.31		6	1.8079	0.35	2.3	0.31		6	2.0041	0.40	2.2	0.32
	8	1.3073	0.50	2.6	0.31		8	1.4808	0.25	2.3	0.32		8	1.6024	0.45	2.2	0.32
	12	1.2713	0.55	2.8	0.30		12	1.1230	0.50	2.4	0.31		12	1.1695	0.50	2.2	0.32
	18	1.2810	0.50	3.0	0.30		18	0.8436	0.55	2.4	0.31		18	0.8888	0.40	2.3	0.31
	25	0.6414	0.50	2.8	0.29		25	0.7018	0.65	2.6	0.30		25	0.7198	0.65	2.3	0.30
400	3	3.2626	0.35	2.6	0.27	700	3	3.7100	0.15	2.3	0.28	1000	3	3.8599	0.25	2.2	0.28
	4	2.5386	0.10	2.5	0.30		4	2.7916	0.25	2.3	0.30		4	3.2353	0.25	2.2	0.30
	5	2.0798	0.30	2.5	0.30		5	2.3184	0.30	2.3	0.31		5	2.2072	0.35	2.1	0.32
	6	1.8051	0.25	2.5	0.31		6	1.8570	0.55	2.3	0.31		6	1.9427	0.25	2.1	0.32
	8	1.3828	0.50	2.5	0.31		8	1.4565	0.40	2.2	0.32		8	1.5544	0.30	2.1	0.32
	12	1.0818	0.50	2.6	0.31		12	1.1371	0.55	2.3	0.32		12	1.2229	0.55	2.2	0.32
	18	0.8015	0.70	2.7	0.30		18	0.9123	0.80	2.5	0.31		18	1.0252	0.55	2.3	0.31
	25	0.6726	0.50	2.5	0.30		25	0.7187	0.55	2.6	0.30		25	0.7552	0.65	2.4	0.31

4.7 Example

A semiconductor factory produces a kind of silicon wafer. The key process characteristic is the oxide thickness x of wafers. Its nominal value and tolerance are specified as $3 \pm 0.1 \mu m$. From the pilot runs, it is found that the probability distribution of x can be very well approximated by a normal one, and the in-control standard deviation σ_0 is estimated as $0.02 \mu m$. The in-control mean μ_0 can be easily adjusted to the nominal value of $3 \mu m$ (i.e., the center between the specification limits). An AL chart is to be built to monitor both mean shift and variance increase of x . The design is carried out as follows:

- 1 The in-control process parameters are known ($\mu_0 = 3 \mu m$ and $\sigma_0 = 0.02 \mu m$). Sample size is set as $n = 5$. τ is specified as 370 that is the ARL_0 value generated by a $3\text{-}\sigma \bar{X}$ chart.
- 2 From Table 4-6, $UCL_{standard}$, ξ , ψ and a_0 for ($n = 5$, $\tau = 400$) are picked. ($\tau = 400$) is closest to the specified value of 370.

$$UCL_{standard} = 2.0798 \mu m, \xi = 0.3, \psi = 2.5, a_0 = 0.3.$$

The status points $P_i(\delta_{\mu,i}, \delta_{\sigma,i})$ under $D_\mu = 1$ are equal to those in (4-29).

In addition, $a_1 = 0.0$, $a_2 = 1.0$, $a_3 = 1.0$, $a_4 = 1.0$, $a_5 = 0.0$.

- 3 $UCL_{standard}$ has to be modified by σ_0 (i.e., multiplied by σ_0^2). The final value of UCL_{AL} for this example is $8.3192 \times 10^{-4} \mu m$.
- 4 Then, the AL control chart is drawn. The control limit is a horizontal line with value of UCL_{AL} , but at last the reduced control limit is activated by the rule p_2 is the greatest value ($P_K = 2$). Therefore, control limit of $\xi \times UCL_{AL} = 2.4958 \times 10^{-4} \mu m$ is used and a signal is generated immediately.

A computer program can also be developed to carry out the design of the *AL* chart. It completes the design in 20s minutes of CPU time in a personal computer (Pentium IV, CPU 2.4 GHz). It is relatively long, because computer simulation is used to evaluate *ARL*. But the design only has to be carried out once, the resultant *AL* chart can be used as long as the specifications have not been changed. The improved detection effectiveness can be appreciated everyday.

Now, the designed *AL* chart can be used to monitor the mean and variance of the diameter x simultaneously. The operators only have to take n observations x_i for each sampling interval h and input them into the computer. A computer program will handle all other manipulation.

As a simulation run shown in Figure 4-2, the simulated process is under control until the time to take the 41st sample, when a shift ($\delta_\mu = 0.4$, $\delta_\sigma = 1.2$) happens. The *AL* chart generates an out-of-control signal upon the 47th sample (see Figure 4-2). At this sample point, it is found that p_2 is the greatest. Therefore, a joint increasing mean shift and increasing variance shift is indicated.

Figure 4-2: An Illustrative Example of the *AL* Chart

Figure 4-2: An Illustrative Example of the AL Chart

4.8 Summary and Discussion

This chapter has brought forth the general idea of the internal adaptive control chart and has developed the *AL* chart as an example. The appealing feature of an internal adaptive chart is its ability to achieve high detection effectiveness without the loss of operational simplicity for an operator.

Unlike the VSI, VSS and VSSI charts with variable sampling features that labor the human operators and do not employ the computers, the *AL* chart releases the human from the troublous operation and makes better use of computing resource. It uses fixed sample size and sampling interval and leaves all the complex adaptations and manipulations to a computer program.

By dynamically varying some charting parameters (i.e. the weighting factor a , the mean shift term $\Delta\bar{x}$ and the actual UCL) based on the information of current process status, the *AL* chart not only significantly outperforms the static $\bar{x}$ & S charts, but also possesses considerably more effectiveness than the 3-*CUSUM* scheme which is thought as very powerful for detecting two-sided mean shifts and increasing variance shift in current literature. Moreover, the *AL* chart is competitive with the adaptive VSSI $\bar{x}$ & S charts in detection effectiveness under the general conditions.

Currently, most of the adaptive charts are developed based on some heuristic ideas. For example, the VSS or VSI chart changes the charting parameters (sample size or sampling interval) by checking whether the last sample point is close to the control limits. Similarly, the *AL* chart is developed based on the underlying idea that adjusts the charting (i.e., a) according to the likely type of the shifts (or the location of the sample

point on the μ and σ domain). More importantly, the AL chart is applicable and advantageous to any SPC applications where sample size and sampling interval are preferred to be fixed and mean shift and increasing variance have to be monitored simultaneously.

The operation of the AL chart seems complicated. However, all the complexity will be handled by a short computer program. For operators, the implementation of this chart is as simple as that for the simplest static $\bar{\bar{x}}\&S$ charts. The design of the AL chart can be computerized and carried out based on the specifications for n , τ and D_μ . We also provided a design table to facilitate the designs.

It is believed that the internal adaptation idea of the AL chart can be applied to the on-line adjustments of the parameters of some other control charts, such as the reference parameter k_{cusum} of a CUSUM chart and the weighting parameter λ_{EWMA} of an EWMA chart.

Chapter 5

Schemes for Monitoring Mean and Variance of Autocorrelated Process

5.1 Introduction

In Chapters 3 and 4, the adaptive control charts (VSSI 3-*CUSUM*, VSSI *WLC*, *AL* charts etc.) are developed for process of independent observations. However, in industrial practice, the successive observations are sometimes serially correlated. The autocorrelation can have great impact on the performance of the control charts designed for independent observations (Knoth *et al.* 2001). This problem also affects the adaptive control charts proposed in previous chapters. It is not appropriate to apply those control charts designed for independent observations to monitoring the autocorrelated process, such as the autoregressive process of order one (AR(1) model) focused on here. Figure 5-1 presents the behavior of the direct application of the $\bar{X}$ &*S* charts and the *AL* chart in terms of *ARL* performance versus different degrees of ϕ (ϕ is the autoregressive parameter in the AR(1) model, which will be discussed shortly). It is noted that when ϕ is equal to zero, the process simply corresponds to the independent case. The left figure shows the in-control condition and the right one is for an out-of-control condition. Obviously for in-control and out-of-control conditions, the influence of ϕ on *ARL*s of both charts are considerable even when a minor degree of the autoregressive structure exists.

Figure 5-1: ARL Behavior of Two Charts versus ϕ ($n = 5$, $\tau = 500$)

There are generally two methods to monitor the processes with autocorrelation in literature, namely, the modified approach and the residuals approach. As suggested by Knoth *et al.* (2001) and Knoth and Schmid (2002), a residual based method must be favored if a variance change is present. Thus the study in this thesis is focused on the residuals approaches. The autocorrelation modeling and shift model are introduced in the next section. Two residuals approaches are compared in Section 5.3. Then, control charts (the AL chart etc.) considered in Chapter 4 are extended to the AR(1) situation and some comparisons are presented. After that, AR(2) and ARMA(1,1) models are investigated. Finally, the summary and discussions are presented.

5.2 AR(1) Modeling and Shift Model

Let $\{X_{i,j}\}$ denote the observed process, where the index i represents the sample number and the index j marks the sequence within the sample. The sample size n is fixed. The objective is to detect deviations from the in-control process, which is denoted by $\{Y_{i,j}\}$.

