Analysis and design of cable-driven parallel kinematic mechanisms

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Analysis and Design of Cable-Driven Parallel Kinematic Mechanisms

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Abstract

This thesis concerns the analysis and design of cable-driven parallel mechanisms (CDPM). Structurally, a CDPM is formed by replacing the supporting legs of a parallel mechanism with active cables. It has the advantages of simple mechanical structure, low moment of inertia, and high speed motion. One distinctive characteristic of CDPMs is the unilateral property of the cables, i.e. cables can only pull but not push. At present, the design and application of CDPMs are limited and there is a lack of systematic analysis methods for CDPMs. This work aims to lay down a framework for the analysis, design and application of fully restrained CDPMs.

The forward kinematics of a CDPM is difficult because of its closed-loop structures. Numerical Newton-Raphson method is employed to obtain the forward displacement solutions. This method is modified to improve its computational aspect by establishing the mapping between the Newton-Raphson matrix and the Jacobian matrix. It is important to ensure suitable tension conditions in CDPMs such that the pose of the moving platform can be fully restrained by at least one appropriate set of positive cable tensions. Three tension conditions for CDPMs are identified and formulated. They are the force-closure condition, the feasible wrench condition and the wrench set condition. A recursive dimension reduction method is proposed to decompose an n -dimensional system into a number of 1-dimensional subsystems. To facilitate the checking of tension conditions, three algorithms FCC, FWC and WSC are developed and computational examples show that they are effective in checking the conditions. Generic strategies of numerically generating and quantifying the workspace are investigated by identifying the geometrical constraints. A tension factor is proposed to be used as a performance index to evaluate the quality of force closure for CDPMs at a specific configuration. Subsequently, a global tension index is defined to evaluate the tension quality over the entire workspace. The results provide valuable insight into the design optimization problems of CDPMs. A novel cable winding mechanism to provide a stationary contacting point and a constant relation between the cable length and actuator rotation is presented. The features are useful in simplifying the system dynamic model as well as in improving the dynamic performance of the system. The formulation of the system dynamic model of CDPMs is derived. The existence of torque solutions can be systematically checked by the feasible wrench condition, enabling the optimal torque values to be obtained using optimization algorithms. The validity of the theoretical analysis such as kinematics, cable tension and workspace is verified by experimental results obtained using a prototype cable-driven planar mechanism.

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List of Notations

A	Structure matrix ($n \times m$)
A[#]	Pseudo inverse of the structure matrix
B	Expression vector representing external wrench
a	Translational acceleration vector of the platform
$a_{i,j}$	Components of the structure matrix
B_i	Connecting point on the base
b_i	Coordinate vector of the suspending point
$b_{i,x}, b_{i,y}, b_{i,z}$	Coordinates of point B_i
b_i	Components of the vector representing external wrench
$c_{f,x}, c_{f,y}, c_{f,z}$	Force coefficients in equivalent wrench expressions
$c_{m,x}, c_{m,y}, c_{m,z}$	Moment coefficients in equivalent wrench expressions
$d_{a,i}$	Vicious damping coefficient of the armature
$d_{m,i}$	Combined vicious damping coefficient of the actuator
$d_{w,i}$	Vicious damping coefficient of the winch
F_p	External wrench vector acting on the platform
f_p	External force vector acting on the platform
$f_{p,x}, f_{p,y}, f_{p,z}$	Optimal force point in equivalent wrench expressions
$f_{p,r}$	Radius of the force set
g	Gravitational acceleration constant
h_i	Distance from the task space origin to surfaces of the convex hull
I	Identity matrix
I_p	Inertial tensor of the moving platform at the centroid P
J_{NR}	Newton-Raphson matrix

$\mathbf{J}_{NR}^{\#}$	Pseudo inverse of the Newton-Raphson matrix
$\mathbf{J}$	Jacobian matrix
$\mathbf{J}_X$	Forward Jacobian matrix
$\mathbf{J}_L$	Inverse Jacobian matrix
$\dot{j}_{a,i}$	Rotational inertial coefficient of the armature
$\dot{j}_{m,i}$	Combined rotational inertial coefficient of the actuator
$\dot{j}_{w,i}$	Rotational inertial coefficient of the winch
$\dot{j}_{1,1}, \dots, \dot{j}_{m,n}$	Components of the Jacobian matrix
$\hat{\mathbf{l}}_i$	Unit vector along the cable from B_i to P_i
l_i	Cable length
$\mathbf{m}_p$	External moment vector acting on the platform
m	Number of cables
m_p	Mass of the moving platform
$m_{p,r}$	Radius of the moment set
$m_{p,x}, m_{p,y}, m_{p,z}$	Optimal moment point in equivalent wrench expressions
$\mathbf{N}, \mathbf{N}_i$	Null vector
n	Number of task-based dimension (degrees of freedom)
n_{ws}	Number of feasible points composing of the workspace
n_o	Number of feasible points covering the range of orientation
n_i	Components of the null vector
P	Point located at the centre of mass of the moving platform
P_i	Connecting point on the moving platform
$\mathbf{p}$	Positional vector of point P
$\mathbf{R}$	Rotation matrix
R_x	Possible displacement range in x direction
R_y	Possible displacement range in y direction
R_ϕ	Possible angular displacement range about z axis
$\mathbf{r}_i$	Coordinate vector of the connecting point (in base frame)
$\mathbf{r}_i^p$	Coordinate vector of the connecting point (in platform frame)
$r_{i,x}^p, r_{i,y}^p, r_{i,z}^p$	Local coordinates of point P_i
$r_{i,x}, r_{i,y}, r_{i,z}$	Global coordinates of point P_i

r_b	Base radius
r_p	Platform radius
$r_{a,i}$	Radius of the driving sheave
$r_{g,i}$	Radius of the driven sheave
$r_{w,i}$	Radius of the winch
$\mathbf{S}$	Derivative of the Jacobian matrix with respect to time
$\mathbf{S}_x, \mathbf{S}_y, \mathbf{S}_z$	Partial derivative matrix of the Jacobian matrix with respect to x, y, z
$\mathbf{S}_\alpha, \mathbf{S}_\beta, \mathbf{S}_\gamma$	Partial derivative matrix of the Jacobian matrix with respect to α, β, γ
$\mathbf{S}_\phi$	Partial derivative matrix of the Jacobian matrix with respect to ϕ
$\mathbf{s}_i$	Column vector of the structure matrix
$\mathbf{T}_{B,P}$	Transformation matrix from the platform frame to the base frame
t	Time
t_i	Cable tension
$t_{i,min}$	Required minimum cable tension
$t_{i,max}$	Allowable maximum cable tension
$\mathbf{X}$	Posture vector of the platform
$\hat{\mathbf{u}}_i$	Unit vector along the cable from P_i to B_i
$\mathbf{v}$	Translational velocity vector of the platform
$\mathbf{Z}$	Arbitrary column vector in homogeneous solution
x, y, z	Coordinates of point P
α	Euler angle about z -axis
α_b	Angle defining the width of the base
α_p	Angle defining the width of the platform
β	Euler angle about y -axis
γ	Euler angle about x -axis
λ, λ_i	Arbitrary scalar
$\boldsymbol{\omega}$	Angular velocity vector of the platform
$\omega_x, \omega_y, \omega_z$	Components of an angular velocity
ϕ	Orientation of the platform in planar cases
τ_i	Actuator torque

List of Abbreviation

CDPM	Cable-Driven Parallel Mechanism
CDPPM	Cable-Driven Planar Parallel Mechanism
CDSPM	Cable-Driven Spatial Parallel Mechanism
CRPM	Completely Restrained Positioning Mechanism
DOF	Degree of Freedom
FCC	Algorithm for “Force-Closure Check”
FCWS	Force-Closure Workspace
FRPM	Fully Restrained Positioning Mechanism
FWC	Algorithm for “Feasible Wrench Check”
FWWS	Feasible Wrench Workspace
GTI	Global Tension Index
GUI	Graphical User Interface
IRPM	Incompletely Restrained Positioning Mechanism
m - n -CDPM	An n -DOF CDPM with m driving cables
TF	Tension Factor
RRPM	Redundantly Restrained Positioning Mechanism
WSC	Algorithm for “Wrench Set Check”

Chapter 1

Introduction

1.1 Motivation

A conventional robot manipulator is a linkage mechanism that consists of a number of rigid links and joints. Based on their kinematics structures, robot manipulators can be classified into two major types: serial and parallel types. A serial mechanism as shown in Fig. 1.1(a) is an open kinematic chain in which the links are connected in series through joints. On the other hand, a parallel mechanism (see Fig. 1.1(b)) is a closed-loop mechanism in which the moving platform is connected to the base through several serial kinematic

(a) Serial type [1]

(b) Parallel type [2]

Figure 1.1: Conventional mechanisms

chains or ‘legs’. The advantages of a serial mechanism are its large workspace and high dexterity. However, because of its cantilever-like structure, a serial mechanism has large moving mass, low stiffness and low loading capacity. In contrast, a parallel mechanism has the advantages of low moment of inertia, high rigidity and high loading capacity. Its disadvantage is the limited workspace due to its inherent closed-loop structure.

Figure 1.2: A cable-driven parallel mechanism

Figure 1.2 shows a cable-driven parallel mechanism (CDPM) formed by replacing all the supporting legs of a parallel mechanism with active cables. It is seen that the CDPM is a closed-loop mechanism in which the moving platform is connected to the base by several cables. The base suspending points can be mounted at the extremities of the base. Therefore, CDPMs can deal with large workspace manipulation tasks. Generally, a CDPM has the following significant advantages:

- Simple lightweight mechanical structure, resulting in low energy consumption,
- Low moment of inertia, high acceleration and high speed motion,
- Large reachable workspace, limited mainly by cable lengths and cable tension constraints,
- Easy reconfigurability by simply relocating platform connecting points and base suspending points.

(a) Oil well fire fighting system [3]

(b) Concept of virtual tennis system [4]

(c) SkyCam end-effector [5]

(d) Studying aircraft behaviour [6]

(e) Rehabilitation system [7]

Figure 1.3: Typical application of CDPMs

These advantages make the CDPM a promising alternative of the rigid-link mechanisms in many applications, such as load lifting and positioning (Fig. 1.3(a)), haptic devices (Fig. 1.3(b)), sport recording (Fig. 1.3(c)), aircraft testing (Fig. 1.3(d)), and robot rehabilitation (Fig. 1.3(e)) etc. However, since cables are well known with a unilateral property (can pull but cannot push moving platforms), the formulations and results obtained for the kinematics analysis, workspace analysis, trajectory planning, etc. of the rigid link mechanisms cannot be directly applied. In addition, apart from unilateral properties of driving cables, some practical limitations of CDPMs include limited orientation workspace, relatively slow stiffness and accuracy, and redundancy of actuators compared to conventional rigid-link mechanisms. The disadvantages make CDPMs not as popular as conventional rigid-link mechanisms. The limitations of orientation workspace, stiffness and accuracy can be improved to a certain extent by design optimization and appropriate control strategies, for which the analysis and design methods proposed in this thesis can be employed.

1.2 Past Research Efforts

The concept of cable-driven parallel mechanism (CDPM) was first applied in the area of crane technology. The operation of an overhead crane, which is normally under-actuated cable-driven mechanism, can cause severe swinging of the load, especially for inexperienced operators. This is obviously dangerous to the operator, and it can result in damage of the load or the facility. Furthermore, swinging can significantly increase the time required to position loads, particularly the time required to manually dampen out the swing before a load can be lowered. A specific example of the combination of robotics and cranes is the ROBOCRANE, which was developed by the National Institute of Standards and Technology (NIST) in the early 1980s (Albus [8]). The ROBOCRANE is a special crane that utilizes six cables to suspend a load platform. It is lightweight, easily assembled and consists of a stable platform supported by six cables suspended from three points on

a fixed or mobile octahedral structure. A conceptual diagram of the ROBOCRANE is shown in Fig. 1.4.

Figure 1.4: ROBOCRANE geometry

Having been designed and developed, ROBOCRANE was recognized to have more benefits than conventional cranes. The major advantage is that it provides sufficient control to allow even a novice operator to locate a load without sway within a few millimetres in x, y and z , and to control orientation without oscillation within one degree in roll, pitch and yaw. Besides, due to its octahedron geometry, the ROBOCRANE requires no counterweight and experiences, no twisting or bending moments. As a result, it can lift at least five times its own weight. This is significantly more than any robot or crane in current use. With the above benefits, and depending on what is suspended from its platform, the ROBOCRANE could perform a variety of tasks such as flexible fixturing, cutting, lifting and positioning.

1.2.1 Classification of CDPMs

Based on the number of cables (m) and the number of degrees of freedom (n), the CDPMs can be classified into two categories, i.e. *incompletely restrained positioning mechanisms* (IRPM, $m < n + 1$) and *fully restrained positioning mechanisms* (FRPM, $m \geq n + 1$) as illustrated in Fig. 1.5. The FRPMs can be further classified into *completely restrained*

positioning mechanisms (CRPM, $m = n + 1$) and redundantly restrained positioning mechanisms (RRPM, $m > n + 1$).

Figure 1.5: Classification of CDPMs

(a) Incompletely restrained positioning mechanisms ($m < n + 1$)

Figure 1.5(a) shows an incompletely restrained positioning mechanism (IRPM). For such mechanisms, only certain degrees of freedom (DOF) of the moving platform can be realized by kinematic constraints. Motion on the other DOFs is governed by the dynamics of moving platform and cables. Hence, external forces like gravity should be applied to get additional (non-kinematic) constraints. This type of mechanisms is typically designed for applications where large workspaces are required. An example is the ROBOTCRANE mentioned above. It is designed for use in tasks such as material handling, inspection, pipe fitting and manufacturing operations such as welding, sawing and grinding (Bostelman [9]). Another type of IRPM which has been designed for transferring cargo to and from ships is the Cable Array Robot. It has been developed at the Pennsylvania State University and is a 4-cable point-mass cable robot (Shiang [10]). Another point-mass manipulator is the SkyCam made by August Design. The SkyCam as shown in Fig. 1.3(c) is a cable robot that moves a video camera for use in stadiums and indoor arenas. The use of IRPM has also been proposed for search and rescue in the event of urban earthquakes (Tadokoro [11]), pose measuring systems (Jeong [12, 13], Ottaviano [14]), haptic devices (Melchiorri [15]), bathroom cleaning robot (Takahashi [16]) and rehabilitation (Homma [17]) as illustrated in Fig. 1.3(e).

(b) Fully restrained positioning mechanisms ($m \geq n + 1$)

For fully restrained positioning mechanisms, all the DOFs can be determined by the kinematic structure of the mechanism. In such a mechanism, there is inherent actuation redundancy. While redundant actuation is an effective approach for enlarging workspace and avoiding singularity, more sophisticated tension distribution scheme need to be devised. The fully restrained positioning mechanisms are further divided into completely restrained positioning mechanisms (CRPM) as shown in Fig. 1.5(b) and redundantly restrained positioning mechanisms (RRPM) as shown in Fig. 1.5(c). A CRPM is a special case of fully restrained positioning mechanisms in which the minimal number of cables ($m = n + 1$) is used. Fully restrained CDPMs have often been designed for applications that require high speed, high acceleration or high stiffness. High speed CDPMs include the WARP mechanism which uses 8 cables to achieve accelerations up to 1g (Tadokoro [18]) and a 7-cable manipulator FALCON that was able to gain accelerations up to 43g (Kawamura [19]). High stiffness CDPMs have been designed for applications in teleoperation (Kawamura [20]), haptic devices (William [21, 22], Gallina [23, 24]), virtual reality as illustrated in Fig. 1.3(b), study of aircraft flight as illustrated in Fig. 1.3(d) and also rehabilitation (Yang [25]).

1.2.2 Research Issues in CDPMs

Currently, the analysis and design of cable-driven mechanisms is a relatively new research area. Since the cables can only carry payloads in tension but not in compression, more driving cables than the number of degrees of freedom are necessary. As a result, the well-developed kinematics, dynamics, design and control methods for conventional rigid-link mechanisms cannot be directly applied to CDPMs. Existing research work published in the literature for CDPMs includes: kinematic analysis (Ming [26], William [27]), tension analysis (Ming [28], Verhoeven [29], Lafourcade [30]), workspace analysis (Tadokoro [31], Pusey [32]), design optimization (Fattah [33], Riechel [34]), and dynamic

analysis (Agrawal [35], kawamura [36]). However, the focus of the research is mainly on incompletely restrained CDPMs (cable-suspended CDPMs) and completely restrained CDPMs with specific configurations. The study of general redundantly restrained CDPMs has not been well addressed and still remains an open problem. The existing null space tension analysis method used by many researchers is based on finding tension solutions using algebraic expressions for completely restrained CDPMs whereby the dimension of the null space is equal to one. However, this approach is almost inapplicable to redundantly restrained CDPMs where the dimension of the null space is greater than one because it is too complicated to obtain explicit algebraic expressions in such cases. As a result, this leads to difficulties and limitations of extending existing design methods to redundantly restrained CDPMs in general. For fully restrained CDPMs, the major challenges include tension analysis (in static as well as in dynamic), tension/torque optimization, and tension-based control in redundant driving systems with bounded positive variables. In this thesis, the detailed review of kinematic analysis, cable tension analysis, workspace analysis together with design optimization, and dynamic analysis of CDPMs will be presented separately in the respective chapters.

1.3 Objective and Research Scope

Past research works were mainly focused on incompletely restrained CDPMs (especially in crane technology) and completely restrained CDPMs with specific configurations. The existing methodologies and algorithms could not be extended to cope with the general fully restrained CDPMs. In particular the tension analysis method, which is the basis for the analysis and design of CDPMs, was not generic and effective enough when extending its application from completely restrained CDPMs ($m = n + 1$) to redundantly restraint mechanisms ($m > n + 1$); and/or from planar mechanisms ($n = 3$) to spatial mechanisms ($n = 6$); and/or from finite positive tension values ($0 \rightarrow +\infty$) to bounded positive tension value ($0 < t_{min} < t_{max} < +\infty$). The objective of this research is to establish

the fundamental and generic tools for systematic analysis of CDPMs, including kinematic analysis, cable tension analysis, workspace analysis, design optimization, and dynamic analysis. Figure 1.6 shows the overall framework.

Figure 1.6: Overview of the thesis framework

The scope of this research is as follows:

- Kinematics analysis of cable-driven parallel mechanisms includes forward and inverse displacement analysis, velocity analysis, and acceleration analysis.
- Cable tension issues will be addressed with the aim of establishing tension conditions for CDPMs such that there is at least one appropriate set of positive cable tensions to maintain the pose of the moving platform.
- Workspace analysis and workspace performance give information to get better designs of mechanisms. Both quantitative and qualitative workspace evaluation will be proposed.

In addition, design optimization problems of CDPMs will also be presented.

- Cable winding methods will be investigated with the aim of proposing a method that could simplify the dynamic model of the cable driving unit. System dynamic model which incorporates both the actuator dynamic model and the mechanism dynamic model will be formulated.
- Torque optimization will be addressed to deal with the multiple torque solutions due to actuation redundancy. A trajectory planning algorithm will be proposed so that the platform can trace the desired trajectory smoothly.
- A cable-driven planar prototype will be developed to verify the theoretical formulation, to validate the proposed tension algorithms experimentally, and to demonstrate the position control scheme for CDPMs.

The fundamental and generic tools developed in this work are for systematic analysis of fully restrained CDPMs. Without loss of generality, two typical configurations as shown in Fig. 1.7 will be used and referred to in computational examples for various analyses throughout the thesis. The 4-cable-driven mechanism with 3 DOFs, denoted

Figure 1.7: Examples of CDPMs used in simulation

as 4-3-CDPPM, is selected to represent completely restrained CDPMs ($m = n + 1$) in planar cases. The 8-cable-driven mechanism with 6 degrees of freedom (8-6-CDSPM) is selected to represent redundantly restrained CDPMs ($m > n + 1$) in spatial cases. These two architectures are adequate to highlight the flexibility of the proposed analyses, which is applicable to any fully restrained CDPM varying from completely restrained CDPMs ($m = n + 1$) to redundantly restraint mechanisms ($m > n + 1$) and from planar mechanisms ($n = 3$) to spatial mechanisms ($n = 6$). In addition, the symmetric design of both configurations is kinematically simple and easy to fabricate. Their kinematic parameters are given in Fig. 1.7.

In practice, the weight of the cables is much lighter than that of the moving platform, and the elongation of the cables is negligible under the working tension range. Hence, the following assumptions, which are adopted commonly in the areas of cable-driven mechanisms, are made for the analyses presented in this report:

- One motor controls exactly the length of one cable.
- All the cables must be in positive tension at all times due to the unilateral property.
- All the cables are assumed to be massless and perfectly stiff so that their inertias and spring stiffness can be ignored.
- The Coulomb friction is also ignored and instead the model of linear viscous friction is taken into account for the frictional losses.

1.4 Thesis Organization

The remaining chapters of this thesis are organized as follows:

In Chapter 2, the kinematic analysis of cable-driven parallel mechanisms is presented. The forward displacement solution of a general CDPM is approached by the numerical method.

Then expressions of velocity and acceleration are derived in detail for the 4-3-CDPPMs and the 8-6-CDSPMs.

In Chapter 3, the cable tension analysis and tension conditions of the CDPMs are addressed. Recursive algorithms are developed to check the tension conditions.

In Chapter 4, the workspace analysis of CDPMs are investigated including workspace generation, quantification, and performance index. A design optimization algorithm based on the complex search method is also introduced.

In Chapter 5, a novel design of cable winding mechanism is proposed. Then, system dynamic model which incorporates both the actuator dynamic model and the mechanism dynamic model is formulated. Torque optimization is also carried out.

In Chapter 6, the development of a cable-driven planar prototype, the control implementation and the experimental results are presented.

Chapter 7 concludes the thesis, summaries the contributions of this work and outlines the future work.

Chapter 2

Kinematic Analysis

2.1 Introduction

This chapter concerns the kinematics issues of cable-driven parallel mechanisms (CDPM). In a CDPM, the position of the moving platform is controlled by the lengths of cables. The purpose of kinematic analysis is therefore to determine the kinematic relations between the cable lengths and the posture of the moving platform. Generally, there are two types of kinematic analysis: the forward kinematics and the inverse kinematics. The forward kinematic analysis deals with the problem of estimating the posture of the moving platform by measuring the length of the cables, which is often used in tracking systems and mechanism calibration procedures. On the other hand, the inverse kinematic analysis deals with the problem of calculating the lengths of cables with the given posture of the moving platform obtained from its trajectory planning, which is important for the motion control of CDPMs. One distinct characteristic of CDPMs is that cables can only pull (exert tension) and cannot push on the moving platform. As a result, kinematic analysis of CDPMs is only valid if the resultant cable tensions are positive. In addition, the forward kinematic problems often have more constraints than the number of variables due to the redundancy of cables. In this chapter, tension of the cables is assumed to be positively maintained, i.e. the cables function like rigid links. Hence, the kinematic analysis of

CDPMs is partly related to that of rigid-link parallel mechanisms.

For parallel mechanisms with rigid links, the forward displacement analysis is very challenging because of the closed loop structures, resulting in a set of highly non-linear equations with multiple solutions (Merlet [37], Nanua [38]). The inverse kinematics on the other hand is relatively straightforward as it can be decoupled for each cable, requiring only direct substitution for a given end-effector pose. Past research on forward kinematics has focused on three approaches: the polynomial-based, the extra-sensor and the numerical-iterative approaches. In the first approach, Gosselin [39] derived a polynomial solution of the direct kinematics of a planar 3-DOF parallel manipulator, leading to a 6th order polynomial equation. For a general 6-6 Stewart platform, the existence of 40 configurations was first demonstrated numerically by Raghavan [40] with a tool known as “polynomial continuation”. Later, Lee [41] presented an algebraic elimination method for the forward kinematics of the general Stewart platform which also directly led to a 40th order univariate equation. Lin [42] presented three specific classes of Stewart platform where each of the mechanisms had the distinguishing feature of six legs meeting either singly or pair-wise at four points in the top and base platforms. The degrees of the polynomials for the three cases are eight, four and twelve. The analytical approach for forward kinematics of a general CDPM is however much more complicated due to the redundancy of cables for fully restrained CDPMs. An extended forward kinematic analysis method was presented by Ming [26]. This is done by obtaining the pose of the n -DOF moving platform from n cable lengths, followed by utilizing inverse kinematics to obtain the lengths of the remaining lengths of $(m - n)$ cables. Generally, the reduction of the set of constraint equations into a univariate high-order polynomial equation in the polynomial-based approach is a complicated process. In addition, a numerical method has to be used to find the solution of a high-ordered polynomial in the end. In the extra-sensor approach, redundant sensors are installed on the passive joint of the manipulator, in addition to those installed on the actuated joints. This simplifies the forward kinematics. Cheok [43] proposed that the forward kinematics of a 6-6 Stewart platform can be

solved in closed form by using three extra sensors. Merlet [44] also proposed that if four extra rotary sensors are added to the general 6-DOF Stewart platform, exact solution can be found. Baron [45] proposed three measurement conditions using a camera in which the forward kinematics of the Stewart platform is linearized, and hence the decoupling of the position and orientation is possible. However, this approach involves high implementation costs and hardware complexities. This makes the numerical-iterative approach attractive because numerical-iterative methods generally result in a faster generation of the forward displacement solution. Wang [46] developed a two-phase algorithm to find kinematic solutions for general parallel robots. Merlet [47] solved the forward kinematics numerically using various methods: iterative method with kinematics Jacobian, iterative method with Euler's angles Jacobian matrix, reduced iterative method and polynomial method. William [27] employed the Newton-Raphson iterative method to study the forward kinematics of a cable-driven planar parallel mechanism. This method was found to be effective due to its property of quadratic convergence. In general, all the numerical approaches face the common problems of reliability, accuracy, sensitivity to the estimated values, and the nature of the resulting constraint equations. This is minimized by using the previous point of the trajectory as an initial guess.

In the following sections, the forward and inverse displacement analysis of the CDPMs is first presented. The Newton-Raphson method is employed to find the solutions for a fully restrained CDPM using the 4-3-CDPPM and the 8-6-CDSPM as examples. Next, an analytical approach is proposed for the 4-3-CDPPM, leading to a 4th order polynomial equation. The symbolic forward displacement solutions are presented. Velocity and acceleration relationships between the task space and the cable space are also important because they are needed in tension analysis, trajectory planing and control. Hence, the instantaneous kinematic relationship between the translational velocity of the driving cables and the velocity of the moving platform is formulated. This is followed by the derivation of equations for acceleration analysis. These equations are expressed in symbolic forms for both the 4-3-CDPPM and the 8-6-CDSPM.

2.2 Displacement Analysis

A schematic diagram of a fully restrained CDPM is shown in Fig. 2.1, in which the moving platform is connected to the base through driving cables, $\mathbf{l}_i = \overrightarrow{B_i P_i}$ ($i = 1, 2, \dots, m$). P_i are connecting points on the moving platform and B_i are suspending points on the base. The displacement analysis is to determine the relationship between the cable lengths l_i and the pose of the moving platform.

Figure 2.1: Kinematic diagram of a CDPM

Let frame $\{B\}$ be the base frame and frame $\{P\}$ be the moving platform frame (attached to the centre of mass P of the moving platform). By convention, all quantities are written in frame $\{B\}$ unless there is a trailing superscript p denoting that it is in frame $\{P\}$. The frame $\{P\}$ is given with respect to the frame $\{B\}$ by the following kinematic transformation matrix $\mathbf{T}_{B,P}$:

$$\mathbf{T}_{B,P} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} \quad (2.1)$$

where

$$\mathbf{p} = \begin{bmatrix} x & y & z \end{bmatrix}^T$$

$$\mathbf{R} = \begin{bmatrix} \cos \alpha \cos \beta & (\cos \alpha \sin \beta \sin \gamma - \sin \alpha \cos \gamma) & (\sin \alpha \sin \gamma + \cos \alpha \sin \beta \cos \gamma) \\ \sin \alpha \cos \beta & (\cos \alpha \cos \gamma + \sin \alpha \sin \beta \sin \gamma) & (\sin \alpha \sin \beta \cos \gamma - \cos \alpha \sin \gamma) \\ -\sin \beta & \cos \beta \sin \gamma & \cos \beta \cos \gamma \end{bmatrix}$$

Note that $\mathbf{p}$ is the position of point P with respect to the frame $\{B\}$. $\mathbf{R}$ is the rotation matrix representing the orientation of frame $\{P\}$ with respect to frame $\{B\}$, in which α, β and γ are the $Z - Y - X$ Euler angles (Craig [48]).

The local coordinates of point P_i (distal ends of the cables) with respect to the frame $\{P\}$ are represented by $\mathbf{r}_i^p = \{r_{i,x}^p \ r_{i,y}^p \ r_{i,z}^p\}^T$, whereas the fixed coordinates of B_i with respect to the frame $\{B\}$ are represented by $\mathbf{b}_i = \{b_{i,x} \ b_{i,y} \ b_{i,z}\}^T$. For each of the cables, the following vector loop-closure equation can be written:

$$\overrightarrow{OB_i} + \overrightarrow{B_iP_i} = \overrightarrow{OP} + \overrightarrow{PP_i} \quad (i = 1, 2, \dots, m) \quad (2.2)$$

It is seen that the cable lengths l_i are related to the pose of the moving platform via the length (Euclidean norm) of the vector $\mathbf{l}_i$ such that

$$\begin{aligned} l_i &= \|\overrightarrow{OP} + \overrightarrow{PP_i} - \overrightarrow{OB_i}\| \\ &= \|\mathbf{p} + \mathbf{R} \cdot \mathbf{r}_i^p - \mathbf{b}_i\| \end{aligned} \quad (2.3)$$

2.2.1 Forward Displacement

The forward displacement analysis is to determine the pose of the moving platform when the lengths of the m cables l_i are known. Equation (2.3) can be squared and written m times, one for each cable, to give m equations in the six unknowns $\mathbf{X} = (x, y, z, \alpha, \beta, \gamma)$:

$$F_i(\mathbf{X}) = (\mathbf{p} + \mathbf{R} \cdot \mathbf{r}_i^p - \mathbf{b}_i)^T (\mathbf{p} + \mathbf{R} \cdot \mathbf{r}_i^p - \mathbf{b}_i) - l_i^2 = 0 \quad (i = 1, 2, \dots, m) \quad (2.4)$$

Since Eqn. (2.4) is a system of highly nonlinear equations, analytical solutions in forward displacement problems are almost unobtainable for a general CDPM except for specific cases such as 4-3-CDPPMs. In this section, a practical approach based on the Newton-Raphson method is employed to numerically solve the forward displacement problem for a general CDPM, in which the partial derivatives, i.e. the Newton-Raphson matrices, can be efficiently computed from the Jacobian matrices.

Newton-Raphson method

For numerical methods, an initial estimated $\mathbf{X}$ is chosen arbitrarily. In general, this will not be the root of Eqn. (2.4). However, there exists a certain $\Delta\mathbf{X}$ which, when added to $\mathbf{X}$, will give the roots in an iterative manner. This can be expressed as follows:

$$F_i(\mathbf{X} + \Delta\mathbf{X}) = F_i(x + \delta x, y + \delta y, \dots, \gamma + \delta\gamma) = 0 \quad (i = 1, 2, \dots, m) \quad (2.5)$$

To find this $\Delta\mathbf{X}$, a linear approximation to this function is obtained by taking the first six terms of its Taylor series expansion about the point $(x, y, z, \alpha, \beta, \gamma)$:

$$\begin{aligned} F_i(\mathbf{X} + \Delta\mathbf{X}) = F_i(\mathbf{X}) &+ \left(\frac{\partial F_i(\mathbf{X})}{\partial x} \right) \delta x + \left(\frac{\partial F_i(\mathbf{X})}{\partial y} \right) \delta y + \left(\frac{\partial F_i(\mathbf{X})}{\partial z} \right) \delta z \\ &+ \left(\frac{\partial F_i(\mathbf{X})}{\partial \alpha} \right) \delta \alpha + \left(\frac{\partial F_i(\mathbf{X})}{\partial \beta} \right) \delta \beta + \left(\frac{\partial F_i(\mathbf{X})}{\partial \gamma} \right) \delta \gamma = 0 \end{aligned} \quad (2.6)$$

Equation (2.6) is a linear system. Hence, the Newton-Raphson Jacobian matrix $\mathbf{J}_{NR}$ can be written in the following form:

$$\mathbf{J}_{NR} = \begin{bmatrix} \frac{\partial F_1(\mathbf{X})}{\partial x} & \frac{\partial F_1(\mathbf{X})}{\partial y} & \dots & \frac{\partial F_1(\mathbf{X})}{\partial \gamma} \\ \frac{\partial F_2(\mathbf{X})}{\partial x} & \frac{\partial F_2(\mathbf{X})}{\partial y} & \dots & \frac{\partial F_2(\mathbf{X})}{\partial \gamma} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial F_m(\mathbf{X})}{\partial x} & \frac{\partial F_m(\mathbf{X})}{\partial y} & \dots & \frac{\partial F_m(\mathbf{X})}{\partial \gamma} \end{bmatrix} \quad (2.7)$$

Starting from an initial guess $\mathbf{X}_0$, the process for step $(k + 1)$ is given by:

- Solve $\mathbf{J}_{NR} \cdot \Delta\mathbf{X}_k = -F(\mathbf{X}_k)$ for $\Delta\mathbf{X}_k$:

$$\Delta\mathbf{X}_k = -\mathbf{J}_{NR}^\# \cdot F(\mathbf{X}_k) \quad (2.8)$$

- Then let $\mathbf{X}_{k+1} = \mathbf{X}_k + \Delta\mathbf{X}_k$.

- Iterate until:

$$\bar{F}(\mathbf{X}) = \sqrt{\sum_{i=1}^m \frac{\left(F_i(\mathbf{X}_{k+1})\right)^2}{m}} \leq \varepsilon \quad (2.9)$$

where

$\mathbf{J}_{NR}^\# = (\mathbf{J}_{NR}^T \mathbf{J}_{NR})^{-1} \mathbf{J}_{NR}^T$ is the pseudo-inverse of $\mathbf{J}_{NR}$

ε is a user-defined tolerance.

Figure 2.2: Newton-Raphson iterative method

Figure 2.2 shows the flowchart to find the forward displacement solution using the Newton-Raphson iterative method. In this diagram, the Newton-Raphson Jacobian matrix $\mathbf{J}_{NR}$ can be obtained by the partial derivatives as given by Eqn. (2.7).

Computational examples: In order to verify the effectiveness of this numerical algorithm, computational results for both 4-3-CDPPMs and 8-6-CDSPMs are obtained. Note that the input values of cable lengths are based on the inverse displacement analysis so as to guarantee the existence of solutions. The value of the tolerance ε is 0.0001 for all cases. Initial values are selected to be zero for position as well as orientation. As shown in Table 2.1, four examples for 4-3-CDPPM and three examples for 8-6-CDSPM have been computed. The results show that with the given cable lengths, all the platform poses are determined successfully. The number of iteration is less than ten in planar cases. The number ranges from 6 to 27 in spatial cases. In terms of errors, the average positional error ($\bar{e}_p$) and the average directional error ($\bar{e}_o$) are close to zero. In addition, the selection of the initial guess value does not seem to affect the results considerably.

