Unsymmetric finite elements for mesh-distortion tolerance performance

Ooi, Ean Tat

2006

Ooi, E. T. (2006). Unsymmetric finite elements for mesh-distortion tolerance performance. Doctoral thesis, Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/5254>

<https://doi.org/10.32657/10356/5254>

Nanyang Technological University

Downloaded on 26 Apr 2023 12:24:15 SGT

UNSYMMETRIC FINITE ELEMENTS FOR MESH- DISTORTION TOLERANCE PERFORMANCE

OOI EAN TAT

School of Mechanical and Aerospace Engineering
Nanyang Technological University
Singapore

A thesis submitted to the Nanyang Technological University
in fulfillment of the requirement for the Degree of
PhD in Engineering

2006

ACKNOWLEDGEMENTS

The past few years have been a period of intense work and wonderful learning experience. I had had the privilege of working, and more-so, learning under the watchful eye of my supervisor, Dr Sellakkutti Rajendran who introduced me to the subject of finite elements. I would like to express, to my supervisor, my sincerest appreciation and thanks for his tutelage, critical reviews of my work and his friendship during this work. Thank you, Dr Rajendran.

I wish also to thank my co-supervisor, Dr Yeo Joon Hock, for his time-to-time reviews regarding my work, and most of all for the workstation he had magnanimously offered. The completion of this work would otherwise be impossible without his support. Thank you, Dr Yeo.

To all my dear friends and colleagues at Nanyang Technological University, especially those in Physiological Mechanics Laboratory, I thank you for contributing to a very pleasant working environment.

Finally, I would like to thank my family for their support, patience and encouragement during this period.

Ooi Ean Tat

July 2005

TABLE OF CONTENTS

LIST OF TABLES.....	ix
LIST OF FIGURES.....	xii
LIST OF SYMBOLS.....	xxiii
PUBLICATIONS.....	xxviii
ABSTRACT.....	xxix

CHAPTER 1. INTRODUCTION

1.1. BACKGROUND.....	1
1.2. OBJECTIVE AND SCOPE.....	2
1.3. ORGANIZATION OF THESIS.....	3

CHAPTER 2. LITERATURE REVIEW

2.1. INTRODUCTION.....	6
2.2. REDUCED INTEGRATION.....	7
2.3. INCOMPATIBLE MODES.....	8
2.4. MULTI-FIELD ELEMENTS.....	11
2.5. ENHANCED ASSUMED STRAIN ELEMENTS.....	19
2.6. DRILLING DEGREES OF FREEDOM.....	20
2.7. PAPKOVITCH-NEUBER ELEMENTS.....	22
2.8. HYBRID-TREFFTZ ELEMENTS.....	24
2.9. AREA COORDINATES.....	26

2.10. MESHLESS METHODS.....	28
2.11. COUPLING OF FINITE ELEMENTS WITH MESHLESS METHODS.....	31
2.12. ELEMENT DISTORTIONS.....	32
2.13. MOTIVATION AND SCOPE OF THE PRESENT APPROACH.....	34

CHAPTER 3. UNSYMMETRIC FORMULATION FOR LINEAR STATIC ANALYSIS

3.1. INTRODUCTION.....	37
3.2. A REVIEW ON THE PRINCIPLE OF VIRTUAL WORK.....	39
3.3. A SUMMARY OF UNSYMMETRIC ELEMENT FORMULATION FOR US-QUAD8	42
3.3.1. ISOPARAMETRIC SHAPE FUNCTIONS.....	44
3.3.2. METRIC SHAPE FUNCTIONS.....	44
3.4. COMPLETENESS AND CONTINUITY OF SHAPE FUNCTIONS.....	47
3.4.1. INTERPRETATION AND VERIFICATION OF COMPLETENESS OF SHAPE FUNCTIONS.....	48
3.4.2. INTERPRETATION AND VERIFICATION OF CONTINUITY OF SHAPE FUNCTIONS.....	51
3.5. ERROR ANALYSIS OF SYMMETRIC AND UNSYMMETRIC FORMULATION.....	57
3.6. COMPUTER IMPLEMENTATION.....	58
3.7. PATCH TEST.....	59
3.7.1. METHODOLOGY.....	60
3.7.2. RESULTS.....	66
3.8. EXTENSION OF UNSYMMETRIC FORMULATION TO THE 20-NODE HEXAHEDRON.....	86

3.9. BENCHMARK TESTS FOR UNSYMMETRIC 20-NODE HEXAHEDRON.....	87
3.9.1. MACNEAL AND HARDER PATCH TEST.....	87
3.9.2. STRAIGHT CANTILEVER BEAMS.....	89
3.9.3. CHEUNG AND CHEN BEAM TESTS.....	90
3.9.4. DISTORTION TESTS.....	92
3.9.5. VOLUMETRIC LOCKING TEST.....	98
3.9.6. SINGLE ELEMENT ASPECT RATIO SENSITIVITY TEST.....	98
3.9.7. CANTILEVER BEAM LOADED IN SHEAR.....	99
3.9.8. RECTANGULAR PLATE.....	103
3.9.9. PINCHED SPHERICAL SHELL.....	107
3.10. COMPUTATIONAL EFFORT.....	108
3.11. CONCLUSIONS.....	110

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR

ANALYSES

4.1. INTRODUCTION.....	112
4.2. ELEMENT FORMULATION FOR US-QUAD8.....	113
4.3. COMPUTER IMPLEMENTATION.....	119
4.4. NUMERICAL EXAMPLES.....	121
4.4.1. LARGE DEFORMATION OF STRAIGHT CANTILEVER BEAM.....	122
4.4.1.1. ANGULAR DISTORTION.....	122
4.4.1.2. CURVED-EDGE DISTORTION.....	123
4.4.1.3. MID-SIDE NODE DISTORTION.....	126

4.4.2. THICK BEAM SUBJECTED TO SHEAR LOAD AT ITS TIP.....	128
4.4.3. COLUMN UNDER ECCENTRIC COMPRESSIVE LOAD.....	129
4.5. COMPUTATIONAL EFFORT.....	131
4.6. EXTENSION TO 20-NODE HEXAHEDRON.....	140
4.6.1. LARGE DEFORMATION OF STRAIGHT CANTILEVER BEAM.....	140
4.6.2. CLAMPED CIRCULAR PLATE SUBJECTED TO A CONCENTRATED LOAD AT THE CENTRE.....	144
4.6.2. HINGED CYLINDRICAL SHELL SUBJECTED TO A CENTRAL POINT LOAD.....	146
4.6.3. HINGED SPHERICAL SHELL SUBJECTED TO A CENTRAL POINT LOAD	148
4.6.4. PULL OUT OF AN OPEN CYLINDER.....	150
4.7. UPDATED LAGRANGIAN IMPLEMENTATION OF UNSYMMETRIC FORMULATION.....	155
4.8. CONCLUSIONS.....	157

CHAPTER 5. UNSYMMETRIC-MIXED FORMULATION FOR INCOMPRESSIBLE

ELASTICITY

5.1. INTRODUCTION.....	159
5.2. VOLUMETRIC LOCKING.....	160
5.3. GOVERNING EQUATIONS.....	163
5.4. SHAPE FUNCTIONS.....	166
5.5. DISCRETIZATION OF THE PRINCIPLE OF VIRTUAL WORK.....	168
5.6. DISCRETIZATION OF THE INCOMPRESSIBILITY CONSTRAINT.....	172

5.7. CONDENSATION OF ELEMENT EQUATIONS.....	174
5.8. COMPUTER IMPLEMENTATION.....	176
5.9. NUMERICAL EXAMPLES.....	178
5.9.1. COOK’S MEMBRANE PROBLEM.....	179
5.9.2. DISTORTION TESTS: CLAMPED BEAM SUBJECTED TO UNIFORM TRANSVERSE LOAD.....	183
5.9.3. DISTORTION TESTS: UNIAXIALLY LOADED SHEET OF SQUARE CROSS SECTION.....	188
5.9.4. STRETCHING OF A RUBBER SHEET WITH A HOLE.....	191
5.9.5. DEFORMATION OF A PLANE BODY WITH A CIRCULAR HOLE.....	193
5.9.6. HOURGLASS INSTABILITY TEST.....	196
5.10. CONCLUSIONS.....	198

CHAPTER 6. ITERATIVE CORRECTIONS OF PARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

6.1. INTRODUCTION.....	200
6.2. ITERATIVE CORRECTIONS OF MESH DISTORTION EFFECTS.....	202
6.3. MATHEMATICAL CONVERGENCE OF ITERATIVE CORRECTIONS.....	204
6.4. ACCELERATION OF ITERATIVE CORRECTIONS.....	206
6.5. ILLUSTRATIVE PROBLEMS.....	208
6.5.1. ANGULAR DISTORTION.....	208
6.5.2. CURVED-EDGE DISTORTION.....	212
6.5.3. MID-SIDE NODE DISTORTION.....	215

6.5.4. THRESHOLD DISTORTION VERSUS SPECTRAL RADIUS.....	216
6.6. PHYSICAL INTERPRETATIONS.....	218
6.7. COMPARISON OF PERFORMANCE.....	221
6.7.1. CANTILEVER BEAM SUBJECTED TO CONSTANT BENDING MOMENT	223
6.7.2. CANTILEVER BEAM SUBJECTED TO LINEAR BENDING MOMENT.....	226
6.7.3. CURVED CANTILEVER BEAM SUBJECTED TO SHEAR LOAD AT ITS END.....	228
6.7.4. APPLICATION TO A PRACTICAL PROBLEM.....	229
6.7.5. COMPUTATIONAL EFFICIENCY AND CONVERGENCE.....	234
6.8. CONCLUSIONS.....	237

CHAPTER 7. CRITICAL DISCUSSIONS

7.1. INTRODUCTION.....	240
7.2. ROTATIONAL FRAME DEPENDENCE.....	240
7.3. SINGULARITY OF P-MATRIX.....	251
7.3.1. DETECTION OF SINGULARITY OF P-MATRIX IN ELEMENT FORMULATION.....	257
7.3.2. AVOIDING SINGULARITY OF P-MATRIX IN ELEMENT FORMULATION	259
7.4. A NOTE ON EXTENSION TO THREE DIMENSIONS.....	262
7.5. CONCLUSIONS.....	263

CHAPTER 8. CONCLUDING REMARKS

8.1. SUMMARY.....	265
-------------------	-----

8.2. MAJOR CONCLUSIONS.....	265
8.3. FUTURE WORK.....	271
8.4. CLOSURE.....	274
REFERENCES.....	275
APPENDIX I.....	297

LIST OF TABLES

3.1	Reproducibility of monomial terms by isoparametric and metric shape functions for typical element distortions.....	50
3.2	Continuity and completeness characteristics of isoparametric and metric shape functions.....	56
3.3a	Theoretical solutions of displacements for patch tests.....	67
3.3b	Theoretical solution of stresses at parametric center for patch tests.....	68
3.4	Results of patch tests.....	69
3.5a	Displacement results of constant strain patch test.....	72
3.5b	Stress results of constant strain patch test.....	73
3.6a	Displacement results of linear strain patch test.....	74
3.6b	Stress results of linear strain patch test.....	75
3.7a	Displacement results of quadratic strain patch test.....	80
3.7b	Stress results of quadratic strain patch test.....	81
3.8	Results of MacNeal and Harder's patch test for linear stress field.....	88
3.9	Results of MacNeal and Harder's straight cantilever beam tests.....	90
3.10a	Results of Cheung and Chen tests for quadratic displacement field.....	92
3.10b	Results of Cheung and Chen tests for cubic displacement field.....	92
3.11	Normalized displacements for element with mid-side node distortion	97
3.12	Results of volumetric locking test for US-HEXA20.....	98
3.13	Results of aspect ratio sensitivity test.....	99
3.14	Normalized tip displacement for spherical shell problem.....	108

3.15	Comparison of computational time for clamped plate with plane distortions	109
4.1	Comparison of computational time of a beam subjected to an end thrust.....	133
4.2a	Estimate of ET for angular distortion.....	139
4.2b	Estimate of ET for curved-edge distortion.....	139
4.2c	Estimate of ET for mid-side node distortion.....	139
4.3a	Percent error from theoretical solution for straight cantilever beam problem.....	143
4.3b	Number of iterations consumed for straight cantilever beam problem.....	143
4.4	Non-dimensional central deflection, w/t of clamped circular plate subjected to a concentrated load, q at its center.....	145
4.5	Percentage of error of computed horizontal and vertical displacements.....	156
5.1	Results of Cook's membrane problem for linear analysis.....	180
5.2	Number of negative eigenvalues and corresponding largest negative eigenvalue as a function of imposed displacements.....	197
6.1	Number of corrective iterations required for angular distortion.....	212
6.2	Number of corrective iterations required for curved-edge distortion.....	214
6.3	Number of corrective iterations required for mid-side node distortion.....	216
6.4	Comparative performance of elements for the constant bending moment problem.....	224
6.5	Comparative performance of elements for the linear bending moment problem.....	226
6.6	Comparative performance of elements for the curved beam problem.....	228
6.7	Number of corrective iterations required for meshes in Figure 6.13.....	231

6.8a	Convergence of results and computational effort for Super-element-I.....	237
6.8b	Convergence of results and computational effort for Super-element-II.....	237
7.1	Relative error of QUAD8 and US-QUAD8 for rotational frame invariance test.....	242
7.2	Relative error of US-QUAD8-CR for rotational frame invariance test.....	246
7.3	Displacement field, $\mathbf{u}$, in element $\{1\}$ for $\delta_p = 3.0$, $\delta_q = 3.0$ and $\delta_r = 0.5$	248
7.4	Displacement field, $\mathbf{u}$, in element $\{1\}$ for $\delta_p = 2.0$, $\delta_q = 2.0$ and $\delta_r = 1.0$	248
7.5	Relative error in solution for near singularity test.....	261

LIST OF FIGURES

3.1	A typical finite element mesh of two 8-node quadrilateral elements.....	51
3.2	Plot of isoparametric shape functions of an 8-node quadrilateral element for (a) typical corner node; (b) typical mid-side node.....	53
3.3	Plot of metric shape functions of an 8-node quadrilateral element with angular distortion for (a) typical corner node; (b) typical mid-side node.....	54
3.4	Plot of metric shape functions of an 8-node quadrilateral element with curved-edge distortion for (a) typical corner node; (b) typical mid-side node.....	54
3.5	Plot metric shape functions of an 8-node quadrilateral element with mid- side node distortion for (a) typical corner node; (b) typical mid-side node.....	55
3.6	Patches of elements considered for patch test; (a) patch with angular distortion; (b) patch with curved-edge distortion; (c) patch with mid-side node distortion; (d) patch of arbitrary geometry.....	64
3.7	Linear stress contour plots of a patch with angular distortions; (a) σ_{xx} stress; (b) σ_{yy} stress; (c) σ_{xy} stress.....	76
3.8	Linear stress contour plots of a patch with curved-edge distortions; (a) σ_{xx} stress; (b) σ_{yy} stress; (c) σ_{xy} stress.....	77
3.9	Linear stress contour plots of a patch with mid-side node distortions; (a) σ_{xx} stress; (b) σ_{yy} stress; (c) σ_{xy} stress.....	78

3.10	Linear stress contour plots of a patch with arbitrary geometry; (a) σ_{xx} stress; (b) σ_{yy} stress; (c) σ_{xy} stress.....	79
3.11	Quadratic stress contour plots of a patch with angular distortions; (a) σ_{xx} stress; (b) σ_{yy} stress; (c) σ_{xy} stress.....	82
3.12	Quadratic stress contour plots of a patch with curved-edge distortions; (a) σ_{xx} stress; (b) σ_{yy} stress; (c) σ_{xy} stress.....	83
3.13	Quadratic stress contour plots of a patch with mid-side node distortions; (a) σ_{xx} stress; (b) σ_{yy} stress; (c) σ_{xy} stress.....	84
3.14	Quadratic stress contour plots of a patch with arbitrary geometry; (a) σ_{xx} stress; (b) σ_{yy} stress; (c) σ_{xy} stress.....	85
3.15	Patch test for three-dimensional elements; outer dimensions: unit cube; $E = 1.0$, $\nu = 0.25$, loading: tension = 1.0 or tip moment $M = 2.0$	88
3.16	Straight cantilever beams; (a) regular; (b) trapezoidal; (c) parallelogram shaped elements; beam dimensions: length = 6.0; width = 0.2; depth = 0.1; $E = 1.0 \times 10^7$; $\nu = 0.3$; mesh = 6×1 ; loading: unit shear load at one end and fixed in the other.....	89
3.17	Cheung and Chen tests; $E = 1500$; $\nu = 0.25$; Loading: tip moment, $M = 4000$ or shear load, $F = 600$	91
3.18	Element meshes used for distortion tests; (a) mid-side node distortion; (b) plane distortion; (c) curved-face distortion; (d) warping; (e) side-curvature	93
3.19a	Normalized displacements for plane distortion sensitivity test.....	94
3.19b	Normalized displacements for mid-side node distortion sensitivity test.....	95

3.19c	Normalized displacement for curved-face distortion sensitivity test.....	95
3.19d	Normalized displacements for warping sensitivity test.....	96
3.19e	Normalized displacements for side-curvature sensitivity test.....	96
3.20	Element meshes used for volumetric locking test; (a) regular and (b) trapezoidal meshes; loading, vertical end shear of 300 units.....	97
3.21	Single-element aspect ratio sensitivity test, $E = 207 \times 10^9 \text{ N/m}^2$, $\nu = 0.25$, tip moment applied, $M = 1656 \text{ Nm}$, $2b = 2c = 0.06\text{m}$, beam fixed at surface ABCD.....	99
3.22	Cantilever beam used for convergence tests; (a) two-element model; (b) eight -element model; (c) super-element used for distorted mesh.....	100
3.23	Convergence results of cantilever beam; (a) convergence of tip displacement; (b) convergence of stress.....	101
3.24	(a) Dimensions for simply supported plate; (b) Super-element mesh with plane distortion; (c) Super-element mesh with curved distortion; (d) Super-element mesh with warping.....	102
3.25a	Convergence of displacements for super-element with plane distortion.....	103
3.25b	Convergence of strain energy for super-element with plane distortion.....	104
3.25c	Convergence of displacements for super-elements with curved-face distortion.....	104
3.25d	Convergence of strain energy for super-elements with curved-face distortion.....	105
3.25e	Convergence of displacements for super-elements with warping.....	105
3.25f	Convergence of strain energy for super-elements with warping.....	106

3.26	Pinched spherical shell problem: Radius = 10.0; thickness = 0.04; $E = 6.825 \times 10^7$; $\nu = 0.3$; $F = 2.0$ on quadrant; hole at free edge = 18°	107
4.1	(a) A cantilever subject to shear load at the tip, $L = 1000$ in; $c = 39.37$ in; thickness = 1.0 in; $EI = 180,000$ kip-in ² ; Super-element with (b) angular; (c) curved-edge; (d) mid-side node distortion.....	121
4.2a	Deviation of vertical displacement from theoretical solution for a mesh with angular distortions, $\delta = 23.622$ in.....	123
4.2b	Deviation of vertical displacement from theoretical solution for a mesh with angular distortions $\delta = 31.496$ in.....	124
4.3a	Deviation of vertical displacement from theoretical solution for a mesh with curved-edge distortions, $\delta = 23.622$ in.....	124
4.3b	Deviation of vertical displacement from theoretical solution for a mesh with curved-edge distortions, $\delta = 31.49$ in.....	125
4.4a	Deviation of vertical displacement from theoretical solution for a mesh with mid-side node distortions, $\delta = 7.874$ in.....	125
4.4b	Deviation of vertical displacement from theoretical solution for a mesh with mid-side node distortions, $\delta = 11.811$ in.....	126
4.5	Thick beam with a shear load at its tip; $L = 20$; $c = 10$; $F = 3$; Poisson's ratio, $\nu = 0.0$; Young's modulus, $E = 12$	126
4.6	(a) Super-element mesh with angular distortion; (b) convergence of vertical displacements at point A	127
4.7	(a) Super-element mesh with curved-edge distortion; (b) convergence of	

	vertical displacements at point A	127
4.8	(a) Super-element mesh with mid-side node distortion; (b) convergence of vertical displacements at point A	128
4.9	A column under an eccentric axial compressive load: (a) geometry; (b) regular mesh; (c) distorted mesh.....	129
4.10a	Horizontal deflection response of column under eccentric compressive load.....	130
4.10b	Vertical deflection response of column under eccentric compressive load	130
4.11	Beam subject to central transverse load and end thrust: Young's modulus, $E = 10$, Poisson's ratio, $\nu = 0.0$, distributed load, $w = 2.0$ applied as a conservative load, point load, $P = 1.0$	131
4.12	Finite element meshes used for beam subjected to end thrust: (a) regular mesh; (b) distorted mesh I (angular distortions); (c) distorted mesh II (curved-edge distortions); (d) distorted mesh III (random perturbation of nodes).....	132
4.13	Stress contour plot of the beam subjected to an end thrust for regular element mesh.....	134
4.14	Stress contour plot of the beam subjected to an end thrust for distorted mesh I.....	135
4.15	Stress contour plot of the beam subjected to an end thrust for distorted mesh II.....	136
4.16	Stress contour plot of the beam subjected to an end thrust for distorted mesh III.....	137

4.17	Super-elements used for discretizing cantilever beam subjected to an end shear load; (a) plane distortion; (b) curved-face distortion.....	141
4.18	Deviation of vertical deflection from theoretical solution for a mesh with plane distortion.....	142
4.19	Deviation of vertical deflection from theoretical solution for a mesh with curved-face distortion.....	142
4.20	A 27-element mesh of one quarter of a clamped circular plate.....	144
4.21	Load-deflection curve of a clamped circular plate subjected to a concentrated load at the center.....	146
4.22	Circular cylindrical shell subjected to a central point load, $R = 2540\text{mm}$, $L = 254\text{mm}$, $t = 6.35\text{mm}$, $\theta = 0.1\text{rad}$, Young's modulus, $E = 3.10275\text{kN/mm}^2$ and Poisson's ratio, $\nu = 0.3$	147
4.23	Load-deflection curve of a hinged cylindrical shell subjected to a central point load.....	148
4.24	A shallow spherical shell subjected to a central point load.....	149
4.25	Load-deflection curve at the center of a hinged spherical shell subjected to a central point load.....	150
4.26	(a) Pull out of an open cylinder; (b) An 8×12 uniform mesh used to discretize one-eighth of the cylinder.....	151
4.27	Load versus central deflection curves of the pull out of an open cylinder problem.....	152
4.28	Configuration of pulled out open cylinder problem at loads; (a) $P = 0\text{ lb}$; (b) $P = 6000\text{ lb}$	153

4.28	Configuration of pulled out open cylinder problem at loads; (c) $P = 120000$ lb; (d) $P = 18000$ lb.....	154
4.29	An $N \times 2$, element mesh of a straight cantilever beam subjected to a concentrated tip load.....	155
5.1	Geometry of Cook's membrane problem.....	178
5.2	Element meshes used for Cook's membrane problem; (a) 8×4 element mapped mesh; (b) 8×4 element distorted mesh.....	179
5.3	Convergence of vertical displacement at point A for Cook's membrane problem.....	180
5.4a	Comparison of contour bands between QUAD9/3 and US-QUAD9/3 for distorted mesh I.....	181
5.4b	Comparison of contour bands between QUAD9/3 and US-QUAD9/3 for distorted mesh II.....	181
5.5	Clamped beam subject to uniform transverse load, $p = 0.04 \text{ N/mm}^2$; $L = 20 \text{ mm}$; $h = 10 \text{ mm}$; Mooney-Rivlin material constants, $C_1 = 0.177 \text{ N/mm}^2$, $C_2 = 0.045 \text{ N/mm}^2$ and $\kappa = 666.0 \text{ N/mm}^2$	183
5.6	Distorted super-elements; (a) angular distortion; (b) curved distortion; (c) mid-side node distortion.....	184
5.7a	Convergence of vertical displacement at point A for super-element with angular distortion, $\delta_p = 1.0$	185
5.7b	Convergence of vertical displacement at point A for super-element with angular distortion, $\delta_p = 2.0$	186
5.8a	Convergence of vertical displacement at point A for super-element with	

	curved-edge distortion, $\delta_p = 1.0$	186
5.8b	Convergence of vertical displacement at point A for super-element with curved-edge distortion, $\delta_p = 2.0$	187
5.9a	Convergence of vertical displacement at point A for super-element with mid-side node distortion; $\delta_p = 0.5$	187
5.9b	Convergence of vertical displacement at point A for super-element with mid-side node distortion; $\delta_l = 1.0$	188
5.10	Square sheet subjected to a longitudinal stretch; $L = 8mm$; Mooney-Rivlin constants, $C_1 = 0.177N/mm^2$, $C_2 = 0.045N/mm^2$, bulk modulus, $\kappa = 666N/mm^2$	189
5.11	S_{yy} -stress contours for stretched square sheet; (a) angular; (b) curved-edge; (c) mid-side-node distortion.....	190
5.12	Stretching of square rubber sheet with a hole; $R = 3$; thickness, $h_0 = 0.75$; imposed horizontal displacement, $u = 5$	191
5.13	Load-deflection response for stretching of a square rubber sheet with a hole.....	192
5.14	Stress contour plots of rubber sheet with a hole at $u = 3$	193
5.15	Incompressible plane body with central hole subjected to concentrated edge force, P , and finite element model of the body.....	194
5.16	Stress contour plot of plane body at different load levels.....	195
5.17	Compression test. The displacement, Δ , is imposed in small increments to mimic a static analysis.....	196

6.1	(a) Cantilever beam subjected to a constant bending moment; Length = 100; Depth = 10; Thickness = 1; Young's modulus = 10^7 ; Poisson's ratio = 0.3; (b) Two-element mesh with angular distortion; (c) Two-element mesh with curved-edge distortion; (d) Two-element mesh with mid-side node distortion.....	209
6.2a	Convergence of displacements for angular distortions without acceleration	210
6.2b	Convergence of σ_{xx} -stress for angular distortions without acceleration.....	211
6.2c	Convergence of displacements for angular distortions with acceleration.....	211
6.2d	Convergence of σ_{xx} -stress for angular distortions with acceleration.....	212
6.3a	Convergence of displacements for curved-edge distortions without acceleration.....	213
6.3b	Convergence of displacements for curved-edge distortions with acceleration.....	213
6.4a	Convergence of displacements for mid-side node distortions without acceleration.....	215
6.4b	Convergence of displacements for mid-side node distortions with acceleration.....	215
6.5	Contour bands of stress with and without corrective iterations.....	217
6.6	Variation of spectral radius with distortion parameter values.....	218
6.7	Material body in space subject to an external force $\mathbf{F}$; (a) original configuration; (b) configuration after deformation.....	219
6.8	Progression of energies with corrective iterations.....	220
6.9	Constant bending moment problem and selected distorted meshes from ref	

[5].....	222
6.10 Linear bending moment problem and selected distorted meshes from ref	
[5].....	225
6.11 A curved cantilever beam subject to a transverse tip load. Inner radius =	
4.12; outer radius = 4.32; thickness = 0.1; arc = 90° ; Young's modulus, $E =$	
1.0×10^7 ; Poisson's ratio, $\nu = 0.25$; unit transverse load at the free end.....	227
6.12 Plate with a central hole subjected to in-plane bending loads; Young's	
modulus, $E = 207 \times 10^9$, Poisson's ration, $\nu = 0.27$; thickness = 1.0,	
pressure, $p = 200$	229
6.13 Finite element meshes of a plate with a hole using 8-node quadrilaterals....	230
6.14 Stress band plots of plate with a hole for QUAD8 and QUAD8+IC.....	232
6.15 Square plate subjected to a uniformly distributed transverse load p ;	
Young's modulus, $E = 15000$; Poisson's ratio, $\nu = 0.0$	234
6.16a Super-elements with angular and curved-edge distortions.....	236
6.16b Typical 4×4 super-element meshes with uniform mesh refinement.....	236
6.17a Convergence of strain energy for super-element I.....	238
6.17b Convergence of strain energy for super-element II.....	238
7.1 Straight cantilever beam meshed subjected to a constant bending moment;	
$L = 10.0$; $c = 5.0$; $M = 1.0$; Young's modulus, $E = 100$; Poisson's ratio, $\nu =$	
0.3.....	241
7.2 Distorted two-element meshes used to model the straight cantilever	
problem.....	242
7.3 Eight node quadrilateral element with an attached local Cartesian	

	coordinate system, (x', y')	244
7.4	Element configurations used to study singularity of P -matrix; (a) single element with angular distortion; (b) single element with curved-edge distortion; (c) single element with mid-side node distortion.....	253
7.5	Singularity contour lines for single-element with angular distortion.....	254
7.6	Singularity contour lines for single-element with curved-edge distortion....	255
7.7	Singularity contour lines for single-element with mid-side node distortion	256
7.8	Successive rotations of coordinate system.....	262

LIST OF SYMBOLS

CONVENTIONS

Vectors and vector fields, tensors – boldface; Examples $\mathbf{F}$, $\boldsymbol{\varphi}$

Material quantities – uppercase

Spatial quantities – lowercase

The symbol $\Leftrightarrow$ means “is equivalent to”

An over-bar, “ $\bar{}$ ” indicates that the quantity is approximated using parametric shape functions

An over-hat, “ $\hat{}$ ” indicates that the quantity is approximated using metric shape functions

OPERATORS

$\rightarrow$ A mapping, $\boldsymbol{\varphi}(\mathbf{X}): \Omega \rightarrow R^3$ reads $\boldsymbol{\varphi}(\mathbf{X})$ maps Ω into R^3

$\mathbf{A}^T$ Transpose of a matrix $\mathbf{A}$

$\cdot$ Inner product of a second order tensor with a vector (i.e., $(\mathbf{A} \cdot \mathbf{b})_i = A_{ij} b_j$)
or a vector with a vector (i.e., $(\mathbf{b} \cdot \mathbf{b})_i = b_i b_i$)

$:$ Inner product of two second order tensors or higher (i.e., $(\mathbf{D}:\mathbf{C})_{ij} = D_{ijkl} C_{kl}$)

$\bigcup_{e=1}^n ()$ Finite element assembly process

MATERIAL CONSTANTS

ν Poisson’s ratio

μ Lamé’s constant

κ	Bulk modulus
E	Young's modulus

SYMBOLS

Chapter 3

u, v, w	Displacement fields
b_x, b_y	Body force intensities
$\epsilon_{xx}, \epsilon_{yy}$	Normal strains
γ_{xy}	Shear strain
σ_{xx}, σ_{yy}	Normal stresses
σ_{xy}	Shear stress
Ω	Domain of the body
Γ	Boundary of the body
N_i	Parametric shape functions
M_i	Metric shape functions
ϵ	Strain field
σ	Stress field
$\mathbf{u}$	Displacement field
$\mathbf{b}$	Body force intensity
$\mathbf{t}$	Surface force intensity
$\mathbf{F}$	Global vector of nodal forces
$\mathbf{U}$	Global vector of nodal displacements
$\mathbf{f}_{error}^{(e)}$	Error in nodal forces

L	Linear operator matrix
N	Matrix of parametric shape functions
M	Matrix of metric shape functions
B	Strain-displacement matrix
D	Constitutive matrix
K	Stiffness matrix

Chapter 4

δW	Virtual work
Ω_o	Reference configuration of the body
Γ_o	Reference configuration of the boundary of the body
S	Second Piola-Kirchhoff stress
E	Green strain
$\mathbf{f}_o$	Body force intensity in the reference configuration
$\mathbf{t}_o$	Surface force intensity in the reference configuration
R	Global vector of residual forces
F	Deformation gradient tensor
$\mathbf{B}_{NL}$	Strain-displacement matrix for geometric nonlinear analysis
$\mathbf{B}_G$	Geometric strain-displacement matrix
$\tilde{\mathbf{S}}^{(e)}$	Stress matrix
C	Constitutive matrix

Chapter 5

${}^t p$	Hydrostatic pressure at time t
${}_o p$	Increment of hydrostatic pressure
W	Strain energy
J	Jacobian of deformation gradient tensor
${}^t J$	Jacobian of deformation gradient tensor at time t
${}_o J$	Increment of Jacobian of deformation gradient tensor
W_{ext}	External work
$\tilde{W}(C)$	Deviatoric component of strain energy
$U(J)$	Volumetric component of strain energy
${}_o \mathbf{u}$	Increment in displacement vector
${}^t \mathbf{S}$	Second Piola-Kirchhoff stress at time t
${}^t \mathbf{E}$	Green strain at time t
${}_o \mathbf{E}$	Increment of Green strain
${}_o \mathbf{e}$	Linear component of incremental Green strain
${}_o \mathbf{\eta}$	Nonlinear component of incremental Green strain
$\mathbf{C}$	Right Cauchy-Green deformation gradient tensor
${}_o {}^t \mathbf{B}_L$	Linear strain-displacement transformation matrix
${}_o {}^t \overline{\mathbf{B}}_{NL}$	Nonlinear strain-displacement transformation matrix
${}_o \tilde{\mathbf{C}}$	Deviatoric component of the material constitutive tensor
${}_o \mathbf{C}$	Material constitutive tensor

Chapter 6

Π	Total potential energy
$\rho(\mathbf{T})$	Spectral radius of a given matrix, $\mathbf{T}$
$\mathbf{K}_s$	Symmetric stiffness matrix using parametric shape functions
$\tilde{\mathbf{K}}_s$	Symmetric stiffness matrix using metric shape functions
$\mathbf{K}_u$	Unsymmetric stiffness matrix using parametric and metric shape functions

Chapter 7

$\vec{\mathbf{K}}$	Local element stiffness matrix
$\mathbf{R}$	Rotation matrix

PUBLICATIONS

The following are some papers published based on this thesis:

- (i) ET Ooi, Rajendran S, JH Yeo. 'A 20-node hexahedron element with enhanced distortion tolerance'. *International Journal for Numerical Methods in Engineering* 2004; **60**: 2501-2530.
- (ii) Rajendran S, ET Ooi, JH Yeo. 'Iterative correction to enhance mesh distortion tolerance of isoparametric QUAD8 element'. *Communications in Numerical Methods in Engineering*; (in press)
- (iii) Rajendran S, ET Ooi. 'Enhancing mesh-distortion tolerance of isoparametric elements'. Presented at the 3rd MIT Conference of Computational Fluid and Solid Mechanics, June 14 – 17, Massachusetts Institute of Technology, Cambridge, MA, USA.
- (iv) ET Ooi, Rajendran S, JH Yeo. 'Mesh-distortion tolerance of US-QUAD8 for geometric nonlinear analysis'. Submitted to *Computers and Structures*
- (v) Rajendran S, ET Ooi, JH Yeo. 'Mesh-distortion immunity analysis of QUAD8 elements by strong-form patch tests'. Submitted to *Communications in Numerical Methods in Engineering* (review)

ABSTRACT

Although the finite element method is a mature field of research and its success in computational mechanics is widely recognized, there is still scope for further improvements. One such area is the development of mesh-distortion tolerant elements that yield accurate solutions even in the presence of mesh distortions. This thesis focuses on the development of a recent finite element formulation, called the unsymmetric formulation, which has high mesh-distortion tolerant properties. The scope of this research includes an extension of the unsymmetric formulation to geometric and material nonlinear problems in solid mechanics, a symmetric implementation of the unsymmetric formulation, and an investigation into some of the drawbacks of the unsymmetric formulation and its remedies.

To begin with, the concept of the unsymmetric formulation, which is based on a reinterpretation of the virtual work principle, is revisited. The rationale behind the use of compatibility- and completeness-fulfilling shape functions for the virtual and trial displacements, respectively, is reviewed. By conducting an elaborate patch test, the unsymmetric formulation is rigorously verified to be immune to mesh distortions. The unsymmetric formulation is then extended to a 20-node hexahedron element and is validated using several test problems.

As a first effort towards extension of the unsymmetric formulation, application to geometric non-linear problems is taken up. The choice of shape functions is rather similar to that of linear problems. Compatibility- and completeness-fulfilling shape functions are used for the incremental virtual and trial displacement fields, respectively. The formulation is then validated using several test problems. The results indicate that the unsymmetric formulation is also distortion tolerant for geometric nonlinear problems.

As a further extension, application to incompressible materials is considered. For this purpose, an amalgamation of the unsymmetric and mixed-formulations is necessary to avoid volumetric locking. The choice of virtual and trial displacement fields is rather similar to that of linear or geometric nonlinear applications. However, in the discretization of the incompressibility constraint, compatibility of the virtual pressure field cannot be maintained due to the discontinuous pressure fields usually employed by the mixed-formulations. Nevertheless, the unsymmetric-mixed formulation generally retains its distortion tolerant properties for most of the test problems albeit errors introduced by the discontinuous virtual pressure field.

Next, a symmetric implementation of the unsymmetric formulation is explored. The motivation is to avoid the need for an unsymmetric equation solver. The idea is to apply corrective iterations to the classical isoparametric elements to rectify the effects of mesh distortions. The otherwise poor performance of the isoparametric elements is progressively improved through application corrective iterations. This technique retains the symmetry of the system matrix, while at the same time, preserves the distortion tolerant capabilities of the unsymmetric formulation. The condition for convergence of corrective iterations as well as a method to accelerate the iterations is also investigated.

Despite its distortion tolerant capabilities, the unsymmetric formulation suffers from rotational frame dependence and occasional singularities in the interpolation matrix. These drawbacks are investigated and remedies are proposed.

Keywords: finite element, mesh-distortion tolerance, distortion sensitivity, higher order completeness, compatibility

CHAPTER ONE. INTRODUCTION

1.1. BACKGROUND

The finite element method is a powerful numerical tool. It aids engineers in analyses of problems involving substantial degree of complexities. Historically, the finite element method was first applied to the design of aircrafts. Its application later expanded into other fields of mechanical and civil engineering. Due to the inadequate computational power at the time of its founding, finite element computations could then be performed only on mainframe computers. Subsequently the rapid progress in computer technology has enabled finite element analyses to be now performed even on PCs. This renders the application of finite element method virtually limitless. To date, the method has found its uses in many engineering disciplines such as structural mechanics, fluid mechanics, thermal problems, acoustics, electrostatics, electromagnetics, biomechanics etc. In principle, any physical problem that can be described by differential equations can be handled using the finite element method.

Despite being already a well-established technique, the finite element method is not without shortcomings. The accuracy of finite element solutions has been known to deteriorate in the presence of mesh distortions even in the early seventies [1 – 4]. Mesh distortions are commonly encountered when the finite element method is used to model complex geometries with curved edges or sharp corners. In nonlinear analyses, the very process of updating the mesh geometry by superimposing the displacements on the

original mesh implicitly introduces geometric distortions. Geometric distortions of elements are thus, generally unavoidable in a finite element analyses.

Typical approaches to improve the accuracy of the finite element solutions involve either increasing the number of elements or using higher order shape functions. These are well known in the literature as the h - and p -type mesh refinement techniques, respectively. These methods have been for long the only way to overcome the ill effects of mesh distortions. Recent approaches attempt to attack the problem at its source, by developing distortion tolerant elements or new numerical techniques. The *meshless methods* are examples of the latter. Meshless methods model the geometry as a set of particles or nodes without the requisite of a mesh to inter-connect them, and thereby cleverly bypass the problem of mesh distortions. Although the meshless methods are effective in circumventing the problem of mesh distortions, they introduce additional complexities such as difficulty in imposing essential boundary conditions, numerical integration and treatment of discontinuities in the medium of interest. The finite element method, on the other hand, is an already established technique and does not experience the aforementioned complexities of meshless methods. Thus, it is motivating to re-explore and reinforce the finite element method in order to solve the problem of mesh distortions at the root.

1.2. OBJECTIVE AND SCOPE

The objective of this research is to explore more thoroughly a recent finite element formulation (called *unsymmetric formulation* [5]) that has demonstrated a high distortion

tolerance for linear elasticity applications, and to extend its application to a few nonlinear problems in elasticity.

The scope of the thesis includes

1. a more in-depth study of the unsymmetric formulation for linear elasticity problems,
2. an extension of the formulation to geometrical nonlinear problems,
3. an extension to the material cum geometric nonlinear problem of hyperelasticity,
4. an alternative implementation of unsymmetric formulation through an iterative solution of classical symmetric formulation, and
5. an investigation into some of the defects of the unsymmetric formulation and their remedies.

For the results reported in this thesis, an unsymmetric finite element [5] is programmed into an in-house FORTRAN code. The code is then developed further to include nonlinear analyses as well as the iterative solution capabilities mentioned above.

1.3. ORGANIZATION OF THESIS

Chapter 2 presents a review of literature on element formulations and other numerical techniques, aimed at improved element performance in the presence of mesh distortions.

The methodology, advantages and limitations of these previous efforts are critically

reviewed. Previous investigations into the mechanisms of element distortions and the roles they play in the loss of accuracy in the finite element solutions are also reviewed.

In Chapter 3, the unsymmetric finite element formulation of Rajendran and Liew [5] is re-examined. Much of the material has been reworked to provide a more detailed presentation. Some key concepts, such as the continuity and completeness of shape functions are reviewed thoroughly. Next, a formal patch test of the strong form is conducted as a critical evaluation of the unsymmetric formulation before taking up further extension of the formulation. Such a rigorous patch test has not been reported in [5]. The extension of the unsymmetric formulation for a 20-node hexahedron is then presented. The performance of the unsymmetric 20-node hexahedron is studied using several benchmark problems.

Extension of the unsymmetric finite element formulation to geometrically-nonlinear elasticity problems is covered in Chapter 4. While the deformation is nonlinear, the material properties considered are still linear elastic. Extension to nonlinear elasticity is implemented for both the 8-node quadrilateral and 20-node hexahedron elements. The elements are implemented based on a Total Lagrangian approach. The performance of both these unsymmetric elements is studied through several test problems.

Chapter 5 presents an unsymmetric-mixed element formulation for applications in finite deformation of incompressible, hyperelastic solids. In this formulation, independent interpolations are used for the displacement and pressure fields. The proposed

unsymmetric-mixed element allow for the response of incompressible materials that would otherwise be impossible with the displacement based unsymmetric element. The unsymmetric-mixed element is then validated using several test problems.

The unsymmetric nature of stiffness matrix in the unsymmetric formulation requires an unsymmetric equation solver, which is rather unusual and demanding in terms of current finite element practices. Hence, an alternative implementation of the unsymmetric formulation through the use of iterative corrections to bypass the requisite of an unsymmetric equation solver is proposed. This is illustrated in Chapter 6. The mathematical convergence of the technique is also established. In addition to still preserving the symmetry of the stiffness matrix, the iterative correction algorithm yields better performances that mimic the maiden unsymmetric elements.

In Chapter 7, some of the shortcomings of the unsymmetric formulation are identified and analyzed thoroughly. This includes the rotational frame dependent behavior of the unsymmetric formulation and possible interpolation failure of the element's shape functions. Some remedies are then proposed to rectify these problems. The treatment of interpolation failure is crucial to ensure a stable element performance.

The final chapter summarizes the strengths and limitations of the unsymmetric element and its applications to linear and nonlinear elasticity problems. Alternative methods for possible improvements of the current method, and possible new unsymmetric formulations are identified. Areas for future research are indicated.

CHAPTER 2. LITERATURE REVIEW

2.1. INTRODUCTION

The finite element method has made an astounding impact among the various numerical methods used to solve engineering problems. It can be applied to any conceivable field in engineering as long as the problem can be modeled by a system of partial differential equations. While the method has been proven to be successful, it is not without demerits. For example, the 4-node isoparametric quadrilateral element, which is one of the earliest finite elements introduced by Taig [6, 7], is vulnerable to shear and volumetric locking, sensitivity to element distortions, and spurious deformation modes.

Engineers use the term *locking* to characterize the excessive stiffness exhibited by finite elements under certain limiting conditions. This problem is attributed to lack of *field consistency* in representing the stress field [8]. Locking attracted the attention of finite element practitioners during the early stages of finite element development. Much effort had been invested in search of a remedy. Among the various methods proposed to eliminate locking are the techniques of *reduced* or *selectively reduced integration* [9, 10], use of incompatible modes [11] and enforcing field consistency [8]. While locking in finite elements was of chief concern at that time, the effects of element distortions tend to be a secondary issue. The field consistent 8-node hexahedral element of Chandra and Prahtap [12] has been observed by Dong and de Freitas [13] to exhibit sensitivity to element distortions.

Observations of sensitivity of finite elements to mesh distortions have been noted as early as in early seventies [1 – 4]. The isoparametric elements were observed to “stiffen” in the presence of mesh distortions, which lead to deterioration in solution accuracies. A series of investigations into the distortion sensitivity of isoparametric elements started with the paper by Stricklin et al. [14]. The elements investigated were the isoparametric 8-node quadrilateral (QUAD8) and the 6-node linear strain triangular element (TRIA6). Bäcklund [15] supplemented these results with the observation that a reduced (2×2) integration enhances the elements performance considerably. Gifford [16] reported the results of the 12-node isoparametric element (QUAD12) and observed that the accuracy loss in QUAD12 due to mesh distortions is of the same degree as that of QUAD8. Since these early observations, many attempts had been devoted to improve the finite element’s performance for distorted meshes. Much attention, however, was devoted to improving the Q4 element of Taig [6, 7] under bending modes. The sections that follow entail a compilation of literature on improving the performance of finite elements in the presence of mesh distortions.

2.2. REDUCED INTEGRATION

In conjunction with the observation of Bäcklund [15], one of the earliest attempts to reduce the finite element’s sensitivity to distortions was the use of reduced or selectively reduced integration [17, 18]. The relevance of such approaches, however, has often been commented as an ad hoc trick towards better solution accuracy since it is not necessarily always successful. Reduced integration techniques may lead to spurious energy modes, also known as *hourglass* or *kinematic* modes that result in instability of the solutions.

Such non-robust elements require stabilization techniques to remove this defect. Flanagan and Belytschko [19, 20] proposed a technique, which they terms as the γ -stabilization, for the 8-node hexahedron and 4-node quadrilateral elements. The γ -stabilization technique was then modified and improved on by other authors [21 – 26]. Stabilized elements were observed to yield better solution accuracy when the elements are distorted.

Naganarayana and Prathap [27] formulated a class of 27-node hexahedral elements through a combination of field consistency and selective reduced integration techniques. Field consistency was imposed on the element to remove the effects of shear and volumetric locking, and spurious energy modes when used with reduced integration without committing any variational crimes. The authors commented on the difficulty of maintaining consistency of the strain fields in distorted elements. In these cases, field consistency can be maintained if uniformly or selectively reduced integration is employed. The field consistent elements with selectively reduced integration schemes were shown to be less sensitive to plane and mid-side node distortions. Except in few isolated cases, the presence of spurious energy modes associated with reduced integration schemes, leads to a poorer performance of the field consistent elements.

2.3. INCOMPATIBLE MODES

The Q4 element of Taig [6, 7] gained much popularity among finite element practitioners due to its simplicity of formulation and computational efficiency. Its practical application, however, was plagued for long by various pathological conditions mentioned previously. In addition, the Q4 element is unable to appropriately represent bending

modes that are crucial in structural analyses. The absence of curvature between two adjacent nodes cripples the element, allowing representation of only a linear displacement or constant strain variation. This is inadequate for bending dominated problems that involve quadratic displacement or linear strain variations. Many efforts have since been invested to improve the performance of Q4 because of its aforementioned merits.

Wilson et al. [11] introduced the method of incompatible modes to rectify the poor performance of the Q4 element for bending behavior. The shape functions of the element are enhanced using two quadratic incompatible modes. These incompatible modes correspond to two additional degrees of freedom that are to vanish at the elements nodes. Their new element, Q6 was capable of solving in-plane bending problems for regular and parallelogram shaped elements. Despite these improvements, Q6, however, fails to pass the constant strain patch test. To remedy this defect, Taylor et al. [28] proposed to evaluate the Jacobian of the transformation matrix of the incompatible displacement vector at the local origin of the element. With this modification, the new element, QM6 is able to pass the constant strain patch test while still capable of representing bending modes for regular shaped elements. Its performance for distorted elements however, was worse compared to its predecessor.

Since its introduction, many attempts had been made to improve on the performance of the Q6 and QM6 elements. The motivation was to formulate a 4-node quadrilateral element that is capable of simultaneously passing the constant strain patch test and represent accurate bending behavior under distorted element geometry. The two-element

skewed cantilever subjected to in-plane bending has been used a standard benchmark to test such classes of elements for their sensitivity to mesh distortions. Emphasis, however, was focused on element formulations that pass the constant strain patch test since this will confirm the convergence of the element in the limit of mesh refinement. Taylor et al. [29] and Wu et al. [30] presented a revised version of the incompatible modes that are capable of passing the constant strain patch test.

Heuck and Wriggers [31] unified the concepts of incompatible modes and stabilization techniques to improve on the then most accurate 5- β element of Pian and Sumihara [32]. The element's shape functions are expanded using a complete first order Taylor series. Taylor expansions of shape functions were first used in conjunction with the γ – stabilization technique by Liu et al [23]. In the approach of Liu et al [23], the strain field was expanded using Taylor series in terms of natural coordinates. Banarach et al. [25] proposed the same idea for the deviatoric part in the displacement gradient operator. In the method of Heuck and Wriggers [31], however, the incompatible modes were incorporated into the bilinear term of shape functions and the product is expanded using a second order Taylor series expansion in physical coordinates at the element's center. The additional degrees of freedom are then condensed out at the element level through the evaluation of an equilibrium constraint to improve computational efficiency. The authors showed that the convergence of their QS6 element is governed by the complete first-order terms of a Taylor series expansion of shape functions. The QS6 element shows better accuracy compared to the 5- β element for distorted meshes.