The modeling assumption made by Knoth *et al.* (2001) is employed, that is, the observations between different samples are independent. This implies that the random vectors $Y_i = (Y_{i,1}, Y_{i,2}, \dots, Y_{i,n})'$, $i \geq 1$ are independent. On the other hand, observations may be correlated within the samples. If each sample is regarded as an individual process, it is assumed that the processes $Y_1, Y_2, \dots$ have the same distribution. Besides, Y_i is required to be weakly stationary with mean μ_0 and autocovariance function $\{\gamma_t\}$, where t is the lag. The values of μ_0 and $\gamma_0, \gamma_1, \dots, \gamma_{n-1}$ are assumed to be known. Here it is assumed that $\{Y_{i,j}\}$ is a weakly stationary autoregressive process of order one (AR(1)) with mean μ_0 and AR(1) parameter ϕ , i.e.

$$Y_{i,j} = \mu_0 + \phi(Y_{i,j-1} - \mu_0) + \varepsilon_{i,j} \quad \text{with } |\phi| < 1 \quad (5-1)$$

The random errors $\varepsilon_{i,j}$ ($j \geq 1, i \geq 1$) are assumed to be independent and normally distributed with mean zero and variance σ_0^2 . Thus we have $E(Y_{i,j}) = \mu_0$ and $\gamma_k = \sigma_0^2 \phi^{|k|} / (1 - \phi^2)$, specially $\gamma_0 = \sigma_0^2 / (1 - \phi^2)$. We assume that the values of μ_0 and σ_0 can be correctly estimated from pilot runs, and the AR(1) structure with ϕ value can be known from model fitting procedure.

The generalized change point model is,

$$X_{i,j} = \begin{cases} Y_{i,j} & \text{for } i < q \\ \delta_\sigma(Y_{i,j} - \mu_0) + \mu_0 + \delta_\mu \sqrt{\gamma_0} & \text{for } i \geq q \end{cases} \quad (5-2)$$

where the integer q stands for the unknown change point. Note that in this model the actual change takes place between two samples or at the first observed value of a sample, but not within the sample. It follows that

$$E(X_{i,j}) = \begin{cases} \mu_0 & \text{for } i < q \\ \mu_0 + \delta_\mu \sqrt{\gamma_0} & \text{for } i \geq q \end{cases} \quad (5-3)$$

and

$$Var(X_{i,j}) = \begin{cases} \gamma_0 & \text{for } i < q \\ \delta_\sigma^2 \gamma_0 & \text{for } i \geq q \end{cases} \quad (5-4)$$

The process is said to be in-control if $\delta_\mu = 0$ and $\delta_\sigma = 1$. It means that the observed process coincides with the target process. If $\delta_\mu \neq 0$ and/or $\delta_\sigma \neq 1$, the process is out-of-control. As in previous chapters, the study is confined to detecting increases in the variance, i.e. $\delta_\sigma > 1$.

5.3 Residuals Approaches

The residuals approach primarily aims at transforming the original autocorrelated observations into independent residuals. To monitor the residuals, the charting parameters can be chosen as in the case of independent variables. Residuals are measured as the difference between the observed value and a predictor for this observation.

Traditionally, residuals approach uses the Minimum Mean Square error (MMS) predictor for all observations, and it is referred as traditional approach here. The predictor of observation $X_{i,j}$ is denoted as $\hat{X}_{i,j}$. Then for the AR(1) process, $\hat{X}_{i,j}$ in the traditional approach can be found.

$$\hat{X}_{i,j} = \mu_0 + \phi(X_{i,j-1} - \mu_0) \quad \text{for all } j \quad (5-5)$$

The residuals of $X_{i,j}$ is denoted as $\hat{\varepsilon}_{i,j}$.

$$\hat{\varepsilon}_{i,j} = (X_{i,j} - \hat{X}_{i,j}) / \sigma_0 \quad (5-6)$$

Thus the residuals are independent and identically distributed (iid) with normal distribution. For the i th sample of size n , the sample mean and the sample variance of the residuals are calculated as

$$\bar{\hat{\varepsilon}}_i = \frac{1}{n} \sum_{j=1}^n \hat{\varepsilon}_{i,j} \quad \text{and} \quad \hat{S}_i^2 = \frac{1}{n-1} \sum_{j=1}^n (\hat{\varepsilon}_{i,j} - \bar{\hat{\varepsilon}}_i)^2 \quad (5-7)$$

The distribution of the $\bar{\hat{\varepsilon}}_i$ and $\hat{S}_i^2$ can be obtained.

$$\bar{\hat{\varepsilon}}_i \sim N\left(\delta_\mu \sqrt{\frac{1-\phi}{1+\phi}}, \frac{\delta_\sigma^2}{n}\right) \quad \text{and} \quad \frac{(n-1)}{\delta_\sigma^2} \hat{S}_i^2 \sim \chi_{n-1}^2 \quad (5-8)$$

where χ_{n-1}^2 denotes the chi-square distribution with $n-1$ degrees of freedom.

More recently, Kramer and Schmid (1998) proposed another residuals approach, which has been adopted by Knoth *et al.* (2001) and Knoth and Schmid (2002). Compared with the traditional approach, this residuals approach is called revised residuals approach here because it makes some modification for the first predictor. Following the previous notations,

$$\hat{X}_{i,j} = \begin{cases} \mu_0 & \text{for } j = 1 \\ \mu_0 + \phi(X_{i,j-1} - \mu_0) & \text{for } j = 2, \dots, n \end{cases} \quad (5-9)$$

Using μ_0 as the predictor for the first observation of the subgroup is based on the assumption of independence between samples. This manipulation also handles the starting problem. It removes the necessity as for the traditional residuals approach to take an additional preceding observation to calculate the first residual. Thus, the revised residuals approach can save one observation in every sample.

$$X_{i,j} - \hat{X}_{i,j} = \begin{cases} \delta_\sigma(Y_{i,1} - \mu_0) + \delta_\mu \sqrt{\gamma_0} & \text{for } j = 1 \\ \delta_\sigma \varepsilon_{i,j} + (1-\phi)\delta_\mu \sqrt{\gamma_0} & \text{for } j = 2, \dots, n \end{cases} \quad (5-10)$$

In the in-control condition, the residuals $X_{i,j} - \hat{X}_{i,j}, j \geq 1$, have mean zero but have unequal variances given nonzero ϕ . Therefore these quantities are normalized by respective in-control standard deviation, $\hat{\varepsilon}_{i,j} = (X_{i,j} - \hat{X}_{i,j}) / \sqrt{\text{Var}_0(X_{i,j} - \hat{X}_{i,j})}$.

$$\hat{\varepsilon}_{i,j} = \begin{cases} \sqrt{1-\phi^2} (X_{i,1} - \hat{X}_{i,1}) / \sigma_0 & \text{for } j = 1 \\ (X_{i,j} - \hat{X}_{i,j}) / \sigma_0 & \text{for } j \geq 2 \end{cases} \quad (5-11)$$

The sample mean and the sample variance of the normalized residuals are calculated by the same formulae as in (5-7). The distribution of the residual mean was given in Knoth *et al.* (2001).

$$\bar{\varepsilon}_i \sim N\left(\frac{\delta_\mu}{n} \left(1 + (n-1)\sqrt{\frac{1-\phi}{1+\phi}}\right), \frac{\delta_\sigma^2}{n}\right) \quad (5-12)$$

From corollary 5.1.3a proposed by Mathai and Provost (1992),

$$\frac{(n-1)}{\delta_\sigma^2} \bar{S}_i^2 \sim \chi_{n-1, \nu}^2 \text{ with } \nu = \frac{n-1}{n} \left(\frac{\delta_\mu}{\delta_\sigma}\right)^2 \left(1 - \sqrt{\frac{1-\phi}{1+\phi}}\right)^2 \quad (5-13)$$

where $\chi_{n-1, \nu}^2$ denotes the noncentral chi-square distribution with $n-1$ degrees of freedom and noncentrality parameter $\nu (\geq 0)$.

By comparing the distributions (5-8, 5-12, 5-13), it can be found that, when there is no change in the mean ($\delta_\mu = 0$), the residuals' distributions from the two approaches are identical to that in the independent situation. However, if there is a change in mean, the expectation of $\bar{\varepsilon}_i$ varies. For ease of exposition, the coefficient of δ_μ in the expectation of $\bar{\varepsilon}_i$ (5-8, 5-12) is called as mean shift ratio. For independent situation ($\phi = 0$), the mean shift ratio is 1. Notably, when sample size approaches infinity, the mean shift ratio of the revised approach approximates that of the traditional approach. The mean shift ratios of the two approaches are illustrated in the Figure 5-2.

Figure 5-2: Mean Shift Ratios of Two Residuals Approaches

The solid curves plot the mean shift ratios of the revised approach, for sample size from 2 to 10. The dashed curve represents the results of the traditional approach, which is equivalent to that obtained from the revised approach for infinite sample size. Obviously, the residuals from positively autocorrelated processes actually produce mean shift of smaller magnitude than those from the independent situation. This makes it difficult for control charts to detect the shift, thus the revised approach that mitigates the impact is preferred. On the other hand, when negative ϕ is in presence, the residuals actually produce mean shift of larger magnitude. The difference between the mean shift ratios of the two approaches is more evident when the ϕ level is high and/or sample size is small.

