Stress investigation on interacting defects in piezoelectric materials

Zhang, Hongxia

2007

Zhang, H. (2007). Stress investigation on interacting defects in piezoelectric materials.
Doctoral thesis, Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/5287>

<https://doi.org/10.32657/10356/5287>

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**STRESS INVESTIGATION ON INTERACTING
DEFECTS IN PIEZOELECTRIC MATERIALS**

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SCHOOL OF MECHANICAL & AEROSPACE ENGINEERING

2007

STRESS INVESTIGATION ON INTERACTING DEFECTS IN PIEZOELECTRIC MATERIALS

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A thesis submitted to Nanyang Technological University
in fulfillment of the requirements for the degree of
Doctor of Philosophy

2007

ACKNOWLEDGEMENTS

My sincere appreciation and gratitude go to Professor Xiao Zhongmin, my supervisor, for his support and guidance during my doctoral program. Interaction with him has inspired me to seek exciting challenges. My sincere appreciation also goes to Dr. Chen Bingjin, for his invaluable assistance, suggestions and discussions. My sincere thanks also go to my friends here in Singapore and the technicians Mr. Koh Hai Tong and Ms. Ho Grace in the Photonics & Microsciences Laboratory.

My deepest thanks go to my parents for their emotional strength and caring concern. I also wish to thank my husband Yin Bohua for his encouragement and support during my stay here at Nanyang Technological University.

I would like to acknowledge financial support from Nanyang Technological University.

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SUMMARY

Dislocation emission mechanisms for micro crack initiation, at the tips of a semi-infinite rigid conducting line for plane case and a finite rigid conducting line for both plane and antiplane cases in a piezoelectric medium, are proposed based on the evaluation of the mechanical strain energy release rate. Due to the highly concentrated stress and electric displacement fields at the tip of a rigid line in a piezoelectric matrix, dislocations of one sign are driven away from the tip, while the stationary dislocations of the opposite sign pile up at the tip. As a result of dislocation pileup at the tip of the rigid line, micro Zener-Stroh cracks are initiated at the tip.

The initiated micro cracks are simulated by continuous distribution of piezoelectric screw and edge dislocations along the crack line for the antiplane and plane cases, respectively. Accordingly, the crack problem is formulated in terms of a set of singular integral equations. The method of Erdogan and Gupta (1972) and Erdogan et al. (1973) is adopted to solve these equations. By evaluating the mechanical strain energy release rate, the extension force for the initiated crack is derived in terms of the strain and electric field intensity factors (at the rigid line tip) and the net Burgers vector inside the crack. A relationship is accordingly obtained between the critical extension force for the rigid line and that for the crack in the same piezoelectric solid. The above investigation has been

performed on the rigid line of either semi-infinite or finite extent. In addition, two critical crack lengths are observed for both antiplane and plane cases. One is stable under the Zener-Stroh mechanism, and the other is unstable under the Griffith mechanism.

In addition to the above mentioned work, the physical problem of interaction between a piezoelectric screw dislocation and collinear rigid conducting/dielectric lines lying on the interface of two dissimilar piezoelectric materials is examined. The dislocation core is subjected to a line force and a line charge. By employing the complex variable method and superposition technique, the general solutions for the field variables are obtained. As a special case, explicit expressions of the field variables for a piezoelectric screw dislocation interacting with a single interfacial rigid line are derived. The energy release rate and the mechanical strain energy release rate are evaluated. The image force on the screw dislocation is calculated. Numerical examples are performed to analyze the effects of Burgers vector and line load on the rigid line extension force. The influence of other parameters, such as the angular position of the screw dislocation, material properties and electromechanical coupling factor, on the image force is also investigated.

Finally, the electroelastic interaction of a piezoelectric mixed dislocation with collinear rigid conducting/dielectric lines along the interface of a piezoelectric bimaterial system is investigated. The general solutions for the field variables are obtained based on the Stroh formalism and the analytical function theory. It is found that the stress and electric displacement fields at the tips of rigid conducting lines possess either a traditional square root singularity or a combination of any two of the three types of singularities: traditional square root, non-square root and oscillatory singularities. Either a square root singularity or a combination of square root and oscillatory singularities is obtained for the stress and

electric displacement fields at the tips of rigid dielectric lines. Singularity types depend on the electroelastic coefficients of the two piezoelectric materials. Energy release rate and mechanical energy release rate for the rigid line are derived in terms of the strain and electric field intensity factors, which is analogous to the stress and electric displacement intensity factors defined for interfacial cracks. For the case of a single interfacial rigid line, the explicit solutions for the field variables are presented. The image force on the mixed dislocation are evaluated and discussed in detail, due to the interface, the interfacial rigid line and other related parameters.

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LIST OF SYMBOLS

Vectors and matrices are shown in **boldface**, scalar magnitudes in lightface. The lower case subscripts for the following variables take on the value 1, 2, and 3, and upper case subscripts take on the value 1, 2, 3, and 4.

$\mathbf{b} = [b_1, b_2, b_3, b_\varphi]^T$	The Burgers vector of a piezoelectric dislocation
$\mathbf{b}^t = [b_i^t, b_\varphi^t]^T$	The total dislocations inside a Zener-Stroh crack
$\mathbf{C}$	The material property for a piezoelectric solid (antiplane case)
c_{ijkl}	The elastic moduli of piezoelectric ceramics
D_i	The electric displacement
E_i	The electric field
e_{kij}	The piezoelectric coefficients of piezoelectric ceramics
F_1, F_2	The image forces acting on a dislocation
$\mathbf{f} = [-p_i, q]^T$	The line load applied to a piezoelectric dislocation core
G	The energy release rate
G_{cr}	The critical crack extension force
G_{cr}^l	The critical rigid line extension force

G_M	The mechanical strain energy release rate
$\mathbf{K}$	The generalized stress intensity factor
K_{IC}	The fracture toughness (Mode I)
$\mathbf{k}$	The generalized strain intensity factor
$P_N^{(a,b)}$	Jacobi polynomial
$\mathbf{u} = [u_i, \varphi]^T$	The generalized displacement
u_i	The elastic displacement
x, y, z	Cartesian coordinates
x_i	Conventional indicial notation
δ	Variational symbol
δ_{JK}	Kronecker delta
ε	The oscillatory singularity index
ε_{ij}	The strain tensor
$[\varepsilon_{ij}, -E_i]^T$	The generalized strain
κ	Non-square root singularity index
κ_{ij}	The dielectric constants of piezoelectric ceramics
μ	The modulus of rigidity of isotropic elastic materials
σ_{ij}	The stress tensor
$[\sigma_{ij}, D_i]^T$	The generalized stress
ν	Poisson's ratio of isotropic elastic materials
$\Phi = [\phi_J]^T$	The generalized stress function
φ	The electric potential

CHAPTER 1

INTRODUCTION

1.1 Backgrounds

Piezoelectricity is a phenomenon of interaction between mechanical and electrical responses of materials. When a piezoelectric material is subjected to mechanical loads, an electric voltage is induced in the material. This effect is usually called the piezoelectric effect. Conversely, when a voltage is applied to such a material, it deforms. This phenomenon is known as the inverse piezoelectric effect. In 1880, the brothers Pierre Curie and Jacques Curie first discovered the piezoelectric effect. They found that certain crystalline materials generate an electric charge proportional to the mechanical stress. The inverse piezoelectric effect was theoretically deduced based on thermodynamic theory by Lippmann in 1881. Subsequently, the Curies experimentally confirmed the inverse effect and obtained a quantitative proof on the complete reversibility of electromechanical deformations in piezoelectric crystals (Qin, 2001).

Due to this intrinsic electromechanical coupling behavior, piezoelectric materials are used widely as high power sonar transducers, electromechanical actuators, and piezoelectric

power supplies. Commonly used piezoelectric materials are ceramics that are formed by conventional ceramic processing. The ceramic is then subjected to a strong DC electric field ($\sim 10^6 \text{ V/m}$) so that the permanent dipole moment can be aligned in the material (Pak, 1992b). This poling process induces piezoelectricity as well as material anisotropy in the ceramic. The resulting anisotropy can be characterized as transversely isotropic with the poling direction normal to the isotropic plane. However, piezoelectric ceramics are very brittle ($K_{IC} \approx 1 \text{ MPa m}^{1/2}$, $G_{cr} \approx 8 \text{ J/m}^2$) and susceptible to fracture (Pohanka and Smith, 1988). Also, defects, such as dislocations, cracks, cavities and inclusions produced during the manufacturing process of piezoelectric ceramics, can adversely influence the performance of such electromechanical devices. To ensure the structural integrity of these devices, it is of great importance to analyze the behavior of various defects under both electrical and mechanical fields in piezoelectric materials.

Despite the fact that electromechanical modeling of piezoelectric ceramics is complicated by electroelastic coupling and anisotropic behaviors, many contributions concerning the dislocation and crack problems in piezoelectric materials have appeared in the literature, see for example the pioneer works of Barnett and Lothe (1975), Parton (1976) and Deeg (1980). Deeg (1980) developed a method of modeling cracks in piezoelectric materials by distributing dislocations and charged dipole lines. Early in 1968, Bilby and Eshelby (1968) showed that a slit-like crack in a linear elastic isotropic medium can be represented by a continuous distribution of straight dislocations. Barnett and Asaro (1972) showed that this method can be extended to analyze cracks in anisotropic materials. In order to employ the method of distributed dislocations, one needs to have an elastic solution for a single dislocation under the same environment. The solutions due to a dislocation or a point load in an infinite medium are known as fundamental solutions. Stroh (1958, 1962) developed

an elegant formalism for treating dislocations in anisotropic linear elastic media. Barnett and Lothe (1975) extended the six-dimensional framework to an eight-dimensional formalism to treat dislocated piezoelectric solids. Later, Pak (1990a), Sosa and Pak (1990), Pak (1992b), Suo et al. (1992) and other researchers presented more detailed research work on this topic. Besides the Green's function approach, analytical methods commonly used in fracture analyses of piezoelectric ceramics include the integral transformation and the potential function method. Among them, the Green's function approach is the most effective tool in the evaluation of the stress and electric displacement fields for crack problems. The finite element method and the boundary element method are the most versatile computational tools to treat fracture problems of piezoelectric materials.

Many contributions have also been made to establish the fracture criteria for piezoelectric ceramics. Generally there are four kinds of fracture criteria for piezoelectric ceramics, namely the stress intensity factor, the energy release rate, the mechanical strain energy release rate and the local energy release rate. It was observed that the contribution of the electric field to the crack behaviors is not considered (Pak, 1990a; Pak, 1992b; Suo et al., 1992) if the stress intensity factor is used as fracture criterion. Pak (1990a) hence used the energy release rate to predict Mode III fracture. Many efforts have also been focused on the use of the energy release rate as fracture criterion. Park and Sun (1995a) proposed an alternative fracture criterion based on the mechanical energy release rate. They argued that since fracture is a mechanical process, it is reasonable to take the mechanical strain energy release rate as fracture criterion. This criterion is found to agree roughly with the measured critical value by Park and Sun (1995b). Later, Gao et al. (1997) proposed a local energy release rate criterion by investigating the problem of electric nonlinearity in fracture of piezoelectric materials.

For the past decades, rigid line inclusions (named anticrack by Dundurs and Markenscoff (1989)), have aroused considerable interest. A rigid line (the term “inclusion” is omitted for simplicity), in a certain sense, is a cut in the material and the cut is then filled with a rigid lamella of negligible thickness. It transmits tractions, but prevents a displacement discontinuity. Wang et al. (1985), Li and Ting (1989) and Fan and Keer (1993) studied a rigid line in a pure-elastic medium and led to the stress intensity factors similar to those for a crack. Hasebe (1984a, 1984b) carried out fracture mechanics analyses for a rigid line in a pure-elastic medium by giving the whole field solutions for various configurations. Xiao and Fan (1996) further proposed a criterion of micro crack initiation at the tip of a rigid line in a pure-elastic material, analogous to a conventional crack propagation. They related the fracture toughness for a rigid line to that for a crack in the same material. In the recent decades, Shi (1997) and Chen et al. (2006, 2007) examined rigid lines in a piezoelectric solid and led to the stress and electric displacement intensity factors. Xiao et al. (2003) put forward a criterion of Mode III micro crack initiation ahead of a semi-infinite rigid line in a piezoelectric medium by a mechanical mechanism, and related the critical toughness for a rigid line to that for a crack.

A number of investigations on the fracture problem of interfacial rigid lines in dissimilar pure-elastic and piezoelectric media have also been conducted. The presence of rigid lines at the interface plays an important role in the fracture behavior, see the works of Ballarini (1990), Fan and Keer (1993), Asundi and Deng (1995) and Wu (1990) for a rigid line in pure-elastic media and the works of Deng and Meguid (1998) and Gao and Fan (2001) for rigid lines in piezoelectric solids under far-field loads. Liu and Fang (2003) discussed the interaction of a screw dislocation with circular rigid conducting lines at the interface of piezoelectric media. It was found that the stresses at the tip of an interfacial rigid line in

elastic media possess a pronounced oscillatory character as in the case of an interfacial crack tip. Deng and Meguid (1998) found that the stress and electric displacement field at the tip of an interfacial rigid conducting line exhibit a combination of any two of the three kinds of singularities: traditional square root singularity, non-square root singularity and oscillatory singularity.

With the increasing usage of piezoelectric composites and multi-layered piezoelectric materials in engineering structures, there is now a strong demand for studying interacting defects at or near the interface, such as interactions between rigid lines and piezoelectric screw/edge dislocations. Moreover, in order to further understand the fracture behavior of piezoelectric ceramics containing rigid lines, it is very important to seek for a possible relationship between the critical extension force for a rigid line and the critical crack extension force in the same piezoelectric solid, based on the evaluation of the mechanical strain energy release rate, as no experimental result on the critical extension force for a rigid line has been reported in the open literature.

1.2 Objectives

The objectives of the current research are to:

- Obtain a comprehensive understanding of Zener-Stroh crack initiation mechanism near the tip of a rigid line in piezoelectric solids.
- Derive a relationship between the critical extension force for a semi-infinite rigid conducting line and that for a crack in the same piezoelectric solid.

- Derive a relationship between the critical extension force for a finite rigid conducting line and that for a crack in the same piezoelectric media.
- Derive a relationship between the antiplane critical extension force for a finite rigid conducting line and that for a crack in the same piezoelectric solid.
- Investigate the interaction of a piezoelectric screw dislocation with collinear rigid conducting/dielectric lines at the interface of two dissimilar piezoelectric materials. The dislocation core is subjected to a line charge and a line force.
- Examine a piezoelectric edge dislocation interacting with collinear interfacial rigid conducting/dielectric lines in two dissimilar piezoelectric media. The dislocation core is subjected to a line charge and a line force.

1.3 Overview of the Thesis

The basic equations for linear electro-elasticity and the general solutions for the plane and antiplane cases in piezoelectric materials are given in Chapter 2. In this chapter, two types of dislocations, complementary crack pairs, and the electric boundary conditions on the crack faces in piezoelectric media are reviewed. The distributed dislocation theory used for solving crack problems and the numerical procedures for solving singular integral equations are also introduced. Some representative works on problems of dislocations, cracks, rigid lines and inclusions are summarized.

A model on a micro crack initiated at the tip of a semi-infinite rigid conducting line in a piezoelectric solid is proposed in Chapter 3. In our study, the micro crack is simulated by distributed piezoelectric dislocations, using the solution arising from the interaction of a

piezoelectric dislocation with a semi-infinite rigid conducting line as the Green's function. Formula for the crack extension force is derived. As results, two critical crack lengths are observed and the critical extension force for a semi-infinite rigid line is related to that for a crack in the same solid.

In Chapter 4, models of micro crack initiation at the tips of a finite rigid conducting line in a piezoelectric solid are formulated for the antiplane and plane cases. Formulae for the crack extension forces for antiplane and plane cases are obtained. As results, two critical crack lengths are observed and relations between the critical extension forces for a rigid line and a crack are obtained. Furthermore, the effect of the length ratio of the initiated crack to the rigid line on the connection constants is analyzed.

In Chapter 5, the electroelastic interaction between a piezoelectric screw dislocation and collinear rigid conducting/dielectric lines at the interface of two dissimilar piezoelectric media is investigated. The general solutions for the field variables are presented. Explicit field solutions are examined in detail for the case of a single interfacial rigid line. The energy release rate and the mechanical strain energy release rate for an interfacial rigid line are formulated. The image force on the screw dislocation is calculated. The effects of some important parameters on the rigid line extension forces and the image force are discussed in detail.

Chapter 6 analyzes the electroelastic interaction of a piezoelectric mixed dislocation with collinear interfacial rigid conducting/dielectric lines in a piezoelectric bimaterial system. The general solutions for the field variables are obtained. The singularity characteristics of the near tip field variables are analyzed. The energy release rate and the mechanical

strain energy release rate for interfacial rigid lines are calculated. The exact field solutions are presented for the case of a single interfacial rigid line. The image force on the mixed dislocation is calculated. The effects of some important parameters on the image force are analyzed in detail.

Finally, conclusions of the thesis and recommendations for future work are given in Chapter 7.

CHAPTER 2

LITERATURE REVIEW AND BASIC THEORY

In this chapter, the linear theory of piezoelectricity, which is the building block for the current study, is first introduced. The characteristics of dislocations and complementary crack pairs, Griffith and Zener-Stroh cracks, and the electric boundary conditions on the crack faces in piezoelectric materials are then reviewed. Following that, the dislocation distributed technique is introduced, with which a crack can be simulated into a series of dislocations and the problem is formulated into singular integral equation. Numerical procedures for handling singular integral equations are also outlined. Finally, some representative works on the problems of cracks and rigid lines in piezoelectric media under remote loads and interaction of a piezoelectric dislocation with interfaces, cracks, inclusions and rigid lines are reviewed.

2.1 Linear Theory of Piezoelectricity

In this section, the basic equations of linear electro-elasticity summarized by Qin (2001) Zhang et al. (2001), and Taya (2005) are first reviewed. General solutions for the plane

and antiplane problems in transversely isotropic piezoelectric materials are introduced respectively. A Cartesian coordinate system (x, y, z) also referred to as (x_1, x_2, x_3) for notational purposes is adopted in this thesis.

2.1.1 Governing Equation

The variational formulation for a static linear piezoelectric solid (with body forces absent and charge free at constant temperature) occupying region V bounded by surface S can be derived from Hamilton's principle as follows (Tiersten, 1996)

$$\delta \int_V (-H) dV + \int_S (T_i \delta u_i - q \delta \varphi) dS = 0, \quad (2.1)$$

where the repeated indices imply summation, H is the electric enthalpy density, T_i is the applied surface traction, q is the applied surface charge, u_i is the displacement, and φ is the electric potential. In this formulation, the displacement and electric potential are taken to be the independent field variables. For linear piezoelectric media, the electric enthalpy density is expressed as

$$H(\varepsilon_{ij}, E_i) = \frac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \frac{1}{2} \kappa_{ij} E_i E_j - e_{ikl} \varepsilon_{kl} E_i. \quad (2.2)$$

In the above equation, c_{ijkl} , e_{kij} , κ_{ij} are the elastic moduli measured in a constant electric field, the piezoelectric constants, and the dielectric constants measured at constant strain, respectively, ε_{ij} is the strain tensor,

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad (2.3)$$

and E_i is the electric field,

$$E_i = -\varphi_{,i}, \quad (2.4)$$

where a comma followed by arguments denotes partial differentiation with respect to the arguments. The first, second and last terms of (2.2) are, respectively, the energy stored in the deformation, the energy stored in the electric fields and the interaction energy. Valid symmetry conditions for the material constants are

$$c_{ijkl} = c_{klij} = c_{ijlk} = c_{jilk}, e_{kij} = e_{kji}, \kappa_{ik} = \kappa_{ki}. \quad (2.5)$$

The variational formulation provides the following result:

– Governing field equations

$$\sigma_{ij,j} = 0, D_{i,i} = 0. \quad (2.6)$$

– Boundary conditions

$$\sigma_{ij}n_j = T_i, D_i n_i = -q. \quad (2.7)$$

– Constitutive equations

$$\sigma_{ij} = \frac{\partial H}{\partial \varepsilon_{ij}} = c_{ijkl}\varepsilon_{kl} - e_{kji}E_k, D_i = -\frac{\partial H}{\partial E_i} = e_{ikl}\varepsilon_{kl} + \kappa_{ik}E_k. \quad (2.8)$$

In these equations, σ_{ij} is the stress tensor, D_i is the electric displacement vector, and n_i is the unit normal vector.

2.1.2 General Solution Based on Stroh Formalism

There exist two powerful formalisms for solving two-dimensional anisotropic elasticity problems in which all the variables depend on x_1 and x_2 only. One is the Lekhnitskii's approach (1963), which begins with the equilibrated stress functions, then compatibility equations. The other is Stroh's formalism, which starts with the displacements, then equilibrium equations. The Lekhnitskii formalism essentially generalizes the approach of

Muskhelishvili (1953) for solving two-dimensional isotropic elastic deformations. Stroh (1958, 1962) elaborated the work of Eshelby et al. (1953) and laid the foundation for the researchers who followed him, and therefore the formalism was named after Stroh. The general solution based on the Stroh formalism resembles that obtained by the Lekhnitskii one. The Stroh formalism is indeed mathematically elegant and technically powerful in solving two-dimensional anisotropic elasticity problems, and Barnett and Lothe (1975) extended this six-dimensional framework to include the piezoelectric effect in which an eight-dimensional formalism had been developed. In the current study we employ the Stroh formalism.

For two-dimensional problems in which all the variables depend on x_1 and x_2 only, a general solution to (2.6) is given by

$$u_J = a_J f(z), \quad J = 1, 2, 3, 4, \quad (2.9)$$

where

$$z = x_1 + px_2, \quad u_4 = \varphi, \quad (2.10)$$

and p and a_J are constants to be determined. In matrix notation,

$$\mathbf{u} = \mathbf{a}f(z). \quad (2.11)$$

Thus $\mathbf{u}$ and $\mathbf{a}$ are four-vectors and $\mathbf{u}$ is called the generalized displacement. By defining

$$\mathbf{Q} = \begin{bmatrix} c_{1k1} & e_{1i1} \\ e_{1k1}^T & -\kappa_{11} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} c_{1k2} & e_{2i1} \\ e_{1k2}^T & -\kappa_{12} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} c_{i2k2} & e_{2i2} \\ e_{2k2}^T & -\kappa_{22} \end{bmatrix}, \quad (2.12)$$

we combine (2.6), (2.8), (2.9) into one equation as

$$[\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T}]\mathbf{a} = \mathbf{0}, \quad (2.13)$$

where the superscript T denotes the transpose, and $\mathbf{Q}$ and $\mathbf{T}$ are the 4×4 matrices. They are symmetric but not positive definite. However, they can be shown to be nonsingular.

Let the generalized stress (stress and electric displacement) function $\Phi = [\phi_1, \phi_2, \phi_3, \phi_4]^T$ be defined as

$$\Phi = \mathbf{b}f(z), \quad \mathbf{b} = (\mathbf{R}^T + p\mathbf{R})\mathbf{a} = -\frac{1}{p}(\mathbf{Q} + p\mathbf{R})\mathbf{a}. \quad (2.14)$$

The second equality in (2.14)₂ is from (2.13). The generalized stresses can be obtained as

$$\sigma_{j1} = -\phi_{j,2}, \quad D_1 = -\phi_{4,2}, \quad \sigma_{j2} = \phi_{j,1}, \quad D_2 = \phi_{4,1}. \quad (2.15)$$

The eight eigenvalues p of (2.13) are all complex and consist of four pairs of complex conjugates. Let

$$p_{\alpha+4} = \bar{p}_\alpha, \quad \text{Im}(p_\alpha) > 0, \quad \alpha = 1, 2, 3, 4, \quad (2.16)$$

$$\mathbf{a}_{\alpha+4} = \bar{\mathbf{a}}_\alpha, \quad \mathbf{b}_{\alpha+4} = \bar{\mathbf{b}}_\alpha, \quad \alpha = 1, 2, 3, 4, \quad (2.17)$$

where Im stands for the imaginary part and the overbar represents the complex conjugate.

The general solutions obtained by superposing eight solutions of the form of (2.11) and (2.14) associated with eight eigenvalues p_α are

$$\mathbf{u} = 2 \text{Re} \left[\sum_{\alpha=1}^4 \mathbf{a}_\alpha f_\alpha(z_\alpha) \right], \quad (2.18)$$

$$\Phi = 2 \text{Re} \left[\sum_{\alpha=1}^4 \mathbf{b}_\alpha f_\alpha(z_\alpha) \right], \quad (2.19)$$

where Re denotes the real part, $f_{\alpha+4} = \bar{f}_\alpha$ is selected and

$$z_\alpha = x_1 + p_\alpha x_2. \quad (2.20)$$

In most applications

$$f_\alpha(z_\alpha) = q_\alpha f(z_\alpha) \quad (\alpha \text{ not summed}) \quad (2.21)$$

is assumed. Hence, equations (2.18) and (2.19) reduce to, in matrix notation,

$$\mathbf{u} = 2 \text{Re}[\mathbf{A}\mathbf{f}(z)] = 2 \text{Re}[\mathbf{A}\langle f(z_\alpha) \rangle \mathbf{q}], \quad (2.22)$$

$$\Phi = 2 \operatorname{Re}[\mathbf{B}\mathbf{f}(z)] = 2 \operatorname{Re}[\mathbf{B}\langle f(z_\alpha) \rangle \mathbf{q}], \quad (2.23)$$

respectively. In the above equation, $\mathbf{A}$ and $\mathbf{B}$ are 4×4 complex matrices given by

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4], \quad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4], \quad (2.24)$$

and $\langle f(z_\alpha) \rangle$ is a 4×4 diagonal matrix

$$\langle f(z_\alpha) \rangle = \operatorname{diag}[f(z_1), f(z_2), f(z_3), f(z_4)]. \quad (2.25)$$

For simplicity, Suo et al. (1992) introduced a one-complex-variable approach which treats each element of $\langle f(z_\alpha) \rangle$ as a function of one complex variable z instead of z_α . However, z should be replaced by z_α , when calculating the α th component of the function. Thus, the generalized strains (strain and electric field) and the generalized stresses are given by

$$\mathbf{u}_{,1} = [u_{j,1}, -E_1]^T = \mathbf{A}\mathbf{F}(z) + \overline{\mathbf{A}\mathbf{F}(z)}, \quad (2.26)$$

$$\Phi_{,1} = [\sigma_{2j}, D_2]^T = \mathbf{B}\mathbf{F}(z) + \overline{\mathbf{B}\mathbf{F}(z)}, \quad (2.27)$$

$$\Phi_{,2} = [\sigma_{1j}, D_1]^T = \mathbf{B}\langle p_\alpha \rangle \mathbf{F}(z) + \overline{\mathbf{B}\langle p_\alpha \rangle \mathbf{F}(z)}, \quad (2.28)$$

respectively, where $\mathbf{F}(z) = d\mathbf{f}(z)/dz$.

Equation (2.14) can be rewritten as

$$\begin{bmatrix} -\mathbf{R}^T & \mathbf{I} \\ -\mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = p \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{R} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}. \quad (2.29)$$

Since $\mathbf{T}^{-1}$ exists, we reduce (2.29) to the following standard eigenrelation in the eight-dimensional space

$$\mathbf{N}\xi = p\xi, \quad (2.30)$$

where

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_4 \end{bmatrix}, \quad \xi = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \quad (2.31)$$

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \mathbf{N}_2 = \mathbf{T}^{-1}, \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q}. \quad (2.32)$$

It is clear that the matrices $\mathbf{N}_2$ and $\mathbf{N}_3$ are symmetric. The vector ξ in (2.30) is a right eigenvector. The left eigenvector η is defined by

$$\eta^T \mathbf{N} = p \eta^T, \mathbf{N}^T \eta = p \eta, \quad (2.33)$$

and it can be shown to be

$$\eta = \begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix}. \quad (2.34)$$

Normalization of ξ_K and η_J , which are orthogonal to each other, gives

$$\eta_J^T \xi_K = \delta_{JK}, \quad (2.35)$$

where δ_{JK} is the Kronecker delta. Making use of (2.17), (2.24), (2.31)₂ and (2.34), we write (2.25) as

$$\begin{bmatrix} \mathbf{B}^T & \mathbf{A}^T \\ \overline{\mathbf{B}}^T & \overline{\mathbf{A}}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \quad (2.36)$$

This is the orthogonality relation. The two 8×8 matrices on the left hand side of (2.36) are the inverses of each other. Their product commutes so that

$$\begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \mathbf{B}^T & \mathbf{A}^T \\ \overline{\mathbf{B}}^T & \overline{\mathbf{A}}^T \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \quad (2.37)$$

This is the closure relation which is equivalent to

$$\mathbf{A}\mathbf{B}^T + \overline{\mathbf{A}}\overline{\mathbf{B}}^T = \mathbf{B}\mathbf{A}^T + \overline{\mathbf{B}}\overline{\mathbf{A}}^T = \mathbf{I}, \mathbf{A}\mathbf{A}^T + \overline{\mathbf{A}}\overline{\mathbf{A}}^T = \mathbf{B}\mathbf{B}^T + \overline{\mathbf{B}}\overline{\mathbf{B}}^T = \mathbf{0}. \quad (2.38)$$

Hence, the three matrices $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$ defined by

$$\mathbf{S} = i(\mathbf{A}\mathbf{B}^T - \mathbf{I}), \mathbf{H} = i2\mathbf{A}\mathbf{A}^T, \mathbf{L} = i2\mathbf{B}\mathbf{B}^T \quad (2.39)$$

are real. The matrices $\mathbf{H}$ and $\mathbf{L}$ are symmetric and nonsingular. Making use of (2.39), one has the following relations

$$\mathbf{M} = -i\mathbf{B}\mathbf{A}^{-1} = i\mathbf{H}^{-1}\mathbf{S} + \mathbf{H}^{-1}, \quad (2.40)$$

$$\mathbf{M}^{-1} = i\mathbf{A}\mathbf{B}^{-1} = \mathbf{L}^{-1} - i\mathbf{S}\mathbf{L}^{-1}. \quad (2.41)$$

The tensors $\mathbf{M}$ and $\mathbf{M}^{-1}$ are Hermitian. Since the imaginary part of a Hermitian is skew-symmetric, we have

$$\mathbf{H}^{-1}\mathbf{S} + \mathbf{S}^T\mathbf{H}^{-1} = \mathbf{0} = \mathbf{S}\mathbf{L}^{-1} + \mathbf{L}^{-1}\mathbf{S}^T. \quad (2.42)$$

In fact we had used (2.42)₁ in writing (2.41). Inserting (2.40) and (2.41) into the relation $\mathbf{M}\mathbf{M}^{-1} = \mathbf{I}$ obtains

$$\mathbf{H}\mathbf{L} - \mathbf{S}\mathbf{S} = \mathbf{I}. \quad (2.43)$$

Equations (2.42) and (2.43) can be rewritten as

$$\begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^T \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^T \end{bmatrix} = - \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \quad (2.44)$$

2.1.3 General Solution for Antiplane Deformation

For a special case of a piezoelectric material representing a transversely isotropic medium belonging to a hexagonal crystal class 6mm, only the out-of-plane displacement and the in-plane electric field are considered such that

$$u_1 = u_2 = 0, \quad u_3 = u_3(x_1, x_2), \quad (2.45)$$

$$E_1 = E_1(x_1, x_2), \quad E_2 = E_2(x_1, x_2), \quad E_3 = 0. \quad (2.46)$$

In this case, the governing equation (2.6) simplify to

$$c_{44}\nabla^2 u_3 + e_{15}\nabla^2 \varphi = 0, \quad e_{15}\nabla^2 u_3 - \kappa_{11}\nabla^2 \varphi = 0, \quad (2.47)$$

where ∇^2 is the two-dimensional Laplacian operator,

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}. \quad (2.48)$$

In matrix notation, we write (2.47) as

$$\mathbf{C}\nabla^2\mathbf{u} = 0, \quad (2.49)$$

where

$$\mathbf{C} = \begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -\kappa_{11} \end{bmatrix}. \quad (2.50)$$

Examination of (2.49) reveals that the field equations can be satisfied if one lets $\mathbf{u}$ be a harmonic function vector (Pak, 1990a). This is achieved by letting $\mathbf{u}$ be the imaginary part of some analytical function vector of $z = x_1 + ix_2$ such that

$$\mathbf{u} = \text{Im}[f_3(z), f_4(z)]^T = \text{Im}[\mathbf{f}(z)]. \quad (2.51)$$

Then the generalized stress function vector Φ is expressed as

$$\Phi = \text{Re}[\mathbf{C}\mathbf{f}(z)]. \quad (2.52)$$

Equations (2.51) and (2.52) are rewritten as (2.22) and (2.23) in the notation of the Stroh formalism, respectively, if one writes

$$\mathbf{A} = -\frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad \mathbf{B} = \frac{1}{2} \mathbf{C} = \frac{1}{2} \begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -\kappa_{11} \end{bmatrix}, \quad (2.53)$$

and note that the eigenvalues are $p_1 = p_2 = i$.

The generalized strains and the generalized stresses can then be expressed by

$$\mathbf{u}_{,1} = [\varepsilon_{13}, -E_1]^T = \text{Im}[\mathbf{F}(z)], \quad (2.54)$$

$$\mathbf{u}_{,2} = [\varepsilon_{23}, -E_2]^T = \text{Re}[\mathbf{F}(z)], \quad (2.55)$$

$$\Phi_{,1} = [\sigma_{23}, D_2]^T = \text{Re}[\mathbf{CF}(z)], \quad (2.56)$$

$$\Phi_{,2} = [\sigma_{13}, D_1]^T = \text{Im}[\mathbf{CF}(z)], \quad (2.57)$$

respectively.

2.2 Dislocations and Cracks

In this section, the fundamental characteristics of two basic types of dislocations, the edge dislocation and the screw dislocation, are first reviewed. The classical Griffith crack is then recalled briefly, followed by an overview of Zener-Stroh crack. The Griffith crack and the Zener-Stroh crack form a complementary pair in view of physical parameters. Furthermore, the critical crack length of a Griffith-Zener-Stroh crack is presented from the energy argument. Finally, some crack boundary conditions in electro-elasticity theory are briefly discussed.

2.2.1 Dislocations and the Burgers Vector

A dislocation is a linear crystallographic defect that defines the boundary between slipped and unslipped portions of the crystal (Hertzberg, 1989). There are two primary types: edge dislocations and screw dislocations. Mixed dislocations combine aspects of both types. An edge dislocation is defined by the edge of an extra half plane of atoms, sketched in Fig. 2.1, where the symbol $\perp$ marks the location of the dislocation line at the core of the dislocation (Guy and Hren, 1974). The dislocation line extends out of the crystal, perpendicular to the plane of the drawing in Fig. 2.1. By introducing this extra half plane, the upper part of the crystal is compressed, while the region below the dislocation experiences considerable dilatation. By convention, the bottom edge of the half plane is defined as a positive edge dislocation. Had the extra half plane been introduced into the lower half of the crystal, the dislocation line would be defined as a negative edge dislocation. This kind of edge dislocation is said to be climb (Hills et al., 1996). The other

kind of edge dislocation, said to be “glide”, are made by shifting the atoms over the cut plane in a direction perpendicular to the line AA , as shown in Fig. 2.2 (Weertman and Weertman, 1992). The quantitative description of a dislocation involves a Burgers circuit. The starting point S and the ending point F are not the same atom for the circuit that encloses the dislocation, Fig. 2.1. Thus, there is a closure failure FS , which is defined as the Burgers vector $\mathbf{b}$, perpendicular to the dislocation line.

The screw dislocation is defined by the line AD in Fig. 2.3 (Hertzberg, 1989), which is generated by shifting the atoms on either side of the cut to a distance of one lattice vector in a direction parallel to the cut surface. The Burgers circuit about the screw dislocation assumes the shape of a helix, like a spiral staircase, Fig. 2.3. The closure failure FS leads to the Burgers vector $\mathbf{b}$ parallel to the dislocation line. A right-handed screw dislocation is defined when a clockwise, 360° rotation causes the helix to advance one lattice vector. The same advance arising from a 360° counterclockwise rotation is a left-handed screw dislocation.

Fig. 2.1 A climb edge dislocation in a crystal structure

Fig. 2.2 A glide edge dislocation in a crystal structure

Fig. 2.3 A screw dislocation in a crystal structure

A mixed dislocation possesses both edge and screw components, within a crystal structure. The Burgers vector of a mixed dislocation can be represented by its components (b_1 , b_2 , b_3). In a piezoelectric medium, the Burgers vector $\mathbf{b}$ of a piezoelectric mixed dislocation is described by the four components (b_1 , b_2 , b_3 , b_φ), in which a nonzero b_φ corresponds to an electric dipole layer along the cut plane (Barnett and Lothe, 1975).

2.2.2 Griffith Crack

A two-dimensional Griffith crack in an elastic medium is shown in Fig. 2.4 (Weertman, 1996). The crack plane is considered to be on the x -axis. The crack tips are located at the points $(-a, 0)$ and $(a, 0)$. By definition, if the crack is stressed in tension, Fig. 2.4(a), it is called a mode I crack. If in in-plane shear parallel to the crack plane and perpendicular to the crack front, Fig. 2.4(b), it is called a mode II crack. If in anti-plane shear parallel to the plane of the crack and the crack front, Fig. 2.4 (c), it is called a mode III crack.

2.2.3 Zener-Stroh Crack

When the applied load is removed from a Griffith crack, the displacement across the crack faces is reduced until it vanishes. A mode I Griffith crack closes up and the crack faces contact each other.

Fig. 2.4 Griffith crack situated between two elastic half spaces

(a) Mode I (b) Mode II (c) Mode III

There is another type of crack whose displacement across the crack plane can never be eliminated. Its crack faces can never be closed if it is a mode I crack. This type of crack is called Zener-Stroh crack (Weertman, 1996).

It was Zener (1948) who first proposed that edge dislocations in a pileup that are stopped at an obstacle, such as a grain boundary, could coalesce into a crack nucleus, as shown in Fig. 2.5. Koehler (1952) independently proposed that edge dislocations might nucleate cracks. Stroh (1954) calculated the number of dislocations required in a dislocation pileup to initiate a crack in detail. He found that 10^3 dislocations are needed in work-hardened copper. Stroh (1955) later identified that a number of the order of 10^2 in one pileup is needed if a stack of dislocations on parallel slip planes exist, and concluded the possibility of a crack initiation instead of fracture around the pile-up, after detailed calculation. This kind of micro crack is generally called Zener-Stroh crack.

Fig. 2.5 Zener's mechanism of crack initiation

Fig. 2.6 Cottrell's model for crack initiation

Fig. 2.7 Anti-Zener-Stroh crack model

Later, many other researchers put forward variants on the Zener-Stroh crack nucleation. Cottrell (1958) proposed that piled-up dislocations on two intersecting slip planes could coalesce into a crack, as shown in Fig. 2.6. Kikuchi et al. (1981) proposed another crack that might be called an anti-Zener-Stroh crack, as shown in Fig. 2.7. Concentrated stress fields exist near the ends of a particle on a grain boundary in a stressed solid. Slip is thus nucleated in these regions. Dislocations of one sign move away from the region, leaving stationary dislocations of the opposite sign behind to form a micro crack near the particle. This is the experimental background for the current study.

The dislocation based Zener-Stroh crack is the crack complementary to the Griffith crack. Physical parameters that are symmetric for the Griffith crack are anti-symmetric for the Zener-Stroh crack, and vice versa. Anti-symmetric dislocation distribution on the Griffith crack (see Section 2.3 in details) induces a symmetric traction stress, while the Zener-Stroh crack has an anti-symmetric stress field from a symmetric dislocation distribution. Stress, displacement, dislocation density and stress intensity factor between the Griffith crack and the Zener-Stroh crack in a homogeneous material were compared by Weertman (1986) in detail. Due to the displacement loading mechanism, the total sum of the Burgers vector of dislocations b' in a Zener-Stroh crack is not equal to zero. The crack tip where the dislocation enters the Zener-Stroh crack is called the blunt tip, while the other tip is called the sharp tip. The crack propagation is always along the sharp tip.