Table 2.1: Computational examples for the forward displacement

	4-3-CDPPM					8-6-CDSPM				
		1	2	3	4		1	2	3	
Input: cable lengths	l_1	0.5315	0.5410	0.6594	0.3858	l_1, l_5	0.5679, 0.8789	0.8676, 0.8676	0.9413, 0.5751	
	l_2	0.5315	0.3925	0.7456	0.5432	l_2, l_6	0.5679, 0.8789	0.6973, 0.6973	1.0026, 0.6913	
	l_3	0.5315	0.5560	0.4907	0.6686	l_3, l_7	0.5679, 0.8789	0.5572, 0.5572	0.9183, 0.5617	
	l_4	0.5315	0.6744	0.3199	0.5734	l_4, l_8	0.5679, 0.8789	0.7709, 0.7009	0.8550, 0.4188	
Output: platform pose	x	0.5	0.6	0.4	0.4	x, γ	0.5, 0	0.7, 0	0.4, 0	
	y	0.5	0.4	0.7	0.4	y, β	0.5, 0	0.6, 0	0.6, 4	
	ϕ	0	2.0	-5.0	8.0	z, α	0.25, 0	0.5, 5.0	0.8, 2.0	
Number of iteration	k	8	6	7	6	k	6	15	27	
Average error	$\overline{e}_p, \overline{e}_o$	0, 0.0	0, 0.0	0, 0.0	0, 0.0	$\overline{e}_p, \overline{e}_o$	0, 0.0	0, 0.0	0, 0.0	

2.2.2 Inverse Displacement

The inverse displacement analysis is to determine the lengths of m cables l_i with the given moving platform pose $\mathbf{T}_{B,P}$. Comparing with the forward displacement analysis, the inverse displacement analysis is simple and straightforward. A unique solution can be determined for each of the cables if the pose of the moving platform is given. Using Eqn. (2.3), the length of each cable can be directly calculated from the given pose $(x, y, z, \alpha, \beta, \gamma)$ of the platform as follows:

$$l_i^2 = \mathbf{p}^T \cdot \mathbf{p} + \mathbf{r}_i^{pT} \cdot \mathbf{r}_i^p + \mathbf{b}_i^T \cdot \mathbf{b}_i + 2\mathbf{p}^T \cdot \mathbf{R} \cdot \mathbf{r}_i^p - 2\mathbf{p}^T \cdot \mathbf{b}_i - 2\mathbf{b}_i^T \cdot \mathbf{R} \cdot \mathbf{r}_i^p \quad (i = 1, 2, \dots, m) \quad (2.10)$$

2.2.3 Analytical Forward Solutions for 4-3-CDPPMs

As mentioned, the analytical solution approach for the forward displacement analysis though preferred is difficult for a general CDPM. However, for a 4-3-CDPPM, the forward displacement analysis can be simplified which leads to solving a 4th order polynomial equation. The solutions can therefore be determined efficiently in symbolic forms and they can be used to determine whether solutions exist which cannot be predicted by numerical solution approaches.

General 4-3-CDPPM

For a general 4-3-CDPPM as shown in Fig. 2.3, the kinematic transformation matrix $\mathbf{T}_{B,P}$ from the base frame $\{B\}$ to the platform frame $\{P\}$ can be written as:

$$\mathbf{T}_{B,P} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & x \\ \sin \phi & \cos \phi & y \\ 0 & 0 & 1 \end{bmatrix} \quad (2.11)$$

Substituting Eqn. (2.11) into Eqn. (2.10) gives:

$$l_i^2 = x^2 + y^2 + u_i x + v_i y + w_i \quad (2.12)$$

Figure 2.3: General 4-3-CDPPM

where

$$u_i = 2(r_{i,x}^p \cos \phi - r_{i,y}^p \sin \phi - b_{i,x})$$

$$v_i = 2(r_{i,x}^p \sin \phi + r_{i,y}^p \cos \phi - b_{i,y})$$

$$w_i = b_{i,x}^2 + b_{i,y}^2 + r_{i,x}^{p^2} + r_{i,y}^{p^2} - 2(r_{i,x}^p b_{i,x} + r_{i,y}^p b_{i,y}) \cos \phi + 2(r_{i,y}^p b_{i,x} - r_{i,x}^p b_{i,y}) \sin \phi$$

Rewriting Eqn. (2.12) for each of the four driving cables, the following four nonlinear equations with three unknowns, i.e. x, y and ϕ , can be obtained:

$$l_1^2 = x^2 + y^2 + u_1x + v_1y + w_1 \quad (2.13)$$

$$l_2^2 = x^2 + y^2 + u_2x + v_2y + w_2 \quad (2.14)$$

$$l_3^2 = x^2 + y^2 + u_3x + v_3y + w_3 \quad (2.15)$$

$$l_4^2 = x^2 + y^2 + u_4x + v_4y + w_4 \quad (2.16)$$

By subtracting the first equation from the second, the third and then the fourth separately, three resulting equations are obtained and can be re-arranged in the following form:

$$\begin{bmatrix} (u_2 - u_1) & (v_2 - v_1) & (w_2 - w_1 - l_2^2 + l_1^2) \\ (u_3 - u_1) & (v_3 - v_1) & (w_3 - w_1 - l_3^2 + l_1^2) \\ (u_4 - u_1) & (v_4 - v_1) & (w_4 - w_1 - l_4^2 + l_1^2) \end{bmatrix} \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix} = 0 \quad (2.17)$$

From the theory of linear algebra, if the coefficient matrix of Eqn. (2.17) is a 3×3 non-singular matrix, then this homogeneous linear equation has only the trivial solution with $x = 0, y = 0$, and $1 = 0$, which is clearly impossible. Hence if the homogeneous system has a non-trivial solution, the coefficient matrix must be a singular matrix, such that:

$$\det \begin{bmatrix} (u_2 - u_1) & (v_2 - v_1) & (w_2 - w_1 - l_2^2 + l_1^2) \\ (u_3 - u_1) & (v_3 - v_1) & (w_3 - w_1 - l_3^2 + l_1^2) \\ (u_4 - u_1) & (v_4 - v_1) & (w_4 - w_1 - l_4^2 + l_1^2) \end{bmatrix} = 0 \quad (2.18)$$

Equation (2.18) has only one unknown variable ϕ . By letting $q = \tan(\phi/2)$, the sine and cosine of the unknown variable ϕ can be expressed in the following standard form:

$$\sin \phi = \frac{2q}{1 + q^2}, \quad \cos \phi = \frac{1 - q^2}{1 + q^2} \quad (2.19)$$

Substituting the above expressions in Eqn. (2.18), a 4th order polynomial equation in q is obtained:

$$c_4 q^4 + c_3 q^3 + c_2 q^2 + c_1 q + c_0 = 0 \quad (2.20)$$

where the coefficients, c_i , are functions of the four cable lengths and the structural kinematic parameters of the 4-3-CDPPM. This 4th order polynomial is significant and very useful because symbolic solutions can always be obtained explicitly. The four symbolic solutions can be found in Appendix A. Finally, the real roots have to be examined by Eqn. (2.13). It can be concluded that at most four sets of forward displacement solutions exist for such a 4-3-CDPPM.

Computational examples: The following kinematic parameters are specially chosen for the 4-3-CDPPM as shown in Fig. 2.3.

$$\begin{cases} B_1 = \{0, 0\} & B_2 = \{15.9, 0\} & B_3 = \{2.5, 9.8\} & B_4 = \{0, 10\} \\ P_1^p = \{0, 0\} & P_2^p = \{17, 0\} & P_3^p = \{16.6, 16.1\} & P_4^p = \{13.2, 16.1\} \\ l_1 = 15 & l_2 = 15.4 & l_3 = 12.6 & l_4 = 12 \end{cases}$$

Here, the base frame $\{B\}$ is attached at B_1 with a horizontal x-axis; the platform frame $\{P\}$ is fixed at P_1 with its x-axis along $P_1 P_2$. With the given dimensions and the coordi-

Figure 2.4: Four forward kinematic solutions of a 4-3-CDPPM

nates of the connecting points, the coefficients of the polynomial Eqn. (2.20) are computed and the four roots are found. From the roots, the orientation angle ϕ can then be calculated using the inverse tangent function and the corresponding values of x and y can also be found using Eqn. (2.17). The four sets of the moving platform pose (x, y, ϕ) must then be verified with Eqn. (2.13). In this example, all four sets of solutions satisfy Eqn. (2.13) and therefore they are the real solutions of the nonlinear system, i.e. Eqns. (2.13)-(2.16). Figure 2.4 illustrates the four platform poses based on the forward displacement solutions.

Symmetric 4-3-CDPPM

In this part, the analytical solutions for a special class of 4-3-CDPPMs that satisfy the ‘symmetric’ condition are derived. A 4-3-CDPPM is symmetric if the shapes formed by the four base points B_i and that formed by the four platform points P_i are both rectangular. It will be shown that the forward displacement analysis of a symmetric 4-3-CDPPM can be greatly simplified. The kinematic parameters of a symmetric 4-3-CDPPM are shown in Fig. 2.5. By substituting them into Eqn. (2.18), the determinant for the forward displacement analysis can be written as:

$$4 \left(2(d_b c_p - c_b d_p) \sin \phi + l_1^2 - l_2^2 + l_3^2 - l_4^2 \right) \left((d_b c_p + c_b d_p) \cos \phi - c_p d_p - c_b d_b \right) = 0 \quad (2.21)$$

Figure 2.5: Symmetric 4-3-CDPPM

This implies that:

$$2(d_b c_p - c_b d_p) \sin \phi + l_1^2 - l_2^2 + l_3^2 - l_4^2 = 0 \quad (2.22)$$

and/or

$$(d_b c_p + c_b d_p) \cos \phi - c_p d_p - c_b d_b = 0 \quad (2.23)$$

It is seen that the orientation of the platform ϕ can be determined directly from Eqns. (2.22) and (2.23) without having to solve a fourth order polynomial as in the general case. Equa-

tion (2.23) can be rewritten as follows:

$$\cos \phi = \frac{c_b d_b + c_p d_p}{d_b c_p + c_b d_p} \quad (2.24)$$

In the above equation, the orientation of the platform ϕ is a constant and does not depend on the cable lengths which is obviously meaningless and can be discarded. Hence, the orientation of the platform ϕ should be computed using Eqn. (2.22) which can be re-arranged as follows:

$$\sin \phi = \frac{l_1^2 - l_2^2 + l_3^2 - l_4^2}{2(c_b d_p - d_b c_p)} \quad (2.25)$$

Equation (2.25) shows that the necessary condition to have the forward solutions is: $|l_1^2 - l_2^2 + l_3^2 - l_4^2| \leq 2|(c_b d_p - d_b c_p)|$. Once the orientation ϕ is obtained, substituting it into Eqn. (2.17) results in the expressions for the position of the platform as follows:

$$\begin{aligned} x = \frac{c_b}{2} &+ \frac{(c_b d_p - d_b c_p)(c_p \sin^2 \phi - d_b \sin \phi + d_p \sin \phi \cos \phi)}{2[c_b d_b + c_p d_p - (c_b d_p + d_b c_p) \cos \phi]} \\ &+ \frac{(l_1^2 - l_2^2)(d_b - d_p \cos \phi) + c_p(l_2^2 - l_3^2) \sin \phi}{2[c_b d_b + c_p d_p - (c_b d_p + d_b c_p) \cos \phi]} \end{aligned} \quad (2.26)$$

$$\begin{aligned} y = \frac{d_b}{2} &+ \frac{(c_b d_p - d_b c_p)(d_p \sin^2 \phi + c_b \sin \phi - c_p \sin \phi \cos \phi)}{2[c_b d_b + c_p d_p - (c_b d_p + d_b c_p) \cos \phi]} \\ &+ \frac{(l_2^2 - l_3^2)(c_b - c_p \cos \phi) + d_p(l_2^2 - l_1^2) \sin \phi}{2[c_b d_b + c_p d_p - (c_b d_p + d_b c_p) \cos \phi]} \end{aligned} \quad (2.27)$$

Computational examples: Two examples are used to find the forward solutions analytically using Eqns. (2.25), (2.26) and (2.27). The input lengths are the same as those used in the second and the third example as shown in Table 2.1. In each case, Eqn. (2.25) gives two orientation angles due to trigonometric properties. The angles correspond to two sets of positional solutions or platform poses. However, only one solution is acceptable because it must satisfy Eqn. (2.13), giving a positive cosine value. This makes sense because the orientation of the platform hardly gets larger than $\pm 90^\circ$ in CDPMs. The solutions of the two examples are shown in Fig. 2.6. Computational results show that the orientation angle is more sensitive to the input cable lengths.

Figure 2.6: Analytical solutions of symmetric 4-3-CDPPMs

2.3 Velocity Analysis

In this section, the instantaneous kinematic relationship between the translational velocity of the driving cables and the velocity of the moving platform is derived. The instantaneous kinematic problem of CDPPMs is generally expressed by:

$$\mathbf{J}_L \dot{\mathbf{L}} = \mathbf{J}_X \dot{\mathbf{X}} \quad (2.28)$$

where

$\dot{\mathbf{L}} = \{\dot{l}_1 \quad \dot{l}_2 \quad \dots \quad \dot{l}_m\}^T$: the vector of cable rates.

$\mathbf{J}_L, \mathbf{J}_X$: Jacobian matrix expressed with respect to the frame $\{B\}$.

$\dot{\mathbf{X}} = \{\mathbf{v} \quad \boldsymbol{\omega}\}^T = \{\dot{x} \quad \dot{y} \quad \dot{z} \quad \omega_x \quad \omega_y \quad \omega_z\}^T$: the vector of translational and rotational velocities of the moving platform, which is also expressed in $\{B\}$.

Differentiating Eqn. (2.2) with respect to time yields

$$l_i \boldsymbol{\omega}_i \times \hat{\mathbf{l}}_i + \dot{l}_i \hat{\mathbf{l}}_i = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}_i \quad (2.29)$$

where $\hat{\mathbf{l}}_i = \mathbf{l}_i / l_i$ is a unit vector along the cable, and $\boldsymbol{\omega}_i$ denotes the angular velocity of the

i th cable with respect to the base frame $\{B\}$. To eliminate ω_i , both sides of Eqn. (2.29) are dot-multiplied by $\hat{\mathbf{l}}_i$ giving:

$$\dot{l}_i = \hat{\mathbf{l}}_i \cdot \mathbf{v} + (\mathbf{r}_i \times \hat{\mathbf{l}}_i) \boldsymbol{\omega} \quad (2.30)$$

From Eqn. (2.30), m scalar equations can be written and assembled in matrix form as in Eqn. (2.28) with:

$$\mathbf{J}_L = \mathbf{I} \quad (m \times m \text{ identity matrix}) \quad (2.31)$$

$$\mathbf{J}_X = \mathbf{J} = \begin{bmatrix} \hat{\mathbf{l}}_1^T & (\mathbf{r}_1 \times \hat{\mathbf{l}}_1)^T \\ \hat{\mathbf{l}}_2^T & (\mathbf{r}_2 \times \hat{\mathbf{l}}_2)^T \\ \vdots & \vdots \\ \hat{\mathbf{l}}_m^T & (\mathbf{r}_m \times \hat{\mathbf{l}}_m)^T \end{bmatrix} \quad (m \times n \text{ Jacobian matrix}) \quad (2.32)$$

2.3.1 Singularity Issues

Equations (2.31) and (2.32) are useful in studying singularity configurations. A singularity configuration of a parallel mechanism refers to a particular configuration in which the mechanism loses or gains one more DOFs instantaneously. It is undesirable and should be excluded from the workspace. Based on the properties of the two Jacobian matrices, $\mathbf{J}_L$ and $\mathbf{J}_X$, the configuration singularities of a CDPM can be classified into three main groups.

Figure 2.7: Examples of singularity configurations of 4-3-CDPPMs

The first (inverse), second (forward), and the third (combined) type of singularities occur, respectively, when matrix $\mathbf{J}_L$ loses rank, matrix $\mathbf{J}_X$ loses rank, and when both matrices $\mathbf{J}_L$ and $\mathbf{J}_X$ simultaneously lose rank. The matrix $\mathbf{J}_L$ is always a diagonal matrix and will not be singular if none of the cable lengths takes zero value. As a result, the inverse and combined singularity configurations do not exist and only the forward singularities need to be considered. Through rank analysis of matrix $\mathbf{J}_X$, the forward ‘kinematics’ singularity configurations can be identified. Figure 2.7 illustrates some singularity configurations of a 4-3-CDPPM in which the moving platform possesses an instantaneous motion in the indicated direction. Such a Jacobian-based singularity analysis approach has two drawbacks when extended to a general CDPM. First, the forward kinematics Jacobian matrix $\mathbf{J}$ is normally a non-square matrix, which makes its rank analysis complicated. Second, matrix $\mathbf{J}$ is meaningful only when all driving cables are in tension. So, if any of the driving cables is slack (which happens to a CDPM), matrix $\mathbf{J}$ becomes invalid. This shows that identifying all of the forward singularity configurations using the Jacobian matrix is not sufficient in CDPMs. Hence, in order to take into account the unilateral driving capability of cables, tension analysis is always required.

2.3.2 Transformation between $\mathbf{J}$ and $\mathbf{J}_{NR}$

When comparing Eqn. (2.6) and Eqn. (2.28), it is noted that the forward Jacobian matrix $\mathbf{J}$ is closely related to the Newton-Raphson matrix $\mathbf{J}_{NR}$. Since the forward Jacobian matrix $\mathbf{J}$ can be derived directly from a specific configuration by vector operations, it can facilitate the formation of the Newton-Raphson matrix rapidly without performing partial differentiation. With the representation of orientation based on $Z - Y - X$ Euler angles, the mapping from the angular velocity $(\omega_x, \omega_y, \omega_z)$ to the instantaneous values of the angle set $(\dot{\alpha}, \dot{\beta}, \dot{\gamma})$ is expressed as follows (Craig [48]):

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} 0 & -\sin \alpha & \cos \alpha \cos \beta \\ 0 & \cos \alpha & \sin \alpha \cos \beta \\ 1 & 0 & -\sin \beta \end{bmatrix} \begin{Bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{Bmatrix} \quad (2.33)$$

By combining Eqns. (2.4),(2.7),(2.28),(2.31),(2.32) and (2.33), the Newton-Raphson Jacobian matrix can be calculated as follows:

$$\mathbf{J}_{NR} = \mathbf{L}_c \cdot \mathbf{J} \cdot \mathbf{\Omega}_c \quad (2.34)$$

where

$$\mathbf{\Omega}_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & & \\ & & & 0 & -\sin \alpha & \cos \alpha \cdot \cos \beta \\ & \mathbf{0} & & 0 & \cos \alpha & \sin \alpha \cdot \cos \beta \\ & & & 1 & 0 & -\sin \beta \end{bmatrix}$$

$$\mathbf{L}_c = 2 \begin{bmatrix} l_1 & & & \mathbf{0} \\ & l_2 & & \\ & & \ddots & \\ \mathbf{0} & & & l_m \end{bmatrix} \quad (m \times m)$$

Equation (2.34) can be used to determine the Newton-Raphson matrix conveniently. This improves the computational aspects of the Newton-Raphson method in the forward displacement analysis.

2.4 Acceleration Analysis

In this section, the relationship between the acceleration of the cables and the acceleration of the moving platform is formulated. The linear accelerations of the cables can be obtained by differentiating Eqn. (2.28) with respect to time directly.

$$\ddot{\mathbf{L}} = \mathbf{J} \cdot \ddot{\mathbf{X}} + \mathbf{S} \cdot \dot{\mathbf{X}} \quad (2.35)$$

where

$\ddot{\mathbf{L}} = \{\ddot{l}_1 \quad \ddot{l}_2 \quad \cdots \quad \ddot{l}_m\}^T$: the vector of cable accelerations.

$\ddot{\mathbf{X}} = \{\mathbf{a} \quad \dot{\boldsymbol{\omega}}\}^T = \{\ddot{x} \quad \ddot{y} \quad \cdots \quad \dot{\omega}_z\}^T$: the vector of translational and rotational

accelerations of the moving platform expressed in $\{B\}$.

$\mathbf{S} = \frac{d\mathbf{J}}{dt}$: derivative of the Jacobian matrix with respect to time.

For completeness and as examples, the analytical formulations of cable accelerations for 4-3-CDPPMs and 8-6-CDSPMs are given in the following sections.

2.4.1 Acceleration Analysis for 4-3-CDPPMs

In 4-3-CDPPMs, four cables are used to realize the three DOFs of the moving platform (x, y and ϕ). From Eqns. (2.3), (2.28) and (2.35), the inverse kinematics is expressed by:

$$\mathbf{L} = \begin{Bmatrix} l_1 \\ \vdots \\ l_4 \end{Bmatrix} = \begin{Bmatrix} \sqrt{(x + r_{1,x} - b_{1,x})^2 + (y + r_{1,y} - b_{1,y})^2} \\ \vdots \\ \sqrt{(x + r_{4,x} - b_{4,x})^2 + (y + r_{4,y} - b_{4,y})^2} \end{Bmatrix} \quad (2.36)$$

$$\dot{\mathbf{L}} = \begin{Bmatrix} \dot{l}_1 \\ \vdots \\ \dot{l}_4 \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{Bmatrix} \quad (2.37)$$

$$\ddot{\mathbf{L}} = \begin{Bmatrix} \ddot{l}_1 \\ \vdots \\ \ddot{l}_4 \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{Bmatrix} + (\mathbf{S}_x \dot{x} + \mathbf{S}_y \dot{y} + \mathbf{S}_\phi \dot{\phi}) \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{Bmatrix} \quad (2.38)$$

where $\mathbf{J}$ is the Jacobian matrix which has the form of

$$\mathbf{J} = \begin{bmatrix} j_{1,1} & j_{1,2} & j_{1,3} \\ j_{2,1} & j_{2,2} & j_{2,3} \\ j_{3,1} & j_{3,2} & j_{3,3} \\ j_{4,1} & j_{4,2} & j_{4,3} \end{bmatrix} \quad (2.39)$$

From Eqn. (2.32), velocity coefficients $j_{i,1}, j_{i,2}, j_{i,3}$ are determined by:

$$j_{i,1} = l_{i,x}/l_i$$

$$j_{i,2} = l_{i,y}/l_i$$

$$j_{i,3} = (r_{i,x}l_{i,y} - r_{i,y}l_{i,x})/l_i$$

and $\mathbf{S}_x, \mathbf{S}_y, \mathbf{S}_\phi$ are the partial derivative matrices of the Jacobian matrix with respect to x, y and ϕ , respectively:

$$\mathbf{S}_x = \begin{bmatrix} \frac{\partial j_{1,1}}{\partial x} & \dots & \frac{\partial j_{1,3}}{\partial x} \\ \vdots & \vdots & \vdots \\ \frac{\partial j_{4,1}}{\partial x} & \dots & \frac{\partial j_{4,3}}{\partial x} \end{bmatrix}; \mathbf{S}_y = \begin{bmatrix} \frac{\partial j_{1,1}}{\partial y} & \dots & \frac{\partial j_{1,3}}{\partial y} \\ \vdots & \vdots & \vdots \\ \frac{\partial j_{4,1}}{\partial y} & \dots & \frac{\partial j_{4,3}}{\partial y} \end{bmatrix}; \mathbf{S}_\phi = \begin{bmatrix} \frac{\partial j_{1,1}}{\partial \phi} & \dots & \frac{\partial j_{1,3}}{\partial \phi} \\ \vdots & \vdots & \vdots \\ \frac{\partial j_{4,1}}{\partial \phi} & \dots & \frac{\partial j_{4,3}}{\partial \phi} \end{bmatrix}$$

with (i=1,2,3,4)

$$\begin{aligned} \frac{\partial j_{i,1}}{\partial x} &= \frac{j_{i,2}^2}{l_i} \\ \frac{\partial j_{i,1}}{\partial y} &= -\frac{j_{i,1}j_{i,2}}{l_i} \\ \frac{\partial j_{i,1}}{\partial \phi} &= -\frac{j_{i,2}(r_{i,x}j_{i,1} + r_{i,y}j_{i,2})}{l_i} \\ \frac{\partial j_{i,2}}{\partial x} &= -\frac{j_{i,1}j_{i,2}}{l_i} \\ \frac{\partial j_{i,2}}{\partial y} &= \frac{j_{i,1}^2}{l_i} \\ \frac{\partial j_{i,2}}{\partial \phi} &= \frac{j_{i,1}(r_{i,x}j_{i,1} + r_{i,y}j_{i,2})}{l_i} \\ \frac{\partial j_{i,3}}{\partial x} &= -\frac{j_{i,2}(r_{i,x}j_{i,1} + r_{i,y}j_{i,2})}{l_i} \\ \frac{\partial j_{i,3}}{\partial y} &= \frac{j_{i,1}(r_{i,x}j_{i,1} + r_{i,y}j_{i,2})}{l_i} \\ \frac{\partial j_{i,3}}{\partial \phi} &= \frac{r_i^2}{l_i} - \frac{j_{i,3}^2}{l_i} - (r_{i,x}j_{i,1} + r_{i,y}j_{i,2}) \end{aligned}$$

2.4.2 Acceleration Analysis for 8-6-CDSPMs

For this type of mechanisms, eight cables are used to manipulate the six DOFs of the moving platform (x, y, z, α, β and γ). The inverse kinematics for this 8-6-CDSPM is slightly different from that of the 4-3-CDPPM because there is a transformation from the angular velocity ($\omega_x, \omega_y, \omega_z$) to the instantaneous values of the angle set ($\dot{\alpha}, \dot{\beta}, \dot{\gamma}$) as shown in Eqn. (2.33). Similarly, the inverse kinematics of spatial mechanisms is expressed by

the followings:

$$\mathbf{L} = \begin{Bmatrix} \sqrt{(x + r_{1,x} - b_{1,x})^2 + (y + r_{1,y} - b_{1,y})^2 + (z + r_{1,z} - b_{1,z})^2} \\ \vdots \\ \sqrt{(x + r_{8,x} - b_{8,x})^2 + (y + r_{8,y} - b_{8,y})^2 + (z + r_{8,z} - b_{8,z})^2} \end{Bmatrix} \quad (2.40)$$

$$\dot{\mathbf{L}} = \mathbf{J} \begin{Bmatrix} \dot{x} \\ \vdots \\ \omega_z \end{Bmatrix} \quad (2.41)$$

$$\ddot{\mathbf{L}} = \mathbf{J} \begin{Bmatrix} \ddot{x} \\ \vdots \\ \dot{\omega}_z \end{Bmatrix} + (\mathbf{S}_x \dot{x} + \mathbf{S}_y \dot{y} + \mathbf{S}_z \dot{z} + \mathbf{S}_\alpha \dot{\alpha} + \mathbf{S}_\beta \dot{\beta} + \mathbf{S}_\gamma \dot{\gamma}) \begin{Bmatrix} \dot{x} \\ \vdots \\ \omega_z \end{Bmatrix} \quad (2.42)$$

where $\mathbf{J}$ is the Jacobian matrix which is given by

$$\mathbf{J} = \begin{bmatrix} j_{1,1} & j_{1,2} & \cdots & j_{1,6} \\ j_{2,1} & j_{2,2} & \cdots & j_{2,6} \\ \vdots & \vdots & \vdots & \vdots \\ j_{8,1} & j_{8,2} & \cdots & j_{8,6} \end{bmatrix} \quad (2.43)$$

The components $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$ are obtained from Eqn. (2.33):

$$\begin{Bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{Bmatrix} = \begin{bmatrix} \frac{\cos \alpha \cdot \sin \beta}{\cos \beta} & \frac{\sin \alpha \cdot \sin \beta}{\cos \beta} & 1 \\ -\sin \alpha & \cos \alpha & 0 \\ \frac{\cos \alpha}{\cos \beta} & \frac{\sin \alpha}{\cos \beta} & 0 \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \quad (2.44)$$

From Eqn. (2.32), velocity coefficients $j_{i,1}, j_{i,2}, \dots, j_{i,6}$ are determined by:

$$\begin{aligned} j_{i,1} &= l_{i,x}/l_i \\ j_{i,2} &= l_{i,y}/l_i \\ j_{i,3} &= l_{i,z}/l_i \\ j_{i,4} &= (r_{i,y}l_{i,z} - r_{i,z}l_{i,y})/l_i \\ j_{i,5} &= (r_{i,z}l_{i,x} - r_{i,x}l_{i,z})/l_i \\ j_{i,6} &= (r_{i,x}l_{i,y} - r_{i,y}l_{i,x})/l_i \end{aligned}$$

and $\mathbf{S}_x, \mathbf{S}_y, \mathbf{S}_z, \mathbf{S}_\alpha, \mathbf{S}_\beta, \mathbf{S}_\gamma$ are the partial derivative matrices of the Jacobian matrix with respect to x, y, z, α, β and γ , respectively:

$$\mathbf{S}_x = \begin{bmatrix} \frac{\partial j_{1,1}}{\partial x} & \cdots & \frac{\partial j_{1,6}}{\partial x} \\ \vdots & \cdots & \vdots \\ \frac{\partial j_{8,1}}{\partial x} & \cdots & \frac{\partial j_{8,6}}{\partial x} \end{bmatrix}; \quad \mathbf{S}_y = \begin{bmatrix} \frac{\partial j_{1,1}}{\partial y} & \cdots & \frac{\partial j_{1,6}}{\partial y} \\ \vdots & \cdots & \vdots \\ \frac{\partial j_{8,1}}{\partial y} & \cdots & \frac{\partial j_{8,6}}{\partial y} \end{bmatrix}; \quad \mathbf{S}_z = \begin{bmatrix} \frac{\partial j_{1,1}}{\partial z} & \cdots & \frac{\partial j_{1,6}}{\partial z} \\ \vdots & \cdots & \vdots \\ \frac{\partial j_{8,1}}{\partial z} & \cdots & \frac{\partial j_{8,6}}{\partial z} \end{bmatrix}$$

$$\mathbf{S}_\alpha = \begin{bmatrix} \frac{\partial j_{1,1}}{\partial \alpha} & \cdots & \frac{\partial j_{1,6}}{\partial \alpha} \\ \vdots & \cdots & \vdots \\ \frac{\partial j_{8,1}}{\partial \alpha} & \cdots & \frac{\partial j_{8,6}}{\partial \alpha} \end{bmatrix}; \quad \mathbf{S}_\beta = \begin{bmatrix} \frac{\partial j_{1,1}}{\partial \beta} & \cdots & \frac{\partial j_{1,6}}{\partial \beta} \\ \vdots & \cdots & \vdots \\ \frac{\partial j_{8,1}}{\partial \beta} & \cdots & \frac{\partial j_{8,6}}{\partial \beta} \end{bmatrix}; \quad \mathbf{S}_\gamma = \begin{bmatrix} \frac{\partial j_{1,1}}{\partial \gamma} & \cdots & \frac{\partial j_{1,6}}{\partial \gamma} \\ \vdots & \cdots & \vdots \\ \frac{\partial j_{8,1}}{\partial \gamma} & \cdots & \frac{\partial j_{8,6}}{\partial \gamma} \end{bmatrix}$$

with (i=1,2,...,8)

$$\begin{aligned} \frac{\partial j_{i,1}}{\partial x} &= \frac{(j_{i,2}^2 + j_{i,3}^2)}{l_i} \\ \frac{\partial j_{i,1}}{\partial y} &= -\frac{j_{i,1}j_{i,2}}{l_i} \\ \frac{\partial j_{i,1}}{\partial z} &= -\frac{j_{i,1}j_{i,3}}{l_i} \\ \frac{\partial j_{i,1}}{\partial \alpha} &= -\frac{r_{i,y}}{l_i} - \frac{j_{i,1}j_{i,6}}{l_i} \\ \frac{\partial j_{i,1}}{\partial \beta} &= \frac{r_{i,x}^{(\beta)}}{l_i} - \frac{j_{i,1}(r_{i,x}^{(\beta)}j_{i,1} + r_{i,y}^{(\beta)}j_{i,2} + r_{i,z}^{(\beta)}j_{i,3})}{l_i} \\ \frac{\partial j_{i,1}}{\partial \gamma} &= \frac{r_{i,x}^{(\gamma)}}{l_i} - \frac{j_{i,1}(r_{i,x}^{(\gamma)}j_{i,1} + r_{i,y}^{(\gamma)}j_{i,2} + r_{i,z}^{(\gamma)}j_{i,3})}{l_i} \\ \\ \frac{\partial j_{i,2}}{\partial x} &= -\frac{j_{i,1}j_{i,2}}{l_i} \\ \frac{\partial j_{i,2}}{\partial y} &= \frac{(j_{i,1}^2 + j_{i,3}^2)}{l_i} \\ \frac{\partial j_{i,2}}{\partial z} &= -\frac{j_{i,2}j_{i,3}}{l_i} \\ \frac{\partial j_{i,2}}{\partial \alpha} &= \frac{r_{i,x}}{l_i} - \frac{j_{i,2}j_{i,6}}{l_i} \\ \frac{\partial j_{i,2}}{\partial \beta} &= \frac{r_{i,y}^{(\beta)}}{l_i} - \frac{j_{i,2}(r_{i,x}^{(\beta)}j_{i,1} + r_{i,y}^{(\beta)}j_{i,2} + r_{i,z}^{(\beta)}j_{i,3})}{l_i} \\ \frac{\partial j_{i,2}}{\partial \gamma} &= \frac{r_{i,y}^{(\gamma)}}{l_i} - \frac{j_{i,2}(r_{i,x}^{(\gamma)}j_{i,1} + r_{i,y}^{(\gamma)}j_{i,2} + r_{i,z}^{(\gamma)}j_{i,3})}{l_i} \end{aligned}$$

