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DISCRETE SPLINE INTERPOLATION AND ITS APPLICATIONS

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Doctor of Philosophy

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Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

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Date

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CHEN FENGMIN

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Abstract

In carrying out continuous spline interpolation of a function, derivatives of the function at some points are always needed. However, in the real world situation, not only that it may be difficult to compute the derivatives of a function, the derivatives may not even exist at some points. In such a situation, the usual continuous spline interpolation will not be suitable. We therefore introduce a discrete interpolation scheme that involves only differences. Since no derivatives are involved, the discrete interpolant can be constructed for a more general class of functions and therefore has a wider range of applications.

In this thesis, we shall develop two kinds of discrete spline via a constructive approach, the first kind of discrete spline involves *forward differences*, while the second kind of discrete spline involves *central differences*.

We recall that a quintic polynomial is a polynomial of degree five. In the first case where *forward differences* are employed, for a function $f(t)$ defined on a discrete interval, we shall develop a class of quintic discrete Hermite interpolant and derive explicit error bounds in ℓ_∞ norm. We also establish, for a two-variable function $f(t, u)$ defined on a discrete rectangle, the biquintic discrete Hermite interpolant and perform the related error analysis. Based on the results of discrete Hermite interpolation, we then define the quintic discrete spline interpolant of the function $f(t)$, formulate its construction, and establish explicit error estimates between $f(t)$ and its spline interpolant. We also tackle the two-variable discrete spline interpolation and the corresponding error analysis for $f(t, u)$. As an application, we solve Fredholm integral equations numerically by using biquintic discrete splines to

degenerate the kernels, and furnish the related error analysis.

In the second case where *central differences* are involved, for a periodic function $f(t)$ defined on a discrete interval, we construct the periodic quintic discrete spline interpolant and obtain the explicit error estimates between the function and its spline interpolant. The treatment is then extended to a periodic function $f(t, u)$ defined on a discrete rectangle, here we establish the two-variable periodic discrete spline interpolant and also provide the error analysis. As applications, we solve second order and fourth order boundary value problems by discrete splines involving central differences. Not only that we tackle the related convergence and error analysis, comparisons with other known methods in the literature are also illustrated by several examples.

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Symbols

For a vector $v = [v_i]$ and a matrix $A = [a_{ij}]$,

$$|v|_0 = \max_i |v_i| \quad \text{and} \quad \|A\| = \max_i \sum_j |a_{ij}|.$$

The following notations will be used in Chapters 2 – 4.

Let a, b, c, d ($b > a$, $d > c$) be integers.

- $N[a, b] = \{a, a + 1, \dots, b\}$.
- Let ρ and ρ' be uniform partitions of $N[a, b]$ and $N[c, d]$ respectively

$$\rho : a = k_1 < k_2 < \dots < k_m = b, \quad k_i \in \mathbf{Z}, \quad 1 \leq i \leq m$$

and

$$\rho' : c = l_1 < l_2 < \dots < l_n = d, \quad l_j \in \mathbf{Z}, \quad 1 \leq j \leq n$$

with step sizes

$$h = k_{i+1} - k_i, \quad 1 \leq i \leq m - 1 \quad \text{and} \quad h' = l_{j+1} - l_j, \quad 1 \leq j \leq n - 1.$$

Let $\tau = \rho \times \rho'$ be a rectangular partition of $N[a, b] \times N[c, d]$.

- Factorial notation $t^{(k)} = \prod_{i=0}^{k-1} (t - i)$, $0^{(0)} = 1$.
- Forward difference operator $\Delta f(t) = f(t + 1) - f(t)$, $\Delta^0 f(t) = f(t)$.
- The largest integer less than or equal to t is $[t]$.

- For a function $f(t)$ defined on $N[a, b + 2]$,

$$\|f\| = \max_{t \in N[a, b+2]} |f(t)|.$$

For a function $f(t, u)$ defined on $N[a, b + 2] \times N[c, d + 2]$,

$$\|f\| = \max_{(t, u) \in N[a, b+2] \times N[c, d+2]} |f(t, u)|.$$

The following notations will be used in Chapters 5 and 6.

Let a, b, c, d be real numbers,

- Let φ and φ' be uniform partitions of $[a, b]$ and $[c, d]$

$$\varphi : a = t_0 < t_1 < \cdots < t_n = b \quad \text{and} \quad \varphi' : c = u_0 < u_1 < \cdots < u_m = d$$

with step sizes $p = (b - a)/n$ and $p' = (d - c)/m$ respectively.

Let $\phi = \varphi \times \varphi'$ be a rectangular partition of $[a, b] \times [c, d]$.

- For a given $0 < h \leq p$, $[a, b]_h = [a, b] \cap \{a, a + h, a + 2h, \dots\}$.
- Let $w(f, p)$ be the modulus of smoothness of f ,

$$w(f, p) = \max\{|f(t) - f(u)| : |t - u| < p, t, u \in [a, b]_h\}.$$

- For a function f defined on $[a, b]_h$

$$\|f\| = \max_{t \in [a, b]_h} |f(t)|.$$

For a function f defined on $[a, b]_h \times [c, d]_h$

$$\|f\| = \max_{(t, u) \in [a, b]_h \times [c, d]_h} |f(t, u)|.$$

- For $t_0 < \cdots < t_k$, let $f[t_0, \dots, t_k]$ denote the usual divided difference of f

$$f[t_0, \dots, t_k] = \sum_{i=0}^k \frac{f(t_i)}{\prod_{j=0, j \neq i}^k (t_i - t_j)}.$$

Chapter 1

Introduction

1.1 Motivation

Hermite interpolation is very well known in the field of numerical mathematics and approximation theory, many research papers have been written on the construction, error estimates and applications of Hermite interpolation, see for example, [47, 55, 66, 68, 108] and the monographs [3, 87] which provide a good documentary of the literature. To recall about continuous Hermite interpolation, in the general case we start with a partition $\pi : a = t_0 < t_1 < \cdots < t_n = b$ and a positive integer m . In each subinterval $[t_{i-1}, t_i]$, $1 \leq i \leq n$, the Hermite interpolant $P_i(t)$ of degree $2m - 1$ is constructed to solve the following interpolation problem

$$P_i^{(j)}(t_{i-1}) = f^{(j)}(t_{i-1}), \quad P_i^{(j)}(t_i) = f^{(j)}(t_i), \quad 0 \leq j \leq m - 1. \quad (1.1)$$

The piecewise Hermite interpolant of degree $2m - 1$ to f is given by

$$s(t) = P_i(t), \quad t_{i-1} \leq t \leq t_i, \quad 1 \leq i \leq n. \quad (1.2)$$

It is clear from the constraints of (1.1) that $s(t) \in C^{m-1}[a, b]$, since derivatives of P_{i-1} and P_i up through the $(m - 1)$ st match at t_i , $1 \leq i \leq n$. In particular, when $m = 2$ we have the piecewise cubic Hermite interpolants that belong to $C^1[a, b]$, and when $m = 3$ we obtain the piecewise quintic Hermite interpolant that belong

to $C^2[a, b]$.

Spline functions arise in a natural way in trying to improve the continuity property of the piecewise Hermite interpolants. The spline approximation in its present form first appeared in a paper by Schoenberg [89], then it is well developed by many authors [5, 37, 49, 76, 87, 90]. Cubic splines which are most often used in approximation theory shall be discussed in the following.

A cubic spline function $s(t)$ is a piecewise cubic polynomial that is twice continuously differentiable, in other words, $s(t) \in C^2[a, b]$. A unique function $s(t)$ satisfying the following interpolatory constraints

$$\begin{aligned} s(t_i) &= f(t_i), \quad 0 \leq i \leq n \\ s'(t_0) &= f'(t_0), \quad s'(t_n) = f'(t_n) \end{aligned} \tag{1.3}$$

is proved to exist.

We can observe from the above Hermite interpolants and spline interpolants that we not only interpolate f at each knot, but also interpolate a given number of derivatives of f at certain knots. For Hermite interpolants, the derivatives of each knot are required while the spline interpolants the derivatives of the two boundary points are needed. Therefore, the given function needs to be sufficiently smooth. However, in many cases this derivative is not available, and a number of other conditions have been suggested. But this often affects the accuracy of the interpolation. Problems of this sort could be effectively avoided by employing discrete Hermite interpolation and discrete spline interpolation in which differences are utilized. There is no guarantee that discrete splines will always be better than continuous splines in practice. However, the numerical results indicate that they are competitive, and sometimes it might be convenient that derivative end conditions become differences in the discrete case.

In 1971 Mangasarian and Schumaker [64] investigated some constrained minimization problems in a real Euclidean space which were discrete analogs of minimization problems in a Banach space. The solutions of these discrete problems exhibited a spline-like structure, and were hence introduced as ‘discrete splines’.

These discrete splines were further found [65] to play a fundamental role in certain best summation formulae for a finite sequence of real numbers. In the field of approximation theory, these discrete splines have been characterized in the work of [19, 39, 40, 59, 61, 88, 90].

There are two basic approaches to developing splines - the *variational approach* wherein splines are defined as the solutions of certain constrained minimization problems, and the *constructive* approach wherein they are defined by piecing together classes of functions at certain ‘knots’. In the very first paper on discrete splines [64], the variational approach has been used and discrete splines are introduced as solutions to constrained minimization problems in real Euclidean space. The constructive approach has been employed in the work of [19, 59, 60, 88]. Both Schumaker [88] and Lyche [59, 60] deal with discrete polynomial splines. In [88], discrete B -splines, which are discrete analogs of the classical B -splines, are explored to give the general construction of discrete splines - here *forward differences* are involved. In comparison, the *cubic* discrete spline discussed in [59] involves *central differences*. Another work by Lyche [60] investigates several discrete spline approximation methods for fitting functions and data, the respective error analysis shows that some results in the continuous case can be obtained from the discrete analog. On a separate note, in [19] discrete L -splines are constructively defined so as to parallel the development of continuous L -splines.

Motivated by the above research on discrete splines and also the recognition that in contrast to continuous splines where derivatives are involved, discrete splines only involve differences and hence have a wider range of applications, in this thesis we shall develop a *quintic discrete spline* via a constructive approach.

1.2 Major Contribution of the Thesis

In this thesis we shall develop two classes of discrete splines involving forward differences and central differences respectively.

For a function defined on integer points, using forward differences we contribute the following:

(1) We develop a class of quintic discrete Hermite interpolants and derive explicit error estimates between the function and its quintic discrete Hermite interpolant; these are extended to two-variable biquintic discrete Hermite interpolants and the related error estimates.

(2) Based on the results of Hermite interpolation, we develop quintic discrete spline interpolation and derive explicit error estimates between the function and its quintic discrete spline interpolant; these are extended to two-variable biquintic discrete spline interpolation and the related error estimates.

(3) As an application, we use biquintic discrete splines to obtain a numerical solution of Fredholm integral equations. Moreover, we establish *a priori* as well as *a posteriori* error bounds between the exact and approximate solutions.

By using central differences, we contribute the following:

(4) We establish a class of periodic quintic discrete splines and derive the error bounds for both one dimensional and two dimensional cases.

(5) As applications, we use the consistency relations derived from cubic discrete splines and deficient cubic discrete splines to solve second order boundary value problems, we also obtain numerical solution of a fourth order boundary value problem based on quintic discrete splines.

1.3 Organization of the Thesis

This thesis is organized as follows. Chapter 2 provides a full treatment of *quintic discrete Hermite interpolation*. To be specific, for a function $f(t)$ defined on $N[a, b+2]$, we define the quintic discrete Hermite interpolant $H_\rho f(t)$, give an explicit expression

of $H_\rho f(t)$, and derive error inequalities of the form

$$\|f - H_\rho f\| \leq c_j \max_{t \in N[a, b+2-j]} |\Delta^j f(t)|, \quad 2 \leq j \leq 6$$

where the constants c_j , $2 \leq j \leq 6$ are explicitly provided. These derived error inequalities require no smoothness assumption on the given function $f(t)$. Using these results, we further establish the two-variable Hermite interpolant $H_\tau f(t, u)$ for a function $f(t, u)$ defined on $N[a, b+2] \times N[c, d+2]$, as well as the related error inequalities. To illustrate our results, six numerical examples are presented where the discrete Hermite interpolants are actually constructed, and the actual errors are also computed to compare with the error bounds obtained. The work in Chapter 2 naturally complements several known results for the continuous case [3, 47, 55, 66, 68, 87, 108], as well as extends the research done on cubic discrete Hermite interpolation [112]. Moreover, the error estimates derived in Chapter 2 improve the results obtained in [2].

Using the results of Chapter 2, we shall tackle *quintic discrete spline interpolation* in Chapter 3. For a function $f(t)$ defined on $N[a, b+2]$, we define the quintic discrete spline interpolant $S_\rho f(t)$, construct $S_\rho f(t)$ by solving some matrix equations, and establish error inequalities of the form

$$\|f - S_\rho f\| \leq d_j \max_{t \in N[a, b+2-j]} |\Delta^j f(t)|, \quad 2 \leq j \leq 6$$

where the constants d_j , $2 \leq j \leq 6$ are explicitly provided. These derived error inequalities require no smoothness assumption on the given function $f(t)$. Moreover, the investigation is extended to the two-variable spline interpolant $S_\tau f(t, u)$ for a function $f(t, u)$ defined on $N[a, b+2] \times N[c, d+2]$, as well as the related error analysis. We also include three numerical examples to illustrate the actual construction of the discrete spline interpolants and to compare the actual errors with the error bounds obtained. Our work naturally complements and extends several known results for the continuous case [3, 4, 36, 47, 56, 86, 87, 93, 100, 108, 109, 111], as well as the research done on cubic discrete spline interpolation [112].

In Chapter 4 we shall present an application – we shall use quintic discrete splines to solve the following Fredholm integral equation

$$u(t) = \int_{\bar{c}}^{\bar{d}} k(t, s)u(s)ds + f(t), \quad t \in [\bar{a}, \bar{b}].$$

From the numerical experiments presented our method is seen to outperform some well known methods in the literature such as Taylor series method, collocation method and Galerkin approximate method. The use of splines in the numerical solution of integral equations has been investigated by many authors [17, 24, 41, 50, 57, 62, 63, 69, 87, 109]. While most of this work involves continuous splines, there are only very few papers that involve discrete splines, only in [7, 105] discrete splines are used to solve the weakly singular Fredholm integral equations. As such our work naturally complements the literature and in particular is applicable to more general integral equations.

So far forward differences have been employed in the discrete splines discussed in Chapters 3 and 4, using central differences we shall discuss another class of discrete splines in Chapters 5 and 6. For a periodic function $f(t)$ defined on the discrete interval $[a - 2h, b + 2h]_h$, using central differences, we construct the periodic quintic discrete spline interpolant $S_\varphi f(t)$, and obtain the explicit error bounds in terms of the modulus of smoothness of f . For the two dimensional case, the periodic biquintic discrete spline interpolant $S_\phi f(t, u)$ is developed to interpolate a given two-variable periodic function $f(t, u)$ defined on the discrete interval $[a - 2h, b + 2h]_h \times [c - 2h, d + 2h]_h$. Our work complements and extends several known results for the discrete splines [39, 40, 59, 60, 80]. We give three applications in Chapter 6, we use cubic splines involving central differences to solve second order boundary value problems and second order obstacle boundary value problems, and quintic splines involving central differences to solve fourth order boundary value problems. Numerical examples are given to compare with other methods in the literature. Since the discrete spline developed contains a parameter, high order accuracy could be achieved by a specific choice of the parameter.

All the computations in this thesis are programmed in MATLAB m code.

1.4 Conclusions and Future Work

Everyone knows that in conducting continuous Hermite/spline interpolation of a function, derivatives of the function at some points are always required – this means that the function should satisfy some smoothness condition. However, it is often realized in the real world situation that this smoothness condition may not be met, or even if it is satisfied, the derivatives may be hard to compute. These difficulties motivate us to develop discrete interpolation schemes that involve only differences. Since derivatives are not needed, the discrete interpolants can be constructed for a more general class of functions and therefore have a wider range of applications.

In this thesis, our main aim is to develop two types of discrete spline interpolation schemes by constructive approach, the first type involves *forward differences* while the second type involves *central differences*.

In the first case where *forward differences* are employed, we consider a function $f(t)$ defined on a discrete interval of integers. Before tackling discrete *spline* interpolation, we first establish discrete *Hermite* interpolation. Here, we have developed a class of *quintic discrete Hermite interpolant* for $f(t)$, obtained its explicit representation, proved its uniqueness, and derived explicit error estimates between $f(t)$ and its quintic discrete Hermite interpolant. These results are then extended to the two-variable case where we have established, for a two-variable function defined on a discrete rectangle, its *two-variable biquintic discrete Hermite interpolant* and the related error analysis. The work done on discrete Hermite interpolation is crucial in developing discrete spline interpolation. We have defined the *quintic discrete spline interpolant* of the function $f(t)$, derived the method of construction of the spline interpolant, showed the uniqueness, and established explicit error estimates between $f(t)$ and its spline interpolant. The work has also been extended to *two-variable biquintic discrete spline interpolation* and the corresponding error estimates. As an application, we have employed biquintic discrete splines to obtain numerical solu-

tions of Fredholm integral equations. This is achieved by using the biquintic discrete spline interpolants of the kernels and the method of kernel degeneration. We have also derived a priori and a posteriori error estimates for our numerical method.

In the second case where *central differences* are involved, we consider a periodic function $f(t)$ defined on a discrete interval of evenly spaced points (need not be integers). We have defined the *periodic quintic discrete spline interpolant* of $f(t)$, obtained the construction of the spline interpolant, proved its uniqueness, and derived explicit error estimates. The work is then extended to a two-variable periodic function defined on a discrete rectangle, here we have developed the *two-variable periodic biquintic discrete spline interpolation* and have also furnished the error analysis. As applications, we have developed three numerical methods for second order and fourth order boundary value problems using discrete splines involving central differences. In all the proposed methods, we have tackled the question of convergence and have performed error analysis. Numerical experiments have shown that our methods compare favourably with other known methods in the literature.

Throughout the thesis, many numerical examples have been provided to illustrate the usefulness and the effectiveness of the results obtained and the numerical methods proposed.

For the future work, we propose the following:

(1) In the field of approximation theory for a given function $f(t) \in C^{2m}[a, b]$, the Lidstone interpolating polynomial $P(t)$ [1] of degree $(2m - 1)$ matches the function value and its $(m - 1)$ even derivatives as follows

$$P(a) = f(a), \quad P^{(2k)}(a) = f^{(2k)}(a), \quad P^{(2k)}(b) = f^{(2k)}(b), \quad 0 \leq k \leq m - 1. \quad (1.4)$$

Without any smoothness conditions on the function $f(t)$, we can develop the discrete Lidstone interpolants involving forward differences satisfying the following conditions

$$P(a) = f(a), \quad \Delta^{2k}P(a) = \Delta^{2k}f(a), \quad \Delta^{2k}(b) = \Delta^{2k}f(b), \quad 0 \leq k \leq m - 1. \quad (1.5)$$

(2) In Chapter 5 we have discussed the periodic quintic discrete spline interpola-

tion involving central differences, we observe that its restriction is that the function considered should be periodic. A possible future work is to remove this restriction.

(3) Another possible future work is to develop discrete Lidstone interpolants involving central differences satisfying the following conditions

$$P(a) = f(a), \quad P^{\{2k\}}(a) = f^{\{2k\}}(a), \quad P^{\{2k\}}(b) = f^{\{2k\}}(b), \quad 0 \leq k \leq m-1 \quad (1.6)$$

where $f^{\{2k\}}$ denotes the $2k$ -th central difference of f .

Chapter 2

Discrete Hermite Interpolation

In contrast to continuous Hermite interpolation where derivatives are involved, discrete Hermite interpolation only involves differences, and hence can be constructed for a more general f and therefore has a wider range of applications. Motivated by this attractive aspect of discrete interpolation, in this chapter we shall develop a class of *quintic discrete Hermite interpolation*.

We shall present quintic discrete Hermite interpolation in this chapter. The plan of this chapter is as follows. In section 2.1, for a function $f(t)$ defined on $N[a, b + 2]$, we shall define its quintic discrete Hermite interpolant $H_\rho f(t)$ and provide an explicit representation of $H_\rho f(t)$. Using discrete Peano's kernel theorem, in section 2.2 we shall derive explicit error bounds for $\|f - H_\rho f\|$. In section 2.3, we shall present three numerical examples to illustrate the construction of $H_\rho f(t)$ and the sharpness of the error bounds obtained in section 2.2. Finally, two-variable discrete Hermite interpolation will be discussed in section 2.4. We shall define the two-variable discrete Hermite interpolant $H_\tau f(t, u)$ for a function $f(t, u)$ defined on $N[a, b + 2] \times N[c, d + 2]$, give its explicit representation, as well as develop the related error estimates. The results are further illustrated by three numerical examples. In this chapter, we assume $h, h' \geq 4$. This chapter is based on the work of [26].

2.1 Definitions and Explicit Representation

Recall the notations listed in the section of Symbols, ρ and ρ' are uniform partitions of $N[a, b]$ and $N[c, d]$ respectively

$$\rho : a = k_1 < k_2 < \cdots < k_m = b, \quad k_i \in \mathbf{Z}, \quad 1 \leq i \leq m$$

and

$$\rho' : c = l_1 < l_2 < \cdots < l_n = d, \quad l_j \in \mathbf{Z}, \quad 1 \leq j \leq n$$

with step sizes

$$h = k_{i+1} - k_i, \quad 1 \leq i \leq m-1 \quad \text{and} \quad h' = l_{j+1} - l_j, \quad 1 \leq j \leq n-1.$$

$\tau = \rho \times \rho'$ is a rectangular partition of $N[a, b] \times N[c, d]$.

In this section, discrete Hermite interpolant will be defined in terms of forward differences and an explicit expression will also be provided.

Definition 2.1. Let $j \in \mathbb{N}$ ($< h$) be fixed. Let $f_i(t)$ be defined on $N[k_i, k_{i+1}+j]$, $1 \leq i \leq m-2$, and $f_{m-1}(t)$ be defined on $N[k_{m-1}, b+2]$. Let $f(t) \equiv \cup_{1 \leq i \leq m-1} f_i(t)$. We say that $f(t) \in D^{(j)}[a, b]$ if

$$\Delta^l f_{i-1}(k_i) = \Delta^l f_i(k_i), \quad 2 \leq i \leq m-1, \quad 0 \leq l \leq j. \quad (2.1)$$

If $j = 0$, (2.1) becomes

$$\Delta^l f_{i-1}(k_i) = \Delta^l f_i(k_i), \quad 2 \leq i \leq m-1, \quad l = 0, \quad (2.2)$$

we recall that $\Delta^0 f(t) = f(t)$, (2.2) becomes

$$f_{i-1}(k_i) = f_i(k_i), \quad 2 \leq i \leq m-1. \quad (2.3)$$

Thus, when $j = 0$, (2.1) is equivalent to (2.3).

If $j = 1$, (2.1) becomes

$$\Delta^l f_{i-1}(k_i) = \Delta^l f_i(k_i), \quad 2 \leq i \leq m-1, \quad 0 \leq l \leq 1, \quad (2.4)$$

when $l = 0$, we can obtain (2.3); when $l = 1$, we recall that $\Delta^1 f(t) = f(t+1) - f(t)$, (2.4) becomes $f_{i-1}(k_i + 1) - f_{i-1}(k_i) = f_i(k_i + 1) - f_i(k_i)$, based on the result of (2.3) we can obtain

$$f_{i-1}(k_i + 1) = f_i(k_i + 1), \quad 2 \leq i \leq m-1. \quad (2.5)$$

Thus, when $j = 1$, (2.1) is equivalent to (2.3) and (2.5).

If $j = 2$, (2.1) becomes

$$\Delta^l f_{i-1}(k_i) = \Delta^l f_i(k_i), \quad 2 \leq i \leq m-1, \quad 0 \leq l \leq 2, \quad (2.6)$$

similarly, based on the results of (2.3) and (2.5) we get

$$f_{i-1}(k_i + 2) = f_i(k_i + 2), \quad 2 \leq i \leq m-1. \quad (2.7)$$

Thus, when $j = 2$, (2.1) is equivalent to (2.3), (2.5) and (2.7).

Therefore, generally speaking, (2.1) is equivalent to

$$f_{i-1}(k_i + l) = f_i(k_i + l), \quad 2 \leq i \leq m-1, \quad 0 \leq l \leq j. \quad (2.8)$$

Hence, the function $f(t) = \cup_{1 \leq i \leq m-1} f_i(t)$ is well defined on $N[a, b+2]$. The set $D^{(p,q)}([a, b] \times [c, d])$ where $p, q \in \mathbb{N}$, $p < h$, $q < h'$, is analogously defined. For any fixed value of u , $f(t, u) \in D^{(p)}[a, b]$, for any fixed value of t , $f(t, u) \in D^{(q)}[c, d]$.

Define the set $H(\rho)$ as

$$H(\rho) = \left\{ g(t) \in D^{(2)}[a, b] : g(t) \text{ is a quintic polynomial in each subinterval } N[k_i, k_{i+1}], 1 \leq i \leq m-2 \text{ and } N[k_{m-1}, b+2] \right\}.$$

Clearly, 6 conditions are needed to construct a quintic polynomial $g(t)$ over each subinterval, altogether $6(m-1)$ conditions are needed since $(m-1)$ intervals are involved. Meanwhile, since $g(t) \in D^{(2)}[a, b]$, it is noted from (2.3), (2.5) and (2.7) that 3 conditions are satisfied at each middle point, therefore we can obtain $3(m-2)$ equations altogether. So, $H(\rho)$ is of dimension $6(m-1) - 3(m-2) = 3m$, and it is the ‘Hermite space’ where the interpolating polynomial lies.

Definition 2.2. For a given function $f(t)$ defined on $N[a, b+2]$, we say $H_\rho f(t)$ is the $H(\rho)$ -interpolant of $f(t)$, also known as the *discrete Hermite interpolant* of $f(t)$, if $H_\rho f(t) \in H(\rho)$ with

$$\Delta^j H_\rho f(k_i) = \Delta^j f(k_i), \quad j = 0, 1, 2, \quad 1 \leq i \leq m. \quad (2.9)$$

To obtain the expression of $H_\rho f(t)$, we first construct the $h_i(t)$, $\bar{h}_i(t)$, $\bar{\bar{h}}_i(t)$, $1 \leq i \leq m$ which are the basic elements of $H(\rho)$ satisfying the following conditions:

$$h_i(k_j) = \delta_{ij}, \quad \Delta h_i(k_j) = \Delta^2 h_i(k_j) = 0, \quad (2.10)$$

$$\Delta \bar{h}_i(k_j) = \delta_{ij}, \quad \bar{h}_i(k_j) = \Delta^2 \bar{h}_i(k_j) = 0, \quad (2.11)$$

$$\Delta^2 \bar{\bar{h}}_i(k_j) = \delta_{ij}, \quad \bar{\bar{h}}_i(k_j) = \Delta \bar{\bar{h}}_i(k_j) = 0. \quad (2.12)$$

As an example, we show how to derive the explicit expression of $h_i(t)$. As the basic element of $H(\rho)$, we know that $h_i(t)$ is a piecewise quintic polynomial defined in each subinterval $N[k_i, k_{i+1}]$, $1 \leq i \leq m-2$ and $N[k_{m-1}, b+2]$. To get the expression of quintic polynomial $h_i(t)$ over each interval, 6 conditions are needed respectively, which are given by (2.10). Therefore, we can conclude that $h_i(t)$ is nonzero over the

two intervals involving the point k_i : $N[k_{i-1}, k_i]$ and $N[k_i, k_{i+1}]$, $h_i(t) \equiv 0$ in other intervals. In the interval $N[k_{i-1}, k_i]$, we obtain the following from (2.10)

$$h_i(k_{i-1}) = 0, \quad h_i(k_i) = 1, \quad \Delta h_i(k_{i-1}) = \Delta^2 h_i(k_{i-1}) = 0, \quad \Delta^2 h_i(k_i) = \Delta^2 h_i(k_i) = 0. \quad (2.13)$$

By observation, we assume the expression of $h_i(t)$ over the interval $N[k_{i-1}, k_i]$ is

$$h_i(t) = \alpha_0 + \alpha_1(t - k_{i-1})^{(1)} + \alpha_2(t - k_{i-1})^{(2)} + \alpha_3(t - k_{i-1})^{(3)} + \alpha_4(t - k_{i-1})^{(4)} + \alpha_5(t - k_{i-1})^{(5)} \quad (2.14)$$

where the parameters α_i , $0 \leq i \leq 5$ can be determined by using the 6 conditions from (2.13). Therefore the expression of $h_i(t)$ over the interval $N[k_{i-1}, k_i]$ can be obtained.

By similar computation, we obtain the following explicit expressions:

$$\begin{aligned} h_i(t) &= \frac{10(t - k_{i-1})^{(3)}}{h(h+1)(h+2)} - \frac{15(t - k_{i-1})^{(4)}}{h(h-1)(h+1)(h+2)} + \frac{6(t - k_{i-1})^{(5)}}{h(h-1)(h-2)(h+1)(h+2)}, \\ &\quad t \in N[k_{i-1}, k_i], \quad 2 \leq i \leq m-1; \quad t \in N[k_{m-1}, b+2], \quad i = m \\ &= \frac{-10(t - k_{i+1})^{(3)}}{h(h-1)(h-2)} - \frac{15(t - k_{i+1})^{(4)}}{h(h-1)(h-2)(h+1)} - \frac{6(t - k_{i+1})^{(5)}}{h(h-1)(h-2)(h+1)(h+2)}, \\ &\quad t \in N[k_i, k_{i+1}], \quad 1 \leq i \leq m-1 \\ &= 0, \quad \text{otherwise;} \\ &\quad (2.15) \\ \bar{h}_i(t) &= \frac{-4(h-3)(t - k_{i-1})^{(3)}}{h(h+1)(h+2)} + \frac{(7h-16)(t - k_{i-1})^{(4)}}{h(h-1)(h+1)(h+2)} - \frac{3(t - k_{i-1})^{(5)}}{h(h-1)(h+1)(h+2)}, \\ &\quad t \in N[k_{i-1}, k_i], \quad 2 \leq i \leq m-1; \quad t \in N[k_{m-1}, b+2], \quad i = m \\ &= \frac{-4(h+3)(t - k_{i+1})^{(3)}}{h(h-1)(h-2)} - \frac{(7h+16)(t - k_{i+1})^{(4)}}{h(h-1)(h-2)(h+1)} - \frac{3(t - k_{i+1})^{(5)}}{h(h-1)(h-2)(h+1)}, \\ &\quad t \in N[k_i, k_{i+1}], \quad 1 \leq i \leq m-1 \\ &= 0, \quad \text{otherwise;} \\ &\quad (2.16) \end{aligned}$$

$$\begin{aligned}
\bar{\bar{h}}_i(t) &= \frac{(h-2)(h-3)(t-k_{i-1})^{(3)}}{2h(h+1)(h+2)} - \frac{(h-3)(t-k_{i-1})^{(4)}}{h(h+1)(h+2)} + \frac{(t-k_{i-1})^{(5)}}{2h(h+1)(h+2)}, \\
&\quad t \in N[k_{i-1}, k_i], \quad 2 \leq i \leq m-1; \quad t \in N[k_{m-1}, b+2], \quad i = m \\
&= \frac{-(h+2)(h+3)(t-k_{i+1})^{(3)}}{2h(h-1)(h-2)} - \frac{(h+3)(t-k_{i+1})^{(4)}}{h(h-1)(h-2)} - \frac{(t-k_{i+1})^{(5)}}{2h(h-1)(h-2)}, \\
&\quad t \in N[k_i, k_{i+1}], \quad 1 \leq i \leq m-1 \\
&= 0, \quad \text{otherwise.}
\end{aligned} \tag{2.17}$$

From the properties of the $h_i(t)$, $\bar{h}_i(t)$ and $\bar{\bar{h}}_i(t)$ which are described as (2.10)-(2.12), the discrete Hermite interpolant $H_\rho f(t)$ satisfying (2.9) can be explicitly expressed as

$$H_\rho f(t) = \sum_{i=1}^m \left[f(k_i) h_i(t) + \Delta f(k_i) \bar{h}_i(t) + \Delta^2 f(k_i) \bar{\bar{h}}_i(t) \right], \tag{2.18}$$

In the following lemma, we shall obtain some estimates on $\bar{h}_i(t)$ and $\bar{\bar{h}}_i(t)$ which are needed in the error analysis of Chapter 3.

Lemma 2.1. *The following equalities hold for $2 \leq i \leq m-2$:*

$$\begin{aligned}
\max_{t \in N[k_i, k_{i+1}]} [|\bar{h}_i(t)| + |\bar{h}_{i+1}(t)|] &= \max \{ \psi_1([T_1]), \psi_1([T_1 + 1]) \} \equiv M_1(h) \\
&\leq \frac{h^3(2h-1)^2(h+2) + h(h-2)^3(2h+1)^2}{12h(h-1)(h-2)(h+1)(h+2)}
\end{aligned} \tag{2.19}$$

where $\psi_1(u) = \frac{u(h-u)}{(h+2)^{(5)}} \{12u^3 + (-h^2 - 18h - 56)u^2 + (h^3 + 6h^2 + 56h + 84)u + h^4 - 9h^2 - 42h - 40\}$ and T_1 is the unique root of $\psi'_1(u) = 0$ in $[2, h]$;

$$\begin{aligned}
\max_{t \in N[k_i, k_{i+1}]} [|\bar{\bar{h}}_i(t)| + |\bar{\bar{h}}_{i+1}(t)|] &= \max \{ \psi_2([T_2]), \psi_2([T_2 + 1]) \} \\
&\equiv M_2(h) \leq \frac{(h-1)(h+1)^2}{32(h-2)}
\end{aligned} \tag{2.20}$$

where $\psi_2(u) = \frac{hu^{(2)}(h+1-u)^{(2)}(h^2+3h+14-6u)}{2(h+2)^{(5)}}$ and T_2 is the unique root of $\psi'_2(u) = 0$ in

$[2, h]$.

