Vibration of beams with delaminations and embedded piezoelectric sensors and actuators

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**Vibration of Beams with Delaminations
and Embedded Piezoelectric
Sensors and Actuators**

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Abstract

Fiber-reinforced composite laminates are increasingly used in the construction of aerospace, civil, marine, automotive and other high performance structures due to their high specific stiffness and strength, excellent fatigue resistance, longer durability as compared to metallic structures, and ability to be tailored for specific applications. Delamination is one of the important failure mechanisms in composite laminates. They may significantly reduce the stiffness and strength of the composites and affect the vibration behavior of the structures.

Smart structures are systems that incorporate particular functions such as sensing, processing and actuation. They have the ability to sense certain stimuli and respond in a controlled manner. Smart structures are important because of their relevance to structural health monitoring, structural vibration control and transportation engineering. Piezoceramic materials are commonly used as sensors and actuators. However, their brittle nature and low tensile strength limit their ability to conform to curved shapes, which can be remedied through the use of arrays of piezoelectric sensors and actuators embedded within the structure.

In this thesis, nondimensional parameters are introduced to study the vibration of bimaterial beams with a single delamination. The use of these parameters provides a better understanding on the effect of the bending stiffness and axial stiffness of the different layers on the vibration of delaminated bimaterial beams. Furthermore, it provides a physical understanding on the ‘constrained mode’ and ‘free mode’ assumptions that are used in delamination vibration. A relative slenderness ratio specific for delamination vibration is further introduced, which is shown to

significantly influence the natural frequency and mode shape of the beam. Global, mixed and local vibration modes occur depending upon the slenderness ratios of the delaminated beams. In addition, an analytical solution for the vibration of three-layer beams with double delaminations is presented. A relative slenderness ratio specific for the analytical solution is introduced, which is shown to dominate the vibration behavior of the beam. The lower bound and upper bound of the natural frequency are proposed by assuming totally ‘free’ and totally ‘constrained’ deformations of the delaminated layers. These bounds are easy to compute and provide useful approximations.

Analytical solutions for the free vibration of composite beams with two enveloping delaminations, two non-overlapping delaminations and two overlapping delaminations are presented. The delaminated beams are modeled as interconnected Euler-Bernoulli beams using the delaminations as their boundaries. The continuity and boundary conditions are satisfied between adjoining beams. Both the ‘constrained mode’ and the ‘free mode’ assumptions are used. A parametric study is performed to investigate the influences of the lengths and locations of the delaminations on the natural frequencies of the composite beams and their corresponding mode shapes. The present study provides useful information for the multiple delamination problems.

A mathematical model for the vibration of beams with embedded piezoelectric sensors and actuators is presented. The piezoelectric sensors and actuators are analyzed as inhomogeneous inclusions in a non-piezoelectric matrix by using Eshelby equivalent inclusion method. The natural frequency of the beam is determined from the variational principle in Rayleigh quotient form, which is expressed as functions of the elastic strain energy and dielectric energy of the piezoelectric inclusions. These

energies are derived using Mura's formulation for inhomogeneous inclusions. In addition, the Euler-Bernoulli beam theory and Rayleigh-Ritz approximation technique are used in the present analysis. A parametric study is conducted to investigate the influences of the volume fraction, aspect ratio and location of the piezoelectric inclusions on the natural frequency of the beam. The influences of the energies due to electro-mechanical coupling of the actuators and mechanical-electrical coupling of the sensors on the natural frequency are also studied.

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Vibration of Beams with Delaminations

A_i	Cross-sectional area of the i th sub-beam
a	Length of delamination
a_o	Length of overlap between two delaminations
a_t	Total axial length of two overlapping delaminations
b	Width of the beam
C_i, S_i, CH_i, SH_i	Arbitrary constants
d	Distance from the center of the beam to the center of the delamination
D_i	Reduced bending stiffness of the i -th composite beam
E	Young's modulus of an isotropic material
E_{11}	Longitudinal Young's modulus of a fiber-reinforced composite lamina
E_{22}	Transverse Young's modulus of a fiber-reinforced composite lamina
EI_i	Bending stiffness i th sub-beam
E_R	Modulus ratio
$\overline{EH}_2, \overline{EH}_3, \overline{EH}_4$	Normalized axial stiffness
$\overline{EI}_2, \overline{EI}_3, \overline{EI}_4$	Normalized bending stiffness
G_{12}	In-plane shear modulus fiber-reinforced composite lamina
H_i	Thickness of the i th sub-beam
H_R	Thickness ratio
L	Length of the beam
L_i	Length of the i th sub-beam
$\lambda_i = \left(\frac{\omega^2 \rho_i A_i}{EI_i} \right)^{\frac{1}{4}} L$	Non-dimensional frequency of the i th sub-beam

P_i	Axial force of the i th sub-beam
ϕ	Angle orientation of the fibers in the lamina
R, R_H	Relative slenderness ratio
ρ_i	Mass density of the i th sub-beam
ν	Poisson's ratio for an isotropic material
ν_{12}	Longitudinal Poisson's ratio of a fiber-reinforced composite lamina
ν_{21}	Longitudinal Poisson's ratio of a fiber-reinforced composite lamina
W_i	Displacement mode shape of the i th sub-beam
ω	Natural frequency of the delaminated beam
ω_0	Natural frequency of the undelaminated beam

Vibration of Beams with Embedded Piezoelectric Sensors and Actuators

Piezoelectric Body

C_{ijmn}	Elastic stiffness tensor
D_i	Electric displacement vector
e_{nij}	Piezoelectric tensor
E_i	Electric field vector
ε_{ij}	Strain tensor
κ_{in}	Permittivity tensor
n_j	Outward pointing unit normal vector
ϕ	Electric potential
u_i	Displacement
ω	Natural frequency

σ_{ij} Stress tensor

Non-piezoelectric matrix with piezoelectric inclusion

C_{ijmn} Elastic moduli of the non-piezoelectric matrix (host beam)
containing the piezoelectric inclusion

C_{ijmn}^* Elastic moduli of the piezoelectric inclusion

d_{kmn} Free-expansion of the piezoelectric actuator

D_i^0 Far fields electric displacement

E_n^0 Far fields electrical field

E_n^p Eigen-electricfield due to the mechanical-electrical coupling

E_n^* Fictitious eigen-electric field

E_n^{**} Total eigen-electric field

ε_{mn}^0 Strain due to the far field stress

ε_{mn} Disturbance strains due to the presence of inhomogeneous
inclusion

ε_{mn}^p Eigenstrain actuation

ε_{mn}^* Uniform fictitious eigenstrain

ε_{mn}^{**} Total eigenstrain

I_{ln}, I_{klmn} Identity tensors

ϕ^0 Far field electric potential

s_{nb} Electric Eshelby tensor

S_{mnab} Elastic Eshelby tensor

σ_{ij} Disturbance stress due to the presence of inhomogeneous inclusion

$u_{i,j}^0$ Displacement due to the far field stress

W_e Elastic strain energy

W_d Dielectric energy

Natural frequency of the beam

a_2, a_3 Dimensions of the piezoelectric ellipsoidal inclusion

e Thickness of the beam

E_H Young's modulus of the host beam

L Length of the beam

n number of sensors/actuators

v_f volume fraction of the piezoelectric inclusion

ω Fundamental frequency of the beam with piezoelectric sensors and actuators

ω_0 Fundamental frequency of the beam without the actuation effects

z Distance between the neutral axis of the beam and the piezoelectric inclusion along the thickness of the beam

Chapter 1

Introduction

This thesis presents studies on the vibration of delaminated composite beams and the vibration of beams with embedded piezoelectric sensors and actuators. The motivation and relevance of these studies are presented in this chapter, together with the objectives and outline of the thesis.

1.1 Motivation

1.1.1 Vibration of delaminated composite beams

Fiber-reinforced composite laminates are increasingly used in the construction of aerospace, civil, marine, automotive and other high performance structures due to their high specific stiffness and strength, excellent fatigue resistance, longer durability as compared to metallic structures, and ability to be tailored for specific applications. However, composites are prone to damage. Delamination is one of the important failure mechanisms in composite laminates due to their weak interlaminar strength (Garg, 1988). Delaminations may arise during fabrication (e.g. incomplete wetting, air entrapment) or during service (e.g. low velocity impact, bird strikes). Delaminations

may not be visible or barely visible on the surface, since they are embedded within the composite structures. However, they may significantly reduce the stiffness and strength of the structures (Tay, 2003). A reduction in the stiffness will affect some design parameters of the structures, such as the vibration parameters (e.g., natural frequency and mode shape). Delaminations reduce the natural frequency, as a direct result of the reduction of stiffness, which may cause resonance if the reduced frequency is close to the working frequency. It is important to understand these changes in the vibration parameters of the composites in a dynamic environment.

Understanding the changes in the vibration parameters of damage composites is also of particular interest to the community of vibration-based structural health monitoring (SHM). SHM, which has emerged as a well recognized field of technology, is defined as an autonomous system for the continuous monitoring, inspection and damage detection of a structure with minimum labor requirement (Kessler, 2001). The basic idea of the vibration-based SHM methods is that the presence of damage causes changes to the physical properties of the structure, such as mass, stiffness and damping. Thus, changes in the physical properties will cause detectable changes to the vibration parameters of the structure (Doebling et al., 1995, 1998; Salawu, 1997).

Although several works on the vibration of delaminated beams have been reported, there are issues that need to be addressed to be able to have a better understanding on the vibration delamination problem. The issues that this thesis will address are discussed in detail in Chapter 2. The study presented in this thesis aims to contribute to the technical community engaged in the analysis and design of composite beams and researchers in the field of vibration-based SHM through an investigation on the vibration of delaminated composite beams.

1.1.2 Vibration of beams with embedded piezoelectric sensors and actuators

Smart structures are systems that incorporate particular functions such as sensing, processing and actuation. They have the ability to sense certain stimuli and respond in a controlled manner (Chung, 2002). Smart structures are important because of their relevance to structural health monitoring, structural vibration control and transportation engineering. A primary focus in the research of smart structures is the use of piezoelectric materials, since these materials can function both as sensors and actuators (Tani et al., 1998).

A major problem in the dynamic operation of structures is undesirable vibrations (Mackerle, 2003). This is why vibration control and active damping are among the most studied areas using smart materials and structures (Chee, 1998; Cao et al., 1999). Extensive research has been done on the vibration control and suppression of structures, as evident from numerous review articles (see for example Rao and Sunar, 1994; Sunar and Rao, 1999; Ahmadian and DeGuillio, 2001; Wetherhold and Aldrahem, 2001; Irschik, 2002). An analysis of these review articles indicates that piezoceramic materials are widely used as sensors and/or actuators. They are either in the form of patches or lamina. The piezoelectric patches are either bonded to or embedded within the structures, whereas the piezoelectric lamina are stacked together with a substrate laminae to form a piezoelectric composite laminate (Chee, 1998).

The above reviews show that the use of piezoceramic PZT materials for structural actuation and sensing is a fairly well-developed area. However, there are several factors that limit the use of piezoceramic materials, such as their brittle nature and low tensile strength, therefore limiting their ability to conform to curved shapes, and the

large add-on mass associated with using typical lead-based piezoceramic (Williams et al., 2002).

The use of active piezoceramic composite actuators has been explored to overcome these limitations. The available types of active composites available commercially or under development at research institutes are: (a) 1–3 composites by Smart Materials Corp. (<http://www.smart-materials.com>); (b) Active fiber composite actuator developed by MIT (Bent, 1997), and; (c) Macro-fiber composite (MFC) actuators constructed at NASA Langley Center (Sodano, 2003; Sodano et al., 2004; Wilkie et al, 2000; Williams, 2004; Williams et al, 2002). An overview and comparison of these actuators can be found in the literature (Nelson, 2002; Williams et al, 2002). The piezoceramic composite actuators take advantage of the fact that lead zirconate titanate can be extruded into very thin wires. Many of these wires can be bonded together to an acrylic base and a flexible insulator. With the addition of copper electrodes, a significantly more durable and flexible sensor/actuator can be made. The flexible nature of the composite patch allows a better, simpler bond to curved surfaces.

The use of arrays of piezoelectric sensors and actuators embedded within the structure would also remedy the above mentioned limitations of piezoceramic materials. Due to their small size, these piezoelectric microdevices have the flexibility to conform to curved shapes, and they add little weight to the structure (Badcock and Birt, 2000). These piezoelectric sensors and actuators can also be tailored to achieve a particular smart structure design.

Few works have been reported for smart beams using arrays of piezoelectric sensors/actuators. This thesis proposes a mathematical model to study the vibration of beams with embedded arrays of piezoelectric sensors and actuators.

1.2 Objectives

1.2.1 New parameters for the vibration of delaminated multilayer beams

A study on the vibration of bimaterial beams with a single delamination and three-layer beams with two equal length delaminations is presented here. This study aims to:

1. introduce nondimensional parameters (axial stiffness ratio, bending stiffness ratio, normalized axial stiffness and normalized bending stiffness) to study the effect of the bending stiffness and axial stiffness of the different layers on the vibration of bimaterial beams with a single delamination;
2. study the ‘constrained mode’ and ‘free mode’ frequencies of the delaminated bimaterial beams by using the nondimensional parameters to provide a physical understanding on the ‘constrained mode’ and ‘free mode’ assumptions, which are used in the study of vibration of delaminated beams;
3. introduce a new parameter, relative slenderness ratio, specific for delamination vibration to be able to predict the delamination opening in the vibration mode shapes of the beams and to characterize the vibration mode shapes;
4. develop an analytical solution to study the vibration of three-layer beams with two equal length delaminations and to propose the lower bound and upper bound solutions of the natural frequency that can provide useful approximations;

1.2.2 Vibration of composite beams with multiple delaminations

A study on the vibration of fiber-reinforced composite beams with multiple delaminations is presented here. This study aims to:

1. develop analytical solutions for the vibration of composite beams with two enveloping delaminations, two non-overlapping delaminations, and two overlapping delaminations that can serve as benchmark solutions for future approximate/numerical analyses;
2. conduct a parametric study to investigate the influence of the sizes and the locations of the delaminations on the natural frequency and mode shape of the composite beams. This study can be used to optimize the design of composite beams and assess the suitability of using the natural frequency and mode shapes of the delaminated beam for damage detection.

1.2.3 Vibration of beams with embedded piezoelectric sensors and actuators

This study on the vibration of beams with embedded piezoelectric sensors and actuators aims to:

1. present a mathematical model for the vibration of beams with embedded arrays of piezoelectric sensors and actuators;
2. conduct a parametric study to investigate the influences of the volume, aspect ratio and location of the piezoelectric sensors/actuators on the natural frequency of the beam;
3. conduct a parametric study to investigate the influences of the energies due to the electromechanical coupling of the actuators and the mechanical-electrical coupling of the sensors on the natural frequency of the beam.

1.3 Thesis Outline

Following this introduction, Chapter 2 presents a background and literature survey, which covers topics related to this research, as well as the current level of understanding in the area of vibration of delaminated composite laminates. This chapter concludes by discussing the issues that this thesis will address.

In Chapter 3, nondimensional parameters are introduced to study the vibration of a bimaterial beam with a single delamination. A relative slenderness ratio, specific for the study of delamination vibration is further introduced. In addition, an analytical solution for three layer beams with two equal length delaminations is presented. The lower bound and upper bound solutions of the natural frequency are proposed.

Chapter 4 presents analytical solutions for the vibrations of composite beams with two enveloping delaminations, two non-overlapping delaminations, and two overlapping delaminations. A parametric study is carried out to investigate the influence of the lengths and locations of the delaminations on the natural frequency and mode shape of the composite beams.

In Chapter 5, a mathematical model for the vibration of beams with embedded piezoelectric sensors and actuators is presented. Parametric studies are conducted to investigate the influences of the volume, shape and location of the piezoelectric sensors/actuators on the natural frequency of the beam. The influences of the energies due to the coupling of the piezoelectric sensors/actuators on the natural frequency are also studied.

Finally, Chapter 6 presents the conclusions and publications resulting from the thesis. Recommendations are made for future works.

Chapter 2

Background and Literature Review

2.1 Introduction

This chapter presents a review of the literature, which covers the areas of vibration of delaminated composite laminates and smart structures using piezoelectric sensors and actuators, and provides a background for the present study. This chapter is outlined as follows. In Section 2.2 the available mathematical models for the vibration of delaminated composite laminates are reviewed and classified according to the fundamental assumptions and theories used. Section 2.3 presents comparisons between the results of the analytical solutions and the results of the mathematical models with experimental data. In Section 2.4, the current level of understanding on the influence of the delamination on the natural frequencies and mode shapes of the composite laminates are discussed. Section 2.5 presents other factors that influence vibration of the delaminated composite laminates. In Section 2.6, the classifications of the available mathematical models for smart structures using piezoelectric sensors/actuators are presented, together with a review on the relevant literature. In

Section 2.7, the summary of the chapter is presented, and the issues that are to be addressed in this thesis are presented and discussed.

2.2 Vibration of Delaminated Composite Laminates

A considerable amount of analytical models and numerical analyses have been reported for the vibration analysis of composite laminates with delaminations. The available mathematical models can be divided into three categories:

1. Region approach models: the delaminated laminate is divided into sub-laminates or segments and the continuity conditions are imposed at the delamination junctions. Each of these sub-laminates is analyzed using the *equivalent single layer* (ESL) theories. The ESL theories consist of the classical laminate theory (CLT), first-order shear deformation theory (FSDT) and the third-order shear deformation theory (TSDT). The ESL theories are derived from the 3-D elasticity theory by making suitable assumptions concerning the deformation of the stress state through the thickness of the laminate (Reddy, 2004).
2. Layerwise models: the laminate is modeled using the *layerwise theories*, which assume separate displacement field expansions within each material layer, thus providing a kinematically correct representation of the strain field in discrete layer laminates, and allowing accurate determination of ply level stresses (Reddy, 2004). The layerwise theories are subdivided into two classes: *partial layerwise* and *full layerwise*. The partial layerwise theories are based on piece-wise linear variation of the inplane displacement components and a constant transverse displacement through the thickness (i.e. the transverse shear strain is piece-wise constant, while

- the transverse normal strain is zero through the thickness). Such layerwise theories are referred to as partial layerwise theories since the transverse normal displacement does not have a layerwise representation. The full layerwise theories use layerwise expansions for all three displacement components, and thus include both discrete layer transverse shear effects and transverse normal effects. The delamination can be modeled as an embedded layer or by introducing discontinuity functions in the displacement fields.
3. Reduced stiffness models: the stiffness of the laminate is considered to be reduced at the delamination region.

2.2.1 Region approach – One-dimensional models

To illustrate the approach for this group of methods, consider a beam with a single through-width delamination as shown in Figure 2.1. The delaminated beam is divided into three spanwise regions, a delamination region and two integral regions (undelaminated). The delamination region is composed of two separate beam segments (delaminated layers), which are joined at their ends to the integral beam segments. The delaminated beam can now be analyzed as four interconnected beams. The equations of motion are formulated for each of the beam segments. By imposing appropriate boundary conditions at the ends of the two integral beam segments and continuity conditions at the delamination junctions, the vibration response of the beam can be obtained.

Since the delamination region is considered as two separate beam segments, there is a shift in the neutral axis of these segments with respect to the reference axis of the integral beam segment. This gives rise to the coupling between the axial and flexural

vibrations. Therefore, apart from the continuity conditions of transverse displacements, slopes, bending moments and shear forces at the delamination junctions, two additional conditions, the continuity of axial forces and displacements need to be satisfied (Figure 2.2). The continuity condition of axial displacement, which produces the differential stretching of the delaminated layers, contributes substantially to flexural stiffness of the delaminated beam.

Figure 2.1. Modeling of a beam with a single through-width delamination (Mujumdar and Suryanarayan (1988)).

The earliest reported model for the vibration analysis of delaminated composite beams was presented by Ramkumar et al. (1979). They modeled the beam with a through-width delamination by using four Timoshenko beams. The coupling effect between the axial and transverse vibrations was neglected in their analysis. The natural frequencies and mode shapes were then solved by a boundary eigenvalue problem.

However, their results showed that the predicted frequencies were consistently much lower than the experimental results.

Figure 2.2. Deformation and stress components in the delamination region: (a) Deformation and stresses with the layers free to slide; (b) Deformation and stresses when the compatibility of axial displacement is satisfied (Mujumdar and Suryanarayan, 1988).

Wang et al. (1981) improved the model of Ramkumar et al. (1979) for an isotropic beam by including the effect of coupling between the axial and transverse vibrations. The classical beam theory was applied to each of the beams. They assumed that the

Chapter 2: Background and Literature Review

delaminated layers deformed ‘freely’ without touching each other. This model was referred to as the ‘free mode’. They found that for delaminated beams with a short and close to midplane delamination, the natural frequencies were close to the experimental values. This implied that the large discrepancy in Ramkumar et al.’s model was mainly due to the neglecting of the coupling effect.

Mujumdar and Suryanarayan (1988) showed that in the case of off-midplane delaminations, the ‘free mode’ model mode shapes are physically inadmissible. This is because the delaminated layers were assumed to deform ‘freely’ without touching each other and thus the delaminated layers will have different transverse deformation. To avoid this kind of incompatibility, they proposed a model based on the assumption that the delaminated layers are ‘constrained’ and thus have identical transverse deformations. In addition, the delaminated layers are assumed to be free to slide over each other in the axial direction except at their ends, which are connected to the integral segments. This model was referred to as the ‘constrained mode’. To ‘constrain’ the delaminated layers, an equal and opposite normal pressure acts on the lower surface of the upper layer and upper surface of the lower delaminated layer (Figure 2.3).

Similar ‘constrained mode’ models were presented by Tracy and Pardo (1989) for a simply-supported composite beam, Hu and Hwu (1995) for composite sandwich beams, Shu and Fan (1996) for bimaterial split beams and Valoor and Chandrashekhara (2000) for thick composite beams.

The ‘constrained mode’ model, however, failed to predict the opening in the mode shapes found in the experiments by Shen and Grady (1992) and Lestari and Hanagud (1999). Shen and Grady (1992) presented an analytical model based on the

Timoshenko beam theory and the cracked beam theory. Two solutions were presented; one uses the ‘constrained mode’ assumption and includes the coupling effect between the axial and transverse vibrations, and the other uses the ‘free mode’ assumption but neglects the coupling effect. The equations of motion were derived through the Hu-Washizu-Barr variational method to include the virtual work done by the inertial forces. The Galerkin procedure was used to determine the natural frequency and the Rayleigh-Ritz method was used to determine the mode shape of the beam. The numerical results were verified using experimental data, which have been widely used as benchmark values by many researchers.

Figure 2.3. Modeling of the delamination region for the ‘constrained mode’: (a) Deformation in the “constrained mode”; (b) Contact pressure between the layers of the delamination region (Mujumdar and Suryanarayan, 1988).

Chapter 2: Background and Literature Review

In the investigations by Hanagud and Luo (1997) and Luo and Hanagud (1997a, b, 2000), they used an analytical model based on the Timoshenko beam theory and piecewise-linear springs to simulate the ‘open’ and ‘closed’ behavior between the delaminated layers. The spring stiffness would then be equal to zero (0) for the ‘free mode’ and infinity (∞) for the ‘constrained mode’.

Wang and Tong (2002) introduced a nonlinear anti-interpenetration constraint model to prevent the overlapping of the delaminated layers during vibration. The Timoshenko beam theory was used. The contact forces between the delaminated layers were expressed as a function of the relative transverse displacements of the layers. Two types of functions were used, namely, a linear spring function and a nonlinear Hertz-type function. The contact forces cause nonlinearity in the equations of motion, which were solved using the finite difference method.

More than one delamination is usually present in composite laminates damaged by low velocity impact (Kutlu and Chang, 1992; Tay, 2003). The available analytical solutions on multiple delaminations were based on the Euler-Bernoulli beam theory. Analytical models based on the ‘constrained mode’ assumption were presented by Wang and Lin (1995) for composite beams and Kim and Hwang (2002) for honeycomb sandwich beams. Analytical models based on the ‘free mode’ assumption were presented by Lee et al. (2003) and Park et al. (2004).

Shu (1995) presented analytical solutions to investigate the vibration of sandwich beams with two delaminations at identical spanwise locations (double delaminations). Two assumptions were considered, the contact mode, ‘free’ and ‘constrained’, between the delaminated layers and the local deformation at the delamination fronts. The local deformation near the two fronts of delamination was investigated by Shu and Mai

(1993). They identified the ‘rigid connector’ and the ‘soft connector’ conditions (Figure 2.4). The cross-section in a ‘rigid connector’ remains perpendicular to the deformed midplane of the beam, and thus takes account of the differential stretching between the delaminated beams. The cross-section of a ‘soft connector’ remains perpendicular to the undeformed midplane of the beam, and thus neglects the differential stretching. The assumptions by Shu (1995) lead to four combinations, namely, the ‘rigid connector’ and ‘constrained mode’ (RC), the ‘rigid connector’ and ‘free mode’ (RF), the ‘soft connector’ and ‘constrained mode’ (SC), and the ‘soft connector’ and ‘free mode’ (SF).

Figure 2.4. Deformation of the section B between delamination and integral regions showing ‘rigid connector’ and ‘soft connector’ (Shu and Mai, 1993).

Lestari and Hanagud (1999) presented an analytical solution for composite beams with multiple delaminations based on the piecewise-linear spring model of Luo and Hanagud (1997a, b, 2000). Experiments were performed to verify the results of their model.

2.2.2 Region approach – Finite element method

2.2.2.1 One-dimensional models

A finite element (FE) model based on the *classical laminate theory* (CLT) for composite beams was presented by Nagesh Babu and Hanagud (1990) by using the ‘free mode’ assumption. Keilers and Chang (1995a, b) presented an FE model for composite beams with piezoelectric sensors and actuators by using the Euler-Bernoulli beam theory and the ‘constrained mode’ assumption.

FE models using the *first-order shear deformation theory* (FSDT) were presented by Chakraborty et al. (2002), Krawczuk et al. (1996a, b, 1997) and Zak et al. (2000). The ‘free mode’ assumption was used. Ju et al. (1994) and Zhu et al. (2005) presented models for composite beams with multiple delaminations. Both the ‘free mode’ and ‘constrained mode’ were considered in their analyses.

FE models using the FSDT for composite beams with piezoelectric sensors/actuators were presented by Islam and Craig (1994) using the ‘free mode’ and Ray and Reddy (2004) using ‘constrained mode’ assumption.

Gummadi and Hanagud (1995) presented FE models based on the FSDT and *third-order shear deformation theory* (TSDT). The ‘free mode’ assumption was used in their analysis. Chattopadhyay et al. (1999a) presented an FE model for composite beams with piezoelectric actuators by using the TSDT and the ‘free mode’ assumption. The piezoelectric actuators were surface bonded on the composite laminates and was treated as another surface layer.

2.2.2.2 Two-dimensional models

‘Free mode’ models for composite plates based on the FSDT were presented by Campanelli and Engblom (1995), Ju et al. (1995), Krawczuk et al. (1997), Zak et al. (2000) and Parhi et al. (1999, 2000, 2001). ‘Constrained mode’ models were presented by Zak et al. (2001) and Chen et al. (2004).

FE models for composite plates based on the TSDT were presented by Chattopadhyay et al. (2000a) and Radu and Chattopadhyay (2002). The ‘free mode’ assumption was used. Hu et al. (2002) presented models using the ‘free mode’ and ‘constrained mode’ assumptions.

TSDT-based ‘free mode’ model for composite plates with piezoelectric actuators were presented Chattopadhyay et al. (1999b, 2000b) and Nam et al. (2000). Thornburg and Chattopadhyay (2002) FE model for composite plates with piezoelectric patches and internal delaminations. They further investigated the contact between the delaminated layers by using fictitious springs, which was assumed to have zero stiffness under tension and stiffness proportional to the transverse modulus of the plate when subjected to compression.

2.2.2.3 Three-dimensional models

Tenek et al. (1993) developed an FE model based on the three-dimensional theory of linear elasticity for composite plates. The ‘free mode’ was used. Yam et al. (2004) presented a three-dimensional FE model for composite plate with an internal delamination. Virtual spring elements were inserted between the delaminated layers to allow the deformation of the layers without penetration.

2.2.3 Layerwise models

These models use the layerwise theories, which are based on piecewise, layer-by-layer approximations of the response quantities in the thickness direction (Noor and Burton, 1990). The layerwise displacement field provides a much more kinematically correct representation of moderate to severe cross sectional warping, associated with the deformation of moderately thick laminates to very thick laminates. The layer-wise linear approximation of displacements through the thickness and the use of Heaviside step functions to model delaminations prove to be an effective approach for an accurate analysis of local effects in laminated composite plates. However, the computational cost of the proposed analysis makes it unattractive for the prediction of global behaviour when compared with conventional theories or ESL theories (Zou et al., 2000). Furthermore, the complexity of the layerwise models increases with increasing number of layers.

2.2.3.1 Full layerwise models

Saravanos and Hopkins (1996) developed an analytical model for predicting natural frequencies, mode shapes and modal damping of delaminated composite beams. The kinematic assumptions of Reddy's layerwise theory (Reddy, 1987) were modified to include the discontinuities in the in-plane and through-the-thickness displacements induced by the delamination (Figure 2.5). The displacement fields are supplemented with Heaviside unit step functions which allow discontinuities in the displacements. Lee (2000) developed an FE model for composite beams with multiple delaminations. The equations of motion were derived using Hamilton's principle. Solutions for open and closed delaminations were presented.

Figure 2.5. Kinematic assumptions of the layerwise theory: (a) laminate; (b) displacements (Saravanos and Hopkins, 1996).

Perel and Palazotto (2002) presented an FE model for dynamics of delaminated composite beam with piezoelectric actuators by using the full layerwise theory. The piezoelectric effect was modeled as an induced strain in the location of the piezoelectric element. The frequencies computed using this method was in agreement with their experimental data.

2.2.3.2 Partial layerwise models

Averill (1994) developed an FE model based on the first-order zigzag theory for moderately thick laminated beams with multiple delaminations. The zigzag theory requires transverse shear stress continuity at the layer interfaces and traction-free boundary conditions at the top and bottom faces of the laminates. To model the delamination, a very thin layer with compliant material stiffness is placed between the laminae. This method is called the compliant layer concept or embedded layer approach, which is valid for modeling delaminations provided that the delaminations remain closed.

Other zigzag models for composite plates with multiple through-width delaminations were presented by Kim et al. (2003a, b) using the first-order zigzag theory and Cho and Kim (2001) using the third-order zigzag theory. The first-order (Kim et al., 2003a, b) or third-order (Cho and Kim, 2001) shear deformation based displacement field was used to address the overall response of the entire laminate and the layerwise functions were used to accommodate the complexity of zigzag-like in-plane deformation through the laminate thickness. Similarly with the above full layerwise models, the multiple delaminations were modeled by supplementing the assumed displacement field with Heaviside unit step functions. These models were implemented using finite elements.

Goshal et al. (2003) developed an FE model using the first-order zigzag theory for delaminated composites and smart composite plates. A coupled piezoelectric-mechanical formulation was used in the development of the constitutive equations to model piezoelectric composite plates. The contact problem of delaminated interfacial surfaces was modeled using linear springs to provide an accurate description of the transient behavior. The accuracy of the results of the model was validated through comparison with experimental data. Chattopadhyay et al. (2004) presented a similar model for composite laminates with embedded or surface piezoelectric actuators and with embedded multiple delaminations.

2.2.4 Reduced stiffness models

The earliest reported investigation on the vibration of delaminated composites was presented by Kulkarni and Frederick (1971). They considered an isotropic clamped-clamped circular cylindrical shell with a circumferentially symmetric crack at the

middle surface. The delamination region was analyzed as a region of reduced bending stiffness. Although their study was a good beginning, their results were limited to provide specific conclusions.