$$\begin{aligned}
 \frac{\partial j_{i,3}}{\partial x} &= -\frac{\dot{j}_{i,1}\dot{j}_{i,3}}{l_i} \\
 \frac{\partial j_{i,3}}{\partial y} &= -\frac{\dot{j}_{i,2}\dot{j}_{i,3}}{l_i} \\
 \frac{\partial j_{i,3}}{\partial z} &= \frac{(j_{i,1}^2 + j_{i,2}^2)}{l_i} \\
 \frac{\partial j_{i,3}}{\partial \alpha} &= -\frac{\dot{j}_{i,3}\dot{j}_{i,6}}{l_i} \\
 \frac{\partial j_{i,3}}{\partial \beta} &= \frac{r_{i,z}^{(\beta)}}{l_i} - \frac{j_{i,3}(r_{i,x}^{(\beta)}j_{i,1} + r_{i,y}^{(\beta)}j_{i,2} + r_{i,z}^{(\beta)}j_{i,3})}{l_i} \\
 \frac{\partial j_{i,3}}{\partial \gamma} &= \frac{r_{i,z}^{(\gamma)}}{l_i} - \frac{j_{i,3}(r_{i,x}^{(\gamma)}j_{i,1} + r_{i,y}^{(\gamma)}j_{i,2} + r_{i,z}^{(\gamma)}j_{i,3})}{l_i}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial j_{i,4}}{\partial x} &= -\frac{\dot{j}_{i,1}\dot{j}_{i,4}}{l_i} \\
 \frac{\partial j_{i,4}}{\partial y} &= -\frac{r_{i,z}}{l_i} - \frac{\dot{j}_{i,2}\dot{j}_{i,4}}{l_i} \\
 \frac{\partial j_{i,4}}{\partial z} &= \frac{r_{i,y}}{l_i} - \frac{\dot{j}_{i,3}\dot{j}_{i,4}}{l_i} \\
 \frac{\partial j_{i,4}}{\partial \alpha} &= r_{i,x}\dot{j}_{i,3} - \frac{r_{i,x}r_{i,z}}{l_i} - \frac{\dot{j}_{i,4}\dot{j}_{i,6}}{l_i} \\
 \frac{\partial j_{i,4}}{\partial \beta} &= r_{i,y}^{(\beta)}j_{i,3} - r_{i,z}^{(\beta)}j_{i,2} - \frac{(r_{i,z}r_{i,y}^{(\beta)} - r_{i,y}r_{i,z}^{(\beta)})}{l_i} - \frac{j_{i,4}(r_{i,x}^{(\beta)}j_{i,1} + r_{i,y}^{(\beta)}j_{i,2} + r_{i,z}^{(\beta)}j_{i,3})}{l_i} \\
 \frac{\partial j_{i,4}}{\partial \gamma} &= r_{i,y}^{(\gamma)}j_{i,3} - r_{i,z}^{(\gamma)}j_{i,2} - \frac{(r_{i,z}r_{i,y}^{(\gamma)} - r_{i,y}r_{i,z}^{(\gamma)})}{l_i} - \frac{j_{i,4}(r_{i,x}^{(\gamma)}j_{i,1} + r_{i,y}^{(\gamma)}j_{i,2} + r_{i,z}^{(\gamma)}j_{i,3})}{l_i}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial j_{i,5}}{\partial x} &= \frac{r_{i,z}}{l_i} - \frac{\dot{j}_{i,1}\dot{j}_{i,5}}{l_i} \\
 \frac{\partial j_{i,5}}{\partial y} &= -\frac{\dot{j}_{i,2}\dot{j}_{i,5}}{l_i} \\
 \frac{\partial j_{i,5}}{\partial z} &= -\frac{r_{i,x}}{l_i} - \frac{\dot{j}_{i,3}\dot{j}_{i,5}}{l_i} \\
 \frac{\partial j_{i,5}}{\partial \alpha} &= r_{i,y}\dot{j}_{i,3} - \frac{r_{i,y}r_{i,z}}{l_i} - \frac{\dot{j}_{i,5}\dot{j}_{i,6}}{l_i} \\
 \frac{\partial j_{i,5}}{\partial \beta} &= r_{i,z}^{(\beta)}j_{i,1} - r_{i,x}^{(\beta)}j_{i,3} - \frac{(r_{i,x}r_{i,z}^{(\beta)} - r_{i,z}r_{i,x}^{(\beta)})}{l_i} - \frac{j_{i,5}(r_{i,x}^{(\beta)}j_{i,1} + r_{i,y}^{(\beta)}j_{i,2} + r_{i,z}^{(\beta)}j_{i,3})}{l_i} \\
 \frac{\partial j_{i,5}}{\partial \gamma} &= r_{i,z}^{(\gamma)}j_{i,1} - r_{i,x}^{(\gamma)}j_{i,3} - \frac{(r_{i,x}r_{i,z}^{(\gamma)} - r_{i,z}r_{i,x}^{(\gamma)})}{l_i} - \frac{j_{i,5}(r_{i,x}^{(\gamma)}j_{i,1} + r_{i,y}^{(\gamma)}j_{i,2} + r_{i,z}^{(\gamma)}j_{i,3})}{l_i}
 \end{aligned}$$

$$\begin{aligned}
\frac{\partial j_{i,6}}{\partial x} &= -\frac{r_{i,y}}{l_i} - \frac{j_{i,1}j_{i,6}}{l_i} \\
\frac{\partial j_{i,6}}{\partial y} &= \frac{r_{i,x}}{l_i} - \frac{j_{i,2}j_{i,6}}{l_i} \\
\frac{\partial j_{i,6}}{\partial z} &= -\frac{j_{i,3}j_{i,6}}{l_i} \\
\frac{\partial j_{i,6}}{\partial \alpha} &= -(r_{i,x}j_{i,1} + r_{i,y}j_{i,2}) + \frac{(r_{i,x}^2 + r_{i,y}^2)}{l_i} - \frac{j_{i,6}^2}{l_i} \\
\frac{\partial j_{i,6}}{\partial \beta} &= r_{i,x}^{(\beta)}j_{i,2} - r_{i,y}^{(\beta)}j_{i,1} - \frac{(r_{i,y}r_{i,x}^{(\beta)} - r_{i,x}r_{i,y}^{(\beta)})}{l_i} - \frac{j_{i,6}(r_{i,x}^{(\beta)}j_{i,1} + r_{i,y}^{(\beta)}j_{i,2} + r_{i,z}^{(\beta)}j_{i,3})}{l_i} \\
\frac{\partial j_{i,6}}{\partial \gamma} &= r_{i,x}^{(\gamma)}j_{i,2} - r_{i,y}^{(\gamma)}j_{i,1} - \frac{(r_{i,y}r_{i,x}^{(\gamma)} - r_{i,x}r_{i,y}^{(\gamma)})}{l_i} - \frac{j_{i,6}(r_{i,x}^{(\gamma)}j_{i,1} + r_{i,y}^{(\gamma)}j_{i,2} + r_{i,z}^{(\gamma)}j_{i,3})}{l_i}
\end{aligned}$$

In above expressions, the terms $r_{i,x}^{(\beta)}, r_{i,y}^{(\beta)}$ and $r_{i,z}^{(\beta)}$ are components of vector $\mathbf{r}_i^{(\beta)}$. Similarly, the terms $r_{i,x}^{(\gamma)}, r_{i,y}^{(\gamma)}$ and $r_{i,z}^{(\gamma)}$ are components of vector $\mathbf{r}_i^{(\gamma)}$. These two vectors are calculated as follows:

$$\begin{aligned}
\mathbf{r}_i^{(\beta)} &= \begin{Bmatrix} -r_{i,x}^p \cos \alpha \sin \beta + r_{i,y}^p \cos \alpha \cos \beta \sin \gamma + r_{i,z}^p \cos \alpha \cos \beta \cos \gamma \\ -r_{i,x}^p \sin \alpha \sin \beta + r_{i,y}^p \sin \alpha \cos \beta \sin \gamma + r_{i,z}^p \sin \alpha \cos \beta \cos \gamma \\ -r_{i,x}^p \cos \beta - r_{i,y}^p \sin \beta \sin \gamma - r_{i,z}^p \sin \beta \cos \gamma \end{Bmatrix} \quad (2.45) \\
\mathbf{r}_i^{(\gamma)} &= \begin{Bmatrix} r_{i,y}^p (\sin \alpha \sin \gamma + \cos \alpha \sin \beta \cos \gamma) + r_{i,z}^p (\sin \alpha \cos \gamma - \cos \alpha \sin \beta \sin \gamma) \\ r_{i,y}^p (\sin \alpha \sin \beta \cos \gamma - \cos \alpha \sin \gamma) - r_{i,z}^p (\cos \alpha \cos \gamma + \sin \alpha \sin \beta \sin \gamma) \\ r_{i,y}^p \cos \beta \cos \gamma - r_{i,z}^p \cos \beta \sin \gamma \end{Bmatrix}
\end{aligned}$$

With the above expressions, the acceleration analysis is presented completely. Although formulations are done for a 4-3-CDPPM and a 8-6-CDSPPM, the same method can be readily extended to other CDPMs.

2.5 Summary

In this chapter, the kinematic analysis of general cable-driven parallel mechanisms is presented. The main results are summarized as follows:

The forward displacement solution of a general CDPM is presented. The numerical Newton-Raphson method is used to obtain the solution for fully restrained CDPMs using

the 4-3-CDPPM and 8-6-CDSPM as examples. It is seen that the existence of mapping between the Newton-Raphson matrix and the Jacobian matrix helps to improve the computational aspect of the Newton-Raphson method. Some computational examples are given in which the number of iterations and the error values demonstrate the effectiveness of this method. The analytical solutions for a general 4-3-CDPPM is also formulated, leading to solving a 4th order polynomial equation. This is significant and very useful because symbolic solutions can always be obtained explicitly. It has been shown that for a 4-3-CDPPM, there exists at most 4 sets of forward displacement solutions. Generally, it should be noted that the solutions may not be acceptable as they do not take into account the unilateral characteristics of the cables. The solutions need to be checked against the tension condition which will be addressed in the next Chapter. Nevertheless, the forward displacement analysis is useful in teaching modes and in mechanism calibration whereby the tension of cables are maintained throughout the motion of the platform.

The inverse displacement solution of a general CDPM is also presented, followed by the derivation of the expressions for the velocity analysis. The velocity relation shows that only forward singularity exists in CDPMs. It is also noted that velocities of the cables are linear combinations of the linear and angular velocities of the platform through the Jacobian matrix. While a Jacobian-based singularity analysis is very important in rigid-link mechanisms, it is seen that such a singularity analysis approach is not sufficient in CDPMs because it is meaningful only when all the cables are in tension. However, it is still useful to identify singularity configurations in which the platform always possesses instantaneous motions although all the cables are in tension. Finally, expressions of acceleration analysis are derived in details, specifically for a 4-3-CDPPM and a 8-6-CDSPM. The formulations will be used for calculating joint kinematics when planning the trajectory of the platform. Acceleration analysis of other configurations with different number of cables can be readily developed from the method presented.

Chapter 3

Tension Analysis

3.1 Introduction

For cable-driven parallel mechanisms (CDPMs), the unilateral characteristic of cables (can only pull but cannot push) makes the cable tension as the most important factor in analyzing the kinetostatic performance of the CDPMs such as workspace and stability. A critical issue of CDPMs is to determine tension conditions in which there is at least one appropriate set of positive cable tensions to maintain the pose of the moving platform. This issue will be addressed in details in this Chapter.

There are two common approaches used to analyze the cable tension: the null space approach through pseudo inverse matrices and the geometrical approach through the construction of convex hulls. In the null space approach, linear algebraic tools are used to find the tension values which are then verified by tension conditions. The homogeneous solution must exist in order to obtain positive tensions in the cables. This happens in a condition that the number of cables (m) must be greater than the number of degrees of freedom (DOF) of the moving platform (n). For completely restrained CDPMs ($m = n + 1$), the homogeneous solution degenerates to a column vector. The components of the vector can be obtained in symbolic forms. Ming [28] employed this approach to cal-

culate the cable tensions of a 3-DOF planar mechanism driven by four cables. Fattah [49], Pusey [50] and Verhoeven [29] used the approach to obtain the optimal workspace and to construct feasible workspaces based on the homogeneous solutions. Tadokoro [51, 52] made use of the homogeneous solution to study the condition of generating force and moment. With this approach, William [53] proved that the workspace of a point-mass mechanism is the convex hull generated by the base. Oh [54] used this approach to form a point-wise feasible region in an $(m - n)$ -dimensional space, where a point in the region is corresponding to a set of positive tension values. However, the method becomes complicated when more cables are employed. Moreover, it is unable to cater for the zero-wrench cases because the region is degenerated to null. In general, it is no longer straightforward to check for positive tensions and to formulate the workspace boundaries using the null space approach as the number of cables increases. For the geometrical approach, the tension vectors will form convex hulls and the external wrench is checked whether it is inside the convex set. This approach directly considers the unilateral property of cables in the formulation because the convexes represent the positive combinations of pulling force. Lafourcade [30] proposed a sketch-based design methodology to study the theoretical workspace. Gosselin [55] used the approach to study the dynamic workspace such that the motion of a moving platform is incorporated into a set of wrenches called a *pseudo-pyramid*. Osumi [56] generated the convex to study the static equilibrium of a manipulator suspended by cables. Similarly, Riechel [34] and Bosscher [57] employed this approach to obtain analytical representations of the workspace boundaries for suspended platforms provided that the required net wrench set is the subset of the available net wrench set. Zlatanov [58] proposed an alternative method of studying geometric features of convex 2-cones and 3-cones in screw spaces. However, the method has been found limited when extended to cones of four or more independent screws. Generally, the geometrical approaches become more complex for mechanisms with higher dimensional task-spaces, i.e. the number of DOFs of the platform is greater than three.

In recent years, the similarity between problems in CDPMs and those in multi-finger

grasping has been highlighted because cables can pull but cannot push whereas finger contacts can push but cannot pull. Ebert-Uphoff [59] employed the antipodal grasp tool to a planar cable robot with two pairs of coincident cables; Kino [60] made use of the force-closure concept in studying the driving condition of cables. Although grasping has been studied in great details (Nguyen [61, 62], Trinkle [63, 64], Yoshikawa [65]), the algorithms developed cannot be applied directly to CDPMs. Firstly, a cable cannot exert a moment about the axis along the cable while a soft finger contact has this property. Furthermore, for frictionless point-contact fingers, the contact normal is always perpendicular to the contacting surface. The issues limit the applications of grasping to CDPMs.

In the following Sections, the equilibrium state of the platform is first presented. The model is expressed in a linear equation system with the cable tensions as variables. Three tension-related issues including force-closure condition, feasible wrench condition and wrench set condition are identified and formulated. To satisfy these conditions, the null space approach and the geometrical approach are presented. Based on the convex property, a generic approach based on recursive dimension reduction technique is proposed to deal with tension conditions, together with tension bound limitation. The algorithms to check whether there exists an appropriate set of positive cable tensions are presented. Simulation results are included to demonstrate the effectiveness of the proposed approach.

3.2 Equilibrium of the Moving Platform

Figure 3.1 shows the free body diagram of a platform subjected to an external wrench. In order to resist the external wrench, the tension forces in all the cables must be large enough to keep the platform in equilibrium. The equilibrium conditions are as follows:

$$\sum_{i=1}^m \mathbf{t}_i + \mathbf{f}_p = 0 \quad (3.1)$$

$$\sum_{i=1}^m \mathbf{r}_i \times \mathbf{t}_i + \mathbf{m}_p = 0 \quad (3.2)$$

Figure 3.1: Free body diagram of the moving platform

where $\mathbf{t}_i = t_i \mathbf{u}_i = -t_i \frac{\mathbf{l}_i}{l_i}$ represents the tension force that acts in the opposite direction of $\mathbf{l}_i$. Substituting $\mathbf{t}_i$ into Eqns. (3.1),(3.2) leads to the following equation:

$$\mathbf{A} \cdot \mathbf{T} = \mathbf{B} \quad (3.3)$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \\ \mathbf{r}_1 \times \mathbf{u}_1 & \mathbf{r}_2 \times \mathbf{u}_2 & \cdots & \mathbf{r}_m \times \mathbf{u}_m \end{bmatrix} \in \mathbb{R}^{n \times m}: \text{ structure matrix}$$

$$\mathbf{T} = \{t_1 \ t_2 \ \cdots \ t_m\}^T \in \mathbb{R}^m: \text{ cable tension}$$

$$\mathbf{B} = - \begin{Bmatrix} \mathbf{f}_p \\ \mathbf{m}_p \end{Bmatrix} \in \mathbb{R}^n: \text{ external wrench.}$$

Equations (3.1) and (3.2) can also be expressed in vector form:

$$\sum_{i=1}^m t_i \mathbf{s}_i = -\mathbf{F}_p \quad (3.4)$$

where

$$\mathbf{s}_i = \begin{Bmatrix} \mathbf{u}_i \\ \mathbf{r}_i \times \mathbf{u}_i \end{Bmatrix}: \text{ the } i\text{th column vector of } \mathbf{A}$$

$$\mathbf{F}_p = \begin{Bmatrix} \mathbf{f}_p \\ \mathbf{m}_p \end{Bmatrix} \in \mathbb{R}^n: \text{ external wrench.}$$

The directional unit vector $\mathbf{u}_i$ ($i = 1, 2, \dots, m$) in the structure matrix $\mathbf{A}$ can be obtained

from the vector loop-closure equation for each cable as:

$$\mathbf{u}_i = -\frac{\mathbf{l}_i}{l_i} = \frac{\overrightarrow{OB_i} - \overrightarrow{OP} - \overrightarrow{PP_i}}{\|\overrightarrow{OB_i} - \overrightarrow{OP} - \overrightarrow{PP_i}\|} = \frac{\mathbf{b}_i - \mathbf{p} - \mathbf{R}\mathbf{r}_i^p}{\|\mathbf{b}_i - \mathbf{p} - \mathbf{R}\mathbf{r}_i^p\|} \quad (3.5)$$

Noted that the structure matrix $\mathbf{A}$ is expressed with respect to and is dependent on the posture of the moving platform. It can be shown that the structure matrix $\mathbf{A}$ and the Jacobian matrix $\mathbf{J}$ are related by $\mathbf{A} = -\mathbf{J}^T$.

3.3 Cable Tension Condition

For CDPMs, the cable tension is the most critical factor and must be considered first before analyzing the workspace, designing or optimization. There are three tension-related questions to be asked at a specific pose of the moving platform:

- Can the platform be restrained properly by the tensions of cables? (Fig. 3.2(a))
- Can the platform be sustained by the cables with finite tensions when subjected to a directed external wrench? (Fig. 3.2(b))
- Can the platform sustain a constant wrench acting from any direction? (Fig. 3.2(c))

Figure 3.2: Tension conditions in CDPMs

The three tension-related questions can be defined and formulated formally in terms of the following three tension conditions: force-closure condition, feasible wrench condition and wrench set condition.

• **Force-closure condition:** The cable mechanism is said to be able to resist any arbitrary wrench (in magnitude and direction) acting on the moving platform (see Fig. 3.2(a)) provided that positive cable tensions can always be maintained. There is no upper limit on the cable tensions. This condition is mathematically described as:

$$\forall \mathbf{F}_p \in \mathbb{R}^n : \exists t_1, t_2, \dots, t_m \in [0, \infty) : \sum_{i=1}^m t_i \mathbf{s}_i = -\mathbf{F}_p \quad (3.6)$$

The above condition makes sure that the cable tensions are able to fully restrain the moving platform. This condition will be used to generate force-closure workspaces.

• **Feasible wrench condition:** Here the external wrench is considered as a static wrench acting on the moving platform, e.g. the gravity of the platform. The tension range of the i th cable is limited from $t_{i,min}$ to $t_{i,max}$. The value $t_{i,min}$ refers to the tension required to keep the cables taut, whereas the value $t_{i,max}$ is limited by the torque of actuators or by the maximum tension without breaking the cables. Hence, the cable mechanism is able to sustain by the cables with finite tensions when subjected to a directed external wrench (see Fig. 3.2(b)) provided that there exists a set of appropriate tensions:

$$\exists t_{i,min} \leq t_i \leq t_{i,max} \ (i = 1, 2, \dots, m) : \sum_{i=1}^m t_i \mathbf{s}_i = -\mathbf{F}_p \in \mathbb{R}^n \quad (3.7)$$

This condition is useful in checking the static equilibrium as well as the dynamic equilibrium of the moving platform where inertial terms are regarded as constants instantaneously. Furthermore, the possible range of the external wrench from a certain direction can be obtained at a specific posture of the platform.

• **Wrench set condition:** Similar to the feasible wrench condition, the i th cable tension is limited from $t_{i,min}$ to $t_{i,max}$. Here, the cable mechanism is able to sustain a certain minimum force $f_{p,r}$ and a certain minimum moment $m_{p,r}$ from any direction (see Fig. 3.2(c))

provided that:

$$\begin{aligned} \forall \mathbf{F}_p \Big| \|\mathbf{f}_p\| = f_{p,r}, \|\mathbf{m}_p\| = m_{p,r} & : \exists t_{i,min} \leq t_i \leq t_{i,max} \ (i = 1, 2, \dots, m) \\ & : \sum_{i=1}^m t_i \mathbf{s}_i = -\mathbf{F}_p \end{aligned} \quad (3.8)$$

From the practical point of view, this condition ensures that the mechanism must possess certain stiffness in translation as well as in torsion. Together with the feasible wrench condition, this condition can be employed to generate the feasible wrench workspace.

3.4 Approaches on Tension Analysis

In this section, the null space approach and the geometrical approach are formulated. Analytical expressions of the null space for completely restrained CDPMs are introduced. In addition, geometrical interpretations are given with mathematical propositions. These formulations are used to derive analytical workspace boundaries for planar cases in the next chapter, which are very useful to get a better understanding of numerical results.

3.4.1 Null Space Approach

From Eqn. (3.3), if the number of unknown variables (m) is less than the number of equations (n), the system is unlikely to have positive tension solution. In the case of $m \geq n$, Nakamura [66] showed that the tension solution of Eqn. (3.3) could be written as follows by using pseudo-inverse matrix:

$$\mathbf{T} = \mathbf{A}^\# \cdot \mathbf{B} + (\mathbf{I} - \mathbf{A}^\# \cdot \mathbf{A})\mathbf{Z} \quad (3.9)$$

where

$\mathbf{A}^\# = \mathbf{A}^T(\mathbf{A} \cdot \mathbf{A}^T)^{-1} \in \mathbb{R}^{m \times n}$: pseudo inverse of the structure matrix $\mathbf{A}$

$\mathbf{I} \in \mathbb{R}^{m \times m}$: Identity matrix

$\mathbf{Z} = \left\{ \begin{matrix} z_1 & z_2 & \cdots & z_m \end{matrix} \right\}^T$: an arbitrary column vector.

In Eqn. (3.9), the first term $\mathbf{A}^\# \cdot \mathbf{B}$ is the particular solution corresponding to a tension solution for achieving any motion while subjecting to an arbitrary wrench $\mathbf{B}$. The second term $(\mathbf{I} - \mathbf{A}^\# \cdot \mathbf{A})\mathbf{Z}$ is the homogeneous solution and is the redundancy that can be utilized to keep cable tensions positive.

Completely restrained CDPMs ($m = n + 1$)

In this case, Eqn. (3.9) can be written as:

$$\mathbf{T} = \mathbf{A}^\# \cdot \mathbf{B} + \lambda \cdot \mathbf{N} \quad (3.10)$$

where the homogeneous solution is expressed as the kernel vector of $\mathbf{A}$ ($\mathbf{N} = \{n_1 \ n_2 \ \dots \ n_m\}^T$) multiplied by an arbitrarily scalar λ . As a result, components of this kernel vector can be found in closed forms. Using Cramer's rule, the i th null component is the determinant of the matrix formed by deleting the i th column of the structure matrix $\mathbf{A}$. From the expression of $\mathbf{A}$ in Eqn. (3.3), the null space components can be expressed as:

$$n_i = (-1)^i \begin{vmatrix} \cdots & \mathbf{u}_{i-1} & \mathbf{u}_{i+1} & \cdots \\ & \mathbf{r}_{i-1} \times \mathbf{u}_{i-1} & \mathbf{r}_{i+1} \times \mathbf{u}_{i+1} & \end{vmatrix} \quad (3.11)$$

If every components of $\lambda \cdot \mathbf{N}$ are positive, then it is always possible to provide an arbitrary positive compensation to the solution satisfying the external wrench $\mathbf{B}$ in Eqns. (3.9),(3.10). This leads to a condition that all n_i must be of the same sign. This is because if $\mathbf{N}$ is a positive vector or a negative vector, the value of λ can be selected to ensure that the tension of each cable is positive. Hence, the sufficient condition to satisfy the positive tension in cables (force-closure condition) is:

$$\begin{cases} \prod_{i=1}^m n_i \neq 0 \\ \sum_{i=1}^m |n_i| = \left| \sum_{i=1}^m n_i \right| \end{cases} \quad (3.12)$$

Redundantly restrained CDPMs ($m > n + 1$)

When $m > n + 1$, $\text{rank}(\mathbf{I} - \mathbf{A}^\# \cdot \mathbf{A}) = m - n$, the tension solution can be written as Eqn. (3.13) which includes $(m - n)$ independent vectors $\mathbf{N}_i$ ($i = 1, 2, \dots, (m - n)$) (Deif [67]), such that:

$$\mathbf{T} = \mathbf{A}^\# \cdot \mathbf{B} + \lambda_1 \cdot \mathbf{N}_1 + \lambda_2 \cdot \mathbf{N}_2 + \dots + \lambda_{m-n} \cdot \mathbf{N}_{m-n} \quad (3.13)$$

Comparing to the case $m = n + 1$, there are more redundant coefficients, λ_i ($i = 1, 2, \dots, m - n$). As a result, it is easier to keep the tension of each cable positive. However, it is no longer straightforward to determine the null vectors $\mathbf{N}_i$ as well as to select coefficients λ_i in Eqn. (3.13) so that the resultant null space is a positive vector.

Examples: Figure 3.3 shows a 4-3-CDPPM and a 8-6-CDSPM at different poses of the moving platform. Their kinematic parameters are given in Fig. 1.7.

The null space approach is used to check the force-closure condition for these configurations. For the 4-3-CDPPMs ($m = n + 1$), the homogeneous solutions are found using Eqn. (3.11). For the 8-6-CDSPMs ($m > n + 1$), the null space is obtained from the reduced row echelon form (Deif [67]). The computational results are shown in Table 3.1. It is seen that the last column in the case of Fig. 3.3(a) can always be set to positive values thus satisfying the force-closure condition whereas the case in Fig. 3.3(b) does not. For the case in Fig. 3.3(c), again the resultant homogeneous solution can always be a positive vector with a suitable selection of λ_1 and λ_2 , e.g. $\lambda_1 = \lambda_2 = 1$. Hence, the force closure condition is satisfied. However for the case in Fig. 3.3(d), no answer can be obtained.

3.4.2 Geometrical Approach

In this approach, the tension conditions can be verified by the interaction of convex hulls, which are defined as the collection of all positive combinations of points in a set. The

Figure 3.3: Examples of force-closure condition

fundamental concepts and theorems on convex analysis are summarized in Appendix B. Using this geometrical approach, the tension vectors, $\sum_{i=1}^m t_i \mathbf{s}_i$, in the left-hand side of Eqn. (3.4) will form convex hulls and the external wrench, $-\mathbf{F}_p$, in the right-hand side of Eqn. (3.4) is checked whether it belongs to the convex set. Hence, the unilateral property of cables in the formulation is directly considered because the convexes represent the positive combinations of pulling force. The following demonstrates this approach for the force-closure condition.

Table 3.1: Computational results using the null space approach

	Structure matrix A	Null space N
In Fig. 3.3(a)	$\begin{bmatrix} -0.6585 & 0.6585 & 0.6585 & -0.6585 \\ -0.7526 & -0.7526 & 0.7526 & 0.7526 \\ 0.0470 & -0.0470 & 0.0470 & -0.3199 \end{bmatrix}$	$\lambda \begin{Bmatrix} 0.0932 \\ 0.0932 \\ 0.0932 \\ 0.0932 \end{Bmatrix}$
In Fig. 3.3(b)	$\begin{bmatrix} -0.8299 & 0.3100 & 0.3990 & -0.7867 \\ -0.5579 & -0.9507 & 0.9170 & 0.6174 \\ -0.0244 & -0.1302 & 0.0756 & -0.0388 \end{bmatrix}$	$\lambda \begin{Bmatrix} 0.1097 \\ 0.0692 \\ 0.1476 \\ -0.0136 \end{Bmatrix}$
In Fig. 3.3(c)	$\begin{bmatrix} -0.5026 & 0.5026 & 0.5026 & -0.5026 & -0.5026 & 0.5026 & 0.5026 & -0.5026 \\ -0.5744 & -0.5744 & 0.5744 & 0.5744 & -0.5744 & -0.5744 & 0.5744 & 0.5744 \\ -0.6462 & -0.6462 & -0.6462 & -0.6462 & 0.6462 & 0.6462 & 0.6462 & 0.6462 \\ 0.0359 & 0.0359 & -0.0359 & -0.0359 & -0.0359 & -0.0359 & 0.0359 & 0.0359 \\ -0.0718 & 0.0718 & 0.0718 & -0.0718 & 0.0718 & -0.0718 & -0.0718 & 0.0718 \\ 0.0359 & -0.0359 & 0.0359 & -0.0359 & 0.0359 & -0.0359 & 0.0359 & -0.0359 \end{bmatrix}$	$\lambda_1 \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} + \lambda_2 \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$
In Fig. 3.3(d)	$\begin{bmatrix} -0.6925 & 0.2176 & 0.2703 & -0.6570 & -0.6925 & 0.2176 & 0.2703 & -0.6570 \\ -0.4655 & -0.6675 & 0.6213 & 0.5156 & -0.4655 & -0.6675 & 0.6213 & 0.5156 \\ -0.5512 & -0.7121 & -0.7355 & -0.5500 & 0.5512 & 0.7121 & 0.7355 & 0.5500 \\ 0.0428 & 0.0223 & -0.0571 & -0.0172 & -0.0428 & -0.0223 & 0.0571 & 0.0172 \\ -0.0396 & 0.1048 & 0.0855 & -0.0565 & 0.0396 & -0.1048 & -0.0855 & 0.0565 \\ -0.0204 & -0.0914 & 0.0512 & -0.0324 & -0.0204 & -0.0914 & 0.0512 & -0.0324 \end{bmatrix}$	$\lambda_1 \begin{Bmatrix} 0.7177 \\ 0.4181 \\ 1.1181 \\ -0.1579 \\ 0.5601 \\ 0.5401 \\ 1 \\ 0 \end{Bmatrix} + \lambda_2 \begin{Bmatrix} 1.5346 \\ -0.4598 \\ 1.5839 \\ -1.1181 \\ -0.5791 \\ 1.1763 \\ 0 \\ 1 \end{Bmatrix}$

In Eqn. (3.6), to resist every external wrench $\mathbf{F}_p \in \mathbb{R}^n$, the set of vectors $t_i \mathbf{s}_i$ must positively span $\mathbb{R}^n$. Subsequently, this set is positively dependent (by choosing $\mathbf{F}_p = 0$) such that:

$$\text{rank}(\mathbf{A}) = n \quad \text{and} \quad (3.14)$$

$$\sum_{i=1}^m t_i \mathbf{s}_i = 0 \quad (3.15)$$

Due to the positive tension constraints, i.e. $t_i > 0$, the left-hand side of Eqn. (3.15) possesses a geometric property from its convex combinations and the zero vector in the right-hand side has a relation to the convex combinations. The following theorem shows that the set of all these convex combinations is actually a convex hull, to which the zero

vector belongs.

Theorem 3.1 *Let $C = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ be a finite collection of points in $\mathbb{R}^n$. Then the convex hull of the $\mathbf{x}_i$ ($i = 1, 2, \dots, m$), is the set of all their convex combinations, i.e.*

$$H = co(C) = \left\{ \mathbf{x} \mid \mathbf{x} = \sum_{i=1}^m \lambda_i \mathbf{x}_i, \quad \sum_{i=1}^m \lambda_i = 1, \quad 0 \leq \lambda_i \in \mathbb{R} \text{ for all } i \right\}$$

In order to make use of Theorem 3.1, Eqn. (3.15) is expressed in the following form:

$$\sum_{i=1}^m \frac{t_i}{\sum_{i=1}^m t_i} \mathbf{s}_i = \sum_{i=1}^m \lambda_i \mathbf{s}_i = 0 \quad \text{with } \lambda_i = \frac{t_i}{\sum_{i=1}^m t_i} \quad (3.16)$$

From Theorem 3.1, it can be stated that Eqn. (3.16) forms a convex hull based on the structure matrix $\mathbf{A}$. Furthermore, the origin belongs to that convex hull because its coordinates are the convex combination of $\mathbf{s}_i$. The following proposition shows a utilization of convex hull in satisfying the force-closure condition.

Proposition 3.1 *A moving platform with n degrees of freedom is fully restrained if and only if the convex hull formed by column vectors of the structure matrix $\mathbf{A}$, $co\{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_m\}$, contains a neighbourhood of the origin.*

Proof. *Necessity:* If the platform is fully restrained, a small external wrench $O(\varepsilon)$ can be expressed by Eqn. (3.6) such that:

$$\sum_{i=1}^m t_i \mathbf{s}_i = O(\varepsilon) \quad (3.17)$$

Dividing Eqn. (3.17) by $\sum_{i=1}^m t_i$ gives:

$$\sum_{i=1}^m \frac{t_i}{\sum_{i=1}^m t_i} \mathbf{s}_i = \frac{O(\varepsilon)}{\sum_{i=1}^m t_i} \simeq O(\varepsilon) \quad (3.18)$$

According to theorem 3.1, it is understood that the neighbourhood of the origin belongs to the convex hull.

Sufficiency: If the neighbourhood of the origin $O(\varepsilon)$ is enclosed by the convex hull, it can be expressed:

$$\sum_{i=1}^m t_i \mathbf{s}_i = O(\varepsilon) \quad \text{with} \quad \sum_{i=1}^m t_i = 1 \quad \text{and} \quad \forall t_i > 0 \quad (3.19)$$

Equation (3.19) implies that there always exists a set of positive tensions being able to sustain a small external wrench in any direction. Therefore, the moving platform is fully restrained. ||

With the geometrical tool developed above, the force-closure condition is implemented by establishing distances from the origin to the surfaces of the convex hull and constraining these distances to be positive. However, this approach becomes difficult for cases which the task-space dimension is larger than three, e.g. cases in Fig. 3.3(c) and Fig. 3.3(d). In addition, as the number of cables increases, the convex hull becomes complicated due to the increase of the number of surfaces.

Examples: The geometrical approach is used to check the force-closure condition for the

Figure 3.4: Checking of force-closure condition by the geometrical method

two planar cases in Fig. 3.3(a) and Fig. 3.3(b). In each case, the size of the structure matrix is 3 by 4 as seen in Table 3.1. Each column represents a vector in a 3-dimensional space (f_x , f_y and m_z). There are four vertices S_i corresponding to the four column vectors. If the origin is included in the convex hull formed by the four vertices, then the configuration satisfies that force-closure condition. Otherwise, the force-closure condition is not satisfied. Figure 3.4(a) and Fig. 3.4(b) shows the convex hulls corresponding to the configurations in Fig. 3.3(a) and Fig. 3.3(b), respectively. It is found that the origin is inside the convex hull for Fig. 3.4(a) whereas the origin does not belong to the convex hull in Fig. 3.4(b). Hence, it can be concluded that the configuration in Fig. 3.3(a) satisfies the force-closure condition while that in Fig. 3.3(b) does not. The results are consistent with those obtained using the null space approach.

3.5 Recursive Dimension Reduction Method

Both the null space approach and the geometrical approach have their limitations in checking the force-closure condition when dealing with high dimensional task-spaces and redundantly restrained mechanisms. They become much more complicated when the tension bound is taken into account. However, with the understanding of the geometrical approach, it is noted that if it is possible to reduce the space dimension, the checking of whether the origin belongs to the convex hull can then be implemented more easily. A novel method of recursive dimension reduction for tension analysis is proposed in this section. It will be shown that a high-dimensional system can be systematically decomposed into 1-dimensional subsystems equivalently. As a result, tension analysis in these subsystems can be readily done. Apart from checking the force-closure condition, the proposed technique can also be directly applied for checking both the feasible wrench condition and the wrench set condition. Subsequently, generic recursive algorithms are developed for the three tension conditions, which are fundamental tension analysis tools facilitating workspace analysis, design optimization and dynamic analysis of fully restrained CDPs.