Proof. For $2 \leq i \leq m-2$ and $t \in N[k_i, k_{i+1}]$, using the explicit expression in (2.16) we have

$$|\bar{h}_i(t)| = \frac{(h-T)(h+1-T)(h+2-T)T(3T+h-5)}{(h+1)^{(4)}} \quad \text{and} \quad |\bar{h}_{i+1}(t)| = \frac{T(h-T)(4h+5-3T)|T-1||T-2|}{(h+2)^{(4)}}$$

where $T = t - k_i \in N[0, h]$. Then, it is easily seen that

$$\max_{T \in N[0,1]} [|\bar{h}_i(t)| + |\bar{h}_{i+1}(t)|] = 1, \quad \max_{T \in N[1,2]} [|\bar{h}_i(t)| + |\bar{h}_{i+1}(t)|] = 2,$$

and

$$\max_{T \in N[2,h]} [|\bar{h}_i(t)| + |\bar{h}_{i+1}(t)|] = \max_{T \in N[2,h]} \psi_1(T) = \max \{\psi_1([T_1]), \psi_1([T_1 + 1])\}$$

where $T_1 \in [2, h]$ is such that $\psi_1'(T_1) = 0$. Note also that $[T_1 + 1] \in N[2, h]$. Combining all the above, we obtain the equality in (2.19). The inequality in (2.19) is the result of maximizing $|\bar{h}_i(t)|$ and $|\bar{h}_{i+1}(t)|$ separately over $T \in N[0, h]$.

To prove (2.20), from (2.17) we have for $2 \leq i \leq m-2$ and $t \in N[k_i, k_{i+1}]$,

$$|\bar{\bar{h}}_i(t)| = \frac{(h-T)(h+1-T)(h+2-T)T|T-1|}{2h(h-1)(h-2)} \quad \text{and} \quad |\bar{\bar{h}}_{i+1}(t)| = \frac{(h-T)(h+1-T)T|T-1||T-2|}{2h(h+1)(h+2)}$$

where $T = t - k_i \in N[0, h]$. Thus, it follows that

$$\max_{T \in N[0,1]} [|\bar{\bar{h}}_i(t)| + |\bar{\bar{h}}_{i+1}(t)|] = 0, \quad \max_{T \in N[1,2]} [|\bar{\bar{h}}_i(t)| + |\bar{\bar{h}}_{i+1}(t)|] = 1,$$

and

$$\max_{T \in N[2,h]} [|\bar{\bar{h}}_i(t)| + |\bar{\bar{h}}_{i+1}(t)|] = \max_{T \in N[2,h]} \psi_2(T) = \max \{\psi_2([T_2]), \psi_2([T_2 + 1])\}$$

where $T_2 \in [2, h]$ is such that $\psi_2'(T_2) = 0$. Note also that $[T_2 + 1] \in N[2, h]$. Combining all the above, we obtain (2.20). $\square$

2.2 Error Analysis

In this section, we shall use the discrete Peano's kernel theorem to analyze the error between the original function f and the quintic discrete Hermite interpolant $H_\rho f$.

Theorem 2.1. [112] (*Discrete Peano's kernel theorem*) Let E be a linear functional and $E(p(t)) = 0$ for all polynomials $p(t)$ of degree $(n-1)$. Then, for any $f(t)$ defined on $\mathbb{N}$,

$$E(f(t)) = \frac{1}{(n-1)!} E_t \left[\sum_{s=a}^{t-n} (t-s-1)_+^{(n-1)} \Delta^n f(s) \right],$$

where $E_t(\cdot)$ means the linear functional E applied to the expression $(\cdot)$ considered as a function of t , and $(t-s-1)_+^{(n-1)} = (t-s-1)^{(n-1)}$ if $t \geq s+1$, and $(t-s-1)_+^{(n-1)} = 0$ if $t < s+1$.

Discrete Peano's kernel theorem is utilized in the following theorem to get the bound of $\|f - H_\rho f\|$.

Theorem 2.2. Let $f(t)$ be defined on $N[a, b+2]$. Then,

$$\|f - H_\rho f\| \leq c_j(h) \max_{t \in N[a, b+2-j]} |\Delta^j f(t)|, \quad 2 \leq j \leq 6 \quad (2.21)$$

where the constants $c_j(h)$, $2 \leq j \leq 6$ are given as follows:

- $c_2(h) = \frac{1}{\sigma} \max\{g_2([t_2]), g_2([t_2+1])\} \leq \frac{1}{\sigma} \gamma_1(\gamma_1-1)(h-\gamma_1)\theta_2(\gamma_2),$

where $\sigma = (h+2)^{(5)}$, $g_2(t) = t(t-1)(h-t)\theta_2(t)$, $\theta_2(t) = 6(h-1)t^3 + (24-9h-12h^2)t^2 + (-26-17h+23h^2+5h^3)t + h^4-8h^3+h^2+20h+4$, t_2 is the root of $g'_2(t) = 0$ in $[3, h-1]$, $\gamma_1 = \frac{1}{3}(1+h+\sqrt{h^2-h+1})$, and $\gamma_2 = (-48+18h+24h^2-6\sqrt{12-30h+25h^2-12h^3+6h^4})[36(h-1)]^{-1}$,

- $c_3(h) = \frac{1}{2\sigma} \max\{g_3([t_3]), g_3([t_3+1])\} \leq \frac{1}{4\sigma} \gamma_1(\gamma_1-1)(h-\gamma_1)\theta_3(\gamma_3),$

where $g_3(t) = \frac{1}{2}t(t-1)(h-t)\theta_3(t)$, $\theta_3(t) = 6t^4 + (2-45h+10h^2)t^3 + (-70+110h+31h^2-20h^3)t^2 + (90-21h-124h^2+27h^3+10h^4)t - 19h^4+28h^3+43h^2-40h-12$, t_3 is the root of $g'_3(t) = 0$ in $[3, h-1]$, and $\gamma_3 = 0.3333h + 1.6779$,

- $c_4(h) = \frac{1}{6\sigma} \max\{g_4([t_4]), g_4([t_4 + 1])\} \leq \frac{3}{384\sigma} (h-2)^2 h^2 \theta_4(\gamma_4),$

where $g_4(t) = \frac{3}{4}(t-1)(t-2)(h+1-t)(h-t)\theta_4(t),$

$\theta_4(t) = t[2t^4 + (2-5h)t^3 + (-6-6h+4h^2)t^2 + (-2-13h+15h^2-3h^3)t + 2h^4-7h^3+9h^2-8h+4],$ t_4 is the root of $g'_4(t) = 0$ in $[3, h-1],$ and $\gamma_4 = 0.5306h + 1.2092,$

- $c_5(h) = \frac{1}{24\sigma} \max\{g_5([t_5]), g_5([t_5 + 1])\} \leq \frac{1}{192\sigma} h^2 (h-1)(h-2)^3 \theta_5(\gamma_5),$

where $g_5(t) = 2(h-1)(h-2)(t-1)(t-2)(h+1-t)(h-t)\theta_5(t),$

$\theta_5(t) = t[-2t^3 + (-4+4h)t^2 + (2+9h-3h^2)t + h^3-3h^2+4h+4],$ t_5 is the root of $g'_5(t) = 0$ in $[3, h-1],$

and $\gamma_5 = \frac{1}{12}(324h^2 + 324h + 12\sqrt{729h^4 + 1134h^3 - 891h^2 - 2700h - 1500})^{1/3} + (3h+5)(324h^2 + 324h + 12\sqrt{1134h^3 - 891h^2 - 2700h - 1500 + 729h^4})^{-1/3} + \frac{1}{2}h - \frac{1}{2},$

- $c_6(h) = \frac{1}{720} \max\{g_6([t_6]), g_6([t_6 + 1])\} \leq \frac{1}{46080} (h+2)^2 h^2 (h-2)^2,$

where $g_6(t) = t(t-1)(t-2)(h+2-t)(h+1-t)(h-t),$ and $t_6 = \frac{h+2}{2}.$

Proof. Without loss of generality, let $\rho : 0 = a = k_1 < k_2 = b = h.$ Then, from (2.18) we have

$$\begin{aligned}
H_\rho f(t) &= \frac{f(0)}{h(h-1)(h-2)} \left[-10(t-h)^{(3)} - \frac{15(t-h)^{(4)}}{h+1} - \frac{6(t-h)^{(5)}}{(h+1)(h+2)} \right] \\
&+ \frac{f(h)}{h(h+1)(h+2)} \left[10t^{(3)} - \frac{15t^{(4)}}{h-1} + \frac{6t^{(5)}}{(h-1)(h-2)} \right] \\
&+ \frac{\Delta f(0)}{h(h-1)(h-2)} \left[-4(h+3)(t-h)^{(3)} - \frac{(7h+16)(t-h)^{(4)}}{h+1} - \frac{3(t-h)^{(5)}}{h+1} \right] \\
&+ \frac{\Delta f(h)}{h(h+1)(h+2)} \left[-4(h-3)t^{(3)} + \frac{(7h-16)t^{(4)}}{h-1} - \frac{3t^{(5)}}{h-1} \right] \\
&+ \frac{\Delta^2 f(0)}{h(h-1)(h-2)} \left[\frac{-(h+2)(h+3)(t-h)^{(3)}}{2} - (h+3)(t-h)^{(4)} - \frac{(t-h)^{(5)}}{2} \right] \\
&+ \frac{\Delta^2 f(h)}{h(h+1)(h+2)} \left[\frac{(h-2)(h-3)t^{(3)}}{2} - (h-3)t^{(4)} + \frac{t^{(5)}}{2} \right].
\end{aligned} \tag{2.22}$$

Hence, on using Theorem 2.1 we obtain

$$f(t) - H_\rho f(t) = \frac{1}{(j-1)!} \sum_{s=0}^{h+2} G_j(t, s) \Delta^j f(s), \quad 2 \leq j \leq 6 \quad (2.23)$$

where

$$\begin{aligned} G_j(t, s) &= (t-s-1)_+^{(j-1)} - \frac{(h-s-1)_+^{(j-1)}}{h(h+1)(h+2)} \left[10t^{(3)} - \frac{15t^{(4)}}{h-1} + \frac{6t^{(5)}}{(h-1)(h-2)} \right] \\ &\quad - (j-1) \frac{(h-s-1)_+^{(j-2)}}{h(h+1)(h+2)} \left[-4(h-3)t^{(3)} + \frac{(7h-16)t^{(4)}}{h-1} - \frac{3t^{(5)}}{h-1} \right] \\ &\quad - (j-1)(j-2) \frac{(h-s-1)_+^{(j-3)}}{h(h+1)(h+2)} \left[\frac{(h-2)(h-3)t^{(3)}}{2} - (h-3)t^{(4)} + \frac{t^{(5)}}{2} \right]. \end{aligned} \quad (2.24)$$

Since $f(t) - H_\rho f(t) = 0$ for $t = 0, 1, 2, h, h+1, h+2$, we have

$$\|f - H_\rho f\| = \max_{t \in N[0, h+2]} |f(t) - H_\rho f(t)| = \max_{t \in N[3, h-1]} |f(t) - H_\rho f(t)|.$$

Further, for $t \in N[3, h-1]$ it is obvious that $G_j(t, h+2) = G_j(t, h+1) = G_j(t, h) = 0$, $2 \leq j \leq 6$. Combining all these, it follows from (2.23) that

$$\|f - H_\rho f\| \leq \frac{1}{(j-1)!} \left(\max_{t \in N[3, h-1]} \sum_{s=0}^{\theta(h)} |G_j(t, s)| \right) \left(\max_{t \in N[0, h+2-j]} |\Delta^j f(t)| \right), \quad 2 \leq j \leq 6 \quad (2.25)$$

where

$$\theta(h) = \begin{cases} h-1, & j=2, 3 \\ h-2, & j=4 \\ h-3, & j=5 \\ h-4, & j=6. \end{cases}$$

We remark that when $h = 4$, the right side of (2.25) is trivially equal to $\frac{1}{(j-1)!}$

$\sum_{s=0}^{\theta(4)} |G_j(3, s)| \cdot \max_{t \in N[0, 6-j]} |\Delta^j f(t)|$, hence in what follows we shall only consider $h \geq 5$.

Case: $j = 2$ From (2.24), we find that

$$G_2(t, s) = \frac{1}{\sigma} \begin{cases} t^{(3)}[\phi_1(t) - (h-1-s)\phi_2(t)], & t \leq s \\ (h+2-t)^{(3)}[\phi_3(t) - s\phi_4(t)], & t \geq s+1 \end{cases} \quad (2.26)$$

where

$$\phi_1(t) = (h-2)(h-t)(4h-3t+5),$$

$$\phi_2(t) = 6t^2 - 15ht - 12t + 10h^2 + 15h + 2,$$

$$\phi_3(t) = (t-1)(3ht + h^2 - 3h + 2),$$

$$\phi_4(t) = 6t^2 + 3ht - 12t + h^2 - 3h + 2.$$

Clearly, $\phi_i(t) \geq 0, 1 \leq i \leq 4$ for $t \in N[3, h-1]$. Further, it is noted that $G_2(t, s)$ changes sign for $t \leq s$ as well as for $t \geq s+1$. We shall make use of the following inequality

$$|a - b| \leq \max\{a, b\}, \quad a, b \geq 0. \quad (2.27)$$

Then, it follows from (2.26) that

$$|G_2(t, s)| \leq \frac{1}{\sigma} \begin{cases} t^{(3)} \max\{\phi_1(t), (h-1-s)\phi_2(t)\}, & t \leq s \\ (h+2-t)^{(3)} \max\{\phi_3(t), s\phi_4(t)\}, & t \geq s+1. \end{cases} \quad (2.28)$$

Subsequently,

$$\begin{aligned} & \sigma \sum_{s=0}^{h-1} |G_2(t, s)| \\ & \leq (h+2-t)^{(3)} \sum_{s=0}^{t-1} \max\{\phi_3(t), s\phi_4(t)\} + t^{(3)} \sum_{s=t}^{h-1} \max\{\phi_1(t), (h-1-s)\phi_2(t)\} \\ & = (h+2-t)^{(3)} \max\left\{t\phi_3(t), \frac{t^{(2)}}{2}\phi_4(t)\right\} + t^{(3)} \max\left\{(h-t)\phi_1(t), \frac{(h-t)^{(2)}}{2}\phi_2(t)\right\}. \end{aligned} \quad (2.29)$$

When $h \leq 8$, we find

$$\sigma \sum_{s=0}^{h-1} |G_2(t, s)| \leq (h+2-t)^{(3)} t \phi_3(t) + t^{(3)}(h-t)\phi_1(t) = g_2(t), \quad 3 \leq t \leq h-1. \quad (2.30)$$

When $h \geq 9$, we get

$$\sigma \sum_{s=0}^{h-1} |G_2(t, s)| \leq \begin{cases} (h+2-t)^{(3)} t \phi_3(t) + \frac{1}{2} t^{(3)}(h-t)^{(2)} \phi_2(t), & 3 \leq t \leq a_1 \\ (h+2-t)^{(3)} t \phi_3(t) + t^{(3)}(h-t)\phi_1(t), & a_1 \leq t \leq a_2 \\ \frac{1}{2} t^{(2)}(h+2-t)^{(3)} \phi_4(t) + t^{(3)}(h-t)\phi_1(t), & a_2 \leq t \leq h-1 \end{cases}$$

where $a_1 = 1 + \frac{3}{4}h - \frac{1}{12}\sqrt{33h^2 + 192}$, $a_2 = 1 + \frac{1}{4}h + \frac{1}{12}\sqrt{33h^2 + 192}$, and further

$$\begin{aligned} \max_{t \in [3, h-1]} \sigma \sum_{s=0}^{h-1} |G_2(t, s)| &\leq \max_{t \in [a_1, a_2]} \{ (h+2-t)^{(3)} t \phi_3(t) + t^{(3)}(h-t)\phi_1(t) \} \\ &= \max_{t \in [a_1, a_2]} g_2(t). \end{aligned} \quad (2.31)$$

It follows from (2.30) and (2.31) that for all h we have

$$\max_{t \in N[3, h-1]} \sigma \sum_{s=0}^{h-1} |G_2(t, s)| \leq \max_{t \in N[3, h-1]} g_2(t) = \max\{g_2([t_2]), g_2([t_2 + 1])\} \quad (2.32)$$

where t_2 is the root of $g'_2(t) = 0$ in $[3, h-1]$.

Moreover, a convenient upper bound for $g_2(t)$ can be obtained as

$$\max_{t \in N[3, h-1]} g_2(t) \leq \left\{ \max_{t \in [3, h-1]} t(t-1)(h-t) \right\} \cdot \left\{ \max_{t \in [3, h-1]} \theta_2(t) \right\} = \gamma_1(\gamma_1-1)(h-\gamma_1)\theta_2(\gamma_2). \quad (2.33)$$

Substituting (2.32) and (2.33) in (2.25) immediately gives the error estimate (2.21) for $j = 2$. The technique used in this case will be applied to the cases $j = 3, 4, 5$, as such our presentation for these cases will be brief.

Case: $j = 3$ Relation (2.24) provides

$$G_3(t, s) = \frac{1}{\sigma} \begin{cases} t^{(3)}[(h-s-1)\phi_5(t) - (h-s-1)^{(2)}\phi_6(t) - \phi_7(t)], & t \leq s \\ (h+2-t)^{(3)}[\phi_8(t) - s\phi_9(t) + s^2\phi_{10}(t)], & t \geq s+1 \end{cases} \quad (2.34)$$

where

$$\phi_5(t) = 2(h-2)(h-t)(4h-3t+5),$$

$$\phi_6(t) = (10h-10)(h-2) - (15h-30)(t-3) + 6(t-3)(t-4),$$

$$\phi_7(t) = (h-1)(h-2)(h-t)(h+1-t),$$

$$\phi_8(t) = (t-1)(t-2)(h-1)(h-2),$$

$$\phi_9(t) = 6ht^2 - 6t^2 + 2h^2t - 15ht + 16t + 9h - 3h^2 - 6,$$

$$\phi_{10}(t) = 6t^2 - 12t + 3ht + h^2 - 3h + 2.$$

We note that $\phi_i(t) \geq 0$, $5 \leq i \leq 10$ for $t \in N[3, h-1]$. Applying inequality (2.27), it follows from (2.34) that

$$|G_3(t, s)| \leq \frac{1}{\sigma} \begin{cases} t^{(3)} \max\{(h-s-1)\phi_5(t), |(h-s-1)^{(2)}|\phi_6(t) + \phi_7(t)\}, & t \leq s \\ (h+2-t)^{(3)} \max\{\phi_8(t) + s^2\phi_{10}(t), s\phi_9(t)\}, & t \geq s+1. \end{cases}$$

Hence, we find

$$\begin{aligned} & \sigma \sum_{s=0}^{h-1} |G_3(t, s)| \\ & \leq (h+2-t)^{(3)} \max \left\{ \sum_{s=0}^{t-1} [\phi_8(t) + s^2\phi_{10}(t)], \sum_{s=0}^{t-1} s\phi_9(t) \right\} \\ & \quad + t^{(3)} \max \left\{ \sum_{s=t}^{h-1} (h-s-1)\phi_5(t), \sum_{s=t}^{h-1} [(h-s-1)^{(2)}|\phi_6(t) + \phi_7(t)] \right\} \end{aligned}$$

$$\begin{aligned}
&= (h+2-t)^{(3)} \max \left\{ t\phi_8(t) + \frac{t^{(2)}(2t-1)}{6}\phi_{10}(t), \frac{t^{(2)}}{2}\phi_9(t) \right\} \\
&\quad + t^{(3)} \max \left\{ \frac{(h-t)^{(2)}}{2}\phi_5(t), \frac{(h-t)^{(3)}}{3}\phi_6(t) + (h-t)\phi_7(t) \right\}.
\end{aligned}$$

When $h \leq 16$, we find

$$\begin{aligned}
&\sigma \sum_{s=0}^{h-1} |G_3(t, s)| \\
&\leq \begin{cases} (h+2-t)^{(3)} \frac{t^{(2)}}{2}\phi_9(t) + t^{(3)} \frac{(h-t)^{(2)}}{2}\phi_5(t), & 3 \leq t \leq a_3 \\ (h+2-t)^{(3)} \left[t\phi_8(t) + \frac{t^{(2)}(2t-1)}{6}\phi_{10}(t) \right] + t^{(3)} \left[\frac{(h-t)^{(3)}}{3}\phi_6(t) + (h-t)\phi_7(t) \right], & a_3 \leq t \leq h-1 \end{cases}
\end{aligned}$$

where $a_3 = \frac{1}{2}(-1 + h + \frac{1}{3}\sqrt{-3 + 3h^2})$.

When $h \geq 17$, we get

$$\begin{aligned}
&\sigma \sum_{s=0}^{h-1} |G_3(t, s)| \\
&\leq \begin{cases} (h+2-t)^{(3)} \left[t\phi_8(t) + \frac{t^{(2)}(2t-1)}{6}\phi_{10}(t) \right] + t^{(3)} \left[\frac{(h-t)^{(3)}}{3}\phi_6(t) + (h-t)\phi_7(t) \right], & 3 \leq t \leq a_4 \\ (h+2-t)^{(3)} \frac{t^{(2)}}{2}\phi_9(t) + t^{(3)} \frac{(h-t)^{(2)}}{2}\phi_5(t), & a_4 \leq t \leq a_3 \\ (h+2-t)^{(3)} \left[t\phi_8(t) + \frac{t^{(2)}(2t-1)}{6}\phi_{10}(t) \right] + t^{(3)} \left[\frac{(h-t)^{(3)}}{3}\phi_6(t) + (h-t)\phi_7(t) \right], & a_3 \leq t \leq h-1 \end{cases}
\end{aligned}$$

where $a_4 = \frac{1}{2}(-1 + h - \frac{1}{3}\sqrt{-3 + 3h^2})$. Hence, for all h we obtain

$$\begin{aligned}
\max_{t \in N[3, h-1]} \sigma \sum_{s=0}^{h-1} |G_3(t, s)| &\leq \max_{t \in N[3, h-1]} \left\{ (h+2-t)^{(3)} \frac{t^{(2)}}{2}\phi_9(t) + t^{(3)} \frac{(h-t)^{(2)}}{2}\phi_5(t) \right\} \\
&= \max_{t \in N[3, h-1]} g_3(t) = \max \{g_3([t_3]), g_3([t_3 + 1])\}
\end{aligned} \tag{2.35}$$

where t_3 is the root of $g'_3(t) = 0$ in $[3, h-1]$.

Also, a convenient upper bound for $g_3(t)$ can be found as

$$\begin{aligned} \max_{t \in N[3, h-1]} g_3(t) &\leq \left\{ \max_{t \in [3, h-1]} \frac{1}{2} t(t-1)(h-t) \right\} \cdot \left\{ \max_{t \in [3, h-1]} \theta_3(t) \right\} \\ &= \frac{1}{2} \gamma_1(\gamma_1 - 1)(h - \gamma_1) \theta_3(\gamma_3). \end{aligned} \quad (2.36)$$

Note that here the maximum of $\theta_3(t)$ occurs *approximately* at γ_3 . The exact point where $\theta_3(t)$ is maximum can be explicitly found, but the expression is too long and therefore for practical purposes the approximation γ_3 is used.

Substituting (2.35) and (2.36) in (2.25) immediately yields (2.21) for $j = 3$.

Case: $j = 4$ Relation (2.24) provides

$$G_4(t, s) = \frac{1}{\sigma} \begin{cases} t^{(3)}[(h-s-1)^{(2)}\phi_{11}(t) - (h-s-1)^{(3)}\phi_{12}(t) - (h-s-1)\phi_{13}(t)], & t \leq s \\ (h+2-t)^{(3)}(s+1)[s^2\phi_{14}(t) - s\phi_{15}(t) + \phi_{16}(t)], & t \geq s+1 \end{cases} \quad (2.37)$$

where

$$\phi_{11}(t) = 3(h-2)(h-t)(4h+5-3t),$$

$$\phi_{12}(t) = 10h^2 + 15h + 2 - 15ht - 12t + 6t^2,$$

$$\phi_{13}(t) = 3(h-1)(h-2)(h-t)(h+1-t),$$

$$\phi_{14}(t) = 6t^2 + (3h-12)t + h^2 - 3h + 2,$$

$$\phi_{15}(t) = (9h-12)t^2 + (3h^2-24h+30)t - 5h^2 + 15h - 10,$$

$$\phi_{16}(t) = 3(h-1)(h-2)(t-1)(t-2).$$

We note that $\phi_i(t) \geq 0$, $11 \leq i \leq 16$ for $t \in N[3, h-1]$. Also,

$$(h-s-1)^{(2)}\phi_{11}(t) - (h-s-1)^{(3)}\phi_{12}(t) \geq 0, \quad t \in N[3, h-2], \quad s \in N[3, h-2], \quad t \leq s$$

and

$$s\phi_{15}(t) - s^2\phi_{14}(t) \geq 0, \quad t \in N[3, h-1], \quad s \in N[0, h-2], \quad t \geq s+1.$$

Thus, applying inequality (2.27), from (2.37) we obtain

$$\sigma|G_4(t, s)| \leq \begin{cases} t^{(3)} \max \{ (h-s-1)^{(2)}\phi_{11}(t) - (h-s-1)^{(3)}\phi_{12}(t), (h-s-1)\phi_{13}(t) \}, & t \leq s \\ (h+2-t)^{(3)}(s+1) \max \{ s\phi_{15}(t) - s^2\phi_{14}(t), \phi_{16}(t) \}, & t \geq s+1. \end{cases}$$

It follows that

$$\begin{aligned} & \sigma \sum_{s=0}^{h-2} |G_4(t, s)| \\ & \leq (h+2-t)^{(3)} \max \left\{ \sum_{s=0}^{t-1} [(s+1)s\phi_{15}(t) - s^2(s+1)\phi_{14}(t)], \sum_{s=0}^{t-1} (s+1)\phi_{16}(t) \right\} \\ & + t^{(3)} \max \left\{ \sum_{s=t}^{h-2} [(h-s-1)^{(2)}\phi_{11}(t) - (h-s-1)^{(3)}\phi_{12}(t)], \sum_{s=t}^{h-2} (h-s-1)\phi_{13}(t) \right\} \\ & = (h+2-t)^{(3)} \max \left\{ \frac{t(t^2-1)}{3}\phi_{15}(t) - \frac{1}{12}t(t-1)(3t-2)(t+1)\phi_{14}(t), \frac{t(t+1)}{2}\phi_{16}(t) \right\} \\ & + t^{(3)} \max \left\{ \frac{(h-t)^{(3)}}{3}\phi_{11}(t) - \frac{(h-t)^{(4)}}{4}\phi_{12}(t), \frac{(h-t)^{(2)}}{2}\phi_{13}(t) \right\}. \end{aligned}$$

When $h \leq 8$, we find

$$\begin{aligned}
& \sigma \sum_{s=0}^{h-2} |G_4(t, s)| \\
& \leq (h+2-t)^{(3)} \left\{ \frac{t(t^2-1)}{3} \phi_{15}(t) - \frac{1}{12} t(t-1)(3t-2)(t+1) \phi_{14}(t) \right\} + t^{(3)} \frac{(h-t)^{(2)}}{2} \phi_{13}(t) \\
& = g_4(t), \quad 3 \leq t \leq h-1.
\end{aligned}$$

When $h \geq 9$, we have

$$\begin{aligned}
& \sigma \sum_{s=0}^{h-2} |G_4(t, s)| \\
& \leq \begin{cases} (h+2-t)^{(3)} \frac{t(t+1)}{2} \phi_{16}(t) + t^{(3)} \left\{ \frac{(h-t)^{(3)}}{3} \phi_{11}(t) - \frac{(h-t)^{(4)}}{4} \phi_{12}(t) \right\}, & 3 \leq t \leq \frac{h-2}{2} \\ (h+2-t)^{(3)} \left\{ \frac{t(t^2-1)}{3} \phi_{15}(t) - \frac{1}{12} t(t-1)(3t-2)(t+1) \phi_{14}(t) \right\} \\ \quad + t^{(3)} \frac{(h-t)^{(2)}}{2} \phi_{13}(t), & \frac{h-2}{2} \leq t \leq h-1. \end{cases}
\end{aligned}$$

Thus, we get for all h ,

$$\max_{t \in N[3, h-1]} \sigma \sum_{s=0}^{h-2} |G_4(t, s)| \leq \max_{t \in N[3, h-1]} g_4(t) = \max \{g_4([t_4]), g_4([t_4 + 1])\} \quad (2.38)$$

where t_4 is the root of $g'_4(t) = 0$ in $[3, h-1]$.

Moreover, a convenient upper bound for $g_4(t)$ can be obtained as

$$\begin{aligned}
\max_{t \in N[3, h-1]} g_4(t) & \leq \left\{ \max_{t \in [3, h-1]} \frac{3}{4} (t-1)(t-2)(h+1-t)(h-t) \right\} \cdot \left\{ \max_{t \in [3, h-1]} \theta_4(t) \right\} \\
& = \frac{3}{64} (h-2)^2 h^2 \theta_4(\gamma_4).
\end{aligned} \quad (2.39)$$

Note that here the maximum of $\theta_4(t)$ occurs *approximately* at γ_4 . The approximation γ_4 is used because the expression of the exact point where $\theta_4(t)$ is maximum is too long and complicated.

Substituting (2.38) and (2.39) in (2.25) immediately gives (2.21) for $j = 4$.

Case: $j = 5$ Relation (2.24) provides

$$G_5(t, s) = \frac{1}{\sigma} \begin{cases} t^{(3)}[(h-s-1)^{(3)}\phi_{17}(t) - (h-s-1)^{(4)}\phi_{18}(t) - (h-s-1)^{(2)}\phi_{19}(t)], & t \leq s \\ (h+2-t)^{(3)}(s+2)^{(2)}[s^2\phi_{20}(t) - s\phi_{21}(t) + \phi_{22}(t)], & t \geq s+1 \end{cases} \quad (2.40)$$

where

$$\phi_{17}(t) = 4(h-2)(h-t)(4h+5-3t),$$

$$\phi_{18}(t) = 6t^2 + (-15h-12)t + 10h^2 + 15h + 2,$$

$$\phi_{19}(t) = 6(h-1)(h-2)(h-t)(h+1-t),$$

$$\phi_{20}(t) = 6t^2 + (3h-12)t + h^2 - 3h + 2,$$

$$\phi_{21}(t) = (12h-18)t^2 + (4h^2-33h+44)t - 7h^2 + 21h - 14,$$

$$\phi_{22}(t) = 6(h-1)(h-2)(t-1)(t-2).$$

We note that $\phi_i(t) \geq 0$, $17 \leq i \leq 22$ for $t \in N[3, h-1]$. Moreover,

$$(h-s-1)^{(3)}\phi_{17}(t) - (h-s-1)^{(4)}\phi_{18}(t) \geq 0, \quad t \in N[3, h-3], \quad s \in N[3, h-3], \quad t \leq s$$

and

$$s\phi_{21}(t) - s^2\phi_{20}(t) \geq 0, \quad t \in N[3, h-1], \quad s \in N[0, h-3], \quad t \geq s+1.$$

Then, using inequality (2.27), from (2.40) we get

$$\begin{aligned} & \sigma |G_5(t, s)| \\ & \leq \begin{cases} t^{(3)} \max \{ (h-s-1)^{(3)} \phi_{17}(t) - (h-s-1)^{(4)} \phi_{18}(t), (h-s-1)^{(2)} \phi_{19}(t) \}, & t \leq s \\ (h+2-t)^{(3)} (s+2)^{(2)} \max \{ s \phi_{21}(t) - s^2 \phi_{20}(t), \phi_{22}(t) \}, & t \geq s+1. \end{cases} \end{aligned}$$

On summing, we find

$$\begin{aligned} & \sigma \sum_{s=0}^{h-3} |G_5(t, s)| \\ & \leq (h+2-t)^{(3)} \max \left\{ \sum_{s=0}^{t-1} [(s+2)^{(3)} \phi_{21}(t) - (s+2)^{(3)} s \phi_{20}(t)], \sum_{s=0}^{t-1} (s+2)^{(2)} \phi_{22}(t) \right\} \\ & \quad + t^{(3)} \max \left\{ \sum_{s=t}^{h-3} [(h-s-1)^{(3)} \phi_{17}(t) - (h-s-1)^{(4)} \phi_{18}(t)], \sum_{s=t}^{h-3} (h-s-1)^{(2)} \phi_{19}(t) \right\} \\ & = (h+2-t)^{(3)} \max \left\{ \frac{(t+2)^{(4)}}{4} \phi_{21}(t) - \frac{1}{20} (t+2)^{(4)} (4t-3) \phi_{20}(t), \frac{(t+2)^{(3)}}{3} \phi_{22}(t) \right\} \\ & \quad + t^{(3)} \max \left\{ \frac{(h-t)^{(4)}}{4} \phi_{17}(t) - \frac{(h-t)^{(5)}}{5} \phi_{18}(t), \frac{(h-t)^{(3)}}{3} \phi_{19}(t) \right\} \\ & = \frac{1}{3} (h+2-t)^{(3)} (t+2)^{(3)} \phi_{22}(t) + \frac{1}{3} t^{(3)} (h-t)^{(3)} \phi_{19}(t) = g_5(t). \end{aligned}$$

Hence,

$$\max_{t \in N[3, h-1]} \sigma \sum_{s=0}^{h-3} |G_5(t, s)| \leq \max_{t \in N[3, h-1]} g_5(t) = \max \{ g_5([t_5]), g_5([t_5 + 1]) \} \quad (2.41)$$

where t_5 is the root of $g'_5(t) = 0$ in $[3, h-1]$.