In contrary to the presentation of autocorrelation impact on the distribution of the mean $\bar{\hat{\varepsilon}}_i$ of sample residuals, the analysis to the impact on distribution of the variance $\hat{S}_i^2$ of sample residuals is more difficult. However, some inferences can be drawn from the statistical properties. Firstly, it is clear that the distribution of $\hat{S}_i^2$ from the traditional approach (5-8) is always the same as that in independent situation. For the results of the revised residuals approach (5-13), comparison on skewness and kurtosis between the chi-square distribution χ_{n-1}^2 and noncentral chi-square distribution $\chi_{n-1,v}^2$ can be carried out. With some manipulation, it can be found that χ_{n-1}^2 is more skewed and peaked than $\chi_{n-1,v}^2$ (Evans *et al.* 1993). Moreover, if let $E(\hat{S}_i^2)_R$ and $Var(\hat{S}_i^2)_R$ denote the expectation and variance of $\hat{S}_i^2$ of the revised residuals approach, and let $E(\hat{S}_i^2)$ and $Var(\hat{S}_i^2)$ denote those in independent situation (or by traditional residuals approach).

$$\frac{E(\hat{S}_i^2)_R}{E(\hat{S}_i^2)} = 1 + \frac{v}{n-1} \quad \text{and} \quad \frac{Var(\hat{S}_i^2)_R}{Var(\hat{S}_i^2)} = 1 + \frac{2v}{n-1} \quad (5-14)$$

For non-zero mean shift and autocorrelation ($v > 0$), above ratios are obviously larger than one. In other words, $\hat{S}_i^2$ of the revised residuals approach has larger expectation and variance than that in the independent situation (or by traditional residuals approach). All these suggest that the probability of $\hat{S}_i^2$ falling out of a fixed control limit increases for revised residuals approach provided that the chi-square distribution is non-central. Therefore, the revised residuals approach is preferred for monitoring $\hat{S}_i^2$ in this situation. It is obvious that the resultant ARL in joint monitoring is decided from both mean aspects and variance aspects. Since the independence between $\hat{S}_i^2$ and $\bar{\hat{\varepsilon}}_i$ still holds, conclusions can be drawn for the Shewhart $\bar{X}$ & S^2 charts that the revised residuals approach is consistently better under positive ϕ provided that the process mean has

shifted. The same conclusion is valid for the WL chart, which is the combination of $\bar{X}$ and S^2 . In addition, it is expected that the revised residuals approach is still preferred for other charts when ϕ is positive. Some numerical tests will be conducted in the next section. For negative ϕ , theoretical analysis is difficult. Since positive autocorrelation is usually much more prevalent than negative autocorrelation in control chart application (Lu and Reynolds 1999), the scope of the study in this chapter is confined to the positive autocorrelation.

5.4 Schemes by Residuals Approach

The implementation of the residuals approach is very easy. Charting parameters designed for independent observations can be directly adopted for monitoring residuals (like in equations 5-6 and 5-11) under the same specifications. The results of ARL can be obtained either by calculation using the residuals distribution or by simulation, depending on whether analytical method is available for the control chart under the independent case.

Firstly, the performances of the two residuals approaches need to be compared by numerical tests to verify the previous conclusion for positive ϕ . ($\phi=0.5$) is used to represent a positively correlated condition. All the control charts compared in Chapter 4 are considered here, namely, the static $\bar{X}$ & S charts, the WL control chart, the VSSI $\bar{X}$ & S charts, the 3- $CUSUM$ scheme and the AL chart. In the numerical tests, the charting parameters developed for independent observations under the same specifications of ($n = 5$, $\tau = 400$, $D_\mu = 1.0$) are all adopted. The ARL results of the static $\bar{X}$ & S charts, the WL control chart and VSSI $\bar{X}$ & S charts are obtained from calculation using the residuals distribution, while the ARL of the 3- $CUSUM$ scheme and the AL chart are evaluated by

simulation with 10,000 iterations. The simulated data of AR(1) model is generated by Routine DRNARM of IMSL Numerical Libraries (1997). The results are listed in Table 5-1, where notations (T) and (R) represent a specific control chart by traditional residuals approach and by revised residuals approach respectively. The charts using revised residuals approach are shaded.

Table 5-1: ARL Values of Two Approaches ($\phi = 0.5$, $n = 5$, $\tau = 400$, $D_\mu = 1$)

		δ_μ										
δ_σ	chart	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
1.0	$\bar{X}S(T)$	400.01	317.98	186.34	99.45	53.29	29.65	17.32	10.67	6.93	4.74	3.42
	$\bar{X}S(R)$	400.01	297.04	154.67	74.89	37.38	19.81	11.24	6.84	4.46	3.11	2.30
	$WL(T)$	399.66	313.61	181.67	97.65	53.28	30.25	18.00	11.24	7.37	5.07	3.66
	$WL(R)$	399.66	291.92	150.03	73.07	36.97	19.86	11.38	6.96	4.55	3.17	2.35
	VSSI(T)	400.92	325.29	175.88	69.70	24.70	9.79	5.07	3.42	2.69	2.30	2.04
	VSSI(R)	400.92	305.53	145.64	51.82	17.30	6.88	3.73	2.62	2.12	1.84	1.67
	3-C(T)	401.95	237.64	85.89	31.79	14.64	8.36	5.63	4.16	3.33	2.81	2.41
	3-C(R)	395.67	210.50	63.42	22.16	10.30	6.13	4.30	3.34	2.71	2.31	2.02
	AL(T)	403.66	224.14	76.15	29.37	14.10	8.28	5.62	4.17	3.26	2.70	2.30
	AL(R)	406.98	188.59	55.65	20.16	10.01	6.09	4.28	3.21	2.61	2.17	1.87
1.2	$\bar{X}S(T)$	52.45	48.36	38.77	28.39	19.92	13.83	9.70	6.94	5.11	3.87	3.02
	$\bar{X}S(R)$	52.45	47.06	35.35	24.08	15.86	10.50	7.13	5.01	3.66	2.79	2.21
	$WL(T)$	47.17	42.77	33.15	23.67	16.49	11.55	8.24	6.03	4.54	3.51	2.80
	$WL(R)$	47.17	41.43	29.95	19.92	13.08	8.77	6.07	4.36	3.26	2.53	2.04
	VSSI(T)	30.74	28.09	21.46	14.07	8.58	5.40	3.77	2.94	2.47	2.19	1.99
	VSSI(R)	30.74	27.48	19.85	12.20	7.10	4.40	3.09	2.43	2.06	1.84	1.68
	3-C(T)	25.48	23.73	18.94	13.62	9.52	6.85	5.15	4.01	3.31	2.82	2.43
	3-C(R)	25.49	23.19	17.27	11.71	7.80	5.47	4.14	3.28	2.72	2.34	2.06
	AL(T)	25.19	22.09	16.34	11.29	7.80	5.80	4.51	3.62	3.00	2.56	2.22
	AL(R)	25.12	21.45	14.29	9.47	6.36	4.72	3.66	2.93	2.44	2.10	1.85
1.4	$\bar{X}S(T)$	14.46	13.97	12.67	10.91	9.06	7.35	5.92	4.76	3.87	3.18	2.65
	$\bar{X}S(R)$	14.46	13.80	12.11	9.98	7.91	6.16	4.80	3.77	3.02	2.47	2.07
	$WL(T)$	13.01	12.46	11.03	9.25	7.51	6.04	4.86	3.94	3.25	2.71	2.31
	$WL(R)$	13.01	12.27	10.47	8.37	6.51	5.05	3.95	3.15	2.57	2.14	1.83
	VSSI(T)	5.56	5.40	4.96	4.36	3.73	3.18	2.75	2.44	2.20	2.03	1.90
	VSSI(R)	5.56	5.36	4.83	4.12	3.43	2.87	2.45	2.15	1.94	1.78	1.66
	3-C(T)	8.34	8.07	7.53	6.73	5.90	4.97	4.21	3.63	3.08	2.71	2.41
	3-C(R)	8.29	8.04	7.43	6.32	5.27	4.38	3.62	3.07	2.64	2.31	2.05
	AL(T)	7.67	7.40	6.79	5.82	4.94	4.14	3.57	3.06	2.65	2.32	2.08
	AL(R)	7.79	7.33	6.47	5.44	4.45	3.66	3.04	2.61	2.26	1.98	1.76