In a method presented by Harrison and Butler (2001), the delaminated beam is assumed to have a reduced shear rigidity. The shear rigidity of the delaminated beam was estimated using a static beam model. This estimated shear rigidity was incorporated to the dynamic stiffness method, which was used to determine the natural frequency and the mode shape of the beam.

Rosettos (1998) developed a model to study the fundamental frequency of a slightly curved sandwich beam with a delamination. A non-dimensional variational approach was developed using the principle of minimum potential energy. The sandwich beam includes strain energy due to initial curvature. The two equal face sheets were assumed to take on the bending and axial strain energies while the core takes only the shear energy. To include the effect of the delamination, the shear strain energy was considered zero at the delamination region since the shear stress is zero at the delamination edge. It was shown that the sensitivity of the natural frequency to the length of delamination is greater for straight beams than for initially curved beams.

2.3 Theoretical and Experimental Results

This section presents a comparison among the results of the different models. Section 2.3.1 presents a comparison between the ‘free mode’ model (Wang et al., 1982) and ‘constrained mode’ model (Mujumdar and Suryanarayan, 1988). Also, the significance of the assumptions on the modes and connectors (Shu, 1995) are

presented and discussed. In Section 2.3.2, the results of the analytical and FE models are compared with experimental data.

2.3.1 Comparison between the ‘free mode’ and ‘constrained mode’ models

Mujumdar and Suryanarayan (1988) compared the frequency results of ‘free mode’ and ‘constrained mode’ models with experimental data. The experiments were carried out on beams specimens made by bonding stainless steel strips with a very thin adhesive layer. The delamination was simulated by bonding only a part of the beams surfaces to create delamination of various sizes in various locations. They showed that for midplane delaminations, the results of the two models are identical and in good agreement with the experimental data. For off-midplane delaminations, for the fundamental frequency, the results of the two models are close with each other and were in good agreement with the experimental data. However, for the second vibration mode, the ‘constrained mode’ results were closer to the experimental values.

For a clamped-clamped isotropic beam, the ‘constrained mode’ model shows a decreasing frequency as the delamination moves towards the midplane (Figure 2.6). This is because the bending stiffness decreases as the delamination moves towards the midplane. In contrast, the ‘free mode’ model shows an increasing frequency. Since the ‘free mode’ model is dependent on the weaker delaminated layer, as the weaker layer strengthens, the natural frequency increases (Shu, 1995). It is further observed that the ‘constrained mode’ and ‘free mode’ results are identical for midplane delamination. This is because the delaminated layers have the same natural frequency and thus vibrate in a ‘constrained’ manner.

Figure 2.6. Influence of the thickness-wise location on the fundamental frequency of a clamped-clamped beam with a central delamination (Mujumdar and Suryanarayan, 1988).

It should be noted that in the above ‘constrained mode’ and ‘free mode’ models, the coupling of the axial and transverse vibrations (or the differential stretching) are considered. The study by Wang et al. (1982) on a clamped-clamped beam showed by neglecting this coupling effect, much lower frequencies were obtained. However, the coupling effect is negligible for very long delaminations (greater than 80 percent of the beam span) and also for the even modes of vibration.

Shu (1995) further investigated the effects of the modes (‘free’ and ‘constrained’) and the connectors (‘soft’ and ‘rigid’) on the natural frequency of the beam (Figure 2.7). The ‘soft connector’ neglects the differential stretching, whereas the ‘rigid connector’ takes account of the differential stretching. For a clamped-clamped beam, the effect of the modes is less significant for short delaminations ($a/L < 0.4$), whereas, the effect of the connector is less significant for very long delaminations ($a/L > 0.8$). For a cantilever beam, the results are dominated by the connectors.

Figure 2.7. Influence of modes and connectors (Shu, 1995).

2.3.2 Comparison with experimental results

The experimental data by Shen and Grady (1992) on a composite beam with a single delamination have been widely used by many researchers to validate their models. The composite beam used in their experiment is made of T300/934 graphite/epoxy cantilever beam with a $[0^\circ/90^\circ]_{2s}$ stacking sequence. The dimensions of the 8-ply beam are $127 \times 12.7 \times 1.016 \text{ mm}^3$. The material properties for the lamina are: $E_{11} = 134 \text{ GPa}$, $E_{22} = 10.3 \text{ GPa}$, $G_{12} = 5 \text{ GPa}$, $\nu_{12} = 0.33$ and $\rho = 1.48 \times 10^3 \text{ kg/m}^3$. The delamination is at midspan, and the delamination lengths considered are 25.4 mm, 50.8 mm, 76.2 mm and 101.6 mm. Figure 2.8 shows the different locations of the delamination along the thicknesswise direction.

The comparison of the results of the analytical solutions with the experimental data is presented in Tables 2.1 – 2.4, whereas the FE results are presented in Tables 2.5 – 2.8. The results of the different models have been shown to be in good agreement with the experimental data, except at Interface 4 (Tables 2.4 and 2.8). It can be seen

that the experimental results are closer to the predicted frequencies that neglect the coupling effect, which might have been an experimental error, as explained by Luo and Hanagud (2000). When the coupling effect is neglected, the predicted frequencies are much lower than the experimental results (Ramkumar et al., 1979). Furthermore, the experimental results by Kim et al. (2003a) showed that the natural frequency decreases more when the delamination is closer to the midplane than to the surface of the laminate, which is in contrast with the Interface 4 results by Shen and Grady (1992).

Figure 2.8. Thicknesswise location of delamination in the composite beam (Shen and Grady, 1992).

Tables 2.1 – 2.8 shows no evident advantage of the shear deformation theories (FSDT and TSDT) over the CLT since the available data is from a slender beam. This indicates that the model based on the CLT provides a reasonable prediction of the natural frequency slender beams.

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Table 2.1. Analytical and experimental results (Interface 1 frequencies, Hz)

Delamination length	Experimental ¹			FSDT ¹		FSDT ²		CLT ³	
	Exp 1	Exp 2	Exp 3	Cons	Free	Cons	Free	Cons	Free
Intact	79.88	79.88	79.75	82.04	82.04	81.86	81.86	81.88	81.88
25.4 mm	78.38	79.13	77.00	80.13	67.36	81.84	81.84	80.72	80.72
50.8 mm	74.38	75.00	76.75	75.29	56.48	76.81	76.81	76.36	76.36
76.2 mm	68.25	66.25	66.38	66.94	67.64	67.64	47.90	68.17	68.17
101.6 mm	57.62	57.50	57.50	57.24	40.59	56.95	56.95	58.35	58.35

¹ Shen and Grady (1992); ² Luo and Hanagud (2000); ³ Della and Shu (2005)**Table 2.2.** Analytical and experimental results (Interface 2 frequencies, Hz)

Delamination length	Experimental ¹			FSDT ¹		FSDT ²		CLT ³	
	Exp 1	Exp 2	Exp 3	Cons	Free	Cons	Free	Cons	Free
Intact	79.88	79.88	79.75	82.04	82.04	81.86	81.86	81.88	81.88
25.4 mm	78.38	78.38	76.63	81.39	68.78	80.86	80.86	81.00	81.00
50.8 mm	75.13	75.25	75.00	78.10	59.44	76.62	76.62	77.45	77.45
76.2 mm	64.00	70.00	69.88	71.16	51.18	68.80	68.80	70.43	70.42
101.6 mm	45.75	49.75	49.50	62.12	43.86	59.34	59.34	61.50	61.49

¹ Shen and Grady (1992); ² Luo and Hanagud (2000); ³ Della and Shu (2005)**Table 2.3.** Analytical and experimental results (Interface 3 frequencies, Hz)

Delamination length	Experimental ¹			FSDT ¹		FSDT ²		CLT ³	
	Exp 1	Exp 2	Exp 3	Cons	Free	Cons	Free	Cons	Free
Intact	79.88	79.88	79.75	82.04	82.04	81.86	81.86	81.88	81.88
25.4 mm	79.63	80.13	80.63	81.46	81.46	82.02	82.01	81.57	81.57
50.8 mm	79.50	81.88	77.88	79.93	79.93	80.79	80.74	80.25	80.23
76.2 mm	75.63	77.13	78.13	76.71	76.71	77.82	77.52	77.27	77.16
101.6 mm	73.38	73.63	70.38	71.66	71.66	73.15	71.73	72.66	72.20

¹ Shen and Grady (1992); ² Luo and Hanagud (2000); ³ Della and Shu (2005)

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Table 2.4. Analytical and experimental results (Interface 4 frequencies, Hz)

Delamination length	Experimental ¹			FSDT ¹		FSDT ²		CLT ³	
	Exp 1	Exp 2	Exp 3	Cons	Free	Cons	Free	Cons	Free
Intact	79.88	79.88	79.75	82.04	82.04	81.86	81.86	81.88	81.88
25.4 mm	75.38	75.25	77.25	81.60	75.83	82.04	82.03	81.72	81.72
50.8 mm	69.38	68.00	69.38	80.38	71.88	80.95	80.87	80.75	80.73
76.2 mm	65.38	59.63	-----	77.70	67.18	78.29	77.61	78.40	78.25
101.6 mm	52.75	57.88	56.25	73.15	61.70	74.05	69.44	74.59	73.81

¹ Shen and Grady (1992); ² Luo and Hanagud (2000); ³ Della and Shu (2005)

Table 2.5. FE and experimental results (Interface 1 frequencies, Hz)

Delamination length	FSDT ⁵		TSDT ⁵		TSDT ⁶	FOZZ ⁷	TOZZ ⁸
	Cons	Free	Cons	Free	Free		
Intact	82.00	82.00	81.87	81.87	82.11	82.13	81.90
25.4 mm					81.37	81.49	81.20
50.8 mm	76.64	76.64	76.52	76.52	76.73	77.19	76.60
76.2 mm					67.57	68.50	67.45
101.6 mm	77.01	77.01	56.56	56.56	56.89	56.94	57.78

⁵ Hu et al. (2002); ⁶ Thornburg and Chattopadhyay (2002); ⁷ Kim et al. (2003a);

⁸ Cho and Kim (2001)

Table 2.6. FE and experimental results (Interface 2 frequencies, Hz)

Delamination length	FSDT ⁵		TSDT ⁵		TSDT ⁶	FOZZ ⁷	TOZZ ⁸
	Cons	Free	Cons	Free	Free		
Intact		82.00	82.00	81.87	81.87	82.11	82.13
25.4 mm						81.42	81.49
50.8 mm	77.02		77.01	76.89	76.89	77.11	77.19
76.2 mm						68.42	68.50
101.6 mm	57.90		57.87	57.69	57.72	58.04	58.11

⁵ Hu et al. (2002); ⁶ Thornburg and Chattopadhyay (2002); ⁷ Kim et al. (2003a);

⁸ Cho and Kim (2001)

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Table 2.7. FE and experimental results (Interface 3 frequencies, Hz)

Delamination length	Cons	FSDT ⁵ Free	Cons	TSDT ⁵ Free	Free	TSDT ⁶	FOZZ ⁷
Intact		82.00	82.00	81.87	81.87	82.11	82.13
25.4 mm						81.92	81.96
50.8 mm		80.56	80.61	80.50	80.45	80.65	80.74
76.2 mm						77.44	77.79
101.6 mm		72.85	71.44	72.61	71.21	71.66	73.12

⁵ Hu et al. (2002); ⁶ Thornburg and Chattopadhyay (2002); ⁷ Kim et al. (2003a);⁸ Cho and Kim (2001)**Table 2.8.** FE and experimental results (Interface 4 frequencies, Hz)

Delamination length	Cons	FSDT ⁵ Free	Cons	TSDT ⁵ Free	Free	TSDT ⁶	FOZZ ⁷
Intact		82.00	82.00	81.87	81.87	82.11	82.13
25.4 mm						81.94	81.97
50.8 mm		80.75	80.74	80.64	80.62	80.83	80.88
76.2 mm						78.04	78.23
101.6 mm		73.78	72.95	73.78	72.72	73.15	73.97

⁵ Hu et al. (2002); ⁶ Thornburg and Chattopadhyay (2002); ⁷ Kim et al. (2003a);⁸ Cho and Kim (2001)

2.4 Influence of the Delamination on the Vibration of Composite

Laminates

In this section, the influence of the delamination on the natural frequencies and mode shapes of the composite laminates are discussed. Some results from the literature are presented to illustrate the points being discussed and not necessarily the only significant contributions of the subject.

2.4.1 Size and locations of the delamination and boundary conditions

In general, delamination decreases the natural frequency and causes changes to the mode shape of the composite laminate. This is due to the reduction of the stiffness caused by the delamination. However, results by Mujumdar and Suryanarayan (1988) showed that a short delamination (less than 25% of the beam length) do not significantly affect the fundamental and second mode frequencies of the beam for all boundary conditions (Figure 2.9). For longer delaminations (greater than 50% of the beam length), the weakening effect of the delamination is dependent on the boundary conditions. In fact, the more restrained is the structure along its edges, the greater is the effect on the natural frequency (Ju et al., 1995). Other studies on composite beams and plates showed that delamination of small size do not significantly affect the lower vibration mode frequencies (Tracy and Pardoen, 1989; Shen and Grady, 1992; Ju et al., 1995; Saravanos and Hopkins, 1996). The experimental investigation by Zak et al. (2000) on cantilever composite beams and plates with a single edge delamination further verified the conclusions of Mujumdar and Suryanarayan (1988).

The experimental investigation of Jian et al. (1997) on thin unidirectional glass-fiber composite plates with circular delamination showed that first seven frequencies were significantly affected by a midplane delamination as small as 0.34% of the plate area. They further showed that the decrease in the natural frequency is more pronounced at higher vibration modes. Other experimental investigations presented similar observations (Tenek et al., 1993; Kim et al., 2003a; Paolozzi and Peroni, 1996; Valdez and Soutis, 1999; Kessler et al., 2002).

Figure 2.9. Influence of the delamination length on the fundamental and second mode frequencies of the beam (Mujumdar and Suryanarayan, 1988).

In contrast, the experimental investigation by Penn et al. (1999) showed that for thick composite plates, the first six frequencies is not significantly affected by delamination sizes of less than 13% of the plate area. The experimental investigation by Hou and Jeronimidis (1999) on circular composite plates with an impact induced delamination showed an increase in the natural frequency of the composite due to the delamination. Further experimental investigation showed that the increase in the frequency is due to the increase in the bending stiffness caused by the local thickening in the delamination region (Hou and Jeronimidis, 2000).

Studies showed that delamination locations also affect the natural frequencies of the laminate. It was observed that a delamination located at midplane and near the

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fixed end of the laminate decreases the natural frequency most (Mujumdar and Suryanarayan, 1988; Krawczuk et al., 1996a, b, 1997; Kim et al., 2003a; Thornburg and Chattopadhyay, 2002).

Experimental investigations have shown that the delamination alters the mode shape of the composites (Luo and Hanagud, 1997a, b). Also, nonlinearities in the mode shapes have been observed. These nonlinearities can be caused by either or both of the following factors:

- opening and closing of the delamination ('delamination breathing'); and
- contact/impact between the delaminated layers during vibration.

In the experiments by Shen and Grady (1992) and Lestari and Hanagud (1999), the 'delamination breathing' was shown to cause nonlinearity in the mode shapes (Figure 2.10). However, the 'delamination breathing' has been shown to have little effect on the natural frequencies of low vibration modes when the delamination is small (Luo and Hanagud, 1997a, b; Thornburg and Chattopadhyay, 2002). Further investigations by Lestari et al. (2000) and Lu et al. (2001) showed that for long delaminations and thin delaminated layers, the nonlinearity of the vibration mode shapes is significant. However, for short delaminations, the nonlinearity is less significant and the nonlinear mode shapes tend to linear mode shapes.

Similarly, for short delamination and lower vibration modes, the nonlinearity due to the contact/impact between the delaminated layers was shown to be less significant and becomes significant for long delamination at higher vibration modes (Luo and Hanagud, 1997; Thornburg and Chattopadhyay, 2002).

(a)

(b)

Figure 2.10. Opening mode of the delaminated beam: (a) fundamental frequency; (b) second bending frequency (Lestari and Hanagud, 1999).

The delamination opening has been shown to be affected by the boundary conditions. The more restrained the beams along its edges, the larger is the delamination opening as shown in Figure 2.11 (Shu, 1995).

Figure 2.11. Delamination opening increases with more restrained at the beam edges (Shu, 1995).

2.4.2 Multiple delaminations

Studies on multiple delaminations showed that the natural frequency decreases as the number of delamination increases (Tenek et al., 1993; Ju et al., 1995a; Lee, 2000; Lee et al., 2003; Zhu et al., 2004). However, the effect of a single long delamination

has been shown to be more significant than the effect of multiple delaminations with comparable length located along the same axial direction (Ju et al., 1994; Lee et al., 2003; Kim et al., 2003b).

2.5 Other Factors that influence the Vibration of Delaminated

Composite Laminates

2.5.1 Laminate stacking sequence

Lee (2000) investigated the effect of the stacking sequence on the fundamental frequency of a delaminated cantilever composite beam, which was made of T300/5208 graphite epoxy. The ‘constrained mode’ assumption was employed. It was observed that as the angle θ increases, the effect of delamination on the fundamental frequency increases (Figure 2.12). Similar results were obtained by Zhu et al. (2004).

Other studies (Kim et al., 2003b, Thornburg and Chattopadhyay, 2002; Parhi et al., 1999, 2001) also showed that changes to the laminate stacking sequence affects the changes in the natural frequency caused by the delamination.

2.5.2 Crack-tip singularity

Thornburg and Chattopadhyay (2003) estimated the effect of the singularity at the delamination crack tip on the natural frequency based on the difference between the delamination end openings of two finite element meshes. Their results showed that for delamination length to laminates thickness ratios greater than five, the effect of the crack tip singularity is minimal. Thus, it can be neglected in the vibration analysis provided that the delamination length to laminate thickness ratio is greater than five.

Figure 2.12. Natural frequencies versus fiber angle for $[0/\theta]_s$ composites containing one and two delaminations (Lee, 2000).

2.5.3 Matrix cracking

Thornburgh and Chattopadhyay (2001, 2003) studied the vibration of composite laminates with delaminations and transverse matrix cracks by using FE analysis. The region with delamination was modeled using the TSDT (Radu and Chattopadhyay, 2002). The matrix cracking was incorporated in the model as a reduction in the ply stiffness. They showed that the presence of matrix cracking further decreases the fundamental frequency. However, the decrease is not significant for small delamination.

2.5.4 Bridging

Brandinelli and Massabo (2003) developed an analytical model to investigate the effects of bridging mechanisms between the delaminated interfaces on the vibration of a delaminated composite beam. The bridging mechanisms could represent the action developed by through-thickness reinforcements, such as stitching or z-pinning. These were modeled as uniform distribution of linear elastic springs that oppose the longitudinal and transversal relative displacements between the delaminated interfaces. Their study on stitched carbon-epoxy laminate showed that low percentages of through-thickness reinforcement can improve the vibration characteristics of delaminated beams.

2.6 Smart Structures using Piezoelectric Sensors and Actuators

Smart or intelligent structures are systems containing multifunctional parts that can perform functions such as sensing, processing/control and actuation (Chee et al., 1998). Spillman et al. (1996) presented a classical definition of the terms smart and intelligent: *intelligent*: the ability to learn or understand or deal with new or trying situation; *smart*: having mental alertness, knowledgeable, shrewd. Thus, while both structures can respond appropriately to external condition, an intelligent structure has the extra ability to learn.

Extensive researches have been conducted in the area of smart/intelligent structures using piezoelectric materials as sensors and actuators. Excellent reviews have been presented by many researchers on the various applications of piezoelectric sensors and actuators. Crawley (1994) presented an overview and assessment on the

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use of intelligent structures for aerospace application. Tani et al. (1998) reviewed the application of smart materials (piezoelectric materials, magnetostrictive materials and shape memory alloys) for intelligent material systems used to control deformation, vibration and fracture of composite materials and structures. Rao and Sunar (1994) and Sunar and Rao (1999) presented reviews on sensing and control of flexible structures using piezoelectric materials. Niezrecki et al. (2001) reviewed the various architectures for piezoelectric actuation. Ahmadian and DeGuillio (2001) reviewed the use of piezoceramics for vibration suppression. Wetherhold and Aldrahem (2001) reviewed the bending and twisting vibration control of flexible structures using piezoelectric materials. Irschik (2002) presented a review on the static and dynamic shape control of structures by piezoelectric actuation.

Review articles on the various mathematical models for laminates and structures with piezoelectric sensors and/or actuators have been presented by Chee et al. (1998), Saravanos and Heyliger (1999), Gopinath et al. (2000), Chopra (2002) and Alzharani and Alghamdi (2003). Brief summaries on the computational models for composite laminates with piezoelectric sensors and actuators can be found in Reddy (1999) and Mota Soares et al. (2000). A bibliographical review on the finite element models for the analysis and simulation of smart materials and structures have been presented by Mackerle (2003). In this literature review, the different classifications of the available mathematical models that have been reported in the above review articles are presented. Furthermore, the relevant literatures are presented and discussed to establish context in the present research.

Chee et al. (1998) classified the models for structures and laminates with piezoelectric sensors/actuators into three categories. The first category consists of

models with piezoelectric sensors/actuators that are bonded to or embedded within the structures as discrete patches. They do not affect the overall properties of the structure when inactivated. The second category consists of structures with distributed piezoelectric sensors/actuators that are bonded on the top or bottom surface of the structure. Similarly with the first category, the piezoelectrics do not affect the structural properties. In the third category, the piezoelectric sensors/actuators are in the form of a layer and are laminated within the structure to form an active laminated composite or *piezoelectric laminate*. A similar classification was presented by Gopinath (2001) where the smart structure models are classified according to the form of the piezoelectric sensor/actuators, as discrete patches or piezoelectric lamina.

Saravanos and Heyliger (1999) presented a more detailed classification for the laminate theories and models for laminates and structures with piezoelectric sensors/actuators. The models were classified based on the fundamental assumptions, the method of representation of piezoelectric layers, and the approximation of the through-the-thickness variation of the electromechanical state variables. The classifications are: (a) *Induced actuation strain models*: these are for structural components containing only piezoelectric actuators. The piezoelectric effect is modeled as an induced strain at the locations of the piezoelectric actuators; (b) *Equivalent single-layer models*: in these models, the piezoelectric laminate is assumed to deform as a single uniform layer (see Section 2.2). Both converse and direct piezoelectric effects are considered. However, the piezoelectric coupling effects are neglected in the equations of motion; (c) *Coupled single-layer models*: these models also assume that the laminate deforms as a single layer while directly representing the piezoelectric coupling effects; (d) *Layerwise models*: these model are based on the

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layerwise theories (see Section 2.2). In addition, these models include the local electromechanical responses of each layer. (e) *Exact solutions*: these are exact solutions of the stress and charge equilibrium equations; (f) *Shell models*: these models are for curved piezoelectric components and shell structures; (g) *Thermal effects*: these model are based on the principle of linear thermopiezoelectricity; (h) *Nonlinear effects*: these models include the nonlinear effects, such as geometry and material nonlinearities.

Specifically, the piezoelectric laminate theories were classified according to the type of kinematic assumptions for the displacement field and the type of representation of the piezoelectric effects (Saravanos and Heyliger, 1999). The classifications are: (a) *Uncoupled single-layer theories*: these theories combine the single-layer kinematic assumptions for the displacements and induced forces and moments in active piezoelectric layers; (b) *Coupled single-layer theories*: these theories combine the single-layer kinematic assumptions for the displacements and both converse and direct piezoelectric effect. The piezoelectric coupling is considered in the formulation of the mechanics of these theories; (c) *Generalized in-plane theories*: these theories are based on the partial layerwise theories (see Section 2.2). They may consider the electric potential as independent state variable and the electromechanical coupling induces by the piezoelectric layers; (d) *Generalized three-dimensional theories*: these theories are based on the full layerwise theories (see Section 2.2). They use the full three-dimensional constitutive relations and equations of motion of a piezoelectric continuum; (e) *Exact piezoelectricity solutions*: these models satisfy exactly the partial differential equations of stress equilibrium and electric displacement conservation

through the laminate thickness, and are exact three-dimensional solutions for a laminated piezoelectric continuum.

Detailed reviews of the induced strain actuation models were presented by Chopra (2002) for beams and plates, and Alzahrani and Alghamdi (2003) for beams. The beam models can be classified as: (a) *Block-force model*: this model represents the piezoelectric actuator in terms of its block force and free strain. The piezoelectric actuator induces a local strain to the beam and in turn experiences an equilibrating force; (b) *Uniform strain model*: in this model, the actuation strain are assumed to be uniform in the piezoelectric actuator, and the induced strain in the beam is also uniform; (c) *Pin force model*: this model assumes that the strain is uniform in the actuator and linear in the beam; (d) *Enhanced pin force model*: this model assumes a linear distribution of strain in both the actuator and the beam; (e) *Bernoulli-Euler model*: this model considers that the beam and actuators as a continuous structure. The Bernoulli-Euler beam theory is used to analyze the structure; (f) *Strain energy model*: this model is based on three assumptions: uniform strain in the actuator, linear strain in the beam, and zero strain at the lower surface of the beam; (g) *Extension-bending-torsion model* (skewed piezo): this model uses the pin-force and Bernoulli-Euler models. However, the piezoelectric actuator can be placed at an arbitrary orientation with respect to the beam axis. The actuator is assumed to induce strain along its length; (h) *Shear-based piezoelectric actuation models*: these models use piezoelectric shear actuators. The application of an electric field in the thickness direction of a piezoelectric actuator that is poled along the longitudinal direction will induce transverse shear strain.

The induced strain plate models include *classical laminate theory* (CLPT), *first-order shear deformation theory* (FSDT), *higher-order shear deformation theory* (HSDT), and *layerwise shear deformation theory* (LWSDT) (Chopra, 2002). All of these theories assume that the actuators and the substrate are integrated as plies of a laminated plate undergoing consistent deformation. The difference in the theories lies in the assumption of the terms of displacement distribution through the thickness of the plate.

Irschik et al. (1998), Krommer and Irschik (1999), and Irschik and Ziegler (2001) analyzed the piezoelectric actuation as an *eigenstrain actuation* for the vibration and shape control of structures. *Eigenstrain* is a generic name originally given by Mura (1987) to nonelastic strains resulting from thermal expansion, phase transformation, initial strains, plastic strains and misfit strains. Eshelby (1957) referred to eigenstrains as stress-free transformation strains. According to Irschik and Ziegler (2001), in the linear piezoelectricity theory, no body forces are produced by the applied electric field. The constitutive relations are the only source of coupling between the electric and mechanical fields. Hence, the electric field acts as an eigenstrain upon the piezoelectric body.

Fan and Qin (1995) analyzed a piezoelectric sensor embedded in a non-piezoelectric elastic matrix by using Eshelby's equivalent inclusion method (Eshelby, 1957; Mura, 1987). By assuming that there is no electromechanical interaction in the matrix material, the piezoelectric inclusion problem was decoupled into two equivalent inclusion problems, namely, an elastic inclusion problem and a dielectric inclusion problem. The two inclusion problems were connected by some *eigenstrain* and *eigen-electricfield*.

An eigenstrain technique for the dynamics of smart beams with embedded arrays of piezoelectric sensors and actuators was presented by Alghamdi and Dasgupta (1993a, 1993b, 2000, 2001). They analyzed the piezoelectric sensors and actuators in the host beam by using Eshelby's equivalent inclusion method. The equations of motion of the structure were derived using Hamilton's principle and the resulting equations were solved using Rayleigh-Ritz method. However, the piezoelectric inclusions were analyzed as elastic inclusions only, thereby neglecting the dielectric effects of the piezoelectric inclusions. The influence of the mechanical-electrical coupling of the piezoelectric sensors on the natural frequency was also neglected in their analyses.

2.7 Summary

The summary of the literature review is presented in this section together with the issues that will be addressed in this thesis.

2.7.1 Vibration of delaminated composite laminates

The available mathematical models for the free vibration of delaminated composite laminates have been reviewed. These models were classified according the fundamental theories and assumptions used. Comparisons between the results of the 'free mode' and 'constrained mode' models, and between the analytical/FE models and experimental data have been presented. This review further showed that a considerable amount of research in area delamination vibration have been undertaken in the recent years. These researches have established a basic understanding on the

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influence of the delamination on the natural frequencies and the mode shapes of the composite laminates. The finite element method has been widely used to study complex problems, such as composite plates with internal delaminations and delaminated beam/plates with piezoelectric sensors/actuators. Whereas, the analytical solutions have been used provide means of physical understanding and comparison with approximate/numerical models. Based on the literature review presented, there are still some issues that need to be addressed to have a better understanding on the vibration of delaminated beams. In this thesis the following issues are addressed:

1. The lack of understanding on delaminated multilayer beams.

Although many studies have been presented for delaminated homogeneous and fiber-reinforced composite beams, studies on bimaterial beams with a single delamination and three-layer beams with two equal length delaminations have not been presented in the literature. To address this issue, analytical solutions are presented and nondimensional parameters are introduced to study the vibration of delaminated bimaterial and three-layer beams.

2. The lack of understanding on the mode shapes of delaminated beams

The delamination opening in the mode shapes of delaminated beams have been observed in experiments, however, a method to predict the delamination opening and to characterize the mode shapes has not been presented in the literature. To address this issue, a new parameter, relative slenderness ratio, is introduced to study the vibration of delaminated multilayer beams.

3. Analytical solutions and parametric studies for the vibration of composite beams with two enveloping, two non-overlapping and two overlapping delaminations have not been presented in the literature

To address this issue, analytical solutions for composite beams with two enveloping, two non-overlapping and two overlapping delaminations are presented, which can serve as benchmark test case for future numerical and approximate solutions. The use of these analytical solutions allows inexpensive simulations of slight variations in the system, such as changes in the physical parameters (delamination lengths and locations) and boundary conditions.

In addition, a parametric study is performed to investigate the influence of the lengths and locations of the delamination on the vibration of the beam. This study can be used to optimize the design of composite beams and assess the effectiveness of using the natural frequency and mode shapes of the delaminated beam for damage detection.

2.7.2 Smart structures using piezoelectric sensors and actuators

The majority of the models presented in the literature for the dynamics of smart beams use piezoceramics as sensors and actuators, which are either in the form of patches or lamina. Few works have been presented for smart structures with embedded arrays of piezoelectric sensor and actuators.

An eigenstrain model for the vibration of beams with embedded array of piezoelectric sensors and actuators was presented by Alghamdi and Dasgupta (1993a, 1993b, 2000, 2001). They analyzed the piezoelectric inclusions as elastic inclusions. However, Fan and Qin (1995) showed that the piezoelectric inclusion problem can be

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decoupled to an elastic inclusion problem and a dielectric inclusion problem. Thus, the dielectric inclusion problem was neglected in Alghamdi and Dasgupta's model (1993a, 1993b). The mechanical-electrical coupling of the piezoelectric sensors was also neglected. To the author's best knowledge, a mathematical model that considers both the elastic and dielectric inclusion problems and the effects of the mechanical-electrical coupling of the piezoelectric sensors has not been reported in the literature.

To address this issue, a mathematical model for the vibration of beams with embedded arrays of piezoelectric sensors and actuators is presented, to include the elastic and dielectric inclusion problems. These piezoelectric inclusion problems will be analyzed using Eshelby's equivalent inclusion method (Eshelby, 1957; Mura, 1987). The natural frequency of the beam is determined using the variational principle in Rayleigh quotient form.