3.5.1 Dimension Reduction

The tension conditions as expressed by Eqns. (3.6)(3.7)(3.8) are transformed into the problem of finding the solution of the linear equation system in Eqn. (3.3) with bounded variables. For fully restrained CDPMs, this is an under-determined system in which there are many sets of tension solutions. However to check the three tension conditions, it is not necessary to obtain all the solutions but important to identify the existence of appropriate solutions. Mathematically, the left-hand side of Eqn. (3.4) is the convex combinations of vectors $\mathbf{s}_i$. For a spatial CDPM, $\mathbf{s}_i$ is a 6-dimensional vector, thus making the geometrical convex analysis approach more difficult. Therefore, the key point in this approach is to reduce the space dimension. It implies that an n -dimensional space is decomposed to some $(n-1)$ -dimensional subspaces through Gaussian eliminations.

Proposition 3.2 *There exists a solution in an original system of n dimensions if and only if there is existence of a solution in all decomposed subsystems of $(n-1)$ dimensions.*

Proof. *Necessity:* Refer to the system as described by Eqn. (3.3), and assume that there is a solution $\mathbf{T}^* = \{t_1^* \ t_2^* \ \cdots \ t_m^*\}^T$. The following equation is obtained. The superscripts refer to the dimension of the space.

$$\begin{bmatrix} a_{1,1}^{(n)} & a_{1,2}^{(n)} & \cdots & a_{1,j}^{(n)} & \cdots & a_{1,m}^{(n)} \\ a_{2,1}^{(n)} & a_{2,2}^{(n)} & \cdots & a_{2,j}^{(n)} & \cdots & a_{2,m}^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n,1}^{(n)} & a_{n,2}^{(n)} & \cdots & a_{n,j}^{(n)} & \cdots & a_{n,m}^{(n)} \end{bmatrix} \begin{bmatrix} t_1^* \\ t_2^* \\ \vdots \\ t_m^* \end{bmatrix} = \begin{bmatrix} b_1^{(n)} \\ b_2^{(n)} \\ \vdots \\ b_n^{(n)} \end{bmatrix} \quad (3.20)$$

According to linear algebra, the solution of above system is not affected under the Gaussian eliminations (Kreyszig [68]). After doing the Gaussian elimination for the column j , Eqn. (3.20) becomes:

$$\begin{bmatrix} a_{1,1}^{(n)} & \cdots & a_{1,j-1}^{(n)} & a_{1,j}^{(n)} & a_{1,j+1}^{(n)} & \cdots & a_{1,m}^{(n)} \\ a_{2,1}^{(n-1)} & \cdots & a_{2,j-1}^{(n-1)} & 0 & a_{2,j+1}^{(n-1)} & \cdots & a_{2,m}^{(n-1)} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n,1}^{(n-1)} & \cdots & a_{n,j-1}^{(n-1)} & 0 & a_{n,j+1}^{(n-1)} & \cdots & a_{n,m}^{(n-1)} \end{bmatrix} \begin{bmatrix} t_1^* \\ t_2^* \\ \vdots \\ t_m^* \end{bmatrix} = \begin{bmatrix} b_1^{(n)} \\ b_2^{(n-1)} \\ \vdots \\ b_n^{(n-1)} \end{bmatrix} \quad (3.21)$$

All the components with superscript $(n-1)$ constitute a subsystem of $(n-1)$ dimensions, $\mathbf{A}_j^{(n-1)} \cdot \mathbf{T}^* = \mathbf{B}_j^{(n-1)}$. It is clear that the solution $\mathbf{T}^*$ is still valid in this subsystem. This is always true for Gaussian elimination of any column. Generally, m columns of the structure matrix $\mathbf{A}$ result in m subsystems of $(n-1)$ dimensions.

Sufficiency: On the other hand, the sufficiency condition can be stated that if at least one subsystem does not have a solution, the original system also does not have the solution.

To do this, the original system $\mathbf{A}^{(n)} \cdot \mathbf{T} = \mathbf{B}^{(n)}$ is written as:

$$\begin{bmatrix} a_{1,1}^{(n)} & a_{1,2}^{(n)} & \cdots & a_{1,m}^{(n)} \\ & \mathbf{A}_j^{(n-1)} & & \end{bmatrix} \cdot \mathbf{T} = \begin{bmatrix} b_1^{(n)} \\ \mathbf{B}_j^{(n-1)} \end{bmatrix} \quad (3.22)$$

With the geometric properties of a linear system, the linear manifold of the subsystem, i.e. the solution set of a finite system of linear equations, is affine (and hence convex). The tension bound $\mathbf{T}$ ($t_{i,min} \leq t_i \leq t_{i,max}$) constitutes a convex hull. The solution of the subsystem, i.e. $\mathbf{A}_j^{(n-1)} \cdot \mathbf{T} = \mathbf{B}_j^{(n-1)}$, is the intersection of the convex set and the convex hull. Therefore, if the intersection is a null, no solution exists in the subsystem. As a result, the whole equivalent system in Eqn. (3.22) does not have a solution and the original system also does not possess a solution. ||

Figure 3.5: Concept of space dimension reduction

The space reduction concept is geometrically illustrated for a 4-3-CDPPM as shown in Fig. 3.5. The original system $\mathbf{A}^{(3)}$ is a 3-dimensional system. The four vectors $(\overrightarrow{OS}_1^{(3)},$

$\overrightarrow{OS}_2^{(3)}$, $\overrightarrow{OS}_3^{(3)}$, and $\overrightarrow{OS}_4^{(3)}$) correspond to the four respective column vectors of the structure matrix. To form the first subspace, a plane that passes through the origin and is perpendicular to the vector $\overrightarrow{OS}_1^{(3)}$ is established. By projecting all the four vectors $\overrightarrow{OS}_i^{(3)}$ on this plane results in only three non-zero vectors $\overrightarrow{OS}_2^{(2)}$, $\overrightarrow{OS}_3^{(2)}$ and $\overrightarrow{OS}_4^{(2)}$. As a result of this projection, a subsystem $\mathbf{A}_1^{(2)}$ has been formed in a 2-dimensional space. If a solution exists in the original system, then this solution must also be valid in this subsystem and therefore the existence of solution is required in this subsystem. Similarly, if the four vectors are projected on planes which are perpendicular to $\overrightarrow{OS}_2^{(3)}$, $\overrightarrow{OS}_3^{(3)}$ and $\overrightarrow{OS}_4^{(3)}$ respectively, the other three 2-dimensional subsystems can also be obtained. The existence of solution is also required in these subsystems.

From the above illustration, it is clear that if the zero column $\overrightarrow{OS}_1^{(2)}$ is eliminated in the subsystem, the subsystem will have one row and one column fewer than the original one. Hence, the same technique can be repeated for each subsystem to further reduce the space dimension. This reduction process can be repeated until the 1-dimensional systems are obtained. This is because checking the existence of solutions in 1-dimensional systems is so simple that only scalar operations are required.

3.5.2 Tension Conditions in 1-Dimensional Systems

In this section, the interpretation of the three tension conditions in 1-dimensional systems are described. In general, the 1-dimensional systems which are decomposed from the original system through Gaussian eliminations, can be expressed as follows:

$$\sum_{k=1}^i \mathbf{f}_k^{(+)} + \sum_{k=1}^j \mathbf{f}_k^{(-)} = \mathbf{f}_p \quad (3.23)$$

where $\mathbf{f}_k^{(+)}$ and $\mathbf{f}_k^{(-)}$ are vectors pointing in the positive direction and the negative direction, respectively. Its magnitude is proportional to the tension range of the respective cable. The right side of Eqn. (3.23), $\mathbf{f}_p$, is an equivalent required wrench in 1-dimensional systems which will be addressed in the next Section.

With the given notations, the checking of force closure in a 1-dimensional system can be represented by Figure 3.6.

Figure 3.6: Checking of force-closure condition

For the force-closure condition, there must be at least two opposite components such that:

$$\sum_{k=1}^j \mathbf{f}_k^{(-)} \cdot \sum_{k=1}^i \mathbf{f}_k^{(+)} < 0 \quad (3.24)$$

The external wrench is not considered under force closure as it is assumed that the cable tensions can always be increased to balance the system.

Figure 3.7: Checking of feasible wrench condition

Similarly, Fig. 3.7 represents the problem of the wrench feasibility. The direction of the equivalent external wrench depends on its sign which is assumed to be positive. It shows that components which are opposite to the external wrench $\mathbf{f}_p$ play actively to resist this external wrench. The other components are inactive but they contribute to the internal equilibrium among the cables. Here, the mathematical condition can be stated as follows:

$$\begin{cases} \sum_{k=1}^j f_{k,max}^{(-)} \geq \sum_{k=1}^i f_{k,min}^{(+)} + f_p \\ \sum_{k=1}^j f_{k,min}^{(-)} \leq \sum_{k=1}^i f_{k,max}^{(+)} + f_p \end{cases} \quad (3.25)$$

Figure 3.8: Checking of wrench set condition

Figure 3.8 shows an example of checking the wrench set condition in 1-dimensional systems. It is expected that the wrench set requirement will be a combination of the force closure and the wrench feasibility:

$$\left\{ \begin{array}{l} \sum_{k=1}^j f_{k,max}^{(-)} \geq \sum_{k=1}^i f_{k,min}^{(+)} + f_p \\ \sum_{k=1}^j f_{k,min}^{(-)} \leq \sum_{k=1}^i f_{k,max}^{(+)} + f_p \\ \sum_{k=1}^i f_{k,max}^{(+)} \geq \sum_{k=1}^j f_{k,min}^{(-)} + f_p \\ \sum_{k=1}^i f_{k,min}^{(+)} \leq \sum_{k=1}^j f_{k,max}^{(-)} + f_p \end{array} \right. \quad (3.26)$$

In this case, the equivalent external wrench exerts a minimum value in both directions. Nevertheless, the checking steps are also based on the scalar calculation and comparison given in Eqn. (3.26).

In Figs. 3.6, 3.7 and 3.8, the direction, the minimum value, and the maximum value of vectors $\mathbf{f}_j^{(-)}$, $\mathbf{f}_i^{(+)}$ are known. Respective conditions in Eqns. (3.24), (3.25) and (3.26) show that checking the force-closure condition, the feasible wrench condition as well as the wrench set condition of 1-dimensional systems is simple and straightforward. Hence, it is recommended that the dimension reduction process is repeated until 1-dimensional systems are obtained. If the original system is an n -dimensional system with m columns, the number of 1-dimensional systems, N_{1-D} , is calculated as follows:

$$N_{1-D} = \frac{m!}{[m - (n - 1)]!} \quad (3.27)$$

Figure 3.9: Number of 1-dimensional systems

For fully restrained CDPMs, the number of 1-dimensional systems is calculated from Eqn. (3.27) and shown in Fig. 3.9. Clearly, systems with higher task-space dimension result in many 1-dimensional systems. Besides, using more cables also considerably increases the number of subsystems in $\mathbb{R}^1$. Typically, for a 4-3-CDPPM, the number of subsystems in $\mathbb{R}^1$ is 12 whereas the number is 6720 for a 8-6-CDSPPM.

3.5.3 Equivalent Required Wrench in 1-Dimensional Systems

During the reduction process, the right side of the system is also changed according to Gaussian eliminations. In order to check the feasible wrench condition in Eqn. (3.25) and the wrench set condition in Eqn. (3.26), the equivalent wrench value f_p must be determined. The original linear system can be written in the following form:

$$\begin{bmatrix} a_{1,1}^{(n)} & a_{1,2}^{(n)} & \cdots & a_{1,n}^{(n)} & \cdots & a_{1,m}^{(n)} \\ a_{2,1}^{(n)} & a_{2,2}^{(n)} & \cdots & a_{2,n}^{(n)} & \cdots & a_{2,m}^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n,1}^{(n)} & a_{n,2}^{(n)} & \cdots & a_{n,n}^{(n)} & \cdots & a_{n,m}^{(n)} \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{Bmatrix} = \begin{Bmatrix} b_1^{(n)} \\ b_2^{(n)} \\ \vdots \\ b_n^{(n)} \end{Bmatrix} \quad (3.28)$$

After decomposition, it can be expressed as:

$$\begin{bmatrix} a_{1,1}^{(n)} & a_{1,2}^{(n)} & \cdots & a_{1,n}^{(n)} & \cdots & a_{1,m}^{(n)} \\ 0 & a_{2,2}^{(n-1)} & \cdots & a_{2,n}^{(n-1)} & \cdots & a_{2,m}^{(n-1)} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_{n,n}^{(1)} & \cdots & a_{n,m}^{(1)} \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{Bmatrix} = \begin{Bmatrix} b_1^{(n)} \\ b_2^{(n-1)} \\ \vdots \\ b_n^{(1)} \end{Bmatrix} \quad (3.29)$$

The last row is the 1-dimensional system and the equivalent wrench is expressed by $b_n^{(1)}$. This component is not considered for the force-closure condition and is a constant for the feasible wrench condition. However, for the wrench set condition, it is a function of force components and moment components and can be expressed as:

$$b_n^{(1)} = (c_{f,x}f_{p,x} + c_{f,y}f_{p,y} + c_{f,z}f_{p,z}) + (c_{m,x}m_{p,x} + c_{m,y}m_{p,y} + c_{m,z}m_{p,z}) \quad (3.30)$$

where $c_{f,x}, c_{f,y}, c_{f,z}, c_{m,x}, c_{m,y}$ and $c_{m,z}$ are coefficients calculated from Gaussian eliminations; $f_{p,x}, f_{p,y}, f_{p,z}, m_{p,x}, m_{p,y}$ and $m_{p,z}$ are values obtained from the required wrench set $(f_{p,r}, m_{p,r})$.

The maximum absolute value of $b_n^{(1)}$ that linearly depends on the required force set and the required moment set needs to be calculated. Figure 3.10 illustrates the required force set as a circle for the planar case or as a sphere for the spatial case of a radius $f_{p,r}$. For the spatial case, the first three components in Eqn. (3.30) defines a plane and the

(a) Planar cases

Figure 3.10: Required force set and equivalent required force

maximum value $f_{p,max}$ is reached when the plane is tangential to the sphere as shown in Fig. 3.10(b). Similarly, the last three components in Eqn. (3.30), which are related to the required moment set, will define the maximum value $m_{p,max}$. Summation of $f_{p,max}$ and $m_{p,max}$ is the equivalent required wrench in the respective 1-dimensional system. Since Eqn. (3.30) is linear, the equivalent wrench value is readily computed for different geometries of the required wrench set.

3.5.4 Recursive Algorithms

The recursive algorithms, for checking the tension conditions based on Gaussian elimination, consist of two major procedures:

- **One-dimension case:** This is the simplest case in which the number of rows is one. In the proposed algorithm, if the tension condition is satisfied, it returns value '1', otherwise it returns value '0'. Equations (3.24), (3.25) and (3.26) are used for each of the conditions, respectively.
- **Higher-dimension case:** In this case, the number of rows is greater than one. Reduc-

ing space dimension is done by Gaussian elimination until 1-dimensional systems. If any subsystem has the returned value '0', the original system does not satisfy the respective condition.

Three algorithms which are called FCC (Force-Closure Check), FWC (Feasible Wrench Check) and WSC (Wrench Set Check) are formulated and coded in Matlab. The flow diagrams of the algorithms are shown in Fig. 3.11, Fig. 3.12 and Fig. 3.13.

Figure 3.11: Recursive algorithm “Force-Closure Check” - FCC

Figure 3.12: Recursive algorithm "Feasible Wrench Check" - FWC

Figure 3.13: Recursive algorithm "Wrench Set Check" - WSC

For ease of programming, the Gaussian elimination is always done for the first column. A column index j is used to indicate the increase of the column order, and the column j is then swapped with the first column. The input information includes the structure matrix $\mathbf{A}$, the task-space configuration $\mathbf{W}_c$, the tension bound $\mathbf{T}_r$, the constant external wrench $\mathbf{F}_p$ and the wrench set requirement $\mathbf{W}_r$. The Matlab scripts for these algorithms are given in Appendix E.

Computational example: The two samples of convex hull as shown Fig. 3.14 are used to demonstrate the process of checking whether the space origin is inside the convex hull.

Figure 3.14: Two samples of convex hull

For Fig. 3.14(a), if the three vectors $(\overrightarrow{OS_i})$ are projected onto the line which is perpendicular to any vector $\overrightarrow{OS_i}$, there always exist both a positive component and a negative component. Hence, the origin belongs to the convex hull in this case. On the other hand for Fig. 3.14(b), if the three vectors are projected onto the line which is perpendicular to $\overrightarrow{OS_1}$ or $\overrightarrow{OS_3}$, all components will have the same sign. This means that the origin does not belong to the convex hull. For both cases, the original convex hulls are in the space of $\mathbb{R}^2$. When projected onto the lines, the original problem is equivalent to the three new convex hulls which are expressed as line segments in the subspace of $\mathbb{R}^1$. The step by step checking of the two examples are shown in Fig. 3.15 and Fig. 3.16.

Application to 4-3-CDPPMs and 8-6-CDSPMs: With the FCC recursive algorithm,

Figure 3.15: Computational steps for the case in Fig. 3.14(a)

Figure 3.16: Computational steps for the case in Fig. 3.14(b)

it shows that the two configurations in Fig. 3.3(a) and 3.3(c) satisfy the force-closure condition whereas the other two in Fig. 3.3(b) and 3.3(d) do not satisfy the condition. With a computer configuration of Pentium 4 and CPU 2.41 GHz, computation time

Table 3.2: Computational results using the recursive dimension reduction approach

	3-D systems	2-D systems	1-D systems	FCC
In Fig. 3.3(a)	$\begin{bmatrix} -0.6585 & 0.6585 & 0.6585 & -0.6585 \\ -0.7526 & -0.7526 & 0.7526 & 0.7526 \\ 0.0470 & -0.0470 & 0.0470 & -0.0470 \end{bmatrix}$	$\begin{bmatrix} 1.3170 & 0.0000 & -1.3170 \\ -0.0940 & 0.0940 & 0.0000 \end{bmatrix}$	$[0.0940 \quad -0.0940]$	1
			$[1.3170 \quad -1.3170]$	
			$[0.0940 \quad -0.0940]$	
		$\begin{bmatrix} -1.3170 & 1.3170 & 0.0000 \\ 0.0940 & 0.0000 & -0.0940 \end{bmatrix}$	$[0.0940 \quad -0.0940]$	
			$[0.0940 \quad -0.0940]$	
			$[1.3170 \quad -1.3170]$	
		$\begin{bmatrix} 1.3170 & 0.0000 & -1.3170 \\ 0.0000 & 0.0940 & -0.0940 \end{bmatrix}$	$[0.0940 \quad -0.0940]$	
			$[1.3170 \quad -1.3170]$	
			$[0.0940 \quad -0.0940]$	
		$\begin{bmatrix} 0.0000 & 1.3170 & -1.3170 \\ -0.0940 & 0.0940 & 0.0000 \end{bmatrix}$	$[1.3170 \quad -1.3170]$	
			$[-0.0940 \quad 0.0940]$	
			$[0.0940 \quad -0.0940]$	
In Fig. 3.3(b)	$\begin{bmatrix} -0.8299 & 0.3100 & 0.3990 & -0.7867 \\ -0.5579 & -0.9507 & 0.9170 & 0.6174 \\ -0.0244 & -0.1302 & 0.0756 & -0.0388 \end{bmatrix}$	$\begin{bmatrix} -1.1591 & 0.6488 & 1.1463 \\ -0.1393 & 0.0639 & -0.0157 \end{bmatrix}$	$[-0.0141 \quad -0.1534]$	0
			$[-0.0252 \quad -0.1285]$	
			$[0.0727 \quad -0.1552]$	
		$\begin{bmatrix} -1.0118 & 0.6980 & -0.5854 \\ 0.0520 & -0.0500 & -0.1234 \end{bmatrix}$	$[-0.0141 \quad -0.1534]$	
			$[-0.0205 \quad -0.1653]$	
			$[-0.1971 \quad 0.2652]$	
		$\begin{bmatrix} 0.7237 & -0.5871 & -1.0553 \\ -0.0518 & 0.0216 & -0.0897 \end{bmatrix}$	$[-0.0205 \quad -0.1653]$	
			$[-0.0252 \quad -0.1285]$	
			$[0.0715 \quad -0.1133]$	
		$\begin{bmatrix} -0.7074 & 1.2301 & -1.2092 \\ -0.1455 & 0.0559 & 0.0165 \end{bmatrix}$	$[-0.1971 \quad 0.2652]$	
			$[-0.1133 \quad 0.0715]$	
			$[0.0727 \quad -0.1552]$	

to complete the FCC algorithm is approximately 1 ms for the 4-3-CDPPM and 1.15 s for the 8-6-CDSPM that the returned result is ‘1’. It is noted that the computing time considerably increases for the 8-6-CDSPM because a large number of 1-dimensional subsystems, i.e. 6720 cases for the 8-6-CDSPM, are checked. However, for configurations unsatisfied the force-closure condition, the computation time is much smaller. In the case of 8-6-CDSPM, there are a large number of 1-dimensional systems (6720) and it is not practical to list the complete computational procedure for this architecture. Hence in Table 3.2, only the subsystems for the two cases of the 4-3-CDPPM are listed step-by-step. It is seen that each 3-dimensional original system is decomposed into 12 1-dimensional

subsystems. The case in Fig. 3.3(a) satisfies the force-closure condition because all 1-dimensional subsystems meet the force-closure condition. For the case in Fig. 3.3(b), the force-closure condition is not satisfied because at least one 1-dimensional subsystem does not meet the force-closure condition. The results are consistent with those obtained by the null space approach and the geometrical approach.

Algorithm remarks: For the recursive algorithms in Figs. 3.11 - 3.13, there is a concern of computational errors in critical situations, e.g. when the configuration is a singularity configuration. In this case, the left-hand side of the last row in Eqn. (3.29), 1-dimensional system, is expected to be zero. To cope with the computational errors, Gaussian elimination with pivoting is employed in these algorithms to minimize the cumulative effect of rounding error in dealing with linear systems (Anton [69]). With this implementation, the algorithms are found to be stable and reliable for checking the tension conditions and detecting singularity configurations.

3.6 Simulation Results

In this section, the algorithms are used to analyze the tension for some special cases. The purpose is to illustrate the versatility of the proposed recursive dimension reduction method.

(a) Point-mass system

Firstly, a point-mass system as shown in Fig. 3.17 is used to illustrate that the force-closure condition can be easily verified with the proposed algorithm FCC. In this system, four cables are used to constrain the point-mass. Here, the directions of the cables will form four column vectors of the structure matrix $\mathbf{A}$, respectively. Similar to the examples in Fig. 3.14, it can be deduced that the force-closure condition is always satisfied as long as point P belongs to the convex polygon formed by the four base points B_i . The convex

Figure 3.17: A point-mass fully restrained system

hull formed by the four base points also corresponds to a workspace which is a set of poses having a force closure. This result is the same as that discussed by William [53]. Generally, this result can be extended to any point-mass CDPM.

(b) Range of external force

The possible range of external force acting on the platform can also be computed using the feasible wrench condition. The 8-6-CDSPM whose kinematic parameters are shown in Fig. 1.7(b) is taken as an example. Given that $t_{min} = 5$ N and $t_{max} = 150$ N are for all cables and the platform weighs 0.5 kg, what is the range of external force acting on the moving platform that can be sustained by the cables?

Figure 3.18 shows the range of external forces (in x and y directions) at various postures of the platform by using the algorithm FWC. It is noted that the possible external force ranges decrease at higher orientation angles and the external force ranges are ‘skewed’ as the platform moves far away from the base centre. This indicates that the workspace will be smaller at higher orientations and that the platform can resist an arbitrary external force better when it is located around the base centre. Furthermore, it is noted that all the boundaries in Fig. 3.18 are lines. This is to be expected as they are the intersection of the force plane (f_x - f_y) and the parallelepiped constructed by the convex combination of $\sum_{i=1}^m t_i \mathbf{s}_i$ whose boundaries are planes.

(a) $x = 0.50, y = 0.50, \alpha = 0^0$

(b) $x = 0.50, y = 0.50, \alpha = 2^0$

(c) $x = 0.75, y = 0.50, \alpha = 0^0$

(d) $x = 0.75, y = 0.50, \alpha = 2^0$

(e) $x = 0.50, y = 0.75, \alpha = 0^0$

(f) $x = 0.50, y = 0.75, \alpha = 2^0$

Figure 3.18: External force ranges on horizontal planes (X-Y) ($\beta = 0^0, \gamma = 0^0, z = 0.5$)

(c) Static tension analysis

So far, the recursive dimension reduction method is applied to fully restrained CDPMs. This example will elaborate its application in IRPMs. Figure 3.19 shows an IRPM in which two cables with tension bounds resist an external downward force to keep the point-mass in balance. The objective is to determine the range of the external force that can be sustained by the two cables B_1P and B_2P .

Figure 3.19: An incompletely restrained system in static balance

Based on the free body diagram at the point mass P, the following equation is obtained:

$$\begin{bmatrix} -\cos \alpha & \cos \alpha \\ \sin \alpha & \sin \alpha \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_p \end{Bmatrix} \quad (3.31)$$

This is a 2-dimensional system and by reducing the space dimension, two 1-dimensional systems are formed for the two column vectors. They are:

$$2t_2 \sin \alpha = f_p \quad \text{and} \quad (3.32)$$

$$2t_1 \sin \alpha = f_p \quad (3.33)$$

With the given tension bounds, it is easy to show that: $2t_{2,\min} \sin \alpha \leq f_p \leq 2t_{1,\max} \sin \alpha$.

(d) Dynamic equilibrium

Figure 3.20 is another example of IRPMs in which the maximum instantaneous horizontal acceleration needs to be determined so that the point-mass can still be dynamically

balanced by the two cables B_1P and B_2P .

Tension bounds:

$$\begin{aligned} t_{1,\min} &= t_{2,\min} = 3 \text{ (N)} \\ t_{1,\max} &= t_{2,\max} = 30 \text{ (N)} \end{aligned}$$

Others: $m = 1 \text{ (kg)}$

$$\alpha = 30^\circ$$

What is the range of a ?

Figure 3.20: An incompletely restrained system in dynamic balance

By analyzing the force relation on the point-mass, the following equation can be obtained.

$$\begin{bmatrix} -\cos \alpha & \cos \alpha \\ \sin \alpha & \sin \alpha \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} = \begin{Bmatrix} -m.a \\ m.g \end{Bmatrix} \quad (3.34)$$

Similarly, reducing space dimension leads to two respective 1-dimension systems:

$$2t_2 \sin \alpha = m.g - m.a. \tan \alpha \quad \text{and} \quad (3.35)$$

$$2t_1 \sin \alpha = m.g + m.a. \tan \alpha \quad (3.36)$$

Figure 3.21: Plot of acceleration with respect to the tension

With the given parameters in Fig. 3.20, Fig. 3.21 shows the change of acceleration with respect to the tension range. In order to satisfy the two conditions in Eqns. (3.35) and (3.36) simultaneously, the acceleration a must be kept in a range from -11.8 to 11.8 m/s^2 .

3.7 Summary

In this chapter, the important issue of CDPMs in determining tension conditions such that there is at least one appropriate set of positive cable tensions to maintain the pose of the moving platform is addressed.

Three tension conditions for CDPMs are identified and formulated. They are the force-closure condition, the feasible wrench condition and the wrench set condition in which the tension bound is taken into account. The null space approach and the geometrical approach are introduced and formulated. It is seen that for CDPMs with lower task-space dimension, the approaches enables geometrical information to be interpreted. However, both approaches becomes very complex when checking the tension conditions in high-dimensional redundant systems. As a result, a recursive dimension reduction method is proposed to decompose an n -dimensional system into 1-dimensional subsystems. Three algorithms FCC, FWC and WSC are developed to check the three tension conditions, respectively. Computational examples show that the proposed algorithms are effective.

In addition, the proposed algorithms can be applied to both incompletely restrained CDPMs ($m < n + 1$) and fully restrained CDPMs ($m \geq n + 1$). The versatility of the proposed recursive dimension reduction technique is demonstrated through simulation results. The tension conditions can also be employed to constrain the platform dynamically in which inertial components are considered as an instantaneous wrench acting on the platform.

In conclusion, the three tension checking algorithms can be used as fundamental tools in analyzing the performance of the CDPMs such as workspace analysis and dynamic analysis.

Chapter 4

Workspace Analysis and Design Optimization

4.1 Introduction

For a parallel mechanism with rigid links, its workspace is the space where the inverse/forward kinematic solutions exist. However for a CDPM, its workspace is the space where sets of positive cable tensions exist because positive cable tensions are required to restrain and drive the moving platform.

With different approaches to the study of tension and depending on the purpose of applications, several terminologies of workspace have been used. For instance, a statically reachable workspace which is defined as the set of all end-effector poses that can be reached statically was addressed by Fattah [70]. A controllable workspace which is a set of postures where forces and torques at the end-effector can be controlled was studied by Verhoeven [71, 72], Tadokoro [31], Zheng [73]. Taking dynamics into consideration, Gosselin [55] defined a dynamic workspace in which its shape depends on the accelerations of the end-effector. Recently, Bosscher [57] and Ebert-Uphoff [59] summarized the basic workspace terminologies for cable-driven robots and also geometrically explored the wrench-feasible workspace by visualizing a net wrench set as a hyper-parallelogram. Be-

cause the tension analysis is not unified, the workspaces are generated differently. In some particular mechanisms, the workspace boundaries can be derived analytically, enabling the workspace to be quantified precisely. However, for a general CDPM, this is almost impossible, especially when the tension bound is included.

Besides workspace quantification, it is also important to determine the quality of the workspace as a mechanism designed for a maximum workspace may lead to undesirable kinematic characteristics such as poor dexterity or manipulability. For rigid-link mechanisms, a variety of workspace quality indices have been used such as manipulability, dexterity, condition number and stiffness. Most of these measures are related to one other as they all depend on the conditioning of the Jacobian matrix. A more commonly used quality index is the global condition index (GCI) which is a measure of the kinematic dexterity of the manipulator over the whole workspace (Gosselin [74], Yang [75]). Large values of GCI ensure good performance with respect to force and velocity transmission. In essence, if the Jacobian matrix of a mechanism is always well-conditioned, i.e. far away from singularities, the mechanism will always have a high quality workspace regardless of the quality indices used. Pusey [32] employed the GCI to study the quality of a 6-6 cable suspended parallel robot. However, the above indices do not reflect the tension state of the cables. A slope-based measure was proposed to study the stability of a pose for an incompletely restrained CDPM (Bosscher [76]). For completely restrained CDPMs, Kurtz [77] proposed a new set of measures based on the unilateral statics including unilateral dexterity, unilateral singularity, unilateral isotropy, unilateral maximum force amplification and unilateral maximum dexterity gradient to replace the commonly used concepts of dexterity, singularity, isotropy, maximum force amplification and maximum dexterity gradient, respectively. Because the workspace of CDPMs are mainly limited by tension conditions, a tension factor, which reflects the tension distribution among cables, is considered as an appropriate criterion to evaluate the performance at a specific pose (Verhoven [78], Shen [79]). However, the tension factor has not thoroughly investigated for fully restrained CDPMs in evaluating the performance over the

entire workspace.

Similar to parallel mechanisms, workspace analysis and workspace performance give information to get better designs of mechanisms. For CDPMs, the unilateral property of cables further restricts the already limited and complex workspace. Various efforts have been conducted to enlarge the workspace by looking at the number of cables, the configurations of the connecting points (Fattah [33], Pham [80], Kurbanhusen [81]). Riechel [34] analyzed the characteristics and trends of the force-feasible workspace under the effects of design parameters. These analyses have not been explored thoroughly for fully restrained CDPMs.

In the following sections, the workspace analysis will be presented first. Both the analytical methods and the numerical methods will be used to generate workspaces. Constraining cable tensions by the null space method and the geometrical method is very useful to get the workspace boundaries analytically in certain cases. The symbolic expressions of the workspace boundaries help designers to have more understanding about workspaces. Then, based on tension conditions, generic algorithms are developed to generate and to quantify the workspace. Tension distribution is also studied, in which a tension factor (TF) is proposed as a performance index to evaluate the quality of force closure for CDPMs at a specific configuration. It is found to be an appropriate measure of the positive tension condition of the structure matrix. In addition, it will be proved that a basically nonlinear optimization of TF can be converted to a linear optimization problem, which can be efficiently computed from linear programming approaches. Subsequently, the criterion of global tension index (GTI) is proposed to evaluate performance of the entire workspace. Finally, design problems of CDPMs with optimized workspace properties are investigated.

4.2 Workspace Analysis

4.2.1 Workspace Definitions

Based on the three tension conditions developed, two types of workspaces namely force-closure workspace and feasible wrench workspace are defined for fully restrained CDPMs.

- Force-closure workspace (FCWS): a set of postures where the force-closure condition is satisfied. This workspace is generated by the force-closure condition. The FCWSs depend only on the geometrical parameters of the base and the moving platform.
- Feasible wrench workspace (FWWS): a set of postures that satisfies both the constant wrench feasibility and the wrench set requirement. This workspace is not only dependent on the geometrical parameters of the mechanism, but also dependent on the tension ranges, gravitational effect, dynamic load and the required wrench set acting on the platform.

For planar mechanisms, the workspace is represented by x, y and ϕ . So, it can be visualized with three dimensional coordinate system. However, for a spatial mechanism with six DOFs (x, y, z, α, β and γ), it is common to use subsets of the six independent parameters for representing the workspace. In addition, CDPMs are well known for large reachable workspaces. Hence, the constant orientation workspace, which is a set of locations of the moving platform when its orientation is fixed, is commonly used. For the purpose of realizing all DOFs, a total orientation workspace is also used. It is the set of locations of the reference point that can be reached with all the orientations among a set defined by ranges on the orientation parameters (Merlet [37]). In this Chapter, both the constant orientation workspace and the total orientation workspace will be employed for the workspace presentation.

4.2.2 Analytical Workspace Generation

In this section, three approaches including the null space approach, the geometrical approach and an instantaneous centre approach are presented to generate the constant orientation force-closure workspace for a 4-3-CDPPM. The first two approaches focus on manipulating ability of the cables on the moving platform, while the other is mainly for cases where the cables lose their constraining ability. Although these approaches cannot be generally applied to any CDPPMs, they enhance the workspace analysis in 4-3-CDPPMs.