Further, a convenient upper bound for $g_5(t)$ can be obtained as

$$\begin{aligned}
& \max_{t \in N[3, h-1]} g_5(t) \\
& \leq 2(h-1)(h-2) \left\{ \max_{t \in [3, h-1]} (t-1)(t-2)(h+1-t)(h-t) \right\} \cdot \left\{ \max_{t \in [3, h-1]} \theta_5(t) \right\} \\
& = \frac{1}{8} h^2 (h-1)(h-2)^3 \theta_5(\gamma_5).
\end{aligned} \tag{2.42}$$

Substituting (2.41) and (2.42) in (2.25) immediately yields (2.21) for $j = 5$.

Case: $j = 6$ From [2, Theorem 5.1] we have

$$\|f - H_\rho f\| \leq \frac{1}{6!} \max_{t \in N[3, h-1]} g_6(t) \cdot \max_{t \in N[0, h-4]} |\Delta^6 f(t)| \tag{2.43}$$

where $g_6(t) = t(t-1)(t-2)(h+2-t)(h+1-t)(h-t)$. Setting $g'_6(t) = 0$, we get $t = t_6 = \frac{h+2}{2} \in [3, h-1]$ which maximizes $g_6(t)$. Note that $[t_6], [t_6 + 1] \in N[3, h-1]$.

Hence,

$$c_6(h) = \frac{1}{6!} \max_{t \in N[3, h-1]} g_6(t) = \frac{1}{6!} \max \{g_6([t_6]), g_6([t_6 + 1])\}. \tag{2.44}$$

Further, it is clear that

$$c_6(h) = \frac{1}{6!} \max_{t \in N[3, h-1]} g_6(t) \leq \frac{1}{6!} \max_{t \in [3, h-1]} g_6(t) = \frac{1}{6!} g_6(t_6) = \frac{1}{46080} (h+2)^2 h^2 (h-2)^2. \tag{2.45}$$

This completes the proof of (2.21) for $j = 6$. $\square$

The explicit presentation of the bound for $\|f - H_\rho f\|$ has been given by (2.21) in terms of $|\Delta^j f(t)|$, $2 \leq j \leq 6$. The sharpness of the bound shall be illustrated by the numerical examples in section 2.3.

Remark 2.1. The case $j = 6$ is given in [2, Theorem 7.3] as follows

$$\|f - H_\rho f\| \leq \frac{1}{46080} h^6 \max_{t \in N[a, b-4]} |\Delta^6 f(t)|.$$

Since

$$c_6(h) \leq \frac{1}{46080} (h+2)^2 h^2 (h-2)^2 \leq \frac{1}{46080} h^6,$$

our result is an improvement.

2.3 Numerical Examples

We shall illustrate the sharpness of the error estimates obtained in Theorem 2.2 by three numerical examples. In each example, we take a function $f(t)$ and construct $H_\rho f(t)$, then we compare the actual error $\|f - H_\rho f\|$ with the respective bound provided in Theorem 2.2.

We remark that the functions considered in the examples are not differentiable at certain points and therefore cannot be approximated by continuous Hermite interpolation (which involves derivatives).

In each example, the steps taken to construct $H_\rho f(t)$ and the related bound are as follows:

- For a function $f(t)$ defined on $N[a, b + 2]$, fix the partition ρ and the step size h .
- Obtain the values $f(k_i), \Delta f(k_i), \Delta^2 f(k_i)$, $1 \leq i \leq m$, then construct $H_\rho f(t)$ in each subinterval $N[k_{i-1}, k_i]$, $2 \leq i \leq m$ as follows:

$$\begin{aligned} H_\rho f(t) = & f(k_i)h_i(t) + \Delta f(k_i)\bar{h}_i(t) + \Delta^2 f(k_i)\bar{\bar{h}}_i(t) + f(k_{i-1})h_{i-1}(t) \\ & + \Delta f(k_{i-1})\bar{h}_{i-1}(t) + \Delta^2 f(k_{i-1})\bar{\bar{h}}_{i-1}(t) \end{aligned}$$

$$\begin{aligned}
&= f(k_i) \left[\frac{10(t-k_{i-1})^{(3)}}{(h+2)^{(3)}} - \frac{15(t-k_{i-1})^{(4)}}{(h+2)^{(4)}} + \frac{6(t-k_{i-1})^{(5)}}{(h+2)^{(5)}} \right] \\
&\quad + \Delta f(k_i) \left[\frac{4(h-3)(t-k_{i-1})^{(3)}}{-(h+2)^{(3)}} + \frac{(7h-16)(t-k_{i-1})^{(4)}}{(h+2)^{(4)}} - \frac{3(t-k_{i-1})^{(5)}}{(h+2)^{(4)}} \right] \\
&\quad + \Delta^2 f(k_i) \left[\frac{(h-2)^{(2)}(t-k_{i-1})^{(3)}}{2(h+2)^{(3)}} - \frac{(h-3)(t-k_{i-1})^{(4)}}{(h+2)^{(3)}} + \frac{(t-k_{i-1})^{(5)}}{2(h+2)^{(3)}} \right] \\
&\quad + f(k_{i-1}) \left[\frac{10(t-k_{i+1})^{(3)}}{-h^{(3)}} - \frac{15(t-k_{i+1})^{(4)}}{(h+1)^{(4)}} - \frac{6(t-k_{i+1})^{(5)}}{(h+2)^{(5)}} \right] \\
&\quad + \Delta f(k_{i-1}) \left[\frac{4(h+3)(t-k_{i+1})^{(3)}}{-h^{(3)}} - \frac{(7h+16)(t-k_{i+1})^{(4)}}{(h+1)^{(4)}} - \frac{3(t-k_{i+1})^{(5)}}{(h+1)^{(4)}} \right] \\
&\quad + \Delta^2 f(k_{i-1}) \left[\frac{(h+3)^{(2)}(t-k_{i+1})^{(3)}}{-2h^{(3)}} - \frac{(h+3)(t-k_{i+1})^{(4)}}{h^{(3)}} - \frac{(t-k_{i+1})^{(5)}}{2h^{(3)}} \right].
\end{aligned}$$

- Compute the actual error $\|f - H_\rho f\| = \max_{t \in N[a, b+2]} |f(t) - H_\rho f(t)|$.
- Obtain the bound $c_j(h) \max_{t \in N[a, b+2-j]} |\Delta^j f(t)|$
 - (i) check that $g'_j(3) > 0$, $g'_j(h-1) < 0$,
 - (ii) solve $g'_j(t) = 0$ to get $t_j \in N[3, h-1]$,
 - (iii) compute $c_j(h)$ which is explicitly given in Theorem 2.2,
 - (iv) compute $\max_{t \in N[a, b+2-j]} |\Delta^j f(t)|$.

In the following three examples, let t_0 be such that $\|f - H_\rho f\| = |f(t_0) - H_\rho f(t_0)|$.

Example 2.1. Consider

$$f(t) = t(t-1)|t-2|(t-3)(t-4)(t-5)(t-6)/10^8$$

with $a = -58$, $b = 62$, $j = 4$ and partitions

$$\{-58, -38, -18, 2, 22, 42, 62\} \quad (m = 7),$$

$$\{-58, -43, -28, -13, 2, 17, 32, 47, 62\} \quad (m = 9),$$

$$\{-58, -48, -38, -28, -18, -8, 2, 12, 22, 32, 42, 52, 62\} \quad (m = 13),$$

$$\{-58, -52, -46, -40, -34, -28, -22, -16, -10, -4, 2, 8, 14, 20, 26, 32, 38, 44, 50, 56, 62\} \\ (m = 21).$$

It is clear that $f(t)$ is not differentiable when $t = 2$, so it is not possible to perform continuous Hermite interpolation with the above partitions.

Table 2.1: Actual error $\|f - H_\rho f\|$ and error bound, $j = 4$

m	7 ($h = 20$)	9 ($h = 15$)	13 ($h = 10$)	21 ($h = 6$)
$\ f - H_\rho f\ $	0.34303500e + 01	0.62318592	0.55440000e - 01	0.22982401e - 02
Bound	0.87385273e + 03	0.25836373e + 03	0.45077073e + 02	0.44349120e + 01
CPU time (s)*	5.97	8.07	10.36	13.93

* Matlab programme is performed on a laptop of FUJITSU LIFEBOOK S Series.

Relative error $\frac{\ f - H_\rho f\ }{ f(t_0) }$	0.44155404e - 03	0.53315534e - 04	0.36584463e - 05	0.11807143e - 06
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Following the steps listed earlier, Hermite interpolant is constructed for different values of m , the results are shown in Table 2.1, Figures 2.1 and 2.2. From Table 2.1, it is evident that as the step size h decreases the error (actual and relative) and the error bound also get smaller, as expected. Obviously, smaller step sizes require more CPU time. With reference to Figure 2.2, we note from the Matlab computations that the maximum error happens at $t = 53$ when $m = 7$ ($h = 20$).

Example 2.2. Consider

$$f(t) = t|t - 1|(t - 2)(t - 3)(t - 4)(t - 5)(t - 6)/10^8$$

with $a = -59$, $b = 61$, $j = 5$ and partitions $m = 7, 9, 13, 21$.

It is clear that $f(t)$ is not differentiable when $t = 1$, so it is not possible to perform continuous Hermite interpolation with the above partitions.

Table 2.2: Actual error $\|f - H_\rho f\|$ and error bound, $j = 5$

m	7 ($h = 20$)	9 ($h = 15$)	13 ($h = 10$)	21 ($h = 6$)
$\ f - H_\rho f\ $	0.34989570e + 01	0.63504000	0.56448000e - 01	0.23385603e - 02
Bound	0.16485638e + 03	0.38726767e + 02	0.51943385e + 01	0.38232000e + 00
CPU time (s)*	7.13	7.46	10.47	15.32

* Matlab programme is performed on a laptop of FUJITSU LIFEBOOK S Series.

Relative error $\frac{\ f - H_\rho f\ }{ f(t_0) }$	0.39200187e - 03	0.47657621e - 04	0.32830187e - 05	0.10635597e - 06
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From Table 2.2, it is noted that the error (actual and relative) and the bound tend to be smaller as the step size h decreases. Obviously, smaller step sizes require more CPU time.

Example 2.3. Consider

$$f(t) = |t|(t-1)(t-2)(t-3)(t-4)(t-5)(t-6)/10^8$$

with $a = -60$, $b = 60$, $j = 6$ and partitions $m = 7, 9, 13, 21$.

It is clear that $f(t)$ is not differentiable when $t = 0$, so it is not possible to perform continuous Hermite interpolation with the above partitions.

Table 2.3: Actual error $\|f - H_\rho f\|$ and error bound, $j = 6$

m	7 ($h = 20$)	9 ($h = 15$)	13 ($h = 10$)	21 ($h = 6$)
$\ f - H_\rho f\ $	0.35675640e + 01	0.64689408	0.57456000e - 01	0.22982401e - 02
Bound	0.41164200e + 01	0.71124480e + 00	0.60480000e - 01	0.24192000e - 02
CPU Time (s)*	4.85	8.04	10.73	14.85

* Matlab programme is performed on a laptop of FUJITSU LIFEBOOK S Series.

Relative error $\frac{\ f - H_\rho f\ }{ f(t_0) }$	0.34881877e - 03	0.426880816e - 04	0.29517855e - 05	0.95974745e - 07
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From Table 2.3, we observe that the error (actual and relative) and the bound become smaller as the step size h becomes smaller. Obviously, smaller step sizes

require more CPU time.

Figure 2.1: Example 2.1 when $m = 7$ ($h = 20$)

Figure 2.2: Enlarged portion of Fig. 2.1 where the error $|f(t) - H_\rho f(t)|$ is large

2.4 Two-variable Discrete Hermite Interpolation

Discrete Hermite interpolation is extended to two-variable Hermite interpolation $H_\tau f(t, u)$ in this section. We shall define $H_\tau f(t, u)$ and obtain the error estimation between f and $H_\tau f(t, u)$ in the following.

Following the continuous case [45, 46], we define

$$\begin{aligned}
 H(\tau) &= H(\rho) \otimes H(\rho') \quad (\text{the tensor product}) \\
 &= \text{Span} \left\{ h_i(t)h_j(u), h_i(t)\bar{h}_j(u), h_i(t)\bar{\bar{h}}_j(u), \bar{h}_i(t)h_j(u), \bar{h}_i(t)\bar{h}_j(u), \right. \\
 &\quad \left. \bar{\bar{h}}_i(t)\bar{\bar{h}}_j(u), \bar{\bar{h}}_i(t)h_j(u), \bar{\bar{h}}_i(t)\bar{h}_j(u), \bar{\bar{h}}_i(t)\bar{\bar{h}}_j(u) \right\}_{i=1}^m \left\{ j=1 \right\}^n \\
 &= \left\{ g(t, u) \in D^{(2,2)}([a, b] \times [c, d]) : g(t, u) \text{ is a two-dimensional} \right. \\
 &\quad \text{polynomial of degree 5 in each variable and in each subrectangle} \\
 &\quad N[k_i, k_{i+1}] \times N[l_j, l_{j+1}], \quad N[k_{m-1}, b+2] \times N[l_j, l_{j+1}], \\
 &\quad N[k_i, k_{i+1}] \times N[l_{n-1}, d+2], \quad 1 \leq i \leq m-2, \quad 1 \leq j \leq n-2, \\
 &\quad \left. \text{and } N[k_{m-1}, b+2] \times N[l_{n-1}, d+2] \right\}.
 \end{aligned}$$

Since $H(\tau)$ is the tensor product of $H(\rho)$ and $H(\rho')$ which are of dimensions $3m$ and $3n$ respectively, $H(\tau)$ is of dimension $9mn$.

Definition 2.3. For a given function $f(t, u)$ defined on $N[a, b+2] \times N[c, d+2]$, we shall denote $f_{i,j}^{\mu,\nu} = \Delta_t^\mu \Delta_u^\nu f(k_i, l_j)$, $\mu, \nu = 0, 1, 2$, $1 \leq i \leq m$, $1 \leq j \leq n$. We say $H_\tau f(t, u)$ is the $H(\tau)$ -interpolant of $f(t, u)$, also known as the *discrete Hermite interpolant* of $f(t, u)$, if $H_\tau f(t, u) \in H(\tau)$ with

$$\Delta_t^\mu \Delta_u^\nu H_\tau f(k_i, l_j) = f_{i,j}^{\mu,\nu}, \quad \mu, \nu = 0, 1, 2, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

Clearly, $H_\tau f(t, u)$ can be explicitly expressed as

$$\begin{aligned}
H_\tau f(t, u) = & \sum_{i=1}^m \sum_{j=1}^n \left[f_{i,j}^{0,0} h_i(t) h_j(u) + f_{i,j}^{0,1} h_i(t) \bar{h}_j(u) + f_{i,j}^{0,2} h_i(t) \bar{\bar{h}}_j(u) \right. \\
& + f_{i,j}^{1,0} \bar{h}_i(t) h_j(u) + f_{i,j}^{1,1} \bar{h}_i(t) \bar{h}_j(u) + f_{i,j}^{1,2} \bar{h}_i(t) \bar{\bar{h}}_j(u) + f_{i,j}^{2,0} \bar{\bar{h}}_i(t) h_j(u) \\
& \left. + f_{i,j}^{2,1} \bar{\bar{h}}_i(t) \bar{h}_j(u) + f_{i,j}^{2,2} \bar{\bar{h}}_i(t) \bar{\bar{h}}_j(u) \right].
\end{aligned} \tag{2.46}$$

The following result provides a characterization of $H_\tau f(t, u)$ in terms of one-variable interpolation schemes.

Lemma 2.2. *Let $f(t, u)$ be defined on $N[a, b+2] \times N[c, d+2]$. Then,*

$$H_\tau f(t, u) = H_{\rho'} H_\rho f(t, u) = H_\rho H_{\rho'} f(t, u). \tag{2.47}$$

Proof. By (2.18) we find

$$\begin{aligned}
H_{\rho'} H_\rho f(t, u) &= H_{\rho'} \left\{ \sum_{i=1}^m \left[f(k_i, u) h_i(t) + \Delta_t f(k_i, u) \bar{h}_i(t) + \Delta_t^2 f(k_i, u) \bar{\bar{h}}_i(t) \right] \right\} \\
&= \sum_{j=1}^n \left\{ \sum_{i=1}^m \left[f_{i,j}^{0,0} h_i(t) + f_{i,j}^{1,0} \bar{h}_i(t) + f_{i,j}^{2,0} \bar{\bar{h}}_i(t) \right] h_j(u) \right. \\
&\quad + \sum_{i=1}^m \left[f_{i,j}^{0,1} h_i(t) + f_{i,j}^{1,1} \bar{h}_i(t) + f_{i,j}^{2,1} \bar{\bar{h}}_i(t) \right] \bar{h}_j(u) \\
&\quad \left. + \sum_{i=1}^m \left[f_{i,j}^{0,2} h_i(t) + f_{i,j}^{1,2} \bar{h}_i(t) + f_{i,j}^{2,2} \bar{\bar{h}}_i(t) \right] \bar{\bar{h}}_j(u) \right\} \\
&= H_\tau f(t, u).
\end{aligned}$$

The proof of the second equality in (2.47) is similar. $\square$

Now let $f(t, u)$ be an arbitrary function defined on $N[a, b+2] \times N[c, d+2]$.

From Lemma 2.2, we have

$$\begin{aligned} f - H_\tau f &= (f - H_\rho f) + H_\rho(f - H_{\rho'} f) \\ &= (f - H_\rho f) + [H_\rho(f - H_{\rho'} f) - (f - H_{\rho'} f)] + (f - H_{\rho'} f) \end{aligned} \quad (2.48)$$

$$= (f - H_\rho f) + [H_{\rho'}(f - H_\rho f) - (f - H_\rho f)] + (f - H_{\rho'} f). \quad (2.49)$$

Using these relations we shall deduce error estimates for two-variable discrete Hermite interpolation.

Theorem 2.3. *Let $f(t, u)$ be defined on $N[a, b+2] \times N[c, d+2]$. Then, $4 \leq j \leq 6$,*

$$\begin{aligned} &\|f - H_\tau f\| \\ &\leq c_j(h) \max_{\substack{t \in N[a, b+2-j] \\ u \in N[c, d+2]}} |\Delta_t^j f(t, u)| + c_j(h') \max_{\substack{t \in N[a, b+2] \\ u \in N[c, d+2-j]}} |\Delta_u^j f(t, u)| + A_j, \end{aligned} \quad (2.50)$$

where

$$A_4 = c_2(h)c_2(h') \max_{\substack{t \in N[a, b] \\ u \in N[c, d]}} |\Delta_t^2 \Delta_u^2 f(t, u)|, \quad (2.51)$$

$$A_5 = \begin{cases} c_2(h)c_3(h') \max_{\substack{t \in N[a, b] \\ u \in N[c, d-1]}} |\Delta_t^2 \Delta_u^3 f(t, u)|, & \text{or} \\ c_3(h)c_2(h') \max_{\substack{t \in N[a, b-1] \\ u \in N[c, d]}} |\Delta_t^3 \Delta_u^2 f(t, u)|. \end{cases} \quad (2.52)$$

$$A_6 = \begin{cases} c_2(h)c_4(h') \max_{\substack{t \in N[a, b] \\ u \in N[c, d-2]}} |\Delta_t^2 \Delta_u^4 f(t, u)|, & \text{or} \\ c_3(h)c_3(h') \max_{\substack{t \in N[a, b-1] \\ u \in N[c, d-1]}} |\Delta_t^3 \Delta_u^3 f(t, u)|, & \text{or} \\ c_4(h)c_2(h') \max_{\substack{t \in N[a, b-2] \\ u \in N[c, d]}} |\Delta_t^4 \Delta_u^2 f(t, u)|. \end{cases} \quad (2.53)$$

Proof. We shall prove the case when $j = 4$, the arguments will be similar for $j = 5, 6$. From (2.48) it follows that

$$\begin{aligned} & |(f - H_\tau f)(t, u)| \\ & \leq |(f - H_\rho f)(t, u)| + |[H_\rho(f - H_{\rho'} f) - (f - H_{\rho'} f)](t, u)| + |(f - H_{\rho'} f)(t, u)|. \end{aligned} \quad (2.54)$$

Applying Theorem 2.2 in (2.54) gives

$$\begin{aligned} & |(f - H_\tau f)(t, u)| \\ & \leq c_4(h) \max_{\substack{t \in N[a, b-2] \\ u \in N[c, d+2]}} |\Delta_t^4 f(t, u)| + c_2(h) \max_{\substack{t \in N[a, b] \\ u \in N[c, d+2]}} |\Delta_t^2 (f - H_{\rho'} f)(t, u)| \\ & \quad + c_4(h') \max_{\substack{t \in N[a, b+2] \\ u \in N[c, d-2]}} |\Delta_u^4 f(t, u)|. \end{aligned} \quad (2.55)$$

Since $\Delta_t^2 H_{\rho'} f = H_{\rho'} \Delta_t^2 f$, using Theorem 2.2 again we get

$$|\Delta_t^2 (f - H_{\rho'} f)(t, u)| \leq c_2(h') \max_{\substack{t \in N[a, b] \\ u \in N[c, d]}} |\Delta_t^2 \Delta_u^2 f(t, u)| \quad (2.56)$$

which on substituting into (2.55) yields (2.50). $\square$

We shall now illustrate the sharpness of the error estimates obtained in The-

orem 2.3 by giving three numerical examples. In each example, we shall construct $H_\tau f(t, u)$ for a partition τ (using (2.46)), then we calculate the actual error $\|f - H_\tau f\|$ as well as the bounds.

For the construction of $H_\tau f(t, u)$, we first obtain the values $f_{i,j}^{0,0}, f_{i,j}^{0,1}, f_{i,j}^{0,2}, f_{i,j}^{1,0}, f_{i,j}^{1,1}, f_{i,j}^{1,2}, f_{i,j}^{2,0}, f_{i,j}^{2,1}, f_{i,j}^{2,2}$, $1 \leq i \leq m$, $1 \leq j \leq n$. Then, in each subinterval $N[k_{i-1}, k_i] \times N[l_{j-1}, l_j]$, $2 \leq i \leq m$, $2 \leq j \leq n$, we compute $H_\tau(t, u)$ as follows:

$$\begin{aligned} & H_\tau f(t, u) \\ &= \sum_{\mu=0}^1 \sum_{\nu=0}^1 \left[f_{i-\mu, j-\nu}^{0,0} h_{i-\mu}(t) h_{j-\nu}(u) + f_{i-\mu, j-\nu}^{0,1} h_{i-\mu}(t) \bar{h}_{j-\nu}(u) + f_{i-\mu, j-\nu}^{0,2} h_{i-\mu}(t) \bar{\bar{h}}_{j-\nu}(u) \right. \\ & \quad + f_{i-\mu, j-\nu}^{1,0} \bar{h}_{i-\mu}(t) h_{j-\nu}(u) + f_{i-\mu, j-\nu}^{1,1} \bar{h}_{i-\mu}(t) \bar{h}_{j-\nu}(u) + f_{i-\mu, j-\nu}^{1,2} \bar{h}_{i-\mu}(t) \bar{\bar{h}}_{j-\nu}(u) \\ & \quad \left. + f_{i-\mu, j-\nu}^{2,0} \bar{\bar{h}}_{i-\mu}(t) h_{j-\nu}(u) + f_{i-\mu, j-\nu}^{2,1} \bar{\bar{h}}_{i-\mu}(t) \bar{h}_{j-\nu}(u) + f_{i-\mu, j-\nu}^{2,2} \bar{\bar{h}}_{i-\mu}(t) \bar{\bar{h}}_{j-\nu}(u) \right]. \end{aligned}$$

The actual error is computed as

$$\|f - H_\tau f\| = \max_{[t,u] \in N[a,b+2] \times N[c,d+2]} |f(t, u) - H_\tau f(t, u)|.$$

The bounds are calculated from (2.50).

In the following three examples, let (t_0, u_0) be such that $\|f - H_\tau f\| = |f(t_0, u_0) - H_\tau f(t_0, u_0)|$.

Example 2.4. $f(t, u) = |t|e^{\frac{tu}{60^2}}$, $a = c = -60$, $b = d = 60$, $j = 4$.

We notice that $f(t, u)$ is not differentiable at the point $(0, u)$, therefore, we cannot perform continuous Hermite interpolation with the partitions below.

Table 2.4: Actual error $\|f - H_\tau f\|$ and error bound, $j = 4$

$m = n$	9 ($h = h' = 15$)	11 ($h = h' = 12$)	13 ($h = h' = 10$)	21 ($h = h' = 6$)
$\ f - H_\tau f\ $	0.44587151e + 01	0.32802058e + 01	0.25489556e + 01	1
Bound	0.59939953e + 03	0.23105216e + 03	0.10457983e + 03	0.10289995e + 02
CPU time (s)*	207.81	212.43	221.01	244.77

* Matlab programme is performed on a laptop of FUJITSU LIFEBOOK S Series.

Relative error $\frac{\ f - H_\tau f\ }{ f(t_0, u_0) }$	0.10962548e + 01	0.10825224e + 01	0.84260086e + 00	1
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The bound in Table 2.4 is calculated using the expression of A_4 (see (2.51)). It is observed from Table 2.4 that as the step sizes become smaller, so are the values of $\|f - H_\tau f\|$ and the error bounds, while more CPU time is required, as expected.

Example 2.5. $f(t, u) = \frac{1}{20}t|t - 1|(t - 2)^{(5)}u|u - 1|(u - 2)^{(5)}$, $a = c = -59$, $b = d = 61$, $j = 5$.

It is noted that $f(t, u)$ is not differentiable at the point (t, u) , where $t = 1$ or $u = 1$. Therefore, we cannot construct continuous Hermite interpolant when the partition includes those points.

Table 2.5: Actual error $\|f - H_\tau f\|$ and error bounds, $j = 5$

$m = n$	9 ($h = h' = 15$)	11 ($h = h' = 12$)	13 ($h = h' = 10$)	21 ($h = h' = 6$)
$\ f - H_\tau f\ $	0.22282287e + 01	0.59574177	0.19806487	0.10284036e - 01
Bound 1	0.34877072e + 04	0.10717326e + 04	0.40546800e + 03	0.26644256e + 02
Bound 2	0.34877072e + 04	0.10717326e + 04	0.40546800e + 03	0.26644256e + 02
CPU Time (s)*	222.04	223.46	227.51	239.70

* Matlab programme is performed on a laptop of FUJITSU LIFEBOOK S Series.

Relative error $\frac{\ f - H_\tau f\ }{ f(t_0, u_0) }$	0.47657622e - 04	0.11203992e - 04	0.32830203e - 05	0.21271387e - 06
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Bound 1 and bound 2 in Table 2.5 are calculated using the expression of A_5 (see(2.52)) respectively. Since the function is symmetric in t and u , so bound 1 and

bound 2 are the same when the step sizes h and h' choose the same value. It is noted from Table 2.5 that as the step sizes become smaller, the values of $\|f - H_\tau f\|$ and the two bounds decrease while more time is needed to execute the programme.

Example 2.6. $f(t, u) = \frac{1}{10^{20}}|t|(t-1)^{(6)}|u|(u-1)^{(6)}$, $a = c = -60$, $b = d = 60$, $j = 6$.

We observe that $f(t, u)$ is not differentiable at the point (t, u) , where $t = 0$ or $u = 0$. Therefore, we cannot construct continuous Hermite interpolant when the partition includes those points.

Table 2.6: Actual error $\|f - H_\tau f\|$ and error bounds, $j = 6$

$m = n$	9 ($h = h' = 15$)	11 ($h = h' = 12$)	13 ($h = h' = 10$)	21 ($h = h' = 6$)
$\ f - H_\tau f\ $	0.25391232e + 01	0.67853974	0.22552046	0.11792749e - 01
Bound 1	0.37971088e + 03	0.92038870e + 02	0.28667973e + 02	0.10072121e + 01
Bound 2	0.91932897e + 03	0.21918705e + 03	0.66095397e + 02	0.22455996e + 01
Bound 3	0.37971088e + 03	0.92038870e + 02	0.28667973e + 02	0.10072121e + 01
CPU Time (s)*	261.44	266.50	268.97	302.28

* Matlab programme is performed on a laptop of FUJITSU LIFEBOOK S Series.

Relative error $\frac{\ f - H_\tau f\ }{f(t_0, u_0)}$	0.42688083e - 04	0.10054244e - 04	0.29517865e - 05	0.19194852e - 06
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The three bounds in Table 2.6 are obtained according to the expressions of A_6 (see (2.53)) respectively. It is noted that bound 1 and bound 3 are closer to the value of $\|f - H_\tau f\|$ compared with bound 2 to the actual error. From Table 2.6 we observe that as the step sizes decrease the values of $\|f - H_\tau f\|$ and the three bounds get smaller but with more time consumed to get the results.

Chapter 3

Discrete Spline Interpolation

Based on the work done in the previous chapter, in this chapter we shall develop a *quintic discrete spline* via a constructive approach. Our definition of discrete spline involves *forward differences* and is in the spirit of that in [88]. However, the method of construction is different - we derive matrix equations which can be solved uniquely to obtain the discrete spline, which, unlike [60,88], is *not* in terms of B -splines. Our approach is parallel to the technique in [59].

The outline of this chapter is as follows. In section 3.1, for a function $f(t)$ defined on $N[a, b+2]$, we shall define its quintic discrete spline interpolant $S_\rho f(t)$. To obtain an explicit expression of $S_\rho f(t)$, we shall show that this amounts to solving some matrix equations. We further provide two representations of $S_\rho f(t)$. Next, using the error bounds for $\|f - H_\rho f\|$ established in Chapter 2, we shall derive error estimates for $\|f - S_\rho f\|$ in section 3.2. To illustrate our results, in section 3.3 we shall present three numerical examples where we show the actual construction of the discrete spline interpolants and also compare the actual errors with the error bounds obtained. Finally, we shall discuss two-variable discrete spline interpolation in section 3.4. For a function $f(t, u)$ defined on $N[a, b+2] \times N[c, d+2]$, we shall define its two-variable discrete spline interpolant $S_\tau f(t, u)$, give an explicit representation of $S_\tau f(t, u)$, as well as perform the related error analysis. This chapter is based on the work of [27].

Throughout this chapter, we assume that the step sizes h and h' of the partitions ρ and ρ' respectively satisfy $h, h' \geq 6$.

3.1 Definition and Explicit Representations

In this section, we shall discuss the existence, uniqueness and the explicit expression of the discrete spline interpolant $S_\rho f$.

To begin, we define the set $S(\rho)$ as

$$S(\rho) = \left\{ g(t) \in D^{(4)}[a, b] : g(t) \text{ is a quintic polynomial in each subinterval } N[k_i, k_{i+1}], 1 \leq i \leq m-2 \text{ and } N[k_{m-1}, b+2] \right\}.$$

Clearly, $S(\rho)$ is of dimension $6(m-1) - 5(m-2) = m+4$.

Definition 3.1. For a given function $f(t)$ defined on $N[a, b+2]$, we say $S_\rho f(t)$ is the $S(\rho)$ -interpolant of $f(t)$, also known as the *discrete spline interpolant of $f(t)$* , if $S_\rho f(t) \in S(\rho)$ with $S_\rho f(k_i) = f(k_i)$, $1 \leq i \leq m$ and $\Delta^j S_\rho f(k_\ell) = \Delta^j f(k_\ell)$, $\ell = 1, m$, $j = 1, 2$.

In the next two results, we shall show that for a given $g(t) \in H(\rho)$ to be in $S(\rho)$, 2 systems of linear equations must be satisfied. This lays the foundation for the existence, uniqueness and construction of the discrete spline interpolant.

Lemma 3.1. Let $2 \leq i \leq m-1$ but fixed, and let $p(t)$, $q(t)$ be two quintic polynomials defined on $N[k_{i-1}, k_i+4]$ and $N[k_i, k_{i+1}+2]$, respectively. Suppose that $\Delta^j p(k_i) = \Delta^j q(k_i) = \Delta^j y_i$, $j = 0, 1, 2$. Then, $\Delta^3 p(k_i) = \Delta^3 q(k_i)$ if and only if

$$\begin{aligned} & 8(h-1)^{(3)} \Delta q(k_{i+1}) - 24(h^2-4) \Delta y_i - 8(h+3)^{(3)} \Delta p(k_{i-1}) \\ & - (h-1)^{(3)}(h-2) \Delta^2 q(k_{i+1}) + 6(h+2)^{(2)}(h-1)^{(2)} \Delta^2 y_i - (h+3)^{(3)}(h+2) \Delta^2 p(k_{i-1}) \\ & = 20(h-1)^{(2)} q(k_{i+1}) - 40(h^2+2) y_i + 20(h+2)^{(2)} p(k_{i-1}). \end{aligned} \tag{3.1}$$

Further, $\Delta^4 p(k_i) = \Delta^4 q(k_i)$ if and only if

$$\begin{aligned}
& (7h+16)(h+2)\Delta p(k_{i-1}) + 16(h^2-4)\Delta y_i + (7h-16)(h-2)\Delta q(k_{i+1}) \\
& + (h+3)^{(3)}\Delta^2 p(k_{i-1}) + 3(h^2-4)\Delta^2 y_i - (h-1)^{(3)}\Delta^2 q(k_{i+1}) \\
& = 15(h-2)q(k_{i+1}) + 60y_i - 15(h+2)p(k_{i-1}).
\end{aligned} \tag{3.2}$$

Proof. Let $G_1(t) = \sum_{j=0}^5 a_j(t-k_i)^{(j)}$ and $G_2(t) = \sum_{j=0}^5 b_j(t-k_i)^{(j)}$ be two quintic polynomials defined on $N[k_{i-1}, k_i+4]$ and $N[k_i, k_{i+1}+2]$, respectively. To have $G_1(t) \equiv p(t)$, we set $G_1(k_{i-1}) = p(k_{i-1})$, $G_1(k_i) = p(k_i) = y_i$, $\Delta G_1(k_{i-1}) = \Delta p(k_{i-1})$, $\Delta G_1(k_i) = \Delta p(k_i) = \Delta y_i$, $\Delta^2 G_1(k_{i-1}) = \Delta^2 p(k_{i-1})$, and $\Delta^2 G_1(k_i) = \Delta^2 p(k_i) = \Delta^2 y_i$ to get an algebraic system of 6 equations that determines the unknowns a_j , $0 \leq j \leq 5$ in terms of $p(k_{i-1})$, $\Delta p(k_{i-1})$, $\Delta^2 p(k_{i-1})$, y_i , Δy_i and $\Delta^2 y_i$.