Motivated by the observation of Lauterstajn and Samuelsson [33] on the then available 4-node quadrilaterals in literature (*viz.*, Q4, Q6, QM6, 5- β , etc.), Huang and Li [34] proposed a series of 4-node quadrilateral elements with quadratic completeness in the Cartesian coordinates. Their objective was to achieve improvements in element performance for distorted meshes in in-plane and axisymmetric analyses. In their proposed element, the authors modified one of the incompatible modes introduced by Wu et al. [35]. This incompatible mode expresses the displacement field as a complete quadratic polynomial in Cartesian coordinates. The proposed elements have the same degrees of freedom as the Q6 element [11] but maintains quadratic completeness even when the shapes of the elements are distorted. The presence of the higher order monomial terms in natural coordinates, however, demands a higher order integration rule for the proposed element. While the proposed element is an improvement over the 5- β element of Pian and Sumihara [32], the Q6 element still yields superior solution accuracy in the two-element cantilever problem with skewed aspect ratio. The authors [34] concluded that satisfaction of completeness conditions in Cartesian coordinates alone is not adequate to guarantee an element performance that is insensitive to mesh distortions.

2.4. MULTI-FIELD ELEMENTS

Multi-field elements are formulated from either the Hellinger-Reissner (displacements and stresses) or Hu-Washizu (displacements, stresses and strains) functionals. The Hellinger-Reissner functional involves the displacements and stresses as field variables while the Hu-Washizu functional involves the displacements, stresses and strains. The hybrid-stress element was first introduced by Pian [36] through the definition of the

complementary energy function. Their proposed element, however, suffers from coordinate frame dependence, satisfaction of equilibrium conditions and sensitivity to mesh distortions. To improve on the original element, Pian and Sumihara [32] reformulated the original hybrid-stress element using the Hellinger-Reissner variational statement. The assumed stresses are redefined in terms of natural coordinates. Geometric perturbations and integral equations are then used to enforce equilibrium of the assumed stresses. The incompatible modes associated with the assumed stresses are eliminated at the element level *a priori*. Pian and Tong [37], and Pian and Wu [38] further improved on the formulation by introducing new constraint equations that require no geometric perturbations. In another attempt, Yuan and Pian [39] used an oblique coordinate system in the stress assumptions so that point-wise equilibrium is enforced.

Chen and Cheung [40] proposed an alternative technique to approximate the assumed stress field in hybrid elements. The authors advocated that the element's sensitivity to distortions could be ameliorated through a rational choice of monomial terms that contains terms of the first order for the assumed stress field. This is achieved by introducing a parameter that takes values between zero and unity in the assumed stresses. The authors proposed an optimum range of values for this parameter for best performances. The performance of their proposed element is superior compared to the classical 4-node quadrilateral and the 5- β element of [32].

Size [41] introduced an admissible matrix formulation for economical construction of hybrid elements. As a companion to the method, a selective scaling technique [42, 43]

has been developed to remedy the elements from locking conditions when modeling slender beams, thin plates and shells. Sze and Fan [44] then combined both the hybrid-stress element proposed by Pian and Tong [37] and his admissible matrix formulation together with a modified scaling technique to improve computational efficiency and solution accuracy. The new element is constructed via orthogonalization of non-constant stress modes with respect to constant ones using the Gram-Schmidt scheme. Parasitic strain components are first identified from the element geometry and are then removed via the modified scaling technique. The modified scaling technique improves over its predecessor [42, 43] by first identifying the extreme ratio of the element based on the element's geometry. This ratio is then scaled to remove locking in their proposed element. Improvements in accuracy and computational efficiency over the classical 8-node hexahedral and 4-node quadrilateral using Wilson's incompatible functions for distorted meshes were achieved.

Improvement in performances of hybrid-stress elements for distorted geometry could also be achieved through the use of internal incompatible displacement functions similar to those in [11]. Wu et al. [45] introduced a rational method to select the incompatible displacement functions. Combining the method of Wu et al. [45] and a modified technique whereby the internal modes are eliminated *a priori*, Chen and Cheung [46] proposed a refined quadrilateral plane element with improved performance compared to the existing quadrilateral elements at that time. While Pian and Sumihara [32] used an *a priori* elimination only for the assumed stresses, Chen and Cheung advocated that this elimination must apply also to the assumed strains. The authors improved on their earlier

effort [47] by introducing a relaxation parameter similar to that in [40] to improve the elements performance in the presence of mesh distortions.

Yeo and Lee [48] introduced new stress assumptions based on those of the 5- β element to improve on the performance of the then available hybrid-stress elements for geometric distortions. A more general form of incompatible modes using two control parameters were introduced into the constraint equations for the stresses. These control parameters, which are computed from the geometry of the element, define a point within the element where a transformation matrix is to be computed. The incompatible modes define the covariant stresses, which is then transformed to physical stress components using the transformation matrix. The authors observed that the incompatible modes play a role in increasing the accuracy of the element under distorted geometry. The authors also showed through their derivations that, unlike their two-control parameters, the constant parameter used by Chen and Cheung [40] is independent of element geometry and orientation. Thus the effects of geometric distortions are not fully accommodated when this constant parameter is used. The new stress assumptions are then applied to the 5- β -quadrilateral and 18- β -hexahedral [37] elements. Both the proposed elements show improvement in performances over the then existing hybrid-stress elements in the presence of element distortions.

Zhou and Nie [49] constructed various schemes of hybrid-stress elements using Wilson's incompatible bubble functions [11] with different choices of stress assumptions. The authors introduced the concept of energy compatibility for the selection of appropriate

stress modes. This energy compatibility condition was then advocated as the key factor for the optimal selection of stress modes. Unlike the $5-\beta$ element [32], the stress field defined using this approach is a coupled field that does not explicitly include a constant stress term. As a consequence, the element passes the constant strain patch test only when the element shapes are parallelograms. The authors, however, showed that the convergence of hybrid-stress elements is independent on whether the combination of the displacement and stress field satisfies the patch test [50] or the inf-sup condition [51]. Rather, the performance of the hybrid-stress elements depends on optimal choices of the stress modes. The 4-node quadrilateral, CH(0-1) formulated based on the proposed condition, improves over existing hybrid-stress elements. While the performance of the method was noteworthy, its mathematical formulation is quite involved.

To improve on the defects of CH(0-1), Xie and Zhou [52] used the same energy compatibility condition to formulate their ECQ4 element. In their derivation, the authors used the compatible shape functions of the 4-node quadrilateral element for the displacement field and the 5-parameter energy compatible hybrid element of CH(0-1) for the assumed stress field. The authors also proposed a new element, LQ6 based on Wilson's incompatible displacements and a 9-parameter linear stress mode. The LQ6 element was shown to be equivalent to the ECQ4 element. In the benchmark problems that the authors considered, both ECQ4 and LQ6 element yield the same accuracy as that of CH(0-1). The mathematical processes involved are however, much simplified compared to the elements predecessor.

Wu and Cheung [53, 54] introduced a penalty equilibrating approach as a companion to the hybrid-stress element [32]. A penalty parameter enforces stress components to satisfy equilibrium in a weak sense. The authors observed that the accuracy of their proposed method increases when it is tested for *trapezoidal locking*. Sze [55], however, explained in his paper that the reason for trapezoidal locking in selective integrated elements is due to *parasitic shear* and not because of deterioration in the equilibrium conditions, as was previously reasoned out by Wu and Cheung [53, 54]. Sze then introduced an artificial scaling technique to scale the strain fields so as to remove trapezoidal locking from the 5- β hybrid-stress element [32]. Sze's method was effective in predicting in-plane bending modes but not for problems that involve more complicated stress fields.

Cao et al. [56] attempted to implement the penalty equilibrating approach in mixed elements that are governed by the Hu-Washizu principle. Prior to element formulation, a penalty parameter is introduced to the Hu-Washizu functional. Element formulation proceeds as usual, along with the additional term attributed to the penalty parameter. The authors investigated the role of the penalty parameter in improving the element's performance for trapezoidal locking. To simplify their analysis, the authors approached the problem by introducing the penalty parameter in the two-dimensional Hellinger-Reissner functional instead of the Hu-Washizu functional. The authors showed that the penalty parameter plays a similar role as that of the artificial scaling parameter introduced by Sze [55]; the penalty parameter acts as a scaling factor to reduce the influence of the parasitic strain or stress when the element shapes are distorted. However, the authors were unable to show the same in the Hu-Washizu formulation due to the mathematical

complexities involved. Nevertheless, they reasoned out that the role of the penalty parameter in both Hellinger-Reissner and Hu-Washizu formulations should be similar. The authors were successful in achieving a mixed element formulation that yields superior performance compared to its predecessors. The same group of authors had also extended this penalty equilibrating approach for elasto-plastic analyses [57].

Cao et al. [58] further improved on their original penalty equilibrium method using a new technique to determine the penalty parameter. In their re-analysis using the Hellinger-Reissner variational statement, the authors found that in their previous implementation, the penalty parameter was not a function of the geometry of the element. The authors reasoned out that the element's performance for distorted geometries could be improved if the penalty parameter is made dependent on the elements' geometry. Incorporating the penalty parameter that relates to the element geometry, the performance of the element's predecessor [56, 57] is improved for distorted elements.

Xie [59] proposed a hybrid-stress macro-element, HQM by combining two triangles into a 4-node quadrilateral to improve the performance of the constant strain triangular element. Compatible linear shape functions of the linear strain triangle using area coordinates are used for the displacement interpolation. The 5-parameter incomplete linear stress modes that satisfy the self-equilibrium equations are used for the assumed stresses in the quadrilateral element. The effects of element distortions are incorporated into the assumed stresses. This improves the performance of the element in the presence of mesh distortions. The redundant stress parameters, if any, are eliminated at the element

level to reduce computational time. The HQM element is superior compared to existing triangular element but inferior to the CH(0-1), ECQ4 and LQ6 elements.

Mijuca [60] proposed a primal-mixed finite element based on the Hellinger-Reissner functional for structural applications that involve geometric and material invariance. Continuous basis functions were introduced to approximate the stress field. The author imposed a constraint on the boundary nodes to facilitate efficient treatment of the stress components. The boundary nodal coordinate surfaces are required to be at least tangent to the local boundary surfaces. The stress approximation is made up of hierarchical bases. The test and trial functions for the stress is chosen from the space of all symmetric tensor fields that are square integrable and have square integrable gradients [61, 62]. In a previous paper, Mijuca and Berkovic [63] showed that the 4-node quadrilateral with 9-stress modes, QC4/9 (4-corner, 4-midside and one bubble) showed distinct improvements over the standard displacement based 4-node quadrilateral. The 4-node quadrilateral with 5-stress modes (4-corner and one bubble), QC4/5, which is equivalent to the 5- β element [32], was shown not to satisfy the stability condition required to pass the patch test [64]. In its three dimensional analogue, the equivalent hexahedral for the QC4/9 element is the HC8/27 element. This element uses eight shape functions to interpolate the displacement, and twenty-seven stress modes (8-corner, 11-mid-side, 6-mid-face and one bubble-central mode). The stability condition of this element was proved analytically [65]. The three dimensional analogue of the QC4/5 element, HC8/9 was also shown not to satisfy the stability condition. In the test problems that the author [60] had performed, the HC8/27 element exhibits excellent performance in the presence of mesh distortions.

2.5. ENHANCED ASSUMED STRAIN ELEMENTS

Simo and Hughes [66] showed that the assumed strain method is consistent with the Hu-Washizu variational statement provided that the stress assumptions are appropriately selected. In the assumed strain method, the strain fields are independent of the displacements and are not derived from the usual strain-displacement relations in classical solid mechanics. However, a rationale for the selection of optimal strain fields that leads to elements with robust performance was not outlined at the time when the method was introduced. This opens up a wide spectrum for formulation of new elements using different assumed strain fields. Of the earliest works was that presented by MacNeal [67], who proposed to improve the performance of the 4-node isoparametric quadrilateral through an appropriate selection of the assumed strain field. His proposed element does not pass the patch test for any arbitrary geometry but improvements in solution accuracy for distorted elements were achieved. Simo and Rifai [68] then proposed a new class of elements by combining both concepts from the Hu-Washizu functional and the assumed strain method. The performance of this so called enhanced assumed strain elements is similar to the 5- β element of Pian and Sumihara [32].

Inspired by the works of MacNeal [67], Stolarski and Chen [69] proposed a different set of assumed strain fields in their element. The core idea of their method depends on the correct identification of the various deformation modes in structural members. These deformation modes are then suitably modified to accommodate the assumed strain field. The authors observed that any improvements in bending behavior depend on whether the assumed strain field is capable of representing the bending modes. The assumed strains

are suitably adapted to simulate the strain field corresponding to the exact solution of trusses under the action of pure bending. The proposed QMOD1 element was able to yield accurate solutions for in-plane bending of a beam regardless of the element distortion. However, QMOD1 passes the patch test only for parallelogram shapes. Convergence is still possible if the shapes of the element approach parallelograms in the limit of mesh refinement.

Korelc and Wriggers [70] proposed a 4-node quadrilateral element (QP6), using the concept of the assumed strain method, enhanced with Taylor expansion of the shape function derivatives. Two additional strain fields are added in conjunction with the standard field in the enhanced assume strain method. The compatible part of the strain field is separated into constant and orthogonal components using Taylor expansion in terms of natural coordinates. Such a procedure yields better element performance compared to the previous works of Hueck and Wriggers [31], whereby the Taylor expansion was performed with respect to Cartesian coordinates. Advantages in computational efficiency were also gained using natural coordinates as the reference since the strain fields now become uncoupled. The Taylor expansion enriches the original shape functions with the inclusion of higher order monomial terms. This leads to the better performance of QP6 under distorted geometry.

2.6. DRILLING DEGREES OF FREEDOM

Drilling degrees of freedom was first employed to improve the poor performance of the constant strain triangular element [71, 72]. Early attempts in such formulations, however,

were not successful until the works of Allman [71] and Carpenter et al [73] were published. The core idea is to include additional rotational degrees of freedom so as to better interpolate the displacement field. By including the rotational degrees of freedom, the element is expected to approach the behavior of distinct structural members. The in-plane rotations also relate to the stiffness matrices in flat shell elements as to avoid singularities [74]. The method of Allman [71] was then extended to displacement based quadrilateral elements [e.g. 75, 76], and multi-field elements [77 – 79]. The performance of these elements is comparable to isoparametric elements having the same number of sides and mid-side nodes.

Cannarozzi and Cannarozzi [80] applied two modified schemes of drilling degrees of freedom to improve the overall performance of isoparametric elements. In their approach, the drilling degrees of freedom are required to match the continuity of displacements at inter-element boundaries to avoid artificial constraints on the deformation. This is achieved by expressing the tangential and normal displacement components using algebraic functions of different degrees. These functions are suitably chosen to allow the the vertex angles to distort, depending on the translational degrees of freedom. The authors formulated two versions of 4-node and 8-node quadrilateral elements (QR4H, QR8H, QR4H0 and QR8H0), the latter versions, being computationally more efficient but less accurate. The QR8H element was able to reproduce an exact quadratic displacement field for angular distortions but not curved distortions.

Choi et al. [81] introduced a direct modification technique based on the concept of hierarchical non-conforming modes. In their observation, the use of these higher order non-conforming modes in addition to the already standard ones, improves the behavior of isoparametric elements under geometric distortions. Evaluation of the correction constants for the derivatives of the non-conforming modes, necessary for the element to pass the patch test is performed analytically. Choi et al. [82] then combined the concepts of the direct modification technique and drilling degrees of freedom to further improve on their previous work [81]. Through different combinations of higher order non-conforming modes, the authors [82] formulated four versions of quadrilaterals and hexahedral elements respectively. The elements corresponding to types II and III yield the best performance. Their proposed elements show a much better performance for distorted elements compared to previous elements adopting drilling degrees of freedom [see e.g. 83 – 85].

2.7. PAPKOVITCH-NEUBER ELEMENTS

Venkatesh and Shrinivasa [86] attempted a variationally-correct, non-conforming solid element, PN34 using assumed displacement fields that satisfy the Navier equations of elasticity exactly. The general solution to the Navier equations is given by [87]. The displacement field is evaluated at each of the nodes in the element to obtain certain constraint conditions. These constraint conditions are solved by successively increasing the orders of multi-nomials in their associated harmonic vector to obtain generic interpolation functions for the displacement vector. These functions are in terms of the vertex displacements with some additional independent variables that are the *natural*

occurring bubble modes. Complete cubic polynomials were chosen for the harmonic vector.

Bassaya et al. [88] used a similar idea to improve the performance of the PN34 element. Their improved element formulation, PN340 is capable of representing higher order stress fields under extreme element distortions. However, the PN340 element fails to represent a constant state of strain, and a constant strain patch test could only be passed in the weak sense. To improve on the deficiencies of PN34 and PN340, Bassaya and Shrinivasa [89] developed a 14-noded hexahedral element (PN5X1) based on the same Papkovitch-Neuber potentials. From a complete displacement field of the fourth order in the harmonic vector, the authors solved twenty-one stress constraint equations to accommodate forty-two degrees of freedom associated with the displacements. To solve for the element shape functions, an additional 9 coefficients from a fourth order strain field is considered. The PN5X1 element passes the constant strain as well as higher order patch tests.

The good performance of the PN-elements can be attributed to the higher order monomial terms in the elements shape functions. While the elements performance is excellent, the mathematics and numerical procedures involved in element formulation is rather complicated. Element formulation involves large sized matrices. The process of obtaining the shape functions is not straightforward and additional constraint has to be considered carefully for the shape functions to exist.

2.8. HYBRID-TREFFTZ ELEMENTS

The hybrid-Trefftz displacement [90] and hybrid-Trefftz stress [91] elements introduced by de Freitas et al. are based on a very rich hierarchical approximation basis. These elements, displacement- or stress- based are derived from the minimum potential energy and complementary energy of the Hellinger-Reissner variational statement, respectively.

The displacement based hybrid-Trefftz element [90] combines both features of the finite element and boundary element methods. Element formulation is based on simultaneous approximation of the displacement field within the element's domain and the traction distribution on the element's boundary. The geometry of the element is characterized using a parametric mapping as in the conventional finite element method. The governing differential equations re-cast into the Navier's equations of elasticity. The hybrid-Trefftz formulation then seeks local satisfaction of the Navier equations of equilibrium.

The weight functions are chosen to identify with the nodal displacements so as to preserve the conforming displacement model as in conventional finite elements. The interpolation functions for displacements are derived from displacement potential functions of plane homogeneous and isotropic elasto-static problems. Harmonic and bi-harmonic potentials with polynomial fields are used to match the problems to the approximations. The local displacement field also includes local effects so that isolated cases of interest (e.g., singular stress fields in crack problems) can be modeled. The tractions on the Dirichlet boundary is approximated with basis functions built on Dirac functions so that the traction vector corresponds to nodal forces. Linearly independent

Chebyshev piecewise continuous polynomials are used to set up the traction approximation matrices.

These approximations give rise to domain integral expressions associated with the elasticity condition, body forces and residual stresses and strains. These integrals are then converted to equivalent boundary integral definitions using the Trefftz constraint. The approximation for the stress field and traction field on the Neumann boundary also satisfies the local equilibrium and elasticity conditions. These are used in the post-processing phases to compute the stress and traction fields associated with the displacement estimate. The hybrid-Trefftz displacement formulation leads to a stiffness matrix that is sparse and symmetric.

The hybrid-Trefftz stress formulation [91] is derived from the Hellinger-Reissner functional. The approximation for the displacement fields follows from the hybrid-Trefftz displacement formulation. In addition to the displacements, the formulation includes an independent stress field. The stress field is approximated in the domain of the element while the displacements, on its boundary. Hierarchical polynomial bases obtained from Papkovitch-Neuber potentials are used to approximate the displacement field. The stress-strain-displacement relations are then used to obtain the approximation basis for the stress field. The Trefftz constraint converts the domain integrals to equivalent boundary integrals. Additional approximation for the strains and tractions aid the setup for the boundary integrals. These also play a role in the post-processing phase.

The hybrid-Trefftz elements have been shown to be insensitive to element distortions if the monomial terms in the approximation basis are extended to include the monomial terms in exact solution. Increasing the degree of approximation basis in the traction field reduces any error due to non-satisfaction of the traction boundary condition. A study using the $L - \infty$ condition number shows that element distortions increases the condition number of the system matrices. When the condition number becomes too large, the system becomes ill-conditioned and the solution becomes unstable.

The good performance of the hybrid-Trefftz elements is due to their high degree of approximation. The shape functions in general, may include all the all the monomial terms required to represent a particular stress field even when the elements are distorted.

2.9. AREA COORDINATES

Area coordinates are popular in the construction of shape functions for triangular elements. Long et al. [92] generalized the method of area coordinates from triangular to quadrilateral elements. Using the techniques outlined by Long et al. [92], Soh et al., [93] constructed two versions of 8-node quadrilaterals, AQ8-I and AQ8-II using the area coordinate method. The compatibility of displacements along adjacent element boundaries is enforced in a weak sense. This requires one of the two conditions: (1) the requisite that displacement fields within the element and the boundary and; (2) the line integral of the displacement field within the element and along each side of the element be of equal magnitudes. The first condition is referred to as the point compatibility condition of nodal displacements at each node while the second is referred to as the line

compatibility condition of average deflections along each side of the element. The area coordinates has the advantage in that the coordinate transformation between the area coordinates and the physical coordinates are always linear. Thus, the area coordinates are insensitive to element distortions, provided that the integration of the stiffness matrix is exact.

The elements AQ8-I and AQ8II [93] are found to yield similar performance as the classical 8-node isoparametric element when element distortions are absent. For element meshes that involve angular distortions both elements using area coordinates show an improvement over the isoparametric element. This is so, since area coordinates satisfy second order completeness in Cartesian coordinates under this distortion type, while the isoparametric elements do not. The elements performance for curved edge distortion, however, is only comparable to the isoparametric elements. Errors associated with curved edge distortions result from the assumption of straight edges when computing the areas of the quadrilaterals. The authors had recommended that a combination of both the 8-node isoparametric quadrilaterals and their proposed elements based on area coordinates to be used when modeling structures with curved boundaries, if better a performance is desired.

Using a similar approach, Chen et al., [94] proposed two 4-node quadrilaterals AGQ6-I and AGQ6-II using the method of area coordinates. Two incompatible internal shape functions are introduced to enable the elements to represent bending modes. These shape functions are associated with the Wilson's incompatible modes [11]. Compatibility of the internal shape functions is enforced in a weak sense using either the point compatibility

or line compatibility conditions. The element AGQ-I is enforced using the latter whereas the AGQ-II element, the former.

Both the proposed elements, AGQ-I and AGQ-II are capable of reproducing the exact solution in the 2-element trapezoidal locking benchmark for any degree of distortion wherein all preceding 4-node quadrilaterals, save for the QMOD1 element of Stolarski and Chen [69], were not successful. The proposed elements [94], however, pass the constant strain patch test only in the weak form. The behavior of these elements are consistent with the theorem proposed by MacNeal [95], which states that 4-node quadrilateral elements with two degrees of freedom per node will either lock in in-plane bending or fail the constant strain patch test when element geometries are in the form of trapezoids.

2.11. MESHLESS METHODS

Various unconventional attempts have been experimented to improve on the drawbacks of the finite element method. In view of the ill effects of geometric distortions in an element mesh, attempts had been made to solve the same boundary value problem without the requisite of an element mesh. The new approach, known as the meshless methods originates from the Smoothed Particle Hydrodynamics method of Gingold and Monaghan [96]. The original method was then suitably modified to solve differential equations pertinent to those commonly encountered engineering problems. To date, many versions of meshless methods have been proposed [e.g. 97 – 101].

In many of the papers published, the meshless methods exhibit improvements in convergence trends compared to finite element methods. The superior performance of the meshless methods is obvious. The shape functions employed by the meshless methods are constructed by fitting an averaged function through the collection of nodes encompassed within a sub-domain. Shape functions constructed in such a manner are very rich in nature, and are capable of accommodating higher order monomial terms. The presence of higher order monomial terms in the meshless shape functions enables it to represent exact solutions of problems more accurately compared to the finite element method.

One compelling advantage that the meshless methods have over the finite elements is in the modeling of hot- or cold-forging processes. In such simulations, the initial finite element mesh may become grossly distorted after a few load steps. This may lead to negative values of Jacobian of coordinate transformation that will halt computations. Re-meshing is unavoidable if computation is to continue. The meshless methods, however, require no re-meshing techniques since integration does not require any nodal-element connectivity as in finite elements.

Despite the advantages the meshless methods have over the finite element method, it is not without drawbacks. Albeit possessing excellent convergence properties, the meshless methods require, in general, more computational time compared to the finite element methods. A major bottleneck of the meshless methods involves the accurate imposition of the Dirichlet boundary conditions. Unlike the finite element method, the shape

functions used in meshless methods lack the Kronecker delta property. Shape functions with Kronecker delta property facilitate the imposition of these boundary conditions efficiently. Various methods to properly impose the Dirichlet boundary conditions have been proposed [e.g. 97, 102 – 104]. However, the use of such methods is often cumbersome and results in more tedious mathematical and numerical treatments. The shape functions of the point interpolation method proposed by Liu and Gu [101] inherit the Kronecker delta property. However, they are not always feasible due to singularities, depending on the choices of the monomial bases used to construct the shape functions. Wang and Liu [105] then proposed the use of radial basis functions to overcome the defects of point interpolation method.

In addition to the difficulty in imposing the Dirichlet boundary conditions, the evaluation of the stiffness matrices in meshless methods are also not straightforward. In the absence of a well-defined boundary, the meshless methods perform the necessary integrations on sub-domains, commonly defined to be in the form of rectangles, discs, plates or spheres, depending on its implementation. Difficulty arises when the integration on such sub-domains are performed on boundaries that splits the sub-domain between two different mediums. In such cases, integration of the stiffness matrix with conventional Gaussian quadrature is not sufficient. Treatment of integrals of such kinds involves techniques to distinguish the boundary interface and then perform integration for the isolated boundary by segments. Such techniques again lead to more complicated mathematics and numerical implementation.

2.11. COUPLING OF FINITE ELEMENTS WITH MESHLESS METHODS

Some researchers have attempted to couple both meshless and finite element methods so as to achieve a formulation that inherits only the benefits of both methods. Idea-wise, the most straightforward of such an approach involves an independent coupling between the finite element and meshless method [106, 107]. In these approaches, the finite elements are used to discretize the boundary of the domain. The inner domain that is not connected to the boundary is discretized with nodes associated with the meshless methods. The motivation of the above approach was to facilitate the ease of imposing the Dirichlet boundary conditions while still able to extract the benefit of higher order convergence with the meshless methods. This approach is capable of removing the difficulty in enforcing the Dirichlet boundary conditions. However, it involves substantial degrees of complexities in the coupling term at the boundaries between both methods.

Coupling of the finite element and meshless methods is also possible if the shape functions used in the finite element method are enriched with the shape functions used in the meshless methods [see e.g., 108]. Strouboulis et al. [109, 110] introduced the Generalized Finite Element Method, which combines the standard finite element and the partition of unity method [111]. The finite element shape functions is augmented by adding special functions, which reflect the known information about the boundary value problem and the input data. This product is then multiplied with a partition of unity corresponding to the standard linear vertex shape functions to construct a conforming approximation. The Generalized Finite Element Method preserves the ease of imposing the Dirichlet boundary conditions as in the finite element method.

It is noteworthy to mention that the approach of both finite element and meshless methods to the solution of boundary-valued problems is similar. Both methods attempt a solution by approximating the field of interest using shape functions. However, the construction and requirements of shape functions in both methods differ from each other. In the finite element method, the shape functions are required to satisfy compatibility between adjacent element edges. This requirement is exploited by the meshless methods, to their advantage, by replacing the element mesh with sub-domains that constitute to a finite number of nodes. Although the unfavorable effects of element distortions can be entirely overcome through a transition from finite elements to meshless methods, the latter introduces additional difficulties that require special treatment.

2.12. ELEMENT DISTORTIONS

Finite element practitioners have observed the unfavorable effects of element distortions since the early 1970's [1 – 4, 14 – 16]. Albeit several observations made, a detailed investigation of the root cause of the deterioration in solution accuracy has been lacking. The common practice to improve the accuracy of solutions of distorted elements has been to increase the number of elements in the mesh or through the use of higher order shape functions. The shape functions thus play a crucial role in determining an element's performance under geometric distortions. Shape functions that can accommodate higher order monomial terms generally lead to better performances.

Lee and Bathe [112] presented their observations from a detailed investigation on element distortions on a class of isoparametric elements. The elements performance was

studied for the possible types of distortions related to the type of element. The authors showed that the presence of element distortions introduces additional terms in the exact displacement field, expressed in terms of natural coordinates. Rajendran and Subramaniam [113] provided a similar interpretation of element distortions using the concept of parametric mapping for an 8-node quadrilateral element. Linear mappings between natural and physical coordinates are associated with element with regular shapes. Angular distortions are associated with bilinear mappings while curved edge distortions include terms of the second and third order in natural coordinates. The sensitivity of the isoparametric elements to geometric distortions is due to its inability to represent all the monomial terms present in the exact displacement field when the elements in the mesh are distorted. Using the results of their analyses, Lee and Bathe [112] showed that the cubic 12-node isoparametric quadrilateral, QUAD12, is unable to represent linear stress fields in the presence of angular distortions. The quadratic 9- and cubic 16-node quadrilateral elements, QUAD9 and QUAD16 with Lagrangian shape functions have been shown to be insensitive to angular distortions due to their ability to represent monomial terms in local coordinates introduced by angular distortions. The authors concluded that an element's sensitivity to geometric distortions could be remedied if higher order monomial terms can be included in the elements shape functions. Martini et al. [114] arrived at a same conclusion through their observation of the behavior of lower and higher order tetrahedral elements.

MacNeal and Harder [115] modified the shape functions of the QUAD9 element and proposed a new 8-node quadrilateral element based on this modified set of shape

functions. The new element is capable of reproducing a general quadratic displacement field under angular distortions. However, the authors did not provide explicit expressions for the shape functions used. Kikuchi et al. [116] used a similar concept as that of MacNeal and Harder [115] and proposed an 8-node quadrilateral element, QUAD8/9 and supplemented their work with explicit expressions for the element shape functions that reproduce quadratic displacement fields under angular distortions.

In other developments Lautersztajn and Samuelsson [33] investigated the distortion sensitivity of a class of incompatible 4-node quadrilateral elements. The authors evaluated the area of the elements after deformation to measure the sensitivity of elements to distortions. Under distorted meshes, the Q4 [6, 7] and QM6 [28] elements have been shown to preserve their original magnitude of areas even after deformation. The authors had attributed this to the poor performance of these elements. In conjunction with the conclusions made by Lee and Bathe [112], Lautersztajn and Samuelsson [33] also concluded that the remedy for element distortions is to have shape functions that are capable of representing all the monomial terms in the exact solution. The analysis made by Lautersztajn and Samuelsson also confirmed MacNeal's theorem on the limit of accuracy that can be achieved through introduction of incompatible modes in the Q4 element [95].

2.13. MOTIVATION AND SCOPE OF THE PRESENT APPROACH

Rajendran and Liew [117] provided an alternative interpretation of the completeness requirements for higher order elements. In addition to the standard linear completeness,

the authors identified additional completeness conditions pertaining to higher order elements. These conditions are satisfied by the isoparametric elements only for regularly shaped elements (e.g., square, rectangles and parallelograms) but not for other shapes in general. Non-satisfaction of these conditions lead to poorer performances when the elements are distorted. Using these conditions to derive element shape functions, the authors proposed a 3-noded bar element for one- and an 8-noded quadrilateral element for two-dimensional analyses. Both these elements are capable of reproducing quadratic displacement fields for mid-side node distortions. Their proposed 8-node quadrilateral element is, however, vulnerable to angular distortions. The authors concluded that their observations are due to the fact that the shape functions of their proposed element do not satisfy inter-element compatibility of displacements in the presence of angular distortions.

Motivated by their previous study [117], Rajendran and Liew [5] proposed an unsymmetric 8-node quadrilateral element (US-QUAD8) using two different sets of shape functions for the virtual and trial displacement fields. The US-QUAD8 element is capable of reproducing a general quadratic displacement field under any angular, curved-edge and mid-side node distortion. Such a formulation belongs to the broad class of the Petrov-Galerkin formulation whereby the test and trial functions are required to satisfy different criteria.

The limitations of their study are as follows:

- The scope was limited to only linear elasticity problems.

- A detailed patch test has not been reported in their publications. While the test problems performed in [5] can be argued out as patch tests, a formal patch test is preferably carried out.
- The stiffness matrix is unsymmetric and hence, requires an unsymmetric equation solver. Thus, the unsymmetric formulation cannot be readily implemented in most commercial finite element packages that have symmetric equation solvers.
- The interpolation matrix $\mathbf{P}$ for the matrix shape functions may be singular for certain configurations. A detailed investigation on the singularity of the $\mathbf{P}$ – matrix has not been reported by them.

The work reported in this thesis is towards extension of the work of Rajendran and Liew [5] so as to improve on these limitations.

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

3.1. INTRODUCTION

Conventional finite element formulations based on the Galerkin approach lead to symmetric stiffness matrices. The Petrov-Galerkin approach, on the other hand, generally leads to unsymmetric stiffness matrices. Nevertheless, Petrov-Galerkin formulations offer more flexibility in terms of selection of shape functions, since both test and trial functions can be drawn from different spaces. Although such formulations are uncommon in structural mechanics, they are well known in fluid mechanics. The upwind Petrov-Galerkin (UWPG) (see e.g. [118]) method is commonly used to remove artificial oscillations in the solution experienced in the classical Galerkin method for flows with Peclet number, $Pe > 1$. The Petrov-Galerkin formulation investigated in this thesis is different from the UWPG formulation. UWPG involves modification of the test functions to include a discontinuous component so as to remove the aforementioned oscillations while the Petrov-Galerkin formulation investigated in this thesis involves replacement of the original trial functions[†] with a different set of shape functions to enforce higher order completeness.

The unsymmetric formulation investigated in this thesis is based on the US-QUAD8 element proposed by Rajendran and Liew [5]. This formulation belongs to the broad class

[†] “Original trial functions” refers to the parametric shape functions otherwise used to construct the trial functions in conventional isoparametric formulations

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

of Petrov-Galerkin formulations. In this chapter, formulation of the US-QUAD8 element is first reviewed in detail. More elaborate patch tests are carried out to verify their claim [5] that the formulation is immune to all types of mesh distortions. The formulation is then extended to a 20-node hexahedron element and validated against typical benchmark problems.

The high distortion tolerance of the unsymmetric formulation crucially depends on appropriate choices of shape functions for the virtual and trial displacement fields in the principle of virtual work. Hence, a review of the principle of virtual work, highlighting the difference in the criteria for the choice of shape functions for the virtual and trial displacement fields is discussed in Section 3.2. A summary of the unsymmetric formulation for US-QUAD8 is then presented in Section 3.3. The completeness and continuity properties of shape functions are discussed in Section 3.4. The implications of inappropriate choice of shape functions for the virtual and trial displacement fields are brought out in Section 3.5. This is then followed by some important discussions on the programming and computational aspects of the formulation in Section 3.6.

Although the unsymmetric formulation has been extensively tested for several benchmark problems [5], a formal patch test has not yet been reported. In Section 3.7, a series of patch tests are conducted to verify the reproducibility of every monomial term in the assumed displacement field for all generic element distortions. The US-QUAD8 element is compared with other 8-node quadrilateral elements in terms of their relative capabilities to pass the patch tests.

The extension of the unsymmetric formulation to the 20-node hexahedron element is detailed in Section 3.8 and its numerical validation is covered in Section 3.9. Some aspects on the computational time required by the unsymmetric formulation are investigated in Section 3.10.

3.2. A REVIEW ON THE PRINCIPLE OF VIRTUAL WORK

The conceptual background of the unsymmetric formulation is reviewed in this section. The unsymmetric formulation of Rajendran and Liew [5] is based on a deeper understanding of the difference of completeness/continuity requirements between the virtual and trial displacement fields in the principle of virtual work:

$$\int_{\Omega} \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\Omega - \int_{\Omega} \delta \mathbf{u}^T \mathbf{b} d\Omega - \int_{\Gamma_t} \delta \mathbf{u}^T \mathbf{t} d\Gamma = 0 \quad (3.1)$$

The vectors $\delta \boldsymbol{\varepsilon}$ and $\delta \mathbf{u}$ denote the virtual strains and virtual displacements respectively. The vector, $\boldsymbol{\sigma}$, represents the real stress field, while $\mathbf{b}$ and $\mathbf{t}$ represent the body and surface force intensities, respectively. The symbols Ω and Γ denote the domain and boundary of the problem considered.

In the application of the finite element method to the principle of virtual work in elasticity problems, it is a well-known fact that the choice for the virtual displacements in Equation (3.1) is arbitrary as long as it is continuous within the domain (i.e., within each element and across element boundaries) and that it satisfies the geometric boundary conditions. Provided that these two conditions are satisfied, any choice of virtual displacement has to yield the same system of equations when substituted in Equation

(3.1). This is implied by the well known “strong form $\Leftrightarrow$ weak form” equivalence (see e.g., [119]). The simplest choice of virtual displacement that satisfies these requirements for continuum problems is a linear polynomial within each element with continuity of displacements across adjacent element boundaries[‡]. This simplest possible choice of a linear polynomial in each element implicitly implies linear completeness of the virtual displacement field. Incidentally, the minimum required smoothness of the virtual displacements based on an argument of integrability of Equation (3.1) also implies linear completeness. This requirement arises by virtue of the fact that first integral of Equation (3.1) involves the first order derivatives of the virtual displacement.

The continuity and completeness requirements of the trial displacement model, however, are rather different from that of the virtual displacements. The minimum requirement is the same as that of virtual displacement, i.e., C^0 continuity and linear completeness. However, finite elements based on such a simple choice of linear virtual and trial displacement models (known as constant strain elements) are known to be less accurate and generally require a very fine mesh for convergence. Thus, going beyond such a simple choice, a natural alternative is to choose a higher order polynomial for the trial displacement model. In other words, the trial displacement model must preferably be complete up to all the monomial terms of the underlying exact displacement field of the problem so as to reproduce the exact solution. The virtual displacements, however, can still be a C^0 continuous linear polynomial model. A more continuous or more complete virtual displacement model is unnecessary, in the principle of virtual work.

[‡] The continuity of displacements across element boundaries is said to be C^0 continuous on the element boundaries.

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

The foregoing discussions suggest differences in the criteria for the choice of virtual and trial displacements; the choice of the former is influenced by continuity (compatibility) requirements while that of the latter by completeness requirements. In the classical Galerkin formulations, this difference is overlooked in favor of symmetry considerations i.e., the virtual displacement is chosen from the same function space as that of the trial displacement model so that the resulting stiffness matrix is symmetric. This over-riding symmetry consideration imposes a severe constraint on the choice of shape functions for the displacement fields, virtual and trial. In other words, the shape functions need to simultaneously satisfy the compatibility conditions as demanded by the virtual displacements as well as completeness requirements of the trial displacement model. The isoparametric shape functions used in the classical symmetric formulations satisfy the former requirement quite naturally but not always the latter [5]. Methods to verify these properties of isoparametric shape functions are discussed in Section 3.4.

The unsymmetric formulation investigated in this thesis is unique in the sense that it uses two different spaces for the choice of virtual and trial displacement models. The virtual displacement is chosen from the space of isoparametric shape functions because they guarantee continuity for any admissible element geometry. The trial displacement model is chosen from the space of metric shape functions because they guarantee higher order completeness irrespective of whether the mesh is distorted or not. Methods for verifying these properties of metric shape functions are discussed in Section 3.4. Formulations based on the above guiding principle for the choice of virtual and trial displacement models lead to the distortion immunity of the elements (e.g., US-QUAD8).

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

The success of an unsymmetric formulation depends on a clever management of the continuity and completeness requirements by an appropriate choice of two different sets of shape functions. Assigning the continuity and completeness requirements to the different sets of shape functions opens up possibilities for a wide spectrum of unsymmetric formulations. The investigations in this thesis are, however, confined to the particular choice of using isoparametric and metric shape functions for constructing the virtual and trial displacements, respectively.

3.3. A SUMMARY OF UNSYMMETRIC ELEMENT FORMULATION FOR US-QUAD8

In their unsymmetric element formulation, Rajendran and Liew [5] discretized the virtual displacements and strain fields using isoparametric shape functions. The trial displacement and strain fields are discretized using metric shape functions.

The isoparametric discretization of the virtual displacement and strain fields can be written as

$$\delta \mathbf{u}^{(e)} = \delta \bar{\mathbf{u}}^{(e)} \equiv \mathbf{N} \delta \bar{\mathbf{u}}_N^{(e)} \quad (3.2)$$

$$\delta \boldsymbol{\varepsilon}^{(e)} = \mathbf{L} \mathbf{N} \delta \bar{\mathbf{u}}_N^{(e)} \equiv \bar{\mathbf{B}} \delta \bar{\mathbf{u}}_N^{(e)} \quad (3.3)$$

Correspondingly, the metric discretization of the trial displacement and strain fields can be written as

$$\mathbf{u}^{(e)} = \hat{\mathbf{u}}^{(e)} \equiv \mathbf{M} \hat{\mathbf{u}}_N^{(e)} \quad (3.4)$$

$$\boldsymbol{\varepsilon}^{(e)} = \mathbf{L} \mathbf{M} \hat{\mathbf{u}}_N^{(e)} \equiv \hat{\mathbf{B}} \hat{\mathbf{u}}_N^{(e)} \quad (3.5)$$

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

The matrices $\mathbf{N}$ and $\mathbf{M}$ refer to the matrices of isoparametric and metric shape functions, $\mathbf{L}$ is the usual linear operator matrix^{††}, and $\bar{\mathbf{B}}$ and $\hat{\mathbf{B}}$ are the strain-displacement matrices corresponding to isoparametric and metric shape functions respectively. An over-bar is used to indicate an isoparametric discretization of the respective quantity while an over-hat refers to a metric discretization. Substituting Equations (3.2) – (3.5) into the principle of virtual work in Equation (3.1) yields:

$$\mathbf{A}_{e=1}^n \left(\int_{\Omega^{(e)}} \delta \bar{\mathbf{u}}_N^{(e)T} (\bar{\mathbf{B}}^T \mathbf{D} \hat{\mathbf{B}}) \hat{\mathbf{u}}_N^{(e)} d\Omega^{(e)} - \int_{\Omega^{(e)}} \delta \bar{\mathbf{u}}_N^{(e)T} \mathbf{N}^T \mathbf{b} d\Omega^{(e)} - \int_{\Gamma_t^{(e)}} \delta \bar{\mathbf{u}}_N^{(e)T} \mathbf{N}^T \mathbf{t} d\Gamma^{(e)} \right) = 0 \quad (3.6)$$

The symbol $\mathbf{A}_{e=1}^n ()$ denotes the usual finite element assembly process. In obtaining Equation (3.6), the generalized Hooke's Law, $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}$, has been used. The matrix, $\mathbf{D}$, represents the constitutive matrix for linear elastic materials.

Invoking the arbitrariness of the virtual displacements and performing the usual finite element assembly process in Equation (3.6) yields the set of linear algebraic equations:

$$\mathbf{K}\mathbf{U} = \mathbf{F} \quad (3.7)$$

where

$$\mathbf{K} = \mathbf{A}_{e=1}^n \left(\int_{\Omega^{(e)}} \bar{\mathbf{B}}^T \mathbf{D} \hat{\mathbf{B}} d\Omega^{(e)} \right) \quad (3.8)$$

$$\mathbf{F} = \mathbf{A}_{e=1}^n \left(\int_{\Omega^{(e)}} \mathbf{N}^T \mathbf{b}^{(e)} d\Omega^{(e)} + \int_{\Gamma_t^{(e)}} \mathbf{N}^T \mathbf{t}^{(e)} d\Gamma^{(e)} \right) \quad (3.9)$$

^{††} $\mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}^T$

and $\mathbf{K}$, $\mathbf{U}$ and $\mathbf{F}$ are the stiffness matrix, the vector of nodal displacements and vector of nodal forces, respectively. The stiffness matrix in Equation (3.8) is, in general, unsymmetric in view of the use of two different sets of shape functions. Explicit expressions for the isoparametric and metric shape functions in the above formulation are presented in Sections 3.3.1 and 3.3.2 respectively.

3.3.1. ISOPARAMETRIC SHAPE FUNCTIONS

The isoparametric shape functions in the unsymmetric formulation are well known and well documented in literature. For an 8-node quadrilateral element, the isoparametric shape functions are of the form

$$N_i = \frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i)(\xi\xi_i + \eta\eta_i - 1) \quad (3.10)$$

$$N_j = \frac{1}{2}(1 - \xi^2)(1 + \eta\eta_j) \quad (3.11)$$

$$N_k = \frac{1}{2}(1 + \xi\xi_k)(1 - \eta^2) \quad (3.12)$$

where i , j , and k refer to the corner nodes, mid-side nodes along the edges $\xi = 0$ and $\eta = 0$, respectively. The matrix of isoparametric shape functions, $\mathbf{N}$ in Equation (3.2), can be expressed as

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & \cdots & 0 & N_8 \end{bmatrix} \quad (3.13)$$

3.3.2. METRIC SHAPE FUNCTIONS

The metric shape functions for an 8-node quadrilateral element is obtained by solving the set of completeness conditions [117]

$$\sum_{i=1}^8 M_i x_i^p y_i^q = x^p y^q \quad (3.14)$$

where M_i is the shape function for node i . The monomial terms $x^p y^q$ in Equation (3.14) stands for a typical monomial term with appropriate values of exponents ($p, q, r = 0, 1, 2$).

For an 8-node quadrilateral element, it is desired to reproduce an exact displacement field of the form

$$\mathbf{u} = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{Bmatrix} a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^2 y + a_7 y^2 x \\ b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 + b_6 x^2 y + b_7 y^2 x \end{Bmatrix} \quad (3.15)$$

where a_i and b_i ($i=1,2,\dots,6$) are constants. The first six monomial terms in the displacement fields of Equation (3.15) correspond to a complete quadratic displacement field. The remaining two monomial terms are suitably chosen so that the selection of these terms does not result in a singular set of basis functions. In terms of metric shape functions, M_i , the finite element approximation to the displacement field in Equation (3.15) can be written as

$$\hat{\mathbf{u}}^{(e)} = \begin{Bmatrix} \hat{u}^{(e)}(x, y) \\ \hat{v}^{(e)}(x, y) \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^8 M_i \hat{u}_i^{(e)} \\ \sum_{i=1}^8 M_i \hat{v}_i^{(e)} \end{Bmatrix} \quad (3.16)$$

where the shape functions, M_i , are explicit functions of (x, y) coordinates, but can also be expressed in terms of local (ξ, η) coordinates using parametric mapping of geometry. For an element to reproduce the displacement field in Equation (3.15), the mathematical condition is:

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

$$\mathbf{u} - \hat{\mathbf{u}}^{(e)} = \mathbf{0} \Leftrightarrow \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{Bmatrix} \hat{u}^{(e)}(x, y) \\ \hat{v}^{(e)}(x, y) \end{Bmatrix}, \quad (x, y) \in \Omega \quad (3.17)$$

The above condition can be shown to yield the completeness conditions [117] in Equation (3.14). Equation (3.14) represents a set of eight simultaneous linear algebraic equations, written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_8 \\ y_1 & y_2 & y_3 & \cdots & y_8 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 y_1^2 & x_2 y_2^2 & x_3 y_3^2 & \cdots & x_8 y_8^2 \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_8 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \\ y \\ \vdots \\ xy^2 \end{Bmatrix} \quad (3.18)$$

or symbolically,

$$\mathbf{P}\mathbf{m} = \mathbf{p}(\mathbf{x}) \quad (3.19)$$

To improve the condition of the $\mathbf{P}$ -matrix in Equation (3.18), the coordinates x_i and y_i (Equation (3.18)) are evaluated with respect to the average of nodal coordinates of the element (x_c, y_c) . Hence, the corresponding vector of monomial terms $\mathbf{p}(\mathbf{x})$, and the $\mathbf{P}$ -matrix are re-defined as

$$\mathbf{p}(\mathbf{x}) = [1 \quad x - x_c \quad y - y_c \quad \cdots \quad (x - x_c)(y - y_c)^2]^T \quad (3.20)$$

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_{1-c} & x_{2-c} & x_{3-c} & \cdots & x_{8-c} \\ y_{1-c} & y_{2-c} & y_{3-c} & \cdots & y_{8-c} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{1-c} y_{1-c}^2 & x_{2-c} y_{2-c}^2 & x_{3-c} y_{3-c}^2 & \cdots & x_{8-c} y_{8-c}^2 \end{bmatrix} \quad (3.21)$$

where

$$\begin{cases} x_{i-c} = x_i - x_c \\ y_{i-c} = y_i - y_c \end{cases} \quad i = 1, 2, 3, \dots, 8 \quad (3.22)$$

Assuming that the $\mathbf{P}$ -matrix is not singular, the metric shape functions can be obtained by inverting the $\mathbf{P}$ -matrix in Equation (3.21). This yields

$$\mathbf{m} = \mathbf{P}^{-1} \mathbf{p}(\mathbf{x}) \quad (3.23)$$

The matrix of metric shape functions, $\mathbf{M}$, in Equation (3.4) is expressed as

$$\mathbf{M} = \begin{bmatrix} M_1 & 0 & M_2 & 0 & \cdots & M_8 & 0 \\ 0 & M_1 & 0 & M_2 & \cdots & 0 & M_8 \end{bmatrix} \quad (3.24)$$

The metric shape functions are derived from the assumed displacement field (Equation (3.15)) by enforcing the condition (3.17). Thus, they are capable of exactly representing this displacement field. The existence of the metric shape functions, however, hinges on the very existence of the inverse of the $\mathbf{P}$ -matrix. The requirement for the same is that the $\mathbf{P}$ -matrix is of full rank. The rank of the $\mathbf{P}$ -matrix, in turn, depends on the choices of monomial terms in Equation (3.15) and the shape of the element. This is an important aspect to be explored and will be taken up for investigation in Chapter 7.