In the last two decades, a number of works have been devoted to micro crack initiation mechanism. Cherepanov (1994) calculated the first critical number of dislocations in a band necessary to nucleate an interfacial micro crack in an isotropic bimaterial system. Gutkin and Ovidko (1994), Wu and Zhou (1996), Wu (1997) and Wu and Yu (2000) analyzed the micro crack nucleation against the non-equilibrium grain boundaries. Xiao and Fan (1996) and Xiao et al. (2003) proposed a dislocation emission mechanism for micro crack initiation at the tip of a rigid line embedded in an elastic medium and a piezoelectric solid, respectively. Many contributions have also been made to investigate the stress fields and the stress intensity factor or the energy release rate at the tip of a Zener-Stroh crack embedded in pure-elastic or piezoelectric media, see, for example, the works of Fan and Xiao (1997), Fan et al. (1998), Suo (2000), Chen (2001), Zhao (2004) and Yan (2006).

2.2.4 Griffith-Zener-Stroh Crack

When a Zener-Stroh crack is loaded by an applied stress σ_a , it becomes a Griffith-Zener-Stroh crack. Thus, the stress intensity factor K at the tip of a plane strain crack of length $2a$ can be expressed as

$$K = K_{ZS} + K_G, \quad (2.58)$$

where

$$K_{ZS} = \frac{b^i \mu}{2(1-\nu)\sqrt{\pi a}} \quad (2.59)$$

for Zener-Stroh crack mechanism, and

$$K_G = \sigma_a \sqrt{\pi a} \quad (2.60)$$

for Griffith crack mechanism. By letting $K = K_c$, the critical crack length a_{cr} is obtained as

$$\sqrt{a_{cr}} = \left(K_c / 2\sigma_a \sqrt{\pi} \right) \left(1 \pm \sqrt{1 - 2\mu\sigma_a b^i / (1-\nu) K_c^2} \right). \quad (2.61)$$

Fig. 2.8 Energy of a Griffith-Zener-Stroh crack under mechanical loading

The minus sign exhibits a stable critical crack length a_{cr-} under the Zener-Stroh crack mechanism, and the plus sign gives an unstable crack length a_{cr+} under the Griffith crack mechanism. Increasing b^t pushes a_{cr-} to a higher value and K_{zs} to a lower value, while increasing σ_a forces a_{cr+} lower and K_G higher. When the applied stress is larger than the critical stress given by

$$\sigma_\alpha^c = (1-\nu)K_c^2/2\mu b^t, \quad (2.62)$$

no critical solution of either kind is possible and the crack propagates catastrophically.

The critical stress was first found by Cottrell (1958) based on an energy argument. The energy stored in this system as a function of crack length, for $\sigma_\alpha < \sigma_\alpha^c$, is schematically depicted in Fig. 2.8. Using an energy argument, the critical length of an interface Griffith-Zener-Stroh crack in an anisotropic bimaterial system was later analyzed by Fan (1994).

2.2.5 Crack Boundary Conditions

Mechanical boundary conditions in piezoelectric media are formulated just the same as those in classical elastic materials. Electric boundary conditions in piezoelectric media are, however, still in debating. Parton (1976) first attempted to define the electric boundary conditions on the crack faces. He assumed that the crack is electrically permeable. An electrically permeable crack requires that the electric potential and the normal component of the electric displacement are continuous across the crack. Mathematically, the electric boundary conditions are

$$\varphi^+ = \varphi^-, D_n^+ = D_n^-, \quad (2.63)$$

where the subscript n stands for the normal component and the superscripts “+” and “-” represent the upper and lower crack faces, respectively.

The above assumption is not physically realistic as there will be a potential drop across a lower capacitance crack. Deeg (1980) and Pak (1990a) proposed another set of electric boundary conditions on the crack faces as

$$D_n^+ = D_n^- = 0. \quad (2.64)$$

The above condition means that the crack surfaces are free of surface charge, and thus the electric displacement vanishes in the environment. This is particularly true for those piezoelectric ceramics with permittivity 10^3 times higher than the environment (e.g. non-conducting gas or vacuum). This result is also given by Suo et al. (1992).

Later, Hao and Shen (1994) considered the electric permeability of air in a crack gap and introduced the following electric boundary conditions on the crack faces

$$D_n^+ = D_n^-, \quad D_n^+(u_n^+ - u_n^-) = -\kappa_\alpha(\varphi^+ - \varphi^-), \quad (2.65)$$

where κ_α is the permittivity of air. Equation (2.65) reduces to the permeable condition (2.63), if $(u_n^+ - u_n^-) = 0$, or to the impermeable condition (2.64), if $\kappa_\alpha = 0$.

By investigating an elliptic flaw in piezoelectric media, Sosa (1991), Dunn (1994), Zhang and Tong (1996), Sosa and Khutoryansky (1996), Zhang et al. (1998), Gao and Fan (1999) and McMeeking (2004) pointed out that the electrically impermeable boundary conditions may not be appropriate for small crack opening in piezoelectric materials.

McMeeking (1987, 1990) modeled cracks in dielectrics containing a conducting fluid.

The boundary condition adopted in his study is

$$\varphi^+ = \varphi^- = 0. \quad (2.66)$$

Similar conducting cracks at the interface of conductor and piezoelectric material have also been modeled by Ru (2000) and Govorukha and Loboda (2000).

We shall adopt (2.64) in this thesis because of its simpler mathematical treatment and the fact that the dielectric constant of a piezoelectric material is three orders of magnitude higher than that of the vacuum or non-conducting gas.

2.3 Modeling of Cracks Using Distributed Dislocations

Distributed dislocation technique (Hills et al., 1996) employs the superposition principle of Bueckner (1958) and models cracks by the distribution of strain nuclei of various kinds along crack lines in otherwise perfect bodies. The crack problem therefore reduces to one of determining the distributed dislocations which generates tractions along the crack line equal and opposite to those produced by the applied loading, so that the crack face still remain traction-free. This method is employed in our study on crack problems.

The distributed dislocation technique is illustrated through a plane strain crack problem: a straight crack, of length $2a$, embedded in an infinite elastic plane loaded by a tensile stress $\sigma_{22}^\infty(x_1)$, perpendicular to the crack, as shown in Fig. 2.9(a). The superposition principle states

$$\sigma_{ij}(x_1, x_2) = \tilde{\sigma}_{ij}(x_1, x_2) + \hat{\sigma}_{ij}(x_1, x_2), \quad (2.67)$$

Fig. 2.9 Application of Bueckner's principle

(a) overall problem (b) stresses in the perfect body (c) separated material

where $\tilde{\sigma}_{ij}(x_1, x_2)$ is the stress field induced in the absence of the crack, as shown in Fig. 2.9(b) and $\hat{\sigma}_{ij}(x_1, x_2)$ is the stress induced by inserting materials to fill the crack without any external loading, as shown in Fig. 2.9(c), provided that the boundary conditions at the traction-free crack faces and at infinity are satisfied.

To obtain the corrective stresses $\hat{\sigma}_{ij}(x_1, x_2)$ by distributing dislocations continuously along the crack line ($x_2 = 0$, $|x_1| < a$), as shown in Fig. 2.10(a), we first consider a distribution of climb dislocations (or glide dislocations for a Mode II crack, or screw dislocations for a Mode III crack) along an infinitesimal element between the points $(\xi, 0)$ and $(\xi + \delta\xi, 0)$ (Fig. 2.10(b)). It is convenient to consider the element as a single concentrated dislocation with infinitesimal Burgers vector $\delta b_2 = B_2(\xi)d\xi$, where $B_2(\xi)$ is the dislocation density at a point $(\xi, 0)$, depicted in Fig. 2.10(c). Due to a single

dislocation with Burgers vector δb_2 at a point $(\xi, 0)$, the normal stress arising along the crack line is given by

$$\hat{\sigma}_{22}(x_1, 0) = \frac{\mu}{2\pi(1-\nu)} \frac{1}{x_1 - \xi} B_2(\xi) d\xi. \quad (2.68)$$

The normal stress due to a continuous distribution of dislocations along the crack line is then given by

$$\hat{\sigma}_{22}(x_1, 0) = \frac{\mu}{2\pi(1-\nu)} \int_{-a}^a \frac{1}{x_1 - \xi} B_2(\xi) d\xi. \quad (2.69)$$

With the boundary conditions, one has

$$-\frac{2(1-\nu)}{\mu} \sigma_{22}^\infty(x_1) = \frac{1}{\pi} \int_{-a}^a \frac{1}{x_1 - \xi} B_2(\xi) d\xi, \quad |x_1| < a. \quad (2.70)$$

Fig. 2.10 Modeling a crack using dislocations

- (a) an array of dislocations (b) an array over a small element
(c) the dislocation density (d) crack opening displacement

The integrand in (2.70) contains two terms, the dislocation density $B_2(\xi)$ and the Cauchy kernel $(x_1 - \xi)^{-1}$. The opening displacement $g(x_1)$ at a point $(x_1, 0)$ (Fig. 2.10(d)) can be related to $B_2(\xi)$ by

$$g(x_1) = -\int_{-a}^{x_1} B_2(\xi) d\xi. \quad (2.71)$$

An additional side condition is imposed as follows

$$\int_{-a}^a B_2(\xi) d\xi = C, \quad (2.72)$$

where C is the total opening displacement. To solve a crack problem using this technique, we need the boundary conditions, the total crack opening displacement and the Green function solution for the stress field of a single dislocation.

2.4 Numerical Solution of Singular Integral Equations

The solution of many mixed boundary value problems in physics and engineering, such as crack problems formulated by distributing dislocations along the crack line, can be reduced to that of a system of singular integral equations of the following form

$$\frac{1}{\pi} \int_{-1}^1 \sum_{j=1}^M Z_{ij} B_j(t) \frac{dt}{t-x} + \int_{-1}^1 \sum_{j=1}^M h_{ij}(x, t) B_j(t) dt = g_i(x), \quad -1 < x < 1, \quad i = 1, \dots, M. \quad (2.73)$$

In the above equation, $B_1(t), \dots, B_M(t)$ are the unknown functions, the constant matrix $\mathbf{Z} = (Z_{ij})$ is nonsingular, the kernels $h_{ij}(x, t)$ ($i, j = 1, \dots, M$) are bounded in the closed domain $-1 \leq (x, t) \leq 1$, and $g_1(x), \dots, g_M(x)$ are the known input functions. The end points $t = \mp 1$ are geometric singularities. By multiplying (2.73) by $\mathbf{Z}^{-1}$ and by following the procedure of Muskhelishvili (1953), it can be shown that the fundamental functions of

(2.73) which characterize the singular behavior of $B_j(t)$ around $t = \mp 1$ are given by

$$w_j(t) = (1+t)^{-1/2+\alpha} (1-t)^{1/2+\beta}, \quad (2.74)$$

where $\alpha, \beta = 0, \pm 1$ and $-1 < -1/2 + \alpha, 1/2 + \beta < 1$. The index of the integral equation is introduced by $k = -(\alpha + \beta)$. If $B_j(t)$ is bounded at both ends,

$$w_j(t) = (1+t)^{1/2} (1-t)^{1/2}, \quad \alpha = 1, \beta = 0, k = -1. \quad (2.75)$$

If $B_j(t)$ has an integrable singularity, then

$$w_j(t) = (1+t)^{-1/2} (1-t)^{-1/2}, \quad \alpha = 0, \beta = -1, k = 1. \quad (2.76)$$

In this case, the solution of (2.73) contains M arbitrary constants, which can be solved subjected to the conditions

$$\int_{-1}^1 B_j(t) dt = C_j, \quad j = 1, \dots, M. \quad (2.77)$$

If $B_j(t)$ is bounded at one end and has an integrable singularity at the other end, we have

$$w_j(t) = (1+t)^{-1/2} (1-t)^{1/2}, \quad \alpha = \beta = 0, k = 0, \quad (2.78)$$

$$w_j(t) = (1+t)^{1/2} (1-t)^{-1/2}, \quad \alpha = 1, \beta = -1, k = 0. \quad (2.79)$$

The unknown function $B_j(t)$ is expressed by

$$B_j(t) = w_j(t) \psi_j(t), \quad (2.80)$$

where the function $\psi_j(t)$ is bounded and continuous in the interval $-1 \leq t \leq 1$.

There are many numerical integration methods for solving (2.73) based on quadratic polynomial approximation of $\psi_j(t)$, e.g., the Lobatto-Chebyshev quadrature developed by Theocaris and Ioakimidis (1977) and the piecewise polynomials adopted by Gerasoulis and Srivastav (1981), Gerasoulis (1982) and Kurtz et al. (1994). The most widely used and easily implemented quadrature is the Gauss-Jacobi quadrature developed by Erdogan

and Gupta (1972) and Erdogan et al. (1973), based on the notion that the fundamental functions $w_j(t)$ ($j = 1, \dots, M$) are the weights of the Jacobi polynomials, and further $\psi_j(t)$ can be effectively obtained by selecting the roots of the proper Jacobi polynomials as the nodal points t_k and x_k in the interval $(-1, 1)$.

To describe the numerical procedure of Erdogan and Gupta (1972) and Erdogan et al. (1973), let us consider the following singular integral

$$S(x) = \frac{1}{\pi} \int_{-1}^1 \frac{1}{t-x} B(t) dt + \int_{-1}^1 h(x, t) B(t) dt, \quad -1 < x < 1, \quad (2.81)$$

where

$$B(t) = w(t)\psi(t), \quad w(t) = (1-t^2)^{-1/2}, \quad (2.82)$$

and $\psi(t)$ is bounded in the interval $-1 \leq t \leq 1$. The solution procedure for different index value k is given as follows.

The case of $k = 1$

In this case, the Jacobi polynomial reduces to the Chebyshev polynomial of the first kind, defined by

$$T_N(t) = \cos N\theta, \quad \cos \theta = t. \quad (2.83)$$

$w(t) = (1-t^2)^{-1/2}$ is the weight of $T_N(t)$. The unknown function $\psi(t)$ are approximated to a sufficient degree of accuracy by the following truncated series

$$\psi(t) = \sum_0^p A_j T_j(t). \quad (2.84)$$

It follows that

$$\frac{1}{\pi} \int_{-1}^1 \frac{1}{t - x_r} B(t) dt = \frac{1}{n} \sum_{l=1}^n \psi(t_l) \frac{1}{t_l - x_r}, \quad (2.85)$$

where

$$T_n(t_l) = 0, \quad t_l = \cos[\pi(2l-1)/2n], \quad l = 1, \dots, n, \quad (2.86)$$

$$U_{n-1}(x_r) = 0, \quad x_r = \cos(\pi r/n), \quad r = 1, \dots, n-1. \quad (2.87)$$

The Gauss-Chebyshev integration formula for a bounded continuous function $h(x, t)\psi(t)$ is given by (Abramowitz and Stegun, 1972)

$$\frac{1}{\pi} \int_{-1}^1 \frac{h(x, t)\psi(t)}{\sqrt{1-t^2}} dt = \sum_{l=1}^n \frac{h(x, t_l)\psi(t_l)}{n}, \quad t_l = \cos[\pi(2l-1)/2n]. \quad (2.88)$$

Thus, Equation (2.85) can be considered as a Gauss-Chebyshev integration formula for the singular integral which is valid only at the discrete set of points x_r given by (2.87).

By using (2.88), the physical condition can be written as

$$\frac{1}{\pi} \int_{-1}^1 B(t) dt = \sum_{l=1}^n \frac{\psi(t_l)}{n} = C, \quad t_l = \cos[\pi(2l-1)/2n]. \quad (2.89)$$

Equations (2.85) and (2.89) provide n algebraic ones to solve the n unknowns $\psi(t_l)$.

The case of $k = -1$

In this case, the Jacobi polynomial reduces to the Chebyshev polynomial of the second kind, defined by

$$U_N(t) = \sin(N+1)\theta / \sin \theta, \quad \cos \theta = t. \quad (2.90)$$

$w(t) = \sqrt{1-t^2}$ is the weight of $U_N(t)$. The following truncated series

$$\psi(t) = \sum_{j=0}^p B_j U_j(t) \quad (2.91)$$

represent $\psi(t)$ with sufficient accuracy. One has

$$\frac{1}{\pi} \int_{-1}^1 \frac{1}{t - x_r} B(t) dt = \sum_{l=1}^n \psi(t_l) \frac{1 - t_l^2}{(n+1)(t_l - x_r)}, \quad (2.92)$$

$$\frac{1}{\pi} \int_{-1}^1 h(x, t) \psi(t) \sqrt{1 - t^2} dt = \sum_{l=1}^n \frac{(1 - t_l^2)}{n+1} h(x, t_l) \psi(t_l), \quad (2.93)$$

where

$$U_n(t_l) = 0, \quad t_l = \cos[\pi l / (n+1)], \quad l = 1, \dots, n, \quad (2.94)$$

$$T_{n+1}(x_r) = 0, \quad x_r = \cos[\pi(2r-1)/(2n+2)], \quad r = 1, \dots, n+1. \quad (2.95)$$

Equations (2.92) and (2.93) provide $n+1$ algebraic ones. It suffices to select n equations.

Usually, n is selected to be an even number and the equation corresponding to $x_{1+n/2}$ is ignored.

The case of $k = 0$, $\alpha = \beta = 0$

In this case, the Jacobi polynomial is defined by

$$P_N^{(1/2, -1/2)}(t) = \frac{\sin[(2N+1)\theta/2]}{\sin(\theta/2)}, \quad \cos \theta = t. \quad (2.96)$$

$w_j(t) = (1+t)^{-1/2}(1-t)^{1/2}$ is the weight of $P_j^{(1/2, -1/2)}(t)$. The unknown function $\psi(t)$ can be approximated with a sufficient degree of accuracy by the following truncated series

$$\psi(t) = \sum_0^p C_j P_j^{(1/2, -1/2)}(t). \quad (2.97)$$

One has

$$\frac{1}{\pi} \int_{-1}^1 \frac{1}{t - x_r} B(t) dt = \sum_{l=1}^n \psi(t_l) \frac{2(1 - t_l)}{(2n+1)(t_l - x_r)}, \quad (2.98)$$

$$\frac{1}{\pi} \int_{-1}^1 \frac{h(x, t) \psi(t) (1-t)^{1/2}}{(1+t)^{1/2}} dt = \sum_{l=1}^n \frac{2(1-t_l) h(x, t_l) \psi(t_l)}{2n+1}, \quad (2.99)$$

where

$$P_n^{(1/2, -1/2)}(t_l) = 0, \quad t_l = \cos[2\pi l / (2n+1)], \quad l = 1, \dots, n, \quad (2.100)$$

$$P_n^{(-1/2, 1/2)}(x_r) = 0, \quad x_r = \cos[\pi(2r-1) / (2n+1)], \quad r = 1, \dots, n. \quad (2.101)$$

Equations (2.98) and (2.99) provide n algebraic ones to solve the n unknowns $\psi(t_l)$.

The case of $k = 0$, $\alpha = 1$, $\beta = -1$

In this case, the Jacobi polynomial is defined by

$$P_N^{(-1/2, 1/2)}(x) = \frac{\cos[(2N+1)\theta/2]}{\cos(\theta/2)}, \quad \cos \theta = x. \quad (2.102)$$

$w_j(t) = (1+t)^{1/2} (1-t)^{-1/2}$ is the weight of $P_j^{(-1/2, 1/2)}(t)$. The unknown function $\psi(t)$ can be

approximated with a sufficient degree of accuracy by the following truncated series

$$\psi(t) = \sum_0^p D_j P_j^{(-1/2, 1/2)}(t), \quad (2.103)$$

and

$$\frac{1}{\pi} \int_{-1}^1 \frac{1}{t-x_r} B(t) dt = \sum_{l=1}^n \frac{2(1+t_l) \psi(t_l)}{(2n+1)(t_l-x_r)}, \quad (2.104)$$

$$\frac{1}{\pi} \int_{-1}^1 \frac{h(x, t) \psi(t) (1+t)^{1/2}}{(1-t)^{1/2}} dt = \sum_{l=1}^n \frac{2(1+t_l) h(x, t_l) \psi(t_l)}{2n+1}, \quad (2.105)$$

where

$$P_n^{(-1/2, 1/2)}(t_l) = 0, \quad t_l = \cos[\pi(2l-1) / (2n+1)], \quad l = 1, \dots, n, \quad (2.106)$$

$$P_n^{(1/2, -1/2)}(x_r) = 0, \quad x_r = \cos[2\pi r / (2n+1)], \quad r = 1, \dots, n. \quad (2.107)$$

Equations (2.106) and (2.107) provide n algebraic ones to solve the n unknowns $\psi(t_l)$.

The numerical method of Erdogan and Gupta (1972) and Erdogan et al. (1973) is used to solve the Cauchy type singular equations formulated in the thesis.

2.5 Crack/Inclusion/Rigid Line Problems

Crack problems in a piezoelectric solid loaded at infinity have been well studied, see for example the works of Pak (1990a) for an impermeable antiplane crack, Zhang and Hack (1992) for a permeable antiplane crack, Pak (1992b), Park and Sun (1995a) and Sosa (1992) for an impermeable plane crack and Sosa and Pak (1990) for a three-dimensional crack. It was identified that the stress, strain, electric displacement and electric field near the tips of an impermeable crack and the stress, strain and electric field ahead of a permeable crack in a piezoelectric medium exhibit a traditional square root singularity, and the singularity of field variables becomes independent of the other respective load parameters. In other words, the electric load alone cannot produce the stress ahead of the crack tip. By using the energy release rate as fracture criterion, Pak (1990a, 1992b) found that the presence of electrical load can either promote or retard crack growth at a given mechanical load, depending on the magnitude, the direction and the type of the electric load, while the crack extension force is always negative in the absence of mechanical load. Park and Sun (1995a) addressed the mechanical strain energy release rate and identified that the strain energy release rate is linearly related to the electric displacement. In other words, by changing the electric load direction, the mechanical strain energy release rate can be made to either increase or decrease. Park and Sun (1995b) conducted Mode I and

mixed mode fracture experiments and found that the mechanical strain energy release rate criterion agrees roughly with the measured critical load. Recently, Zhu and Yang (1999) predicted the crack kinking direction based on the maximum Mode I fracture toughness. Qin and Mai (2000) proposed that kinking is favored when the ratio of the energy release rate for kinking out of the interface to that for interfacial cracking is greater than the ratio of substrate toughness to interface toughness.

The antiplane problems of an interfacial circular crack in piezoelectric materials have also been investigated by Zhong and Meguid (1997) and Deng and Meguid (1999b) for an impermeable model and Wang and Zhong (2002) for a conducting model. Gao and Wang (2001a) and Chue and Ou (2005) studied permeable antiplane cracks along the interface and perpendicular to the interface, respectively. All the field variables around the tip of an interfacial conducting crack show oscillatory singularities, different from those associated with an impermeable one. Numerous works have also been carried out on the generalized two-dimensional interfacial crack problems in piezoelectric solids, such as Kuo and Barnett (1991), Suo et al. (1992) and Herrmann et al. (2001) for an impermeable model, Hao and Shen (1994), Gao and Wang (2000), Beom (2003), Gao and Balke (2003b), Gao et al. (2004) and Hausler et al. (2004) for a permeable model, and Govorukha and Loboda (2000) and Beom and Atluri (2002) for a conducting model. Wang and Shen (2002) gave a general treatment on various interfacial defects. Kuo and Barnett (1991) and Suo et al. (1992) first discovered the type of singularities of order $r^{-1/2 \pm i\varepsilon}$ and $r^{-1/2 \mp \kappa}$ (ε and κ are real) at the tip of an impermeable interfacial crack. Later, Ou and Wu (2003) proved that either $\varepsilon = 0$ or $\kappa = 0$ for an impermeable model in a transversely isotropic piezoelectric bimaterial system and therefore the near-tip stress and electric displacement exhibit non-oscillatory when $\varepsilon = 0$.

Antiplane and plane inclusion models in piezoelectric materials have also been developed by Pak (1992a), Meguid and Zhong (1997), Kattis et al. (1998), Dunn and Taya (1993), Chung and Ting (1996), and Dunn and Wienecke (1997), Pan (2004a, b), among others. Problems of cylindrical and spheroidal inclusions in a transversely piezoelectric material have been treated by Mikata (2000, 2001). Problems of rigid lines in pure-elastic media subjected to remote loads have been well analyzed recently. It was identified that the tip stress for a rigid line (Wang et al., 1985, 1986; Hao and Wu, 1989a, b, c; Li and Ting, 1989; Jiang, 1991; Liu and Jiang, 1994; Hurtado et al., 1996) exhibits a traditional square root singularity and that of an interfacial rigid line (Ballarini, 1990; Wu, 1990; Jiang and Liu, 1992; Asundi and Deng, 1995; Jiang and Cheung, 1995) possesses a pronounced oscillatory character. The negative extension force on the rigid line in elastic solids was observed by Wang et al. (1985, 1986) and Wu (1990). In the recent decades, rigid line problems in piezoelectric media have attracted researchers' attention. Shi (1997) studied collinear rigid lines in a piezoelectric medium subjected to antiplane mechanical loading and in-plane electric loading at infinity. It was found that the square root singularity of all the field variables around the tips of a rigid conducting line is independent of other respective loading parameters, similar to that for an impermeable crack. For the case of a rigid dielectric line, the square root singularity of the near tip stress, strain and electric field depends on mechanical and electrical loads, while the electric displacement at the tip is a constant. The negative driving force on the rigid line in a piezoelectric solid can be observed, by calculating the J -integral value. Later, Deng and Meguid (1998) and Gao and Fan (2001) analyzed the electro-elastic fields for the two bonded dissimilar piezoelectric half-planes with rigid conducting and dielectric lines at the interface under remote loads, respectively. All the possible singularity cases of the near tip stress and electric displacement were discussed in detail by Deng and Meguid (1998).

2.6 Singularity-Crack/Inclusion/Interface/Rigid Line Interaction

The interaction of a dislocation with such defects as cracks, inclusions and rigid lines in piezoelectric media is an important topic in the fracture mechanics. A great number of works have been carried out on the interactions of a screw dislocation (Lee et al., 2000; Chen et al., 2002; Chen et al., 2004a; Chen et al., 2004c) and an edge dislocation (Gao and Balke, 2003a; Chen et al., 2004b; Chen et al., 2005a, b) with cracks in a piezoelectric solid. Efforts have also been devoted to the interactions of a screw dislocation (Meguid and Deng, 1998; Deng and Meguid, 1999a; Liu et al., 2004; Xiao et al., 2004; Liu et al., 2006) and an edge dislocation (Huang and Kuang, 2001) with inclusions in piezoelectric media. Attempts have also been made to study the interactions of a screw dislocation (Soh et al., 2002; Wu et al., 2003; Liu and Fang, 2004; Fang et al., 2005; Zheng et al., 2005) and an edge dislocation (Gao and Wang, 2001b; Gao et al., 2005) with interfacial cracks in piezoelectric materials.

The interactions of a dislocation and load-induced singularity with rigid lines in isotropic elastic solids have been well studied by many researchers, e.g., Dundurs and Markenscoff (1989), Jiang (1991), Markenscoff et al. (1994), Jiang and Cheung (1995) and Liu et al. (2005). Fan and Keer (1993) investigated an edge dislocation interacting with interfacial rigid lines in anisotropic media. In a piezoelectric solid, recently, Chen et al. (2007) and (2006a) addressed a screw dislocation and an edge dislocation interacting with collinear rigid lines, respectively, where the rigid line is considered to be either conducting or dielectric. Liu and Fang (2003) investigated the interaction between a screw dislocation and rigid conducting lines along the inclusion in piezoelectric solids. Pan (2002) analyzed the problem of a point load (force and electric charge) in a piezoelectric half plane with

general surface conditions. Pan and Han (2004) investigated a point load in piezoelectric multilayered half spaces. Pan and Tonon (2000), Pan and Yuan (2000) and Pan and Han (2005) presented three-dimensional extended Green's displacements in a piezoelectric solid, bimaterial system, and multilayered half space, respectively.

In the above works, the force on a dislocation was calculated, and the effects of material properties on the force and the equilibrium position of the dislocation were analyzed. The singularity of the stress and electric displacement field at the tips of cracks and rigid lines were identified and led to corresponding intensity factors. However, no attempt has been made to study the interaction of a screw or an edge dislocation with interfacial rigid lines in dissimilar piezoelectric media, which is often encountered during electric packaging and manufacturing of intelligent composites. These practical problems motivated the current research work reported in the thesis.

CHAPTER 3

MICRO CRACK INITIATION AT THE TIP OF A SEMI-INFINITE RIGID CONDUCTING LINE IN A PIEZOELECTRIC SOLID

3.1 Introduction

It is known that the stress field at the tip of a rigid line has a square root singularity, according to a near tip asymptotic expansion (Williams, 1952). A square root singularity of the generalized stresses at the tip of a rigid conducting line in a piezoelectric medium was also identified (Deng and Meguid, 1998), and led to the generalized stress intensity factors. However, the effect of the electric field on the overall fracture toughness of the matrix material is not included, if the generalized stress intensity factor is used as the fracture criterion (Pak, 1990a; Suo et al., 1992). Pak (1992b) and other authors therefore used the energy release rate as a fracture criterion. However, the crack extension force calculated as the energy release rate is always negative in the absence of the mechanical load. Later Park and Sun (1995a) proposed the mechanical strain energy release rate. The mechanical strain energy release rate is linearly dependent on the electric displacement

and can thus be made to either increase or decrease depending on the direction of the electric load. This assumption agrees roughly with the experiment result obtained by Park and Sun (1995b) as well.

In the current study, a criterion of micro crack initiation at the tip of a semi-infinite rigid conducting line in a piezoelectric solid, in analogy to conventional crack propagation, is proposed as

$$G_M^l = G_{cr}^l, \quad (3.1)$$

where G_M^l is the mechanical strain energy release rate when a semi-infinite rigid conducting line extends, and is referred to as the rigid line extension force; the superscript l is added in order to distinguish it from the crack extension force. The critical value G_{cr}^l at the right hand side of (3.1) is deemed to be a material constant that can be determined from tests. However, based on the author's knowledge, no such experimental results can be found in open literature so far. It is our conjecture that G_{cr}^l can be correspondingly related to the critical crack extension force G_{cr} in the same material, since both cases associate with the square root singularity.

To search for a possible correlation between G_{cr}^l and G_{cr} , for example,

$$G_{cr}^l = CG_{cr}, \quad (3.2)$$

where C is a constant to be determined through a model of micro crack initiation ahead of the tip of a semi-infinite rigid conducting line in a piezoelectric medium, proposed in the following sections. The mechanism of the model is schematically shown in Figs. 3.1(a) and (b), where a semi-infinite rigid conducting line l_i near its tip is loaded with the

generalized stress fields characterized by $\mathbf{K}^I = [K_{II}^I, K_I^I, K_{III}^I, K_D^I]^T$, with K_{II}^I , K_I^I , K_{III}^I being the in-plane shear, tensile, out of plane shear stress intensity factors, respectively, and K_D^I being the electric displacement intensity factor. Due to the highly concentrated stress and electric displacement fields along certain slip planes, dislocations of one sign are driven away from the tip. Since dislocations are generated in pairs, the dislocations with the opposite sign pile up at the tip of the rigid line. Strain energy is built up gradually at the tip. One possible way to release the high strain energy is to initiate a Zener-Stroh crack in front of the rigid line. It is worth noting that this crack initiation mechanism was observed first by Kikuchi et al. (1981) and was called anti-Zener-Stroh crack by Weertman (1986) later on.

It is worth to mention that other crack initiation mechanisms (Xia and Shih, 1995) ahead of a rigid line are possible. However, in the current study we seek to extend the models put forward by Xiao and Fan (1996) and Xiao et al. (2003), which have been confirmed experimentally by Kikuchi (1981), into the plane problem of a piezoelectric solid, rather than to compare all the possible failure mechanisms. After all, it is the first analytical work for the micro crack initiation for this configuration, according to the author's best knowledge.

3.2 Formulation

Consider a semi-infinite rigid conducting line l_l lies on $x_2 = 0$, $x_1 < 0$ in a homogeneous piezoelectric medium as shown in Fig. 3.1(a), whose poling direction is along the x_2 axis.

Fig.3.1 Schematic of micro crack initiation mechanism at the tip of a semi-infinite rigid conducting line in a piezoelectric solid

Based on the micro crack initiation mechanism discussed in the previous section and the distribution characteristics of the generalized stresses and deformations ahead of the rigid line, a Zener-Stroh crack is assumed to be initiated at the tip of the rigid line, as shown in Fig. 3.1(b). We assume that the initiated micro crack is still loaded with the $\mathbf{K}^I$ -field. The

distributions of the generalized stress fields around the rigid line tip are approximated by $\mathbf{K}^l / \sqrt{2\pi x_1}$. Let the net Burgers vector or total dislocations inside the micro crack be $\mathbf{b}^t = [b_1^t, b_2^t, b_3^t, b_\phi^t]^T$, with $b_4^t = b_\phi^t$. In this section, we analyze how the micro crack of length $2a$ initiated from the rigid line tip is affected by the exerted load $\mathbf{K}^l$ and the net Burgers vector $\mathbf{b}^t$ inside the micro crack.

The initiated micro crack is to be simulated by a continuous distribution of dislocations along the crack line. To obtain the Green's function for such a simulation, we begin with a piezoelectric mixed dislocation with Burgers vector $\mathbf{b}(\xi) = [b_1(\xi), b_2(\xi), b_3(\xi), b_\phi(\xi)]^T$ at $(\xi, 0)$ ahead of a semi-infinite rigid conducting line, based on the work of Chen et al. (2004). The constitutive relations for piezoelectric ceramics with the x_2 axis along the poling direction are listed in Appendix A. The generalized stresses arising along the line $x_2 = 0$ due to the piezoelectric mixed dislocation $\mathbf{b}(\xi)$ interacting with the semi-infinite rigid conducting line are obtained as

$$\begin{aligned} & [\sigma_{21}(x_1, 0), \sigma_{22}(x_1, 0), \sigma_{23}(x_1, 0), D_2(x_1, 0)]^T \\ &= \frac{1}{4\pi} \left[\mathbf{L} \mathbf{b}(\xi) \frac{1}{\sqrt{x_1}(\sqrt{x_1} - \sqrt{\xi})} \right] + \frac{1}{4\pi} \left[(\mathbf{H}^{-1} - \mathbf{H}^{-1} \mathbf{S}^2) \mathbf{b}(\xi) \frac{1}{\sqrt{x_1}(\sqrt{x_1} + \sqrt{\xi})} \right], \end{aligned} \quad (3.3)$$

where the real matrices $\mathbf{S}$, $\mathbf{H}$ and $\mathbf{L}$ are defined in (2.39). The generalized stresses, due to a continuous distribution of dislocations along the crack line, are then given by

$$\begin{aligned} & [\sigma_{21}(x_1, 0), \sigma_{22}(x_1, 0), \sigma_{23}(x_1, 0), D_2(x_1, 0)]^T \\ &= \frac{1}{4\pi} \left\{ \mathbf{L} \int_0^{2a} \hat{\mathbf{B}}(\xi) \frac{1}{x_1 - \sqrt{x_1} \xi} d\xi + [\mathbf{H}^{-1} - \mathbf{H}^{-1} \mathbf{S}^2] \int_0^{2a} \hat{\mathbf{B}}(\xi) \frac{1}{x_1 + \sqrt{x_1} \xi} d\xi \right\}, \end{aligned} \quad (3.4)$$

where $\hat{\mathbf{B}}(\xi)$, the dislocation density along the crack line, takes the form

$$\hat{\mathbf{B}}(\xi) = [\hat{B}_1(\xi), \hat{B}_2(\xi), \hat{B}_3(\xi), \hat{B}_\phi(\xi)]^T, \quad \hat{B}_4(\xi) = \hat{B}_\phi(\xi). \quad (3.5)$$

The crack surface is assumed to be free from charge (Deeg, 1980; Pak, 1990). Since the micro crack is still loaded by the near tip generalized stress fields $\mathbf{K}^I / \sqrt{2\pi x_1}$, one has

$$\begin{aligned} & \frac{1}{4\pi} \left\{ \mathbf{L} \int_0^{2a} \hat{\mathbf{B}}(\xi) \frac{1}{x_1 - \sqrt{x_1 \xi}} d\xi + [\mathbf{H}^{-1} - \mathbf{H}^{-1} \mathbf{S}^2] \int_0^{2a} \hat{\mathbf{B}}(\xi) \frac{1}{x_1 + \sqrt{x_1 \xi}} d\xi \right\}, \\ & = -\frac{\mathbf{K}^I}{\sqrt{2\pi x_1}}, \quad 0 < x_1 < 2a. \end{aligned} \quad (3.6)$$

Note that, since the net Burgers vector $\mathbf{b}^t$ inside the micro crack exists, the dislocation density must satisfy

$$\int_0^{2a} \hat{\mathbf{B}}(\xi) d\xi = \mathbf{b}^t. \quad (3.7)$$

To express (3.6) in a standard Fredholm-type integral equation, a term at the inner part is expressed as follows

$$\frac{1}{x_1 - \sqrt{x_1 \xi}} = \frac{2}{x_1 - \xi} - \frac{1}{\sqrt{x_1}(\sqrt{x_1} + \sqrt{\xi})}. \quad (3.8)$$

The integral over $[0, 2a]$ can be normalized to $[-1, 1]$ by the following substitutions

$$t = \frac{x_1}{a} - 1, \quad s = \frac{\xi}{a} - 1. \quad (3.9)$$

Let $g(t, s)$ denote the regular part of the kernel

$$g(t, s) = \frac{1}{t + 1 + \sqrt{(t + 1)(s + 1)}}. \quad (3.10)$$

Equation (3.6) reads

$$\begin{aligned} & \frac{1}{4\pi} \sum_{j=1}^4 L_{ij} \int_{-1}^1 \hat{B}_j(s) \frac{2}{t - s} ds + \frac{1}{4\pi} \sum_{j=1}^4 W_{ij} \int_{-1}^1 \hat{B}_j(s) g(t, s) ds \\ & = -\frac{K_i^I}{\sqrt{2\pi a}} \frac{1}{\sqrt{1+t}}, \quad -1 < t < 1, \quad i = 1, 2, 3, 4, \end{aligned} \quad (3.11)$$

where

$$K_1^I = K_{II}^I, \quad K_2^I = K_I^I, \quad K_3^I = K_{III}^I, \quad K_4^I = K_D^I, \quad (3.12)$$

L_{ij} denotes the element of matrix $\mathbf{L}$, and W_{ij} denotes the element of matrix $\mathbf{W}$, given by

$$\mathbf{W} = \mathbf{H}^{-1} - \mathbf{H}^{-1}\mathbf{S}^2 - \mathbf{L}. \quad (3.13)$$

Equation (3.7) has also been changed to

$$\int_{-1}^1 \hat{B}_i(s) ds = \frac{b_i^t}{a}, \quad i = 1, 2, 3, 4, \quad (3.14)$$

accordingly.

3.3 Numerical Procedure

Equations (3.11) and (3.14) are in the standard form of a singular Fredholm integral equation, which can be solved numerically by following the method of Erdogan and Gupta (1972) and Erdogan et al. (1973). Let $\psi(s)$ be a bounded function in $[-1, 1]$, and

$$\hat{\mathbf{B}}(s) = \psi(s)(1+s)^\alpha(1-s)^\beta, \quad (3.15)$$

where α and β are the singularities at the left and right tips of the crack, respectively, and

$$\psi(s) = [\psi_1(s), \psi_2(s), \psi_3(s), \psi_\varphi(s)]^T, \quad \psi_4(s) = \psi_\varphi(s). \quad (3.16)$$

The discrete forms of (3.11) and (3.14) are given by

$$\frac{1}{4\pi} \sum_{m=1}^n \sum_{j=1}^4 \hat{W}_m \left[\frac{2L_{ij}}{t_r - s_m} + W_{ij} g(t_r, s_m) \right] \psi_j(s_m) = -\frac{K_i^l}{\sqrt{2\pi a}} \frac{1}{\sqrt{1+t_r}}, \quad i = 1, 2, 3, 4, \quad (3.17)$$

$$\sum_{m=1}^n \hat{W}_m \psi_i(s_m) = \frac{b_i^t}{a}, \quad i = 1, 2, 3, 4, \quad (3.18)$$

respectively, where s_m and t_r are the roots of Jacobi polynomials

$$P_n^{(\alpha, \beta)}(s_m) = 0, \quad P_{n+\alpha+\beta}^{(\alpha+1, \beta+1)}(t_r) = 0, \quad -1 < \text{Re}(\alpha, \beta) < 0, \quad m = 1, \dots, n, \quad r = 1, \dots, n-1, \quad (3.19)$$

and $\hat{W}_m$ is given by

$$\hat{W}_m = - \frac{(2n + \alpha + \beta + 1)\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)2^{\alpha+\beta}}{\pi(n+1)!(n + \alpha + \beta + 1)\Gamma(n + \alpha + \beta + 1)P_{n+1}^{(\alpha,\beta)}(s_m)} \frac{d}{dt} P_n^{(\alpha,\beta)}(s_m) \quad (3.20)$$

with $\Gamma(z)$ being the well known Gamma function.