1.6	$\bar{X}S(T)$	6.32	6.22	5.92	5.48	4.96	4.41	3.87	3.38	2.96	2.59	2.28
	$\bar{X}S(R)$	6.32	6.18	5.78	5.22	4.58	3.95	3.38	2.89	2.48	2.15	1.89
	$WL(T)$	5.79	5.67	5.33	4.85	4.31	3.78	3.30	2.88	2.52	2.23	1.98
	$WL(R)$	5.79	5.63	5.18	4.59	3.96	3.37	2.88	2.47	2.14	1.87	1.67
	VSSI(T)	2.78	2.75	2.68	2.58	2.46	2.32	2.19	2.07	1.97	1.87	1.79
	VSSI(R)	2.78	2.74	2.65	2.52	2.36	2.20	2.05	1.92	1.80	1.70	1.62
	3-C(T)	4.88	4.81	4.68	4.43	4.11	3.78	3.41	3.11	2.80	2.53	2.31
	3-C(R)	4.85	4.81	4.65	4.25	3.91	3.51	3.11	2.79	2.47	2.21	2.00
	AL(T)	4.34	4.22	4.05	3.82	3.48	3.12	2.82	2.56	2.29	2.10	1.91
1.8	AL(R)	4.29	4.22	3.97	3.59	3.27	2.90	2.57	2.27	2.04	1.85	1.67
	$\bar{X}S(T)$	3.67	3.64	3.54	3.39	3.20	2.99	2.77	2.55	2.33	2.14	1.96
	$\bar{X}S(R)$	3.67	3.63	3.50	3.30	3.06	2.79	2.54	2.29	2.07	1.88	1.71
	$WL(T)$	3.43	3.39	3.28	3.11	2.90	2.68	2.46	2.26	2.07	1.90	1.75
	$WL(R)$	3.43	3.37	3.22	3.01	2.76	2.50	2.25	2.03	1.84	1.68	1.54
	VSSI(T)	2.12	2.11	2.09	2.05	2.01	1.96	1.91	1.85	1.80	1.74	1.69
	VSSI(R)	2.12	2.11	2.07	2.03	1.97	1.90	1.83	1.76	1.69	1.63	1.57
	3-C(T)	3.56	3.53	3.49	3.38	3.20	3.05	2.86	2.68	2.51	2.33	2.18
	3-C(R)	3.58	3.56	3.44	3.30	3.10	2.92	2.69	2.47	2.29	2.09	1.94
2.0	AL(T)	2.98	2.93	2.91	2.79	2.64	2.46	2.34	2.17	2.01	1.88	1.75
	AL(R)	2.99	2.95	2.84	2.71	2.54	2.34	2.18	1.99	1.85	1.70	1.58
	$\bar{X}S(T)$	2.55	2.54	2.50	2.43	2.35	2.25	2.15	2.04	1.93	1.82	1.72
	$\bar{X}S(R)$	2.55	2.53	2.48	2.39	2.28	2.16	2.03	1.90	1.78	1.66	1.56
	$WL(T)$	2.41	2.40	2.35	2.28	2.19	2.09	1.98	1.87	1.76	1.66	1.57
	$WL(R)$	2.41	2.39	2.33	2.24	2.12	1.99	1.87	1.74	1.63	1.52	1.43
	VSSI(T)	1.84	1.84	1.83	1.82	1.79	1.77	1.74	1.71	1.68	1.65	1.61
	VSSI(R)	1.84	1.84	1.82	1.80	1.77	1.73	1.69	1.65	1.61	1.57	1.53
	3-C(T)	2.86	2.86	2.84	2.78	2.69	2.62	2.50	2.39	2.24	2.14	2.03
	3-C(R)	2.90	2.87	2.83	2.73	2.65	2.53	2.38	2.23	2.11	1.98	1.86
	AL(T)	2.28	2.28	2.25	2.21	2.12	2.03	1.96	1.87	1.79	1.70	1.61
	AL(R)	2.32	2.29	2.23	2.15	2.08	1.98	1.87	1.77	1.66	1.57	1.48

From the ARL values displayed in the table, we can find that the revised residuals approach is always better than the traditional residuals approach for nonzero mean shift. And the advantages of the revised residuals approach are more evident for larger and dominant mean shifts. Based on these findings, more numerical test results are presented using the style of Table 4-2. The measure of $MRARL$ as in (4-31) is used. The values of 0.3, 0.6 and 0.9 are chosen to represent different levels of ϕ from low to high.

Table 5-2: *MRARL* Values for Four Charts ($\phi = 0.3, D_\mu = 1$)

Chart	Small shift ($0 \leq \delta_\mu \leq 1, 1 \leq \delta_\sigma \leq 1.6$, excluding $\delta_\mu = 0$ & $\delta_\sigma = 1$)			Whole shift ($0 \leq \delta_\mu \leq 2, 1 \leq \delta_\sigma \leq 2$ excluding $\delta_\mu = 0$ & $\delta_\sigma = 1$)		
	δ_μ (1)	δ_σ (2)	All shift (3)	δ_μ (4)	δ_σ (5)	All shift (6)
RUN(0): $n = 5, \tau = 400$						
Static $\bar{X}\&S$	2.9579	1.8182	2.0742	2.1890	1.5582	1.5064
<i>WL</i>	2.9680	1.6440	1.8782	2.2300	1.4250	1.3814
VSSI $\bar{X}\&S$	1.6620	0.8655	1.0368	1.2810	0.8208	0.9534
3- <i>C</i>	1.0579	1.0904	1.1348	1.0613	1.1414	1.1680
RUN(1): $n = 3, \tau = 200$						
Static $\bar{X}\&S$	2.5394	1.4420	1.7531	2.0389	1.3193	1.3819
<i>WL</i>	2.4917	1.3446	1.6376	2.0175	1.2361	1.2966
VSSI $\bar{X}\&S$	1.5978	0.8516	1.0656	1.3329	0.8495	1.0557
3- <i>C</i>	0.9338	1.2346	1.1221	0.9688	1.2636	1.1363
RUN(2): $n = 3, \tau = 600$						
Static $\bar{X}\&S$	4.1813	1.8341	2.4692	3.1395	1.5976	1.7482
<i>WL</i>	4.1539	1.6477	2.2770	3.1488	1.4413	1.6151
VSSI $\bar{X}\&S$	2.0412	0.8788	1.1798	1.5440	0.8396	1.0823
3- <i>C</i>	1.0221	1.2470	1.1765	1.0414	1.2753	1.1867
RUN(3): $n = 7, \tau = 200$						
Static $\bar{X}\&S$	2.0587	1.4897	1.6176	1.6027	1.3127	1.2685
<i>WL</i>	2.1242	1.3526	1.4741	1.6756	1.2112	1.1820
VSSI $\bar{X}\&S$	1.3592	0.8531	0.9570	1.1382	0.8280	0.9177
3- <i>C</i>	0.9722	1.1028	1.0997	0.9848	1.1261	1.1036
RUN(4): $n = 7, \tau = 600$						
Static $\bar{X}\&S$	3.0986	1.9583	2.1869	2.1983	1.6316	1.5291
<i>WL</i>	3.2058	1.7608	1.9573	2.3122	1.4883	1.3949
VSSI $\bar{X}\&S$	1.3818	0.8512	0.9499	1.1607	0.8422	0.9649
3- <i>C</i>	1.2325	1.0639	1.1815	1.1267	1.1109	1.1581

Table 5-3: *MRARL* Values for Four Charts ($\phi = 0.6$, $D_\mu = 1$)

Chart	Small shift ($0 \leq \delta_\mu \leq 1$, $1 \leq \delta_\sigma \leq 1.6$, excluding $\delta_\mu = 0$ & $\delta_\sigma = 1$)			Whole shift ($0 \leq \delta_\mu \leq 2$, $1 \leq \delta_\sigma \leq 2$ excluding $\delta_\mu = 0$ & $\delta_\sigma = 1$)		
	δ_μ (1)	δ_σ (2)	All shift (3)	δ_μ (4)	δ_σ (5)	All shift (6)
RUN(0): $n = 5$, $\tau = 400$						
Static $\bar{X}\&S$	3.0126	1.8050	2.1246	2.5595	1.5487	1.6326
WL	2.9285	1.6320	1.9060	2.5037	1.4164	1.4718
VSSI $\bar{X}\&S$	2.1101	0.8593	1.1602	1.4888	0.8162	0.9585
3-C	1.0952	1.0689	1.1403	1.0669	1.1261	1.1761
RUN(1): $n = 3$, $\tau = 200$						
Static $\bar{X}\&S$	2.5288	1.4506	1.7547	2.2614	1.3170	1.4517
WL	2.4394	1.3527	1.6310	2.1658	1.2338	1.3457
VSSI $\bar{X}\&S$	1.9786	0.8567	1.1688	1.5629	0.8470	1.0744
3-C	0.9473	1.2420	1.1361	0.9681	1.2582	1.1545
RUN(2): $n = 3$, $\tau = 600$						
Static $\bar{X}\&S$	4.1198	1.8623	2.4770	3.5407	1.6142	1.8774
WL	3.9880	1.6730	2.2575	3.4034	1.4562	1.7017
VSSI $\bar{X}\&S$	2.7306	0.8931	1.3765	1.9557	0.8479	1.1376
3-C	1.0450	1.2692	1.2004	1.0369	1.2844	1.2053
RUN(3): $n = 7$, $\tau = 200$						
Static $\bar{X}\&S$	2.1374	1.4778	1.6601	1.8316	1.3043	1.3461
WL	2.1290	1.3420	1.4992	1.8543	1.2036	1.2336
VSSI $\bar{X}\&S$	1.5996	0.8455	1.0159	1.2034	0.8225	0.8951
3-C	1.0131	1.0941	1.1112	0.9951	1.1194	1.1179
RUN(4): $n = 7$, $\tau = 600$						
Static $\bar{X}\&S$	3.2303	1.9620	2.2806	2.6369	1.6321	1.6821
WL	3.2016	1.7640	2.0155	2.6641	1.4885	1.5045
VSSI $\bar{X}\&S$	1.7564	0.8527	1.0500	1.2752	0.8420	0.9406
3-C	1.3034	1.0593	1.1983	1.1715	1.1025	1.1770