3.4. COMPLETENESS AND CONTINUITY OF SHAPE FUNCTIONS

The specific choice of shape functions leading to a successful unsymmetric element formulation relies on two important properties, *viz.*, completeness and continuity. Both isoparametric and metric shape functions exhibit different potentials in terms of satisfaction of these properties. Accordingly, one offers a more appropriate choice than the other for serving as the virtual or trial displacement fields, respectively. The completeness and continuity of these shape functions are reviewed in this section for an 8-node quadrilateral element. Methods for verifying these properties are also discussed.

3.4.1. INTERPRETATION AND VERIFICATION OF COMPLETENESS OF SHAPE FUNCTIONS

The eight completeness conditions in Equation (3.14) can be categorized into three sets.

The first set of completeness conditions given by

$$\sum_{i=1}^8 M_i = 1 \quad ; \quad \sum_{i=1}^8 M_i x_i = x \quad ; \quad \sum_{i=1}^8 M_i y_i = y \quad (3.25 - 3.27)$$

are the well-known linear completeness requirements for a typical finite element formulation. Satisfaction of these conditions ensures convergence of the finite element solution in the limit of mesh refinement. Equation (3.25) represents the conditions to be satisfied for the finite element shape functions to reproduce rigid body motions while Equations (3.26) and (3.27) represents the conditions to reproduce constant strain fields.

The second set of completeness conditions is given by

$$\sum_{i=1}^8 M_i x_i^2 = x^2 \quad ; \quad \sum_{i=1}^8 M_i x_i y_i = xy \quad ; \quad \sum_{i=1}^8 M_i y_i^2 = y^2 \quad (3.28 - 3.30)$$

The higher order completeness requirements given by Equations (3.28) – (3.30) are required, in addition to the linear order completeness, to enable the element to reproduce exactly a complete quadratic displacement field. For an element to reproduce a complete quadratic displacement field that is independent of element distortions, all the completeness requirements in Equations (3.28) – (3.30) must be satisfied for any admissible geometry of the element. The third set of completeness conditions are

$$\sum_{i=1}^8 M_i x_i^2 y_i = x^2 y \quad ; \quad \sum_{i=1}^8 M_i y_i^2 x_i = y^2 x \quad (3.31 \text{ \& } 3.32)$$

Shape functions that satisfy Equations (3.31) and (3.32) for any admissible element geometry ensures reproduction of monomial terms $x^2 y$ and xy^2 regardless of element distortion.

The reproducibility of all eight monomial terms in the assumed displacement field in Equation (3.15) can be numerically verified for any shape function using the completeness conditions (3.25 – 3.32). These conditions can be checked for any arbitrary element shape. The shape functions evaluated at any point within the element, (x, y) , can then be checked, term by term, for satisfaction of completeness conditions (3.25 – 3.32). Such a numerical check for the metric shape functions, however, is trivial since the metric shape functions are derived from the very same set of completeness conditions. The metric shape functions, thus, reproduces all the monomial terms $1, x, y, x^2, xy, y^2, x^2y$ and xy^2 for any admissible element geometry.

Alternatively, all the completeness conditions can also be individually verified using the approach of Lee and Bathe [112]. This procedure starts by considering the parametric mapping of a distorted element (e.g., angularly distorted element)

$$x = a_0 + a_1\xi + a_2\eta + a_3\xi\eta \quad (3.33)$$

$$y = b_0 + b_1\xi + b_2\eta + b_3\xi\eta \quad (3.34)$$

Next, a typical monomial term in the displacement field (3.15), $u = y^2$ for instance, is expanded in terms of local coordinates as

$$u = y^2 = c_0 + c_1\xi + c_2\eta + c_3\xi^2 + c_4\xi\eta + c_5\eta^2 + c_6\xi^2\eta + c_7\xi\eta^2 + c_8\xi^2\eta^2 \quad (3.35)$$

where a_i , b_i and c_i are constants. An isoparametric interpolation of the displacement field, u , in terms of local coordinates is then written as

$$u = \sum_{i=1}^8 N_i(\xi, \eta) \bar{u}_i = d_0 + d_1\xi + d_2\eta + d_3\xi^2 + d_4\xi\eta + d_5\eta^2 + d_6\xi^2\eta + d_7\xi\eta^2 \quad (3.36)$$

Table 3.1. Reproducibility of monomial terms by isoparametric and metric shape functions for typical element distortions^{†††}

Monomial Terms	Distortion Type	Isoparametric	Metric
$1, x, y$	Regular, Aspect Ratio	√	√
	Angular	√	√
	Curved-edge	√	√
	Mid-side node	√	√
x^2, xy, y^2	Regular, Aspect Ratio	√	√
	Angular	×	√
	Curved-edge	×	√
	Mid-side node	×	√
x^2y, xy^2	Regular, Aspect Ratio	√	√
	Angular	×	√
	Curved-edge	×	√
	Mid-side node	×	√

To faithfully reproduce the term $u = y^2$, the isoparametric interpolation in Equation (3.36) must be able to represent all the monomial terms of Equation (3.35). Comparing Equations (3.35) and (3.36), it is observed that the isoparametric interpolation (Equation (3.36)) lacks the monomial term $\xi^2\eta^2$. This would mean that the isoparametric shape functions are incapable of reproducing the term y^2 under angular distortions. A similar verification can be repeated for other monomial terms in Equation (3.15) for other types of distortions. Such verifications for the metric shape functions would again be trivial. The results of numerical verification of the completeness conditions for typical element distortion types are shown in Table 3.1.

^{†††} The symbol “√” indicates that the shape functions is capable to reproduce the monomial terms. The symbol “×” denotes that the shape functions fails to reproduce the monomial terms

3.4.2. INTERPRETATION AND VERIFICATION OF CONTINUITY OF SHAPE FUNCTIONS

The continuity of shape functions needs to be assessed in terms of the intra- and inter-element continuity. Intra-element continuity refers to the continuity of shape functions within the element. Since the stiffness matrix involves the strains, which are the first derivatives of the displacements, the shape functions must ensure intra-element continuity of at least $C^1(\Omega^{(e)})$ within the element, that is, a linear interpolation. The minimum requirement of $C^1(\Omega^{(e)})$ continuity also implies linear completeness. A linear polynomial is required to ensure representation of at least a constant strain field within each element, and is crucial for the convergence in the limit of mesh refinement. Both isoparametric and metric shape functions satisfy the linear completeness requirements (see Table 3.1) for any admissible element geometry. Thus both the isoparametric and metric shape functions meet the minimum requirements of intra-element continuity.

Figure 3.1. A typical finite element mesh of two 8-node quadrilateral elements

Inter-element continuity refers to the continuity of displacements across adjacent element boundaries. The minimum requirement of inter-element continuity differs for structural and continuum elements. In continuum elements, only inter-element continuity of $C^0(\Gamma^{(e)})$ order across the elements boundaries is necessary. This simply implies continuous representation of the displacement fields across the element boundaries.

As an illustration, inter-element continuity of an 8-node quadrilateral element requires that the displacement interpolations based on element $\{1\}$ and $\{2\}$ to be the same along the common edge 2-6-3 (see Figure 3.1), i.e., $u^{(1)} = u^{(2)}$. In terms of shape functions, ψ_i , the displacements $u^{(1)}$ and $u^{(2)}$ are written as

$$u^{(1)} = \psi_1 u_1 + \psi_2 u_2 + \psi_3 u_3 + \psi_4 u_4 + \psi_5 u_5 + \psi_6 u_6 + \psi_7 u_7 + \psi_8 u_8 \quad (3.37)$$

$$u^{(2)} = \psi_2 u_2 + \psi_3 u_3 + \psi_6 u_6 + \psi_9 u_9 + \psi_{10} u_{10} + \psi_{11} u_{11} + \psi_{12} u_{12} + \psi_{13} u_{13} \quad (3.38)$$

To enforce equality of displacements $u^{(1)}$ and $u^{(2)}$ along the common edge, one possibility is to make the shape functions ψ_1 , ψ_4 , ψ_5 , ψ_7 and ψ_8 in element $\{1\}$, and ψ_9 , ψ_{10} , ψ_{11} , ψ_{12} and ψ_{13} in element $\{2\}$ vanish from Equations (3.37) and (3.38). In such a case, the resulting displacement interpolations in both elements become a linear combination of the shape functions ψ_2 , ψ_3 and ψ_6 with u_2 , u_3 and u_6 as coefficients. To ensure the above, the shape functions, ψ_i must be compatible shape functions, i.e., they are to be designed such that they take a value of unity at node i and zero at all edges not containing the node. The aforementioned property will hereafter be referred to as the

node-edge Kronecker delta property. This guarantees continuity of shape functions of $C^0(\Gamma^{(e)})$ order across element boundaries.

Figure 3.2. Plot of isoparametric shape functions of an 8-node quadrilateral element for (a) typical corner node; (b) typical mid-side node

The isoparametric shape functions inherently satisfy the node-edge Kronecker delta property for all element geometries. As an illustration, Figure 3.2 shows the plots of the isoparametric shape functions for two typical nodes. It is seen from the plots that the isoparametric shape functions indeed satisfy the node-edge Kronecker delta property. It is trivial to verify that shape functions of other nodes also satisfy this property.

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

Figure 3.3. Plot of metric shape functions of an 8-node quadrilateral element with angular distortion for (a) typical corner node; (b) typical mid-side node

Figure 3.4. Plot of metric shape functions of an 8-node quadrilateral element with curved-edge distortion for (a) typical corner node; (b) typical mid-side node

Figure 3.5. Plot metric shape functions of an 8-node quadrilateral element with mid-side node distortion for (a) typical corner node; (b) typical mid-side node

The metric shape functions satisfy the node-edge Kronecker delta property only for regular element shapes. They generally loose this property when the elements are distorted. Figures 3.3 – 3.5 show the plots of metric shape functions in (ξ, η) of two typical nodes for three types of distortions *viz.*, angular, curved-edge and mid-side node. The shape function plots reveals that the metric shape functions violate the node-edge Kronecker delta property in the presence of angular and curved-edge distortion. However, the metric shape functions are observed to satisfy the node-edge Kronecker delta property for mid-side node distortions.

The conditions under which the isoparametric and metric shape functions satisfy the continuity and completeness conditions discussed in Sections 3.4.1 and 3.4.2 are summarized in Table 3.2. An outstanding observation from Table 3.2 is that the

isoparametric shape functions form a good choice for continuity while the metric shape functions provide a good choice for completeness. In view of the continuity and completeness requirements based on the virtual work principle in Section 3.2, these functions offer themselves as natural choices for constructing the virtual and trial displacement models, respectively.

Table 3.2. Continuity and completeness characteristics of isoparametric and metric shape functions

	Isoparametric	Metric
Satisfaction of all continuity conditions	All element geometries	Regular element shapes
Satisfaction of all higher order completeness conditions	Regular element shapes	All element geometries

As a summary, the guidelines for appropriate choices of shape functions leading to a successful unsymmetric formulation can be stated in terms of the following two criteria:

Criterion 1. *For all element geometries, distorted or not, the shape functions for discretizing the virtual displacement field must satisfy intra-element continuity of at least $C^1(\Omega^{(e)})$ order within the element domain, and inter-element displacement continuity of at least $C^0(\Gamma^{(e)})$ order (i.e., node-edge Kronecker delta property).*

Criterion 2. *For all element geometries, distorted or not, the shape functions used to model the trial displacement field must satisfy linear as well as higher order completeness requirements [117] so as to enable the reproduction of assumed polynomial displacement fields.*

Element formulations satisfying both *Criteria 1* and *2* are immune to all distortions for problems where the exact solution involves the monomial terms of the assumed polynomial displacement field.

3.5. ERROR ANALYSIS OF SYMMETRIC AND UNSYMMETRIC FORMULATION

Physical interpretations as to why the unsymmetric formulation performs better than the classical symmetric formulation is analyzed in terms of error in nodal forces. The discussions pertinent to this section assume that the exact displacement field of the problem takes a form as that in Equation (3.15). Further, it is assumed that the body and surface force intensities, $\mathbf{b}$ and $\mathbf{t}$, can be interpolated exactly using either isoparametric or metric shape functions so that the solution will not be affected by any errors in the nodal lumping of the body and surface force intensities.

The approach for error analyses of the symmetric and unsymmetric element formulations is presented as follows:

- (i) Symmetric formulation: Both virtual and trial displacement fields are discretized using isoparametric shape functions. The error in nodal forces can be expressed as

$$\mathbf{f}_{error}^{(e)} = \int_{\Omega^{(e)}} \bar{\mathbf{B}}^T \mathbf{D} \mathbf{L} (\bar{\mathbf{u}}^{(e)} - \mathbf{u}^{(e)}) d\Omega^{(e)} \quad (3.39)$$

The isoparametric shape functions satisfy the node-edge Kronecker delta property for any element geometry. Consequently, their use for the virtual displacement field complies with *Criterion 1*. However, they do not satisfy *Criterion 2*. Thus, the isoparametric shape functions cannot guarantee the condition $\bar{\mathbf{u}}^{(e)} - \mathbf{u}^{(e)} = \mathbf{0}$ for any element shape. As a result, the error in nodal forces in Equation (3.39)

will not, in general, vanish due to non-satisfaction of higher order completeness.

This leads to deterioration in solution accuracy of the symmetric formulation in the presence of mesh distortions.

- (ii) Unsymmetric formulation: The virtual displacements are discretized using isoparametric shape functions while the trial displacements are discretized using metric shape functions. The error in nodal forces is

$$\mathbf{f}_{error}^{(e)} = \int_{\Omega^{(e)}} \overline{\mathbf{B}}^T \mathbf{D} \mathbf{L} (\hat{\mathbf{u}}^{(e)} - \mathbf{u}^{(e)}) d\Omega^{(e)} \quad (3.40)$$

The choice of isoparametric shape functions for the virtual displacements guarantees the node-edge Kronecker delta property for any element shape, thus satisfying *Criterion 1*. The choice of metric shape functions to model the trial displacements guarantees the condition $\hat{\mathbf{u}}^{(e)} - \mathbf{u}^{(e)} = \mathbf{0}$ for any element shape by virtue of Equation (3.17). This complies with *Criterion 2*. Thus, the error in nodal forces (3.40) vanishes when the exact solution involves only the monomial terms in Equation (3.15). The unsymmetric formulation is immune to mesh distortions with respect to every monomial term in the assumed displacement field.

3.6. COMPUTER IMPLEMENTATION

The unsymmetric element has been implemented in an in-house FORTRAN code. In addition to the isoparametric shape functions, computation of the metric shape functions and their derivatives are required at each Gauss point. Since the stiffness matrix is, in general, unsymmetric, both upper and lower triangular portions of the stiffness matrices need to be computed and stored. An in-house unsymmetric version of the frontal solver is used for solving the global equations.

A pseudo-code for the element stiffness routine is given below:

1. Loop over elements in the domain
 - a) Form the $\mathbf{P}$ -matrix in Equation (3.21) and compute its inverse
2. Loop over integration points
 - a) Compute the isoparametric shape functions, $\mathbf{N}$, and their derivatives as in the case of the classical isoparametric element.
 - b) Compute the metric shape functions, $\mathbf{m}$, using Equation (3.23), and their derivatives using the equations, $\mathbf{m}_{,x} = \mathbf{P}^{-1}\mathbf{p}_{,x}$ and $\mathbf{m}_{,y} = \mathbf{P}^{-1}\mathbf{p}_{,y}$, which are the differentiated versions of Eq. (3.23).
 - c) Compute the matrices $\bar{\mathbf{B}} = \mathbf{L}\mathbf{N}$ and $\hat{\mathbf{B}} = \mathbf{L}\mathbf{M}$.
 - d) From the material constants, set up the elastic constitutive matrix, $\mathbf{D}$.
 - e) Compute the unsymmetric stiffness matrix using Equation (3.8)
3. Terminate loop over integration points.
4. Terminate loop over elements.

The computation the vector of nodal forces, $\mathbf{F}$, subsequent finite element assembly and solution of equations processes follow from conventional finite element practices.

3.7. PATCH TEST

The patch test was originally introduced by Irons et al. [120] to verify if an arbitrary patch of elements is capable of reproducing a constant strain field, as the size of elements in the patch become infinitesimally small. Elements satisfying the patch test are deemed to be able to converge to the exact solution in the limit of mesh refinement. Since its

introduction, the patch test had been modified and redefined by many authors [29, 121 – 124]. In general, the patch tests can be performed in three different ways *viz.*, patch test *A*, patch test *B* and patch test *C* as described by Zienkiewicz [125].

Rajendran and Liew [5] tested the performance of the US-QUAD8 element using a series of benchmark problems. However, a formal patch test was not reported in their work. As a part of critical evaluation of their element for the present work, an elaborate patch test is conducted. The patch test is performed in the strong form, i.e., considering a patch of four finite-sized elements without resorting to mesh refinement. Constant, linear and quadratic strain patch tests are conducted conforming to patch test type *B* [125]. These tests serve to verify if all monomial terms in Equation (3.15) are indeed reproduced exactly by US-QUAD8. The patches considered are designed to include all the generic distortions that an 8-node quadrilateral may experience in practice *viz.*, angular, curved-edge and mid-side node distortions.

3.7.1. METHODOLOGY

The monomial terms in Equation (3.15) can be grouped into monomial terms of the first (x, y) , second (x^2, xy, y^2) and third (x^2y, xy^2) order. In this section, the reproduction of monomial terms in each of these groups will be tested independently.

In solid mechanics, the differential equations of stress equilibrium cast in terms of displacement derivatives is given by the Navier equations of elasticity. For the special case of plane strain, these equations are

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{1}{1-2\nu} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + \frac{b_x}{\mu} = 0 \quad (3.41)$$

$$\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{1}{1-2\nu} \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{b_y}{\mu} = 0 \quad (3.42)$$

where ν is the Poisson's ratio, μ is the Lamé constant, b_x and b_y are the body force intensities in the x – and y – directions, respectively.

For the patch tests performed in this section, the displacement fields that are imposed on the patch of elements are carefully chosen to satisfy the Navier equations of elasticity so as to ensure that the fields are physically possible fields. This is essential in order to ascertain that the solution errors obtained during the patch tests are entirely due to elements defects, if any, and not due to non-satisfaction of the equilibrium equations.

With such carefully selected imposed fields, an element free of completeness or compatibility defects will pass the patch tests in the strong form (e.g., US-QUAD8). An element satisfying only linear completeness and free of compatibility defects will pass the constant strain patch test in the strong form and higher order patch tests in the weak form (e.g., QUAD8). An element satisfying only linear or even higher order completeness but with compatibility defects may or may not be able to pass the patch tests even in the weak form (e.g., MM-QUAD8^{††}, other incompatible elements). In other words, any errors in the finite element solution result only from the inability of the element to reproduce the monomial terms in the imposed displacement field.

^{††} MM-QUAD8 is the symmetric 8-node quadrilateral element using metric shape functions for both virtual and trial displacement fields.

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

For the linear displacement patch test, the choice of the imposed field is rather easy. Assuming for simplicity, that body force intensities are zero, any displacement field of the form

$$u = a_1x + a_2y \quad (3.43)$$

$$v = b_1x + b_2y \quad (3.44)$$

with arbitrary choices for a_i 's and b_i 's can be shown to satisfy both the Navier equations of elasticity simultaneously. The strains resulting from these displacement fields can be computed via the strain-displacement relations

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (3.45, 3.46 \text{ \& } 3.47)$$

The stresses can then be computed using the generalized Hooke's law. A patch test that involves displacement fields in Equations (3.43) and (3.44) is a constant strain patch test.

In the quadratic displacement patch test, the displacement fields of interest are chosen to be of the form

$$u = a_3x^2 + a_4xy + a_5y^2 \quad (3.48)$$

$$v = b_3x^2 + b_4xy + b_5y^2 \quad (3.49)$$

This is known as a linear strain patch test. Unlike the constant strain patch test, the constants, a_i 's and b_i 's in Equations (3.48) and (3.49) cannot be chosen independent of each other. The relation between the constants can be obtained by solving Equations (3.41) and (3.42) for a_i 's and b_i 's by substituting u and v from Equations (3.48) and (3.49). For simplicity, zero body force intensities are again assumed here. Since there are two (Navier) equations, they can be solved for any two arbitrary constants a_i 's and b_i 's

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

(a_4 and b_4 for example). This yields expressions for these constants in terms of other a_i 's and b_i 's.

The cubic displacement patch test considered in this section is concerned with the reproduction of displacement fields of the form

$$u = a_6 x^2 y + a_7 xy^2 \quad (3.50)$$

$$v = b_6 x^2 y + b_7 xy^2 \quad (3.51)$$

The monomial terms in Equations (3.50) and (3.51) do not involve all the third order terms of the Pascal triangle. The patch test with the displacement fields in Equations (3.50) and (3.51) is a quadratic strain patch test.

As with the linear strain patch test, the constants a_i 's and b_i 's cannot be chosen independent of each other. If a procedure similar to the previous is adopted, the solution for any two arbitrary constants will be dependent on the positions x and y . This violates the presumption that a_i 's and b_i 's are constants. Appropriate body force intensities must be included to ensure that the displacement fields take the forms of Equations (3.50) and (3.51) with a_i 's and b_i 's being only constants. Hence, a different procedure needs to be adopted here. By assuming any arbitrary values of constants a_6 , a_7 , b_6 and b_7 , the expressions for u and v given by Equations (3.50) and (3.51) are substituted into the Navier equations and solved for the unbalanced body force intensities b_x and b_y . The negative of these forces are then applied as body forces and are suitably integrated to obtain the lumped nodal forces to be applied at the nodes of the patch.

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

Four patches of elements shown in Figures 3.6a – 3.6d, each consisting of four elements is considered. The patch of elements considered involves typical distortions that are admissible in an 8-node quadrilateral, viz., angular, curved edge and midside-node distortions.

Figure 3.6. Patches of elements considered for patch test; (a) patch with angular distortion; (b) patch with curved-edge distortion; (c) patch with mid-side node distortion; (d) patch of arbitrary geometry

A patch of arbitrary element geometry that includes the effects of all three distortions previously mentioned is also considered. The test is performed according to patch test type B of Zienkiewicz [125]. For each patch, the displacements are applied at the

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

boundary nodes according to the order of patch test performed, constant, linear or quadratic. The lumped body forces, if applicable, are applied at the interior nodes. The accuracy of displacements at the interior nodes and the stresses at the parametric center (ξ, η) in each element are observed.

Three elements, *viz.*, the 8-node isoparametric quadrilateral (QUAD8), the symmetric 8-node quadrilateral with metric shape functions (MM-QUAD8) and the unsymmetric 8-node quadrilateral (US-QUAD8) are considered. For reference purposes, the theoretical displacements at the interior nodes and stresses of each element at the parametric center of elements are listed in Tables 3.3a and 3.3b, respectively, for all the four patches according to the order of patch test (constant, linear or quadratic). The stresses are computed using material constants $\mu = 3846.15$ and $\nu = 0.3$.

The constant strain patch test is performed by imposing displacement fields of the forms

$$u = 0.01x + 0.02y \quad (3.52)$$

$$v = 0.03x + 0.01y \quad (3.53)$$

on the four patches of elements in Figures 3.5a – 3.5d.

In the linear strain patch test, the Navier equations of elasticity are solved for the constants a_4 and b_4 in Equations (3.48) and (3.49). This yields

$$a_4 = -0.4(2b_3 + 7b_5) \quad (3.54)$$

$$b_4 = -0.4(7a_3 + 2a_5) \quad (3.55)$$

Choosing arbitrary values for constants for a_3 , a_5 , b_3 and b_5 , the displacement field used for the linear strain patch test in this section is

$$u = 0.01x^2 - 0.1xy + 0.02y^2 \quad (3.56)$$

$$v = 0.02x^2 - 0.044xy + 0.03y^2 \quad (3.57)$$

The displacements and body force intensities imposed on the patches of elements for the quadratic strain patch test are

$$u = 0.001x^2y + 0.002xy^2 \quad (3.58)$$

$$v = 0.003x^2y + 0.001xy^2 \quad (3.59)$$

The body force intensities are obtained by substituting the displacement fields (3.58) and (3.59) into the Navier equations. This yields

$$b_x = 3846.15(-0.004x + 2.5(-0.006x - 0.004y) - 0.02y) \quad (3.60)$$

$$b_y = 3846.15(-0.02x - 0.06y - 0.01(x + y)) \quad (3.61)$$

3.7.2. RESULTS

Table 3.4 summarizes the conclusions from the patch tests. The details of the test results are given in Tables 3.5a – 3.7b. The higher order monomial terms in the quadratic strain patch test demands a higher order integration of the stiffness matrix than usually required. The results in Tables 3.7a and 3.7b are obtained using 5×5 Gaussian quadrature rule. Figures 3.7 – 3.14 show the σ_{xx} , σ_{yy} and σ_{xy} stress contour plots of all four element patches for the linear and quadratic strain patch tests. The US-QUAD8 element passes the patch tests for all the patches of elements considered.

Table 3.3a. Theoretical solutions of displacements for patch tests

Node	dof	Theoretical displacements for constant strain patch test				Theoretical displacements for linear strain patch test				Theoretical displacements for quadratic strain patch test			
		Angular	Curved	Mid-side	Arbitrary	Angular	Curved	Mid-side	Arbitrary	Angular	Curved	Mid-side	Arbitrary
12	u	0.095000	0.075000	0.078000	0.142310	-0.747500	-0.437500	-0.496600	0.410792	0.099750	0.015625	0.054600	0.071702
	v	0.135000	0.100000	0.109000	0.089839	0.053000	0.037500	0.036300	1.153055	0.141750	0.027344	0.076300	0.045265
8	u	0.095000	0.050000	0.081000	0.105000	-0.628750	-0.218750	-0.476950	-0.617500	0.099750	0.046875	0.062775	0.105000
	v	0.197500	0.087500	0.165500	0.115000	0.349875	0.034375	0.231075	0.165000	0.207375	0.062500	0.128263	0.115000
7	u	0.145000	0.125000	0.122000	0.156076	-1.647500	-1.187500	-1.156600	-1.841100	0.369750	0.421875	0.219600	0.414173
	v	0.285000	0.250000	0.241000	0.203196	0.593000	0.487500	0.432300	0.176552	0.726750	0.738281	0.433800	0.539214
18	u	0.145000	0.150000	0.119000	0.195000	-1.828750	-1.968750	-1.236950	-1.397500	0.369750	0.234375	0.205275	0.546000
	v	0.222500	0.262500	0.184500	0.185000	0.139875	0.309375	0.098075	0.933000	0.567375	0.468750	0.318263	0.518000
6	u	0.140000	0.100000	0.100000	0.150000	-1.715000	-0.875000	-0.875000	-0.990000	0.343000	0.125000	0.125000	0.270000
	v	0.245000	0.175000	0.175000	0.150000	0.269500	0.137500	0.137500	0.468000	0.600250	0.218750	0.218750	0.270000

Table 3.3b. Theoretical solution of stresses at parametric center for patch tests

Elem	Theoretical displacements for constant strain				Theoretical displacements for linear strain patch test				Theoretical displacements for quadratic strain patch test			
	Angular	Curved	Mid-side	Arbitrary	Angular	Curved	Mid-side	Arbitrary	Angular	Curved	Mid-side	Arbitrary
1	σ_{xx}	192.31	192.31	192.31	-1376.92	-1153.85	-1359.23	-3961.37	1629.81	1018.63	317.00	558.80
	σ_{yy}	192.31	192.31	192.31	-3469.23	-2692.31	-940.77	534.60	3072.12	1896.03	448.91	330.71
	σ_{xy}	192.31	192.31	192.31	-1869.23	-1461.54	-633.08	-246.17	716.35	456.73	177.24	185.75
2	σ_{xx}	192.31	192.31	192.31	-1453.85	-1230.77	-1286.92	-3730.49	389.42	129.21	1383.54	1393.81
	σ_{yy}	192.31	192.31	192.31	-1084.62	-307.69	-3182.31	-1264.64	562.50	141.23	2600.06	1636.97
	σ_{xy}	192.31	192.31	192.31	-715.38	-307.69	-1717.69	-1094.64	216.35	72.12	611.09	789.25
3	σ_{xx}	192.31	192.31	192.31	-3876.92	-3615.38	-3486.92	-8143.66	2769.23	2965.75	2301.62	4266.74
	σ_{yy}	192.31	192.31	192.31	-2892.31	-3307.69	-2674.62	-886.93	4000.00	4516.23	3348.91	4011.84
	σ_{xy}	192.31	192.31	192.31	-1907.69	-2076.92	-1751.54	-1475.21	1538.46	1610.58	1275.32	2242.63
4	σ_{xx}	192.31	192.31	192.31	-3953.85	-3692.31	-3559.23	-8166.05	1048.08	1162.86	837.39	2010.60
	σ_{yy}	192.31	192.31	192.31	-507.69	-923.08	-433.08	1577.97	1009.62	1271.03	800.06	991.86
	σ_{xy}	192.31	192.31	192.31	-753.85	-923.08	-666.92	-275.21	557.69	649.04	443.78	462.33

Table 3.4. Results of patch tests^{††††}

	QUAD8	MM-QUAD8	US-QUAD8
CONSTANT STRAIN PATCH TEST			
Angular	√	√	√
Curved-edge	√	×	√
Mid-side node	√	√	√
Arbitrary patch	√	×	√
LINEAR STRAIN PATCH TEST			
Angular	×	×	√
Curved-edge	×	×	√
Mid-side node	×	√	√
Arbitrary patch	×	×	√
QUADRATIC STRAIN PATCH TEST			
Angular	×	×	√
Curved-edge	×	×	√
Mid-side node	×	×	√
Arbitrary patch	×	×	√

QUAD8 fails all the linear and quadratic patch tests. This is due to lack of higher order completeness of the isoparametric shape functions. The MM-QUAD8 element fails the patch tests except for the constant strain patch test with angular and mid-side node distortion, and the linear strain patch tests with mid-side node distortion. Unlike the QUAD8 element, the failure MM-QUAD8 to pass the patch tests is due to non-satisfaction of the inter-element continuity across the element boundaries. The reason for MM-QUAD8, passing the constant strain and linear strain patch tests for mid-node distortion can be explained. Numerical check on node-edge Kronecker delta property reveals that MM-QUAD8 does indeed satisfy the inter-element continuity for these two cases. This leads to the conclusion that the inter-element continuity is affected by the order of monomial terms present in the assumed displacement fields, as well as whether the edges of the elements are straight or curved. The effect of the latter appears less dominating compared to the former.

^{††††} The symbol “√” indicates that the element passes the patch test. The symbol “×” denotes that the element fails the patch test.

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

To have an element performance that is unaffected by errors due to inter-element discontinuity, the shape functions for the virtual displacement field must possess the node-edge Kronecker delta property for any admissible element geometry. For an element to reproduce a typical monomial term for any admissible element geometry, the shape functions for the trial displacement field must preserve the completeness condition associated with that particular monomial term for any admissible element geometry. The US-QUAD8 element satisfies both the above criteria and is hence capable of passing all the patch tests considered.

As a minimum requirement, an element is required to pass the patch test in the limit of mesh refinement *viz.*, in the weak form. It can be easily shown that both elements, QUAD8 and MM-QUAD8, pass the patch tests as the mesh is progressively refined. However, in the patch tests conducted above, mesh refinement has not been considered and the patch test has been conducted in the *strong form*, i.e., the elements were deemed to have failed the test if they fail to yield the exact solutions with just fewer elements in the patch.

In principle, a pass of strong form patch test, with respect to each of the monomial terms present in the trial displacement field along with various possible mesh distortion types, certifies the *distortion immunity* of the element with respect to the monomial terms considered. This means that, whatever be the type of mesh distortion, the element will be able to reproduce the exact displacement field as long as it involves only these monomial terms. Although the US-QUAD8 element does give distortion-immune performance in

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

this respect, there exists a maximum limit on the extent of distortion that it can handle. This limit does not depend on the choice of shape functions for the virtual and trial displacement fields, but rather, the Jacobian of geometric mapping. The element continues to give distortion-immune performance as long as the Jacobian is positive at the numerical integration points.

Table 3.5a. Displacement results of constant strain patch test

[illegible]

Table 3.6a. Displacement results of linear strain patch test

Node	dof	% Error in computed nodal disp. for angular distortion			% Error in computed nodal disp. for curved-edge distortion			% Error in computed nodal disp. for midside node distortion			% Error in computed nodal disp. for general (arbitrary) distortion		
		QUAD8	MM- QUAD8	US- QUAD8	QUAD8	MM- QUAD8	US- QUAD8	QUAD8	MM- QUAD8	US- QUAD8	QUAD8	MM- QUAD8	US- QUAD8
12	u	0.605	1.645	0.000	2.037	11.278	0.000	1.879	0.000	0.000	9.327	14.831	0.000
	v	0.997	22.200	0.000	54.949	92.227	0.000	9.147	0.001	0.000	2.991	1.890	0.000
8	u	0.729	1.412	0.000	14.736	8.878	0.000	2.430	0.000	0.000	4.709	12.598	0.000
	v	0.204	0.384	0.000	178.735	56.420	0.000	8.746	0.001	0.000	4.351	17.189	0.000
7	u	0.186	0.246	0.000	0.751	1.828	0.000	0.807	0.000	0.000	1.484	2.581	0.000
	v	0.119	1.405	0.000	4.227	4.575	0.000	0.768	0.000	0.000	11.166	3.917	0.000
18	u	0.181	0.136	0.000	1.637	3.041	0.000	0.937	0.000	0.000	4.380	1.104	0.000
	v	0.563	0.028	0.000	19.859	8.863	0.000	20.607	0.002	0.000	5.553	2.854	0.000
6	u	0.440	0.333	0.000	2.762	2.988	0.000	2.831	0.000	0.000	1.789	5.127	0.000
	v	0.454	2.094	0.000	69.178	7.627	0.000	27.996	0.001	0.000	9.835	12.325	0.000

Table 3.6b. Stress results of linear strain patch test

	Elem	% Error in computed stresses for angular distortion			% Error in stresses for curved-edge distortion			% Error in computed stresses for midside node distortion			% Error in computed stresses for general (arbitrary) distortion		
		QUAD8	MM-QUAD8	US-QUAD8	QUAD8	MM-QUAD8	US-QUAD8	QUAD8	MM-QUAD8	US-QUAD8	QUAD8	MM-QUAD8	US-QUAD8
1	σ_{xx}	1.074	3.332	0.000	18.365	7.153	0.000	7.515	0.000	0.000	14.618	8.540	0.000
	σ_{yy}	0.272	1.470	0.000	22.458	4.687	0.000	14.511	0.000	0.000	60.751	19.204	0.000
	σ_{xy}	0.307	0.323	0.000	31.677	3.426	0.000	10.082	0.000	0.000	27.304	4.185	0.000
2	σ_{xx}	0.552	2.886	0.000	36.020	13.606	0.000	1.367	0.000	0.000	5.412	6.166	0.000
	σ_{yy}	0.175	5.654	0.000	228.740	62.253	0.000	3.195	0.000	0.000	10.841	3.821	0.000
	σ_{xy}	0.634	2.264	0.000	191.426	29.835	0.000	0.229	0.000	0.000	1.863	7.094	0.000
3	σ_{xx}	0.306	1.071	0.000	12.262	9.900	0.000	2.929	0.000	0.000	9.295	0.648	0.000
	σ_{yy}	0.006	2.408	0.000	21.278	6.686	0.000	5.104	0.000	0.000	22.074	2.805	0.000
	σ_{xy}	0.367	0.627	0.000	28.359	2.995	0.000	3.644	0.000	0.000	4.818	0.078	0.000
4	σ_{xx}	0.270	0.995	0.000	5.739	4.285	0.000	0.494	0.000	0.000	8.009	1.170	0.000
	σ_{yy}	1.313	16.239	0.000	65.502	17.826	0.000	23.479	0.001	0.000	11.430	8.210	0.000
	σ_{xy}	1.047	3.100	0.000	50.155	6.057	0.000	0.590	0.000	0.000	29.469	35.557	0.000

Figure 3.7. Linear stress contour plots of a patch with angular distortions; (a) σ_{xx} stress; (b) σ_{yy} stress; (c) σ_{xy} stress

Figure 3.8. Linear stress contour plots of a patch with curved-edge distortions; (a) σ_{xx} stress; (b) σ_{yy} stress; (c) σ_{xy} stress

Figure 3.9. Linear stress contour plots of a patch with mid-side node distortions; (a) σ_{xx} stress; (b) σ_{yy} stress; (c) σ_{xy} stress

Figure 3.10. Linear stress contour plots of a patch with arbitrary geometry; (a) σ_{xx} stress; (b) σ_{yy} stress; (c) σ_{xy} stress

Table 3.7a. Displacement results of quadratic strain patch test

Node	dof	% Error in computed nodal disp. for angular distortion			% Error in computed nodal disp. for curved-edge distortion			% Error in computed nodal disp. for midside node distortion			% Error in computed nodal disp. for general (arbitrary) distortion		
		QUAD8	MM-QUAD8	US-QUAD8	QUAD8	MM-QUAD8	US-QUAD8	QUAD8	MM-QUAD8	US-QUAD8	QUAD8	MM-QUAD8	US-QUAD8
12	u	1.670	28.283	0.000	17.880	222.947	0.000	11.769	18.294	0.000	9.709	29.145	0.000
	v	1.249	17.216	0.000	34.729	63.941	0.000	0.841	4.876	0.000	23.588	77.299	0.000
8	u	0.889	25.043	0.000	47.518	59.544	0.000	18.606	18.294	0.000	8.140	0.526	0.000
	v	3.339	5.169	0.000	0.727	62.874	0.000	8.996	13.930	0.000	11.013	15.283	0.000
7	u	0.563	8.630	0.000	3.804	34.819	0.000	4.203	0.199	0.000	1.765	18.011	0.000
	v	0.431	4.985	0.000	4.986	33.729	0.000	0.184	1.999	0.000	0.141	7.729	0.000
18	u	0.772	4.393	0.000	0.600	100.961	0.000	4.744	4.319	0.000	1.346	18.696	0.000
	v	0.023	0.527	0.000	4.325	63.636	0.000	1.069	5.304	0.000	5.880	12.261	0.000
6	u	4.046	4.850	0.000	9.428	35.319	0.000	17.790	33.114	0.000	3.601	3.193	0.000
	v	2.160	2.915	0.000	13.363	9.810	0.000	6.440	15.285	0.000	1.062	9.112	0.000

Table 3.7b. Stress results for quadratic strain patch test

	Elem	% Error in computed stresses for angular distortion			% Error in stresses for curved-edge distortion			% Error in computed stresses for midside node distortion			% Error in computed stresses for general (arbitrary) distortion		
		QUAD8	MM-QUAD8	US-QUAD8	QUAD8	MM-QUAD8	US-QUAD8	QUAD8	MM-QUAD8	US-QUAD8	QUAD8	MM-QUAD8	US-QUAD8
1	σ_{xx}	0.497	52.661	0.000	36.058	14.446	0.000	6.754	34.123	0.000	12.628	24.114	0.000
	σ_{yy}	0.632	4.995	0.000	44.698	12.133	0.000	1.344	73.810	0.000	3.944	8.571	0.000
	σ_{xy}	1.033	81.197	0.000	20.329	40.285	0.000	6.595	55.155	0.000	5.037	111.731	0.000
2	σ_{xx}	3.656	100.238	0.000	65.914	20.694	0.000	3.042	2.614	0.000	3.109	195.560	0.000
	σ_{yy}	2.963	98.811	0.000	160.998	76.052	0.000	1.321	39.716	0.000	1.722	140.678	0.000
	σ_{xy}	7.762	98.028	0.000	9.443	127.809	0.000	2.313	6.316	0.000	3.575	137.188	0.000
3	σ_{xx}	0.442	2.234	0.000	12.830	48.645	0.000	0.481	8.052	0.000	8.225	12.713	0.000
	σ_{yy}	0.458	12.586	0.000	22.692	45.865	0.000	0.990	27.610	0.000	3.498	24.579	0.000
	σ_{xy}	0.192	2.610	0.000	5.678	42.462	0.000	1.718	10.811	0.000	3.104	21.336	0.000
4	σ_{xx}	0.754	102.626	0.000	18.431	7.681	0.000	1.059	17.733	0.000	8.234	99.933	0.000
	σ_{yy}	0.869	179.849	0.000	35.753	0.380	0.000	0.611	61.462	0.000	11.653	135.525	0.000
	σ_{xy}	1.739	130.446	0.000	13.369	2.164	0.000	4.395	69.964	0.000	2.041	211.504	0.000

Figure 3.11. Quadratic stress contour plots of a patch with angular distortions; (a) σ_{xx} stress; (b) σ_{yy} stress; (c) σ_{xy} stress

Figure 3.12. Quadratic stress contour plots of a patch with curved-edge distortions; (a) σ_{xx} stress; (b) σ_{yy} stress; (c) σ_{xy} stress

Figure 3.13. Quadratic stress contour plots of a patch with mid-side node distortions; (a) σ_{xx} stress; (b) σ_{yy} stress; (c) σ_{xy} stress

Figure 3.14. Quadratic stress contour plots of a patch with arbitrary geometry; (a) σ_{xx} stress; (b) σ_{yy} stress; (c) σ_{xy} stress

3.8. EXTENSION OF UNSYMMETRIC FORMULATION TO THE 20-NODE HEXAHEDRON

The excellent performance of the unsymmetric 8-node quadrilateral element under distorted meshes motivates extension of the same idea to 3-D solid elements. In this section, the unsymmetric element formulation for the 20-node hexahedron element is explored. The displacement fields, pertinent to a 20-node hexahedron involve

$$u = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5xy + a_6y^2 + a_7yz + a_8z^2 + a_9zx + a_{10}x^2y + a_{11}x^2z + a_{12}y^2x + a_{13}y^2z + a_{14}z^2x + a_{15}z^2y + a_{16}xyz + a_{17}x^2yz + a_{18}xy^2z + a_{19}xyz^2 \quad (3.62)$$

$$v = b_0 + b_1x + b_2y + b_3z + b_4x^2 + b_5xy + b_6y^2 + b_7yz + b_8z^2 + b_9zx + b_{10}x^2y + b_{11}x^2z + b_{12}y^2x + b_{13}y^2z + b_{14}z^2x + b_{15}z^2y + b_{16}xyz + b_{17}x^2yz + b_{18}xy^2z + b_{19}xyz^2 \quad (3.63)$$

$$w = c_0 + c_1x + c_2y + c_3z + c_4x^2 + c_5xy + c_6y^2 + c_7yz + c_8z^2 + c_9zx + c_{10}x^2y + c_{11}x^2z + c_{12}y^2x + c_{13}y^2z + c_{14}z^2x + c_{15}z^2y + c_{16}xyz + c_{17}x^2yz + c_{18}xy^2z + c_{19}xyz^2 \quad (3.64)$$

The first ten monomial terms in Equations (3.62) – (3.64) correspond to a complete quadratic displacement field. The remaining ten terms must be chosen carefully to avoid a singular basis as in the case of the 8-node quadrilateral. The metric shape functions for a 20-node hexahedron are obtained by solving the set of twenty completeness conditions

$$\sum_{i=1}^{20} M_i x_i^p y_i^q z_i^r = x^p y^q z^r \quad (3.65)$$

wherein $x^p y^q z^r$ in Equation (3.65) stands for a typical monomial term in Equations (3.62) – (3.64), with appropriate values of exponents ($p, q, r = 0, 1$ and 2). The metric shape functions for the 20-node hexahedron can be obtained following a similar procedure as outlined in Equations (3.18) – (3.23).

The isoparametric shape functions for a 20-node hexahedron element are well known in literature and can be expressed as

$$N_i = \frac{1}{8}(1 + \xi_0)(1 + \eta_0)(1 + \zeta_0)(\xi_0 + \eta_0 + \zeta_0 - 2) \quad (3.66)$$

for a typical corner node, with $\xi_0 \equiv \xi\xi_i$, $\eta_0 \equiv \eta\eta_i$ and $\zeta_0 \equiv \zeta\zeta_i$. For a typical mid-side node located at $\xi_i = 0$, $\eta_i = \pm 1$ and $\zeta_i = \pm 1$, the isoparametric shape functions can be expressed as

$$N_i = \frac{1}{4}(1 + \xi^2)(1 + \eta_0)(1 + \zeta_0) \quad (3.67)$$

Element formulation proceeds as usual from Equations (3.2) – (3.9).

3.9. BENCHMARK TESTS FOR UNSYMMETRIC 20-NODE HEXAHEDRON

The performance of the unsymmetric 20-node hexahedron (US-HEXA20) is studied using several benchmark problems. Results are compared with the classical isoparametric 20-node hexahedron (HEXA20), the 20-node hexahedron element of ANSYS and some hexahedron elements in literature. In the sections to follow, results of the benchmark problems obtained with the aforementioned elements are presented and discussed.

3.9.1. MACNEAL AND HARDER PATCH TEST

The element mesh for the patch test proposed by MacNeal and Harder [126] is shown in Figure 3.15. A bending moment is applied to induce a linear stress field. A patch test of this form may be viewed as a patch test of Type C according to Zienkiewicz [125]. Table 3.8 shows the normalized deflections at the tip of load application and stresses at the root. All three elements, US-HEXA20, HEXA20 and the 20-node hexahedron of ANSYS, pass

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

the patch test considering only two decimal place accuracy. However, US-HEXA20 reproduces the exact solution with a higher precision, up to at least six decimal places (Table 3.8). All the three elements reproduce exact displacements and stresses when the patch test is repeated for a constant stress field. Hence these results are not shown.

Figure 3.15. Patch test for three-dimensional elements; outer dimensions: unit cube;

$E = 1.0$, $\nu = 0.25$, loading: tension = 1.0 or tip moment $M = 2.0$

Table 3.8. Results of MacNeal and Harder's patch test for linear stress field

Element	Normalized tip deflection	
	Linear Displacement Field	Quadratic Displacement Field
ANSYS	1.000000	0.992750
HEXA20	1.000000	0.990030
US-HEXA20	1.000000	1.000000

Figure 3.16. Straight cantilever beams; (a) regular; (b) trapezoidal; (c) parallelogram shaped elements; beam dimensions: length = 6.0; width = 0.2; depth = 0.1; $E = 1.0 \times 10^7$; $\nu = 0.3$; mesh = 6×1 ; loading: unit shear load at one end and fixed in the other

3.9.2. STRAIGHT CANTILEVER BEAMS

The straight cantilever beams proposed by MacNeal and Harder [126] tests the elements for their principle deformation modes and curvature. Regular, parallel-o-piped and trapezoidal shaped elements are considered. These meshes are shown in Figure 3.16. Table 3.9 shows the computed displacements normalized with respect to the theoretical solution given by [126]. The performance of all the elements considered is comparable.

Table 3.9. Results of MacNeal and Harder's straight cantilever beam tests

Element shape	Load type	Normalized tip deflection				
		[†] HEX20	[‡] HEX20(R)	ANSYS	HEXA20	US-HEXA20
Regular	In-plane shear	0.970	0.984	0.994	0.994	0.994
	Out-of plane shear	0.961	0.972	0.992	0.992	0.992
Trapezoidal	In-plane shear	0.886	0.964	0.966	0.965	0.986
	Out-of plane shear	0.920	0.964	0.987	0.978	0.989
Parallelogram	In-plane shear	0.967	0.994	0.989	0.989	0.990
	Out-of plane shear	0.941	0.961	0.987	0.987	0.988

For regular shaped elements, the performance of US-HEXA20 is similar to HEXA20. US-HEXA20 is observed to give consistently good results for all the three meshes. None of the elements are able to reproduce the exact solution for this problem even with a six-element mesh. The reason for this is that the exact solution involves a cubic polynomial with monomial terms x^3, y^3 , etc., that cannot be represented by Equations (3.62) – (3.64).

3.9.3. CHEUNG AND CHEN BEAM TESTS

Seven beam configurations proposed by Cheung and Chen [127], viz., A1 – A7 in Figure 3.17 are considered. Quadratic and cubic displacement fields are imposed on all beam configurations A1 – A7 through the action of an applied bending moment and a shear force at the tip of the cantilever beam. The element is tested for the effects of aspect ratio and geometric distortion. Table 3.10 summarizes the results.

[†] Standard 20-node hexahedral element tested by MacNeal and Harder (1985)

[‡] Standard 20-node hexahedral element with reduced order integration tested by MacNeal and Harder (1985)

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

Table 3.10a. Results of Cheung and Chen tests for quadratic displacement field

Mesh	Normalized displacements			Normalized stresses	
	ANSYS	HEXA20	US-HEXA20	HEXA20	US-HEXA20
A1	1.000000	1.000000	1.000000	1.000000	1.000000
A2	1.000000	1.000000	1.000000	1.000000	1.000000
A3	0.999858	0.999869	1.000000	1.001498	1.000000
A4	0.999452	0.999484	1.000000	0.994302	1.000000
A5	0.999606	0.999586	1.000000	0.993162	1.000000
A6	0.665333	0.578220	1.000000	0.592707	1.000000
A7	0.997209	0.996520	1.000000	0.992097	1.000000

Table 3.10b. Results of Cheung and Chen tests for cubic displacement field

Mesh	Normalized displacements			Normalized stresses	
	ANSYS	HEXA20	US-HEXA20	HEXA20	US-HEXA20
A1	0.839201	0.839038	0.839038	1.012000	1.012000
A2	0.983236	0.982114	0.982659	1.014800	1.014800
A3	0.978002	0.977721	0.978366	1.007773	1.006979
A4	0.967740	0.967629	0.972540	0.914554	0.925336
A5	0.967822	0.967863	0.973236	1.015871	1.033429
A6	0.618807	0.541773	0.926064	0.773830	1.204363
A7	1.005029	1.004382	1.011587	1.028746	1.027677

US-HEXA20 predicts the solutions exactly for all meshes with tip moment loading, for which the exact displacement field involves a quadratic polynomial. However, HEXA20 and the 20-node hexahedron of ANSYS do so only for the undistorted meshes A1 and A2. In the case of tip shear loading, the performance of US-HEXA20 is, in general, comparable to that of the other elements. For mesh A6, wherein the element distortion is severe, US-HEXA20 exhibits a superior performance.