Similarly, the unknowns b_j , $0 \leq j \leq 5$ are computed by requiring $G_2(k_i) = q(k_i) = y_i$, $G_2(k_{i+1}) = q(k_{i+1})$, $\Delta G_2(k_i) = \Delta q(k_i) = \Delta y_i$, $\Delta G_2(k_{i+1}) = \Delta q(k_{i+1})$, $\Delta^2 G_2(k_i) = \Delta^2 q(k_i) = \Delta^2 y_i$ and $\Delta^2 G_2(k_{i+1}) = \Delta^2 q(k_{i+1})$ such that $G_2(t) \equiv q(t)$.

Now, $\Delta^3 p(k_i) = \Delta^3 q(k_i)$ if and only if $a_3 = b_3$, which is the same as (3.1). Moreover, $\Delta^4 p(k_i) = \Delta^4 q(k_i)$ if and only if $a_4 = b_4$, which is the same as (3.2).

□

Lemma 3.2. For a given $g(t) \in H(\rho)$, we define $c_i = g(k_i)$, $\Delta c_i = \Delta g(k_i)$, $\Delta^2 c_i = \Delta^2 g(k_i)$, $1 \leq i \leq m$. Then, $g(t) \in S(\rho)$ if and only if the vectors $\Delta c = [\Delta c_i]_{i=2}^{m-1}$ and $\Delta^2 c = [\Delta^2 c_i]_{i=2}^{m-1}$ satisfy the matrix equations

$$B^1(\Delta c) = w^1 \quad \text{and} \quad B^2(\Delta^2 c) = w^2, \tag{3.3}$$

where $B^1 = [b_{ij}^1]_{i,j=1}^{m-2}$, $B^2 = [b_{ij}^2]_{i,j=1}^{m-2}$ are $(m-2) \times (m-2)$ matrices and $w^1 = [w_i^1]_{i=2}^{m-1}$, $w^2 = [w_i^2]_{i=2}^{m-1}$ are $(m-2) \times 1$ vectors given as follows:

$$\begin{aligned}
b_{11}^1 &= (h+1)(h-2)(2h-1)(227h^3 - 149h^2 - 618h + 216), \\
b_{12}^1 &= 2(h-1)^{(2)}(h+1)(79h^3 - 343h^2 + 429h - 108), \\
b_{13}^1 &= 3(h-1)^{(4)}(h-2)(2h-1),
\end{aligned}$$

$$\begin{aligned}
b_{m-2,m-4}^1 &= 3(h+4)^{(4)}(h+2)(2h+1), \\
b_{m-2,m-3}^1 &= 2(h-1)(h+2)^{(2)}(79h^3 + 343h^2 + 429h + 108), \\
b_{m-2,m-2}^1 &= (h-1)(h+2)(2h+1)(227h^3 + 149h^2 - 618h - 216), \\
b_{11}^2 &= (h-1)^{(2)}(708h^4 - 1511h^3 - 2481h^2 + 2294h + 3096), \\
b_{12}^2 &= 2(h-1)^{(2)}(201h^4 - 1102h^3 + 1518h^2 + 553h - 1548), \\
b_{13}^2 &= (h-1)^{(4)}(h-2)(16h-43), \\
b_{m-2,m-4}^2 &= (h+4)^{(4)}(h+2)(43+16h), \\
b_{m-2,m-3}^2 &= 2(h+2)^{(2)}(201h^4 + 1102h^3 + 1518h^2 - 553h - 1548), \\
b_{m-2,m-2}^2 &= (h+2)^{(2)}(708h^4 + 1511h^3 - 2481h^2 - 2294h + 3096),
\end{aligned}$$

for $3 \leq i \leq m-2$,

$$\begin{aligned}
b_{i-1,i-3}^1 &= b_{i-1,i-3}^2 = (h+4)^{(4)}, \\
b_{i-1,i-2}^1 &= b_{i-1,i-2}^2 = 2(h-1)(h+2)^{(2)}(13h+24), \\
b_{i-1,i-1}^1 &= b_{i-1,i-1}^2 = 6(h+1)(h-1)(11h^2-24), \\
b_{i-1,i}^1 &= b_{i-1,i}^2 = 2(h+1)(h-1)^{(2)}(13h-24), \\
b_{i-1,i+1}^1 &= b_{i-1,i+1}^2 = (h-1)^{(4)},
\end{aligned}$$

and all other b_{ij}^1 's and b_{ij}^2 's are zeros (B^1 and B^2 are 5-band diagonal matrices), note also that b_{20}^1 , $b_{m-3,m-1}^1$, b_{20}^2 , $b_{m-3,m-1}^2$ though not elements of B^1 or B^2 are also defined above, they are used in the elements of w^1 and w^2 below;

$$\begin{aligned}
w_2^1 &= 15(h-1)^{(3)}(h-2)(2h-1)c_4 + 10(h-1)^{(2)}(31h^3 - 20h^2 - 78h + 27)c_3 \\
&\quad + 5(h+1)(2h-1)(13h^3 + 143h^2 - 372h + 108)c_2 \\
&\quad - 10(h+1)(47h^4 - 36h^3 - 146h^2 + 99h - 18)c_1 \\
&\quad - h^3(h+1)(h+3)(32h^2 - 113)\Delta^2 c_1 \\
&\quad - 2(h+1)(111h^5 + 152h^4 - 516h^3 - 485h^2 + 378h - 72)\Delta c_1, \\
w_3^1 &= -b_{20}^1 \Delta c_1 + 5(h-1)^{(3)}c_5 + 10(h-1)(5h^2 - h - 12)c_4 \\
&\quad + 180(h+1)(h-1)c_3 - 10(h+1)(5h^2 + h - 12)c_2 - 5(h+3)^{(3)}c_1,
\end{aligned}$$

$$w_i^1 = 5(h-1)^{(3)}c_{i+2} + 10(h-1)(5h^2 - h - 12)c_{i+1} + 180(h+1)(h-1)c_i \\ - 10(h+1)(5h^2 + h - 12)c_{i-1} - 5(h+3)^{(3)}c_{i-2}, \quad 4 \leq i \leq m-3,$$

$$w_{m-2}^1 = -b_{m-3,m-1}^1 \Delta c_m + 5(h-1)^{(3)}c_m + 10(h-1)(5h^2 - h - 12)c_{m-1} \\ + 180(h+1)(h-1)c_{m-2} - 10(h+1)(5h^2 + h - 12)c_{m-3} \\ - 5(h+3)^{(3)}c_{m-4},$$

$$w_{m-1}^1 = 10(h-1)(47h^4 + 36h^3 - 146h^2 - 99h - 18)c_m \\ - 10(h+2)^{(2)}(31h^3 + 20h^2 - 78h - 27)c_{m-2} \\ - 5(h-1)(13h^3 - 143h^2 - 372h - 108)(2h+1)c_{m-1} \\ - 15(h+3)^{(3)}(h+2)(2h+1)c_{m-3} \\ - 2(h-1)(111h^5 - 152h^4 - 516h^3 + 485h^2 + 378h + 72)\Delta c_m \\ + h^3(h-1)(h-3)(32h^2 - 113)\Delta^2 c_m,$$

$$w_2^2 = 20(h-1)^{(2)}(h-2)(16h-43)c_4 + 120(h-1)^{(2)}(3h^2 + 7h - 43)c_3 \\ - 60(36h^4 - 131h^3 + 49h^2 + 164h - 172)c_2 \\ + 40(37h^4 - 129h^3 + 76h^2 + 21h - 86)c_1 + 120h^3(4h^2 - 31)\Delta c_1 \\ + 2(h+1)(h+3)(h-1)(23h^3 - 19h^2 - 2h - 344)\Delta^2 c_1,$$

$$w_3^2 = -b_{20}^2 \Delta^2 c_1 + 20(h-1)^{(2)}c_5 + 40(h-1)(h+4)c_4 - 120(h^2 - 2)c_3 \\ + 40(h+1)(h-4)c_2 + 20(h+2)^{(2)}c_1,$$

$$w_i^2 = 20(h-1)^{(2)}c_{i+2} + 40(h-1)(h+4)c_{i+1} - 120(h^2 - 2)c_i \\ + 40(h+1)(h-4)c_{i-1} + 20(h+2)^{(2)}c_{i-2}, \quad 4 \leq i \leq m-3,$$

$$w_{m-2}^2 = -b_{m-3,m-1}^2 \Delta^2 c_m + 20(h-1)^{(2)}c_m + 40(h-1)(h+4)c_{m-1} \\ - 120(h^2 - 2)c_{m-2} + 40(h+1)(h-4)c_{m-3} + 20(h+2)^{(2)}c_{m-4},$$

$$w_{m-1}^2 = 40(37h^4 + 129h^3 + 76h^2 - 21h - 86)c_m - 120h^3(4h^2 - 31)\Delta c_m \\ - 60(36h^4 + 131h^3 + 49h^2 - 164h - 172)c_{m-1} \\ + 120(h+2)^{(2)}(3h^2 - 7h - 43)c_{m-2} \\ + 20(h+2)^{(2)}(h+2)(43 + 16h)c_{m-3} \\ + 2(h-1)(h-3)(h+1)(23h^3 + 19h^2 - 2h + 344)\Delta^2 c_m.$$

Moreover, B^1 and B^2 are strictly diagonally dominant matrices, and so from (3.3) the unknowns Δc_i , $\Delta^2 c_i$, $2 \leq i \leq m-1$ can be obtained uniquely in terms of c_i , $1 \leq i \leq m$, Δc_1 , Δc_m , $\Delta^2 c_1$ and $\Delta^2 c_m$.

Proof. From Lemma 3.1, the ‘continuity’ of $\Delta^3 g(t)$ and $\Delta^4 g(t)$ is equivalent to (3.1) and (3.2), which are better written as follows:

$$\begin{aligned} P_i : & -(h-1)^{(3)}(h-2)\Delta^2 c_{i+1} + 6(h+2)^{(2)}(h-1)^{(2)}\Delta^2 c_i - (h+3)^{(3)}(h+2)\Delta^2 c_{i-1} \\ & = 20(h-1)^{(2)}c_{i+1} - 40(h^2+2)c_i + 20(h+2)^{(2)}c_{i-1} - 8(h-1)^{(3)}\Delta c_{i+1} \\ & \quad + 24(h^2-4)\Delta c_i + 8(h+3)^{(3)}\Delta c_{i-1}, \end{aligned}$$

$$\begin{aligned} Q_i : & -(h-1)^{(3)}\Delta^2 c_{i+1} + 3(h^2-4)\Delta^2 c_i + (h+3)^{(3)}\Delta^2 c_{i-1} \\ & = 15(h-2)c_{i+1} + 60c_i - 15(h+2)c_{i-1} - (7h+16)(h+2)\Delta c_{i-1} \\ & \quad - 16(h^2-4)\Delta c_i - (7h-16)(h-2)\Delta c_{i+1}. \end{aligned}$$

Consider the operation $O_1 : r_1 P_{i+1} + r_2 P_i + r_3 P_{i-1} + r_4 Q_{i+1} + r_5 Q_i + r_6 Q_{i-1}$ where r_j , $1 \leq j \leq 6$ are numbers to be determined. Set the coefficients of $\Delta^2 c_{i-2}$, $\Delta^2 c_{i-1}$, $\Delta^2 c_i$, $\Delta^2 c_{i+1}$, $\Delta^2 c_{i+2}$ to zero, this gives a system of 5 equations. Let $r_3 = 1$, we can solve the system of 5 equations to get $r_1 = \frac{-(h-1)^{(3)}}{(h+3)^{(3)}}$, $r_2 = \frac{3(h-2)}{(h+3)(h+1)}$, $r_4 = \frac{(h-1)^{(3)}(h-2)}{(h+3)^{(3)}}$, $r_5 = \frac{-6(h-1)^{(2)}}{(h+3)}$, $r_6 = h+2$.

Then, the operation O_1 gives

$$\begin{aligned} b_{i-1,i+1}^1 \Delta c_{i+2} + b_{i-1,i}^1 \Delta c_{i+1} + b_{i-1,i-1}^1 \Delta c_i + b_{i-1,i-2}^1 \Delta c_{i-1} + b_{i-1,i-3}^1 \Delta c_{i-2} &= w_i^1, \\ 3 \leq i \leq m-2. \end{aligned} \tag{3.4}$$

Next, consider the operation $O_2 : q_1 P_3 + q_2 P_2 + q_3 Q_3 + q_4 Q_2$ where q_j , $1 \leq j \leq 4$ are numbers to be determined so that the coefficients of $\Delta^2 c_2$, $\Delta^2 c_3$, $\Delta^2 c_4$ are zero. Let $q_1 = 1$, then we can solve a system of 3 equations to get $q_2 = \frac{2h^4-18h^3-11h^2+72h-18}{3(h-1)(h-2)(h-3)(2h-1)}$, $q_3 = -(h-2)$, $q_4 = \frac{2(h+1)(17h^2+27h-9)}{3(2h-1)(h-3)}$.

The operation O_2 then yields

$$b_{11}^1 \Delta c_2 + b_{12}^1 \Delta c_3 + b_{13}^1 \Delta c_4 = w_2^1. \quad (3.5)$$

We also consider the operation $O_3 : z_1 P_{m-1} + z_2 P_{m-2} + z_3 Q_{m-1} + z_4 Q_{m-2}$ where $z_j, 1 \leq j \leq 4$ are numbers to be determined so that the coefficients of $\Delta^2 c_{m-3}, \Delta^2 c_{m-2}, \Delta^2 c_{m-1}$ are zero. Let $z_2 = 1$, then we can solve a system of 3 equations to get $z_1 = \frac{2h^4+18h^3-11h^2-72h-18}{3(h+1)(h+2)(h+3)(2h+1)}, z_3 = \frac{-2(h-1)(17h^2-27h-9)}{3(2h+1)(h+3)}, z_4 = h+2$. The operation O_3 then provides

$$b_{m-2,m-4}^1 \Delta c_{m-3} + b_{m-2,m-3}^1 \Delta c_{m-2} + b_{m-2,m-2}^1 \Delta c_{m-1} = w_{m-1}^1. \quad (3.6)$$

Now, (3.4)–(3.6) can be written as the matrix equation $B^1(\Delta c) = w^1$. Moreover, it can be checked that B^1 is strictly diagonally dominant for $h \geq 6$. Hence, we can solve the matrix equation and obtain $\Delta c_i, 2 \leq i \leq m-1$ uniquely in terms of $c_i, 1 \leq i \leq m, \Delta c_1, \Delta c_m, \Delta^2 c_1$ and $\Delta^2 c_m$.

The derivation of $B^2(\Delta^2 c) = w^2$ uses a similar technique. We rewrite P_i and Q_i as follows:

$$\begin{aligned} R_i : & 8(h-1)^{(3)} \Delta c_{i+1} - 24(h^2-4) \Delta c_i - 8(h+3)^{(3)} \Delta c_{i-1} \\ &= 20(h-1)^{(2)} c_{i+1} - 40(h^2+2) c_i + 20(h+2)^{(2)} c_{i-1} + (h-1)^{(3)} (h-2) \Delta^2 c_{i+1} \\ &\quad - 6(h+2)^{(2)} (h-1)^{(2)} \Delta^2 c_i + (h+3)^{(3)} (h+2) \Delta^2 c_{i-1}, \\ S_i : & (7h-16)(h-2) \Delta c_{i+1} + 16(h^2-4) \Delta c_i + (7h+16)(h+2) \Delta c_{i-1} \\ &= 15(h-2) c_{i+1} + 60 c_i - 15(h+2) c_{i-1} - (h+3)^{(3)} \Delta^2 c_{i-1} - 3(h^2-4) \Delta^2 c_i \\ &\quad + (h-1)^{(3)} \Delta^2 c_{i+1}. \end{aligned}$$

Then, the operation $r_1 R_{i+1} + r_2 R_i + r_3 R_{i-1} + r_4 S_{i+1} + r_5 S_i + r_6 S_{i-1}$ with $r_1 = \frac{(h-2)(7h-16)}{(h+2)}, r_2 = 16(h-2), r_3 = 7h+16, r_4 = \frac{-8(h-1)^{(3)}}{(h+2)}, r_5 = 24(h-2), r_6 =$

$8(h+1)(h+3)$ gives

$$b_{i-1,i+1}^2 \Delta^2 c_{i+2} + b_{i-1,i}^2 \Delta^2 c_{i+1} + b_{i-1,i-1}^2 \Delta^2 c_i + b_{i-1,i-2}^2 \Delta^2 c_{i-1} + b_{i-1,i-3}^2 \Delta^2 c_{i-2} = w_i^2, \quad 3 \leq i \leq m-2. \quad (3.7)$$

Next, the operation $q_1 R_3 + q_2 R_2 + q_3 S_3 + q_4 S_2$ with $q_1 = -(7h-16)$, $q_2 = \frac{-2(79h^3-232h^2-211h+688)}{(h-2)(16h-43)}$, $q_3 = 8(h-1)(h-3)$, $q_4 = \frac{-8(14h^4-8h^3-173h^2+152h+258)}{(h-2)(16h-43)}$ leads to

$$b_{11}^2 \Delta^2 c_2 + b_{12}^2 \Delta^2 c_3 + b_{13}^2 \Delta^2 c_4 = w_2^2. \quad (3.8)$$

Finally, the operation $z_1 R_{m-1} + z_2 R_{m-2} + z_3 S_{m-1} + z_4 S_{m-2}$ with

$$z_1 = \frac{2(79h^3+232h^2-211h-688)}{(16h+43)(h+2)}, \quad z_2 = 7h+16, \quad z_3 = \frac{-8(14h^4+8h^3-173h^2-152h+258)}{(16h+43)(h+2)}, \quad z_4 = 8(h+1)(h+3)$$

yields

$$b_{m-2,m-4}^2 \Delta^2 c_{m-3} + b_{m-2,m-3}^2 \Delta^2 c_{m-2} + b_{m-2,m-2}^2 \Delta^2 c_{m-1} = w_{m-1}^2. \quad (3.9)$$

Now, (3.7)–(3.9) can be written as the matrix equation $B^2(\Delta^2 c) = w^2$. It can be checked that B^2 is strictly diagonally dominant for $h \geq 6$. Hence, $\Delta^2 c_i$, $2 \leq i \leq m-1$ can be solved uniquely in terms of c_i , $1 \leq i \leq m$, Δc_1 , Δc_m , $\Delta^2 c_1$ and $\Delta^2 c_m$. $\square$

Remark 3.1. To solve (3.3) for Δc and $\Delta^2 c$, the condition $h \geq 6$ may not be necessary as long as B^1, B^2 are invertible.

We are now ready to show the existence and uniqueness of the discrete spline interpolant.

Theorem 3.1. *For a given function $f(t)$ defined on $N[a, b+2]$, $S_\rho f(t)$ exists and is unique.*

Proof. For any given function $g(t)$ defined on $N[a, b+2]$, $H_\rho g(t)$ exists and is unique. Further, by Lemma 3.2 for the given set of numbers $c_i = f(k_i)$, $1 \leq i \leq m$, $\Delta^j c_\ell = \Delta^j f(k_\ell)$, $\ell = 1, m$, $j = 1, 2$, there exist unique $\Delta c_i, \Delta^2 c_i$, $2 \leq i \leq m-1$ satisfying (3.3). Now, let $g(t)$ be such that $g(k_i) = c_i$, $\Delta g(k_i) = \Delta c_i$, $\Delta^2 g(k_i) = \Delta^2 c_i$, $1 \leq i \leq m$. Then, again by Lemma 3.2, $H_\rho g(t) \in S(\rho)$. However, from Definition 3.1

this $H_\rho g(t)$ is actually the same as $S_\rho f(t)$. $\square$

We shall now provide two representations of $S_\rho f(t)$ in the following remarks.

Remark 3.2. From the proof of Theorem 3.1 and (2.4), it is clear that $S_\rho f(t)$ can be expressed as

$$\begin{aligned} S_\rho f(t) = & \sum_{i=1}^m f(k_i) h_i(t) + \Delta f(k_1) \bar{h}_1(t) + \Delta f(k_m) \bar{h}_m(t) + \sum_{i=2}^{m-1} \Delta c_i \bar{h}_i(t) \\ & + \Delta^2 f(k_1) \bar{\bar{h}}_1(t) + \Delta^2 f(k_m) \bar{\bar{h}}_m(t) + \sum_{i=2}^{m-1} \Delta^2 c_i \bar{\bar{h}}_i(t), \end{aligned}$$

where $\Delta c_i, \Delta^2 c_i, 2 \leq i \leq m-1$ satisfy (3.3).

Remark 3.3. We can describe a basis for $S(\rho)$, namely the ‘cardinal splines’, $\{s_i(t)\}_{i=1}^{m+4}$, which are defined by the following interpolating conditions:

$$\begin{aligned} s_i(k_j) &= \delta_{ij}, \quad \Delta s_i(a) = \Delta s_i(b) = 0, \quad \Delta^2 s_i(a) = \Delta^2 s_i(b) = 0, \quad 1 \leq i, j \leq m \\ s_{m+1}(k_i) &= 0, \quad \Delta s_{m+1}(a) = 1, \quad \Delta s_{m+1}(b) = 0, \quad \Delta^2 s_{m+1}(a) = \Delta^2 s_{m+1}(b) = 0, \\ s_{m+2}(k_i) &= 0, \quad \Delta s_{m+2}(a) = 0, \quad \Delta s_{m+2}(b) = 1, \quad \Delta^2 s_{m+2}(a) = \Delta^2 s_{m+2}(b) = 0, \\ s_{m+3}(k_i) &= 0, \quad \Delta^2 s_{m+3}(a) = 1, \quad \Delta^2 s_{m+3}(b) = 0, \quad \Delta s_{m+3}(a) = \Delta s_{m+3}(b) = 0, \\ s_{m+4}(k_i) &= 0, \quad \Delta^2 s_{m+4}(a) = 0, \quad \Delta^2 s_{m+4}(b) = 1, \quad \Delta s_{m+4}(a) = \Delta s_{m+4}(b) = 0, \\ & 1 \leq i \leq m. \end{aligned}$$

Obviously, $S_\rho f(t)$ can be explicitly expressed as

$$S_\rho f(t) = \sum_{i=1}^m f(k_i) s_i(t) + \Delta f(a) s_{m+1}(t) + \Delta f(b) s_{m+2}(t) + \Delta^2 f(a) s_{m+3}(t) + \Delta^2 f(b) s_{m+4}(t).$$

The following lemma is needed in the next section.

Lemma 3.3. [70] *Let A be a square matrix such that $\|A\| < 1$. Then, $(I + A)$ is nonsingular and*

$$\|(I + A)^{-1}\| \leq \frac{1}{1 - \|A\|}$$

where I is the identity matrix.

3.2 Error Analysis

Based on the results obtained in Chapter 2 and the discrete spline interpolant $S_\rho f$ developed in section 3.1, we shall perform the error estimation between the original function f and its discrete spline interpolant $S_\rho f$ in this section.

Let $f(t)$ be an arbitrary function defined on $N[a, b + 2]$. Throughout, we shall denote $f_i^j = \Delta^j f(k_i)$, $1 \leq i \leq m$, $j = 0, 1, 2$. We begin with the equality

$$(f - S_\rho f)(t) = (f - H_\rho f)(t) + (H_\rho f - S_\rho f)(t). \quad (3.10)$$

In (3.10) the term $(H_\rho f - S_\rho f)(t)$ belongs to $H(\rho)$, and $(H_\rho f - S_\rho f)(k_i) = 0$, $1 \leq i \leq m$, and $\Delta^j(H_\rho f - S_\rho f)(k_\ell) = 0$, $\ell = 1, m$, $j = 1, 2$. Hence, it follows from (2.18) that

$$\begin{aligned} (H_\rho f - S_\rho f)(t) &= \sum_{i=2}^{m-1} \Delta(H_\rho f - S_\rho f)(k_i) \cdot \bar{h}_i(t) + \sum_{i=2}^{m-1} \Delta^2(H_\rho f - S_\rho f)(k_i) \cdot \bar{\bar{h}}_i(t) \\ &= \sum_{i=2}^{m-1} [\Delta f_i - \Delta S_\rho f(k_i)] \bar{h}_i(t) + \sum_{i=2}^{m-1} [\Delta^2 f_i - \Delta^2 S_\rho f(k_i)] \bar{\bar{h}}_i(t) \\ &= \sum_{i=2}^{m-1} e_i^1 \bar{h}_i(t) + \sum_{i=2}^{m-1} e_i^2 \bar{\bar{h}}_i(t), \end{aligned} \quad (3.11)$$

where $e_i^1 = \Delta f_i - \Delta S_\rho f(k_i)$ and $e_i^2 = \Delta^2 f_i - \Delta^2 S_\rho f(k_i)$. Denoting $e^1 = [e_i^1]_{i=2}^{m-1}$ and $e^2 = [e_i^2]_{i=2}^{m-1}$, to recall the definition for a vector norm, $|e^1|_0 = \max_{2 \leq i \leq m-1} |e_i^1|$,

$|e^2|_0 = \max_{2 \leq i \leq m-1} |e_i^2|$. The relation (3.11) leads to

$$\begin{aligned}
|(H_\rho f - S_\rho f)(t)| &\leq |e^1|_0 \max_{t \in N[a, b+2]} \sum_{i=2}^{m-1} |\bar{h}_i(t)| + |e^2|_0 \max_{t \in N[a, b+2]} \sum_{i=2}^{m-1} |\bar{\bar{h}}_i(t)| \\
&= |e^1|_0 \max_{\substack{t \in N[k_i, k_{i+1}] \\ 2 \leq i \leq m-2}} [|\bar{h}_i(t)| + |\bar{h}_{i+1}(t)|] \\
&\quad + |e^2|_0 \max_{\substack{t \in N[k_i, k_{i+1}] \\ 2 \leq i \leq m-2}} [|\bar{\bar{h}}_i(t)| + |\bar{\bar{h}}_{i+1}(t)|] \tag{3.12}
\end{aligned}$$

where we have used the fact that $|\bar{h}_i(t)|$ is nonzero only in the interval $N[k_{i-1}, k_{i+1}]$, $2 \leq i \leq m-1$. Using (3.12), it now follows from (3.10) that

$$\begin{aligned}
|(f - S_\rho f)(t)| &\leq |(f - H_\rho f)(t)| + |e^1|_0 \max_{\substack{t \in N[k_i, k_{i+1}] \\ 2 \leq i \leq m-2}} [|\bar{h}_i(t)| + |\bar{h}_{i+1}(t)|] \\
&\quad + |e^2|_0 \max_{\substack{t \in N[k_i, k_{i+1}] \\ 2 \leq i \leq m-2}} [|\bar{\bar{h}}_i(t)| + |\bar{\bar{h}}_{i+1}(t)|]. \tag{3.13}
\end{aligned}$$

In the right side of (3.13) we can apply Theorem 2.2 and Lemma 2.1, thus it remains to compute upper bounds for $|e^1|_0$ and $|e^2|_0$, which we shall tackle in Lemma 3.4 and Lemma 3.5.

We define the following functions and constants which will be used in the next results:

$$\begin{aligned}
\eta_1(z) &= \sum_{t=0}^z t = \frac{1}{2}z(z+1), \quad \eta_2(z) = \sum_{t=0}^z t^2 = \frac{1}{6}z(z+1)(2z+1), \\
\eta_3(z) &= \sum_{t=0}^z t^3 = \frac{1}{4}z^2(z+1)^2, \quad \eta_4(z) = \sum_{t=0}^z t^4 = \frac{1}{30}z(z+1)(2z+1)(3z^2+3z-1), \\
\eta_5(z) &= \sum_{t=0}^z t^5 = \frac{1}{12}z^2(z+1)^2(2z^2+2z-1),
\end{aligned}$$

$$\beta = 2(h+2)(145h^5 - 339h^4 - 1276h^3 + 51h^2 + 699h + 180),$$

$$\kappa = 10(h-1)(h-2)(29h^4 + 88h^3 - 632h^2 + 269h + 516),$$

$$\alpha_1 = \frac{h-4}{5}, \alpha_2 = \frac{2(67h^3-277h^2+327h-72)}{5(68h^2-141h+36)}, \alpha_3 = \frac{2(62h^5-166h^4-107h^3+490h^2-234h+36)}{5(47h^4-36h^3-146h^2+99h-18)},$$

$$\alpha_4 = \frac{2(h-2)}{5}, \alpha_5 = \frac{3h-4}{5}, \alpha_6 = \frac{4(h-1)}{5}, \alpha_7 = \frac{111h^5-152h^4-516h^3+485h^2+378h+72}{5(47h^4+36h^3-146h^2-99h-18)},$$

$$\alpha_8 = \frac{206h^3+151h^2-474h-144}{5(68h^2+141h+36)},$$

$$\alpha_9 = 0.1775h - 0.5640, \alpha_{10} = 0.7672h - 0.4540, \alpha_{11} = 0.2859h - 0.4144,$$

$$\alpha_{12} = 0.9259h - 0.2105, \alpha_{13} = \frac{3(h-1)}{5}, \alpha_{14} = 0.4028h - 0.1821, \alpha_{15} = 0.3515h + 0.0280,$$

$$\alpha_{16} = 0.6876h - 0.0423, \alpha_{17} = \frac{2(h-1)}{5}, \alpha_{18} = 0.6700h + 0.0957, \alpha_{19} = 0.3819h + 0.1533,$$

$$\alpha_{20} = \frac{h-1}{5}, \alpha_{21} = \frac{3h^3(4h^2-31)}{37h^4+129h^3+76h^2-21h-86}, \alpha_{22} = 0.1169h - 0.4861,$$

$$\alpha_{23} = 0.5317h - 0.7620, \alpha_{24} = 0.4805h - 0.6486, \alpha_{25} = h + \frac{1}{2} - \frac{1}{10}\sqrt{10h^2 + 70h + 145},$$

$$\alpha_{26} = 0.2625h + 0.0578, \alpha_{27} = 0.7105h - 0.4300, \alpha_{28} = 0.1627h - 0.2206,$$

$$\alpha_{29} = 0.8369h - 0.1875, \alpha_{30} = h + \frac{3}{2} - \frac{1}{10}\sqrt{30h^2 + 210h + 385}, \alpha_{31} = 0.4301h + 0.6558,$$

$$\alpha_{32} = 0.8672h - 0.1520, \alpha_{33} = 0.4950h + 0.0968, \alpha_{34} = h + \frac{5}{2} - \frac{1}{10}\sqrt{60h^2 + 420h + 745},$$

$$\alpha_{35} = \frac{23h^5+130h^4+362h^3+553h^2-61h-602}{37h^4+129h^3+76h^2-21h-86}.$$

Lemma 3.4. *Let $f(t)$ be defined on $N[a, b+2]$. Then,*

$$|e^1|_0 \leq a_j(h) \max_{t \in N[a, b+2-j]} |\Delta^j f(t)|, \quad 2 \leq j \leq 6 \quad (3.14)$$

where the constants $a_j(h)$, $2 \leq j \leq 6$ are given as follows:

- $\beta a_2(h)$

$$= 2(h-1)(111h^5 - 152h^4 - 516h^3 + 485h^2 + 378h + 72)(2[\alpha_7] + 2 - h)$$

$$- 10(h-1)(47h^4 + 36h^3 - 146h^2 - 99h - 18)[2\eta_1([\alpha_7]) - \eta_1(h-1)]$$

$$+ (h-1)(h+2)^{(2)}\{(2[\alpha_8] + 2 - h)(206h^3 + 151h^2 - 474h - 144)$$

$$+ [2\eta_1([\alpha_8]) - \eta_1(h-1)](-340h^2 - 705h - 180)\}$$

$$+ 3(2h+1)(h+3)^{(3)}(h+2)\{(2[\alpha_6] + 2 - h)(4h-4) - 5[2\eta_1([\alpha_6]) - \eta_1(h-1)]\},$$
- $\beta a_3(h)$ (when $h \leq 11$)
$$= -10(h-1)(47h^4 + 36h^3 - 146h^2 - 99h - 18)[-2\eta_2([\alpha_9]) + 2\eta_2([\alpha_{10}]) - \eta_2(h-1)]$$

$$\begin{aligned}
& + 6(h-1)(74h^5 - 23h^4 - 284h^3 + 80h^2 + 87h + 18)[-2\eta_1([\alpha_9]) + 2\eta_1([\alpha_{10}]) \\
& \quad - \eta_1(h-1)] - 2h^3(h-1)(h-3)(32h^2 - 113)(-2[\alpha_9] + 2[\alpha_{10}] - h) \\
& + (h+2)^{(2)}(h-1)\{-5(68h^2 + 141h + 36)[-2\eta_2([\alpha_{11}]) + \eta_2(h-1)] \\
& \quad - 18h(5h^3 + 2h^2 - 15h - 3)[-2([\alpha_{11}] + 1) + h] \\
& \quad + (412h^3 + 642h^2 - 243h - 108)[-2\eta_1([\alpha_{11}]) + \eta_1(h-1)]\} \\
& + 3(2h+1)(h+3)^{(3)}(h+2)\{-5[-2\eta_2([\alpha_{13}]) + \eta_2(h-1)] \\
& \quad + (8h-3)[-2\eta_1([\alpha_{13}]) + \eta_1(h-1)] - 3h^{(2)}([\alpha_{13}] + 1 - h)\},
\end{aligned}$$