Chapter 3

New Parameters for the Vibration of Delaminated Multilayered Beams

3.1 Introduction

Many studies have been presented for delaminated homogeneous and fiber-reinforced composite beams. However, studies on bimaterial beams with a single delamination and three-layer beams with two equal length delaminations have not been presented in the literature. In this chapter, nondimensional parameters of *normalized axial stiffness*, *normalized bending stiffness*, *axial stiffness ratio* and *bending stiffness ratio* are introduced to study the vibration of bimaterial beams with a *single delamination*. The use of these parameters provides a better understanding on the influence of the bending stiffness and axial stiffness of the different layers on the vibration of delaminated bimaterial beams. Furthermore, it provides a physical understanding on the ‘constrained mode’ and ‘free mode’ assumptions, which are used in the study of vibration of delaminated beams. A *relative slenderness ratio* specific for delamination vibration is further introduced, which is shown to significantly influence the natural frequency and mode shape of the beam. Global, mixed and local

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vibration modes occur depending upon the slenderness ratios of the delaminated beams.

An analytical solution for the vibration of three-layer beams with two equal length delaminations is presented. Both, the ‘free mode’ and ‘constrained mode’ assumptions in delamination vibration are used in the present analysis. The beam is analyzed as five interconnected beams using the delaminations as their boundaries. The continuity and boundary conditions are satisfied between adjoining beams. The Euler-Bernoulli beam theory is applied to each of the beams. A relative slenderness ratio specific for the analytical solution is introduced, which is shown to dominate the vibration behavior of the beam.

The lower bound and upper bound of the natural frequency are proposed by assuming a totally ‘free’ and totally ‘constrained’ deformations of the delaminated layers. These bounds are easy to compute and provide useful approximations. Using these bounds, the vibration of sandwich beams with two equal length delaminations is studied using the normalized axial stiffness, normalized bending stiffness and relative slenderness ratio.

3.2 Formulations and Introduction on New Parameters

3.2.1 Bimaterial beams with a single delamination

The analytical solutions reported by Shu (1995) on a ‘free mode’ and Shu and Fan (1996) on a ‘constrained mode’ and are used in the present study. Figure 3.1 shows a beam with length L and thickness H_1 . The beam is made of two distinct layers, with Young’s modulus E_2 and E_3 , and thickness H_2 and H_3 . The beam is separated along

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the interface by a delamination with length a and located at a distance d from the center of the beam. The beam can then be subdivided into three spanwise regions, a delamination region and two integral regions. The delamination region is comprised of two segments (delaminated layers), beam 2 and beam 3, which are joined at their ends to the integral segments, beam 1 and beam 4. Each of the four beams are treated as Euler-Bernoulli beams, thus the analytical solutions are valid provided that $L_i \gg H_i$, where $i = 1 - 4$ (Shu and Fan, 1996).

Two assumptions were made based on the literature: (a) the first assumption concerns the complicated changing contact between the delaminated layers. Wang et al. (1982) assumed that the delaminated layers deformed ‘freely’ without touching each other (‘free mode’), which was shown to be physically inadmissible. Mujumdar and Suryanarayan (1988) then proposed a ‘constrained mode’ where the delaminated layers are assumed to be in touch along their whole length all the time, but are allowed to slide over each other; (b) the second assumption concerns the influence of the coupling between the axial and bending vibrations, which is due to the shifting of the reference axis of the delaminated layers with respect to the reference axis of the undelaminated beam. Ramkumar et al. (1979) neglected the coupling effect in their analysis and their frequency results were consistently much lower than the experimental values. Wang et al. (1982) included the coupling effect and their results showed that for short and midplane delaminations, the predicted frequencies were close to the experimental values. Similarly, Mujumdar and Suryanarayan (1988) considered the coupling effect by including the effect of the differential stretching between the delaminated layers and their results were in good agreement with the

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experimental values. In the present research, the ‘free mode’ and ‘constrained mode’ assumptions and the differential stretching are considered.

Figure 3.1. (a) A bimaterial beam is delaminated along the interface; (b) Beams 2 and 3 are connected at their ends to the integral beams 1 and 4; (c) The delaminated beam is analyzed as four interconnected beams.

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In addition, the present analysis is simplified by the following assumptions: (a) the delamination is through-the-width; (b) the two layers of the bimaterial beam are homogeneous, isotropic and linearly elastic; (c) the effect of coupling due to the difference in the material properties of the two layers is neglected; and (d) the densities of the two layers are identical. The last assumption is only relevant for the ‘free mode’ model, since the densities do not influence the ‘constrained mode’ frequency (Shu and Fan, 1996).

3.2.1.1 ‘Free mode’

For the ‘free mode’ model, the governing equation for beam i is

$$EI_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \quad (i = 1 - 4) \quad (3.1)$$

where EI_i is the bending stiffness of beam i , $w_i(x,t)$ is the midplane deflection of the beam, ρ_i is the mass density and $A_i = H_i$ is the cross-sectional area of the beam per unit width. For plane stress problem, E is the Young’s modulus, while for plane strain problem, E would be replaced by an equivalent Young’s modulus $E = E/(1-\nu^2)$, where ν is the Poisson’s ratio. The bending stiffnesses per unit width of beam 2 and beam 3 are $EI_2 = E_2 H_2^3/12$ and $EI_3 = E_3 H_3^3/12$ for rectangular cross-sections, respectively. The bending stiffness of bimaterial beam 1 and beam 4 is determined by $EI_1 = EI_2 + EI_3 + (E_2 H_2 H_3^2 + E_3 H_3 H_2^2)/4$. The ρA for beam 2 and beam 3 are $\rho_2 H_2$ and $\rho_3 H_3$, and $(\rho_2 H_2 + \rho_3 H_3)$ for beam 1 and beam 4.

Equation (3.1) is valid for each of the beams. For beam 1, the governing equation is

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$$\left(\frac{E_2 H_2^3}{12} + \frac{E_3 H_3^3}{12} + \frac{E_2 H_2 H_3^2 + E_3 H_3 H_2^2}{4} \right) \frac{\partial^4 w_1}{\partial x^4} + (\rho_2 H_2 + \rho_3 H_3) \frac{\partial^2 w_1}{\partial t^2} = 0 \quad (3.2)$$

Substituting $w_i = W_i(x)\sin(\omega t)$ for natural vibration in eqn. (3.2) and eliminating the trivial solution $\sin(\omega t) = 0$, it follows that

$$\left(\frac{E_2 H_2^3}{12} + \frac{E_3 H_3^3}{12} + \frac{E_2 H_2 H_3^2 + E_3 H_3 H_2^2}{4} \right) W_1'''' - (\rho_2 H_2 + \rho_3 H_3) \omega^2 W_1 = 0 \quad (3.3)$$

where ω is the natural frequency, $W_i(x)$ is the mode shape and prime ' denotes differentiation with respect to the x -coordinate. Similarly for beam 2 and beam 3

$$\frac{E_2 H_2^3}{12} W_2'''' - \rho_2 H_2 \omega^2 W_2 = 0 \quad (3.4)$$

$$\frac{E_3 H_3^3}{12} W_3'''' - \rho_3 H_3 \omega^2 W_3 = 0 \quad (3.5)$$

Hereafter all the boundary and continuity conditions are written out in terms of $W_i(x)$ ($i = 1 - 4$) instead of $w_i(x, t)$. The generalized solutions of eqns. (3.3) – (3.5) are,

$$W_i(x) = C_i \cos(\lambda_i \frac{x}{L}) + S_i \sin(\lambda_i \frac{x}{L}) + CH_i \cosh(\lambda_i \frac{x}{L}) + SH_i \sinh(\lambda_i \frac{x}{L}) \quad (3.6)$$

where

$$\lambda_i^4 = \frac{\omega^2 \rho_i A_i}{EI_i} L^4 \quad (3.7)$$

and where λ_i is the non-dimensional frequency. The 16 unknown coefficients C_i , S_i , CH_i and SH_i ($i = 1 - 4$) are determined from the 4 boundary conditions and 12 continuity conditions.

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The appropriate boundary conditions that can be applied at the supports, $x = x_1$ and $x = x_4$, are $W_i = 0$ and $W_i' = 0$, if the end of the beam is clamped; $W_i = 0$ and $W_i'' = 0$, if simply supported; $W_i' = 0$ and $W_i''' = 0$, if free, where $i = 1$ and 4 .

The continuity conditions for deflection, slope, bending moment and shear at the delamination junction $x = x_2$ are

$$W_1 = W_2 = W_3 \quad (3.8)$$

$$W_1' = W_2' = W_3' \quad (3.9)$$

$$EI_1 W_1'' + \frac{(E_2 H_2 E_3 H_3) H_1^2}{4a(E_2 H_2 + E_3 H_3)} (W_1' - W_4') = EI_2 W_2'' + EI_3 W_3'' \quad (3.10)$$

$$EI_1 W_1''' = EI_2 W_2''' + EI_3 W_3''' \quad (3.11)$$

The second term on the left-hand side in eqn. (3.10) represents the contribution to the bending moment from the differential stretching between beam 2 and beam 3. Equations (3.8) – (3.11) provide six equations. The same continuity conditions at $x = x_3$ result in another six equations.

The total number of boundary and continuity conditions is 16, which is equal to the total number of unknown coefficients C_b , S_b , CH_i and SH_i . A non-trivial solution exists only when the determinant of the coefficient matrix vanishes. The natural frequencies can be obtained as eigenvalues and the mode shapes, as eigenvectors.

3.2.1.2 ‘Constrained mode’

For the ‘constrained mode’ model, the governing equation for beam 1 and beam 4 is identical to eqn. (3.3). For beam 2 and beam 3, the governing equation is

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$$\left(\frac{E_2 H_2^3}{12} + \frac{E_3 H_3^3}{12} \right) W_2'''' - (\rho_2 H_2 + \rho_3 H_3) \omega^2 W_2 = 0 \quad (3.12)$$

The generalized solutions of the ‘constrained mode’ are identical in form with the ‘free mode’. The unknown coefficients C_i , S_i , CH_i and SH_i , however, are reduced to 12 coefficients which can be determined from the 4 boundary conditions and 8 continuity conditions.

The boundary conditions for the ‘constrained mode’ are the same with the boundary conditions for the ‘free mode’. The continuity conditions for deflection, slope, bending moment and shear at $x = x_2$ are,

$$W_1 = W_2 \quad (3.13)$$

$$W_1' = W_2' \quad (3.14)$$

$$EI_1 W_1'' + \frac{(E_2 H_2 E_3 H_3) H_1^2}{4a(E_2 H_2 + E_3 H_3)} (W_1' - W_4') = (EI_2 + EI_3) W_2'' \quad (3.15)$$

$$EI_1 W_1''' = (EI_2 + EI_3) W_2''' \quad (3.16)$$

where $(EI_2 + EI_3)$ is the bending stiffness at the delamination region. Similarly, we can derive the continuity conditions at $x = x_3$. For the ‘constrained mode’, the boundary conditions and continuity conditions provide 12 homogeneous equations for the 12 unknown coefficients C_i , S_i , CH_i and SH_i .

3.2.1.3 Stiffness ratios and normalized stiffnesses

In this section, the axial stiffness ratio, bending stiffness ratio, normalized axial stiffness and normalized bending stiffness are introduced. The modulus ratio E_R is

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defined as $E_R = E_2/E_3$ and the thickness ratio H_R is $H_R = H_2/H_3$. The axial stiffness ratio and bending stiffness ratio are defined, respectively, as

$$\frac{E_2 H_2}{E_3 H_3} = E_R H_R \quad (3.17)$$

$$\frac{EI_2}{EI_3} = (E_R H_R) H_R^2 \quad (3.18)$$

The normalized axial stiffnesses are

$$\overline{EH}_2 = \frac{E_2 H_2}{E_2 H_2 + E_3 H_3} = \frac{E_R H_R}{1 + E_R H_R} \quad (3.19)$$

$$\overline{EH}_3 = \frac{E_3 H_3}{E_2 H_2 + E_3 H_3} = \frac{1}{1 + E_R H_R} \quad (3.20)$$

and the normalized bending stiffnesses are

$$\overline{EI}_2 = \frac{EI_2}{EI_1} = \frac{E_R H_R (H_R)^2}{1 + E_R H_R (H_R)^2 + 3(E_R H_R + H_R^2)} \quad (3.21)$$

$$\overline{EI}_3 = \frac{EI_3}{EI_1} = \frac{1}{1 + E_R H_R (H_R)^2 + 3(E_R H_R + H_R^2)} \quad (3.22)$$

The normalized bending stiffness at the delamination region is $\overline{EI}_2 + \overline{EI}_3$, which can be expressed in terms of stiffness ratio E_R and thickness ratio H_R as

$$\begin{aligned} \overline{EI}_2 + \overline{EI}_3 &= \frac{1 + E_R H_R (H_R)^2}{1 + E_R H_R (H_R)^2 + 3(E_R H_R + H_R^2)} \\ &= \frac{1 + EI_2/EI_3}{1 + \frac{EI_2}{EI_3} + 3\left(E_R H_R + \frac{EI_2/EI_3}{E_R H_R}\right)} \end{aligned} \quad (3.23)$$

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From eqns. (3.18) and (3.23), we can conclude that as the axial stiffness ratio or the bending stiffness ratio increases, the normalized bending stiffness at the delamination region $\overline{EI}_2 + \overline{EI}_3$ decreases. By substituting eqn. (3.19) in eqn. (3.21) and rearranging, the thickness ratio can be expressed in terms of $\overline{EI}_2$ and $\overline{EH}_2$ as

$$H_R = \sqrt{\frac{\overline{EI}_2 (1 + 2\overline{EH}_2)}{\overline{EH}_2 - \overline{EI}_2 (3 - 2\overline{EH}_2)}} \quad (3.24)$$

Substituting eqn. (3.19) in eqn. (3.23), the bending stiffness at the delamination region can be expressed in terms of $\overline{EH}_2$ and H_R as

$$\overline{EI}_2 + \overline{EI}_3 = \frac{1 - (1 - H_R^2) \overline{EH}_2}{1 + 2\overline{EH}_2 + (3 - 2\overline{EH}_2) H_R^2} \quad (3.25)$$

Therefore, the range of $\overline{EH}_2$ and $\overline{EI}_2$ is limited by $\left[1 + 2\overline{EH}_2 + (3 - 2\overline{EH}_2) H_R^2\right] > 0$.

Furthermore, from eqns. (3.24) and (3.25), we can conclude that as the normalized bending stiffness $\overline{EI}_2$ increases, the normalized bending stiffness at the delamination region $\overline{EI}_2 + \overline{EI}_3$ decreases.

3.2.1.4 Relative slenderness ratio

In this section, the relative slenderness ratio R is introduced. The relative slenderness ratio is defined as the ratio of the frequency of the undelaminated beam to that of the frequency of the weaker delaminated layer (beam 2 or beam 3) in the delamination region

$$R = \omega_0 / \text{MIN}(\omega_2, \omega_3) \quad (3.26)$$

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where $\omega_0 = (\lambda_0 / L)^2 \sqrt{EI_0 / \rho_0 A_0}$ is the fundamental frequency of the undelaminated beam and $\omega_2 = (\lambda_2 / a)^2 \sqrt{EI_2 / \rho_2 A_2}$ and $\omega_3 = (\lambda_3 / a)^2 \sqrt{EI_3 / \rho_3 A_3}$ are the fundamental frequencies of beam 2 and beam 3, respectively. In the analysis, beam 2 and beam 3 are considered to be clamped along their edges, and thus, $\lambda_2 = \lambda_3 = 4.73$.

3.2.2 Three-layer beams with double delaminations

Figure 3.2(a) shows a beam made of three distinct layers with Young's modulus E_2 , E_3 and E_4 , and thickness H_2 , H_3 and H_4 . The two delaminations are of identical length a and located at a distance d from the center of the beam. For brevity, this delamination configuration will be referred to as *double delaminations*. The beam is analyzed as five interconnected beams (Figure 3.2(b)). The delaminated layers, beams 2, 3 and 4, are connected at their ends to the integral beams 1 and 5.

Figure 3.2. (a) Geometry of a beam with double delaminations; (b) The beam with is modeled as five interconnected beams.

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When the beam vibrates, the weakest amongst beams 2, 3 and 4 tends to have a higher magnitude of deformation than the stronger ones. However, the weakest beam may be ‘constrained’ by the others, and they have to vibrate together in a ‘constrained mode’. This greatly complicates the problem, for homogeneous beams, the following cases are then considered:

(1) ‘Free mode’: $H_4 > H_3 > H_2$ during upward motion, or $H_2 > H_3 > H_4$ during downward motion. Without loss of generality, $H_4 > H_3 > H_2$ is chosen (Figure 3.3(a)). In this case, the delaminated beams 2, 3 and 4 vibrate ‘freely’ and have different transverse deformations.

(2) ‘Partially constrained mode’: $H_4 > H_2 > H_3$ during upward motion, or $H_2 > H_4 > H_3$ during downward motion. $H_4 > H_2 > H_3$ is chosen (Figure 3.3(b)). Although the thinnest beam 3 vibrates first, it will impinge on and be stopped by the thicker beam 2. The two beams vibrate together which will be followed by the thickest beam 4.

(3) ‘Constrained mode’: $H_2 > H_3 > H_4$ or $H_2 > H_4 > H_3$ during upward motion, or $H_4 > H_3 > H_2$ or $H_4 > H_2 > H_3$ during downward motion. $H_4 > H_3 > H_2$ (Figure 3.3(c)) and $H_4 > H_2 > H_3$ are chosen and the results will be compared with the results of cases 1 and 2. The two thinner beams 2 and 3 are ‘constrained’ by the thickest beam 4 and the three beams vibrate together. The formulation of the ‘partially constrained mode’ is similar to the ‘free mode’ and the ‘constrained mode’ and is omitted for the sake of brevity. The following sections present the formulation of the ‘free mode’ and the ‘constrained mode’ models for the vibration of a beam with double delaminations.

Figure 3.3. The beam can either vibrate ‘freely’ or in a ‘constrained’ manner. (a) ‘Free mode’; (b) ‘Partially constrained mode’; (c) ‘Constrained mode’.

3.2.2.1 ‘Free mode’

For the ‘free mode’ model, the governing equations for the free vibration of a tri-layered beam with double delaminations are

$$EI_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \quad (i = 1 - 5) \quad (3.27)$$

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The generalized solutions are identical in form with the single delamination (eqn. (3.6)). There are 20 unknown coefficients C_i , S_i , CH_i and SH_i and can be determined from the 4 boundary conditions and 16 continuity conditions.

The boundary conditions are the same with the single delamination. The continuity conditions for deflection and slope at $x = x_2$ are

$$W_1 = W_2 = W_3 = W_4 \quad (3.28)$$

$$W_1' = W_2' = W_3' = W_4' \quad (3.29)$$

From Figure 3.4, the continuity for shear and bending moments at $x = x_2$ are

$$V_1 = V_2 + V_3 + V_4 \quad (3.30)$$

$$M_1 = M_2 + M_3 + M_4 + P_2 \left(\frac{H_2}{2} \right) + P_3 \left(H_2 + \frac{H_3}{2} \right) + P_4 \left(H_2 + H_3 + \frac{H_4}{2} \right) \quad (3.31)$$

where

$$V_i = -EI_i W_i''' \quad (3.32)$$

$$M_i = -EI_i W_i'' \quad (i = 1 - 4) \quad (3.33)$$

Figure 3.4. Continuity of shear and moment at the delamination boundary $x = x_2$.

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The axial forces P_i can be solved from the compatibility between the stretching/shortening of the delaminated layers and axial equilibrium (Mujumdar and Suryanarayan, 1988), thus

$$\frac{P_3 a}{E_3 A_3} - \frac{P_2 a}{E_2 A_2} = (W_1'(x_2) - W_5'(x_3)) \frac{H_2 + H_3}{2} \quad (3.34)$$

$$\frac{P_4 a}{E_4 A_4} - \frac{P_3 a}{E_3 A_3} = (W_1'(x_2) - W_5'(x_3)) \frac{H_3 + H_4}{2} \quad (3.35)$$

$$P_1 = P_2 + P_3 + P_4 = 0 \quad (3.36)$$

and eqns. (3.30) and (3.31) can be expressed as

$$EI_1 W_1''' = EI_2 W_2''' + EI_3 W_3''' + EI_4 W_4''' \quad (3.37)$$

$$EI_1 W_1'' + \frac{(H + H_3)^2 E_2 A_2 E_4 A_4 + (H_2 + H_3)^2 E_2 A_2 E_3 A_3 + (H_3 + H_4)^2 E_3 A_3 E_4 A_4}{4a(E_2 A_2 + E_3 A_3 + E_4 A_4)} \times (W_1'(x_2) - W_5'(x_3)) = EI_2 W_2'' + EI_3 W_3'' + EI_4 W_4'' \quad (3.38)$$

where

$$EI_1 = EI_2 + EI_3 + EI_4 + \frac{E_2 H_2 (H_3 + H_4)^2 + E_3 H_3 (H_2 - H_4)^2 + E_4 H_4 (H_2 + H_3)^2}{4} \quad (3.39)$$

The second term on the left-hand side in eqn. (3.38) represents the contribution to the bending moment from the differential stretching between beams 2, 3 and 4 and contributes to the bending stiffness of the beam.

Similarly, we can derive the continuity conditions at $x = x_3$. The boundary conditions and the continuity conditions provide 20 homogeneous equations for 20 unknown coefficients C_i , S_i , CH_i and SH_i . A non-trivial solution for the coefficients

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exists only when the determinant of the coefficient matrix vanishes. The frequencies and mode shapes can be obtained as eigenvalues and eigenvectors, respectively.

A FORTRAN program is developed to solve the eigenvalue problem. An LU decomposition, which is a procedure for decomposing a square matrix into a product of *lower triangular matrix* L and an *upper triangular matrix* U, is performed to determine the equation for the determinant of the coefficient matrix, where all the components are expressed in terms of the natural frequency ω . A root bisection method is then performed to determine the roots of the determinant, where the roots are the eigenvalues. The input of the expressions in the 20 x 20 matrix and the subsequent LU decomposition has to be done with care to ensure successful operation.

3.2.2.2 ‘Constrained mode’

For the ‘constrained mode’ model, the governing equations are

$$EI_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \quad (i = 1 \text{ and } 5) \quad (3.40)$$

For delaminated beams 2, 3 and 4, we have

$$(EI_2 + EI_3 + EI_4) \frac{\partial^4 w_2}{\partial x^4} + (\rho_2 A_2 + \rho_3 A_3 + \rho_4 A_4) \frac{\partial^2 w_2}{\partial t^2} = 0 \quad (3.41)$$

The generalized solutions for the ‘constrained mode’ are identical in form with the ‘free mode’. The unknown coefficients C_i , S_i , CH_i and SH_i , however, are reduced to 12 coefficients which can be determined from the 4 boundary conditions and 8 continuity conditions.

The continuity conditions for deflection, slope, shear and bending moments at $x = x_2$ are

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$$W_1 = W_2 \quad (3.42)$$

$$W_1' = W_2' \quad (3.43)$$

$$EI_1 W_1''' = (EI_2 + EI_3 + EI_4) W_2''' \quad (3.44)$$

$$EI_1 W_1'' + \frac{(H + H_3)^2 E_2 A_2 E_4 A_4 + (H_2 + H_3)^2 E_2 A_2 E_3 A_3 + (H_3 + H_4)^2 E_3 A_3 E_4 A_4}{4a(E_2 A_2 + E_3 A_3 + E_4 A_4)} \times (W_1'(x_2) - W_2'(x_3)) = (EI_2 + EI_3 + EI_4) W_2'' \quad (3.45)$$

Similarly, we can derive the continuity conditions at $x = x_3$. For the ‘constrained mode’, the boundary conditions and the continuity conditions provide 12 homogeneous equations for 12 unknown coefficients C_i , S_i , CH_i and SH_i .

3.2.3 Sandwich beams with double delaminations

For the sandwich beam, the thickness and elastic modulus of the two outer layers are identical ($H_2 = H_4$, $E_2 = E_4$). The normalized axial stiffnesses are

$$\overline{EH}_2 = \overline{EH}_4 = \frac{E_2 H_2}{E_2 H_2 + E_3 H_3 + E_4 H_4} \quad (3.46)$$

$$\overline{EH}_3 = \frac{E_3 H_3}{E_2 H_2 + E_3 H_3 + E_4 H_4} \quad (3.47)$$

The normalized bending stiffnesses are

$$\overline{EI}_2 = \overline{EI}_4 = EI_2 / EI_1 \quad (3.48)$$

$$\overline{EI}_3 = EI_3 / EI_1 \quad (3.49)$$

3.3 Lower Bound and Upper Bound of the Natural Frequency

This section proposes to estimate the natural frequency with upper and lower bounds. The upper bound is first considered. In general, one of the beams 2, 3 and 4 tends to vibrate first. For homogeneous beams, when $H_4 > H_2 > H_3$, although the weakest beam 3 vibrates first, it will impinge on and be stopped by the stronger beam 2. The two beams vibrate together which will be followed by the beam 4. However, such analysis is tedious and complex. For multiple delaminations, the solution will be very lengthy and the order of complexity will render an exact solution impossible. An upper bound can be obtained if all of the beams are assumed to vibrate simultaneously together ('constrained mode'). Constraints strengthen the delaminated beam and increase its natural frequency. This simple solution avoids the tedious consideration of sequence vibration amongst beams 2, 3 and 4. The lower bound solution assumes that beams 2, 3 and 4 vibrates 'freely' ('free mode'), which ignores the interaction or constraints amongst the beams and thus reduces the natural frequency.

3.4 Results and Discussions

3.4.1 Verification

To verify the accuracy of the present results, a comparison with published results on a single delamination is made. The first three non-dimensional natural frequencies of a clamped-clamped beam with a midplane and central delamination having various lengths are compared with the analytical results of Wang et al. (1982) and FEM results

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of Lee (2000). Tables 3.1 – 3.3 show good agreement between the present results and the analytical and FEM results.

Table 3.1.

Non-dimensional fundamental frequency (λ^2) of a clamped-clamped isotropic beam with a midplane delamination.

Delamination length, a/L	Present Constrained and Free	Analytical Wang et al. (1982)	FEM Lee(2000)
0.00	22.37	22.39	22.36
0.10	22.37	22.37	22.36
0.20	22.36	22.35	22.35
0.30	22.24	22.23	22.23
0.40	21.83	21.83	21.82
0.50	20.89	20.88	20.88
0.60	19.30	19.29	19.28
0.70	17.23	17.23	17.22
0.80	15.05	15.05	15.05
0.90	13.00	13.00	12.99

Table 3.2

Non-dimensional mode 2 frequency (λ^2) of a clamped-clamped isotropic beam with a midplane delamination

Delamination length, a/L	Present Constrained and Free	Analytical Wang et al. (1982)	FEM Lee (2000)
0.00	61.67	61.67	61.61
0.10	60.81	60.76	60.74
0.20	56.00	55.97	55.95
0.30	49.00	49.00	48.97
0.40	43.89	43.87	43.86
0.50	41.52	41.45	41.50
0.60	41.04	40.93	41.01
0.70	40.82	40.72	40.80
0.80	39.07	39.01	39.04
0.90	35.39	35.38	35.38

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Table 3.3

Non-dimensional mode 3 frequency (λ^2) of a clamped-clamped isotropic beam with a midplane delamination.

Delamination length, a/L	Present Constrained and Free	Analytical Wang et al. (1982)	FEM Lee (2000)
0.00	120.90	120.91	120.68
0.10	120.83	120.81	120.62
0.20	118.87	118.76	118.69
0.30	109.16	109.04	109.03
0.40	93.59	93.57	93.51
0.50	82.29	82.29	82.23
0.60	77.69	77.64	77.64
0.70	77.18	77.05	77.12
0.80	75.43	75.33	75.39
0.90	69.19	69.17	69.16

3.4.2 Bimaterial beams with a single delamination

3.4.2.1 Stiffness ratios and normalized stiffnesses

Figure 3.5 shows the variation of the fundamental frequency ω/ω_o with the bending stiffness ratio EI_2/EI_3 of a clamped-clamped beam with a central delamination ($d/L = 0$). The fundamental frequency of the delaminated beam ω is normalized with respect to the fundamental frequency of the undelaminated beam ω_o . The axial stiffness ratio is $E_2H_2/E_3H_3 = 0.5$. The fundamental frequency varies in three phases. For the ‘constrained mode’, in phase I, ω/ω_o decreases rapidly as EI_2/EI_3 increases. In phase II, ω/ω_o decreases slowly. In phase III, ω/ω_o displays an almost constant frequency. The following explanation is offered: The fundamental frequency ω/ω_o is significantly influenced by the bending stiffness in the delamination region, $\overline{EI}_2 + \overline{EI}_3$. When the axial stiffness ratio, E_2H_2/E_3H_3 or E_RH_R , is constant, the change in $\overline{EI}_2 + \overline{EI}_3$ is dependent on the bending stiffness ratio, EI_2/EI_3 (see eqn. (3.23)). In

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phase I, $EI_2/EI_3 \leq E_R H_R$ and increasing EI_2/EI_3 will cause a drastic decrease in $\overline{EI}_2 + \overline{EI}_3$ and thus, ω/ω_o decreases drastically. In phase III, $EI_2/EI_3 \gg E_R H_R$ and $\overline{EI}_2 + \overline{EI}_3$ is almost constant, and ω/ω_o is almost constant.

Figure 3.5. Variation of the fundamental frequency ω/ω_o with the bending stiffness ratio EI_2/EI_3 of a clamped-clamped beam ($E_2H_2/E_3H_3 = 0.5$).

For the ‘free mode’, for $a/L > 0.5$, ω/ω_o increases in phase I, stabilizes in phase II and decreases in phase III. The following explanation is offered: for the ‘free mode’, the natural frequency is dominated by the weaker delaminated layer. In phases I and II, beam 2 is weaker than beam 3 ($\omega_2 < \omega_3$). In phase I, as EI_2/EI_3 increases, beam 2 strengthens and thus ω/ω_o increases. In phase II, the frequency of beam 2 is almost the same as beam 3, $\omega_2 \approx \omega_3$, and an almost constant frequency is observed. In phase III, as EI_2/EI_3 further increases, beam 3 weakens and thus ω/ω_o decreases.

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As expected, for both the ‘free mode’ and ‘constrained mode’, ω/ω_o decreases as the delamination length a/L increases. This can be explained by the decrease in the moment contribution from the differential stretching, which is represented by the second term on the left-hand side of eqns. (3.10) and (3.15), as a/L increases. In phases II and III, the decrease in ω/ω_o due to the increase in a/L is almost constant. This indicates that for $EI_2/EI_3 > 0.5$, the decrease in the natural frequency due to the increase of delamination length is not significantly influenced by the bending stiffness ratio. The two modes predict the same frequency at $EI_2/EI_3 = 2.0$ ($EI_2/EI_3 = E_3H_3/E_2H_2$). This is because the frequencies of beam 2 and beam 3 are identical, $\omega_2 = \omega_3$, and the two beams vibrate in a ‘constrained’ manner.

For $E_2H_2/E_3H_3 = 1.0$, the ‘constrained mode’ frequency is almost constant for $a/L > 0.5$ (Figure 3.6) (only half is shown due to the symmetry the results, i.e. ω/ω_o is the same when $EI_2/EI_3 = EI_3/EI_2$). This is because the bending stiffness at the delaminated region is constant ($\overline{EI}_2 + \overline{EI}_3 = 0.25$, see eqn. (3.23)). For $a/L < 0.5$, lower frequencies are observed for $EI_2/EI_3 < 0.5$. This can be explained by decrease in the differential stretching due to the uneven distribution of the bending stiffnesses. The moment contribution from the differential stretching is proportional to the difference between the slopes at the delamination junctions, $(W_1' - W_4')$ (see eqn. (3.15)). This difference is reduced due to the uneven distribution of the bending stiffnesses. For a clamped-clamped beam, the influence of $(W_1' - W_4')$ decreases as the delamination length increases (Shu, 1995). This explains the almost constant frequency for $a/L > 0.4$. For the ‘free mode’, ω/ω_o increases monotonically with increasing bending stiffness ratio. The ‘free’ and ‘constrained’ modes predict the same frequency at

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$EI_2/EI_3 = 1.0$, where $\omega_2 = \omega_3$. It should be noted that $E_2H_2/E_3H_3 = 1.0$ and $EI_2/EI_3 = 1.0$ corresponds to a homogeneous beam with a midplane delamination.