3.9.4. DISTORTION TESTS

The unsymmetric element is now tested for its sensitivity to generic distortion modes that a 20-node hexahedron element may experience. A straight cantilever beam with dimensions $10 \times 0.1 \times 1$, and material properties, $E = 1.0 \times 10^7$ and $\nu = 0.0$ is considered.

The distortion parameter δ is introduced into the mesh to characterize the extent of distortion in the elements. A unit constant bending moment, M at the free end of the cantilever causes the elements in the mesh to experience a quadratic displacement field.

Figure 3.18. Element meshes used for distortion tests; (a) mid-side node distortion; (b) plane distortion; (c) curved-face distortion; (d) warping; (e) side-curvature

Five types of distortions shown in Figure 3.18a – 3.18e viz., mid-side node, plane, curved-face, warping and side-curvature [134], are considered. For the mesh that involves

side-curvature, the distortion is described using two parameters, R and θ . R denotes the Euclidean distance of the displaced node from its original un-displaced position and the angle θ is as shown in Figure 3.18e. The parameter R is fixed at a value of 0.1 and the angle θ is varied over a range from $0^\circ - 90^\circ$.

US-HEXA20 reproduces the exact solution for all types and extents of distortions considered. For all the distortions, HEXA20 and the 20-node hexahedron of ANSYS show rapid increase in error as the distortion parameter is increased. This is evident from Figures 3.19a – 3.19e. For mid-side node distortions, Table 3.11 lists the deflections at the mid-side nodes in comparison with the 27-node field-consistent hexahedron element of Nagarayana and Prathap [27]. The performance of the field consistent HEXA27.1* and HEXA27.2* elements, despite exhibiting slight distortion sensitivity, are comparable with US-HEXA20.

Figure 3.19a. Normalized displacements for plane distortion sensitivity test

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

Figure 3.19b. Normalized displacements for mid-side node distortion sensitivity test

Figure 3.19c. Normalized displacement for curved-face distortion sensitivity test

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

Figure 3.19d. Normalized displacements for warping sensitivity test

Figure 3.19e. Normalized displacements for side-curvature sensitivity test

Table 3.11. Normalized displacements for element with mid-side node distortion

Element	Normalized deflection at mid-node			
	$\Delta_m = 0.0$	$\Delta_m = 0.1$	$\Delta_m = 1.0$	$\Delta_m = 2.5$
ANSYS	1.000	0.918	0.233	-0.011
[†] HEXA27.0	-	0.918	0.244	0.021
[‡] HEXA27.1,2	-	0.319	0.005	0.001
[♂] HEXA27.1*,2*	-	1.000	1.009	1.037
HEXA20	1.000	0.918	0.244	-0.021
US-HEXA20	1.000	1.000	1.000	1.000

Figure 3.20. Element meshes used for volumetric locking test; (a) regular and (b) trapezoidal meshes; loading, vertical end shear of 300 units

[†] 27-node element using Lagrangian interpolation functions proposed by Naganarayana and Prathap (1991)

[‡] HEXA27.1-field consistent element, HEXA27.2-solid element with assumed strain functions for the constrained fields with reference to Nagarayana and Prathap (1991)

[♂] HEXA27.1*-selectively integrated element and HEXA 27.2*-uniformly reduce integrated element with reference to Nagarayana and Prathap (1991)

3.9.5. VOLUMETRIC LOCKING TEST

A cantilever beam is modeled using the element meshes in Figure 3.20, and tests an element for the effects of volumetric locking when the Poisson's ratio approaches a value of 0.5 [129]. The meshes considered include trapezoidal shaped elements. The results, normalized with respect to the exact solution are tabulated in Table 3.12. The results for HEXA20 are excluded since this test checks only for the locking tendency of the present element when modeling incompressible materials. The present element does not exhibit any significant locking up to Poisson's ratio $\nu = 0.4999999$.

Table 3.12. Results of volumetric locking test for US-HEXA20

Poisson's ratio	Normalized tip deflection	
	Regular mesh	Trapezoidal mesh
0.3	1.002	1.003
0.4	1.001	1.001
0.49	1.001	0.999
0.499	0.997	0.997
0.4999	0.997	0.999
0.49999	0.997	0.998
0.499999	0.997	0.998
0.4999999	0.997	0.998

3.9.6. SINGLE ELEMENT ASPECT RATIO SENSITIVITY TEST

Robinson [130] proposed this problem as a stringent benchmark test to check if solid elements can undergo pure bending without experiencing parasitic shear. Details of the test problem are shown in Figure 3.21. The elements are tested for locking as the aspect ratio is increased from 1 – 16. No tendency of locking was observed in US-HEXA20 as in the case with the HEXA20 element. Both elements are free from the ill effects of parasitic shear. The results are detailed in Table 3.13.

Figure 3.21. Single-element aspect ratio sensitivity test, $E = 207 \times 10^9 \text{ N/m}^2$, $\nu = 0.25$, tip moment applied, $M = 1656 \text{ Nm}$, $2b = 2c = 0.06\text{m}$, beam fixed at surface ABCD

Table 3.13. Results of aspect ratio sensitivity test

Element	Aspect ratio				
	1.0	2.0	4.0	8.0	16.0
ANSYS	1.000	1.000	1.000	1.000	1.000
HEXA20	1.000	1.000	1.000	1.000	1.000
US-HEXA20	1.000	1.000	1.000	1.000	1.000

3.9.7. CANTILEVER BEAM LOADED IN SHEAR

A cantilever beam subjected to a unit vertical shear load, P with dimensions $10.0 \times 0.5 \times 1.0$, $E = 1 \times 10^5$ and $\nu = 0.3$ shown in Figure 3.22 is considered. The theoretical solutions for the vertical tip displacement u_y , and σ_{xx} - stress at the root of the cantilever, can be computed as $u_y = \frac{PL^3}{3EI}$ and $\sigma_{xx} = \frac{PLh}{2I}$. Here, I represents the second moment of cross sectional area, L is the length of the cantilever and h , the thickness of the beam.

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

The mesh is deliberately distorted by moving all the four mid-side nodes at the interfaces of the elements along the length of the cantilever, by a distance $\delta = 1.0$. The cantilever is fixed at one end. The unit shear load applied at the cantilever's tip induces a linear bending moment along the cantilever's length. The convergence of the tip displacements and stresses were observed and the results are shown in Figures 3.23a and 3.23b.

Figure 3.22. Cantilever beam used for convergence tests; (a) two-element model; (b) eight -element model; (c) super-element used for distorted mesh

Figure 3.23a shows that US-HEXA20 converges to the correct displacements while HEXA20 and the 20-node hexahedron of ANSYS converge to a normalized displacement of 0.4. The convergence of US-HEXA20 is, however, not monotonic. Figure 3.23b shows that US-HEXA20 converges to the correct stress values with less number of elements compared to HEXA20 and the 20-node hexahedron of ANSYS.

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

Figure 3.23. Convergence results of cantilever beam; (a) convergence of tip displacement; (b) convergence of stress

Figure 3.24. (a) Dimensions for simply supported plate; (b) Super-element mesh with plane distortion; (c) Super-element mesh with curved distortion; (d) Super-element mesh with warping

3.9.8. RECTANGULAR PLATE

A simply supported rectangular plate with dimensions shown in Figure 3.24a and a thickness of 0.02 units, subjected to a uniformly distributed load $p = 5\text{kPa}$ is considered. This problem tests the convergence of the elements under various distorted meshes. Symmetry of the plate's geometry and its boundary conditions is exploited by modeling only a quarter of the plate. The distorted super-elements considered for this problem are shown in Figures 3.24b – 3.24d. Three types of distortion are considered *viz.*, plane, curved-faced and warping. The mesh is uniformly refined using the aforementioned super-elements.

The plate is solved for its elastic strain energy and transverse deflection at its mid-point. The theoretical solution for this problem is in the form of a Fourier series and can be obtained from standard texts dealing with deflection of plates (see e.g., [131]).

Figure 3.25a. Convergence of displacements for super-element with plane distortion

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

Figure 3.25b. Convergence of strain energy for super-element with plane distortion

Figure 3.25c. Convergence of displacements for super-elements with curved-face distortion

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

Figure 3.25d. Convergence of strain energy for super-elements with curved-face distortion

Figure 3.25e. Convergence of displacements for super-elements with warping

Figure 3.25f. Convergence of strain energy for super-elements with warping

Figures 3.25a – 3.25f show the convergence plots of strain energy and transverse deflection at mid-point of the plate for all the super-elements considered. The variable h refers to the size of the super-elements. For super-elements with plane distortions, the convergence plots for all three elements, US-HEXA20, HEXA20 and the 20-node hexahedron of ANSYS show a similar slope. Thus, the conventional practice of assessing the relative superiority of the elements based on the slope in the convergence plot is not very revealing. If the errors in displacement and strain energy are used to assess the performance of the elements, US-HEXA20 is superior compared to the other two elements. Alternatively, the area under the convergence curve can be used to monitor the errors, thus comparing the relative superiority of the elements. In such a case, a curve that yields a lower area is indicative of a superior element performance, as is the case of US-HEXA20. The better performance of US-HEXA20 is more pronounced for meshes with

curved-face distortion and warping. For both these distortions, the convergence trend of the present element is not linear as is the case with Figure 3.25a and 3.25b.

3.9.10. PINCHED SPHERICAL SHELL

The hemispherical shell with an 18° hole in Figure 3.26 proposed by MacNeal and Harder [126] is considered. This test problem assesses the performance of the elements for doubly curved structures where membrane and bending strains contribute to the radial displacements at the load points. The mesh is progressively refined by increasing the number of elements in orders of $(N \times N)$.

Figure 3.26. Pinched spherical shell problem: Radius = 10.0; thickness = 0.04; $E = 6.825 \times 10^7$; $\nu = 0.3$; $F = 2.0$ on quadrant; hole at free edge = 18°

Table 3.14 details the convergence of the normalized displacements at the point of load application (normalized with respect to 0.094). For a coarse 4×4 element mesh, the

accuracy of US-HEXA20 is inferior to HEXA20 and the 20-node hexahedron of ANSYS. However, the accuracy improves and surpasses the other two elements when the mesh is refined, and approaches the exact solution with less number of elements. In general, the performance of US-HEXA20 is bad for thin shell structures with a coarse mesh. Nevertheless, it improves rapidly with further refinement.

Table 3.14. Normalized tip displacement for spherical shell problem

Normalized tip displacements			
Mesh	ANSYS	HEXA20	US-HEXA20
4×4	0.021433	0.021217	0.014711
8×8	0.258461	0.257849	0.611636
10×10	0.457268	0.456715	0.779302
16×16	0.838645	0.838503	0.943067
20×20	0.921272	0.921223	0.989548

3.10. COMPUTATIONAL EFFORT

There are additional coding efforts in implementing the unsymmetric element compared to the classical isoparametric element. Two sets of shape functions and their derivatives, isoparametric and metric, need to be computed at each integration point whereas only the former is required for the isoparametric element. The inversion of the $\mathbf{P}$ -matrix, however, is required only once per element as it is a constant matrix for a given element. The x - and y - derivatives of the metric shape functions are obtained by using differentiated versions of Equation (3.23), viz., $\mathbf{m}_{,x} = \mathbf{P}^{-1}(\partial \mathbf{P} / \partial x)$ and $\mathbf{m}_{,y} = \mathbf{P}^{-1}(\partial \mathbf{P} / \partial y)$ respectively, which just involve matrix multiplications. Evaluation of the $\mathbf{P}$ -matrix increases the computational time.

In contrast to HEXA20, US-HEXA20 requires an unsymmetric equation solver, which doubles the storage requirement of the global stiffness matrix. Also, the computational time is increased due to additional operations related to the computation of all lower and upper triangular entries of the element stiffness matrix and the additional operations involved when an unsymmetric solver is in use.

Table 3.15. Comparison of computational time for clamped plate with plane distortions

	Computational time (Unix Platform)					
	System time			User time		
	HEXA20	US-HEXA20	Approx. increase (%)	HEXA20	US-HEXA20	Approx. increase (%)
Element formulation	0.053	0.101	91	0.119	0.142	19
Equation solution	4.83	5.05	5	9.87	21.33	116

Table 3.15 shows a comparison of computation time for the clamped plate problem with central point load of Mac Neal and Harder [126]. All computations were performed on Unix platform, with an out-of-core frontal solver. The plate is meshed using 36 hexahedral elements with angular distortions. Comparison of *system time* (Table 3.15) shows that the element stiffness computation requires approximately 91% more computation time than the isoparametric element, while equation solution requires only 5% more time. However, comparison of *user time* shows that the element stiffness computation requires approximately 19% more time, and the equation solution requires 116% more time. The 116% increase in the *user time* is due to the storage of additional stiffness entries of the unsymmetric stiffness matrix and the associated increase in the I/O operations characteristics of an out-of-core frontal solver. The implementation would be more efficient if an in-core solver were used.

3.11. CONCLUSIONS

The concept of the unsymmetric formulation for the 8-node quadrilateral element [5] has been reviewed. It is based on a reinterpretation of the principle of virtual work. The idea is to use compatibility- and completeness-fulfilling shape functions for the virtual and trial displacement fields, respectively. This approach leads to the distortion tolerant performance of the unsymmetric formulation.

An extensive patch test of the strong form has been performed to verify the reproduction of all the monomial terms $1, x, y, x^2, xy, y^2, x^2y$ and xy^2 . The patches of elements considered involves generic distortions that an 8-node quadrilateral element may experience in practice viz., angular, curved-edge and mid-side node. US-QUAD8 has been found to satisfy all the patch tests (constant-, linear- and quadratic-strain), necessary to reproduce the aforementioned monomial terms for all the types of distortions considered. This confirms the claim that the unsymmetric formulation is indeed immune to mesh distortions with respect to all the monomial terms in the assumed displacement field, viz., $1, x, y, x^2, xy, y^2, x^2y$ and xy^2 .

The unsymmetric formulation has been extended to a 20-node hexahedron element. The performance of US-HEXA20 is superior compared to the classical isoparametric 20-node hexahedron whenever the elements in the mesh are distorted. However, the performance of US-HEXA20 is similar to the classical isoparametric 20-node hexahedron whenever the elements in the mesh are of regular shapes. The unsymmetric formulation requires more computational time due to additional mathematical operations required in

CHAPTER 3. UNSYMMETRIC FINITE ELEMENT FORMULATION FOR LINEAR STATIC ANALYSIS

computing two sets of shape functions and also the use of an unsymmetric equation solver. The increase in computational time due to formulation of element stiffness is more pronounced compared to the increase of computational time due to an unsymmetric equation solver.

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

4.1. INTRODUCTION

The design of structures such as those used in aerospace and automotive applications often utilize lightweight polymeric materials that undergo large deformations without exceeding their elastic limit. In view of large deformations, the response of these structures under static or dynamic loading conditions is inherently nonlinear. Such classes of problems are said to exhibit geometric nonlinearity and the complexity of treating such nonlinear responses usually demands a numerical solution approach (e.g., Kohnke [132]; Fertis and Afonta [133]). Another class of problem that involves geometric non-linearity is the problem of instability such as the buckling of beams and plates, and the theory of such problems is well documented in literature (e.g., see Timoshenko and Gere [87]).

In the previous chapter both US-QUAD8 and US-HEXA20 elements have shown promising performances in the presence of mesh distortions for linear static analysis. In this chapter, the performance of both elements is investigated for geometric nonlinear static analyses. The core idea behind both element formulations (for both linear and nonlinear analyses) is very similar. However, computer implementation is more involved in geometric nonlinear problems. Two shape functions *viz.*, isoparametric and metric will again be used as for the virtual and trial displacement fields based on *Criterion 1* and *Criterion 2* stated in the previous chapter on Page 56. The metric shape functions are

obtained by solving the completeness conditions in Equation (3.14) and Equation (3.65) for US-QUAD8 and US-HEXA20 elements respectively.

Derivations that lead to the element formulation of US-QUAD8 for geometric nonlinear analyses are detailed in Section 4.2. Some practical aspects on computational and numerical implementation of US-QUAD8 for geometric nonlinear analyses are discussed in Section 4.3. The efficacy of US-QUAD8 in terms of accuracy and computational time is assessed using several test problems, described in Section 4.4. The extension of US-HEXA20 element to geometric nonlinear analyses is outlined in Section 4.6. Its performance is assessed using several test problems.

4.2. ELEMENT FORMULATION FOR US-QUAD8

The governing linearized equation is implemented using a Total Lagrangian scheme wherein all the static and kinematic variables are referred to the initial configuration. The element formulation is presented here for the implementation of the unsymmetric 8-node quadrilateral element. A similar approach is applicable to the implementation of other continuum elements. Uppercase fonts are used to denote the nodal coordinates $\mathbf{X} \equiv (X, Y)$ of the initial configuration. The metric shape functions are evaluated from the completeness conditions in Equation (3.14) with (x, y) replaced by (X, Y) .

The principle of virtual work confirming to the Total Lagrangian formulation is written as [183]

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

$$\delta W = \int_{\Omega_0} \mathbf{S} : \delta \mathbf{E} d\Omega_0 - \int_{\Omega_0} \mathbf{f}_0 \cdot \delta \mathbf{u} d\Omega_0 - \int_{\Gamma_0} \mathbf{t}_0 \cdot \delta \mathbf{u} d\Gamma_0 = 0 \quad (4.1)$$

where Ω_0 and Γ_0 indicate the domain and boundary of the problem referred to initial configuration, $\mathbf{f}_0$ and $\mathbf{t}_0$ are the body and surface force intensities in the initial configuration, $\mathbf{S}$ is the second Piola-Kirchoff stress tensor, $\delta \mathbf{E}$ is the virtual Green strain tensor, $\delta \mathbf{u}$ is the virtual displacement field and δW is the virtual work. In Equation (4.1) the operation “:” represents the inner product of two second order tensors (or higher). The second Piola-Kirchoff stress is related to the Green strain, $\mathbf{E}$ through the material constitutive tensor, $\mathbf{C}$ via $\mathbf{S} = \mathbf{C} : \mathbf{E}$.

Equation (4.1) can be linearized in the direction of an incremental displacement, $\Delta \mathbf{u}$, as

$$\delta W + D(\delta W)[\Delta \mathbf{u}] = 0 \quad (4.2)$$

The term $D(\delta W)[\Delta \mathbf{u}]$ can be split into internal and external work components. Assuming the body and surface forces are conservative, the external virtual work components in Equation (4.2) vanish. The remaining internal work component in the term $D(\delta W)[\Delta \mathbf{u}]$ can be expanded with the product rule and upon rearranging terms, Equation (4.2) becomes

$$\int_{\Omega_0} D\mathbf{E}[\delta \mathbf{u}] : \mathbf{S}[\Delta \mathbf{u}] d\Omega_0 + \int_{\Omega_0} \mathbf{S} : \left[\left(\frac{\partial \Delta \mathbf{u}}{\partial \mathbf{X}} \right)^T \left(\frac{\partial \delta \mathbf{u}}{\partial \mathbf{X}} \right) \right] d\Omega_0 = - \int_{\Omega_0} \mathbf{S} : \delta \mathbf{E} d\Omega_0 + \int_{\Omega_0} \mathbf{f}_0 \cdot \delta \mathbf{u} d\Omega_0 + \int_{\Gamma_0} \mathbf{t}_0 \cdot \delta \mathbf{u} d\Gamma_0 \quad (4.3)$$

The virtual strains, $\delta \mathbf{E}$ and virtual displacements, $\delta \mathbf{u}$ are constituents of the test functions. They must satisfy compatibility of displacement at the boundaries of adjacent

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

elements $\Gamma_o^{(e)}$, and hence, must ensure inter-element continuity of at least $C^0(\Gamma^{(e)})$ order across the element boundaries. With reference to the requirements of *Criterion 1* in Chapter 3, the isoparametric shape functions, N_i , form a good choice. The virtual displacement and strain fields are written as

$$\delta \mathbf{u}^{(e)} = \delta \bar{\mathbf{u}}^{(e)} \equiv \mathbf{N} \delta \bar{\mathbf{u}}_n^{(e)} \quad (4.4)$$

$$\delta \mathbf{E}^{(e)} = \bar{\mathbf{B}}_{NL} \delta \bar{\mathbf{u}}_n^{(e)} \quad (4.5)$$

Substituting Equations (4.4) and (4.5) into Equation (4.3) and recasting into matrix form yields

$$\begin{aligned} \mathbf{A}_{e=1}^n \left[\delta \bar{\mathbf{u}}_n^{(e)T} \left(\int_{\Omega_0^{(e)}} \bar{\mathbf{B}}_{NL}^T \Delta \mathbf{S}^{(e)} d\Omega_0 + \int_{\Omega_0^{(e)}} \bar{\mathbf{B}}_G^T \tilde{\mathbf{S}}^{(e)} \left(\frac{\partial \Delta \mathbf{u}^{(e)}}{\partial \mathbf{X}} \right) d\Omega_0 \right) \right] = \\ \mathbf{A}_{e=1}^n \left[\delta \bar{\mathbf{u}}_n^{(e)T} \left(- \int_{\Omega_0^{(e)}} \bar{\mathbf{B}}_{NL}^T \mathbf{S}^{(e)} d\Omega_0 + \int_{\Omega_0^{(e)}} \mathbf{N}^T \mathbf{f}_0 d\Omega_0 + \int_{\Gamma_0^{(e)}} \mathbf{N}^T \mathbf{t}_0 d\Gamma_0 \right) \right] \end{aligned} \quad (4.6)$$

The shape function matrix, $\mathbf{N}$ and non-linear strain-displacement matrix in Equation (4.6) are

$$\mathbf{N} = [\mathbf{N}_1 \quad \mathbf{N}_2 \quad \mathbf{N}_3 \quad \cdots \quad \mathbf{N}_8] \quad (4.7)$$

and

$$\bar{\mathbf{B}}_{NL} = [(\mathbf{L}\mathbf{N}_1)\bar{\mathbf{F}}^T \quad (\mathbf{L}\mathbf{N}_2)\bar{\mathbf{F}}^T \quad (\mathbf{L}\mathbf{N}_3)\bar{\mathbf{F}}^T \quad \cdots \quad (\mathbf{L}\mathbf{N}_8)\bar{\mathbf{F}}^T] \quad (4.8)$$

respectively, where

$$\mathbf{N}_i = \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix} \quad (4.9)$$

$$\bar{\mathbf{F}} = \mathbf{I} + \frac{\partial \bar{\mathbf{u}}}{\partial \mathbf{X}} \quad (4.10)$$

The term $\bar{\mathbf{F}}$ in Equation (4.10) is the deformation gradient tensor associated with the isoparametric shape functions and $\mathbf{L}$ is the usual linear-strain-displacement differential operator matrix. The geometric strain-displacement matrix, $\bar{\mathbf{B}}_G$ is

$$\bar{\mathbf{B}}_G^T = \begin{bmatrix} \frac{\partial \mathbf{N}}{\partial X} & \frac{\partial \mathbf{N}}{\partial Y} & 0 & 0 \\ 0 & 0 & \frac{\partial \mathbf{N}}{\partial X} & \frac{\partial \mathbf{N}}{\partial Y} \end{bmatrix} \quad (4.11)$$

The stress matrix, $\tilde{\mathbf{S}}^{(e)}$ is expressed as

$$\tilde{\mathbf{S}}^{(e)} = \begin{bmatrix} S_{XX} & S_{XY} & 0 & 0 \\ S_{XY} & S_{YY} & 0 & 0 \\ 0 & 0 & S_{XX} & S_{XY} \\ 0 & 0 & S_{XY} & S_{YY} \end{bmatrix} \quad (4.12)$$

where S_{XX} , S_{YY} and S_{XY} are the components of the Second Piola-Kirchoff stresses. If the isoparametric shape functions are also employed to model the trial displacement field $\mathbf{u}^{(e)}$, the *error nodal forces* (i.e., the nodal forces resulting from the error when the isoparametric shape functions are used to model the trial displacement field) can be expressed as

$$\mathbf{f}_{error}^{(e)} = \int_{\Omega_0^{(e)}} \bar{\mathbf{B}}_{NL}^T (\Delta \bar{\mathbf{S}}^{(e)} - \Delta \mathbf{S}^{(e)}) d\Omega_0 + \int_{\Omega_0^{(e)}} \bar{\mathbf{B}}_G^T \tilde{\mathbf{S}}^{(e)} \left(\frac{\partial (\Delta \bar{\mathbf{u}}^{(e)} - \Delta \mathbf{u}^{(e)})}{\partial \mathbf{X}} \right) d\Omega_0 \quad (4.13)$$

In Chapter 3, the isoparametric shape functions have been shown to violate the higher order completeness conditions in the presence of geometric distortions. Higher order completeness is required to guarantee the reproduction of higher order monomial terms viz., X^2 , XY , Y^2 , X^2Y and XY^2 . Hence, the magnitude of error in Equation (4.13) is expected to increase when the shapes of the element are distorted.

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

However, if the metric shape functions are used to model the trial displacement field, $\mathbf{u}^{(e)}$ as:

$$\mathbf{u}^{(e)} = \hat{\mathbf{u}}^{(e)} \equiv \mathbf{M} \hat{\mathbf{u}}_n^{(e)} \quad (4.14)$$

$$\Delta \mathbf{E}^{(e)} = \hat{\mathbf{B}}_{NL} \Delta \hat{\mathbf{u}}_n^{(e)} \quad (4.15)$$

The error in nodal forces can then be expressed as

$$\mathbf{f}_{error}^{(e)} = \int_{\Omega_0^{(e)}} \bar{\mathbf{B}}_{NL}^T (\Delta \hat{\mathbf{S}}^{(e)} - \Delta \mathbf{S}^{(e)}) d\Omega_0 + \int_{\Omega_0^{(e)}} \bar{\mathbf{B}}_G^T \tilde{\mathbf{S}}^{(e)} \left(\frac{\partial (\Delta \hat{\mathbf{u}}^{(e)} - \Delta \mathbf{u}^{(e)})}{\partial \mathbf{X}} \right) d\Omega_0 \quad (4.16)$$

The metric shape functions, on the other hand, have been shown (in Chapter 3) to satisfy all completeness conditions for any admissible element geometry. This guarantees the reproducibility of the monomial terms, $1, X, Y, X^2, XY, Y^2, X^2Y$ and XY^2 by virtue of Equation (3.17). Thus, the magnitude of error given by Equation (4.16) is expected to be less, compared to the magnitude of error given by Equation (4.14). In view of this, the metric shape functions forms a better choice as basis functions to construct the trial displacement model compared to their isoparametric counterparts even for geometric nonlinear problems. This expectation emerges naturally from the observations in Chapter 3 that such a choice does indeed lead to distortion-tolerant performance for linear problems. The numerical examples presented in Section 4.4 support these choices of shape functions for geometric nonlinear problems.

Use of the metric interpolation for the trial solutions in Equation (4.6) yields

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

$$\begin{aligned} \mathbf{A}_{e=1}^N \left[\delta \bar{\mathbf{u}}_n^{(e)T} \left(\int_{\Omega_0^{(e)}} \bar{\mathbf{B}}_{NL}^T \mathbf{C} \hat{\mathbf{B}}_{NL} d\Omega_0 + \int_{\Omega_0^{(e)}} \bar{\mathbf{B}}_G^T \tilde{\mathbf{S}}^{(e)} \hat{\mathbf{B}}_G d\Omega_0 \right) \Delta \mathbf{u}_n^{(e)} \right] = \\ \mathbf{A}_{e=1}^N \left[\delta \bar{\mathbf{u}}_n^{(e)T} \left(- \int_{\Omega_0^{(e)}} \bar{\mathbf{B}}_{NL}^T \mathbf{S}^{(e)} d\Omega_0 + \int_{\Omega_0^{(e)}} \mathbf{N}^T \mathbf{f}_0 d\Omega_0 + \int_{\Gamma_0^{(e)}} \mathbf{N}^T \mathbf{t}_0 d\Gamma_0 \right) \right] \end{aligned} \quad (4.17)$$

where the stress-strain relation, $\Delta \mathbf{S} = \mathbf{C} \Delta \mathbf{E}$. The matrix, $\mathbf{C}$ is the elastic constitutive matrix. The expressions for the shape function matrix, $\mathbf{M}$, non-linear strain-displacement matrix, $\hat{\mathbf{B}}_{NL}$ and geometric strain-displacement matrix, $\hat{\mathbf{B}}_G$ associated with the metric shape functions are

$$\mathbf{M} = [\mathbf{M}_1 \quad \mathbf{M}_2 \quad \mathbf{M}_3 \quad \cdots \quad \mathbf{M}_8] \quad (4.18)$$

and

$$\hat{\mathbf{B}}_{NL} = [(\mathbf{L}\mathbf{M}_1)\hat{\mathbf{F}}^T \quad (\mathbf{L}\mathbf{M}_2)\hat{\mathbf{F}}^T \quad (\mathbf{L}\mathbf{M}_3)\hat{\mathbf{F}}^T \quad \cdots \quad (\mathbf{L}\mathbf{M}_8)\hat{\mathbf{F}}^T] \quad (4.19)$$

respectively, where

$$\mathbf{M}_i = \begin{bmatrix} M_i & 0 \\ 0 & M_i \end{bmatrix} \quad (4.20)$$

$$\hat{\mathbf{F}} = \mathbf{I} + \frac{\partial \hat{\mathbf{u}}}{\partial \mathbf{X}} \quad (4.21)$$

$$\hat{\mathbf{B}}_G^T = \begin{bmatrix} \frac{\partial \mathbf{M}}{\partial X} & \frac{\partial \mathbf{M}}{\partial Y} & 0 & 0 \\ 0 & 0 & \frac{\partial \mathbf{M}}{\partial X} & \frac{\partial \mathbf{M}}{\partial Y} \end{bmatrix} \quad (4.22)$$

The term $\hat{\mathbf{F}}$ given by Equation (4.21) is the deformation gradient tensor associated with the metric shape functions.

Invoking the arbitrariness of the virtual displacements in Equation (4.17) yields the system of linear simultaneous equations

$$\mathbf{K}\Delta\mathbf{U} = \mathbf{R} \quad (4.23)$$

where

$$\mathbf{R} = \mathbf{A} \sum_{e=1}^N \left(- \int_{\Omega_0^{(e)}} \bar{\mathbf{B}}_{NL}^T \mathbf{S}^{(e)} d\Omega_0^{(e)} + \int_{\Omega_0^{(e)}} \mathbf{N}^T \mathbf{f}_0 d\Omega_0^{(e)} + \int_{\Gamma_0^{(e)}} \mathbf{N}^T \mathbf{t}_0 d\Gamma_0^{(e)} \right) \quad (4.24)$$

$$\mathbf{K} = \mathbf{A} \sum_{e=1}^N \left(\int_{\Omega_0^{(e)}} \bar{\mathbf{B}}_{NL}^T \mathbf{C} \hat{\mathbf{B}}_{NL} d\Omega_0^{(e)} + \int_{\Omega_0} \bar{\mathbf{B}}_G^T \tilde{\mathbf{S}}^{(e)} \hat{\mathbf{B}}_G d\Omega_0 \right) \quad (4.25)$$

and $\Delta\mathbf{U}$ is the global vector of the incremental nodal displacements. At the initiation of the solution process, the force term, $\mathbf{R}$ in Equation (4.23) involves only the second and third terms on the right-hand-side of Equation (4.24). As the equilibrium iteration proceeds, the residual forces $\mathbf{R}$ involve all the terms on the right-hand-side of Equation (4.24). Equilibrium iterations are terminated when the difference in the norm of the current and previous residual forces satisfies a certain tolerance value. The tangent stiffness matrix, $\mathbf{K}$ in Equation (4.25) is unsymmetric in view of the two different shape functions used for virtual and trial displacement fields.

4.3. COMPUTER IMPLEMENTATION

The element formulation of US-QUAD8 extended for geometric nonlinear analyses has been implemented on an in-house FORTRAN code. Programming efforts was in large, involved in implementation of the existing linear source code in Chapter 3 for nonlinear capabilities. The element subroutine is to some extent similar to that in linear analyses with the addition of the auxiliary terms associated with geometric nonlinear analyses

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

(viz., deformation gradient tensors, $\bar{\mathbf{F}}$ and $\hat{\mathbf{F}}$, geometric stiffness matrix, etc). An in-house unsymmetric version of the frontal solver is used for solving the global equations.

The pseudo-code for the formulation of the element stiffness matrix is given as follows:

1. Loop over elements in the domain
 - a. Form the $\mathbf{P}$ -matrix in Equation (3.21) and compute its inverse
2. Loop over integration points
 - a. Compute the isoparametric shape functions, $\mathbf{N}$, and their derivatives as in the case of the classical QUAD8 element
 - b. Compute the metric shape functions, $\mathbf{m}$, using Equation (3.23), and their derivatives using the equations, $\mathbf{m}_{,X} = \mathbf{P}^{-1}\mathbf{p}_{,X}$ and $\mathbf{m}_{,Y} = \mathbf{P}^{-1}\mathbf{p}_{,Y}$, which are the differentiated versions of Equation (3.23)
 - c. Compute the deformation gradient tensors $\bar{\mathbf{F}}$ using Equation (4.10) and $\hat{\mathbf{F}}$ using Equation (4.21)
 - d. The matrices $\bar{\mathbf{B}}_{NL}$, $\bar{\mathbf{B}}_G$, $\hat{\mathbf{B}}_{NL}$ and $\hat{\mathbf{B}}_G$ are computed according to Equations (4.8), (4.11), (4.19) and (4.22) respectively
 - e. Read the values of the current stresses and form the $\tilde{\mathbf{S}}^{(e)}$ matrix using Equation (4.12)
 - f. From the material constants, set up the elastic constitutive matrix, $\mathbf{C}$
 - g. Compute the unsymmetric stiffness matrix using Equation (4.25)
3. Terminate loop over integration points
4. Terminate loop over elements

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

The assembly of element stiffness matrices and force vectors, and solution of the system equations adopting the Newton-Raphson scheme follow from conventional finite element practices. The iterative procedure in the present implementation is provided in Appendix I for the sake of completion.

Figure 4.1 (a) A cantilever subject to shear load at the tip, $L = 1000$ in; $c = 39.37$ in; thickness = 1.0 in; $EI = 180,000$ kip-in²; Super-element with (b) angular; (c) curved-edge; (d) mid-side node distortion

4.4. NUMERICAL EXAMPLES

Several benchmark problems of structures undergoing large deflections are considered. The results obtained with US-QUAD8 are compared with that obtained with the classical 8- and 9-node isoparametric quadrilateral elements. The element stiffness matrix is integrated using 3×3 Gaussian quadrature rule. The 8- and 9-node isoparametric

quadrilateral elements will, hereinafter, be referred to as QUAD8 and QUAD9, respectively. An L_2 -norm of residual forces ≤ 0.001 is used as the convergence criterion.

4.4.1. LARGE DEFORMATION OF STRAIGHT CANTILEVER BEAM

A cantilever beam with dimensions and material properties shown in Figure 4.1a is subjected to a shear load at its tip and solved as a geometrically nonlinear problem. The shear load at the tip is applied as a conservative load. The geometry is discretized into ten super-elements along the length of the beam. The super-elements used are illustrated in Figure 4.1b – d.

A distortion parameter, δ , which controls the extent of mesh distortion, is deliberately introduced into the initial undistorted super-element mesh. The test problem is intended to test the element formulation for its sensitivity to angular, curved-edge and mid-side node distortions. The problem is solved for the vertical deflection at the tip of the cantilever and the results are compared with the theoretical solution [134].

4.4.1.1. ANGULAR DISTORTION

Figures 4.2a and 4.2b show the %-deviation of the computed deflections from the theoretical solution for the super-elements with angular distortion (Fig. 4.1b). This deviation is plotted against the applied load, considering two typical values of distortion parameter, δ . Unlike the QUAD8 element, the deviation of computed deflections in the case of US-QUAD8 is considerably low. The performance of QUAD9 is comparable to that of US-QUAD8 although the latter yields slightly better results. A comparison of

these two plots reveals that QUAD8 exhibits sensitivity to an increase in the distortion parameter, δ . However, both QUAD9 and US-QUAD8 elements do not exhibit sensitivity to an increase in the distortion parameter.

4.4.1.2. CURVED EDGE DISTORTION

The deviation of the computed solution, from that obtained theoretically [134] for the super-element mesh with mid-side node distortion (Fig. 4.1c) is shown in Figures 4.3a and 4.3b. The plots reveal that this type of distortion least affects the performance of US-QUAD8 while both QUAD8 and QUAD9 exhibit high sensitivity. However, QUAD9 is observed to perform somewhat better than QUAD8. Increasing the value of δ further deteriorates the performance of both QUAD8 and QUAD9 while the performance of US-QUAD8 remains unaffected.

Figure 4.2a. Deviation of vertical displacement from theoretical solution for a mesh with angular distortions, $\delta = 23.622\text{in}$

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

Figure 4.2b. Deviation of vertical displacement from theoretical solution for a mesh with angular distortions $\delta = 31.496\text{in}$

Figure 4.3a. Deviation of vertical displacement from theoretical solution for a mesh with curved-edge distortions, $\delta = 23.622\text{in}$

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

Figure 4.3b. Deviation of vertical displacement from theoretical solution for a mesh with curved-edge distortions, $\delta = 31.49\text{in}$

Figure 4.4a. Deviation of vertical displacement from theoretical solution for a mesh with mid-side node distortions, $\delta = 7.874\text{in}$

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

Figure 4.4b. Deviation of vertical displacement from theoretical solution for a mesh with mid-side node distortions, $\delta = 11.811$ in

4.4.1.3. MID-SIDE NODE DISTORTION

Figures 4.4a and 4.4b show the %-deviation of the computed solution for a mesh with curved-edge distortion (Fig. 4.2d). The performance of US-QUAD8 element is least affected by the presence of mid-side node distortion. The solution accuracy of US-QUAD8 is superior compared to both QUAD8 and QUAD9 elements.

Figure 4.5. Thick beam with a shear load at its tip; $L = 20$; $c = 10$; $F = 3$; Poisson's ratio, $\nu = 0.0$; Young's modulus, $E = 12$

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

Figure 4.6. (a) Super-element mesh with angular distortion; (b) convergence of vertical displacements at point A

Figure 4.7. (a) Super-element mesh with curved-edge distortion; (b) convergence of vertical displacements at point A

Figure 4.8. (a) Super-element mesh with mid-side node distortion; (b) convergence of vertical displacements at point A

4.4.2. THICK BEAM SUBJECTED TO SHEAR LOAD AT ITS TIP

Figure 4.5 shows a thick beam under plane stress condition subjected to a transverse shear load at its free end. All the nodes at the fixed end of the cantilever are restrained in both x - and y -directions. This problem tests the convergence of the three elements QUAD8, QUAD9 and US-QUAD8, employing uniform mesh refinement with distorted meshes. An over-kill solution is obtained by solving the problem with a very fine mesh without geometric distortion. The displacement at point $A(L, c/2)$ given by this over-killed solution is used as a reference solution to compare the elements performance.

Figures 4.6a, 4.7a and 4.8a show the distorted super-elements used to discretize the beam. The convergence of the vertical displacement at point A for the three elements is shown in Figures 4.6b, 4.7b and 4.8b. The performances of all the three types of elements are comparable for angular distortions (Figure 4.6b), although QUAD9 shows a slightly

superior performance for coarse meshes. For curved-edge as well as mid-side node distortions (Figures 4.7b and 4.8b), US-QUAD8 element converges faster compared to both QUAD8 and QUAD9 elements.

Figure 4.9. A column under an eccentric axial compressive load: (a) geometry; (b) regular mesh; (c) distorted mesh

4.4.3. COLUMN UNDER ECCENTRIC COMPRESSIVE LOAD

A column with an eccentric compressive load in Figure 4.9a is modelled with five 8-node quadrilaterals elements with and without mesh distortion shown in Figures 4.9b and 4.9c, respectively. Plane stress conditions are assumed and the column is solved for the horizontal and vertical tip displacements at the point of loading. All degrees of freedom at the fixed end of the cantilever are restrained. The response is simulated for a total load

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

$P = 4P_{cr}$, where $P_{cr} = \frac{\pi^2 EI}{4L^2}$, is the critical buckling load in the absence of eccentricity of loading.

Figure 4.10a. Horizontal deflection response of column under eccentric compressive load

Figure 4.10b. Vertical deflection response of column under eccentric compressive load

The non-dimensional horizontal and vertical deflections are shown in Figures 4.10a and 4.10b, respectively. This test problem is employed to study the sensitivity of load-deflection response to mesh distortion for a typical instability problem.

The load-deflection curve for an element that is insensitive to mesh distortion should not display any deviation of response whether or not the mesh is distorted. The plots (Figures 4.10a and 4.10b) reveal that the load-deflection curves of US-QUAD8 element do not deviate much compared to QUAD8. It can be concluded that US-QUAD8 exhibits far less sensitivity to mesh distortion compared to the QUAD8 element for this problem.

Figure 4.11. Beam subject to central transverse load and end thrust: Young's modulus, $E = 10$, Poisson's ratio, $\nu = 0.0$, distributed load, $w = 2.0$ applied as a conservative load, point load, $P = 1.0$

4.5. COMPUTATIONAL EFFORT

In order to assess the increase in computational effort, we consider a beam subjected to a combination of a central point force and an end thrust as shown in Figure 4.11. The beam is discretized using 20×2 elements and solved with 100 load increments. The nodes on

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

the left end of the beam are restrained in both horizontal and vertical directions while the other end is restrained only in the horizontal direction. The meshes (distorted and undistorted) used for the purpose are shown in Figures 4.12. While the increase in computational effort has been investigated in the case of linear analyses, the present investigates the additional computational effort for geometric nonlinear problems.

Figure 4.12. Finite element meshes used for beam subjected to end thrust: (a) regular mesh; (b) distorted mesh I (angular distortions); (c) distorted mesh II (curved-edge distortions); (d) distorted mesh III (random perturbation of nodes)

Table 4.1 shows a comparison of computation time of both QUAD8 and US-QUAD8 elements. The results indicate that US-QUAD8 requires 17-38% more computational

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

time compared to QUAD8 for all element meshes considered. The increase in computational time due to the necessity to compute the **P**-matrix, the metric shape functions, and their derivatives is more dominant compared to that due to the necessity to use an unsymmetric equation solver. A similar observation had been made in the case of linear analyses in Chapter 3, Section 3.10.

The increase in computational time is more for distorted meshes compared to the undistorted meshes. The computational time for QUAD8 was observed to reduce in the presence of mesh distortions. This reduction in computational time offered by QUAD8 element is, however, is at the expense of a decrease in accuracy of the computed results, *viz.*, compare the smoothness of contour lines in Figures 4.15a and 4.15b.

Table 4.1. Comparison of computational time of a beam subjected to an end thrust

QUAD8			US-QUAD8		
Mesh	No. Iterations	Time (CPU s)	No. Iterations	Time (CPU s)	% Increase in time
Regular	287	224.80	286	262.67	16.8
Distorted mesh I	286	200.63	287	261.33	30.3
Distorted mesh II	263	190.36	288	263.23	38.3
Distorted mesh III	269	216.49	285	277.77	28.3

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

(a) QUAD8

(b) US-QUAD8

Figure 4.13. Stress, σ_{zz} contour plot of the beam subjected to an end thrust for regular element mesh

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

(a) QUAD8

(b) US-QUAD8

Figure 4.14. Stress, σ_{zz} contour plot of the beam subjected to an end thrust for distorted mesh I

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

(a) QUAD8

(b) US-QUAD8

Figure 4.15. Stress, σ_{zz} contour plot of the beam subjected to an end thrust for distorted mesh II

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

(a) QUAD8

(b) US-QUAD8

Figure 4.16. Stress, σ_{zz} contour plot of the beam subjected to an end thrust for distorted mesh III

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

Figures 4.13 – 4.16 show the contour plots of σ_{xx} - Cauchy stress for all the meshes considered. For both regular mesh as well as distorted mesh I, US-QUAD8 and QUAD8 yield contour plots of comparable smoothness (see Figure 4.13 and 4.14). However, for mesh II and mesh III, US-QUAD8 yields smoother contour lines compared to QUAD8 (see Figure 4.15 and 4.16). The smoother contour lines at the inter-element boundaries suggests that US-QUAD8 yields more accurate results for distorted meshes despite requiring more computational time. For regular meshes, however, both elements yield similar results, US-QUAD8 requiring more computational time.

While the present element requires more computational time, this additional effort is reimbursed with improved solution accuracy for distorted element meshes. Thus, a comparison of efficiency of the elements based on computational time alone does not seem fair. For a more objective assessment of the elements efficiencies, both computational time and accuracy need to be considered.

The product of the relative error in the solution and the computational time (ET) is proposed here as a measure of the computational efficiency. Elements with lower values of the product, ET , are deemed to be computationally more effective. Tables 4.2a – 4.2c list the relative error, computational time and the product ET for the straight cantilever problem in Figure 4.1a with 10 super-elements.

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

Table 4.2a. Estimate of ET for angular distortion

$\delta = 0.6$				$\delta = 0.8$		
Element	%Error	Time (s)	ET	% Error	Time (s)	ET
QUAD8	17.8559	34.34	613.1706	34.5631	36.21	1251.5292
QUAD9	3.0643	37.83	115.9216	4.6600	38.43	179.0848
US-QUAD8	2.4544	56.93	139.7306	3.4161	57.89	197.7605

Table 4.2b. Estimate of ET for curved-edge distortion

$\delta = 0.6$				$\delta = 0.8$		
Element	%Error	Time (s)	ET	% Error	Time (s)	ET
QUAD8	87.6095	33.14	2903.3800	91.8282	31.45	2887.9959
QUAD9	57.1480	37.33	2133.3363	71.3022	36.59	2608.9480
US-QUAD8	1.2397	56.75	70.3519	1.8928	57.18	108.2323

Table 4.2c. Estimate of ET for mid-side node distortion

$\delta = 0.6$				$\delta = 0.8$		
Element	%Error	Time (s)	ET	% Error	Time (s)	ET
QUAD8	26.2618	32.78	860.8610	51.9191	33.00	1713.3294
QUAD9	26.2601	37.32	980.0266	51.9107	37.50	1946.6497
US-QUAD8	0.6771	56.95	38.5620	1.2599	66.59	83.8977

Comparison of the accuracy of solution is similar to that discussed in Section 4.4.1. In terms of computational time, however, US-QUAD8 is computationally the most expensive. The computational time for QUAD9 and QUAD8 are comparable. Comparison of ET values shows that the present element is computationally more efficient compared to QUAD8 and QUAD9 for meshes with curved-edge and mid-side node distortion. In the case of angular distortion, QUAD9 yields better ET values. The ET values of US-QUAD8 for angular distortions are higher but still at a competitive level

when compared to QUAD9. On an overall basis, the present element can be concluded to be more efficient compared to both QUAD8 and QUAD9.

4.6. EXTENSION TO 20-NODE HEXAHEDRON

The unsymmetric formulation of US-QUAD8 applied to geometric nonlinear analyses is also extended to include US-HEXA20. In its implementation, the dimensions of the stress matrix, $\tilde{\mathbf{S}}^{(e)}$, the shape function matrices, $\mathbf{N}$ and $\mathbf{M}$, the strain displacement matrices, $\bar{\mathbf{B}}_{NL}$, $\hat{\mathbf{B}}_{NL}$, $\bar{\mathbf{B}}_G$ and $\hat{\mathbf{B}}_G$ are suitably adjusted to accommodate the additional entries required for the 20-node hexahedron. The formulation of the stiffness matrix and evaluation of the residual load vector follow the steps detailed in Section 4.2.

This section presents the results of US-HEXA20 applied to a few numerical examples that involve large deflection of plate and shell structures. The results of US-HEXA20 are compared with results obtained from the commercial finite element package, ANSYS, the classical 20-node hexahedron element, HEXA20 and some plate or shell elements available in literature.

4.6.1. LARGE DEFORMATION OF STRAIGHT CANTILEVER BEAM

The static deformation of the straight cantilever beam in Section 4.4.1 subjected to an end shear force is used here to assess the distortion tolerance of the US-HEXA20 element. The cantilever is discretized using five super-elements along its length, each super-element being of distorted geometries shown in Figure 4.17 below.

Figure 4.17. Super-elements used for discretizing cantilever beam subjected to an end shear load; (a) plane distortion; (b) curved-face distortion.

The distortion parameter, δ , is deliberately introduced into each of the super-elements to assess the elements distortion sensitivity. The cantilever is solved for its vertical tip deflection, and the results obtained from both elements HEXA20 and US-HEXA20 are compared with the theoretical solution [134].

Figures 4.18 and 4.19 show the %-deviation of the computed deflections from the theoretical solution for the super-elements with plane and curved-face distortions, respectively. The results indicate that the US-HEXA20 element, like its US-QUAD8 counterpart is distortion tolerant with respect to the tested distortions while the HEXA20 element is not. An increase in the magnitude of distortion parameter δ does not very much affect the solutions of US-HEXA20 as it does in HEXA20. These observations confirm the distortion sensitivity of US-HEXA20.

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

Figure 4.18. Deviation of vertical deflection from theoretical solution for a mesh with plane distortion.