• $\beta a_3(h)$ (when $h \geq 12$)

$$\begin{aligned}
& = -10(h-1)(47h^4 + 36h^3 - 146h^2 - 99h - 18)[-2\eta_2([\alpha_9]) + 2\eta_2([\alpha_{10}]) - \eta_2(h-1)] \\
& \quad + 6(h-1)(74h^5 - 23h^4 - 284h^3 + 80h^2 + 87h + 18)[-2\eta_1([\alpha_9]) + 2\eta_1([\alpha_{10}]) \\
& \quad \quad - \eta_1(h-1)] - 2h^3(h-1)(h-3)(32h^2 - 113)(-2[\alpha_9] + 2[\alpha_{10}] - h) \\
& + (h+2)^{(2)}(h-1)\{-5(68h^2 + 141h + 36)[-2\eta_2([\alpha_{11}]) + 2\eta_2([\alpha_{12}]) - \eta_2(h-1)] \\
& \quad + (412h^3 + 642h^2 - 243h - 108)[-2\eta_1([\alpha_{11}]) + 2\eta_1([\alpha_{12}]) - \eta_1(h-1)] \\
& \quad - 18h(5h^3 + 2h^2 - 15h - 3)[-2[\alpha_{11}] + 2[\alpha_{12}] - h]\} \\
& + 3(2h+1)(h+3)^{(3)}(h+2)\{-5[-2\eta_2([\alpha_{13}]) + \eta_2(h-1)] \\
& \quad + (8h-3)[-2\eta_1([\alpha_{13}]) + \eta_1(h-1)] - 3h^{(2)}([\alpha_{13}] + 1 - h)\},
\end{aligned}$$

• $\beta a_4(h)$

$$\begin{aligned}
& = -10(h-1)(47h^4 + 36h^3 - 146h^2 - 99h - 18)[-2\eta_3([\alpha_{14}]) + \eta_3(h-1)] \\
& + 6(h-1)(111h^5 + 83h^4 - 336h^3 - 245h^2 - 117h - 18)[-2\eta_2([\alpha_{14}]) + \eta_2(h-1)] \\
& - 2(h-1)(96h^6 + 45h^5 - 325h^4 - 171h^3 - 5h^2 + 144h + 36)[-2\eta_1([\alpha_{14}]) + \eta_1(h-1)] \\
& + (h+2)^{(2)}(h-1)\{-5(68h^2 + 141h + 36)[2\eta_3([\alpha_{15}]) - 2\eta_3([\alpha_{16}]) + \eta_3(h-1)] \\
& \quad + 3(206h^3 + 491h^2 + 231h + 36)[2\eta_2([\alpha_{15}]) - 2\eta_2([\alpha_{16}]) + \eta_2(h-1)] \\
& \quad - (270h^4 + 726h^3 + 323h^2 - 174h - 72)[2\eta_1([\alpha_{15}]) - 2\eta_1([\alpha_{16}]) + \eta_1(h-1)] \\
& \quad + 2h(2h^4 + 18h^3 - 11h^2 - 72h - 18)(2[\alpha_{15}] - 2[\alpha_{16}] + h)\} \\
& + 3(2h+1)(h+3)^{(3)}(h+2)\{-5[2\eta_3([\alpha_{17}]) - \eta_3(h-1)] + 2(h+1)^{(3)}(2[\alpha_{17}] + 2 - h) \\
& \quad + 3(4h+1)[2\eta_2([\alpha_{17}]) - \eta_2(h-1)] - (3h+2)(3h-1)[2\eta_1([\alpha_{17}]) - \eta_1(h-1)]\},
\end{aligned}$$

• $\beta a_5(h)$

$$\begin{aligned}
& = -10(h-1)(47h^4 + 36h^3 - 146h^2 - 99h - 18)[-2\eta_4([\alpha_{18}]) + \eta_4(h-1)] \\
& + 4(h-1)(222h^5 + 401h^4 - 492h^3 - 1220h^2 - 729h - 126)[-2\eta_3([\alpha_{18}]) + \eta_3(h-1)]
\end{aligned}$$

$$\begin{aligned}
& -2(h-1)(192h^6 + 756h^5 + 83h^4 - 2178h^3 - 2210h^2 - 909h - 126) \\
& \quad \times [\eta_2(h-1) - 2\eta_2([\alpha_{18}])] \\
& + 4(h-1)(96h^6 + 156h^5 - 242h^4 - 507h^3 - 250h^2 + 27h + 18)[-2\eta_1([\alpha_{18}]) + \eta_1(h-1)] \\
& + (h+2)^{(2)}(h-1)\{-5(68h^2 + 141h + 36)[2\eta_4([\alpha_{19}]) - \eta_4(h-1)] \\
& \quad + 2(412h^3 + 1322h^2 + 1167h + 252)[2\eta_3([\alpha_{19}]) - \eta_3(h-1)] \\
& \quad - (540h^4 + 2688h^3 + 3932h^2 + 1743h + 252)[2\eta_2([\alpha_{19}]) - \eta_2(h-1)] \\
& \quad + 2(8h^5 + 342h^4 + 888h^3 + 526h^2 - 15h - 36)[2\eta_1([\alpha_{19}]) - \eta_1(h-1)] \\
& \quad + 2(h+2)^{(4)}(17h^2 - 27h - 9)(2[\alpha_{19}] + 2 - h)\} \\
& + 3(2h+1)(h+3)^{(3)}(h+2)\{-5[-2\eta_4([\alpha_{20}]) + \eta_4(h-1)] \\
& \quad + 2(8h+7)[-2\eta_3([\alpha_{20}]) + \eta_3(h-1)] - (18h^2 + 30h + 7)[-2\eta_2([\alpha_{20}]) + \eta_2(h-1)] \\
& \quad + (8h^3 + 18h^2 + 6h - 2)[-2\eta_1([\alpha_{20}]) + \eta_1(h-1)] + (h+2)^{(4)}(2[\alpha_{20}] + 2 - h)\},
\end{aligned}$$

• $\beta a_6(h)$

$$\begin{aligned}
& = -10(h-1)\{(47h^4 + 36h^3 - 146h^2 - 99h - 18)\eta_5(h-1) \\
& \quad - 3(h+2)(37h^4 + 32h^3 - 116h^2 - 93h - 18)\eta_4(h-1) \\
& \quad + (64h^6 + 474h^5 + 507h^4 - 1158h^3 - 2200h^2 - 1197h - 198)\eta_3(h-1) \\
& \quad - 3(h+2)(64h^5 + 87h^4 - 174h^3 - 266h^2 - 123h - 18)\eta_2(h-1) \\
& \quad + 2(h+1)^{(2)}(64h^4 + 77h^3 - 195h^2 - 243h - 54)\eta_1(h-1)\} \\
& + 5(h-1)(h+2)^{(2)}\{(-68h^2 - 141h - 36)\eta_5(h-1) \\
& \quad + (206h^3 + 831h^2 + 936h + 216)\eta_4(h-1) \\
& \quad - (180h^4 + 1308h^3 + 2746h^2 + 1983h + 396)\eta_3(h-1) \\
& \quad + (8h^5 + 612h^4 + 2438h^3 + 3153h^2 + 1440h + 216)\eta_2(h-1) \\
& \quad + 2h(17h^5 + 3h^4 - 296h^3 - 693h^2 - 522h - 108)\eta_1(h-1)\} \\
& - 15(2h+1)(h+3)^{(3)}(h+2)\{-\eta_5(h-1) + (4h+6)\eta_4(h-1) \\
& \quad - (6h^2 + 18h + 11)\eta_3(h-1) + 2(2h+3)(h^2 + 3h + 1)\eta_2(h-1) - (h+3)^{(4)}\eta_1(h-1)\}.
\end{aligned}$$

Proof. Let $r^1 = [r_i^1(f)]_{i=2}^{m-1}$ be an $(m-2) \times 1$ vector defined by

$$r^1 = B^1 e^1, \quad (3.15)$$

where the matrix B^1 is given in Lemma 3.2. Then, it follows that

$$r^1 = B^1(\Delta f) - w^1, \quad (3.16)$$

where $\Delta f = [\Delta f_i]_{i=2}^{m-1}$ is an $(m-2) \times 1$ vector, and w^1 is defined in Lemma 3.2 with $c_i = f_i$, $1 \leq i \leq m$, $\Delta^j c_\ell = \Delta^j f_\ell$, $\ell = 1, m$, $j = 1, 2$.

From (3.16), we have the first component of r^1 as

$$r_2^1(f) = b_{13}^1 \Delta f_4 + b_{12}^1 \Delta f_3 + b_{11}^1 \Delta f_2 - w_2^1. \quad (3.17)$$

Since $r_2^1(p) = 0$ for all polynomials $p(t)$ of degree $(j-1)$, $2 \leq j \leq 6$, it follows from Theorem 2.1 that

$$r_2^1(f) = \frac{1}{(j-1)!} \sum_{s=k_1}^{k_4+1} (r_2^1)_t(t-s-1)_+^{(j-1)} \Delta^j f(s). \quad (3.18)$$

Using (3.17), we have

$$\begin{aligned} & (r_2^1)_t(t-s-1)_+^{(j-1)} \\ &= (j-1) \left[b_{13}^1 (k_4-s-1)_+^{(j-2)} + b_{12}^1 (k_3-s-1)_+^{(j-2)} + b_{11}^1 (k_2-s-1)_+^{(j-2)} \right] - \hat{w}_2^1 \end{aligned} \quad (3.19)$$

where $\hat{w}_2^1$ is simply w_2^1 with $f(t) = (t-s-1)_+^{(j-1)}$. It is obvious from (3.19) that $(r_2^1)_t(t-s-1)_+^{(j-1)} = 0$ for $s = k_4, k_4+1$. Thus, (3.18) reduces to

$$r_2^1(f) = \frac{1}{(j-1)!} \sum_{s=k_1}^{k_4-1} (r_2^1)_t(t-s-1)_+^{(j-1)} \Delta^j f(s). \quad (3.20)$$

We shall continue the proof only for $j = 2$ as the proof for other cases is similar. From (3.19), we shall find the explicit expressions of $(r_2^1)_t(t-s-1)_+$ for different subintervals of s , namely, $N[k_1, k_2-1]$, $N[k_2, k_3-1]$ and $N[k_3, k_4-1]$, and then sum $|(r_2^1)_t(t-s-1)_+|$ over the intervals. To illustrate, for $s \in N[k_3, k_4-1]$, we have

$$(r_2^1)_t(t-s-1)_+ = 3(h-1)^{(3)}(h-2)(2h-1)(h-4-5T) \equiv \phi(T)$$

where $T = k_4 - s - 1 \in N[0, h-1]$. Note that $\phi(T)$ changes sign at $T = \alpha_1$, thus

we find

$$\begin{aligned}
& \sum_{s=k_3}^{k_4-1} |(r_2^1)_t(t-s-1)_+| \\
&= 3(h-1)^{(3)}(h-2)(2h-1) \left\{ 2 \sum_{T=0}^{[\alpha_1]} (h-4-5T) - \sum_{T=0}^{h-1} (h-4-5T) \right\} \\
&= 3(h-1)^{(3)}(h-2)(2h-1) \{ (2[\alpha_1] + 2 - h)(h-4) - 10\eta_1([\alpha_1]) + 5\eta_1(h-1) \}.
\end{aligned}$$

Using a similar method, we get eventually

$$\begin{aligned}
& \sum_{s=k_1}^{k_4-1} |(r_2^1)_t(t-s-1)_+| \\
&= \sum_{s=k_3}^{k_4-1} |(r_2^1)_t(t-s-1)_+| + \sum_{s=k_2}^{k_3-1} |(r_2^1)_t(t-s-1)_+| + \sum_{s=k_1}^{k_2-1} |(r_2^1)_t(t-s-1)_+| \\
&= 3(h-1)^{(3)}(h-2)(2h-1) \{ (2[\alpha_1] + 2 - h)(h-4) - 10\eta_1([\alpha_1]) + 5\eta_1(h-1) \} \\
&\quad + (h-2)(h^2-1) \{ (2[\alpha_2] + 2 - h)(134h^3 - 144 - 554h^2 + 654h) \\
&\quad + [-2\eta_1([\alpha_2]) + \eta_1(h-1)](340h^2 - 705h + 180) \} \\
&\quad + 2(h+1) \{ (2[\alpha_3] + 2 - h)(124h^5 + 980h^2 - 332h^4 - 468h - 214h^3 + 72) \\
&\quad - [-2\eta_1([\alpha_3]) + \eta_1(h-1)](-235h^4 + 180h^3 + 730h^2 - 495h + 90) \}.
\end{aligned} \tag{3.21}$$

Next, for $3 \leq i \leq m-2$, the relations corresponding to (3.17), (3.20) and (3.21) are respectively

$$\begin{aligned}
r_i^1(f) &= b_{i-1,i+1}^1 \Delta f_{i+2} + b_{i-1,i}^1 \Delta f_{i+1} + b_{i-1,i-1}^1 \Delta f_i + b_{i-1,i-2}^1 \Delta f_{i-1} + b_{i-1,i-3}^1 \Delta f_{i-2} - w_i^1, \\
r_i^1(f) &= \frac{1}{(j-1)!} \sum_{s=k_{i-2}}^{k_{i+2}-1} (r_i^1)_t(t-s-1)_+^{(j-1)} \Delta^j f(s)
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
& \sum_{s=k_{i-2}}^{k_{i+2}-1} |(r_i^1)_t(t-s-1)_+| \\
&= (h-1)^{(3)} \{ (2[\alpha_1] + 2 - h)(h-4) - 5[2\eta_1([\alpha_1]) - \eta_1(h-1)] \} \\
& \quad + (h^2-1)(11h-18) \{ (2[\alpha_4] + 2 - h)(2h-4) - 5[2\eta_1([\alpha_4]) - \eta_1(h-1)] \} \\
& \quad + (h^2-1)(11h+18) \{ (2[\alpha_5] + 2 - h)(3h-4) - 5[2\eta_1([\alpha_5]) - \eta_1(h-1)] \} \\
& \quad + (h+3)^{(3)} \{ (2[\alpha_6] + 2 - h)(4h-4) - 5[2\eta_1([\alpha_6]) - \eta_1(h-1)] \}.
\end{aligned} \tag{3.23}$$

Finally, for the last component of r^1 , the relations corresponding to (3.17), (3.20) and (3.21) are respectively

$$\begin{aligned}
r_{m-1}^1(f) &= b_{m-2,m-4}^1 \Delta f_{m-3} + b_{m-2,m-3}^1 \Delta f_{m-2} + b_{m-2,m-2}^1 \Delta f_{m-1} - w_{m-1}^1, \\
r_{m-1}^1(f) &= \frac{1}{(j-1)!} \sum_{s=k_{m-3}}^{k_m-1} (r_{m-1}^1)_t(t-s-1)_+^{(j-1)} \Delta^j f(s)
\end{aligned} \tag{3.24}$$

and

$$\sum_{s=k_{m-3}}^{k_m-1} |(r_{m-1}^1)_t(t-s-1)_+| = a_2(h) \cdot \beta. \tag{3.25}$$

Comparing the right sides of (3.21), (3.23) and (3.25), the one in (3.25) is the largest and so it follows from (3.20), (3.22) and (3.24) that

$$\max_{2 \leq i \leq m-1} |r_i^1(f)| \leq \beta a_2(h) \max_{t \in N[a, b+2-j]} |\Delta^j f(t)|. \tag{3.26}$$

Now, we multiply both sides of (3.15) by the diagonal matrix $L = [l_{ij}]$, where $l_{ii} = 1/a$, $a \in \mathbb{R}^+$ to obtain $Lr^1 = LB^1e^1$. This implies that

$$|e^1|_0 \leq \|(LB^1)^{-1}\| \cdot \|Lr^1\|. \tag{3.27}$$

Let $LB^1 = I + A$ where A is an $(m-2) \times (m-2)$ matrix with $\|A\| < 1$, from Lemma

3.3 it follows that $\|(I + A)^{-1}\| \leq (1 - \|A\|)^{-1}$. Using this and (3.26) in (3.27) gives

$$|e^1|_0 \leq \frac{\|Lr^1\|}{1 - \|A\|} \leq \frac{1}{a(1 - \|A\|)} \max_{2 \leq i \leq m-1} |r_i^1(f)| \leq \frac{\beta a_2(h)}{a(1 - \|A\|)} \max_{t \in N[a, b+2-j]} |\Delta^j f(t)|. \quad (3.28)$$

To obtain the smallest bound in (3.28), we shall maximize $(1 - \|A\|)a$ over $a \in \mathbb{R}^+$. For this, from $A = LB^1 - I$ we find

$$\begin{aligned} \|A\| &= \max \left\{ \frac{1}{a}(b_{12}^1 + b_{13}^1) + \left| \frac{b_{11}^1}{a} - 1 \right|, \frac{1}{a}(b_{m-2, m-4}^1 + b_{m-2, m-3}^1) + \left| \frac{b_{m-2, m-2}^1}{a} - 1 \right|, \right. \\ &\quad \left. \frac{1}{a}(b_{i-1, i-3}^1 + b_{i-1, i-2}^1 + b_{i-1, i}^1 + b_{i-1, i+1}^1) + \left| \frac{b_{i-1, i-1}^1}{a} - 1 \right| \right\} \\ &= \frac{1}{a}(b_{m-2, m-4}^1 + b_{m-2, m-3}^1) + \left| \frac{b_{m-2, m-2}^1}{a} - 1 \right|. \end{aligned} \quad (3.29)$$

For $a \geq b_{m-2, m-2}^1$, we always have $\|A\| < 1$ (since B^1 is strictly diagonally dominant); while for $0 < a \leq b_{m-2, m-2}^1$, the condition $\|A\| < 1$ is equivalent to $a^1 < a \leq b_{m-2, m-2}^1$ where $a^1 = 2(h+2)(309h^5 + 410h^4 - 336h^3 - 14h^2 + 135h + 36)$. Hence, using the expression of (3.29) we get

$$\max_{a \in \mathbb{R}^+, \|A\| < 1} (1 - \|A\|)a = \max \left\{ \max_{a^1 < a \leq b_{m-2, m-2}^1} (1 - \|A\|)a, \max_{a \geq b_{m-2, m-2}^1} (1 - \|A\|)a \right\} = \beta,$$

which upon substituting into (3.28) gives (3.14) when $j = 2$. $\square$

Lemma 3.5. *Let $f(t)$ be defined on $N[a, b+2]$. Then,*

$$|e^2|_0 \leq b_j(h) \max_{t \in N[a, b+2-j]} |\Delta^j f(t)|, \quad 2 \leq j \leq 6 \quad (3.30)$$

where the constants $b_j(h)$, $2 \leq j \leq 6$ are given as follows:

- $\kappa b_2(h) = -40(37h^4 + 129h^3 + 76h^2 - 21h - 86)[2\eta_1([\alpha_{21}]) - \eta_1(h-1)]$
 $+ 120h^3(4h^2 - 31)(2[\alpha_{21}] + 2 - h)$
 $- 20(h+2)^{(2)}\{(34h^2 + 33h - 172)\eta_1(h-1) - 2h^2(25h^2 + 54h - 43)\}$
 $+ 20(16h + 43)(h+1)(h+2)^2[h^2 - \eta_1(h-1)],$
- $\kappa b_3(h)$
 $= -40(37h^4 + 129h^3 + 76h^2 - 21h - 86)[-2\eta_2([\alpha_{22}]) + 2\eta_2([\alpha_{23}]) - \eta_2(h-1)]$

$$\begin{aligned}
& + 40(24h^5 + 37h^4 - 57h^3 + 76h^2 - 21h - 86)[-2\eta_1([\alpha_{22}]) + 2\eta_1([\alpha_{23}]) - \eta_1(h-1)] \\
& - 4(h^2 - 1)(h-3)(23h^3 + 19h^2 - 2h + 344)(-2[\alpha_{22}] + 2[\alpha_{23}] - h) \\
& + 2(h+2)^{(2)}\{10(34h^2 + 33h - 172)[2\eta_2([\alpha_{24}]) - \eta_2(h-1)] \\
& \quad - 10(100h^3 + 250h^2 - 139h - 172)[2\eta_1([\alpha_{24}]) - \eta_1(h-1)] \\
& \quad + (h-1)(3h+4)(134h^2 + 185h - 516)(2[\alpha_{24}] + 2 - h)\} \\
& + 2(16h + 43)(h+1)(h+2)^2\{10[2\eta_2([\alpha_{25}]) - \eta_2(h-1)] \\
& \quad - 10(2h+1)[2\eta_1([\alpha_{25}]) - \eta_1(h-1)] + 3(3h+4)(h-1)(2[\alpha_{25}] + 2 - h)\},
\end{aligned}$$

• $\kappa b_4(h)$

$$\begin{aligned}
& = -40(37h^4 + 129h^3 + 76h^2 - 21h - 86)[-2\eta_3([\alpha_{26}]) + 2\eta_3([\alpha_{27}]) - \eta_3(h-1)] \\
& + 120(h+1)(12h^4 + 25h^3 + 11h^2 + 65h - 86)[-2\eta_2([\alpha_{26}]) + 2\eta_2([\alpha_{27}]) - \eta_2(h-1)] \\
& - 4(69h^6 + 210h^5 + 494h^4 + 990h^3 - 1399h^2 - 1470h + 1376) \\
& \quad \times [-2\eta_1([\alpha_{26}]) + 2\eta_1([\alpha_{27}]) - \eta_1(h-1)] \\
& + 2(h+2)^{(2)}\{10(34h^2 + 33h - 172)[-2\eta_3([\alpha_{28}]) + 2\eta_3([\alpha_{29}]) - \eta_3(h-1)] \\
& \quad - 30(50h^3 + 142h^2 - 53h - 172)[-2\eta_2([\alpha_{28}]) + 2\eta_2([\alpha_{29}]) - \eta_2(h-1)] \\
& \quad + (1206h^4 + 3567h^3 - 1777h^2 - 5688h + 2752)[-2\eta_1([\alpha_{28}]) + 2\eta_1([\alpha_{29}]) - \eta_1(h-1)] \\
& \quad - 2h^{(2)}(79h^3 + 232h^2 - 211h - 688)(-2[\alpha_{28}] + 2[\alpha_{29}] - h)\} \\
& + 2(16h + 43)(h+1)(h+2)^2\{10[-2\eta_3([\alpha_{30}]) + \eta_3(h-1)] - 30(h+1) \\
& \quad \times [-2\eta_2([\alpha_{30}]) + \eta_2(h-1)] \\
& \quad + (27h^2 + 39h - 16)[-2\eta_1([\alpha_{30}]) + \eta_1(h-1)] - h(7h^2 + 9h - 16)(-2[\alpha_{30}] - 2 + h)\},
\end{aligned}$$

• $\kappa b_5(h)$

$$\begin{aligned}
& = -40(37h^4 + 129h^3 + 76h^2 - 21h - 86)[-2\eta_4([\alpha_{31}]) + 2\eta_4([\alpha_{32}]) - \eta_4(h-1)] \\
& + 240(8h^5 + 37h^4 + 67h^3 + 76h^2 - 21h - 86)[-2\eta_3([\alpha_{31}]) + 2\eta_3([\alpha_{32}]) - \eta_3(h-1)] \\
& - 8(69h^6 + 570h^5 + 1789h^4 + 2715h^3 + 1261h^2 - 2205h - 1634) \\
& \quad \times [-2\eta_2([\alpha_{31}]) + 2\eta_2([\alpha_{32}]) - \eta_2(h-1)] \\
& + 24(23h^6 + 110h^5 + 288h^4 + 450h^3 - 213h^2 - 560h + 172)[-2\eta_1([\alpha_{31}]) + 2\eta_1([\alpha_{32}]) - \\
& \quad \eta_1(h-1)] + 4(h+2)^{(2)}\{5(34h^2 + 33h - 172)[-2\eta_4([\alpha_{33}]) + \eta_4(h-1)] \\
& \quad - 10(100h^3 + 318h^2 - 73h - 516)[-2\eta_3([\alpha_{33}]) + \eta_3(h-1)] \\
& \quad + (1206h^4 + 5067h^3 + 2653h^2 - 7113h - 3268)[-2\eta_2([\alpha_{33}]) + \eta_2(h-1)] \\
& \quad - (316h^5 + 1818h^4 + 2295h^3 - 3466h - 2265h^2 + 1032)[-2\eta_1([\alpha_{33}]) + \eta_1(h-1)] \\
& \quad - 2h^{(2)}(14h^4 + 8h^3 - 173h^2 - 152h + 258)(-2[\alpha_{33}] - 2 + h)\}
\end{aligned}$$

$$\begin{aligned}
& + 4(16h + 43)(h + 1)(h + 2)^2 \{ 5[\eta_4([\alpha_{34}]) - \eta_4(h - 1)] - 10(2h + 3) \\
& \quad \times [\eta_3([\alpha_{34}]) - \eta_3(h - 1)] + (27h^2 + 69h + 19)[\eta_2([\alpha_{34}]) - \eta_2(h - 1)] \\
& \quad - (14h^3 + 45h^2 + 17h - 6)[\eta_1([\alpha_{34}]) - \eta_1(h - 1)] \\
& \quad + 2(h + 1)^{(2)}(h^2 + 2h - 3)(2[\alpha_{34}] + 2 - h) \},
\end{aligned}$$

• $\kappa b_6(h)$

$$\begin{aligned}
& = -40(37h^4 + 129h^3 + 76h^2 - 21h - 86)[-2\eta_5([\alpha_{35}]) + \eta_5(h - 1)] \\
& \quad + 200(h + 2)(12h^4 + 50h^3 + 65h^2 + 22h - 86)[-2\eta_4([\alpha_{35}]) + \eta_4(h - 1)] \\
& \quad - 40(23h^6 + 310h^5 + 1213h^4 + 2125h^3 + 1687h^2 - 1085h - 1978) \\
& \quad \quad \times [-2\eta_3([\alpha_{35}]) + \eta_3(h - 1)] \\
& \quad + 40(h + 2)(69h^5 + 372h^4 + 860h^3 + 815h^2 - 749h - 602)[-2\eta_2([\alpha_{35}]) + \eta_2(h - 1)] \\
& \quad - 80h(23h^5 + 130h^4 + 362h^3 + 553h^2 - 61h - 602)[-2\eta_1([\alpha_{35}]) + \eta_1(h - 1)] \\
& \quad - 20(h + 2)^{(2)} \{ (34h^2 + 33h - 172)\eta_5(h - 1) - 10(h + 2)(25h^2 + 38h - 86)\eta_4(h - 1) \\
& \quad \quad + (402h^4 + 2189h^3 + 2531h^2 - 2681h - 3956)\eta_3(h - 1) \\
& \quad \quad - (158h^5 + 1512h^4 + 3931h^3 + 989h^2 - 5472h - 2408)\eta_2(h - 1) \\
& \quad \quad - 2h(14h^5 - 85h^4 - 736h^3 - 975h^2 + 758h + 1204)\eta_1(h - 1) \} \\
& \quad + 20(16h + 43)(h + 1)(h + 2)^2 \{ \eta_5(h - 1) - 5(h + 2)\eta_4(h - 1) \\
& \quad \quad + (9h^2 + 33h + 23)\eta_3(h - 1) - (7h^3 + 36h^2 + 48h + 14)\eta_2(h - 1) \\
& \quad \quad + (2h + 7)(h + 2)^{(3)}\eta_1(h - 1) \}.
\end{aligned}$$

Proof. Let $r^2 = [r_i^2(f)]_{i=2}^{m-1}$ be an $(m - 2) \times 1$ vector defined by $r^2 = B^2 e^2$, where the matrix B^2 is given in Lemma 3.2. Then, it follows that $r^2 = B^2(\Delta^2 f) - w^2$, where $\Delta^2 f = [\Delta^2 f_i]_{i=2}^{m-1}$ is an $(m - 2) \times 1$ vector, and w^2 is defined in Lemma 3.2 with $c_i = f_i$, $1 \leq i \leq m$, $\Delta^j c_\ell = \Delta^j f_\ell$, $\ell = 1, m$, $j = 1, 2$. The rest of the proof follows a similar technique as the proof of Lemma 3.4. $\square$

In view of (3.13), we obtain the error estimates for spline interpolation as follows.

Theorem 3.2. *Let $f(t)$ be defined on $N[a, b + 2]$. Then*

$$\|f - S_\rho f\| \leq d_j(h) \max_{t \in N[a, b+2-j]} |\Delta^j f(t)|, \quad 2 \leq j \leq 6 \quad (3.31)$$

where

$$d_j(h) = c_j(h) + a_j(h)M_1(h) + b_j(h)M_2(h),$$

and $c_j(h)$, $M_1(h)$, $M_2(h)$, $a_j(h)$ and $b_j(h)$ are given in Theorem 2.2 and Lemma 2.1, Lemma 3.4 and Lemma 3.5 respectively.

3.3 Numerical Examples

We shall illustrate the sharpness of the error estimates obtained in Theorem 3.2 by two numerical examples. In each example, we take a function $f(t)$ defined on $N[a, b+2]$ and construct $S_\rho f(t)$ for a partition ρ , then we calculate the actual error $\|f - S_\rho f\|$ as well as the respective bound in (3.31).

We remark that the functions considered in the examples are not differentiable at certain points and therefore cannot be approximated by continuous spline interpolation (which involves derivatives). In each example, the steps taken to construct $S_\rho f(t)$ and the related bound are as follows:

- For a function $f(t)$ defined on $N[a, b+2]$, fix the partition ρ and the step size h .
- Obtain the values $f(k_i)$, $1 \leq i \leq m$ and $\Delta^j f(k_\ell)$, $\ell = 1, m$, $j = 1, 2$. In (3.3), with $c_i = f(k_i)$, $1 \leq i \leq m$ and $\Delta^j c_\ell = \Delta^j f(k_\ell)$, $\ell = 1, m$, $j = 1, 2$, check that B^1 and B^2 are invertible and solve for $\Delta c = [\Delta c_i]_{i=2}^{m-1}$ and $\Delta^2 c = [\Delta^2 c_i]_{i=2}^{m-1}$.
- Using Remark 3.2, construct $S_\rho f(t)$ in each subinterval $N[k_{i-1}, k_i]$, $2 \leq i \leq m$ as follows:

$$\begin{aligned} S_\rho f(t) = & f(k_i)h_i(t) + \Delta c_i \bar{h}_i(t) + \Delta^2 c_i \bar{\bar{h}}_i(t) + f(k_{i-1})h_{i-1}(t) + \Delta c_{i-1} \bar{h}_{i-1}(t) \\ & + \Delta^2 c_{i-1} \bar{\bar{h}}_{i-1}(t). \end{aligned}$$

- Compute the actual error

$$\|f - S_\rho f\| = \max_{t \in N[a, b+2]} |f(t) - S_\rho f(t)|.$$

- Obtain the bound in the right side of (3.31) for $j = 6$.

In the following examples, let t_0 be such that $\|f - S_\rho f\| = |f(t_0) - S_\rho f(t_0)|$.

Example 3.1. Consider

$$f(t) = |t|(t-1)^{(7)}|t-8|(t-9)^{(6)}/10^{24}$$

with $a = 0$ and $b = 80$.

$f(t)$ is not differentiable at $t = 0$ and $t = 8$, therefore, it cannot be approximated by continuous spline interpolation. Discrete spline interpolation involving forward differences is constructed following the steps listed earlier. The results are shown in Table 3.1, the spline interpolant and the function are plotted in Figures 3.1 and 3.2 below.

Table 3.1: Actual error $\|f - S_\rho f\|$ and error bounds, $j = 6$

m	9 ($h = 10$)	11 ($h = 8$)
$\ f - S_\rho f\ $	0.38219141e + 01	0.15739690e + 01
Bound	0.24712798e + 03	0.87403452e + 02
CPU time (s)	15.0	22.2

* Matlab programme is performed on a laptop of FUJITSU LIFEBOOK S Series.

Relative error $\frac{\ f - S_\rho f\ }{ f(t_0) }$	0.10890599e - 01	0.21232796e - 02
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From Table 3.1 we observe that the value of $\|f - S_\rho f\|$ and the bound tend to be smaller as the step size h gets smaller. It is reasonable that it takes Matlab more time to run the programme since more piecewise splines have to be constructed when h becomes smaller. With reference to Figure 3.2, we note from the Matlab programme that the maximum error happens at $t = 69$ when $m = 11$ ($h = 8$).

Example 3.2. Consider

$$f(t) = |t|(t^5 - 3t + 1)(t - 8)|t - 6|\ln(t + 1)/10^9$$

with $a = 0$ and $b = 60$.

Continuous spline interpolation cannot be utilized to approximate this example since $f(t)$ is not differentiable at $t = 0$ and $t = 6$, therefore, discrete spline interpolation involving forward differences is employed. The results are listed in Table 3.2.

Table 3.2: Actual error $\|f - S_\rho f\|$ and error bounds, $j = 6$

m	7 ($h = 10$)	11 ($h = 6$)
$\ f - S_\rho f\ $	0.19143081e+01	0.13903024
Bound	0.46240560e + 02	0.54135637e + 01
CPU time (s)*	13.6	17.6

* Matlab programme is performed on a laptop of FUJITSU LIFEBOOK S Series.

Relative error $\frac{\ f - S_\rho f\ }{ f(t_0) }$	0.34525834e - 03	0.46458026e - 05
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We notice from Table 3.2 that as h decreases, the value of $\|f - S_\rho f\|$ and the bound tend to be smaller while more time is needed to run the Matlab programme.