Figure 3.6. Variation of the fundamental frequency ω/ω_0 with the bending stiffness ratio EI_2/EI_3 of a clamped-clamped beam ($E_2H_2/E_3H_3 = 1.0$).

The difference between the ‘free mode’ and ‘constrained mode’ frequencies largely depend on the opening of the mode shapes using the ‘free mode’. Figure 3.7 shows the mode shapes for the circled geometry in Figure 3.6. Three types of vibration modes are observed, global, mixed and local vibration modes. In a global vibration mode, the amplitudes of beam 2 and beam 3 are almost equal (Figure 3.7(a)), in a mixed vibration mode, the amplitudes of the beam 2 and beam 3 are comparable (Figure 3.7(b)), and in a local vibration mode, beam 2 will have a much higher amplitude than beam 3 (Figure 3.7(c)). The opening in the mode shape decreases as EI_2/EI_3 increases, which corresponds to the decreasing difference between the ‘constrained mode’ and the ‘free mode’ frequencies (Figure 3.6).

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Figure 3.7. Vibration modes at $a = 0.7L$, $d = 0.0$ and $E_2H_2/E_3H_3 = 1.0$ for various bending stiffness ratios: (a) $EI_2/EI_3 = 1.0$; (b) $EI_2/EI_3 = 0.4$; (c) $EI_2/EI_3 = 0.1$.

Figure 3.8 shows the influence of the axial stiffness ratio E_2H_2/E_3H_3 on the fundamental frequency ω/ω_o of a clamped-clamped beam. The spanwise location of the delamination is central ($d/L = 0$) and the bending stiffness ratio is $EI_2/EI_3 = 0.5$. For the ‘constrained mode’, ω/ω_o varies in three phases. In phase I, ω/ω_o increases as the axial stiffness ratio increases up to $E_2H_2/E_3H_3 = 0.75$, where $EI_2 + EI_3$ is maximum. In phase II, ω/ω_o decreases rapidly and in phase III, ω/ω_o decreases relatively slower. ω/ω_o further decreases as the delamination length a/L increases, which is almost constant for $a/L > 0.4$ in phases II and III. This indicates that the decrease in ω/ω_o due to the increase in delamination length a/L is not significantly influenced by the axial stiffness ratio for $E_2H_2/E_3H_3 > 0.75$ and $a/L > 0.4$. A similar trend is observed for the ‘free mode’. The two modes predict the same frequency when $E_2H_2/E_3H_3 = 2.0$ ($E_2H_2/E_3H_3 = EI_3/EI_2$), where $\omega_2 = \omega_3$. A large difference between

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the frequencies of the two modes is observed when the ‘constrained mode’ frequency is maximum. This indicates that for a weak surface beam 2, a higher ‘constrained mode’ frequency results in a larger delamination opening.

In Figure 3.9 ($EI_2/EI_3 = 1.0$), for both the ‘constrained mode’ and ‘free mode’, the fundamental frequency varies in three phases (only half is shown due to the symmetry of the results, i.e. ω/ω_o is the same when $E_2H_2/E_3H_3 = E_3H_3/E_2H_2$). In phase I, ω/ω_o increases rapidly as E_2H_2/E_3H_3 increases. In phase II, ω/ω_o increases relatively slower, and in phase III, ω/ω_o displays an almost constant frequency. In phases II and III, the decreases in ω/ω_o due to the increase of delamination length a/L is not significantly affected by the axial stiffness ratio for long delaminations ($a/L > 0.4$), since the decrease is almost constant.

Figure 3.10 shows the variation of the fundamental frequency ω/ω_o with the normalized bending stiffness of the surface layer $\overline{EI}_2$ of a clamped-clamped beam with a central delamination ($d/L = 0$). The normalized axial stiffness is $\overline{EH}_2 = 0.3$. For the ‘constrained mode’, ω/ω_o decreases as $\overline{EI}_2$ increases. This is because, the bending stiffness at the delamination region, $\overline{EI}_2 + \overline{EI}_3$, decreases as $\overline{EI}_2$ increases, as shown by eqns. (3.24) and (3.25). For the ‘free mode’, ω/ω_o first increases up to about $\overline{EI}_2 = 0.11$ and then decreases.

Figure 3.6 is repeated in Figure 3.11, where ω/ω_o is with respect to the normalized bending stiffness. For $\overline{EH}_2 = 0.5$, the ‘constrained mode’ predicts an almost constant frequency for $a/L > 0.5$ (only half is presented due to symmetry of the results, i.e. $\overline{EI}_2 = 0.0 \rightarrow 0.125$ and $\overline{EI}_2 = 0.125 \rightarrow 0.25$). With $a/L < 0.4$, slightly lower frequencies are observed at $\overline{EI}_2 < 0.08$. For the ‘free mode’, ω/ω_o increases

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monotonically with $\overline{EI}_2$. The ‘free mode’ predicts the same frequency as the ‘constrained mode’ at $\overline{EI}_2 = 0.125$, where $\omega_2 = \omega_3$. It should be noted that $\overline{EH}_2 = 0.5$ and $\overline{EI}_2 = 0.125$ corresponds to a homogeneous beam with a midplane delamination.

Figure 3.8. Variation of the fundamental frequency ω/ω_0 with the axial stiffness ratio E_2H_2/E_3H_3 of a clamped-clamped beam ($EI_2/EI_3 = 0.5$).

Figure 3.9. Variation of the fundamental frequency ω/ω_0 with the axial stiffness ratio E_2H_2/E_3H_3 of a clamped-clamped beam ($EI_2/EI_3 = 1.0$).

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Figure 3.10. Variation of the fundamental frequency ω/ω_o with the normalized bending stiffness $\overline{EI}_2$ of a clamped-clamped beam with a central delamination ($d/L = 0$) at $\overline{EH}_2 = 0.3$.

Figure 3.11. Variation of the fundamental frequency ω/ω_o with the normalized bending stiffness $\overline{EI}_2$ of a clamped-clamped beam with a central delamination ($d/L = 0$) at $\overline{EH}_2 = 0.5$.

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Figures 3.12 and 3.13 show the variation of the fundamental frequency ω/ω_o with the normalized axial stiffness $\overline{EH}_2$ of a clamped-clamped beam. The spanwise location of the delamination is central ($d/L = 0$). Figure 3.12 shows a typical result for $\overline{EI}_2 < 0.125$ and Figure 3.13 for $\overline{EI}_2 > 0.125$. In Figure 3.12, for the ‘constrained mode’, ω/ω_o varies in two phases. In phase I, ω/ω_o increases up to about $\overline{EH}_2 = 0.375$, where $\overline{EI}_2 + \overline{EI}_3$ is maximum. In phase II, ω/ω_o decreases monotonically with increasing $\overline{EH}_2$, since $\overline{EI}_2 + \overline{EI}_3$ decreases as $\overline{EH}_2$ increases. ω/ω_o further decreases as the delamination length a/L increases, which is almost constant for long delaminations ($a/L > 0.4$). For the ‘free mode’, ω/ω_o first increases up to about $\overline{EH}_2 = 0.625$, where $\omega_2 \approx \omega_3$, and then decreases. A large difference between the frequencies of the two modes is observed when the ‘constrained mode’ frequency is maximum, which indicates that for a much weaker surface beam 2, a higher ‘constrained mode’ frequency results in a large delamination opening.

For $\overline{EI}_2 = 0.15$ (Figure 3.13), the ‘constrained mode’ frequency varies slightly as $\overline{EH}_2$ increases. ω/ω_o decreases with increasing delamination length a/L , which is almost the same for any $\overline{EH}_2$. For the ‘free mode’, ω/ω_o increases up to about $\overline{EH}_2 = 0.5$ and then decreases. The difference between the frequencies of the two modes is less significant for short delaminations ($a/L < 0.4$), while it becomes significant at $\overline{EH}_2 < 0.4$ and $\overline{EH}_2 > 0.6$ for long delaminations ($a/L > 0.5$).

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Figure 3.12. Variation of the fundamental frequency ω/ω_o with the normalized axial stiffness $\overline{EH}_2$ of a clamped-clamped beam with a central delamination ($d/L = 0$) at $\overline{EI}_2 = 0.075$.

Figure 3.13. Variation of the fundamental frequency ω/ω_o with the normalized axial stiffness $\overline{EH}_2$ of a clamped-clamped beam with a central delamination ($d/L = 0$) at $\overline{EI}_2 = 0.15$.

Figures 3.14 and 3.15 shows the variation of the fundamental frequency ω/ω_o with EI_2/EI_3 of a cantilever beam with a central delamination ($d/L = 0$). In Figure 3.14

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($E_2H_2/E_3H_3 = 0.5$), ω/ω_o decreases in three phases. ω/ω_o decreases rapidly as EI_2/EI_3 increases in phase I, ω/ω_o decreases slowly in phase II, and ω/ω_o stabilizes in phase III. In Figure 3.15 ($E_2H_2/E_3H_3 = 1.0$), ω/ω_o increases up to about $EI_2/EI_3 = 0.5$, after which ω/ω_o displays an almost constant frequency. This indicates that for a cantilever beam, the decrease in the difference between the slopes ($W_1' - W_4'$) (see eqns. 3.10 and 3.15) due to the uneven distribution of the stiffness significantly influence the natural frequency. In both figures, the ‘free’ and ‘constrained’ modes predict almost the same frequencies.

Figures 3.16 and 3.17 show the variation of the fundamental frequency ω/ω_o with the normalized bending stiffness $\overline{EI}_2$ of a cantilever beam with a central delamination ($d/L = 0$). In Figure 3.16 ($\overline{EH}_2 = 0.3$), for both ‘constrained’ and ‘free’ modes, ω/ω_o decreases monotonically with increasing $\overline{EI}_2$ and increasing a/L . Figure 3.15 is repeated in Figure 3.17, where ω/ω_o is with respect to the normalized bending stiffness. For $\overline{EH}_2 = 0.5$, ω/ω_o increases up to about $\overline{EI}_2 = 0.075$, after which ω/ω_o displays an almost constant value.

For a large delamination opening, the difference between the ‘free mode’ and ‘constrained mode’ frequencies is large, and for a small delamination opening, the difference is small. In both Figures 3.16 and 3.17, the two modes predict almost the same frequencies. However, a slight difference between the two frequencies is observed for $\overline{EI}_2 < 0.0625$ and $a/L > 0.6$. This indicates that opening in the mode shape will occur when the surface layer is much weaker than the other and when the delamination is long (Figure 3.18).

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Figure 3.14. Variation of the fundamental frequency ω/ω_0 with the bending stiffness ratio EI_2/EI_3 of a cantilever beam ($E_2H_2/E_3H_3 = 0.5$) for the 'constrained' and 'free' modes.

Figure 3.15. Variation of the fundamental frequency ω/ω_0 with the bending stiffness ratio EI_2/EI_3 of a cantilever beam ($E_2H_2/E_3H_3 = 1.0$).

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Figure 3.16. Variation of the fundamental frequency ω/ω_o with the normalized bending stiffness $\overline{EI}_2$ of a cantilever beam with a central delamination ($d/L = 0$) at $\overline{EH}_2 = 0.3$.

Figure 3.17. Variation of the fundamental frequency ω/ω_o with the normalized bending stiffness $\overline{EI}_2$ of a cantilever beam with a central delamination ($d/L = 0$) at $\overline{EH}_2 = 0.5$.

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Figure 3.18. Opening in the mode shape occur when the delamination is long ($a/L > 0.6$) and the surface layer is much weaker than the other, $\overline{EH}_2 = 0.3$, $\overline{EI}_2 = 0.0125$, $H_2/H_1 = 0.21$.

3.4.2.2 Relative slenderness ratio

Figure 3.19 shows the fundamental frequency with the surface layer thickness, H_2 , for a homogeneous beam ($E_2/E_3 = 1$). For homogeneous beams $R = (a/L)^2 / (H_2/H_1)$. $R < 0.5$ represents a global vibration mode (Figure 3.20(a)), $0.5 < R < 1.0$, a mixed vibration mode (Figure 3.20(b)), and $R > 1.0$, a local vibration mode (Figure 3.20(c)). In Figure 3.19, for the ‘constrained mode’, the fundamental frequency ω/ω_o does not change perceptibly until the delamination depth decreases H_2/H_1 is greater than 15%. ω/ω_o decreases rapidly after a threshold value H_2 of about $0.2H_1$. With a higher R , ω/ω_o decreases more rapidly. For the ‘free mode’, ω/ω_o decreases with H_2/H_1 and curve $R = 1$ has the largest decrease. At higher slenderness ratio ($R > 1.25$), R determines and almost constant frequency. The ‘free mode’ and ‘constrained mode’ frequencies are identical at $H_2 = 0.5 H_1$ (midplane delamination), which results in a global vibration mode.

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Figure 3.19. The fundamental frequency ω/ω_o varies with delamination depth H_2/H_1 of a clamped homogeneous beam.

Figure 3.20. Vibration mode shapes for various slenderness ratios R at $H_2 = 0.25 H_1$: (a) Global; (b) Mixed; (c) Local.

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Figure 3.21 shows the variation of the fundamental frequency of a bimaterial beam with the normalized bending stiffness EI_2/EI_1 for $E_2H_2/E_1H_1 = 0.3$ (Figure 3.21(a)) and $E_2H_2/E_1H_1 = 0.5$ (Figure 3.21(b)). Only the results when beam 2 is weaker than beam 3 are presented, since the results are identical when beam 3 is weaker than beam 2. In Figure 3.21(a), for the ‘constrained mode’, ω/ω_o decreases with increasing EI_2/EI_1 and it decreases further with increasing R . For the ‘free mode’, for curves $R < 1$, ω/ω_o decreases slightly. At higher slenderness ratio ($R > 1$), R determines an almost constant frequency. In Figure 3.21(b), for the ‘constrained mode’, ω/ω_o is almost constant for $R = 0.5$, while for at $R > 0.5$, ω/ω_o decreases monotonically with increasing EI_2/EI_1 and increasing R . For the ‘free mode’, ω/ω_o is almost constant. Figures 3.21(a,b) further show that the difference between the frequencies of the two modes is small at $R = 0.5$. This indicates that for delaminated bimaterial beams, global vibration modes occur for low slenderness ratios $R < 0.5$.

Figure 3.22 shows the dominating influence of the slenderness ratio R on the ‘free mode’ frequency of the homogeneous beam with a central delamination ($d/L = 0.0$). The $(H_2 H_3)$ couples label the five curves by their order. For higher slenderness ratio, local vibration mode dominates and, thus, R determines the fundamental frequency. For lower R , global vibration mode prevails and the influence of the delaminations decreases. It is possible to obtain a good approximation of the fundamental frequency at higher slenderness ratio ($R > 1.25$), since the curves are close to each other. In addition, the fundamental frequency ω/ω_0 is shown to be slightly affected by the location of the delamination along the thickness of the beam.

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Figure 3.21. The fundamental frequency ω/ω_0 varies with normalized bending stiffness EI_2/EI_1 of a clamped bimaterial beam: (a) $E_2H_2/E_1H_1 = 0.3$; (b) $E_2H_2/E_1H_1 = 0.5$.

Figure 3.23 shows the significant influence of the slenderness ratio on the ‘free mode’ frequency of a delaminated bimaterial beam for $EI_2/EI_1 = 0.05$ (Figure 3.23(a)) and $EI_2/EI_1 = 0.1$ (Figure 3.23(b)) with various normalized axial stiffness E_2H_2/E_1H_1 . As R increases ω/ω_0 decreases and difference between the frequencies for different E_2H_2/E_1H_1 decreases. A good approximation of the fundamental frequency can be obtained for $R > 1$.

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Figure 3.22. Slenderness ratio dominates the ‘free mode’ frequency of a clamped homogeneous beam.

Figure 3.23. Slenderness ratio dominates the ‘free mode’ frequency of a clamped bimaterial beam.

3.4.3 Homogeneous beams with double delaminations

This section presents the results obtained using the analytical model described in Section 3.2.2 to study a clamped-clamped homogeneous beam with double delaminations. Figure 3.24 shows the influence of the surface layer thickness, H_2 , on the fundamental frequency of the beam. The delaminations are symmetrically located about the beam center ($d = 0.0$). For the ‘constrained mode’, ω/ω_o decreases in phase I, with the lowest frequency occurring at $H_2 = H/4$. This is because, the bending stiffness in the delamination region, $EI_2 + EI_3 + EI_4$, decreases in phase I and has the lowest value when $H_2 = H_1/4$. The decrease, however, is less significant for $a/L < 0.5$. In phase II, ω/ω_o increases which is due to the increasing bending stiffness of the delaminated beams. For the ‘free mode’ (phase I) with thin surface layers ($H_3 > H_2$), ω/ω_o increases rapidly. A kink is observed for the curves $a > 0.4L$ at a location near $H_2 = H_1/4$, after which ω/ω_o increases relatively slowly (phase II). The ‘partially constrained mode’ and the ‘constrained mode’ frequencies converge at $H_2 = H_1/2$. This is because, when $H_2 = H_1/2$, $H_3 = 0$. This results in a single midplane delamination where the ‘free mode’ and the ‘constrained mode’ frequencies are identical.

The ‘free mode’ mode shapes ($H_2 < H_1/4$) and the ‘partially constrained mode’ mode shapes ($H_2 > H_1/4$) for the circled geometries in Figure 3.24 are computed and shown in Figure 3.25. Three types of vibration modes are observed, local, mixed and global vibration modes. When the delamination is located before the kink in Figure 3.24, a local vibration mode is observed (Figure 3.25(a)). When the delamination is located near the kink but before the transition point $H_2 = H/4$, a mixed vibration mode is observed (Figure 3.25(b)), where beams 2, 3 and 4 vibrates ‘freely’ and have different transverse deformations. When the delamination is located near the kink but

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after the transition point, a mixed vibration mode is observed (Figure 3.25(c)), however, beam 3 is ‘constrained’ by the thicker beam 2 and the two beams vibrate in a ‘constrained’ manner. When the delamination is near midplane, a global vibration mode is observed (Figure 3.25(d)), this is because the thickness of beams 2 and 4 are almost the same.

Figure 3.24. Variation of the fundamental frequency ω/ω_o with the surface delamination depth H_2 in two phases, $H_4 = H_1/2$.

Figure 3.26 shows the variation of the fundamental frequency with the surface layer thickness, H_2 , for a clamped-clamped beam. A new slenderness ratio, $R_H = \omega_o / \omega_2$, specific for the analytical solution presented in Section 3.2.2 is introduced, where $\omega_o = (4.73/L)^2 \sqrt{EI_o / \rho_o A_o}$ is the fundamental frequency of the undelaminated beam and $\omega_2 = (4.73/a)^2 \sqrt{EI_2 / \rho_2 A_2}$ is the fundamental frequency of the surface beam 2 (see Figure 3.2). Beam 2 is considered to be clamped along its edges. The subscripts 2 and 0 denote the surface beam 2 and the undelaminated beam,

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respectively. For homogeneous beams, $R_H = (a/L)^2 / (H_2/H_1)$. $0.6 < R_H < 1$ represents the case when the vibration mode is mixed, for $R_H > 0.6$, a global vibration mode and for $R_H > 1$, a local vibration mode. For the ‘constrained mode’, ω/ω_0 decreases rapidly after a threshold value H_2 of about $0.1 H_1$ and decreases slowly after about $0.3 H_1$. With a higher R_H , ω/ω_0 decreases more rapidly. For the ‘free’ and ‘partially constrained’ modes, the decrease in the frequency varies in three phases. In phase I (‘free mode’), beams 2, 3 and 4 vibrate ‘freely’, and R determines an almost constant ω/ω_0 . In the transition phase II (‘partially constrained mode’), $H_2 > H_3$, the bending stiffness of beams 2 and 3 ($EI_2 + EI_3$) decreases, thus ω/ω_0 decreases. In phase III, $EI_2 + EI_3$ increases and ω/ω_0 increases.

Figure 3.25. Vibration modes at $a = 0.4L$ and $d = 0.0$ for various delamination depth H_2/H_1 : (a) $H_2 = 0.1 H_1$; (b) $H_2 = 0.2 H_1$; (c) $H_2 = 0.3 H_1$; (d) $H_2 = 0.45 H_1$.

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Figure 3.27 shows the dominating influence of the slenderness ratio R_H on the fundamental frequency of the beam. The $(H_3 H_4)$ couples label the crowded five curves by their order. There are two groups of curves $H_3 > H_2$ and $H_2 > H_3$, where curves in group $H_3 > H_2$ represent the ‘free mode’ and curves in group $H_2 > H_3$ represent the ‘partially constrained mode’. The two groups of curves correspond to the two phases in Figure 3.24, where phase I (‘free mode’) and phase II (‘partially constrained mode’) have different characteristics. For low slenderness ratio ($R_H < 0.5$), global vibration modes dominate and the influence of the delaminations vanishes. A good approximation of ω/ω_0 can be obtained for high slenderness ratio ($R_H > 1.5$). The normalized frequency ω/ω_0 can be assumed to depend on the slenderness ratio only within each group. It is worth noting, however, that the dependence is different for the two groups.

Figure 3.26. Fundamental frequency varies with delamination depth, $H_4 = H_1/2$.

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Figure 3.27. Slenderness ratio dominates the fundamental frequency for both ‘free’ and ‘partially constrained’ modes, $H_4 = H_1/2$.

3.4.4 Sandwich beams with double delaminations

In this section, the vibration of sandwich beam with double delaminations is studied using the normalized axial stiffness and normalized bending stiffness presented in Section 3.2.1.3, and the relative slenderness ratio presented in Section 3.2.1.4. The lower and upper bounds of the natural frequency are used by assuming a totally ‘free’ and totally ‘constrained’ deformation of the delaminated layers, respectively.

Figures 3.28 and 3.29 show the variation of the fundamental frequency ω/ω_0 with the normalized axial stiffness $\overline{EH}_2 = \overline{EH}_4$ of a clamped-clamped beam. At $\overline{EI}_2 = 0.025$ (Figure 3.28), for the ‘constrained mode’, ω/ω_0 decreases monotonically with increasing $\overline{EH}_2$ and increasing delamination length a/L . However, the decrease is less significant for $a/L < 0.3$. For the ‘free mode’, ω/ω_0 first increases slightly up to about $\overline{EH}_2 = 0.37$, where $\omega_2 \approx \omega_3$, and then decreases rapidly. At $\overline{EI}_2 = 0.075$ (Figure 3.29), the ‘constrained mode’ frequency decreases up to about $\overline{EH}_2 = 0.21$, after which ω/ω_0 displays an almost constant frequency. For $a/L < 0.4$, $\overline{EH}_2$ do not

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significantly influence the fundamental frequency. For the ‘free mode’, a similar trend is observed as in Figure 3.28.

Figure 3.28. Variation of the fundamental frequency ω/ω_o with the normalized axial stiffness $\overline{EH}_2$ of a clamped-clamped beam with a central delaminations ($d/L = 0$) at $\overline{EI}_2 = 0.025$.

Figure 3.30 shows the fundamental frequency ω/ω_o with the normalized bending stiffness of the $\overline{EI}_2 = \overline{EI}_4$ for different normalized axial stiffnesses $\overline{EH}_2 = \overline{EH}_4$. For the ‘constrained mode’, the difference between the frequencies decreases as $\overline{EI}_2 = \overline{EI}_4$ increases. The curves display almost the same frequencies at $\overline{EI}_2 = \overline{EI}_4 > 0.08$. This is because as the bending stiffness of the two outer layers increases, the bending stiffness of the middle layer decreases and thus its influence decreases. In contrast, for the ‘free mode’, the difference between the frequencies increases as $\overline{EI}_2 = \overline{EI}_4$ increases. ω/ω_o first increases until it predicts the same frequency as the ‘constrained mode’ and then ω/ω_o decreases rapidly.

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Figure 3.29. Variation of the fundamental frequency ω/ω_o with the normalized axial stiffness $\overline{EH}_2$ of a clamped-clamped beam with a central delaminations ($d/L = 0$) at $\overline{EI}_2 = 0.075$.

Figure 3.30. Variation of the fundamental frequency ω/ω_o with the normalized bending stiffness $\overline{EI}_2$ of a clamped-clamped beam with a central delaminations ($d/L = 0$).

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Figure 3.31 shows the influence of the slenderness ratio on the fundamental frequency of a clamped homogeneous beam ($E_2 = E_3 = E_4$) with central delaminations ($d/L = 0.0$) for the ‘free mode’. The thickness of the outer layers are identical ($H_2 = H_4$). The variation of the fundamental frequency ω/ω_o with the relative slenderness ratio R displays the essential features of Figure 3.22.

Figure 3.31. Slenderness ratio influence the ‘free mode’ frequency of a clamped-clamped homogeneous beam with central delaminations ($d/L = 0$).

3.4 Concluding Remarks

Nondimensional parameters have been introduced to study the vibration of bimaterial beams with a single delamination and three-layer beams with two equal length delaminations. The use of these parameters was shown to provide a better understanding on the vibration of delaminated multilayer beams. In addition, an analytical solution for beams with two equal length delaminations (double delaminations) was presented. The lower and upper bounds of the natural frequency

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were identified. Using these bounds, the vibration of sandwich beams with double delaminations was studied. Based on the results, the following conclusions are made:

1. The ‘constrained mode’ frequency is significantly influenced by the bending stiffness at the delamination region and the moment contribution due to the differential stretching between the delaminated layers. For the ‘free mode’, in addition to the influence of the differential stretching, the frequency is significantly influenced by the weakest delaminated layer.
2. At higher axial stiffness and bending stiffness ratios, the decrease in the natural frequency due to the increase in the delamination length is not significantly influenced by both ratios.
3. As the axial stiffness ratio or the bending stiffness ratio increases, the normalized bending stiffness at the delamination region decreases and thus, the natural frequency of the delaminated beam decreases.
4. The difference between the ‘free mode’ and the ‘constrained mode’ frequencies largely depend on the opening of the ‘free mode’ mode shapes. As the opening in the mode shape increases, the difference between the two frequencies increases.
5. The relative slenderness ratio was shown to influence the natural frequency and mode shape of the delaminated beam. Global, mixed and local vibration modes occur depending upon the slenderness ratio. At higher slenderness ratio R , local vibration mode dominates and thus, R determines the ‘free mode’ frequency. At lower R , global vibration mode prevails and the influence of the delamination decreases.

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6. For homogeneous beams with a single delamination, for the ‘constrained mode’, the fundamental frequency does not change perceptibly until the delamination depth is greater than 15%. For the ‘free mode’, the fundamental frequency is slightly affected by the delamination depth.
7. For beams with double delaminations, the beam was found to vibrate together in a ‘constrained mode’, or independently in a ‘free mode’, or in a mixed ‘partially constrained mode’ depending upon the relative thickness of the delaminated layers of the beam.

Chapter 4

Vibration of Composite Beams with Multiple Delaminations

4.1 Introduction

In this chapter, analytical solutions for the free vibration of fiber-reinforced composite beams with two enveloping delaminations, two non-overlapping delaminations and two overlapping delaminations are presented. The delaminated beams are modeled as interconnected Euler-Bernoulli beams using the delaminations as their boundaries. The continuity and boundary conditions are satisfied between adjoining beams. Both, the ‘constrained mode’ and the ‘free mode’ assumptions are used. Results are verified with published analytical and experimental results to verify the validity of the present solutions. Apart from being benchmark solutions for numerical/approximate solutions, these analytical solutions allow inexpensive simulations of slight variations in the system, such as changes in the delamination lengths and locations and boundary conditions.

To provide a better understanding on the vibration of composite beams with multiple delaminations, a parametric study is conducted to investigate the influence of

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the lengths and locations of the delaminations on the natural frequencies of the composite beams and their corresponding mode shapes. This study can be used to optimize the design of simple composite beams, and to assess the effectiveness of using the natural frequency and mode shape to detect delaminations in composite beams.

Delaminations introduce additional coupling between the axial stiffness and bending stiffness terms (bending-extension coupling) due to nonsymmetric laminate configuration of the delaminated layers. The influence of the bending-extension coupling on the natural frequency is also studied in this chapter.

This chapter is reported as follows: First, the vibrations of composite beams with multiple delaminations are formulated. Second, the results of the present model are compared with the analytical and experimental data reported in the literature to verify the validity of the present model. Third, the influence of the bending-extension coupling on the natural frequencies of the beams is examined. Finally, a parametric study on the influence of the lengths and locations of the delaminations on the natural frequency and mode shape of the composite beam is presented.

4.2 Formulations

This section presents the formulations for the vibration of fiber-reinforced composite beams with multiple delaminations. The delaminations are assumed to be through-the-width. Delamination growth and the interlaminar shear stresses between the plies of the composite beams are not taken into account in the present analysis.

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Four assumptions are made based on the literature: (a) The first assumption concerns the complicated changing contact between the delaminated layers, ‘free mode’ (Wang et al, 1982) and ‘constrained mode’ (Mujumdar and Suryanarayan, 1988); (b) The second assumption concerns the influence of the coupling between the axial and bending vibrations. Ramkumar et al. (1979) neglected the coupling effect in their analysis and their frequency results were consistently much lower than the experimental values. Wang et al. (1982) included the coupling effect and their results showed that for short and midplane delaminations, the predicted frequencies were close to the experimental values. Similarly, Mujumdar and Suryanarayan (1988) considered the coupling effect by including the effect of the differential stretching between the delaminated layers and their results were in good agreement with the experimental values.; (c) The third assumption concerns the coupling of the axial stiffness and the bending stiffness of the composite beams or the bending-extension coupling, which is important for highly non-symmetric laminates (Tracy and Pardoen 1996); (d) The fourth assumption concerns the lower bound and upper bound of the natural frequency (see Section 3.3). In this research, the ‘free’ and ‘constrained’ modes, the ‘rigid connector’ and the bending-extension coupling are considered in the analysis of the composite beam. The lower bound and the upper bound of the natural frequency are considered by assuming a totally ‘free’ and totally ‘constrained’ deformations of the delaminated layers.

4.2.1 Two enveloping delaminations

Figure 4.1(a) shows a beam with length L and thickness H_1 with two delaminations. The longer delamination ‘envelops’ the shorter delamination. This

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delamination configuration will be referred to as enveloping delaminations. The shorter and longer delaminations are of lengths a_1 and a_2 , and located at distances d_1 and d_2 away from the center of the beam, respectively. The two delaminations divide the beam into three layers with thickness H_3 , H_4 and H_5 . The delaminated beam is analyzed as seven interconnected beams (Figure 4.1(b)). These beams are treated as Euler-Bernoulli beams.

Figure 4.1. (a) A model of a beam with two enveloping delaminations; (b) The beam with is modeled as seven interconnected beams.