Figure 4.19. Deviation of vertical deflection from theoretical solution for a mesh with curved-face distortion.

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

Table 4.3 shows the error in computed displacements and the number of iterations for HEXA20 and US-HEXA20. The results are shown for two different numbers of loadsteps. For an undistorted mesh, both HEXA20 and US-HEXA20 yield the same error and consume the same amount of equilibrium iterations for 10 as well as 200 loadsteps.

Table 4.3a. Percent error from theoretical solution for straight cantilever beam problem.

Distortion Type	No. Loadsteps	% Error	
		HEXA20	US-HEXA20
Undistorted	10	7.39	7.39
	200	1.65	1.65
Plane, $\delta = 1.0$	10	1.97	6.72
	200	5.59	2.23
Curved Face, $\delta = 1.0$	10	54.46	7.14
	200	54.88	1.87

Table 4.3b. Number of iterations consumed for straight cantilever beam problem.

Distortion Type	No. Loadsteps	Number of iterations	
		HEXA20	US-HEXA20
Undistorted	10	32	32
	200	400	400
Plane, $\delta = 1.0$	10	32	33
	200	400	400
Curved Face, $\delta = 1.0$	10	38	32
	200	509	400

For a mesh with plane distortions ($\delta = 1$), HEXA20 yields an error of 1.97% when 10 loadsteps are used, compared to 6.72% using US-HEXA20. However, the error in HEXA20 increases from 1.97% to 5.59% as the number of loadsteps is increased from 10 to 200 whilst, the error in US-HEXA20 decreases from 6.72% to 2.23%. The number of

equilibrium iterations consumed by both HEXA20 and US-HEXA20 are about the same for 10 as well as 200 loadsteps. For a mesh with curved-face distortion ($\delta = 1.0$), HEXA20 yields an error of 54.46% when 10 loadsteps are used and this error increases to 54.88% when 200 loadsteps are used. US-HEXA20 on the other hand, yields an error of 7.14% and 1.86% for ten and two hundred loadsteps, respectively. HEXA20 consumes more equilibrium iterations compared to US-HEXA20 for both loadsteps considered.

4.6.2. CLAMPED CIRCULAR PLATE SUBJECTED TO A CONCENTRATED LOAD AT THE CENTRE

A large deformation analysis of a clamped circular plate subjected to a central point load, q with Poisson's ratio, $\nu = 0.3$, Young's modulus, $E = 10^7$, radius, $R = 100$, and thickness, $t = 2$ is considered. Due to the symmetry of the plate, only a quarter of the plate is modeled using a 27-element mesh as shown in Figure 4.20.

Figure 4.20. A 27-element mesh of one quarter of a clamped circular plate

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

Table 4.4 shows the dimensionless central deflection of the circular plate, w/t , of US-HEXA20 in comparison with the analytical solution of Chia [135]. The results obtained with the 20-node hexahedron element of ANSYS, the triangular plate/shell element of Zhang and Cheung [136] and the QS plate element of Pica et al. [137] are also shown. The triangular plate-bending element of Zhang and Cheung [136] is a combination of Allman's triangular plate element with vertex degrees of freedom [71] and the refined non-conforming triangular plate-bending element RT9 [185]. The 8-node serendipity plate element of Pica et al. [137] is based on the Mindlin plate theory and is integrated using 2×2 quadrature rule for all bending, membrane and shear terms in the stiffness matrix. The 20-node hexahedron element of ANSYS is integrated using a 14-point quadrature rule.

Table 4.4. Non-dimensional central deflection, w/t , of clamped circular plate subjected to a concentrated load, q , at its center

$\frac{qR^2}{Et^4}$	w/t				
	Nonlinear Analytical Solution [135]	24 – triangular plate/shell elements [136]	12 – QS elements [137]	ANSYS	US-HEXA20
1	0.2129	0.2057	0.2044	0.2009	0.2031
2	0.4049	0.3892	0.3908	0.3841	0.3877
3	0.5695	0.5483	0.5528	0.5420	0.5470
4	0.7098	0.6868	0.6930	0.6785	0.6838
5	0.8309	0.8057	0.8152	0.7973	0.8026
6	0.9372	0.9176	0.9237	0.9045	0.9074

Figure 4.18 shows the results of Table 4.3 graphically. The linear solution obtained from the triangular plate/shell element of Zhang and Cheung [136] is also included in Figure

4.21 for reference. As observed from Table 4.4 and Figure 4.21, the performance of US-HEXA20 is comparable with the other elements in consideration. Any advantages or disadvantages of US-HEXA20 over the other elements in consideration are not obvious since element distortions in the mesh (Figure 4.17) are not apparent.

Figure 4.21. Load-deflection curve of a clamped circular plate subjected to a concentrated load at the center

4.6.3. HINGED CYLINDRICAL SHELL SUBJECTED TO A CENTRAL POINT LOAD

A circular cylindrical shell with geometric and material properties shown in Figure 4.22 is subjected to a central point load on its convex side. The longitudinal edges are constrained while the curved boundaries are completely free. Symmetry of the model is exploited and only a quarter of the structure is modeled.

The cylindrical shell problem is solved for its central vertical deflection using a 6×6 element mesh of US-HEXA20 elements. The nonlinear relationship between the load and central deflection is shown in Figure 4.23. Also included in Figure 4.23 are the results

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

reproduced from a 5×5 mesh of the triangular flat shell element of Hsiao [138], a 4×4 mesh of a 24 degree of freedom triangular flat shell element by Meek and Tan [139], a 6×6 mesh of triangular plate/shell element by Zhang and Cheung [136] and a 6×6 mesh of 20-node hexahedron elements of ANSYS. The flat shell element of Hsiao [138] is based on the flat shell element of Bathe and Ho [186], modified to eliminate rigid rotations, whilst the triangular-faceted shell element of Meek and Tan [136] is capable of representing linear stress variations and includes normal rotations along the element sides.

Figure 4.22. Circular cylindrical shell subjected to a central point load, $R = 2540\text{mm}$, $L = 254\text{mm}$, $t = 6.35\text{mm}$, $\theta = 0.1\text{rad}$, Young's modulus, $E = 3.10275\text{kN/mm}^2$ and Poisson's ratio, $\nu = 0.3$

The converged solution of a 20×20 mesh of the 20-node hexahedron element of ANSYS is used as a reference solution for this problem. The performance of the present element is comparable with the solutions yielded by ANSYS and the elements of Hsiao [138], and Zhang and Cheung [136]. As with the previous test problem, the distortions experienced by the element in the mesh are not distinct and hence, the advantage of US-HEXA20 element is not obvious.

Figure 4.23. Load-deflection curve of a hinged cylindrical shell subjected to a central point load

4.6.3. HINGED SPHERICAL SHELL SUBJECTED TO A CENTRAL POINT LOAD

A shallow spherical shell subjected to a central point load, with geometric and material parameters shown in Figure 4.24 is considered. All the edges of the shell are hinged. Due to the symmetry of the structure, only a quarter of the shell is modeled. A mesh consisting of 4×4 US-HEXA20 elements is used to solve for the central deflection at the point of load application.

Figure 4.24. A shallow spherical shell subjected to a central point load.

Figure 4.25 shows the load-deflection curves of some shell elements available in literature, viz., 4×4 mesh of triangular plate/shell elements of Zhang and Cheung [136], the 2×2 parabolic curved shell elements of Surana [140] and the series solution of Leicester [141]. The parabolic curved shell element of Surana [140] is formulated by retaining the nonlinear rotation terms in the definition of the displacement field, consistent with derivation of element properties, thus enabling large rotations to be simulated.

The solution given by the 20-node hexahedron of ANSYS for a 4×4 element mesh is also included. The solution of a 20×20 mesh of 20-node hexahedrons of ANSYS is used as a reference solution to represent the converged solution in the limit of mesh refinement. The load-deflection response of US-HEXA20 element lies in between the

solutions given by Surana [140] and Leicester [141] but lies closest to the reference solution obtained with a 20×20 mesh given by ANSYS.

Figure 4.25. Load-deflection curve at the center of a hinged spherical shell subjected to a central point load

4.6.4. PULL OUT OF AN OPEN CIRCULAR CYLINDER

An open circular cylinder as shown in Figure 4.26a is pulled by a pair of concentrated loads applied at the midplane of the cylinder. The cylinder is of radius, $R = 4.953$ in., length, $L = 10.35$ in. and thickness $t = 0.094$ in. The material properties used for this problem are $E = 10.5 \times 10^6$ psi and $\nu = 0.3125$. Symmetry of the model is exploited and one eighth of the cylinder is modeled using an 8×12 mesh as shown in Figure 4.26b. Symmetry of the model is exploited and one eighth of the cylinder is modeled using an 8×12 mesh as shown in Figure 4.26b.

Figure 4.26. (a) Pull out of an open cylinder; (b) An 8×12 uniform mesh used to discretize one-eighth of the cylinder.

The problem is solved for the vertical deflections at the points of load application, up to a maximum load of $P = 18000$ lb. The load-deflection curves predicted by HEXA20, the hybrid-stress element (8×12 mesh) of Sze et al [181] and US-HEXA20 are shown along with the results obtained by Gruttmann et al. [182] in Figure 4.27. The HS element of Sze [181] is derived from an extended Hellinger-Reissner functional using generalized stresses arising from a modified generalized laminate stiffness matrix. The assumed stresses are chosen according to the HS solid element of Pian [187].

The figure shows that the load-deflection curve given by US-HEXA20 coincides with the solution of Gruttmann et al. [182] and is comparable with the solution of the HS element of Sze [181]. Figures 4.28a – 4.28d show the deformed configurations, viewed from the open edge of the cylinder at loads, $P = 0$ lb, 6,000 lb, 12,000 lb and 18,000 lb, respectively.

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

Figure 4.27. Load versus central deflection curves of the pull out of an open cylinder problem.

Figure 4.28. Configuration of pulled out open cylinder problem at loads; (a) $P = 0 \text{ lb}$; (b) $P = 6000 \text{ lb}$.

Figure 4.28. Configuration of pulled out open cylinder problem at loads; (c) $P = 120000$ lb; (d) $P = 18000$ lb.

4.7. UPDATED LAGRANGIAN IMPLEMENTATION OF UNSYMMETRIC FORMULATION

The implementation of unsymmetric formulation has so far been based on a total Lagrangian implementation. Another alternative is the updated Lagrangian implementation. Unlike the total Lagrangian counterpart, the updated Lagrangian approach involves continuous updating of the mesh's geometry. Depending on the problem, the process of mesh updating process introduces distortions into the mesh. The resulting distortions are naturally occurring distortions, and these can introduce errors in the solution. A brief investigation on the updated Lagrangian implementation of the unsymmetric formulation is carried out in this section.

Figure 4.29. An $N \times 2$, element mesh of a straight cantilever beam subjected to a concentrated tip load.

A straight cantilever beam subjected to a concentrated tip load, $P = 0.32N$ with dimensions shown in Figure 4.29 is considered. The beam is made of material having a Young's modulus of $E = 12$ and Poisson's ratio, $\nu = 0.3$. An $N \times 2$ mesh of rectangular/square shaped elements is used to discretize the cantilever and the error in the computed displacements at point $C(100,10)$ is monitored. Table 4.5 shows the horizontal

and vertical displacement errors obtained using an updated Lagrangian of QUAD8 and US-QUAD8. An over-killed solution using a 40×4 element mesh of QUAD8 elements is used as a basis for comparison.

Table 4.5. Percentage of error of computed horizontal and vertical displacements[¶]

N	QUAD8		US-QUAD8	
	UX	UY	UX	UY
5	3.7717	1.7942	2.5889	1.1890
10	0.9270	0.4835	0.8371	0.4357
15	0.4730	0.2520	0.4582	0.2436
20	0.2789	0.1526	0.2764	0.1512

[¶] Overkill solution: UX = -20.60087; UY = -62.37092

The solution error of US-QUAD8, as seen from Table 4.5 is less, compared to that of QUAD8 for all element mesh sizes considered e.g., for $N = 5$, QUAD8 yields a solution error for UY of 1.7942% while US-QUAD8 yields a solution error of 1.1890%. The improvement in solution accuracy is marginal since the naturally occurring distortions in this problem are not severe. A more pronounced improvement in solution accuracy is likely if the meshes become grossly distorted due to mesh updating procedure. Such is the case in metal forming processes, e.g., hot and cold forging. However, simulations of these problems require the development of contact algorithms, along with the consideration of plastic flow. Currently, the computer code developed has no facilities to handle plastic deformations and contact algorithms and hence such problems could not be solved. Application of the unsymmetric formulation for metal forming applications by

itself is an interesting and challenging research problem. In view of time constraint, however, it is reserved for future work.

4.8. CONCLUSIONS

The unsymmetric formulation has been extended to problems involving geometric nonlinearity. Unsymmetric versions of the 8-node quadrilateral and 20-node hexahedron have been developed. Element formulation is implemented based on a total Lagrangian approach. The choice of shape functions for the incremental virtual and trial displacement fields follows from *Criterion 1* and *Criterion 2* (Chapter 3), as with linear analysis.

In the present implementation, the deformation gradient tensor, $\mathbf{F}$, is computed using isoparametric or metric shape functions depending on whether it appears in the virtual or trial displacement field. Alternatively, one may invariably use the isoparametric shape functions regardless of whether $\mathbf{F}$ appears in the virtual or trial displacement field. No significant difference in the computed results has been observed between these two choices.

The performance of US-QUAD8 for geometric nonlinear problems has been studied using several test problems. The performance of US-QUAD8 has been found to be superior compared to that of QUAD8 for all angular, curved-edge and mid-side node distortions, whereas it is similar to that of QUAD9 for meshes involving angular distortions. US-QUAD8 has been found to yield superior performance compared to both QUAD8 and QUAD9 for meshes involving curved-edge and mid-side node distortions.

CHAPTER 4. UNSYMMETRIC FORMULATION FOR GEOMETRIC NONLINEAR ANALYSES

In terms of computational time, US-QUAD8 is more expensive. Nevertheless, an assessment of overall computational efficiency, measured as a product of solution error and computational time (ET), has revealed that US-QUAD8 is comparable to QUAD9 for meshes involving angular distortions and is superior to both QUAD8 and QUAD9 for curved-edge and mid-side node distortions.

Similarly, the performance of US-HEXA20 has also been found to be superior compared to that of HEXA20 for test problems involving plane and curved-faced distortions. The number of Newton Raphson iterations consumed by both HEXA20 and US-HEXA20 are the same for both undistorted and plane distorted elements. However, US-HEXA20 requires fewer number of iterations compared to HEXA20 for elements with curved-faced distortions. For test problems involving shell structures, US-HEXA20 is comparable to the other shell and plate elements in literature [136 – 140]. In some instances, US-HEXA20 has been observed to yield solutions that are closest to that obtained from an “over-killed” solution using a very fine mesh (see example, Section 4.6.3).

The results of both US-QUAD8 and US-HEXA20 in the test problems considered in this chapter indicate that the unsymmetric formulation is effective in the treatment of mesh distortions for geometric nonlinear analyses.

CHAPTER 5. UNSYMMETRIC-MIXED FORMULATION FOR INCOMPRESSIBLE ELASTICITY

5.1. INTRODUCTION

For linear elasticity applications, Rajendran and Liew [5] tested the displacement-based US-QUAD8 element extensively for various pathological conditions using a series of test problems. These problems include a volumetric locking test. US-QUAD8 did not exhibit any symptoms of volumetric locking in the test problem they considered. Based on this observation, a direct implementation of the US-QUAD8 element for finite deformation problems was first attempted for an incompressible neo-Hookean material. The response of the element under incompressible material behavior, however, contradicts what was initially observed for linear elasticity problems. Under plane strain conditions of materials defined by incompressible strain energy functions, US-QUAD8 was observed to exhibit symptoms of volumetric locking. Hence, possible methods to remove volumetric locking in the displacement-based unsymmetric element are first explored.

A literature search revealed that successful treatment of volumetric locking in finite elements involves the *mixed formulations*. Mixed formulations employ two separate interpolations for the displacement and pressure fields. In this chapter, both concepts *viz.*, unsymmetric and mixed formulations are combined in search of an element formulation that is free of volumetric locking, yet still yields accurate solutions in the presence of mesh distortions. This amalgamation, which in this thesis is called the unsymmetric-

mixed element formulation, is then validated using several test problems. Only static analyses will be considered in this chapter.

The phenomenon of volumetric locking, its effects, and common treatment in literature are briefly reviewed in Section 5.2. Governing equations that set up the platform for the unsymmetric-mixed element formulation are subsequently presented in Section 5.3. Element formulation for an unsymmetric-mixed, 9-node quadrilateral element with three independent pressure variables is detailed in Sections 5.4 – 5.7. Computational and numerical aspects related to implementation of the element are discussed in Section 5.8. The proposed element is then validated using several test problems in Section 5.9.

5.2. VOLUMETRIC LOCKING

Displacement based finite elements suffer from volumetric locking when they are used to model nearly or fully incompressible materials (e.g. natural rubber and metals undergoing plastic deformation). Due to the large ratio of bulk to shear modulus in such materials, the displacement based finite elements overestimate the amount of energy stored in volumetric deformations [142]. This fact was reported by Sussman and Bathe [143] as the rationale for the poor performance of displacement-based finite elements in modeling incompressible materials. Prathap [144] provided an alternative explanation of the volumetric locking phenomena using his field-consistency approach. Research into formulations to remove volumetric locking for incompressible materials has been extensive. The survey performed by Galada [145] summarizes the early works devoted to modeling responses of incompressible materials with finite elements.

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

The basic approach in treatment of volumetric locking employs separate interpolations for the displacement, and the stress component that is related to the hydrostatic pressure. The potential energy is augmented using an incompressibility constraint, imposed using Lagrange multipliers. This approach falls under the category of mixed methods and was successfully implemented by pioneering authors [146 – 149] in the late 1960's. The use of Lagrange multipliers introduces additional degrees of freedom to the final assembly of the finite element equations. These additional degrees of freedom can be removed from the system of equations through the use of discontinuous pressure interpolations across the element boundaries. This approach, where the Lagrange multipliers are condensed out of the system of equations, is the *consistent penalty method*. This technique is used in view of the computational advantages it offers [143, 150 – 153].

Another well-known approach to alleviate volumetric locking is the enhanced strain method introduced by Simo and Rifai [68]. This technique involves augmenting the space of discrete strains with suitable local functions. In large strain problems, the enhanced strain method, however, experiences an undesirable mode of deformation known as the hourglass mode. This has prompted research into formulations to remove such deformations [154 – 155]. Pantuso and Bathe [156] concluded that enhanced strain elements satisfying the inf-sup test [see e.g. 157] in linear analyses do not guarantee good performance in large strain analyses due to hourglass deformations. The $9/3$, displacement/pressure quadrilateral element was shown to be free of hourglass modes. This element and its three-dimensional counterpart, the $27/4$, displacement/pressure hexahedron have been advocated as the most effective elements for large strain analyses.

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

In this chapter, an attempt will be made to implement the unsymmetric element through the mixed formulation so as to avoid volumetric locking when modeling incompressible materials. The unsymmetric formulation has been shown to yield good performance in the presence of mesh distortions. Thus, when it is combined with the mixed formulation, the resulting element is expected to inherit the advantageous of both techniques. The proposed approach will be experimented on a 9-node quadrilateral with three independent pressure degrees of freedom instead of the 8-node quadrilateral, despite discussions in previous chapters centering about the 8-node quadrilateral.

The switch from eight to nine nodes is necessary for the following reason: when the mixed formulation was employed in literature, it was found that a formulation using equal displacement and pressure degrees of freedom was not effective [143]. Hughes [119] summarizes the effective combinations of displacement and pressure degrees of freedom in mixed elements. He concluded that the 8-node quadrilateral is effective when it is used with two pressure degrees of freedom. However, such an implementation is not recommended since the monomial terms in the pressure interpolation necessitates preference in only a particular direction[†]. The 9-node quadrilateral with three pressure degrees of freedom, on the other hand, has been mathematically proven to be effective in

[†] A two-degree of freedom interpolation of the pressure field, $p(\xi, \eta)$ would involve either

$$p(\xi, \eta) = a_0 + a_1 \xi \quad (5.1)$$

or

$$p(\xi, \eta) = a_0 + a_1 \eta \quad (5.2)$$

In either equations (5.1) or (5.2), the linear variation of pressures can be captured in only the ξ or η direction, depending on the orientation of the element. The pressure field is thus orientation dependent.

mixed formulations [143]. A linear interpolation involving a constant and the monomial terms ξ and η in the local coordinate system, precludes preferences in any directions. The 9-node quadrilateral is thus used for this investigation.

5.3. GOVERNING EQUATIONS

Hyperelastic materials are characterized by the existence of a potential function that represents the strain energy per unit volume in the undeformed body. Rubber is an example of such a material. Commonly used potential functions that characterize the behavior of rubber are that of Rivlin and Saunders [158], Ogden [159] and Arruda and Boyce [160]. These models describe rubber as an incompressible material.

The incompressibility becomes an additional constraint equation to be considered, and has to be treated properly in order to avoid volumetric locking. Under such conditions it is useful to employ an additive decomposition of the strain energy function, W , into a purely volumetric and a purely deviatoric counterpart. This decomposition expresses the strain energy as

$$W = \tilde{W}(\mathbf{C}) + \kappa U(J) \quad (5.3)^\dagger$$

where $\tilde{W}(\mathbf{C})$ is a constituent of the deviatoric component of the strain energy and κ represents the bulk modulus that is independent of the deformation. $U(J)$ is the volumetric component of the strain energy, and is a function of the Jacobian,

[†] Unless mentioned otherwise, the strain energy function used throughout this chapter will be of the Mooney-Rivlin description

$$\tilde{W}(\mathbf{C}) = C_1(I_1 - 3) + C_2(I_2 - 3)$$

where C_1 and C_2 are the Mooney-Rivlin constants, and I_1 and I_2 are the first and second invariants of the right Cauchy-Green deformation tensor.

$J = \det \mathbf{F} = (\det \mathbf{C})^{1/2}$. The deformation gradient tensor is represented as $\mathbf{F}$, and $\mathbf{C}$ is the right Cauchy-Green deformation tensor, which is computed as $\mathbf{C} = \mathbf{F}^T \mathbf{F}$.

The function used for the volumetric potential $U(J)$ must be convex and reaches a minimum value when $J = 1$. A reasonable choice for $U(J)$ that is adopted for the present implementation is given by [143 and 153] as

$$U(J) = \frac{1}{2}(J - 1)^2 \quad (5.4)$$

Other forms of the volumetric strain energy function have also been proposed and their effectiveness is discussed in detail in [161].

The governing linearized equation is implemented using a Total Lagrangian approach wherein all kinematic variables are referred to the initial configuration. The second Piola-Kirchoff stress, ${}^t\mathbf{S}$ at time t is evaluated by taking the derivative of Equation (5.3) with respect to the right Cauchy-Green deformation tensor, $\mathbf{C}$

$${}^t\mathbf{S} = 2 \frac{\partial W}{\partial \mathbf{C}} = 2\kappa U'(J) \frac{\partial {}^tJ}{\partial \mathbf{C}} + \frac{\partial \tilde{W}}{\partial \mathbf{C}} \quad (5.5)$$

The quantity $\kappa U'(J)$ in Equation (5.5) is nothing, but the hydrostatic pressure at time t , ${}^t p$, which is also the incompressibility constraint. Making this substitution, the second Piola-Kirchoff stress can be expressed as

$${}^t\mathbf{S} = 2 {}^t p \frac{\partial {}^tJ}{\partial \mathbf{C}} + 2 \frac{\partial \tilde{W}}{\partial \mathbf{C}} \quad (5.6)$$

The incremental virtual work expression for an incompressible hyperelastic material under the action of external loads can be expressed as [162]

$$\begin{aligned} & \int_{\Omega_o} \left(4 \frac{\partial^2 \tilde{W}}{\partial \mathbf{C} \partial \mathbf{C}} + 4 {}^t p \frac{\partial^2 ({}^t J)}{\partial \mathbf{C} \partial \mathbf{C}} \right) {}_o \mathbf{e} \delta {}_o \mathbf{e} d\Omega_o + \int_{\Omega_o} 2 \frac{\partial ({}^t J)}{\partial \mathbf{C}} {}_o p \delta {}_o \mathbf{e} d\Omega_o \\ & + \int_{\Omega_o} \left(2 \frac{\partial \tilde{W}}{\partial \mathbf{C}} + 2 \frac{\partial ({}^t J)}{\partial \mathbf{C}} {}^t p \right) \delta {}_o \boldsymbol{\eta} d\Omega_o = W_{ext} - \int_{\Omega_o} \left(2 \frac{\partial \tilde{W}}{\partial \mathbf{C}} + 2 \frac{\partial ({}^t J)}{\partial \mathbf{C}} {}^t p \right) \delta {}_o \mathbf{e} d\Omega_o \end{aligned} \quad (5.7)$$

where ${}_o p$ is the incremental hydrostatic pressure. Adopting the notation of Bathe [162], the incremental Green strain, ${}_o \mathbf{E}$, has been decomposed into linear (${}_o \mathbf{e}$) and nonlinear (${}_o \boldsymbol{\eta}$) components:

$${}_o \mathbf{E} = {}_o \mathbf{e} + {}_o \boldsymbol{\eta} \quad (5.8)$$

The superscript ‘ t ’ indicates that the quantity is evaluated at ‘time’ t . Here, Ω_o and Γ_o refer to the domain and boundary of the problem at the initial configuration, the symbol δ denotes the first variation of a quantity, and W_{ext} represents the external work done by external forces acting on the body. The right-subscript ‘ o ’ is used to indicate that the quantity is referred to its initial configuration.

The incompressibility constraint, ${}^t p = \kappa U'(J)$, is enforced in a weak sense by taking the first derivative of Equation (5.4) with respect to the Jacobian, J . This results in $U'(J) = J - 1$. Substituting this into the incompressibility constraint yields

$${}^t p = \kappa(J - 1) \quad (5.9)$$

Equation (5.9) is then multiplied with a test function (virtual pressure field, $\delta {}_o p$) and then integrated over the entire domain to yield

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

$$\int_{\Omega_o} \left\{ J - 1 - \frac{1}{\kappa} {}^t p \right\} \delta_o p d\Omega_o = 0 \quad (5.10)$$

The incremental form of the Jacobian, J in Equation (5.10) can be written as $J = {}^t J + {}_o J$,

wherein the incremental quantity ${}_o J$, is

$${}_o J = \frac{\partial({}^t J)}{\partial {}^t \mathbf{E}} {}_o \mathbf{E} = 2 \frac{\partial({}^t J)}{\partial \mathbf{C}} {}_o \mathbf{E} \quad (5.11)$$

Making use of the incremental form of the Jacobian, J and substituting Equation (5.11) into Equation (5.10) yields

$$\int_{\Omega_o} 2 \frac{\partial({}^t J)}{\partial \mathbf{C}} {}_o \mathbf{E} \delta_o p d\Omega_o = \int_{\Omega_o} \left(1 - {}^t J + \frac{1}{\kappa} {}^t p \right) \delta_o p d\Omega_o \quad (5.12)$$

Both Equations (5.7) and (5.12) constitute the weak form of the boundary value problem in finite elasticity for an incompressible material.

5.4. SHAPE FUNCTIONS

Two sets of shape functions, isoparametric and metric, will be used for the virtual and trial displacement/pressure fields, respectively according to *Criterion 1 and Criterion 2* in Chapter 3. A typical 9-node quadrilateral element is expected to reproduce displacement fields of the form

$$u(X, Y) = a_1 + a_2 X + a_3 Y + a_4 X^2 + a_5 XY + a_6 Y^2 + a_7 X^2 Y + a_8 XY^2 + a_9 X^2 Y^2 \quad (5.13)$$

$$v(X, Y) = b_1 + b_2 X + b_3 Y + b_4 X^2 + b_5 XY + b_6 Y^2 + b_7 X^2 Y + b_8 XY^2 + b_9 X^2 Y^2 \quad (5.14)$$

where a_i and b_i , ($i=1,2,3,\dots,9$) are arbitrary constants. The metric shape functions of a 9-node quadrilateral element are obtained by solving the set of nine simultaneous equations resulting from the completeness conditions

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

$$\sum_{i=1}^9 M_i X_i^p Y_i^q = X^p Y^q \quad (5.15)$$

The terms $X^p Y^q$ in Equation (5.15) refer to monomial terms in Equations (5.13) and (5.14) with appropriate indices $p, q = 0, 1$ or 2 . Capital fonts, X and Y are used to indicate that the monomial terms are referred to the initial configuration of the body. Equation (3.23) with $\mathbf{P}^{-1}$ and $\mathbf{p}(\mathbf{x})$, suitably modified to include the nine monomial terms, $1, X, Y, \dots, X^2 Y^2$ in Equations (5.13) and (5.14) is then used to compute the metric shape functions.

The isoparametric shape functions for the 9-node quadrilateral can be expressed as (see e.g. [163]):

$$N_i = \frac{1}{4} \xi_i \xi \eta_i \eta (1 + \xi_i \xi) (1 + \eta_i \eta) \quad (5.16)$$

$$N_j = \frac{1}{2} \eta_j \eta (1 + \eta_j \eta) (1 - \xi^2) \quad (5.17)$$

$$N_k = \frac{1}{2} \xi_k \xi (1 + \xi_k \xi) (1 - \eta^2) \quad (5.18)$$

$$N_l = (1 - \xi^2) (1 - \eta^2) \quad (5.19)$$

where i refers to the corner nodes, j and k refer to the mid-side nodes along the edges $\xi = 0$ and $\eta = 0$, respectively, and l refers to the node at the parametric center of the element.

5.5. DISCRETIZATION OF THE PRINCIPLE OF VIRTUAL WORK

The principle of virtual work given by Equation (5.7) is now discretized using isoparametric and metric shape functions. The virtual strains in Equation (5.7) are first considered. These involve the first derivatives of incremental virtual displacements $\delta_o \mathbf{u}$, which are the test functions. The choice of shape functions for the incremental virtual displacements is to comply with *Criterion 1* in Chapter 3 i.e., the shape functions must satisfy the node-edge Kronecker delta property for any admissible element geometry. For this purpose, the isoparametric shape functions form a good choice to represent the incremental virtual displacement field. The isoparametric discretization of the virtual displacement field can be written as

$$\delta_o \mathbf{u}(\xi, \eta) \approx \delta_o \bar{\mathbf{u}} = \mathbf{N} \delta_o \bar{\mathbf{u}}_n \quad (5.20)$$

where $\mathbf{N}$ is the matrix of isoparametric shape functions.

Adopting the notations of Bathe [162], the linear and nonlinear components of the virtual strain increments can be expressed in terms of the incremental virtual displacements using the transformation matrices ${}^t_o \bar{\mathbf{B}}_L$ and ${}^t_o \bar{\mathbf{B}}_{NL}$. The over-bar above the strain-displacement transformation matrices indicates that these matrices are computed using isoparametric shape functions. The isoparametric interpolation of the linear and nonlinear virtual strains can be expressed as

$$\delta_o \mathbf{e} \approx \delta_o \bar{\mathbf{e}} \equiv {}^t_o \bar{\mathbf{B}}_L \delta_o \bar{\mathbf{u}}_n \quad (5.21)$$

$$\delta_o \boldsymbol{\eta} \approx \delta_o \bar{\boldsymbol{\eta}} \equiv {}^t_o \bar{\mathbf{B}}_{NL} \delta_o \bar{\mathbf{u}}_n \quad (5.22)$$

Substituting the expressions for the incremental virtual strains into Equation (5.7) and recasting the equation in matrix form yields, for a typical element

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

$$\begin{aligned} & \delta_o \bar{\mathbf{u}}_n^T \left\{ \int_{\Omega_o} {}^t \bar{\mathbf{B}}_L^T \left({}_o \tilde{\mathbf{C}} + 4 {}^t p \frac{\partial^2 ({}^t J)}{\partial \mathbf{C} \partial \mathbf{C}} \right) {}_o \mathbf{e} d\Omega_o \right\} + \delta_o \bar{\mathbf{u}}_n^T \left\{ \int_{\Omega_o} {}^t \bar{\mathbf{B}}_{NL}^T \left({}^t \tilde{\mathbf{S}} + 2 {}^t p \frac{\partial ({}^t J)}{\partial \mathbf{C}} \right) {}_o \boldsymbol{\eta} d\Omega_o \right\} \\ & + \delta_o \bar{\mathbf{u}}_n^T \left\{ \int_{\Omega_o} {}^t \bar{\mathbf{B}}_L^T \left(2 \frac{\partial ({}^t J)}{\partial \mathbf{C}} \right) {}_o p d\Omega_o \right\} = W_{ext} - \delta_o \bar{\mathbf{u}}_n^T \left\{ \int_{\Omega_o} {}^t \bar{\mathbf{B}}_L^T \left({}^t \tilde{\mathbf{S}} + 2 \frac{\partial ({}^t J)}{\partial \mathbf{C}} \right) {}_o p d\Omega_o \right\} \end{aligned} \quad (5.23)$$

In Equation (5.23), ${}^t \tilde{\mathbf{S}}$ and ${}_o \tilde{\mathbf{C}}$ are the deviatoric components of the stress matrix and material constitutive tensor respectively, given by

$${}^t \tilde{\mathbf{S}} = 2 \frac{\partial \tilde{W}}{\partial \mathbf{C}} \quad (5.24)$$

$${}_o \tilde{\mathbf{C}} = 4 \frac{\partial^2 \tilde{W}}{\partial \mathbf{C} \partial \mathbf{C}} \quad (5.25)$$

The material constitutive tensor is a function of the deformation and its evaluation requires the computation of the right Cauchy-Green deformation gradient tensor, $\mathbf{C}$. The right Cauchy-Green tensor is evaluated with respect to the metric shape functions as $\hat{\mathbf{C}} = \hat{\mathbf{F}}^T \hat{\mathbf{F}}$. Computation of the deformation gradient tensor, $\hat{\mathbf{F}}$ follows from Equation (4.21) in Chapter 4.

The incremental hydrostatic pressure is then approximated using a linear interpolation in terms of local coordinates, (ξ, η)

$${}_o p(\xi, \eta) \approx {}_o \bar{p}(\xi, \eta) \equiv \mathbf{H} \Delta \mathbf{p} \quad (5.26)$$

where

$$\mathbf{H} = \begin{bmatrix} 1 & \xi & \eta \end{bmatrix} \quad (5.27)$$

$$\Delta \mathbf{p} = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}^T \quad (5.28)$$

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

If the isoparametric shape functions are used to discretize the trial incremental displacements (and linear and nonlinear strains), ${}_o\mathbf{u}$ (${}_o\mathbf{e}$ and ${}_o\boldsymbol{\eta}$) in linearized principle of virtual work (5.23), the error nodal forces can be expressed as

$$\begin{aligned} \mathbf{f}_{error}^{(e)} = & \int_{\Omega_o} {}^t\overline{\mathbf{B}}_L^T \left({}_o\tilde{\mathbf{C}} + 4 {}^t p \frac{\partial^2({}^t J)}{\partial \mathbf{C} \partial \mathbf{C}} \right) ({}_o\mathbf{e} - {}_o\bar{\mathbf{e}}) d\Omega_o + \int_{\Omega_o} {}^t\overline{\mathbf{B}}_{NL}^T \left({}^t\tilde{\mathbf{S}} + 2 {}^t p \frac{\partial({}^t J)}{\partial \mathbf{C}} \right) ({}_o\boldsymbol{\eta} - {}_o\bar{\boldsymbol{\eta}}) d\Omega_o \\ & \int_{\Omega_o} {}^t\overline{\mathbf{B}}_L^T \left(2 \frac{\partial({}^t J)}{\partial \mathbf{C} \partial \mathbf{C}} \right) ({}_o p - {}_o\bar{p}) d\Omega_o \end{aligned} \quad (5.29)$$

Instead of the isoparametric shape functions, metric shape functions can alternatively be used to discretize the incremental trial displacement and corresponding linear and nonlinear strain fields:

$${}_o\mathbf{u}(X, Y) \approx {}_o\hat{\mathbf{u}} \equiv \mathbf{M}_o \hat{\mathbf{u}}_n \quad (5.30)$$

$${}_o\mathbf{e} \approx {}_o\hat{\mathbf{e}} \equiv {}^t\hat{\mathbf{B}}_L {}_o\hat{\mathbf{u}}_n \quad (5.31)$$

$${}_o\boldsymbol{\eta} \approx {}_o\hat{\boldsymbol{\eta}} \equiv {}^t\hat{\mathbf{B}}_{NL} {}_o\hat{\mathbf{u}}_n \quad (5.32)$$

The use of metric shape functions for the incremental trial strain fields results in error nodal forces that can be expressed as

$$\begin{aligned} \mathbf{f}_{error}^{(e)} = & \int_{\Omega_o} {}^t\overline{\mathbf{B}}_L^T \left({}_o\tilde{\mathbf{C}} + 4 {}^t p \frac{\partial^2({}^t J)}{\partial \mathbf{C} \partial \mathbf{C}} \right) ({}_o\mathbf{e} - {}_o\hat{\mathbf{e}}) d\Omega_o + \int_{\Omega_o} {}^t\overline{\mathbf{B}}_{NL}^T \left({}^t\tilde{\mathbf{S}} + 2 {}^t p \frac{\partial({}^t J)}{\partial \mathbf{C}} \right) ({}_o\boldsymbol{\eta} - {}_o\hat{\boldsymbol{\eta}}) d\Omega_o \\ & \int_{\Omega_o} {}^t\overline{\mathbf{B}}_L^T \left(2 \frac{\partial({}^t J)}{\partial \mathbf{C} \partial \mathbf{C}} \right) ({}_o p - {}_o\bar{p}) d\Omega_o \end{aligned} \quad (5.33)$$

The difference in magnitude of the error nodal forces in Equations (5.29) and (5.33) depends on the differences between the isoparametric ($({}_o\mathbf{e} - {}_o\bar{\mathbf{e}})$ and $({}_o\boldsymbol{\eta} - {}_o\bar{\boldsymbol{\eta}})$) and

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

metric $((_o\mathbf{e}-_o\hat{\mathbf{e}})$ and $(_o\boldsymbol{\eta}-_o\hat{\boldsymbol{\eta}}))$ representations of the trial strain fields. These are intrinsic functions of the displacements $_o\bar{\mathbf{u}}$ or $_o\hat{\mathbf{u}}$. The isoparametric shape functions of a 9-node quadrilateral element satisfy all the completeness conditions necessary to reproduce the monomial terms X^2, XY, Y^2, X^2Y, XY^2 and X^2Y^2 , only for regular and angularly distorted meshes. However, they cannot reproduce these terms for meshes of other distortion types (curved-edge and mid-side node distortions [112]) and consequently, violate *Criterion 2* governing the choice of shape functions to construct the trial displacement model. On the other hand, the metric shape functions satisfy all the completeness conditions necessary to reproduce all the monomial terms $1, X, Y, XY, X^2, XY, Y^2$ and X^2Y^2 for any admissible element geometry. Thus, they satisfy *Criterion 2*, governing the choice of shape functions to construct the trial displacement model.

Recalling the discussions on the principle of virtual work in Chapter 3, the choice of shape functions for the trial displacement field should preferably be as complete as possible so as to better represent the exact solution (i.e., leading to lower error nodal forces). Thus, in terms of the completeness (reproducibility) properties, the metric shape functions serve as a better choice for constructing the incremental trial displacement model, compared to the isoparametric shape functions.

Comparing the completeness properties of both isoparametric and metric shape functions, the magnitudes of error nodal forces in Equations (5.29) and (5.33) are expected to be of the same degree if the elements in the mesh were undistorted. On the other hand, the magnitude of error nodal forces in Equation (5.33) are expected to be less compared to

the magnitude of error nodal forces in Equation (5.29) in the presence of curved-edge and mid-side node distortions. Most of the test problems presented in Section 5.9 generally stand in support of these speculations. In this respect, the metric shape functions are better suited to represent the trial displacement field compared to the isoparametric shape functions. Using the metric shape functions for the trial displacement field, accompanied by a linear interpolation for the hydrostatic pressure as in Equation (5.26), Equation (5.23) becomes

$$\begin{aligned} & \delta_o \bar{\mathbf{u}}_n^T \left\{ \int_{\Omega_o} {}^t \bar{\mathbf{B}}_L^T {}_o \mathbf{C} {}^t \hat{\mathbf{B}}_L d\Omega_o + \int_{\Omega_o} {}^t \bar{\mathbf{B}}_{NL}^T {}^t \mathbf{S} {}^t \hat{\mathbf{B}}_{NL} d\Omega_o \right\} {}_o \hat{\mathbf{u}}_n + \delta_o \bar{\mathbf{u}}_n^T \left\{ \int_{\Omega_o} {}^t \bar{\mathbf{B}}_L^T \left(2 \frac{\partial({}^t J)}{\partial \mathbf{C}} \right) \mathbf{H} d\Omega_o \right\} \Delta \mathbf{p} \\ & = W_{ext} - \delta_o \bar{\mathbf{u}}_n^T \left\{ \int_{\Omega_o} {}^t \bar{\mathbf{B}}_L^T {}^t \mathbf{S} d\Omega_o \right\} \end{aligned} \quad (5.34)$$

where

$${}_o \mathbf{C} = {}_o \tilde{\mathbf{C}} + 4 {}^t p \frac{\partial^2({}^t J)}{\partial \mathbf{C} \partial \mathbf{C}} \quad (5.35)$$

$${}^t \mathbf{S} = {}^t \tilde{\mathbf{S}} + 2 {}^t p \frac{\partial({}^t J)}{\partial \mathbf{C}} \quad (5.36)$$

5.6. DISCRETIZATION OF THE INCOMPRESSIBILITY CONSTRAINT

Discretization of the incompressibility constraint (Equation (5.12)) involves the incremental trial strains, ${}_o \mathbf{E}$, and the increment of virtual hydrostatic pressure $\delta_o p$. The test functions are the incremental virtual hydrostatic pressure, $\delta_o p$. The classical implementation of the mixed formulation for a 9-node quadrilateral uses a similar linear interpolation as in Equation (5.26) for the pressure field, and is discontinuous across

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

element boundaries. Such an implementation, however, violates the inter-element continuity requirement of shape functions advocated by *Criterion 1* in Chapter 3. However, Sussman and Bathe [143] showed that formulations with equal displacement and pressure degrees of freedom, wherein the hydrostatic pressure is continuous across the element boundaries are ineffective in preventing volumetric locking. Thus, violation of *Criterion 1* for selection of shape functions for the virtual pressure field appears to be inevitable in this case. The numerical examples presented in Section 5.9 do seem to reveal malicious effects of this violation (Figure 5.7) on the accuracy of the computed solution. Exploring a solution to this problem by experimenting on various means of ensuring that the virtual hydrostatic pressure satisfies *Criterion 1* may be a rewarding direction of research but has not been attempted for the present studies. Thus, the virtual incremental hydrostatic pressure is interpolated linearly in terms of local coordinates as

$$\delta_o p = \mathbf{H} \delta(\Delta \mathbf{p}) \quad (5.37)$$

Substituting Equation (5.37) into Equation (5.12) and recasting the resulting equation into matrix form yields

$$\delta(\Delta \mathbf{p}) \left\{ \int_{\Omega_o} \mathbf{H}^T \left(2 \frac{\partial({}^t J)}{\partial \mathbf{C}} \right)_o \mathbf{E} d\Omega_o \right\} = \delta(\Delta \mathbf{p}) \left\{ \int_{\Omega_0} \mathbf{H}^T \left(1 - {}^t J + \frac{1}{\kappa} {}^t p \right) d\Omega_o \right\} \quad (5.38)$$

The choice of shape functions to construct the incremental trial strains, ${}_o \mathbf{E}$, follow from the satisfaction of all completeness conditions advocated by *Criterion 2* in Chapter 3. Since the metric shape functions inherently satisfy all the completeness conditions in Equation (5.15) for any element geometry, it serves as a more appropriate choice to

represent the trial strain field compared to the isoparametric shape functions. Equation (5.38) then becomes

$$\delta(\Delta \mathbf{p}) \left\{ \int_{\Omega_o} \mathbf{H}^T \left(2 \frac{\partial({}^t J)}{\partial \mathbf{C}} \right) {}^t \hat{\mathbf{B}}_L d\Omega_o \right\} {}_o \hat{\mathbf{u}}_n = \delta(\Delta \mathbf{p}) \left\{ \int_{\Omega_0} \mathbf{H}^T \left(1 - {}^t J + \frac{1}{\kappa} {}^t p \right) d\Omega_o \right\} \quad (5.39)$$

5.7. CONDENSATION OF ELEMENT EQUATIONS

Invoking the arbitrariness of the virtual displacements and pressures in Equations (5.34) and (5.39), respectively, the system of equations for the unsymmetric-mixed displacement/pressure formulation for an element can be represented concisely in matrix form as

$$\begin{bmatrix} \mathbf{K}_u & \bar{\mathbf{K}}_p \\ \hat{\mathbf{K}}_p & \mathbf{0} \end{bmatrix} \begin{Bmatrix} {}_o \hat{\mathbf{u}}_n \\ \Delta \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_u \\ \mathbf{F}_p \end{Bmatrix} \quad (5.40)$$

where

$$\mathbf{K}_u = \int_{\Omega_o} {}^t \bar{\mathbf{B}}_L^T {}^t \mathbf{C} {}^t \hat{\mathbf{B}}_L d\Omega_o + \int_{\Omega_o} {}^t \bar{\mathbf{B}}_{NL}^T {}^t \mathbf{S} {}^t \hat{\mathbf{B}}_{NL} d\Omega_o \quad (5.41)$$

$$\bar{\mathbf{K}}_p = \int_{\Omega_0} {}^t \bar{\mathbf{B}}_L^T \left(2 \frac{\partial({}^t J)}{\partial \mathbf{C}} \right) \mathbf{H} d\Omega_o \quad (5.42)$$

$$\hat{\mathbf{K}}_p = \int_{\Omega_o} \mathbf{H}^T \left(2 \frac{\partial({}^t J)}{\partial \mathbf{C}} \right) {}^t \hat{\mathbf{B}}_L d\Omega_o \quad (5.43)$$

$$\mathbf{F}_u = \mathbf{R} - \int_{\Omega_o} {}^t \bar{\mathbf{B}}_L^T {}^t \mathbf{S} d\Omega_o \quad (5.44)$$

$$\mathbf{F}_p = \int_{\Omega_0} \mathbf{H}^T \left(1 - {}^t J + \frac{1}{\kappa} {}^t p \right) d\Omega_o \quad (5.45)$$

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

Making use of the near incompressibility condition, a further simplification may be made.

As the bulk modulus tends to infinity, $\kappa \rightarrow \infty$, the Jacobian approaches unity, $J \rightarrow 1$.

Thus, the contribution of the load vector, $\mathbf{F}_p$ may be neglected and instead of Equation (5.40), the system linear equations becomes

$$\begin{bmatrix} \mathbf{K}_u & \bar{\mathbf{K}}_p \\ \hat{\mathbf{K}}_p & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}}_n \\ \Delta \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_u \\ \mathbf{0} \end{Bmatrix} \quad (5.46)$$

Since the hydrostatic pressure is discontinuous across element boundaries, the pressure coefficients from the second set of equations in Equation (5.46) can be condensed out of the equation. This removes the additional degrees of freedom associated with the pressure modes and thus, reduces computational time. However, the presence of the zero diagonal term in Equation (5.46) denies the option to eliminate the pressure coefficients effectively. To proceed, a perturbed Lagrangian form can be [152] adopted. This involves augmenting Equation (5.46) as

$$\begin{bmatrix} \mathbf{K}_u & \bar{\mathbf{K}}_p \\ \hat{\mathbf{K}}_p & \varepsilon \mathbf{Q} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}}_n \\ \Delta \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_u \\ \mathbf{0} \end{Bmatrix} \quad (5.47)$$

where ε is a small number suitably chosen to make Equation (5.47) nearly equivalent to Equation (5.46) and $\mathbf{Q}$ is a square matrix, known as the *pressure-mass matrix*. Its size depends on the number of pressure coefficients, (3×3) , in the present implementation. The pressure mass matrix may also be chosen as $\mathbf{Q} = \mathbf{I}$ [152].

The second set of equations in Equation (5.47) may first be solved for the vector of pressure increments $\Delta \mathbf{p}$. This yields

$$\Delta \mathbf{p} = -\frac{1}{\varepsilon} \mathbf{Q}^{-1} \left[\left(\hat{\mathbf{K}}_p \right)_o \hat{\mathbf{u}}_n \right] \quad (5.48)$$

Subsequently, substitution of Equation (5.48) into Equation (5.47) yields the condensed stiffness matrix, $\mathbf{K}$ and the load vector $\mathbf{F}_u$:

$$\left(\mathbf{K}_u - \frac{1}{\varepsilon} \bar{\mathbf{K}}_p \mathbf{Q}^{-1} \hat{\mathbf{K}}_p \right)_o \hat{\mathbf{u}}_n = \mathbf{K} \left(\hat{\mathbf{u}}_n \right) = \mathbf{F}_u \quad (5.49)$$

The stiffness matrix $\mathbf{K}$ in Equation (5.49) is unsymmetric in view of the two different shape functions used in $\mathbf{K}_u$, and so is the product of matrices $\bar{\mathbf{K}}_p \mathbf{Q}^{-1} \hat{\mathbf{K}}_p$. The assembly and solution of system equations follows the same procedure as with conventional mixed finite elements.