(a) Instantaneous centre approach

This approach is used to analyze the out-of-control configurations in which the cables are unable to constrain the moving platform at a specific pose under external wrenches. In an out-of-control configuration, the moving platform will still possess instantaneous motions in certain directions even when all the cables are locked. The approach provides an insightful geometrical description of such configurations and it will be shown that the workspace of the cable-driven mechanism can also be obtained. The instantaneous centre is defined as a common point of two rigid bodies (or the two rigid bodies' extended parts) at which the two bodies have the same velocity at an instant of time. Hence, the two rigid bodies have no relative motion at their instantaneous centre. It follows that if the platform has no instantaneous centre with respect to the base, then it has no instantaneous mobility and the cables are able to constrain the platform.

If the line equation for each cable l_i , which goes through the suspending point B_i and the connecting point P_i , is given by: $a_i u + b_i v + c_i = 0$ ($i = 1, 2, 3, 4$), the force-closure workspace of a 4-3-CDPPM is a set postures that satisfy the following expressions (see Appendix C for more details):

$$\begin{cases} \prod_{i=1}^4 \Delta_i \neq 0 \\ \sum_{i=1}^4 |\Delta_i| = \left| \sum_{i=1}^4 \Delta_i \right| \end{cases} \quad (4.1)$$

where

$$\Delta_1 = - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}, \Delta_3 = - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{vmatrix}, \Delta_4 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(b) Null space approach

Using the null space approach, the force-closure condition has been formulated as given by Eqn. (3.12). Therefore, for 4-3-CDPPMs, the force-closure workspace can also be constructed by applying the same condition:

$$\begin{cases} \prod_{i=1}^4 n_i \neq 0 \\ \sum_{i=1}^4 |n_i| = \left| \sum_{i=1}^4 n_i \right| \end{cases} \quad (4.2)$$

where n_i ($i = 1, 2, 3, 4$) is the determinant of the matrix formed by deleting the i th column of the structural matrix $\mathbf{A}$. They can be expressed as follows for 4-3-CDPPMs:

$$\begin{aligned} n_1 &= - \begin{vmatrix} u_{2,x} & u_{2,y} & (r_{2,x}u_{2,y} - r_{2,y}u_{2,x}) \\ u_{3,x} & u_{3,y} & (r_{3,x}u_{3,y} - r_{3,y}u_{3,x}) \\ u_{4,x} & u_{4,y} & (r_{4,x}u_{4,y} - r_{4,y}u_{4,x}) \end{vmatrix}; & n_2 &= \begin{vmatrix} u_{1,x} & u_{1,y} & (r_{1,x}u_{1,y} - r_{1,y}u_{1,x}) \\ u_{3,x} & u_{3,y} & (r_{3,x}u_{3,y} - r_{3,y}u_{3,x}) \\ u_{4,x} & u_{4,y} & (r_{4,x}u_{4,y} - r_{4,y}u_{4,x}) \end{vmatrix} \\ n_3 &= - \begin{vmatrix} u_{1,x} & u_{1,y} & (r_{1,x}u_{1,y} - r_{1,y}u_{1,x}) \\ u_{2,x} & u_{2,y} & (r_{2,x}u_{2,y} - r_{2,y}u_{2,x}) \\ u_{4,x} & u_{4,y} & (r_{4,x}u_{4,y} - r_{4,y}u_{4,x}) \end{vmatrix}; & n_4 &= \begin{vmatrix} u_{1,x} & u_{1,y} & (r_{1,x}u_{1,y} - r_{1,y}u_{1,x}) \\ u_{2,x} & u_{2,y} & (r_{2,x}u_{2,y} - r_{2,y}u_{2,x}) \\ u_{3,x} & u_{3,y} & (r_{3,x}u_{3,y} - r_{3,y}u_{3,x}) \end{vmatrix} \end{aligned}$$

(c) Geometrical approach

Using this approach, the frame origin for a particular pose of the platform must be inside the convex hull of a set of S_i ($i = 1, 2, 3, 4$) so that the force-closure condition is satisfied. Firstly, assume that the platform is in a particular pose so that the frame origin is inside the convex hull as shown in Fig. 4.1. The convex hull is created by the column vectors $\mathbf{s}_i$ of the structure matrix $\mathbf{A}$. The boundaries of the FCWS are then defined by the four perpendicular distances from the frame origin to the four surfaces of the convex hull. Hence, if the frame origin falls on the surface, the edge or the vertices of the convex hull, then the moving platform is on the boundary of the FCWS.

Figure 4.1: Convex hull formed at a particular pose of the moving platform

Distance vectors are expressed as $\mathbf{h}_i = h_i \mathbf{n}_i$, whose magnitude h_i represents the distance from the origin to a surface and the unit normal vector $\mathbf{n}_i$ is pointing outwards from the respective surfaces. Let the vertices be denoted by $S_i = (s_{i,x}, s_{i,y}, s_{i,z})$ ($i = 1, 2, 3, 4$) and their coordinates are defined with respect to the axis f_x, f_y and m_z . With these notations, the four normal vectors can be written as:

$$\begin{aligned} \mathbf{n}_1 &= \frac{\overrightarrow{S_2S_4} \times \overrightarrow{S_2S_3}}{\|\overrightarrow{S_2S_4} \times \overrightarrow{S_2S_3}\|} : \text{unit normal vector of the surface } (S_2S_3S_4) \\ \mathbf{n}_2 &= \frac{\overrightarrow{S_1S_3} \times \overrightarrow{S_1S_4}}{\|\overrightarrow{S_1S_3} \times \overrightarrow{S_1S_4}\|} : \text{unit normal vector of the surface } (S_1S_3S_4) \\ \mathbf{n}_3 &= \frac{\overrightarrow{S_1S_4} \times \overrightarrow{S_1S_2}}{\|\overrightarrow{S_1S_4} \times \overrightarrow{S_1S_2}\|} : \text{unit normal vector of the surface } (S_1S_2S_4) \\ \mathbf{n}_4 &= \frac{\overrightarrow{S_1S_2} \times \overrightarrow{S_1S_3}}{\|\overrightarrow{S_1S_2} \times \overrightarrow{S_1S_3}\|} : \text{unit normal vector of the surface } (S_1S_2S_3) \end{aligned} \quad (4.3)$$

Subsequently, distance h_i is the length of the vector projection of any $\overrightarrow{OS_j}$ ($j \neq i$) on $\mathbf{n}_i$.

For instance:

$$h_1 = \overrightarrow{OS_2} \cdot \mathbf{n}_1; \quad h_2 = \overrightarrow{OS_1} \cdot \mathbf{n}_2; \quad h_3 = \overrightarrow{OS_1} \cdot \mathbf{n}_3; \quad h_4 = \overrightarrow{OS_1} \cdot \mathbf{n}_4 \quad (4.4)$$

It is noted that the denominator of h_i is the vector magnitude and is always greater than zero. Therefore, with $h_i \geq 0$, Eqns. (4.3) and (4.4) can be simplified into the following

expressions:

$$\begin{aligned}
 h_1 &= s_{2x}(s_{4y}s_{3z} - s_{4z}s_{3y}) + s_{2y}(s_{4z}s_{3x} - s_{4x}s_{3z}) + s_{2z}(s_{4x}s_{3y} - s_{4y}s_{3x}) \geq 0 \\
 h_2 &= s_{1x}(s_{4z}s_{3y} - s_{4y}s_{3z}) + s_{1y}(s_{4x}s_{3z} - s_{4z}s_{3x}) + s_{1z}(s_{4y}s_{3x} - s_{4x}s_{3y}) \geq 0 \quad (4.5) \\
 h_3 &= s_{1x}(s_{4y}s_{2z} - s_{4z}s_{2y}) + s_{1y}(s_{4z}s_{2x} - s_{4x}s_{2z}) + s_{1z}(s_{4x}s_{2y} - s_{4y}s_{2x}) \geq 0 \\
 h_4 &= s_{1x}(s_{2y}s_{3z} - s_{2z}s_{3y}) + s_{1y}(s_{2z}s_{3x} - s_{2x}s_{3z}) + s_{1z}(s_{2x}s_{3y} - s_{2y}s_{3x}) \geq 0
 \end{aligned}$$

The above inequalities can be used to determine whether the pose of the platform is inside or outside the FCWS. It is noted that there is a relation between the conditions in Eqns. (4.1), (4.2) and (4.5). If one of the distances in Eqn. (4.5) is zero e.g. $h_1 = 0$, then the three vectors $\overrightarrow{OS_2}$, $\overrightarrow{OS_3}$ and $\overrightarrow{OS_4}$ are on the same plane. Consequently, the three vectors are also linearly dependent, making the component Δ_1 in Eqn. (4.1) and the component n_1 in Eqn. (4.2) to be zero. Hence, if any of the three terms Δ_i , n_i and h_i is zero, the others will also be zero.

All vertices S_i of the convex hull are dependent on locations of B_i , P_i and the pose of the moving platform $(\mathbf{p}, \mathbf{R})$. By expanding Eqn. (4.5), the distances h_i can be expressed in terms of x , y and ϕ . After rearranging, each limiting case has the quadratic form of x and y as follows:

$$h_i = c_{i,0}x^2 + c_{i,1}xy + c_{i,2}y^2 + c_{i,3}x + c_{i,4}y + c_{i,5} \geq 0 \quad (4.6)$$

where the coefficients $c_{i,0}, c_{i,1}, \dots, c_{i,5}$ are functions of the base geometry, the platform geometry and the orientation angle of the moving platform. At a given orientation, the expressions for the FCWS boundaries can be obtained by setting each $h_i = 0$ in turn.

(d) Examples

The constant orientation FCWSs of the 4-3-CDPPM (see Fig. 1.7(a)) is shown in Figure 4.2 with the small black bullets representing the base points. The four curves can be plotted using the set of four equations of either Δ_i , n_i , or h_i ($i = 1, 2, 3, 4$). The shaded

area is the constant orientation FCWS which satisfies the four inequalities. It is seen that the workspace is limited by $h_2 = 0$ and $h_4 = 0$ at positive orientations ($\phi > 0^\circ$), and by $h_1 = 0$ and $h_3 = 0$ at negative orientations ($\phi < 0^\circ$). Furthermore, the workspace areas are the same for the two opposite orientations.

Figure 4.2: Analytical force-closure workspace

The results also show that the workspace area gets smaller at a larger orientation. The workspace eventually approaches to a line passing the diagonal of the base as the orientation is increased to a certain value. For a symmetric 4-3-CDPPM (Fig. 2.5), the

largest rotation range is limited by an angular difference between the base angle and the platform angle, corresponding to the orientation when two opposite cables are collinear. Hence, if the platform is force closure at a specific orientation, it still be force closure at smaller orientations. The x - y motion range of the platform is found to be maximum at zero-orientation. For symmetric 4-3-CDPPMs, the maximum motion ranges and the maximum orientation can be obtained using the following equations:

$$\begin{aligned} R_\phi &= 2|\arctan(d_b/c_b) - \arctan(d_p/c_p)| \\ R_x &= |c_b - c_p| \\ R_y &= |d_b - d_p| \end{aligned} \quad (4.7)$$

Another interesting result is that three of the four workspace boundaries always intersect at one point. For instance, as shown in Fig. 4.2(a), the three boundaries defined by $h_2 = 0$, $h_3 = 0$ and $h_4 = 0$ meet at a point whose coordinates can be calculated as follows:

$$x_{234} = \frac{(c_p \cos \phi - d_p \sin \phi)}{2}; \quad y_{234} = \frac{(c_p \sin \phi + d_p \cos \phi)}{2} \quad (4.8)$$

Similarly, the intersecting points when $h_1 = h_3 = h_4 = 0$, $h_1 = h_2 = h_4 = 0$, and $h_1 = h_2 = h_3 = 0$ can also be determined, respectively:

$$\left\{ \begin{array}{l} x_{134} = c_b + \frac{(c_p \cos \phi - d_p \sin \phi)}{2}; \quad y_{134} = -\frac{(c_p \sin \phi - d_p \cos \phi)}{2} \\ x_{124} = c_b - \frac{(c_p \cos \phi - d_p \sin \phi)}{2}; \quad y_{124} = d_b - \frac{(c_p \sin \phi + d_p \cos \phi)}{2} \\ x_{123} = \frac{(c_p \cos \phi + d_p \sin \phi)}{2}; \quad y_{123} = d_b + \frac{(c_p \sin \phi - d_p \cos \phi)}{2} \end{array} \right. \quad (4.9)$$

It is noted that

$$x_{234}^2 + y_{234}^2 = \frac{(c_p^2 + d_p^2)}{4} : \text{constant} \quad (4.10)$$

This implies that the locus of the intersecting point falls on a circle centred at the base point B_1 . Similarly, the other three intersecting points also fall on circles with the same radius $\sqrt{c_p^2 + d_p^2}/2$ and centred at the base points B_2, B_3 and B_4 respectively. These results together with the analytical boundaries, enable the workspace area to be calculated accurately.

4.2.3 Numerical Workspace Generation

The analytical approaches become complex as the number of cables increases and cannot be used to find the FWWS with bounded tensions. Hence, a numerical approach is developed to generate the two types of workspace in a systematic way.

Figure 4.3 shows the flow diagram of generating the FCWS and the FWWS. The possible motion ranges are first discretized into a number of points. The points are checked one by one and those satisfying the tension condition will be included into the workspace. Clearly if a large motion range is used, it will be time consuming to search for all the points. For CDPMs, if the platform lies outside the convex hull formed by the base, the configuration is unlikely to satisfy the force-closure condition. This is because the platform will be pulled inwards when the cable are in tension. This helps to limit possible motion ranges used for discretization. In following examples, the motion range of 4-3-CDPPMs is a square enclosing a circle that circumscribes the base. Similarly, the motion range of 8-6-CDSPMs is a cube enclosing a sphere that circumscribes the base. Once the positional data of the platform is obtained, then the orientation of the platform is taken into consideration. For the total orientation workspace, only the upper bound and the lower bound of the orientation range need to be checked at each position. As discussed, the force-closure condition is used to generate the force-closure workspace. For the feasible wrench workspace, the feasible wrench condition and the wrench set condition are combined so that the constant wrench and required wrench sets are considered simultaneously. In this case, the equivalent wrench set is shifted by an amount equivalent to the constant wrench.

Figure 4.4 shows the constant orientation FCWS generated for the 4-3-CDPPM and the 8-6-CDSPM (their kinematic parameters are shown in Fig. 1.7). It is seen that the FCWSs of the 4-3-CDPPM in Fig. 4.4(a) and Fig. 4.4(b) are the same as those generated by the analytical method as shown in Fig. 4.2(a) and Fig. 4.2(b). For the 8-6-CDSPM, the

Figure 4.3: Diagram of generating force-closure workspace and feasible wrench workspace

FCWS of four different layers are shown in Fig. 4.4(c) and Fig. 4.4(d). The shape of the workspaces on the layers are similar to those of planar ones.

(a) $\phi = +2^0$

(b) $\phi = -2^0$

(c) $\alpha = +2^0, \beta = \gamma = 0^0$

(d) $\alpha = -2^0, \beta = \gamma = 0^0$

Figure 4.4: Force-closure workspace of the 4-3-CDPPM (a,b) and the 8-6-CDSPM (c,d)

(a) $\phi = +2^0, f_{p,r} = 20 \text{ N}$

(b) $\phi = +2^0, m_{p,r} = 2 \text{ N.m}$

Figure 4.5: Feasible wrench workspace of the 4-3-CDPPM (a,b) and the 8-6-CDSPM (c,d)

The FWWS for the two mechanisms are shown in Fig. 4.5. Here, it is assumed that the cable tension ranges from $t_{i,min} = 3 \text{ N}$ to $t_{i,max} = 54 \text{ N}$ and that there is no constant wrench. The figures on the left are sets of postures which can sustain an arbitrary force up to 20 N and those on the right are sets of postures which can sustain an arbitrary moment up to 2 N.m at a constant orientation. Clearly the FWWS with tension bound is a subset of the FCWS where the tension is not limited.

4.2.4 Workspace Quantification

Discretization algorithms can be used to quantify the workspace. The steps involve discretizing the 3-dimensional space, solving the inverse kinematics at each pose, and verifying the constraints that limit the workspace. The discretization algorithms are used by many researchers and can also be applied here to any type of CDPM. Similar to the way of generating the workspace, the translational motion ranges are separated into segments $(\Delta x, \Delta y, \Delta z)$. The volume of each elementary cube is determined by $\Delta V = \Delta x \cdot \Delta y \cdot \Delta z$. Therefore, the volume of the entire workspace can be approximated numerically by summing up the volumes of all the cubes which belong to the respective workspace.

$$V = \sum_{i=1}^{n_{ws}} \Delta V = n_{ws} \cdot \Delta V \quad (4.11)$$

Figure 4.6: Diagram of quantifying force-closure workspace and feasible wrench workspace

where n_{ws} is the number of points composing of the workspace. The flow diagram to calculate the workspace volume is shown in Fig. 4.6.

4.3 Workspace Performance Index

Apart from workspace quantification, it is also important to determine the quality of the workspace as a mechanism designed for a maximum workspace may have undesirable kinematic characteristics. Various performance indices such as the velocity/force ellipsoid, the condition number and the manipulability measure have been devised for assessing kinematic performance of robotics manipulators. Among them, the condition number is commonly used as a measure of dexterity. However, it cannot be applied directly for CDPMs. This is illustrated by the examples in Fig. 4.7. It is seen that the mechanism shown in Fig. 4.7(a) cannot resist the force f_p . Likewise, the mechanism shown in Fig. 4.7(c) is not able to counter the applied moment m_p . This is because in both cases, they are at the limits of the workspace. Hence, they are poorly conditioned configurations compared to the one in Fig 4.7(b) in terms of tension distribution. However, this may not be reflected by the condition number (k_c), which is the ratio of the largest singular value to the smallest singular value of the Jacobian matrix (Nakamura [66]). From the values of the condition number, the configuration in Fig. 4.7(c) is better conditioned than the one in Fig 4.7(b).

Figure 4.7: Performance index of CDPMs based on the condition number k_c

4.3.1 Tension Factor (TF)

For CDPMs, a tension factor is proposed to be used as a performance index to evaluate the quality of force closure at a specific configuration. The tension factor is defined as the minimum tension over the maximum tension of the cables. If $\mathbf{N}$ is the homogeneous solution of the cable tensions, then:

$$\text{TF} = \frac{\min(\mathbf{N})}{\max(\mathbf{N})} \quad (4.12)$$

The tension factor (TF) is a measure of the positive tension condition of the structure matrix. It reflects the relative tension distribution among the cables for a specific platform pose inside the force-closure workspace. The range of the TF is from zero to one. When the TF approaches to zero, one of the cable tension is close to zero, i.e the platform is located near to the workspace boundary. Hence if the TF approaches one, the platform is positioned far from the workspace boundary. The CDPM is called isotropic if its TF attains one. A larger TF is more favourable because there is a better tension balance among the cables. The TF can be geometrically expressed by the vector of homogeneous solution. This concept can be illustrated using a 3-2-CDPPM, i.e. a point-mass on a plane driven by three cables, as an example. From Eqn. (3.3), to achieve a unity force in all directions, i.e. $\mathbf{B}^T \mathbf{B} = \mathbf{I}$, the cable tensions must satisfy:

$$\begin{aligned} (\mathbf{A}\mathbf{T})^T \mathbf{A}\mathbf{T} &= \mathbf{I} \quad \text{or} \\ \mathbf{T}^T \mathbf{A}^T \mathbf{A}\mathbf{T} &= \mathbf{I} \end{aligned} \quad (4.13)$$

Since the structure matrix $\mathbf{A}$ is of rank two, the quantity $(\mathbf{A}^T \mathbf{A})$ is a symmetric 3×3 matrix of rank two. Equation (4.13) therefore describes a cylinder with its elements oblique to the coordinate axes in the cable tension space as shown in Fig. 4.8. The principal axes of this tension ellipsoid, i.e. the cylinder, coincide with the eigenvectors of $(\mathbf{A}^T \mathbf{A})$ and the length of its principal axes are equal to the reciprocals of the square roots of the eigenvalues. Since $(\mathbf{A}^T \mathbf{A})$ is a symmetric 3×3 matrix of rank two, this quantity

always has an eigenvalue of zero magnitude, resulting in a principal axis of infinite length. This axis defines the direction of the homogeneous solution $\mathbf{N}$.

Figure 4.8: Cable tension ellipsoid

From the geometric point of view, each element on the cylindrical surface in the cable tension space maps onto a point in the force space. This implies that cable tensions can be proportionally increased along the direction of its homogeneous solution without affecting the resultant forces. If the direction of the homogeneous solution is not in the first quadrant, then increasing cable tensions along the longitudinal direction of the cylindrical surface will result in some negative tensions. In addition, if the structure matrix is not of full rank, the cylinder is degenerated to a plane in which at least one cable tension is always zero. However as long as the force-closure condition is satisfied, the two situations mentioned above do not arise.

From the above discussion, it can be concluded that the direction of the longitudinal axis plays an important role in the distribution of cable tensions. The more the longitudinal axis is skewed towards any axis, the larger the differences in tensions among the cables. It is therefore desirable to have the longitudinal axis of the cylinder making equal angles with the coordinate axes. The vector which makes equal angles with all coordinate axes is called the isotropic vector. In an n -dimensional space, the unit isotropic vector is given by $\{\sqrt{1/n} \quad \sqrt{1/n} \quad \dots \quad \sqrt{1/n}\}^T$.

For CRPMs, the homogeneous solution, i.e. the longitudinal axis of the cylinder, is obtained for a specific pose. Hence, the TF determined using Eqn. (4.12) is unique. However, for RRPMS, the homogeneous solution spans the whole null space. In this case, the TF is determined by the homogeneous solution which is nearest to the isotropic axis. Geometrically, this solution is obtained by projecting the isotropic vector onto the space spanned by the null vectors $(\mathbf{N}_1, \mathbf{N}_2, \dots, \mathbf{N}_{m-n})$. Computationally, this is an optimization process of finding the homogeneous solution with the largest TF, i.e. the optimal null solution. Let $\mathbf{T}_{iso}$ and $\mathbf{T}_{opt}$ be the isotropic tension vector and the optimal null vector, respectively. Their relative angle is given by:

$$\begin{aligned} \cos(\mathbf{T}_{iso}, \mathbf{T}_{opt}) &= \frac{\mathbf{T}_{iso} \cdot \mathbf{T}_{opt}}{\|\mathbf{T}_{iso}\| \cdot \|\mathbf{T}_{opt}\|} \\ &= \frac{\sum_{i=1}^m t_{iso,i} \cdot t_{opt,i}}{\sqrt{\sum_{i=1}^m t_{iso,i}^2} \sqrt{\sum_{i=1}^m t_{opt,i}^2}} = \frac{(t_{opt,1} + t_{opt,2} + \dots + t_{opt,m})}{\sqrt{m} \sqrt{t_{opt,1}^2 + t_{opt,2}^2 + \dots + t_{opt,m}^2}} \end{aligned} \quad (4.14)$$

In order to make $\mathbf{T}_{opt}$ to be the optimal null vector, i.e. the closest one to the isotropic vector $\mathbf{T}_{iso}$, the right side of Eqn. (4.14) should be maximized. It means that the optimal homogeneous solution is determined by:

$$\begin{aligned} &\text{Maximize} \quad \frac{\sum_{i=1}^m t_i}{\sqrt{\sum_{i=1}^m t_i^2}} \\ &\text{subject to} \quad \mathbf{A} \cdot \mathbf{T} = 0 \\ &\quad t_i > 0 \quad (i = 1, 2, \dots, m) \end{aligned} \quad (4.15)$$

The optimization problem stated in Eqn. (4.15) is a nonlinear optimization with linear constraints. This nonlinear optimization can be converted to the linear optimization as described below.

Proposition 4.1 *A null vector is optimal if and only if the summation of all its components is minimum.*

Proof. *Necessity:* This is to verify that an optimal null vector will have the smallest summation of all its components. Equation (4.15) shows that if $\mathbf{T}_{opt}$ satisfies the linear constraints, i.e. $\mathbf{A} \cdot \mathbf{T}_{opt} = 0$, then $\lambda \cdot \mathbf{T}_{opt}$ also satisfies these constraints, i.e. $\mathbf{A} \cdot (\lambda \cdot \mathbf{T}_{opt}) = 0$, where λ is a scalar. If the cable tension is bounded with the positive minimum tension t_{min} ($t_{min} > 0$), every component of $\mathbf{T}_{opt}$ must be greater than or equal to t_{min} . If $t_{opt,i}$ is assumed the smallest among the components of $\mathbf{T}_{opt}$, λ can be chosen as $t_{min}/t_{opt,i}$ such that the equivalent optimal vector $\mathbf{T}_{opt}^*$ is obtained as follows:

$$\mathbf{T}_{opt}^* = \frac{t_{min}}{t_{opt,i}} \begin{pmatrix} t_{opt,1} \\ \vdots \\ t_{opt,i} \\ \vdots \\ t_{opt,m} \end{pmatrix} \quad (4.16)$$

This is an equivalent optimal null vector because $\mathbf{T}_{opt}^*$ results in the same value of the objective function in Eqn. (4.15) as $\mathbf{T}_{opt}$. It is seen that the i th component gets the value t_{min} while the others are greater than or equal to the minimum tension value. It is noted that any λ which is smaller than $t_{min}/t_{opt,i}$ will make the i th component of $\mathbf{T}_{opt}^*$ smaller than the minimum tension t_{min} . Subsequently, this leads to:

$$\lambda \sum_{j=1}^m t_{opt,j} \geq \frac{t_{min}}{t_{opt,i}} \sum_{j=1}^m t_{opt,j} \quad (4.17)$$

Equation (4.17) means that the summation of components of $\mathbf{T}_{opt}^*$ is minimum because any scalar which is greater than $t_{min}/t_{opt,i}$ will always proportionally increase all the components. Therefore, this leads to a larger summation of components.

Sufficiency: On the other hand, it is needed to verify that if $\mathbf{T}_{opt}$ is a positive null vector with the smallest summation of its components, this vector will possess the largest tension factor. Let the objective function in Eqn. (4.15) be f . It can then be expressed as follows:

$$f = \frac{s}{\sqrt{s^2 - c}} \quad (4.18)$$

where

$$s = \sum_{i=1}^m t_i > 0$$

$$c = \sum_{i,j=1; i \neq j}^m t_i t_j > 0$$

Differentiating Eqn. (4.18) with respect to s leads to:

$$f' = \frac{-c}{\sqrt{(s^2 - c)^3}} \quad (4.19)$$

In Eqn. (4.19), f' is always negative for positive null vectors because $c > 0$ and $\sqrt{(s^2 - c)^3} > 0$. It implies that f is a decreasing function which means that increasing s decreases f . So, if $\mathbf{T}_{opt}$ has the smallest summation of its components, substituting its components into Eqn. (4.18) results in the largest function value, i.e. the largest tension factor. ||

Using the above proposition, the nonlinear optimization problem in Eqn. (4.15) can be converted to a linear optimization problem given by:

$$\begin{aligned} & \text{Minimize} && \sum_{i=1}^m t_i && (4.20) \\ & \text{subject to} && \mathbf{A} \cdot \mathbf{T} = 0 \\ & && t_i \geq t_{min} > 0 \quad (i = 1, 2, \dots, m) \end{aligned}$$

This optimization problem in Eqn. (4.20) can be solved by using linear optimization algorithms and the optimal tension factor can be determined efficiently.

4.3.2 Simulation Results

Figure 4.9 shows the TF of the 4-3-CDPPMs within its force-closure workspace at three different platform orientations. The TF is plotted by making the size of the squares proportional to its values at discrete positions. The biggest square at the centre in Fig. 4.9(a) has the value '1', corresponding to the case that the tensions in the cables are equal. This is only attainable when the platform is located at the position of (0.5, 0.5) with zero orientation.

Figure 4.9: Tension factor of 4-3-CDPPMs

Figure 4.9(b) and Fig. 4.9(c) show the distribution of TF in the workspaces for $\phi = +2^\circ$ and $\phi = -2^\circ$, respectively. Again, it is noted that the nearer the platform is to the boundary of the workspace, the smaller the TF becomes. Similarly, the TF in four different layers of the 8-6-CDSPM is shown in Fig. 4.10, with the moving platform fixed at an orientation of $\alpha = \beta = \gamma = 0^\circ$. The simulation results also show that the nearer the platform is to the base centre, i.e. $x = y = z = 0.5$, the larger the TF becomes.

In order to have a better sense about the TF, Fig. 4.11 shows the tension distribution among the cables of a 4-3-CDPPM having a platform dimensions of 0.1 m by 0.06 m. The tension values are represented by arrows pointing along the cables towards the suspending points of the base. The shortest arrow represents the minimum pretension force. It is important to note that for the same location, the TF may change significantly with the

Figure 4.10: Tension factor of 8-6-CDSPMs with $\alpha = \beta = \gamma = 0^0$

orientation of the platform. Hence, when determining the total orientation workspace, a point which is excluded from the total orientation workspace may have a good TF at positive orientations but have a poor TF at negative orientations.

Figure 4.11: Tension distribution among the cables

4.3.3 Global Tension Index (GTI)

The TF is a local measure because it characterizes the tension distribution at a given posture of the moving platform. In order to evaluate the quality of the whole workspace, the global tension index (GTI) is proposed. By integrating the local TF over the workspace, the GTI can be obtained as follows:

$$\text{GTI} = \frac{\int_V \text{TF}(\mathbf{A}) dV}{\int_V dV} \quad (4.21)$$

where V denotes the workspace volume of the mechanism.

According to the finite workspace generation, the GTI value can be computed numerically. Equation (4.21) can be written separately for both the constant orientation workspace and the total orientation workspace as:

$$\text{GTI} = \begin{cases} \frac{\sum_{i=1}^{n_{ws}} \text{TF}_i(\mathbf{A})}{n_{ws}} & \text{for constant orientation workspaces} \\ \frac{\sum_{i=1}^{n_{ws}} \sum_{j=1}^{n_o} \text{TF}_{i,j}(\mathbf{A})}{n_o \cdot n_{ws}} & \text{for total orientation workspaces} \end{cases} \quad (4.22)$$

where

$TF_i(\mathbf{A})$ represents the TF of the structure matrix $\mathbf{A}$ computed from every posture i in the constant orientation workspace.

$TF_{i,j}(\mathbf{A})$ represents the TF of the structure matrix $\mathbf{A}$ computed from every position i at every orientation j in the total orientation workspace.

n_{ws} is the number of points composing of the workspace.

n_o is the number of points covering the range of possible orientations.

4.4 Design Optimization

To design a CDPM, the most critical issue is to determine its geometrical parameters. As shown in Fig. 4.12, the geometry of the base and the platform are related to the tension bounds $(t_{i,min}, t_{i,max})$, the feasible wrenches $(f_{p,r}, m_{p,r})$ and the resultant feasible wrench workspaces. In general, the tension bounds and the feasible wrench are specified, and the geometrical parameters are to be optimized in order to achieve the required feasible wrench workspace.

Figure 4.12: Relationship between design attributes for CDPMs

In this section, the design optimization is done for a class of 4-3-CDPPMs with symmetric property as shown in Fig. 4.13. A 4-3-CDPPM is symmetric if the shapes formed by the four suspending points of the base (B_1, B_2, B_3, B_4) and that formed by the four

connecting points at the platform (P_1, P_2, P_3, P_4) are both rectangular. The base and the platform are characterized by (r_b, α_b) and (r_p, α_p) respectively. In practice, a 4-3-CDPPM with a symmetric design is often adopted because it is kinematically simple with fewer geometrical parameters and easy to fabricate.

Figure 4.13: Symmetric 4-3-CDPPMs

4.4.1 Design Problem Statement

Two types of design optimization problems are described in the following.

(a) Configuration optimization

The objective is to get the optimal CDPM configuration when the size of the base is confined to a maximum base radius $r_{b,max}$. Both the suspending points B_i and the connecting points P_i are to be determined so that the workspace is maximum with the given tension ranges, orientation ranges and required wrench sets. In this case, the objective function, design variables and constraints are as follows:

- **Objective function:** maximizing the feasible wrench workspace

- **Constraints:**

- Satisfying tension ranges: $t_{i,min} \leq t_i \leq t_{i,max}$ ($i = 1, 2, 3, 4$)
- Possessing a minimum wrench from any direction: $\|\mathbf{f}_p\| \geq f_{p,r}, \|\mathbf{m}_p\| \geq m_{p,r}$
- Having a certain rotation range at every positions: $\phi_{min} \leq \phi \leq \phi_{max}$

- **Design variables and boundaries:**

- Range of base radius: $0 < r_b \leq r_{b,max}$
- Range of radius ratios: $0 < k_r \leq 1$ ($k_r = r_p/r_b$)
- Range of base angles: $0^0 \leq \alpha_b \leq 90^0$
- Range of platform angles: $0^0 \leq \alpha_p \leq 90^0$

(b) Task-Based optimization

The objective is to design an optimal CDPM to satisfy a given trajectory of the moving platform. For example, in ship-building industry, plasma cutting is widely used to cut steel plates into different shapes. In this application, a platform which carries the plasma torch must trace the required trajectories exactly. Normally the platform radius is confined to a maximum $r_{p,max}$ and the base radius is not limited. Here, it is desired to determine the smallest base area so that the resultant workspace can cover all points of the trajectory when the tension ranges, required wrench sets and orientation ranges are given. For such an optimization problem, the objective function, design variables and constraints are described below:

- **Objective function:** minimizing the base dimensions

- **Constraints:**

- Satisfying tension ranges: $t_{i,min} \leq t_i \leq t_{i,max}$ ($i = 1, 2, 3, 4$)

- Possessing a minimum wrench from any direction: $\|\mathbf{f}_p\| \geq f_{p,r}, \|\mathbf{m}_p\| \geq m_{p,r}$
- Satisfying the required rotation range: $\phi_{min} \leq \phi \leq \phi_{max}$
- Working in the feasible wrench workspace: $\text{trajectory} \subset \text{FWWS}$

• **Design variables and boundaries:**

- Range of platform radii: $0 < r_p \leq r_{p,max}$
- Range of radius ratios: $0 < k_r \leq 1$ ($k_r = r_p/r_b$)
- Range of base angles: $0^\circ \leq \alpha_b \leq 90^\circ$
- Range of platform angles: $0^\circ \leq \alpha_p \leq 90^\circ$

4.4.2 Optimization Algorithm

Since the objective function and constraints of the optimization models are highly non-linear, a direct optimization algorithm will be appropriate. In this case, the complex optimization algorithm (Deb [82]) is used to search for the solution for the design optimization. The algorithm begins with a number of feasible points created randomly. If a point is found to be infeasible, a new point is created using the previously-generated feasible points. Usually, the infeasible point is pushed towards the centroid of the previously-found feasible points. Once a set of feasible points is found, the worst point is reflected about the centroid of the rest of the points to find a new point. Depending on the feasibility and function value of the new point, the point is further modified and accepted. If the new point falls outside the variable boundaries, the point is modified to fall on the violated boundary. If the new point is infeasible, the point is retracted towards the feasible points. The worst point in the simplex is replaced by this new feasible point and the algorithm continues for the next iteration. The steps of the complex search algorithm for maximum problems are outlined below:

- **Step 1:** Set a bound for design variable vector $\mathbf{x}(\mathbf{x}_L, \mathbf{x}_U)$, a reflection parameter ν

and termination parameters ε , and δ . A point $\mathbf{x}$ consists of k design variables.