Equations (3.17) and (3.18) provide a system of $4n$ linear algebraic equations for the $4n$ unknowns $\psi_i(s_m)$, $i = 1, 2, 3, 4$. We denote $\psi_1^{(j)}(s)$ and $\psi_3^{(j)}(s)$ ($j = 1, 2$) as the solutions of the first set of linear system (3.21), and $\psi_2^{(j)}(s)$ and $\psi_4^{(j)}(s)$ ($j = 1, 2, 3, 4$) the solutions of the second set of linear system (3.22)

$$\begin{cases} \sum_{m=1}^n \hat{W}_m \left[\frac{2L_{ii}}{t_r - s_m} + W_{ii}g(t_r, s_m) \right] \psi_i^{(j)}(s_m) = \frac{\delta_{ij}}{\sqrt{1+t_r}}, \\ \sum_{m=1}^n \hat{W}_m \psi_i^{(j)}(s_m) = \delta_{ij}, \end{cases} \quad i = 1, 3, \quad j = 1, 2, \quad (3.21)$$

$$\begin{cases} \sum_{m=1}^n \hat{W}_m \left[\frac{2L_{22}}{t_r - s_m} + W_{22}g(t_r, s_m) \right] \psi_2^{(j)}(s_m) + \sum_{m=1}^n \hat{W}_m \left[\frac{2L_{24}}{t_r - s_m} + W_{24}g(t_r, s_m) \right] \psi_4^{(j)}(s_m) \\ = \frac{\delta_{1j}}{\sqrt{1+t_r}}, \\ \sum_{m=1}^n \hat{W}_m \left[\frac{2L_{42}}{t_r - s_m} + W_{42}g(t_r, s_m) \right] \psi_2^{(j)}(s_m) + \sum_{m=1}^n \hat{W}_m \left[\frac{2L_{44}}{t_r - s_m} + W_{44}g(t_r, s_m) \right] \psi_4^{(j)}(s_m) \\ = \frac{\delta_{2j}}{\sqrt{1+t_r}}, \\ \sum_{m=1}^n \hat{W}_m \psi_2^{(j)}(s_m) = \delta_{3j}, \\ \sum_{m=1}^n \hat{W}_m \psi_4^{(j)}(s_m) = \delta_{4j}, \end{cases} \quad j = 1, 2, 3, 4, \quad (3.22)$$

respectively, where the symbol δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1 & (i = j), \\ 0 & (i \neq j). \end{cases} \quad (3.23)$$

Thus, one obtains

$$\psi_1(s) = -\sqrt{\frac{8\pi}{a}} K_{\text{II}}^{(I)} \psi_1^{(1)}(s) + \frac{b_1^t}{a} \psi_1^{(2)}(s), \quad (3.24)$$

$$\psi_2(s) = -\sqrt{\frac{8\pi}{a}} K_{\text{I}}^I \psi_2^{(1)}(s) - \sqrt{\frac{8\pi}{a}} K_{\text{D}}^I \psi_2^{(2)}(s) + \frac{b_2^t}{a} \psi_2^{(3)}(s) + \frac{b_4^t}{a} \psi_2^{(4)}(s), \quad (3.25)$$

$$\psi_3(s) = -\sqrt{\frac{8\pi}{a}} K_{\text{III}}^I \psi_1^{(1)}(s) + \frac{b_3^t}{a} \psi_3^{(2)}(s), \quad (3.26)$$

$$\psi_4(s) = -\sqrt{\frac{8\pi}{a}} K_{\text{I}}^I \psi_4^{(1)}(s) - \sqrt{\frac{8\pi}{a}} K_{\text{D}}^I \psi_4^{(2)}(s) + \frac{b_2^t}{a} \psi_4^{(3)}(s) + \frac{b_4^t}{a} \psi_4^{(4)}(s). \quad (3.27)$$

For the current problem, there is a square root singularity at the right tip of the micro crack, so $\beta = -1/2$ is taken in calculation. At the left tip of the micro crack, oscillatory singularities can be caused by the incompatibility of piezoelectric medium and rigid line. However they are physically invalid. The oscillatory displacement implies that the upper and lower crack surfaces interpenetrate each other. Fortunately, the oscillatory singularity is removed, and a square root singularity is restored by introducing the contact zone (Comninou, 1977). Based on these arguments, we take $\alpha = -1/2$ in calculation. This assumption may cause some approximation of the field solutions at the left tip. However, we are really interested in the solutions at the right tip, which are responsible for crack propagation. Moreover, based on the work of He et al. (1991), the effect of approximation of field variables at the left tip on the right tip is negligible.

With the numerical solution of the dislocation density functions, the generalized stress intensity factors $\mathbf{K} = [K_{\text{II}}, K_{\text{I}}, K_{\text{III}}, K_{\text{D}}]^T$ at the micro crack tip are given as follows (see Appendix B for a brief description):

$$\mathbf{K} = \sqrt{\frac{\pi a}{4}} \mathbf{L} \Psi(1), \quad (3.28)$$

where K_{II} , K_I , K_{III} are the in-plane shear, tensile, out of plane shear stress intensity factors, respectively, and K_D is the electric displacement intensity factor. In the following, we focus on K_{II} , K_I and K_D for the plane case.

Substituting (3.24), (3.25) and (3.27) into (3.28) leads to

$$K_{II} = \sqrt{2\pi} d_{21} K_I^l - \sqrt{\frac{\pi}{4a}} d_{22} b_1^t, \quad (3.29)$$

$$K_I = \sqrt{2\pi} d_{11} K_I^l + \sqrt{2\pi} d_{12} K_D^l - \sqrt{\frac{\pi}{4a}} (d_{13} b_2^t + d_{14} b_4^t), \quad (3.30)$$

$$K_D = \sqrt{2\pi} d_{41} K_I^l + \sqrt{2\pi} d_{42} K_D^l - \sqrt{\frac{\pi}{4a}} (d_{43} b_2^t + d_{44} b_4^t), \quad (3.31)$$

where

$$d_{2i} = -L_{11} \psi_1^{(i)}(1), \quad d_{1j} = -L_{22} \psi_2^{(j)}(1) - L_{24} \psi_4^{(j)}(1), \quad d_{4j} = -L_{42} \psi_2^{(j)}(1) - L_{44} \psi_4^{(j)}(1),$$

$$i = 1, 2, \quad j = 1, 2, 3, 4. \quad (3.32)$$

The unknowns $\psi_1^{(1)}(1)$ and $\psi_1^{(2)}(1)$ are determined from (3.21), and $\psi_2^{(j)}(1)$ and $\psi_4^{(j)}(1)$ ($j = 1, 2, 3, 4$) from (3.22). Since $\psi_i^{(j)}(1) \approx \psi_i^{(j)}(s_1)$ for large n , we take $n = 100$ in the following calculations, and select PZT-5H, PZT-5, PZT-4 and PZT-7A ceramics for our numerical examples, with the material coefficients listed in Table 3.1 (Pak, 1992; Dunn and Taya, 1994). It is worth mentioning that the stress intensity factor K_{II} at the crack tip is independent of the K_D^l -field, when the poling direction is along the x_2 axis.

Table 3.1
Material coefficients for piezoelectric ceramics

		PZT-5H	PZT-5	PZT-4	PZT-7A
c_{11}	10^{10} Nm^{-2}	12.6	12.1	13.9	14.8
c_{12}	10^{10} Nm^{-2}	5.50	7.54	7.78	7.62
c_{13}	10^{10} Nm^{-2}	5.30	7.52	7.43	7.42
c_{33}	10^{10} Nm^{-2}	11.7	11.1	11.5	13.1
c_{44}	10^{10} Nm^{-2}	3.53	2.11	2.56	2.54
e_{31}	Cm^{-2}	-6.50	-5.40	-5.20	-2.10
e_{33}	Cm^{-2}	23.3	15.8	15.1	9.50
e_{15}	Cm^{-2}	17.0	12.3	12.7	9.20
κ_{11}	$10^{-10} \text{ C(Vm)}^{-1}$	151	81.1	64.6	40.7
κ_{33}	$10^{-10} \text{ C(Vm)}^{-1}$	130	73.5	56.2	20.8

The poling direction is along the x_3 axis.

According to the work of Park and Sun (1995a), the crack extension force is calculated as the mechanical strain energy released in propagating the crack an infinitesimal distance, i.e.,

$$G_M = \lim_{\chi \rightarrow 0} \frac{1}{\chi} \int_0^\chi \sigma_{2i}(x_1, 0) u_i(\chi - x_1, 0) dx_1, \quad i = 1, 2, \quad (3.33)$$

where χ is the assumed crack extension. Thus, the extension force for the micro crack, in terms of the generalized stress intensity factors, is obtained as

$$G_M = \frac{1}{2} \hat{\mathbf{K}}^T \hat{\mathbf{L}} \tilde{\mathbf{K}}, \quad (3.34)$$

where $\hat{\mathbf{K}} = [K_{II}, K_I]^T$, $\tilde{\mathbf{K}} = [K_{II}, K_I, K_D]^T$, and $\hat{\mathbf{L}}$ is the 2×3 matrix obtained by deleting the third and fourth rows and the third column of the inverse of matrix $\mathbf{L}$. Substituting the values given by (3.29)-(3.31) into (3.34), we find

$$G_M = \frac{1}{2} 2\pi^2 (\tilde{\mathbf{K}}^*)^T \mathbf{X} \tilde{\mathbf{K}}^* + \frac{\pi\sqrt{\pi}}{\sqrt{2a}} (\tilde{\mathbf{K}}^*)^T \mathbf{Y} \mathbf{b}^t + \frac{\pi}{8a} (\mathbf{b}^t)^T \mathbf{Z} \mathbf{b}^t, \quad (3.35)$$

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where $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are the 3×3 matrices. The elements of $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are denoted by X_{ij} , Y_{ij} , and Z_{ij} , respectively. Expressions of the elements X_{ij} , Y_{ij} , and Z_{ij} are listed in Appendix C, and the nonzero values of them are given in Table 3.2.

The extension force for a semi-infinite rigid conducting line loaded by a $\tilde{\mathbf{K}}^l$ -field, based on the mechanical strain energy release rate, is obtained as

$$G_M^l = -\frac{1}{2}(\hat{\mathbf{k}}^l)^T \hat{\mathbf{H}} \tilde{\mathbf{k}}^l, \quad (3.36)$$

where $\tilde{\mathbf{K}}^l = [K_1^l, K_2^l, K_4^l]^T$, $\tilde{\mathbf{k}}^l = [k_1^l, k_2^l, k_\phi^l]^T$ are the generalized strain intensity factors at the rigid line tip, $\hat{\mathbf{k}}^l = [k_1^l, k_2^l]^T$, and $\hat{\mathbf{H}}$ is the 2×3 matrix obtained by deleting the third and fourth rows and the third column of the inverse of matrix $\mathbf{H}$. The elements of $\hat{\mathbf{H}}$ are denoted by $\hat{H}_{ij}$. Also we have

$$\tilde{\mathbf{K}}^l = \tilde{\mathbf{S}}^T \tilde{\mathbf{H}}^{-1} \tilde{\mathbf{k}}^l, \quad (3.37)$$

where $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{H}}$ are the 3×3 matrices obtained by deleting the third row and the third column of matrices $\mathbf{S}$ and $\mathbf{H}$, respectively. Noting that

$$k_2^l = -\frac{\hat{H}_{24}}{\hat{H}_{22}} k_\phi^l, \quad (3.38)$$

Equation (3.37) can be rewritten in terms of G_M^l as

$$G_M = -\frac{2\pi^2 \mathbf{s}^T \tilde{\mathbf{X}} \mathbf{s}}{\hat{H}_{11}} G_M^l + \pi^2 X_{11} (K_{II}^l)^2 + \frac{\pi\sqrt{\pi}}{\sqrt{2a}} (\tilde{\mathbf{K}}^l)^T \mathbf{Y} \mathbf{b}^t + \frac{\pi}{8a} (\mathbf{b}^t)^T \mathbf{Z} \mathbf{b}^t. \quad (3.39)$$

In the above equation, $\tilde{\mathbf{X}}$ is a 2×2 matrix obtained by deleting the first row and the first column of $\mathbf{X}$, and $\mathbf{s} = [\hat{S}_{21}, \hat{S}_{31}]^T$, where $\hat{S}_{ij}$ stands for the element of matrix $\hat{\mathbf{S}} = \tilde{\mathbf{S}}^T \tilde{\mathbf{H}}^{-1}$.

The values of $\hat{S}_{21}$ and $\hat{S}_{31}$ for the above example materials are given in Table 3.2.

Table 3.2

The values of the elements X_{ij} , Y_{ij} , Z_{ij} and $\hat{S}_{ij}$

	PZT-5H	PZT-5	PZT-4	PZT-7A
X_{11}	3.621×10^{-13}	4.827×10^{-13}	3.964×10^{-13}	3.785×10^{-13}
X_{22}	3.140×10^{-13}	4.168×10^{-13}	3.512×10^{-13}	3.381×10^{-13}
$X_{23} (X_{32})$	1.378×10^{-4}	2.300×10^{-4}	2.393×10^{-4}	2.863×10^{-4}
X_{33}	5.430×10^3	5.782×10^3	9.384×10^3	2.148×10^4
Y_{11}	4.538×10^{-2}	4.309×10^{-2}	4.337×10^{-2}	4.353×10^{-2}
Y_{22}	4.229×10^{-2}	4.019×10^{-2}	4.028×10^{-2}	4.214×10^{-2}
Y_{23}	6.848×10^{-12}	6.490×10^{-12}	5.487×10^{-12}	3.206×10^{-12}
Y_{32}	1.653×10^7	1.898×10^7	2.342×10^7	3.218×10^7
Y_{33}	4.900×10^{-3}	5.953×10^{-3}	6.107×10^{-3}	4.377×10^{-3}
Z_{11}	5.687×10^9	3.842×10^9	4.744×10^9	5.006×10^9
Z_{22}	5.621×10^9	3.791×10^9	4.514×10^9	5.198×10^9
$Z_{23} (Z_{32})$	0.992	0.688	0.691	0.425
Z_{33}	8.425×10^{-11}	5.462×10^{-11}	4.916×10^{-11}	1.788×10^{-11}
$\hat{S}_{21}$	-1.419×10^{10}	-4.013×10^9	-6.424×10^9	-8.437×10^9
$\hat{S}_{31}$	-14.189	-10.740	-10.794	-7.290

Equation (3.39) provides the relationship among the micro crack extension force G_M , the rigid line extension force G_M^l , the generalized stress intensity factor $\tilde{\mathbf{K}}^l$ at the rigid line tip, the total dislocations inside the micro crack $\mathbf{b}^l$ and the micro crack length $2a$.

3.4 Discussion

3.4.1 Critical Crack Length

Equation (3.39) suggests that the initiated micro crack tends to reach its critical value at

$$G_{cr} = -\frac{2\pi^2 \mathbf{s}^T \tilde{\mathbf{X}} \mathbf{s}}{\hat{H}_{11}} G_M^l + \pi^2 X_{11} (K_{II}^l)^2 + \frac{\pi \sqrt{\pi}}{\sqrt{2a_{cr}}} (\tilde{\mathbf{K}}^l)^T \mathbf{Y} \mathbf{b}^t + \frac{\pi}{8a_{cr}} (\mathbf{b}^t)^T \mathbf{Z} \mathbf{b}^t. \quad (3.40)$$

For a general Zener-Stroh crack loaded with external stress and nonzero net dislocations, there are two critical crack lengths: one is stable and the other unstable (Fan, 1994). This particular physical phenomenon is also shown in (3.40). The smaller critical crack length is a stable one under the Zener-Stroh mechanism. The crack propagates until it reaches its second critical crack length, which is under the Griffith crack mechanism. Obviously, the applied $\tilde{\mathbf{K}}^l$ -field near the rigid line tip and the net Burgers vectors $\mathbf{b}^t$ inside the micro crack determine the two critical crack lengths.

The shear stress intensity factor K_{II}^l does not exist at the tip of the rigid line embedded in an isotropic elastic medium subjected to the remote load, according to the solutions by Wang et al. (1985) and Li and Ting (1989). Combined with the matrices $\mathbf{S}$, $\mathbf{L}$ and $\mathbf{H}$ for isotropic elastic materials given by Ting and Hwu (1988), equation (3.40) reduces to

$$G_{cr|iso} = -\left[\sqrt{2\pi} d_{11}(1)\right]^2 \frac{(1-2\nu)^2}{3-4\nu} G_M^l|_{iso} + \frac{\pi \sqrt{\pi}}{\sqrt{2a_{cr}}} d_{11}(1) \psi_2^{(3)}(1) K_1^l b_2^t + \frac{\pi}{8a_{cr}} \frac{\mu}{1-\nu} \left[\psi_2^{(3)}(1)\right]^2 (b_2^t)^2, \quad (3.41)$$

where

$$d_{11}(1) = \psi_2^{(1)}(1) \frac{\mu}{1-\nu} \quad (3.42)$$

is independent of the shear modulus in fact, and

$$G_M^l|_{iso} = -\frac{1-\nu}{2\mu} \frac{3-4\nu}{(1-2\nu)^2} (K_1^l)^2, \quad 1 < 3-4\nu < 3. \quad (3.43)$$

The numerical values of $d_{11}(1)$ and $\psi_2^{(3)}(1)$ are listed in Table 3.3 for different Poisson's ratio ν . It is worth noting that these values are consistent with those listed in Table 1 of

the work of Xiao and Fan (1996). The results show that the effect of ν on $d_{11}(1)$ is quite small.

3.4.2 Connection Constant

Consider a very brittle piezoelectric material that no dislocations are emitted from the tip of the rigid line and electric dislocations can be negligible, e.g., $b_2^t = b_1^t = b_\varphi^t = 0$. Also if the rigid line is not loaded by the tip shear stress field K_{II}^I , (3.40) reduces to

$$G_{cr}^I = CG_{cr}, \quad (3.44)$$

where

$$C = -\frac{1}{2\pi^2} \frac{\hat{H}_{11}}{\mathbf{s}^T \tilde{\mathbf{X}} \mathbf{s}}. \quad (3.45)$$

Equation (3.45) partially confirmed our initial conjecture in (3.1), though only for very brittle materials under the tensile stress and electric displacement fields. The numerical values of the coefficient C for PZT-5H, PZT-4, PZT-5 and PZT-7A ceramics are listed in Table 3.4. The C value also indicates that the micro crack initiation is catastrophic.

For isotropic elastic materials, one finds

$$G_{cr|iso}^I = C_{|iso} G_{cr|iso}, \quad (3.46)$$

in which

$$C_{|iso} = -\frac{3-4\nu}{(\sqrt{2}\pi d_{11}(1))^2 (1-2\nu)^2}. \quad (3.47)$$

It is worth to mention that for the incompressible solid, $\nu = 0.5$, the numerical value of $C_{|iso}$ approaches infinity, which indicates that the crack initiation is catastrophic.

Table 3.3
The coefficients for isotropic elastic materials

ν	$d_{11}(1)$	$\psi_2^{(3)}(1)$
1/4	0.140453	0.337617
1/3	0.141730	0.328747
1/2	0.143271	0.318310

Table 3.4
The connection constants C

	PZT-5H	PZT-5	PZT-4	PZT-7A
C	-23.723	-70.605	-48.633	-41.418

For ductile piezoelectric ceramics, the net Burgers vector $\mathbf{b}^t$ inside the micro crack is nonzero. In this case, G_{cr}^I relies on the total shear dislocations b_1^t , the total climbing dislocations b_2^t , and the total electric dislocations b_ϕ^t inside the micro crack as shown in (3.40). We would like to assume that b_1^t , b_2^t , and b_ϕ^t are material dependent constants, which measure the magnitudes of the ductility and dielectricity of the matrix material, respectively. Also, it is very likely that no more dislocations enter the micro crack after a micro crack is initiated, because the concentrated stress and electric displacement fields ahead of the tip of the rigid line have been released. Nonetheless, all these assumptions and conjecture need experimental support to pursue further theoretical investigations.

3.5 Conclusion

A dislocation emission mechanism for micro crack initiation at the tip of a semi-infinite rigid conducting line in piezoelectric materials was proposed, based on the evaluation of the mechanical strain energy release rate for the rigid line. For a rigid line embedded in a

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piezoelectric matrix, dislocations of one sign are driven away from the tip due to highly concentrated stress and electric displacement fields, while the stationary dislocations of the opposite sign are left behind. As a result of dislocation pileup, a micro Zener-Stroh crack is initiated at the tip of the rigid line.

The initiated micro crack was simulated by distributing piezoelectric mixed dislocations along the crack line. Accordingly, the crack problem was formulated into a system of singular integral equations. These equations were solved by following the method of Erdogan and Gupta (1972) and Erdogan et al. (1973). It was identified that the crack extension force can be expressed in terms of the strain and electric field intensity factors at the tip of the semi-infinite rigid conducting line and the net Burgers vector inside the crack. As a result, the relationship was obtained between the critical extension force for a semi-infinite rigid conducting line and the critical crack extension force for the plane case in the same piezoelectric solid. In addition, two critical crack lengths were obtained. One is stable under the Zener-Stroh mechanism, and the other is unstable under the Griffith mechanism.

CHAPTER 4

MICRO CRACK INITIATION AT THE TIP OF A FINITE RIGID CONDUCTING LINE IN A PIEZOELECTRIC SOLID

4.1 Introduction

Similar to the analyses given in Chapter 3, the criteria of micro crack initiation at the tip of a finite rigid conducting line in a piezoelectric solid can be set as

$$G_-^l = G_{-cr}^l, \quad (4.1)$$

$$G_+^l = G_{+cr}^l, \quad (4.2)$$

for antiplane and plane cases, respectively, where the rigid line extension force G is calculated based on the mechanical strain energy release rate, and the subscripts “-” and “+” denote the antiplane and plane cases, respectively. The two critical values G_{-cr}^l and G_{+cr}^l at the right hand sides of (4.1) and (4.2) are deemed to be material constants that can be determined from tests. However, there are no such experimental results in the open literature so far. It is our conjecture that the two constants G_{-cr}^l and G_{+cr}^l for a finite rigid

conducting line can be correspondingly related to G_{-cr} and G_{+cr} for a crack in the same solid, since the cases of rigid lines and cracks associate with the square root singularity.

We denote by G_{-cr} and G_{+cr} the critical crack extension forces of piezoelectric solids for antiplane and plane cases, respectively. The objective is to search for possible relations between G_{-cr}^I and G_{-cr} , as well as between G_{+cr}^I and G_{+cr} , for example,

$$G_{-cr}^I = C_1 G_{-cr}, \quad (4.3)$$

$$G_{+cr}^I = C_2 G_{+cr}, \quad (4.4)$$

where C_1 and C_2 are constants to be determined. To perform the investigation, models of micro crack initiation at the tip of a rigid line proposed by Xiao and Fan (1996) and Xiao et al. (2003) are extended to the current problem.

Let a finite rigid conducting line of length $2a$ be located at $x_2 = 0$, $|x_1| < a$, embedded in a homogeneous piezoelectric medium, whose poling direction is along the x_3 axis for the antiplane case (Fig. 1(a)), and along the x_2 axis for the plane case, as shown in Fig. 2(a). For the antiplane case, the finite rigid conducting line is loaded around its tips with the generalized stress fields characterized by $\mathbf{K}_-^I = [K_{III}^I, K_D^I]^T$, where K_{III}^I and K_D^I are the antiplane shear stress and electric displacement intensity factors, respectively. For the plane case, the finite rigid conducting line is loaded near its tips with the generalized stress fields characterized by $\mathbf{K}_+^I = [K_{II}^I, K_I^I, K_D^I]^T$, where K_{II}^I , K_I^I and K_D^I are the plane shear stress, tensile stress, and electric displacement intensity factors, respectively. Based on the dislocation emission mechanism at the rigid line tip proposed by Xiao and Fan (1996) and Xiao et al. (2003), dislocations of one sign are driven away from the tip of the

finite rigid conducting line due to the concentrated fields along certain slip planes, and the dislocations with the opposite sign pile up at the rigid line tip. Strain energy is built up gradually at the tip. As a result, Zener-Stroh cracks are initiated at the tip as shown in Figs. 1(b) and 2(b) for the antiplane and plane cases, respectively. This is a possible way to release the high strain energy.

Fig. 4.1 Schematic of micro crack initiation mechanism at the tips of a finite rigid conducting line in a piezoelectric solid (antiplane case)

Fig. 4.2 Schematic of micro crack initiation mechanism at the tips of a finite rigid conducting line in a piezoelectric solid (plane case)

4.2 Formulation

4.2.1 Antiplane Case

As discussed in the previous section, due to the high stress and electric displacement fields (Shi, 1997), a pair of micro cracks of length $2c$ are assumed to be initiated along $x_1 = \pm a$, $0 < |x_2| < 2c$, at the tip of the finite rigid conducting line loaded by a $\mathbf{K}_-^l$ -field (Fig. 1(b)). The micro cracks are assumed to be still loaded by the $\mathbf{K}_-^l$ -field when they are very small. The pair of micro cracks at one tip has the same field variables as that at the other tip, and the deformations of upper and lower cracks in the x_3 direction are anti-symmetric about the x_1 axis. Here the upper-right crack is analyzed. The distributions of generalized stresses near the rigid line tip are approximated by $\mathbf{K}_-^l / \sqrt{2\pi x_2}$ ($x_2 > 0$). We study how a micro crack of length $2c$ is affected the net Burgers vectors $\tilde{\mathbf{b}}^t = [b_3^t, b_\phi^t]^T$, with $b_4^t = b_\phi^t$, inside the micro crack, and the $\mathbf{K}_-^l$ -field.

To employ the Green's function method for the study, we begin with two piezoelectric screw dislocations with Burgers vector $\mathbf{b} = [b_3, b_\phi]^T$, one at (a, ζ) ($\zeta > 0$) and the other at $(a, -\zeta)$. Due to the interaction of the two dislocations with a finite rigid conducting line, the generalized stress fields induced at a point (a, x_2) ($x_2 > 0$) take the form (Chen et al., 2006)

$$\begin{aligned} & [\sigma_{13}(a, x_2), D_1(a, x_2)]^T \\ &= \frac{1}{4\pi} \mathbf{C} \left[-\frac{1}{x_2 - \zeta} - \frac{1}{x_2 + \zeta} \sqrt{\frac{\zeta}{x_2}} h_1(x_2, \zeta, a) + h_5(x_2, \zeta, a) \right] \mathbf{b}(\zeta), \end{aligned} \quad (4.5)$$

where the material property matrix $\mathbf{C}$ is given in (2.50), and $h_1(x_2, \zeta, a)$ and $h_5(x_2, \zeta, a)$ are given in (D.1) and (D.5) (Appendix D), respectively.

By using the above solution as the Green's function, the generalized stress fields at a point (a, x_2) , $x_2 > 0$, due to a continuous distribution of piezoelectric screw dislocations along the crack line, are then given by

$$\begin{aligned} & [\sigma_{13}(a, x_2), D_1(a, x_2)]^T \\ &= \frac{1}{4\pi} \mathbf{C} \int_0^{2c} \left[-\frac{1}{x_2 - \zeta} - \frac{1}{x_2 - \zeta} \sqrt{\frac{\zeta}{x_2}} h_1(x_2, \zeta, a) + h_5(x_2, \zeta, a) \right] \tilde{\mathbf{B}}(\zeta) d\zeta, \end{aligned} \quad (4.6)$$

where $\tilde{\mathbf{B}}(\zeta) = [B_3(\zeta), B_\varphi(\zeta)]^T$ with $\tilde{B}_4(\zeta) = \tilde{B}_\varphi(\zeta)$ is the density vector of piezoelectric screw dislocations.

With the assumption that the micro crack faces are free from electric charges (Deeg, 1980; Pak, 1990a), one has

$$\begin{aligned} & \int_0^{2c} \mathbf{C} \left[\frac{1}{x_2 - \zeta} - \frac{1}{x_2 - \zeta} \sqrt{\frac{\zeta}{y}} h_1(x_2, \zeta, a) - h_5(x_2, \zeta, a) \right] \tilde{\mathbf{B}}(\zeta) d\zeta \\ &= \mathbf{K}^I \sqrt{\frac{8\pi}{x_2}}, \quad 0 < x_2 < 2c. \end{aligned} \quad (4.7)$$

Since the net Burgers vector $\tilde{\mathbf{b}}^t$ inside the micro crack is nonzero, the dislocation density must satisfy

$$\int_0^{2c} \tilde{\mathbf{B}}(\zeta) d\zeta = \tilde{\mathbf{b}}^t. \quad (4.8)$$

Introduce the following substitutions

$$u = \frac{x_2}{c} - 1, \quad r = \frac{\zeta}{c} - 1. \quad (4.9)$$

Equations (4.7) and (4.8) with (4.6) are rewritten as

$$\begin{aligned} & \frac{1}{4\pi} \sum_{j=3}^4 C_{(i-2)(j-2)} \int_{-1}^1 \tilde{B}_j(r) \left[\frac{1}{u-r} + \frac{1}{u-r} \tilde{h}_1(u, r, c/a) - \tilde{h}_5(u, r, c/a) \right] dr, \\ & = \frac{K_i^l}{\sqrt{2\pi c}} \frac{1}{\sqrt{u+1}}, \quad -1 < u < 1, \quad i = 3, 4, \end{aligned} \quad (4.10)$$

$$\int_{-1}^1 \tilde{B}_i(r) dr = \frac{b_i^t}{c}, \quad i = 3, 4, \quad (4.11)$$

respectively, where

$$K_3^l = K_{\text{III}}^l, \quad K_4^l = K_{\text{D}}^l, \quad (4.12)$$

C_{ij} ($i, j = 1, 2$) stands for the element of $\mathbf{C}$, and $\tilde{h}_1(u, r, c/a)$ and $\tilde{h}_5(u, r, c/a)$ are given in (D.6) and (D.7) (Appendix D), respectively.

4.2.2 Plane Case

As shown in Fig. 2(a), the finite rigid conducting line is loaded with the $\mathbf{K}_+^l$ -field. Due to the concentrated fields mentioned and referred to that was obtained by Deng and Meguid (1998), dislocations of one sign move away from the rigid line tips, and the left-behind dislocations form a micro crack of length $2c$ along $x_2 = 0$ at each tip, as shown in Fig. 2(b). We assume that the initiated micro cracks are still under the tip $\mathbf{K}_+^l$ -field control. Since the physical properties of the two micro cracks are the same, we only study the one on the right hand side. The distributions of generalized stresses near the right tip of the rigid line are approximated by $\mathbf{K}_+^l / \sqrt{2\pi(x_1 - a)}$ ($x_1 > a$). We study how a micro crack located at $x_2 = 0$, $a < x_1 < a + 2c$, is affected by the $\mathbf{K}_+^l$ -field and the net Burgers vectors of piezoelectric edge dislocations $\hat{\mathbf{b}}^t = [b_1^t, b_2^t, b_\phi^t]^T$ with $b_4^t = b_\phi^t$, inside the micro crack.

The Green's function method is employed for the study. The generalized stresses arising along the crack line due to a piezoelectric edge dislocation $\mathbf{b} = [b_1, b_2, b_\phi]^T$ at $(2a + \xi, 0)$ interacting with a finite rigid conducting line (Chen et al., 2007), are given by

$$\begin{aligned} & [\sigma_{21}(x_1, 0), \sigma_{22}(x_1, 0), D_2(x_1, 0)]^T \\ &= \frac{1}{4\pi} \left\{ \frac{1}{x_1 - \xi} \left[\hat{\mathbf{W}} - \check{\mathbf{W}} \sqrt{\frac{\xi^2 - a^2}{x_1^2 - a^2}} \right] - \frac{1}{\sqrt{x_1^2 - a^2}} \check{\mathbf{W}} \right\} \mathbf{b}(\xi), \end{aligned} \quad (4.13)$$

where $\hat{\mathbf{W}}$ and $\check{\mathbf{W}}$ are the 3×3 matrices given by

$$\hat{\mathbf{W}} = \tilde{\mathbf{H}}^{-1} - \tilde{\mathbf{H}}^{-1} \tilde{\mathbf{S}}^2 + \tilde{\mathbf{L}}, \quad \check{\mathbf{W}} = \tilde{\mathbf{H}}^{-1} - \tilde{\mathbf{H}}^{-1} \tilde{\mathbf{S}}^2 - \tilde{\mathbf{L}}. \quad (4.14)$$

In the above equation, $\tilde{\mathbf{H}}$, $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{L}}$ are the 3×3 matrices obtained by deleting the third row and the third column of $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{L}$ defined in (2.39), respectively. The elements of $\tilde{\mathbf{H}}$, $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{L}}$ are denoted by $\tilde{H}_{ij}$, $\tilde{S}_{ij}$ and $\tilde{L}_{ij}$, respectively.

The generalized stresses arising due to a continuous distribution of piezoelectric edge dislocations along the crack line are then given by

$$\begin{aligned} & [\sigma_{21}(x_1, 0), \sigma_{22}(x_1, 0), D_2(x_1, 0)]^T \\ &= \frac{1}{4\pi} \int_a^{a+2c} \left\{ \frac{1}{x_1 - \xi} \left[\hat{\mathbf{W}} - \check{\mathbf{W}} \sqrt{\frac{\xi^2 - a^2}{x_1^2 - a^2}} \right] - \frac{1}{\sqrt{x_1^2 - a^2}} \check{\mathbf{W}} \right\} \hat{\mathbf{B}}(\xi) d\xi, \end{aligned} \quad (4.15)$$

in which $\hat{\mathbf{B}}(\xi) = [\hat{B}_1(\xi), \hat{B}_2(\xi), \hat{B}_\phi(\xi)]^T$ with $\hat{B}_4(\xi) = \hat{B}_\phi(\xi)$ is the density vector of piezoelectric edge dislocations.

Assuming the crack faces are free from electric charges, one has

$$\int_a^{a+2c} \left\{ \frac{\hat{\mathbf{B}}(\xi)}{x_1 - \xi} \left[\hat{\mathbf{W}} - \check{\mathbf{W}} \sqrt{\frac{\xi^2 - a^2}{x_1^2 - a^2}} \right] - \frac{\check{\mathbf{W}}}{\sqrt{x_1^2 - a^2}} \right\} d\xi = -\mathbf{K}_+^I \sqrt{\frac{8\pi}{x_1 - a}}, \quad 0 < x_1 - a < 2c. \quad (4.16)$$

Considering the net Burgers vectors $\hat{\mathbf{b}}^t$ inside the micro crack, one has

$$\int_a^{a+2c} \hat{\mathbf{B}}(\xi) d\xi = \hat{\mathbf{b}}^t. \quad (4.17)$$

The integral over $[a, a+2c]$ is normalized to $[-1, 1]$ by the following substitutions

$$t = \frac{x_1 - a}{c} - 1, \quad s = \frac{\xi - a}{c} - 1. \quad (4.18)$$

Equations (4.16) and (4.17) with (4.15) now read

$$\begin{aligned} & \sum_{i=1}^3 \left[\bar{W}_{mi} \int_{-1}^1 \frac{\hat{B}_i(s)}{t-s} ds - \check{W}_{mi} \int_{-1}^1 \frac{g_1(t, c/a) \hat{B}_i(s)}{g_3(s, c/a)(t-s)} ds - \check{W}_{mi} \int_{-1}^1 g_1(t, c/a) \hat{B}_i(s) ds \right] \\ &= -\sqrt{\frac{8\pi}{c}} g_2(t) K_m^l, \quad -1 < t < 1, \quad m = 1, 2, 3, \end{aligned} \quad (4.19)$$

$$\int_{-1}^1 \hat{B}_i(s) ds = \frac{\check{b}_i}{c}, \quad i = 1, 2, 3, \quad (4.20)$$

respectively, where

$$K_1^l = K_{II}^l, \quad K_2^l = K_I^l, \quad K_3^l = K_D^l, \quad (4.21)$$

$$\check{b}_1 = b_1^t, \quad \check{b}_2 = b_2^t, \quad \check{b}_3 = b_\phi^t, \quad (4.22)$$

$$g_1(t, c/a) = \frac{1}{\sqrt{1+t}\sqrt{1+t+2a/c}}, \quad g_2(t) = \frac{1}{\sqrt{1+t}}, \quad g_3(s, c/a) = \frac{1}{\sqrt{1+s}\sqrt{1+s+2a/c}}, \quad (4.23)$$

4.3 Numerical Procedure

Let $\Xi(r)$ and $\Psi(s)$ be bounded functions in $[-1, 1]$, and the piezoelectric screw and edge dislocation densities can be written as

$$\tilde{\mathbf{B}}(r) = \Xi(r)(1+r)^\lambda (1-r)^\gamma, \quad (4.24)$$

$$\hat{\mathbf{B}}(s) = \boldsymbol{\Psi}(s)(1+s)^\alpha (1-s)^\beta, \quad (4.25)$$

respectively, with

$$\boldsymbol{\Psi}(s) = [\psi_1(s), \psi_2(s), \psi_3(s)]^T, \quad \psi_3(s) = \psi_\varphi(s), \quad (4.26)$$

$$\boldsymbol{\Xi}(r) = [\Xi_3(r), \Xi_4(r)]^T, \quad \Xi_4(r) = \Xi_\varphi(r). \quad (4.27)$$

4.3.1 Antiplane Case

By following the method of Erdogan and Gupta (1972) and Erdogan et al. (1973), the discrete forms of (4.10) and (4.11) are written as

$$\begin{aligned} & \frac{1}{4\pi} \sum_{j=1}^n \tilde{W}_j(r_j) \sum_{k=3}^4 C_{(m-2)(k-2)} \Xi_k(r_j) \left[-\frac{1 + \tilde{h}_1(u_i, r_j, c/a)}{u_i - r_j} + \tilde{h}_5(u_i, r_j, c/a) \right] \\ & = -\frac{K_m^l}{\sqrt{2\pi c}} \frac{1}{\sqrt{1+u_i}}, \quad m = 3, 4, \end{aligned} \quad (4.28)$$

$$\sum_{j=1}^n \tilde{W}_j(r_j) \Xi_r(r_j) = \frac{b_r^l}{c}, \quad r = 3, 4, \quad (4.29)$$

respectively, where r_j and u_i are the roots of the Jacobi polynomials,

$$P_n^{(\lambda, \gamma)}(r_j) = 0, \quad P_{n+\lambda+\gamma}^{(\lambda+1, \gamma+1)}(u_i) = 0, \quad j = 1, \dots, n, \quad i = 1, \dots, n-1, \quad (4.30)$$

and $\tilde{W}_j(r_j)$ ($j = 1, \dots, n$) can be obtained from (3.21) by replacing s_m , α , β with r_j , λ , γ , respectively.

Equations (4.28) and (4.29) provide a system of $2n$ linear algebraic equations for the $2n$ unknowns $\Xi_i(r_j)$ ($i = 3, 4$, $j = 1, 2, \dots, n$). If $\Xi_i^{(1)}(r)$ and $\Xi_i^{(2)}(r)$ are the solutions of the following set of system

$$\begin{cases} \sum_{j=1}^n \tilde{W}_j(r_j) \Xi_m^{(k)}(r_j) \left[-\frac{1 + \tilde{h}_1(u_i, r_j, c/a)}{u_i - r_j} + \tilde{h}_5(u_i, r_j, c/a) \right] = \frac{1}{\sqrt{1+u_i}} \delta_{1k}, \\ \sum_{j=1}^n \tilde{W}_j(r_j) \Xi_m^{(k)}(r_j) = \delta_{2k}, \\ m = 3, 4, \quad k = 1, 2, \end{cases} \quad (4.31)$$

we then obtain

$$\Xi_3^{(1)}(r) = \Xi_4^{(1)}(r), \quad \Xi_3^{(2)}(r) = \Xi_4^{(2)}(r), \quad (4.32)$$

$$\Xi_3(r) = -\sqrt{\frac{8\pi}{c}} \frac{\kappa_{11} K_{II}^I + e_{15} K_D^I}{c_{44} \kappa_{11} + e_{15}^2} \Xi_3^{(1)}(r) + \frac{b_3^t}{c} \Xi_3^{(2)}(r), \quad (4.33)$$

$$\Xi_4(r) = -\sqrt{\frac{8\pi}{c}} \frac{e_{15} K_{II}^I - c_{44} K_D^I}{c_{44} \kappa_{11} + e_{15}^2} \Xi_4^{(1)}(r) + \frac{b_\phi^t}{c} \Xi_4^{(2)}(r). \quad (4.34)$$

For the current problem, there is a square root singularity at the upper tip of the micro crack, so $\gamma = -1/2$ is taken. Similar to that analysis in Section 3.3, Chapter 3, the contact zone introduced by Comninou (1977) is adopted at the lower tip of the micro crack. Thus, there is a square root singularity at the lower tip, and we take $\lambda = -1/2$. Based on the work of He et al. (1991), the effect of field approximation at the lower tip on the upper tip is negligible. Moreover, we are really interested in the solutions at the upper tip, which are responsible for crack propagation.