5.8. COMPUTER IMPLEMENTATION

The US-QUAD9/3 element has been implemented on an in-house FORTRAN routine. The coding effort is more involved compared to geometric nonlinear analyses due to the additional pressure degrees of freedom, and that the constitutive matrix is now a function of the deformation. As in the implementation of the unsymmetric formulation for geometric nonlinear analyses in Chapter 4, the deformation gradient tensor appearing in the trial displacement model, $\hat{\mathbf{F}}$, can also be computed using isoparametric shape functions as $\bar{\mathbf{F}}$ according to Equation (4.10). The performance of either elements using $\hat{\mathbf{F}}$ or $\bar{\mathbf{F}}$ for the trial displacement model are, in general, similar. The pseudo-code describing element formulation of US-QUAD9/3 is given as follows:

1. Loop over elements in the domain
 - a. Form the $\mathbf{P}$ -matrix in Equation (3.21) and compute its inverse
2. Loop over integration points

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

- a. Compute the isoparametric shape functions, $\mathbf{N}$, and their derivatives.
 - b. Compute the metric shape functions, $\mathbf{m}$, using Equation (3.23), and their derivatives using the equations, $\mathbf{m}_{,X} = \mathbf{P}^{-1}\mathbf{p}_{,X}$ and $\mathbf{m}_{,Y} = \mathbf{P}^{-1}\mathbf{p}_{,Y}$, which are the differentiated versions of Eq. (3.23).
 - c. Compute the deformation gradient tensors $\bar{\mathbf{F}}$ and $\hat{\mathbf{F}}$ associated with the isoparametric and metric shape functions respectively.
 - d. Compute the Cauchy-Green deformation tensor, $\hat{\mathbf{C}} = \hat{\mathbf{F}}^T \hat{\mathbf{F}}$.
 - e. Compute the linear and nonlinear strain-displacement matrices, ${}^t_o \bar{\mathbf{B}}_L$, ${}^t_o \hat{\mathbf{B}}_L$, ${}^t_o \bar{\mathbf{B}}_{NL}$ and ${}^t_o \hat{\mathbf{B}}_{NL}$.
 - f. Read the values of the current deviatoric stresses, ${}^t \tilde{\mathbf{S}}$ and hydrostatic pressure, ${}^t p$.
 - g. Form the ${}^t \mathbf{S}$ matrix according to Equation (5.36).
 - h. Evaluate the material constitutive matrix, ${}_o \mathbf{C}$ according to Equation (5.35)
 - i. Set up the pressure interpolation matrix, $\mathbf{H}$.
 - j. Evaluate the components of the element stiffness matrix, $\mathbf{K}_u$, $\bar{\mathbf{K}}_p$ and $\hat{\mathbf{K}}_p$ according to Equations (5.41), (5.42) and (5.43) respectively.
3. Terminate loop over integration points
 4. Condense out the degrees of freedom related to the hydrostatic pressure wherein the modified stiffness matrix is now, $\mathbf{K} = \left(\mathbf{K}_u - \frac{1}{\varepsilon} \bar{\mathbf{K}}_p \mathbf{Q}^{-1} \hat{\mathbf{K}}_p \right)$
 5. Terminate loop over elements

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

The assembly of element stiffness matrices and force vectors, and solution of the system equations adopting the Newton-Raphson scheme follow from conventional finite element practices. The iterative procedure in the present implementation is provided in Appendix I for the sake of completion.

Figure 5.1. Geometry of Cook's membrane problem

5.9. NUMERICAL EXAMPLES

In this section, the performance of the unsymmetric-mixed 9/3, displacement/pressure, element (US-QUAD9/3) is investigated for volumetric locking defects and its sensitivity to mesh distortions using several benchmark problems. Comparison is made with the classical isoparametric 9/3, displacement/pressure, element (QUAD9/3) to check for any improvement in performance. All elements are integrated using 3×3 quadrature rule. For the solutions reported in this section, the iterative procedure follows from the standard full Newton Raphson algorithm.

Figure 5.2. Element meshes used for Cook's membrane problem; (a) 8×4 element mapped mesh; (b) 8×4 element distorted mesh

5.9.1. COOK'S MEMBRANE PROBLEM

The Cook's membrane under the action of a unit vertical load, F (Figure 5.1), is a bending dominated benchmark problem often used to assess the performance of finite elements for volumetric locking. The material properties used are $\mu = 0.375 \text{ N/mm}^2$, $\lambda = 0.75 \times 10^7 \text{ N/mm}^2$ for linear analysis [164], and $\mu = 0.8 \text{ N/mm}^2$, $\kappa = 8000 \text{ N/mm}^2$ [153] for nonlinear analysis, with a volumetric function defined by Equation (5.4).

For the linear analysis, the membrane is solved for its displacements at point P and the total strain energy of deformation. Starting from a coarse mesh, the membrane is progressively refined to check for the convergence of the displacements and strain energy. To study the effects of geometric distortions on the element, three distorted

meshes consisting of 32 elements are generated by randomly perturbing those nodes lying within the boundaries of the membrane of the 8×4 mapped mesh shown in Figure 5.2a. A typical distorted mesh resulting from random perturbation of the nodes is shown in Figure 5.2b.

Table 5.1. Results of Cook's membrane problem for linear analysis[†]

Element	Output	Mesh			Distorted Meshes		
		4×2	8×4	16×8	Distorted I	Distorted II	Distorted III
QUAD9/3	$u_1(P)$	-7.196	-7.234	-7.243	-7.150	-7.216	-7.159
	$u_2(P)$	16.489	16.433	16.435	16.240	16.355	16.321
	$E (\times 10^{-2})$	16.467	16.463	16.482	16.269	16.380	16.358
US-QUAD9/3	$u_1(P)$	-7.188	-7.241	-7.242	-7.198	-7.241	-7.215
	$u_2(P)$	16.549	16.428	16.434	16.414	16.440	16.428
	$E (\times 10^{-2})$	16.508	16.467	16.481	16.447	16.484	16.447

[†]Exact solutions: $u_1(P) = -7.2840$; $u_2(P) = 16.442$; $E \times 10^{-2} = 16.491$

Figure 5.3. Convergence of vertical displacement at point A for Cook's membrane problem

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

Figure 5.4a. Comparison of S_{XX} -contour bands between QUAD9/3 and US-QUAD9/3 for distorted mesh I

Figure 5.4b. Comparison of S_{XX} -contour bands between QUAD9/3 and US-QUAD9/3 for distorted mesh II

The results obtained using both US-QUAD9/3 and QUAD9/3 elements are shown in Table 5.1. The exact solutions reported by Auricchio et al. [164] are used as a reference. Both elements show comparable performances for all mesh sizes in the case of mapped meshes. US-QUAD9/3, however, yields better solution accuracy compared to QUAD9/3 for the distorted meshes considered.

For the nonlinear analysis, the membrane is solved for the vertical displacement at point P . Convergence of the vertical displacement at point P is monitored, as the mesh is progressively refined. The converged vertical displacement at point P using 1000 QUAD9/3 elements agrees with that of reference [153]. This will be used as a reference to compare the performances of both US-QUAD9/3 and QUAD9/3. Figure 5.4 shows the convergence of vertical displacement at point P . Both QUAD9/3 and US-QUAD9/3 show a similar convergence trend. Nevertheless, the error of US-QUAD9/3 is lower compared to that of QUAD9/3, suggesting a better performance of the former.

To investigate the performance of US-QUAD9/3 for distorted element geometry, two meshes consisting of 72 distorted elements are used to model the membrane. Figures 5.4a and 5.4b show the plot of the S_{xx} - stress obtained with QUAD9/3 and US-QUAD9/3. The smoother contour bands exhibited by US-QUAD9/3 for these meshes suggest better accuracy of US-QUAD9/3 compared to QUAD9/3.

Figure 5.5. Clamped beam subject to uniform transverse load, $p = 0.04 \text{ N/mm}^2$; $L = 20 \text{ mm}$; $h = 10 \text{ mm}$; Mooney-Rivlin material constants, $C_1 = 0.177 \text{ N/mm}^2$, $C_2 = 0.045 \text{ N/mm}^2$ and $\kappa = 666.0 \text{ N/mm}^2$

5.9.2. DISTORTION TESTS: CLAMPED BEAM SUBJECTED TO UNIFORM TRANSVERSE LOAD

Figure 5.5 shows the geometry of a thick beam and its corresponding material properties considering a Mooney-Rivlin material model. The beam is discretized using super-elements with angular, curved and mid-side node distorted geometry as shown in Figures 5.6a – 5.6c.

The distortion parameter, δ_p , is measured on a super-element size, $h = 10.0$. This problem assesses the convergence of the results with uniform mesh refinement using distorted meshes in Figure 5.6. An over-kill solution of the vertical displacement at point $A(L/2, h/2)$, obtained from a mesh of 968 regular shaped QUAD9/3 elements, is used as a reference for comparison of performances.

Figure 5.6. Distorted super-elements; (a) angular distortion; (b) curved distortion; (c) mid-side node distortion

Figures 5.7a and 5.7b show the convergence of the vertical displacement at point A for the super-element mesh with angular distortions. In contrast to what was expected of US-QUAD9/3, its accuracy is inferior compared to QUAD9/3. The possibility of US-QUAD9/3 behaving in this manner was mentioned in Section 5.6. An increase in the distortion parameter δ_p increases the error in the solution of US-QUAD9/3. QUAD9/3, however, is practically unaffected by angular distortions [112]. The dominant error in the case of angular distortions appears to result from the discontinuity of virtual pressure

interpolations across the boundary of the element, although the incompatible nature of the metric shape functions in US-QUAD9/3 may have its share of error.

Figure 5.7a. Convergence of vertical displacement at point A for super-element with angular distortion, $\delta_p = 1.0$

The convergence of the vertical displacement for the super-element mesh with curved-edge distortions is shown in Figures 5.8a and 5.8b for two different values of distortion parameters, $\delta_p = 0.1$ and $\delta_p = 0.2$ respectively. For both distortion parameter values, US-QUAD9/3 is seen to perform better. Although errors caused by the discontinuities in the virtual pressure fields are still expected to be present, the results in Figures 5.8a and 5.8b seem to suggest that the dominant error for this mesh results from inaccuracies in representing the trial displacement fields under curved-edge distortions. QUAD9/3, which uses isoparametric shape functions for the trial displacement and pressure fields do not satisfy the higher order completeness conditions under curved-edge distortions. US-QUAD9/3 on the other hand, satisfies the higher order completeness under curved-edge distortions. This leads to a better performance of the element.

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

Figure 5.7b. Convergence of vertical displacement at point A for super-element with angular distortion, $\delta_p = 2.0$

Figure 5.8a. Convergence of vertical displacement at point A for super-element with curved-edge distortion, $\delta_p = 1.0$

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

Figure 5.8b. Convergence of vertical displacement at point A for super-element with curved-edge distortion, $\delta_p = 2.0$

Figure 5.9a. Convergence of vertical displacement at point A for super-element with mid-side node distortion; $\delta_p = 0.5$

Figure 5.9b. Convergence of vertical displacement at point A for super-element with mid-side node distortion; $\delta_I = 1.0$

A similar convergence trend as that of curved-edge distortion is observed for the super-element mesh with mid-side node distortion (see Figures 5.9a and 5.9b). Errors due to non-satisfaction of higher order completeness in the presence of mid-side node distortions again appears to be the reason for the poor performance of QUAD9/3. US-QUAD9/3 is observed to perform far better compared to QUAD9/3.

5.9.3. DISTORTION TESTS: UNIAXIALLY LOADED SHEET OF SQUARE CROSS-SECTION

A square sheet with dimensions $8 \times 8 \text{ mm}^2$ is clamped at both ends and is subjected to axial elongation. Exploiting symmetry of the geometry and boundary conditions of the problem, only the upper right quarter of the sheet is modeled. A Mooney-Rivlin material model is employed and the specifications of the material constants are detailed in Figure 5.10. Plane strain conditions are assumed and the distorted super-elements in Figures 5.5a – 5.5c are used to discretize the sheet. The sheet is discretized with 6×6 super-elements.

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

The values of distortion parameters used are $\delta_p = 0.6$ for angular distortions, $\delta_p = 0.25$ for curved distortions and $\delta_p = 0.2$ for mid-side node distortion, measured on a super-element size, $h = 10.0$. The prescribed displacement at the clamped end of the specimen stretches it to twice its original length.

Figure 5.10. Square sheet subjected to a longitudinal stretch; $L = 8\text{mm}$; Mooney-Rivlin constants, $C_1 = 0.177\text{N/mm}^2$, $C_2 = 0.045\text{N/mm}^2$, bulk modulus, $\kappa = 666\text{N/mm}^2$

Figures 5.11a – 5.11c show the S_{yy} -stress contour plot of the specimen for the three distorted meshes. For a mesh with angular distortion (Figure 5.11a), both elements QUAD9/3 and US-QUAD9/3 yield very similar contours. This is to be expected since QUAD9/3 is not affected by angular distortions [112]. For a mesh with curved-edge and mid-side node distortions (Figures 5.11b and 5.11c), US-QUAD9/3 yields smoother stress contours compared to QUAD9/3. This observation suggests that US-QUAD9/3 yields better accuracy compared to QUAD9/3 for these two types of distortion.

Figure 5.1.1. S_{yy} -stress contours for stretched square sheet; (a) angular; (b) curved-edge; (c) mid-side-node distortion

5.9.4. STRETCHING OF A RUBBER SHEET WITH A HOLE

This problem involves the stretching of a square rubber sheet with a circular hole, which has been previously analyzed by Parisch [165] and Betsch et al. [166]. The material of the rubber sheet is of a Mooney-Rivlin model with constants $C_1 = 25$ and $C_2 = 7$. The geometry of the rubber sheet is shown in Figure 5.12. Due to symmetry of the plate and its loading conditions, only a quarter of the sheet is modeled using 111 elements.

Figure 5.12. Stretching of square rubber sheet with a hole; $R = 3$; thickness, $h_0 = 0.75$; imposed horizontal displacement, $u = 5$

Betsch et al. [166] solved the problem using their proposed 4-node enhanced strain shell element. In their model, extension of the rubber sheet results in simultaneous reduction of the sheet's thickness. The corresponding change in thickness Δh is computed via $\Delta h = h_0 \Delta \lambda$, wherein h_0 is the initial thickness and $\Delta \lambda$ is the stretch increment. In this test problem, the response of the sheet is computed for a stretch increment value,

$\Delta\lambda = 0.5$. To account for the change in thickness in the analysis of Betsch et al. [166], the thickness of the rubber sheet in this two-dimensional analysis is assumed to be half of the total change in thickness throughout the total stretch increment, $\Delta\lambda$. The thickness of the rubber sheet in this analysis is thus assumed to be 0.75.

Figure 5.13. Load-deflection response for stretching of a square rubber sheet with a hole

Figure 5.13 shows the load deflection responses of US-QUAD9/3 and QUAD9/3 elements in comparison with the shell element of Betsch et al. [166]. The performance of both QUAD9/3 and US-QUAD9/3 are similar. The deviation of the load-deflection curves from that of Betsch et al. [166] is due to the assumption of a constant plate thickness. Figure 5.14 shows the S_{xx} -stress contours at $\Delta\lambda = 0.3$ of both QUAD9/3 and US-QUAD9/3 elements.

Figure 5.14. Stress, S_{XX} -contour plots of rubber sheet with a hole at $u = 3$

5.9.5. DEFORMATION OF A PLANE BODY WITH CIRCULAR HOLE

The deformation of a finite-plane strain incompressible body with a central hole subjected to a concentrated edge force will be investigated. Dimensions of the undeformed plate are 10×10 in. The diameter of the central hole is initially 4 in. The body is subjected to an external load P of 2000 lb. The material is of a Mooney-Rivlin model with material constants $C_1 = 80$ psi and $C_2 = 20$ psi. Figure 5.15 shows the geometry and finite element model of the plane body. The vertical edges of the body are clamped. Due to symmetry of the model, only half the plane body is modeled.

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

The deformed profiles and S_{yy} -stress contours of the plane body at various load levels are shown in Figure 5.16 (shown here for the US-QUAD9/3 element). The principal “diameter” of the distorted heart-shaped hole reached 4.93 in when $P = 2000$ lb. Similar results were obtained when QUAD9/3 is used instead. The results show close agreement with that reported by Oden [167], in which the principal diameter of the heart-shaped hole reached 5 in. For this particular problem, the performance of US-QUAD9/3 is as good as QUAD9/3 since the distortions in the mesh considered are not severe.

Figure 5.15. Incompressible plane body with central hole subjected to concentrated edge force, P , and finite element model of the body

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

Figure 5.16. Stress S_{YY} - contour plot of plane body with a hole at different load

5.9.6. HOURGLASS INSTABILITY TEST

The final test problem involves the compression test to check for hourglass instability modes in US-QUAD9/3. Hourglass modes have been associated with the enhanced strain elements [68]. Despite their robust applications to solve elastic and non-elastic problems in solid mechanics, a standard application of these elements to simple problems sometimes leads to some complications. An example of this problem involves a homogeneous compression of a rectangular block undergoing finite elastic deformations [184]. Due to the strong compression, loss of stability is expected to take place at some level of deformation and as a consequence, negative eigenvalues may appear. For this problem, the solution of the enhanced strain elements exhibits hourglass deformations that are characterized by very large negative eigenvalues in the tangent matrix ($>>1000$) [156]. The presence of hourglass deformation modes prematurely terminates the finite element solution process.

Figure 5.17. Compression test. The displacement, Δ , is imposed in small increments to mimic a static analysis.

CHAPTER 5. UNSYMMETRIC-MIXED ELEMENT FOR INCOMPRESSIBLE ELASTICITY

The geometry of the compression test used by Pantuso and Bathe [156] to check for hourglass instabilities in the QUAD9/3 and 4/4-c/6 enhanced strain element [162] is as shown in Figure 5.17. Here, it is used to check for hourglass instabilities in the US-QUAD9/3 element.

Due to symmetry of the model, only one half of the rectangular block is modeled using a 4×4 mesh of US-QUAD9/3 elements. The number of negative eigenvalues, if any, and the corresponding largest negative eigenvalue for each deformed configuration are shown in Table 5.2. The largest negative eigenvalue from Table 5.2 does not appear to correspond to hourglass modes because it is too small compared to the 4/4-c/6 element analyzed by Pantuso and Bathe [156]. Hence, US-QUAD9/3 does not show any tendency of hourglassing for this problem.

Table 5.2. Number of negative eigenvalues and corresponding largest negative eigenvalue as a function of imposed displacements

Δ	No. of negative eigenvalues	Largest negative eigenvalue
0.1	0	-
0.2	0	-
0.3	0	-
0.4	2	-8.2×10^{-5}
0.5	4	-3.1×10^{-4}
0.6	4	-3.1×10^{-4}
0.7	4	-3.1×10^{-3}

5.10. CONCLUSIONS

A straightforward attempt to directly implement US-QUAD8 to model the response of incompressible neo-Hookean materials has been found to be unsuccessful. US-QUAD8 has been found to experience volumetric locking when modeling incompressible materials under plane strain conditions. To remove volumetric locking, an implementation of the unsymmetric formulation through the mixed-formulations is necessary. This has been attempted for a 9-node quadrilateral with a linear pressure interpolation (US-QUAD9/3). The same attempt for an 8-node quadrilateral element was not attempted since a combination of 8 displacements and 3 pressure modes will still lock, whilst the optimum combination of 8 displacements and 2 pressure modes will introduce directionality in the pressure field. Isoparametric and metric shape functions have been used to discretize the virtual and trial displacement/pressure fields, respectively.

Discretization of the linearized principle of virtual work follows a similar approach as that for geometric nonlinear problems in Chapter 4. However, in the discretization of the incompressibility constraint, it has been found that *Criterion 1* in Chapter 3 cannot be satisfied. This is a consequence of the discontinuous interpolation of the virtual pressure field employed in the mixed formulations. Due to this discontinuity, the performance of US-QUAD9/3 has been observed to be inferior compared to QUAD9/3 for the test problem involving angular distortions in Section 5.9.2 (see Figures 5.7a and 5.7b). The overall performance of US-QUAD9/3, however, has been observed to be superior compared to QUAD9/3. This is particularly so, for element meshes that involve curved-

edge and mid-side node distortions. Investigations into the tendency of hourglassing in US-QUAD9/3 have revealed that such a possibility does not exist.

This thesis considers the application of the unsymmetric formulation to incompressible elasticity for two-dimensional elements only. The formulation can be readily applied to three-dimensional elements as well. A straightforward extension of the idea to a three-dimensional element like a 27-noded hexahedron element would involve four pressure modes. The pressure modes represent a linear pressure interpolation in three dimensions, $p = 1 + \xi + \eta + \zeta$. The virtual and trial displacement fields in the principle of virtual work are discretized using isoparametric and metric shape functions respectively. Static condensation can then be performed after element formulation to reduce the number of degrees of freedom associated with the pressure modes. An unsymmetric 27-noded hexahedron element with four pressure modes is expected to behave in a similar manner as that of US-QUAD9/3.

Further improvements on the performance of US-QUAD9/3 may be achieved if the errors due to the discontinuities of the virtual pressure field can be reduced or eliminated. However, an effective way to handle this problem is apparently not obvious.

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

6.1. INTRODUCTION

The unsymmetric element formulation is capable of reproducing the assumed Cartesian quadratic displacement field under any admissible element geometry. It belongs to the broad family of Petrov-Galerkin formulations, where two different sets of shape functions are used as the test and trial functions. Despite its excellent performance, the element matrix is unsymmetric. An unsymmetric system matrix is not preferable in current finite element practice since it demands additional space for data storage and also increases the computational time required in the solution process. In addition to these, most commercial finite element packages do not have unsymmetric equation solvers. Thus, a straightforward implementation of the unsymmetric formulation in commercial finite element packages would appear cumbersome.

As an alternative, an iterative approach to enhance the performance of isoparametric elements in the presence of mesh distortions has been developed. This approach still mimics the excellent performance of the unsymmetric element. The central idea is to use the classical isoparametric element (in which, the stiffness matrix is symmetric) and improve its solution iteratively by applying corrections to the load vector. The correction load is calculated using the unsymmetric stiffness matrix. Its magnitude depends on the severity of distortions present in the mesh, and it varies progressively with iterations.

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION
TOLERANCE

Iterative methods for solution of simultaneous linear equations have been known for long [168]. However, the present iterative method is rather different from these methods in the sense that it is meant to improve on the accuracy of the solution already obtained from a direct solver. Historically, iterative methods have been used in finite element computations ever since the initial developments of the method (e.g., see [169]). Iterative solvers generally require less computer memory compared to the direct solvers. However, because the solution time of iterative solvers cannot be predicted precisely, they were almost abandoned during the 1960s and 1970s in favor of the direct solvers. In course of time, as larger and larger sized problems had to be solved, the direct solvers were found to demand higher computer memory compared to their iterative counterparts. This revived the interest in iterative solvers. Development of conjugate gradient (CG) and preconditioning conjugate gradient (PCG) techniques gave new impetus to iterative methods (e.g., see [170, 171 and 172]). The CG and PCG techniques are widely used in eigen-solution algorithms and in structural re-analysis after small changes in the stiffness or mass parameters of the system (e.g., see [173 – 175]). The combined approximations (CA) method of Kirsch et al. [175], which is equivalent to the PCG method, provides an insightful approach to solve re-analysis problems.

The present iterative correction algorithm may be looked upon as a structural re-analysis problem. The effect of mesh distortion on the stiffness matrix is treated analogous to the effect of structural changes. Iterations are employed to correct the solution error caused by the mesh distortions. The solution for the distorted mesh, as given by the direct solver, is used as the starting solution for the iterations. Iterative improvement is then effected

through modifications of the load vector so as to account for the effects of mesh distortion. The improved solution is obtained using this modified load vector.

The algorithm for the iterative corrections is described in Section 6.2. Aspects on its mathematical convergence are established in Section 6.3. An acceleration scheme is proposed to speed up the convergence of the corrective iterations. This is detailed in Section 6.4. The effectiveness of the iterative corrections and the acceleration scheme is then demonstrated using simple illustrative problems in Section 6.5. Physical interpretations on the mechanisms of the iterative corrections are discussed in Section 6.6. The efficacy of the iterative correction algorithm is then compared with other existing quadrilateral elements using several numerical benchmarks in Section 6.7.

6.2. ITERATIVE CORRECTIONS OF MESH DISTORTION EFFECTS

In the Galerkin weighted residual formulation leading to the QUAD8 element, the finite element assembly results in global stiffness equations of the form

$$\mathbf{K}_s \mathbf{u}_s = \mathbf{F} \quad (6.1)$$

where $\mathbf{K}_s$ is a symmetric stiffness matrix, $\mathbf{F}$ is the nodal load vector and $\mathbf{u}_s$ is the corresponding nodal displacement vector. The subscript, s , in Equation (6.1) indicates that the respective quantities correspond to the symmetric formulation. The counterpart of Equation (6.1) is the unsymmetric formulation leading to the US-QUAD8 element [5]

$$\mathbf{K}_u \mathbf{u}_u = \mathbf{F} \quad (6.2)$$

The subscript, u , in Equation (6.2) indicates that the respective quantities correspond to unsymmetric formulation. The stiffness matrices $\mathbf{K}_s$ and $\mathbf{K}_u$ are obtained as

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

$$\mathbf{K}_s = \mathbf{A} \left(\mathbf{K}_s^{(e)} \right) \quad (6.3a)$$

$$\mathbf{K}_u = \mathbf{A} \left(\mathbf{K}_u^{(e)} \right) \quad (6.3b)$$

where

$$\mathbf{K}_s^{(e)} = \int_{\Omega^{(e)}} \bar{\mathbf{B}}^T \mathbf{D} \bar{\mathbf{B}} d\Omega^{(e)} \quad (6.4)$$

$$\mathbf{K}_u^{(e)} = \int_{\Omega^{(e)}} \bar{\mathbf{B}}^T \mathbf{D} \hat{\mathbf{B}} d\Omega^{(e)} \quad (6.5)$$

$$\bar{\mathbf{B}} = \begin{bmatrix} N_{1,x} & 0 & N_{2,x} & 0 & N_{3,x} & 0 & \dots & N_{8,x} & 0 \\ 0 & N_{1,y} & 0 & N_{2,y} & 0 & N_{3,y} & \dots & 0 & N_{8,y} \\ N_{1,y} & N_{1,x} & N_{2,y} & N_{2,x} & N_{3,y} & N_{3,x} & \dots & N_{8,y} & N_{8,x} \end{bmatrix}, \quad (6.6)$$

$$\hat{\mathbf{B}} = \begin{bmatrix} M_{1,x} & 0 & M_{2,x} & 0 & M_{3,x} & 0 & \dots & M_{8,x} & 0 \\ 0 & M_{1,y} & 0 & M_{2,y} & 0 & M_{3,y} & \dots & 0 & M_{8,y} \\ M_{1,y} & M_{1,x} & M_{2,y} & M_{2,x} & M_{3,y} & M_{3,x} & \dots & M_{8,y} & M_{8,x} \end{bmatrix}, \quad (6.7)$$

N_i and M_i ($i=1,2,3,\dots,8$) are the isoparametric and metric shape functions, respectively,

and $N_{i,x}$, $N_{i,y}$, $M_{i,x}$ and $M_{i,y}$ are their Cartesian derivatives.

The objective now is to implement Equation (6.2) implicitly using the QUAD8 element.

Using the substitution, $\mathbf{K}_u = \mathbf{K}_s + (\mathbf{K}_u - \mathbf{K}_s)$, Equation (6.2) can be rewritten as

$$\mathbf{K}_s \mathbf{u}_u = \mathbf{F} + (\mathbf{K}_s - \mathbf{K}_u) \mathbf{u}_u \quad (6.8)$$

Next, an iterative solution for Equation (6.8) is sought for using the following sequence of iterations:

$$\mathbf{K}_s \mathbf{u}_{i+1} = \mathbf{F} + (\mathbf{K}_s - \mathbf{K}_u) \mathbf{u}_i \quad (6.9)$$

where i is the iteration number. The starting vector $\mathbf{u}_1$ is chosen as $\mathbf{u}_s$. The second term on the right side of Equation (6.9) represents the correction load vector. Its magnitude varies with the extent of distortion and iteration number.

The criterion for convergence of the iterative correction is based on the computed strain energy. The corrective iterations are terminated when the differences between the strain energy of the current and previous iteration is less than some specified tolerance value.

$$\frac{\mathbf{u}_{i+1}^T \mathbf{K}_s \mathbf{u}_{i+1} - \mathbf{u}_i^T \mathbf{K}_s \mathbf{u}_i}{\mathbf{u}_i^T \mathbf{K}_s \mathbf{u}_i} < \varepsilon \quad (6.10)$$

Alternatively, the convergence criterion may be based on the computed norm of displacements. In such a case, the convergence criterion is

$$\|\mathbf{u}_{i+1} - \mathbf{u}_i\| < \varepsilon \quad (6.11)$$

Problems involving quadratic displacement fields can be solved exactly by QUAD8 within the scope of the classical solution procedure (Equation (6.1)), provided that the mesh are of regular geometries. In this special case, the correction load vector turns out to be a zero vector. For a distorted mesh, the correction load vector is, in general, nonzero.

6.3. MATHEMATICAL CONVERGENCE OF ITERATIVE CORRECTIONS

For low to moderate mesh distortions, the iterations of Equation (6.9) converge to the exact solution. However, the algorithm has been observed to diverge for severe distortions. Nevertheless, the mathematical conditions for convergence are easy to explore. Subtracting Equation (6.8) from Equation (6.9), yields

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

$$\mathbf{K}_s(\mathbf{u}_{i+1} - \mathbf{u}_u) = (\mathbf{K}_s - \mathbf{K}_u)(\mathbf{u}_i - \mathbf{u}_u) \quad (6.12)$$

For a wellposed problem, $\mathbf{K}_s^{-1}$ exists, and hence, Equation (6.12) may be written as

$$(\mathbf{u}_{i+1} - \mathbf{u}_u) = [\mathbf{K}_s^{-1}(\mathbf{K}_s - \mathbf{K}_u)] (\mathbf{u}_i - \mathbf{u}_u) \quad (6.13)$$

Equation (6.13) is of the form

$$\boldsymbol{\varphi}_{i+1} = \mathbf{T}\boldsymbol{\varphi}_i \quad (6.14)$$

where $\boldsymbol{\varphi}_i \equiv (\mathbf{u}_i - \mathbf{u}_u)$, $\boldsymbol{\varphi}_{i+1} \equiv (\mathbf{u}_{i+1} - \mathbf{u}_u)$ and $\mathbf{T}: R^n \rightarrow R^n$. In terms of appropriate norms, Equation (6.14) is written as

$$\|\boldsymbol{\varphi}_{i+1}\| \leq \|\mathbf{T}\| \|\boldsymbol{\varphi}_i\| \quad (6.15)$$

Ideally, it is desired that $\|\boldsymbol{\varphi}_i\| \rightarrow 0$ as $i \rightarrow \infty$, or alternatively, $\|\boldsymbol{\varphi}_{i+1}\| \leq \|\boldsymbol{\varphi}_i\|$. This is possible from Equation (6.15) only if $\|\mathbf{T}\| \leq 1$, which means that the mapping has to be *contractive*. The condition for the convergence for such a mapping follows from the *Banach fixed-point theorem* (e.g., see Reddy [176]), which is reproduced below (without proof):

Theorem 6.1. Banach fixed-point theorem

Let (X, d) be a metric space. If $T: X \rightarrow X$ is a contractive mapping, i.e., there is a real number $0 \leq q < 1$ such that

$$d(Tx, Ty) \leq qd(x, y) \quad (6.16)$$

for all $x, y \in X$, then,

- a. the map T admits one and only one fixed point x^* in X , i.e., the fixed point is unique, and

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

- b. starting with an arbitrary element $x_\epsilon \in X$, the iterations defined by the sequence $x_n = Tx_{n-1}$ with $n = 1, 2, 3, \dots$, converges to the unique fixed point x^* .

By the Banach fixed-point theorem, the condition for convergence is $q < 1$. Comparing Equations (6.15) and (6.16), and noting that the norm is always positive, this condition becomes

$$0 \leq \|T\| \equiv \|K_s^{-1}(K_s - K_u)\| < 1 \quad (6.17)$$

While any matrix norm may in principle be used in Equation (6.17), a norm of small magnitude is preferred. A larger norm is too restrictive in the sense that it underestimates the maximum allowable distortion to guarantee convergence. Hence, for the present work, the spectral radius, which is defined as the absolute maximum of eigenvalues of the given matrix, T is used:

$$\rho(T) = \max\{|\lambda| : \lambda \in \text{eigenspectrum of } T\} \quad (6.18)$$

Thus, the condition for convergence may be stated as Lemma 6.1.

Lemma 6.1 Condition for the convergence of iterative correction algorithm

The iterations of Equation (6.14) converge to the solution of Equation (6.2)

$$\forall \mathbf{u}_o, \mathbf{F} \in R^n \text{ if } \rho(K_s^{-1}(K_s - K_u)) < 1.$$

6.4. ACCELERATION OF ITERATIVE CORRECTIONS

Direct application of iterative corrections may sometimes demand large number of iterations, particularly when the distortions present in the finite element mesh are severe.

Hence, an acceleration technique is desirable to render the procedure more attractive for general applications. The acceleration scheme proposed here adopts a similar approach as that of Rajendran et al. [177], which was developed for improving the convergence rate of the inverse iteration method.

In the proposed acceleration scheme, an optimum linear combination of the current and preceding iteration vector is sought for, by minimizing the total potential energy. Considering the iteration sequence in (6.9), the notation $\mathbf{u}_{i+1}$ is replaced by $\tilde{\mathbf{u}}_{i+1}$ and the equation is rewritten as

$$\mathbf{K}_s \tilde{\mathbf{u}}_{i+1} = \mathbf{F} + (\mathbf{K}_s - \mathbf{K}_u) \mathbf{u}_i \quad (6.19)$$

The optimum linear combination of successive iteration vectors that is sought for is written as

$$\mathbf{u}_{i+1} = \tilde{\mathbf{u}}_{i+1} + \alpha \mathbf{u}_i \quad (6.20)$$

where, the scalar α is chosen to minimize the total potential, Π :

$$\Pi = \frac{1}{2} \mathbf{u}_{i+1}^T \tilde{\mathbf{K}}_s \mathbf{u}_{i+1} - \mathbf{u}_{i+1}^T \mathbf{F} \quad (6.21)$$

In Equation (6.21), the expression $\frac{1}{2} \mathbf{u}_{i+1}^T \tilde{\mathbf{K}}_s \mathbf{u}_{i+1}$ corresponds to the strain energy and the matrix $\tilde{\mathbf{K}}_s$ corresponds to the symmetric stiffness matrix of metric shape functions

$$\tilde{\mathbf{K}}_s^{(e)} = \int_{\Omega^{(e)}} \hat{\mathbf{B}}^T \mathbf{D} \hat{\mathbf{B}} d\Omega^{(e)} \quad (6.22)$$

Substituting Equation (6.20) into Equation (6.21) for $\mathbf{u}_{i+1}$, the expression for Π becomes

$$\Pi = \frac{1}{2} \left((\tilde{\mathbf{u}}_{i+1} + \alpha \mathbf{u}_i)^T \tilde{\mathbf{K}}_s (\tilde{\mathbf{u}}_{i+1} + \alpha \mathbf{u}_i) \right) - (\tilde{\mathbf{u}}_{i+1} + \alpha \mathbf{u}_i)^T \mathbf{F} \quad (6.23)$$

Equating the first derivative of Π with respect to α to zero, the condition for the stationarity of Π is obtained as

$$\frac{1}{2}(\mathbf{u}_i^T \tilde{\mathbf{K}}_s \tilde{\mathbf{u}}_{i+1} + \tilde{\mathbf{u}}_{i+1}^T \tilde{\mathbf{K}}_s \mathbf{u}_i) + \alpha \mathbf{u}_i^T \tilde{\mathbf{K}}_s \mathbf{u}_i - \mathbf{u}_i^T \mathbf{F} = 0 \quad (6.24)$$

Solving for α from Equation (6.24) yields

$$\alpha = \frac{\mathbf{u}_i^T \mathbf{F} - \tilde{\mathbf{u}}_{i+1}^T \tilde{\mathbf{K}}_s \mathbf{u}_i}{\mathbf{u}_i^T \tilde{\mathbf{K}}_s \mathbf{u}_i} \quad (6.25)$$

Upon evaluating the scalar α , the new displacement vector $\mathbf{u}_{i+1}$ is computed with Equation (6.20), and used on the right side of Equation (6.19) for the next iteration. The corrective iterations ceases when the difference of strain energy between the current and previous iteration meets the convergence criterion of Equation (6.10) or (6.11).

6.5. ILLUSTRATIVE PROBLEMS

The performance of the iterative corrections is first studied for a straight cantilever beam subjected to an in-plane bending moment shown in Figure 6.1a. The objective is to assess the ability of iterative corrections to improve the otherwise poor performance of QUAD8 in the presence of mesh distortions. The theoretical solution for the vertical displacement at the point (100,0) is -0.012 while that of the σ_{xx} -stress at (0,0) is -120 . Three types of distortions are considered. The convergence of the iterative corrections is studied by varying the degree of distortions in the element meshes.

6.5.1. ANGULAR DISTORTION

Two 8-node quadrilateral elements are used to model the cantilever beam. The extent of angular distortion is controlled by varying the magnitude of distortion parameter, δ . The

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

cantilever beam is solved using the iterative correction algorithm for $\delta = 5, 10, 15, 20$ and 25 . At each value of δ , the mid-side nodes are repositioned to occupy the mid-point of the elements respective sides.

Figure 6.1(a) Cantilever beam subjected to a constant bending moment; Length = 100; Depth = 10; Thickness = 1; Young's modulus = 10^7 ; Poisson's ratio = 0.3; (b) Two-element mesh with angular distortion; (c) Two-element mesh with curved-edge distortion; (d) Two-element mesh with mid-side node distortion

The convergence of the vertical displacement at point (100,0) and σ_{xx} -stress at point (0,0) is shown in Figures 6.2a and 6.2b, respectively. These figures show that the displacements and stresses indeed converge towards the theoretical solution as the number of iterations is increased. The more severe the mesh distortion, the slower the

convergence. The number of iterations required for the solution to converge is shown in Table 6.1 for the values of δ considered.

Figures 6.2c and 6.2d show the convergence of the displacement and stress when the acceleration scheme is used. The results indicate that the acceleration scheme is effective in reducing the number of iterations, particularly when the distortions are severe. For example, for a value of $\delta = 25$, the acceleration scheme is capable of reducing the number of iterations from 94 to 32.

The iterations diverge when the distortion parameter δ exceeds a value of 25. This is, in conformance with Lemma 6.1, i.e., the spectral radius of the iteration matrix, $\mathbf{T}$ exceeds the permissible value. Hence the iterations diverge.

Figure 6.2a. Convergence of displacements for angular distortions without acceleration

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

Figure 6.2b. Convergence of σ_{xx} -stress for angular distortions without acceleration

Figure 6.2c. Convergence of displacements for angular distortions with acceleration

Figure 6.2d. Convergence of σ_{xx} -stress for angular distortions with acceleration

Table 6.1. Number of corrective iterations required for angular distortion

Distortion parameter, δ	Number of iterations	
	Iterative corrections without acceleration technique	Iterative corrections with acceleration technique
5	4	5
10	7	6
15	12	10
20	23	14
25	94	32

6.5.2. CURVED-EDGE DISTORTION

The beam is modeled using two elements with curved-edge distortion shown in Figure 6.1c. The model is solved for $\delta = 5, 10, 15$ and 20 . The convergence of vertical displacement at point $(100,0)$ is shown in Figure 6.3a and 6.3b. The convergence of σ_{xx} -stress at point $(0,0)$ yields a similar trend and is not shown here. Figure 6.3a indicates that

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

the iterations converge to the exact solution for the all values of distortion parameters, δ considered. Larger values of δ demand more iterations for convergence.

Figure 6.3a. Convergence of displacements for curved-edge distortions without acceleration

Figure 6.3b. Convergence of displacements for curved-edge distortions with acceleration

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

When used with the acceleration scheme, the iterative corrections converge with fewer numbers of iterations (Figure 6.3b). The acceleration technique is capable of reducing the number of corrective iterations by approximately 50%. For a value of $\delta = 15$, the number of iterations is reduced from 331 to 162, while for a value of $\delta = 20$, the number of iterations is reduced from 610 to 308. Table 6.2 shows the number of iterations required of the iterative corrections to converge.

Table 6.2. Number of corrective iterations for curved-edge distortion

Distortion parameter, δ	Number of iterations	
	Iterative corrections without acceleration technique	Iterative corrections with acceleration technique
5	43	12
10	146	65
15	331	162
20	610	308

The number of iterations required for convergence is observed to be more than that required for angular distortions (comparison between Tables 6.1 and 6.2). For example, a value of $\delta = 20$, requires approximately 600 iterations for convergence, while the same value of δ in the case of angular distortions require only 90 iterations. As with the case of angular distortions, the iterative corrections diverge when the distortion parameter δ exceeds a value of 35, which corresponds to a value of $\rho > 1$.

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

Figure 6.4a. Convergence of displacements for mid-side node distortions without acceleration

Figure 6.4b. Convergence of displacements for mid-side node distortions with acceleration

6.5.3. MID-SIDE NODE DISTORTION

The cantilever beam is modeled using two quadrilateral elements with mid-side node distortion, introduced by moving the mid-side node on the top surface of each element by

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

a value δ shown in Figure 6.1d. The problem is solved for values of $\delta = 5, 8, 10$ and 12. The convergence of the displacement at (100,0) is shown in Figure 6.4a. The convergence of displacement using the acceleration scheme is shown in Figure 6.4b. Compared to curved-edge distortions, the number of iterations required for convergence is much less.

Table 6.3. Number of corrective iterations for mid-side node distortion

Distortion parameter, δ	Number of iterations	
	Iterative corrections without acceleration technique	Iterative corrections with acceleration technique
5	11	8
8	20	12
10	29	16
12	42	22

The convergence of iterations discussed previously, involve the displacement and stress results at specific locations of the cantilever. In order to ascertain that the results indeed converge to exact solution at every point of the cantilever, the contour plots of σ_{xx} are shown in Figure 6.5 for the three types of distortions. The plots reveal that the stresses converge correctly as evidenced by the stress bands becoming straight and varying smoothly across the element interfaces.

6.5.4. THRESHOLD DISTORTION VERSUS SPECTRAL RADIUS

By numerical experiments, the threshold value of distortion parameter for angular and curved-edge distortions is found from the previous sections to be around 25 and 35, respectively. The usefulness of Lemma 6.1 in predicting these threshold values will now

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

be verified. By varying the distortion parameter in the range $0 \leq \delta \leq 40$, the spectral radius is computed for both distorted meshes, and the results are shown in Figure 6.6. The plot shows that the spectral radius indeed crosses unity at these threshold values of distortion (25 and 35) as observed in sections 6.5.1 and 6.5.2, and in conformance with Lemma 6.1. Thus, the spectral radius is a faithful measure of the threshold value of distortion that marks the onset of divergence.

Figure 6.5. Stress σ_{xx} -contour bands of stress with and without corrective iterations

Although the computation of spectral radius is relatively easy for the test problems presented, it would be prohibitive for large problems. A more practical condition to sense

the onset of divergence would be to check if $\|\boldsymbol{\varphi}_{i+1}\| \leq \|\boldsymbol{\varphi}_i\|$ or, in other words, to check if the ratio $\tilde{\rho} \equiv \|\boldsymbol{\varphi}_{i+1}\| / \|\boldsymbol{\varphi}_i\| \leq 1$ for every iteration, and abort the iterative process if $\tilde{\rho} > 1$. (Note that for convergent iterations, $\tilde{\rho} \rightarrow 1$ as $i \rightarrow \infty$.) For this purpose, the redefinitions, $\boldsymbol{\varphi}_i \equiv (\mathbf{u}_{i-1} - \mathbf{u}_i)$ and $\boldsymbol{\varphi}_{i+1} \equiv (\mathbf{u}_i - \mathbf{u}_{i+1})$, are necessary because the earlier definition given below Equation (6.14) involves $\mathbf{u}_u$ which is unknown.

Figure 6.6. Variation of spectral radius with distortion parameter values

6.6. PHYSICAL INTERPRETATIONS

Further insights into the mechanisms of the iterative correction algorithm can further be explored by considering the deformation of a generic finite element mesh subjected to external loads. Figure 6.7a shows a material body in space subjected to an external load $\mathbf{F}$ that results in a displacement field of the form given by Equation (3.15). Assuming that the isoparametric elements used to discretize the body is free of geometric distortions, the action of $\mathbf{F}$ causes the material body to deform to a configuration indicated by the bold

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

line in Figure 6.7b. The work done by the external force is stored inside the material body in the form of strain energy, $\frac{1}{2} \bar{\mathbf{u}}^T \mathbf{K}_s \bar{\mathbf{u}}$, where $\bar{\mathbf{u}}$ is the vector of nodal displacements obtained by solving Equation (6.1). The displacements $\bar{\mathbf{u}}$ in this case are exact.

Figure 6.7. Material body in space subject to an external force $\mathbf{F}$; (a) original configuration; (b) configuration after deformation

In the case where the isoparametric elements in the mesh are distorted, the stiffness matrix becomes stiffer [14 – 16] e., $\mathbf{K}_s \rightarrow \mathbf{K}_s^+$). Under the action of the same magnitude of external load $\mathbf{F}$, the material body deforms only to occupy the configuration shown by the dashed line in Figure 6.7b. Consequently, the norm of displacements reduces, $\|\bar{\mathbf{u}}\| < \|\mathbf{u}\|$, where $\mathbf{u}$ is the exact displacement field and the displacements are thus under-predicted. The total potential, $\bar{\Pi}$, of the isoparametric element can be written as

$$\bar{\Pi} = \frac{1}{2} \bar{\mathbf{u}}^T \mathbf{K}_s \bar{\mathbf{u}} - \bar{\mathbf{u}}^T \mathbf{F} \quad (6.26)$$

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

In the iterative correction method, the total potential, $\hat{\Pi}$, can be obtained by integrating Equation (6.8) with respect to the displacements to yield

$$\hat{\Pi} = \frac{1}{2} \hat{\mathbf{u}}^T \mathbf{K}_s \hat{\mathbf{u}} - \hat{\mathbf{u}}^T \mathbf{F} - \frac{1}{2} \hat{\mathbf{u}}^T (\mathbf{K}_s - \mathbf{K}_U) \hat{\mathbf{u}} \quad (6.27)$$

When the elements in the mesh are undistorted, and the action of the external load $\mathbf{F}$ results in a displacement field of the form in Equation (3.15), the displacement fields, $\mathbf{u}$, $\bar{\mathbf{u}}$ and $\hat{\mathbf{u}}$, stiffness matrices, $\mathbf{K}_s$ and $\mathbf{K}_U$, and total potentials, $\bar{\Pi}$ and $\hat{\Pi}$ in this special case, coincide (i.e., $\mathbf{u} = \bar{\mathbf{u}} = \hat{\mathbf{u}}$, $\mathbf{K}_s = \mathbf{K}_U$ and $\bar{\Pi} = \hat{\Pi} = \Pi_{theory}$).

Figure 6.8. Progression of energies with corrective iterations

On the other hand, in the presence of mesh distortions, only conditions $\mathbf{u} = \hat{\mathbf{u}}$ and $\hat{\Pi} = \Pi_{theory}$ hold. The iterative corrections can thus be viewed as the external work done by the corrective load vector so that the now stiffer system is able to deform as if it is free of mesh distortions, under the action of the same load $\mathbf{F}$. This amount of external work is

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

represented by the term $\frac{1}{2} \hat{\mathbf{u}}^T (\mathbf{K}_s - \mathbf{K}_u) \hat{\mathbf{u}} \equiv W_{IC}$ in Equation (6.27). As the iterations progress, the nodal displacements increase in magnitude and approach the value of displacements corresponding to the distortion-free mesh. The strain energy calculated with respect to the “stiffer” stiffness matrix $\mathbf{K}_s \rightarrow \mathbf{K}_s^+$ will progressively increase as $\bar{\mathbf{u}} \rightarrow \hat{\mathbf{u}}$, to $U_{IC} = \frac{1}{2} \hat{\mathbf{u}}^T \mathbf{K}_s \hat{\mathbf{u}}$, before the solution converges. If the product

$$U_{\text{Recovered}} = \frac{1}{2} \hat{\mathbf{u}}^T \mathbf{K}_s \hat{\mathbf{u}} - W_{IC} = U_{IC} - W_{IC} \quad (6.28)$$

is evaluated, the strain energy corresponding to a regular mesh can be recovered when the iterations converge. This is shown in Figure 6.8 for the case of angular distortion with $\delta = 25$. Thus, the iterative corrections may be viewed as a process of recovering the strain energy correctly, by applying the corrective load at the nodes

6.7. COMPARISON OF PERFORMANCE

The performance of the iterative correction method is now compared with that of QUAD8 and US-QUAD8. A 3 x 3 Gaussian quadrature of the stiffness integral is employed for all the elements. For convenience of reference, the following acronyms are used to earmark the various elements considered:

- QUAD8 : Classical 8-node isoparametric quadrilateral element
- US-QUAD8 : Unsymmetric 8-node quadrilateral element
- QUAD8+IC : QUAD8 with Iterative Corrections
- QUAD8+AIC : QUAD8 with Accelerated Iterative Corrections

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

(a) A cantilever beam subject to a constant bending moment. $L = 100$; $c = 10$; thickness = 1.0; Young's modulus, $E = 1.0 \times 10^7$; Poisson's ratio, $\gamma = 0.3$; $M = 20c^2$ distributed as $f_x = 240y/c - 120$ acting at the free and fixed end

(b) Selected distorted meshes from ref. [5]

Figure 6.9. Constant bending moment problem and selected distorted meshes from ref. [5]

6.7.1. CANTILEVER BEAM SUBJECTED TO CONSTANT BENDING MOMENT

Figure 6.9a shows the geometry of a straight cantilever beam under plane stress conditions introduced by Lee and Bathe [112]. The five different distorted meshes shown in Figure 6.9b were used in reference [5] to assess the distortion sensitivity of the elements under investigation. Here, they are used to study the distortion sensitivity of the iterative correction method.