- **Step 2:** Generate an initial set of $2k$ feasible points randomly: $\mathbf{x}_i = \mathbf{x}_L + \chi(\mathbf{x}_H - \mathbf{x}_L)$ where χ is a random factor from 0 to 1. Then, calculate the respective function values $(f_1, f_2, \dots, f_{2k})$.

If $\mathbf{x}_i$ is infeasible, calculate the centroid of the current set of points, retract the point $\mathbf{x}_i$ half the distance to the centroid. This is repeated until it is feasible.

- **Step 3:** Carry out the reflection step:

(a) Select $\mathbf{x}_{min}$ such that $f(\mathbf{x}_{min}) = \min(f_1, f_2, \dots, f_{2k}) = f_{min}$; and $\mathbf{x}_{max}$ such that $f(\mathbf{x}_{max}) = \max(f_1, f_2, \dots, f_{2k}) = f_{max}$

(b) Calculate the centroid $\mathbf{x}_c$ (of points except $\mathbf{x}_{min}$):

$$\mathbf{x}_c = \left(\sum_{i=1}^{2k} \mathbf{x}_i - \mathbf{x}_{min} \right) / (2k - 1)$$

and determine the new point $\mathbf{x}_m$: $\mathbf{x}_m = \mathbf{x}_c + \nu(\mathbf{x}_c - \mathbf{x}_{min})$

(c) If $\mathbf{x}_m$ is feasible and $f(\mathbf{x}_m) \leq f_{min}$, retract half the distance to the centroid $\mathbf{x}_c$. Continue this until $f(\mathbf{x}_m) > f_{min}$; Else if $\mathbf{x}_m$ is feasible and $f(\mathbf{x}_m) > f_{min}$, go to Step 5; Else if $\mathbf{x}_m$ is infeasible, go to step 4.

- **Step 4:** Check for feasibility of the solution

(a) Reset violated variable bounds:

$$\begin{cases} \text{If } \mathbf{x}_m < \mathbf{x}_L \text{ set } \mathbf{x}_m = \mathbf{x}_L \\ \text{If } \mathbf{x}_m > \mathbf{x}_U \text{ set } \mathbf{x}_m = \mathbf{x}_U \end{cases}$$

(b) If the resulting $\mathbf{x}_m$ is infeasible, retract $\mathbf{x}_m$ half distance to the centroid. Continue this until $\mathbf{x}_m$ is feasible. Go to step 3(c).

- **Step 5:** Replace $\mathbf{x}_{min}$ by $\mathbf{x}_m$. Check for termination.

(a) Calculate $\bar{f} = \frac{1}{2k} \sum_{i=1}^{2k} f(\mathbf{x}_i)$ and $\bar{\mathbf{x}} = \frac{1}{2k} \sum_{i=1}^{2k} \mathbf{x}_i$.

(b) if $\sqrt{\sum_{i=1}^{2k} (f(\mathbf{x}_i) - \bar{f})^2} \leq \varepsilon$ and $\sqrt{\sum_{i=1}^{2k} \|\mathbf{x}_i - \bar{\mathbf{x}}\|^2} \leq \delta$; terminate else continue doing optimization (go to Step 3(a)).

Algorithm remarks: In the original complex search method, the centroid point calculated in step 3(b) is always assumed to be better than the worst point. However, for CDPMs, it is realized from Eqn. (4.7) that the feasible region of design variables is not convex. As a result, the resultant centroid may be not only worse than the worst point but also infeasible, i.e. violating the constraints. In such cases, the centroid is modified to be retracted half the distance to the best point until it is feasible and better than the worst point. The computational time mainly depends on the calculation of satisfactory tension condition. Therefore, this optimization problem is quite time consuming, especially for problems of configuration optimization in terms of area/volume because the tension conditions are checked through numerical algorithms. When the optimization algorithm is applied to different mechanisms, the computational complexity can be estimated via the number of 1-dimensional subsystems as shown in Fig. 3.9, e.g. the computation time for 8-6-CDSPMs is at least a few hundred times as many as that for 4-3-CDPPMs.

4.4.3 Optimization Results

For CDPMs, the correlation between two parameters α_b and α_p considerably affect the objective values. This does not guarantee the centroid to be a feasible point or a better point in comparison with the worst point. Therefore, the complex optimization algorithm needs to be slightly modified according to the design problems in order to get the optimum point effectively.

(a) Configuration optimization: For this problem, all constraints are taken into account to calculate the objective function. The optimization is done with various ranges of design parameters as well as required wrench sets. Many optimization results are obtained and six representative cases are listed in Table 4.1. For these cases, the weight of the platform causes a constant external wrench of $f_x = 0$, $f_y = -1.962$ N and $m_z = 0$. The tension range of the cables is from 3 N to 54 N.

Table 4.1: Optimization results

No.	r_b	k_r	α_b	α_p	ϕ	$f_{p,r}$	$m_{p,r}$	r_b	α_b	r_p	α_p	A_{FWWS}
1	0 – 1.0	0 – 1.0	00 – 90	00 – 90	± 2	5	1.0	1.0	90.0^0	1.00	00.0^0	2.792
2	0 – 1.0	0 – 1.0	00 – 90	00 – 90	± 5	8	2.0	1.0	90.0^0	1.00	00.1^0	2.358
3	0 – 0.8	0 – 0.1	00 – 90	00 – 45	± 2	1	0.1	0.8	45.7^0	0.04	00.2^0	0.864
4	0 – 0.8	0 – 0.1	00 – 90	00 – 45	± 2	1	0.5	0.8	46.9^0	0.08	00.0^0	0.723
5	0 – 0.5	0 – 0.5	10 – 80	10 – 45	± 5	3	0.1	0.5	51.4^0	0.03	10.0^0	0.240
6	0 – 0.5	0 – 0.5	20 – 70	20 – 40	± 5	8	0.1	0.5	54.0^0	0.03	20.8^0	0.168

The results in Table 4.1 show that the base radius always gets its maximum value and the platform angle always gets its minimum value. The third case indicates that the platform radius tends to be very small when the external wrenches are small. When the external wrenches are increased as in the other cases, the platform radius gets the value with the maximum radius ratio. The resultant optimal workspaces show that the optimal configuration is mainly determined by the geometrical parameters rather than the wrench constraints of the mechanism. It means that with the same wrench constraints, the configuration that results in a larger force-closure workspace will definitely result in a larger feasible wrench workspace. It is obvious that for the full range cases, i.e. case 1 and case 2, the optimal configuration is: $r_b = r_p = r_{b,max}$ and $\alpha_b = 0^0, \alpha_p = 90^0$ as shown

(a) Optimal configuration

(b) $\phi^+ = f(x, y)$

Figure 4.14: Global optimal mechanism

in Fig. 4.14(a). The moving platform becomes a line. From Eqn. (4.7), it can be easily verified that with these values, the possible range of translation and rotation is maximum. The optimal force-closure workspace in which the positive orientation is expressed as a function of translations is illustrated in Fig. 4.14(b). It is seen that the positive orientation angle can be approximately obtained up to nearly $+90^\circ$. These results indicate that the effect of configuration parameters and orientations of the moving platform on the workspace of fully restrained CDPMs is similar to that of incompletely restrained CDPMs as studied by Pusey [32]. Generally, the following common features of workspace analysis can be found in every cable-driven mechanism:

- The higher the radius ratio ($0 < k_r \leq 1$), the larger the workspace;
- The more the similarity between the base and the platform, i.e. $|\alpha_b - \alpha_p| \rightarrow 0$, the smaller the workspace; and
- The higher the orientation of the moving platform requires, the smaller the workspace becomes.

(b) Task-Based optimization: In this case, the optimal CDPM configuration must possess a feasible wrench workspace covering the required trajectory. The objective function value is only meaningful for the feasible point. To check the feasibility of a configuration with a given trajectory, all the points in the trajectory must be coordinated relatively to the base, e.g. around the base centre. This design problem is illustrated through the following examples.

Example 1: In making steel pipe fitting adapters.

Figure 4.15(a) shows a steel pipe adapter used to connect a larger pipe to a smaller pipe. This adapter consists of three separate parts as shown in Fig. 4.15(b). When deployed on a plane, the middle part has a shape in the form of the letter ‘C’. To cut this part from a large steel plate, a plasma torch must follow the part’s contour precisely. The plasmas

Figure 4.15: Application of the 4-3-CDPPM in making pipe adapters

torch is mounted on the platform. The design task is then to locate the four suspending points and the four connecting points of a 4-3-CDPPM so that the platform which carries the plasma torque can trace along the contour with the base occupying the smallest area. This design problem is stated as follows:

- Weight of the platform: 1 kg.
- Actuators and cables are supplied with $t_{i,min} = 3$ N and $t_{i,max} = 54$ N.
- Connecting points can be arranged on a circle of radius from 0.1 to 0.2 m.
- Maximum desired acceleration of the platform: 4.5 N/m^2 .
- Geometrical parameters of the part: $h_1 = 0.36$, $h_2 = 0.54$ and $\zeta = 199.7^\circ$.

There is no constant wrench acting on the platform, i.e. $f_x = f_y = m_z = 0$ as the mechanism is placed on a horizontal plane. The desired acceleration of the moving platform, together with its weight, gives the minimum force set acting on the platform: $f_{p,r} = m_p \cdot a = 4.5$ N, $m_{p,r} = 0$. The orientation of the platform can be kept at a constant e.g. $\phi = 0^\circ$ since the plasma torch focuses its energy on a spot and rotation of the platform is not necessary. For the part, it can be placed symmetrically about the y-axis as shown in Fig. 4.15(c). Point M is the mid-point between O and N, and is chosen to

be coincident with the base centre. With the given dimensions, point data of the trajectory can be easily created by five points A,B,C,D and O whose local coordinates are $(-0.355,-0.242)$, $(-0.533,-0.273)$, $(0.533,-0.273)$, $(0.355,-0.242)$ and $(0,-0.180)$ with respect to M, respectively. These data are used to check the feasibility of a new design in the optimization algorithm. By inputting these known parameters and the design constraints to the optimization algorithm, the results obtained are shown in Table 4.2.

Table 4.2: The optimal result of the pipe adapter

Design parameters	Constraints	Optimal result
$r_p = 0.1-0.2$	$\phi = 0^0$	$r_b = 0.88$
$k_r = 0-1$	$f_x = 0, f_y = 0, m_z = 0$	$\alpha_b = 28^0$
$\alpha_b = 0^0-90^0$	$f_{p,r} = 4.5, m_{p,r} = 0$	$r_p = 0.16$
$\alpha_p = 0^0-90^0$	$t_{i,min} = 3, t_{i,max} = 54 \quad (i = 1, \dots, 4)$	$\alpha_p = 0^0$

The workspace corresponding to the optimal configuration is shown in Fig. 4.16(b). It is noted that the resultant workspace encloses the whole trajectory.

(a) Optimal configuration

(b) Resultant workspace

Figure 4.16: Task-based design of the pipe adapter

Example 2: An automatic painting system.

Figure 4.17 shows a paint spraying system using a cable-driven platform carrying a spray gun to paint workpieces on a conveyor. The moving platform needs to have 3 DOFs on the vertical plane. Hence, the design task is to determine the optimal geometrical parameters of a 4-3-CDPPM in terms of the base area. Inputs of this problem is given as follows:

- Weight of the platform including the spray gun: 2 kg.
- Actuators and cables are supplied with $t_{i,min} = 3$ N and $t_{i,max} = 150$ N.
- Connecting points are changeable with $r_p = 0.2-0.4$ and $\alpha_p = 5^0-45^0$.
- Parts on the conveyor have dimensions of 0.3×0.1 m.
- The spray nozzle is supplied with an aperture of 30^0 .
- Suspending points B_1 and B_2 are at the same height with the conveyor belt.

Figure 4.17: Application of the 4-3-CDPPM in automatic painting systems

Since the mechanism is placed on a vertical plane, therefore the weight of the platform causes a constant downward force: $f_x = m_z = 0$ and $f_y = -9.81m_p = -19.62$ N. To obtain good spraying quality, the trajectory of the spray gun is assumed to follow an arc ABC . It needs to move from left to right and back ($A \rightarrow C \rightarrow A$) and change its orientation in the process. The range of rotation needed is chosen to be from -5^0 to $+5^0$. The spray nozzle must be kept at a distance of at least 0.186 (m) from the part surface so that the paint trace can cover the 0.1 width of the part. Point A and C have coordinates of $(-0.12, 0.03)$ and $(0.12, 0.03)$ with respect to point B , respectively. The base angle is ranged from 0^0 to 45^0 . The reacting wrenches on the platform due to spraying

is assumed to be $f_{p,r} = 6$ N, $m_{p,r} = 2$ N.m for maintaining stability. Inputting these known parameters and constraints to the optimization algorithm gives the optimal result as shown in Table 4.3.

Table 4.3: The optimal result of the painting system

Design parameters	Constraints	Optimal result
$r_p = 0.2-0.4$	$\phi = \pm 5^0$	$r_b = 0.42$
$k_r = 0-1$	$f_x = 0, f_y = -19.62, m_z = 0$	$\alpha_b = 26^0$
$\alpha_b = 0^0-45^0$	$f_{p,r} = 6, m_{p,r} = 2$	$r_p = 0.2$
$\alpha_p = 5^0-45^0$	$t_{i,min} = 3, t_{i,max} = 150 \quad (i = 1, \dots, 4)$	$\alpha_p = 5^0$

The workspace corresponding to the optimal configuration is shown in Fig. 4.18. It is seen that the resultant workspace envelops the required trajectory.

Figure 4.18: Task-based design of the painting system

4.5 Summary

In this chapter, the workspace analyses and the design optimization of CDPMs are addressed. The main results are as follows:

Based on the three tension conditions, the force-closure workspace and the feasible wrench workspace are proposed and defined. From the force-closure property of a CDPM, three analytical approaches including the null space method, the geometrical method and

the instantaneous centre method are presented and used to generate the force-closure workspace of a 4-3-CDPPM. The results show that the force-closure workspace is bounded by quadratic curves. The analytical boundary expressions derived also reveals that the platform reaches its workspace boundaries when three out of the four cables intersect at one point. Due to the limitation of the analytical approaches, generic strategies for numerically generating and quantifying the workspace of a general m - n -CDPM are proposed. Geometrical constraints are primarily identified from the analytical boundaries. This helps to limit the search space and therefore makes the numerical method more efficient. This approach has been illustrated on the 4-3-CDPPM as well as the 8-6-CDSPM.

A tension factor is proposed to be used as a performance index to evaluate the quality of force closure for CDPMs at a specific configuration. The tension factor (TF) is a measure of the positive tension condition of the structure matrix. It reflects the relative tension distribution among the cables for a specific platform pose inside the force-closure workspace. A linear programming approach is proposed to optimize the TF, which is computationally efficient. From the simulation results, it has been shown that the nearer the platform is to the boundary of the workspace, the smaller the TF becomes. The global tension index is proposed to evaluate the quality of tension distribution over the entire workspace. Subsequently, two design problems are proposed to meet practical needs. An optimization methodology for addressing these design problems has been presented. Simulation results show the effectiveness of the optimization approach.

Chapter 5

Dynamic Analysis

5.1 Introduction

Dynamic analysis plays an important role in performance analysis and control implementation of CDPMs. Dynamic equations relate the motion of a system to the applied force/torque. In this Chapter, the dynamic modelling of a CDPM system is presented.

There are two types of dynamic analysis: the inverse dynamics and the forward dynamics. The inverse dynamic analysis concerns the determination of the required actuator's torque when the motion of the moving platform and the external wrench are known. On the other hand, the forward dynamic analysis calculates the motion of the moving platform with the given actuator's torque and the external wrench. The inverse dynamics is used for trajectory planning and model-based motion control, while the forward dynamics is employed for motion simulation.

Due to the unilateral property of the cable actuation, the dynamic modelling of CDPMs is different from that of a conventional parallel mechanism as addressed by Yamamoto [83], Shiang [84], Agrawal [35], Kino [85], William [86], Kawamura [36]. It was noted by these researchers that dynamic modelling and analysis are critical to CDPMs because of

the positive tension requirement of the cables. This is even more critical for incompletely restrained CDPMs because the platform driven by this type of mechanism such as conventional cranes is prone to swinging. Hence, it is important to implement dynamic control strategies to control the motion of the platform precisely. Various control algorithms have been proposed for this type of mechanisms. Gorman [87] designed a sliding mode control system to track a given trajectory based on the dynamic model of a point-mass system. A flatness-based control system for a cable suspension manipulator CABLEV was implemented by Maier [88] to achieve asymptotically stable tracking behaviour through cascaded feedback of state variables. Various control approaches such as feedback controllers based on Lyapunov theory, feedback linearization and sliding mode control were used by Agrawal [89, 90, 91, 92] to study control performances of a cable-suspended robot and a dual-stage cable robot. Yanai also developed anti-sway control methods based on dynamics compensation [93] and feedback control with exact linearization for a cable-suspended mechanism [94]. For a fully restrained CDPM, the kinematic constraints are enough to obtain deterministic motion of the moving platform as long as tensions are maintained in cables. Therefore, practical control approaches can be adopted for this type of mechanisms. For example, Ming [28] implemented a PID controller in which the elongation of the cables is taken into account and compensated for a 4-3-CDPPM. A PD controller which includes a nonlinear, coupled dynamic model was proposed for a point-mass CDPPM by William [95]. Beside the positional feedback loop, a force feedback loop is also necessary in teleoperation applications. This enables the master devices to have the force reflecting ability. Kawamura [96] developed a tension control system which is applied in virtual sport training systems.

For a CDPM, cable driving units are required. A few cable driving units were developed by Morizono [97], Maeda [98] and Hiller [99] in which the cables were found to oscillate in their perpendicular directions when they were being wound or unwound by the actuators. The reason is that the contacting point from which the cable extends is not a stationary point. Clearly, the unwanted oscillating phenomenon will affect the dynamic performance

of the system.

In this chapter, the cable winding methods are first investigated. In order to obtain a stationary contacting point for the driving cables, a novel cable winding mechanism is proposed. This is followed by the formulation of the system dynamic model incorporating the dynamic models of both the cable-driven mechanism and the cable driving unit. Based on the developed dynamic model, the feasible wrench condition will be employed to check the existence of torque solution at a specific pose of the moving platform. This takes into account the practical limitation of torque ranges (from $\tau_{i,min}$ to $\tau_{i,max}$). An optimal torque distribution method is proposed and formulated. The motor torques of a 4-3-CDPPM as its platform traces both the linear trajectory and circular trajectory are presented.

5.2 Cable Winding Mechanism

In a CDPM, each cable is usually driven by a rotary actuator through a pulley system consisting of a cable winding mechanism and a guiding roller as shown in Fig. 5.1.

Figure 5.1: Main components in CDPMs

The function of the cable winding mechanism (at A) is to provide means of winding and unwinding the cables. The guiding roller (at B) is used to direct the cable to the connecting point of the moving platform at P .

5.2.1 Winding of Cables

Figure 5.2 shows two conventional ways of winding cables on a winch.

Figure 5.2: Two conventional methods of winding cable

In the first method (Fig. 5.2(a)), the cables overlap on the winch, resulting in changes in the winding radius r_w and shifting of the contacting point vertically. Hence, the kinematic relationship between the changes in cable length Δl and the angular displacement of the actuator $\Delta\theta$ varies as the cables are winding and unwinding:

$$\Delta l = r_w \Delta\theta \quad (5.1)$$

In the second method (Fig. 5.2(b)), the cables are arranged side by side thus maintaining a constant ratio of $\Delta l/\Delta\theta$, i.e. r_w is a constant. However, the contacting point is not stationary but shifts in the horizontal direction.

Both the above methods of winding with varying contacting point and/or changing ratio $\Delta l/\Delta\theta$ are not desirable and will complicate control. For control purposes, it is desirable that the cable winding mechanism meets the following design requirements:

- The contacting point in the cable winding mechanism from which the cable extends must remain stationary during winding and unwinding.
- The relationship between the change in cable length Δl and the angular displacement of the actuator $\Delta\theta$ is a constant.

5.2.2 New Cable Driving Mechanism

In order to fulfil the design requirements, it is necessary that the winch must possess a translational motion and a rotational motion simultaneously so that a constant $\Delta l/\Delta\theta$ and a stationary contacting point are maintained. The proposed winding mechanism is shown in Fig. 5.3.

Figure 5.3: Proposed winding method

In this design, the winch which arranges cables side by side is rotated about a fixed screw shaft. In this way, the winch is made movable and will translate along the linear guides while rotating. This enables the translation and rotation motions to be synchronized by one actuator through a timing belt drive. To achieve a stationary contacting point, the pitch of the winch must be equal to that of the screw. For this winding mechanism, the relationship between the change in cable length Δl and the angular displacement of the actuator $\Delta\theta$ is as follows:

$$\Delta l = \frac{r_w r_a}{r_g} \Delta\theta \quad (5.2)$$

where r_a, r_g and r_w are the radii of the driving sheave, the driven sheave and the winch, respectively. With a stationary contacting point, the actual displacement of the cable length Δl can be easily computed from the joint angle using Eqn. (5.2), thus simplifying the kinematic and dynamic model of the proposed cable winding mechanism.

5.3 Dynamic Modelling

5.3.1 Mechanism Dynamic Model

The schematic diagram of a CDPM is shown in Figure 5.4.

Figure 5.4: Free body diagram of the moving platform

Based on the Newton-Euler equations, the dynamic model of the CDPM can be written as follows:

$$\sum_{i=1}^m \mathbf{t}_i + \mathbf{f}_p + m_p \mathbf{g} - m_p \dot{\mathbf{v}} = 0 \quad (5.3)$$

$$\sum_{i=1}^m \mathbf{r}_i \times \mathbf{t}_i + \mathbf{m}_p - \mathbf{I}_p \dot{\boldsymbol{\omega}} - \boldsymbol{\omega} \times (\mathbf{I}_p \boldsymbol{\omega}) = 0 \quad (5.4)$$

where

m_p : mass of the moving platform.

$\mathbf{g}$: gravitational acceleration.

$\mathbf{I}_p$: inertial tensor of the moving platform at the centroid P .

$\mathbf{v}$, $\boldsymbol{\omega}$: velocity and angular velocity at the centroid.

Substituting $\mathbf{t}_i = t_i \mathbf{u}_i$ into Eqns. (5.3),(5.4) leads to the following equation:

$$\mathbf{A} \cdot \mathbf{T} = \mathbf{B} \quad (5.5)$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \\ \mathbf{r}_1 \times \mathbf{u}_1 & \mathbf{r}_2 \times \mathbf{u}_2 & \cdots & \mathbf{r}_m \times \mathbf{u}_m \end{bmatrix} \in \mathbb{R}^{n \times m}: \text{the structure matrix}$$

$$\mathbf{T} = \{t_1 \ t_2 \ \dots \ t_m\}^T \in \mathbb{R}^m: \text{cable tension}$$

$$\mathbf{B} = \left\{ \begin{array}{c} m_p \dot{\mathbf{v}} - m_p \mathbf{g} - \mathbf{f}_p \\ \mathbf{I}_p \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_p \boldsymbol{\omega}) - \mathbf{m}_p \end{array} \right\} \in \mathbb{R}^n: \text{dynamic wrench acting on the platform}$$

5.3.2 Actuator Dynamic Model

Figure 5.5 shows the proposed cable driving unit (CDU) where the cable winding mechanism is driven by a servo motor through a timing belt drive. The schematic representation of the cable driving unit for dynamic modelling is shown in Fig. 5.6.

Figure 5.5: Proposed cable driving unit

Figure 5.6: Schematic diagram of the cable driving unit

The dynamic equation of the CDU can then be written as:

$$j_{m,i}\ddot{\theta}_i + d_{m,i}\dot{\theta}_i + r_{m,i}t_i = \tau_i \quad (5.6)$$

with

$$\begin{aligned} j_{m,i} &= j_{a,i} + j_{w,i} (r_{a,i}/r_{g,i})^2 \\ d_{m,i} &= d_{a,i} + d_{w,i} (r_{a,i}/r_{g,i})^2 \\ r_{m,i} &= r_{w,i}r_{a,i}/r_{g,i} \end{aligned}$$

where

$(j_{a,i}, d_{a,i})$: rotational inertial and viscous damping coefficient of the armature.

$(j_{w,i}, d_{w,i})$: rotational inertial and viscous damping coefficient of the winch.

It is assumed that the torque on each motor is sufficient to maintain the positive cable tension. Equation (5.6) is written in the following form for m actuators:

$$\mathbf{T} = \mathbf{R}_m^{-1} \left(\boldsymbol{\tau} - \mathbf{J}_m \ddot{\boldsymbol{\theta}} - \mathbf{D}_m \dot{\boldsymbol{\theta}} \right) \quad (5.7)$$

where

$$\begin{aligned} \mathbf{T} &= [t_1 \quad t_2 \quad \dots \quad t_m]^T \\ \boldsymbol{\tau} &= [\tau_1 \quad \tau_2 \quad \dots \quad \tau_m]^T \\ \boldsymbol{\theta} &= [\theta_1 \quad \theta_2 \quad \dots \quad \theta_m]^T \end{aligned}$$

$$\mathbf{R}_m = \text{diag}(r_{m,1}, r_{m,2}, \dots, r_{m,m})$$

$$\mathbf{J}_m = \text{diag}(j_{m,1}, j_{m,2}, \dots, j_{m,m})$$

$$\mathbf{D}_m = \text{diag}(d_{m,1}, d_{m,2}, \dots, d_{m,m})$$

5.3.3 System Dynamic Model

The dynamic model of the overall system is derived by combining Eqns. (5.5) and (5.7). An inverse kinematic mapping is needed to express the angular displacement of the motor θ_i as a function of the platform posture $\mathbf{X}$. Let all θ_i be zero when the platform is located at the origin of the base frame $\{\mathbf{B}\}$ with zero orientation. If a clockwise rotation of the motor shaft is assumed to be positive, a positive angle θ_i will cause a negative change Δl_i in cable length i : $\theta_i \cdot r_{m,i} = -\Delta l_i$. The cable length displacement is $\Delta l_i = l_i - l_{i0}$.

Based on the kinematic equations presented in Chapter 2, the general cable length $\|\mathbf{l}_i\| = \|\overrightarrow{B_i P_i}\|$ ($i = 1, 2, \dots, m$) can be calculated from the inverse kinematics as follows:

$$l_i = \|\mathbf{p} + \mathbf{R}\mathbf{r}_i^p - \mathbf{b}_i\| \quad \text{and} \quad (5.8)$$

$$l_{i0} = \|\mathbf{r}_i - \mathbf{b}_i\| \quad (5.9)$$

This leads to the following joint displacement:

$$\boldsymbol{\theta} = \mathbf{R}_m^{-1} \begin{Bmatrix} \|\mathbf{r}_1 - \mathbf{b}_1\| - \|\mathbf{p} + \mathbf{R}\mathbf{r}_1^p - \mathbf{b}_1\| \\ \vdots \\ \|\mathbf{r}_m - \mathbf{b}_m\| - \|\mathbf{p} + \mathbf{R}\mathbf{r}_m^p - \mathbf{b}_m\| \end{Bmatrix} \quad (5.10)$$

Successive time derivatives of Eqn. (5.10) yield:

$$\dot{\boldsymbol{\theta}} = \frac{\partial \boldsymbol{\theta}}{\partial \mathbf{X}} \dot{\mathbf{X}} \quad (5.11)$$

$$\ddot{\boldsymbol{\theta}} = \frac{d}{dt} \left(\frac{\partial \boldsymbol{\theta}}{\partial \mathbf{X}} \right) \dot{\mathbf{X}} + \frac{\partial \boldsymbol{\theta}}{\partial \mathbf{X}} \ddot{\mathbf{X}} \quad (5.12)$$

Substituting Eqns. (5.11) and (5.12) into Eqn. (5.7) leads to the following:

$$\mathbf{T} = \mathbf{R}_m^{-1} \left[\boldsymbol{\tau} - \mathbf{J}_m \left(\frac{d}{dt} \left(\frac{\partial \boldsymbol{\theta}}{\partial \mathbf{X}} \right) \dot{\mathbf{X}} + \frac{\partial \boldsymbol{\theta}}{\partial \mathbf{X}} \ddot{\mathbf{X}} \right) - \mathbf{D}_m \frac{\partial \boldsymbol{\theta}}{\partial \mathbf{X}} \dot{\mathbf{X}} \right] \quad (5.13)$$

Finally, by combining Eqn. (5.5) and (5.13), the dynamic equation of motion of the CDPM can be expressed in a standard form as:

$$\mathbf{H}\ddot{\mathbf{X}} + \mathbf{h}(\mathbf{X}, \dot{\mathbf{X}}) = \mathbf{A}\mathbf{R}_m^{-1}\boldsymbol{\tau} \quad (5.14)$$

where

$$\begin{aligned} \mathbf{H} &= \mathbf{A}\mathbf{R}_m^{-1}\mathbf{J}_m\frac{\partial\boldsymbol{\theta}}{\partial\mathbf{X}} + \begin{bmatrix} m_p\mathbf{I} & 0 \\ 0 & \mathbf{I}_p \end{bmatrix} \\ \mathbf{h}(\mathbf{X}, \dot{\mathbf{X}}) &= \mathbf{A}\mathbf{R}_m^{-1} \left[\mathbf{J}_m\frac{d}{dt} \left(\frac{\partial\boldsymbol{\theta}}{\partial\mathbf{X}} \right) + \mathbf{D}_m\frac{\partial\boldsymbol{\theta}}{\partial\mathbf{X}} \right] \dot{\mathbf{X}} + \left\{ \begin{array}{l} -m_p\mathbf{g} - \mathbf{f}_p \\ \boldsymbol{\omega} \times (\mathbf{I}_p\boldsymbol{\omega}) - \mathbf{m}_p \end{array} \right\} \end{aligned}$$

In practical application, the desired position, velocity and acceleration of the moving platform, together with external wrenches are known and hence the left-hand side of Eqn. (5.14) is defined. In addition, if all the CDUs are the same, Eqn. (5.14) can be simplified into the following equation with variables τ_i :

$$\mathbf{A}\boldsymbol{\tau} = \mathbf{C} \quad (5.15)$$

Equation (5.15) is an under-determined linear system. It is in a form similar to Eqn. (3.3) with the variables as torque values and the dynamic terms on the right-hand side. At an instantaneous time, the component $\mathbf{C}$ on the right-hand side is a constant and is considered as an instantaneous external wrench. Moreover, the torque values are also bounded. As a result, the feasible wrench condition can still be used to check for the existence of torque solutions.

5.4 Optimal Torque Resolver

Due to actuation redundancy, there are many sets of torque solutions. This could result in dangerously high torque solutions unless the redundancy is properly addressed. The smallest torque is preferred because this will correspond to lesser stress of the cables,

smaller actuator's current and lower power consumption. It should be noted that under the effect of the dynamic terms, it is possible to balance the moving platform without satisfying the force-closure condition. This situation is undesirable and should be avoided because the system will be out of control if a certain disturbance appears in the direction against the dynamic wrench. Hence, it is important that the force-closure condition is satisfied so that the platform is fully restrained by the cables. In this case, when an external wrench is applied to the moving platform, the external wrench will be resisted by a number of active cables. The tension in the other cables has the effect of maintaining certain tension distribution among the cables.

In addition, because the joint torques are non-negative, the objective function in this torque optimization is to minimize $\tau = \sum_{i=1}^m \tau_i$ when subject to constraints in Eqn. (5.15) and the constraints of bounded variables. In this case, the objective function is linear and the constraints are also linear. Hence, the torque optimization is a linear programming problem which is computationally efficient. As discussed in Chapter 4, if the instantaneous external wrench $\mathbf{C}$ is significantly smaller than the minimum torque requirement, the homogeneous solution dominates the torque distribution. In such cases, minimizing the summation of torques ($\sum_{i=1}^m \tau_i$) also implies that the torque distribution is uniformly optimized. Normally, the standard form of linear programs deals with non-negative variables. So an algorithm is needed to convert bounded torque variables into non-negative variables. This is done by replacing each variable τ_i by two new variables $\tau_i^* \geq 0$ and $\tau_i^{**} \geq 0$ and introducing a linear equality constraint. They are expressed as follows:

$$\tau_i = \tau_i^* + \tau_{i,min} \quad (5.16)$$

$$\tau_i^* + \tau_i^{**} = \tau_{i,max} - \tau_{i,min} \quad (5.17)$$

Substituting each τ_i by the two new variables τ_i^* and τ_i^{**} leads to a new optimization problem expressed in the standard linear programming form:

$$\text{Minimize } \tau = \sum_{i=1}^m (\tau_i^* + \tau_{i,min}) \quad (5.18)$$

subject to

$$\begin{cases} \sum_{i=1}^m a_{1,i} \tau_i^* = c_1 - \sum_{i=1}^m a_{1,i} \tau_{i,min} \\ \sum_{i=1}^m a_{2,i} \tau_i^* = c_2 - \sum_{i=1}^m a_{2,i} \tau_{i,min} \\ \vdots \\ \sum_{i=1}^m a_{n,i} \tau_i^* = c_n - \sum_{i=1}^m a_{n,i} \tau_{i,min} \end{cases}$$

$$\begin{cases} \tau_1^* + \tau_1^{**} = \tau_{1,max} - \tau_{1,min} \\ \tau_2^* + \tau_2^{**} = \tau_{2,max} - \tau_{2,min} \\ \vdots \\ \tau_m^* + \tau_m^{**} = \tau_{m,max} - \tau_{m,min} \end{cases}$$

$$\tau_1^*, \tau_2^*, \dots, \tau_m^*, \tau_1^{**}, \tau_2^{**}, \dots, \tau_m^{**} \geq 0$$

For this standard linear programming, there are existing effective algorithms developed from two basic approaches. One approach is the simplex method, while the other is the interior-point approach. The simplex approach starts with one extreme point, hops to a better neighbouring extreme point along the boundary, and finally stops at an optimal extreme point. Unlike the simplex method, the interior-point method stays in the interior of feasible domain and tries to position a current solution as the ‘centre of universe’ in finding a better direction for the next move. In this section, a standard Matlab function ‘linprog’, which is based on the algorithm of ‘Linear Interior Point Solver’ (Yang [100]), is employed because it is found to be efficient and stable in searching for the optimal torque solution.

Computational examples: The optimal torques are computed for both 4-3-CDPPMs and 8-6-CDSPMs (see Fig. 1.7). The weight of the moving platform is 0.5 kg with an inertial moment of $2.08 \cdot 10^{-5}$, $4.17 \cdot 10^{-5}$ and $5.41 \cdot 10^{-5}$ kg.m² about x -axis, y -axis and z -axis, respectively. The 4-3-CDPPM is placed on a vertical plane so that the platform weight is also taken into account. The same cable driving units are used. Figures 5.7(a), 5.7(b) and 5.7(c) show the optimal torques obtained at various postures of the platform in the planar case. The platform is assumed to move starting with a zero velocity and an

acceleration of $\ddot{x} = \ddot{y} = 1 \text{ m/s}^2$ and $\dot{\omega}_z = 1 \text{ rad/s}^2$ instantaneously. Figure. 5.7(d), 5.7(e) and 5.7(f) show the optimal torques calculated at various postures of the platform in the spatial case. Here, the platform is also assumed to move starting with zero velocity and an acceleration of $\ddot{x} = \ddot{y} = \ddot{z} = 1 \text{ m/s}^2$ and $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 1 \text{ rad/s}^2$. To keep the cables in tension, the minimum torque of 0.02 N.m is needed. Simulation results show that the optimal torque solutions will have at least one motor torque with the minimum value for the planar cases whereas at least two minimum torques are obtained in the spatial cases.