By following the procedure of Muskhelishvili (1975) and in view of (4.6) and (4.24), the generalized stress intensity factors $\mathbf{K}_- = [K_{III}, K_D]^T$ at the upper tip of the upper-right micro crack are obtained as

$$\mathbf{K}_- = \lim_{x_2 \rightarrow 2c^+} \sqrt{2\pi(x_2 - 2c)} [\sigma_{13}(a, x_2), D_1(a, x_2)]^T = -\sqrt{\frac{\pi c}{4}} \mathbf{C} \Xi(1), \quad (4.35)$$

where K_{III} and K_D are the antiplane shear stress and electric displacement intensity factors, respectively. Substituting (4.33) and (4.34) into (4.35) leads to

$$K_{III} = \sqrt{2\pi}\Xi_3^{(1)}(1)K_{III}^I - \frac{\sqrt{\pi}}{2\sqrt{c}} \left[c_{44}\Xi_3^{(2)}(1)b_3^t + e_{15}\Xi_4^{(2)}(1)b_\phi^t \right], \quad (4.36)$$

$$K_D = \sqrt{2\pi}\Xi_4^{(1)}(1)K_D^I - \frac{\sqrt{\pi}}{2\sqrt{c}} \left[e_{15}\Xi_3^{(2)}(1)b_3^t - \kappa_{11}\Xi_4^{(2)}(1)b_\phi^t \right]. \quad (4.37)$$

According to the works of Park and Sun (1995a; 1995b), the antiplane crack extension force, based on the mechanical energy release rate, can be defined as

$$G_{-M} = \lim_{\chi \rightarrow 0} \frac{1}{\chi} \int_0^\chi \sigma_{13}(a, x_2) u_3(a, \chi - x_2) dx_2. \quad (4.38)$$

Thus the antiplane crack extension force is expressed in terms of the generalized stress intensity factors as

$$G_{-M} = \frac{\kappa_{11}(K_{III})^2 + e_{15}K_{III}K_D}{2(c_{44}\kappa_{11} + e_{15}^2)}. \quad (4.39)$$

The antiplane rigid line extension force based on the mechanical energy release rate is obtained as

$$G_{-M}^I = -\frac{\kappa_{11}(K_{III}^I)^2 + e_{15}K_{III}^IK_D^I}{2(c_{44}\kappa_{11} + e_{15}^2)}. \quad (4.40)$$

In view of (4.36), (4.37) and (4.40), equation (4.39) is rewritten as

$$\begin{aligned} G_{-M} = & -\left[\sqrt{2\pi}\Xi_3^{(1)}(1) \right]^2 G_{-M}^I \\ & - \frac{\pi\sqrt{\pi}}{\sqrt{2c}} \Xi_3^{(1)}(1)\Xi_3^{(2)}(1) \left[K_{III}^I b_3^t + \frac{(c_{44}b_3^t + e_{15}b_\phi^t)(\kappa_{11}K_{III}^I + e_{15}K_D^I)}{2(c_{44}\kappa_{11} + e_{15}^2)} \right] \\ & + \frac{\pi[\Xi_3^{(2)}(1)]^2}{8c} [c_{44}(b_3^t)^2 + e_{15}b_3^t b_\phi^t]. \end{aligned} \quad (4.41)$$

Table 4.1

The antiplane coefficients under the ratio $c/a = 10^{-4}$

$\Xi_3^1(1) (\Xi_4^1(1))$	$\Xi_3^2(1) (\Xi_4^2(1))$
0.117764	0.415922

Note that $\Xi_j^{(i)}(1) \approx \Xi_j^{(i)}(r_1)$ ($i = 1, 2$) for large n . In our calculation, we take $n = 100$. It is observed from (4.31) that the coefficients $\Xi_3^{(1)}(1)$, $\Xi_4^{(1)}(1)$, $\Xi_3^{(2)}(1)$ and $\Xi_4^{(2)}(1)$ depend on the ratio of the micro crack length $2c$ to the rigid line length $2a$, but are independent of the material property constants. Therefore, it is possible to find the constant C_1 from (4.41) that exhibits the relationship among the crack extension force G_{-M} , the rigid line extension force G_{-M}^l , the generalized stress intensity factor $\mathbf{K}_-^l$, the net Burgers vector $\mathbf{b}^t$ and the micro crack length $2c$. The coefficients $\Xi_3^{(1)}(1)$, $\Xi_4^{(1)}(1)$, $\Xi_3^{(2)}(1)$ and $\Xi_4^{(2)}(1)$ corresponding to the ratio $c/a = 10^{-4}$ are listed in Table 4.1. Since these coefficients rely on the ratio c/a , the constant C_1 could be a function of it.

4.3.2 Plane Case

Employing the method of Erdogan and Gupta (1972) and Erdogan et al. (1973), we write the discrete forms of (4.19) and (4.20) as

$$\begin{aligned} & \frac{1}{4\pi} \sum_{r=1}^n \sum_{i=1}^3 \tilde{W}_r(s_r) \left[\frac{\tilde{W}_{mi}}{t_u - s_r} - \frac{\tilde{W}_{mi}}{t_u - s_r} \frac{g_1(t_u, c/a)}{g_3(s_r, c/a)} - \tilde{W}_{mi} g_1(t_u, c/a) \right] \psi_i(s_r) \\ &= -\frac{K_m^l}{\sqrt{2\pi c}} g_2(t_u), \quad m = 1, 2, 3, \end{aligned} \quad (4.42)$$

$$\sum_{r=1}^n \tilde{W}_r(s_r) \psi_i(s_r) = \frac{\tilde{b}_i}{c}, \quad i = 1, 2, 3, \quad (4.43)$$

respectively, where s_r and t_u are the roots of the Jacobi polynomials,

$$P_n^{(\alpha, \beta)}(s_r) = 0, \quad P_{n+\alpha+\beta}^{(\alpha+1, \beta+1)}(t_u) = 0, \quad r = 1, \dots, n, \quad u = 1, \dots, n-1, \quad (4.44)$$

and $\tilde{W}_r(s_r)$ can be obtained from (3.21) by replacing s_m with s_r .

Equations (4.42) and (4.43) provide a system of $3n$ linear algebraic equations for the $3n$ $\psi_i(s_r)$ ($i = 1, 2, 3$, $r = 1, 2, \dots, n$). If we denote $\psi_m^{(j)}(s_r)$ as the solutions of the system

$$\begin{cases} \sum_{r=1}^n \sum_{i=1}^3 \tilde{W}_r(s_r) \left[\tilde{W}_{mi} g_1(t_u, c/a) - \frac{\tilde{W}_{mi} g_1(s_r, c/a) - \tilde{W}_{mi} g_1(t_u, c/a)}{(t_u - s_r) g_3(s_r, c/a)} \right] \psi_i^{(j)}(s_r) = g_2(t_u) \delta_{mj}, \\ \sum_{r=1}^n \tilde{W}_r(s_r) \psi_m^{(j)}(s_r) = \delta_{(m+4)j}, \\ m = 1, 2, 3, \quad j = 1, 2, \dots, 6, \end{cases} \quad (4.45)$$

where the Kronecker delta δ_{ij} is given in (3.24), one then has

$$\psi_1(s) = \sqrt{\frac{8\pi}{c}} K_{II}^I \psi_1^{(1)}(s) + \frac{\tilde{b}_1}{c} \psi_1^{(4)}(s), \quad (4.46)$$

$$\psi_2(s) = \sqrt{\frac{8\pi}{c}} [K_I^I \psi_2^{(2)}(s) + K_D^I \psi_2^{(3)}(s)] + \frac{b_2^I}{c} \psi_2^{(5)}(s) + \frac{b_\phi^I}{c} \psi_2^{(6)}(s), \quad (4.47)$$

$$\psi_3(s) = \sqrt{\frac{8\pi}{c}} [K_I^I \psi_3^{(2)}(s) + K_D^I \psi_3^{(3)}(s)] + \frac{b_2^I}{c} \psi_3^{(5)}(s) + \frac{b_\phi^I}{c} \psi_3^{(6)}(s). \quad (4.48)$$

Since a square root singularity exists at the right tip of the micro crack, $\beta = -1/2$ is taken in calculation. Oscillatory singularities at the left tip of the crack can be caused by the incompatibility of piezoelectric material and rigid line. Using the contact zone introduced by Comninou (1997), we can take $\alpha = -1/2$ at the left tip. Based on the work of He et al. (1991), the effect of the field approximation at the left tip on the right tip is negligible. Moreover, we are really interested in the solutions at the right tip, which are responsible for crack propagation.

The generalized stress intensity factors $\mathbf{K}_+ = [K_{II}, K_I, K_D]^T$ at the right tip of the micro crack are defined as

$$\mathbf{K}_+ = \lim_{x_1 \rightarrow a+2c^+} \sqrt{2\pi(x_1 - 2a - c)} [\sigma_{21}(x_1, 0), \sigma_{22}(x_1, 0), D_2(x_1, 0)]^T, \quad (4.49)$$

where K_{II} , K_I and K_D are the plane shear stress, tensile stress and electric displacement

intensity factors, respectively. Substituting (4.15) into (4.49) with (4.25) and following the procedure of Muskhelishvili (1975), one obtains

$$\mathbf{K}_+ = \frac{\sqrt{\pi c}}{4} (\hat{\mathbf{W}} - \check{\mathbf{W}}) \boldsymbol{\psi}(1). \quad (4.50)$$

Substituting (4.46)-(4.48) into (4.50), we have

$$K_{II} = \sqrt{2} \pi f_{11} K_{II}^I + \frac{\sqrt{\pi}}{2\sqrt{c}} f_{12} b_1^I, \quad (4.51)$$

$$K_I = \sqrt{2} \pi f_{21} K_I^I + \sqrt{2} \pi f_{22} K_D^I + \frac{\sqrt{\pi}}{2\sqrt{c}} (f_{23} b_2^I + f_{24} b_\phi^I), \quad (4.52)$$

$$K_D = \sqrt{2} \pi f_{31} K_I^I + \sqrt{2} \pi f_{32} K_D^I + \frac{\sqrt{\pi}}{2\sqrt{c}} (f_{33} b_2^I + f_{34} b_\phi^I), \quad (4.53)$$

where

$$f_{1i} = \tilde{L}_{11} \psi_1^{(i^2)}(1), \quad i = 1, 2, \quad (4.54)$$

$$f_{ki} = \tilde{L}_{k2} \psi_2^{(i+1)}(1) + \tilde{L}_{k3} \psi_3^{(i+1)}(1), \quad k = 2, 3, \quad i = 1, 2, \quad (4.55)$$

$$f_{ki} = \tilde{L}_{k2} \psi_2^{(i+2)}(1) + \tilde{L}_{k3} \psi_3^{(i+2)}(1), \quad k = 2, 3, \quad i = 3, 4. \quad (4.56)$$

The coefficients f_{1i} , f_{2i} and f_{3i} can be obtained from the numerical solutions $\psi_i^{(k)}(1)$, with $\psi_i^{(k)}(1) \approx \psi_i^{(k)}(s_1)$ for large number of n . It is also observed from (4.45) that these coefficients depend on the ratio of the micro crack length $2c$ to the rigid line length $2a$. We take $n = 100$ in the following calculations, and select PZT-5H, PZT-5, PZT-4 and PZT-7A ceramics for our numerical examples (see Table 3.1).

Based on the mechanical strain energy release rate, the crack extension force, in terms of the generalized stress intensity factors, is thus expressed as

$$G_{+M} = \frac{1}{2} (\hat{\mathbf{K}}_+)^T \hat{\mathbf{L}} \mathbf{K}_+, \quad (4.57)$$

where $\hat{\mathbf{K}}_+ = [K_{II}, K_I]^T$ and $\hat{\mathbf{L}}$ is the 2×3 matrix obtained by deleting the third row of the inverse of matrix $\tilde{\mathbf{L}}$. $\hat{L}_{ij}$ represents the element of $\hat{\mathbf{L}}$. Substituting (4.50) into (4.57) with (4.51)-(4.53), one has

$$G_{+M} = \frac{1}{2} 2\pi^2 (\mathbf{K}_+^I)^T \mathbf{X} \mathbf{K}_+^I + \frac{\pi\sqrt{\pi}}{\sqrt{2c}} (\mathbf{K}_+^I)^T \mathbf{Y} \hat{\mathbf{b}}^t + \frac{\pi}{8c} (\hat{\mathbf{b}}^t)^T \mathbf{Z} \hat{\mathbf{b}}^t, \quad (4.58)$$

where $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are the 3×3 matrices. The elements of matrices $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are denoted by X_{ij} , Y_{ij} , and Z_{ij} , respectively, which are given in Appendix E. The values of elements X_{ij} , Y_{ij} , and Z_{ij} for the ratio $c/a = 10^{-4}$ are listed in Table 4.2.

Table 4.2

The values of the elements X_{ij} , Y_{ij} , Z_{ij} and $\hat{S}_{ij}$ under the ratio $c/a = 10^{-4}$

	PZT-5H	PZT-5	PZT-4	PZT-7A
X_{11}	3.621×10^{-13}	4.827×10^{-13}	3.964×10^{-13}	3.785×10^{-13}
X_{22}	3.140×10^{-13}	4.168×10^{-13}	3.512×10^{-13}	3.381×10^{-13}
$X_{23} (X_{32})$	1.378×10^{-4}	2.300×10^{-4}	2.393×10^{-4}	2.863×10^{-4}
X_{33}	5.429×10^3	5.782×10^3	9.384×10^3	2.148×10^4
Y_{11}	4.538×10^{-2}	4.311×10^{-2}	4.341×10^{-2}	4.357×10^{-2}
Y_{22}	4.227×10^{-2}	4.019×10^{-2}	4.027×10^{-2}	4.213×10^{-2}
Y_{23}	6.826×10^{-12}	6.473×10^{-12}	5.471×10^{-12}	3.195×10^{-12}
Y_{32}	1.652×10^7	1.898×10^7	2.342×10^7	3.218×10^7
Y_{33}	4.891×10^{-3}	5.949×10^{-3}	6.102×10^{-3}	4.372×10^{-3}
Z_{11}	5.688×10^9	3.851×10^9	4.754×10^9	5.015×10^9
Z_{22}	5.615×10^9	3.790×10^9	4.512×10^9	5.195×10^9
$Z_{23} (Z_{32})$	0.989	0.687	0.689	0.425
Z_{33}	8.307×10^{-11}	5.390×10^{-11}	4.850×10^{-11}	1.764×10^{-11}
$\hat{S}_{21}$	-1.419×10^{10}	-4.013×10^9	-6.424×10^9	-8.437×10^9
$\hat{S}_{31}$	-14.189	-10.740	-10.794	-7.290

The extension force for a finite rigid conducting line loaded by a tip $\mathbf{K}^l$ -field, based on the mechanical energy release rate, is obtained as

$$G_{+M}^l = -\frac{1}{2}(\hat{\mathbf{k}}_+^l)^T \hat{\mathbf{H}} \mathbf{k}_+^l, \quad (4.59)$$

where $\mathbf{k}_+^l = [k_1^l, k_2^l, k_\phi^l]^T$ are the generalized strain intensity factors at the rigid line tip,

$\hat{\mathbf{k}}_+^l = [k_1^l, k_2^l]^T$, and $\hat{\mathbf{H}}$ is the 2×3 matrix obtained by deleting the third row from the

inverse of matrix $\tilde{\mathbf{H}}$. The elements of $\hat{\mathbf{H}}$ are denoted by $\hat{H}_{ij}$. We have

$$\mathbf{K}_+^l = \tilde{\mathbf{S}}^T \tilde{\mathbf{H}}^{-1} \mathbf{k}_+^l. \quad (4.60)$$

In terms of G_{+M}^l , (4.58) can be rewritten as

$$\begin{aligned} G_{+M} = & -\frac{2\pi^2 \mathbf{s}^T \tilde{\mathbf{X}} \mathbf{s}}{\hat{H}_{11}} G_{+M}^l \\ & + \pi^2 X_{11} (K_{II}^l)^2 + \frac{\pi\sqrt{\pi}}{\sqrt{2c}} (\mathbf{K}_+^l)^T \mathbf{Y} \hat{\mathbf{b}}^t + \frac{\pi}{8c} (\hat{\mathbf{b}}^t)^T \mathbf{Z} \hat{\mathbf{b}}^t. \end{aligned} \quad (4.61)$$

In the above equation, $\tilde{\mathbf{X}}$ is a 2×2 matrix obtained by deleting the first row and the first column from matrix $\mathbf{X}$, and $\mathbf{s} = [\hat{S}_{21}, \hat{S}_{31}]^T$, in which $\hat{S}_{ij}$ stands for the element of matrix $\hat{\mathbf{S}} = \tilde{\mathbf{S}}^T \tilde{\mathbf{H}}^{-1}$. The values of elements $\hat{S}_{21}$ and $\hat{S}_{31}$ for the ratio $c/a = 10^{-4}$, with the example materials, are given in Table 4.2. In deriving (4.61), equations (3.39) and (4.60) have been used.

Equation (4.61) exhibits the relationship among the micro crack extension force G_{+M} , the finite rigid line extension force G_{+M}^l , the generalized stress intensity factors $\mathbf{K}_+^l$, the net Burgers vector $\hat{\mathbf{b}}^t$ inside the micro crack and the micron crack length $2c$.

4.4 Discussion

4.4.1 Antiplane Case

Equation (4.41) suggests that the upper-right micro crack tends to reach its critical value at

$$\begin{aligned}
 G_{-cr} = & -\left[\sqrt{2}\pi\Xi_3^{(1)}(1)\right]^2 G_{-M}^l \\
 & - \frac{1}{\sqrt{2c_{cr}}} \pi \sqrt{\pi}\Xi_3^{(1)}(1)\Xi_3^{(2)}(1) \left[K_{III}^l b_3^t + \frac{(c_{44}b_3^t + e_{15}b_\phi^t)(\kappa_{11}K_{III}^l + e_{15}K_D^l)}{2(c_{44}\kappa_{11} + e_{15}^2)} \right] \\
 & + \frac{1}{8c_{cr}} \pi \left[\Xi_3^{(2)}(1)\right]^2 \left[c_{44}(b_3^t)^2 + e_{15}b_3^t b_\phi^t \right], \tag{4.62}
 \end{aligned}$$

The two critical crack lengths can be obtained from (4.62), determined by the antiplane tip $\mathbf{K}_-^l$ -fields and the net Burgers vector $\tilde{\mathbf{b}}^t$ of piezoelectric screw dislocations inside the micro crack. The smaller critical crack length is a stable one under the Zener-Stroh mechanism. The crack propagates until it reaches its second critical crack length which is under the Griffith crack mechanism.

Equation (4.62) can be reduced to

$$\begin{aligned}
 G_{-cr|iso} = & -\left[\sqrt{2}\pi\Xi_3^{(1)}(1)\right]^2 G_{-|iso}^l \\
 & - \frac{1}{\sqrt{2c_{cr}}} \pi \sqrt{\pi}\Xi_3^{(1)}(1)\Xi_3^{(2)}(1) K_{III}^l b_3^t + \frac{1}{8c_{cr}} \pi \left[\Xi_3^{(2)}(1)\right]^2 \mu (b_3^t)^2 \tag{4.63}
 \end{aligned}$$

for isotropic elastic media, where

$$G_{-|iso}^l = -\frac{1}{2\mu} (K_{III}^l)^2. \tag{4.64}$$

Table 4.3
The connection constants C_1 and C_2 for PZT-5H

c/a	C_1	C_2
10^{-8}	-3.653	-23.7226
10^{-7}	-3.653	-23.7226
10^{-6}	-3.653	-23.7226
10^{-5}	-3.653	-23.7226
10^{-4}	-3.65299	-23.7226
5×10^{-4}	-3.65295	-23.7226
10^{-3}	-3.6529	-23.7226
5×10^{-3}	-3.65247	-23.7229
10^{-2}	-3.65193	-23.7233
10^{-1}	-3.64158	-23.7292
1	-3.56074	-23.7644

Consider a very brittle piezoelectric material so that no dislocations are emitted from the tip of the rigid line and electric dislocations can be negligible, i.e., $b_3^t = b_\phi^t = 0$. Under this circumstance, equations (4.62) and (4.63) can be reduced to

$$G_{-cr}^I = C_1 G_{-cr}, \quad (4.65)$$

with

$$C_1 = -\frac{1}{(\sqrt{2}\pi\Xi_3^{(1)}(1))^2}. \quad (4.66)$$

Equation (4.65) partially confirmed the initial conjecture in (4.1), for both very brittle piezoelectric and isotropic elastic materials. Since the coefficient $\Xi_3^{(1)}(1)$ is a result of the ratio c/a , the coefficient C_1 is a function of the ratio c/a . The numerical values of the coefficient C_1 for the different ratio c/a are listed in Table 4.3. It shows that the absolute value of the coefficient C_1 decreases with increasing c/a value, or with the micro crack propagation. It is observed that the C_1 value is almost constant, when the micro crack is at the very initial stage ($c/a < 10^{-3}$). The variation of the C_1 value for the different ratio

c/a is so small that the coefficient C_1 can be considered as a constant. The C_1 value also indicated that the crack initiation will be catastrophic.

4.4.2 Plane Case

Equation (4.61) suggests that the right micro crack tends to reach its critical value at

$$G_{+M} = -\frac{2\pi^2 \mathbf{s}^T \tilde{\mathbf{X}} \mathbf{s}}{\hat{H}_{11}} G_{+M}^l + \pi^2 X_{11} (K_{II}^l)^2 + \frac{\pi \sqrt{\pi}}{\sqrt{2c_{cr}}} (\mathbf{K}_+^l)^T \mathbf{Y} \hat{\mathbf{b}}^t + \frac{\pi}{8c_{cr}} (\hat{\mathbf{b}}^t)^T \mathbf{Z} \hat{\mathbf{b}}^t. \quad (4.67)$$

Similar to the previous discussions, two critical crack lengths can be obtained from (4.67).

The tip $\mathbf{K}_+^l$ -field and the net Burgers vector $\hat{\mathbf{b}}^t$ of piezoelectric edge dislocations inside the micro crack determine the two values for the two different physical mechanisms.

Since the shear stress intensity factor K_{II}^l does not exist at the tip of the rigid line in an isotropic elastic material subjected to remote loads, according to the solutions by Wang et al. (1985) and Li and Ting (1989), using the matrices $\mathbf{S}$, $\mathbf{L}$ and $\mathbf{H}$ for isotropic elastic materials given by Ting and Hwu (1988), equation (4.67) can be reduced to

$$G_{+cr|iso} = -\left[\sqrt{2}\pi f_{21}(1)\right]^2 \frac{(1-2\nu)^2}{3-4\nu} G_{+iso}^l + \frac{\pi \sqrt{\pi}}{\sqrt{2c_{cr}}} f_{21}(1) \psi_2^{(5)}(1) K_1^l b_2^t + \frac{\pi}{8c_{cr}} \frac{\mu}{1-\nu} \left[\psi_2^{(5)}(1)\right]^2 (b_2^t)^2, \quad (4.68)$$

where

$$f_{21}(1) = \psi_2^{(2)}(1) \frac{\mu}{1-\nu}, \quad (4.69)$$

$$G_{+iso}^l = -\frac{1-\nu}{2\mu} \frac{3-4\nu}{(1-2\nu)^2} (K_1^l)^2, \quad 1 < 3-4\nu < 3. \quad (4.70)$$

In view of (4.45) and (4.69), it can be found that the coefficients $f_{21}(1)$ and $\psi_2^{(5)}(1)$ are functions of Poisson's ratio ν and the ratio c/a only, for isotropic elastic materials. Table 4.4 provides the numerical values of $f_{21}(1)$ and $\psi_2^{(5)}(1)$ for the ratio $c/a = 10^{-4}$ with different Poisson's ratio ν . It is worth noting that these values are consistent with those listed in Table 1 of the work of Xiao and Fan (1996). The results show that the effect of ν on $f_{21}(1)$ and $\psi_2^{(5)}(1)$ is negligible.

Consider a very brittle piezoelectric material so that no dislocations are emitted from the rigid line tip and the electric dislocations are negligible, i.e., $b_1^t = b_2^t = b_\phi^t = 0$. Without the tip shear stress field K_{II}^I , equation (4.67) is reduced to

$$G_{+cr}^I = C_2 G_{+cr}, \quad (4.71)$$

where

$$C_2 = -\frac{1}{2\pi^2} \frac{\hat{H}_{11}}{\mathbf{s}^T \tilde{\mathbf{X}} \mathbf{s}}. \quad (4.72)$$

Table 4.4

The coefficients for isotropic elastic materials under the ratio $c/a = 10^{-4}$

ν	$f_{21}(1)$	$\psi_2^{(5)}(1)$
1/4	0.140453	0.337617
1/3	0.141730	0.328747
1/2	0.143271	0.318310

Table 4.5

The coefficient C_2 under the ratio $c/a = 10^{-4}$

	PZT-5H	PZT-5	PZT-4	PZT-7A
C_2	-23.723	-70.605	-48.633	-41.418

Equation (4.72) partially confirmed the initial conjecture in (4.2), although it is only for very brittle piezoelectric materials under the tensile stress and electric displacement fields. We select PZT-5H as the medium for a numerical example. The calculated values of the coefficient C_2 for different c/a value are listed in Table 4.3. It can be observed that the coefficient C_2 is constant, when the micro crack is at the very initial stage ($c/a < 10^{-3}$). With the further propagation of the micro crack, the absolute value of the coefficient C_2 increases so slightly that we can still consider it a constant in the same medium. The C_2 value also indicates that the micro crack initiation is catastrophic. The numerical values of C_2 for PZT-5H, PZT-4, PZT-5 and PZT-7A ceramics corresponding to $c/a = 10^{-4}$ are listed in Table 4.5. It is worth to mention that the values are consistent with that obtained in Chapter 3.

For isotropic elastic materials, one finds

$$G_{+cr|iso}^l = C_{2|iso} G_{+cr|iso}, \quad (4.73)$$

in which

$$C_{2|iso} = -\frac{3-4\nu}{(\sqrt{2}\pi f_{21}(1))^2(1-2\nu)^2}. \quad (4.74)$$

It is worth to point out that for the incompressible solid, $\nu = 0.5$, the numerical value of $C_{2|iso}$ approaches infinity, which indicates the crack initiation is catastrophic.

For ductile piezoelectric materials, the net Burgers vectors $\tilde{\mathbf{b}}^t$ and $\hat{\mathbf{b}}^t$ are nonzero. In this case, G_{-cr}^l depends on the total shear dislocations b_3^t and the total electric dislocations b_φ^t inside the micro crack as shown in (4.62); G_{+cr}^l relies on the total shear dislocations b_1^t ,

the total climbing dislocations b_2^t and the total electric dislocations b_ϕ^T inside the micro crack as shown in (4.67). We would like to assume that b_1^t , b_2^t , b_3^t and b_ϕ^t are material dependent constants, which measure the magnitudes of the ductility and dielectricity of the matrix material. It is also very likely that after a micro crack is initiated, no more dislocations enter the crack because the concentrated stress ahead of the rigid line tip has been released. Nonetheless, all these assumptions and conjecture need experimental support to pursue further theoretical investigations.

4.5 Conclusion

Criteria of a pair of micro cracks initiation and one micro crack initiation at each tip of a finite rigid conducting line in a piezoelectric medium were proposed for antiplane and plane cases, respectively, based on the evaluation of the mechanical strain energy release rates. The initiated micro cracks were simulated by distributed piezoelectric screw and edge dislocations along the crack line for antiplane and plane cases, respectively. The antiplane and plane crack problems were accordingly formulated into a respective set of singular integral equations. The method of Erdogan and Gupta (1972) and Erdogan et al. (1973) was used to solve these equations. The expressions of crack extension forces for antiplane and plane cases were obtained in terms of the corresponding strain and electric field intensity factors at the tip of a finite rigid conducting line and the corresponding net Burgers vector of the dislocations inside the crack. As a result, the critical extension forces for a finite rigid conducting line for antiplane and plane cases were related to the corresponding critical extension forces for a same size crack in the same piezoelectric

solid. The connection constant for plane case is consistent with that obtained in chapter 3, when the crack is very small compared to the finite rigid conducting line. Two critical crack lengths were also obtained for both antiplane and plane cases. The stable length is under the Zener-Stroh mechanism, and the unstable one is under the Griffith mechanism.

CHAPTER 5

INTERACTION OF A SCREW DISLOCATION WITH COLLINEAR INTERFACIAL RIGID LINES IN PIEZOELECTRIC MEDIA

5.1 Introduction

In this chapter the physical problem of a piezoelectric screw dislocation interacting with collinear rigid lines at the interface of two dissimilar piezoelectric media is studied. Rigid lines are considered to be either conducting or dielectric. The screw dislocation core is subjected to a line force and a line charge. By using the complex variable method and the superposition technique, the general solutions for the field variables are presented first. Explicit expressions of the field variables are then derived for the special case of a single interfacial rigid conducting/dielectric line. Furthermore, energy release rates, mechanical strain energy release rates and the image force on the piezoelectric screw dislocation are calculated for this special case. Finally, numerical examples are performed to show the effects of some important parameters on rigid line extension forces and the image force in detail.

5.2 Statement of the Problem

In a Cartesian coordinate system x_i ($i=1,2,3$), let two different piezoelectric materials bonded together along $x_2 = 0$, loaded uniformly at infinity. The materials in $x_2 > 0$ and $x_2 < 0$ are designated as s_1 and s_2 , respectively, Fig. 5.1. Consider a piezoelectric screw dislocation located at a point $z_d(x_{1d}, x_{2d})$ in s_1 , which is assumed to be straight and infinitely long in the x_3 direction, suffering a finite discontinuity in the displacement b_3 and the electric potential b_ϕ across the slip plane. Its core is subjected to a line force p_3 in the x_3 direction and a line charge q . At the interface, assume that there are n collinear rigid lines l_j ($j=1,2,\dots,n$). Let L_l and L_b be the unions of the interfacial rigid lines and the remaining bonded parts, respectively.

Fig. 5.1 A piezoelectric screw dislocation near collinear rigid lines
at the interface of two dissimilar piezoelectric media

As shown in Fig. 5.1, it is assumed that there is no external load acting on the interfacial rigid lines. Therefore, the mechanical boundary condition and the equilibrium condition of the interfacial rigid lines are

$$u_{3(1)}^+(x_1) = u_{3(2)}^-(x_1) = u_{0j}, \text{ on } l_j \ (j = 1, 2, \dots, n), \quad (5.1)$$

$$\int_{l_j} [\sigma_{32(1)}^+(x_1) - \sigma_{32(2)}^-(x_1)] dx_1 = 0, \text{ on } l_j \ (j = 1, 2, \dots, n), \quad (5.2)$$

where the superscripts + and – refer to the boundary values from s_1 and s_2 , respectively, and the subscripts (1) and (2) stand for s_1 and s_2 , respectively. From (5.1), one obtains

$$u'_{3(1)}^+(x_1) = 0, \ u'_{3(2)}^-(x_1) = 0, \text{ on } L_l, \quad (5.3)$$

where the prime represents the derivative with respect to the argument x_1 .

We assume that the thickness of rigid lines is zero. Thus the electric boundary conditions can be described as (Shi, 1997)

$$\varphi_{(1)}^+(x_1) = \varphi_{(2)}^-(x_1) = \varphi_{0j}, \text{ on } l_j \ (j = 1, 2, \dots, n), \quad (5.4)$$

$$\int_{l_j} [D_{2(1)}^+(x_1) - D_{2(2)}^-(x_1)] dx_1 = 0, \text{ on } l_j \ (j = 1, 2, \dots, n) \quad (5.5)$$

for the case of rigid conducting lines, and

$$\varphi_{(1)}^+(x_1) = \varphi_{(2)}^-(x_1), \text{ on } l_j \ (j = 1, 2, \dots, n), \quad (5.6)$$

$$D_{2(1)}^+(x_1) = D_{2(2)}^-(x_1), \text{ on } l_j \ (j = 1, 2, \dots, n) \quad (5.7)$$

for the case of rigid dielectric lines. From (5.4), one obtains

$$\varphi_{(1)}'^+(x_1) = 0, \ \varphi_{(2)}'^-(x_1) = 0, \text{ on } L_l. \quad (5.8)$$

The boundary conditions on the perfectly bonded parts, in matrix notation, are

$$\mathbf{u}_{(1)}^+(x_1) = \mathbf{u}_{(2)}^-(x_1), \text{ on } L_b, \quad (5.9)$$

$$\mathbf{t}_{2(1)}^+(x_1) = \mathbf{t}_{2(2)}^-(x_1), \text{ on } L_b, \quad (5.10)$$

where $\mathbf{u} = [u_3, \varphi]^T$ are the generalized displacements and $\mathbf{t}_{2(i)} = [\sigma_{23(i)}, D_{2(i)}]^T$ ($i = 1, 2$) are the generalized stresses. In view of (2.51) and $\mathbf{F}(z) = d\mathbf{f}(z)/dz$, the problem has been reduced to finding the two analytical functions $\mathbf{F}_{(i)} = [F_{3(i)}(z), F_{4(i)}(z)]^T$ ($i = 1, 2$), which must satisfy the boundary conditions (5.2), (5.3), (5.9), (5.10) with either (5.5) and (5.8) or (5.6) and (5.7) for the conducting and dielectric cases, respectively. In the above $z = x_1 + ix_2$ is a complex variable, where $i = \sqrt{-1}$.

5.3 Formulation and Solution

Based on the superposition principle, the two complex vectors $\mathbf{F}_{(1)}(z)$ and $\mathbf{F}_{(2)}(z)$ can be expressed as

$$\mathbf{F}_{(i)}(z) = \mathbf{F}_{(i)}^\infty + \mathbf{F}_{(i)}^a(z) + \mathbf{F}_{(i)}^b(z), \quad i = 1, 2, \quad (5.11)$$

where $\mathbf{F}_{(i)}^\infty = [F_{3(i)}^\infty, F_{4(i)}^\infty]^T$ are constant vectors to be determined by the remote load in the absence of rigid lines, $\mathbf{F}_{(i)}^a = [F_{3(i)}^a, F_{4(i)}^a]^T$ are the complex functions due to the interaction between a screw dislocation and a perfectly bonded interface, and the complex potentials $\mathbf{F}_{(i)}^b = [F_{3(i)}^b, F_{4(i)}^b]^T$ correspond to the perturbed field due to the collinear interfacial rigid lines. It is required that $\mathbf{F}_{(i)}^a(\infty) = 0$ and $\mathbf{F}_{(i)}^b(\infty) = 0$.

We consider the perfectly bonded interface sub-problem in the absence of rigid lines first.

The remote generalized stresses $\mathbf{t}_2^\infty = [\sigma_{32}^\infty, D_2^\infty]^T$ and strains $\boldsymbol{\varepsilon}_1^\infty = [\varepsilon_{31}^\infty, -E_1^\infty]^T$ are applied

so that the continuous conditions on the generalized stresses and displacements along the entire x_1 axis are satisfied. Thus from (2.54) and (2.56), one has

$$\frac{1}{2}[\mathbf{F}_{(i)}^\infty - \overline{\mathbf{F}_{(i)}^\infty}] = i\boldsymbol{\varepsilon}_1^\infty, \quad i = 1, 2, \quad -\infty < x_1 < \infty, \quad (5.12)$$

$$\frac{1}{2}\mathbf{C}_{(i)}[\mathbf{F}_{(i)}^\infty + \overline{\mathbf{F}_{(i)}^\infty}] = \mathbf{t}_2^\infty, \quad i = 1, 2, \quad -\infty < x_1 < \infty. \quad (5.13)$$

From (5.12) and (5.13), we have

$$\mathbf{F}_{(i)}^\infty = \mathbf{C}_{(i)}^{-1}\mathbf{t}_2^\infty + i\boldsymbol{\varepsilon}_1^\infty, \quad i = 1, 2, \quad (5.14)$$

where $\mathbf{C}_{(1)}$ and $\mathbf{C}_{(2)}$ are the material property matrices of piezoelectric media occupying s_1 and s_2 , respectively. With constitutive equation (2.8), $\boldsymbol{\varepsilon}_1^\infty$ can be expressed as

$$\boldsymbol{\varepsilon}_1^\infty = \mathbf{C}_{(1)}^{-1}\mathbf{t}_{1(1)}^\infty = \mathbf{C}_{(2)}^{-1}\mathbf{t}_{1(2)}^\infty, \quad (5.15)$$

where $\mathbf{t}_{1(1)}^\infty = [\sigma_{31(1)}^\infty, D_{1(1)}^\infty]^\text{T}$ and $\mathbf{t}_{1(2)}^\infty = [\sigma_{31(2)}^\infty, D_{1(2)}^\infty]^\text{T}$ are the generalized stresses applied at infinity to s_1 and s_2 , respectively.