Table 6.4 shows the normalized solution of stresses and displacements at typical points of the cantilever. The number of iterations required for the iterative correction algorithm to converge is also shown. The exact solution is listed in the last column of Table 6.4. The results in Table 6.4 suggest that the distortion-stricken results obtained with the QUAD8 element are much improved to approach the results of US-QUAD8 with the aid of iterative corrections. The iterative correction method in its maiden form is capable of reproducing all the exact solutions for all the five meshes in Figure 6.9b. US-QUAD8 yields similar results. When used with the acceleration technique the number of iterations is reduced for meshes 2, 5, 7 and 8. However, the algorithm diverges for Mesh 3. In other words, with the use of the acceleration technique, the maximum distortion that can be successfully handled is reduced. Thus, the convergence of the iterative corrections with the proposed acceleration technique is no longer governed by Lemma 6.1. A criterion more stringent than that stated by Lemma 6.1 is required. Investigation into this aspect is less motivating, as the maximum distortion limit is reduced.

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION
TOLERANCE

Table 6.4. Comparative performance of elements for the constant bending
moment problem

Normalized solution								
		QUAD8	US- QUAD8	QUAD8+ IC	No. of iterations	QUAD8+ AIC	No. of iterations	Exact solution
Mesh 2	$\sigma_x(0,0)$	0.7161	1.0000	1.0000		1.0000		-120
	$\sigma_x(0,10)$	0.4844	1.0000	1.0000	40	1.0000	22	120
	$\nu(100,0)$	0.2046	1.0000	1.0000		1.0000		-0.012
Mesh 3	$\sigma_x(0+,0)$	0.1190	1.0000	1.0000		-		-120
	$\sigma_x(0,0+)$	0.0476	1.0000	1.0000		-		-120
	$\sigma_x(0,10-)$	0.0435	1.0000	1.0000	341	-	-	120
	$\sigma_x(0+,10)$	0.1141	1.0000	1.0000		-		120
	$\nu(100,0)$	0.0397	1.0000	1.0000		-		-0.012
Mesh 5	$\sigma_x(0,0)$	1.0014	1.0000	1.0001		1.0000		-120
	$\sigma_x(0,10)$	1.0014	1.0000	1.0000	4	1.0000	4	120
	$\nu(20,0)$	0.9191	1.0000	1.0000		1.0000		- 0.00048
Mesh 7	$\sigma_x(0,0)$	1.0225	1.0000	1.0000		1.0000		-120
	$\sigma_x(0,20)$	0.9759	1.0000	1.0000	8	1.0000	8	120
	$\nu(10,0)$	1.0031	1.0000	1.0000		1.0000		- 0.00006
Mesh 8	$\sigma_x(0,0)$	1.2510	1.0000	1.0000		1.0000		-120
	$\sigma_x(0,10)$	-0.4740	1.0000	1.0000	20	1.0000	15	120
	$\nu(100,0)$	0.7134	1.0000	1.0000		1.0000		-0.012

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

(a) A cantilever beam subject to a linear bending moment. $L = 100$; $c = 10$; thickness = 1.0; Young's modulus, $E = 1.0 \times 10^7$; Poisson's ratio, $\nu = 0.3$; $P = 20c^2/L$ distributed as $f_y = 240y/c - 120$ acting at the free and fixed end

$$\hat{x}(y) = 10 - 8(y/10)(1 - y/10)$$

Mesh 5

$$\hat{x}(x) = x + 40(x/100)(1 - x/100)$$

$$\hat{y}(x) = 10 - 8(x/10)(1 - x/10)$$

Mesh 7

(b) Selected distorted meshes from ref. [5]

Figure 6.10. Linear bending moment problem and selected distorted meshes from ref [5]

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

Table 6.5. Comparative performance of elements for the linear bending moment problem

		Normalized solution						
		QUAD8	US- QUAD8	QUAD8+ IC	No. of iterations	QUAD8+ AIC	No. of iterations	Exact solution
Mesh 2	σ_{xx} at G ₁	0.2977	0.6701	0.6712	87	0.6740	53	64.0748
	σ_{xx} at G ₂	0.1994	0.6224	0.6286		0.6188		-59.8483
	$v(100,0)$	0.2002		0.7973		0.7840		0.008046
Mesh 3	σ_{xx} at G ₁	0.0521	0.5978	0.5977	298	0.5974	31	53.3626
	σ_{xx} at G ₂	0.0533	0.5969	0.5969		0.5966		-53.5031
	$v(100,0)$	0.0562	0.7821	0.7820		0.6740		64.0748
Mesh 5	σ_{xx} at G ₁	1.0231	0.9853	0.9854	6	0.9863	6	62.9376
	σ_{xx} at G ₂	1.0231	0.9853	0.9854		0.9863		-62.9376
	$v(100,0)$	0.9132	0.9719	0.9727		0.9727		0.000366
Mesh 7	σ_{xx} at G ₁	0.9633	0.9878	0.9880	4	0.9874	4	77.3077
	σ_{xx} at G ₂	0.9824	0.9932	0.9933		0.9939		-71.9743
	$v(100,0)$	0.9709	0.9785	0.9773		0.9773		0.000132
Mesh 8a	σ_{xx} at G ₁	0.2486	0.8468	0.8468	13	0.8789	8	59.2295
	σ_{xx} at G ₂	0.4254	0.8313	0.8313		0.8909		-61.2295
	$v(100,0)$	0.4114	0.8734	0.8734		0.9146		0.008046

6.7.2. CANTILEVER BEAM SUBJECTED TO LINEAR BENDING MOMENT

The cantilever beam in Figure 6.9a is now subjected to a linear bending moment, effected by a shear force at its free end as shown in Figure 6.10a. The exact solution for this problem is reported in [112] and the problem is solved for the displacement at the lower tip of the cantilever and stresses at specific points marked on the five different meshes of

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

elements in Figure 6.10b. This test problem tests the distortion sensitivity of elements under a cubic displacement field. The results are summarized in Table 6.5.

Comparing the performance of the iterative correction algorithm with that of US-QUAD8 reveals that the iterative corrections are capable of closely reproducing the solutions of US-QUAD8. The acceleration scheme reduces the number of iterations required for the algorithm to converge, in particular, for severely distorted meshes (Meshes 2, 3 and 8a). For the meshes where the distortions are relatively small, the effectiveness of the acceleration technique, is again, not apparent (Meshes 5 and 7).

Fig. 6.11. A thin cantilever beam subjected to a transverse tip load. Inner radius = 4.12; outer radius = 4.32; thickness = 0.1; arc = 90° ; Young's modulus, $E = 1.0 \times 10^7$; Poisson's ratio, $\nu = 0.25$; unit transverse load at the free end

6.7.3. CURVED BEAM SUBJECTED TO SHEAR LOAD AT ITS END

A curved cantilever beam, subjected to a shear load at its free end shown in Figure 6.11 is considered. The cross section of the beam is modeled with one element and the problem is solved under plane stress conditions for the vertical displacement at the tip with mesh refinement along the length of the cantilever. The exact solution for the vertical tip displacement along the direction of the load is available in Timoshenko and Goodier [87]. For the problem with corresponding geometric and material parameters shown in the figure, the solution is 0.0866 (rounded to four decimal places).

Table 6.6. Comparative performance of elements for the curved beam problem

Mesh	QUAD8	US-QUAD8	QUAD8+ IC	No. of iterations	QUAD8+ AIC	No. of iterations
2×1	0.0888	0.0905	0.1114	5	0.1859	32
4×1	0.5817	0.8397	0.8394	15	0.7007	13
5×1	0.7672	0.8494	0.8492	8	0.8225	7
6×1	0.8688	0.8957	0.8958	8	0.8926	10
10×1	0.9769	0.9799	0.9799	5	0.9798	5
20×1	0.9963	0.9964	0.9965	3	0.9965	4

Table 6.6 summarizes the normalized finite element solution for this problem. The results in Table 6.6 reveal that the iterative correction algorithm reproduces the results of US-QUAD8. The convergence trend of the solution of QUAD8+IC is similar to that of US-QUAD8. The number of corrective iterations is observed to decrease as the number of element increases. This is so, since an increase in the number of elements indirectly reduces the extent of distortions in each element in the finite element mesh.

Figure 6.12. Plate with a central hole subjected to in-plane bending loads; Young's modulus, $E = 207 \times 10^9$, Poisson's ratio, $\nu = 0.27$; thickness = 1.0, pressure, $p = 200$

6.7.4. APPLICATION TO A PRACTICAL PROBLEM

The problem of a plate with a hole subjected to an in-plane bending moment shown in Figure 6.12 is considered. The distribution of σ_{yy} -stress is computed with and without application of corrective iterations. This problem demonstrates how the iterative corrections improve the accuracy of solution.

The plate is initially solved with a coarse model consisting of three QUAD8 elements shown in Figure 6.13a. The stress contour band-plot for this mesh, shown in Figure 6.14a, exhibits large discontinuities of stresses, in particular at the boundaries adjacent elements.

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION
TOLERANCE

Figure 6.13. Finite element meshes of a plate with a hole using 8-node quadrilaterals

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

Application of the iterative correction for this mesh leads to considerable improvement in the continuity of the stress contours (Figure 6.14b).

Next, the mesh is selectively refined, in particular, around the hole where the stress gradients are expected to be high. The three stages of refined meshes are shown in Figures 6.13b – 6.13d. Their corresponding σ_{yy} -stress contour plots are shown in Figures 6.14b – 6.14d. Comparison of the smoothness of the stress bands reveals that the iterative correction yields consistent improvement in the solution. In particular, the stress contour band plot given by QUAD8+IC for the coarser mesh in Figure 6.14c, is smoother than that the stress contour band plot given by QUAD8 for a finer mesh, Figure 6.14d. It can be concluded that the improved accuracy of the iterative corrections allows the problem to be solved with a coarser mesh.

Table 6.7 shows the number of corrective iterations that are required for the element meshes in Figure 6.13a – 6.13d. The number of iterations in general, reduces as the size of the element decreases.

Table 6.7. Number of corrective iterations required for meshes in Figure 6.13

Mesh	No. of iterations
Figure 6.13a	17
Figure 6.13b	7
Figure 6.13c	5
Figure 6.13d	7

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

(a) Stress, σ_{yy} band plot for the mesh shown in Figure 6.13a

(b) Stress, σ_{yy} band plot for the mesh shown in Figure 6.13b

Figure 6.14. Stress band plots of plate with a hole for QUAD8 and QUAD8+IC

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

(c) Stress, σ_{yy} band plot for the mesh shown in Figure 6.13c

(d) Stress, σ_{yy} band plot for the mesh shown in Figure 6.13d

Figure 6.14. Stress, σ_{yy} band plots obtained with and without iterative correction

6.7.5. COMPUTATIONAL EFFICIENCY AND CONVERGENCE

In order to assess the additional computational time required of the iterative correction algorithm, the problem shown in Figure 6.15 is discretized with the distorted super-element meshes shown in Figure 6.16a. Uniform refinement is employed to study the convergence of the algorithm. Typical super-element meshes are shown in Figure 6.16b. The problem is solved for its vertical displacement at point A along with uniform mesh refinement. The un-accelerated iterative correction algorithm is used. The computational time required of US-QUAD8 is also reported for comparison.

Figure 6.15. Square plate subjected to a uniformly distributed transverse load p ;

Young's modulus, $E = 15000$; Poisson's ratio, $\nu = 0.0$

The results are summarized in Tables 6.8. Comparison of columns 3 and 6 suggests that the computational time spent on the corrective iterations increases rapidly as the mesh is refined. However, the iterative correction yields further improved accuracies compared to

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

the classical element (see columns 2 and 5). The product of %-error in displacement and computational time (ET) is shown in columns 4 and 7. This product ET may perhaps be looked upon as a measure of the *error-computational-time* or the *wasteful-computational-time*. If comparison is made using this product, the iterative correction seems computationally more efficient for these problems. The results obtained with US-QUAD8 are shown in columns 9 and 10. The % error is not listed, as it is the same as for QUAD8+IC (column 5). Although QUAD8+IC and US-QUAD8 yield the same % error, QUAD8+IC requires more computational time than US-QUAD8 (columns 6 and 9). This is so since computation of QUAD8+IC involves additional arithmetic operations necessary to compute the correction stiffness matrix. The higher frequency of calling the solver routine also contributes to this increase in computational time.

The convergence of strain energy obtained using the corrective iterations is plotted in Figure 6.17. For both distorted meshes, the iterative correction yields more accurate solutions compared to QUAD8. Nevertheless, for super-element mesh I, the error curve for QUAD8+IC is not decreasing steadily (Figure 6.17a) with element size h . This is perhaps due to the non-conforming nature of the metric shape functions in US-QUAD8.

Despite the increase in computational time required by QUAD8+IC compared to US-QUAD8, the iterative correction method is still attractive, since it requires only a symmetric solver. This facilitates easy implementation of the unsymmetric element in an already available commercial finite element code that employs symmetric solvers. Another possible advantage of the iterative correction method is in the analysis of free

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

vibration problems. In such problems, direct implementation of unsymmetric finite element (US-QUAD8) may lead to complex eigenvalues because the stiffness matrix is unsymmetric. These complex eigenvalues may be avoided by an iterative correction implementation (e.g., QUAD8+IC) as this involves only a symmetric stiffness matrix.

Super-element mesh I

Super-element mesh II

Figure 6.16a. Super-elements with angular and curved-edge distortions

Super-element mesh I

Super-element mesh II

Figure 6.16b. Typical 4×4 super-element meshes with uniform mesh refinement

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION TOLERANCE

Table 6.8a. Convergence of results and computational effort for Super-element-I[§]

QUAD8				QUAD8+IC				US-QUAD8	
h	% Error	Time	ET	% Error	Time	ET	No. Iter	Time	ET
10.00	4.9104	2.38	11.68	0.1650	2.86	0.47	11	2.58	0.43
5.00	1.1210	2.78	3.12	0.2257	3.84	0.87	11	2.84	0.64
2.50	0.2452	3.19	0.78	0.0668	8.42	0.56	10	3.34	0.22
1.25	0.0492	13.36	0.66	0.0073	48	0.35	7	14.18	0.10

Table 6.8b. Convergence of results and computational effort for Super-element-II[§]

QUAD8				QUAD8+IC				US-QUAD8	
h	% Error	Time	ET	% Error	Time	ET	No. Iter	Time	ET
10.00	8.2844	2.05	16.98	1.9186	2.76	5.30	12	2.10	4.03
5.00	3.1304	2.37	7.42	0.2014	2.95	0.59	8	2.61	0.53
2.50	0.9749	3.08	3.00	0.0591	5.03	0.30	6	3.33	0.20
1.25	0.2737	10.90	2.98	0.0119	26.54	0.32	4	12.47	0.15

6.8. CONCLUSIONS

An iterative correction algorithm has been developed to enhance the mesh-distortion tolerance of the classical 8-node isoparametric quadrilateral element. The results from the test problems reveal that the proposed algorithm gives a distortion-tolerant performance close to that of US-QUAD8. The mathematical criterion for convergence of the iterative correction algorithm has been developed (Lemma 6.1). For the test problems considered,

[§] The error obtained using US-QUAD8 is of the same degree as that of QUAD8+IC and is hence not shown

the iterations converge to the exact solution for small to moderate mesh distortions, and diverges (violates Lemma 6.1) only when the extent of mesh distortions are too severe.

Figure 6.17a. Convergence of strain energy for super-element I

Figure 6.17b. Convergence of strain energy for super-element II

CHAPTER 6. ITERATIVE CORRECTIONS OF ISOPARAMETRIC ELEMENTS TO ENHANCE MESH-DISTORTION
TOLERANCE

An acceleration scheme has been proposed to speed up the convergence of iterations. The effectiveness of acceleration is most pronounced when the mesh distortions are severe. For mild distortions, however, the effectiveness of acceleration is not apparent.

Although QUAD8+IC requires more computational time than QUAD8, the additional computational time is reimbursed in terms of the increased accuracy of the solution as evidenced by the test problem of section 6.7.5. Thus, the iterative correction technique promises to be a computationally efficient method.

The concept of iterative corrections can be easily extended to the three-dimensional elements. Computation of the corrective stiffness matrix is similar to what has been discussed in this chapter, i.e., taking the difference between the unsymmetric stiffness-matrix of US-HEXA20 and the symmetric stiffness of HEXA20. Iterative corrections then proceed as usual according to Equation (6.9), suitably modified to accommodate the 20-noded elements.

CHAPTER 7. CRITICAL DISCUSSIONS

7.1. INTRODUCTION

In previous chapters, the unsymmetric formulation has been explored for linear and nonlinear static applications. The unsymmetric elements have demonstrated superior performance compared to the classical isoparametric elements in the presence of mesh distortions. However, the unsymmetric element is not without drawbacks. The possibility of interpolation failure due to singularity of the $\mathbf{P}$ -matrix has already been recognized [5, 178]. However, the conditions under which this might occur have not been explored thus far. In addition to the possibility of a singular $\mathbf{P}$ -matrix, the unsymmetric elements have also been observed to exhibit rotational frame dependent behavior and Poisson's ratio sensitivity under certain conditions. Both these aspects will be explored in this chapter. The conditions under which the unsymmetric element exhibits rotational frame dependent behavior are investigated in Section 7.2. The possibility of singular $\mathbf{P}$ -matrices is investigated, and a remedy is then proposed. These aspects are presented in Section 7.3.

7.2. ROTATIONAL FRAME DEPENDENCE

Despite its excellent performance in the presence of severe mesh distortions, the US-QUAD8 element has been observed to exhibit rotational frame dependent performance under certain conditions. One such instance involves prescribing zero displacements for all the degrees of freedom on one edge of an element with non-zero values of Poisson's ratio. To illustrate this defect, a 2-element mesh of a straight cantilever beam subjected to a constant bending moment and support conditions as shown in Figure 7.1 is considered.

Figure 7.1. Straight cantilever beam meshed subjected to a constant bending moment; $L = 10.0$; $c = 5.0$; $M = 1.0$; Young's modulus, $E = 100$; Poisson's ratio, $\nu = 0.3$

All the degrees of freedom on the edge at $x = 0$ are restrained. The geometry is then rotated counterclockwise from 0° to 90° in steps of 10° , and the problem is solved for the displacements at each angle. The magnitude of displacement, i.e., $\sqrt{u^2 + v^2}$ at point A is monitored to study the rotational frame dependent behavior. The test problem is then repeated using the three types of distorted element meshes shown in Figure 7.2.

Table 7.1 shows the results obtained with QUAD8 and US-QUAD8 elements. The magnitude of displacements, for a mesh without distortion and $\theta = 0^\circ$, obtained using an “overkill” of 8-node quadrilateral elements of ANSYS is used as a reference to compare the solutions. The values of distortion parameters associated with those shown in Figure 7.2 are chosen as $\delta_p = 3.0$, $\delta_q = 3.0$ and $\delta_r = 0.5$. Despite exhibiting superior solution accuracy compared to QUAD8, US-QUAD8 exhibits a rotational dependence performance even for regular element shapes. This is evident from the fluctuations observed in the magnitude of displacements as the geometry is being rotated.

Figure 7.2. Distorted two-element meshes used to model the straight cantilever problem

Table 7.1. Relative error of QUAD8 and US-QUAD8 for rotational frame invariance test

	$\delta_p = \delta_q = \delta_r = 0$		$\delta_p = 3.0$		$\delta_q = 3.0$		$\delta_r = 0.5$	
θ	QUAD8	US-QUAD8	QUAD8	US-QUAD8	QUAD8	US-QUAD8	QUAD8	US-QUAD8
0	-2.1111	-2.1111	-3.2037	-3.2037	-43.8148	-9.7778	-3.7963	-3.7963
10	-2.1111	-2.1296	-3.2037	-0.5185	-43.8148	-9.7593	-3.7963	-2.1111
20	-2.1111	-2.1296	-3.2037	-0.5370	-43.8148	-9.7222	-3.7963	-2.0185
30	-2.1111	-2.0926	-3.2037	-0.5000	-43.8148	-9.8148	-3.7963	-1.8148
40	-2.1111	-2.0185	-3.2037	-0.3704	-43.8148	-10.5556	-3.7963	-1.5370
50	-2.1111	-2.0000	-3.2037	-0.2407	-43.8148	-11.0185	-3.7963	-1.6296
60	-2.1111	-2.0370	-3.2037	-0.6296	-43.8148	-10.0741	-3.7963	-1.9259
70	-2.1111	-2.0741	-3.2037	-0.6481	-43.8148	-9.8148	-3.7963	-2.0741
80	-2.1111	-2.0926	-3.2037	-0.5370	-43.8148	-9.7778	-3.7963	-2.1296
90	-2.1111	-2.1111	-3.2037	-3.2037	-43.8148	-9.7778	-3.7963	-3.7963

[†] “Overkill” solution = 0.054

In the presence of mesh distortions, the performance of QUAD8 deteriorates, as expected. However, the error in the solution remains constant when the geometry is rotated, irrespective of the angle of rotation. US-QUAD8 again exhibits a performance

that depends on the angle of rotation. The accuracy of the solutions also reduces in the presence of distortions. Albeit exhibiting rotational frame dependence, the displacement solution given by US-QUAD8 is still more accurate compared to QUAD8 for all distorted element meshes considered.

Although the rotational frame dependent behavior of US-QUAD8 is observed to marginally improve the accuracy of the solutions (Table 7.1) for some rotation angles, it is a discouraging aspect since it suggests an erratic element performance. Thus, the increase in solution accuracy attained via rotational frame dependent behavior as seen in this example cannot be relied on. Nevertheless, US-QUAD8 exhibits rotational frame dependence only when all the degrees of freedom on the left edge of the cantilever are restrained. If only the degrees of freedom of the two corner nodes are restrained, US-QUAD8 does not exhibit the frame dependent behavior (results not shown). This problem also vanishes when the value of Poisson's ratio, $\nu = 0.0$, even if all the degrees of freedom at the left edge of the cantilever are restrained.

The rotational frame dependent behavior of US-QUAD8 for this particular problem can be remedied by attaching a local Cartesian coordinate system onto the element as shown in Figure 7.3. The orientation of the local Cartesian coordinate system follows the orientation of the element. One possibility is to make one of the axes of the local coordinate system coincide with one of the edges of the element. Referring to Figure 7.3, the local x' -axis is chosen to coincide with the edge 1-5-2. This axis is oriented at an

arbitrarily chosen angle, θ , to the global x -axis. The local y' -axis is perpendicular to the local x' -axis.

Figure 7.3. Eight node quadrilateral element with an attached local Cartesian coordinate system, (x', y')

To implement this *co-rotational* local coordinate system, it is first necessary to transform the global nodal coordinates of the element, (x_i, y_i) , into local coordinates (x'_i, y'_i) using the relation

$$\begin{Bmatrix} x'_i \\ y'_i \end{Bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \end{Bmatrix} = [R_\theta]^T \begin{Bmatrix} x_i \\ y_i \end{Bmatrix} \quad (7.1)$$

The average of nodal coordinates, (x'_c, y'_c) are computed with reference to the local coordinates (x'_i, y'_i) . The metric shape functions are then evaluated according to Equation (3.23), wherein the matrices, $\mathbf{p}(\mathbf{x})$ and $\mathbf{P}$ now take the forms

$$\mathbf{p}(\mathbf{x}) = \begin{bmatrix} 1 & x' - x'_c & y' - y'_c & \cdots & (x' - x'_c)(y' - y'_c)^2 \end{bmatrix}^T \quad (7.2)$$

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x'_{1-c} & x'_{2-c} & x'_{3-c} & \cdots & x'_{8-c} \\ y'_{2-c} & y'_{2-c} & y'_{3-c} & \cdots & y'_{8-c} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x'_{1-c} y'^2_{1-c} & x'_{2-c} y'^2_{2-c} & x'_{3-c} y'^2_{3-c} & \cdots & x'_{8-c} y'^2_{8-c} \end{bmatrix} \quad (7.3)$$

where

$$\left. \begin{aligned} x'_{i-c} &= x'_i - x'_c \\ y'_{i-c} &= y'_i - y'_c \end{aligned} \right\} \quad i = 1, 2, 3, \dots, 8 \quad (7.4)$$

The derivatives of metric shape functions are evaluated with respect to the local Cartesian coordinates (x'_i, y'_i) , as $\partial \mathbf{M} / \partial x'_i$ and $\partial \mathbf{M} / \partial y'_i$. In a similar manner, the isoparametric shape functions and its derivatives, $\partial \mathbf{N} / \partial x'_i$ and $\partial \mathbf{N} / \partial y'_i$, are evaluated with reference to the local coordinates (x'_i, y'_i) . These local derivatives are now inputs in the strain-displacement matrices, $\bar{\mathbf{B}}$ and $\hat{\mathbf{B}}$ in Equations (3.3) and (3.5), respectively. Formulation of the element stiffness matrix then proceeds following the procedure described in Section 3.3 of Chapter 3. This element stiffness matrix is, however, evaluated with reference to the local coordinates (x'_i, y'_i) . Prior to element assembly, the *local element stiffness matrix* is then transformed back into the global coordinate system according to the relation

$$\bar{\mathbf{K}} = [\mathbf{R}] \mathbf{K} [\mathbf{R}]^T \quad (7.5)$$

where the element stiffness matrix, $\mathbf{K}$, is

$$\mathbf{K} = \int_{\Omega^{(e)}} \bar{\mathbf{B}}^T \mathbf{D} \hat{\mathbf{B}} d\Omega^{(e)} \quad (7.6)$$

the 16×16 rotation matrix, $[\mathbf{R}]$, is defined as

$$[\mathbf{R}] = \begin{bmatrix} [R_\theta] & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & [R_\theta] & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & [R_\theta] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & [R_\theta] \end{bmatrix} \quad (7.7)$$

Using the co-rotational implementation of US-QUAD8, the test problem in Figure 7.1 is solved again using the meshes in Figure 7.2. The acronym US-QUAD8-CR is used to indicate the co-rotational implementation of US-QUAD8. Table 7.2 shows the magnitude displacements at $(L,0)$ obtained with US-QUAD8-CR.

Table 7.2. Relative error of US-QUAD8-CR for rotational frame invariance test

θ	$\delta_p = \delta_q = \delta_r = 0$	$\delta_p = 3.0$	$\delta_q = 3.0$	$\delta_r = 0.5$
0	-2.1111	-3.2037	-9.7778	-3.7963
10	-2.1111	-3.2037	-9.7778	-3.7963
20	-2.1111	-3.2037	-9.7778	-3.7963
30	-2.1111	-3.2037	-9.7778	-3.7963
40	-2.1111	-3.2037	-9.7778	-3.7963
50	-2.1111	-3.2037	-9.7778	-3.7963
60	-2.1111	-3.2037	-9.7778	-3.7963
70	-2.1111	-3.2037	-9.7778	-3.7963
80	-2.1111	-3.2037	-9.7778	-3.7963
90	-2.1111	-3.2037	-9.7778	-3.7963

[†] “Overkill” solution = 0.054

The results in Table 7.2 indicate that the attachment of a local Cartesian coordinate system is effective in eliminating the rotational frame dependent behavior of US-QUAD8 for this particular problem.

The rotational frame dependent behavior of US-QUAD8 can be explained by investigating of the expression for the assumed displacement field when different

boundary conditions are imposed on the edge $x = 0.0$ of the cantilever. The general displacement field that can be represented by US-QUAD8 is of the form

$$\mathbf{u} = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{Bmatrix} a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^2y + a_7y^2x \\ b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 + b_6x^2y + b_7y^2x \end{Bmatrix} \quad (7.8)$$

For illustration, the two-element mesh of Figure 7.1 oriented at $\theta = 0^\circ$ is considered. Using the nodal coordinates, (x_i, y_i) , and nodal displacements, (u, v) , of element $\{1\}$, the constants, a_i and b_i in Equation (7.8) are determined for the following three cases for each of the distortion types in Figure 7.2:

Case I

Non-zero Poisson's ratio, ($\nu > 0.0$) and all degrees of freedom at only the two corner nodes at $x = 0.0$ are fixed.

Case II

Zero Poisson's ratio, ($\nu = 0.0$) and all degrees of freedom at all three nodes for the three nodes at $x = 0.0$ are fixed.

Case III

Non-zero Poisson's ratio, ($\nu > 0.0$) and all degrees of freedom at all three nodes for the three nodes at $x = 0.0$ are fixed.

In cases *I* and *II*, substitution of the nodal displacements into Equation (7.8) and solving for constants, a_i 's and b_i 's yield the same horizontal and vertical displacement field, $\mathbf{u}$, as

$$\mathbf{u} = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{Bmatrix} -0.0024x + 0.00096xy \\ 0.00072y - 0.00048x^2 - 0.000144y^2 \end{Bmatrix} \quad (7.9)$$

Equation (7.9) is valid for all the distortions considered in Figure 7.2, regardless of the distortion parameters δ_p , δ_q and δ_r .

Following a similar procedure for case *III*, three different sets of displacement fields are obtained corresponding to the three types of distortions considered *viz.*, angular, curved-edge and mid-side node. These displacement fields are summarized in Table 7.3 for $\delta_p = 3.0$, $\delta_q = 3.0$ and $\delta_r = 0.5$.

Table 7.3. Displacement field, $\mathbf{u}$, in element $\{1\}$ for $\delta_p = 3.0$, $\delta_q = 3.0$ and $\delta_r = 0.5$

Distortion	Displacement Field, $\mathbf{u}$
Angular, $\delta_p = 3.0$	$u(x, y) = -0.00223605x - 0.00003967x^2 + 0.00091452xy + 0.00001342x^2y$ $v(x, y) = -0.000234394x - 0.00047206x^2 + 0.000271015xy - 0.000043691xy^2$
Curved-edge, $\delta_q = 3.0$	$u(x, y) = -0.00211448x + 0.00084579xy$ $v(x, y) = -0.000447765x - 0.00038717x^2 + 0.000147302xy - 0.00002946xy^2$
Mid-side node, $\delta_r = 0.5$	$u(x, y) = -0.0021926x - 0.00002505x^2 + 0.00087807xy + 0.00001005x^2y$ $v(x, y) = -0.00080632x - 0.00049038x^2 + 0.000170511xy - 0.000035865xy^2$

Table 7.4. Displacement field, $\mathbf{u}$, in element $\{1\}$ for $\delta_p = 2.0$, $\delta_q = 2.0$ and $\delta_r = 1.0$

Distortion	Displacement Fields
Angular, $\delta_p = 2.0$	$u(x, y) = -0.00221269x - 0.00003383x^2 + 0.00089425xy + 0.00001216x^2y$ $v(x, y) = -0.000187522x - 0.000468106x^2 + 0.000233545xy - 0.000040463xy^2$
Curved-edge, $\delta_q = 2.0$	$u(x, y) = -0.0021485x - 0.00001213x^2 + 0.00085714xy$ $v(x, y) = -0.000325526x - 0.000414857x^2 + 0.000159973xy - 0.000031995xy^2$
Mid-side node, $\delta_r = 1.0$	$u(x, y) = -0.00219184x - 0.00002473x^2 + 0.00087905xy$ $v(x, y) = -0.00051847x - 0.000474254x^2 + 0.000159183xy - 0.000035403xy^2$

The displacement fields in Case *III* are different, for different values of the distortion parameters δ_p , δ_q and δ_r . For example, Table 7.4 shows the displacement field, $\mathbf{u}$, pertaining to values of distortion parameters, $\delta_p = 2.0$, $\delta_q = 2.0$ and $\delta_r = 1.0$.

The frame invariance behavior of US-QUAD8 in cases *I* and *II* can be explained in terms of the monomial terms that can be exactly reproduced by US-QUAD8. The nodal coordinates resulting from a rotation of the element mesh by some arbitrary angle, θ , can be expressed in terms of the rotated coordinate system (x', y') as

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} x' \\ y' \end{Bmatrix} = [R_\theta] \begin{Bmatrix} x' \\ y' \end{Bmatrix} \quad (7.10)$$

If Equation (7.10) is substituted into Equation (7.9), it can be shown that the resulting displacement field is quadratic with only the monomial terms 1, x' , y' , $(x')^2$, $x'y'$ and $(y')^2$ in terms of the local coordinates x' and y' . In Chapter 3, US-QUAD8 has been shown to satisfy the completeness conditions required to reproduce these quadratic terms for any arbitrary configuration of the element. This explains the rotational frame invariance of US-QUAD8 with respect to the monomial terms 1, x , y , xy , x^2 , and y^2 pertaining to cases *I* and *II*.

On the other hand, in case *III*, wherein US-QUAD8 experiences rotational frame dependent behavior, the exact solution includes the incomplete monomial terms of the third order, x^2y and xy^2 (Tables 7.3 and 7.4) Following a similar approach as in cases *I* and *II*, Equation (7.10) is substituted for the monomial terms (x, y) in the displacement fields in Tables 7.4 and 7.5. The resulting displacement fields in terms of local

coordinates, (x', y') , now involves the cubic terms $(x')^3$ and $(y')^3$ in addition to the eight monomial terms, $1, x', y', (x')^2, x'y', (y')^2, (x')^2 y',$ and $x'(y')^2$. Since the exact solution now involves the cubic terms, $(x')^3$ and $(y')^3$, and that US-QUAD8 does not satisfy the completeness conditions necessary for the reproduction of these terms, US-QUAD8 cannot be expected to reproduce the displacement fields in Tables 7.3 and 7.4 for any arbitrary rotation angle, θ . The solution of US-QUAD8 can only be an approximation of the exact solution.

A change in the rotation angle, θ , would result in a subsequent change in the coefficients in the displacement field, $\mathbf{u}$ in Tables 7.3 and 7.4. Recalling that the solution of US-QUAD8 is only approximate with respect to the cubic terms $(x')^3$ and $(y')^3$, the accuracy of US-QUAD8 in representing the displacement fields at any given angle, θ , depends on how close is, the metric shape functions in representing these displacement fields. As a consequence, the solution accuracy that can be achieved by US-QUAD8 depends on the orientation of the element. This explains the rotational frame dependent behavior of US-QUAD8 when the exact solution involves the monomial terms $x^2 y$ and xy^2 . Similarly, extending the same argument to exact displacement fields that involve the higher order monomial terms $x^3, y^3, x^4, x^3 y$, etc., it can be shown that US-QUAD8 will also exhibit rotational frame dependent behavior when the exact displacement fields involve these monomial terms.

Although the solution given by a co-rotational formulation will be frame invariant with respect to a “fixed” orientation of the local coordinate system, it is not a complete remedy

to the rotational frame dependence problem of US-QUAD8. The decision to align the local coordinate system to an arbitrary “side” of the element is rather ad hoc and may not be the optimum choice that will yield the most accurate solution. However, a more rational approach to locate the best orientation of the local coordinate system is currently not obvious.

7.3. SINGULARITY OF **P**-MATRIX

In Chapter 3, a passing mention was made on possibilities of the **P**-matrix in Equation (3.21) becoming singular. Singularities of **P**-matrix may be caused by improper selection of monomial terms in Equation (3.15 or 7.8) or by the very geometry of the element itself i.e., the nodal positions of the element inherently result in a singular **P**-matrix.

The choices of appropriate monomial terms that lead to a singular **P**-matrix were discussed by MacNeal [178]. The monomial terms must be so selected such that they complete a full order in the Pascal triangle in the case of two-dimensional elements, or, equivalently, the Pascal pyramid in the case of three-dimensional elements. Additional monomial terms, if any, must be selected to preserve the symmetry of the Pascal triangle or pyramid. Kidger [179] encountered a similar situation when formulating the shape functions for the 14-node hexahedron. Upon successfully making appropriate choices for the first ten monomial terms to represent a quadratic displacement field in three dimensions, the choices for the remaining four monomial terms remained non-unique. Different versions of the 14-node hexahedron were then proposed. These were termed Type I, II, III and IV. It was found that only the version of element with monomial terms

that preserves the symmetry of the Pascal pyramid constitute a generally trouble-free element, whereas the other three are plagued with problems such as rank deficiencies and frame dependence.

MacNeal's discussions [178], along with Kidger's observations [179] suggest that selection of monomial terms that preserves the symmetry of the Pascal's pyramid (Pascal's triangle in two dimensions) would guarantee bases that are of full rank. However, this is not always the case. For example, if the last two monomial terms in Equation (3.15) are replaced with x^3 and y^3 , the entire collection of monomial terms can in principle, represent these cubic terms. Unfortunately, the metric shape functions for these combinations of monomial terms may not, in general, exist. For example, when the shape of the element is regular (viz., squares or rectangles), the choice of such monomial terms become linearly dependent and contributes to the rank deficiency in the **P**-matrix.

Liu and Gu [101] encountered similar problems of singularities when they derive the shape functions for their mesh-less point interpolation method. The authors explored several remedies for the problem such as a slight perturbation of nodal coordinates or, a rotation of the coordinate system by some arbitrary small angle. However, these approaches are sometimes not convenient in practice. Nodal perturbations may not cause problems when it is introduced inside the domain of the element but if this is introduced for the nodes on the boundary of the problem domain, it directly modifies the entire geometry of the problem and thereby, the general statement of the problem itself.

Coordinate transformation by some arbitrary angle appears feasible, but this ad hoc method introduces some complications in the post-processing phase.

Figure 7.4. Element configurations used to study singularity of $\mathbf{P}$ -matrix; (a) single element with angular distortion; (b) single element with curved-edge distortion; (c) single element with mid-side node distortion

Singularities of the $\mathbf{P}$ -matrix related to geometry are often encountered when the element is rotated 45° from its original Cartesian x -axis. In such cases, the $\mathbf{P}$ -matrix becomes nearly singular and leads to solutions with large errors. To alleviate this problem, the methods proposed by Liu and Gu [101] in their point interpolation mesh-less method is still applicable. Near or exact-singularities of the $\mathbf{P}$ -matrix have not been encountered in any of the test problems presented in Chapter 3. Nevertheless, considering all admissible

positions that the nodes in the element can occupy in the Cartesian space of real numbers, the possibility of the **P**-matrix becoming singular is still very real.

This section investigates the singularity of **P**-matrix for the three typical element configurations shown in Figure 7.4. The objective of this study is to obtain a relation between the parameters h , L and δ (Figure 7.4) that correspond to singular **P**-matrices for rotational angles in the range $0^\circ \leq \theta \leq 90^\circ$ in counterclockwise direction with respect to the x -axis in Figure 7.4a.

Figure 7.5. Singularity contour lines for single-element with angular distortion

The **P**-matrix is evaluated according to Equation (3.21). The nodal coordinates are expressed in terms of δ , h and L . In each case considered, the distortion parameter, δ , and the height of the element, h , are normalized with respect to the element's length, L .

Singularity of the $\mathbf{P}$ -matrix is then investigated in terms of parameters δ/L and h/L by solving for the roots of the equation $\det(\mathbf{P})=0$, for δ/L in terms of h/L at rotation angles $0^\circ \leq \theta \leq 90^\circ$.

For the element configuration in Figure 7.4a, the determinant of the $\mathbf{P}$ -matrix is zero only when $\delta/L = -1$ or 1 for all h/L and $0^\circ \leq \theta \leq 90^\circ$. The results are plotted in Figure 7.5 and the singularity corresponds to the case of trivial distortion associated with collapsing a quadrilateral element into a triangular element, wherein all the three nodes at edges $y = 0$ or $y = h$ coincide.

Figure 7.6. Singularity contour lines for single-element with curved-edge distortion

Figure 7.6 shows the *singularity contour lines* for the element with curved-edge distortions in Figure 7.4b at $\theta = 0^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ$, and 90° . The plot reveals that

P-matrix ceases to become singular for values of $h/L > 0.5$ (and $\delta/L < -1$). Singularity of the **P**-matrix at $\delta/L = -1$ corresponds to the case of trivial distortion wherein, both mid-side nodes of two opposing faces coincides.

Figure 7.7. Singularity contour lines for single-element with mid-side node distortion

Figure 7.7 shows the relation between δ/L and h/L that correspond to singular **P**-matrices for the single element with mid-side node distortion in Figure 7.4c, at several angles of rotations, θ . Values of $\|\delta/L\| > 0.5$ are inadmissible and the singularity contour lines are shown for the range $-0.5 \leq \delta/L \leq 0.5$. As in the case of curved edge distortions, the possibility of a singular **P**-matrix vanishes when the aspect ratio of the element, h/L exceeds a certain limit (and $-0.5 < \delta/L < 0.5$), for each orientation, θ .

In numerical computations, however, the **P**-matrix rarely becomes exactly singular. The combination of the parameters δ/L , h/L and θ that correspond to the singularity contour lines in Figures 7.5 – 7.7 are real numbers. Thus it is very unlikely that the nodal configurations of the elements correspond exactly to values of these parameters. However, when the nodal positions are close enough to render the resulting **P**-matrix nearly singular, the metric shape functions resulting from this near singular **P**-matrix lead to large numerical errors when used in the finite element equations.

7.3.1. DETECTION OF SINGULARITY OF P-MATRIX IN ELEMENT FORMULATION

Exact singularity of the **P**-matrix in numerical implementation is rarely encountered except in isolated cases wherein two or three nodes in the element coincide. This results in two or three identical columns in the **P**-matrix that results in rank deficiency. The metric shape functions do not exist when the **P**-matrix is exactly singular.

However, a more common phenomenon that is encountered in numerical implementation involves the case when the combination of nodal locations in the element is such that it constitutes to a near singular **P**-matrix. Under such instances, the inverse of the **P**-matrix still exists but may be ill conditioned. This results in large errors in the metric shape functions and their derivatives, which in turn leads to large solution errors. This section presents an algorithm to detect instances of near singular **P**-matrices. The section that follows details a remedy for errors caused by near singularity of this matrix.

Although the errors caused by near singularity of the **P**-matrix can only be visualized after the finite element solution is obtained, it is still possible to detect singularity prior to element stiffness calculations. *A priori* detection of near singularities of **P**-matrix is essential not only to prevent unnecessary computations that would lead to solutions that are in error, but also to enable the necessary steps to be taken to prevent the error resulting from a near singular **P**-matrix to affect the solution.

Numerical experiments that involve configurations of elements that lead to near singular **P**-matrices reveal that the metric shape functions loose their interpolating properties at every point in the element domain. In other words, all the completeness conditions in Equation (3.14) will be violated. Thus, it is useful to check for the satisfaction of the completeness condition corresponding to $p = q = r = 0$ at any arbitrary point in the element domain at the instant the **P**-matrix is computed, i.e.

$$\sum_{i=1}^N M_i = 1 \quad (7.11)$$

In the present implementation, satisfaction of Equation (7.11) is verified at the parametric center $(\xi = 0, \eta = 0)$ of the element. Due to the rounding off errors in evaluating the inverse of the **P**-matrix, Equation (7.11) may not be satisfied exactly. Rounding off errors is often seen to occur in the fifteenth decimal place when the sum of shape functions is evaluated. This requires the need to set upper and lower bounds to Equation (7.11) in order to numerically check for near singularities of the **P**-matrix. Numerical results thus far revealed that $\sum_{i=1}^N M_i \gg 1$ at any point in the element whenever the condition is not

satisfied. Nevertheless, to be conservative, the upper and lower limits to check for satisfaction of Equation (7.11) are set to

$$1.0 - (1.0 \times 10^{-12}) < \sum_{i=1}^N M_i < 1.0 + (1.0 \times 10^{-12}) \quad (7.12)$$

7.3.2. AVOIDING SINGULARITY OF P-MATRIX IN ELEMENT FORMULATION

Having developed a method to detect the near-singularity of the **P**-matrix, an algorithm to avoid such instances is explored. A straightforward but imprudent approach would be to re-mesh the initial element mesh whenever near singularities of the **P**-matrix is detected. However, this would be a too time-consuming approach. An alternative approach would be to apply a transformation of coordinate system similar to the co-rotational formulation implemented in Section 7.2.

In the proposed approach, once near singularity of the **P**-matrix is detected, the nodal coordinates that make up the element are rotated by some arbitrary angle, θ_R . This arbitrary angle is obtained using a random number generator. Upon rotating the nodal coordinates by θ_R according to Equation (7.1), the **P**-matrix is recomputed and satisfaction of condition (7.12) is again checked. This is repeated until Equation (7.12) is finally satisfied. Thus far, a single rotation is found to be sufficient to avoid near singularities.

The stiffness matrix is then evaluated based on the new coordinates according to the element formulation detailed in Section 3.3 in Chapter 3. Upon computing the element stiffness, the stiffness matrix is re-rotated back into its original configuration according to

Equation (7.5). Coordinate transformation need only be applied to elements that constitute to near singular $\mathbf{P}$ -matrices. The pseudo code for the implementation of coordinate transformation is described as follows:

1. Loop over elements in the domain
 - a) Extract the nodal coordinates of the element
 - b) Form the $\mathbf{P}$ -matrix and compute its inverse
 - c) Evaluate the metric shape functions, $\mathbf{M}$ using Equation (3.25) at the parametric center, $(\xi = 0, \eta = 0)$
 - d) Check for satisfaction of condition (7.12)
 - e) Proceed to Step 2 if condition (7.12) is satisfied, continue if the condition is not satisfied
 - f) Rotate the nodal coordinates by some arbitrary angle, θ_R , generated by a random number generator
 - g) Return to Step (1b)
2. Compute the element stiffness in the usual manner as outlined in Section 3.3 in Chapter 3
3. Terminate loop over integration points
4. If the nodal coordinates have been rotated in Step 1, apply the coordinate transformation by the total angle on which the element has been rotated according to Equation (7.5)
5. Terminate loop over elements

To illustrate the effectiveness of the proposed algorithm, the two-element cantilever shown in Figure 7.1 with dimensions and material properties, $L = 10.0$, $c = 0.5$, $M = 0.5$, $E = 100$ and $\nu = 0.3$, and a distortion parameter of $\delta_q = -1.25$, according to the definition in Figure 7.2 is considered. This nodal configuration results in a near singular $\mathbf{P}$ -matrix in element $\{1\}$ when the cantilever is rotated by an angle $\theta = 45^\circ$ counterclockwise from the x -axis. All degrees of freedom at the corner nodes $(0,0)$ and $(0,c)$ are restrained. The solution to such a problem involves a purely quadratic displacement field. Any errors in the solution yielded by the unsymmetric element will thus be due to the near singularity of the $\mathbf{P}$ -matrix since the US-QUAD8 element has been shown to be capable of reproducing all the quadratic terms in the Pascal's triangle. Table 7.5 shows the relative errors in the displacement magnitudes at $A(L,0)$ when near singularity of $\mathbf{P}$ -matrix is encountered and the remedied solutions for typical random rotation angles, θ_R .

Table 7.5. Relative error in solution for near singularity test

θ_R (rad)	Relative error (%)
0.000000	-70.1690
-0.872400	0.0000
-0.654036	0.0000
-0.560191	0.0000
-0.218012	0.0000
0.383852	0.0000
0.535501	0.0000

When coordinate transformation is not applied, $\theta_R = 0$, near singularity of the $\mathbf{P}$ -matrix results in about -70% error in the solution. Coordinate transformation by any random

angle $\theta_R \neq 0$, in Table 7.5 indicate that the proposed algorithm is capable of rectifying the possible errors caused by near singularity of the **P**-matrix.

7.4. A NOTE ON EXTENSION TO THREE DIMENSIONS

The rotational frame dependence and singularity discussed in previous sections also affect the US-HEXA20 element. The co-rotational formulation employed to remove these defects from US-QUAD8 can also be applied to US-HEXA20. In general, two successive rotations about any two of the three (x -, y - and z -) coordinate axes may be necessary although rotation about one of these axes, alone, may be sufficient for some problems (Figure 7.8).

Figure 7.8. Successive rotations of coordinate system.

The transformation from the original configuration to the new configuration is defined as

$$\begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix} = \mathbf{R}_y(\theta) \cdot \mathbf{R}_x(\phi) \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \quad (7.13)$$

where

$$\mathbf{R}_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix} \quad (7.14)$$

$$\mathbf{R}_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \quad (7.15)$$

The occurrence of near singularity of the $\mathbf{P}$ -matrix in US-HEXA20 can be traced using the same condition as in US-QUAD8. Equation (7.11) followed by Equation (7.12) can still be used for this purpose, with the only difference that $N = 20$, the number of nodes in US-HEXA20. This condition may be evaluated at the parametric center, $(\xi = 0, \eta = 0, \zeta = 0)$.

7.5. CONCLUSIONS

Although the unsymmetric formulation is quite successful in handling mesh distortions, it does have limitations. Two drawbacks, *viz.*, the rotational frame dependence and the possibility of near singularity of the $\mathbf{P}$ -matrix have been identified and some remedies have been proposed.

The US-QUAD8 element has been observed to be rotationally frame invariant with respect to the first six monomial terms in the Pascal pyramid ($1, x, y, x^2, xy$ and y^2). However, the element is frame variant with respect to the higher order monomial terms x^2y, xy^2, x^3, y^3 , etc. The US-QUAD8 element is frame dependent with respect to the first six monomial terms because the metric shape functions are complete up to the second order in the Pascal triangle. Although US-QUAD8 can reproduce the monomial terms x^2y

and xy^2 , it exhibits rotational frame dependence with respect to these terms. This results as a consequence of the incomplete third order monomial terms associated with the metric shape functions in the Pascal triangle. As a result, a rotation of the element mesh by some arbitrary angle will introduce additional cubic terms in the rotated coordinate system, (x', y') , that cannot be reproduced by the metric shape functions. This leads to the frame dependent behavior of the element, which can also be associated with the lack of geometric isotropy of the metric shape functions. If the exact solution involves monomial terms that are not present in the assumed displacement field (7.9), US-QUAD8 not only fails to reproduce the exact solution, but also exhibits frame dependent behavior with respect to these monomial terms.