4.2.1.1 'Free mode'

The governing equations for the free vibration of a delaminated beam using the Euler-Bernoulli beam theory are

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$$D_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \quad (i = 1 - 7) \quad (4.1)$$

where D_i is the reduced bending stiffness of the i -th beam, ρ_i is the mass density and A_i is the cross sectional area of the beam. The mechanical properties for the composite beams are determined using the classical laminate theory (Jones, 1999). Thus

$$D_i = D_{11}^{(i)} - \frac{\left(B_{11}^{(i)}\right)^2}{A_{11}^{(i)}} \quad (4.2)$$

where

$$A_{11}^{(i)} = b \sum_{k=1}^{n_i} (\bar{Q}_{11}^k)_k (z_k - z_{k-1}) \quad (4.3)$$

$$B_{11}^{(i)} = \frac{b}{2} \sum_{k=1}^{n_i} (\bar{Q}_{11}^k)_k (z_k^2 - z_{k-1}^2) \quad (4.4)$$

$$D_{11}^{(i)} = \frac{b}{3} \sum_{k=1}^{n_i} (\bar{Q}_{11}^k)_k (z_k^3 - z_{k-1}^3) \quad (4.5)$$

$$\bar{Q}_{11}^k = Q_{11}^k \cos^4 \phi + Q_{22}^k \sin^4 \phi + 2(Q_{12}^k + 2Q_{66}^k) \cos^2 \phi \sin^2 \phi \quad (4.6)$$

$$Q_{11} = \frac{E_{11}}{1 - \nu_{12}\nu_{21}} \quad (4.7)$$

$$Q_{22} = \frac{E_{22}}{1 - \nu_{12}\nu_{21}} \quad (4.8)$$

$$Q_{12} = \frac{\nu_{12}E_{22}}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_{11}}{1 - \nu_{12}\nu_{21}} \quad (4.9)$$

$$Q_{66} = G_{12} \quad (4.10)$$

$$\nu_{21} = \frac{\nu_{12}E_{22}}{E_{11}} \quad (4.11)$$

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and where $A_{11}^{(i)}$ is the extensional stiffness, $B_{11}^{(i)}$ is the coupling stiffness, $D_{11}^{(i)}$ is the bending stiffness, $\bar{Q}_{11}^k$ is the coefficient stiffness of the lamina, b is the width, n_i is the number of plies, E_{11} and E_{22} are the longitudinal and transverse Young's moduli, respectively, G_{12} is the in-plane shear modulus, ν_{12} and ν_{21} are the longitudinal and transverse Poisson's ratio, respectively, ϕ is the angle of the k -th lamina orientation and z_k and z_{k-1} are the locations of the k -th lamina with respect to the midplane of the i -th beam (Figure 4.2). The second term on the right-hand side of eqn. (4.2) takes account of the reduction of stiffness due to the bending-extension coupling (Jones, 1999).

Figure 4.2. The i -th beam laminate.

For free vibrations

$$w_i(x_i, t) = W_i(x_i) \sin(\omega t) \quad (4.12)$$

where ω is the natural frequency and W_i is the mode shape. Substituting eqn. (4.3) in eqn. (4.1) and eliminating the trivial solution $\sin(\omega t) = 0$, one can obtain the general solutions of the differential equation in eqn. (4.1) as

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$$W_i(x) = C_i \cos(\lambda_i \frac{x}{L}) + S_i \sin(\lambda_i \frac{x}{L}) + CH_i \cosh(\lambda_i \frac{x}{L}) + SH_i \sinh(\lambda_i \frac{x}{L}) \quad (4.13)$$

where

$$\lambda_i^4 = \frac{\omega^2 \rho_i A_i}{D_i} L^4 \quad (4.14)$$

and where λ_i is the non-dimensional frequency. The 28 unknown coefficients C_i , S_i , CH_i and SH_i are determined from the 4 boundary conditions and 24 continuity conditions.

Hereafter, all the boundary and continuity conditions are written out in terms of $W_i(x)$ instead of $w_i(x,t)$. The appropriate boundary conditions that can be applied at the supports, $x = x_1$ and $x = x_6$, are $W_i = 0$ and $W_i' = 0$, if the end of the beam is clamped; $W_i = 0$ and $W_i'' = 0$, if simply supported; $W_i'' = 0$ and $W_i''' = 0$, if free where $i = 1$ and 7 and prime ' denotes differentiation with respect to the x -coordinate .

The continuity conditions for deflection, slope and shear at $x = x_2$ are

$$W_1 = W_2 = W_3 \quad (4.15)$$

$$W_1' = W_2' = W_3' \quad (4.16)$$

$$V_1 = V_2 + V_3 \quad (4.17)$$

$$M_1 = M_2 + M_3 - P_2 \left(\frac{H_1}{2} - \frac{H_2}{2} \right) + P_3 \left(\frac{H_1}{2} - \frac{H_3}{2} \right) \quad (4.18)$$

where

$$V_i = -D_i W_i''' \quad (4.19)$$

$$M_i = -D_i W_i'' \quad (i = 1 - 3) \quad (4.20)$$

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The axial forces P_i can be solved from the compatibility between the stretching/shortening of the delaminated layers and axial equilibrium (Mujumdar and Suryanarayan, 1988), thus

$$\frac{P_3 a_2}{A_{11}^{(3)}} - \frac{P_2 a_2}{A_{11}^{(2)}} = (W_1'(x_2) - W_7'(x_5)) \frac{H_1}{2} \quad (4.21)$$

$$P_1 = P_2 + P_3 = 0 \quad (4.22)$$

By substituting eqn. (4.19) in eqn. (4.17), and eqns. (4.20) to (4.22) in eqn. (4.18), the continuity of shear and bending moment can be expressed as

$$D_1 W_1''' = D_2 W_2''' + D_3 W_3''' \quad (4.23)$$

$$D_1 W_1'' + \frac{H_1^2}{4a_2} \left(\frac{A_{11}^{(2)} A_{11}^{(3)}}{A_{11}^{(2)} + A_{11}^{(3)}} \right) (W_1'(x_2) - W_7'(x_5)) = D_2 W_2'' + D_3 W_3'' \quad (4.24)$$

Similarly, we can derive the continuity conditions at $x = x_3$, $x = x_4$ and $x = x_5$. The boundary conditions and continuity conditions provide 28 homogeneous equations for 28 unknown coefficients C_i , S_i , CH_i and SH_i . A non-trivial solution for the coefficients exists only when the determinant of the coefficient matrix vanishes. The frequencies and mode shapes can be obtained as eigenvalues and eigenvectors, respectively.

4.2.1.2 ‘Constrained mode’

For the ‘constrained mode’ model, the delaminated beam is analyzed as five beam segments I – V (Figure 4.1). The governing equations for beam segments I – V are

$$D_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \quad (i = \text{I and V}) \quad (4.25)$$

$$(D_2 + D_3) \frac{\partial^4 w_{II}}{\partial x^4} + (\rho_2 A_2 + \rho_3 A_3) \frac{\partial^2 w_{II}}{\partial t^2} = 0 \quad (4.26)$$

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$$(D_3 + D_4 + D_5) \frac{\partial^4 w_{III}}{\partial x^4} + (\rho_3 A_3 + \rho_4 A_4 + \rho_5 A_5) \frac{\partial^2 w_{III}}{\partial t^2} = 0 \quad (4.27)$$

$$(D_3 + D_6) \frac{\partial^4 w_{IV}}{\partial x^4} + (\rho_3 A_3 + \rho_6 A_6) \frac{\partial^2 w_{IV}}{\partial t^2} = 0 \quad (4.28)$$

where $D_I = D_V = D_1 = D_7$ and $\rho_I A_I = \rho_V A_V = \rho_1 A_1 = \rho_7 A_7$. The generalized solutions for the ‘constrained mode’ are identical in form to the ‘free mode’. The unknown coefficients C_i , S_i , CH_i and SH_i , however, are reduced to 20 coefficients which can be determined from the 4 boundary conditions and 16 continuity conditions.

The boundary conditions for the ‘constrained mode’ are similar to the boundary conditions for the ‘free mode’. The continuity conditions for deflection, slope, shear and bending moments at $x = x_2$ are,

$$W_I = W_{II} \quad (4.29)$$

$$W_I' = W_{II}' \quad (4.30)$$

$$D_1 W_I''' = (D_2 + D_3) W_{II}''' \quad (4.31)$$

$$D_1 W_I'' + \frac{H_1^2}{4a_2} \left(\frac{A_{11}^{(2)} A_{11}^{(3)}}{A_{11}^{(2)} + A_{11}^{(3)}} \right) (W_I'(x_2) - W_V'(x_5)) = (D_2 + D_3) W_{II}'' \quad (4.32)$$

Similarly, we can derive the continuity conditions at $x = x_3$, $x = x_4$ and $x = x_5$. For the ‘constrained mode’, the boundary conditions and continuity conditions provide 20 homogeneous equations for 20 unknown coefficients C_i , S_i , CH_i and SH_i .

4.2.2 Two non-overlapping delaminations

Figure 4.3 shows a composite beam with length L and thickness H_1 having two non-overlapping delaminations with lengths a_1 and a_2 located at distances d_1 and d_2 from the center of the beam, respectively. The delaminated beam can be viewed as a

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combination of seven beams, each beam having a thickness H_i ($i = 1 - 7$), connected at the delamination boundaries $x = x_2, x = x_3, x = x_4$ and $x = x_5$.

4.2.2.1 'Free mode'

The governing equations are

$$D_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \quad (i = 1 - 7) \quad (4.33)$$

Figure 4.3. (a) A model of a beam with two non-overlapping delaminations; (b) The delaminated beam is modeled by seven interconnected beams.

The general solutions of the differential equation in eqn. (4.33) are

$$W_i(x) = C_i \cos(\lambda_i \frac{x}{L}) + S_i \sin(\lambda_i \frac{x}{L}) + CH_i \cosh(\lambda_i \frac{x}{L}) + SH_i \sinh(\lambda_i \frac{x}{L}) \quad (4.34)$$

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There are 28 unknown coefficients C_i , S_i , CH_i and SH_i , which can be determined from the 4 boundary conditions and 24 continuity conditions.

The boundary conditions are similar to the boundary conditions for the two enveloping delaminations. The continuity conditions for deflection, slope, shear bending moments at $x = x_2$ are

$$W_1 = W_2 = W_3 \quad (4.35)$$

$$W_1' = W_2' = W_3' \quad (4.36)$$

$$D_1 W_1''' = D_2 W_2''' + D_3 W_3''' \quad (4.37)$$

$$D_1 W_1'' = D_2 W_2'' + D_3 W_3'' + P_2 \left(\frac{H_1}{2} - \frac{H_2}{2} \right) - P_3 \left(\frac{H_1}{2} - \frac{H_3}{2} \right) \quad (4.38)$$

The axial forces P_i can be solved from the compatibility between the stretching/shortening of the delaminated layers and axial equilibrium (Mujumdar and Suryanarayan, 1988), thus

$$\frac{P_3 a_1}{A_{11}^{(3)}} - \frac{P_2 a_1}{A_{11}^{(2)}} = (W_1'(x_2) - W_4'(x_3)) \frac{H_1}{2} \quad (4.39)$$

$$P_1 = P_2 + P_3 = 0 \quad (4.40)$$

Substituting eqns. (4.39) and (4.40) in eqns. (4.38), we have

$$D_1 W_1'' + \frac{H_1^2}{4a_1} \left(\frac{A_{11}^{(2)} A_{11}^{(3)}}{A_{11}^{(2)} + A_{11}^{(3)}} \right) (W_1'(x_2) - W_4'(x_3)) = D_2 W_2'' + D_3 W_3'' \quad (4.41)$$

Similarly, we can derive the continuity conditions at $x = x_3$, $x = x_4$ and $x = x_5$. The boundary conditions and continuity conditions provide 28 homogeneous equations for 28 unknown coefficients C_i , S_i , CH_i and SH_i .

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4.2.2.2 ‘Constrained mode’

The delaminated beam is analyzed as five beam segments I – V (Figure 4.3). The governing equations for beam segments I – V are

$$D_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \quad (i = \text{I, III, V}) \quad (4.42)$$

$$(D_2 + D_3) \frac{\partial^4 w_{II}}{\partial x^4} + (\rho_2 A_2 + \rho_3 A_3) \frac{\partial^2 w_{II}}{\partial t^2} = 0 \quad (4.43)$$

$$(D_5 + D_6) \frac{\partial^4 w_{IV}}{\partial x^4} + (\rho_5 A_5 + \rho_6 A_6) \frac{\partial^2 w_{IV}}{\partial t^2} = 0 \quad (4.44)$$

where $D_I = D_{III} = D_V = D_1 = D_4 = D_7$ and $\rho_I A_I = \rho_{III} A_{III} = \rho_V A_V = \rho_1 A_1 = \rho_4 A_4 = \rho_7 A_7$.

The generalized solutions for the ‘constrained mode’ are identical in form to ‘free mode’. The unknown coefficients, however, are reduced to 20 coefficients which can be determined from the 4 boundary conditions and 16 continuity conditions.

The boundary conditions for the ‘constrained mode’ are identical to the boundary conditions of the ‘free mode’. The continuity conditions for deflection, slope, shear and bending moments at $x = x_2$ are,

$$W_I = W_{II} \quad (4.45)$$

$$W_I' = W_{II}' \quad (4.46)$$

$$D_I W_I''' = (D_2 + D_3) W_{II}''' \quad (4.47)$$

$$D_I W_I'' + \frac{H_1^2}{4a_1} \left(\frac{A_{11}^{(2)} A_{11}^{(3)}}{A_{11}^{(2)} + A_{11}^{(3)}} \right) (W_I'(x_2) - W_{III}'(x_3)) = (D_2 + D_3) W_{II}'' \quad (4.48)$$

Similarly, we can derive the continuity conditions at $x = x_3$, $x = x_4$ and $x = x_5$. For the ‘constrained mode’, the boundary conditions and continuity conditions provide 20 homogeneous equations for 20 unknown coefficients C_i , S_i , CH_i and SH_i .

4.2.3 Two overlapping delaminations

Figure 4.4 shows a beam with length L and thickness H_1 with two overlapping delaminations. The two overlapping delaminations with lengths a_1 and a_2 are located at spanwise distances d_1 and d_2 from the center of the beam, respectively. The delaminated beam is treated as seven Euler-Bernoulli beams connected at the delamination boundaries. The two delaminations divide the beam into three layers with thicknesses H_3 , H_4 and H_5 .

Figure 4.4. (a) A model of a beam with two overlapping delaminations; (b) The delaminated beam is modeled by seven interconnected beams.

4.2.3.1 'Free mode'

The governing equations are

$$D_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \quad (i = 1 - 7) \quad (4.49)$$

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The general solutions of the differential equation in eqn. (4.49) are

$$W_i(x) = C_i \cos(\lambda_i \frac{x}{L}) + S_i \sin(\lambda_i \frac{x}{L}) + CH_i \cosh(\lambda_i \frac{x}{L}) + SH_i \sinh(\lambda_i \frac{x}{L}) \quad (4.50)$$

and where λ_i is the non-dimensional frequency. The 28 unknown coefficients C_i , S_i , CH_i and SH_i ($i = 1 - 7$) are determined from the 4 boundary conditions and 24 continuity conditions.

The boundary conditions are similar to the boundary conditions for the two enveloping delaminations. The continuity conditions for deflection, slope, shear and bending moment at the delamination boundary $x = x_2$ are

$$W_1 = W_2 = W_3 \quad (4.51)$$

$$W_1' = W_2' = W_3' \quad (4.52)$$

$$D_1 W_1''' = D_2 W_2''' + D_3 W_3''' \quad (4.53)$$

$$D_1 W_1'' = D_2 W_2'' + D_3 W_3'' - P_3 \left(\frac{H_1}{2} - \frac{H_3}{2} \right) + P_2 \left(\frac{H_1}{2} - \frac{H_2}{2} \right) \quad (4.54)$$

Each of the delamination boundaries $x = x_3$, $x = x_4$ and $x = x_5$ provides 6 equations and a total of 24 equations can be set up for all the delamination boundaries.

Five axial forces P_2 , P_3 , P_4 , P_5 and P_6 that appear in the moment continuity condition are still unknown, so additional equations are needed. The derivation of the additional equations for the axial forces P_i ($i = 2 - 6$) for the two overlapping delamination configuration takes a different route and is more complicated as compared to a single delamination, two enveloping delaminations, and two non-overlapping delaminations.

At $x = x_2$ and $x = x_5$, from the axial force balance, we have

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$$P_1 = P_2 + P_3 = 0; \quad P_3 = -P_2 \quad (4.55)$$

$$P_7 = P_4 + P_6 = 0; \quad P_4 = -P_6 \quad (4.56)$$

At $x = x_3$ and $x = x_4$, from the axial force balance, we have

$$P_2 = P_4 + P_5 \quad (4.57)$$

$$P_6 = P_3 + P_5 \quad (4.58)$$

If the beam does not vibrate, P_2, P_3, P_4, P_5 and P_6 are all zero, because they are defined as the axial forces present during the vibration of the beam. The axial length of the beam between $x = x_2$ and $x = x_5$ remains equal before and during vibration (Figure 4.5). This leads to the following equations.

Figure 4.5. The vibration configuration between $x=x_2$ and $x=x_5$. The lengths of the central lines remain unchanged.

$$\left(\frac{H_1}{2} - \frac{H_2}{2} \right) W_1'(x_2) + L_2 + \left(\frac{H_2}{2} - \frac{H_4}{2} \right) W_2'(x_3) + L_4 - \left(\frac{H_7}{2} - \frac{H_4}{2} \right) W_7'(x_5) = L_{2o} + L_{4o} \quad (4.59)$$

$$\begin{aligned} \left(\frac{H_1}{2} - \frac{H_2}{2} \right) W_1'(x_2) + L_2 + \left(\frac{H_5}{2} - \frac{H_2}{2} \right) W_2'(x_3) + L_5 - \left(\frac{H_6}{2} - \frac{H_5}{2} \right) W_6'(x_4) \\ + L_6 - \left(\frac{H_6}{2} - \frac{H_7}{2} \right) W_7'(x_5) = L_{2o} + L_{5o} + L_{6o} \end{aligned} \quad (4.60)$$

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$$\left(\frac{H_3}{2} - \frac{H_1}{2}\right)W_1'(x_2) + L_3 - \left(\frac{H_3}{2} - \frac{H_6}{2}\right)W_6'(x_4) + L_6 - \left(\frac{H_6}{2} - \frac{H_7}{2}\right)W_7'(x_5) = L_{3o} + L_{6o} \quad (4.61)$$

where L_{io} represents the length of the i -th beam segment before vibration, H_i is the thickness of the i -th beam segment, and W_i' is the slope. If L_i represents the length of the i -th beam segment, then

$$L_i - L_{io} = \frac{PL_{io}}{A_{11}^{(i)}} \quad (i = 2 - 6) \quad (4.62)$$

Substituting eqn. (4.62) into eqns. (4.59) to (4.61) leads to

$$\frac{P_2L_{2o}}{A_{11}^{(2)}} + \frac{P_4L_{4o}}{A_{11}^{(4)}} = -\left(\frac{H_1}{2} - \frac{H_2}{2}\right)W_1'(x_2) - \left(\frac{H_2}{2} - \frac{H_4}{2}\right)W_2'(x_3) + \left(\frac{H_7}{2} - \frac{H_4}{2}\right)W_7'(x_5) \quad (4.63)$$

$$\begin{aligned} \frac{P_2L_{2o}}{A_{11}^{(2)}} + \frac{P_5L_{5o}}{A_{11}^{(5)}} + \frac{P_6L_{6o}}{A_{11}^{(6)}} = & -\left(\frac{H_1}{2} - \frac{H_2}{2}\right)W_1'(x_2) - \left(\frac{H_5}{2} - \frac{H_2}{2}\right)W_2'(x_3) \\ & + \left(\frac{H_6}{2} - \frac{H_5}{2}\right)W_6'(x_4) + \left(\frac{H_6}{2} - \frac{H_7}{2}\right)W_7'(x_5) \end{aligned} \quad (4.64)$$

$$\frac{P_3L_{3o}}{A_{11}^{(3)}} + \frac{P_6L_{6o}}{A_{11}^{(6)}} = -\left(\frac{H_3}{2} - \frac{H_1}{2}\right)W_1'(x_2) + \left(\frac{H_3}{2} - \frac{H_6}{2}\right)W_6'(x_4) + \left(\frac{H_6}{2} - \frac{H_7}{2}\right)W_7'(x_5) \quad (4.65)$$

Thus we have the number of equations to solve the simultaneous equations.

4.3.2.2 ‘Constrained mode’

The delaminated beam is analyzed as five beam segments I – V (Figure 4.4). The governing equations for beam segments I – V are

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$$D_1 \frac{\partial^4 w_I}{\partial x^4} + \rho_1 A_1 \frac{\partial^2 w_I}{\partial t^2} = 0 \quad (4.66)$$

$$(D_2 + D_3) \frac{\partial^4 w_{II}}{\partial x^4} + (\rho_2 A_2 + \rho_3 A_3) \frac{\partial^2 w_{II}}{\partial t^2} = 0 \quad (4.67)$$

$$(D_3 + D_4 + D_5) \frac{\partial^4 w_{III}}{\partial x^4} + (\rho_3 A_3 + \rho_4 A_4 + \rho_5 A_5) \frac{\partial^2 w_{III}}{\partial t^2} = 0 \quad (4.68)$$

$$(D_4 + D_6) \frac{\partial^4 w_{IV}}{\partial x^4} + (\rho_4 A_4 + \rho_6 A_6) \frac{\partial^2 w_{IV}}{\partial t^2} = 0 \quad (4.69)$$

$$D_7 \frac{\partial^4 w_V}{\partial x^4} + \rho_7 A_7 \frac{\partial^2 w_V}{\partial t^2} = 0 \quad (4.70)$$

where $D_I = D_V = D_1 = D_7$ and $\rho_I A_I = \rho_V A_V = \rho_1 A_1 = \rho_7 A_7$. The generalized solutions for the ‘constrained mode’ are identical in form to the ‘free mode’. However, the unknown coefficients C_b , S_b , CH_i and SH_i are reduced to 20, which can be determined from the 4 boundary conditions and 16 continuity conditions.

The continuity conditions for deflection, slope, shear and bending moments at $x = x_2$ are,

$$W_I = W_{II} \quad (4.71)$$

$$W_I' = W_{II}' \quad (4.72)$$

$$D_1 W_I''' = (D_2 + D_3) W_{II}''' \quad (4.73)$$

$$D_1 W_I'' + P_3 \left(\frac{H_1}{2} - \frac{H_3}{2} \right) - P_2 \left(\frac{H_1}{2} - \frac{H_2}{2} \right) = (D_2 + D_3) W_{II}'' \quad (4.74)$$

Similarly, we can derive the continuity conditions at $x = x_3$, $x = x_4$ and $x = x_5$. For the ‘constrained mode’, the boundary conditions and continuity conditions provide 20 homogeneous equations for the 20 unknown coefficients C_b , S_b , CH_i and SH_i .

4.3 Results and Discussions

This section presents the results for the analytical solutions presented in Section 4.2. The results of the present model are first verified with published analytical and experimental results on a composite beam with a single delamination. Next, the influence of the bending-extension coupling on the natural frequencies of the beams is examined. The results for multiple delaminations are then presented and discussed. For two enveloping delaminations, results are first presented for clamped-clamped and cantilever homogeneous beams. Then, the results for composite beams with two enveloping delaminations are shown. For two non-overlapping delaminations and two overlapping delaminations, results for cantilever composite beams are presented.

4.3.1 Verifications

4.3.1.1 Simply supported composite beam with a single delamination

The first verification for the vibration of delaminated beams is performed on an AS/3501-6 graphite/epoxy beam with a $[90^\circ/\pm 45^\circ/0^\circ]_{2s}$ stacking sequence studied by Tracy and Pardoen (1989). The beam test specimens are 287 cm long by 25.4 mm wide. The delamination is through-width and is located at midspan and midplane. The delamination lengths are 25.4 mm, 50.8 mm and 101.6 mm. The normalized bending stiffness of the upper and lower delaminated layers are $D_2 / D_1 = D_3 / D_1 = 0.164$ and the normalized axial stiffness of the upper and lower delaminated layers are $A_{11}^{(3)} / A_{11}^{(1)} = A_{11}^{(2)} / A_{11}^{(1)} = 0.5$. Using the ‘constrained mode’, the first four frequencies of a simply supported composite beam were determined and compared with the experimental results. The natural frequencies as a function of the delamination length

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are shown in Figure 4.6. It can be seen that the frequencies predicted by the present model show good agreement with the experimental data. Furthermore, it is observed that for the lower vibration modes (first three modes), there is no significant effect on the natural frequency even when the delamination length is 35% of the beam length.

Figure 4.6. Comparison of predicted natural frequencies of a delaminated simply supported composite beam with experimental data.

4.3.1.2 Cantilever composite beam with a single delamination

The second verification of the present solution is performed on a cantilever composite beam with a single delamination, which was studied by Shen and Grady (1992). The beam is made of T300/934 graphite/epoxy cantilever beam with a $[0^\circ/90^\circ]_{2s}$ stacking sequence. The dimensions of the 8-ply beam are $127 \times 12.7 \times 1.016$ mm³. The material properties for the lamina are: $E_{11} = 134$ GPa, $E_{22} = 10.3$ GPa, $G_{12} = 5$ GPa, $\nu_{12} = 0.33$ and $\rho = 1.48 \times 10^3$ kg/m³. All the delaminations are at midspan and the lengths are 25.4 mm, 50.8 mm, 76.2 mm and 101.6 mm. The locations of the delaminations along the thicknesswise direction are shown in Figure 4.7. The fundamental frequencies of the ‘free mode’ and ‘constrained mode’ are

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shown in Tables 4.1 – 4.4. Good agreement is obtained between the frequencies predicted by the present solution and the experimental and analytical ('constrained mode') results by Shen and Grady (1992) and analytical results by Luo and Hanagud (2000). In Table 4.4, a difference between the experimental results and theoretical results, which might have been an experimental error, as explained in Section 2.3.2. In Tables 4.1 to 4.4, it is observed that only the delamination length greater than 50.8 mm (40% of the beam length) can introduce a significant drop in the fundamental frequency of the beam. This may require the use of the higher vibration modes in order to detect delaminations shorter than 25.4 cm (20% of the beam length).

Figure 4.7. Locations of the delamination at various interfaces for a $[0/90]_{2s}$ Graphite/Epoxy composite laminate (Shen and Grady 1992).

Table 5 shows the second bending (Mode 2) frequencies of the delaminated beam, with various midplane delamination lengths (Interface 1). Again, good agreement was obtained between the present results and the results obtained using finite element models based on the higher-order theory and 3D NASTRAN (Radu and Chattopadhyay, 2002). These correlations further validate the capability of the present model for reliable frequency prediction of composite beams with multiple delaminations.

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To verify the mode shape results of the present model, the ‘free mode’ mode shapes for the beam with delamination located at Interface 4 are determined and compared with the results of Hu et al. (2002). It is observed that the mode shapes of the present solution (Figure 4.8) are in good agreement with the mode shapes of the finite element model using the third-order shear deformation theory (Figure 4.9).

Table 4.1
Interface 1 Fundamental frequencies (Hz)

Delamination length	Present ^a		Present ^b		Experimental ¹			FSDT ¹		FSDT ²	
	Cons	Free	Cons	Free	Exp 1	Exp 2	Exp 3	Cons	Cons	Cons	Free
Intact	81.88	81.88	81.88	81.88	79.88	79.88	79.75	82.04	81.86	81.86	81.86
25.4 mm	80.72	80.72	80.47	80.47	78.38	79.13	77.00	80.13	81.84	81.84	81.84
50.8 mm	76.36	76.36	75.36	75.36	74.38	75.00	76.75	75.29	76.81	76.81	76.81
76.2 mm	68.17	68.17	66.14	66.13	68.25	66.25	66.38	66.94	67.64	67.64	67.64
101.6 mm	58.35	58.35	55.67	55.67	57.62	57.50	57.50	57.24	56.95	56.95	56.95

¹ Shen and Grady (1992); ² Luo and Hanagud (2000)

^a Without bending-extension coupling; ^b With bending-extension coupling

Table 4.2
Interface 2 Fundamental frequencies (Hz)

Delamination length	Present ^a		Present ^b		Experimental ¹			FSDT ¹		FSDT ²	
	Cons	Free	Cons	Free	Exp 1	Exp 2	Exp 3	Cons	Cons	Cons	Free
Intact	81.88	81.88	81.88	81.88	79.88	79.88	79.75	82.04	81.86	81.86	81.86
25.4 mm	81.00	81.00	80.58	80.58	78.38	78.38	76.63	81.39	80.86	80.86	80.86
50.8 mm	77.45	77.45	75.81	75.81	75.13	75.25	75.00	78.10	76.62	76.62	76.62
76.2 mm	70.43	70.42	67.05	67.05	64.00	70.00	69.88	71.16	68.80	68.80	68.80
101.6 mm	61.50	61.49	56.86	56.86	45.75	49.75	49.50	62.12	59.34	59.34	59.34

¹ Shen and Grady (1992); ² Luo and Hanagud (2000)

^a Without bending-extension coupling; ^b With bending-extension coupling

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Table 4.3
Interface 3 Fundamental frequencies (Hz)

Delamination length	Present ^a		Present ^b		Experimental ¹			FSDT ¹	FSDT ²	
	Cons	Free	Cons	Free	Exp 1	Exp 2	Exp 3	Cons	Cons	Free
Intact	81.88	81.88	81.88	81.88	79.88	79.88	79.75	82.04	81.86	81.86
25.4 mm	81.57	81.57	81.53	81.53	79.63	80.13	80.63	81.46	82.02	82.01
50.8 mm	80.25	80.23	80.13	80.09	79.50	81.88	77.88	79.93	80.79	80.74
76.2 mm	77.27	77.16	77.03	76.75	75.63	77.13	78.13	76.71	77.82	77.52
101.6 mm	72.66	72.20	72.28	70.92	73.38	73.63	70.38	71.66	73.15	71.73

¹ Shen and Grady (1992); ² Luo and Hanagud (2000)

^a Without bending-extension coupling; ^b With bending-extension coupling

Table 4.4.
Interface 4 Fundamental frequencies (Hz)

Delamination length	Present ^a		Present ^b		Experimental ¹			FSDT ¹	FSDT ²	
	Cons	Free	Cons	Free	Exp 1	Exp 2	Exp 3	Cons	Cons	Free
Intact	81.88	81.88	81.88	81.88	79.88	79.88	79.75	82.04	81.86	81.86
25.4 mm	81.72	81.72	81.57	81.57	75.38	75.25	77.25	81.60	82.04	82.03
50.8 mm	80.75	80.73	80.33	80.31	69.38	68.00	69.38	80.38	80.95	80.87
76.2 mm	78.40	78.25	77.56	77.41	65.38	59.63	-----	77.70	78.29	77.61
101.6 mm	74.59	73.81	73.22	72.51	52.75	57.88	56.25	73.15	74.05	69.44

¹ Shen and Grady (1992); ² Luo and Hanagud (2000)

^a Without bending-extension coupling; ^b With bending-extension coupling

Table 4.5
Mode 2 frequencies (Hz) of a composite beam with a midplane delamination (Interface 1)

Delamination length	Present ^a Cons and Free	Present ^b Cons and Free	Higher-order ¹ theory	NASTRAN 3D ¹
Intact	513.20	513.20	513.30	510.70
25.4 mm	498.38	495.09	509.24	504.18
50.8 mm	473.77	465.96	469.02	478.66
76.2 mm	413.30	399.07	369.08	399.36
101.6 mm	332.36	315.50	325.79	305.75

¹ Radu and Chattopadhyay (2002), ^a Without coupling; ^b With bending-extension coupling

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Figure 4.8. Interface 4 mode shapes: (a) 25.4 mm delamination; (b) 50.8 mm delamination; (c) 76.2 mm delamination; (d) 101.6 mm delamination.