Now we consider the interaction between a piezoelectric screw dislocation $\mathbf{b} = [b_3, b_\varphi]^\text{T}$ with a line load $\mathbf{f} = [-p_3, q]^\text{T}$ and a perfectly bonded interface. For this sub-problem, the generalized stresses and the generalized displacements are continuous along the entire x_1 axis, so that

$$[\mathbf{F}_{(1)}^a(x_1) - \overline{\mathbf{F}_{(1)}^a(x_1)}]^\dagger = [\mathbf{F}_{(2)}^a(x_1) - \overline{\mathbf{F}_{(2)}^a(x_1)}], \quad -\infty < x_1 < \infty, \quad (5.16)$$

$$\mathbf{C}_{(1)}[\mathbf{F}_{(1)}^a(x_1) + \overline{\mathbf{F}_{(1)}^a(x_1)}]^\dagger = \mathbf{C}_{(2)}[\mathbf{F}_{(2)}^a(x_1) + \overline{\mathbf{F}_{(2)}^a(x_1)}], \quad -\infty < x_1 < \infty. \quad (5.17)$$

The complex potentials have the form:

$$\mathbf{F}_{(i)}^a(z) = \begin{cases} \mathbf{F}_{10}(z) + \mathbf{F}_{1d}(z), & z \in s_1, \\ \mathbf{F}_{20}(z), & z \in s_2, \end{cases} \quad (5.18)$$

where $\mathbf{F}_{10}(z)$ and $\mathbf{F}_{20}(z)$ are the holomorphic functions in s_1 and s_2 , respectively, and $\mathbf{F}_{1d}(z)$ is the complex potential of an infinite half plane subjected to a screw dislocation and is given by (Pak, 1990b)

$$\mathbf{F}_{1d}(z) = \frac{1}{2\pi(z - z_d)} [\mathbf{b} + i\mathbf{C}_{(1)}^{-1}\mathbf{f}]. \quad (5.19)$$

Rearranging (5.16) and using standard analytic continuation arguments, one reaches

$$\mathbf{F}_{10}(z) - \bar{\mathbf{F}}_{1d}(z) + \bar{\mathbf{F}}_{20}(z) = 0, \quad z \in s_1. \quad (5.20)$$

With the same arguments, rearranging (5.17) leads to

$$\mathbf{C}_{(1)}\mathbf{F}_{10}(z) + \mathbf{C}_{(1)}\bar{\mathbf{F}}_{1d}(z) - \mathbf{C}_{(2)}\bar{\mathbf{F}}_{20}(z) = 0, \quad z \in s_1. \quad (5.21)$$

Combining (5.20) with (5.21), one finds

$$\mathbf{F}_{(1)}^a(z) = \mathbf{\Theta}^{-1}[\mathbf{C}_{(2)} - \mathbf{C}_{(1)}]\bar{\mathbf{F}}_{(1)}^d(z) + \mathbf{F}_{1d}(z), \quad z \in s_1, \quad (5.22)$$

$$\mathbf{F}_{(2)}^a(z) = 2\mathbf{\Theta}^{-1}\mathbf{C}_{(1)}\mathbf{F}_{1d}(z), \quad z \in s_2, \quad (5.23)$$

where

$$\mathbf{\Theta} = \mathbf{C}_{(1)} + \mathbf{C}_{(2)}. \quad (5.24)$$

The following task is to consider the rigid line disturbed sub-problem. One has from (5.3), (5.9), (5.12) and (5.16) that

$$[\mathbf{F}_{(1)}^c(x_1) + \bar{\mathbf{F}}_{(2)}^c(x_1)]^+ = [\bar{\mathbf{F}}_{(1)}^c(x_1) + \mathbf{F}_{(2)}^c(x_1)]^-, \quad (5.25)$$

where

$$\mathbf{F}_{(i)}^c(z) = \mathbf{F}_{(i)}^\infty(z) + \mathbf{F}_{(i)}^b(z), \quad i = 1, 2. \quad (5.26)$$

Define an analytical function as

$$\mathbf{I}(z) = \begin{cases} \mathbf{F}_{(1)}^c(z) + \bar{\mathbf{F}}_{(2)}^c(z), & z \in s_1, \\ \mathbf{F}_{(2)}^c(z) + \bar{\mathbf{F}}_{(1)}^c(z), & z \in s_2. \end{cases} \quad (5.27)$$

According to the Liouville's theorem, considering (5.11) and (5.14), one obtains

$$\mathbf{I}(z) = \mathbf{\Theta}^{-1} \mathbf{t}_2^\infty. \quad (5.28)$$

Let us introduce a generalized stress jump function:

$$\Delta \mathbf{t}_2(x_1) = [\Delta \sigma_{23}(x_1), \Delta D_2(x_1)]^T = \mathbf{t}_{2(1)}^+(x_1) - \mathbf{t}_{2(2)}^-(x_1). \quad (5.29)$$

Inserting (5.11) into (5.29) and using (2.56), (5.17) and (5.27) result in

$$2\Delta \mathbf{t}_2(x_1) = \mathbf{\Theta}[\mathbf{J}^+(x_1) - \mathbf{J}^-(x_1)], \quad (5.30)$$

where

$$\mathbf{J}(z) = [J_3(z), J_\varphi(z)]^T = \begin{cases} \mathbf{F}_{(1)}^c(z), & z \in s_1, \\ \mathbf{F}_{(2)}^c(z) + [\mathbf{C}_{(1)}^{-1} - \mathbf{C}_{(2)}^{-1}] \mathbf{t}_2^\infty, & z \in s_2. \end{cases} \quad (5.31)$$

Since $\Delta \mathbf{t}_2(x_1) = 0$ on the perfectly bonded part, (5.30) shows that $\mathbf{J}(z)$ is analytic in the entire plane except along the rigid lines. One has from (2.51) (5.11) and (5.31) that

$$2i\mathbf{u}'_{(1)} = \mathbf{J}^+(x_1) + \mathbf{J}^-(x_1) + \mathbf{J}_0(x_1), \quad (5.32)$$

where

$$\mathbf{J}_0(x_1) = [J_{03}(x_1), J_{0\varphi}(x_1)]^T = -2\mathbf{C}_{(1)}^{-1} \mathbf{t}_2^\infty + \mathbf{\Theta}^{-1} \mathbf{C}_{(1)} \left[\frac{\mathbf{b} + i\mathbf{C}_{(1)}^{-1} \mathbf{f}}{\pi(x_1 - z_d)} - \frac{\mathbf{b} - i\mathbf{C}_{(1)}^{-1} \mathbf{f}}{\pi(x_1 - \bar{z}_d)} \right]. \quad (5.33)$$

5.3.1 Collinear Rigid Conducting Lines

Considering (5.3) and (5.8), one uses (5.32) to obtain

$$\mathbf{J}^+(x_1) + \mathbf{J}^-(x_1) = -\mathbf{J}_0(x_1), \text{ on } L_l. \quad (5.34)$$

Based on the theory of Muskhelishvili (1975), the solution of (5.34) can be obtained as

$$\mathbf{J}(z) = X_0(z) \left[-\frac{1}{2\pi i} \int_{L_l} \frac{1}{X_0^+(x_1)(x_1 - z)} \mathbf{J}_0(x_1) dx_1 + \mathbf{P}(z) \right], \quad (5.35)$$

where

$$X_0(z) = \prod_{j=1}^n (z - a_j)^{-1/2} (z - b_j)^{-1/2}, \quad (5.36)$$

$$\mathbf{P}(z) = [P_3(z), P_4(z)]^T = \mathbf{p}_n z^n + \mathbf{p}_{n-1} z^{n-1} + \cdots + \mathbf{p}_0, \quad \mathbf{p}_n = [c_n, d_n]^T. \quad (5.37)$$

The single value branch in (5.36) will be determined when the condition $\lim_{z \rightarrow \infty} [z^n X_0(z)] = 1$

is satisfied. Taking the limit $z \rightarrow \infty$ in (5.31) and using (5.35)-(5.37) lead to

$$\mathbf{p}_n = \mathbf{C}_{(1)}^{-1} \mathbf{t}_2^\infty + i \mathbf{e}_1^\infty. \quad (5.38)$$

From (5.2), (5.5) and (5.30), one can obtain the closed path integral

$$\oint_{\Lambda_j} \Theta \mathbf{J}(z) dz = \mathbf{0}, \text{ on } \Lambda_j \quad (j = 1, 2, \dots, n), \quad (5.39)$$

where Λ_j is the contour surrounding the interfacial rigid line l_j in clockwise direction.

Substituting (5.35) and (5.38) into (5.39), one finds the $2n$ constants $c_{n-1}, c_{n-2}, \dots, c_0$ and

$$d_{n-1}, d_{n-2}, \dots, d_0.$$

5.3.2 Collinear Rigid Dielectric Lines

It follows from (5.3), (5.31), (5.32), (5.34) and (5.35) that

$$J_3(z) = -X_0(z) \left[\frac{1}{2\pi i} \int_{L_l} \frac{1}{X_0^+(x_1)(x_1 - z)} J_{03}(x_1) dx_1 + P_3(z) \right], \quad (5.40)$$

$$c_n = \frac{1}{c_{44(1)} \kappa_{11(1)} + e_{15(1)}^2} [\kappa_{11(1)} \sigma_{32}^\infty + e_{15(1)} D_2^\infty] + i e_{31}^\infty. \quad (5.41)$$

From (5.2), (5.7) and (5.30), one has the closed path integral

$$\oint_{\Lambda_j} \Theta^{(1)} \mathbf{J}(z) dz = \mathbf{0}, \text{ on } \Lambda_j \quad (j = 1, 2, \dots, n), \quad (5.42)$$

where $\Theta^{(1)}$ is the first row of matrix Θ , given by

$$\Theta^{(1)} = [c_{44(1)} + c_{44(2)}, e_{15(1)} + e_{15(2)}]. \quad (5.43)$$

Considering (5.7), we can obtain from (5.30) that

$$\Theta^{(2)} \mathbf{J}(z) = \Theta^{(2)} \mathbf{J}(\infty), \quad (5.44)$$

where $\Theta^{(2)}$ is the second row of matrix Θ , given by

$$\Theta^{(2)} = [e_{15(1)} + e_{15(2)}, -\kappa_{11(1)} - \kappa_{11(2)}]. \quad (5.45)$$

Substituting (5.40) and (5.41) into (5.42) and (5.44) yields the n constants $c_{n-1}, c_{n-2}, \dots, c_0$.

Once $J_3(z)$ is obtained, $J_\varphi(z)$ can be found from (5.44).

After $\mathbf{J}(z)$ is obtained, the generalized strain and stress intensity factors can be evaluated.

Noting from (5.9) that the strain $\varepsilon_{31}(x_1, 0)$ and the electric field $E_1(x_1, 0)$ are continuous on the bonded interface, we evaluate the generalized strain intensity factors at the right tip of the j th rigid line as

$$\mathbf{k} = [k_{\varepsilon_1}, -k_{E_1}]^T = \lim_{x_1 \rightarrow b_j} \sqrt{2\pi(x_1 - b_j)} \operatorname{Im}[\mathbf{J}(x_1)]. \quad (5.46)$$

Noting from (5.10) that the generalized stresses $[\sigma_{23(i)}(x_1, 0), D_{2(i)}(x_1, 0)]^T$ are continuous on the bonded interface, we evaluate the generalized stress intensity factors at the right tip of the j th rigid line as

$$\mathbf{K} = [K_{\sigma_{32}}, K_{D_2}]^T = \lim_{x_1 \rightarrow b_j} \sqrt{2\pi(x_1 - b_j)} \mathbf{C}_{(1)} \operatorname{Re}[\mathbf{J}(x_1)]. \quad (5.47)$$

It should be pointed out that the solutions can be reduced to those obtained by Shi (1997) for the problem of collinear rigid lines in a homogeneous piezoelectric solid under a far-field load and those derived by Chen et al. (2006) for the electroelastic interaction of a screw dislocation with collinear rigid lines in a homogeneous piezoelectric medium.

5.4 Field Singularity at the Tip of a Single Rigid Line

In this section, the field variables will be particularly discussed for the interaction of a piezoelectric screw dislocation with a single interfacial rigid line in the interval $(-a, a)$ under a uniform remote load. In this case, we have

$$X_0(z) = (z^2 - a^2)^{-1/2}, \quad (5.48)$$

$$\mathbf{P}(z) = [\mathbf{C}_{(1)}^{-1} \mathbf{t}_2^\infty + i \mathbf{\varepsilon}_1^\infty] z + \mathbf{p}_0 \quad (5.49)$$

for the case of rigid conducting line, and

$$P_4(z) = \left\{ \frac{1}{c_{44(1)} \kappa_{11(1)} + e_{15(1)}^2} [\kappa_{11(1)} \sigma_{32}^\infty + e_{15(1)} D_2^\infty] + i \varepsilon_{31}^\infty \right\} z + c_0 \quad (5.50)$$

for the rigid dielectric line case.

5.4.1 A Single Rigid Conducting Line

Substituting (5.33), (5.48) and (5.49) into (5.35) and (5.39), applying the residue theorem and neglecting the constant of integration, one has

$$\begin{aligned} \mathbf{J}(z) = & \frac{a^2}{2\pi\sqrt{z^2 - a^2}} \mathbf{\Theta}^{-1} \left\{ \frac{1}{(\sqrt{z^2 - a^2} + z)(\sqrt{\bar{z}_d^2 - a^2} + z) - a^2} [\mathbf{C}_{(1)} \mathbf{b} - i \mathbf{f}] \right. \\ & + \frac{1}{(\sqrt{z^2 - a^2} + \bar{z}_d)(\sqrt{\bar{z}_d^2 - a^2} + \bar{z}_d) - a^2} [\mathbf{C}_{(1)} \mathbf{b} - i \mathbf{f}] \\ & - \frac{1}{(\sqrt{z^2 - a^2} + z)(\sqrt{z_d^2 - a^2} + z) - a^2} [\mathbf{C}_{(1)} \mathbf{b} + i \mathbf{f}] \\ & \left. - \frac{1}{(\sqrt{z^2 - a^2} + z_d)(\sqrt{z_d^2 - a^2} + z_d) - a^2} [\mathbf{C}_{(1)} \mathbf{b} + i \mathbf{f}] + \frac{z}{\sqrt{z^2 - a^2}} i \varepsilon_1^\infty \right\}. \end{aligned} \quad (5.51)$$

With reference to (5.22), (5.23), (5.26) and (5.31), one can obtain

$$\mathbf{F}_{(1)}(z) = \frac{1}{\sqrt{z^2 - a^2}} \left\{ \Theta^{-1} [\mathbf{b} - i\mathbf{C}_{(1)}^{-1}\mathbf{f}] \frac{\mathbf{C}_{(2)}\sqrt{z^2 - a^2} - \mathbf{C}_{(1)}\left(\sqrt{\bar{z}_d^2 - a^2} + z - \bar{z}_d\right)}{2\pi(z - \bar{z}_d)} + \Theta^{-1} [\mathbf{b} + i\mathbf{C}_{(1)}^{-1}\mathbf{f}] \frac{\mathbf{C}_{(2)}\sqrt{z^2 - a^2} + \mathbf{C}_{(1)}\left[\sqrt{z_d^2 - a^2} + z - z_d\right]}{2\pi(z - z_d)} + i\boldsymbol{\varepsilon}_1^\infty z \right\}, \quad (5.52)$$

$$\mathbf{F}_{(2)}(z) = \frac{1}{\sqrt{z^2 - a^2}} \left\{ \Theta^{-1} [\mathbf{C}_{(1)}\mathbf{b} - i\mathbf{f}] \frac{\sqrt{z^2 - a^2} - \sqrt{\bar{z}_d^2 - a^2}}{2\pi(z - \bar{z}_d)} + \Theta^{-1} [\mathbf{C}_{(1)}\mathbf{b} + i\mathbf{f}] \frac{\sqrt{z^2 - a^2} + \sqrt{z_d^2 - a^2}}{2\pi(z - z_d)} + i2\Theta^{-1}\mathbf{f} + i\boldsymbol{\varepsilon}_1^\infty z \right\}. \quad (5.53)$$

Substituting (5.52) and (5.53) into (2.56) and (2.57), one finds the generalized stresses in the upper and lower media, respectively. Since we are particularly interested in the field variables near the tip, it is convenient to write the coordinate in polar form as $z_d = r_d e^{i\theta_d}$ (see Fig. 5.1). By substituting (5.51) into (5.46) and (5.47), the generalized strain and stress intensity factors ahead of the rigid conducting line tip are obtained as

$$\mathbf{k} = \sqrt{\pi a} \boldsymbol{\varepsilon}_1^\infty - \frac{1}{\sqrt{\pi a}} \Theta^{-1} \left\{ \sqrt{\frac{r_1}{r_2}} \sin\left(\frac{\theta_1 - \theta_2}{2}\right) \mathbf{C}_{(1)}\mathbf{b} + \left[\sqrt{\frac{r_1}{r_2}} \cos\left(\frac{\theta_1 - \theta_2}{2}\right) - 1 \right] \mathbf{f} \right\}, \quad (5.54)$$

$$\mathbf{K} = \mathbf{0}, \quad (5.55)$$

respectively. At a distance r ahead of the tip, one obtains

$$\mathbf{t}_{1(i)} = [\sigma_{13(i)}, D_{1(i)}]^T = (2\pi r)^{-1/2} \mathbf{C}_{(i)} \mathbf{k}, \quad x_2 = 0, \quad i = 1, 2. \quad (5.56)$$

5.4.2 A Single Rigid Dielectric Line

Substituting (5.48) and (5.50) into (5.42) and (5.44) with (5.31) and (5.51), applying the residue theorem and neglecting the constants of integration, one obtains

$$J_3(z) = h(z), \quad J_\varphi(z) = -\frac{\Theta_{21}}{\Theta_{22}} h(z), \quad (5.57)$$

where

$$h(z) = \frac{1}{2\pi\sqrt{z^2 - a^2}} \left[(A_1 - iA_2) \left(\frac{\sqrt{z^2 - a^2} - \sqrt{\bar{z}_d^2 - a^2}}{z - \bar{z}_d} - 1 \right) - (A_1 + iA_2) \left(\frac{\sqrt{z^2 - a^2} - \sqrt{z_d^2 - a^2}}{z - z_d} - 1 \right) \right] + \frac{1}{\sqrt{z^2 - a^2}} i\epsilon_{31}^\infty z, \quad (5.58)$$

$$A_1 = [\Theta^{-1} \mathbf{C}_{(1)} \mathbf{b}]_1, \quad A_2 = [\Theta^{-1} \mathbf{f}]_1. \quad (5.59)$$

In the above equations, the subscript 1 outside the bracket stands for the first element of the vector and Θ_{ij} denotes the component of matrix Θ .

With reference to (5.22), (5.23), (5.26) and (5.31), one can express the complex functions $\mathbf{F}_{(i)}(z)$ ($i = 1, 2$) as

$$\mathbf{F}_{(1)}(z) = \frac{1}{2\pi(z - \bar{z}_d)} \Theta^{-1} [\mathbf{C}_{(2)} - \mathbf{C}_{(1)}] [\mathbf{b} - i\mathbf{C}_{(1)}^{-1} \mathbf{f}] + \frac{1}{2\pi(z - z_d)} [\mathbf{b} + i\mathbf{C}_{(1)}^{-1} \mathbf{f}] + h(z) \mathbf{a}, \quad (5.60)$$

$$\mathbf{F}_{(2)}(z) = \frac{1}{\pi(z - z_d)} \Theta^{-1} \mathbf{C}_{(1)} [\mathbf{b} + i\mathbf{C}_{(1)}^{-1} \mathbf{f}] + h(z) \mathbf{a}, \quad (5.61)$$

where

$$\mathbf{a} = \left[1, -\frac{\Theta_{21}}{\Theta_{22}} \right]^T. \quad (5.62)$$

Substituting (5.60) and (5.61) into (2.56) and (2.57), one obtains the generalized stresses in the upper and lower piezoelectric media, respectively. By substituting (5.57) into (5.46) and (5.47) in the polar coordinate system as shown in Fig. 5.1, the generalized strain and stress intensity factors at the rigid dielectric line tip are obtained as

$$\mathbf{k} = \boldsymbol{\alpha} \left\{ \sqrt{\pi a} \varepsilon_{31}^{\infty} - \frac{1}{\sqrt{\pi a}} \left[A_1 \sqrt{\frac{r_1}{r_2}} \sin\left(\frac{\theta_1 - \theta_2}{2}\right) + A_2 \sqrt{\frac{r_1}{r_2}} \cos\left(\frac{\theta_1 - \theta_2}{2}\right) - A_2 \right] \right\}, \quad (5.63)$$

$$\mathbf{K} = \mathbf{0}, \quad (5.64)$$

respectively. At a distance r ahead of the tip, one obtains

$$\mathbf{t}_{1(i)} = [\sigma_{13(i)}, D_{1(i)}]^T = (2\pi r)^{-1/2} \mathbf{C}_{(i)} \mathbf{k}, \quad x_2 = 0, \quad i = 1, 2. \quad (5.65)$$

It is seen from (5.56) and (5.65) that the generalized stresses $\mathbf{t}_{1(i)} = [\sigma_{13(i)}, D_{1(i)}]^T$ ($i = 1, 2$) exhibit a square root singularity at the tips of an interfacial rigid line, either conducting or dielectric. It is also found that the singularity of the electric displacement $D_{1(i)}$ vanishes at the tips of a rigid dielectric line in a homogeneous medium, which is a special case where $\mathbf{C}_{(1)} = \mathbf{C}_{(2)}$. It is worth to point out that the field singularity at the tips of the two types of rigid line is independent of far-field load $\mathbf{t}_2^{\infty} = [\sigma_{32}^{\infty}, D_2^{\infty}]^T$, which agrees with the result obtained by Shi (1997) for a single rigid conducting/dielectric line in a homogeneous piezoelectric medium loaded at infinity. Hence, the far-field load $\mathbf{t}_2^{\infty}$ is not considered in the following analyses.

5.5 Rigid Line Extension Forces

According to the field variables obtained for a piezoelectric screw dislocation interacting with a single interfacial rigid conducting/dielectric line in Section 5.4, we will analyze the rigid conducting/dielectric extension forces by evaluating the energy release rate as well as the mechanical strain energy release rate in this section.

5.5.1 A Single Rigid Conducting Line

The generalized stress jumps across the rigid conducting line can be obtained from (5.30) and (5.51) as

$$\Delta \mathbf{t}_2(x_1) = \frac{1}{\pi \sqrt{a^2 - x_1^2}} \left\{ \pi \mathbf{\Theta} \mathbf{\epsilon}_1^\infty x_1 + \mathbf{f} - \text{Im} \left[(\mathbf{C}_{(1)} \mathbf{b} - i \mathbf{f}) \frac{\sqrt{\bar{z}_d^2 - a^2}}{x_1 - \bar{z}_d} \right] \right\}, \quad |x_1| < a. \quad (5.66)$$

With the expression of $\mathbf{k}$ in (5.54), at a distance r behind the tip of the interfacial rigid conducting line, equation (5.66) arrives at

$$\Delta \mathbf{t}_2(r) = \frac{1}{\sqrt{2\pi r}} \mathbf{\Theta} \mathbf{k}. \quad (5.67)$$

From (5.32) and (5.51), the generalized displacement gradients ahead of the right tip of the rigid conducting line along $x_2 = 0$ are obtained as

$$\mathbf{u}' = \frac{1}{\pi \sqrt{x_1^2 - a^2}} \left\{ \pi \mathbf{\epsilon}_x^\infty x_1 + \mathbf{\Theta}^{-1} \mathbf{f} - \text{Im} \left[\mathbf{\Theta}^{-1} (\mathbf{C}_{(1)} \mathbf{b} - i \mathbf{f}) \frac{\sqrt{\bar{z}_d^2 - a^2}}{x_1 - \bar{z}_d} \right] \right\}, \quad x_1 > a. \quad (5.68)$$

At a distance r ahead of the rigid conducting line tip along $x_2 = 0$, equation (5.68) yields

$$\mathbf{u} = \mathbf{k} \sqrt{2r/\pi}. \quad (5.69)$$

Thus, by analogy with the calculation of the energy released in propagating a crack an infinitesimal distance in piezoelectric materials (Pak, 1990a), the energy release rate for a rigid line can be evaluated by

$$G = \lim_{\chi \rightarrow 0} -\frac{1}{2\chi} \int_0^\chi \Delta \mathbf{t}_2^\top(r) \mathbf{u}(\chi - r, 0) dr, \quad (5.70)$$

where $\Delta \mathbf{t}_2(r)$ stand for the generalized stress jumps behind the tip and $\mathbf{u}$ represent the generalized displacements ahead of the tip. Substituting (5.67) and (5.69) into (5.70) finds

$$G = -\frac{1}{4} \mathbf{k}^\top \mathbf{\Theta} \mathbf{k}. \quad (5.71)$$

According to the work of Park and Sun (1995a), the mechanical strain energy release rate for a rigid line can be calculated by

$$G_M = \lim_{\chi \rightarrow 0} -\frac{1}{2\chi} \int_0^\chi \Delta \sigma_{23}(r) u_3(\chi - r, 0) dr. \quad (5.72)$$

In view of (5.46), (5.67), (5.69) and (5.72), one has

$$G_M = -\frac{1}{4} (\Theta_{11} k_{\varepsilon_1}^2 - \Theta_{12} k_{\varepsilon_1} k_{E_1}). \quad (5.73)$$

5.5.2 A Single Rigid Dielectric Line

Similarly, from (5.30) and (5.57), one obtains

$$\Delta \sigma_{23}(x_1) = \frac{|\Theta|}{\pi \Theta_{22} \sqrt{a^2 - x_1^2}} \left\{ \pi \varepsilon_{31}^\infty x_1 + A_2 - \operatorname{Im} \left[(A_1 - iA_2) \frac{\sqrt{\bar{z}_d^2 - a^2}}{x_1 - \bar{z}_d} \right] \right\}, \quad |x_1| < a, \quad (5.74)$$

where $|\cdot|$ represents the determinant of a matrix. With the expression of $\mathbf{k}$ in (5.63), at a distance r behind the rigid dielectric line tip, equation (5.74) yields

$$\Delta \sigma_{23}(r) = \frac{1}{\sqrt{2\pi r}} \frac{|\Theta|}{\Theta_{22}} k_{\varepsilon_1}. \quad (5.75)$$

From (5.32) and (5.57), at a distance r ahead of the tip along the interface

$$\begin{aligned} \mathbf{u}' = & \frac{1}{\pi \sqrt{a^2 - x_1^2}} \left\{ \pi \varepsilon_{31}^\infty x_1 + A_2 - \operatorname{Im} \left[(A_1 - iA_2) \frac{\sqrt{\bar{z}_d^2 - a^2}}{x_1 - \bar{z}_d} \right] \right\} \mathbf{a} \\ & + \frac{1}{\pi} \operatorname{Im} \left[\frac{A_1 \mathbf{a} - \Theta^{-1} \mathbf{C}_{(1)} \mathbf{b} - i(A_2 \mathbf{a} - \Theta^{-1} \mathbf{f})}{x_1 - \bar{z}_d} \right] - \varepsilon_{31}^\infty \mathbf{a} + \varepsilon_1^\infty, \quad x_1 > a. \end{aligned} \quad (5.76)$$

With the expression of $\mathbf{k}$ in (5.63), at a distance r ahead of the tip along the interface, equation (5.76) yields

$$\mathbf{u} = \mathbf{k} \sqrt{2r/\pi}. \quad (5.77)$$

Considering the condition $\Delta D_2(x_1) = 0$ along the rigid dielectric line, we can find that the energy release rate calculated in (5.70) is the same as the mechanical strain energy release rate in (5.72) for an interfacial rigid dielectric line. Thus, substituting (5.75) and (5.77) with (5.62) into (5.70) or (5.72) gives

$$G = G_M = -\frac{1}{4} \frac{|\Theta|}{\Theta_{22}} k_{\varepsilon_1}^2. \quad (5.78)$$

It is worth to point out that (5.71), (5.73) and (5.78) can be easily reduced to that for the simpler case, a rigid line in a homogeneous piezoelectric solid.

5.6 Force on the Piezoelectric Screw Dislocation

In this section, we evaluate the forces acting on the piezoelectric screw dislocation with a line load due to the fields generated by a single interfacial rigid conducting/dielectric line. The forces can be obtained from the generalized Peach-Koehler formula (Pak, 1990b):

$$F_1 - iF_2 = [\mathbf{b}^T \mathbf{C}_{(1)} + i\mathbf{f}^T] \mathbf{F}_{(1)}^L, \quad (5.79)$$

where $\mathbf{F}_{(1)}^L$ can be obtained by subtracting the fields generated by the screw dislocation from (5.52) for the conducting case or (5.60) for the dielectric case with $z \rightarrow z_d$.

5.6.1 A Single Rigid Conducting Line

From (5.52) in a polar coordinate system as shown in Fig. 5.1, one obtains

$$\begin{aligned} \mathbf{F}_{(1)}^L = & \frac{1}{2\pi a} \boldsymbol{\Theta}^{-1} \left\{ \left[\mathbf{I} - \mathbf{C}_{(2)} \mathbf{C}_{(1)}^{-1} \right] \mathbf{f} d_1 + \mathbf{f} \sum_{j=2}^3 (-1)^j d_j - \mathbf{C}^{(1)} \mathbf{b} \sum_{j=4}^5 d_j \right. \\ & \left. + i \mathbf{f} \sum_{j=4}^5 (-1)^j d_j + i \mathbf{C}_{(1)} \mathbf{b} \sum_{j=1}^3 d_j - i \mathbf{C}_{(2)} \mathbf{b} d_1 \right\} + \boldsymbol{\varepsilon}_1^\infty [d_6 + i d_7], \end{aligned} \quad (5.80)$$

where $\mathbf{I}$ is a 4×4 unit matrix, and

$$d_1 = \frac{1}{\hat{r}_1 \sin \theta_1 + \hat{r}_2 \sin \theta_2}, \quad (5.81a)$$

$$d_2 = \frac{1}{\sqrt{\hat{r}_1 \hat{r}_2}} \sin \left(\frac{\theta_1 + \theta_2}{2} \right) - \frac{\hat{r}_1 \cos \theta_1 + \hat{r}_2 \cos \theta_2}{2 \hat{r}_1 \hat{r}_2} \tan \left(\frac{\theta_1 + \theta_2}{2} \right), \quad (5.81b)$$

$$d_3 = -\frac{1}{\sqrt{\hat{r}_1 \hat{r}_2}} \sin \left(\frac{\theta_1 + \theta_2}{2} \right) + \frac{\hat{r}_1 \sin \theta_2 + \hat{r}_2 \sin \theta_1}{2 \hat{r}_1 \hat{r}_2}, \quad (5.81c)$$

$$d_4 = \frac{1}{\sqrt{\hat{r}_1 \hat{r}_2}} \cos \left(\frac{\theta_1 + \theta_2}{2} \right) - \frac{\hat{r}_1 \cos \theta_1 + \hat{r}_2 \cos \theta_2}{2 \hat{r}_1 \hat{r}_2}, \quad (5.81d)$$

$$d_5 = -\frac{1}{\sqrt{\hat{r}_1 \hat{r}_2}} \cos \left(\frac{\theta_1 + \theta_2}{2} \right) + \frac{\hat{r}_1 \cos \theta_2 + \hat{r}_2 \cos \theta_1}{2 \hat{r}_1 \hat{r}_2}, \quad (5.81e)$$

$$d_6 = \frac{\hat{r}_2 - \hat{r}_1}{2 \sqrt{\hat{r}_1 \hat{r}_2}} \sin \left(\frac{\theta_1 - \theta_2}{2} \right), \quad (5.81f)$$

$$d_7 = \frac{\hat{r}_1 + \hat{r}_2}{2 \sqrt{\hat{r}_1 \hat{r}_2}} \cos \left(\frac{\theta_1 - \theta_2}{2} \right), \quad (5.81g)$$

with

$$\hat{r}_1 = r_1 / a, \quad \hat{r}_2 = r_2 / a. \quad (5.82)$$

Substituting (5.80) into (5.79), one has

$$\begin{aligned} F_1 = & -\frac{1}{2\pi a} \left[\mathbf{b}^T \mathbf{C}_{(1)} \boldsymbol{\Theta}^{-1} \mathbf{C}_{(1)} \mathbf{b} \sum_{j=4}^5 d_j + \mathbf{b}^T \mathbf{C}_{(1)} \boldsymbol{\Theta}^{-1} \mathbf{f} (2d_3) \right. \\ & \left. + \mathbf{f}^T \boldsymbol{\Theta}^{-1} \mathbf{f} \sum_{j=4}^5 (-1)^j d_j \right] + \mathbf{b}^T \mathbf{C}_{(1)} \boldsymbol{\varepsilon}_1^\infty d_6 - \mathbf{f}^T \boldsymbol{\varepsilon}_1^\infty d_7, \end{aligned} \quad (5.83)$$

$$F_2 = \frac{1}{2\pi a} \left\{ \mathbf{f}^T \boldsymbol{\Theta}^{-1} [\mathbf{C}_{(2)} \mathbf{C}_{(1)}^{-1} - \mathbf{I}] \mathbf{f} d_1 - \mathbf{f}^T \boldsymbol{\Theta}^{-1} \mathbf{f} \sum_{j=2}^3 (-1)^j d_j + \mathbf{b}^T \boldsymbol{\Omega}^{-1} \mathbf{b} d_1 \right. \\ \left. - \mathbf{b}^T \mathbf{C}_{(1)} \boldsymbol{\Theta}^{-1} \mathbf{C}_{(1)} \mathbf{b} \sum_{j=1}^3 d_j + \mathbf{b}^T \mathbf{C}_{(1)} \boldsymbol{\Theta}^{-1} \mathbf{f} (2d_5) \right\} - \mathbf{b}^T \mathbf{C}_{(1)} \boldsymbol{\varepsilon}_1^\infty d_7 - \mathbf{f}^T \boldsymbol{\varepsilon}_1^\infty d_6, \quad (5.84)$$

where

$$\boldsymbol{\Omega} = \mathbf{C}_{(1)}^{-1} + \mathbf{C}_{(2)}^{-1}. \quad (5.85)$$

5.6.2 A Single Rigid Dielectric Line

In a polar coordinate system as shown in Fig. 5.1, it is found from (5.60) that

$$\mathbf{F}_{(1)}^L = \frac{1}{2\pi a} \left\{ \boldsymbol{\Theta}^{-1} [\mathbf{C}_{(1)} - \mathbf{C}_{(2)}] [\mathbf{C}_{(1)}^{-1} \mathbf{f} + i\mathbf{b}] d_1 + \boldsymbol{\alpha} \left[A_2 \sum_{j=2}^3 (-1)^j d_j - A_1 \sum_{j=4}^5 d_j \right. \right. \\ \left. \left. + iA_2 \sum_{j=4}^5 (-1)^j d_j + iA_1 \sum_{j=2}^3 d_j \right] \right\} + \boldsymbol{\alpha} \boldsymbol{\varepsilon}_{31}^\infty (d_6 + id_7). \quad (5.86)$$

Substituting (5.86) into (5.79), one obtains

$$F_1 = \frac{1}{2\pi a} \left\{ \mathbf{b}^T [\mathbf{C}_{(1)} - \mathbf{C}_{(2)}] \boldsymbol{\Theta}^{-1} \mathbf{f} d_1 - \mathbf{f}^T \boldsymbol{\Theta}^{-1} (\mathbf{C}_{(1)} - \mathbf{C}_{(2)}) \mathbf{b} d_1 - \mathbf{f}^T \boldsymbol{\alpha} \left[A_2 \sum_{j=4}^5 (-1)^j d_j \right. \right. \\ \left. \left. + A_1 \sum_{j=2}^3 d_j \right] - \mathbf{b}^T \mathbf{C}_{(1)} \boldsymbol{\alpha} \left[A_1 \sum_{j=4}^5 d_j - A_2 \sum_{j=2}^3 (-1)^j d_j \right] \right\} + \mathbf{b}^T \mathbf{C}_{(1)} \boldsymbol{\alpha} \boldsymbol{\varepsilon}_{31}^\infty d_6 - \mathbf{f}^T \boldsymbol{\alpha} \boldsymbol{\varepsilon}_{31}^\infty d_7, \quad (5.87)$$

$$F_2 = \frac{1}{2\pi a} \left\{ \mathbf{b}^T [\boldsymbol{\Omega}^{-1} - \mathbf{C}_{(1)} \boldsymbol{\Theta}^{-1} \mathbf{C}_{(1)}] \mathbf{b} d_1 - \mathbf{b}^T \mathbf{C}_{(1)} \boldsymbol{\alpha} \left[A_1 \sum_{j=2}^3 d_j + A_2 \sum_{j=4}^5 (-1)^j d_j \right] \right. \\ \left. - \mathbf{f}^T \boldsymbol{\Theta}^{-1} [\mathbf{I} - \mathbf{C}_{(2)} \mathbf{C}_{(1)}^{-1}] \mathbf{f} d_1 - \mathbf{f}^T \boldsymbol{\alpha} \left[A_2 \sum_{j=2}^3 (-1)^j d_j - A_1 \sum_{j=4}^5 d_j \right] \right\} \\ - \mathbf{b}^T \mathbf{C}_{(1)} \boldsymbol{\alpha} \boldsymbol{\varepsilon}_{31}^\infty d_7 - \mathbf{f}^T \boldsymbol{\alpha} \boldsymbol{\varepsilon}_{31}^\infty d_6. \quad (5.88)$$

5.7 Numerical Examples and Discussions

In this section, numerical examples are performed to demonstrate the effects of various parameters on energy release rates and mechanical strain energy release rates derived in Section 5.5 and the image forces on the screw dislocation obtained in Section 5.6, in the absence of the far-field load. In a polar coordinate system (Fig. 5.1), we assume $r_d = 1.2a$, and the Burgers vector of the screw dislocation and the line load along its core have the values (Lee et al., 2000)

$$b_3 = 10^{-9} m, b_\varphi = 1.0 V, p_3 = 10 N/m, q = 10^{-8} C/m. \quad (5.89)$$

Let us introduce parameters

$$\alpha = c_{44(2)}/c_{44(1)}, \beta = e_{15(2)}/e_{15(1)}, \gamma = \kappa_{11(2)}/\kappa_{11(1)}, \quad (5.90)$$

and the electromechanical coupling factor

$$k_e = e_{15}/\sqrt{c_{44}\kappa_{11}}. \quad (5.91)$$

5.7.1 Rigid Line Extension Forces

As an example, PZT-6B is considered for s_1 , and a kind of piezoelectric ceramic for s_2 , whose material coefficients c_{44} and κ_{11} are the same as those for PZT-6B, and e_{15} is four times as much as that for PZT-6B, i.e., $\alpha = 1$, $\gamma = 1$ and $\beta = 4$ are assumed. The material coefficients for PZT-6B are listed in Table 5.1 (Lee et al., 2000; Soh et al., 2002). Rigid line extension forces are normalized by the factors

$$G_0(b_3) = \frac{c_{44(1)}b_3^2}{4\pi a}, G_0(b_\varphi) = \frac{\kappa_{11(1)}b_\varphi^2}{4\pi a}, G_0(p_3) = \frac{p_3^2}{4\pi a c_{44(1)}}, G_0(q) = \frac{q^2}{4\pi a \kappa_{11(1)}} \quad (5.92)$$

Table 5.1
Material coefficients for piezoelectric ceramics

	c_{44} 10^{10} Nm^{-2}	e_{15} Cm^{-2}	κ_{11} $10^{-10} \text{ C(Vm)}^{-1}$	k_e
PZT-6B	27.1	4.6	36	0.4657
PZT-5H	35.3	17	151	0.7363
PZT-4	25.6	12.7	64.6	0.9876

due to b_3 , b_φ , p_3 and q , respectively. For example, normalized extension forces $G^*(b_3)$ and $G_M^*(p_3)$ are calculated as $G(b_3)/G_0(b_3)$ and $G_M(p_3)/G_0(p_3)$, respectively. $G(b_3)$ is the energy release rate due to b_3 , and $G_M(p_3)$ is the mechanical strain energy release rate due to p_3 . Normalized extension forces G^* versus the angular position θ_d of a screw dislocation are plotted in Fig. 5.2 and Fig. 5.3. Figures 5.4 and 5.5 depict the variations of the normalized extension forces G_M^* with θ_d . In all the figures, the symbols “C” and “D” denote the conducting and dielectric cases, respectively.

Fig.5.2 The effect of the Burgers vector on the energy release rate

Fig. 5.3 The effect of the line load on the energy release rate

It is observed from Figs. 5.2 and 5.3 that normalized extension forces $G_D^*(b_3)$, $G_D^*(b_\varphi)$, $G_D^*(p_3)$ and $G_D^*(q)$ for a given θ_d are nonpositive. The negative driving force on the rigid dielectric line favors a contraction in the length of the segment. It is also seen from Figs. 5.2 and 5.3 that the normalized forces $G_C^*(b_3)$ and $G_C^*(p_3)$ are always nonpositive; however b_φ and q result in positive driving forces on the rigid conducting line. It is also possible that the combination of b_3 , b_φ , p_3 and q induces a negative driving force on the rigid conducting line for a given θ_d . It is worth noting that for a given θ_d , the value of the driving force on the rigid dielectric line is less than that on the rigid conducting line due to b_3 , b_φ , and q , respectively, and the line force p_3 leads to the same driving force. The same phenomenon can be observed from Figs. 5.4 and 5.5, based on the evaluation of the mechanical energy release rate, except that q does not cause driving force on the rigid conducting line.

Fig. 5.4 The effect of the Burgers vector on the mechanical strain energy release rate

Fig. 5.5 The effect of the line load on the mechanical strain energy release rate

From Figs. 5.2-5.5, the difference can be observed between the energy release rate and the mechanical strain energy release rate for an interfacial rigid conducting line. However, the difference is very small between the energy release rate and the mechanical strain energy release rate for an interfacial rigid dielectric line. By differentiating G^* and G_M^* with respect to θ_d , we find the angular position of the screw dislocation at which the magnitude of rigid line extension forces becomes maximum: $\theta_d \approx 18^\circ$ with the Burgers vector and $\theta_d = 0^\circ$ with the line load, while the extension forces are zero when $\theta_d = 0^\circ$ with the Burgers vector and $\theta_d = 61.6^\circ$ with the line load.

5.7.2 Image Force on the Piezoelectric Screw Dislocation

Force F_2 on the screw dislocation is normalized by the parameter

$$F_0 = \frac{c_{44(1)} b_3^2}{4\pi a}. \quad (5.93)$$

Normalized force F_{20} is calculated as F_2/F_0 . Here we discuss the effects of the material coefficients of piezoelectric ceramic for s_2 on F_{20} , by analyzing the effects of parameters α , β and γ on F_{20} in view of (5.90). As an example, PZT-6B is considered for s_1 .

Normalized force F_{20} as a function of the angular position θ_d of a screw dislocation for different α and $\beta, \gamma = 1$ is plotted in Fig. 5.6. It is observed that the “harder” the medium s_2 relative to s_1 , (i.e., the higher the modulus ratio α), the larger the value of F_{20} for a given θ_d . When $\alpha = 5$, the value of F_{20} becomes always positive, which indicates that

the screw dislocation is always repelled by the interfacial rigid conducting/dielectric lines in the x_2 direction, and therefore a screw dislocation has no equilibrium position in the x_2 direction. It is also observed that there are two equilibrium positions for a screw dislocation in the x_2 direction when $\alpha = 0.5$.