Figure 3.1: Example 3.1 when $m = 11$ ($h = 8$) Figure 3.2: Enlarged portion of Fig. 3.1 where the error $|f(t) - S_\rho f(t)|$ is large

3.4 Two-variable Discrete Spline Interpolation

In this section, we shall develop the two-variable discrete spline interpolant $S_\tau f(t, u)$, and provide the error estimates of $\|f - S_\tau f\|$, the sharpness of the results is illustrated by a numerical example in the end.

We define

$$\begin{aligned}
 S(\tau) &= S(\rho) \otimes S(\rho') \quad (\text{the tensor product}) \\
 &= \text{Span} \{s_i(t)s_j(u)\}_{i=1}^{m+4} \quad \text{ }_{j=1}^{n+4} \quad (\text{see Remark 3.3}) \\
 &= \left\{ g(t, u) \in D^{(4,4)}([a, b] \times [c, d]) : g(t, u) \text{ is a two-dimensional polynomial of} \right. \\
 &\quad \text{degree 5 in each variable and in each subrectangle } N[k_i, k_{i+1}] \times N[l_j, l_{j+1}], \\
 &\quad N[k_{m-1}, b+2] \times N[l_j, l_{j+1}], \quad N[k_i, k_{i+1}] \times N[l_{n-1}, d+2], \\
 &\quad \left. 1 \leq i \leq m-2, \quad 1 \leq j \leq n-2, \text{ and } N[k_{m-1}, b+2] \times N[l_{n-1}, d+2] \right\}.
 \end{aligned}$$

Since $S(\tau)$ is the tensor product of $S(\rho)$ and $S(\rho')$ which are of dimensions $(m+4)$ and $(n+4)$ respectively, $S(\tau)$ is of dimension $(m+4)(n+4)$.

Definition 3.2. For a given function $f(t, u)$ defined on $N[a, b+2] \times N[c, d+2]$, we shall denote $f_{i,j}^{\mu,\nu} = \Delta_t^\mu \Delta_u^\nu f(k_i, l_j)$, $\mu, \nu = 0, 1, 2$, $1 \leq i \leq m$, $1 \leq j \leq n$. We say $S_\tau f(t, u)$ is the $S(\tau)$ -interpolant of $f(t, u)$, also known as the *discrete spline interpolant* of $f(t, u)$, if $S_\tau f(t, u) \in S(\tau)$ with $\Delta_t^\mu \Delta_u^\nu S_\tau f(k_i, l_j) = f_{i,j}^{\mu,\nu}$ where μ, ν, i and j satisfy

- (1) if $\mu = \nu = 0$, then $1 \leq i \leq m$, $1 \leq j \leq n$;
- (2) if $\mu = 1, 2$, $\nu = 0$, then $i = 1, m$, $1 \leq j \leq n$;
- (3) if $\mu = 0$, $\nu = 1, 2$, then $1 \leq i \leq m$, $j = 1, n$; and
- (4) if $\mu = 1, 2$, $\nu = 1, 2$, then $(i, j) = (1, 1), (1, n), (m, 1), (m, n)$.

(3.32)

Remark 3.4. Since $S(\tau) \subset H(\tau)$, $S_\tau f(t, u)$ can be explicitly expressed as

$$\begin{aligned}
& S_\tau f(t, u) \\
&= \sum_{i=1}^m \sum_{j=1}^n \left[S_\tau f(k_i, l_j) h_i(t) h_j(u) + \Delta_u S_\tau f(k_i, l_j) h_i(t) \bar{h}_j(u) + \Delta_u^2 S_\tau f(k_i, l_j) h_i(t) \bar{\bar{h}}_j(u) \right. \\
&\quad + \Delta_t S_\tau f(k_i, l_j) \bar{h}_i(t) h_j(u) + \Delta_t \Delta_u S_\tau f(k_i, l_j) \bar{h}_i(t) \bar{h}_j(u) + \Delta_t \Delta_u^2 S_\tau f(k_i, l_j) \bar{h}_i(t) \bar{\bar{h}}_j(u) \\
&\quad \left. + \Delta_t^2 S_\tau f(k_i, l_j) \bar{\bar{h}}_i(t) h_j(u) + \Delta_t^2 \Delta_u S_\tau f(k_i, l_j) \bar{\bar{h}}_i(t) \bar{h}_j(u) + \Delta_t^2 \Delta_u^2 S_\tau f(k_i, l_j) \bar{\bar{h}}_i(t) \bar{\bar{h}}_j(u) \right]. \tag{3.33}
\end{aligned}$$

In (3.33), the values $\Delta_t^\mu \Delta_u^\nu S_\tau f(k_i, l_j)$ where μ, ν, i and j do not fulfill (3.32) exist uniquely. Indeed, this is an immediate consequence of Lemma 3.2 and is stated as follows.

Lemma 3.6. *For a given $g(t, u) \in H(\tau)$, we define $c_{i,j}^{\mu,\nu} = \Delta_t^\mu \Delta_u^\nu g(k_i, l_j)$, $\mu, \nu = 0, 1, 2$, $1 \leq i \leq m$, $1 \leq j \leq n$. The function $g(t, u) \in S(\tau)$ if and only if $c_{i,j}^{\mu,\nu}$, where μ, ν, i and j are such that*

$$\begin{aligned}
& (1) \text{ if } \mu = 1, 2, \nu = 0, \text{ then } 2 \leq i \leq m-1, 1 \leq j \leq n; \\
& (2) \text{ if } \mu = 0, \nu = 1, 2, \text{ then } 1 \leq i \leq m, 2 \leq j \leq n-1; \text{ and} \\
& (3) \text{ if } \mu = 1, 2, \nu = 1, 2, \text{ then } 2 \leq i \leq m-1, j = 1, n, \\
& \quad \text{and } 1 \leq i \leq m, 2 \leq j \leq n-1
\end{aligned} \tag{3.34}$$

satisfy the matrix equations

$$B^1(\Delta \bar{c}) = \bar{w}^1, \quad B^2(\Delta^2 \bar{c}) = \bar{w}^2, \quad B_h^1(\Delta \hat{c}) = \hat{w}^1 \quad \text{and} \quad B_{h'}^2(\Delta^2 \hat{c}) = \hat{w}^2, \tag{3.35}$$

where

- the vectors $\Delta \bar{c} = [c_{i,j}^{1,\nu}]$ and $\Delta^2 \bar{c} = [c_{i,j}^{2,\nu}]$ are such that if $\nu = 0$, then $2 \leq i \leq m-1$, $1 \leq j \leq n$; and if $\nu = 1, 2$, then $2 \leq i \leq m-1$, $j = 1, n$;
- the vectors $\Delta \hat{c} = [c_{i,j}^{\mu,1}]$ and $\Delta^2 \hat{c} = [c_{i,j}^{\mu,2}]$ are such that for $\mu = 0, 1, 2$, $2 \leq j \leq n-1$, $1 \leq i \leq m$;

- the matrices B^1 and B^2 are given in Lemma 3.2, whereas $B_{h'}^1$ and $B_{h'}^2$ are simply B^1 and B^2 with h replaced by h' ;
- the vectors $\bar{w}^1$ and $\bar{w}^2$ are simply w^1 and w^2 (given in Lemma 3.2) with c_i , Δc_i and $\Delta^2 c_i$ replaced by $c_{i,j}^{0,\nu}$, $c_{i,j}^{1,\nu}$ and $c_{i,j}^{2,\nu}$ respectively; whereas the vectors $\hat{w}^1$ and $\hat{w}^2$ are simply w^1 and w^2 with c_j , Δc_j and $\Delta^2 c_j$ replaced by $c_{i,j}^{\mu,0}$, $c_{i,j}^{\mu,1}$ and $c_{i,j}^{\mu,2}$ respectively.

Moreover, from (3.35) the unknowns $c_{i,j}^{\mu,\nu}$, where μ , ν , i and j satisfy (3.34), can be obtained uniquely in terms of $c_{i,j}^{\mu,\nu}$, where μ , ν , i and j fulfill (3.32).

Theorem 3.3. For any function $f(t, u)$ defined on $N[a, b+2] \times N[c, d+2]$, $S_\tau f(t, u)$ exists and is unique.

Proof. The proof is similar to that of Theorem 3.1. $\square$

Remark 3.5. In view of Remark 3.2, $S_\tau f(t, u)$ can be explicitly expressed in terms of cardinal splines as

$$\begin{aligned}
S_\tau f(t, u) = & \sum_{i=1}^m \sum_{j=1}^n f_{i,j}^{0,0} s_i(t) s_j(u) \\
& + \sum_{i=1}^m [f_{i,1}^{0,1} s_{n+1}(u) + f_{i,n}^{0,1} s_{n+2}(u) + f_{i,1}^{0,2} s_{n+3}(u) + f_{i,n}^{0,2} s_{n+4}(u)] s_i(t) \\
& + \sum_{i=1}^n [f_{1,i}^{1,0} s_{m+1}(t) + f_{m,i}^{1,0} s_{m+2}(t) + f_{1,i}^{2,0} s_{m+3}(t) + f_{m,i}^{2,0} s_{m+4}(t)] s_i(u) \\
& + f_{1,1}^{1,1} s_{m+1}(t) s_{n+1}(u) + f_{1,n}^{1,1} s_{m+1}(t) s_{n+2}(u) + f_{m,1}^{1,1} s_{m+2}(t) s_{n+1}(u) \\
& + f_{m,n}^{1,1} s_{m+2}(t) s_{n+2}(u) + f_{1,1}^{2,1} s_{m+3}(t) s_{n+1}(u) + f_{1,n}^{2,1} s_{m+3}(t) s_{n+2}(u) \\
& + f_{m,1}^{2,1} s_{m+4}(t) s_{n+1}(u) + f_{m,n}^{2,1} s_{m+4}(t) s_{n+2}(u) + f_{1,1}^{1,2} s_{m+1}(t) s_{n+3}(u) \\
& + f_{1,n}^{1,2} s_{m+1}(t) s_{n+4}(u) + f_{m,1}^{1,2} s_{m+2}(t) s_{n+3}(u) + f_{m,n}^{1,2} s_{m+2}(t) s_{n+4}(u) \\
& + f_{1,1}^{2,2} s_{m+3}(t) s_{n+3}(u) + f_{1,n}^{2,2} s_{m+3}(t) s_{n+4}(u) + f_{m,1}^{2,2} s_{m+4}(t) s_{n+3}(u) \\
& + f_{m,n}^{2,2} s_{m+4}(t) s_{n+4}(u).
\end{aligned}$$

As a direct consequence of Remarks 3.2 and 3.5, we have the following result which provides an important characterization of $S_\tau f(t, u)$ in terms of one-variable interpolation schemes.

Lemma 3.7. *Let $f(t, u)$ be defined on $N[a, b + 2] \times N[c, d + 2]$. Then,*

$$S_\tau f(t, u) = S_{\rho'} S_\rho f(t, u) = S_\rho S_{\rho'} f(t, u). \quad (3.36)$$

Proof. The proof is similar to that of Lemma 2.2 in Chapter 2. $\square$

Now let $f(t, u)$ be an arbitrary function defined on $N[a, b + 2] \times N[c, d + 2]$. From Lemma 3.7, we have

$$\begin{aligned} f - S_\tau f &= (f - S_\rho f) + S_\rho(f - S_{\rho'} f) \\ &= (f - S_\rho f) + [S_\rho(f - S_{\rho'} f) - (f - S_{\rho'} f)] + (f - S_{\rho'} f) \end{aligned} \quad (3.37)$$

$$= (f - S_\rho f) + [S_{\rho'}(f - S_\rho f) - (f - S_\rho f)] + (f - S_{\rho'} f). \quad (3.38)$$

Using these relations and Theorem 3.2 we shall deduce error estimates for two-dimensional discrete spline interpolation.

Theorem 3.4. *Let $f(t, u)$ be defined on $N[a, b + 2] \times N[c, d + 2]$. Then,*

$$\begin{aligned} &\|f - S_\tau f\| \\ &\leq d_j(h) \max_{\substack{t \in N[a, b + 2 - j] \\ u \in N[c, d + 2]}} |\Delta_t^j f(t, u)| + d_j(h') \max_{\substack{t \in N[a, b + 2] \\ u \in N[c, d + 2 - j]}} |\Delta_u^j f(t, u)| + A_j, \\ &\hspace{25em} 4 \leq j \leq 6 \end{aligned} \quad (3.39)$$

where

$$A_4 = d_2(h) d_2(h') \max_{\substack{t \in N[a, b] \\ u \in N[c, d]}} |\Delta_t^2 \Delta_u^2 f(t, u)|,$$

$$A_5 = \begin{cases} d_2(h)d_3(h') \max_{\substack{t \in N[a, b] \\ u \in N[c, d-1]}} |\Delta_t^2 \Delta_u^3 f(t, u)|, & \text{or} \\ d_3(h)d_2(h') \max_{\substack{t \in N[a, b-1] \\ u \in N[c, d]}} |\Delta_t^3 \Delta_u^2 f(t, u)| \end{cases}$$

and

$$A_6 = \begin{cases} d_2(h)d_4(h') \max_{\substack{t \in N[a, b] \\ u \in N[c, d-2]}} |\Delta_t^2 \Delta_u^4 f(t, u)|, & \text{or} \\ d_3(h)d_3(h') \max_{\substack{t \in N[a, b-1] \\ u \in N[c, d-1]}} |\Delta_t^3 \Delta_u^3 f(t, u)|, & \text{or} \\ d_4(h)d_2(h') \max_{\substack{t \in N[a, b-2] \\ u \in N[c, d]}} |\Delta_t^4 \Delta_u^2 f(t, u)|. \end{cases}$$

Proof. We shall prove the case when $j = 4$, the arguments will be similar for $j = 5, 6$. From (3.37) we get

$$\begin{aligned} |(f - S_\tau f)(t, u)| &\leq |(f - S_\rho f)(t, u)| + |S_\rho(f - S_{\rho'} f) - (f - S_{\rho'} f)(t, u)| \\ &\quad + |(f - S_{\rho'} f)(t, u)|. \end{aligned} \tag{3.40}$$

Applying Theorem 3.2 in (3.40) gives

$$\begin{aligned} &|(f - S_\tau f)(t, u)| \\ &\leq d_4(h) \max_{\substack{t \in N[a, b-2] \\ u \in N[c, d+2]}} |\Delta_t^4 f(t, u)| + d_2(h) \max_{\substack{t \in N[a, b] \\ u \in N[c, d+2]}} |\Delta_t^2 (f - S_{\rho'} f)(t, u)| \\ &\quad + d_4(h') \max_{\substack{t \in N[a, b+2] \\ u \in N[c, d-2]}} |\Delta_u^4 f(t, u)|. \end{aligned} \tag{3.41}$$

Since $\Delta_t^2 S_{\rho'} f = S_{\rho'} \Delta_t^2 f$, using Theorem 3.2 again we get

$$|\Delta_t^2(f - S_{\rho'} f)(t, u)| \leq d_2(h') \max_{\substack{t \in N[a, b] \\ u \in N[c, d]}} |\Delta_t^2 \Delta_u^2 f(t, u)| \quad (3.42)$$

which on substituting into (3.41) yields (3.39) when $j = 4$. $\square$

We shall now illustrate the sharpness of the error estimates obtained in Theorem 3.4 by a numerical example.

Example 3.3. Consider

$$f(t, u) = (1 - e^{tu/400}) |t|/100$$

with $a = c = 0$ and $b = d = 48$.

We remark that $f(t, u)$ is not differentiable at some points and therefore it is not appropriate to use continuous spline interpolation. For a fixed partition τ , we shall obtain $S_\tau f(t, u)$, the biquintic spline interpolant of $f(t, u)$. Then, we calculate the actual error $\|f - S_\tau f\|$ as well as the bounds in (3.39) for $j = 6$.

To construct $S_\tau f(t, u)$, in view of Remark 3.5 we need only to construct the cardinal splines $s_i(t)$, $1 \leq i \leq m + 4$ and $s_j(u)$, $1 \leq j \leq n + 4$. To compute a particular cardinal spline say $s_1(t)$, from Remark 3.2 we know exactly the values of $c_i = s_1(k_i)$, $1 \leq i \leq m$ and $\Delta^j c_\ell = \Delta^j s_1(k_\ell)$, $\ell = 1, m$, $j = 1, 2$, substitute these into the two matrix equations in (3.3) and solve for the values Δc_i and $\Delta^2 c_i$, $2 \leq i \leq m - 1$. Then, noting Remark 3.1 the cardinal spline s_1 has the expression

$$\begin{aligned} s_1(t) = & \sum_{i=1}^m s_1(k_i) h_i(t) + \Delta s_1(k_1) \bar{h}_1(t) + \Delta s_1(k_m) \bar{h}_m(t) + \sum_{i=2}^{m-1} \Delta c_i \bar{h}_i(t) \\ & + \Delta^2 s_1(k_1) \bar{\bar{h}}_1(t) + \Delta^2 s_1(k_m) \bar{\bar{h}}_m(t) + \sum_{i=2}^{m-1} \Delta^2 c_i \bar{\bar{h}}_i(t). \end{aligned}$$

Indeed, from the expressions of h_i , $\bar{h}_i$ and $\bar{\bar{h}}_i$, we see that in each subinterval

$N[k_{i-1}, k_i]$, $2 \leq i \leq m$,

$$\begin{aligned} s_1(t) &= s_1(k_i)h_i(t) + s_1(k_{i-1})h_{i-1}(t) + \Delta c_i \bar{h}_i(t) + \Delta c_{i-1} \bar{h}_{i-1}(t) \\ &\quad + \Delta^2 c_i \bar{\bar{h}}_i(t) + \Delta^2 c_{i-1} \bar{\bar{h}}_{i-1}(t). \end{aligned}$$

Then, we compute the actual error

$$\|f - S_\tau f\| = \max_{(t,u) \in N[a,b+2] \times N[c,d+2]} |f(t, u) - S_\tau f(t, u)|$$

as well as the bounds in (3.39) for $j = 6$. Let (t_0, u_0) be such that $\|f - S_\tau f\| = |f(t_0, u_0) - S_\tau f(t_0, u_0)|$. The results are tabulated as follows:

Table 3.3: Actual error $\|f - S_\tau f\|$ and error bounds, $j=6$

$m (= n)$	7 ($h = h' = 8$)	9 ($h = h' = 6$)
$\ f - S_\tau f\ $	$0.66187125e - 02$	$0.21483362e - 02$
Bound 1	$0.13637518e + 02$	$0.73463482e + 01$
Bound 2	$0.27812870e + 02$	$0.15152823e + 02$
Bound 3	$0.16876111e + 02$	$0.89818595e + 01$
CPU time (s)*	606.4	811.5

* Matlab programme is performed on a laptop of FUJITSU LIFEBOOK S Series.

Relative error $\frac{\ f - S_\tau f\ }{ f(t_0, u_0) }$	$0.21352939e - 03$	$0.24991480e - 04$
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It is noted from Table 3.3 that as the step sizes get smaller the values of $\|f - S_\tau f\|$ and the three bounds become smaller while more time is required. Moreover, it is obvious that bound 1 is closer to $\|f - S_\tau f\|$ compared with the other two bounds.

To illustrate graphically, in the following figures we shall plot the case $m = n = 9$. Figure 3.3 shows the original function and its spline interpolant, due to the close approximation the graphs are presented separately, otherwise they would just appear as one graph. Figure 3.4 shows the absolute error between original function $f(t, u)$ and discrete spline interpolant $S_\tau f(t, u)$, note that the maximum error occurs

at $(t, u) = (44, 48)$ when $m = n = 9$ ($h = h' = 6$).

Figure 3.3: Example 3.3 when $m = n = 9$ ($h = h' = 6$)

Figure 3.4: Example 3.3 absolute error between original function and spline interpolant when $m = n = 9$ ($h = h' = 6$)

Chapter 4

Solving Fredholm Integral Equation

4.1 Introduction

In this chapter, we consider the following Fredholm integral equation of the second kind

$$u(t) = \int_{\bar{c}}^{\bar{d}} K(t, s)u(s)ds + f(t), \quad t \in [\bar{a}, \bar{b}]. \quad (4.1)$$

This equation appears in many applications, for instance transport theory [25, 107], potential theory [94, 96], fracture mechanics and elasticity [44, 95], just to name a few.

Our main objective is to solve (4.1) by numerical means. Indeed, if the kernel $K(t, s)$ is *degenerate*, i.e., $K(t, s)$ can be expressed as a sum of products of functions of t and functions of s ($\sum_i \sum_j a_{ij} F_i(t) G_j(s)$), then solving (4.1) is reduced to solving a system of algebraic equations, which can be accomplished numerically. To degenerate the kernels many techniques are known, however our interest in particular here is the work of Arthur [17], Netravali and de Figueiredo [69], Schultz [87] and Wong and Agarwal [109]. In their work, bicubic and biquintic splines [3, 108] have been employed to degenerate kernels that are at least differentiable in $[\bar{a}, \bar{b}] \times [\bar{c}, \bar{d}]$. However, in the case when the kernels are not differentiable, then it is not appropriate

to use the splines in [17, 69, 87, 109] which involve derivatives, rather *discrete splines* which involve only differences can be applied.

In this chapter, we shall employ the discrete spline developed in Chapter 3 to degenerate the kernel $K(t, s)$ in (4.1) as well as perform error analysis of our method.

The outline of this chapter is as follows. In section 4.2, the biquintic discrete spline is used to degenerate the kernel $K(t, s)$ and obtain the approximate solution of (4.1). Moreover, we establish *a priori* as well as *posteriori* error bounds between the exact and approximate solutions of (4.1). Finally, in section 4.3 we present a numerical example to illustrate our method and the error bounds obtained, a comparison with some well known methods in the literature is also included. This chapter is based on work of [28].

4.2 Main Results

In this section, the kernel $K(t, s)$ will be first converted into $L(x, y)$ through some transformation, then the biquintic discrete spline interpolant $S_\tau L(x, y)$ is used to approximate $L(x, y)$, accordingly, an approximate solution $\bar{v}(x)$ of (4.1) will be obtained to approximate the exact solution $v(x)$. Finally, two bounds are provided for $\|v - \bar{v}\|$.

We shall consider the Fredholm integral equation

$$u(t) = \int_{\bar{c}}^{\bar{d}} K(t, s)u(s)ds + f(t), \quad t \in [\bar{a}, \bar{b}] \quad (4.2)$$

where $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in \mathbb{R}$, $K(t, s)$ is defined on $[\bar{a}, \bar{b}] \times [\bar{c}, \bar{d}]$ and $f(t) \in C[\bar{a}, \bar{b}]$.

In general the kernel $K(t, s)$ in (4.2) is not degenerate, i.e., it cannot be expressed as $K(t, s) = \sum_i \sum_j a_{ij} F_i(t) G_j(s)$. However, it is not appropriate to approximate $K(t, s)$ by its continuous spline interpolant unless $K(t, s)$ is at least differentiable (which is not our assumption), as the continuous spline interpolant involves derivatives. Therefore, through some transformation we shall degenerate $K(t, s)$ by biquintic discrete spline.

Let $r > 0$ be a fixed constant such that $r\bar{a}$, $r\bar{b}$, $r\bar{c}$ and $r\bar{d}$ are integers. (Note

that it is always possible to find r , since there exist constants $m_i > 0$, $i = 1, 2, 3, 4$ such that $m_1\bar{a}$, $m_2\bar{b}$, $m_3\bar{c}$ and $m_4\bar{d}$ are integers, we can take r to be the least common multiple of m_i 's.)

Define

$$x = rt, \quad y = rs, \quad a = r\bar{a}, \quad b = r\bar{b}, \quad c = r\bar{c}, \quad d = r\bar{d}.$$

Then, $x \in [a, b]$, $y \in [c, d]$, and (4.2) is equivalent to

$$u\left(\frac{x}{r}\right) = \frac{1}{r} \int_c^d K\left(\frac{x}{r}, \frac{y}{r}\right) u\left(\frac{y}{r}\right) dy + f\left(\frac{x}{r}\right), \quad x \in [a, b]. \quad (4.3)$$

Denoting

$$v(x) = u\left(\frac{x}{r}\right), \quad L(x, y) = K\left(\frac{x}{r}, \frac{y}{r}\right), \quad g(x) = f\left(\frac{x}{r}\right),$$

the equation (4.3) becomes

$$v(x) = \frac{1}{r} \int_c^d L(x, y)v(y)dy + g(x), \quad x \in [a, b]. \quad (4.4)$$

Clearly, solving equation (4.4) is equivalent to solving (4.2), since the solution of (4.4) is $v(x) = v(rt) = u(t)$. Further, the kernel $L(x, y)$ in (4.4) is now ready to be approximated by its biquintic discrete spline interpolant $S_\tau L(x, y)$, where τ is a given partition of $N[a, b] \times N[c, d]$. With this approximation, the resulting integral equation appears as

$$\bar{v}(x) = \frac{1}{r} \int_c^d S_\tau L(x, y) \cdot \bar{v}(y)dy + g(x), \quad x \in [a, b] \quad (4.5)$$

which determines an approximate solution $\bar{v}(x)$ of (4.4).

Noting Remark 3.5, for simplicity we shall assume that the biquintic discrete spline interpolant of $L(x, y)$ can be written as

$$S_\tau L(x, y) = \sum_{l=1}^M \sum_{k=1}^N a_{lk} F_l(x) G_k(y). \quad (4.6)$$

Then, (4.5) takes the form

$$\bar{v}(x) = \sum_{l=1}^M F_l(x)\theta_l + g(x), \quad x \in [a, b] \quad (4.7)$$

where

$$\theta_l = \frac{1}{r} \int_c^d \sum_{k=1}^N a_{lk} G_k(y) \bar{v}(y) dy, \quad 1 \leq l \leq M. \quad (4.8)$$

From (4.7) the approximate solution $\bar{v}(x)$ can be obtained if we can determine the $M \times 1$ vector $\theta = [\theta_l]$. For this, we substitute (4.6) and (4.7) in (4.5) to obtain

$$\theta_l = \frac{1}{r} \sum_{k=1}^N \int_c^d a_{lk} G_k(y) \sum_{j=1}^M \theta_j F_j(y) dy + \frac{1}{r} \sum_{k=1}^N \int_c^d a_{lk} G_k(y) g(y) dy, \quad 1 \leq l \leq M \quad (4.9)$$

which in system form can be written as $\theta = Q\theta + z$, or

$$(I - Q)\theta = z, \quad (4.10)$$

where $Q = [q_{ij}]$ is an $M \times M$ matrix with

$$q_{ij} = \frac{1}{r} \sum_{k=1}^N \int_c^d a_{ik} G_k(y) F_j(y) dy, \quad 1 \leq i, j \leq M \quad (4.11)$$

and $z = [z_i]$ is an $M \times 1$ vector with

$$z_i = \frac{1}{r} \sum_{k=1}^N \int_c^d a_{ik} G_k(y) g(y) dy, \quad 1 \leq i \leq M. \quad (4.12)$$

It is clear from (4.7) and (4.10) that a unique $\bar{v}(x)$ exists if and only if the matrix $(I - Q)$ is nonsingular. To provide sufficient conditions for the existence of a unique $\bar{v}(t)$, we introduce the operators R and S on $C[a, b]$ as follows

$$R[v] = \frac{1}{r} \int_c^d L(x, y) v(y) dy$$

and

$$S[\bar{v}] = \frac{1}{r} \int_c^d S_\tau L(x, y) \cdot \bar{v}(y) dy$$

so that (4.4) and (4.5) in operator form can be written respectively as

$$(I - R)[v] = g \quad (4.13)$$

and

$$(I - S)[\bar{v}] = g. \quad (4.14)$$

Definition 4.1. Let $T : C[a, b] \rightarrow C[a, b]$ be an operator defined by $T[v] = g$. We say that the operator T is *invertible* if $T[v] = g$ has a unique solution $v \in C[a, b]$ for each $g \in C[a, b]$.

The next result ensures that (4.5) has a unique solution $\bar{v}(x)$.

Lemma 4.1. *If $(I - R)$ is invertible and*

$$\gamma = (\bar{d} - \bar{c}) \|L - S_\tau L\| \cdot \|(I - R)^{-1}\| < 1, \quad (4.15)$$

then $(I - S)$ is invertible, i.e., (4.5) has a unique solution $\bar{v}(x)$.

Proof. We have to show that $(I - S)[\bar{v}] = g$ has a unique solution $\bar{v} \in C[a, b]$ for each $g \in C[a, b]$. For an arbitrary $g \in C[a, b]$, since $(I - R)$ is invertible, we let $v_0 \in C[a, b]$ be the unique solution of $(I - R)[v_0] = g$. Clearly, it suffices to consider

$$(I - R)^{-1}(I - S)[\bar{v}] = v_0 \quad (4.16)$$

and to show that this has a unique solution $\bar{v} \in C[a, b]$.

Let $P = (I - R)^{-1}(S - R)$. We have

$$\begin{aligned} (I - R)^{-1}(I - S) &= I - (I - R)^{-1}[(I - R) - (I - S)] \\ &= I - (I - R)^{-1}(S - R) = I - P \end{aligned}$$

and so (4.16) is the same as

$$(I - P)[\bar{v}] = v_0,$$

or

$$\bar{v} = P[\bar{v}] + v_0 \equiv T[\bar{v}]. \quad (4.17)$$

We shall show that the operator T is a contraction mapping on $C[a, b]$.

For this, first we note that

$$(R - S)[v] = \frac{1}{r} \int_c^d (L - S_\tau L)(x, y) \cdot v(y) dy$$

and so

$$\|(R - S)[v]\| \leq \frac{1}{r} (d - c) \|L - S_\tau L\| \cdot \|v\| = (\bar{d} - \bar{c}) \|L - S_\tau L\| \cdot \|v\|.$$

It follows that

$$\|R - S\| \leq (\bar{d} - \bar{c}) \|L - S_\tau L\|. \quad (4.18)$$

Next, in view of (4.18) we find

$$\begin{aligned} \|P\| &= \|(I - R)^{-1}(S - R)\| \leq \|(I - R)^{-1}\| \cdot \|S - R\| \\ &\leq \|(I - R)^{-1}\| (\bar{d} - \bar{c}) \|L - S_\tau L\| \\ &= \gamma < 1. \end{aligned} \quad (4.19)$$

Now, the operator T maps $C[a, b]$ into itself and for all $\phi, \varphi \in C[a, b]$ using (4.19) we have

$$\|T[\phi] - T[\varphi]\| = \|P[\phi - \varphi]\| \leq \|P\| \cdot \|\phi - \varphi\| \leq \gamma \|\phi - \varphi\|$$

where $\gamma < 1$. Hence, T is a contraction mapping on $C[a, b]$. Thus, in conclusion (4.17) and consequently (4.16) has a unique solution $\bar{v}$. $\square$

Lemma 4.2. *A solution $\bar{v}$ of (4.5) has the following bound*

$$\|\bar{v}\| \leq \frac{1}{1 - \gamma} \|(I - R)^{-1}\| \cdot \|g\|. \quad (4.20)$$

Proof. From the proof of Lemmas 4.1 and 3.3, we can obtain a bound for $\|\bar{v}\|$ as follows

$$\begin{aligned}
 \|\bar{v}\| &\leq \|(I - P)^{-1}\| \cdot \|v_0\| \\
 &\leq \|(I - P)^{-1}\| \cdot \|(I - R)^{-1}\| \cdot \|g\| \\
 &\leq \frac{1}{1 - \|P\|} \|(I - R)^{-1}\| \cdot \|g\| \\
 &\leq \frac{1}{1 - \gamma} \|(I - R)^{-1}\| \cdot \|g\|. \quad \square
 \end{aligned}$$

The next result gives *a priori* and *posteriori* bounds for $\|v - \bar{v}\|$.

Theorem 4.1. *If $(I - R)$ is invertible and (4.15) holds, then*

$$\|v - \bar{v}\| \leq \frac{\gamma}{1 - \gamma} \|v\| \quad (4.21)$$

and

$$\|v - \bar{v}\| \leq \gamma \|\bar{v}\|. \quad (4.22)$$

Proof. To prove (4.21), from (4.14) and (4.13) we find

$$\begin{aligned}
 \bar{v} &= (I - S)^{-1}[g] = (I - S)^{-1}(I - R)[v] \\
 &= (I - S)^{-1}[(I - S) + (S - R)][v] \\
 &= [I + (I - S)^{-1}(S - R)][v] \\
 &= v + (I - S)^{-1}(S - R)[v].
 \end{aligned}$$

This implies

$$\|v - \bar{v}\| \leq \|(I - S)^{-1}(S - R)\| \cdot \|v\|. \quad (4.23)$$

Next, we write

$$(I - R)^{-1}(I - S) = (I - R)^{-1}(I - R + R - S) = I - (I - R)^{-1}(S - R)$$

or

$$(I - R)^{-1} = (I - S)^{-1} - (I - R)^{-1}(S - R)(I - S)^{-1}.$$

Then, we have

$$(I - R)^{-1}(S - R) = (I - S)^{-1}(S - R) - (I - R)^{-1}(S - R)(I - S)^{-1}(S - R)$$

or

$$P = Z - PZ$$

where $P = (I - R)^{-1}(S - R)$ and $Z = (I - S)^{-1}(S - R)$. It follows that

$$\|P\| \geq \|Z\| - \|PZ\| \geq \|Z\| - \|P\| \cdot \|Z\| = \|Z\|(1 - \|P\|).$$

However, from (4.19), $\|P\| \leq \tau < 1$ and hence we find that

$$\|Z\| = \|(I - S)^{-1}(S - R)\| \leq \frac{\|P\|}{1 - \|P\|} \leq \frac{\gamma}{1 - \gamma}. \quad (4.24)$$

Using (4.24) in (4.23) gives (4.21).