Figure 4.9. Interface 4 mode shapes: (a) 25.4 mm delamination; (b) 50.8 mm delamination; (c) 76.2 mm delamination; (d) 101.6 mm delamination (Hu et al., 2002).

4.3.2 Bending-extension coupling

The natural frequency of the delaminated beams further decreases due to the influence of the bending-extension coupling (Tables 4.1 – 4.4). The influence of the coupling on the natural frequency of a cantilever beam with a single delamination and double delaminations is shown in Figure 4.10. ω_{wo} denotes the frequency of the beam without coupling, ω_{wc} denotes the frequency of the beam with coupling, and ω_o is the frequency of an undelaminated beam. For the single delamination, the delamination is located at midplane (interface 1), whereas for the double delaminations, the delaminations are located at midplane and interface 3. For the ‘constrained mode’ (Figure 4.10(a)), the influence of coupling increases as the delamination length a/L increases. The influence further increases for double delaminations. A slight difference between the fundamental and the mode 2 frequencies is observed. For the ‘free mode’ (Figure 4.10(b)), the influence of the coupling on the natural frequencies is similar to the ‘constrained mode’, since the ‘free mode’ and ‘constrained mode’ frequencies are identical for midplane delaminations. The influence further increases for double delaminations, with the fundamental and mode 2 frequencies having a significant difference.

The results show that the bending-extension coupling significantly influences the frequency of the delaminated beam. Hereafter, all the numerical results for composite beams will include the coupling effect. The notation ω_{wc} , which previously denotes the frequency of the delaminated beam with the coupling will simply be denoted as ω .

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Figure 4.10. Influence of the coupling between the axial and bending stiffnesses of the delaminated layers on the natural frequencies of a cantilever beam with a single delamination and double delaminations: (a) 'Constrained mode'; (b) 'Free mode'.

4.3.3 Two enveloping delaminations

4.3.3.1 Homogeneous beams

This section presents the results for a homogeneous and isotropic beam with two enveloping delaminations. For homogeneous and isotropic beams, the bending stiffness is $D_i = EI_i$. For plain-stress problem, E is the Young's modulus. For plain

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strain problem, E would be replaced by an equivalent Young's modulus $E = E/(1 - \nu)$, where ν is the Poisson's ratio.

Results of three delamination configurations: single delamination, two enveloping delaminations and double delaminations, are compared. For enveloping delaminations, the length of the longer delamination a_2 is twice the length of the shorter delamination a_1 ($a_2 = 2a_1$), while the spanwise locations of the delaminations are the identical, $d_1 = d_2$. For double delaminations, the lengths and the spanwise locations of the delaminations are identical.

Figure 4.11 shows the influence of the delamination lengths on the fundamental and mode 2 frequencies of a clamped-clamped beam. The natural frequency, ω , is normalized with respect to the frequency of an undelaminated beam, ω_0 . As expected the frequencies decreases as the delamination increases, this is because the bending stiffness of the beam decreases as the delamination length increases. Figure 4.11(a) shows that the reduction of the fundamental frequency is negligible for short delaminations ($a < 0.3L$) for all the three delamination configurations. The change remains to be small for delamination length of up to $a = 0.5L$ for single delamination and enveloping delaminations. The frequency decreases significantly for longer delaminations ($a > 0.5L$), which is more pronounced for double delaminations. It is further observed that the difference between the 'constrained mode' frequencies of the single delamination and the enveloping delaminations is negligible. This indicates that the fundamental frequency is highly influenced by the longer delamination.

The influence of the delamination length is more pronounced on the mode 2 frequency (Figure 4.11(b)). The influence, however, is negligible for $a < 0.1L$. The difference between the mode 2 frequencies of the single delamination and the

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enveloping delaminations is more obvious as compared to the fundamental frequencies (Figure 4.11(a)). Based on the above results, for a clamped-clamped beam, the sensitivity of the fundamental frequency is high for the longer delamination but low for the second short delamination, while, the sensitivity of the mode 2 frequency is high for both the long and short delaminations.

Figure 4.11. Natural frequency decreases as the delamination length increases for a clamped-clamped beam: (a) Fundamental frequency; (b) Mode 2 frequency.

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The difference between the frequencies of the ‘constrained mode’ and the ‘free mode’ largely depend on the opening of the mode shapes using the ‘free mode’. Figures 4.12 and 4.13 show that the ‘free mode’ mode shape displays a large opening for a long delamination ($a = 0.6L$), but hardly displays an opening for a short delamination ($a = 0.2L$). This explains the increasing difference between the frequencies of the two modes as the delamination lengths increase. Furthermore, when the changes in the mode shape is small, the changes in the natural frequency is small, such as when $a = 0.2L$ (see Figure 4.11). It should be noted that overlapping in the mode shape occur in the second mode of vibration due to the ‘free mode’ assumption that the beams deforms ‘freely’ and have different transverse deformation.

Figure 4.14 presents the frequencies obtained for a cantilever beam with various delamination lengths. Figure 4.14(a) shows that the reduction in the fundamental frequency of the three delamination configurations is negligible for $a < 0.2L$ and remains to be small for delamination length of up to $a = 0.4L$ for single delamination and enveloping delaminations. A small difference between the ‘constrained mode’ frequencies of the single delamination and the enveloping delaminations is observed for $a > 0.6L$. Figure 4.14(b) shows the reduction of the mode 2 frequency is negligible when $a < 0.2L$ and remains to be small for delamination length of up to $a = 0.4L$. The difference between the ‘constrained mode’ frequencies of the single delamination and the enveloping delaminations is insignificant. Based on the above results, for a cantilever beam, the sensitivity of both the fundamental and mode 2 frequencies is high for the longer delamination and low for the second short delamination.

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Figure 4.12. The ‘free mode’ mode shape of a clamped-clamped beam with two enveloping delaminations: (a) Fundamental mode shape; (b) Mode 2 mode shape.

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Figure 4.13. The 'free mode' mode shape of a clamped-clamped beam with double delaminations: (a) Fundamental mode shape; (b) Mode 2 mode shape.

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Figure 4.14(a), further shows a slight difference between the fundamental frequencies of the ‘constrained mode’ and the ‘free mode’ of the enveloping delaminations for $a > 0.5L$, whereas for the double delamination, the differences are less significant. In Figure 4.14(b), the difference between mode 2 frequencies of the ‘constrained mode’ and ‘free mode’ of the enveloping delaminations becomes significant only when $a > 0.8L$, whereas for the double delaminations, significant difference can be observed when $a > 0.3L$. The difference between the ‘constrained mode’ and ‘free mode’ frequencies can be explained by the opening in the mode shapes. As shown in Figures 4.15 and 4.16, as the opening in the mode shape becomes larger, the difference between the two frequencies increases (Figures 4.14).

Figure 4.17 shows the variation of the ‘constrained mode’ frequencies due to the thicknesswise locations of the delaminations for a clamped-clamped beam and a cantilever beam. For single delamination, the lowest frequency occurs when the delamination is at midplane (Figure 4.17(a)). Similarly for enveloping delaminations, the lowest frequency occurs when the longer delamination is at midplane. In Fact, the difference between the frequencies of the single and the enveloping delaminations is negligible which further shows that the fundamental frequency is largely dependent on the longer delamination. For double delaminations, minimum frequency occurs when the lower delamination is off-midplane, which is due to the influence of the second delamination. The mode 2 frequency shows that for single delamination, the frequency is lowest for midplane delamination. However, for enveloping delaminations, minimum frequency occurs when the longer delamination is off-midplane which is due to the significant influence of the second shorter delamination. As expected, for double delaminations, minimum frequency occurs when the lower delamination is off-

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midplane. For a cantilever beam (Figure 4.17(b)), the effect of the delaminations is more pronounced on the fundamental frequency than the mode 2 frequency for the three delamination configurations. For enveloping delaminations, the fundamental frequency displays a higher sensitivity to the second short delamination than the mode 2 frequency.

Figure 4.14. Natural frequency decreases as the delamination length increases for a cantilever beam: (a) Fundamental frequency; (b) Mode 2 frequency.

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Figure 4.15. The ‘free mode’ mode shape of a cantilever beam with two enveloping delaminations: (a) Fundamental mode shape; (b) Mode 2 mode shape.

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Figure 4.16. The 'free mode' mode shape of a cantilever beam with double delaminations: (a) Fundamental mode shape; (b) Mode 2 mode shape.

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Figure 4.17. The variation of the natural frequency with thicknesswise location of the longer delamination: (a) Clamped-clamped beam; (b) Cantilever beam.

Figure 4.18 shows the influence of the spanwise location of the delamination on the fundamental and mode 2 frequencies for a clamped-clamped beam. Due to the symmetry of the problem, only $d > 0.0$ is shown. Figure 4.18(a) shows that the influence of the delamination on the fundamental frequency increases as the delaminations moves towards the beam ends. This is due to the influence of the

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transverse shear force (the third derivative of the displacement mode shape, W'''). As shown in Figure 4.19, for mode 1 (fundamental frequency), the shear force is minimum at the beam midspan and maximum at the beams end. The shear force increases the influence of the delamination on the natural frequency, since the frequency decreases as the delaminations move from a lower shear region to a higher shear region (midspan to beam ends), as shown in Figure 4.18(a). Furthermore, when the delaminations are at midspan, the difference between the frequencies of the single delamination and enveloping delaminations is negligible, while the difference between the frequencies of the single delamination and double delaminations is significant. As the delaminations move towards the beam end, the difference between the frequencies of the single delamination and enveloping delaminations increases while the enveloping and double delaminations frequencies converges. Figure 4.18(b) shows that the delaminations influence the mode 2 frequency most when the delaminations are at midspan, which corresponds to a high shear region (Figure 4.19), and least when the delaminations are located at about $d = 0.2L$, which corresponds to a low shear region. Furthermore, the difference between the frequencies of the single delamination and the enveloping delaminations decreases as the delaminations move from a higher to a lower shear region. The same is observed for the single delamination and double delaminations. For the cantilever beam (Figure 4.20), the variations in the fundamental and mode 2 natural frequencies due to the spanwise locations of the delaminations are significantly influenced by the shear force (Figure 4.21).

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Figure 4.18. The variation of the natural frequency with spanwise locations of the delaminations of a clamped-clamped beam: (a) Fundamental frequency; (b) Mode 2 frequency.

Figure 4.19. Shear force of an intact clamped-clamped beam.

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Figure 4.20. The variation of the natural frequency with spanwise locations of the delaminations of a cantilever beam: (a) Fundamental frequency; (b) Mode 2 frequency.

Figure 4.21. Shear force of an intact cantilever beam.

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Figure 4.22. Natural frequencies of composite beams with multiple delaminations: (a) Clamped-clamped beam; (b) Cantilever beam.

4.3.3.2 Composite beam

This section presents the results for composite beam with single delamination, enveloping delaminations and double delaminations. The beam is made of T300/934 graphite/epoxy with a $[0^\circ/90^\circ]_{2s}$ stacking sequence. The dimensions of the 8-ply beam

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are $127 \times 12.7 \times 1.016 \text{ mm}^3$. The material properties for the lamina are: $E_{11} = 134 \text{ GPa}$, $E_{22} = 10.3 \text{ GPa}$, $G_{12} = 5 \text{ GPa}$, $\nu_{12} = 0.33$ and $\rho = 1.48 \times 10^3 \text{ kg/m}^3$. For enveloping and double delaminations, the delaminations are located at Interface 1 (midplane) and Interface 3 (see Figure 4.7). The results for the fundamental and mode 2 frequencies of clamped-clamped beam and cantilever beam are shown in Figure 4.22. It can be seen that the variations of the fundamental and mode 2 frequencies displays the essential features of Figures 4.11 and 4.14 for the homogeneous beam.

4.3.4 Two non-overlapping delaminations

This section presents the results for two non-overlapping delaminations. The beam material is identical to the composite beam studied by Shen and Grady (1992) (see Section 4.3.1.2). Figure 4.23 shows the influence of the delamination lengths on the fundamental frequency ω/ω_0 of a cantilever beam. In Figure 4.23(a), the longer delamination with length a_2 is at midplane and the shorter delamination with length a_1 is off-midplane, while in Figure 4.23(b), both the delaminations are off-midplane. The frequency decreases as the delamination lengths, a_1 and a_2 , increases. This is due to the decrease in the bending stiffness of the beam as the delamination lengths increase. The influence of the short delamination length a_1 on the fundamental frequency is significant, however, the influence decreases as a_2 increases. When the longer delamination is at midplane, the frequencies predicted using ‘constrained mode’ and the ‘free mode’ are almost the same. When the longer delamination is off-midplane, the ‘free mode’ predicts a lower frequency than the ‘constrained mode’ for $a_2 > 0.4L$.

Figure 4.24 shows the influence of the shorter delamination length a_1 and thicknesswise location H_2 on the fundamental frequency ω/ω_0 of the beam. The longer

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delamination, $a_2 = 0.5L$, is at midplane ($H_5 = 0.5H_1$) and $d_2 = 0.2L$. The frequency decreases as a_2 increases and the frequency decreases further as the shorter delamination moves closer towards the midplane. This is due to the decreasing bending stiffness at the delamination region. It is further observed that the frequencies of the ‘constrained’ and the ‘free’ modes are almost the same.

Figure 4.23. Influence of the delamination lengths, a_1 and a_2 , on the fundamental frequency of a cantilever beam: (a) The longer delamination is at midplane and the shorter delamination is off-midplane; (b) The two delaminations are off-midplane.

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Figure 4.24. Influence of the shorter delamination length, a_1 , on the fundamental frequency of a cantilever beam.

Figure 4.25 shows the influence of the spanwise location, d_2 , of the longer delamination, $a_2 = 0.4L$, on the fundamental frequency ω/ω_0 for different delamination length a_1 . In Figure 4.25(a), the longer delamination is at midplane and the shorter delamination is off-midplane, while, in Figure 4.25(b), both the delaminations are off-midplane. For both cases, it is observed that the frequency increases as the longer delamination moves towards the free end of the beam. However, the frequency decreases as a_1 increases. The decrease in frequency due to a_1 appears to be constant for any spanwise location d_2 . The increase in frequency as the longer delamination moves towards the free end of the beams is greater when the longer delamination is at midplane. When the longer delamination a_2 is at midplane ($H_5 = 0.5H_1$), the frequencies of the ‘constrained mode’ and the ‘free mode’ are almost the same (Figure 25(a)). When the longer delamination is off-midplane, a difference between the ‘constrained mode’ and the ‘free mode’ frequencies is observed and this difference increases as the longer delamination moves away from the center of the beam (Figure 4.25(b)).

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Figure 4.25. Influence of the spanwise location of the longer delamination, d_2 , and the length of the shorter delamination, a_1 , on the fundamental frequency of a cantilever beam: (a) The longer delamination is at midplane and the shorter delamination is off-midplane; (b) The two delaminations are off-midplane.

The mode shapes of the composite cantilever beam with symmetric delaminations with various delamination lengths are determined using the ‘free mode’. Two extreme cases are considered, when the two delaminations are at midplane (Figure 4.26) and between the two outermost plies (Figure 4.27). The two delaminations are at located at one-fourth and three-fourths of the beam length ($d_1 = d_2 = 0.25L$). Figure 4.26 shows

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that the presence of the delaminations cause changes to the mode shape and is significant for $a_1 = a_2 > 0.3L$. In addition, the change in the mode shape is more obvious at the delamination region near the fixed end of the beam. In Figure 4.27, the mode shape displays a slight opening for $a_1 = a_2 > 0.3L$ and that the opening is more obvious at the delamination region near the free end of the beam.

Figure 4.26. Fundamental 'free mode' mode shapes for cantilever composite beam with two non-overlapping delaminations located at midplane at $d_1 = d_2 = 0.25L$: (a) $a_1 = a_2 = 0.10L$; (b) $a_1 = a_2 = 0.20L$; (c) $a_1 = a_2 = 0.30L$; (d) $a_1 = a_2 = 0.40L$.

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Figure 4.27. Fundamental 'free mode' mode shapes for cantilever composite beam with two non-overlapping delaminations located between two outermost plies and at $d_1 = d_2 = 0.25L$: (a) $a_1 = a_2 = 0.10L$; (b) $a_1 = a_2 = 0.20L$; (c) $a_1 = a_2 = 0.30L$; (d) $a_1 = a_2 = 0.40L$.

4.3.5 Two overlapping delaminations

This section presents the results for two overlapping delaminations. Figure 4.28 shows the influence of the total axial delamination length a_t and the overlapping length a_o on the fundamental frequency of the cantilever beam. Where a_t is the

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distance between the extreme ends of the two delaminations along the beam length. The lengths of the two delaminations are equal, $a_1 = a_2$, and $a_t + a_o = a_1 + a_2$. The overlap a_o lies symmetrically about the center of the beam. $a_o = 0.0$ represents two non-overlapping delaminations, where the frequency is approximated by having $d_1 = d_2 = a_1/2 - 5 \times 10^{-5} L$. $a_o = a_t$ represents two fully overlapping delaminations. For both the ‘free mode’ and ‘constrained mode’, the decrease in ω/ω_0 due to a_t is less significant for $a_t/L < 0.2$ and remains to be small up to about $a_t/L = 0.4$. The decrease becomes significant for $a_t/L > 0.4$. It is further observed that, ω/ω_0 decreases as the overlap, a_o increases.

Figure 4.28. Influence of the total axial delamination length, a_t , and the overlapping length, a_o , on the fundamental frequency of a cantilever beam, $H_4 = H_5 = 0.25H_1$, $H_3 = 0.5H_1$.

Figure 4.29 shows the influence of the total axial delamination length a_t and overlapping length a_o on the mode 2 frequency of the cantilever beam. For the ‘constrained mode’, the decrease in the frequency ω/ω_0 is less significant for $a_t/L < 0.4$ and becomes significant for $a_t/L > 0.4$. The presence of an overlap, a_o ,

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further decreases the frequency. However, the frequencies for $a_o/a_t = 0.2, 0.4, 0.6$ are close to each other, which indicates that the influence of two overlapping delaminations with $a_o/a_t = 0.2, 0.4, 0.6$ are almost the same. Similarly, for the ‘free mode’, influence of a_t on ω/ω_0 becomes significant for $a_t/L > 0.4$. However, a significant decrease in the frequency is observed as a_o increases.

Figure 4.29. Influence of the total axial delamination length, a_t , and the overlapping length, a_o , on the mode 2 frequency of a cantilever beam, $H_4 = H_5 = 0.25H_1$, $H_3 = 0.5H_1$.

The ‘free mode’ mode shapes corresponding to the fundamental frequencies for $a_t/L = 0.3, 0.5, 0.6, 0.8$ at $a_o/a_t = 0.6$ are computed and shown in Figure 4.30. Figure 4.30 shows an increasing delamination opening as a_t increases. It is further observed that overlapping in the mode shapes occur. This is because the midplane delamination near the clamped end of the beam causes significant change to the mode shape, while the surface delamination near the free end of the beam does not significantly influence the mode shape of the beam (see Figures 4.26 and 4.27 for non-overlapping delaminations).

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Figure 4.30. 'Free mode' mode shape corresponding to the fundamental frequency of a cantilever beam with two overlapping delaminations at $a_o = 0.6a_t$: (a) $a_t = 0.3L$; (b) $a_t = 0.5L$; (c) $a_t = 0.6L$; (d) $a_t = 0.8L$.

Figure 4.31 shows the 'free mode' mode shape for the second vibration mode for $a_t/L = 0.6, 0.8$ at $a_o/a_t = 0.6$. The delamination opening increases as a_t increases, which correspond to the increasing difference between the 'free mode' and 'constrained mode' frequencies (Figure 4.29). This difference between the two

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frequencies further increases as a_o increases, which indicates that the delamination opening increases as the overlapping of the surface delamination and the midplane delamination increases.

(a)

(b)

Figure 4.31. 'Free mode' mode shape corresponding to the mode 2 frequency of a cantilever beam with two overlapping delaminations at $a_o = 0.6a_t$: (a) $a_t = 0.6L$; (b) $a_t = 0.8L$.

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Figure 4.32 and 4.33 show the influence of the surface delamination length, a_2 , on the fundamental and mode 2 frequencies of the beam, respectively. Delamination a_1 is at midplane ($H_3 = 0.5H_1$) and midspan ($d_1 = 0.0$), while the surface delamination is at $H_4 = 0.25H_1$ and spanwisely located at the right end of the midplane delamination ($d_2 = 0.5a_1$). For the fundamental frequency (Figure 4.32), the decrease in ω/ω_0 due to the surface delamination a_2 is less significant for $a_2 < 0.5L$. For the ‘free mode’, the decrease in ω/ω_0 is negligible for $a_2 < 0.2L$. The decrease remains to be small up to about $a_2 = 0.4L$ and becomes significant for $a_2 > 0.4L$. For the mode 2 frequency (Figure 4.33), the influence of a_2 on ‘constrained mode’ frequency is less significant for $a_2 < 0.3L$ and remains to be small for up to about $a_2 = 0.4L$. A significant decrease in ω/ω_0 is observed for $a_2 > 0.4L$. For the ‘free mode’, the decrease in ω/ω_0 is significant for $a_2 > 0.1L$.

Figure 4.32. Influence of the overlapping delamination length, a_2 , on the fundamental frequency of a cantilever beam, $H_4 = H_5 = 0.25H_1$, $H_3 = 0.5H_1$.

Figure 4.33. Influence of the overlapping delamination length, a_2 , on the mode 2 frequency of a cantilever beam, $H_4 = H_5 = 0.25H_1$, $H_3 = 0.5H_1$.

4.4 Concluding Remarks

Analytical solutions for the free vibration of composite beams with two enveloping delaminations, two non-overlapping delaminations and two overlapping delaminations have been presented. Both ‘free mode’ and ‘constrained mode’ solutions were presented. The results of the present solutions were shown to agree well with the analytical results and experimental data presented in the literature. These solutions can serve as benchmark solutions for future numerical/approximate solutions. The influence of the coupling between the axial and bending stiffnesses of the delaminated layers on the natural frequency was investigated, which was shown to influence the natural frequency of the beam. In addition, a parametric study showed that the frequencies and mode shapes of the composite beams are significantly influenced by

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the lengths and the locations of the delaminations. Based on the results obtained from the study, the following conclusions are made:

4.4.1 Two enveloping delaminations

1. For a clamped-clamped beam, the fundamental frequency showed a high sensitivity for the longer delamination but a low sensitivity for the second short delamination, whereas for the mode 2 frequency, the sensitivity is high for both delaminations. For a cantilever beam, the fundamental and mode 2 frequencies show a high sensitivity to the longer delamination but low sensitivity to the second short delamination.
2. Short delaminations ($a < 0.3L$) do not significantly influence the fundamental frequency and the mode shape of a beam, while long delaminations ($a > 0.5L$) significantly reduce the frequency and alter the mode shape.
3. The ‘free mode’ mode shapes displays a large opening for long delaminations ($a > 0.5L$) but hardly displays an opening for short delaminations ($a < 0.3L$). These results in a large difference between the ‘constrained mode’ and the ‘free mode’ frequencies for long delaminations and a small difference for short delaminations.

4.4.2 Two non-overlapping delaminations

1. For a composite cantilever beam with two non-equal length delaminations, the effect of the shorter delamination on the fundamental frequency of the beam decreases as the length of the longer delamination increases. When the longer delamination is at midplane, the predicted frequencies by the ‘constrained mode’ and the ‘free mode’ are either the same or very close to each other.

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2. For symmetric delaminations, when the delaminations are at midplane, changes in the mode shape is observed when the delamination lengths are greater than 30% of the beams length ($a_1 = a_2 > 0.3L$) and the change is more obvious at the delamination region near the fixed end of the beam. When the delaminations are near the beam surface, the mode shape displays an opening when the delaminations lengths are $a_1 = a_2 > 0.3L$ and that the opening is more obvious at the delamination region near the free end of the beam.

4.4.3 Two overlapping delaminations

1. The total axial delamination length a_t significantly influence the natural frequency of the beam. The presence of an overlap, a_o , further decreases the frequency. However, the influence is less significant when $a_t < 0.3L$. In addition, the difference between the ‘free mode’ and ‘constrained mode’ frequencies largely depend on the opening in ‘free mode’ mode shapes. When the mode shape displays a large opening, the difference between two frequencies is large. When the mode shape displays a small opening, the difference between two frequencies is small.
2. A short overlapping delamination ($a_2 < 0.4L$) does not significantly influence the frequency of the beam. Results show a dominating influence of the longer delamination on the natural frequency of the beam.

Chapter 5

Vibration of Beams with Embedded Piezoelectric Sensors and Actuators

5.1 Introduction

An eigenstrain technique for the dynamics of smart beams with embedded arrays of piezoelectric sensors and actuators was presented by Alghamdi and Dasgupta (1993a, 1993b, 2000, 2001). They analyzed the piezoelectric sensors and actuators in the host beam by using Eshelby's equivalent inclusion method. However, the piezoelectric inclusions were analyzed as elastic inclusions only, thereby neglecting the dielectric effects of the piezoelectric inclusions. The influence of the mechanical-electrical coupling of the piezoelectric sensors on the natural frequency was also neglected in their analyses. In this chapter, a mathematical model for the vibration of beams with embedded arrays of piezoelectric sensors and actuators is presented. The piezoelectric sensors and actuators are analyzed as inhomogeneous ellipsoidal inclusions in a non-piezoelectric matrix (host beam) by using Eshelby's equivalent inclusion method (Eshelby, 1957; Mura, 1987). The formulation for the piezoelectric inclusion problem is decoupled into two equivalent inclusion problems, an elastic

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problem and a dielectric problem. An equation for the natural frequency of the beam is determined using the variational principle in Rayleigh quotient form, which is expressed as functions of the elastic strain energy and dielectric energy of the piezoelectric inclusions. These energies are derived using Mura's (1987) formulation for inhomogeneous inclusions. The Euler-Bernoulli beam theory and Rayleigh-Ritz approximation technique are used in the present analysis. In addition, this chapter presents parametric studies on the influences of the volume fraction, aspect ratio and location of the piezoelectric inclusions on the natural frequency of the beam. The influences of the energies due to the piezoelectric coupling on the natural frequency are also studied.

This chapter is reported as follows: first, the mathematical modeling is presented in Section 5.2, which begins with the derivation of the equation for the natural frequency of a piezoelectric body in Section 5.2.1. The problem of piezoelectric ellipsoidal inclusions in a non-piezoelectric matrix is analyzed using Eshelby's equivalent inclusion method in Section 5.2.2. The energies of the piezoelectric inclusions are formulated in Section 5.2.3 and an explicit solution for the natural frequency of a beam with piezoelectric inclusions is obtained in Section 5.2.4. Next, the results of the mathematical model is presented and discussed in Section 5.3. The influence of the volume fraction, aspect ratio and location of the piezoelectric ellipsoidal inclusions on the natural frequency of the beam is investigated. In addition, the influences of the energies due to piezoelectric coupling on the natural frequency are studied. Finally, the concluding remarks for this chapter are presented in Section 5.4.

5.2 Mathematical Modeling

5.2.1 Variational principle in Rayleigh quotient form

Let Ω be a region occupied by a piezoelectric body and S be boundary surface of Ω . The constitutive relations for a linear piezoelectric material are

$$\sigma_{ij} = C_{ijmn}\varepsilon_{mn} - e_{nij}E_n \quad \text{in } \Omega \quad (5.1)$$

$$D_i = e_{imn}\varepsilon_{mn} + \kappa_{in}E_n \quad \text{in } \Omega \quad (5.2)$$

where σ_{ij} , ε_{ij} , E_i and D_i are the stress tensor, strain tensor, electric field vector and the electric displacement vector, respectively. C_{ijmn} , e_{nij} and κ_{in} are the elastic stiffness tensor, the piezoelectric tensor and permittivity tensor, respectively. The strain and electric field are derivable from the mechanical and electric potential as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (5.3)$$

$$E_i = -\phi_{,i} \quad (5.4)$$

where u_i is the displacement and ϕ is the electric potential. For the time-harmonic free vibration of a piezoelectric body with circular frequency ω , the governing equations and boundary conditions in rectangular Cartesian coordinates are (Tiersten, 1969)

$$\sigma_{ij,j} = (C_{ijmn}u_{m,n} + e_{nij}\phi_{,n})_{,j} = \rho\omega^2 u_i \quad \text{in } \Omega \quad (5.5)$$

$$D_{i,i} = (e_{imn}u_{m,n} - \kappa_{in}\phi_{,n})_{,i} = 0 \quad \text{in } \Omega \quad (5.6)$$

$$u_i = 0 \quad \text{on } S \quad (5.7)$$

$$\sigma_{ij}n_j = (C_{ijmn}u_{m,n} + e_{nij}\phi_{,n})n_j = 0 \quad \text{on } S \quad (5.8)$$

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$$\phi = 0 \quad \text{on } S \quad (5.9)$$

$$D_i n_i = (e_{imn} u_{m,n} - \kappa_{in} \phi_{,n}) n_i = 0 \quad \text{on } S \quad (5.10)$$

where n_j is the outward pointing unit normal vector. Conventional indicial notation is utilized where repeated subscripts are summed over the range 1–3 and the comma denotes partial differentiation. In the MKS system, the variables of piezoelectricity have the following units (Dunn and Taya, 1993):

$$[\sigma_{ij}] = \text{N m}^{-2}, \quad [D_i] = \text{NV}^{-1} \text{m}^{-1}, [\varepsilon_{ij}] = \text{m m}^{-1}, [E_i] = \text{V m}^{-1}, [u_i] = \text{m}$$

$$[C_{ijmn}] = \text{N m}^{-2}, \quad [e_{imn}] = \text{NV}^{-1} \text{m}^{-1} = \text{C m}^{-2}, [\kappa_{in}] = \text{NV}^{-2} = \text{C}^2 \text{N}^{-1} \text{m}^{-2}, [\phi] = \text{V}$$

Equations (5.5) and (5.6) are the Euler equations and eqns. (5.7) – (5.10) are the boundary conditions of the following stationary expression G (Eernisse, 1967)

$$\begin{aligned} G &= \frac{1}{2} \int_V (\sigma_{ij} u_{i,j} + E_i D_i - \rho \omega^2 u_i u_i) dV - \int_S (\sigma_{ij} n_j u_i - \phi D_i n_i) dS \\ &= \int_V \left(\frac{1}{2} C_{ijmn} u_{i,j} u_{m,n} - \frac{1}{2} \kappa_{in} \phi_{,i} \phi_{,n} + e_{imn} u_{m,n} \phi_{,i} - \frac{1}{2} \rho \omega^2 u_i u_i \right) dV \\ &\quad - \int_S u_i (C_{ijmn} u_{m,n} + e_{nij} \phi_{,n}) n_j dS + \int_S \phi (e_{imn} u_{m,n} - \kappa_{in}^* \phi_{,n}) n_i dS \end{aligned} \quad (5.11)$$

Since the variation of G is zero ($\delta G = 0$), then $G = 0$ (Eernisse, 1967; Tiersten, 1969).