Normalized force F_{20} as a function of the angular position θ_d of a screw dislocation for different β and $\alpha, \gamma = 1$ are shown in Fig. 5.7. At a given θ_d , it is observed that the value of F_{20} becomes positive when $\beta \geq 2$ for the case of an interfacial rigid dielectric line and when $\beta = 6$ for an interfacial rigid conducting line case, respectively. As a result, a screw dislocation has no equilibrium position in the x_2 direction for such cases. When $\beta = 0.1$, two equilibrium positions for a screw dislocation are found in the x_2 direction.

Fig. 5.6 Normalized force F_{20} versus θ_d for different α with $\beta = 1$ and $\gamma = 1$

Fig. 5.7 Normalized force F_{20} versus θ_d for different β with $\alpha = 1$ and $\gamma = 1$

Fig. 5.8 Normalized force F_{20} versus θ_d for different γ with $\alpha = 1$ and $\beta = 1$

Normalized force F_{20} as a function of the angular position θ_d of a screw dislocation for different γ and $\alpha, \beta = 1$ are plotted in Fig. 5.8. It can be found that the smaller the value of γ , the larger the value of F_{20} for a given θ_d . When $\gamma = 0.1$, the value of F_{20} becomes always positive. With an increase in γ , the value of F_{20} decreases and becomes always negative when $\gamma = 5$. Therefore, a screw dislocation has no equilibrium position in the x_2 direction when $\gamma = 0.1$ and $\gamma = 5$.

It is worth noting from Figs. 5.6-5.8 that the repulsive force for a given θ_d is stronger for the dielectric case than that for the conducting case with the same α , β and γ ; however, the opposite phenomenon is observed for the attractive force for a given θ_d . It is also seen that the absolute value of F_{20} increases dramatically as the screw dislocation approaches the interface except for the special case in which $\mathbf{C}_{(1)} = \mathbf{C}_{(2)}$.

Now we discuss the effect of the electromechanical coupling factor k_e of piezoelectric ceramic for s_1 on the normalized image force. As examples, three kinds of piezoelectric ceramics PZT-6B, PZT-5H and PZT-4 are considered for s_1 , and s_2 is assumed to be a pure-elastic material whose elastic stiffness constant c_{44} is correspondingly the same as that for PZT-6B, PZT-5H or PZT-4, i.e. $\alpha = 1$ is assumed. The material coefficients and the electromechanical coupling factor k_e for PZT-6B, PZT-5H and PZT-4 are listed in Table 5.1. Here we only consider b_3 , taking $b_3 = p_3 = q = 0$. Normalized force $F_2^*(b_3)$ is calculated as $F_2(b_3)/F_0$, where $F_2(b_3)$ is the image force acting on a screw dislocation only due to b_3 .

Normalized image force $F_2^*(b_3)$ versus the angular position θ_d of a screw dislocation for different k_e is plotted in Fig. 5.9. It is observed that the stronger the electromechanical coupling effect k_e , the larger is the value of $F_2^*(b_3)$ for a given θ_d for both conducting and dielectric cases. It is also found that for a given k_e , there exist two angular positions θ_{d1}^* and θ_{d2}^* ($\theta_{d2}^* = 180^\circ - \theta_{d1}^*$) at which the values of $F_2^*(b_3)$ for the conducting case are the same as those for the dielectric case. It is also noted that for a given k_e , the value of $F_2^*(b_3)$ is larger for the conducting case than that for the dielectric case, when the angular position is in the ranges $0^\circ < \theta_d < \theta_{d1}^*$ and $\theta_{d2}^* < \theta_d < 180^\circ$. The opposite phenomenon can be observed for the angular position in the range $\theta_{d1}^* < \theta_d < \theta_{d2}^*$. θ_{d1}^* and θ_{d2}^* can be obtained from equations (5.84) and (5.88).

Fig. 5.9 Normalized force $F_2^*(b_3)$ versus θ_d for different k_e with $\alpha = 1$

5.8 Conclusion

A theoretical analysis was performed for the interaction between a piezoelectric screw dislocation and collinear rigid conducting/dielectric lines at the interface of two dissimilar piezoelectric media subjected to far-field antiplane mechanical and in-plane electric loads. The piezoelectric screw dislocation is applied with line force and line charges at its core. General solutions were obtained for n finite rigid conducting/dielectric lines. Explicit expressions of the field variables were derived for the special case of a single interfacial rigid conducting/dielectric line and the classical square root singularity was identified for the near tip stress and electric displacement fields. For this special case, energy release rates, mechanical strain energy release rates and the force acting on the screw dislocation were also calculated. It was found for a rigid dielectric line that the energy release rate coincides with the mechanical strain energy releaser rate. Numerical examples were given to show the effects of Burgers vector and line load on energy release rate and mechanical strain energy release rate, and the effects of material coefficients and electromechanical coupling factor on the image force in the x_2 direction.

CHAPTER 6

INTERACTION OF A MIXED DISLOCATION WITH COLLINEAR INTERFACIAL RIGID LINES IN PIEZOELECTRIC MEDIA

6.1 Introduction

This chapter investigates the generalized two-dimensional problem of a mixed dislocation interacting with collinear interfacial rigid conducting/dielectric lines in two piezoelectric solids loaded at infinity. Based on the Stroh formalism and the analytical function theory, the general solutions for the field variables are derived first. It is identified that the stress and electric displacement fields present either a square root singularity or a combination of any two of the three types of singularities: traditional square root, oscillatory, and non-square root singularities, at the tips of rigid conducting lines, and either a square root singularity or a combination of square root and oscillatory singularities, at the tips of rigid dielectric lines. Singularities rely on the electroelastic coefficients of the two piezoelectric media. Energy release rate and mechanical strain energy release rate are then expressed in terms of the strain and electric field intensity factors, in analogy to the stress and electric

displacement intensity factors defined for interfacial cracks. Furthermore, the exact field solutions for a single interfacial rigid line are presented, as a special example. The image force on the mixed dislocation is given in detail for this special case. Finally, numerical examples are performed to demonstrate the effects of some important parameters on the image force.

6.2 Statement of the Problem

Referred to a Cartesian coordinate system x_i ($i = 1, 2, 3$), examine an infinite space which consists of two half-infinite planes: $x_2 > 0$ (s_1) and $x_2 < 0$ (s_2). The half planes are occupied by two different piezoelectric materials bonded together along the x_1 axis. The whole space is subjected to uniform loading at infinity. Consider a piezoelectric mixed dislocation with Burgers vector $\mathbf{b} = [b_1, b_2, b_3, b_\phi]^T$ at a point $z_d = x_{1d} + ix_{2d}$ in s_1 . The mixed dislocation has a line load $\mathbf{f} = [-p_1, -p_2, -p_3, q]^T$ at its core. It is assumed that n collinear rigid lines l_j ($j = 1, 2, \dots, n$) with the end points a_j and b_j are at the interface. Let L_l and L_b stand for the unions of rigid lines and perfectly bonded parts along the interface, respectively (see Fig. 6.1).

The mechanical boundary and equilibrium conditions of the interfacial rigid lines are

$$u_{m,1}^+(x_1) = u_{m,1}^-(x_1) = \delta_{m2} \omega_j, \quad m = 1, 2, 3, \quad j = 1, 2, \dots, n, \quad \text{on } L_l, \quad (6.1)$$

$$\int_{L_l} [\sigma_{2m}^+(x_1) - \sigma_{2m}^-(x_1)] dx_1 = 0, \quad m = 1, 2, 3, \quad \text{on } L_l, \quad (6.2)$$

$$\int_{L_l} [\sigma_{22}^+(x_1) - \sigma_{22}^-(x_1)] x_1 dx_1 = 0, \quad \text{on } L_l, \quad (6.3)$$

6.3 Formulation and Solution

Let $\mathbf{F}_1(z)$ and $\mathbf{F}_2(z)$ be the complex potentials in s_1 and s_2 , respectively. Based on the superposition principle, we can express

$$\mathbf{F}_k(z) = \mathbf{F}_k^\infty + \mathbf{F}_{ka}(z) + \mathbf{F}_{kb}(z), \quad k = 1, 2, \quad (6.10)$$

where constant vectors $\mathbf{F}_k^\infty$ are determined by the uniform far-field load in the absence of rigid lines, $\mathbf{F}_{ka}(z)$ are the analytical functions due to the interaction of a dislocation with a perfectly bonded interface with $\mathbf{F}_{ka}(\infty) = 0$, and complex potentials $\mathbf{F}_{kb}(z)$ correspond to the perturbed field due to the collinear interfacial rigid lines with $\mathbf{F}_{kb}(\infty) = 0$.

We begin with the perfectly bonded interface sub-problem in the absence of rigid lines.

The uniform far-field loads $\mathbf{\Phi}_{,1}^\infty = [\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, D_2^\infty]^\text{T}$ and $\mathbf{u}_{,1}^\infty = [\varepsilon_{11}^\infty, \varepsilon_{12}^\infty + \omega^\infty, 2\varepsilon_{13}^\infty, -E_1^\infty]^\text{T}$ are applied so that the continuous conditions on tractions and displacements along the interface are satisfied. Thus from equations (2.54) and (2.56), we have

$$\mathbf{A}_k \mathbf{F}_k^\infty + \overline{\mathbf{A}}_k \overline{\mathbf{F}}_k^\infty = \mathbf{u}_{,1}^\infty, \quad \mathbf{B}_k \mathbf{F}_k^\infty + \overline{\mathbf{B}}_k \overline{\mathbf{F}}_k^\infty = \mathbf{\Phi}_{,1}^\infty, \quad k = 1, 2, \quad -\infty < x_1 < \infty. \quad (6.11)$$

In view of (2.36) and from (6.11), one has

$$\mathbf{F}_k^\infty = \mathbf{A}_k^\text{T} \mathbf{\Phi}_{,1}^\infty + \mathbf{B}_k^\text{T} \mathbf{u}_{,1}^\infty, \quad k = 1, 2. \quad (6.12)$$

Now we consider the sub-problem of a mixed dislocation interacting with an interface of two bonded piezoelectric media. The continuous conditions of the generalized stresses and displacements along the entire x_1 axis require

$$\mathbf{u}_{a,1}^+(x_1) = \mathbf{u}_{a,1}^-(x_1), \quad \mathbf{\Phi}_{a,1}^+(x_1) = \mathbf{\Phi}_{a,1}^-(x_1), \quad -\infty < x_1 < \infty, \quad (6.13)$$

where $\mathbf{u}_{a,1}$ and $\Phi_{a,1}$ represent the generalized strains and stresses, respectively. For this sub-problem, the solution is given by

$$\mathbf{F}_a(z) = \begin{cases} \mathbf{F}_{1d}(z) + \mathbf{F}_{10}(z) & z \in s_1, \\ \mathbf{F}_{20}(z) & z \in s_2, \end{cases} \quad (6.14)$$

where $\mathbf{F}_{1d}(z)$ is the solution for a dislocation at z_d in an infinite plane:

$$\mathbf{F}_{1d}(z) = \frac{1}{2\pi i} \left\langle \frac{1}{z - z_{d\alpha}} \right\rangle (\mathbf{B}_1^T \mathbf{b} + \mathbf{A}_1^T \mathbf{f}). \quad (6.15)$$

The bracket $\langle \rangle$ above denotes a diagonal matrix. Following the arguments in Suo et al. (1992), one finds

$$\mathbf{F}_{10}(z) = \mathbf{A}_1^{-1} \mathbf{Y}^{-1} (\overline{\mathbf{M}}_1 - \overline{\mathbf{M}}_2) \overline{\mathbf{A}}_1 \overline{\mathbf{F}}_{1d}(z), \quad (6.16)$$

$$\mathbf{F}_{20}(z) = 2\mathbf{A}_2^{-1} \overline{\mathbf{Y}}^{-1} \mathbf{H}_1^{-1} \mathbf{A}_1 \mathbf{F}_{1d}(z), \quad (6.17)$$

where Hermitian matrix $\mathbf{Y}$ involving bimaterial properties is defined as

$$\mathbf{Y} = \mathbf{M}_1 + \overline{\mathbf{M}}_2. \quad (6.18)$$

Matrices $\mathbf{M}$ and $\mathbf{H}$ are expressed in (2.41) and (2.39), respectively, and the subscripts 1 and 2 distinguish the two materials. This notation should not cause confusion if one notes the context in which it appears. Substitution of (6.16) and (6.17) into (6.14) gives the complete solution.

The remaining task is to consider the rigid line disturbed sub-problem. From (6.1), (6.8) and either (6.4) or (6.6), the continuity of the generalized displacements along the x_1 axis, with (2.54), (6.10), (6.11)₁ and (6.13)₁, gives

$$\mathbf{A}_1 \mathbf{F}_{1b}^+(x_1) + \overline{\mathbf{A}}_1 \overline{\mathbf{F}}_{1b}^-(x_1) = \mathbf{A}_2 \mathbf{F}_{2b}^-(x_1) + \overline{\mathbf{A}}_2 \overline{\mathbf{F}}_{2b}^+(x_1), \quad -\infty < x_1 < \infty. \quad (6.19)$$

The analytic continuity argument leads to

$$\begin{cases} \mathbf{A}_1 \mathbf{F}_{1b}(z) = \overline{\mathbf{A}}_2 \overline{\mathbf{F}}_{2b}(z) & z \in s_1, \\ \mathbf{A}_2 \mathbf{F}_{2b}(z) = \overline{\mathbf{A}}_1 \overline{\mathbf{F}}_{1b}(z) & z \in s_2. \end{cases} \quad (6.20)$$

With such representations, the generalized stress jumps across rigid lines are obtained as

$$\Delta \Phi_{,1}(x_1) = i\mathbf{Y}[\mathbf{I}^+(x_1) - \mathbf{I}^-(x_1)], \quad (6.21)$$

where

$$\mathbf{I}(z) = \begin{cases} \mathbf{A}_1 \mathbf{F}_{1b}(z) & z \in s_1, \\ \mathbf{Y}^{-1} \overline{\mathbf{Y}} \mathbf{A}_2 \mathbf{F}_{2b}(z) & z \in s_2. \end{cases} \quad (6.22)$$

Since $\Delta \Phi_{,1}(x_1) = 0$ on L_b from (6.9), (6.21) shows that $\mathbf{I}(z) = [I_1(z), I_2(z), I_3(z), I_4(z)]^T$

is analytic in the entire plane except at the rigid lines.

Substituting (6.10) into (2.54) with (6.11)₁ gives

$$\mathbf{u}_{,1} = \mathbf{u}_{,1}^\infty + \mathbf{u}_{a,1} + \mathbf{I}^+(x_1) + \overline{\mathbf{Y}}^{-1} \mathbf{Y} \mathbf{I}^-(x_1), \quad (6.23)$$

where

$$\mathbf{u}_{a,1} = \frac{2}{\pi} \text{Im} \left[\mathbf{N} \left\langle \frac{1}{x - z_{d\alpha}} \right\rangle \mathbf{d} \right]. \quad (6.24)$$

In the above equation, $\mathbf{N} = \overline{\mathbf{Y}}^{-1} \mathbf{H}_1^{-1} \mathbf{A}_1$ and $\mathbf{d} = \mathbf{B}_1^T \mathbf{b} + \mathbf{A}_1^T \mathbf{f}$ are defined for simplicity, whose elements are represented by N_{ij} and d_i ($i, j = 1, 2, 3, 4$), respectively. Once $\mathbf{I}(z)$ is found, so is the full field.

The solution is simple if $\mathbf{Y}$ is real. When collinear rigid lines are in a homogeneous solid, $\mathbf{Y}$ is indeed real. $\mathbf{Y}$ can be real for a bimaterial having certain symmetry. In the sub-sections 6.3.1 and 6.3.2, real $\mathbf{Y}$ is assumed, and the case that $\mathbf{Y}$ is complex is examined in the sub-sections 6.3.3 and 6.3.4. Before the following analyses, let us introduce a 3×3 bimaterial matrix $\hat{\mathbf{Y}}$ with the elements $\hat{Y}_{ij}$, given by

$$\hat{Y}_{ij} = Y_{ij} - \frac{1}{Y_{44}} Y_{i4} Y_{4j}, \quad (6.25)$$

where Y_{ij} denote the elements of $\mathbf{Y}$. It is easily shown from (6.25) that

$$\hat{Y}_{ij}^{-1} = Y_{ij}^{-1}, \quad (6.26)$$

where $\hat{Y}_{ij}^{-1}$ and Y_{ij}^{-1} stands for the elements of matrices $\hat{\mathbf{Y}}^{-1}$ and $\mathbf{Y}^{-1}$, respectively. It is shown that $\hat{\mathbf{Y}}$ is a positive definite Hermitian matrix by using the property of $\mathbf{Y}$.

6.3.1 Rigid Conducting Lines: Square Root Singularity Fields

With real $\mathbf{Y}$, from (6.1), (6.4) and (6.23), one has

$$\mathbf{I}^+(x_l) + \mathbf{I}^-(x_l) = \mathbf{\Omega} - \mathbf{u}_{,1}^\infty - \mathbf{u}_{a,1} \text{ on } L_l. \quad (6.27)$$

where

$$\mathbf{\Omega} = \begin{cases} [0, \omega_1, 0, 0]^T & \text{on } l_1, \\ \dots\dots\dots \\ [0, \omega_n, 0, 0]^T & \text{on } l_n. \end{cases} \quad (6.28)$$

Following the method of Muskhelishvili (1975), we can obtain the general solution of (6.27) as

$$\begin{aligned} \mathbf{I}(z) = & \frac{1}{2} X(z) \left\{ \oint_{\eta} \frac{\mathbf{\Omega}}{X(\xi)(\xi - z)} d\xi - \mathbf{u}_{,1}^\infty \left[\frac{1}{X(z)} - x(\infty) \right] \right. \\ & - \frac{1}{\pi i} \mathbf{N} \left\langle \frac{1}{(z_\alpha - z_{d\alpha}) X(z_\alpha)} - 1 - \frac{1}{(z_\alpha - z_{d\alpha}) X(z_{d\alpha})} \right\rangle \mathbf{d} \\ & \left. + \frac{1}{\pi i} \overline{\mathbf{N}} \left\langle \frac{1}{(z_\alpha - \bar{z}_{d\alpha}) X(z_\alpha)} - \frac{1}{(z_\alpha - \bar{z}_{d\alpha}) X(\bar{z}_{d\alpha})} - 1 \right\rangle \overline{\mathbf{d}} + \mathbf{P}^a(z) \right\}, \end{aligned} \quad (6.29)$$

where

$$\mathbf{P}^a(z) = \mathbf{c}_{n-1}^a z^{n-1} + \dots + \mathbf{c}_0^a, \quad (6.30)$$

$$X(z) = \prod_{j=1}^n (z - a_j)^{-1/2} (z - b_j)^{-1/2}, \quad (6.31)$$

$$x(\infty) = \prod_{j=1}^n [z - (a_j + b_j)/2], \quad (6.32)$$

in which η is the union of the n contours $\eta_1, \dots, \eta_n$, surrounding the rigid lines $L_1, \dots, L_n$ in clockwise direction. The single value branch in (6.31) will be determined when the condition $\lim_{z \rightarrow \infty} [z^n X(z)] = 1$ is satisfied.

From (6.2), (6.5), and (6.21), we obtain the closed path integral

$$\oint_{\eta_j} \mathbf{I}(z) dz = \mathbf{0}, \quad j = 1, 2, \dots, n. \quad (6.33)$$

From (6.3) and (6.21), we obtain the closed path integral

$$\mathbf{Y}^{(2)} \oint_{\eta_j} \mathbf{I}(z) dz = \mathbf{0}, \quad j = 1, 2, \dots, n, \quad (6.34)$$

where $\mathbf{Y}^{(2)} = [Y_{21}, Y_{22}, Y_{23}, Y_{24}]$. The n vectors $\mathbf{c}_{j-1}^a$ and the n constants ω_j ($j = 1, 2, \dots, n$) can be found by substituting (6.29) into (6.33) and (6.34) with (6.30)-(6.32). $\mathbf{F}_{kb}(z)$ can be found by using (6.22).

6.3.2 Rigid Dielectric Lines: Square Root Singularity Fields

With (6.21), the condition (6.7) along rigid dielectric lines leads to

$$\mathbf{Y}^{(4)} \mathbf{I}(z) = \mathbf{0}. \quad (6.35)$$

One has from (6.35) that

$$I_4(z) = -\frac{Y_{4j}}{Y_{44}} I_j(z), \quad j = 1, 2, 3, \quad (6.36)$$

$$\mathbf{I}(z) = (\hat{\mathbf{I}} - \mathbf{E}) \hat{\mathbf{I}}(z), \quad (6.37)$$

where $\hat{\mathbf{I}}(z) = [I_1(z), I_2(z), I_3(z)]^T$, $\hat{\mathbf{I}}$ is obtained by deleting the fourth column from the 4×4 identity matrix $\mathbf{I}$, and $\mathbf{E} = [\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3]$ with $\mathbf{E}_i = [0, 0, 0, Y_{4i}/Y_{44}]^T$. It is worth to note that the relations in (6.35)-(6.37) are valid for real and complex $\mathbf{Y}$.

With real $\mathbf{Y}$, equations (6.1), (6.23) and (6.26) lead to

$$\hat{\mathbf{I}}^+(x_1) + \hat{\mathbf{I}}^-(x_1) = \hat{\mathbf{\Omega}} - \hat{\mathbf{u}}_{,1}^\infty - \hat{\mathbf{u}}_{a,1}, \quad (6.38)$$

where

$$\hat{\mathbf{\Omega}} = \begin{cases} [0, \omega_1, 0]^T & \text{on } l_1, \\ \dots\dots\dots & \\ [0, \omega_n, 0]^T & \text{on } l_n, \end{cases} \quad (6.39)$$

$$\hat{\mathbf{u}}_{,1}^\infty = [u_{1,1}^\infty, u_{2,1}^\infty, u_{3,1}^\infty]^T, \quad (6.40)$$

$$\hat{\mathbf{u}}_{a,1} = [u_{1,1}^a, u_{2,1}^a, u_{3,1}^a]^T. \quad (6.41)$$

The general solution of (6.38) is given by

$$\begin{aligned} \hat{\mathbf{I}}(z) = & \frac{1}{2} X(z) \left\{ \oint_{\eta} \frac{\hat{\mathbf{\Omega}}}{X(\xi)(\xi - z)} d\xi - \hat{\mathbf{u}}_{,1}^\infty \left[\frac{1}{X(z)} - x(\infty) \right] \right. \\ & - \frac{1}{\pi i} \hat{\mathbf{N}} \left\langle \frac{1}{(z_\alpha - z_{d\alpha})X(z_\alpha)} - 1 - \frac{1}{(z_\alpha - z_{d\alpha})X(z_{d\alpha})} \right\rangle \mathbf{d} \\ & \left. + \frac{1}{\pi i} \hat{\mathbf{N}} \left\langle \frac{1}{(z_\alpha - \bar{z}_{d\alpha})X(z_\alpha)} - \frac{1}{(z_\alpha - \bar{z}_{d\alpha})X(\bar{z}_{d\alpha})} - 1 \right\rangle \bar{\mathbf{d}} + \mathbf{P}^b(z) \right\}, \end{aligned} \quad (6.42)$$

where $\hat{\mathbf{N}}$ is a 3×4 matrix obtained by deleting the fourth row from matrix $\mathbf{N}$ and

$$\mathbf{P}^b(z) = \mathbf{c}_{n-1}^b z^{n-1} + \dots + \mathbf{c}_0^b. \quad (6.43)$$

Using (6.2), (6.21) and (6.26), we can obtain the closed path integral

$$\oint_{\eta_j} \hat{\mathbf{I}}(z) dz = \mathbf{0}, \quad j = 1, 2, \dots, n. \quad (6.44)$$

From (6.3), (6.21) and (6.26), one has the closed path integral

$$\hat{\mathbf{Y}}^{(2)} \oint_{\eta_j} \hat{\mathbf{I}}(z) z dz = 0, \quad j = 1, 2, \dots, n, \quad (6.45)$$

where $\hat{\mathbf{Y}}^{(2)} = [\hat{Y}_{21}, \hat{Y}_{22}, \hat{Y}_{23}]$. Substituting (6.42) into (6.44) and (6.45) with (6.31), (6.32) and (6.43), we obtain the n vectors $\mathbf{c}_{j-1}^b$ and the n constants ω_j ($j = 1, 2, \dots, n$). Then, we can find $\mathbf{F}_{kb}(z)$ using (6.22) and (6.37).

For both conducting and dielectric cases, the intensity factors of the generalized strains $[u_{m,1}(x_1, 0), -E_1(x_1, 0)]^T$ ($m = 1, 2, 3$) and stresses $[\sigma_{2m}(x_1, 0), D_2(x_1, 0)]^T$ at the right tip of the j th rigid line can be evaluated as

$$\mathbf{k} = [k_1, k_2, k_3, k_4]^T = \lim_{x_1 \rightarrow b_j} \sqrt{2\pi(x_1 - b_j)} [\mathbf{I}(x_1) + \bar{\mathbf{I}}(x_1)], \quad (6.46)$$

$$\mathbf{K} = [K_{II}, K_I, K_{III}, K_D]^T = \lim_{x_1 \rightarrow b_j} \sqrt{2\pi(x_1 - b_j)} [i\mathbf{M}_I \mathbf{I}(x_1) - i\bar{\mathbf{M}}_I \bar{\mathbf{I}}(x_1)], \quad (6.47)$$

respectively. For the dielectric case, equation (6.46) with (6.37), can be rewritten as

$$\mathbf{k} = (\hat{\mathbf{I}} - \mathbf{E})\hat{\mathbf{k}}, \quad (6.48)$$

where

$$\hat{\mathbf{k}} = [k_1, k_2, k_3]^T = \lim_{x_1 \rightarrow b_j} \sqrt{2\pi(x_1 - b_j)} [\hat{\mathbf{I}}(x_1) + \bar{\hat{\mathbf{I}}}(x_1)]. \quad (6.49)$$

At the interface a distance r , the generalized displacements ahead of the tips, and the generalized stress jumps behind the tips, of the j th rigid conducting/dielectric line, are given by

$$\mathbf{u} = (2r/\pi)^{1/2} \mathbf{k}, \quad \Delta\Phi_{,1}(r) = (1/2\pi r)^{1/2} \mathbf{Y}\mathbf{k}, \quad (6.50)$$

respectively. For the dielectric case, equation (6.50)₂ can be further rewritten as

$$[\Delta\sigma_{21}(r), \Delta\sigma_{22}(r), \Delta\sigma_{23}(r)]^T = (1/2\pi r)^{1/2} \hat{\mathbf{Y}}\hat{\mathbf{k}}, \quad \Delta D_2(r) = 0. \quad (6.51)$$

From (6.47), at the interface a distance r ahead of the rigid conducting/dielectric line tip, the generalized stresses are given by

$$[\sigma_{2j}, D_2]^T = (2\pi r)^{-1/2} \mathbf{K}. \quad (6.52)$$

It is interesting to note that the structure of the generalized strains in (6.50)₁ is the same as that of the generalized stresses ahead of an impermeable crack tip and the structure of the generalized stresses in (6.50)₂ is the same as that of the generalized displacement jumps behind the crack tip (Suo et al., 1992).

The energy release rate for the j th rigid conducting/dielectric line can be evaluated by

$$G = \lim_{\Lambda \rightarrow 0} -\frac{1}{2\Lambda} \int_0^\Lambda \Delta \Phi_{,1}^T(r) \mathbf{u}(\Lambda - r, 0) dr. \quad (6.53)$$

Substituting (6.50) into (6.53) leads to

$$G = -\frac{1}{4} \mathbf{k}^T \mathbf{Y} \mathbf{k}. \quad (6.54)$$

For the dielectric case, equation (6.54) can be simplified as

$$G = -\frac{1}{4} \hat{\mathbf{k}}^T \hat{\mathbf{Y}} \hat{\mathbf{k}}. \quad (6.55)$$

The mechanical strain energy release rate for the j th rigid conducting/dielectric line can be computed by the closure integral

$$G_M = \lim_{\Lambda \rightarrow 0} -\frac{1}{2\Lambda} \int_0^\Lambda \Delta \sigma_{2j}(r) u_j(\Lambda - r, 0) dr, \quad j = 1, 2, 3. \quad (6.56)$$

Substituting (6.50) into (6.56) leads to

$$G_M = -\frac{1}{4} Y_{ij} k_j k_i, \quad i = 1, 2, 3, \quad j = 1, 2, 3, 4. \quad (6.57)$$

For the dielectric case, equation (6.57) can be rewritten as (6.55).

It is worth to point out that all the solutions in the sub-sections 6.3.1 and 6.3.2 can be applied to the corresponding problems of collinear rigid conducting/dielectric lines in a homogeneous piezoelectric medium, where $\mathbf{M}_1 = \mathbf{M}_2$ and $\mathbf{Y} = 2\mathbf{H}_1^{-1}$.

6.3.3 Rigid Conducting Lines: Square Root, Oscillatory and Non-square Root Singularity Fields

For complex $\mathbf{Y}$, equations (6.1), (6.4) and (6.23) lead to

$$\mathbf{I}^+(x_1) + \bar{\mathbf{Y}}^{-1} \mathbf{Y} \mathbf{I}^-(x_1) = \mathbf{\Omega} - \mathbf{u}_{,1}^\infty - \mathbf{u}_{a,1} \text{ on } L_l. \quad (6.58)$$

Assuming that the eigenvalues of $\bar{\mathbf{Y}}^{-1} \mathbf{Y}$ take the form $-e^{2\pi i \delta_\alpha}$ with $\delta_\alpha = -1/2 + i\varepsilon_\alpha$, one has

$$\varepsilon_{1,2} = \pm \varepsilon, \quad \varepsilon_{3,4} = \mp i\kappa, \quad (6.59)$$

where

$$\varepsilon = \tanh^{-1}(\beta_1)/\pi, \quad \kappa = \tan^{-1}(-i\beta_2)/\pi, \quad (6.60)$$

$$\beta_1 = [(b^2 - c)^{1/2} - b]^{1/2}, \quad \beta_2 = i[(b^2 - c)^{1/2} + b]^{1/2}, \quad (6.61)$$

$$b = \frac{1}{4} \text{tr}[(\mathbf{D}^{-1} \mathbf{U})^2], \quad c = \|\mathbf{D}^{-1} \mathbf{U}\|, \quad \mathbf{D} = \text{Re}[\mathbf{Y}], \quad \mathbf{U} = \text{Im}[\mathbf{Y}]. \quad (6.62)$$

Let Λ_α be the eigenvector of $\bar{\mathbf{Y}}^{-1} \mathbf{Y}$ corresponding to the eigenvalue $-e^{2\pi i \delta_\alpha}$. Suo et al. (1992) assumed that both values of ε and κ are non-zero and four linearly independent eigenvectors would be obtained for an impermeable crack in a piezoelectric bimaterial system. Three possible cases of the combination of ε and κ are considered here:

- case 1: $\varepsilon \neq 0$ and $\kappa \neq 0$ (i.e. $b \neq 0$ and $c \neq 0$),
 - case 2: $\varepsilon \neq 0$ and $\kappa = 0$ (i.e. $b < 0$ and $c = 0$),
 - case 3: $\varepsilon = 0$ and $\kappa \neq 0$ (i.e. $b > 0$ and $c = 0$).
- (6.63)

It is identified that the eigenvectors Λ_1 and Λ_2 are complex conjugates, i.e. $\Lambda_2 = \bar{\Lambda}_1$, and Λ_3 and Λ_4 are real for cases 1 and 2, and all of them are real for case 3 (Deng and Meguid, 1998). Ou and Wu (2003) confirmed that cases 2 and 3 apply to the problem of an impermeable interfacial crack in a transversely isotropic piezoelectric bimaterial. Four

linearly independent eigenvectors can be obtained for the three cases. For simplicity, we define

$$\Lambda_A = [\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4], \Lambda_B = \begin{cases} [\bar{\Lambda}_1, \Lambda_1, \Lambda_4, \Lambda_3] & \text{for case 1,} \\ [\bar{\Lambda}_1, \Lambda_1, \Lambda_3, \Lambda_4] & \text{for case 2,} \\ [\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_3] & \text{for case 3,} \end{cases} \quad (6.64)$$

which satisfy

$$\Lambda_B^T \mathbf{D} \Lambda_A = \langle v_1, v_2, v_3, v_4 \rangle, \begin{cases} v_1 = v_2, v_3 = v_4 & \text{for case 1,} \\ v_1 = v_2 & \text{for case 2,} \\ v_3 = v_4 & \text{for case 3.} \end{cases} \quad (6.65)$$

In the above three cases, the general solution of (6.58) can be given as

$$\mathbf{I}(z) = \sum_{\alpha=1}^4 \lambda_{\alpha}(z) \Lambda_{\alpha}, \quad (6.66)$$

where

$$\lambda_{\alpha}(x_1) = \frac{e^{-\pi \varepsilon_{\alpha}} k'_{\alpha} X_{\alpha}(x_1)}{2\sqrt{2\pi} \cos(i\pi \varepsilon_{\alpha})}, \quad (6.67)$$

$$X_{\alpha}(z) = \prod_{j=1}^n (z - a_j)^{-1/2+i\varepsilon_{\alpha}} (z - b_j)^{-1/2-i\varepsilon_{\alpha}}. \quad (6.68)$$

In the above equation, the factors k'_1 and k'_2 are complex conjugates, i.e. $k'_2 = \bar{k}'_1$, and k'_3 and k'_4 are real for cases 1 and 2, and all of them are real for case 3. The arrangement of k'_{α} ensures that the strain and the electric field are real (see (6.74) below).

Multiplying the two sides of (6.58) by $\mathbf{W} = \Lambda_B^T \mathbf{D}$, we obtain

$$[\lambda_{\alpha}(x_1)]^+ + e^{2\pi \varepsilon_{\alpha}} [\lambda_{\alpha}(x_1)]^- = \mathbf{W}^{\alpha} \mathbf{I}^0(x_1) / v_{\alpha}, \quad \alpha = 1, 2, 3, 4, \quad (6.69)$$

where

$$\mathbf{I}^0(x_1) = \mathbf{\Omega} - \mathbf{u}_{,1}^{\infty} - \mathbf{u}_{a,1}, \quad (6.70)$$

and $\mathbf{W}^{\alpha} = [W_{\alpha 1}, W_{\alpha 2}, W_{\alpha 3}, W_{\alpha 4}]$, in which $W_{\alpha j}$ stands for the element of matrix $\mathbf{W}$.

The solution to (6.69) is given as

$$\begin{aligned} \lambda_\alpha(z) = & \frac{X_\alpha(z)}{(1+e^{2\pi i \varepsilon_\alpha})v_\alpha} \left\{ \oint_\eta \frac{\mathbf{W}^\alpha \boldsymbol{\Omega}}{X_\alpha(\xi)(\xi-z)} d\xi - \mathbf{W}^\alpha \mathbf{u}_{,1}^\infty \left[\frac{1}{X_\alpha(z)} - x_\alpha(\infty) \right] \right. \\ & + \frac{1}{i\pi} \sum_{j=1}^4 W_{\alpha j} \sum_{k=1}^4 \bar{N}_{jk} \bar{d}_k \left[\frac{1}{(z-\bar{z}_{dk})X_\alpha(z)} - \frac{1}{(z-\bar{z}_{dk})X_\alpha(\bar{z}_{dk})} - 1 \right] + P_\alpha^c(z) \\ & \left. - \frac{1}{i\pi} \sum_{j=1}^4 W_{\alpha j} \sum_{k=1}^4 N_{jk} d_k \left[\frac{1}{(z-z_{dk})X_\alpha(z)} - \frac{1}{(z-z_{dk})X_\alpha(z_{dk})} - 1 \right] \right\}, \quad (6.71) \end{aligned}$$

where

$$P_\alpha^c(z) = c_{\alpha n-1}^c z^{n-1} + \cdots + c_{\alpha 0}^c, \quad (6.72)$$

$$x_\alpha(\infty) = \prod_{j=1}^n [z - (a_j + b_j)/2 - i\varepsilon_\alpha(b_j - a_j)]. \quad (6.73)$$

Substituting (6.66) into (6.33) and (6.34) with (6.68) and (6.71)-(6.73), we obtain the $4n$ constants $c_{\alpha k}^c$ ($\alpha = 1, 2, 3, 4$, $k = 0, 1, \dots, n-1$) and the n constants ω_j ($j = 1, 2, \dots, n$). The function $\mathbf{F}_{kb}(z)$ can be obtained through (6.22).

The generalized strains along the interface can be written as

$$\mathbf{u}_{,1}(x_1) = \frac{1}{\sqrt{2\pi}} [k'_1 X_1(x_1) \boldsymbol{\Lambda}_1 + k'_2 X_2(x_1) \boldsymbol{\Lambda}_2 + k'_3 X_3(x_1) \boldsymbol{\Lambda}_3 + k'_4 X_4(x_1) \boldsymbol{\Lambda}_4]. \quad (6.74)$$

It reads that the components $k'_1 X_1(x_1)$ and $k'_2 X_2(x_1)$ present the singularities of $r^{-1/2+i\varepsilon_\alpha}$, and $k'_3 X_3(x_1)$ and $k'_4 X_4(x_1)$ exhibit the singularities of $r^{-1/2+\kappa}$, at the interface a distance r ahead of the tips of rigid conducting lines. In cases 1 and 2, $k'_1 X_1(x_1)$ and $k'_2 X_2(x_1)$ are in the plane spanned by $\text{Re}[\boldsymbol{\Lambda}_1]$ and $\text{Im}[\boldsymbol{\Lambda}_1]$, and $k'_3 X_3(x_1)$ and $k'_4 X_4(x_1)$ are along $\boldsymbol{\Lambda}_3$ and $\boldsymbol{\Lambda}_4$, respectively, and in case 3, $k'_i X_i(x_1)$ ($i = 1, 2, 3, 4$) are along $\boldsymbol{\Lambda}_i$, respectively. In case 1, the components are analogous to the corresponding tractions and charge for an impermeable crack at the interface of piezoelectric media, where $\varepsilon \neq 0$ and $\kappa \neq 0$ are assumed (Suo et al., 1992).

At the interface, the generalized stresses disturbed by rigid conducting lines are given by

$$[\sigma_{2j}, D_2]^T = 2 \operatorname{Re} \left[\sum_{\alpha=1}^4 \lambda_{\alpha}(x_1) \mathbf{B}_1 \mathbf{A}_1^{-1} \mathbf{\Lambda}_{\alpha} \right]. \quad (6.75)$$

At the tips of rigid conducting lines, the generalized stresses present oscillatory and non-square root singularities in case 1, oscillatory and square root singularities in case 2, and square root and non-square root singularities in case 3, respectively.

The generalized strain intensity factors at the right tip of the j th rigid conducting line are defined as

$$\begin{aligned} \tilde{k}_{\alpha} &= \lim_{x_1 \rightarrow b_j} k'_{\alpha} \prod_{i=1}^n (x_1 - a_i)^{-1/2+i\varepsilon_{\alpha}} \prod_{i=1, i \neq j}^n (x_1 - b_i)^{-1/2-i\varepsilon_{\alpha}} \\ &= \lim_{x_1 \rightarrow b_j} 2\sqrt{2\pi} \cos(i\pi\varepsilon_{\alpha}) e^{\pi\varepsilon_{\alpha}} (x_1 - b_j)^{1/2+i\varepsilon_{\alpha}} \lambda_{\alpha}(x_1). \end{aligned} \quad (6.76)$$

It reads that $\tilde{k}_1$ and $\tilde{k}_2$ are conjugate complex, and $\tilde{k}_3$ and $\tilde{k}_4$ are real for cases 1 and 2.

All of them are real for case 3.

At the interface a distance r ahead of the j th rigid conducting line tips

$$\mathbf{u} = \sqrt{\frac{r}{2\pi}} \sum_{\alpha=1}^4 \frac{2}{1-i2\varepsilon_{\alpha}} \tilde{k}_{\alpha} r^{-i\varepsilon_{\alpha}} \mathbf{\Lambda}_{\alpha}. \quad (6.77)$$

At a distance r behind the j th rigid conducting line tips

$$\Delta \Phi_{,1}(r) = \mathbf{D} \sqrt{\frac{1}{2\pi r}} \sum_{\alpha=1}^4 \frac{1}{\cos(i\pi\varepsilon_{\alpha})} \tilde{k}_{\alpha} r^{-i\varepsilon_{\alpha}} \mathbf{\Lambda}_{\alpha}. \quad (6.78)$$

It is worth noting that the structure of the generalized displacements in (6.77) is the same as that of the generalized stresses ahead of an impermeable crack tip and the structure of the generalized stress jumps in (6.68) is the same as that of the generalized displacement jumps behind the crack tip (Suo et al., 1992).