Table 5-4: *MRARL* Values for Four Charts ($\phi = 0.9$, $D_\mu = 1$)

Chart	Small shift ($0 \leq \delta_\mu \leq 1$, $1 \leq \delta_\sigma \leq 1.6$, excluding $\delta_\mu = 0$ & $\delta_\sigma = 1$)			Whole shift ($0 \leq \delta_\mu \leq 2$, $1 \leq \delta_\sigma \leq 2$ excluding $\delta_\mu = 0$ & $\delta_\sigma = 1$)		
	δ_μ (1)	δ_σ (2)	All shift (3)	δ_μ (4)	δ_σ (5)	All shift (6)
RUN(0): $n = 5$, $\tau = 400$						
Static $\bar{X}\&S$	2.5774	1.8261	2.0267	3.0905	1.5654	1.7799
WL	2.3841	1.6511	1.8066	2.6571	1.4316	1.5505
VSSI $\bar{X}\&S$	2.3826	0.8690	1.2388	2.2685	0.8243	1.0844
3-C	1.1539	1.0895	1.1332	1.1454	1.1463	1.1866
RUN(1): $n = 3$, $\tau = 200$						
Static $\bar{X}\&S$	2.3109	1.4393	1.7015	2.5109	1.3142	1.5226
WL	2.1774	1.3421	1.5752	2.2297	1.2313	1.3822
VSSI $\bar{X}\&S$	2.3277	0.8496	1.2639	2.2741	0.8456	1.1960
3-C	0.9655	1.2336	1.1581	0.9910	1.2607	1.1867
RUN(2): $n = 3$, $\tau = 600$						
Static $\bar{X}\&S$	3.7156	1.8551	2.3797	4.1015	1.6142	2.0259
WL	3.4463	1.6663	2.1360	3.5373	1.4561	1.7642
VSSI $\bar{X}\&S$	3.6381	0.8867	1.6175	3.4361	0.8472	1.4053
3-C	1.0939	1.2641	1.2166	1.0801	1.2850	1.2315
RUN(3): $n = 7$, $\tau = 200$						
Static $\bar{X}\&S$	1.8644	1.4744	1.5939	2.1921	1.3022	1.4447
WL	1.7532	1.3390	1.4318	1.9480	1.2017	1.2834
VSSI $\bar{X}\&S$	1.6549	0.8434	1.0336	1.5368	0.8212	0.9334
3-C	1.0651	1.0912	1.1114	1.0782	1.1199	1.1410
RUN(4): $n = 7$, $\tau = 600$						
Static $\bar{X}\&S$	2.6595	1.9536	2.1500	3.3836	1.6281	1.8852
WL	2.4407	1.7569	1.8840	2.8925	1.4852	1.6129
VSSI $\bar{X}\&S$	2.2438	0.8491	1.1811	2.0381	0.8404	1.0504
3-C	1.3786	1.0645	1.1793	1.3549	1.1065	1.2025

In Tables (4-2, 5-2 to 5-4), $MRARL(n)$ represents the $MRARL$ values in the n th column.

From the data on these tables, several findings can be summarized.

1. Ratios in variance shift ($MRARL(2)$ and $MRARL(5)$) stay all the same, i.e. unaffected by ϕ .
2. The rankings in the ratios of overall shift relative to the AL chart ($MRARL(6)$) of the static $\bar{X}$ & S , the WL chart and the 3- $CUSUM$ scheme stay the same for $\phi(0.3, 0.6, 0.9)$ as for the independent case. For the ϕ values considered here, the $MRARL(6)$ ratios increase with the increase of the ϕ value. It means that the AL chart becomes more powerful in overall performance as $AR(1)$ degree becomes higher. It is mainly because the actual mean shift becomes smaller for higher degree of positive ϕ value. The $MRARL(6)$ ratios indicate that all of the three static charts are still inferior to the AL chart.
3. By examining the $MRARL(6)$ ratios of independent and autocorrelated conditions, the VSSI $\bar{X}$ & S charts perform better for larger sample size in $RUN(3, 4)$ and are less effective for smaller sample size in $RUN(1, 2)$. Generally, the VSSI $\bar{X}$ & S charts have smaller ARL for dominant variance shift and/or medium size mean shift, but the AL chart takes advantage in all other situations. It is evident that the VSSI $\bar{X}$ & S charts perform poorly for high degree of positive ϕ (Table 5-4) which actually produces smaller mean shift.

The VSSI WLC and VSSI 3- $CUSUM$ schemes using residuals approach can be studied in the same way, and their performance under different degrees of positive ϕ is mainly affected by their effectiveness to detect mean shifts of smaller magnitudes in the independent cases. However, numerical experiments for the VSSI WLC and VSSI 3- $CUSUM$ schemes have not been carried out in this thesis.

5.5 Other Autocorrelation Models

Autoregressive model of order two (AR(2)) and first-order autoregressive and first-order moving average ARMA(1,1) model are two other time series models of considerable practical importance (Box and Jenkins 1976). In this section, the revised residuals approach for AR(2) is developed first. Then some issues for ARMA(1,1) are addressed.

With a few exceptions, the notations used for AR(1) model are still adopted here. For the AR(2) process, model (5-1) need to be modified as follows.

$$Y_{i,j} = \mu_0 + \phi_1(Y_{i,j-1} - \mu_0) + \phi_2(Y_{i,j-2} - \mu_0) + \varepsilon_{i,j} \quad (5-15)$$

where ϕ_2 is the second order autoregressive parameter. The stationary condition is to jointly satisfy: $\phi_1 + \phi_2 < 1$, $\phi_1 - \phi_2 > -1$ and $-1 < \phi_2 < 1$.

The shift model (5-2) is still valid, except the variance of the in-control process (Box and Jenkins 1976)

$$\gamma_0 = \left(\frac{1 - \phi_2}{1 + \phi_2} \right) \left[\frac{\sigma_0^2}{(1 - \phi_2)^2 - \phi_1^2} \right] \quad (5-16)$$

Some other equations for the AR(2) process are derived below, corresponding to equations (5-9) to (5-13).

$$\hat{X}_{i,j} = \begin{cases} \mu_0 & \text{for } j = 1 \\ \mu_0 + \phi_1(X_{i,j-1} - \mu_0) + \phi_2(X_{i,j-2} - \mu_0) & \text{for } j = 2, \dots, n \end{cases} \quad (5-17)$$

$$X_{i,j} - \hat{X}_{i,j} = \begin{cases} \delta_\sigma(Y_{i,j} - \mu_0) + \delta_\mu\sqrt{\gamma_0} & \text{for } j = 1 \\ \delta_\sigma\varepsilon_{i,j} + \delta_\mu\sqrt{\gamma_0}(1 - \phi_1 - \phi_2) & \text{for } j = 2, \dots, n \end{cases} \quad (5-18)$$

$$\hat{\varepsilon}_{i,j} = \begin{cases} \sqrt{\left(\frac{1+\phi_2}{1-\phi_2}\right)\left[(1-\phi_2)^2 - \phi_1^2\right]}(X_{i,1} - \hat{X}_{i,1})/\sigma_0 & \text{for } j = 1 \\ (X_{i,j} - \hat{X}_{i,j})/\sigma_0 & \text{for } j = 2, \dots, n \end{cases} \quad (5-19)$$

$$\bar{\varepsilon}_i \sim N\left(\frac{\delta_\mu}{n} \left[1 + (n-1) \sqrt{\frac{(1-\phi_2-\phi_1)(1-\phi_2)}{(1-\phi_2+\phi_1)(1+\phi_2)}}\right], \frac{\delta_\sigma^2}{n}\right) \quad (5-20)$$

$$\frac{(n-1)}{\delta_\sigma^2} \hat{S}_i^2 \sim \chi_{n-1}^2 \text{ with } v = \frac{n-1}{n} \left(\frac{\delta_\mu}{\delta_\sigma}\right)^2 \left[1 - \sqrt{\frac{(1-\phi_2-\phi_1)(1-\phi_2)}{(1-\phi_2+\phi_1)(1+\phi_2)}}\right]^2 \quad (5-21)$$

The predictor of traditional residuals approach is,

$$\hat{X}_{i,j} = \mu_0 + \phi_1(Y_{i,j-1} - \mu_0) + \phi_2(Y_{i,j-2} - \mu_0) \quad \text{for all } j \quad (5-22)$$

The distribution of $\bar{\varepsilon}_i$ and $\hat{S}_i^2$ are obtained below.

$$\bar{\varepsilon}_i \sim N\left(\delta_\mu \sqrt{\frac{(1-\phi_2-\phi_1)(1-\phi_2)}{(1-\phi_2+\phi_1)(1+\phi_2)}}, \frac{\delta_\sigma^2}{n}\right) \text{ and } \frac{(n-1)}{\delta_\sigma^2} \hat{S}_i^2 \sim \chi_{n-1}^2 \quad (5-23)$$

Some conclusions can be drawn by following the same procedure of analysis for AR(1).