The Rayleigh quotient for ω^2 can be written as

$$\begin{aligned} \omega^2 &= \frac{\int_V \left(\frac{1}{2} C_{ijmn} u_{i,j} u_{m,n} - \frac{1}{2} \kappa_{in} \phi_{,i} \phi_{,n} + e_{imn} u_{m,n} \phi_{,i} \right) dV}{\int_V \frac{1}{2} \rho u_i u_i dV} \\ &\quad + \frac{\int_S u_i (C_{ijmn} u_{m,n} + e_{nij} \phi_{,n}) n_j dS - \int_S \phi (e_{imn} u_{m,n} - \kappa_{in} \phi_{,n}) n_i dS}{\int_V \frac{1}{2} \rho u_i u_i dV} \end{aligned} \quad (5.12)$$

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By adding and subtracting $\frac{1}{2} \kappa_{in} \phi_{,i} \phi_{,n}$ to the integrand of the numerator in the first term on the right-hand side in eqn. (5.12), we obtain

$$\omega^2 = \frac{\int_V \left[\frac{1}{2} C_{ijmn} u_{i,j} u_{m,n} + \frac{1}{2} \kappa_{in} \phi_{,i} \phi_{,n} + D_i \phi_{,i} \right] dV}{\int_V \frac{1}{2} \rho u_i u_i dV} + \frac{\int_S u_i \left(C_{ijmn} u_{m,n} + e_{nij} \phi_{,n} \right) n_j dS - \int_S \phi \left(e_{imn} u_{m,n} - \kappa_{in} \phi_{,n} \right) n_i dS}{\int_V \frac{1}{2} \rho u_i u_i dV} \quad (5.13)$$

By integration by parts and applying the divergence theorem, we obtain

$$\int_V D_i \phi_{,i} dV = \int_V (D_i \phi)_{,i} dV - \int_V D_{i,i} \phi dV = \int_S D_i \phi n_i dS - \int_V D_{i,i} \phi dV \quad (5.14)$$

Substituting eqns. (5.6) – (5.10) and (5.14) in eqn. (5.13), for stationary solutions, the values of the Rayleigh quotient can be expressed as

$$\omega^2 = \frac{\int_V \left(\frac{1}{2} C_{ijmn} u_{i,j} u_{m,n} + \frac{1}{2} \kappa_{in} \phi_{,i} \phi_{,n} \right) dV}{\int_V \frac{1}{2} \rho u_i u_i dV} = \frac{\frac{1}{2} \int_V C_{ijmn} \varepsilon_{ij} \varepsilon_{mn} dV + \frac{1}{2} \int_V \kappa_{in} E_i E_n dV}{\frac{1}{2} \int_V \rho u_i u_i dV} \quad (5.15)$$

which was obtained by Eernisse (1967) and Yang and Batra (1993).

The numerator in eqn. (5.15) is the internal energy of the piezoelectric body, which is the sum of the elastic energy and the dielectric energy. For beams with piezoelectric inclusions, these energies will be formulated using the method presented by Mura (1987) for inhomogeneous inclusions in Section 5.2.3.

5.2.2 Non-piezoelectric matrix with piezoelectric inclusions

According to Mura (1987), *eigenstrain* is a generic name for nonelastic strains resulting from thermal expansion, phase transformation, initial strains, plastic strains

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and misfit strains. An inclusion is a sub-domain Ω in a domain D , where eigenstrain is prescribed in Ω and is zero in the matrix $D-\Omega$. The elastic moduli of the inclusion are assumed to be the same as the matrix. When the sub-domain Ω with prescribed eigenstrain has different elastic moduli with the matrix, Ω is called an inhomogeneous inclusion.

With the above definitions, let us consider an infinite non-piezoelectric matrix $D-\Omega$ with elastic moduli C_{ijmn} containing a piezoelectric ellipsoidal inclusion, perfectly bonded to the matrix, with domain Ω and elastic moduli C_{ijmn}^* (Figure 5.1). Since the elastic moduli of the piezoelectrics are different to that of the host beam, they can be analyzed as inhomogeneous inclusions.

Figure 5.1. A homogeneous and isotropic beam with embedded piezoelectric sensors and actuators.

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The constitutive equation for the piezoelectric inclusion is given in eqns. (5.1) and (5.2) and is rewritten as follows:

$$\sigma_{ij} = C_{ijmn}^* \varepsilon_{mn} - e_{nij}^* E_n \quad \text{in } \Omega \quad (5.16)$$

$$D_i = e_{imn} \varepsilon_{mn} + \kappa_{in}^* E_n \quad \text{in } \Omega \quad (5.17)$$

Since there is no electromechanical coupling in the matrix, the constitutive equation for the matrix is

$$\sigma_{ij} = C_{ijmn} \varepsilon_{mn} \quad \text{in } D - \Omega \quad (5.18)$$

$$D_i = \kappa_{in} E_n \quad \text{in } D - \Omega \quad (5.19)$$

The above constitutive equations, eqns. (5.16) – (5.19), can be written in a more generalized form. The constitutive equation for a non-piezoelectric matrix with a piezoelectric inclusion can be written as (Jiang et al., 1997, 1999)

$$\sigma_{ij} = C_{ijmn} \varepsilon_{mn} + \hat{C}_{ijkl} I_{klmn} \varepsilon_{mn} - \hat{e}_{nkl} I_{klj} E_n \quad \text{in } D \quad (5.20)$$

$$D_i = \kappa_{in} E_n + \hat{e}_{ikl} I_{klmn} \varepsilon_{mn} + \hat{\kappa}_{il} I_{ln} E_n \quad \text{in } D \quad (5.21)$$

where

$$\begin{aligned} \hat{C}_{ijkl} &= C_{ijkl}^* - C_{ijkl} \\ \hat{e}_{ikl} &= e_{ikl}^* - e_{ikl} = e_{ikl}^* \\ \hat{\kappa}_{il} &= \kappa_{il}^* - \kappa_{il} \end{aligned} \quad (5.22)$$

The superscript ‘*’ refers to the material property of the piezoelectric inclusions. I_{klmn}

and I_{ln} are fourth order and second order identity tensors, i.e.

$$\begin{aligned} I_{klmn} &= \begin{cases} \frac{1}{2}(\delta_{km} \delta_{ln} - \delta_{kn} \delta_{lm}) & \text{in } \Omega \\ 0 & \text{otherwise} \end{cases} \\ I_{ln} &= \begin{cases} \delta_{ln} & \text{in } \Omega \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (5.23)$$

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The assumption of no electro-mechanical interaction in the matrix material partially decouples the original piezoelectric inclusion problem into two inhomogeneous inclusion problems, namely, an elastic inclusion problem and a dielectric inclusion problem (Fan and Qin, 1995).

5.2.2.1 Elastic inclusion problem

For the elastic inclusion problem, the Hooke's law is expressed as (Eshelby, 1957; Mura, 1987)

$$\sigma_{ij}^0 + \sigma_{ij} = C_{ijmn}^* (\varepsilon_{mn}^0 + \varepsilon_{mn}) - e_{nij} (E_n^0 + E_n) \quad (5.24)$$

or

$$\sigma_{ij}^0 + \sigma_{ij} = C_{ijmn}^* (\varepsilon_{mn}^0 + \varepsilon_{mn} - \varepsilon_{mn}^p) \quad (5.25)$$

where

$$C_{ijmn}^* \varepsilon_{mn}^p = e_{nij} (E_n^0 + E_n) \quad (5.26)$$

and where σ_{ij}^0 is the uniform far field loading, ε_{mn}^0 is the strain corresponding to the far field loading, σ_{ij} and ε_{mn} are the disturbance stress and strains due to the presence of inhomogeneous inclusions, respectively, and ε_{mn}^p is the eigenstrain due to the electromechanical coupling of the piezoelectric material or the *eigenstrain actuation* (Irschik, et al., 1998; Krommer and Irschik, 1999; Irschik and Ziegler, 2001).

According to Eshelby's equivalent inclusion method, one can convert the inhomogeneous inclusion to an equivalent inclusion with the elastic constant of the matrix and a uniform fictitious eigenstrain ε_{mn}^* . With this concept, eqn. (5.25) can be written as

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$$\sigma_{ij}^0 + \sigma_{ij} = C_{ijmn} (\varepsilon_{mn}^0 + \varepsilon_{mn} - \varepsilon_{mn}^p - \varepsilon_{mn}^*) \quad (5.27)$$

By equating eqns. (5.25) and (5.27), we have

$$C_{ijmn}^* (\varepsilon_{mn}^0 + \varepsilon_{mn} - \varepsilon_{mn}^p) = C_{ijmn} (\varepsilon_{mn}^0 + \varepsilon_{mn} - \varepsilon_{mn}^{**}) \quad (5.28)$$

where the total eigenstrain ε_{mn}^{**} is defined as

$$\varepsilon_{mn}^{**} = \varepsilon_{mn}^p + \varepsilon_{mn}^* \quad (5.29)$$

The strain disturbance ε_{mn} can be related to the total eigenstrain by the elastic Eshelby tensor S_{mnab} as

$$\varepsilon_{mn} = S_{mnab} \varepsilon_{ab}^{**} \quad (5.30)$$

The elastic Eshelby tensor S_{mnab} is only a function of the matrix Poisson's ratio and the ellipsoidal aspect ratio (Mura, 1987). For a cylindrical inclusion with elliptical cross-section ($a_3 \rightarrow \infty$), the components of the Eshelby tensor are

$$\begin{aligned} S_{1111} &= 0 \\ S_{2222} &= \frac{1}{2(1-\nu)} \left\{ \frac{a_3^2 + 2a_2a_3}{(a_2 + a_3)^2} + (1-2\nu) \frac{a_3}{a_2 + a_3} \right\} \\ S_{3333} &= \frac{1}{2(1-\nu)} \left\{ \frac{a_2^2 + 2a_2a_3}{(a_2 + a_3)^2} + (1-2\nu) \frac{a_2}{a_2 + a_3} \right\} \\ S_{1122} &= 0 \\ S_{1133} &= 0 \\ S_{2211} &= \frac{1}{2(1-\nu)} \frac{2\nu a_3}{a_2 + a_3} \\ S_{2233} &= \frac{1}{2(1-\nu)} \left\{ \frac{a_3^2}{(a_2 + a_3)^2} - (1-2\nu) \frac{a_3}{a_2 + a_3} \right\} \\ S_{3311} &= \frac{1}{2(1-\nu)} \frac{2\nu a_2}{a_2 + a_3} \end{aligned}$$

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$$\begin{aligned}
 S_{3322} &= \frac{1}{2(1-\nu)} \left\{ \frac{a_2^2}{(a_2 + a_3)^2} - (1-2\nu) \frac{a_2}{a_2 + a_3} \right\} \\
 S_{1212} &= \frac{a_3}{2(a_2 + a_3)} \\
 S_{1313} &= \frac{a_2}{2(a_2 + a_3)} \\
 S_{2323} &= \frac{1}{2(1-\nu)} \left\{ \frac{a_2^2 + a_3^2}{(a_2 + a_3)^2} + \frac{(1-2\nu)}{2} \right\}
 \end{aligned} \tag{5.31}$$

and all other components are zero. Substituting eqn. (5.30) in eqn. (5.28) and rearranging, ε_{mn}^{**} can be expressed as

$$\varepsilon_{mn}^{**} = - \left[\hat{C}_{ijkl} S_{klmn} + C_{ijmn} \right]^{-1} \left[\hat{C}_{ijrs} \varepsilon_{rs}^0 - C_{ijrs}^* \varepsilon_{rs}^p \right] \tag{5.32}$$

5.2.2.2 Dielectric inclusion problem

The above equivalent inclusion method is repeated here to for the dielectric inclusion problem. Equation (5.17) is rewritten as

$$D_i^0 + D_i = \kappa_{in}^* (E_n^0 + E_n - E_n^p) \tag{5.33}$$

where

$$-\kappa_{in}^* E_n^p = e_{imn} (\varepsilon_{mn} + \varepsilon_{mn}^0) \tag{5.34}$$

where D_i^0 and E_n^0 are the known far fields, and E_n^p is caused by the mechanical-electrical coupling. Similarly with the elastic inclusion problem, we obtain the relationship

$$\kappa_{in}^* (E_n^0 + E_n - E_n^p) = \kappa_{in} (E_n^0 + E_n - E_n^p - E_n^*) \tag{5.35}$$

where E_n^* is the fictitious eigen-electric field. By defining a total eigen-electric field

E_n^{**} as

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$$E_n^{**} = E_n^p + E_n^* \quad (5.36)$$

and an electric Eshelby tensor, s_{nb} , the electric field disturbance E_n can be written as

$$E_n = s_{nb} (E_b^p + E_b^*) = s_{nb} E_b^{**} \quad (5.37)$$

The electric Eshelby tensor, s_{nb} , is only a function of the ellipsoidal aspect ratio (Hatta and Taya, 1986; Fan and Qin, 1995). For a cylindrical inclusion with ellipsoidal cross-section ($a_3 \rightarrow \infty$), the components of the electric Eshelby tensor are

$$\begin{aligned} s_{nb} &= 0 & n \neq b \\ s_{11} &= 0 \\ s_{22} &= \frac{a_2}{a_2 + a_3} \\ s_{33} &= \frac{a_3}{a_2 + a_3} \end{aligned} \quad (5.38)$$

Substituting eqn. (5.37) in eqn. (5.35) and rearranging, E_n^{**} can be expressed as

$$E_n^{**} = \left[-\hat{\kappa}_{im} s_{mn} - \kappa_{in} \right]^{-1} \left[\hat{\kappa}_{ik} E_k^0 - \kappa_{ik}^* E_k^p \right] \quad (5.39)$$

5.2.3 Energies of piezoelectric inclusions

5.2.3.1 Elastic strain energy

The elastic strain energy of the piezoelectric inclusion is formulated using the method presented by Mura (1987). When a body D , containing an inhomogeneous inclusion Ω with eigenstrain ε_{ij}^p , is subjected to an external force F_i , the elastic strain energy is

$$W_e = \frac{1}{2} \int_D (\sigma_{ij}^0 + \sigma_{ij}) (u_{i,j}^0 + u_{i,j} - \varepsilon_{ij}^p) dD \quad (5.40)$$

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where the displacement field is denoted by $u_{i,j}^0 + u_{i,j}$ and the stress field by $\sigma_{ij}^0 + \sigma_{ij}$.

$u_{i,j}^0$ and σ_{ij}^0 are the displacement and stress when F_i acts in the absence of Ω .

Furthermore, we have

$$\sigma_{ij}^0 = C_{ijmn} u_{m,n}^0 = C_{ijmn} \varepsilon_{mn}^0 \quad (5.41)$$

and

$$\sigma_{ij}^0 + \sigma_{ij} = C_{ijmn}^* (\varepsilon_{mn}^0 + \varepsilon_{mn} - \varepsilon_{mn}^p) \quad (5.42)$$

By integration by parts

$$\begin{aligned} \int_D \left[\sigma_{ij} (u_i^0 + u_i) \right]_{,j} dD &= \int_D \sigma_{ij} (u_i^0 + u_i)_{,j} dD + \int_D \sigma_{ij,j} (u_i^0 + u_i) dD \\ &= \int_D \sigma_{ij} (u_{i,j}^0 + u_{i,j}) dD + \int_D \sigma_{ij,j} (u_i^0 + u_i) dD \end{aligned} \quad (5.43)$$

and

$$\int_D \sigma_{ij} (u_{i,j}^0 + u_{i,j}) dD = \int_D \left[\sigma_{ij} (u_i^0 + u_i) \right]_{,j} dD - \int_D \sigma_{ij,j} (u_i^0 + u_i) dD \quad (5.44)$$

Applying the divergence theorem

$$\int_D \sigma_{ij} (u_{i,j}^0 + u_{i,j}) dD = \int_S \sigma_{ij} (u_i^0 + u_i) n_j dS - \int_D \sigma_{ij,j} (u_i^0 + u_i) dD \quad (5.45)$$

Since $\sigma_{ij,j} = 0$ in D (equilibrium conditions) and $\sigma_{ij} n_j = 0$ on S (boundary conditions),

we obtain

$$\int_D \sigma_{ij} (u_{i,j}^0 + u_{i,j}) dD = 0 \quad (5.46)$$

Furthermore, we have

$$\begin{aligned} \sigma_{ij}^0 (u_{i,j} - \varepsilon_{ij}^p) &= C_{ijmn} u_{m,n}^0 (\varepsilon_{ij} - \varepsilon_{ij}^p) \\ &= C_{ijmn} \varepsilon_{mn}^0 (\varepsilon_{ij} - \varepsilon_{ij}^p - \varepsilon_{ij}^* + \varepsilon_{ij}^*) \\ &= \varepsilon_{mn}^0 C_{ijmn} (\varepsilon_{ij} - \varepsilon_{ij}^p - \varepsilon_{ij}^*) + C_{ijmn} \varepsilon_{mn}^0 \varepsilon_{ij}^* \\ &= \sigma_{ij} \varepsilon_{ij}^0 + \sigma_{ij}^0 \varepsilon_{ij}^* \end{aligned} \quad (5.47)$$

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By integration by parts and applying the divergence theorem

$$\int_D \sigma_{ij} u_{i,j}^0 dD = \int_S \sigma_{ij} u_i^0 n_j dS - \int_D \sigma_{ij,j} u_i^0 dD = 0 \quad (5.48)$$

and so we have

$$\int_V \sigma_{ij}^0 (\varepsilon_{ij} - \varepsilon_{ij}^p) dD = \int_\Omega \sigma_{ij}^0 \varepsilon_{ij}^* dD \quad (5.49)$$

From the above eqns. (5.40) – (5.49), the elastic strain energy of the piezoelectric inclusion can be expressed as

$$W_e = \frac{1}{2} \int_D \sigma_{ij}^0 \varepsilon_{ij}^0 dD + \frac{1}{2} \int_\Omega \sigma_{ij}^0 \varepsilon_{ij}^* dD - \frac{1}{2} \int_\Omega \sigma_{ij} \varepsilon_{ij}^p dD \quad (5.50)$$

where ε_{ij}^* can be determined from

$$C_{ijmn}^* (\varepsilon_{mn}^0 + S_{mnab} \varepsilon_{ab}^*) = C_{ijmn} (\varepsilon_{mn}^0 + S_{mnab} \varepsilon_{ab}^* - \varepsilon_{mn}^*) \quad (5.51)$$

and σ_{ij} can be determined from

$$\sigma_{ij} = C_{ijmn}^* (S_{mnab} \varepsilon_{ab}^{**} - \varepsilon_{ab}^p) = C_{ijmn} (S_{mnab} \varepsilon_{ab}^{**} - \varepsilon_{ab}^{**}) \quad (5.52)$$

5.2.3.2 Dielectric energy

Similarly, the dielectric energy of the piezoelectric inclusion can be determined using the above method. The dielectric energy is given by

$$W_d = \frac{1}{2} \int_\Omega (D_i^0 + D_i) (E_i^0 + E_i - E_i^p) dD \quad (5.53)$$

where

$$D_i^0 = -\kappa_{in}^* \phi_{,n}^0 = \kappa_{in}^* E_n^0 \quad (5.54)$$

and

$$D_i^0 + D_i = \kappa_{in}^* (E_n^0 + E_n - E_n^p) \quad (5.55)$$

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By integration by parts

$$\begin{aligned} \int_{\Omega} \left[D_i (-\phi^0 - \phi) \right]_{,i} dD &= \int_{\Omega} D_i (-\phi^0 - \phi)_{,i} dD + \int_{\Omega} D_{i,i} (-\phi^0 - \phi) dD \\ &= \int_{\Omega} D_i (-\phi_{,i}^0 - \phi_{,i}) dD + \int_{\Omega} D_{i,i} (-\phi^0 - \phi) dD \end{aligned} \quad (5.56)$$

and

$$\int_{\Omega} D_i (-\phi_{,i}^0 - \phi_{,i}) dD = \int_{\Omega} \left[D_i (-\phi^0 - \phi) \right]_{,i} dD - \int_{\Omega} D_{i,i} (-\phi^0 - \phi) dD \quad (5.57)$$

Applying the divergence theorem, we have

$$\int_{\Omega} D_i (-\phi_{,i}^0 - \phi_{,i}) dD = \int_S D_i (-\phi^0 - \phi) n_i dS - \int_{\Omega} D_{i,i} (-\phi^0 - \phi) dD \quad (5.58)$$

Since $D_{i,i} = 0$ in D (Gauss' law of electrostatic) and $D_i n_i = 0$ on S (electrical boundary conditions), we obtain

$$\int_{\Omega} D_i (-\phi_{,i}^0 - \phi_{,i}) dD = \int_{\Omega} D_i (E_i^0 + E_i) dD = 0 \quad (5.59)$$

Furthermore, we have

$$\begin{aligned} D_i^0 (E_i - E_i^p) &= \kappa_{in}^* E_n^0 (E_i - E_i^p) \\ &= E_n^0 \kappa_{in}^* (E_i - E_i^p) \\ &= D_i E_i^0 \end{aligned} \quad (5.60)$$

By integration by parts and applying the divergence theorem

$$\int_{\Omega} D_i E_i^0 dD = \int_{\Omega} D_i (-\phi_{,i}^0) dD = \int_S D_i (-\phi^0) n_i dS - \int_{\Omega} D_{i,i} (-\phi^0) dD = 0 \quad (5.61)$$

From the above eqns. (5.53) – (5.61), the dielectric energy of the piezoelectric inclusion is determined by

$$W_d = \frac{1}{2} \int_{\Omega} D_i^0 E_i^0 dD - \frac{1}{2} \int_{\Omega} D_i E_i^p dD \quad (5.62)$$

where D_i can be determined from

$$D_i = \kappa_{in}^* (s_{na} E_a^{**} - E_i^p) \quad (5.63)$$

5.2.4 Natural frequency of a beam with embedded piezoelectric sensors and actuators

Figure 5.1 shows a beam with embedded arrays of piezoelectric sensors and actuators. The piezoelectric materials in the one of the rows are considered to act as sensors and the other row, as actuators. Bending the beam induces strains in the sensors and it produces an output, which is used in a closed-loop constant-feedback-gain control circuit to activate the corresponding actuators. The actuation induces strain along the length of the beam causing an effect opposite to that caused by the initial bending. This results in a stiffening of the beam and an accompanying increase in the natural frequency (Alghamdi and Dasgupta, 1993 a, 1993b).

In order for the Eshelby's equivalent inclusion method to be applicable to the present problem, the following two conditions must be satisfied: (a) the piezoelectric sensors/actuators should be embedded at about 20% of the beam thickness from the top and bottom faces of the beam; and (b) the distance between neighboring sensors/actuators should be larger than 3.6 times the width, a_2 (see Figure 5.2), of the sensor/actuator so that the interactions among the sensors/actuators can be neglected. To be able to satisfy the above conditions, the volume of the piezoelectric materials must be small relative to the volume of the host beam, which can be about less than five percent of the volume of the host beam (Alghamdi and Dasgupta, 1993a, 1993b).

The Rayleigh quotient in eqn. (5.15) presents an explicit solution to estimate the natural frequency of a piezoelectric body. We extend the use eqn. (5.15) for a non-piezoelectric beam with piezoelectric inclusions. Using the elastic strain energy of the

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piezoelectric inclusion in eqn. (5.50) and the dielectric energy in eqn. (5.62), the Rayleigh quotient can be written as

$$\omega^2 = \frac{\frac{1}{2} \int_D C_{ijmn} \varepsilon_{ij}^0 \varepsilon_{mn}^0 dD + \frac{1}{2} \int_{\Omega} C_{ijmn} \varepsilon_{ij}^* \varepsilon_{mn}^0 dD - \frac{1}{2} \int_{\Omega} C_{ijmn}^* \varepsilon_{ij}^p (S_{mnab} \varepsilon_{ab}^{**} - \varepsilon_{mn}^p) dD}{\frac{1}{2} \int_V \rho u_i u_i dV} + \frac{\frac{1}{2} \int_{\Omega} \kappa_{in}^* E_i^0 E_n^0 dD - \frac{1}{2} \int_{\Omega} \kappa_{in}^* E_i^p (s_{na} E_a^{**} - E_n^p) dD}{\frac{1}{2} \int_V \rho u_i u_i dV} \quad (5.64)$$

It should be noted that the elastic strain energy, which consists of first three terms in the numerator on the right-hand side in eqn. (5.64), was derived using Mura's method (1987) and is a simpler expression as compared with that presented by Alghamdi and Dasgupta (1993a, 1993b, 2000, 2001). The third term in the numerator represents the elastic strain energy due to the electromechanical coupling of the piezoelectric actuators, which will be referred to as *eigenstrain actuation energy*. The fifth term represents the dielectric energy due to the mechanical-electrical coupling of the piezoelectric sensors, which was not included in the analyses of Alghamdi and Dasgupta. This energy will be referred to as *eigen-electric energy*. If the eigen-electric energy is neglected, then the first two terms in the numerator comprise the energies of an unactuated beam.

In order to perform the integration in the above equation, all that remains now is to assume explicit representations for the applied flexural strain field and the eigenstrain actuation. We make further assumptions:

- a. The Euler-Bernoulli beam theory and plane stress assumption are used.

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- b. Each of the embedded piezoelectric sensors/actuators is assumed to be a piezoelectric cylinder with elliptical cross-section, whose polarization is oriented along the thickness of the beam (x_3 axis).
- c. The actuators will both have actuation and fictitious eigenstrains, whereas the sensors will only have fictitious eigenstrain.
- d. Only the sensors will have real eigen-electricfield due to the direct piezoelectric effect (applied stress on piezoelectric materials induces electric polarization).
- e. Since the electric field E_n^0 is applied to the piezoelectric inclusions (actuators), the disturbance electric field E_n is zero and thus $E_n^* = 0$ for both the actuators and sensors.

Using the Rayleigh-Ritz technique, for a simply-supported beam, the transverse displacement function is given by

$$u_3 = \sum_{m=1}^{\infty} a_m \sin(\omega_n t) \sin\left(\frac{m\pi x_2}{L}\right) \quad (5.65)$$

where the x_2 -axis is oriented along the length of the beam, u_3 is the transverse deflection in the x_3 direction, ω_n and a_n are the natural frequency and amplitude of the n th mode, respectively. Only the fundamental frequency ($m = 1$) is of interest in this study. In view of the Euler-Bernoulli beam theory and plane stress assumptions, the only non-zero term in the bending strain field ε_{ij}^0 is ε_2^0 and is determined by

$$\varepsilon_2^0 = -z \frac{\partial^2 u_3}{\partial x_2^2} = z \frac{\pi^2}{L^2} a_1 \sin(\omega_1 t) \sin\left(\frac{\pi x_2}{L}\right) \quad (5.66)$$

where z is the distance of the piezoelectric device from the neutral axis of the beam.

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The only non-zero component of the actuation voltage vector is E_3^0 and is proportional to the bending strain ε_2^0 . The eigenstrain actuation ε_{ij}^p is determined from eqn. (5.26)

$$\varepsilon_{mn}^p = d_{kmn} E_k^0 \quad (5.67)$$

where

$$d_{kmn} = [C_{ijmn}^*]^{-1} e_{kij} \quad (5.68)$$

and where d_{kmn} is the free-expansion of the piezoelectric actuator for a unit applied electric field. The non-zero terms of the electromechanical eigenstrain ε_{mn}^p expressed in contracted Voight notation is

$$\varepsilon_q^p = d_{3q} E_3^0 \quad (q = 1 - 6) \quad (5.69)$$

The eigen-electricfield due to the mechanical-electrical coupling E_k^p is determined from eqn. (5.34)

$$E_k^p = -[\kappa_{ik}^*]^{-1} e_{imn} (\varepsilon_{mn} + \varepsilon_{mn}^0) \quad (5.70)$$

and the non-zero terms are

$$E_3^p = -\frac{e_{3q}}{\kappa_{33}^*} (\varepsilon_q + \varepsilon_q^0) \quad (q = 1 - 6) \quad (5.71)$$

5.3 Results and Discussions

This section presents the results obtained using the mathematical model described in Section 5.2 to study the vibration of a simply supported isotropic beam with embedded piezoelectric sensors and actuators. For the purpose of verification, the present results are compared with the results of Alghamdi and Dasgupta (1993b). The

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host beam is made of Alplex plastic and piezoelectric sensors/actuators are made of PZT-5H. Table 5.1 presents the properties of Alplex and PZT-5H, which is poled along the length of the beam (x_2 -axis). The dimensions of the beam and the piezoelectric materials are shown in Figure 5.2, where the piezoelectric sensors and actuators is located between the surface and neutral axis of the beam ($z = e/4$). The normalized fundamental frequency ω/ω_0 as a function of the normalized eigenstrain actuation $\varepsilon_2^p/\varepsilon_2^0$ is shown in Figure 5.3, where ω_0 is the fundamental frequency of the beam without the actuation effect. The number of actuators, n , is identical to the number of sensors. It can be seen that the present results compared well with the results of Alghamdi and Dasgupta (1993b). It should be noted that in the present method, the elastic strain energy was derived using Mura's method for inhomogeneous inclusions, and the elastic strain energy expression is simpler as compared with that presented by Alghamdi and Dasgupta (1993a, 1993b, 2000, 2001). The dielectric energy of the piezoelectric inclusion, which was neglected in Alghamdi and Dasgupta's works, is included in the present method.

Figure 5.2. Dimensions of the beam and piezoelectric materials.

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Table 5.1. Properties of the host beam and the piezoelectric material poled along x_2 -axis (Alghamdi and Dasgupta, 1993).

	E (GPa)	ν	d_{21} (10^{-12} m/V)	d_{22} (10^{-12} m/V)	d_{16}	κ_{11} (10^{-10} C/Vm)	κ_{22} (10^{-10} C/Vm)
PZT-5H	64	0.39	-274	593	741	150	66
ALPLEX	3.5	0.35	—	—	—	—	—

Table 5.2. Properties of the piezoelectric material poled along x_3 -axis.

	E (GPa)	ν	d_{31} (10^{-12} m/V)	d_{33} (10^{-12} m/V)	d_{15}	κ_{11} (10^{-10} C/Vm)	κ_{33} (10^{-10} C/Vm)
PZT-5H	64	0.39	-274	593	741	150	66

Figure 5.3. Comparison of predicted fundamental frequencies, ω/ω_0 , with the theoretical results of Alghamdi and Dasgupta (1993b).

Figure 5.4 shows the influence of the poling direction of the piezoelectric actuators on the fundamental frequency of the beam, ω/ω_0 . The properties of PZT-5H poled along the thickness of the beam (x_3 -axis) are given in Table 5.2. As expected, the

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fundamental frequency increases more when the actuator is poled along the thickness direction, since the actuator induces more strain along length of the beam, this result in a greater increase of stiffness of the beam and the natural frequency. It is further observed that ω/ω_0 increases with increasing eigenstrain actuation $\varepsilon_2^p/\varepsilon_2^0$ and with increasing number of sensors/actuators, n . Similarly, Figure 5.5 shows ω/ω_0 increases with increasing eigenstrain actuation $\varepsilon_2^p/\varepsilon_2^0$ and with increasing piezoelectric volume fraction, v_f .

Figure 5.4. Influence of the poling direction of the piezoelectric materials on the fundamental frequency of the beam, ω/ω_0 .

Figure 5.6 shows the influence of the aspect ratio of the ellipsoidal piezoelectric inclusions a_2/a_3 on the fundamental frequency of the beam for different piezoelectric volume fraction v_f . It is observed that the use of a flat piezoelectric device increases the natural frequency. Furthermore, the influence of a_2/a_3 becomes more obvious for higher v_f and $\varepsilon_2^p/\varepsilon_2^0$.

Figure 5.5. Influence of the eigenstrain actuation $\varepsilon_2^p / \varepsilon_2^0$ and the piezoelectric volume fraction v_f on the fundamental frequency of the beam ω / ω_0 .

Figure 5.7 shows that frequency ω/ω_0 increases as the actuator is located nearer to the beam surface and further increases with increasing v_f and $\varepsilon_2^p / \varepsilon_2^0$. In addition, increasing a_2/a_3 further increases the natural frequency (Figure 5.8). However, the influence of the aspect ratio a_2/a_3 is less significant for $\varepsilon_2^p / \varepsilon_2^0 = 1$.