Figure 5.7: Optimal dynamic torques of 4-3-CDPPMs (a,b,c) and 8-6-CDSPM (d,e,f)

5.5 Trajectory Planning

It is desirable to move the platform smoothly from a starting posture to the end posture following a certain trajectory. In general, a quintic polynomial can be used as the time

history of postures to avoid vibrations caused by rough and jerky motion, i.e. to satisfy the required velocity as well as acceleration at the starting posture and the ending posture. A quintic has the form:

$$s(t) = c_5 t^5 + c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0 \quad (5.19)$$

and the velocity and acceleration of the platform along the path are:

$$\dot{s}(t) = 5c_5 t^4 + 4c_4 t^3 + 3c_3 t^2 + 2c_2 t + c_1 \quad (5.20)$$

$$\ddot{s}(t) = 20c_5 t^3 + 12c_4 t^2 + 6c_3 t + 2c_2 \quad (5.21)$$

Coefficients c_i ($i = 0, 1, \dots, 5$) are determined by six constraints of position, velocity and acceleration at both the starting posture and the ending posture. If the kinematic parameters at the starting posture are $s_s, \dot{s}_s, \ddot{s}_s$ and the kinematic parameters at the ending posture are $s_e, \dot{s}_e, \ddot{s}_e$, then substituting them into Eqns. (5.19), (5.20) and (5.21) results in six equations from which the six coefficients can be calculated:

$$\begin{aligned} c_5 &= \frac{[12(s_e - s_s) - 6(\dot{s}_e + \dot{s}_s)t_f + (\ddot{s}_e - \ddot{s}_s)t_f^2]}{2t_f^5} \\ c_4 &= \frac{[-30(s_e - s_s) + 2(7\dot{s}_e + 8\dot{s}_s)t_f - (2\ddot{s}_e - 3\ddot{s}_s)t_f^2]}{2t_f^4} \\ c_3 &= \frac{[20(s_e - s_s) - 4(2\dot{s}_e + 3\dot{s}_s)t_f + (\ddot{s}_e - 3\ddot{s}_s)t_f^2]}{2t_f^3} \\ c_2 &= \frac{\ddot{s}_s}{2} \\ c_1 &= \dot{s}_s \\ c_0 &= s_s \end{aligned} \quad (5.22)$$

where t_f is the time duration for trajectory.

With these coefficients, the instant position, velocity and acceleration are calculated from Eqns. (5.19), (5.20) and (5.21), respectively. These values are substituted into the dynamic equation in which the optimal torque values are obtained. Basically, any trajectory can

be approximately interpolated into lines and circles. The following illustrate a linear planning and a circular planning for 4-3-CDPPMs

5.5.1 Linear Trajectory

Figure 5.8 shows a linear trajectory that the platform of a 4-3-CDPPM should follow.

Figure 5.8: Linear trajectory

If the velocity and the acceleration are assumed to be zero at the starting posture and the ending posture, then the intermediate postures in Eqn. (5.23) can be expressed as:

$$\begin{Bmatrix} x(t) \\ y(t) \\ \phi(t) \end{Bmatrix} = \begin{Bmatrix} x_s \\ y_s \\ \phi_s \end{Bmatrix} + \left[6 \left(\frac{t}{t_f} \right)^5 - 15 \left(\frac{t}{t_f} \right)^4 + 10 \left(\frac{t}{t_f} \right)^3 \right] \begin{Bmatrix} x_e - x_s \\ y_e - y_s \\ \phi_e - \phi_s \end{Bmatrix} \quad (5.23)$$

5.5.2 Circular Trajectory

Similarly for the circular trajectory as shown in Fig. 5.9, the intermediate postures are calculated from Eqns. (5.24), (5.25) and (5.26) if the requirement of velocity and acceleration is zero at the starting posture and the ending posture.

$$x(t) = x_c + r_t \cos \left(\zeta_s + \left[6 \left(\frac{t}{t_f} \right)^5 - 15 \left(\frac{t}{t_f} \right)^4 + 10 \left(\frac{t}{t_f} \right)^3 \right] (\zeta_e - \zeta_s) \right) \quad (5.24)$$

$$y(t) = y_c + r_t \sin \left(\zeta_s + \left[6 \left(\frac{t}{t_f} \right)^5 - 15 \left(\frac{t}{t_f} \right)^4 + 10 \left(\frac{t}{t_f} \right)^3 \right] (\zeta_e - \zeta_s) \right) \quad (5.25)$$

$$\phi(t) = \phi_s + \left[6 \left(\frac{t}{t_f} \right)^5 - 15 \left(\frac{t}{t_f} \right)^4 + 10 \left(\frac{t}{t_f} \right)^3 \right] (\phi_e - \phi_s) \quad (5.26)$$

Figure 5.9: Circular trajectory

5.5.3 Simulation Results

Figure 5.10 shows two sample tasks for the 4-3-CDPPM. In the first task (Fig. 5.10(a)), the moving platform keeps constantly at 0^0 -orientation while tracing a linear trajectory. For the second task (Fig. 5.10(b)), the platform traces a circular trajectory, reaching 2^0 -orientation at the ending posture. The tasks complete in 20 seconds. Planned kinematic parameters for the two trajectories are shown in Fig. 5.10(c) and Fig. 5.10(d), respectively.

Using the same dynamic parameters as in previous section, the respective computed torques for the two trajectories are shown in Fig. 5.10(e) and Fig. 5.10(f). The torque results show that all the torques are within the allowable ranges (from 0.02 to 0.37) as specified and at least one of the torques is at the minimum value. In Fig. 5.10(e), the total torque increases when the platform is being pulled upward with the third cable having the largest torque. Similarly in Fig. 5.10(f), the total torque value is higher when the

(a) Linear trajectory

(b) Circular trajectory

(c) Task space planning

(d) Task space planning

(e) Optimal torque

(f) Optimal torque

Figure 5.10: Dynamic torques as the platform tracks along the trajectories

platform moves upwards against the direction of the gravitational force. It is noticed that the platform is oriented at 2° at the final position (0.3, 0.5) which is opposite to the initial position (0.7, 0.5) about the base centre (0.5, 0.5). Hence, the resultant torque is higher than that at the initial posture.

5.6 Summary

The dynamic analysis of fully restrained CDPMs is presented this chapter. The main results are summarized as follows:

A novel cable winding mechanism is first presented. The winch of the mechanism is designed to have a combined translational and rotational motion so that a constant $\Delta l / \Delta \theta$ and a stationary contacting point are achieved. The features are useful in simplifying the system dynamic model as well as in improving the dynamic performance of the system. The dynamic model of the cable driving unit where the cable winding mechanism is driven by a servo motor through a timing belt is developed.

The dynamic equation of motion of a CDPM, including the dynamic model of both the mechanism and the actuator, is formulated. It is shown that by using the proposed cable-driving units for actuation, the dynamic model can be expressed in a linear system for specified dynamic terms. In addition, the torques can also be systematically checked by the feasible wrench condition. A method of replacing a bounded torque variable by two nonnegative variables enables the torque optimization problem to be expressed in a standard linear programming form with the objective function as finding the minimum summation of torques. The optimal torque values always exist as long as the feasible wrench condition is satisfied. The motor torques of a CDPPM as its platform traces both the linear trajectory and circular trajectory are computed. The results show that the proposed torque optimization scheme is effective. It is seen that the resultant torques are within the specified range with the trajectory profile represented as a quintic polynomial.

Chapter 6

Prototype Development and Control Implementation

6.1 Introduction

For a fully restrained CDPM, kinematic constraints are enough to obtain deterministic motion of the moving platform as long as tensions are maintained in the cables. Therefore, practical control approaches, such as the independent joint control approach, can be adopted for such mechanisms. In this approach, a PID control law is used to stabilize as well as to satisfy the joint performance.

In this chapter, the development of a 4-3-CDPPM prototype for the purpose of verifying the theoretical analysis is first presented. The experiments to assess the performance of the cable winding mechanism in terms of time response and winding accuracy are described. Other issues related to the prototype such as dimensions, cable selection and home initialization are also discussed. This is followed by a position control scheme for CDPMs so that a desired path can be traced satisfactorily. Experiments to assess the operation of the 4-3-CDPPM prototype in terms of maintaining positive cable tensions (torques) to keep the platform fully constrained are then described. The positioning capability of this prototype, e.g. accuracy, repeatability and tracking error, is also addressed.

6.2 Prototype Development

Four CDUs are needed to realize three DOFs of the moving platform. Firstly, the fabrication of the CDU, together with its verification, is mentioned. Then other issues related to the development of the whole 4-3-CDPPM system are presented.

6.2.1 Cable Driving Unit

Figure 6.1 shows the prototype of the cable driving unit, comprising the cable winding mechanism and a servo motor. The detailed drawings of mechanical parts can be found in Appendix D.

Figure 6.1: Cable driving unit

In order to achieve a compact CDU and to shorten the development time, a servo motor that comes with a built-in motion control system is preferred. The SmartMotorTM (model SM2310) from ANIMATICS[®] [101] is selected to drive the winding mechanism through a timing belt drive. It consists of a brushless D.C. motor, a PID micro-controller, a power amplifier, an optical encoder, and a RS232/485 communication port. Various motion

control strategies can be implemented using the low level control commands provided. Position control mode is the default mode of operation for the SmartMotor. However, it is also possible to operate in open-loop torque control mode which is useful in initializing positive cable tensions in the home position.

(a) Time response

The time response of the CDU to the step input signals is shown in Fig. 6.2. One revolution of the SmartMotor is equal to 2000 counts. Coefficients of the PID controller are set at the default values. The response time to the two different step inputs is about at 0.3 s.

(a) Step input: 200

(b) Step input: 500

Figure 6.2: Time response of the CDU to step inputs with PID turning values: $K_P = 42$, $K_I = 28$ and $K_D = 550$

(b) Winding accuracy

Two experiments are conducted in order to evaluate the tolerances in the vertical and horizontal directions of the winding point on the winch. As shown in Fig. 6.3, a probe is placed at the lowest point of the groove to measure the error in vertical direction. For the horizontal direction, it is placed so that it is just touching with the side of the groove.

The motor has a resolution of 2000 counts/rev. The speed reducing ratio of the CDU is 4. Hence, each revolution of the winch corresponds to 8000 counts. The procedure to collect the data are described below:

Figure 6.3: Configuration of testing the cable driving unit

- A location on the winch is chosen as the reference point.
- The motor is activated at a unit step of 500 counts. The values shown in the reader are recorded.
- The previous step is repeated until the count of 63000 is reached.

The procedure is repeated for both cases in measuring the errors in the vertical and the horizontal directions. The setup of the experiments are shown in Fig. 6.4. A probe (Mahr Millitron 1240) with a resolution of $0.1 \mu\text{m}$ is used to sense for variations on the groove as the motor is rotated in steps of 500 counts.

(a) Vertical measurement setup

(b) Horizontal measurement setup

Figure 6.4: Experimental setup for the testing of CDU

The plots of the data points collected for both the vertical and the horizontal directions

are shown in Fig. 6.5(a) and Fig. 6.5(c) respectively. They are found to be in cyclic movements due to cyclic manufacturing disturbance as can be seen from the curve fitted on the data points (Fig. 6.5(b) and Fig. 6.5(d)). From the plots, the tolerance in the vertical direction is found to be $\pm 101.4 \mu\text{m}$ and that in the horizontal direction is $\pm 131.7 \mu\text{m}$. The CDU winds and unwinds a cable from a roller. The distance from the CDU to the roller is often a lot larger than the measured tolerances. Hence, the tolerances in the vertical and horizontal directions are acceptable and can be ignored.

Figure 6.5: Tolerance of the cable driving unit

6.2.2 4-3-CDPPM Prototype

The prototype of a 4-3-CDPPM system is shown in Fig. 6.6.

Figure 6.6: The 4-3-CDPPM system

A few issues related to the development of the prototype are described below:

- Geometrical dimensions:** Four identical CDUs are used and located at the corners of the aluminum frame. The moving platform has a dimension of 0.087 m by 0.073 m. The platform, with roller balls underneath is able to move smoothly on an aluminium plate. The cables are routed using vertical rollers from the CDU to the suspending points on the corners of the moving platform. The vertical rollers at the four corners define the base of the 4-3-CDPPM. The dimension of rectangular base is 0.46 m by 0.75 m.
- Cable selection:** The selection of cables is based on the results obtained from tensile test. Figure 6.7 shows the plots from tensile tests conducted on several types of cables using the Instron machine. The sample lengths of the cables are 150 mm. It is desirable to select a cable which is flexible enough to be routed and has negligible elongation under load. Verhoeven [29] noted that the cable which has the ratio of the maximum tension over the stiffness per reciprocal unit length smaller than 0.5% is satisfactory. Based on

the plots, the nylon coated wire with $\varnothing = 0.8$ mm is chosen for the 4-3-CDPPM.

Figure 6.7: Tensile tests of various cables

- **Control program:** A graphical user interface (GUI) is developed in Visual C++ 6.0 to control the system. The main functions are to issue high-level coordinated motion commands and to synchronize the four CDUs so that the moving platform can track the specified trajectories smoothly and closely. A simulation option is also included for the users to test the planning data before implementing the trajectory.

- **Hardware connection:** With the computer as a host, the four motors are connected to a single RS-232 port of the host in a daisy chain arrangement as shown in Fig. 6.8. The connection box functions as an electrical circuit interface between the host and the motors. It enables the users to select one of the motors or all four motors at the same time for testing during the experiment. The solid line indicates the double pole, double throw switch in “ON” position whereas the dashed line indicates the switch in “OFF” position. To activate the motor, the respective switch would be set to the “ON” position, establishing connections between the host and the motors.

Figure 6.8: Connection diagram from the host to the SmartMotors

• **Home position:** A swing arm mechanism as shown in Fig. 6.6 is used to set the ‘home’ position of the moving platform. This will be the reference position for the moving platform. When the command “Home Initialization” is executed in the GUI, all the motors are switched to the torque mode. Under the torque mode, the motor shaft will be applied a specified torque independent of the platform position in a direction that causes the cable to be in tension. The home mechanism is then used to determine the initial pose of the platform by aligning the platform relative to the mechanism. This is achieved while the four cables are in tension as the motors are run under the torque mode. Once the alignment is done, the command “Home Initialization” is executed again changing the operation of the motors to the position mode. All the values of position and orientation are set to zero.

6.3 Position Control of CDPMs

Figure 6.9 shows the position control scheme for CDPMs. The objective is to control the motors in moving the platform from a current pose to the desired pose.

In this independent joint control approach, although the system dynamics is not taken into account in the PID control laws, it is necessary to ensure that the resultant torques do not

exceed the maximum torque of the actuators as the platform tracks planned trajectories.

Figure 6.9: Control scheme for CDPMs

As shown in Fig. 6.9, the trajectory planning, the kinematic calculation and the torque optimization are carried out off-line. All the points on the trajectory must satisfy the tension condition to ensure that the four CDUs are able to drive the platform properly at run time. Once the appropriate trajectory is planned, the values of joint kinematics obtained from the inverse kinematics are distributed to the CDUs. This is carried out by the host computer where the current kinematics of the CDUs are monitored based on the data from the rotary encoders. Hence for coordinated motions, the function of the host computer is to perform the high level tasks such as trajectory planning, torque optimization, and motor synchronization, whereas the closed-loop servo control is executed by the PID micro-controller inside the motors.

6.4 Experimental Results

Experiments are carried out to assess the operation of the 4-3-CDPPM prototype in terms of maintaining cable tensions (torques) and to estimate the positioning capability of this prototype in terms of accuracy, repeatability and tracking error.

6.4.1 Tension Measurements

The experimental setup to measure the tension of the cable is shown in Fig. 6.10. A strain gage sensor roller is placed between two fixed guide rollers. The tension in the cable can be determined based on the deflection of the roller picked up by the strain gage. The output signal from the tension sensor is processed and the tension is displayed on the meter.

Figure 6.10: Experimental setup for measuring cable tension

For the 4-3-CDPPM prototype, the base frame is located at the centre of the base (origin). The platform is initially positioned at the base centre - “Home Position”, and the four cables are set at a pre-tension of 3 N. During the experiment, the platform is programmed

(a) Tension values, $x = 0, \phi = 0^\circ$

(b) Tension values, $x = 0, \phi = +3^\circ$

(c) Tension values, $x = -0.1, \phi = 0^0$

Figure 6.11: Plots of cable tensions and their tension factors

to move along the y -axis from 0 to 0.2 m in steps of 0.04 m before returning to the home position. Likewise, the platform is then moved from 0 to -0.2 m in steps of -0.04 m. The tension values in the cables are measured statically at every incremental step of the platform. This is carried out for three different initial poses of the platform: $(x = 0, \phi = 0^0)$, $(x = 0, \phi = +3^0)$, and $(x = -0.1, \phi = 0^0)$.

Based on the measured tension values, the tension factor (TF) at each position can also be calculated. Figure 6.11 shows the plots of the tension values and the tension factors (TF). The experimental results show that the tension of the four cables is always kept positive for the three cases. In particular, the tension values can be adjusted within the allowable tension range from 3 to 54 N according to the specified minimum tension of cables and maximum torque of actuators. It is seen that the tension factor decreases as the platform is moved further from the base centre which is consistent with the analysis. The experimental results also show that the tension factor is smaller when the orientation

of the platform increases. On the whole, it can be concluded that the experimental tension factor follows closely the trend predicted by the analytical analysis. The difference between the two set of results is expected as the positional control system is implemented without the cable tension feedback. Any positional error will result in a different set of cable tensions, thus affecting the tension factor. Besides, the accuracy of actuators also contribute to this difference.

6.4.2 Accuracy, Repeatability and Tracking Errors

(a) Errors in position and orientation

For a 4-3-CDPPM, the positional errors are represented by the accuracy in position (x, y) and orientation (ϕ) . In this experiment, an articulated type Coordinate Measuring Machine known as Faro arm (Fig. 6.12(a)), which has a resolution of 0.001 mm, is used to measure the position and the orientation of the moving platform statically. Note that the position of the platform is measured at its centre point whereas two discrete points are needed to determine the orientation as shown in Fig. 6.12(b).

(a)

(b)

Figure 6.12: Faro arm used for static measurement

(a) Positional error, $\phi = 0^\circ$

(b) Positional error, $\phi = +3^\circ$

(c) Positional error, $\phi = -3^\circ$

(d) Directional error, $x = 0$

Figure 6.13: Static errors in position (a,b,c) and in orientation (d)

To measure the positional error, the platform is programmed to move in steps of 40 mm along the y axis while keeping the x coordinate fixed. This is repeated for x values of -80 , -40 , 0 , $+40$ and $+80$ mm for three different orientations of the platform: 0° , $+3^\circ$ and -3° . The experimental results are plotted as shown in Figs. 6.13(a), 6.13(b) and 6.13(c). The results show that the positional error of the 4-3-CDPPM prototype is about ± 0.4 mm. Similarly, Fig. 6.13(d) shows the directional errors as the platform is programmed to move for various sets of y and ϕ along the line $x = 0$. Note that large orientations cannot be achieved at poses which is far from the origin ($y = 0$). For each pose, coordinates of two points are recorded and then combined to obtain the orientation. From the experimental results, the error in orientation for the 4-3-CDPPM prototype is approximately $\pm 0.5^\circ$.

(b) Repeatability

In this experiment, the platform is moved from the origin to 7 points located closely to the workspace boundary as shown in Fig. 6.14.

Figure 6.14: Sample points

For each point, the procedure is repeated for 20 times and the actual displacement data X_i ($i = 1, 2, \dots, 20$) are measured by the Faro arm (Fig. 6.12(a)). The repeatability of the 4-3-CDPPM prototype is evaluated by the standard deviation:

$$\sigma = \sqrt{\frac{1}{N-1} \sum_i^N (X_i - \bar{X})^2} \quad (6.1)$$

where N is the number of the same action and $\bar{X}$ is the mean value:

$$\bar{X} = \frac{\sum_i^N X_i}{N} \quad (6.2)$$

The repeatability evaluation is shown in Table 6.1. Because of the symmetric design, these sample points in the first quadrant are representative for the whole workspace. As the result of the experiment, the repeatability of the 4-3-CDPPM prototype is about ± 0.1 mm.

Table 6.1: The repeatability at sample points

Point	S_1	S_2	S_3	S_4	S_5	S_6	S_7
σ (mm)	0.117	0.097	0.113	0.111	0.099	0.092	0.090

(c) Tracking errors

In this experiment, a pointed device is fixed at the centre of the platform to trace its trajectory. The trajectory is then scanned and processed to obtain the coordinates of the path. The platform is programmed to move in circles with radii (10, 20, 30, 40 and 50 mm) and angular velocities (from 0.56 to 16.76 with an increment of 0.56 rad/s).

Figure 6.15 demonstrates a trajectory tracking when the platform traces along a circle with radius of 30 mm and at an angular velocity of 4.48 rad/s and Fig. 6.16 shows the radial tracking error between the desired circle and the actual circle. The results indicate that an increment in radius results in a higher error. In addition, a higher angular velocity also leads to a higher error. In particular, this error is considerably higher at a high velocity

Figure 6.15: Trajectory tracking at an angular velocity of 4.48 rad/s

with a large radius. This is expected as the present control algorithm is based only on kinematic constraints. In general, if the platform moves slowly, e.g. at an angular velocity of 0.56 rad/s, the dynamic effect will be insignificant and the resultant error is less than 0.5 mm.

Figure 6.16: Tracking errors with respect to trajectory radii and angular velocities

6.5 Summary

In this Chapter, the development of a 4-3-CDPPM prototype for the purpose of verifying the analytical analysis is described.

From the experiments, the vertical tolerance and the horizontal tolerance of the grooves for the CDU prototype are found to be $\pm 101.4 \mu\text{m}$ and $\pm 131.7 \mu\text{m}$, respectively. The tolerances are acceptable and can be ignored. A simple independent joint control approach is used for position control of fully restrained CDPPMs. In this control scheme, the trajectory planning, the kinematic calculation and the torque optimization are carried out offline before the joint displacements are controlled by the four cable driving units.

Experimental results show that the 4-3-CDPPM prototype can achieve an accuracy of $\pm 0.4 \text{ mm}$ in position and $\pm 0.5^\circ$ in orientation with the repeatability of $\pm 0.1 \text{ mm}$ while maintaining appropriate cable tensions. It is also shown that the experimental tension factors follows closely the trend predicted by the analytical analysis. Hence, it can be concluded that the simple control scheme is successfully implemented for position control of a 4-3-CDPPM. The prototype can be used as the basis for industrial and military applications that requiring large working areas but not high accuracy, such as security patrol and traffic monitoring using cable-driven cameras, equipment identification, and plate cutting etc. Due to the hardware limitation, the dynamic control is not implemented in the current prototype. This needs to be investigated in future studies.

Chapter 7

Conclusion and Future Work

In this thesis, the fundamental research issues of cable-driven parallel mechanisms (CDPM) including kinematic analysis, tension analysis, workspace analysis, design optimization, dynamic analysis and torque optimization have been addressed. In this chapter, the main contributions of this work are first summarized, and then the future research directions are outlined.

7.1 Contributions

With the objective of establishing the fundamental and generic tools for systematic analysis of CDPMs, the achievements of this research work can be summarized as follows:

- **A generic approach for kinematic formulation of CDPMs.**

Due to the kinematic redundancy in CDPMs, the analytical displacement solution for a general CDPM is hardly obtainable. In this thesis, the forward displacement solution of a general CDPM is computed by using the numerical Newton-Raphson method. This method is generic and is effective due to its property of quadratic convergence. However, to obtain the Newton-Raphson matrix, much computational

effort is needed to calculate partial derivatives in a conventional procedure. An explicit relation between the Newton-Raphson matrix and the Jacobian matrix is proposed to improve the Newton-Raphson method in terms of computational efficiency. Simulation examples are given to show the effectiveness of this method in terms of the number of iterations and the error values. The modified procedure of Newton-Raphson method is applicable to both the completely restrained CDPM ($m = n + 1$) and the redundantly restrained CDPM ($m > n + 1$). In addition, a generic approach together with the formulations for acceleration analysis, which is essential to build the system dynamic model, is presented for CDPMs. The expressions of acceleration analysis are derived in details for a 4-3-CDPPM and a 8-6-CDSPM.

- **Formulation of three tension conditions and a recursive dimension reduction method for tension analysis.**

For cable-driven parallel mechanisms (CDPMs), the unilateral characteristic of cables (can only pull but not push) makes the cable tension the most important factor in analyzing the kinetostatic performance of the CDPMs such as workspace and stability. Three tension conditions for CDPMs are identified and formulated. They are the force-closure condition, the feasible wrench condition and the wrench set condition. The force-closure condition is primarily for fully restrained CDPM. The feasible wrench condition and the wrench set condition are more practical because the tension bound i.e. the range of allowable cable tensions, is taken into account. The three tension conditions are fundamental tools in analyzing and designing CDPMs. Both the null space and geometric approaches become very complex when checking the tension conditions in high-dimensional redundant systems. A recursive dimension reduction method is proposed to decompose an n -dimensional system into 1-dimensional subsystems. It enables the tension conditions to be checked in 1-dimensional system requiring only scalar operations. Moreover, this approach is able to cope with any external wrench set which appears in the wrench set condi-

tion as long as this wrench set is specified. Three algorithms FCC, FWC and WSC are developed to check the three tension conditions, respectively. Computational examples show that the proposed algorithms are effective.

- **A generic strategy for generating and quantifying the workspace, a global tension index for evaluating the workspace performance, and a methodology for design optimization of CDPMs.**

Based on the three tension conditions, the force-closure workspace and the feasible wrench workspace are proposed and defined. Due to the limitation of the analytical approaches, generic strategies for numerically generating and quantifying the workspace of a general m - n -CDPM are proposed. Geometrical constraints are primarily identified from the analytical boundaries. This helps to limit the search space and therefore makes the numerical method more efficient. The proposed numerical workspace analysis method with three tension conditions embedded can be used for the analysis and design of any kind of CDPM. A tension factor (TF) is proposed to be used as a performance index to evaluate the quality of force closure for CDPMs at a specific configuration. It reflects the relative tension distribution among the cables for a specific platform pose inside the force-closure workspace. From the simulation results, it has been shown that the nearer the platform is to the boundary of the workspace, the smaller the TF becomes. The global tension index is proposed to evaluate the quality of tension distribution over the entire workspace. Two design problems to meet practical needs are described. An optimization methodology for addressing these design problems has been presented.

- **A modular cable driving unit with a novel cable winding mechanism, the formulation of the system dynamic model, and an efficient torque optimization approach.**

A novel cable winding mechanism is presented. The winch of the mechanism is designed to have a combined translational and rotational motion so that a constant $\Delta l/\Delta\theta$ and a stationary contacting point are achieved. The features are useful in

simplifying the system dynamic model as well as in improving the dynamic performance of the system. The winding mechanism is integrated with a ‘smart’ servo motor consisting of a brushless D.C. motor, a PID micro-controller, a power amplifier, an optical encoder and a communication port to form the cable driving unit. The cable driving unit is compact and can be used as a standard module for building CDPMs. The dynamic equations of motion of a CDPM, including the dynamic models of both the cable-driven mechanism and the cable driving unit, is formulated. It is shown that by using the proposed cable-driving units for actuation, the dynamic model can be expressed in a linear system for specified dynamic terms. In addition, the torques can also be systematically checked by the feasible wrench condition. A method of replacing a bounded torque variable by two nonnegative variables enables the torque optimization problem to be expressed in a standard linear programming form with the objective function as finding the minimum summation of torques. The optimal torque values always exist as long as the feasible wrench condition is satisfied. The motor torques of a 4-3-CDPPM, as the platform traces a linear trajectory and a circular trajectory, are computed. The results show that the proposed torque optimization scheme is effective. It is seen that the resultant torques are within the specified range with the trajectory profile represented as a quintic polynomial. The system dynamic model as well as the optimal torque solution have been verified for 4-3-CDPPMs and 8-6-CDSPMs.

- **A prototype 4-3-CDPPM and experimental studies.**

A prototype of the 4-3-CDPPM has been designed and built for the purpose of verifying theoretical analysis such as kinematics, cable tension and workspace. A computer program providing the GUIs for the computation and control of CDPMs are also developed (Appendix E). The performance of the cable winding mechanism is found to be satisfactory in achieving a stationary contacting point and a constant ratio of $\Delta l / \Delta \theta$. Experimental results show that the 4-3-CDPPM prototype can achieve an accuracy of ± 0.4 (mm) in position and $\pm 0.5^\circ$ in orientation while main-

taining appropriate cable tensions. The repeatability of the mechanism is found to be ± 0.1 mm. It is also shown that the experimental tension factors follows closely the trend predicted by the analytical analysis. A simple control scheme is successfully implemented for position control of a 4-3-CDPPM. The prototype can be used as the basis for industrial and military applications that requiring large working areas but not high accuracy, such as security patrol and traffic monitoring using cable-driven cameras, equipment identification, and plate cutting.

From the achievements presented above, the main contributions of this work are:

- A recursive dimension reduction method - a systematic approach for tension analysis in verifying tension conditions.
- A generic procedure to generate and quantify two types of workspace, namely force-closure workspace and feasible wrench workspace.
- A proposal of tension factor and global tension index - performance indices of evaluating the force-closure property of fully restrained CDPMs.
- A novel design of the cable driving unit.

In brief, the research work has presented a general framework for the analysis and design of fully restrained CDPMs.

7.2 Future Work

The future research directions for cable-driven parallel mechanisms are outlined below:

- In our analyses, the suspending point is considered as a stationary point. For planar cases, this can be validated by using a roller mechanism as shown in Fig. 7.1, i.e.

point B is stationary. For spatial mechanisms in general, it is difficult to ensure that suspending points are really “stationary points”. Therefore, in order to control cable length precisely, pulley radii should be accommodated in all formulation.

Figure 7.1: A roller mechanism to obtain a stationary suspending point

- A practical kinematic control algorithm has been implemented. To make CDPMs applicable in applications with high speed, high precision tracking, appropriate control strategies which include the system dynamic model should be developed. In addition, tension control loop should be incorporated to smoothly distribute tension among the cables.
- One of the main concerns in manipulation systems is the positioning accuracy of the end-effector. Practically, the stiffness of cables is relatively low compared to that of rigid links. This definitely affects the precision of CDPMs. Therefore, identifying how the cable stiffness would dynamically affect the positioning accuracy is necessary. In addition, calibration models need to be developed to improve the position accuracy of CDPMs.

Appendix A

Symbolic Solutions of the 4th Order Polynomial Equation

The general fourth order polynomial equation can be written as:

$$x^4 + ax^3 + bx^2 + cx + d = 0 \quad (\text{A.1})$$

Four solutions can be found in symbolic forms as follows:

$$x_1 = -\frac{a}{4} - \frac{V}{2} - \frac{1}{2}\sqrt{2K - \frac{\sqrt[3]{2}L}{3U} - \frac{U}{3\sqrt[3]{2}} - \frac{N}{4V}} \quad (\text{A.2})$$

$$x_2 = -\frac{a}{4} - \frac{V}{2} + \frac{1}{2}\sqrt{2K - \frac{\sqrt[3]{2}L}{3U} - \frac{U}{3\sqrt[3]{2}} - \frac{N}{4V}} \quad (\text{A.3})$$

$$x_3 = -\frac{a}{4} + \frac{V}{2} - \frac{1}{2}\sqrt{2K - \frac{\sqrt[3]{2}L}{3U} - \frac{U}{3\sqrt[3]{2}} + \frac{N}{4V}} \quad (\text{A.4})$$

$$x_4 = -\frac{a}{4} + \frac{V}{2} + \frac{1}{2}\sqrt{2K - \frac{\sqrt[3]{2}L}{3U} - \frac{U}{3\sqrt[3]{2}} + \frac{N}{4V}} \quad (\text{A.5})$$

with

$$U = \sqrt[3]{M + \sqrt{-4L^3 + M^2}} \quad (\text{A.6})$$

$$V = \sqrt{K + \frac{\sqrt[3]{2}L}{3U} + \frac{U}{3\sqrt[3]{2}}} \quad (\text{A.7})$$

Where

$$K = \frac{a^2}{4} - \frac{2b}{3} \quad (\text{A.8})$$

$$L = b^2 - 3ac + 12d \quad (\text{A.9})$$

$$M = 2b^3 - 9abc + 27c^2 + 27a^2d - 72bd \quad (\text{A.10})$$

$$N = -a^3 + 4ab - 8c \quad (\text{A.11})$$

Appendix B

Fundamental Theorems of Convex Analysis

This section summarizes fundamental concepts related to convex analysis which is useful for tension analysis of CDPMs. Proofs of respective theorems are also given (Panik [102]).

B.1 Convex Sets

A subset C of $\mathbb{R}^n$ is said to be convex if for points $\mathbf{x}_1, \mathbf{x}_2 \in C$, their convex combination (or internal average)

$$\mathbf{x} = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2, \quad \lambda_1 + \lambda_2 = 1, \quad \text{and } 0 \leq \lambda_1, \lambda_2 \in \mathbb{R} \quad (\text{B.1})$$

is also a member of C . Geometrically, this definition implies that the closed line segment joining $\mathbf{x}_1$ and $\mathbf{x}_2$ lies entirely in C as illustrated in Fig. B.1. A subset C of $\mathbb{R}^n$ containing only a single element is taken to be (trivially) convex.

Theorem B.1 *Let $\{C_i\}$ be a family of convex sets in $\mathbb{R}^n$. Then their intersection $\cap_i C_i$ is a convex set.*

Proof: Let $\mathbf{x}_1, \mathbf{x}_2 \in \cap_i C_i$. Then $\mathbf{x}_1, \mathbf{x}_2 \in C_i$ for all i . Since each C_i is convex, their convex combination $[\mathbf{x}_1, \mathbf{x}_2] \in C_i$ for all i . Thus $[\mathbf{x}_1, \mathbf{x}_2] \in \cap_i C_i$ so that the intersection is convex. ||

Figure B.1: Illustration of convexity

Theorem B.2 *A set C in $\mathbb{R}^n$ is convex if and only if it contains all the convex combinations of its elements.*

Proof (necessity): Given that C is convex, it is desired to demonstrate that each convex combination of points from C belongs to C . The proof proceeds by induction on the number m of points from C used to form the convex combination.

For $m = 1$, $\mathbf{x}_1 \in C$ trivially.

For $m = 2$, $[\mathbf{x}_1, \mathbf{x}_2] \in C$ by definition.