The possibility of the **P**-matrix becoming singular for certain nodal configurations has been studied for generic element geometries. Three typical cases have been considered and expressions for singularity of **P**-matrix have been derived, relating the aspect ratio, distortion parameter and rotation angle of the elements. It is to be noted that apart from element configurations presented in this chapter, other nodal configurations leading to singularity of the **P**-matrix are also possible. A method to detect possible singularities of the **P**-matrix for any element configuration prior to element formulation has been proposed. Upon detection of near singularity, a coordinate transformation is to be applied to rectify the problem. The method, although ad hoc by nature, has been found to be successful in preventing errors resulting from near singularities. A permanent cure to the singularity problem that is mathematically respectable is yet to appear in the literature although such singularity problems have been known to exist for decades [180].

CHAPTER 8. CONCLUDING REMARKS

8.1. SUMMARY

Mesh distortions and their malicious effects on the performance of isoparametric elements have been known since their introduction. Conventional approaches to treat the ill-effects of mesh distortions have been to employ either the h - or p -refinement techniques. Parallel to these, there have been efforts to develop distortion tolerant elements. The recently published unsymmetric formulation [5] is one among these. The objective of this research has been to critically investigate the unsymmetric formulation and extend it to newer applications. The research work carried out has focussed on

1. an extensive literature search on techniques to tackle mesh distortion effects,
2. a critical review of, and further investigations into the original unsymmetric formulation for linear elasticity problems [5],
3. extension of unsymmetric formulation to nonlinear elasticity problems,
4. development of an iterative technique to bypass the necessity of an unsymmetric equation solver, which is otherwise required for an unsymmetric element, and
5. a detailed study of some drawbacks of unsymmetric formulation such as the origin of singularity of interpolation matrix, rotational frame invariance, and Poisson ratio sensitivity.

8.2. MAJOR CONCLUSIONS

The review of literature has revealed that symmetric finite element formulations based on the Galerkin approach have been widely adopted over the past four decades. However,

unsymmetric formulations based on the Petrov-Galerkin approach have been a less explored research area for solid and structural mechanics applications. The central idea of the unsymmetric formulation [5] is to use compatibility fulfilling shape functions and completeness fulfilling shape functions for the left and right **B**-matrices, respectively, appearing in the element stiffness matrix integral. Such a judicious choice of shape functions leads to an (unsymmetric) element that is highly tolerant to mesh distortions. The literature review reveals that this is a unique approach to develop mesh-distortion tolerant elements. Another advantage of the unsymmetric formulation is its simplicity as compared to other distortion-tolerant formulations; except for the choice of shape functions, the rest of the formulation is the same as that of the well-established isoparametric formulations.

As a part of further investigations into the unsymmetric formulation for linear elasticity problems, an elaborate patch test (in the strong form) has been performed to verify the reproduction of the monomial terms, $1, x, y, x^2, xy, y^2, x^2y$ and xy^2 . Three types of 8-node quadrilateral elements (QUAD8, US-QUAD8 and MM-QUAD8) have been tested for generic element distortions that an 8-node quadrilateral may experience. The results have revealed that only the unsymmetric element (US-QUAD8) is capable of reproducing all these monomial terms. This is attributed to the clever choice of the left and right shape functions in order to satisfy the compatibility and completeness requirements, respectively. QUAD8 satisfies the patch test only for a linear displacement field, while MM-QUAD8 does so only for selected patches of elements, depending on the imposed displacement fields. Thus, the present study confirms the claim of the source paper [5]

that US-QUAD8 is capable of reproducing a complete quadratic displacement field irrespective of the type of mesh distortion.

As a second part of further investigations into linear elasticity applications, the unsymmetric formulation has been extended to the 20-node hexahedron element (US-HEXA20). Results from the benchmark problems have shown that US-HEXA20 possesses similar distortion tolerant properties as that of US-QUAD8. A study on the computational time has revealed that US-HEXA20 is computationally more expensive than HEXA20. The computational time has been observed to increase by 91% for element formulation, and 5% for equation solution when an in-core solver is used. When an out-of-core solver is used, the computational time for equation solution is increased by 116%. This 116% increase in computational time is due to the additional reading/writing operations associated with the storage of the unsymmetric system matrix in the hard disk. These estimates are, however, only representative and may vary depending on how the storage is handled in the particular implementation in question.

Towards extension of the unsymmetric formulation to other applications, the first problem considered was geometric nonlinear static analysis. Both US-QUAD8 and US-HEXA20 elements have been developed for the purpose. Conceptually, the element formulation is very similar to that in linear analyses. However, in geometric nonlinear analyses, the element formulation involves two additional aspects *viz.*, the geometric stiffness matrix and the deformation gradient tensor. The derivation of the geometric stiffness matrix is straightforward with the application of *Criterion 1* and *Criterion 2* for

the selection of shape functions for the virtual and trial displacement fields, respectively. The deformation gradient tensor appearing in the trial displacement model, however, can be computed using either metric or isoparametric shape functions. Element formulations using either of these shape functions have been found to yield similar performances. Nevertheless, for the present work, the metric shape functions have been used. Results from the benchmark problems have revealed that the unsymmetric formulation is capable of distortion tolerant performances for geometric nonlinear analyses. The accuracies of US-QUAD8 and QUAD9 have been observed to be comparable for meshes with angular distortions. The accuracy of US-QUAD8, however, has been observed to be superior to that of QUAD8 and QUAD9 in the presence of curved-edge and mid-side node distortions. In terms of computational time, US-QUAD8 has been observed to be computationally more expensive compared to QUAD8 and QUAD9. However, an assessment of element's effectiveness based on the product of relative error and computational time, (ET), has revealed that US-QUAD8 is comparable with QUAD9 in the presence of angular distortions and superior to both QUAD8 and QUAD9 for curved-edge and mid-side node distortions.

As a next effort to extend the unsymmetric formulation to other applications, the problem of modeling incompressible hyperelastic materials was considered. A straightforward extension of the displacement-based unsymmetric formulation to incompressible hyperelastic materials was unsuccessful. Investigations have revealed that the element suffers from volumetric locking. To remove volumetric locking, an implementation of the unsymmetric formulation in conjunction with the mixed-formulations has been found to

be necessary. Mixed-formulations use independent interpolation for the displacement and pressure fields. The proposed unsymmetric-mixed formulation was implemented for a 9-node quadrilateral element with three independent pressure degrees of freedom. Discretization of the principle of virtual work has been observed to be somewhat similar to that in geometric nonlinear analyses. However, the discontinuity in the pressure interpolations poses difficulty in satisfying *Criterion 1* with regard to discretization of the incompressibility constraint. Results from benchmark problems have shown that the unsymmetric-mixed formulation, like its symmetric counterpart, is effective in removing volumetric locking. However, in view of the discontinuous pressure across element boundaries, US-QUAD9/3 has been observed to exhibit inferior performance compared to QUAD9/3 for certain problems, in particular for element meshes with angular distortions. However, the performance of US-QUAD9/3 has been observed to be superior compared to QUAD9/3 for curved-edge and mid-side node distortions.

One of the major drawbacks of the unsymmetric formulation is that the resulting element matrix is unsymmetric. This leads to an unsymmetric system matrix, and its solution requires an unsymmetric equation solver. A method to circumvent the necessity of an unsymmetric equation solver while still matching the distortion tolerant performance of the unsymmetric formulation has been explored. This led to the development of an iterative correction algorithm for the classical isoparametric elements. Conceptually, the iterative corrections transfer the effect of mesh distortions to the load vector and thereby, progressively improve the solution's accuracy with successive iterations. However, in its maiden version, the iterative corrections have been observed to converge rather slowly

for some test problems. To speed up the iterations, an acceleration scheme based on an optimum combination of the current and previous iteration vector has been implemented and has been observed to be effective in reducing the number of iterations. A study on the mathematical condition for the convergence of the corrective iterations shows that the algorithm will diverge when the spectral radius of the iteration matrix is larger than unity, i.e., $\rho([\mathbf{K}_s^{-1}(\mathbf{K}_s - \mathbf{K}_u)]) > 1$. Although the iterative corrections require more computational time than QUAD8, its computational effectiveness, measured as the product of relative error and computational time has been observed to be superior for distorted meshes.

As a last problem for this thesis, an investigation into two drawbacks of the unsymmetric formulation have been taken up, *viz.*, the rotational frame dependent behavior and the possibility of singularities of the interpolation ($\mathbf{P}$) matrix. Detailed investigations have been performed on some instances where the unsymmetric formulation exhibits rotational frame dependence. It has been verified that the unsymmetric formulation does not exhibit any frame dependent behavior when the exact displacement field involves a complete quadratic field, i.e., only the first six monomial terms $1, x, y, x^2, xy$ and y^2 . The unsymmetric formulation, however, exhibits rotational frame dependence when the exact displacement field involves the incomplete monomial terms of the third order, x^2y and xy^2 , and the higher order monomial terms, such as x^3, y^3, x^4, x^3y , etc., that are not present in the assumed displacement field (Equation (3.15)). A co-rotational implementation of the unsymmetric element has been proposed to avoid the rotational frame dependent behavior of the element.

In addition to its rotational frame dependent behavior, the performance of the unsymmetric formulation has been found to rely on condition of the $\mathbf{P}$ -matrix. Whenever the $\mathbf{P}$ -matrix becomes exactly singular, the algorithm has to abort. However, exact singularity is rarely experienced in practice. On the other hand, near singular $\mathbf{P}$ -matrices are practical possibilities and this leads to large solution errors. Detailed investigations into the origin of singularities of the $\mathbf{P}$ -matrix have been conducted for three typical element configurations. The study has revealed that there are theoretically unlimited possibilities, by which the $\mathbf{P}$ -matrix may become singular. Further investigations into the singularity of the $\mathbf{P}$ -matrix revealed a promising tip to handle near singularities: all the completeness conditions in Equation (3.14) are violated when the $\mathbf{P}$ -matrix becomes nearly singular. The violation of these conditions has been used to detect the presence of near singularities prior to element formulation. Upon detection, a coordinate transformation is then performed on the nodal coordinates of the element in order to rectify their defect. This technique has been observed to be effective in treating near singularities of $\mathbf{P}$ -matrices. Although an ideal solution to the problem is to avoid the very possibility of singularity of the $\mathbf{P}$ -matrix, such a solution is currently not obvious.

8.3. FUTURE WORK

There are several potential directions for further improvement with regard to the performance and application of the unsymmetric finite element formulation. In particular, this thesis has considered only combinations of isoparametric and metric shape functions to realize the present unsymmetric formulation. Since a successful unsymmetric formulation just requires the use of two different shape functions satisfying *Criterion 1*

and *Criterion 2* (Chapter 3), a vast variety of unsymmetric elements can be envisaged. Use of shape functions derived from area coordinates [93] in place of metric shape functions is one such possibility. This is expected to remove any possible singularities related to the **P**-matrix, yet, still yield distortion tolerant performance similar to the unsymmetric elements investigated in this thesis. Use of other shape functions may also be explored. Employing singularity functions in place of the metric interpolation so as to model crack problems in fracture mechanics is another possibility.

The distortion tolerance of unsymmetric elements investigated in this thesis relies on two different criteria to be satisfied by the virtual and trial displacement fields in the principle of virtual work. If a single set of shape functions can be found to satisfy both requirements of compatibility and higher order completeness simultaneously, it would then be possible to achieve a symmetric distortion-tolerant element. However, construction of such shape functions may be a challenging task.

The effectiveness of the proposed acceleration technique for the corrective iterations can be improved using a larger combination of previous iteration vectors. In this way, convergence may be possible with only a few iterations. Although the technique may be successful in accelerating the convergence, it is expected to be more restrictive in the sense of the maximum extent of distortion that the corrective iterations can handle. This trend has already been observed in a preliminary investigation. Hence, methods to circumvent this problem may have to be explored.

A direct application of the unsymmetric formulation to free vibration problems may possibly lead to complex eigenvalues because of its unsymmetric system matrix. Complex eigenvalues are difficult to handle/interpret in dynamic analyses. Thus, it may be useful to extend the implementation of the iterative corrections to free vibration problems. Complex eigenvalues may possibly be avoided through such an implementation. The actual implementation of the iterative corrections to free vibration problems may depend very much on the solution method used for the eigen-analysis, $K_U \phi_i = \lambda_i M_U \phi_i$. Where, K_U is the unsymmetric stiffness matrix, M_U is the corresponding unsymmetric mass matrix, ϕ_i is the eigenvector and λ_i is the eigenvalue corresponding to ϕ_i . If the inverse iteration method were used for instance, the iterative sequence proceeds according to

$$K_U x_{i+1} = \lambda_i M_U x_i \quad (8.1)$$

In order to implement the iterative correction method, K_U in Equation (8.1) may be decomposed into a symmetric and skew-symmetric part. Equation (8.1) then becomes

$$K_S x_{i+1} = \lambda_i M_U x_i - (K_U - K_S) x_i \quad (8.2)$$

where K_S is the symmetric stiffness matrix. Equation (8.2) can, in principle, be solved for the eigenvectors, x_i and corresponding eigenvalues, λ_i . The eigenvalues, λ_i are real in view of the symmetric stiffness matrix. The solution to Equation (8.2) may not be straightforward the right side of Equation (8.2) involves the as yet undetermined eigenvalue, λ_i .

Finally, it would be of interest to develop the unsymmetric elements for other applications such as 3-D solids, plates, shells and beams. Extension to plasticity is another possible direction. However, such an extension requires a good contact algorithm to enable solution of a broader range of problems.

8.4. CLOSURE

Investigated in this thesis is the performance of an unsymmetric finite element formulation in the presence of mesh distortions. The major effort has been to extend the original implementation of the unsymmetric element to nonlinear static analyses and to improve on some of the limitations of the element. It is hoped that this work will provide a strong foundation for further development of unsymmetric elements for other potential applications.

REFERENCES:

1. Fried I. 'Accuracy of complex finite elements'. *AIAA Journal* 1972; **10**: 347 – 349.
2. Henshell RD, Walters D, Warburton GB. 'A new family of curved elements for vibration and stability'. *Journal of Sound and Vibration* 1972; **20**, 381-397.
3. Fried I. 'Possible loss of accuracy in curved (isoparametric) finite elements - comments on a paper by Henshell, Walters and Warburton'. *Journal of Sound and Vibration* 1972; **23**, 507-510.
4. Henshell RD, Walters D, Warburton GB. 'On possible loss of accuracy in curved finite elements'. *Journal of Sound and Vibration* 1972; **23**, 510-513.
5. Rajendran S, Liew KM. 'A novel unsymmetric 8-node plane element immune to mesh distortion under a quadratic displacement field'. *International Journal for Numerical Methods in Engineering* 2003; **58**: 1713 – 1748.
6. Taig IC. 'Structural analysis by the matrix displacement method' *English Electric Aviation Limited Report S.O 17* 1961.
7. Taig IC, Kerr RI. 'Some problems in the discrete element representation of aircraft structures', in BF Veubeke (ed.), *Matrix Methods of Structural Analysis*, Pergamon Press, Oxford, 1964: 267 – 315.
8. Prathap G. 'The finite element method in structural mechanics'. Kluwer Academic Press: Dordrecht 1993.
9. Irons BM. 'Engineering application of numerical integration in stiffness methods', *AIAA Journal* 1966; **14**: 2035 – 2037.

10. Hughes TJR. 'Equivalence of finite elements for nearly incompressible elasticity'. *ASME Journal of Applied Mechanics* 1977; **44**: 181 – 183.
11. Wilson EL, Taylor RI, Doherty WP, Ghaboussi J. 'Incompatible displacement models' in Fenves SJ et al. (eds.), *Numerical and Computer Methods in Structural Mechanics*, Academic Press, New York, 1973; 43 – 57.
12. Chandra S, Prathap G. 'A field-consistent formulation for the eight-noded solid finite element'. *Computers & Structures* 1989; **33**(2): 345 – 355.
13. Dong YF, de Freitas JAT. 'An efficient eight node incompatible solid element with stress interpolation'. *Computers & Structures* 1992; **22**: 773 – 781.
14. Stricklin JA, Ho WS, Richardson EQ, Haisler WE. 'On isoparametric vs. linear strain triangular elements'. *International Journal for Numerical Methods in Engineering* 1977; **11**, 1041-1043.
15. Bäcklund J. 'On isoparametric elements'. *International Journal for Numerical Methods in Engineering* 1978; **12**, 731-732.
16. Gifford LN. 'More on distorted isoparametric elements'. *International Journal for Numerical Methods in Engineering* 1979; **14**, 290-291.
17. Zienkiewicz OC, Taylor RL, Too JM. 'Reduced integration technique in general analysis of plates and shells'. *International Journal for Numerical Methods in Engineering* 1971; **3**: 275-290.
18. Prathap G, Bhashyam GR. 'Reduced integration and shear flexible beam element'. *International Journal for Numerical Methods in Engineering* 1982; **18**: 195-210.

19. Flanagan DP, Belytschko T, 'A uniform strain hexahedron and quadrilateral with orthogonal hourglass control'. *International Journal for Numerical Methods in Engineering* 1981; **17**: 679 – 706.
20. Correction of article by DP Flanagan and Belytschko T. *International Journal for Numerical Methods in Engineering* 1983; **19**: 689 – 710.
21. Belytschko T, Ong JS-J, Liu WK, Kennedy JM. 'Hourglass control in linear and nonlinear problems'. *Computer Methods and Applied Mechanics in Engineering* 1984; **43**: 251 - 276.
22. Schulz JC. 'Finite element hourglassing control'. *International Journal for Numerical Methods in Engineering* 1985; **21**: 1039 - 1048.
23. Liu WK, Ong JS-J, Uras RA. 'Finite element stabilization matrices - a unification approach'. *Computer Methods and Applied Mechanics in Engineering* 1985; **53**: 13 - 46.
24. Jacquotte OP, Oden JT, Becker EB. 'Numerical control of the hourglass instability'. *International Journal for Numerical Methods in Engineering* 1986; **22**: 219 - 228.
25. Bachrach WE, Liu WK, Uras RA. 'A consolidation of various approaches in developing naturally based quadrilaterals'. *Computer Methods and Applied Mechanics in Engineering* 1986; **55**: 43 - 62.
26. Belytschko T, Binderman LP. 'Assumed strain stabilization of the 4-node quadrilateral with 1-point quadrature for nonlinear problems'. *Computer Methods and Applied Mechanics in Engineering* 1991; **88**: 311 - 340.

27. Naganarayana BP, Prathap G. 'Field consistency analysis of the 27-noded hexahedral elements for constrained media elasticity'. *Finite Elements in Analysis and Design* 1991; **9**: 149 – 168.
28. Taylor RL, Beresford PJ, Wilson EL. 'A non-conforming element for stress analysis' *International Journal for Numerical Methods in Engineering* 1976; **10**: 1211 – 1219.
29. Taylor RL, Simo JC, Zienkiewicz OC, Chan AHC. 'The patch test – a condition for assessing FEM convergence'. *International Journal for Numerical Methods in Engineering* 1986; **22**: 39 – 62.
30. Wu CC, Huang MG, Pian THH, 'Consistency condition and convergence criteria of incompatible elements: general formulation of incompatible functions and its application'. *Computers & Structures* 1987; **27**: 639 – 644.
31. Heuck U, Wriggers P. 'A formulation for the 4-node quadrilateral element'. *International Journal for Numerical Methods in Engineering* 1995; **38**: 3007 – 3037.
32. Pian THH, Sumihara K. 'Rational approach for assumed stress finite elements'. *International Journal for Numerical Methods in Engineering* 1984; **20**: 1685 – 1695.
33. Lautersztajn SN, Samuelsson A, 'A further discussion on four-node isoparametric quadrilaterals in plane bending'. *International Journal for Numerical Methods in Engineering* 2000; **47**: 129 – 140.

34. Huang YQ, Li QS. 'Four node incompatible plane axisymmetric elements with quadratic completeness in physical space'. *International Journal for Numerical Methods in Engineering* 2004; **61**: 1603 – 1624.
35. Wu CC, Huang YQ, Ramm E. 'A further study on incompatible models: revise-stiffness approach and completeness of trial functions'. *Computer Methods in Applied Mechanics and Engineering* 2001; **190**: 5923 – 5934.
36. Pian THH. 'Derivation of element stiffness matrices by assumed stress distributions'. *AIAA Journal* 1964; **2**: 1333 – 1336.
37. Pian THH, Tong P. 'Relation between incompatible displacement model and hybrid stress model'. *International Journal for Numerical Methods in Engineering* 1986; **22**: 173 – 181.
38. Pian THH, Wu CC. 'A rational approach for choosing stress terms for hybrid element formulation'. *International Journal for Numerical Methods in Engineering* 1988; **26**: 2331 – 2343.
39. Yuan KY, Pian THH. 'New strategy for assumed stress for 4-node hybrid-stress membrane element'. *International Journal for Numerical Methods in Engineering* 1993; **36**: 1747 – 1763.
40. Chen WJ, Cheung YK. 'A new approach for the hybrid element method'. *International Journal for Numerical Methods in Engineering* 1987; **24**: 1697 – 1709.
41. Sze KY. 'Efficient formulation of robust hybrid elements using orthogonal stress/strain interpolants and admissible matrix formulation'. *International Journal for Numerical Methods in Engineering* 1992; **35**: 1 – 20.

42. Sze KY, Ghali A. 'A two-fold solid element suiting thin-mesh analysis by admissible matrix formulation'. *Engineering Computations* 1992; **9**: 649 – 668.
43. Sze KY, Ghali A. 'A hexahedral element for plates, shells and beams by selective scaling'. *International Journal for Numerical Methods in Engineering* 1993; **36**: 1519 – 1540.
44. Sze KY, Fan H. 'An economical assumed stress brick element and its implementation'. *Finite Elements in Analysis and Design* 1996; **21**: 179 – 200.
45. Wu CC, Huang MG, Pian THH. 'Consistency condition and convergence criteria of incompatible elements: general formulation of incompatible functions and its application'. *Computers & Structures* 1987; **27**: 639 – 644.
46. Chen W, Cheung YK. 'A robust refined quadrilateral plane element'. *International Journal for Numerical Methods in Engineering* 1995; **38**: 649 – 666.
47. Cheung YK, Chen W. 'Refined hybrid method for plane isoparametric element using an orthogonal approach'. *Computers & Structures* 1992; **42**(5): 638 – 694.
48. Yeo ST, Lee BC. 'New stress assumption for hybrid stress elements and refined four-node plane and eight node brick elements'. *International Journal for Numerical Methods in Engineering* 1997; **40**: 2933 – 2952.
49. Zhou TX, Nie YF. 'Combined hybrid approach to finite element schemes with high performance'. *International Journal for Numerical Methods in Engineering* 2001; **51**: 181 – 202.
50. Irons BM. 'Numerical integration applied to finite element methods'. *Conference on Use of Digital Computers in Structural Engineering* 1966; University of Newcastle.

51. Brezzi F, Fortin M. *Mixed and Hybrid Finite Element Methods*. Springer: Berlin, 1991.
52. Xie XP, Zhou TX. 'Optimization of stress modes by energy compatibility'. *International Journal for Numerical Methods in Engineering* 2004; **59**: 293 – 313.
53. Wu CC, Cheung YK. 'On optimization approach of hybrid stress elements'. *Finite Elements in Analysis and Design* 1996; **26**: 111 – 128.
54. Wu CC, Cheung YK. 'Penalty equilibrating approach and an innovative formulation of the 4-noded hybrid stress elements'. *Communications in Numerical Methods in Engineering* 1996; **12**: 707 – 717.
55. Sze KY. 'On immunizing the 5-beta hybrid stress element models from 'trapezoidal locking' in practical analyses'. *International Journal for Numerical Methods in Engineering* 2000; **47**: 907 – 920.
56. Cao YP, Hu N, Lu J, Fukunaga H, Yao ZH. 'A 3D brick element based on Hu-Washizu variational principle for mesh distortion'. *International Journal for Numerical Methods in Engineering* 2002; **53**: 2529 – 2548.
57. Cao YP, Hu N, Fukunaga H, Lu J, Yao ZH. 'A highly accurate brick element based on a three-field variational principle for elasto-plastic analysis'. *Finite Elements in Analysis and Design* 2003; **39**: 1156 – 1171.
58. Cao YP, Hu N, Lu J, Fukunaga H, Yao ZH. 'A new scheme for designing the penalty factor in 3-D penalty equilibrating mixed elements'. *International Journal for Numerical Methods in Engineering* 2004; **20**: 455 – 464.
59. Xie XP. 'An accurate hybrid macro-element with linear displacements'. *Communications in Numerical Methods in Engineering* 2005; **21**: 1 – 12.

60. Mijuca D. 'On hexahedral finite element HC8/27 in elasticity', *Computational Mechanics* 2004; **33**(6): 466 – 480.
61. Mirza FA, Olson MD. 'The mixed finite element in plane elasticity', *International Journal for Numerical Methods in Engineering* 1980; **15**: 273 – 289.
62. Berkovic M, Draškovic Z. 'On essential mechanical boundary conditions in two field finite element approximations'. *Computer Methods and Applied Mechanics in Engineering* 1991; **91**: 1339 – 1355.
63. Mijuca D, Berkovic M. 'On the efficiency of the primal-mixed finite element scheme'. *Advanced Computer and Structural Mechanics* 1998; Civil-Comp Press: 61 – 69.
64. Berkovic M, Mijuca D. 'On the main properties of primal-mixed finite element formulation'. *Facta Universitatis Series Mechanics, Automatic Control and Robotics* 1999; **2**(9): 903 – 920.
65. Mijuca D. 'On the reliability of the new finite element HC8/27'. *Facta Universitatis Series Mechanics, Automatic Control and Robotics* 2002; **3**(12): 393 – 404.
66. Simo JC, Hughes TJR. 'On the variational foundations of assumed strain methods'. *ASME Journal of Applied Mechanics* 1986; **53**: 51 -54.
67. MacNeal RH. 'Derivation of element stiffness matrices by assumed strain distribution'. *Nuclear Engineering and Design* 1982; **70**: 3 - 12.
68. Simo JC, Rifai MS. 'A class of mixed assumed strain methods and the method of incompatible modes'. *International Journal for Numerical Methods in Engineering* 1990; **29**: 1595 - 1638.

69. Stolarski HK, Chen YI. 'Assumed strain formulation for the four-node quadrilateral with improved in-plane bending behavior'. *International Journal for Numerical Methods in Engineering* 1995; **38**: 1287 – 1305.
70. Korelc J, Wriggers P. 'Improved enhanced strain four-node element with Taylor expansion of the shape functions'. *International Journal for Numerical Methods in Engineering* 1997; **40**: 407 – 421.
71. Allman DJ. 'A compatible triangular element including vertex rotations for plane elasticity analysis'. *Computers & Structures* 1984; **19**: 1 – 8.
72. Cook RD. 'A plane hybrid element with rotational d.o.f. and adjustable stiffness'. *International Journal for Numerical Methods in Engineering* 1987; **24**: 1499 – 1508.
73. Carpenter N, Stolarski H, Belytschko T. 'A flat triangular shell element with improved membrane interpolation'. *Communications in Applied Numerical Methods* 1985; **1**: 161 – 168.
74. Tocher JL, Hartz BJ. 'Higher order finite element for plane stress'. *ASCE Journal of Engineering Mechanis* 1967; **93**(8): 5402 -5415.
75. Allman DJ. 'A quadrilateral finite element including vertex rotations for plane elasticity analysis'. *International Journal for Numerical Methods in Engineering* 1988; **26**: 717 - 730.
76. MacNeal RH, Harder RL. 'A refined four-noded membrane element with rotational degrees of freedom'. *Computers & Structures* 1988; **28**(1): 75 - 88.

77. Aminpour MA. 'An assumed-stress hybrid 4-node shell element with drilling degrees of freedom'. *International Journal for Numerical Methods in Engineering* 1992; **33**: 19 - 38.
78. Yunus SM, Saigal S, Cook RD. 'On improved hybrid finite elements with rotational degrees of freedom'. *International Journal for Numerical Methods in Engineering* 1989; **28**: 785 - 800.
79. Sze KY, Chen W, Cheung YK. 'An efficient quadrilateral plane element with drilling degrees of freedom using orthogonal stress modes'. *Computers & Structures* 1992; **42**: 695 - 705.
80. Cannarozzi AA, Cannarozzi M. 'A displacement scheme with drilling degrees of freedom for plane elements'. *International Journal for Numerical Methods in Engineering* 1995; **38**: 3433 - 3452.
81. Choi CK, Chung KY, Lee TY. 'A direct modification method for strains de to non-conforming modes'. *Structural Engineering and Mechanics* 2001; **11**(3): 325 – 340.
82. Choi CK, Lee TY, Chung KY. 'Direct modification for non-conforming elements with drilling DOF'. *International Journal for Numerical Methods in Engineering* 2002; **55**: 1463 – 1476.
83. Ibrahimbegovic A, Taylor RL, Wilson EL. 'A robust quadrilateral membrane element with rotational degrees of freedom'. *International Journal for Numerical Methods in Engineering* 1990; **30**: 445 – 457.

84. Ibrahimbegovic A, Wilson EL. 'Thick shell and solid finite elements with independent rotation fields'. *International Journal for Numerical Methods in Engineering* 1991; **31**: 1393 – 1414.
85. Iura M, Atluri SN. 'Formulation of a membrane element with drilling degrees of freedom'. *Computational Mechanics* 1992; **9**: 417 – 428.
86. Venkatesh DN, Shrinivasa U. 'Generation of an eight node brick element using Papcovitch-Neuber functions'. *Computers & Structures* 1995; **54**(6): 1077 – 1084.
87. Timoshenko SP, Goodier JN. '*Theory of Elasticity*'. McGraw-Hill: New York 1951.
88. Bassaya K, Bhattacharya K, Shrinivasa U. 'Eight-node brick PN340 represents constant stress field exactly'. *Computers & Structures* 2000; **74**: 441 – 460.
89. Bassaya K, Shrinivasa U. 'A 14-node brick element, PN5X1, exactly representing linear stress fields'. *Computers & Structures* 2000; **74**: 147 – 165.
90. Teixeira de Freitas JA, Cismaşiu C. 'Numerical implementation of hybrid-Trefftz displacement elements'. *Computers and Structures* 1999; **73**: 207 – 225.
91. Teixeira de Freitas JA, Bussamra FLS. 'Three-dimensional hybrid-Trefftz stress elements'. *International Journal for Numerical Methods in Engineering* 2000; **47**: 927 – 950.
92. Long YQ, Li JX, Long ZF, Cen S. 'Area co-ordinates used in quadrilateral elements'. *Communications in Numerical Methods in Engineering* 1999; **15**(8): 533 – 545.

93. Soh AK, Long YQ, Cen S. 'Development of eight-node quadrilateral membrane elements using the area coordinates method'. *Computational Mechanics* 2000; **25**: 376 – 384.
94. Chen XM, Cen S, Long Y, Yao ZH. 'Membrane elements insensitive to distortion using the quadrilateral area coordinate method'. *Computers & Structures* 2004; **82**: 35 – 54.
95. MacNeal RH. 'A theorem regarding the locking of tapered four-noded membrane elements'. *International Journal for Numerical Methods in Engineering* 1987; **24**: 1793 – 1799.
96. Gingold RA, Monaghan JJ. 'Smoothed particle hydrodynamics - Theory and application to non-spherical stars'. *Royal Astronomical Society, Monthly Notices* 1977; **181**: 375 – 389.
97. Belytschko T, Gu L, Lu YY. 'Element free Galerkin methods'. *International Journal for Numerical Methods in Engineering* 1994; **37**: 229 – 256.
98. Liu WK, Jun S, Zhang YF. 'Reproducing kernel particle methods'. *International Journal for Numerical Methods in Engineering* 1996; **20**: 1081 – 1106.
99. Alturi SN, Zhu TL. 'The meshless local Petrov-Galerkin (MLPG) approach for solving problems in elasto-statics'. *Computational Mechanics* 2000; **25**: 169 – 179.
100. De S, Bathe KJ. 'The method of finite spheres'. *Computational Mechanics* 2000; **25**: 329 – 345.
101. Liu GR, Gu YT. 'A point interpolation method for two-dimensional solids'. *International Journal for Numerical Methods in Engineering* 2001; **50**: 937 – 951.

102. Lu YY, Belytschko T, Gu L. 'A new implementation of the element free Galerkin method'. *Computer Methods in Applied Mechanics and Engineering* 1994; **113**: 397 – 414.
103. Kaljevic I, Saigal S. 'An improved element free Galerkin formulation'. *International Journal for Numerical Methods in Engineering* 1997; **40**: 2953 – 2974.
104. Alturi SN, Zhu TL. 'A new meshless local Petrov-Galerkin (MLPG) approach in computational mechanics'. *Computational Mechanics* 1998; **22**: 117 – 127.
105. Wang JG, Liu GR. 'A point interpolation meshless method based on radial basis functions'. *International Journal for Numerical Methods in Engineering* 2002; **54**: 1623 – 1648.
106. Krongauz Y, Belytschko T. 'Enforcement of essential boundary conditions in meshless approximations using finite elements'. *Computer Methods and Applied Mechanics in Engineering* 1997; **163**: 205 – 230.
107. Liu GR, Gu YT. 'Meshless local Petrov-Galerkin (MLPG) method in combination with finite element and boundary element approaches'. *Computational Mechanics* 2000; **26**: 536 – 546.
108. Huerta A, Fernández-Méndez S. 'Enrichment and coupling of the finite element and meshless methods'. *International Journal for Numerical Methods in Engineering* 2000; **48**: 1615 – 1636.
109. Strouboulis T, Copps K, Babuška I. 'The design and analysis of the Generalized Finite Element Method'. *Computer Methods and Applied Mechanics in Engineering* 2000; **181**: 43 – 69.

110. Strouboulis T, Copps K, Babuška I. 'The Generalized Finite Element Method'. *Computer Methods and Applied Mechanics in Engineering* 2001; **190**: 4081 – 4193.
111. Babuška I, Melenk JM. 'The partition of unity finite element method'. *International Journal for Numerical Methods in Engineering* 1997; **40**: 727 – 758.
112. Lee NS, Bathe KJ. 'Effects of element distortions on the performance of isoparametric elements'. *International Journal for Numerical Methods in Engineering* 1993; **36**: 3553 – 3576.
113. Rajendran S, Subramaniam S. 'Mesh distortion sensitivity of 8-node plane elasticity elements based on parametric, metric, parametric-metric, and metric-parametric formulations'. *Structural Engineering and Mechanics* 2004; **17**(6): 767 – 788.
114. Martini E, Selleri S, Pelosi G. '3-D FEM robustness to mesh distortions versus basis function order'. *Microwave and Optical Technology Letters* 2000; **27**: 17 – 20.
115. MacNeal RH, Harder RL. 'Eight nodes or nine?', *International Journal for Numerical Methods in Engineering* 1992; **33**: 1049 – 1058.
116. Kikuchi F, Okabe M, Fujio H. 'Modification of the 8-node serendipity element'. *Computer Methods and Applied Mechanics in Engineering* 1999; **179**: 91 – 109.
117. Rajendran S, Liew KM. 'Completeness requirements of shape functions for higher order elements'. *Structural Engineering and Mechanics* 2000; **10**(2): 93 – 110.

118. Belytschko T, Liu WK, Moran B. '*Nonlinear Finite Elements for Continua and Structures*'. John Wiley & Sons Ltd: New York 2000.
119. Hughes TJR. '*The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*'. Prentice Hall: New Jersey 1987.
120. Irons BM. 'Numerical integration applied to finite element methods'. *Conference on Use of Digital Computers in Structural Engineering 1966*; University of Newcastle.
121. Fraeijs de Veubeke B. 'Variational principles and the patch test'. *International Journal for Numerical Methods in Engineering* 1974; **8**: 783 – 801.
122. Sander G, Beckers P. 'The influence of the choice connectors in the finite element method'. *International Journal of Numerical Methods in Engineering* 1977; **11**: 1491 – 1505.
123. de Arantes Oliveira ER. 'The patch test and the general convergence criteria of the finite element method'. *International Journal of Solids and Structures* 1977; **13**: 159 – 178.
124. Zienkiewicz OC, Taylor RL. 'The finite element patch test revisited. A computer test for convergence, validation and error estimators'. *Computer Methods in Applied Mechanics and Engineering* 1997; **149**: 223 – 254.
125. Zienkiewicz OC, Taylor RL. '*The Finite Element Method Vol. 1: Basic Formulation and Linear Problems (4th ed.)*'. McGraw-Hill Book Company: London 1989.
126. MacNeal RH, Harder RL. 'A proposed standard set of problems to test finite element accuracy'. *Finite Elements in Analysis and Design* 1985; **1**: 3 – 20.
127. Cheung YK, Chen W. 'Isoparametric hybrid hexahedral elements for three-dimensional stress analysis'. *International Journal for Numerical Methods in Engineering* 1988; **26**: 677 – 693.
128. Hellen TK. 'A finite element mesh quality assessment program'. In *Accuracy, Reliability and Training in FEM Technology*, Robinson J et al. (eds),

- Proceedings of the Fourth World Congress on Finite Element Methods*, Interlaken, Switzerland, 1984.
129. Mallikarjuna RK, Shrinivasa U. 'A set of pathological tests to validate new finite elements'. *Sadhana* 2001; **26**(6): 549 – 590.
130. Robinson J. 'Early FEM user project-single element test for aspect ratio sensitivity of solids (Part I). *Finite Element News* 1986; **1**: 26 – 32.
131. Ugural AC. '*Stresses in Plates and Shells*'. McGraw-Hill: New York 1981.
132. Kohnke PC. 'Large deflection analysis of frame structures by fictitious forces'. *International Journal for Numerical Methods in Engineering* 1978; **12**: 1279-1294.
133. Fertis DG, Afonta AO. 'Pseudolinear and equivalent systems for large deflections of members'. *ASCE Journal of Engineering Mechanics* 1989; **115** (11), 2440.
134. Fertis DG, '*Nonlinear mechanics*', CRC Press: Florida 1993.
135. Chia CY. *Non-linear Analyses of Plates*. McGraw-Hill: New York, 1980.
136. Zhang YX, Cheung YK. 'A refined non-linear non-conforming triangular plate/shell element'. *International Journal for Numerical Methods in Engineering* 2003; **56**: 2387 – 2408.
137. Pica A, Wood RD, Hinton E. 'Finite element analysis of geometrically nonlinear plate behavior using a Mindlin formulation'. *Computers & Structures* 1980; **11**: 203 – 215.
138. Hsiao KM. 'Nonlinear analysis of general shell structures by flat triangular shell elements'. *Computers & Structures* 1987; **25**: 665 – 675.
139. Meek JL, Tan JS. 'Instability analysis of thin plates and arbitrary shells using a faceted shell element with Loof nodes'. *Computer Methods in Applied Mechanics and Engineering* 1986; **57**: 147 – 170.

140. Surana KS. 'Geometrically nonlinear formulation for the curved shell elements'. *International Journal for Numerical Methods in Engineering* 1983; **19**: 581 – 615.
141. Leicester RH. 'Finite deformations of shallow shells'. *Journal of the Engineering Mechanics Division (ASCE)* 1968; **94**:1409 – 1423.
142. Penn RW. 'Volume changes accompanying the extension of rubber'. *Trans. Soc. Rheology* 1970; **14**, 509 – 517.
143. Sussman T, Bathe KJ. 'A finite element formulation for nonlinear incompressible elastic and inelastic analysis'. *Computers & Structures* 1987; **26(1/2)**, 357-409.
144. Prathap G. '*Finite element analysis: Principles and Practice*'. National Aerospace Laboratories, Special Publication, Bangalore, India: 1993.
145. Galada MS. 'Numerical solutions of nonlinear problems of continua-II. Survey of incompressibility constraints and software aspects'. *Computers & Structures* 1986; **22(5)**, 841-855.
146. Herrmann LR. 'Elasticity equations for incompressible and nearly incompressible materials by a variational theorem'. *AIAA Journal* 1965; **3**, 1896 – 1900.
147. Oden JT, Key JE. 'Numerical analysis of finite axisymmetric deformations of incompressible elastic solids of revolution'. *International Journal of Solids and Structures* 1970; **6**, 497 – 518.
148. Nagtegaal JC, Parks DM, Rice JR. 'On numerically accurate finite element solutions in the fully plastic range'. *Computational Methods in Applied Mechanics and Engineering* 1974; **4**, 153 – 177.

149. Taylor RL, Pister KS, Herrmann LR. 'On a variational theorem for incompressible and nearly incompressible orthotropic elasticity'. *International Journal of Solids and Structures* 1968; **4**, 875 – 883.
150. Häggblad B and Sundberg JA. 'Large strain solutions of rubber components'. *Computers and Structures* 1983; **5-6**, 835-845.
151. Brádaign CMÓ and Pipes RB. 'A finite element formulation for highly anisotropic incompressible elastic solids'. *Int. J. Numer. Methods Eng* 1992; **33**, 1573-1596.
152. Srinivasan R and Perucchio R. 'Finite element analysis of anisotropic nonlinear incompressible elastic solids by a mixed model'. *Int. J. Numer. Methods Eng* 1994; **37**, 3075-3092.
153. Brink U and Stein E. 'On some mixed finite element methods for incompressible and nearly incompressible finite elasticity'. *Computational Mechanics* 1996; **19**, 105-119.
154. Belytschko T and Bindeman LP. 'Assumed strain stabilization of the 4-node quadrilateral with 1-point quadrature for nonlinear analysis'. *Comput. Methods Appl. Mech. Eng* 1991; **88**, 311-340.
155. Crisfield MA, Moita GF, Jelenić G and Lyons LPR, 'Enhanced lower-order element formulations for large strains'. *Computational. Mechanics* 1995; **17(1-2)**, 62-73.
156. Pantuso D and Bathe KJ. 'On the stability of mixed finite elements in large strain analysis of incompressible solids'. *Finite Elements in Analysis and Design* 1997; **28**, 83-104.

157. Chapelle D and Bathe KJ. 'The inf-sup test'. *Computers & Structures* 1993; **47**, 537-545.
158. Rivlin RS and Saunders DW. 'Large elastic deformations of isotropic materials VII. Experiments on the deformation of rubber'. *Philosophical Transaction of the Royal Society of London Series A* 1951; **243**, 251-288.
159. Ogden RW. 'Large deformation isotropic elasticity - on the correlation of theory and experiment of incompressible rubber like solids'. *Proceedings of Royal Society of London Series A* 1972; **326**, 565-584.
160. Arruda EM and Boyce MC. 'A three dimensional constitutive model for the large stretch behavior of rubber elastic materials'. *Journal of Mechanics and Physics of Solids* 1993; **41**, 398-412.
161. Hartmann S and Neff P. 'Polyconvexity of generalized polynomial-type hyperelastic strain energy functions for near-incompressibility'. *International Journal of Solids and Structures* 2003; **40**, 2767-2791.
162. Bathe KJ. '*Finite Element Procedures*'. Prentice Hall, Englewood Cliffs, New Jersey: 1996.
163. Becker EB, Carey GF and Oden JT. '*Finite Elements: An Introduction, Volume I*'. Prentice Hall, New Jersey, 1981.
164. Auricchio F, Beirão d Veiga L, Lovadina C, Reali A. 'An analysis of some mixed-enhanced finite element for plane linear elasticity'. *Computer Methods in Applied Mechanics and Engineering* 2004; **194 (27 – 29)**, 2947 – 2968.
165. Parisch H. 'Efficient non-linear finite element shell formulation involving large strains'. *Engineering Computations* 1986; **3**, 121 – 128.

166. Betsch P, Gruttmann F, Stein E. 'A 4-node shell element for the implementation of general hyperelastic 3D-elasticity at finite strains'. *Computer Methods in Applied Mechanics and Engineering* 1996; **130**, 57 – 79.
167. Oden JT. '*Finite Elements of Nonlinear Continua*'. McGraw-Hill Inc, New York, 1972.
168. Seidel L. 'Über ein verfahren die gleichungen auf welche die methode der kleinsten quadrate führt, sowie lineare gleichungen überhaupt durch successive annäherung aufzulösen'. *Abhandlungen Bayerische Akademie der Wissenschaften* 1875; **11**. 81-108.
169. Clough RW, Wilson EL. 'Stress analysis of a gravity dam by the finite element method'. *Proceedings, Symposium on the Use of Computers in Civil Engineering, Laboratorio Nacional de Engenharia Civil, Lisbon, Portugal, Oct. 1962*; 29.1-20.22.
170. Hestenes MR, Stiefel E. 'Methods of conjugate gradients for solving linear systems'. *Journal of Research of the National Bureau of Standards* 1952; **49**, 409-436
171. Reid JK. 'On the method of conjugate gradients for the solution of large sparse systems of linear equations'. *Conference on Large Sets of Linear Equations, St. Catherine's College, Oxford, 1970*; 231-254.
172. Arora JS. 'Survey of Structural reanalysis techniques'. *Journal of the Structural Division ASCE* 1976; **102**, 783-802.

173. Yang X, Sarkar TK, Arvas E. A survey of conjugate gradient algorithms for solution of extreme eigen-problems of a symmetric matrix. *IEEE Transactions on Acoustics, Speech and Signal Processing* 1989; **37**, 1550-1556.
174. Feng TY, Owen DRJ. Conjugate gradient methods for solving the smallest eigenpair of large symmetric eigenvalue problems. *International Journal for Numerical Methods in Engineering* 1996; **39**, 2209-2229.
175. Kirsch U, Kocvara M, Zowe, J. Accurate reanalysis of structures by a preconditioned conjugate gradient method. *International Journal for Numerical Methods in Engineering* 2002; **55**, 233-251.
176. Reddy JN. *Applied Functional Analysis and Variational Methods in Engineering*. McGraw-Hill Book Company: New York, 1970.
177. Rajendran S, Raghavan MR, Narasimhan MV. 'Improving the convergence rate of the inverse iteration method'. *Computers & Structures* 1995; **55**. 709-715.
178. MacNeal RH. *Finite Elements: Their Design and Performance*. Marcel Dekker, Inc: New York 1994.
179. Kidger DJ. 'The fourteen-node brick element'. *2nd International Robert J. Melosh Medal Paper Competition 1990*, Duke University, NC.
180. Irons BM, Ahmad S. *Techniques of Finite Elements*. E. Hoorwood, Halsted Press: New York 1980.
181. Sze KY, Chan WK, Pian THH. 'An eight-node hybrid-stress solid-shell element for geometric non-linear analysis of elastic shells'. *International Journal for Numerical Methods in Engineering* 2002; **55**: 853-878.

182. Gruttmann F, Stein E, Wriggers P. 'Theory and numerics of thin elastic shells with finite rotation'. *Ing. – Archiv.* 1989; **59**: 54-67.
183. Crisfield MA. *Non-linear finite element analysis of solids and structures. Vol 1.* Chichester; Wiley: New York 1991.
184. Wriggers P, Reese S. 'A note on enhanced strain methods for large deformations', *Computer Methods in Applied Mechanics and Engineering* 1996; **135**: 201-209.
185. Cheung YK, Chen W. 'Refined nine-parameter triangular thin plate bending element by using refined direct stiffness methods', *International Journal for Numerical Methods for Engineering* 1995; **38**: 283-298.
186. Bathe KJ, Ho LW. 'A simple and effective element for analysis of general shell structures'. *Computers & Structures* 1981; **13**: 673-681.
187. Pian THH. 'Finite elements based on consistently assumed stresses and displacements', *Finite Elements in Analysis and Design* 1992; 1985; **1**: 131-140.

APPENDIX I

In all the nonlinear problems solved throughout this thesis, the standard Newton Raphson iteration algorithm has been adopted. The iterative procedures involved are described as follows:

1. Set up initial displacement, $\mathbf{U}_0$, usually $\mathbf{U}_0 = \mathbf{0}$
2. Initiate loop over number of loadsteps
 - a. Form the element stiffness matrix, $\mathbf{K}^{(e)}$ and assemble the global stiffness matrix, $\mathbf{K}$
 - b. Solve the equilibrium equation $\mathbf{K}\Delta\mathbf{U} = \mathbf{F}^\dagger$, for the predictor displacements, $\Delta\mathbf{U}$
 - c. Update the total displacements, $\mathbf{U} = \mathbf{U}_0 + \Delta\mathbf{U}$
 - d. Initiate loop over equilibrium iterations
 - i. Compute the internal forces and obtain the residual forces[‡]
 - ii. Check for convergence of equilibrium iterations^{††}
 - iii. If convergence criterion is met, go to Step (e)
 - iv. If the full Newton Raphson iteration scheme is used, re-form the element stiffness matrix, $\mathbf{K}^{(e)}$ and re-assemble the global stiffness matrix, $\mathbf{K}$ using the current displacements, $\mathbf{U}$
 - v. Solve the equilibrium equation $\mathbf{K}\mathbf{U}^{\text{iter}} = \mathbf{R}$, where $\mathbf{R}$ is the residual force
 - vi. Update the total displacements, $\mathbf{U} = \mathbf{U} + \mathbf{U}^{\text{iter}}$

[†] $\mathbf{F}$ here is representative of the current load increment, which may be due to applied nodal loads, body loads or surface forces

[‡] The residual forces are obtained as in Equation (4.24) for geometric nonlinear analyses in Chapter 4 and as in Equation (5.44) for the finite strain problems in Chapter 5

^{††} An L2 norm of previous and current residual forces is used to check for convergence

- vii. Go to Step (i)
 - e. Terminate loop over equilibrium iterations
 - 3. Terminate loop over number of loadsteps if no more loadsteps are available, else return to Step (2)