Substituting (6.77) and (6.78) into (6.53) leads to the energy release rate for the j th rigid conducting line

$$G = \begin{cases} -\frac{\nu_1}{2 \cosh^2 \pi \varepsilon} |\tilde{k}_1|^2 - \frac{\nu_3}{2 \cos^2 \pi \kappa} \tilde{k}_3 \tilde{k}_4, & \text{for case 1,} \\ -\frac{\nu_1}{2 \cosh^2 \pi \varepsilon} |\tilde{k}_1|^2 - \frac{\nu_3}{4} \tilde{k}_3^2 - \frac{\nu_4}{4} \tilde{k}_4^2, & \text{for case 2,} \\ -\frac{\nu_1}{4} \tilde{k}_1^2 - \frac{\nu_2}{4} \tilde{k}_2^2 - \frac{\nu_3}{2 \cos^2 \pi \kappa} \tilde{k}_3 \tilde{k}_4, & \text{for case 3.} \end{cases} \quad (6.79)$$

However, substituting (6.77) and (6.78) into (6.56) cannot give the explicit expression of the mechanical strain energy release rate for the j th rigid conducting line, as it depends on the integral path.

6.3.4 Rigid Dielectric Lines: Square Root and Oscillatory Singularity Fields

For complex $\mathbf{Y}$, equations (6.1), (6.23) and (6.26) lead to

$$\hat{\mathbf{I}}^+(x_1) + \overline{\hat{\mathbf{Y}}}^{-1} \hat{\mathbf{Y}} \hat{\mathbf{I}}^-(x_1) = \hat{\mathbf{\Omega}} - \hat{\mathbf{u}}_{,1}^\infty - \hat{\mathbf{u}}_{a,1} \text{ on } L_l. \quad (6.80)$$

Similarly, the eigenvalues of $\overline{\hat{\mathbf{Y}}}^{-1} \hat{\mathbf{Y}}$ are assumed to be the form $e^{2\pi\varepsilon_\alpha}$. One has

$$\varepsilon_{1,2} = \pm \varepsilon, \quad \varepsilon_3 = 0, \quad (6.81)$$

where

$$\varepsilon = \frac{1}{2\pi} \ln \frac{1+\eta}{1-\eta}, \quad \eta = \sqrt{-\frac{1}{2} \text{tr}[(\hat{\mathbf{D}}^{-1} \hat{\mathbf{U}})^2]}, \quad \hat{\mathbf{D}} = \text{Re}[\hat{\mathbf{Y}}], \quad \hat{\mathbf{U}} = \text{Im}[\hat{\mathbf{Y}}]. \quad (6.82)$$

The associated eigenvectors $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are complex conjugates, i.e. $\mathbf{\Lambda}_2 = \overline{\mathbf{\Lambda}_1}$, and $\mathbf{\Lambda}_3$ is real. We define

$$\mathbf{\Lambda}_A = [\mathbf{\Lambda}_1, \overline{\mathbf{\Lambda}_1}, \mathbf{\Lambda}_3], \quad \mathbf{\Lambda}_B = [\overline{\mathbf{\Lambda}_1}, \mathbf{\Lambda}_1, \mathbf{\Lambda}_3], \quad (6.83)$$

which satisfy

$$\Lambda_B^T \hat{\mathbf{D}} \Lambda_A = \langle v_1, v_2, v_3 \rangle, \quad v_1 = v_2. \quad (6.84)$$

The general solution of (6.79) can be given as

$$\hat{\mathbf{I}}(z) = \sum_{\alpha=1}^3 \lambda_{\alpha}(z) \Lambda_{\alpha}, \quad (6.85)$$

where $\lambda_{\alpha}(z)$ is evaluated by (6.67) with the factors k'_1 and k'_2 being complex conjugates and k'_3 being real. Equation (6.68) has been used. A similar procedure to that adopted in the preceding section permits us to obtain

$$\begin{aligned} \lambda_{\alpha}(z) = & \frac{X_{\alpha}(z)}{(1 + e^{2\pi i \epsilon_{\alpha}})v_{\alpha}} \left\{ \oint_{\eta} \frac{\hat{\mathbf{W}}^{\alpha} \hat{\mathbf{\Omega}}}{X_{\alpha}(\xi)(\xi - z)} d\xi - \hat{\mathbf{W}}^{\alpha} \hat{\mathbf{u}}_{,1}^{\infty} \left[\frac{1}{X_{\alpha}(z)} - x_{\alpha}(\infty) \right] \right. \\ & + \frac{1}{i\pi} \sum_{j=1}^3 \hat{W}_{\alpha j} \sum_{k=1}^4 \bar{N}_{jk} \bar{d}_k \left[\frac{1}{(z - \bar{z}_{dk})X_{\alpha}(z)} - \frac{1}{(z - \bar{z}_{dk})X_{\alpha}(\bar{z}_{dk})} - 1 \right] + P_{\alpha}^d(z) \\ & \left. - \frac{1}{i\pi} \sum_{j=1}^3 \hat{W}_{\alpha j} \sum_{k=1}^4 N_{jk} d_k \left[\frac{1}{(z - z_{dk})X_{\alpha}(z)} - \frac{1}{(z - z_{dk})X_{\alpha}(z_{dk})} - 1 \right] \right\}, \end{aligned} \quad (6.86)$$

where $\hat{\mathbf{W}}^{\alpha}$ denotes the first α row of the matrix $\hat{\mathbf{W}} = \hat{\Lambda}_B^T \hat{\mathbf{D}}$ with the elements $\hat{W}_{ij}$, and

$$P_{\alpha}^d(z) = c_{\alpha n-1}^d z^{n-1} + \cdots + c_{\alpha 0}^d. \quad (6.87)$$

Substituting (6.85) into (6.44) and (6.45) with (6.68), (6.73), (6.86) and (6.87), we have the $3n$ constants $c_{\alpha k}^d$ ($\alpha = 1, 2, 3$, $k = 0, 1, \dots, n-1$) and the n constants ω_j ($j = 1, 2, \dots, n$).

The functions $\mathbf{F}_{kb}(z)$ are obtained through (6.22) and (6.37).

At the interface, the generalized stresses disturbed by rigid dielectric lines are obtained as

$$[\sigma_{2j}, D_2]^T = 2 \operatorname{Re} \left[\mathbf{B}_1 \mathbf{A}_1^{-1} (\hat{\mathbf{I}} - \mathbf{E}) \sum_{\alpha=1}^3 \lambda_{\alpha}(x_1) \Lambda_{\alpha} \right]. \quad (6.88)$$

It is noted that the near tip stress and electric displacement fields present oscillatory and square root singularities, similar to the fields near the tips of a permeable interfacial crack in piezoelectric media (Beom, 2003).

The strains along the interface can be written as

$$\hat{\mathbf{u}}_{,1}(x_1) = \frac{1}{\sqrt{2\pi}} [k'_1 X_1(x_1) \mathbf{\Lambda}_1 + \bar{k}'_1 \bar{X}_1(x_1) \bar{\mathbf{\Lambda}}_1 + k'_3 X_3(x_1) \mathbf{\Lambda}_3]. \quad (6.89)$$

It reads that the strains are decomposed into two components: one is in the plane spanned by $\text{Re}[\mathbf{\Lambda}_1]$ and $\text{Im}[\mathbf{\Lambda}_1]$, and the other is along the $\mathbf{\Lambda}_3$, in analogy to the tractions for a crack in anisotropic elastic media (Suo, 1990). The strain intensity factors $\tilde{k}_\alpha$ ($\alpha = 1, 2, 3$) defined in (6.76) for the conducting case are used here to characterize square root and oscillatory singularities at the j th rigid dielectric line tip. The electric field at the bonded interface is given by

$$E_1(x_1) = 2\text{Re}[\mathbf{e}^T \hat{\mathbf{I}}(x)/Y_{44}], \quad (6.90)$$

with $\mathbf{e} = [-Y_{41}, -Y_{42}, -Y_{43}]^T$, and it also shows oscillatory and square root singularities.

At the interface a distance r ahead of the j th rigid dielectric line tip, the displacements are given by

$$\hat{\mathbf{u}} = \sqrt{\frac{r}{2\pi}} \sum_{\alpha=1}^3 \frac{2}{1-i2\varepsilon_\alpha} \tilde{k}_\alpha r^{-i\varepsilon_\alpha} \mathbf{\Lambda}_\alpha. \quad (6.91)$$

The stress jumps at a distance r behind the j th rigid dielectric line tip are written as

$$[\Delta\sigma_{21}(r), \Delta\sigma_{22}(r), \Delta\sigma_{23}(r)]^T = \hat{\mathbf{D}} \sqrt{\frac{1}{2\pi r}} \sum_{\alpha=1}^3 \frac{1}{\cos(i\pi\varepsilon_\alpha)} \tilde{k}_\alpha r^{-i\varepsilon_\alpha} \mathbf{\Lambda}_\alpha. \quad (6.92)$$

By substituting (6.91) and (6.92) into (6.53) or (6.56), the energy release rate and the mechanical strain energy release rate for the j th rigid dielectric line can be obtained as

$$G = G_M = -\frac{v_1}{2\cosh^2 \pi\varepsilon} |\tilde{k}_1|^2 - \frac{v_3}{4} \tilde{k}_3^2. \quad (6.93)$$

For both conducting and dielectric cases, it should be pointed out that it is impossible to find specific normalizing constants for the eigenvectors $\mathbf{\Lambda}_\alpha$ such that the generalized

strain intensity factors defined in (6.76) can be reduced to the classical generalized strain intensity factors at the tip of a rigid line in a homogeneous piezoelectric material, as the two materials becomes the same.

6.4 A Single Rigid Line: Square Root Singularity Fields

In this section we consider a single interfacial rigid line of length $2a$ centered along the x_1 axis for the case that $\mathbf{Y}$ is real. The rotation of the rigid line is assumed to be ω . One has

$$\mathbf{\Omega} = [0, \omega, 0, 0]^T, \quad (6.94)$$

$$\hat{\mathbf{\Omega}} = [0, \omega, 0]^T, \quad (6.95)$$

$$X(z) = (z + a)^{-1/2} (z - a)^{-1/2}, \quad (6.96)$$

$$x(\infty) = z. \quad (6.97)$$

6.4.1 A Single Rigid Conducting Line

In this case, equation (6.29) becomes

$$\begin{aligned} \mathbf{I}(z) = & \frac{1}{2} (\mathbf{\Omega} - \mathbf{u}_{,1}^\infty) \left[1 - \frac{z}{\sqrt{z^2 - a^2}} \right] - \frac{1}{2\pi i} \mathbf{N} \left\langle \frac{1}{z_\alpha - z_{d\alpha}} \left[1 - \sqrt{\frac{z_{d\alpha}^2 - a^2}{z_\alpha^2 - a^2}} \right] - \frac{1}{\sqrt{z_\alpha^2 - a^2}} \right\rangle \mathbf{d} \\ & + \frac{1}{2\pi i} \bar{\mathbf{N}} \left\langle \frac{1}{z_\alpha - z_{d\alpha}} \left[1 - \sqrt{\frac{z_{d\alpha}^2 - a^2}{z_\alpha^2 - a^2}} \right] - \frac{1}{\sqrt{z_\alpha^2 - a^2}} \right\rangle \bar{\mathbf{d}} + \frac{z}{2\sqrt{z^2 - a^2}} \mathbf{c}_0^a(z). \end{aligned} \quad (6.98)$$

Substituting (6.98) into (6.33) and (6.34) and applying the residue theorem, one has

$$\mathbf{c}_0^a = \mathbf{0}, \quad (6.99)$$

$$\omega = \frac{1}{Y_{22}} \left\{ \mathbf{Y}^{(2)} \mathbf{u}_{,1}^\infty + \frac{4}{\pi a^2} \operatorname{Im} \left[\mathbf{Y}^{(2)} \mathbf{N} \left\langle \sqrt{z_{d\alpha}^2 - a^2} - z_{d\alpha} \right\rangle \mathbf{d} \right] \right\}. \quad (6.100)$$

In view of (6.22) and (6.98) with (6.99) and (6.100), using (6.10) and (6.14)-(6.17), and neglecting constant items, we have

$$\begin{aligned} \mathbf{F}_1(z) = & \frac{1}{2\pi i} \mathbf{A}_1^{-1} \left\{ \mathbf{A}_1 \left\langle \frac{1}{z_\alpha - z_{d\alpha}} \right\rangle \mathbf{d} - \mathbf{N} \left\langle \frac{1}{z_\alpha - z_{d\alpha}} \left[1 - \sqrt{\frac{z_{d\alpha}^2 - a^2}{z_\alpha^2 - a^2}} \right] - \frac{1}{\sqrt{z_\alpha^2 - a^2}} \right\rangle \mathbf{d} \right. \\ & + \bar{\mathbf{A}}_1 \left\langle \frac{1}{z_\alpha - \bar{z}_{d\alpha}} \right\rangle \bar{\mathbf{d}} - \bar{\mathbf{N}} \left\langle \frac{1}{z_\alpha - \bar{z}_{d\alpha}} \left[1 + \sqrt{\frac{\bar{z}_{d\alpha}^2 - a^2}{z_\alpha^2 - a^2}} \right] + \frac{1}{\sqrt{z_\alpha^2 - a^2}} \right\rangle \bar{\mathbf{d}} \\ & \left. + (\mathbf{u}_{,1}^\infty - \mathbf{\Omega}) \frac{i\pi z}{\sqrt{z^2 - a^2}} \right\}, \quad (6.101) \end{aligned}$$

$$\begin{aligned} \mathbf{F}_2(z) = & \frac{1}{2\pi i} \mathbf{A}_2^{-1} \left\{ \mathbf{N} \left\langle \frac{1}{z_\alpha - z_{d\alpha}} \left[1 + \sqrt{\frac{z_{d\alpha}^2 - a^2}{z_\alpha^2 - a^2}} \right] + \frac{1}{\sqrt{z_\alpha^2 - a^2}} \right\rangle \mathbf{d} \right. \\ & \left. + \bar{\mathbf{N}} \left\langle \frac{1}{z_\alpha - \bar{z}_{d\alpha}} \left[1 - \sqrt{\frac{\bar{z}_{d\alpha}^2 - a^2}{z_\alpha^2 - a^2}} \right] - \frac{1}{\sqrt{z_\alpha^2 - a^2}} \right\rangle \bar{\mathbf{d}} + (\mathbf{u}_{,1}^\infty - \mathbf{\Omega}) \frac{i\pi z}{\sqrt{z^2 - a^2}} \right\}. \quad (6.102) \end{aligned}$$

Substitution of (6.101) and (6.102) into (2.56) yields the complete solutions of the stress and electric displacement fields in s_1 and s_2 , respectively.

Substituting (6.98) into (6.46) with (6.94), (6.99) and (6.100), one obtains the generalized strain intensity factors at the tips of the rigid conducting line

$$\mathbf{k} = \sqrt{\pi a} (\mathbf{u}_{,1}^\infty - \mathbf{\Omega}) + \frac{2}{\sqrt{\pi a}} \operatorname{Im} \left[\mathbf{N} \left\langle 1 - \sqrt{\frac{z_{d\alpha} + a}{z_{d\alpha} - a}} \right\rangle \mathbf{d} \right]. \quad (6.103)$$

By substituting (6.98) into (6.47) with (6.94), (6.99) and (6.100), the generalized stress intensity factors at the tips of the rigid conducting line are obtained as

$$\mathbf{K} = \mathbf{S}_1^T \mathbf{H}_1^{-1} \mathbf{k} = \mathbf{S}_2^T \mathbf{H}_2^{-1} \mathbf{k}, \quad (6.104)$$

where $\mathbf{S}_1^T \mathbf{H}_1^{-1} = \mathbf{S}_2^T \mathbf{H}_2^{-1}$ for real $\mathbf{Y}$.

Substituting (6.103) into (6.54) yields the energy release rate

$$G = -\frac{\pi a}{4} (\mathbf{u}_{,1}^\infty - \mathbf{\Omega})^T \mathbf{Y} (\mathbf{\Omega} - \mathbf{u}_{,1}^\infty) - (\mathbf{u}_{,1}^\infty - \mathbf{\Omega})^T \mathbf{Y} \operatorname{Im} \left[\mathbf{N} \left\langle 1 - \sqrt{\frac{z_{d\alpha} + a}{z_{d\alpha} - a}} \right\rangle \mathbf{d} \right] \\ - \frac{1}{\pi a} \operatorname{Im} \left[\mathbf{d}^T \left\langle 1 - \sqrt{\frac{z_{d\alpha} + a}{z_{d\alpha} - a}} \right\rangle \mathbf{N}^T \right] \mathbf{Y} \operatorname{Im} \left[\mathbf{N} \left\langle 1 - \sqrt{\frac{z_{d\alpha} + a}{z_{d\alpha} - a}} \right\rangle \mathbf{d} \right] \quad (6.105)$$

for the rigid conducting line. Similarly, the substitution of (6.103) into (6.57) will lead to the mechanical strain energy release rate G_M . This solution procedure is not given in detail here.

6.4.2 A Single Rigid Dielectric Line

In this case, equation (6.42) reads

$$\hat{\mathbf{I}}(z) = (\hat{\mathbf{\Omega}} - \hat{\mathbf{u}}_{,1}^\infty) \left[\frac{1}{2} - \frac{z}{2\sqrt{z^2 - a^2}} \right] - \frac{1}{2\pi i} \hat{\mathbf{N}} \left\langle \frac{1}{z_\alpha - z_{d\alpha}} \left[1 - \sqrt{\frac{z_{d\alpha}^2 - a^2}{z_\alpha^2 - a^2}} \right] - \frac{1}{\sqrt{z_\alpha^2 - a^2}} \right\rangle \mathbf{d} \\ + \frac{1}{2\pi i} \bar{\hat{\mathbf{N}}} \left\langle \frac{1}{z_\alpha - \bar{z}_{d\alpha}} \left[1 - \sqrt{\frac{\bar{z}_{d\alpha}^2 - a^2}{z_\alpha^2 - a^2}} \right] - \frac{1}{\sqrt{z_\alpha^2 - a^2}} \right\rangle \bar{\mathbf{d}} + \frac{z}{2\sqrt{z^2 - a^2}} \mathbf{c}_0^b. \quad (6.106)$$

Substituting (6.106) into (6.44) and (6.45) and applying the residue theorem, we have

$$\mathbf{c}_0^b = 0, \quad (6.107)$$

$$\omega = \frac{1}{\hat{Y}_{2j}} \left\{ \sum_{j=1}^3 \hat{Y}_{2j} u_{j,1}^\infty + \frac{4}{\pi a^2} \operatorname{Im} \left[\hat{\mathbf{Y}} \hat{\mathbf{N}} \left\langle \sqrt{z_{d\alpha}^2 - a^2} - z_{d\alpha} \right\rangle \mathbf{d} \right] \right\}. \quad (6.108)$$

With the aid of (6.22), (6.37), and (6.106)-(6.108), using (6.10) and (6.14)-(6.17), and neglecting constants, we have

$$\begin{aligned} \mathbf{F}_1(z) = & \mathbf{A}_1^{-1} \left\{ \frac{1}{2\pi i} \mathbf{A}_1 \left\langle \frac{1}{z_\alpha - z_{d\alpha}} \right\rangle \mathbf{d} - \frac{1}{2\pi i} (2\bar{\mathbf{N}} - \bar{\mathbf{A}}_1) \left\langle \frac{1}{z_\alpha - \bar{z}_{d\alpha}} \right\rangle \bar{\mathbf{d}} \right\} \\ & - \frac{1}{2\pi i} \mathbf{A}_1^{-1} (\hat{\mathbf{I}} - \mathbf{E}) \left\{ \hat{\mathbf{N}} \left\langle \frac{1}{z_\alpha - z_{d\alpha}} \left[1 - \sqrt{\frac{z_{d\alpha}^2 - a^2}{z_\alpha^2 - a^2}} \right] - \frac{1}{\sqrt{z_\alpha^2 - a^2}} \right\rangle \mathbf{d} \right. \\ & \left. - \bar{\hat{\mathbf{N}}} \left\langle \frac{1}{z_\alpha - \bar{z}_{d\alpha}} \left[1 - \sqrt{\frac{\bar{z}_{d\alpha}^2 - a^2}{z_\alpha^2 - a^2}} \right] - \frac{1}{\sqrt{z_\alpha^2 - a^2}} \right\rangle \bar{\mathbf{d}} + (\hat{\boldsymbol{\Omega}} - \hat{\mathbf{u}}_{,1}^\infty) \frac{i\pi z}{\sqrt{z^2 - a^2}} \right\}. \quad (6.109) \end{aligned}$$

$$\begin{aligned} \mathbf{F}_2(z) = & \mathbf{A}_2^{-1} \left\{ \frac{1}{2\pi i} \mathbf{N} \left\langle \frac{2}{z_\alpha - z_{d\alpha}} \right\rangle \mathbf{d} + (\hat{\mathbf{I}} - \mathbf{E}) \left[-\frac{1}{2\pi i} \hat{\mathbf{N}} \left\langle \frac{1}{z_\alpha - z_{d\alpha}} \left[1 - \sqrt{\frac{z_{d\alpha}^2 - a^2}{z_\alpha^2 - a^2}} \right] \right. \right. \right. \\ & \left. \left. - \frac{1}{\sqrt{z_\alpha^2 - a^2}} \right\rangle \mathbf{d} + \frac{1}{2\pi i} \bar{\hat{\mathbf{N}}} \left\langle \frac{1}{z_\alpha - \bar{z}_{d\alpha}} \left[1 - \sqrt{\frac{\bar{z}_{d\alpha}^2 - a^2}{z_\alpha^2 - a^2}} \right] - \frac{1}{\sqrt{z_\alpha^2 - a^2}} \right\rangle \bar{\mathbf{d}} \right. \right. \\ & \left. \left. - (\hat{\boldsymbol{\Omega}} - \hat{\mathbf{u}}_{,1}^\infty) \frac{z}{2\sqrt{z^2 - a^2}} \right\}. \quad (6.110) \end{aligned}$$

The substitution of the above equations (6.109) and (6.110) into (2.56) finds the complete stress and electric displacement field solutions in s_1 and s_2 , respectively.

Substituting (6.106) into (6.49) with (6.95), (6.107) and (6.108), and using (6.48), one has the generalized strain intensity factors at the tips of the rigid dielectric line

$$\mathbf{k} = \sqrt{\pi a} (\hat{\mathbf{I}} - \mathbf{E}) (\hat{\mathbf{u}}_{,1}^\infty - \hat{\boldsymbol{\Omega}}) + \frac{2}{\sqrt{\pi a}} (\hat{\mathbf{I}} - \mathbf{E}) \operatorname{Im} \left[\hat{\mathbf{N}} \left\langle 1 - \sqrt{\frac{z_{d\alpha} + a}{z_{d\alpha} - a}} \right\rangle \mathbf{d} \right]. \quad (6.111)$$

Substituting (6.106) into (6.47) with (6.95), (6.107) and (6.108), one finds the generalized stress intensity factors at the tips of the rigid dielectric line

$$\mathbf{K} = \mathbf{S}_1^T \mathbf{H}_1^{-1} \mathbf{k} = \mathbf{S}_2^T \mathbf{H}_2^{-1} \mathbf{k}. \quad (6.112)$$

Substituting (6.111) into (6.54) leads to the energy release rate and the mechanical strain energy release rate for the rigid dielectric line

$$G = G_M = -\frac{\pi a}{4} (\hat{\mathbf{u}}_{,1}^\infty - \hat{\mathbf{\Omega}})^\top \hat{\mathbf{Y}} (\hat{\mathbf{u}}_{,1}^\infty - \hat{\mathbf{\Omega}}) - (\hat{\mathbf{u}}_{,1}^\infty - \hat{\mathbf{\Omega}})^\top \hat{\mathbf{Y}} \operatorname{Im} \left[\hat{\mathbf{N}} \left\langle 1 - \sqrt{\frac{z_{d\alpha} + a}{z_{d\alpha} - a}} \right\rangle \mathbf{d} \right] \\ - \frac{1}{\pi a} \operatorname{Im} \left[\mathbf{d}^\top \left\langle 1 - \sqrt{\frac{z_{d\alpha} + a}{z_{d\alpha} - a}} \right\rangle \hat{\mathbf{N}}^\top \right] \hat{\mathbf{Y}} \operatorname{Im} \left[\hat{\mathbf{N}} \left\langle 1 - \sqrt{\frac{z_{d\alpha} + a}{z_{d\alpha} - a}} \right\rangle \mathbf{d} \right]. \quad (6.113)$$

It is observed that the electric field applied at infinity cannot induce the rotation of the interfacial rigid dielectric line and makes no contribution to the singularities of the stress and electric displacement fields at the tips of the interfacial rigid dielectric line in the two half-infinite planes.

6.5 A Single Rigid Line: Square Root, Oscillatory and Non-square Root Singularity Field

In this section, let us consider a rigid line of length $2a$ centered along the x_1 axis, when $\mathbf{Y}$ is complex. The rotation of the rigid line is assumed to be ω . Equations (6.28) and (6.39) become (6.94) and (6.95), respectively. Equations (6.68) and (6.73) become

$$X_\alpha(z) = (z + a)^{-1/2+i\varepsilon_\alpha} (z - a)^{-1/2-i\varepsilon_\alpha}, \quad (6.114)$$

$$x_\alpha(\infty) = z - i2a\varepsilon_\alpha, \quad (6.115)$$

respectively. For simplicity, we define

$$q_1(z, \alpha) = \frac{z - i2a\varepsilon_\alpha}{\sqrt{z^2 - a^2}} \left[\frac{z + a}{z - a} \right]^{i\varepsilon_\alpha}, \quad (6.116)$$

$$q_2(z, z_{d\beta}, \alpha) = \frac{1}{z - z_{d\beta}} \left\{ 1 - \sqrt{\frac{z_{d\beta}^2 - a^2}{z^2 - a^2}} \left[\frac{z + a}{z - a} \frac{z_{d\beta} - a}{z_{d\beta} + a} \right]^{i\varepsilon_\alpha} \right\} - \frac{1}{\sqrt{z^2 - a^2}} \left[\frac{z + a}{z - a} \right]^{i\varepsilon_\alpha}. \quad (6.117)$$

6.5.1 A Single Rigid Conducting Line

Substituting (6.66) into (6.33) and (6.34) with (6.71), (6.72) ($n=1$), and (6.114)-(6.117), and applying the residue theorem, we have

$$\lambda_\alpha(z) = (\mathbf{W}^\alpha \mathbf{u}_{,1}^\infty - W_{\alpha 2} \omega) q_1(z, \alpha) - \frac{1}{i\pi} \sum_{j=1}^4 W_{\alpha j} \left[\sum_{\beta=1}^4 N_{j\beta} d_\beta q_2(z, z_{d\beta}, \alpha) - \sum_{k=1}^4 \bar{N}_{j\beta} \bar{d}_\beta q_2(z, \bar{z}_{d\beta}, \alpha) \right], \quad (6.118)$$

$$\omega = \rho_1 \sum_{k=1}^4 v_k \left\{ \sum_{j=1}^4 \frac{1}{i\pi a^2} W_{kj} \left[\sum_{r=1}^4 N_{jr} d_r \left[i2a\varepsilon_k - z_{dr} + (z_{dr} + a)^{\frac{1-i\varepsilon_k}{2}} (z_{dr} - a)^{\frac{1+i\varepsilon_k}{2}} \right] - \sum_{r=1}^4 \bar{N}_{jr} \bar{d}_r \left[i2a\varepsilon_k - \bar{z}_{dr} + (\bar{z}_{dr} + a)^{\frac{1-i\varepsilon_k}{2}} (\bar{z}_{dr} - a)^{\frac{1+i\varepsilon_k}{2}} \right] \right\} + \mathbf{W}^k \mathbf{u}_{j,1}^\infty \left(\frac{1}{2} + 2\varepsilon_k^2 \right) \right\}, \quad (6.119)$$

where

$$\rho_1 = 1 / \sum_{k=1}^4 v_k W_{k2} (1/2 + 2\varepsilon_k^2), \quad v_k = \frac{\Xi_k}{(1 + e^{2\pi\varepsilon_k}) v_k}, \quad (6.120)$$

with Ξ_k being the element of the 1×4 matrix $\Xi = \mathbf{Y}^{(2)} \mathbf{\Lambda}_A$. In view of (6.22) and (6.66)

with (6.116)-(6.120), using (6.10) and (6.14)-(6.17), and neglecting constants, we have

$$\mathbf{F}_1(z) = \frac{1}{2\pi i} \left[\left\langle \frac{1}{z_\alpha - z_{d\alpha}} \right\rangle \mathbf{d} - \mathbf{A}_1^{-1} (2\bar{\mathbf{N}} - \bar{\mathbf{A}}_1) \left\langle \frac{1}{z_\alpha - \bar{z}_{d\alpha}} \right\rangle \bar{\mathbf{d}} \right] + \sum_{\alpha=1}^4 \frac{\lambda_\alpha \mathbf{A}_1^{-1} \mathbf{\Lambda}_\alpha}{(1 + e^{2\pi\varepsilon_\alpha}) v_\alpha}, \quad (6.121)$$

$$\mathbf{F}_2(z) = \frac{1}{\pi i} \mathbf{A}_2^{-1} \mathbf{N} \left\langle \frac{1}{z_\alpha - z_{d\alpha}} \right\rangle \mathbf{d} + \mathbf{A}_2^{-1} \bar{\mathbf{Y}}^{-1} \mathbf{Y} \sum_{\alpha=1}^4 \frac{\lambda_\alpha \mathbf{\Lambda}_\alpha}{(1 + e^{2\pi\varepsilon_\alpha}) v_\alpha}. \quad (6.122)$$

The substitution of (6.121) and (6.122) into (2.56) yields the complete stress and electric displacement field solutions in s_1 and s_2 , respectively.

By substituting (6.118) into (6.76) with the aid of (6.116), (6.117), (6.119) and (6.120), the generalized strain intensity factors at the tips of the rigid conducting line are obtained as

$$\begin{aligned} \tilde{k}_\alpha = \frac{(2a)^{i\varepsilon_\alpha}}{v_\alpha} \left\{ \frac{1}{i\sqrt{\pi a}} \sum_{j=1}^4 W_{\alpha j} \left\{ \sum_{k=1}^4 N_{jk} d_k \left[1 - \sqrt{\frac{z_{dk}+a}{z_{dk}-a}} \left(\frac{z_{dk}-a}{z_{dk}+a} \right)^{i\varepsilon_\alpha} \right] \right. \right. \\ \left. \left. - \sum_{k=1}^4 \bar{N}_{jk} \bar{d}_k \left[1 - \sqrt{\frac{\bar{z}_{dk}+a}{\bar{z}_{dk}-a}} \left(\frac{\bar{z}_{dk}-a}{\bar{z}_{dk}+a} \right)^{i\varepsilon_\alpha} \right] \right\} + \sqrt{\pi a} \mathbf{W}^\alpha (\mathbf{u}_{,1}^\infty - \mathbf{\Omega})(1 - i2\varepsilon_\alpha) \right\}. \quad (6.123) \end{aligned}$$

The expression of the energy release rate G for the rigid conducting line can be obtained by substituting (6.123) into (6.78). This solution procedure is not given in detail here.

6.5.2 A Single Rigid Dielectric Line

Substituting (6.85) into (6.44) and (6.45) with (6.86), (6.87) ($n=1$), and (6.114)-(6.117), and applying the residue theorem, we have

$$\begin{aligned} \lambda_\alpha(z) = (\hat{\mathbf{W}}^\alpha \hat{\mathbf{u}}_{,1}^\infty - \hat{W}_{\alpha 2} \omega) q_1(z, \alpha) \\ - \frac{1}{i\pi} \sum_{j=1}^3 \hat{W}_{\alpha j} \left[\sum_{\beta=1}^4 N_{j\beta} d_\beta q_2(z, z_{d\beta}, \alpha) - \sum_{k=1}^4 \bar{N}_{j\beta} \bar{d}_\beta q_2(z, \bar{z}_{d\beta}, \alpha) \right], \quad (6.124) \\ \omega = \rho_2 \sum_{k=1}^3 \mathcal{G}_k \left\{ \frac{1}{i\pi a^2} \sum_{j=1}^3 \hat{W}_{kj} \left\{ \sum_{r=1}^4 N_{jr} d_r [i2a\varepsilon_k - z_{dr} + (z_{dr} + a)^{1/2-i\varepsilon_k} (z_{dr} - a)^{1/2+i\varepsilon_k}] \right. \right. \\ \left. \left. - \sum_{r=1}^4 \bar{N}_{jr} \bar{d}_r [i2a\varepsilon_k - \bar{z}_{dr} + (\bar{z}_{dr} + a)^{1/2-i\varepsilon_k} (\bar{z}_{dr} - a)^{1/2+i\varepsilon_k}] \right\} + \sum_{j=1}^3 \hat{W}_{kj} \hat{\mathbf{u}}_{j,1}^\infty (1/2 + 2\varepsilon_k^2) \right\}, \quad (6.125) \end{aligned}$$

where

$$\rho_2 = 1 / \sum_{k=1}^3 \mathcal{G}_k \hat{W}_{k2} (1/2 + 2\varepsilon_k^2), \quad \mathcal{G}_k = \frac{Q_k}{(1 + e^{2\pi\varepsilon_k}) v_k}, \quad (6.126)$$

with Q_k being the elements of the 1×3 matrix $\mathbf{Q} = \hat{\mathbf{Y}}^{(2)} \mathbf{\Lambda}_A$. In view of (6.22), (6.37) and (6.85) with (6.124), (6.116), (6.117), (6.125), and (6.126), using (6.10) and (6.14)-(6.17), and neglecting constants, we have

$$\begin{aligned} \mathbf{F}_1(z) = & \frac{1}{2\pi i} \left[\left\langle \frac{1}{z_\alpha - z_{d\alpha}} \right\rangle \mathbf{d} - \mathbf{A}_1^{-1} (2\bar{\mathbf{N}} - \bar{\mathbf{A}}_1) \left\langle \frac{1}{z_\alpha - \bar{z}_{d\alpha}} \right\rangle \bar{\mathbf{d}} \right] \\ & + \mathbf{A}_1^{-1} (\hat{\mathbf{I}} - \mathbf{E}) \sum_{\alpha=1}^3 \frac{\lambda_\alpha \mathbf{\Lambda}_\alpha}{(1 + e^{2\pi \varepsilon_\alpha}) v_\alpha}, \end{aligned} \quad (6.127)$$

$$\mathbf{F}_2(z) = \frac{1}{\pi i} \mathbf{A}_2^{-1} \mathbf{N} \left\langle \frac{1}{z_\alpha - z_{d\alpha}} \right\rangle \mathbf{d} + \mathbf{A}_2^{-1} \bar{\mathbf{Y}}^{-1} \mathbf{Y} (\hat{\mathbf{I}} - \mathbf{E}) \sum_{\alpha=1}^3 \frac{\lambda_\alpha \mathbf{\Lambda}_\alpha}{(1 + e^{2\pi \varepsilon_\alpha}) v_\alpha}. \quad (6.128)$$

The substitution of (6.127) and (6.128) into (2.56) yields the complete stress and electric displacement field solutions in s_1 and s_2 , respectively.

By substituting (6.124) into (6.76) with the aid of (6.116), (6.117), (6.125) and (6.126), the generalized strain intensity factors at the rigid dielectric line tip are obtained as

$$\begin{aligned} \tilde{k}_\alpha = & \frac{(2a)^{i\varepsilon_\alpha}}{v_\alpha} \left\{ \frac{1}{i\sqrt{\pi a}} \sum_{j=1}^3 \hat{W}_{\alpha j} \left\{ \sum_{k=1}^4 N_{jk} d_k \left[1 - \sqrt{\frac{z_{dk} + a}{z_{dk} - a}} \left(\frac{z_{dk} - a}{z_{dk} + a} \right)^{i\varepsilon_\alpha} \right] \right. \right. \\ & \left. \left. - \sum_{k=1}^4 \bar{N}_{jk} \bar{d}_k \left[1 - \sqrt{\frac{\bar{z}_{dk} + a}{\bar{z}_{dk} - a}} \left(\frac{\bar{z}_{dk} - a}{\bar{z}_{dk} + a} \right)^{i\varepsilon_\alpha} \right] + \sqrt{\pi a} \hat{\mathbf{W}}^\alpha (\hat{\mathbf{u}}_{,1}^\infty - \hat{\mathbf{\Omega}}) (1 - i2\varepsilon_\alpha) \right\} \right\}, \quad \alpha = 1, 2, 3. \end{aligned} \quad (6.129)$$

The expression of the energy release rate G and the mechanical strain release rate G_M for the interfacial rigid dielectric line can be obtained by substituting (6.129) into (6.93).

Similar to the case in which $\mathbf{Y}$ is real, the electric field applied at infinity does not affect the rotation of the single interfacial rigid dielectric line, as well as the stress and electric displacement fields in the two half-infinite planes with the single interfacial rigid dielectric line.

6.6 Image Force on the Dislocation

The image forces in the x_1 and x_2 directions on the dislocation can be obtained from the generalized Peach-Koehler formula (Pak, 1990b) as

$$F_1 = \sigma_{i2}^d b_i = \mathbf{b}^T \boldsymbol{\Phi}_{,1}^d, \quad (6.130)$$

$$F_2 = -\sigma_{i1}^d b_i = \mathbf{b}^T \boldsymbol{\Phi}_{,2}^d, \quad (6.131)$$

where the generalized stresses $\sigma_{i2}^d (\boldsymbol{\Phi}_{,1}^d)$ and $\sigma_{i1}^d (-\boldsymbol{\Phi}_{,2}^d)$ can be obtained by subtracting the fields generated from the dislocation when $z \rightarrow z_d$. Here we consider the case of a single rigid line of length $2a$ centered along the interface of two piezoelectric media in the absence of far-field loads.

6.6.1 A Single Rigid Conducting Line

For the case in which $\mathbf{Y}$ is real, from (6.101), (2.56) and (2.57), one has

$$\begin{aligned} \boldsymbol{\Phi}_{,1}^d = & \frac{1}{\pi} \text{Im}[\mathbf{B}_1 \mathbf{A}_1^{-1} \bar{\mathbf{A}}_1 \langle g_1(z_{d\alpha}) \rangle \bar{\mathbf{B}}_1^T \mathbf{b} - \mathbf{B}_1 \mathbf{A}_1^{-1} \mathbf{N} \langle g_2(z_{d\alpha}) \rangle \mathbf{B}_1^T \mathbf{b} \\ & - \mathbf{B}_1 \mathbf{A}_1^{-1} \bar{\mathbf{N}} \langle g_3(z_{d\alpha}) \rangle \bar{\mathbf{B}}_1^T \mathbf{b}] - \text{Re}[\mathbf{B}_1 \mathbf{A}_1^{-1} \boldsymbol{\Omega} g_4(z_d)], \end{aligned} \quad (6.132)$$

$$\begin{aligned} \boldsymbol{\Phi}_{,2}^d = & \frac{1}{\pi} \text{Im}[\mathbf{B}_1 \mathbf{A}_1^{-1} \bar{\mathbf{A}}_1 \langle p_\alpha g_1(z_{d\alpha}) \rangle \bar{\mathbf{B}}_1^T \mathbf{b} - \mathbf{B}_1 \mathbf{A}_1^{-1} \mathbf{N} \langle p_\alpha g_2(z_{d\alpha}) \rangle \mathbf{B}_1^T \mathbf{b} \\ & - \mathbf{B}_1 \mathbf{A}_1^{-1} \bar{\mathbf{N}} \langle p_\alpha g_3(z_{d\alpha}) \rangle \bar{\mathbf{B}}_1^T \mathbf{b}] - \text{Re}[\mathbf{B}_1 \langle p_\alpha \rangle \mathbf{A}_1^{-1} \boldsymbol{\Omega} g_4(z_d)], \end{aligned} \quad (6.133)$$

where

$$\begin{aligned} g_1(z_{d\alpha}) = & \frac{1}{z_{d\alpha} - \bar{z}_{d\alpha}}, \quad g_2(z_{d\alpha}) = \frac{z_{d\alpha} - \sqrt{z_{d\alpha}^2 - a^2}}{z_{d\alpha}^2 - a^2}, \\ g_3(z_{d\alpha}) = & \frac{z_{d\alpha} - \bar{z}_{d\alpha} + \sqrt{z_{d\alpha}^2 - a^2} + \sqrt{\bar{z}_{d\alpha}^2 - a^2}}{(z_{d\alpha} - \bar{z}_{d\alpha})\sqrt{z_{d\alpha}^2 - a^2}}, \quad g_4(z_d) = \frac{z_d}{\sqrt{z_d^2 - a^2}}, \end{aligned} \quad (6.134)$$

and the non-zero component ω of $\mathbf{\Omega}$ is obtained by deleting the first term from (6.100).