The revised residuals approach is consistently better than or equally powerful as the

traditional residuals approach when $\sqrt{\frac{(1-\phi_2-\phi_1)(1-\phi_2)}{(1-\phi_2+\phi_1)(1+\phi_2)}}$ in (5-23) is smaller than one.

The condition to satisfy this is,

$$-\frac{1}{4} < \phi_1 < 1 - \phi_2, \quad \frac{1 - \sqrt{1 + 4\phi_1}}{2} < \phi_2 < \frac{1 + \sqrt{1 + 4\phi_1}}{2} \text{ and } -1 < \phi_2 < 1. \quad (5-24)$$

It has been verified with numerical tests. Another issue to be noted is the starting

problem. Since the predictor of $\hat{X}_{i,j}$ is composed of observations $X_{i,j-1}$ and $X_{i,j-2}$,

therefore, an additional observation need to be made to calculate the second residual in

the revised residuals approach. The traditional residuals approach needs two additional

observations.

For the ARMA(1,1) process, Knoth's shift model can still be adopted with some modification on γ_0 . The ARMA(1,1) time series model for the shift model (5-2) is expressed as follows

$$Y_{i,j} = \mu_0 + \phi(Y_{i,j-1} - \mu_0) + (1 - \theta B)\varepsilon_{i,j}. \quad (5-25)$$

where B is a backshift operator such that $B\varepsilon_{i,j} = \varepsilon_{i,j-1}$, θ is the moving average parameter, and $\varepsilon_{i,j}$'s are independent normal random errors with variance σ_0^2 . In the literature, some other models for ARMA(1,1) are also available. Here they are briefly introduced and their relationship with the Knoth's model is clarified.

The first one is the standard model by (Box and Jenkins 1976), which can be written as

$$X_{i,j} = \mu_i + \phi(X_{i,j-1} - \mu_i) + (1 - \theta B)\rho_{i,j} \quad \text{and} \quad \mu_i = \begin{cases} \mu_0 & \text{for } i < q \\ \mu_\delta & \text{for } i \geq q \end{cases} \quad (5-26)$$

where μ_δ is the shifted process mean, $\rho_{i,j}$'s are the independent normal random variables with variance σ_ρ^2 , where

$$\sigma_\rho^2 = \begin{cases} \sigma_0^2 & \text{for } i < q \\ (\delta_\sigma \sigma_0)^2 & \text{for } i \geq q \end{cases}. \quad (5-27)$$

The second model called AR(1) plus a random error model is also used in a number of papers (MacGregor and Harris (1993) and Lu and Reynolds (1999)). Some additional details and extensions pertaining to this AR(1) plus error model are given in Reynolds *et al.* (1996). Briefly, the model can be expressed as

$$X_{i,j} = \zeta_{i,j} + \iota_{i,j}, \quad \zeta_{i,j} = (1 - \phi)\mu_i + \phi\zeta_{i,j-1} + \pi_{i,j} \quad \text{and} \quad \mu_i = \begin{cases} \mu_0 & \text{for } i < q \\ \mu_\delta & \text{for } i \geq q \end{cases} \quad (5-28)$$

where $\zeta_{i,j}$ follows an AR(1) process, $t_{i,j}$'s are independent normal random errors with a mean of zero and variance σ_t^2 . $\pi_{i,j}$'s are independent normal random variables with mean of zero and variance σ_π^2 .

Some studies (Reynolds *et al.* (1996) and Lu and Reynolds (1999a, b)) have shown that the AR(1) process with an additional random error is equivalent to an ARMA(1,1) process and relatively simple equations are available for transformation between each other. Here the equivalence of Knoth's model and the standard model (5-26) is proved for the traditional residuals approach. These two models use the same Minimum Mean Square error (MMS) predictor,

$$\hat{X}_{i,j} = \mu_0 + \phi(X_{i,j-1} - \mu_0) - \theta(X_{i,j-1} - \hat{X}_{i,j-1}) \quad (5-29)$$

Residuals from Knoth's model is

$$\begin{aligned} X_{i,j} - \hat{X}_{i,j} &= \delta_\sigma(Y_{i,j} - \mu_0) + \mu_0 + \delta_\mu\sqrt{\gamma_0} - [\mu_0 + \phi(X_{i,j-1} - \mu_0) - \theta(X_{i,j-1} - \hat{X}_{i,j-1})] \\ &= \delta_\sigma(Y_{i,j} - \mu_0) + \mu_0 + \delta_\mu\sqrt{\gamma_0} \\ &\quad - \left\{ \mu_0 + \phi[\delta_\sigma(Y_{i,j-1} - \mu_0) + \mu_0 + \delta_\mu\sqrt{\gamma_0} - \mu_0] - \theta(X_{i,j-1} - \hat{X}_{i,j-1}) \right\} \\ &= \delta_\sigma[Y_{i,j} - \mu_0 - \phi(Y_{i,j-1} - \mu_0)] + \delta_\mu\sqrt{\gamma_0}(1 - \phi) + \theta(X_{i,j-1} - \hat{X}_{i,j-1}) \\ &= \delta_\sigma(\varepsilon_{i,j} - \theta\varepsilon_{i,j-1}) + \delta_\mu\sqrt{\gamma_0}(1 - \phi) + \theta(X_{i,j-1} - \hat{X}_{i,j-1}) \\ &= \delta_\sigma(1 - \theta B)\varepsilon_{i,j} + \delta_\mu\sqrt{\gamma_0}(1 - \phi) + \theta(X_{i,j-1} - \hat{X}_{i,j-1}) \\ \varepsilon_{i,j} &\sim N(0, \delta_\sigma^2) \end{aligned}$$

Residuals from standard model is

$$\begin{aligned} X_{i,j} - \hat{X}_{i,j} &= X_{i,j} - [\mu_0 + \phi(X_{i,j-1} - \mu_0) - \theta(X_{i,j-1} - \hat{X}_{i,j-1})] \\ &= \mu_\delta + \phi(X_{i,j-1} - \mu_\delta) + \rho_{i,j} - \theta\varepsilon_{i,j-1} \\ &\quad - [\mu_0 + \phi(X_{i,j-1} - \mu_0) - \theta(X_{i,j-1} - \hat{X}_{i,j-1})] \\ &= \rho_{i,j} - \theta\rho_{i,j-1} + (\mu_\delta - \mu_0)(1 - \phi) + \theta(X_{i,j-1} - \hat{X}_{i,j-1}) \\ &= (1 - \theta B)\rho_{i,j} + (\mu_\delta - \mu_0)(1 - \phi) + \theta(X_{i,j-1} - \hat{X}_{i,j-1}) \\ \rho_{i,j} &\sim N(0, \delta_\sigma^2\delta_0^2) \end{aligned}$$

As $(\mu_\delta - \mu_0) = \delta_\mu \sqrt{\gamma_0}$ in default, the results of the traditional residuals approach by using the two models are equivalent. Hence the Knoth's model is also equivalent to the AR(1) plus error model.

Now we can use this relationship to make some inferences for the Knoth's model. As for the AR(1) plus error model, it was proved by Lu and Reynolds (1999b) that a shift in the variance will make the residuals still correlated. Thus the same conclusion can be inferred for Knoth's model. This causes complication to the analytical method to obtain *ARL*. It is also noted that the moving average term brings certain complexity to current residuals approaches.

5.6 Summary and Discussion

In this chapter the application of control charts for both mean and variance shifts under autocorrelation was studied. Two residuals approaches are focused here, namely, the traditional residuals approach and the revised residuals approach. For the AR(1) process, the two approaches are compared from the residuals distribution and numerical experiments for five control schemes. It is shown that the revised residuals approach is better than the traditional residuals approach for positive AR(1) parameter as long as a mean shift exists. For the variance shift both approaches have the same performance. In addition, the revised residuals approach can handle the starting problem. However, the independence assumption between samples still has to be carefully checked. Based on the residuals distribution, the condition to choose between the residuals approach is also derived for AR(2) process. Moreover, the Knoth's model for ARMA(1,1) process is associated with the standard ARMA(1,1) model.

The residuals approach can be easily implemented for all control charts considered here. The results of the comparative studies for the AR(1) parameter show that the *AL* chart still extensively outperforms the static control charts considered, i.e. the static $\bar{X}$ &*S*, the *WL* chart and the 3-*CUSUM* scheme. Moreover, the degree of superiority of the *AL* chart increases with the increase of the magnitude of the positive AR(1) parameter. On the other hand, the overall performance of the VSSI $\bar{X}$ &*S* charts under positive AR(1) is better for larger sample size and poorer for smaller sample size than that of the *AL* chart, like in the independent cases. Generally, the VSSI $\bar{X}$ &*S* charts have smaller *ARL* for dominant variance shift and/or medium size mean shift, but the *AL* chart takes advantage in all other situations. It is evident that the VSSI $\bar{X}$ &*S* charts perform poorly for high degree of positive ϕ which actually produces a smaller mean shift. In summary, the effectiveness of the *AL* chart in monitoring AR(1) processes is still favorable.

Chapter 6

Conclusions

6.1 Conclusions

This thesis presents some advances in control chart techniques for joint monitoring process mean and variance. These techniques are developed based on the WL chart and the WLC (WL -CUSUM) chart.