The variation of the fundamental frequency ω/ω_0 with the stiffness of the host beam E_H is shown in Figures 5.9 and 5.10, where the sensors and actuators are located between the neutral axis and the beam surface. It is observed that ω/ω_0 decreases drastically up to about $E_H = 2.5$ GPa, ω/ω_0 then decreases slowly up to about $E_H = 7$ GPa, after which ω/ω_0 displays an almost constant frequency. This can be explained by the decreasing influence of the actuation as the host stiffness increases. Furthermore, the natural frequency ω/ω_0 decreases more for higher volume fraction v_f (Figure 5.9) and higher $\varepsilon_2^p / \varepsilon_2^0$ (Figure 5.10).

Figure 5.6. Influence of the aspect ratio a_2/a_3 of the piezoelectric sensors and actuators on the fundamental frequency ω/ω_0 .

Figures 5.11 – 5.14 shows the influence of the eigenstrain actuation energy on the natural frequency of the beams. The influence of this energy is studied by considering the difference between the frequencies when all the energy terms are included in Eqn. (5.64) and when the eigenstrain actuation energy (third term in the numerator) is neglected, where $(\Delta\omega)_{em}$ is the difference between the two frequencies. It can be seen that the trends are similar to that of Figures 5.5 – 5.10, which indicates that the

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influence of the energy due to electromechanical coupling on the natural frequency is significant.

Figure 5.7. Influence of the actuator location, d/e , on the fundamental frequency of the beam, ω/ω_0 , with various piezoelectric volume fraction v_f .

Figure 5.8. Influence of the actuator location, d/e , on the fundamental frequency of the beam, ω/ω_0 , with various aspect ratio a_2/a_3 of the piezoelectric sensors and actuators.

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Figure 5.9. Variation of the fundamental frequency, ω/ω_0 , with the host stiffness for different volume fractions of the piezoelectric sensors and actuators, v_f .

Figure 5.10. Variation of the fundamental frequency, ω/ω_0 , with the host stiffness for various eigenstrain actuation $\varepsilon_2^p/\varepsilon_2^0$.

Figure 5.11. Changes in the natural frequency due to energy of the electromechanical coupling, $(\Delta\omega)_{em}/\omega_0$, with eigenstrain actuation, $\varepsilon_2^p/\varepsilon_2^0$, for different volume fraction, v_f .

Figures 5.16 – 5.19 shows the influence of the eigen-electric energy on the natural frequency of the beam, where $(\Delta\omega)_{me}$ is the difference between the frequencies when all the energy terms are included in Eqn. (5.64) and when the energy due to the mechanical-electrical coupling (fifth term in the numerator) is neglected. Figure 5.16 shows that the influence of the eigen-electric energy on the natural frequency of the beam decreases with increasing eigenstrain actuation $\varepsilon_2^p/\varepsilon_2^0$. This influence increases as aspect ratio a_2/a_3 increases (Figure 5.17), however, it decreases as $\varepsilon_2^p/\varepsilon_2^0$ increases.

In Figures 5.18 and 5.19, the influence of the eigen-electric energy on the natural frequency increases with increasing host stiffness and increases with increasing volume fraction v_f (Figure 5.18) and increasing eigenstrain actuation $\varepsilon_2^p/\varepsilon_2^0$ (Figure 5.19). However, the maximum influence is less than 1% of ω_0 .

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Figure 5.12 Changes in the natural frequency due to energy of the electromechanical coupling, $(\Delta\omega)_{em}/\omega_0$, with aspect ratio, a_2/a_3 , for different volume fraction, v_f :
 (a) $\varepsilon_2^p/\varepsilon_2^0 = 1$; (b) $\varepsilon_2^p/\varepsilon_2^0 = 2$.

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Figure 5.13. Changes in the natural frequency due to energy of the electromechanical coupling, $(\Delta\omega)_{em}/\omega_0$, with actuator location, d/e .

Figure 5.14. Changes in the natural frequency due to energy of the electromechanical coupling, $(\Delta\omega)_{em}/\omega_0$, with the host stiffness for different volume fraction, v_f .

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Figure 5.15. Changes in the natural frequency due to energy of the electromechanical coupling, $(\Delta\omega)_{em}/\omega_0$, with the host stiffness for different eigenstrain actuation, $\varepsilon_2^p/\varepsilon_2^0$.

Figure 5.16. Changes in the natural frequency due to energy of the mechanical-electrical coupling, $(\Delta\omega)_{me}/\omega_0$, with eigenstrain actuation, $\varepsilon_2^p/\varepsilon_2^0$, for different volume fraction, ν_f .

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Figure 5.17. Changes in the natural frequency due to energy of the mechanical-electrical coupling, $(\Delta\omega)_{me}/\omega_0$, with the aspect ratio, a_2/a_3 .

Figure 5.18. Changes in the natural frequency due to energy of the mechanical-electrical coupling, $(\Delta\omega)_{me}/\omega_0$, with the host stiffness for different volume fraction, v_f .

Figure 5.19. Changes in the natural frequency due to energy of the mechanical-electrical coupling, $(\Delta\omega)_{me}/\omega_0$, with the host stiffness for different eigenstrain actuation, $\varepsilon_2^p/\varepsilon_2^0$.

5.4 Concluding Remarks

A mathematical model for the vibration of a beam with embedded piezoelectric sensors and actuators has been presented. This model was used to study the influence of the volume fraction, aspect ratio and location of the piezoelectric sensors/actuators on the natural frequency of the beam. The influence of the energies due to the coupling of the piezoelectric sensors/actuators on the natural frequency was studied. Based on these studies, the following conclusions are made:

1. The natural frequency of the beam increases as the eigenstrain actuation and the piezoelectric volume fraction increase. Flat piezoelectric materials further increases the natural frequency due to the increasing influence of the eigenstrain actuation energy on the natural frequency as the aspect ratio increases.

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2. The natural frequency of the beam decreases as the host stiffness increases. Furthermore, the natural frequency decreases more as the piezoelectric volume fraction and the eigenstrain actuation increases. This is because the influence of the eigenstrain actuation energy on the natural frequency of the beam increases with increasing eigenstrain actuation and decreases with increasing host stiffness
3. The eigenstrain actuation energy has been shown to significantly influence the natural frequency of the beam, while the eigen-electric energy has less significant influence.
4. Piezoelectric materials with high aspect ratio increase the influence of the eigen-electric energy on the natural frequency. The influence of the eigen-electric energy decreases with increasing eigenstrain actuation and increasing host stiffness.

Chapter 6

Conclusions, Publications and Recommendations for Future Work

6.1 Conclusions

Studies on the vibration of delaminated composite beams and the vibration of beams with embedded arrays of piezoelectric sensors and actuators have been presented. The following are the conclusions of this thesis:

6.1.1 New parameters for the vibration of delaminated multilayered beams

Nondimensional parameters of normalized axial stiffness, normalized bending stiffness, axial stiffness ratio and bending stiffness ratio, have been introduced to study the vibration of delaminated bimaterial beams. The use of these parameters have been shown to provide a better understanding on the effect of the bending stiffness and axial stiffness of the different layers on the vibration of delaminated bimaterial beams and a physical understanding on the ‘constrained mode’ and ‘free mode’ assumptions used in delamination vibration. In addition, a new slenderness ratio has been introduced, which have been shown to characterize the mode shapes of the delaminated beams.

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Global, mixed and local vibration modes occur depending upon the relative slenderness ratio of the beam.

The vibration of three layer beams with two equal length delaminations has been solved analytically without resorting to numerical approximation. The beam was found to vibrate together in a ‘constrained mode’, or independently in a ‘free mode’, or in a mixed ‘partially constrained mode’ depending upon the relative thickness of the delaminated layers of the beam. A new slenderness ratio specific for the analytical solution was introduced, which was shown to dominate the vibration behavior of the beam. However, the dominance is different for the ‘constrained mode’, the ‘free mode’, and the ‘partially constrained mode’.

The lower and upper bounds of the natural frequency were obtained by assuming a totally ‘constrained’ and totally ‘free’ deformations of the delaminated layers. Apart from providing useful approximations, the proposed lower bound and upper bound are general and can be easily applied to multiple delamination problems.

6.1.2 Vibration of composite beams with multiple delaminations

Analytical solutions for the free vibration of composite beams with two enveloping, two non-overlapping and overlapping delaminations have been presented. The results of the present solution were shown to agree well with the analytical and experimental data reported in the literature. Apart from being the solutions of a multiple delamination vibration problem, the analytical solutions can serve as a benchmark test case for general numerical/approximation schemes for vibration of beams with multiple delaminations. The use of these analytical solutions allows

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inexpensive simulations of slight variations in the system, such as changes in the delamination lengths and locations, and boundary conditions.

The influence of the coupling between the axial and bending stiffnesses of the delaminated layers on the natural frequency was investigated and was shown to significantly influence the natural frequencies of the beams. In addition, a parametric study was conducted to investigate the influence of the lengths and locations of the delaminations on the natural frequencies and the corresponding mode shapes of the beams. The study can be used to optimize the design of composite beams in a dynamic environment. However, the first two frequencies of the beam are not sensitive to delamination lengths that less than 10% of the beam length and thus not suitable for damage detection. There is a need to study higher vibration modes if the natural frequencies are to be used for damage detection.

6.1.3 Vibration of beams with embedded piezoelectric sensors and actuators

A mathematical model for the vibration of beams with embedded piezoelectric sensors and actuators has been presented. Through the use of the variational principle in Rayleigh quotient form, Eshelby's equivalent inclusion method and Mura's formulation for the energies of inhomogeneous inclusions, an explicit solution for the natural frequency of the beam was obtained. A parametric study was conducted to investigate the influence of the volume fraction, aspect ratio and location of the piezoelectric sensors/actuators on the natural frequency of the beam. In addition, the influence of the energies due to the electromechanical coupling of the actuators and the mechanical-electrical coupling of the sensors on the natural frequency was also studied.

6.2 Publications Resulting from the Thesis

- Shu, D. and Della, C.N. (2004). Vibrations of multiple delaminated beams. *Composite Structures* 64(3-4), 467–477.
- Shu, D. and Della, C.N. (2004). Free vibration analysis of composite beams with two non-overlapping delaminations. *International Journal of Mechanical Sciences* 46(4), 509–526.
- Della, C.N., Shu, D. and Parlapalli, M.S.R. (2004). Vibrations of beams with two overlapping delaminations. *Composite Structures* 66(1-4), 101–108. (Presented at 12th International Conference on Composite Structures (ICCS/12). Monash University-Australia- 17th-19th Nov. 2003).
- Della, C.N., Shu, D. and Zhao, Y.P (2005). Vibration of composite beams with two overlapping delaminations. *Acta Mechanica Sinica* (English Series) 21(1), 47–55.
- Della, C.N. and Shu, D. (2005). Free vibration analysis of composite beams with overlapping delaminations. *European Journal of Mechanics-A* 24(3), 491–503.
- Della, C.N. and Shu, D. (2005). Vibration of beams with double delaminations. *Journal of Sound and Vibration* 282(3-5), 919–935.
- Della, C.N. and Shu, D. (2006). Vibration of delaminated multilayer beams. *Composites: Part B* 37(2-3), 227–236.
- Della, C.N. and Shu, D. Vibration of beams with embedded piezoelectric sensors and actuators. *Smart Materials and Structures* 15(2), 529–537.
- Della, C.N. and Shu, D. Vibration of delaminated composite laminates: a review. *Applied Mechanics Reviews* (Accepted).

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- Della, C.N. and Shu, D. Free vibration analysis of bimaterial beams. *Composite Structures* (Accepted).
- Della, C.N. and Shu, D. Vibration of beams with piezoelectric inclusions. *International Journal of Solids and Structures* (In review).
- Della, C.N., Shu, D. and Parlapalli, M.S.R. Vibration of prebuckled beams with multiple delaminations. *International Journal of Mechanical Sciences* (In review).
- Della, C.N., Shu, D. Vibration of beams with two overlapping delaminations in prebuckled states. *Composites: Part B* (In review).
- Della, C.N., Shu, D. and Parlapalli, M.S.R. Free vibration analysis of prebuckled beams with two enveloping delaminations. *Composites Science and Technology* (In review).
- Della, C.N. and Shu, D. Vibration of prebuckled beams with two non-overlapping delaminations. (In preparation).

6.3 Future Work

1. Improve the present model for multiple delaminations to include the shear deformation and rotary inertia effects, and to use the model to investigate the natural frequencies and mode shapes of the delaminated composite beams at higher vibration modes (mode 3 and above).
2. Perform experiments to further verify the results of the multiple delamination models to be able to identify the limitations of the models and propose modifications and improvement for practical applications.

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3. Improve the mathematical model for beams with embedded piezoelectric sensors and actuators to include the disturbance energy within the vicinity of the piezoelectric inclusions.
4. Extend the mathematical model for beams with embedded piezoelectric sensors and actuators to include cracks in the host beam.

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APPENDIX A

Governing Equation for the Free Vibration of Composite Beams

In this section, the governing equation for the free vibration of composite beams is formulated. The strain-displacement equation for beams in bending can be expressed as

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad (\text{A.1})$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \quad (\text{A.2})$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \quad (\text{A.3})$$

where ε_x and ε_z are the normal strains in the x and z directions, respectively; u and w denotes the displacements in the x and z directions, respectively; and γ_{xz} is the transverse shear strain. Integrating the eqn. (A.3) gives

$$u = u_o + z \left(\gamma_{xz} - \frac{\partial w}{\partial x} \right) \quad (\text{A.4})$$

where u_o is the midplane axial displacement. Differentiating eqn. (A.4) with respect to x , we can write an equation for the axial strain as

$$\varepsilon_x = \varepsilon_{x_o} + z k_x \quad (\text{A.5})$$

where

Appendix A: Governing Equation for the Free Vibration of Composite Beams

$$\varepsilon_{x_o} = \frac{\partial u_o}{\partial x} \quad (\text{A.6})$$

$$k_x = \frac{\partial}{\partial x} \left(\gamma_{xz} - \frac{\partial w}{\partial x} \right) \quad (\text{A.7})$$

and where ε_{x_o} and k_x are the midplane axial strain and curvature, respectively.

The equations of motion for the beam (Hu and Hwu, 1995) is expressed as

$$\frac{\partial N_x}{\partial x} = 0 \quad (\text{A.8})$$

$$\frac{\partial Q_x}{\partial x} + N_x \frac{\partial^2 w}{\partial x^2} + q = \rho H \frac{\partial^2 w}{\partial t^2} \quad (\text{A.9})$$

$$\frac{\partial M_x}{\partial x} = Q_x + I \frac{\partial^2 \beta_x}{\partial t^2} \quad (\text{A.10})$$

where

$$\beta_x = \gamma_{xz} - \frac{\partial w}{\partial t} \quad (\text{A.11})$$

and where β_x , I , ρ and H are the rotation angle, moment of inertia (with respect to the midplane), mass density and the thickness per unit width of the composite beam, respectively; q is the normal pressure distributed over the top or bottom surface of the composite beam. The resultant force, N_x and resultant moment, M_x of the beam can be defined using the classical lamination theory as

$$N_x = \varepsilon_{x_o} A_{11} + k_x B_{11} \quad (\text{A.12})$$

$$M_x = \varepsilon_{x_o} B_{11} + k_x D_{11} \quad (\text{A.13})$$

and Q_x can be expressed as

$$Q_x = S \gamma_{xz} \quad (\text{A.14})$$

where

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$$A_{11}^{(i)} = b \sum_{k=1}^{n_i} (\bar{Q}_{11}^k)_k (z_k - z_{k-1}) \quad (\text{A.15})$$

$$B_{11}^{(i)} = \frac{b}{2} \sum_{k=1}^{n_i} (\bar{Q}_{11}^k)_k (z_k^2 - z_{k-1}^2) \quad (\text{A.16})$$

$$D_{11}^{(i)} = \frac{b}{3} \sum_{k=1}^{n_i} (\bar{Q}_{11}^k)_k (z_k^3 - z_{k-1}^3) \quad (\text{A.17})$$

$$\bar{Q}_{11}^k = Q_{11}^k \cos^4 \phi + Q_{22}^k \sin^4 \phi + 2(Q_{12}^k + 2Q_{66}^k) \cos^2 \phi \sin^2 \phi \quad (\text{A.18})$$

$$Q_{11} = \frac{E_{11}}{1 - \nu_{12}\nu_{21}} \quad (\text{A.19})$$

$$Q_{22} = \frac{E_{22}}{1 - \nu_{12}\nu_{21}} \quad (\text{A.20})$$

$$Q_{12} = \frac{\nu_{12}E_{22}}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_{11}}{1 - \nu_{12}\nu_{21}} \quad (\text{A.21})$$

$$Q_{66} = G_{12} \quad (\text{A.22})$$

$$\nu_{21} = \frac{\nu_{12}E_{22}}{E_{11}} \quad (\text{A.23})$$

and where $A_{11}^{(i)}$ is the extensional stiffness, $B_{11}^{(i)}$ is the coupling stiffness, $D_{11}^{(i)}$ is the bending stiffness, $\bar{Q}_{11}^k$ is the coefficient stiffness of the lamina, b is the width, n_i is the number of plies, E_{11} and E_{22} are the longitudinal and transverse Young's moduli, respectively, G_{12} is the in-plane shear modulus, ν_{12} and ν_{21} are the longitudinal and transverse Poisson's ratio, respectively, ϕ is the angle of the k -th lamina orientation and z_k and z_{k-1} are the locations of the k -th lamina with respect to the midplane of the i -th beam. Equation (A.8) reveals that N_x is a constant throughout the beam and is equal to an axial load P applied at the end of the beam, i.e.

Appendix A: Governing Equation for the Free Vibration of Composite Beams

$$N_x = -P \quad (\text{A.24})$$

Substituting eqns. (A.14) and (A.24) in eqn. (A.9), we have

$$\frac{\partial \gamma_{xz}}{\partial x} = \frac{P}{S} \frac{\partial^2 w}{\partial x^2} + \frac{\rho H}{S} \frac{\partial^2 w}{\partial t^2} - \frac{q}{S} \quad (\text{A.25})$$

An expression for the curvature, k_x can be obtained with the substitution of eqn. (A.25) in eqn. (A.7), thus

$$k_x = -\left(1 - \frac{P}{S}\right) \frac{\partial^2 w}{\partial x^2} + \frac{\rho H}{S} \frac{\partial^2 w}{\partial t^2} - \frac{q}{S} \quad (\text{A.26})$$

Substituting eqns. (A.12) and (A.24) in eqn. (A.13), we have

$$M_x = \left(D_{11} - \frac{B_{11}^2}{A_{11}}\right) k_x - \frac{PB_{11}}{A_{11}} \quad (\text{A.27})$$

Substituting eqns. (A.14) and (A.27) in eqn. (A.10) and differentiating both sides with respect to x , we have

$$D \frac{\partial^2 k_x}{\partial x^2} = S \frac{\partial \gamma_{xz}}{\partial x} + I \frac{\partial^2 k_x}{\partial t^2} \quad (\text{A.28})$$

where

$$D = D_{11} - \frac{B_{11}^2}{A_{11}} \quad (\text{A.29})$$

and where D is the reduced bending stiffness. Finally, by substituting eqns. (A.25) and (A.26) in eqn. (A.28), the equation of motion for the composite beam can be expressed as

$$\begin{aligned} & \left(1 - \frac{P}{S}\right) D \frac{\partial^4 w}{\partial x^4} - \left[I \left(1 - \frac{P}{S}\right) + \frac{\rho H D}{S} \right] \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho H I}{S} \frac{\partial^4 w}{\partial t^4} + \frac{\rho H}{S} \frac{\partial^2 w}{\partial t^2} \\ & = \frac{D}{S} \frac{\partial^2 q}{\partial x^2} - P \frac{\partial^2 w}{\partial x^2} - \frac{I}{S} \frac{\partial^2 q}{\partial t^2} + q \end{aligned} \quad (\text{A.30})$$

For free vibrations, $P = q = 0$ and eqn. (A.30) can be reduced to

Appendix A: Governing Equation for the Free Vibration of Composite Beams

$$D \frac{\partial^4 w}{\partial x^4} - \left[I + \frac{\rho H D}{S} \right] \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho H I}{S} \frac{\partial^4 w}{\partial t^4} + \rho H \frac{\partial^2 w}{\partial t^2} = 0 \quad (\text{A.31})$$

For classical beams, the second and the third terms are negligible, thus the governing equation for vibration of classical beams is

$$D \frac{\partial^4 w}{\partial x^4} + \rho H \frac{\partial^2 w}{\partial t^2} = 0 \quad (\text{A.30})$$

Appendix B: Homogeneous Equations for the Vibration of a Beam with a Single Delamination

APPENDIX B

Homogeneous Equations for the Vibration of a Beam with a Single Delamination

In this section, the 12 homogeneous equations for the ‘constrained mode’ model to determine the eigenvalue (nondimensional frequency), λ , of a beam with a single delamination is presented. The generalized solution for the governing equation (eqn A.30) is

$$W_i(x) = C_i \cos(\lambda_i x) + S_i \sin(\lambda_i x) + CH_i \cosh(\lambda_i x) + SH_i \sinh(\lambda_i x)$$

where:

$$\lambda_i^4 = \frac{\omega^2 \rho_i H_i}{D_i}$$

1. Equations (1): boundary condition for a clamped end at $x = x_1$,

$$W_1(x = x_1) = 0$$

$$C_1 \cos(\lambda_1 x_1) + S_1 \sin(\lambda_1 x_1) + CH_1 \cosh(\lambda_1 x_1) + SH_1 \sinh(\lambda_1 x_1) = 0$$

2. Equations (2): boundary condition for a clamped end at $x = x_1$,

$$W_1'(x = x_1) = 0$$

$$-C_1 \lambda_1 \sin(\lambda_1 x_1) + S_1 \lambda_1 \cos(\lambda_1 x_1) + CH_1 \lambda_1 \sinh(\lambda_1 x_1) + SH_1 \lambda_1 \cosh(\lambda_1 x_1) = 0$$

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3. Equation (3): continuity conditions for the deflection at $x=x_2$,

$$W_1(x = x_2) = W_2(x = x_2)$$

$$C_1 \cos(\lambda_1 x_2) + S_1 \sin(\lambda_1 x_2) + CH_1 \cosh(\lambda_1 x_2) + SH_1 \sinh(\lambda_1 x_2) \\ - C_2 \cos(\lambda_2 x_2) - S_2 \sin(\lambda_2 x_2) - CH_2 \cosh(\lambda_2 x_2) - SH_2 \sinh(\lambda_2 x_2) = 0$$

4. Equation (4): continuity conditions for the slope at $x=x_2$,

$$W_1'(x = x_2) = W_2'(x = x_2)$$

$$-\lambda_1 C_1 \sin(\lambda_1 x_2) + \lambda_1 S_1 \cos(\lambda_1 x_2) + \lambda_1 CH_1 \sinh(\lambda_1 x_2) + \lambda_1 SH_1 \cosh(\lambda_1 x_2) \\ + \lambda_2 C_2 \sin(\lambda_2 x_2) - \lambda_2 S_2 \cos(\lambda_2 x_2) - \lambda_2 CH_2 \sinh(\lambda_2 x_2) - \lambda_2 SH_2 \cosh(\lambda_2 x_2) = 0$$

5. Equation (5): continuity conditions for the deflection at $x=x_3$,

$$W_2(x = x_3) = W_4(x = x_3)$$

$$C_2 \cos(\lambda_2 x_3) + S_2 \sin(\lambda_2 x_3) + CH_2 \cosh(\lambda_2 x_3) + SH_2 \sinh(\lambda_2 x_3) \\ - C_4 \cos(\lambda_4 x_3) - S_4 \sin(\lambda_4 x_3) - CH_4 \cosh(\lambda_4 x_3) - SH_4 \sinh(\lambda_4 x_3) = 0$$

6. Equation (6): continuity conditions for the slope at $x=x_3$,

$$W_2'(x = x_3) = W_4'(x = x_3)$$

$$-\lambda_2 C_2 \sin(\lambda_2 x_3) + \lambda_2 S_2 \cos(\lambda_2 x_3) + \lambda_2 CH_2 \sinh(\lambda_2 x_3) + \lambda_2 SH_2 \cosh(\lambda_2 x_3) \\ + \lambda_4 C_4 \sin(\lambda_4 x_3) - \lambda_4 S_4 \cos(\lambda_4 x_3) - \lambda_4 CH_4 \sinh(\lambda_4 x_3) - \lambda_4 SH_4 \cosh(\lambda_4 x_3) = 0$$

7. Equations (7): boundary condition for a clamped end at $x = x_4$,

$$W_4(x = x_4) = 0$$

$$C_4 \cos(\lambda_4 x_4) + S_4 \sin(\lambda_4 x_4) + CH_4 \cosh(\lambda_4 x_4) + SH_4 \sinh(\lambda_4 x_4) = 0$$

7'. Equations (7): boundary condition for a free end (cantilever beam) at $x = x_4$,

$$W_4''(x = x_4) = 0$$

$$-\lambda_4^2 C_4 \cos(\lambda_4 x_4) - \lambda_4^2 S_4 \sin(\lambda_4 x_4) + \lambda_4^2 CH_4 \cosh(\lambda_4 x_4) + \lambda_4^2 SH_4 \sinh(\lambda_4 x_4) = 0$$

8. Equations (8): boundary condition for a clamped end at $x = x_4$,

$$W_4'(x = x_4) = 0$$

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$$-C_4 \lambda_4 \sin(\lambda_4 x_4) + S_4 \lambda_4 \cos(\lambda_4 x_4) + CH_4 \lambda_4 \sinh(\lambda_4 x_4) + SH_4 \lambda_4 \cosh(\lambda_4 x_4) = 0$$

8'. Equations (8): boundary condition for a free end (cantilever beam) at $x = x_4$,

$$W_4'''(x = x_4) = 0$$

$$\lambda_4^3 C_4 \sin(\lambda_4 x_4) - \lambda_4^3 S_4 \cos(\lambda_4 x_4) + \lambda_4^3 CH_4 \sinh(\lambda_4 x_4) + \lambda_4^3 SH_4 \cosh(\lambda_4 x_4) = 0$$

9. Equation (9): continuity condition for the shear at $x = x_2$,

$$D_1 W_1'''(x = x_2) = D_2 W_2'''(x = x_2)$$

$$\begin{aligned} D_1 \lambda_1^3 C_1 \sin(\lambda_1 x_2) - S_1 \lambda_1^3 B_1 \cos(\lambda_1 x_2) + D_1 \lambda_1^3 CH_1 \sinh(\lambda_1 x_2) + D_1 \lambda_1^3 SH_1 \cosh(\lambda_1 x_2) \\ - D_2 \lambda_2^3 C_2 \sin(\lambda_2 x_2) + D_2 \lambda_2^3 S_2 \cos(\lambda_2 x_2) - D_2 \lambda_2^3 CH_2 \sinh(\lambda_2 x_2) - D_2 \lambda_2^3 SH_2 \cosh(\lambda_2 x_2) = 0 \end{aligned}$$

10. Equation (10): continuity condition for the shear at $x = x_3$,

$$D_2 W_2'''(x = x_3) = D_4 W_4'''(x = x_3)$$

$$\begin{aligned} D_2 \lambda_2^3 C_2 \sin(\lambda_2 x_3) - D_2 \lambda_2^3 S_2 \cos(\lambda_2 x_3) + D_2 \lambda_2^3 CH_2 \sinh(\lambda_2 x_3) + D_2 \lambda_2^3 SH_2 \cosh(\lambda_2 x_3) \\ - D_4 \lambda_4^3 C_4 \sin(\lambda_4 x_3) + D_4 \lambda_4^3 S_4 \cos(\lambda_4 x_3) - D_4 \lambda_4^3 CH_4 \sinh(\lambda_4 x_3) - D_4 \lambda_4^3 SH_4 \cosh(\lambda_4 x_3) = 0 \end{aligned}$$

11. Equation (11): continuity condition for the moment at $x = x_2$,

$$M_1 = M_2 + P_3 \frac{H}{2}$$

$$-D_1 W_1''(x = x_2) = -D_2 W_2''(x = x_2) + P_3 \frac{H}{2}$$

$$D_1 W_1''(x = x_2) + \frac{H^2}{4a_1} \left(\frac{E_2 H_2 E_3 H_3}{E_2 H_2 + E_3 H_3} \right) [W_1'(x = x_2) - W_4'(x = x_3)] = D_2 W_2''(x = x_2)$$

$$\text{Let } G = \frac{H^2}{4a} \left(\frac{E_2 H_2 E_3 H_3}{E_2 H_2 + E_3 H_3} \right)$$

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$$\begin{aligned}
& [-D_1 \lambda_1^2 \cos(\lambda_1 x_2) - G \lambda_1 \sin(\lambda_1 x_2)] C_1 \\
& + [-D_1 \lambda_1^2 \sin(\lambda_1 x_2) + G \lambda_1 \cos(\lambda_1 x_2)] S_1 \\
& + [D_1 \lambda_1^2 \cosh(\lambda_1 x_2) + G \lambda_1 \sinh(\lambda_1 x_2)] CH_1 \\
& + [D_1 \lambda_1^2 \sinh(\lambda_1 x_2) + G \lambda_1 \cosh(\lambda_1 x_2)] SH_1 \\
& + D_2 \lambda_2^2 C_2 \cos(\lambda_2 x_2) + D_2 \lambda_2^2 S_2 \sin(\lambda_2 x_2) - D_2 \lambda_2^2 CH_2 \cosh(\lambda_2 x_2) - D_2 \lambda_2^2 SH_2 \sinh(\lambda_2 x_2) \\
& + G \lambda_4 C_4 \sin(\lambda_4 x_3) - G \lambda_4 S_4 \cos(\lambda_4 x_3) - G \lambda_4 CH_4 \sinh(\lambda_4 x_3) - G \lambda_4 SH_4 \cosh(\lambda_4 x_3) = 0
\end{aligned}$$

12. Equation (12): continuity condition for the moment at $x=x_3$,

$$M_4 = M_2 + P_3 \frac{H}{2}$$

$$-D_4 W_4''(x = x_3) = -D_2 W_2''(x = x_3) + P_3 \frac{H}{2}$$

$$D_4 W_4''(x = x_3) + P_3 \frac{H}{2} = D_2 W_2''(x = x_3)$$

$$D_4 W_4''(x = x_3) + \frac{H^2}{4a} \left(\frac{E_2 H_2 E_3 H_3}{E_2 H_2 + E_3 H_3} \right) [W_1'(x = x_2) - W_4'(x = x_3)] = D_2 W_2''(x = x_3)$$

$$\begin{aligned}
& + G \lambda_1 C_1 \sin(\lambda_1 x_2) - G \lambda_1 S_1 \cos(\lambda_1 x_2) - G \lambda_1 CH_1 \sinh(\lambda_1 x_2) - G \lambda_1 SH_1 \cosh(\lambda_1 x_2) \\
& - D_2 \lambda_2^2 C_2 \cos(\lambda_2 x_3) - D_2 \lambda_2^2 S_2 \sin(\lambda_2 x_3) + D_2 \lambda_2^2 CH_2 \cosh(\lambda_2 x_3) + D_2 \lambda_2^2 SH_2 \sinh(\lambda_2 x_3) \\
& + [D_4 \lambda_4^2 \cos(\lambda_4 x_3) - G \lambda_4 \sin(\lambda_4 x_3)] C_4 \\
& + [D_4 \lambda_4^2 \sin(\lambda_4 x_3) + G \lambda_4 \cos(\lambda_4 x_3)] S_4 \\
& + [-D_4 \lambda_4^2 \cosh(\lambda_4 x_3) + G \lambda_4 \sinh(\lambda_4 x_3)] CH_4 \\
& + [-D_4 \lambda_4^2 \sinh(\lambda_4 x_3) + G \lambda_4 \cosh(\lambda_4 x_3)] SH_4 = 0
\end{aligned}$$