Next, make the induction hypothesis that C is closed under taking all convex combinations of fewer than m vectors, i.e. $m-1$ points from C . Hence, constructing a linear combination of m points from C , i.e. for $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m \in C$ forms:

$$\mathbf{x} = \sum_{i=1}^m \lambda_i \mathbf{x}_i, \quad \sum_{i=1}^m \lambda_i = 1, \quad \lambda_i \geq 0 \text{ for all } i \quad (\text{B.2})$$

if $\lambda_i = 1$, then $\mathbf{x} = \mathbf{x}_m \in C$. If $\lambda_i < 1$, then $\sum_{i=1}^{m-1} \lambda_i = 1 - \lambda_m > 0$ so that

$$\mathbf{x} = \sum_{i=1}^{m-1} \lambda_i \mathbf{x}_i + \lambda_m \mathbf{x}_m = (1 - \lambda_m) \sum_{i=1}^{m-1} \left(\frac{\lambda_i}{1 - \lambda_m} \right) \mathbf{x}_i + \lambda_m \mathbf{x}_m \quad (\text{B.3})$$

$$= (1 - \lambda_m) \bar{\mathbf{x}} + \lambda_m \mathbf{x}_m \quad (\text{B.4})$$

By the above assertion (the induction hypothesis), $\bar{\mathbf{x}} \in C$. Since $\mathbf{x}$ is a convex combination of $\bar{\mathbf{x}}$ and $\mathbf{x}_m$, it follows that $\mathbf{x} \in C$.

(sufficiency) That every convex combination of points from C implies that C is convex from the definition of convexity. ||

The next theorem considers a key property of "the set of all convex combinations of a finite number of points".

Theorem B.3 *Let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ be a finite collection of points in $\mathbb{R}^n$. Then the set of all convex combinations of the $\mathbf{x}_i$, $i = 1, \dots, m$,*

$$C = \left\{ \mathbf{x} \mid \mathbf{x} = \sum_{i=1}^m \lambda_i \mathbf{x}_i, \quad \sum_{i=1}^m \lambda_i = 1, \quad 0 \leq \lambda_i \in \mathbb{R} \text{ for all } i \right\}$$

is convex

Proof: Let $\mathbf{u}, \mathbf{v}$ be any vectors expressible as convex combinations of the $\mathbf{x}_i$, $i = 1, \dots, m$, so that

$$\mathbf{u} = \sum_{i=1}^m \lambda'_i \mathbf{x}_i, \quad \sum_{i=1}^m \lambda'_i = 1, \quad \lambda'_i \geq 0 \text{ for all } i; \quad (\text{B.5})$$

$$\mathbf{v} = \sum_{i=1}^m \lambda''_i \mathbf{x}_i, \quad \sum_{i=1}^m \lambda''_i = 1, \quad \lambda''_i \geq 0 \text{ for all } i. \quad (\text{B.6})$$

Let us form

$$\lambda \mathbf{u} + (1 - \lambda) \mathbf{v} = \sum_{i=1}^m \left[\lambda \lambda'_i + (1 - \lambda) \lambda''_i \right] \mathbf{x}_i \quad (\text{B.7})$$

$$\sum_{i=1}^m \left[\lambda \lambda'_i + (1 - \lambda) \lambda''_i \right] = \lambda \sum_{i=1}^m \lambda'_i + (1 - \lambda) \sum_{i=1}^m \lambda''_i = 1, \quad \left[\lambda \lambda'_i + (1 - \lambda) \lambda''_i \right] \geq 0 \text{ for all } i$$

Since $\lambda \mathbf{u} + (1 - \lambda) \mathbf{v}$ is clearly a convex combination of the $\mathbf{x}_i$ for $0 \leq \lambda \leq 1$, it follows that $\lambda \mathbf{u} + (1 - \lambda) \mathbf{v} = [\mathbf{u}, \mathbf{v}] \in C$ and thus C is a convex set. ||

It is briefly noted that if $C_1, C_2, \dots, C_m$ are each convex sets in $\mathbb{R}^n$, then their convex combination $C = \sum_{i=1}^m \lambda_i C_i$, $\sum_{i=1}^m \lambda_i = 1$, $0 \geq \lambda_i \in \mathbb{R}$ is also convex.

B.2 Convex Hull

The convex hull (or convex closure) of a set S in $\mathbb{R}^n$, denoted by $\text{co}(S)$, is the collection of all convex combinations of points from S or:

$$\text{co}(S) = \left\{ \mathbf{x} \mid \mathbf{x} = \sum_{i=1}^m \lambda_i \mathbf{x}_i, \mathbf{x}_i \in S, \sum_{i=1}^m \lambda_i = 1, 0 \leq \lambda_i \in \mathbb{R} \text{ for all } i \right\} \quad (\text{B.8})$$

Figure B.2 illustrates the convex hull of a set S and of a finite collection of points in $\mathbb{R}^2$.

Figure B.2: Illustration of convex hull

Theorem B.4 *A set H in $\mathbb{R}^n$ is the smallest convex set containing an arbitrary set S in $\mathbb{R}^n$ if and only if $H = \text{co}(S)$.*

Proof (Sufficiency): It is first demonstrated that if $H = \text{co}(S)$, then $\text{co}(S)$ is the smallest convex set containing S . By theorem B.3, $\text{co}(S)$ is convex. Moreover, for every convex set C in $\mathbb{R}^n$ containing S , $\text{co}(S) \subseteq C$. This inclusion statement must be true since C is convex and, given $S \subseteq C$, it follows from the theorem B.2 that every convex combination of elements of S belongs to C .

(necessity): It is next demonstrated that if H is the smallest convex set containing S , then $H = \text{co}(S)$. Now, it is known that $S \subseteq H$ and $H \subseteq C$ for every convex set C containing S . Since $\text{co}(S)$ is a convex set containing S , it follows that $H \subseteq \text{co}(S)$. However, from the first part of the proof and from $S \subseteq H$, it leads to $\text{co}(S) \subseteq H$ and thus $H = \text{co}(S)$. $\parallel$

How should the word "smallest" in the preceding theorem be interpreted? The answer is provided by the following theorem.

Theorem B.5 *The convex hull of a set S in $\mathbb{R}^n$ is the intersection of all convex sets C_i in $\mathbb{R}^n$ which contains S , i.e. $\text{co}(S) = \cap_i C_i$, where $S \subset C_i$ for all i .*

Proof: Clearly $\cap_i C_i$ must contain S and, by theorem B.1, is convex. In addition, for any convex set C_i containing S , $\cap_i C_i$ is the smallest convex set which contains S . Hence, by the theorem B.4, $\cap_i C_i = \text{co}(S)$. ||

Theorem B.6 *Let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ be a finite collection of points in $\mathbb{R}^n$. Then the convex hull of the $\mathbf{x}_i$, $(i = 1, 2, \dots, m)$ is the set of all their convex combinations, i.e.*

$$H = \text{co}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\} = \left\{ \mathbf{x} \mid \mathbf{x} = \sum_{i=1}^m \lambda_i \mathbf{x}_i, \quad \sum_{i=1}^m \lambda_i = 1, \quad 0 \leq \lambda_i \in \mathbb{R} \text{ for all } i \right\}$$

Proof: This can be proved by induction on the number of points m . The objective is to demonstrate that every convex set containing $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$ also contains $\text{co}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$.

For $m = 1$, the theorem is true since $\mathbf{x} = \mathbf{x}_1$ for $\lambda_1 = 1$.

It is assumed that the theorem is true for $m - 1$ so that

$$H_1 = \text{co}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{m-1}\} = \left\{ \mathbf{x} \mid \mathbf{x} = \sum_{i=1}^{m-1} \nu_i \mathbf{x}_i, \quad \sum_{i=1}^{m-1} \nu_i = 1, \quad 0 \leq \nu_i \in \mathbb{R} \text{ for all } i \right\}$$

looking next to the convex hull of $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$, it should be evident that $\mathbf{x}_m$ as well as every element of H_1 must be a member of H . Let the set of convex combinations of $\mathbf{x}_m$ and points in H_1 be represented as

$$H_2 = \left\{ \mathbf{x} \mid \mathbf{x} = \lambda \mathbf{x}_m + (1 - \lambda) \sum_{i=1}^{m-1} \nu_i \mathbf{x}_i, \quad 0 \leq \lambda \leq 1 \right\}$$

If $\lambda_i = (1 - \lambda)\nu_i$ ($i = 1, 2, \dots, m - 1$) and $\lambda_m = \lambda$, then $\sum_{i=1}^m \lambda_i = \sum_{i=1}^{m-1} (1 - \lambda)\nu_i + \lambda = 1$, $\lambda_i \geq 0$ for all i . Hence H_2 must be the set of all convex combinations of $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$ and consequently represents the convex hull of $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ given that H_1 is the convex hull of $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{m-1}\}$. So by induction, the convex hull of $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ is the set of all convex combinations of these points. ||

Appendix C

Tension Constraints for 4-3-CDPPMs Using Instantaneous Centre Approach

The instantaneous centre method can be applied to geometrically study the instantaneous mobility of the moving platform. The instantaneous centre is defined as a common point of two rigid bodies (or the two rigid bodies' extended parts) at which the two bodies have the same velocity at an instant of time. Hence, the two rigid bodies have no relative motion at their instantaneous centre. It follows that if the platform has no instantaneous centre with respect to the base, then it has no instantaneous mobility and the cables are able to constrain the platform. However, if a cable-driven planar platform has instantaneous centres relative to the base when all cables are locked, then the platform will gain instantaneous mobility or DOFs. Furthermore, the platform is said to have gained one, two or three instantaneous DOF if the instantaneous centre(s) has a fixed location, form a line or a line segment, or form an area or part of a plane respectively.

Assume that the cable i in Fig. C.1(a) is locked and in tension, the connecting point P_i can only have infinitesimal motion along the line e-e. Hence, the possible velocity of point P_i will be either v_1 or v_4 and therefore the instantaneous centres of the moving platform should lie along line f-f. If the cable i becomes slack, point P_i can then move in any directions in the lower half plane of line e-e, e.g. v_2 or v_3 . If the moving platform is assumed to have a counter-clockwise rotation, the locations of the instantaneous centres will then lie in the right half plane of line f-f. Combining these two situations, the locations of the instantaneous centres will be the shaded area on the right half plane of line f-f

Figure C.1: Instantaneous centre analysis

(including the line) denoted by $S_{i,R}$, as shown in Fig. C.1(b). Similarly, if the moving platform is assumed to have a clockwise rotation, then the locations of the instantaneous centres will be the shaded area on the left half plane of line f-f (including the line) denoted by $S_{i,L}$, as shown in Fig. C.1(c).

Figure C.2: Instantaneous centre area

A 4-3-CDPPM has four cables and therefore once a possible rotation (counter-clockwise or clockwise) of the moving platform is given, four such half planes ($S_{i,R}$ or $S_{i,L}$) defined by the cable lines B_iP_i ($i = 1, 2, 3, 4$) can be identified (see Fig. C.2). For each of the two possible rotations, its resultant instantaneous centre area will be the intersection of the corresponding four half planes. For counter-clockwise rotation, the location of instantaneous centres is given by: $S_R = S_{1,R} \cap S_{2,R} \cap S_{3,R} \cap S_{4,R}$. For clockwise rotation,

its instantaneous centres is given by: $S_L = S_{1,L} \cap S_{2,L} \cap S_{3,L} \cap S_{4,L}$. By combining the two cases, the final location of instantaneous centres of the moving platform is the union of S_R and S_L , i.e. $S_U = S_R \cup S_L$. Clearly S_U can be null, a point, a line, or an area. The platform is said to be completely restrained if and only if S_U is null. Based on this method, a workspace analysis algorithm can be readily formulated. Mathematically, the line equation for each cable l_i is given by:

$$a_i u + b_i v + c_i = 0 \quad (i = 1, 2, 3, 4) \quad (\text{C.1})$$

For each line, the coefficients a_i , b_i and c_i can be computed from positional vectors of the suspending points B_i ($b_{i,x}$, $b_{i,y}$) and the connecting points P_i ($r_{i,x}$, $r_{i,y}$) as follows:

$$\begin{aligned} a_i &= b_{i,y} - r_{i,y} \\ b_i &= r_{i,x} - b_{i,x} \\ c_i &= b_{i,x} r_{i,y} - r_{i,x} b_{i,y} \end{aligned} \quad (\text{C.2})$$

Without loss of generality, the two half planes defined by the cable line $B_i P_i$ (for counter-clockwise and clockwise rotations) can be expressed by the following two inequalities:

$$a_i u + b_i v + c_i \geq 0 \quad (\text{counter-clockwise}) \quad (\text{C.3})$$

$$a_i u + b_i v + c_i \leq 0 \quad (\text{clockwise}) \quad (\text{C.4})$$

By introducing nonnegative slack parameters λ_i ($i = 1, 2, 3, 4$) and a variable w , such that $\lambda_i \geq 0$ and $w \in \mathbb{R}$, Eqns. (C.3) and (C.4) can be uniformly written as:

$$a_i u + b_i v + c_i + \lambda_i w = 0 \quad (i = 1, 2, 3, 4) \quad (\text{C.5})$$

Equation (C.5) is equivalent to Eqn. (C.3) if $w \leq 0$ and it is equivalent to Eqn. (C.4) if $w \geq 0$. Now repeatedly writing Eqn. (C.5) for each of the four cables, it yields:

$$\begin{cases} a_1 u + b_1 v + c_1 + \lambda_1 w = 0 \\ a_2 u + b_2 v + c_2 + \lambda_2 w = 0 \\ a_3 u + b_3 v + c_3 + \lambda_3 w = 0 \\ a_4 u + b_4 v + c_4 + \lambda_4 w = 0 \end{cases} \quad (\text{C.6})$$

Since w can only be assumed to take one value at a time, Eqn. (C.6) can be used to represent both the counter-clockwise rotation (when $w \geq 0$) and the clockwise rotation

(when $w \leq 0$) in a concise and uniform manner. Equation (C.6) can be expressed in matrix form:

$$\begin{bmatrix} a_1 & b_1 & c_1 & \lambda_1 \\ a_2 & b_2 & c_2 & \lambda_2 \\ a_3 & b_3 & c_3 & \lambda_3 \\ a_4 & b_4 & c_4 & \lambda_4 \end{bmatrix} \begin{Bmatrix} u \\ v \\ 1 \\ w \end{Bmatrix} = 0 \quad (\text{C.7})$$

Clearly the instantaneous centre of the moving platform with respect to the base will exist if Eqn. (C.7) has non-trivial solutions for variables u, v and w . Equation (C.7) will have non-trivial solutions if and only if:

$$\det \begin{bmatrix} a_1 & b_1 & c_1 & \lambda_1 \\ a_2 & b_2 & c_2 & \lambda_2 \\ a_3 & b_3 & c_3 & \lambda_3 \\ a_4 & b_4 & c_4 & \lambda_4 \end{bmatrix} = 0 \quad (\text{C.8})$$

Expanding Eqn. (C.8) leads to:

$$\lambda_1 \Delta_1 + \lambda_2 \Delta_2 + \lambda_3 \Delta_3 + \lambda_4 \Delta_4 = 0 \quad (\text{C.9})$$

where

$$\Delta_1 = - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}, \Delta_3 = - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{vmatrix}, \Delta_4 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Using Eqn. (C.9), all the out-of-control configurations can be identified for 4-3-CDPPMs as follows:

- For $\prod \Delta_i = 0$, at least one set of λ_i ($i = 1, 2, 3, 4$) with $\lambda_i \geq 0$ and $\sum \lambda_i^2 \neq 0$ can be identified to satisfy Eqn. (C.9). For example, if $\Delta_1 = 0$, a positive λ_1 can be assumed and let the other λ_i take the zero value to satisfy Eqn. (C.9). Under this condition, the 4-3-CDPPM will be out-of-control. Geometrically, the condition $\prod \Delta_i = 0$ means that at least three cables intersect at a point or parallel to one another. When all $\Delta_i = 0$ ($i = 1, 2, 3, 4$), then the four cables will intersect at a point or are parallel.
- For $\prod \Delta_i \neq 0$ and $|\sum \Delta_i| < \sum |\Delta_i|$, not all Δ_i will have the same sign. Hence, at least one set of λ_i ($i = 1, 2, 3, 4$) with $\lambda_i \geq 0$ and $\sum \lambda_i^2 \neq 0$ can be identified to satisfy Eqn. (C.9). Hence, the mechanism is also out-of-control in this case.

- For $\prod \Delta_i \neq 0$ and $|\sum \Delta_i| = \sum |\Delta_i|$, all Δ_i will have the same sign. Equation (C.9) can only be satisfied if every $\lambda_i = 0$, implying that all the cables are regarded as rigid links. This is because Eqn. (C.5) will be degenerated to Eqn. (C.1). However this is impossible. Therefore, there is no instantaneous centre in this case and the moving platform is said to be fully constrained by the cables.

Based on the above, the force-closure condition of the 4-3-CDPPM at a specific posture is given by:

$$\begin{cases} \prod_{i=1}^4 \Delta_i \neq 0 \\ \sum_{i=1}^4 |\Delta_i| = \left| \sum_{i=1}^4 \Delta_i \right| \end{cases} \quad (\text{C.10})$$

This condition shows that the force-closure workspace of a 4-3-CDPPM can be indirectly generated by investigating the mobility of the moving platform.

Appendix D

Design Drawings for the Cable Driving Unit

In this section, details of the mechanical design of the cable driving unit (CDU) is presented. Figure D.1 shows the mechanical design of the CDU and Fig. D.2 shows the exploded assembly of the CDU with individual part numbers. Some important design features of the CDU are described as follows:

- A trapezoidal screw thread (10) is rigidly fixed to the base by shaft holders (11) with the usage of set screws (12) at both ends. The trapezoidal screw thread nut (16) would be located rigidly inside the winch (15). This is achieved by using a steel plate cap (17) with bolts (19) to restrict lateral displacement and set screws (20) to restrict rotational displacement between the trapezoidal screw thread nut and the winch. Hence, this configuration would allow the winch to move spirally along the trapezoidal screw thread. Note that the pitch of the winch must be equal to that of the trapezoidal screw thread.
- A timing belt drive is used for transmitting power from the motor shaft to the winch. The driving sheave (2) is connected directly to the motor shaft. The driven sheave (21) is supported by a bearing (with holder) (23) so that it can freely rotate on one of the non-threaded ends of trapezoidal screw thread. Another bearing with holder (9) would be fixed to the other non-threaded end of the trapezoidal screw thread.
- Four guiding shafts (14) would connect the two bearings (with holder), the winch and the driven sheave. Thus the shafts here function to transmit the rotation from the driven sheave to the winch to enable the winch to move spirally along the

trapezoidal screw thread. Linear bushings (18) allow the shafts to slide smoothly along the interior holes created on the winch to accommodate these shafts. The two bearings at the two ends of the trapezoidal screw thread locate the four shafts and the driven sheave via the use of three retaining rings (24) on each of the four shafts. Hence, the sequence of torque/force transmission is from motor shaft, to the driving sheave, to the timing belt (25), to the driven sheave, to the four shafts and finally to the spiral motion of the winch.

- For assembling the timing belt on the two sheaves, the two sheaves must initially be brought closer together so that the timing belt can be placed on them. Subsequently, the centre distance between the two sheaves would be increased until the timing belt is in the desired tension. Thus the design must allow the user to change the centre distance between the two sheaves. To this end, it is the work bracket (4) with the motor (8) that is allowed to translate perpendicularly to the centre line of sheaves through the use of hexagon bolts (6) and nuts (7). Thus, the base plate (1) is designed to accommodate the hexagon nuts.

Table D.1 give the individual parts that were selected to make up the design of the CDU. These parts were selected from standard mechanical components for the sake of low costs. Some parts need to be modified to match the desired dimensions. Subsequently, detailed drawings of major parts are attached.

In this design, the speed reduction ratio from the chosen belt drive is 4:1. If it is in need of a larger speed reduction ratio, this design gives an option for the integration of the Animatics's planetary gearheads without any need to change the original design. Furthermore, one design feature that is not shown here is the use and placement of the limit switches in the gaps between the winch and the two bearings. The limit switches would function to cut off the power supply to the motor as the winch translates beyond its operating limit. This is to avoid any possibility of collision occurring between the winch with either one of the two bearings.

Figure D.1: Final assembly of the CDU

Figure D.2: Exploded assembly of the CDU with individual part numbers

Table D.1: Part list

Part No.	Item description	Reference pages	Qty	Specification code (in accordance to catalogue)	D.No.
1	Steel plate ^{p,*}	[1]-1274	1	SSA-A140-BC140-T8	PL21
2	Pulley - driving sheave ^{p,*}	[1]-969	1	HTPA15 S3M060-K-P6.35	SH21
3	Set screws (for driving sheave) ^p	[1]-657	2	MSSF M4-L6	-
4	Work bracket (L-shaped for motor) ^p	[1]-1333	1	KPAW U T6-AC70-B40-L60-X30-Y40-F10-G25-V40-Q47-R47-M4-NA5-DC38	WB11
5	Socket head cap screws ^p	[1]-623	4	CB M4-L12	-
6	Hexagon bolts ^p	[1]-629	2	RSCB M4-L15	-
7	Aluminum nuts ^p	[1]-630	2	CBALN M4	-
8	Smart motor ^p	[2]-11	1	SM2310D	-
9	Bearing with holder (single type) ^{p,*}	[1]-323	1	BGRA 6001ZZ	BH21
10	Trapezoidal screw thread ^p	[1]-250	1	MTSBRX D16-L135-F12-V10-T23-G11-Q12-S12-E10	ST11
11	Holders for screw shaft ^p	[1]-111	2	SHKLA D10-H40	HS11
12	Set screws (for screw holder) ^p	[1]-657	2	MSSF M4-L6	-
13	Socket button head cap screws ^p	[1]-633	4	BCB M4-L12	-
14	Guiding shafts ^{p,*}	[1]-291	4	SFRR D4-L115-B2-S2	GS11
15	Winch ^f	-	1	-	WH11
16	Nut for trapezoidal screw thread ^p	[1]-253	1	MTSSR d16	-
17	Locking plate (to lock nut) ^{p,*}	[1]-1280	1	SSFB A45.0-B20.0-T6.0	PL22
18	Oil-free bushings - LF type ^p	[1]-154	4	LFZF d4-L6	-
19	Socket head cap screws (for winch) ^p	[1]-623	2	CB M3-L14	-
20	Set screws (for winch) ^p	[1]-657	3	MSSF M4-L12	-
21	Pulley - driven sheave ^{p,*}	[1]-969	1	HTPA60 S3M060-A-P34	SH22
22	Set screws (for driven sheave) ^p	[1]-657	2	MSSF M5-L10	-
23	Bearing with holder (double type) ^{p,*}	[1]-327	1	BGRB 6001ZZ-L25	BH22
24	E type stopper rings ^p	[1]-664	12	NETW d4	-
25	High torque synchronous belt ^p	[1]-977	1	HTUN 264S3M-60	-

* Modification done on the original part.

p The part is purchased.

f The part is fabricated.

[1] FACE - Mechanical Standard Components for Factory Automation, 2001.5 → 2002.4, Misumi Corp.

[2] ANIMATICS - Integrated Motion Control Products, Animatics Corp.

Appendix E

Computational and Control GUIs

E.1 Computational GUI - Matlab

Figure E.1: GUI for the computation

A graphical user interface (GUI) based on Matlab has been developed as shown in Fig. E.1. This GUI is basically coded for four specific CDPMs including two planar mechanisms (4-2-CDPPM and 4-3-CDPPM) and two spatial mechanisms (7-6-CDSPM and 8-6-CDSPM). Research issues are coded with respective programs which are based the three tension conditions. The following are Matlab scripts for the three tension conditions.

```
=====
% PHAM CONG BANG
% Program: Force-closure condition
% Date: 08 Oct 2004

function f=force_closure_check(maxo) % Original matrix

% check if it is non-zero matrix
if (maxo == 0)
    condition = 0;
else
    % Eliminate zero columns
    no_col = size(maxo,2);
    i = 0;
    for j=1:1:no_col
        if (max(maxo(:,j))~=0)~=0)
            i = i+1;
            maxe(:,i)=maxo(:,j);
        end
    end % No zero column exists
    % Check satisfaction in each row
    [nrow,ncol] = size(maxe);
    pos = (maxe > 0);
    neg = (maxe < 0);
    condition = 1;
    for i=1:nrow
        if ~((max(pos(i,:)) == 1) & (max(neg(i,:)) == 1))
            condition = 0;
        end
    end
end

% Beginning the algorithm
```



```

if condition
    % The base case
    if (nrow == 1)
        f = 1;
    else % The recursive part
        csign = 1;
        idc = 1; % start from the first column
        while ((csign == 1) & (idc <= ncol))
            maxa = maxe; % Use original matrix
            icol = maxa(:,1); % Swap two column: 1 - idc
            maxa(:,1) = maxa(:,idc);
            maxa(:,idc) = icol;
            % Select the pivot entry
            pivot = max(abs(maxa(:,1)));
            swaps = 0; % swap status
            idr = 1; % index row start from row 2
            while (swaps == 0)
                if (abs(maxa(idr,1)) == pivot)
                    irow = maxa(1,:); % Swap two row: 1 - idr
                    maxa(1,:) = maxa(idr,:);
                    maxa(idr,:) = irow;
                    swaps = 1;
                end
                idr = idr + 1;
            end
            % Eliminate the first column
            for i=2:nrow
                heso = maxa(i,1)/maxa(1,1);
                for j=1:ncol
                    maxa(i,j) = -maxa(1,j)*heso + maxa(i,j);
                end
            end
            maxa;
            maxb = maxa(2:nrow,2:ncol);
            csign = csign*force_closure_check(maxb);
            idc = idc + 1; % go to the next column
        end
    end
    f = csign;

```

```

    end
else
    f = 0;
end
=====

=====

% PHAM CONG BANG
% Program: Feasible wrench condition
% Date: 08 Oct 2004

function f=wrench_feasible_check(maxo,fo,to)

% Original structure matrix: maxo
% Original external force: fo
% Original tension range: to (1st row = tmin, 2nd row = tmax)

% check if it is non-zero matrix
if (maxo == 0)
    condition = 0;
else
    vo = [0; 0];
    % Eliminate zero columns
    no_col = size(maxo,2);
    i = 0;
    for j=1:1:no_col
        if (max(maxo(:,j)~=0)~=0)
            i = i+1;
            maxe(:,i)=maxo(:,j);
            tten(:,i)=to(:,j); % temporary tension
        end
    end % No zero column exists
    to = tten; % Remove the respective tension
    [nrow,ncol] = size(maxe);
    condition = 1;
end

% Beginning the algorithm
if condition

```

```

% The base case
if (nrow == 1)
    maxplus = maxe.*(maxe > 0).*to(2,:); % max of positive components
    minplus = maxe.*(maxe > 0).*to(1,:); % min of positive components
    maxminus = -maxe.*(maxe < 0).*to(2,:); % max of negative components
    minminus = -maxe.*(maxe < 0).*to(1,:); % min of negative components
    if (fo == 0)
        if ((sum(maxplus) >= sum(minminus))
            & (sum(minplus) <= sum(maxminus)))
            f = 1;
        else
            f = 0;
        end
    elseif (fo < 0)
        if ((sum(maxplus) >= (sum(minminus) - fo)) ...
            & (sum(minplus) <= (sum(maxminus) - fo)))
            f = 1;
        else
            f = 0;
        end
    else % (fo > 0)
        if ((sum(maxminus) >= (sum(minplus) + fo)) ...
            & (sum(minminus) <= (sum(maxplus) + fo)))
            f = 1;
        else
            f = 0;
        end
    end
end
else % The recursive part
    % Include the external vector + tension range vector
    maxe = [maxe fo; to vo];
    csign = 1;
    idc = 1; % start from the first column
    while ((csign == 1) & (idc <= ncol))
        maxa = maxe; % Use original matrix
        icol = maxa(:,1); % Swap two column: 1 - idc
        maxa(:,1) = maxa(:,idc);
        maxa(:,idc) = icol;
    end
end

```

```

% Select the pivot entry
pivot = max(abs(maxa(1:nrow,1)));
swaps = 0; % swap status
idr = 1; % index row start from row 2
while (swaps == 0)
    if (abs(maxa(idr,1)) == pivot)
        irow = maxa(1,:); % Swap two row: 1 - idr
        maxa(1,:) = maxa(idr,:);
        maxa(idr,:) = irow;
        swaps = 1;
    end
    idr = idr + 1;
end
% Eliminate the first column
for i=2:nrow
    heso = maxa(i,1)/maxa(1,1);
    for j=1:(ncol+1)
        maxa(i,j) = -maxa(1,j)*heso + maxa(i,j);
    end
end
maxa;
maxb = maxa(2:nrow,2:ncol);
fe = maxa(2:nrow,ncol+1); % edited external force
te = maxa(nrow+1:nrow+2,2:ncol); % swapped tension range
csign = csign*wrench_feasible_check(maxb,fe,te);
idc = idc + 1; % go to the next column
end
f = csign;
end
else
    if (fo ~= 0)
        f = 0;
    else
        f = 1;
    end
end
end

```

=====

```

=====
% PHAM CONG BANG
% Program: Wrench set condition
% Date: 08 Oct 2004

function f=wrench_set_check(maxo,wc,to,wr)

% Original structure matrix: maxo
% Wrench configuration: wc = [fx 0 0 ]
%                               [0 fy ..
%                               [.....]
% Wrench radius: wr = [rf nf]
%                               [rm nm]
% Original tension range: to (1st row = tmin, 2nd row = tmax)

% check if it is non-zero matrix
if (maxo == 0)
    condition = 0;
else
    % Assign matrix
    maxw = wc;
    % Create zero matrix
    vo = zeros(2,wr(1,2)+wr(2,2));
    % Eliminate zero columns
    no_col = size(maxo,2);
    i = 0;
    for j=1:1:no_col
        if (max(maxo(:,j)~=0)~=0)
            i = i+1;
            maxe(:,i)=maxo(:,j);
            tten(:,i)=to(:,j); % temporary tension
        end
    end % No zero column exists
    to = tten; % Remove the respective tension
    [nrow,ncol] = size(maxe);
    condition = 1;
end

```

```

% Beginning the algorithm
if condition
    % The base case
    if (nrow == 1)
        maxplus = maxe.*(maxe > 0).*to(2,:); % tension max of positive components
        minplus = maxe.*(maxe > 0).*to(1,:); % tension min of positive components
        maxminus = -maxe.*(maxe < 0).*to(2,:); % tension max of negative components
        minminus = -maxe.*(maxe < 0).*to(1,:); % tension min of negative components
        % Calculate maximum force
        unit_force_magnitude = 0;
        for i=1:wr(1,2)
            unit_force_magnitude = unit_force_magnitude + maxw(i)^2;
        end
        unit_force_magnitude = sqrt(unit_force_magnitude);
        if (unit_force_magnitude == 0)
            max_force = 0;
        else % Optimum force
            force_point = wr(1,1)*maxw(1:wr(1,2))/unit_force_magnitude;
            max_force = maxw(1:wr(1,2)).*force_point;
        end
        % Calculate maximum moment (if any)
        if (wr(2,2) > 0)
            unit_moment_magnitude = 0;
            for i=(wr(1,2)+1):(wr(1,2)+wr(2,2))
                unit_moment_magnitude = unit_moment_magnitude + maxw(i)^2;
            end
            unit_moment_magnitude = sqrt(unit_moment_magnitude);
            if (unit_moment_magnitude == 0)
                max_moment = 0;
            else
                % Optimum moment
                moment_point = wr(2,1)*maxw((wr(1,2)+1):(wr(1,2)+wr(2,2))) ...
                    /unit_moment_magnitude;
                max_moment = maxw((wr(1,2)+1):(wr(1,2)+wr(2,2))).*moment_point;
            end
        else
            max_moment = 0;
        end
    end
end

```

```

max_wrench = sum(max_force) + sum(max_moment);
if ((sum(maxplus) >= (sum(minminus) + max_wrench)) ...
    & (sum(minplus) <= (sum(maxminus) + max_wrench)) ...
    & (sum(maxminus) >= (sum(minplus) + max_wrench)) ...
    & (sum(minminus) <= (sum(maxplus) + max_wrench)))
    f = 1;
else
    f = 0;
end
else % The recursive part
    % Include the external vector + tension range vector
    maxe = [maxe maxw; to vo];
    csign = 1;
    idc = 1; % start from the first column
    while ((csign == 1) & (idc <= ncol))
        maxa = maxe; % Use original matrix
        icol = maxa(:,1); % Swap two column: 1 - idc
        maxa(:,1) = maxa(:,idc);
        maxa(:,idc) = icol;
        % Select the pivot entry
        pivot = max(abs(maxa(1:nrow,1)));
        swaps = 0; % swap status
        idr = 1; % index row start from row 2
        while (swaps == 0)
            if (abs(maxa(idr,1)) == pivot)
                irow = maxa(1,:); % Swap two row: 1 - idr
                maxa(1,:) = maxa(idr,:);
                maxa(idr,:) = irow;
                swaps = 1;
            end
            idr = idr + 1;
        end
        % Eliminate the first column
        for i=2:nrow
            heso = maxa(i,1)/maxa(1,1);
            for j=1:(ncol+size(vo,2))
                maxa(i,j) = -maxa(1,j)*heso + maxa(i,j);
            end
        end
    end
end

```

```

end
maxa;
maxb = maxa(2:nrow,2:ncol);
wce = maxa(2:nrow,ncol+1:(ncol+size(vo,2))); % edited wrench configuration
te = maxa(nrow+1:nrow+2,2:ncol); % swapped tension range
csign = csign*wrench_set_check(maxb,wce,te,wr);
idc = idc + 1; % go to the next column
end
f = csign;
end
else
f = 0;
end
=====

```

E.2 Control GUI - Visual C++

Figure E.2: GUI for the control

Figure E.2 shows a GUI which has been developed in Visual C++ 6.0 to control the planar prototype as mentioned in Chapter 5. This control program is built on the Component Object Model (COM) library, SMIEngine.dll, which is developed by Animatics Corporation. This library, which implements three component objects: SMiHost, SMIScan and SMIMotor, is briefly introduced in Fig. E.3.

Figure E.3: SMIEngine library

SMiHost Object: This object publishes three interfaces: ISMIComm, ISMIPath, and ISMICMotion. The ISMIComm interface provides basic functionality for sending commands to the motors. The ISMIPath provides a few functions to generate trajectory points for a line or arc path. The trajectory points can then be used by the Coordinated Motion interface (ISMICMotion). The ISMICMotion interface allows the user to control the motors during Coordinated Motion.

SMIScan Object: This object has two interfaces: ISMIScan and ISMIErrors. The ISMIScan interface has methods to scan an SMI program, or make a downloadable file. The ISMIErrors interface has methods such as GoFirst and GoNext to navigate through errors, and properties such as Description, Line, and column to get information about the current Error.

SMIMotor Object: This object encapsulates the methods and properties related to a SmartMotorTM. The only interface in this object, ISMIMotor, provides some properties and methods such as GoTo that makes it very easy to communicate with motors.

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