In the category of the variable sampling schemes, the VSSI WLC scheme and the VSSI 3-CUSUM scheme are proposed and their designs are described in detail. As the special cases of VSSI WLC scheme, the VSI WLC scheme and the VSS WLC scheme are also discussed. The optimization design is carried out to minimize the out-of-control ATS at a specified design point, meanwhile, using the in-control ATS_0 and inspection rate R as the constraints. The VSSI 3-CUSUM, VSSI WLC and VSI WLC schemes are generally more powerful than the static and adaptive $\bar{X}$ & S charts in detecting the process shifts in mean and/or variance. However, it is strongly recommended to use the VSI WLC scheme, because it is not only much easier to design and operate than the VSSI 3-CUSUM scheme but also has a slight edge over the latter in ATS performance from an overall viewpoint. On the other hand, the VSI WLC scheme is almost as powerful as the VSSI WLC scheme, and the former does not need the adaptation of sample size. Furthermore, it has been shown that the VSI WLC and VSSI WLC schemes are more capable for diagnosing the joint mean and variance process shift than the $\bar{X}$ & S charts, and the VSSI 3-CUSUM scheme is the most capable for pure mean or variance shifts.

An internal adaptive control chart, the *AL* chart, for monitoring the process shifts in mean and variance simultaneously is also developed. The *AL* chart uses fixed sampling interval and sample size, but adapts weighting factor a by using computer software. The *AL* chart is preferred from an operational and managerial viewpoint because of the ease of use. The overall effectiveness of the *AL* chart is not only substantially higher than that of the static $\bar{X}$ & S charts but also better than that of the static 3-*CUSUM* scheme, and even competitive with the VSSI $\bar{X}$ & S . For diagnosing shift type, the *AL* chart shows a good balance between diagnosing joint shift and pure mean or variance shifts.

The application of some static and adaptive control charts to autocorrelated process with underlying AR(1) model is studied for joint monitoring purpose. Two residuals approaches are compared by both statistical analysis and numerical experiments. It is found that the revised residuals approach is to be favored for positive AR(1) process. Through extensive numerical studies, it is found that the overall performance of the *AL* chart varies to some extent under different degrees of positive AR(1) autocorrelation, but its leading position compared to other charts is unchanged. The situations for AR(2) and AR(1, 1) models are also investigated.

6.2 Contributions

This thesis has made following original contributions to the literature and practice of quality engineering.

Variable sampling schemes:

- (1) It is the first time to apply adaptive features (VSI, VSS and VSSI) to *CUSUM* control charts for monitoring process mean and variance simultaneously. Compared

with the VSSI $\bar{X}$ & S charts, all of the VSSI 3-CUSUM, VSSI WLC and VSI WLC schemes are able to improve the detection effectiveness to a substantial degree without increasing the false alarm rate and the inspection rate.

- (2) Since the VSSI WLC scheme only uses a single chart to monitor the process mean and variance, it significantly simplifies the design and operation of the chart compared to the combined CUSUM scheme.
- (3) The design procedure of the VSSI WLC scheme can be applied to the designs of the VSS WLC scheme and the VSI WLC scheme with minor modifications. It is discovered that the VSI WLC scheme also has very high detection power and is relatively easier to implement, and therefore may be the most desirable adaptive chart for joint monitoring two-sided mean shifts and increasing variance shift.

Internal adaptive control chart:

It is the first time that an internal adaptive control chart (the AL chart) is developed for monitoring both sample mean and variance. Compared with the variable sampling schemes, the adaptation of the AL chart can be conducted on-line by a software system and, thus, makes adaptive scheme more applicable. In comparison with the other static control charts with fixed sample size and sampling interval (the $\bar{X}$ & S charts, the WL chart and the static 3-CUSUM scheme), the superiority of AL chart is notable in terms of ARL performance. Moreover, it is even competitive with the VSSI $\bar{X}$ & S charts, which requires the changes of both sampling intervals and sample sizes from time to time. The adaptation and diagnosis methods of the AL chart are also original in the field.

Studies for autocorrelated data:

It is the first time the two kind of residuals approaches under AR(1) and AR(2) models are compared for joint monitoring process mean and variance from the statistical analysis. The analytical conclusions for the two models have also been verified by numerical studies. The guidelines of selection of the two approaches under AR(1) and AR(2) models are given. Using the residuals approach, the adaptive control charts developed in this thesis can also be used to monitor autocorrelated process. Specifically, the performance of the *AL* chart is compared with that of some other charts under different degrees of positive AR(1) autocorrelation. The results show that the *AL* chart still outperforms other static charts in these studies.

6.3 Future Research

There are some areas for future research. These areas include a scheme integrating the CUSUM technique with the *AL* chart, the diagnosis method of the *AL* chart for change point problem, the investigation of the between-sample independence assumption made in the revised residuals approach and the concerns of decreasing variance shift.

1. From the results in Chapter 4, it is obvious that the 3-*CUSUM* scheme (stemming from the static $\bar{X}$ & *S* charts) is significantly more powerful than the static $\bar{X}$ & *S* charts (as shown in Tables 4-1 and 4-2). It brings about a prospect that a CUSUM chart based on *AL* (called the *AL*-CUSUM chart) may be more effective than the *AL* chart. In fact, since even the basic *AL* chart is considerably more effective than the 3-*CUSUM* scheme, the *AL*-CUSUM chart is likely to excel any existing chart with fixed sample size and sampling interval for monitoring the two-sided mean shifts and increasing variance shift. Generally speaking, we can design the *AL*-CUSUM

chart by using AL as the statistic for the cumulative sum procedure. The updated information from updating procedure (4-8) can be utilized to adjust not only the weighting factor a and control limit but also the reference value in order to achieve an even better performance.

2. There has been very little discussion about change point problem for control chart jointly monitoring mean and variance, except a recent one by Park and Park (2004). They developed a joint maximum likelihood estimator, which is applicable to all control charts. On the other hand, the Bayesian method can be used for estimation of change point (e.g. Smith (1975)). The performance of the Bayesian inference is partly affected by its prior probability. From our studies on the AL chart, we find the probabilities of the status points at the time of signaling can be used to estimate the process status. If this information is utilized as the prior probabilities of the Bayesian change point estimator, it might bring forth a new method for the diagnosis of change point.
3. In Chapter 5, the revised residuals approach is developed based on the assumption of the independency between samples. However, if this assumption is not satisfied, there will be some degree of autocorrelation in the residuals from the revised residuals approach, because of the starting predictor and its residual. But we have not found any available method to judge the validity of this assumption in practice. In the paper by Alwan and Radson (1992), they discussed the correlation of within-sample and between-sample. To handle the between-sample correlation, they recommended that the samples can be taken far enough apart so that they are “practically” independent, so-called practical independence. They defined a measure

for the practical independence and used it for the process mean of modified approach. It would be an interesting topic to develop a measure of practical independence for process mean and variance of the revised residuals approach.

4. Like most of the S chart and R chart, the WL statistic based control charts (VSSI WLC , AL chart, etc) only handle the increasing variance shift in this thesis. However, decreasing shift may be of interest in some applications. The WL chart is able to detect decreasing δ_σ (because WL becomes smaller for decreasing δ_σ), but not quite effective. It is a common problem for any other chart, due to the skewness of the distribution of S . It is recommended that effort be made to take into account of the two-sided variance shifts.
5. The performance of the control charts developed in this thesis is based on the normality assumption for observations. However, it's not the case in some situations. The sensitivity studies for other practical distributions will be investigated.
6. In addition to just the ARL or ATS measure, features such as variance or upper quantiles of the Run Length distributions will be considered to further evaluate the performance of the control charts developed.
7. For the time depended processes, the underlying autocorrelated structure may also shift due to the physical changes of a process. In response to this observation, the AR parameters may shift to other levels. In the future, out-of-control conditions can be extended to consider these shifts.

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Appendix A

Relationship between Standard Condition and General Condition

The relationship between these two conditions for the WL chart was proved by Tian (2003).

$$WL = \lambda s^2 + (1 - \lambda)(\bar{x} - \mu_0)^2 \quad (A1)$$

The WL formula can be written as,

$$WL = \sigma_0^2 w\ell \quad (A2)$$

$$w\ell = \lambda \left(\frac{s}{\sigma_0} \right)^2 + (1 - \lambda) \left(\frac{\bar{x} - \mu_0}{\sigma_0} \right)^2 \quad (A3)$$

Let z denote standard condition statistic, and x denote the general condition statistic.