To show (4.22), from (4.13) and (4.14) we have

$$\begin{aligned} v &= (I - R)^{-1}[g] = (I - R)^{-1}(I - S)[\bar{v}] \\ &= (I - R)^{-1}(I - R + R - S)[\bar{v}] \\ &= \bar{v} + (I - R)^{-1}(R - S)[\bar{v}] = \bar{v} - P[\bar{v}] \end{aligned}$$

which leads to

$$\|v - \bar{v}\| \leq \|P\| \cdot \|\bar{v}\| \leq \gamma \|\bar{v}\|. \quad \square$$

Finally, the next theorem combines earlier results to guarantee a unique solution of (4.5) as well as to give *a priori* and *posteriori* error bounds.

Theorem 4.2. *Let $(I - R)$ be invertible, $L(x, y) = K\left(\frac{x}{r}, \frac{y}{r}\right)$ be defined on $N[a, b + 2] \times N[c, d + 2]$, and the partition τ of $N[a, b + 2] \times N[c, d + 2]$ be such that $\gamma_1 < 1$*

or $\gamma_2 < 1$ or $\gamma_3 < 1$, where

$$\begin{aligned} \gamma_1 = & [d_6(h)\|\Delta_x^6 L(x, y)\| + d_2(h)d_4(h')\|\Delta_x^2 \Delta_y^4 L(x, y)\| + d_6(h')\|\Delta_y^6 L(x, y)\|] \\ & \times (\bar{d} - \bar{c}) \|(I - R)^{-1}\|, \end{aligned}$$

$$\begin{aligned} \gamma_2 = & [d_6(h)\|\Delta_x^6 L(x, y)\| + d_3(h)d_3(h')\|\Delta_x^3 \Delta_y^3 L(x, y)\| + d_6(h')\|\Delta_y^6 L(x, y)\|] \\ & (\bar{d} - \bar{c}) \|(I - R)^{-1}\| \end{aligned}$$

and

$$\begin{aligned} \gamma_3 = & [d_6(h)\|\Delta_x^6 L(x, y)\| + d_4(h)d_2(h')\|\Delta_x^4 \Delta_y^2 L(x, y)\| + d_6(h')\|\Delta_y^6 L(x, y)\|] \\ & \times (\bar{d} - \bar{c}) \|(I - R)^{-1}\|. \end{aligned}$$

Then, $(I - S)$ is invertible. Moreover, (4.21) and (4.22) hold with γ replaced by appropriate γ_1 or γ_2 or γ_3 .

Proof. From (3.39), we get

$$\|L - S_\tau L\| \leq d_6(h)\|\Delta_x^6 L(x, y)\| + d_2(h)d_4(h')\|\Delta_x^2 \Delta_y^4 L(x, y)\| + d_6(h')\|\Delta_y^6 L(x, y)\|.$$

It follows that

$$\gamma = (\bar{d} - \bar{c}) \|L - S_\tau L\| \cdot \|(I - R)^{-1}\| \leq \gamma_1 < 1. \quad (4.25)$$

Thus, an application of Lemma 4.1 leads to $(I - S)$ is invertible.

Further, using Theorem 4.1 and the fact that $\gamma \leq \gamma_1$, we find

$$\|v - \bar{v}\| \leq \frac{\gamma}{1 - \gamma} \|v\| \leq \frac{\gamma_1}{1 - \gamma_1} \|v\| \quad (4.26)$$

and

$$\|v - \bar{v}\| \leq \gamma \|\bar{v}\| \leq \gamma_1 \|\bar{v}\|. \quad (4.27)$$

The other cases are obtained in a similar way. This completes the proof. $\square$

4.3 Numerical Example

In this section, we shall present a numerical example to illustrate the results obtained in section 4.2.

Consider the integral equation

$$u(t) = \int_0^1 \frac{1}{2} |t| e^{(t-1)s} u(s) ds + \frac{1}{2} (e^t + 1), \quad t \in [0, 1] \quad (4.28)$$

whose exact solution is known to be $u(t) \equiv e^t$. Problems with kernels similar to that of (4.28) but differentiable can be found in [51, 69, 109].

Pick $r = 90$. Then, with $t = \frac{x}{90}$, $s = \frac{y}{90}$, (4.28) becomes

$$u\left(\frac{x}{90}\right) = \frac{1}{90} \int_0^{90} \frac{1}{2} \left| \frac{x}{90} \right| e^{(\frac{x}{90}-1)\frac{y}{90}} u\left(\frac{y}{90}\right) dy + \frac{1}{2} (e^{\frac{x}{90}} + 1), \quad x \in [0, 90]. \quad (4.29)$$

Denoting $v(x) = u\left(\frac{x}{90}\right)$, (4.29) is the same as

$$v(x) = \frac{1}{90} \int_0^{90} \frac{1}{2} \left| \frac{x}{90} \right| e^{(\frac{x}{90}-1)\frac{y}{90}} v(y) dy + \frac{1}{2} (e^{\frac{x}{90}} + 1), \quad x \in [0, 90]. \quad (4.30)$$

Comparing (4.30) with (4.4), we have $g(x) = \frac{1}{2} (e^{\frac{x}{90}} + 1)$, and the kernel is given by

$$L(x, y) = \left| \frac{x}{180} \right| e^{(\frac{x}{90}-1)\frac{y}{90}}.$$

Note that in (4.28) if $|t|$ is replaced by t , the solution does not change over the considered range $t \in [0, 1]$, but now the kernel is differentiable. Thus, alternatively we can also use continuous splines as in [17, 69, 87, 109] to degenerate the kernel, here we use discrete splines as an alternative way to degenerate the kernel.

To proceed, it is noted that the operator R is given by

$$R[v] = \frac{1}{90} \int_0^{90} \left| \frac{x}{180} \right| e^{(\frac{x}{90}-1)\frac{y}{90}} v(y) dy.$$

We find

$$\begin{aligned}
\|R[v]\| &\leq \frac{1}{90} \max_{x \in [0,90]} \int_0^{90} \left| \frac{x}{180} e^{(\frac{x}{90}-1)\frac{y}{90}} \right| dy \cdot \|v\| \\
&= \max_{x \in [0,90]} \frac{1}{2} \frac{x(e^{(\frac{x}{90}-1)} - 1)}{x - 90} \|v\| \\
&= \frac{1}{2} \|v\|,
\end{aligned}$$

which yields

$$\|R\| \leq \frac{1}{2} < 1.$$

Using Lemma 3.3, we then have

$$\|(I - R)^{-1}\| \leq \frac{1}{1 - \|R\|} \leq \frac{1}{1 - \frac{1}{2}} = 2. \quad (4.31)$$

Noting (4.31), in γ_1 , γ_2 and γ_3 (Theorem 4.2) we may replace the term $\|(I - R)^{-1}\|$ by 2. Clearly, with this modification the two error bounds (4.21) and (4.22) (with γ replaced by appropriate γ_1 or γ_2 or γ_3) will be larger.

We shall find $\bar{v}(x)$, the approximate solution of (4.30) obtained by replacing $L(x, y)$ with its biquintic discrete spline interpolant, as well as the actual value of $\|v - \bar{v}\|$, which is then compared with the modified error bounds (4.21) and (4.22).

Step 1. We shall choose n and m (i.e., fix the partition τ) such that in Theorem 4.2, the quantity γ_1 (or γ_2 or γ_3) < 1 .

Step 2. We shall obtain $S_\tau L(x, y)$, the biquintic discrete spline interpolant of the kernel $L(x, y)$. For this, in view of Remark 3.5 we need only to construct the cardinal splines $s_i(x)$, $1 \leq i \leq m + 4$ and $s_j(y)$, $1 \leq j \leq n + 4$. To compute a particular cardinal spline say $s_1(x)$, from Remark 3.3 we know exactly the values of $c_i = s_1(k_i)$, $1 \leq i \leq m$ and $\Delta^j c_\ell = \Delta^j s_1(k_\ell)$, $\ell = 1, m$, $j = 1, 2$, substitute these into the two matrix equations in (3.3) and solve for the values Δc_i and $\Delta^2 c_i$, $2 \leq i \leq m - 1$.

Then, noting Remark 3.2 the cardinal spline s_1 has the expression

$$\begin{aligned} s_1(x) = & \sum_{i=1}^m s_1(k_i)h_i(x) + \Delta s_1(k_1)\bar{h}_1(x) + \Delta s_1(k_m)\bar{h}_m(x) + \sum_{i=2}^{m-1}(\Delta c_i)\bar{h}_i(x) \\ & + \Delta^2 s_1(k_1)\bar{\bar{h}}_1(x) + \Delta^2 s_1(k_m)\bar{\bar{h}}_m(x) + \sum_{i=2}^{m-1}(\Delta^2 c_i)\bar{\bar{h}}_i(x). \end{aligned}$$

Indeed, from the expressions of h_i , $\bar{h}_i$ and $\bar{\bar{h}}_i$, we see that in each subinterval $N[k_{i-1}, k_i]$, $2 \leq i \leq m$,

$$\begin{aligned} s_1(x) = & s_1(k_i)h_i(x) + (\Delta c_i)\bar{h}_i(x) + (\Delta^2 c_i)\bar{\bar{h}}_i(x) \\ & + s_1(k_{i-1})h_{i-1}(x) + (\Delta c_{i-1})\bar{h}_{i-1}(x) + (\Delta^2 c_{i-1})\bar{\bar{h}}_{i-1}(x). \end{aligned}$$

Step 3. To find $\bar{v}(x)$, we need to solve the system (4.10) to obtain the vector θ , then from (4.7) it follows that

$$\bar{v}(x) = \sum_{l=1}^{m+4} s_l(x)\theta_l + \frac{1}{2} \left(e^{\frac{x}{90}} + 1 \right), \quad x \in [0, 90]. \quad (4.32)$$

This gives immediately an approximate solution $\bar{u}$ of (4.28) as

$$\bar{u}\left(\frac{x}{90}\right) = \bar{v}(x), \quad x \in [0, 90]. \quad (4.33)$$

In Table 4.1 we present the values of θ_l , $1 \leq l \leq m+4$ for several different values of n and m . Table 4.2 gives the actual value of $\|v - \bar{v}\|$ and the modified error bounds.

From Table 4.2 we observe the following:

- (i) As n and m increase, the values of $\|v - \bar{v}\|$, $\frac{\gamma_1}{1-\gamma_1}\|v\|$ and $\gamma_1\|\bar{v}\|$ decrease.
- (ii) A *posteriori* error bound $\gamma_1\|\bar{v}\|$ is smaller than a *a priori* error bound $\frac{\gamma_1}{1-\gamma_1}\|v\|$ as it should be.

Table 4.1: Values of $\theta_l, 1 \leq l \leq m + 4$

$n = m$	7	10	11
θ_1	0.00000000e + 00	0.00000000e + 00	0.00000000e + 00
θ_2	0.90680206e - 01	0.58759534e - 01	0.52585459e - 01
θ_3	0.19780621e + 00	0.12442443e + 00	0.11070138e + 00
θ_4	0.32436064e + 00	0.19780621e + 00	0.17492940e + 00
θ_5	0.47386702e + 00	0.27981175e + 00	0.24591235e + 00
θ_6	0.65048795e + 00	0.37145450e + 00	0.32436064e + 00
θ_7	0.85914091e + 00	0.47386702e + 00	0.41105940e + 00
θ_8	0.55865344e - 02	0.58831497e + 00	0.50687635e + 00
θ_9	0.15185775e - 01	0.71621273e + 00	0.61277046e + 00
θ_{10}	0.62418733e - 04	0.85914091e + 00	0.72980156e + 00
θ_{11}	0.16967171e - 03	0.55865344e - 02	0.85914091e + 00
θ_{12}		0.15185775e - 01	0.55865344e - 02
θ_{13}		0.62418733e - 04	0.15185775e - 01
θ_{14}		0.16967171e - 03	0.62418733e - 04
θ_{15}			0.16967171e - 03

Table 4.2: Actual error and error bounds

$n = m$	7	10	11
$\ v - \bar{v}\ $	0.21659283e - 08	0.16594060e - 09	0.79829921e - 10
$\frac{\gamma_1}{1 - \gamma_1} \ v\ $	0.41425748e - 05	0.64166792e - 06	0.40023514e - 06
$\gamma_1 \ \bar{v}\ $	0.41425685e - 05	0.64166777e - 06	0.40023508e - 06
CPU time (s) *	273.3	547.4	731.7

* Matlab programme is performed on a laptop of FUJITSU LIFEBOOK S Series.

(iii) From (4.22), (4.15) and (4.31), we get

$$\|v - \bar{v}\| \leq \gamma \|\bar{v}\| = (\bar{d} - \bar{c}) \|L - S_\tau L\| \cdot \|(I - R)^{-1}\| \cdot \|\bar{v}\| \leq 2 \|L - S_\tau L\| \cdot \|\bar{v}\|.$$

Hence, the bounds for $\|L - S_\tau L\|$ can be used to predict the performance of the method. Since the constants $d_i(h)$, $2 \leq i \leq 6$ (given in Theorem 3.2) are of $O(h^i)$, it is clear from Theorem 3.4 ($j = 6$) that the bounds are of $O(\hat{h}^6)$ where $\hat{h} = \max\{h, h'\}$. Let $E_{\hat{h}}$ be the actual error $\|v - \bar{v}\|$ obtained by using

a partition τ of step sizes h and h' . We observe the following which agree well with the fact that the bounds for $\|L - S_\tau L\|$ are of $O(\hat{h}^6)$.

$n = m = 7, \hat{h} = 15$	$\ v - \bar{v}\ \equiv E_{15}$
$n = m = 10, \hat{h} = 10$	$\ v - \bar{v}\ \equiv E_{10} \approx \left(\frac{10}{15}\right)^6 E_{15} = 0.19e - 09$
$n = m = 11, \hat{h} = 9$	$\ v - \bar{v}\ \equiv E_9 \approx \left(\frac{9}{10}\right)^6 E_{10} = 0.88e - 10$

Moreover, to illustrate graphically, in Figure 4.1 we show the original kernel $L(x, y)$ and its spline interpolant $S_\tau L(x, y)$, due to the close approximation the graphs are presented separately, otherwise they would just appear as one graph. In Figure 4.2, we plot the approximate solution $\bar{v}(x)$ (see (4.32)) and the exact solution $v(x) = u\left(\frac{x}{90}\right) = e^{\frac{x}{90}}$.

Figure 4.1: Comparison of the original kernel and its spline interpolant when $n = m = 7$

Figure 4.2: Comparison of the exact solution v and the approximate solution $\bar{v}$ when $n = m = 7$

Finally, we shall apply three well known methods in the literature to the integral equation (4.30) and compare the results with our method.

Taylor series method (see [51], Section 5.1.3). Here, we approximate the kernel $L(x, y)$ by the first n terms of its Taylor series, i.e., we use

$$e^{(\frac{x}{90}-1)\frac{y}{90}} \approx 1 + \left(\frac{x}{90} - 1\right) \frac{y}{90} + \cdots + \frac{1}{(n-1)!} \left[\left(\frac{x}{90} - 1\right) \frac{y}{90}\right]^{n-1}. \quad (4.34)$$

The resulting approximate kernel is degenerate. We then use the degenerate kernel method described in section 4.3 to get an approximate solution $\bar{v}$ of (4.30). The actual errors are tabulated in Table 4.3. Note that when $n = 6$, we get a *biquintic* approximation in (4.34) which gives a *much larger error* than our *biquintic* spline method.

Collocation method (see [51], p.163). In this method, the approximate solution $\bar{v}$ of (4.30) is given by

$$\bar{v}(x) = \sum_{k=0}^{n-1} a_k \left(\frac{x}{90}\right)^k. \quad (4.35)$$

Table 4.3: The Taylor series method

n	6	7	10
$\ v - \bar{v}\ $	$0.12500446e - 04$	$0.13773201e - 05$	$0.10324981e - 08$
CPU time (s)*	5.0	5.3	8.2

* Matlab programme is performed on a laptop of FUJITSU LIFEBOOK S Series.

The constants a_k are obtained by requiring that

$$\bar{v}(x_\ell) = \frac{1}{90} \int_0^{90} L(x_\ell, y) \bar{v}(y) dy + g(x_\ell), \quad 1 \leq \ell \leq n \quad (4.36)$$

where x_ℓ 's are taken to be the n evenly spaced points in $[0, 90]$ with $x_1 = 0$ and $x_n = 90$. The actual errors are tabulated in Table 4.4. Note that our spline interpolant of the kernel is also exact at x_ℓ , $1 \leq \ell \leq n$. In this sense we compare the collocation method with our method and find that the collocation method gives *much larger errors*.

Table 4.4: The Collocation method

n	7	10	11
$\ v - \bar{v}\ $	$0.11915863e - 06$	$0.43859316e - 09$	$0.11877583e - 08$
CPU time (s)*	2.9	3.1	4.3

* Matlab programme is performed on a laptop of FUJITSU LIFEBOOK S Series.

Galerkin approximate method (see [51], p.167). Here, the approximate solution $\bar{v}$ of (4.30) is given by (4.35) where the constants a_k are obtained by requiring that

$$\int_0^{90} \left(\frac{x}{90} \right)^j \left[\bar{v}(x) - \frac{1}{90} \int_0^{90} L(x, y) \bar{v}(y) dy - g(x) \right] dx = 0, \quad 0 \leq j \leq n-1. \quad (4.37)$$

The actual errors are tabulated in Table 4.5, once again we note that this method gives *much larger errors* than our method.

Table 4.5: The Galerkin approximate method

n	7	10	11
$\ v - \bar{v}\ $	$0.98192165e - 07$	$0.23719739e - 03$	$0.47476537e - 02$
CPU time (s)*	5.8	8.2	9.7

* Matlab programme is performed on a laptop of FUJITSU LIFEBOOK S Series.

From all the discussions above, we see that our method of approximating the kernel by its biquintic discrete spline interpolant gives the best approximate solution $\bar{v}$ whose explicit expression is given in (4.32).

Chapter 5

Periodic Quintic Discrete Spline Interpolation

Discrete splines are piecewise polynomials where continuity of differences rather than derivatives are satisfied at the joining points of the polynomial pieces. Discrete splines were first introduced by Mangasarian and Schumaker [64] in 1971 as solutions to constrained minimization problems in real Euclidean space, which are discrete analogs of minimization problems in Banach space whose solutions are generalized splines. Thereafter Schumaker [88] studied constructive aspects of these discrete splines in terms of discrete B -splines, Astor and Duris [19] introduced discrete L -splines, and Lyche [59, 60] discussed cubic discrete splines involving central differences. Following Lyche's research on *cubic* discrete splines involving *central differences*, many papers appeared in this area, just to mention a few – in [80] the error estimate is established for the first difference of the cubic discrete spline interpolation proposed in [59, 60]; in [39, 40] the periodic cubic discrete spline which interpolates a given function at one interior point of each mesh interval is discussed; in [91] the cubic discrete spline satisfying certain averaging interpolatory conditions is investigated; in [81] the cubic deficient discrete spline is studied. Other related work on *non-cubic* discrete splines involving central differences includes the quadratic discrete splines discussed in [79].

Motivated by the above research, in this chapter we shall develop a class of

periodic quintic discrete spline interpolant involving *central differences* and establish the related existence, uniqueness and error estimates. The *two-variable* case will also be tackled. Our work naturally extends the literature and especially complements and/or extends the above mentioned work of [39, 40, 59, 60, 80] on *one-variable cubic* discrete splines. We also extend the research of [3, 108] from the continuous case to the discrete case, as well as complement the work of [27, 112] on cubic and quintic discrete splines involving *forward differences*.

The outline of the chapter is as follows. In section 5.1, we shall develop periodic quintic discrete spline interpolation for a periodic function $f(t)$, and establish the existence and uniqueness of the discrete spline interpolant. The error analysis of the periodic discrete spline interpolation is discussed in section 5.2. A numerical example is presented in section 5.3 to illustrate the interpolation procedure as well as the error estimates obtained. Finally, in section 5.4 we shall discuss the two-variable case, in particular the periodic biquintic discrete spline interpolant of a periodic function $f(t, u)$ is defined, its existence, uniqueness as well as the error estimates are established, and an illustrative example is also presented. This chapter is based on the work of [30].

5.1 Periodic Discrete Spline Interpolation

For a given $h > 0$, we recall the *central difference operator* D_h applying to a function $F(t)$ gives

$$\begin{aligned} D_h^{\{0\}} F(t) &= F(t); \quad D_h^{\{1\}} F(t) = \frac{F(t+h) - F(t-h)}{2h}; \quad D_h^{\{2\}} F(t) = \frac{F(t+h) - 2F(t) + F(t-h)}{h^2}; \\ D_h^{\{3\}} F(t) &= \frac{F(t+2h) - 2F(t+h) + 2F(t-h) - F(t-2h)}{2h^3}; \\ D_h^{\{4\}} F(t) &= \frac{F(t+2h) - 4F(t+h) + 6F(t) - 4F(t-h) + F(t-2h)}{h^4}. \end{aligned}$$

We also use the basic polynomials $t^{\{j\}}$ introduced by [59]

$$t^{\{j\}} = t^j, \quad j = 0, 1, 2; \quad t^{\{3\}} = t(t^2 - h^2), \quad x^{\{4\}} = t^2(t^2 - h^2), \quad x^{\{5\}} = t(t^2 - h^2)(t^2 - 4h^2).$$

It is noted that $D_h^{\{1\}} t^{\{j\}} = jt^{j-1}$, $j = 0, 1, 2, 3, 5$ and $D_h^{\{1\}} t^{\{4\}} = 2t(2t^2 + h^2)$.

Throughout this chapter, let $0 < h \leq p$, and we assume that p and p' are

multiples of h . Then it is clear that t_i 's are in $[a, b]_h$ and u_i 's are in $[c, d]_h$.

With reference to the uniform partition φ of $[a, b]$, we shall now develop a class of discrete spline interpolation. Similar treatment holds for the uniform partition φ' of $[c, d]$.

Definition 5.1. A function $S(t; \varphi, h)$ is called a *quintic discrete spline* if its restriction $S_i(t)$ on $[t_{i-1}, t_i]$ is a quintic polynomial for $i = 1, 2, \dots, n$ and

$$D_h^{\{\mu\}} S_i(t_i) = D_h^{\{\mu\}} S_{i+1}(t_i), \quad 1 \leq i \leq n-1, \quad \mu = 0, 1, 2, 3, 4. \quad (5.1)$$

For a positive number P_0 , we say a function $g(t)$ is P_0 -periodic if $g(t) = g(t + P_0)$. We shall now introduce *periodic quintic* discrete spline. In the spirit of [39, 40] where *periodic cubic* discrete spline is studied, let

$$S_h(\varphi) = \left\{ S(t; \varphi, h) : S(t; \varphi, h) \text{ is a quintic discrete spline and it is } (b-a)\text{-periodic} \right\}.$$

Definition 5.2. For a $(b-a)$ -periodic function $f(t)$ defined on $[a-2h, b+2h]_h$, we say $S_\varphi f(t)$ is the $S_h(\varphi)$ -interpolant of $f(t)$, also known as the *periodic quintic discrete spline* of $f(t)$, if $S_\varphi f(t) \in S_h(\varphi)$ with

$$S_\varphi f(t_i) = f(t_i), \quad 0 \leq i \leq n-1. \quad (5.2)$$

Remark 5.1. In Definition 5.2, it actually suffices to have the periodic function $f(t)$ defined on the uniform partition φ . However, for the error analysis in the next section, we require the periodic function $f(t)$ to be defined on $[a-2h, b+2h]_h$. To be consistent, we therefore impose throughout that the $(b-a)$ -periodic function $f(t)$ is defined on $[a-2h, b+2h]_h$.

Let the functions $g_i(t)$, $\bar{g}_i(t)$, $\bar{\bar{g}}_i(t)$ satisfy the following for $0 \leq i, j \leq n-1$:

$$g_i(t_j) = \delta_{ij}, \quad D_h^{\{2\}} g_i(t_j) = D_h^{\{4\}} g_i(t_j) = 0,$$

$$D_h^{\{2\}} \bar{g}_i(t_j) = \delta_{ij}, \quad \bar{g}_i(t_j) = D_h^{\{4\}} \bar{g}_i(t_j) = 0,$$

$$D_h^{\{4\}} \bar{\bar{g}}_i(t_j) = \delta_{ij}, \quad \bar{\bar{g}}_i(t_j) = D_h^{\{2\}} \bar{\bar{g}}_i(t_j) = 0.$$

By direct computation, we have the explicit expressions:

$$\begin{aligned} g_i(t) &= \frac{t-t_{i-1}}{p}, \quad t \in [t_{i-1}, t_i], \quad 1 \leq i \leq n \\ &= \frac{t_{i+1}-t}{p}, \quad t \in [t_i, t_{i+1}], \quad 0 \leq i \leq n-1 \\ &= 0, \quad \text{otherwise;} \\ \bar{g}_i(t) &= \frac{(t-t_{i-1})^{\{3\}}}{6p} - \frac{(p^2-h^2)(t-t_{i-1})}{6p}, \quad t \in [t_{i-1}, t_i], \quad 1 \leq i \leq n \\ &= \frac{(t_{i+1}-t)^{\{3\}}}{6p} - \frac{(p^2-h^2)(t_{i+1}-t)}{6p}, \quad t \in [t_i, t_{i+1}], \quad 0 \leq i \leq n-1 \\ &= 0, \quad \text{otherwise;} \\ \bar{\bar{g}}_i(t) &= \frac{(t-t_{i-1})^{\{5\}}}{120p} - \frac{(p^2-h^2)(t-t_{i-1})^{\{3\}}}{36p} + \frac{(t-t_{i-1})(p^2-h^2)(7p^2+2h^2)}{360p}, \\ &\quad t \in [t_{i-1}, t_i], \quad 1 \leq i \leq n \\ &= \frac{(t_{i+1}-t)^{\{5\}}}{120p} - \frac{(p^2-h^2)(t_{i+1}-t)^{\{3\}}}{36p} + \frac{(t_{i+1}-t)(p^2-h^2)(7p^2+2h^2)}{360p}, \\ &\quad t \in [t_i, t_{i+1}], \quad 0 \leq i \leq n-1 \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Remark 5.2. Let $M_i = D_h^{\{2\}} S_\varphi f(t_i)$, $F_i = D_h^{\{4\}} S_\varphi f(t_i)$, $0 \leq i \leq n$. Then, $S_\varphi f(t)$ can be written as

$$S_\varphi f(t) = \sum_{i=0}^{n-1} [f_i g_i(t) + M_i \bar{g}_i(t) + F_i \bar{\bar{g}}_i(t)], \quad t \in [a, b]. \quad (5.3)$$

In particular, for $t \in [t_{i-1}, t_i]$, $1 \leq i \leq n$, the spline $S_\varphi f(t)$ has the expression

$$S_\varphi f(t) = (S_\varphi f)_i(t) = f_{i-1} g_{i-1}(t) + f_i g_i(t) + M_{i-1} \bar{g}_{i-1}(t) + M_i \bar{g}_i(t) + F_{i-1} \bar{\bar{g}}_{i-1}(t) + F_i \bar{\bar{g}}_i(t). \quad (5.4)$$

We shall now prove the existence and uniqueness of $S_\varphi f(t)$.

Theorem 5.1. Let $f(t)$ be a given $(b-a)$ -periodic function defined on $[a-2h, b+2h]_h$. Then, there exists a unique periodic quintic discrete spline $S_\varphi f(t)$.

Proof. We shall show that the expression (5.3) exists and is unique. Using the expression (5.4), the ‘continuity’ requirement $D_h^{\{1\}}(S_\varphi f)_i(t_i) = D_h^{\{1\}}(S_\varphi f)_{i+1}(t_i)$, $1 \leq i \leq n-1$ leads to the equation

$$\begin{aligned} P_i : (p^2 - h^2)M_{i-1} + 2(h^2 + 2p^2)M_i + (p^2 - h^2)M_{i+1} &= 6(f_{i-1} - 2f_i + f_{i+1}) \\ &+ \frac{(p^2 - h^2)}{60} \{ (2h^2 + 7p^2)F_{i-1} + 4(4p^2 - h^2)F_i + (2h^2 + 7p^2)F_{i+1} \}. \end{aligned} \quad (5.5)$$

Further, the ‘continuity’ requirement $D_h^{\{3\}}(S_\varphi f)_i(t_i) = D_h^{\{3\}}(S_\varphi f)_{i+1}(t_i)$, $1 \leq i \leq n-1$ yields

$$Q_i : M_{i-1} - 2M_i + M_{i+1} = \frac{1}{6} [(p^2 - h^2)F_{i-1} + 2(h^2 + 2p^2)F_i + (p^2 - h^2)F_{i+1}]. \quad (5.6)$$

Now, the operation $(P_i - \frac{p^2 - h^2}{10}Q_i)$ provides

$$\begin{aligned} \frac{p^2(2p^2 + 7h^2)}{10}F_i &= -\frac{3(p^2 - 4h^2)}{10}M_{i-1} - \frac{6(9p^2 + 4h^2)}{10}M_i - \frac{3(p^2 - 4h^2)}{10}M_{i+1} \\ &+ 6(f_{i-1} - 2f_i + f_{i+1}). \end{aligned} \quad (5.7)$$

From (5.7) we can obtain the expressions of F_{i-1} , F_i , and F_{i+1} and then substitute into the relation P_i (or Q_i) to get the ‘M-equation’ as follows

$$\begin{aligned} &a_1M_{i-2} + a_2M_{i-1} + a_3M_i + a_2M_{i+1} + a_1M_{i+2} \\ &= \frac{1}{6} [(p^2 - h^2)f_{i-2} + 2(2h^2 + p^2)f_{i-1} - 6(h^2 + p^2)f_i + 2(2h^2 + p^2)f_{i+1} + (p^2 - h^2)f_{i+2}] \end{aligned} \quad (5.8)$$

where

$$a_1 = \frac{(p^2 - h^2)(p^2 - 4h^2)}{120}, \quad a_2 = \frac{2(p^2 - h^2)(8h^2 + 13p^2)}{120}, \quad a_3 = \frac{6(4h^4 + 5h^2p^2 + 11p^4)}{120}. \quad (5.9)$$

Note that $a_3 > 2(|a_1| + a_2)$.

Next, the operation $[P_i - (p^2 - h^2)Q_i]$ gives

$$M_i = \frac{f_{i+1} - 2f_i + f_{i-1}}{p^2} - \frac{p^2 - h^2}{120p^2} [(p^2 - 4h^2)F_{i-1} + 8(p^2 + h^2)F_i + (p^2 - 4h^2)F_{i+1}]. \quad (5.10)$$

From (5.10) we can obtain the expressions of M_{i-1} , M_i , and M_{i+1} and then substitute into the relation P_i (or Q_i) to get the 'F-equation' as follows

$$a_1F_{i-2} + a_2F_{i-1} + a_3F_i + a_2F_{i+1} + a_1F_{i+2} = f_{i-2} - 4f_{i-1} + 6f_i - 4f_{i+1} + f_{i+2}. \quad (5.11)$$

Since f and $S_\varphi f$ are $(b-a)$ -periodic, we have $f_{-2} = f_{n-2}$, $f_{-1} = f_{n-1}$, $f_0 = f_n$, $f_{n+1} = f_1$, $M_{-2} = M_{n-2}$, $M_{-1} = M_{n-1}$, $M_0 = M_n$, $M_{n+1} = M_1$, $F_{-2} = F_{n-2}$, $F_{-1} = F_{n-1}$, $F_0 = F_n$ and $F_{n+1} = F_1$. so the equations (5.8) and (5.11) actually hold for $0 \leq i \leq n-1$.

Clearly, the existence and uniqueness of $S_\varphi f(t)$ depend on the existence and uniqueness of the solutions of M_i and F_i from the systems (5.8) and (5.11) respectively. It is easy to observe that the coefficient matrices of both systems (5.8) and (5.11) are strictly diagonally dominant and hence invertible. Therefore, solving (5.8) and (5.11) give unique M_i and F_i , $0 \leq i \leq n-1$. Then $S_\varphi f(t)$ can be written as (5.3) or (5.4). $\square$

Remark 5.3. It is possible to describe a basis for $S_h(\varphi)$, namely the 'cardinal splines', $\{s_i(t)\}_{i=0}^{n-1}$, defined by the following interpolation conditions

$$s_i(t_j) = \delta_{ij}^* = \begin{cases} 1, & \text{if } j = i + nk (k \in \mathbf{Z}) \\ 0, & \text{otherwise.} \end{cases}$$

Obviously $S_\varphi f(t)$ can be expressed as

$$S_\varphi f(t) = \sum_{i=0}^{n-1} f_i s_i(t).$$

5.2 Error Analysis

For convenience, we shall denote $g^{\{\mu\}}(t) = D_h^{\{\mu\}}g(t)$, and with respect to the uniform partition φ , denote $g_i^{\{\mu\}} = D_h^{\{\mu\}}g(t_i)$.

In this section, we shall perform the error analysis for the periodic discrete spline interpolation developed in section 5.1. The following lemmas are needed.

Lemma 5.1. [59] *Let α, β be given real numbers such that $\alpha < \beta$ and $\beta \in \{\alpha + h, \alpha + 2h, \dots\}$ for some $h > 0$. Let $g : [\alpha - h, \beta + h]_h \rightarrow \mathbb{R}$ be a given function. Let the operators L and U be defined by*

$$(Lg)(t) = \frac{t - \alpha}{\beta - \alpha}g(\beta) + \frac{\beta - t}{\beta - \alpha}g(\alpha), \quad (Ug)(t) = g(t) - (Lg)(t).$$

Then, we have,

$$\|Ug\| \leq w(g, \beta - \alpha), \quad (5.12)$$

$$\|Ug\| \leq \frac{(\beta - \alpha)^2}{8} \|g^{\{2\}}\| \quad (5.13)$$

and

$$\|D_h^{\{1\}}Ug\| \leq \frac{\beta - \alpha}{2} \|g^{\{2\}}\|. \quad (5.14)$$

Lemma 5.2. [59] *Let $\{a_i\}_{i=1}^I$ and $\{b_j\}_{j=1}^J$ be given sequences of nonnegative real numbers such that $\sum_{i=1}^I a_i = \sum_{j=1}^J b_j$. Then for any real valued function g defined on a discrete interval $[\alpha, \beta]_h$, we have,*

$$\left| \sum_{i=1}^I a_i g[t_{i0}, t_{i1}, \dots, t_{ik}] - \sum_{j=1}^J b_j g[u_{j0}, u_{j1}, \dots, u_{jk}] \right| \leq \frac{1}{k!} \left(\sum_{i=1}^I a_i \right) w(g^{\{k\}}, |\beta - \alpha - kh|) \quad (5.15)$$

where $t_{i\ell}$'s, $u_{j\ell}$'s are in $[\alpha, \beta]_h$.