For the case in which $\mathbf{Y}$ is complex, we have, from (6.121), (2.56) and (2.57),

$$\Phi_{,1}^d = \frac{1}{\pi} \text{Im}[\mathbf{B}_1 \mathbf{A}_1^{-1} (\bar{\mathbf{A}}_1 - 2\bar{\mathbf{N}}) \langle g_1(z_{d\alpha}) \rangle \bar{\mathbf{B}}_1^T \mathbf{b}] + 2 \text{Re} \left[\mathbf{B}_1 \mathbf{A}_1^{-1} \sum_{\alpha=1}^4 \frac{\lambda_{\alpha}^d(z_d) \mathbf{\Lambda}_{\alpha}}{(1 + e^{2\pi \varepsilon_{\alpha}}) v_{\alpha}} \right], \quad (6.135)$$

$$\Phi_{,2}^d = \frac{1}{\pi} \text{Im}[\mathbf{B}_1 \mathbf{A}_1^{-1} (\bar{\mathbf{A}}_1 - 2\bar{\mathbf{N}}) \langle p_{\alpha} g_1(z_{d\alpha}) \rangle \bar{\mathbf{B}}_1^T \mathbf{b}] + 2 \text{Re} \left[\mathbf{B}_1 \langle p_{\alpha} \rangle \mathbf{A}_1^{-1} \sum_{\alpha=1}^4 \frac{\lambda_{\alpha}^d(z_d) \mathbf{\Lambda}_{\alpha}}{(1 + e^{2\pi \varepsilon_{\alpha}}) v_{\alpha}} \right], \quad (6.136)$$

where $\lambda_{\alpha}^d(z_d)$ can be found from (6.118) by letting $z \rightarrow z_d$ and the non-zero component ω of $\mathbf{\Omega}$ is obtained by deleting the last term from (6.119).

6.6.2 A Single Rigid Dielectric Line

For the case in which $\mathbf{Y}$ is real, we obtain, from (6.108), (2.56) and (2.57)

$$\begin{aligned} \Phi_{,1}^d &= \frac{1}{\pi} \text{Im}[\mathbf{B}_1 \mathbf{A}_1^{-1} (\bar{\mathbf{A}}_1 - 2\bar{\mathbf{N}}) \langle g_1(z_{d\alpha}) \rangle \bar{\mathbf{B}}_1^T \mathbf{b} - \mathbf{B}_1 \mathbf{A}_1^{-1} (\hat{\mathbf{I}} - \mathbf{E}) \hat{\mathbf{N}} \langle g_2(z_{d\alpha}) \rangle \bar{\mathbf{B}}_1^T \mathbf{b} \\ &\quad + \mathbf{B}_1 \mathbf{A}_1^{-1} (\hat{\mathbf{I}} - \mathbf{E}) \bar{\hat{\mathbf{N}}} \langle g_5(z_{d\alpha}) \rangle \bar{\mathbf{B}}_1^T \mathbf{b}] - \text{Re}[\mathbf{B}_1 \mathbf{A}_1^{-1} (\hat{\mathbf{I}} - \mathbf{E}) \hat{\mathbf{\Omega}} g_4(z_d)], \end{aligned} \quad (6.137)$$

$$\begin{aligned} \Phi_{,2}^d &= \frac{1}{\pi} \text{Im}[\mathbf{B} \mathbf{A}_1^{-1} (\bar{\mathbf{A}}_1 - 2\bar{\mathbf{N}}) \langle p_{\alpha} g_1(z_{d\alpha}) \rangle \bar{\mathbf{B}}_1^T \mathbf{b} - \mathbf{B} \mathbf{A}_1^{-1} (\hat{\mathbf{I}} - \mathbf{E}) \hat{\mathbf{N}} \langle p_{\alpha} g_2(z_{d\alpha}) \rangle \bar{\mathbf{B}}_1^T \mathbf{b} \\ &\quad + \mathbf{B} \mathbf{A}_1^{-1} (\hat{\mathbf{I}} - \mathbf{E}) \bar{\hat{\mathbf{N}}} \langle p_{\alpha} g_5(z_{d\alpha}) \rangle \bar{\mathbf{B}}_1^T \mathbf{b}] - \text{Re}[\mathbf{B} \langle p_{\alpha} \rangle \mathbf{A}_1^{-1} (\hat{\mathbf{I}} - \mathbf{E}) \hat{\mathbf{\Omega}} g_4(z_d)], \end{aligned} \quad (6.138)$$

where

$$g_5(z_{d\alpha}) = \frac{z_{d\alpha} + \bar{z}_{d\alpha} - \sqrt{z_{d\alpha}^2 - a^2} - \sqrt{\bar{z}_{d\alpha}^2 - a^2}}{(\sqrt{z_{d\alpha}^2 - a^2} + \sqrt{\bar{z}_{d\alpha}^2 - a^2}) \sqrt{z_{d\alpha}^2 - a^2}}, \quad (6.139)$$

and the non-zero element ω of $\hat{\mathbf{\Omega}}$ is obtained by deleting the first term from (6.108).

For the case in which $\mathbf{Y}$ is complex, from (6.127), (2.56) and (2.57), one has

$$\Phi_{,1}^d = \frac{1}{\pi} \text{Im} \left[\mathbf{B}_1 \mathbf{A}_1^{-1} (\bar{\mathbf{A}}_1 - 2\bar{\mathbf{N}}) \langle g_1(z_{d\alpha}) \rangle \bar{\mathbf{B}}_1^T \mathbf{b} \right] + 2 \text{Re} \left[\mathbf{B}_1 \mathbf{A}_1^{-1} (\hat{\mathbf{I}} - \mathbf{E}) \sum_{\alpha=1}^3 \frac{\lambda_{\alpha}^d(z_d) \mathbf{\Lambda}_{\alpha}}{(1 + e^{2\pi\epsilon_{\alpha}}) v_{\alpha}} \right], \quad (6.140)$$

$$\begin{aligned} \Phi_{,2}^d &= \frac{1}{\pi} \text{Im} \left[\mathbf{B}_1 \mathbf{A}_1^{-1} (\bar{\mathbf{A}}_1 - 2\bar{\mathbf{N}}) \langle p_{\alpha} g_1(z_{d\alpha}) \rangle \bar{\mathbf{B}}_1^T \mathbf{b} \right] \\ &+ 2 \text{Re} \left[\mathbf{B}_1 \langle p_{\alpha} \rangle \mathbf{A}_1^{-1} (\hat{\mathbf{I}} - \mathbf{E}) \sum_{\alpha=1}^3 \frac{\lambda_{\alpha}^d(z_d) \mathbf{\Lambda}_{\alpha}}{(1 + e^{2\pi\epsilon_{\alpha}}) v_{\alpha}} \right], \end{aligned} \quad (6.141)$$

where $\lambda_{\alpha}^d(z_d)$ can be found from (6.124) by letting $z \rightarrow z_d$ and the non-zero ω of $\hat{\mathbf{\Omega}}$ is obtained by deleting the last term from (6.125).

6.7 Numerical Examples and Discussions

In this section, numerical examples are performed to show the singularity types of the near-tip stress and electric displacement field, as well as to examine the effects of such parameters as the Burgers vector components b_1 , b_2 , b_{φ} and the position of the mixed dislocation on the image force. As an example, BaTiO₃ is considered for s_1 and PZT-6B for s_2 , whose material coefficients are listed in Table 6.1 (Ou and Wu, 2003). Poling directions of the two materials are assumed to be along the x_2 axis. Based on the material constants given, we obtain

$$\epsilon = 0, \quad \kappa = 0.03528, \quad (6.142)$$

$$\mathbf{\Lambda}_A = \begin{bmatrix} 0 & 0 & 3.506 \times 10^{-10} & -3.506 \times 10^{-10} \\ 0 & -1.6807 \times 10^{-9} & -2.245 \times 10^{-10} & -2.245 \times 10^{-10} \\ 1 & 0 & 0 & 0 \\ 0 & 0.9999 & 0.9999 & 0.9999 \end{bmatrix} \quad (6.143)$$

for the conducting case, and

$$\varepsilon = 0.02439, \quad (6.144)$$

$$\Lambda_A = \begin{bmatrix} 0.7315 & 0.7315 & 0 \\ 0.6818i & -0.6818i & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (6.145)$$

for the dielectric case. For BaTiO₃/PZT-6B bimaterial, it is found from (6.59) and (6.144) that the stress and electric displacement fields at the rigid conducting line tips present square root and non-square root singularities. It is also seen from (6.81) and (6.144) that the stress and electric displacement fields at the rigid dielectric line tips have square root and oscillatory singularities, for BaTiO₃/PZT-6B and other piezoelectric materials.

In a polar coordinate system (r, θ, x_3) , we write $z_d = r_d e^{i\theta_d}$. The cylindrical components of the image force are given by

$$F_t = -F_1 \sin[\theta_d] + F_2 \cos[\theta_d], \quad (6.146)$$

$$F_r = F_1 \cos[\theta_d] + F_2 \sin[\theta_d]. \quad (6.147)$$

Table 6.1
Material coefficients for piezoelectric ceramics

		PZT-6B	BaTiO ₃
c_{11}	10^{10} Nm^{-2}	16.8	15.0
c_{12}	10^{10} Nm^{-2}	6.00	6.60
c_{13}	10^{10} Nm^{-2}	6.00	6.60
c_{33}	10^{10} Nm^{-2}	16.3	14.6
c_{44}	10^{10} Nm^{-2}	2.71	4.40
e_{31}	Cm^{-2}	-0.90	-4.35
e_{33}	Cm^{-2}	7.10	17.5
e_{15}	Cm^{-2}	4.60	11.4
κ_{11}	$10^{-10} \text{ C(Vm)}^{-1}$	36.0	98.7
κ_{33}	$10^{-10} \text{ C(Vm)}^{-1}$	34.0	112

The poling direction is along the x_3 axis.

The normalizing factors are taken as

$$F_0(b_i) = \frac{1}{4\pi a} \mathbf{b}^T \mathbf{L}_1 \mathbf{b}, \quad i = 1, 2, 4, \quad (6.148)$$

for non-zero b_i . Thus the normalized tangential and radial components of the image force are given by

$$F_t^*(b_i) = F_t(b_i)/F_0(b_i), \quad F_r^*(b_i) = F_r(b_i)/F_0(b_i), \quad i = 1, 2, 4, \quad (6.149)$$

where a positive term contributes to a repulsive force and a negative term to an attractive one. Normalized forces $F_t^*(b_1)$, $F_r^*(b_1)$, $F_t^*(b_2)$, $F_r^*(b_2)$, $F_t^*(b_\varphi)$ and $F_r^*(b_\varphi)$ versus r_d/a are depicted in Figs. 6.2-6.7, respectively, when $\theta_d = \pi/8$. Normalized forces $F_t^*(b_1)$, $F_r^*(b_1)$, $F_t^*(b_2)$, $F_r^*(b_2)$, $F_t^*(b_\varphi)$ and $F_r^*(b_\varphi)$ versus θ_d , when $r_d = 1.5a$, are plotted in Figs. 6.8-6.13, respectively.

In the tangential direction, it is seen from Fig. 6.2 that the interface and the rigid dielectric line attract the dislocation with b_1 at a point (r_d, θ_d) when $r_d/a < \hat{r}_1^*$, and repel the dislocation with b_1 at a point (r_d, θ_d) when $r_d/a > \hat{r}_1^*$. The same phenomenon can be observed when $\hat{r}_1^*$ is replaced by $\hat{r}_2^*$ ($\hat{r}_2^* > \hat{r}_1^*$) for the conducting case. It is also found that the attractive force is stronger and the repulsive force is weaker for the conducting case than that for the dielectric case. The opposite phenomenon above is observed from Fig. 6.3 in the radial direction, where $\hat{r}_1^*$ and $\hat{r}_2^*$ are replaced by $\hat{r}_3^*$ and $\hat{r}_4^*$ ($\hat{r}_4^* > \hat{r}_3^*$), respectively.

As shown in Fig. 6.4, in the tangential direction, the interface and the rigid conducting line always attract the dislocation with b_2 . However, the interface and the rigid dielectric

line repel the dislocation with b_2 at a point (r_d, θ_d) when $r_d/a < \hat{r}_5^*$ or $r_d/a > \hat{r}_6^*$ ($\hat{r}_6^* > \hat{r}_5^*$), and attract the dislocation with b_2 at a point (r_d, θ_d) when $\hat{r}_5^* < r_d/a < \hat{r}_6^*$, in the tangential direction. Moreover from Fig. 6.4, it is noted that the attractive force on a dislocation with b_2 is stronger for the conducting case than that for the dielectric case, along the tangential direction. We can observe from Fig. 6.5 that the interface and the rigid dielectric line always repel the dislocation with b_2 , in the radial direction. However, the interface and the rigid conducting line repel the dislocation with b_2 at a point (r_d, θ_d) when $r_d/a < \hat{r}_7^*$, and attract the dislocation with b_2 at a point (r_d, θ_d) when $r_d/a > \hat{r}_7^*$, in the radial direction. Also it can be observed that the rigid dielectric line repels the dislocation with b_2 more strongly than the rigid conducting line does.

Fig. 6.2 Normalized tangential force $F_t^*(b_1)$ versus r_d/a for $\theta_d = \pi/8$

Fig. 6.3 Normalized radial force $F_r^*(b_1)$ versus r_d/a for $\theta_d = \pi/8$

Fig. 6.4 Normalized tangential force $F_t^*(b_2)$ versus r_d/a for $\theta_d = \pi/8$

Fig. 6.5 Normalized radial force $F_r^*(b_2)$ versus r_d/a for $\theta_d = \pi/8$

In the tangential direction, it is clear from Fig. 6.6 that the rigid conducting line induces stronger attractive force on the dislocation with b_ϕ than the rigid dielectric line does. In the radial direction, it is observed from Fig. 6.7 that the interface and the rigid dielectric line always attract the dislocation with b_ϕ . However, the interface and the rigid conducting line repel the dislocation with b_ϕ at a point (r_d, θ_d) when $r_d/a < \hat{r}_8^*$ and attract the dislocation with b_ϕ at a point (r_d, θ_d) when $r_d/a > \hat{r}_8^*$. Further we observe from Fig. 6.7 that the rigid dielectric line always attracts the dislocation with b_ϕ more strongly than the rigid conducting line does in the radial direction.

It is seen from Fig. 6.8 that the value of the tangential force on a dislocation with b_1 away from the interface is always larger for the dielectric case than that for the conducting case, which means that the repulsive force is stronger and the attractive force is weaker due to

the rigid dielectric line than that due to the rigid conducting line. From Fig. 6.9, it is observed that the value of the radial force on a dislocation with b_1 at a point (r_d, θ_d) in which $\theta_d < 90^\circ$ is always larger for the conducting case than that for the dielectric case. However, the opposite phenomenon is observed when $90^\circ < \theta_d < 180^\circ$.

The force on dislocation with b_2 which repels the dislocation away from the interface, as is seen from Fig. 6.10, is stronger due to a rigid dielectric line than that due to a rigid conducting line in the tangential direction. The same phenomenon is observed from Fig. 6.11 for the case $\theta_d < 90^\circ$ in the radial direction. It is also observed that the value of the radial force increases with the increasing θ_d ($\theta_d < 90^\circ$ or $\theta_d > 90^\circ$).

Fig. 6.6 Normalized tangential force $F_t^*(b_\varphi)$ versus r_d/a for $\theta_d = \pi/8$

Fig. 6.7 Normalized radial force $F_r^*(b_\varphi)$ versus r_d/a for $\theta_d = \pi/8$

Fig. 6.8 Normalized tangential force $F_t^*(b_1)$ versus θ_d for $r_d = 1.5a$

Fig. 6.9 Normalized radial force $F_r^*(b_1)$ versus θ_d for $r_d = 1.5a$

It is observed from Fig. 6.12 that the attractive force on a dislocation with b_φ at a point (r_d, θ_d) when $\theta_d < 90^\circ$ is always stronger for the rigid conducting line case than that for the rigid dielectric line case, in the tangential direction. It is also observed from Fig. 6.12, the rigid dielectric line repels a dislocation with b_φ at a point (r_d, θ_d) more strongly than the rigid conducting line does, when $90^\circ < \theta_d < \theta_d^*$. The opposite phenomenon can be observed at a point (r_d, θ_d) , when $\theta_d^* < \theta_d < 180^\circ$. It is worth noting that the value of the tangential force increases with the increase in θ_d . Figure 6.13 clearly shows that the interface and the rigid dielectric line always attract a dislocation with b_φ in the radial direction, and the value of the radial force is always larger for the conducting case than that for the dielectric case.

It is worth noting from Figs. 6.2-6.7 that, the image forces on the dislocation, which moves it away from the interface along the radial direction, are approaching the same value gradually and become zero ultimately for the two cases. This is due to the fact that the remarkable distance between the dislocation and the interface is weakening the interaction between the dislocation and the rigid lines. It can be observed from Figs. 6.8, 6.10, and 6.12 that the tangential forces on the dislocation near the interface approach the same for the two cases, which indicates that the interaction between the dislocation and the interface is dominant. From Figs. 6.11 and 6.13, the great change of the radial force acting on a dislocation near the point $z^*(1.5a, 90^\circ)$ is observed due to the electroelastic interaction. The dislocation with b_1 at $z^*(1.5a, 90^\circ)$ for the dielectric and conducting cases being in equilibrium is also observed in the examples.

Fig. 6.10 Normalized tangential force $F_t^*(b_2)$ versus θ_d for $r_d = 1.5a$

Fig. 6.11 Normalized radial force $F_r^*(b_2)$ versus θ_d for $r_d = 1.5a$

Fig. 6.12 Normalized tangential force $F_t^*(b_\varphi)$ versus θ_d for $r_d = 1.5a$

Fig. 6.13 Normalized radial force $F_r^*(b_\varphi)$ versus θ_d for $r_d = 1.5a$

6.8 Conclusion

The Stroh formalism was employed to analyze the interaction between a piezoelectric dislocation and the collinear rigid lines, which are either conducting or dielectric, at the interface of two piezoelectric media. The general solutions for the field variables were obtained. A square root singularity was identified for the stress and electric displacement fields at the tips of interfacial rigid conducting/dielectric lines in the two piezoelectric media whose material coefficients make the matrix $\mathbf{Y}$ real. It was further found that the stress and electric displacement fields near the tips of interfacial rigid conducting lines present either a combination of square root and oscillatory singularities or a combination of square root and non-square root singularities, in transversely isotropic piezoelectric

media where the matrix $\mathbf{Y}$ is complex, or a combination of oscillatory and non-square root singularities, in piezoelectric media whose material coefficients do not match so suitably that the real numbers $\varepsilon \neq 0$ and $\kappa \neq 0$. The fields at the tips of rigid dielectric lines have square root and oscillatory singularities, at the interface of a piezoelectric bimaterial system where the matrix $\mathbf{Y}$ is complex. The strain and electric field intensity factors were introduced to characterize the near-tip field singularity. The expressions of the rigid line extension forces, based on the energy release rate and the mechanical energy release rate, were derived. The exact field solutions for the case of a single interfacial rigid line were presented. The effects of the Burgers vector and the dislocation position on the image force were analyzed in detail.

CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS

7.1 Conclusions

Four physical problems have been solved in this thesis. 1) Micro crack initiation at the tip of a semi-infinite rigid conducting line in a piezoelectric solid. 2) Micro crack initiation at the tips of a finite rigid conducting line in a piezoelectric medium. 3) Interaction of a piezoelectric screw dislocation with collinear interfacial rigid conducting/dielectric lines in piezoelectric media. 4) Interaction of a piezoelectric mixed dislocation with collinear rigid conducting/dielectric lines at the interface of a piezoelectric bimaterial system.

In the first problem, a criterion of micro crack initiation at the tip of a semi-infinite rigid conducting line in a piezoelectric medium was proposed for plane case, based on the evaluation of the mechanical strain energy release rate for a rigid line. The initiated micro crack was represented by a continuous distribution of piezoelectric dislocations along the crack line. The crack problem was accordingly formulated into a set of singular integral equations. These equations were solved by following the method of Erdogan and Gupta (1972) and Erdogan et al. (1973). The analysis provided the crack extension force in

terms of the strain and electric field intensity factors at the tip of the semi-infinite rigid conducting line and the net Burgers vector inside the crack. Accordingly, an important result is that the critical extension force for the semi-infinite rigid conducting line is related to the critical crack extension force in the same piezoelectric solid for plane case. In addition, two critical crack lengths were observed. One is stable under the Zener-Stroh mechanism, and the other is unstable under the Griffith mechanism.

Similarly, the criteria of micro crack initiation at the tips of a finite rigid conducting line in a piezoelectric solid for both antiplane and plane cases were proposed in the solution of the second physical problem, respectively. A similar analysis gave the crack extension forces for the antiplane and plane cases in terms of the corresponding strain and electric field intensity factors at the tip of the finite rigid line and the corresponding net Burgers vector inside the crack. As a result, the connections were obtained between the critical extension force of a finite rigid conducting line and that of a same size crack in the same piezoelectric medium, for both antiplane and plane cases, respectively. The connection constant for the plane case is consistent with that obtained in the first problem, when the crack is very small compared to the finite rigid line. Also two critical crack lengths were observed. One is stable under the Zener-Stroh mechanism, and the other is unstable under the Griffith mechanism.

In the third physical problem, a theoretical analysis was performed for the interaction of a piezoelectric screw dislocation with collinear rigid lines at the interface of a piezoelectric bimaterial system under far-field antiplane stress and in-plane electric displacement. The dislocation core is subjected to a line force and a line charge. The rigid lines are either conducting or dielectric. By using the complex potential method, the general solutions for

the field variables were obtained. It was identified that the near tip fields exhibit a square root singularity. The explicit solutions of the field variables were presented for the special case of a single interfacial rigid line. The driving forces on the rigid line were calculated, based on the energy release rate and the mechanical strain energy release rate. The image force acting on the screw dislocation was evaluated. Numerical examples were performed to analyze the influence of the Burgers vector and the line load on the rigid line extension force. The effects of such parameters as the angular position of the dislocation, material coefficients and electromechanical coupling factor on the image force were also studied.

The fourth physical problem was on the interaction of a piezoelectric mixed dislocation with collinear rigid conducting/dielectric lines at the interface of two piezoelectric media. Based on the Stroh formalism, the general solutions for the field variables were obtained. The stress and electric displacement fields at the tip of rigid conducting/dielectric lines are shown to have a square root singularity, at the interface of the two piezoelectric media having certain symmetry so that the matrix $\mathbf{Y}$ is real. It was further identified that the stress and electric displacement fields at the tips of rigid conducting lines exhibit either a combination of square root and non-square root singularities or a combination of square root and oscillatory singularities, at the interface of a transversely isotropic piezoelectric bimaterial system where the matrix $\mathbf{Y}$ is complex, or a combination of non-square root and oscillatory singularities, at the interface of two piezoelectric media whose material properties do not match so suitably that the real numbers $\varepsilon \neq 0$ and $\kappa \neq 0$. The fields at the tips of interfacial rigid dielectric lines have square root and oscillatory singularities, in a piezoelectric bimaterial system where the matrix $\mathbf{Y}$ is complex. The strain and electric field intensity factors were introduced to characterize the near-tip field singularity. The expressions of the rigid line extension forces based on the evaluations of the energy

release rate and the mechanical strain energy release rate were derived. The exact field solutions for the special case of a single interfacial rigid line were presented. Numerical analysis was given for the effects of the Burgers vector and the dislocation position on the image force.

7.2 Recommendations

It is known that the fracture behavior of the piezoelectric ceramics is quite complex as it involves piezoelectricity and internal strain. A number of theoretical and experimental investigations on the problems of defects and defect interactions in piezoelectric solids have been conducted in recent decades. However, there is no single fracture criterion accepted by the researchers in this field so far. In order to provide a better understanding of the fracture behavior of piezoelectric ceramics, following recommendations are made for possible future studies.

Fig. 7.1 Interaction of a crack with interfacial rigid lines in piezoelectric solids

Noting that micro crack initiation at some angle may occur at the tips of a rigid line in a piezoelectric medium under the mixed loads for the plane case, the first possible problem is on kinked micro crack initiation at the tips of a finite and semi-infinite rigid line in a piezoelectric medium. The rigid line is considered to be either conducting or dielectric. The approach in Chapters 3 and 4 can be employed to analyze the kinked micro crack, if the rigid line extension force is evaluated as the mechanical strain energy release rate of per infinitesimal translation in the inclined direction.

The second possible problem is to examine the interaction between a crack and collinear rigid lines at the interface of two piezoelectric materials, as shown in Fig. 7.1. The rigid lines are considered to be either conducting or dielectric. Both the Zener-Stroh crack and the Griffith crack can be considered, since they represent different fracture mechanisms in predicting crack propagation. The crack can be modeled as an electrically impermeable, permeable or conducting slit, and simulated by distributing piezoelectric screw and edge dislocations along the crack line for both antiplane and plane cases, respectively. Thus, the solutions for a piezoelectric screw and edge dislocation interacting with interfacial rigid lines obtained in Chapters 5 and 6 can be used as the Green's function in integral equation formulation of cracks for the antiplane and plane cases, respectively. Also, the interaction between an interfacial crack (Zener-Stroh and Griffith crack) and interfacial rigid lines should be investigated. It is known that the presence of rigid lines and cracks at the interface play an important role in fracture behavior.

Associated with the traditional square root, oscillatory and non-square root singularities of the field variables at the tip of an interfacial rigid line, the third possible problem is to analyze a micro crack kinking out of an interface with the aim of providing the fracture

mechanics needed to assess whether an interfacial micro crack will be initiated along the interface, or at some angle kinking out of the interface, as shown in Fig. 7.2. In addition, micro crack initiation at the tip of an interfacial rigid line for antiplane case is worth to study. Similarly, the initiated micro crack can be simulated by a continuous distribution of piezoelectric screw and edge dislocations along the crack line for both antiplane and plane cases, respectively. The Green's functions have been obtained in Chapters 5 and 6.

Although there are many experimental measurements in open literature, the available experimental data are scattered and limited. Therefore, more experimental measurements are needed to verify the fracture criteria of piezoelectric ceramics, such as the stress and electric displacement intensity factors, the energy release rate and the mechanical strain energy release rate, combined with the electric boundary conditions. Also the results for the critical rigid line extension forces obtained in Chapters 3 and 4 need experimental support for further investigation.

Fig. 7.2 A kinked micro crack initiated at the tip of an interfacial rigid line in piezoelectric media

In addition, the three-dimensional problems of interactions between a dislocation and cracks/inclusions in a transversely isotropic piezoelectric bimaterial system and multi-layer media are of great importance and worth to study. Inclusions can be used to model penny-shaped electrode or other piezoelectric/dielectric medium. Penny-shaped cracks can be considered to be electrically impermeable, permeable or conducting. Penny-shaped cracks and inclusions can be considered to be either normal or parallel to the poling axis of piezoelectric materials.

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APPENDIX A

In a rectangular coordinate system (x_1, x_2, x_3) , the x_2 axis is set along the poling axis.

Based on the transformation laws for tensors (Nye, 1985), the constitutive relations for poled piezoelectric ceramic with the x_2 axis as the poling axis are of the form:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{21} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{13} & c_{12} & 0 & 0 & 0 \\ c_{13} & c_{33} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{13} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(c_{11}-c_{12})}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{32} \\ 2\varepsilon_{31} \\ 2\varepsilon_{21} \end{bmatrix} - \begin{bmatrix} 0 & e_{31} & 0 \\ 0 & e_{33} & 0 \\ 0 & e_{31} & 0 \\ 0 & 0 & e_{15} \\ 0 & 0 & 0 \\ e_{15} & 0 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \quad (\text{A.1})$$

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & e_{15} \\ e_{31} & e_{33} & e_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{32} \\ 2\varepsilon_{31} \\ 2\varepsilon_{21} \end{bmatrix} - \begin{bmatrix} \kappa_{11} & 0 & 0 \\ 0 & \kappa_{33} & 0 \\ 0 & 0 & \kappa_{11} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \quad (\text{A.2})$$

where the transformation matrix is

$$(\Omega_{ij}) = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \quad (\text{A.3})$$

APPENDIX B

Once the dislocation density functions are ascertained, the singular behavior of the generalized stresses can be established very well. The generalized stresses are given by

$$\begin{aligned} & [\sigma_{21}(at+a,0), \sigma_{22}(at+a,0), \sigma_{23}(at+a,0), D_2(at+a,0)]^T \\ &= \frac{1}{4\pi} \int_{-1}^1 \mathbf{L}\hat{\mathbf{B}}(s) \frac{2}{t-s} ds + \frac{1}{4\pi} \int_{-1}^1 \mathbf{W}\hat{\mathbf{B}}(s) g(t,s) ds + \frac{1}{\sqrt{2\pi a}} \frac{1}{\sqrt{1+t}} \mathbf{K}', |t| > 1. \end{aligned} \quad (\text{B.1})$$

Let

$$\frac{1}{\pi} \int_{-1}^1 \frac{1}{t-s} \hat{\mathbf{B}}(s) ds = \mathbf{h}(t), \quad |t| < 1, \quad -1 < s < 1, \quad (\text{B.2})$$

where $\mathbf{h}(t) = [h_1(t), h_2(t), h_3(t), h_4(t)]^T$. In view of (3.16), the inversion of (B.2) is then given by

$$\hat{\mathbf{B}}(s) = -\frac{1}{\pi} \frac{1}{\sqrt{1-s^2}} \int_{-1}^1 \mathbf{h}(r) \frac{1}{s-r} \sqrt{1-r^2} dr + \hat{\mathbf{C}} \frac{1}{\sqrt{1-s^2}}. \quad (\text{B.3})$$

For a Zener-Stroh crack, when the dislocation density function $\hat{B}_i(s)$ ($i=1,2,3,4$) is symmetric, the corresponding function $h_i(r)$ ($i=1,2,3,4$) is anti-symmetric (Weertman, 1996). Therefore, substituting the first term on the right hand side of (B.3) into (3.16) make no contribution to the integral result. One then finds

$$\hat{\mathbf{B}}(s) = -\frac{1}{\pi} \frac{1}{\sqrt{1-s^2}} \int_{-1}^1 \mathbf{h}(r) \frac{1}{s-r} \sqrt{1-r^2} dr + \frac{1}{\pi a} \frac{1}{\sqrt{1-s^2}} \mathbf{b}', \quad (\text{B.4})$$

which leads to

$$\begin{aligned}
 & \frac{1}{\pi} \int_{-1}^1 \frac{\hat{\mathbf{B}}(s)}{t-s} ds \\
 &= \frac{1}{\pi} \left\{ \int_{-1}^1 \left[-\frac{1}{\pi \sqrt{1-s^2} (t-s)(r-s)} \right] ds \left[\int_{-1}^1 \mathbf{h}(r) \sqrt{1-r^2} dr \right] + \int_{-1}^1 \mathbf{b}^t \frac{1}{\pi a \sqrt{1-s^2} (t-s)} ds \right\} \\
 &= -\frac{1}{\pi} \frac{1}{\sqrt{t^2-1}} \int_{-1}^1 \mathbf{h}(r) \frac{1}{t-r} \sqrt{1-r^2} dr + \frac{1}{\pi} \frac{1}{\sqrt{t^2-1}} \frac{\mathbf{b}^t}{a} \\
 &= \frac{\Psi(t)}{\sqrt{t^2-1}}, \quad |t| > 1. \tag{B.5}
 \end{aligned}$$

Defining the generalized stress intensity factor in a conventional manner, we have

$$\mathbf{K} = \lim_{x_1 \rightarrow 2a} \sqrt{2\pi(x_1 - 2a)} [\sigma_{21}(x_1, 0), \sigma_{22}(x_1, 0), \sigma_{23}(x_1, 0), D_2(x_1, 0)]^T, \quad |x_1| > 2a. \tag{B.6}$$

The singularity of the crack tip stress is contributed by the first term on the right hand side of (B.1). With the aid of (B.5), (B.6) reads,

$$\mathbf{K} = \lim_{t \rightarrow 1^-} \sqrt{\frac{\pi a}{2}} \sqrt{1-t} \mathbf{L} \hat{\mathbf{B}}(t). \tag{B.7}$$

APPENDIX C

$$X_{11} = \hat{L}_{11}(d_{21})^2. \quad (\text{C.1})$$

$$X_{22} = \hat{L}_{22}(d_{11})^2 + \hat{L}_{23}d_{11}d_{41}. \quad (\text{C.2})$$

$$X_{23} = X_{32} = \hat{L}_{22}d_{11}d_{12} + \frac{1}{2}\hat{L}_{23}(d_{12}d_{41} + d_{11}d_{42}). \quad (\text{C.3})$$

$$X_{33} = \hat{L}_{22}(d_{12})^2 + \hat{L}_{23}d_{12}d_{42}. \quad (\text{C.4})$$

$$X_{12} = X_{13} = X_{21} = X_{31} = 0. \quad (\text{C.5})$$

$$Y_{11} = \hat{L}_{11}d_{21}d_{22}. \quad (\text{C.6})$$

$$Y_{22} = \hat{L}_{22}d_{11}d_{13} + \frac{1}{2}\hat{L}_{23}(d_{11}d_{43} + d_{41}d_{13}). \quad (\text{C.7})$$

$$Y_{23} = \hat{L}_{22}d_{11}d_{14} + \frac{1}{2}\hat{L}_{23}(d_{11}d_{44} + d_{41}d_{14}). \quad (\text{C.8})$$

$$Y_{32} = \hat{L}_{22}d_{12}d_{13} + \frac{1}{2}\hat{L}_{23}(d_{12}d_{43} + d_{42}d_{13}). \quad (\text{C.9})$$

$$Y_{33} = \hat{L}_{22}d_{12}d_{14} + \frac{1}{2}\hat{L}_{23}(d_{12}d_{44} + d_{42}d_{14}). \quad (\text{C.10})$$

$$Y_{12} = Y_{13} = Y_{21} = Y_{31} = 0. \quad (\text{C.11})$$

$$Z_{11} = \hat{L}_{11}(d_{22})^2. \quad (\text{C.12})$$

$$Z_{22} = \hat{L}_{22}(d_{13})^2 + \hat{L}_{23}d_{13}d_{43}. \quad (\text{C.13})$$

$$Z_{23} = Z_{32} = \hat{L}_{22}d_{13}d_{14} + \frac{1}{2}\hat{L}_{23}(d_{13}d_{44} + d_{14}d_{43}). \quad (\text{C.14})$$

$$Z_{33} = \hat{L}_{22}(d_{14})^2 + \hat{L}_{33}(d_{44})^2 + \hat{L}_{23}d_{14}d_{44}. \quad (\text{C.15})$$

$$Z_{12} = Z_{13} = Z_{21} = Z_{31} = 0. \quad (\text{C.16})$$

APPENDIX D

$$h_1(x_2, \zeta, a) = \sqrt{\frac{1}{2(x_2^2 + 4a^2)}} [\zeta x_2 + 4a^2 + P(x_2, \zeta, a)], \quad (\text{D.1})$$

$$h_2(x_2, \zeta, a) = \sqrt{\frac{1}{2(x_2^2 + 4a^2)}} [\zeta x_2 - 4a^2 + P(x_2, \zeta, a)], \quad (\text{D.2})$$

$$h_3(x_2, \zeta, a) = -\sqrt{\frac{1}{2(x_2 + \zeta)[(x_2 + \zeta)^2 + 4a^2]}} \zeta [\zeta x_2 + \zeta^2 + 4a^2 + Q(x_2, \zeta, a)], \quad (\text{D.3})$$

$$h_4(x_2, \zeta, a) = -\sqrt{\frac{1}{2(x_2 + \zeta)[(x_2 + \zeta)^2 + 4a^2]}} \zeta [\zeta x_2 + \zeta^2 - 4a^2 + Q(x_2, \zeta, a)], \quad (\text{D.4})$$

$$h_5(x_2, \zeta, a) = -\frac{1 - h_3(x_2, \zeta, a)}{x_2} - \frac{1}{x_2 + \zeta} \left[1 - \sqrt{\frac{\zeta}{x_2}} h_2(x_2, \zeta, a) \right] - \frac{1 + h_4(x_2, \zeta, a)}{x_2 + 2\zeta}, \quad (\text{D.5})$$

$$\tilde{h}_1(u, r, c/a) = \sqrt{\frac{r+1}{u+1}} h_1(u+1, r+1, a/c), \quad (\text{D.6})$$

$$\tilde{h}_5(u, r, c/a) = \frac{1}{c} h_5(u+1, r+1, a/c), \quad (\text{D.7})$$

with

$$P(x_2, \zeta, a) = \sqrt{16a^4 + 4(x_2^2 + \zeta^2)a^2 + \zeta^2 x_2^2}, \quad (\text{D.8})$$

$$Q(x_2, \zeta, a) = \sqrt{16a^4 + 4(x_2^2 + 2\zeta^2 + 2x_2\zeta)a^2 + \zeta^2(x_2 + \zeta)^2}. \quad (\text{D.9})$$

APPENDIX E

$$X_{11} = \hat{L}_{11}(f_{11})^2. \quad (\text{E.1})$$

$$X_{22} = \hat{L}_{22}(f_{21})^2 + \hat{L}_{23}f_{21}f_{31}. \quad (\text{E.2})$$

$$X_{23} = X_{32} = \hat{L}_{22}f_{21}f_{22} + \frac{1}{2}\hat{L}_{23}(f_{22}f_{31} + f_{21}f_{32}). \quad (\text{E.3})$$

$$X_{33} = \hat{L}_{22}(f_{22})^2 + \hat{L}_{23}f_{22}f_{32}. \quad (\text{E.4})$$

$$X_{12} = X_{13} = X_{21} = X_{31} = 0. \quad (\text{E.5})$$

$$Y_{11} = \hat{L}_{11}f_{11}f_{12}. \quad (\text{E.6})$$

$$Y_{22} = \hat{L}_{22}f_{21}f_{23} + \frac{1}{2}\hat{L}_{23}(f_{21}f_{33} + f_{31}f_{23}). \quad (\text{E.7})$$

$$Y_{23} = \hat{L}_{22}f_{21}f_{24} + \frac{1}{2}\hat{L}_{23}(f_{21}f_{34} + f_{31}f_{24}). \quad (\text{E.8})$$

$$Y_{32} = \hat{L}_{22}f_{22}f_{23} + \frac{1}{2}\hat{L}_{23}(f_{22}f_{33} + f_{32}f_{23}). \quad (\text{E.9})$$

$$Y_{33} = \hat{L}_{22}f_{22}f_{24} + \frac{1}{2}\hat{L}_{23}(f_{22}f_{34} + f_{32}f_{24}). \quad (\text{E.10})$$

$$Y_{12} = Y_{13} = Y_{21} = Y_{31} = 0. \quad (\text{E.11})$$

$$Z_{11} = \hat{L}_{11}(f_{12})^2, \quad (\text{E.12})$$

$$Z_{22} = \hat{L}_{22}(f_{23})^2 + \hat{L}_{23}f_{23}f_{33}. \quad (\text{E.13})$$

$$Z_{23} = Z_{32} = \hat{L}_{22} f_{23} f_{24} + \frac{1}{2} \hat{L}_{23} (f_{23} f_{34} + f_{24} f_{33}). \quad (\text{E.14})$$

$$Z_{33} = \hat{L}_{22} (f_{24})^2 + \hat{L}_{33} (f_{34})^2 + \hat{L}_{23} f_{24} f_{34}. \quad (\text{E.15})$$

$$Z_{12} = Z_{13} = Z_{21} = Z_{31} = 0. \quad (\text{E.16})$$

PUBLICATIONS

1. Micro-crack initiation at the tip of a semi-infinite rigid line inhomogeneity in piezoelectric solids. *International Journal of Engineering Science*, 2005, 43(15-16), 1223-1233.
2. Microcrack initiation at the tip of a finite rigid conducting line in piezoelectric media. *Journal of Mechanics of Materials and Structures*, 2006, 1(3), 559-579.
3. A piezoelectric screw dislocation interacts with interfacial collinear rigid lines in piezoelectric bimetals. *International Journal of Solids and Structures*, 2007, 44(1), 255-271.
4. Interaction of a dislocation with collinear rigid lines at the interface of piezoelectric media. *Journal of Mechanics of Materials and Structures*, accepted.