Since the distributions of z and x are

$$z \sim N(\delta_\mu, \delta_\sigma^2), x \sim N(\mu_0 + \delta_\mu \sigma_0, \delta_\sigma^2 \sigma_0^2) \quad (A4)$$

The variable x and $\bar{x}$ can be expressed by z and $\bar{z}$ respectively as follows,

$$\begin{aligned} x &= \left(\frac{z - \delta_\mu}{\delta_\sigma} \right) \delta_\sigma \sigma_0 + (\mu_0 + \delta_\mu \sigma_0) \\ &= (z - \delta_\mu) \sigma_0 + \mu_0 + \delta_\mu \sigma_0 \\ &= z \times \sigma_0 + \mu_0 \end{aligned} \quad (A5)$$

$$\bar{x} = \bar{z} \times \sigma_0 + \mu_0 \quad (A6)$$

Let s^2 denote the variance of general condition statistic, and s_z^2 for standard condition statistic. Therefore, the relationship between s^2 and s_z^2 are found to be

$$\begin{aligned}
 s^2 &= \frac{\sum (x - \bar{x})^2}{n-1} \\
 &= \frac{\sum [(z \times \sigma_0 + \mu_0) - (\bar{z} \times \sigma_0 + \mu_0)]^2}{n-1} \\
 &= \frac{\sigma_0^2 \sum (z - \bar{z})^2}{n-1} = \sigma_0^2 s_z^2
 \end{aligned} \tag{A7}$$

By substituting (A6) and (A7) into equation (A3), the statistic $w\ell$ is changed to

$$\begin{aligned}
 w\ell &= \lambda \left(\frac{s}{\sigma_0} \right)^2 + (1-\lambda) \left(\frac{\bar{x} - \mu_0}{\sigma_0} \right)^2 \\
 &= \lambda \left(\frac{\sigma_0^2 s_z^2}{\sigma_0^2} \right) + (1-\lambda) \left(\frac{(\bar{z} - \delta_\mu) \sigma_0 + \mu_0 + \delta_\mu \sigma_0 - \mu_0}{\sigma_0} \right)^2 \\
 &= \lambda s_z^2 + (1-\lambda) \bar{z}^2 \\
 &= \lambda s_z^2 + (1-\lambda) (\bar{z} - \mu_{z0})^2
 \end{aligned} \tag{A8}$$

where $\mu_{z0} = 0$. Hence, the $w\ell$ is the standard condition form of WL . According to equation (A2), the relationship between the control limits under these two conditions can be obtained.

$$UCL_{WL} = \sigma_0^2 UCL_{standard}$$

In other words, $w\ell$ is a special case of WL under the standard condition ($\mu_0 = 0$) and ($\sigma_0 = 1$). Consequently, there exists an equation between the cumulative probability function $F(\cdot)$ for WL (under the general condition: $\mu_0 \neq 0$ or/and $\sigma_0 \neq 1$) and the cumulative probability function $F^*(\cdot)$ for $w\ell$.

$$F(y) = F^*(y/\sigma_0^2) \tag{A9}$$

Based on above derivations, this kind of relationship is further proved for the VSSI WLC scheme and the AL chart.

For the VSSI *WLC* scheme, we firstly assume (A10) is valid, and then prove this assumption results the same *ATS* values under both general and standard conditions.

$$w_A' = w_A / \sigma_0^2, k_A' = k_A / \sigma_0^2, H_A' = H_A / \sigma_0^2 \quad (\text{A10})$$

Let d' be the interval width of each state of the statistic C_i under the standard condition,

$$d' = H_A' / M = d / \sigma_0^2 \quad (\text{A11})$$

As aforementioned, the in-control states can be further divided into states in central zone (states 0 to g) and states in warning zone (states $g+1$ to $M-1$), where g is the integer closest to w_A / d . Thus, the g for standard condition (w_A' / d') is equal to that of the general condition. Therefore, it is also the same that sample size n_1 should be used if $i \leq g$ and n_2 is used if $i > g$.

Next, let p_{ij}' be the transition probability under the standard condition. When $j = 0$ (see equation 3-25),

$$\begin{aligned} p_{ij}' &= F'[(0.5 - i)d' + k_A'] \\ &= F'\left[\frac{(0.5 - i)d}{\sigma_0^2} + \frac{k_A}{\sigma_0^2}\right] = F[(0.5 - i)d + k_A] = p_{ij} \end{aligned} \quad (\text{A12})$$

It can be shown similarly that p_{ij}' is equal to p_{ij} too, when $j > 0$. It means that the transition probability matrix $\mathbf{R}$ (or $\mathbf{R}_0$) for both standard and general conditions is the same. As a result, the value of *ATS* (or ATS_0) under both conditions is also identical for given δ_μ and δ_σ .

For the *AL* chart, if the a value can be proved to remain unchanged under standard condition and general condition, the relationship like (A2) also holds.

Substituting (A6) into (4-10), and (A7) into (4-11).

$$f_{\bar{x},i}(\bar{x}) = f_{\bar{x},i}(\bar{z}) / \sigma_0 \quad (\text{A13})$$

$$f_{s,i}(s) = f_{s,i}(s_z) \quad (\text{A14})$$

Because of the term c in equation (4-8), the $p_i(t)$ ($i = 0, 1, \dots, 5$) values will be same for standard and for general conditions. Hence, the point with maximum probability (P_k), remains the same, so does the weighting factor a . Finally, it is obvious that (A15) is valid.

$$UCL_{AL} = \sigma_0^2 UCL_{standard} \quad (\text{A15})$$

Appendix B

Glossary

(Roughly in order of appearance)

VSS = variable sample size

VSI = variable sampling interval

$VSSI$ = variable sample size and sampling interval

WL = weighted loss function statistic (or its control chart)

WLC = CUSUM statistic of weighted loss function (or its control chart)

$3-CUSUM$ = combined triple CUSUM scheme

$VSS\ WLC$ = the WLC control chart with VSS feature

$VSI\ WLC$ = the WLC control chart with VSI feature

$VSSI\ WLC$ = the WLC control chart with $VSSI$ feature

$VSSI\ 3-CUSUM$ = the $3-CUSUM$ scheme with $VSSI$ feature

AL = adaptive loss function statistic (or its control chart)

$ARMA$ = autoregressive and moving average models (time series models)

ARL = expected number of samples required to signal (esp. out of control)

ARL_0 = average run length while in control

ATS = average time required to signal (esp. out of control)

ATS_0 = average time required to signal while in control

$MRATS$ = average of relative ATS ratios with respect to the $VSSI\ WLC$ scheme
in the comparisons of Chap. 3

$MRARL$ = average of relative ARL ratios with respect to the AL chart in the
comparisons of Chap. 4 and Chap. 5

UCL = upper control limit of various non-CUSUM charts (UCL_{AL} etc)

LCL = lower control limit of Shewhart mean charts ($LCL_{\bar{x}}$, LCL_Z)

H = control limit of various CUSUM charts (H_I , H_D , H_V , H_A)

w = warning limit

k = reference parameter of various CUSUM charts (k_I , k_D , k_V , k_A)

τ = allowed minimum ATS_0 in Chap. 3

= allowed minimum ARL_0 in Chap. 4 and Chap. 5

R = allowed maximum in-control average inspection rate in design

r = resultant in-control average inspection rate

μ_0 = in-control process mean

σ_0 = in-control process standard deviation

δ_μ = size of mean shift in terms of σ_0

δ_σ = size of standard deviation shift in terms of σ_0

d_μ = minimum mean shift to be protected against in Chap. 3

d_σ = minimum standard deviation shift to be protected against in Chap. 3

D_μ = minimum mean shift to be protected against, also used to calculate other values of status point P_i in Chap. 4 and Chap. 5.

η = factor used to allocate the power between monitoring mean and variance in the design of combined charts, called allocation factor

h = sampling interval

h_1 = long sampling interval (for relax sample in VSI and VSSI charts)

h_2 = short sampling interval (for alert sample in VSI and VSSI charts)

h_{min} = allowed minimum sampling interval in design

n = sample size

n_1 = small sample size (for relax sample in VSS and VSSI charts)

n_2 = large sample size (for alert sample in VSS and VSSI charts)

n_{max} = allowed maximum sample size in design

λ = fixed weighting factor in WL and WLC statistics

a = adaptive weighting factor in AL statistic

$P_i(\delta_{\mu,i}, \delta_{\sigma,i})$ = the i th status point in AL chart ($i = 1, 2, \dots, 5$), denoting a process with mean shift $\delta_{\mu,i}$ and standard deviation shift of $\delta_{\sigma,i}$

p_i = the prior or posterior probability of P_i , depending on its use

$p_i(t)$ = p_i at time t

P_k = the P_i with the largest p_i after a round of updates, called key point

a_i = the weighting factor associated with P_i

a_k = the weighting factor associated with P_k , i.e. a used in AL calculation

ζ = reducing factor in AL chart, used to adapt control limit UCL_{AL}

ψ = power parameter in AL chart, used to accelerate the increase of p_k

AR(1) = autoregressive model of order one

AR(2) = autoregressive model of order two

ARMA(1,1) = first-order autoregressive and first-order moving average model

ϕ = AR(1) parameter

ϕ_1 = the first autoregressive parameter in AR(2) model

ϕ_2 = the second autoregressive parameter in AR(2) model

θ = moving average parameter in ARMA(1,1)

γ_0 = the variance of an autocorrelated process

$\varepsilon_{i,j}$ = random error in various time series models

v = noncentrality parameter of non-central Chi-square distributions

$\{X_{i,j}\}$ = observed autocorrelated process

$X_{i,j}$ = the j th observation in the i th sample of process $\{X_{i,j}\}$

$\hat{X}_{i,j}$ = predictor of $X_{i,j}$, depending on different residuals approaches

$\{Y_{i,j}\}$ = in-control autocorrelated process

$\hat{\varepsilon}_{i,j}$ = residual obtained from $X_{i,j}$ and $\hat{X}_{i,j}$, procedure varies

$\bar{\varepsilon}_i$ = the mean of the i th sample residuals

$\hat{S}_i^2$ = the variance of the i th sample residuals