Let $f(t)$ be a $(b-a)$ -periodic function defined on $[a-2h, b+2h]_h$, and $S_\varphi f(t)$ be the unique periodic quintic discrete spline interpolant of $f(t)$. Let the error $e(t) = S_\varphi f(t) - f(t)$. We are now ready to derive upper bounds for $\|e^{\{\mu\}}\|$, $\mu = 0, 1, 2, 3, 4$. Here, the norm $\|\cdot\|$ is taken over the discrete interval $[a, b]_h$.

Theorem 5.2. *Let $f(t)$ be a $(b-a)$ -periodic function defined on $[a-2h, b+2h]_h$, and $e(t) = S_\varphi f(t) - f(t)$. Then, we have*

$$\|e^{\{4\}}\| \leq (1 + \gamma)w(f^{\{4\}}, p), \quad (5.16)$$

$$\|e^{\{3\}}\| \leq \frac{p}{2}\|e^{\{4\}}\| + \frac{2}{p}\gamma w(f^{\{2\}}, p) \leq \frac{p}{2}(1 + \gamma)w(f^{\{4\}}, p) + \frac{2}{p}\gamma w(f^{\{2\}}, p), \quad (5.17)$$

$$\|e^{\{2\}}\| \leq \frac{p^2}{8}\|e^{\{4\}}\| + \gamma w(f^{\{2\}}, p) \leq \frac{p^2}{8}(1 + \gamma)w(f^{\{4\}}, p) + \gamma w(f^{\{2\}}, p), \quad (5.18)$$

$$\|e^{\{1\}}\| \leq \frac{p}{2}\|e^{\{2\}}\| \leq \frac{p^3}{16}(1 + \gamma)w(f^{\{4\}}, p) + \frac{p}{2}\gamma w(f^{\{2\}}, p) \quad (5.19)$$

and

$$\|e\| \leq \frac{p^2}{8}\|e^{\{2\}}\| \leq \frac{p^4}{64}(1 + \gamma)w(f^{\{4\}}, p) + \frac{p^2}{8}\gamma w(f^{\{2\}}, p) \quad (5.20)$$

Where $\gamma = \frac{40p^4}{(4h^2+p^2)(h^2+p^2)}$.

Proof. First, we shall prove (5.16). To begin, we have $e^{\{4\}}(t_i) = (S_\varphi f)^{\{4\}}(t_i) - f^{\{4\}}(t_i) = F_i - f_i^{\{4\}}$, or $F_i = e_i^{\{4\}} + f_i^{\{4\}}$. So the ‘F-equation’ (5.11) can be rewritten as

$$\begin{aligned} & a_1 e_{i-2}^{\{4\}} + a_2 e_{i-1}^{\{4\}} + a_3 e_i^{\{4\}} + a_2 e_{i+1}^{\{4\}} + a_1 e_{i+2}^{\{4\}} \\ &= -a_1 f_{i-2}^{\{4\}} - a_2 f_{i-1}^{\{4\}} - a_3 f_i^{\{4\}} - a_2 f_{i+1}^{\{4\}} - a_1 f_{i+2}^{\{4\}} + (f_{i-2} - 4f_{i-1} + 6f_i - 4f_{i+1} + f_{i+2}) \triangleq G_i \end{aligned}$$

$$0 \leq i \leq n-1. \quad (5.21)$$

To recall the definition for vector norm, $|G|_0 = \max_{0 \leq i \leq n-1} |G_i|$, $|e^{\{4\}}|_0 = \max_{0 \leq i \leq n-1} |e_i^{\{4\}}|$, it follows that

$$\begin{aligned} a_3 |e_i^{\{4\}}| &\leq |G_i| + a_1 |e_{i-2}^{\{4\}}| + a_2 |e_{i-1}^{\{4\}}| + a_2 |e_{i+1}^{\{4\}}| + a_1 |e_{i+2}^{\{4\}}| \\ &\leq |G|_0 + 2(a_1 + a_2) |e^{\{4\}}|_0 \end{aligned}$$

which leads to

$$a_3 |e^{\{4\}}|_0 \leq |G|_0 + 2(a_1 + a_2) |e^{\{4\}}|_0$$

or

$$|e^{\{4\}}|_0 \leq \frac{10}{(4h^2 + p^2)(h^2 + p^2)} |G|_0. \quad (5.22)$$

We shall now obtain a bound for $|G|_0$. Noting the relation between fourth order central difference and divided difference,

$$f^{\{4\}}(t) = 24f[t - 2h, t - h, t, t + h, t + 2h] \triangleq Q_f(t, h),$$

we can rewrite G_i as

$$\begin{aligned} G_i = & -a_1 Q_f(t_{i-2}, h) - a_2 Q_f(t_{i-1}, h) - a_3 Q_f(t_i, h) - a_2 Q_f(t_{i+1}, h) - a_1 Q_f(t_{i+2}, h) \\ & + p^4 Q_f(t_i, p). \end{aligned} \quad (5.23)$$

Since $2(a_1 + a_2) + a_3 = p^4$, we apply Lemma 5.2 with $g = f$, $\alpha = t_{i-2} - 2h$, $\beta = t_{i+2} + 2h$ and $k = 4$ to get

$$|G_i| \leq \frac{1}{4!} (p^4) 24w(f^{\{4\}}, 4p) \leq 4p^4 w(f^{\{4\}}, p)$$

which implies

$$|G|_0 \leq 4p^4 w(f^{\{4\}}, p). \quad (5.24)$$

Then, using (5.24) in (5.22) provides

$$|e^{\{4\}}|_0 \leq \frac{40p^4}{(4h^2 + p^2)(h^2 + p^2)} w(f^{\{4\}}, p) = \gamma w(f^{\{4\}}, p). \quad (5.25)$$

Now we shall apply Lemma 5.1 with $g = e^{\{4\}}$, $\alpha = t_{i-1}$, and $\beta = t_i$. Here, we have

$$(Le^{\{4\}})(t) = \frac{t - t_{i-1}}{p} e^{\{4\}}(t_i) + \frac{t_i - t}{p} e^{\{4\}}(t_{i-1}) \quad (5.26)$$

and

$$\begin{aligned} (Ue^{\{4\}})(t) &= (U(S_\varphi f)^{\{4\}})(t) - (Uf^{\{4\}})(t) \\ &= (S_\varphi f)^{\{4\}}(t) - (L(S_\varphi f)^{\{4\}})(t) - (Uf^{\{4\}})(t) = - (Uf^{\{4\}})(t) \end{aligned} \quad (5.27)$$

where in the last equality, it is noted that $(S_\varphi f)^{\{4\}}(t)$ is linear and for $t \in [t_{i-1}, t_i]$,

$$(S_\varphi f)^{\{4\}}(t) = \frac{t - t_{i-1}}{p} F_i + \frac{t_i - t}{p} F_{i-1} = (L(S_\varphi f)^{\{4\}})(t).$$

Since $e^{\{4\}} = Ue^{\{4\}} + Le^{\{4\}}$, noting (5.27) we have

$$\|e^{\{4\}}\| \leq \|Ue^{\{4\}}\| + \|Le^{\{4\}}\| = \|Uf^{\{4\}}\| + \|Le^{\{4\}}\| \leq w(f^{\{4\}}, p) + \|Le^{\{4\}}\| \quad (5.28)$$

where we have used (5.12) in the last inequality. It is clear from (5.26) that

$$(Le^{\{4\}})(t) \leq |e^{\{4\}}|_0 \left(\frac{t - t_{i-1}}{p} + \frac{t_i - t}{p} \right) = |e^{\{4\}}|_0$$

and so, together with (5.25), we have

$$\|Le^{\{4\}}\| \leq |e^{\{4\}}|_0 \leq \gamma w(f^{\{4\}}, p). \quad (5.29)$$

Using (5.29) in (5.28) immediately gives (5.16). (We observe that the norms $\|\cdot\|$ in (5.28) and (5.29) are over the interval $[t_{i-1}, t_i]_h$. Since the eventual bound obtained for $\|e^{\{4\}}\|$ is the same for all $[t_{i-1}, t_i]_h$, $1 \leq i \leq n$, it is also the bound for $\|e^{\{4\}}\| = \max_{t \in [a, b]_h} |e^{\{4\}}(t)|$. This observation also prevails for the rest of the proof.)

Next, we shall obtain the bound for $\|e^{\{2\}}\|$. Noting that $M_i = e^{\{2\}}(t_i) + f^{\{2\}}(t_i) = e_i^{\{2\}} + f_i^{\{2\}}$, the ‘M-equation’ (5.8) can be written as

$$\begin{aligned} & a_1 e_{i-2}^{\{2\}} + a_2 e_{i-1}^{\{2\}} + a_3 e_i^{\{2\}} + a_2 e_{i+1}^{\{2\}} + a_1 e_{i+2}^{\{2\}} \\ &= -a_1 f_{i-2}^{\{2\}} - a_2 f_{i-1}^{\{2\}} - a_3 f_i^{\{2\}} - a_2 f_{i+1}^{\{2\}} - a_1 f_{i+2}^{\{2\}} \\ &+ \frac{1}{6} [(p^2 - h^2)f_{i-2} + 2(2h^2 + p^2)f_{i-1} - 6(h^2 + p^2)f_i + 2(2h^2 + p^2)f_{i+1} + (p^2 - h^2)f_{i+2}] \triangleq H_i \end{aligned}$$

$$0 \leq i \leq n-1. \quad (5.30)$$

Using an earlier argument as in (5.22), it follows from (5.30) that

$$|e^{\{2\}}|_0 \leq \frac{10}{(4h^2 + p^2)(h^2 + p^2)} |H|_0. \quad (5.31)$$

We shall now establish a bound for $|H|_0$. Noting the relation between second order central difference and divided difference.

$$D_h^{\{2\}} f(t) = \frac{f(t+h) - 2f(t) + f(t-h)}{h^2} = 2f[t-h, t, t+h] \triangleq P_f(t, h),$$

we find

$$\begin{aligned} & (p^2 - h^2)f_{i+2} + 2(2h^2 + p^2)f_{i+1} - 6(h^2 + p^2)f_i + 2(2h^2 + p^2)f_{i-1} + (p^2 - h^2)f_{i-2} \\ &= [(p^2 - h^2)f_{i+2} - 2(p^2 - h^2)f_{i+1} + (p^2 - h^2)f_i] \\ &+ [2(h^2 + 2p^2)f_{i+1} - 4(h^2 + 2p^2)f_i + 2(h^2 + 2p^2)f_{i-1}] \\ &+ [(p^2 - h^2)f_i - 2(p^2 - h^2)f_{i-1} + (p^2 - h^2)f_{i-2}] \\ &= 2p^2(p^2 - h^2)f[t_i, t_{i+1}, t_{i+2}] + 4p^2(h^2 + 2p^2)f[t_{i-1}, t_i, t_{i+1}] \\ &+ 2p^2(p^2 - h^2)f[t_{i-2}, t_{i-1}, t_i] \\ &= p^2(p^2 - h^2)P_f(t_{i+1}, h) + 2p^2(h^2 + 2p^2)P_f(t_i, h) + p^2(p^2 - h^2)P_f(t_{i-1}, h). \end{aligned} \tag{5.32}$$

Then, using (5.32) in (5.30) gives

$$\begin{aligned} H_i &= -a_1P_f(t_{i-2}, h) - a_2P_f(t_{i-1}, h) - a_3P_f(t_i, h) - a_2P_f(t_{i+1}, h) - a_1P_f(t_{i+2}, h) \\ &+ \frac{1}{6}[p^2(p^2 - h^2)P_f(t_{i+1}, h) + 2p^2(h^2 + 2p^2)P_f(t_i, h) + p^2(p^2 - h^2)P_f(t_{i-1}, h)]. \end{aligned} \tag{5.33}$$

Applying Lemma 5.2 with $g = f$, $\alpha = t_{i-2} - h$, $\beta = t_{i+2} + h$ and $k = 2$, we find

$$|H_i| \leq \frac{1}{2!}(p^4)2w(f^{\{2\}}, 4p) \leq 4p^4w(f^{\{2\}}, p)$$

which implies

$$|H|_0 \leq 4p^4w(f^{\{2\}}, p). \tag{5.34}$$

Hence using (5.34) in (5.31) gives

$$|e^{\{2\}}|_0 \leq \gamma w(f^{\{2\}}, p). \quad (5.35)$$

Now applying Lemma 5.1 with $g = e^{\{2\}}$, $\alpha = t_{i-1}$, $\beta = t_i$, we obtain $\|Ue^{\{2\}}\| \leq \frac{p^2}{8} \|e^{\{4\}}\|$. Further, we have

$$(Le^{\{2\}})(t) = \frac{t - t_{i-1}}{p} e^{\{2\}}(t_i) + \frac{t_i - t}{p} e^{\{2\}}(t_{i-1}) \quad (5.36)$$

which leads to $\|Le^{\{2\}}\| \leq |e^{\{2\}}|_0$. Since

$$e^{\{2\}} = Ue^{\{2\}} + Le^{\{2\}}, \quad (5.37)$$

we get

$$\|e^{\{2\}}\| \leq \|Ue^{\{2\}}\| + \|Le^{\{2\}}\| \leq \frac{p^2}{8} \|e^{\{4\}}\| + |e^{\{2\}}|_0$$

which, upon substitution of (5.16) and (5.35), gives (5.18) immediately.

Next, we shall prove (5.17). Taking central difference of (5.37), we obtain

$$e^{\{3\}} = D_h^{\{1\}}(Ue^{\{2\}}) + D_h^{\{1\}}(Le^{\{2\}})$$

which on using (5.14) yields,

$$\|e^{\{3\}}\| \leq \frac{p}{2} \|e^{\{4\}}\| + \|D_h^{\{1\}}(Le^{\{2\}})\|. \quad (5.38)$$

It is clear from (5.36) that

$$D_h^{\{1\}}(Le^{\{2\}})(t) = \frac{1}{p} [e^{\{2\}}(t_i) - e^{\{2\}}(t_{i-1})]$$

and so

$$\|D_h^{\{1\}}(Le^{\{2\}})\| \leq \frac{2}{p} |e^{\{2\}}|_0. \quad (5.39)$$

Using (5.16), (5.39) and (5.35) in (5.38) gives (5.17) immediately.

Finally we shall prove (5.20) and (5.19). Applying Lemma 5.1 with $g = e$, $\alpha =$

t_{i-1} , and $\beta = t_i$, we see that $Lg = 0$ and therefore $Ue = e$. Thus, it follows from (5.13) that

$$\|e\| = \|Ue\| \leq \frac{p^2}{8} \|e^{\{2\}}\|. \quad (5.40)$$

Further, since $e = Ue$, an application of (5.14) leads to

$$\|e^{\{1\}}\| = \|D_h^{\{1\}}(Ue)\| \leq \frac{p}{2} \|e^{\{2\}}\|. \quad (5.41)$$

An application of (5.18) in (5.40) and (5.41) gives (5.20) and (5.19) respectively.

□

5.3 Numerical Example

In this section, we shall present an example to illustrate the periodic discrete spline interpolation developed in section 5.1 as well as the error bounds obtained in section 5.2.

Example 5.1. Consider the function

$$f(t) = \frac{1}{100} \left[\sin^2(\pi t) + \frac{19}{20} \cos^2(\pi t) \right], \quad t \in [0, 1].$$

In this example, we have $[a, b] = [0, 1]$ and the steps taken to obtain the periodic quintic discrete spline interpolant $S_\varphi f(t)$ and the errors are listed as follows.

- Fix the uniform partition φ (i.e., step size p) and choose a value for h .
- Solve the systems (5.8) and (5.11) to get M_i 's and F_i 's respectively. Then, $S_\varphi f(t)$ can be constructed in each subinterval $[t_{i-1}, t_i]$ following (5.4).
- Compute the actual errors

$$\|e^{\{\mu\}}\| = \|f^{\{\mu\}} - (S_\varphi f)^{\{\mu\}}\| = \max_{t \in [0, 1]_h} |f^{\{\mu\}}(t) - (S_\varphi f)^{\{\mu\}}(t)|, \quad \mu = 0, 1, 2, 3, 4.$$

- Compute the bounds (5.16) – (5.20) given in Theorem 5.2.

Let $t_0 = t_0(\mu)$ be such that $\|e^{\{\mu\}}\| = |f^{\{\mu\}}(t_0) - (S_\varphi f)^{\{\mu\}}(t_0)|$. The actual errors and the error bounds are presented in Table 5.1. The error bounds are calculated according to Theorem 5.2 respectively. To illustrate graphically, we have plotted $S_\varphi f(t)$ and $f(t)$ in Figure 5.1.

Table 5.1: Actual errors and error bounds for Example 5.1

	$p = \frac{1}{10}, h = p$	$p = \frac{1}{8}, h = \frac{p}{4}$	$p = \frac{1}{10}, h = \frac{p}{4}$	$p = \frac{1}{15}, h = \frac{p}{4}$
$\ e\ $	0	$0.82039194e - 08$	$0.20521179e - 08$	$0.17547366e - 09$
Bound	$0.96628467e - 05$	$0.47809205e - 03$	$0.24082627e - 03$	$0.70158943e - 04$
$\ e^{\{1\}}\ $	0	$0.13126271e - 06$	$0.41042359e - 07$	$0.52642099e - 08$
Bound	$0.38651387e - 03$	$0.15298946e - 01$	$0.96330507e - 02$	$0.42095366e - 02$
$\ e^{\{2\}}\ $	0	$0.88913504e - 05$	$0.35473364e - 05$	$0.68230315e - 06$
Bound	$0.77302774e - 02$	$0.63964914e + 00$	$0.19266101e + 00$	$0.12628610e + 00$
$\ e^{\{3\}}\ $	0	$0.25175638e - 03$	$0.12575208e - 3$	$0.36340402e - 04$
Bound	$0.16331306e + 00$	$0.42046640e + 01$	$0.40397862e + 01$	$0.38724574e + 00$
$\ e^{\{4\}}\ $	0	$0.19214105e - 01$	$0.12196109e - 01$	$0.53762787e - 02$
Bound	$0.34830056e + 00$	$0.92202866e + 01$	$0.74626360e + 01$	$0.50324713e + 01$
CPU time (s)*	14.2	19.9	32.7	71.2

* Matlab programme is performed on a laptop of FUJITSU LIFEBOOK S Series.

Relative error $\frac{\ e\ }{ f(t_0) }$	0	$0.86184404e - 06$	$0.21573455e - 06$	$0.17547366e - 07$
$\frac{\ e^{\{1\}}\ }{ f(t_0) }$	0	$0.138101443e - 04$	$0.43188490e - 05$	$0.52649310e - 06$
$\frac{\ e^{\{2\}}\ }{ f(t_0) }$	0	$0.93593163e - 03$	$0.37340383e - 03$	$0.71821384e - 04$
$\frac{\ e^{\{3\}}\ }{ f(t_0) }$	0	$0.26487278e - 01$	$0.13232774e - 01$	$0.38247541e - 02$
$\frac{\ e^{\{4\}}\ }{ f(t_0) }$	0	$0.19214105e + 01$	$0.12196109e + 01$	$0.56592408e + 00$

From the above table, we observe the following:

- (i) It is noted that as the step size p decreases, the error (actual and relative) and the bound also become smaller while more time is needed.
- (ii) For fixed values of p and h , $\|e^{\{j+1\}}\|$ is closer to the bound compared with $\|e^{\{j\}}\|$ to the corresponding bound, $j = 0, 1, 2, 3$.

(iii) We recall that φ is the uniform partition of $[a, b]$, $\varphi : a = t_0 < t_1 < \dots < t_n = b$ with the step size $p = \frac{(b-a)}{n}$. For a given h , the discrete interval $[a, b]_h$ is defined as $[a, b]_h = [a, b] \cap \{a, a + h, a + 2h, \dots\}$. It is obvious that when $h = p$, the partition φ is the same as $[a, b]_h$. Then using the definition of central differences and the periodic conditions of f and $S_\varphi f$, it is easy to deduce that all the $e^{\{j\}} = 0$, $j = 0, 1, 2, 3, 4$.

(iv) For a fixed value $h = \frac{p}{4}$, the values of $e^{\{j\}}$, $j = 0, 1, 2, 3, 4$ and the corresponding bounds become smaller as p gets smaller. With a smaller value of p , more piecewise splines have to be constructed, which leads to longer time for Matlab to run the programme.

Figure 5.1: f and $S_\varphi f$ when $h = p = \frac{1}{10}$

5.4 Two-variable Periodic Discrete Spline Interpolation

For any positive numbers P_1 and P_2 , we say a function $g(t, u)$ is (P_1, P_2) -periodic if $g(t + P_1, u) = g(t, u)$, $g(t, u + P_2) = g(t, u)$ and $g(t + P_1, u + P_2) = g(t, u)$. For convenience, we shall denote $g^{\{\mu, \nu\}}(t, u) = D_{h,t}^{\{\mu\}} D_{h,u}^{\{\nu\}} g(t, u)$, and with respect to the partition $\phi = \varphi \times \varphi'$, denote $g_{i,j}^{\{\mu, \nu\}} = D_{h,t}^{\{\mu\}} D_{h,u}^{\{\nu\}} g(t_i, u_j)$.

In this section, we shall develop two-variable periodic discrete spline interpolation based on the results of one-variable case established in previous sections. To begin, we define $S_h(\phi)$ as

$$\begin{aligned}
 S_h(\phi) &= S_h(\varphi) \otimes S_h(\varphi') \text{ (the tensor product)} \\
 &= \text{Span}\{s_i(t)s_j(u)\}_{i=0, j=0}^{n-1, m-1} \quad (\text{see Remark 5.3}) \\
 &= \left\{ S(t, u) : S(t, u) \text{ is a two-dimensional polynomial of degree 5 in each} \right. \\
 &\quad \text{variable, its restriction } S_{ij}(t, u) \text{ on } [t_{i-1}, t_i] \times [u_{j-1}, u_j], \quad 1 \leq i \leq n, \\
 &\quad \left. 1 \leq j \leq m \text{ is biquintic,} \right. \\
 &\quad S_{ij}^{\{\mu, \nu\}}(t_i, u_j) = S_{i+1,j}^{\{\mu, \nu\}}(t_i, u_j) = S_{i,j+1}^{\{\mu, \nu\}}(t_i, u_j) = S_{i+1,j+1}^{\{\mu, \nu\}}(t_i, u_j), \\
 &\quad 1 \leq i \leq n-1, \quad 1 \leq j \leq m-1, \quad \mu, \nu = 0, 1, 2, 3, 4, \\
 &\quad \left. \text{and } S(t, u) \text{ is } (b-a, d-c)\text{-periodic} \right\}.
 \end{aligned}$$

Definition 5.3. For a $(b-a, d-c)$ -periodic function $f(t, u)$ defined on $[a-2h, b+2h]_h \times [c-2h, d+2h]_h$, we say $S_\phi f(t, u)$ is a $S_h(\phi)$ -interpolant of $f(t, u)$, also known as a *periodic biquintic discrete spline*, if $S_\phi f(t, u) \in S_h(\phi)$ with

$$S_\phi f(t_i, u_j) = f(t_i, u_j), \quad 0 \leq i \leq n-1, \quad 0 \leq j \leq m-1. \quad (5.42)$$

Remark 5.4. In Definition 5.3, it actually suffices to have the periodic function $f(t, u)$ defined on the partition ϕ . However, the subsequent error analysis requires the periodic function $f(t, u)$ to be defined on $[a - 2h, b + 2h]_h \times [c - 2h, d + 2h]_h$. To be consistent, we therefore impose throughout that the $(b - a, d - c)$ -periodic function $f(t, u)$ is defined on $[a - 2h, b + 2h]_h \times [c - 2h, d + 2h]_h$.

Remark 5.5. Let $c_{i,j}^{\{\mu,\nu\}} = S_\phi^{\{\mu,\nu\}} f(t_i, u_j)$, $0 \leq i \leq n$, $0 \leq j \leq m$, $\mu, \nu \in \{0, 2, 4\}$ (note that $c_{i,j}^{\{0,0\}} = f_{ij}$). Then, $S_\phi f(t, u)$ can be written as

$$\begin{aligned}
 S_\phi f(t, u) = & \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[f_{ij} g_i(t) g_j(u) + c_{i,j}^{\{0,2\}} g_i(t) \bar{g}_j(u) + c_{i,j}^{\{0,4\}} g_j(t) \bar{\bar{g}}_i(u) \right. \\
 & + c_{i,j}^{\{2,0\}} \bar{g}_i(t) g_j(u) + c_{i,j}^{\{2,2\}} \bar{g}_i(t) \bar{g}_j(u) + c_{i,j}^{\{2,4\}} \bar{g}_i(t) \bar{\bar{g}}_j(u) \\
 & \left. + c_{i,j}^{\{4,0\}} \bar{\bar{g}}_i(t) g_j(u) + c_{i,j}^{\{4,2\}} \bar{\bar{g}}_i(t) \bar{g}_j(u) + c_{i,j}^{\{4,4\}} \bar{\bar{g}}_i(t) \bar{\bar{g}}_j(u) \right]. \tag{5.43}
 \end{aligned}$$

In (5.43), the $c_{i,j}^{\{\mu,\nu\}}$'s exist uniquely. Indeed, this is an immediate consequence from the proof of Theorem 5.1 and is stated as follows.

Theorem 5.3. Let $f(t, u)$ be a given $(b - a, d - c)$ -periodic function defined on $[a - 2h, b + 2h]_h \times [c - 2h, d + 2h]_h$. Then, there exists a unique periodic biquintic discrete spline interpolant $S_\phi f(t, u)$.

Proof. We shall show that the $c_{i,j}^{\{\mu,\nu\}}$'s in (5.43) exist uniquely. From the proof of Theorem 5.1 (notably (5.8) and (5.11)), the $c_{i,j}^{\{\mu,\nu\}}$'s satisfy the following relations:

$$\begin{aligned}
 a_1 c_{i-2,j}^{\{2,\nu\}} + a_2 c_{i-1,j}^{\{2,\nu\}} + a_3 c_{i,j}^{\{2,\nu\}} + a_2 c_{i+1,j}^{\{2,\nu\}} + a_1 c_{i+2,j}^{\{2,\nu\}} = & \frac{1}{6} \left[(p^2 - h^2) c_{i-2,j}^{\{0,\nu\}} \right. \\
 & \left. + 2(2h^2 + p^2) c_{i-1,j}^{\{0,\nu\}} - 6(h^2 + p^2) c_{i,j}^{\{0,\nu\}} + 2(2h^2 + p^2) c_{i+1,j}^{\{0,\nu\}} + (p^2 - h^2) c_{i+2,j}^{\{0,\nu\}} \right], \tag{5.44}
 \end{aligned}$$

$$\begin{aligned}
 a_1 c_{i-2,j}^{\{4,\nu\}} + a_2 c_{i-1,j}^{\{4,\nu\}} + a_3 c_{i,j}^{\{4,\nu\}} + a_2 c_{i+1,j}^{\{4,\nu\}} + a_1 c_{i+2,j}^{\{4,\nu\}} = & (c_{i-2,j}^{\{0,\nu\}} - 4c_{i-1,j}^{\{0,\nu\}} + 6c_{i,j}^{\{0,\nu\}} - 4c_{i+1,j}^{\{0,\nu\}} + c_{i+2,j}^{\{0,\nu\}}), \tag{5.45}
 \end{aligned}$$

Further, we have

$$\begin{aligned} a'_1 c_{i,j-2}^{\{\mu,2\}} + a'_2 c_{i,j-1}^{\{\mu,2\}} + a'_3 c_{i,j}^{\{\mu,2\}} + a'_2 c_{i,j+1}^{\{\mu,2\}} + a'_1 c_{i,j+2}^{\{\mu,2\}} &= \frac{1}{6} \left[(p^2 - h^2) c_{i,j-2}^{\{\mu,0\}} \right. \\ &\quad \left. + 2(2h^2 + p^2) c_{i,j-1}^{\{\mu,0\}} - 6(h^2 + p^2) c_{i,j}^{\{\mu,0\}} + 2(2h^2 + p^2) c_{i,j+1}^{\{\mu,0\}} + (p^2 - h^2) c_{i,j+2}^{\{\mu,0\}} \right], \end{aligned} \quad (5.46)$$

$$a'_1 c_{i,j-2}^{\{\mu,4\}} + a'_2 c_{i,j-1}^{\{\mu,4\}} + a'_3 c_{i,j}^{\{\mu,4\}} + a'_2 c_{i,j+1}^{\{\mu,4\}} + a'_1 c_{i,j+2}^{\{\mu,4\}} = (c_{i,j-2}^{\{\mu,0\}} - 4c_{i,j-1}^{\{\mu,0\}} + 6c_{i,j}^{\{\mu,0\}} - 4c_{i,j+1}^{\{\mu,0\}} + c_{i,j+2}^{\{\mu,0\}}) \quad (5.47)$$

where a'_i , $i = 1, 2, 3$ are just a_i (see (5.9)) with p replaced by p' . In (5.44)–(5.47) we have $\mu, \nu \in \{0, 2, 4\}$, $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$ (Noting the periodicity conditions).

As observed in the proof of Theorem 5.1, the systems (5.44)–(5.47) can be solved to give unique solutions of $c_{i,j}^{\{\mu,\nu\}}$'s. In fact,

- for $\nu = 0$, from (5.44) and (5.45) we can get $c_{i,j}^{2,0}$ and $c_{i,j}^{4,0}$, $0 \leq i \leq n-1$, $0 \leq j \leq m-1$;
- for $\mu = 0$, from (5.46) and (5.47) we can get $c_{i,j}^{0,2}$ and $c_{i,j}^{0,4}$, $0 \leq i \leq n-1$, $0 \leq j \leq m-1$;
- for $\nu = 2, 4$, from (5.44) we can obtain $c_{i,j}^{2,2}$ and $c_{i,j}^{2,4}$, $0 \leq i \leq n-1$, $0 \leq j \leq m-1$;
- for $\nu = 2, 4$, from (5.45) we can obtain $c_{i,j}^{4,2}$ and $c_{i,j}^{4,4}$, $0 \leq i \leq n-1$, $0 \leq j \leq m-1$.

Hence, $S_\phi f(t, u)$ is unique and it can be written explicitly as (5.43). $\square$

Remark 5.6. In view of Remark 5.3, $S_\phi f(t, u)$ can be expressed in terms of cardinal splines as

$$S_\phi f(t, u) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f_{ij} s_i(t) s_j(u). \quad (5.48)$$

Lemma 5.3. *Let $f(t, u)$ be a $(b-a, d-c)$ -periodic function defined on $[a-2h, b+2h]_h \times [c-2h, d+2h]_h$. Then,*

$$S_\phi f(t, u) = S_{\phi'} S_\phi f(t, u) = S_\phi S_{\phi'} f(t, u). \quad (5.49)$$

Proof. From Remark 5.3, we obtain

$$S_{\varphi'} S_{\varphi} f(t, u) = S_{\varphi'} \left[\sum_{i=0}^{n-1} f(t_i, u) s_i(t) \right] = \sum_{j=0}^{m-1} \left[\sum_{i=0}^{n-1} f_{ij} s_i(t) \right] s_j(u) = S_{\phi} f(t, u).$$

The proof of the second equality is similar. $\square$

We shall now derive error estimates for two-variable periodic discrete spline interpolation. Here, the *modulus of smoothness with respect to the variable t* is defined as

$$w_t(g, r) = \max \{ |g(t, u) - g(t', u)| : |t - t'| < r, \ t, t' \in [\alpha, \beta]_h, \ u \in [\xi, \eta]_h \},$$

while the *modulus of smoothness with respect to the variable u* is defined as

$$w_u(g, r) = \max \{ |g(t, u) - g(t, u')| : |u - u'| < r, \ u, u' \in [\xi, \eta]_h, \ t \in [\alpha, \beta]_h \}.$$

Theorem 5.4. *Let $f(t, u)$ be a $(b-a, d-c)$ -periodic function defined on $[a-2h, b+2h]_h \times [c-2h, d+2h]_h$. Then, we have*

$$\begin{aligned} \|f - S_{\phi} f\| &= \max_{(t,u) \in [a,b]_h \times [c,d]_h} |f(t, u) - S_{\phi} f(t, u)| \\ &\leq \frac{p^4}{64} (1 + \gamma) w_t(f^{\{4,0\}}, p) + \frac{p^2}{8} \gamma w_t(f^{\{2,0\}}, p) \\ &\quad + \frac{(p')^4}{64} (1 + \gamma') w_u(f^{\{0,4\}}, p') + \frac{(p')^2}{8} \gamma' w_u(f^{\{0,2\}}, p') \\ &\quad + \frac{p^4}{32} (1 + \gamma) \left[\frac{(p')^4}{64} (1 + \gamma') w_u(f^{\{4,4\}}, p') + \frac{(p')^2}{8} \gamma' w_u(f^{\{4,2\}}, p') \right] \\ &\quad + \frac{p^2}{4} \gamma \left[\frac{(p')^4}{64} (1 + \gamma') w_u(f^{\{2,4\}}, p') + \frac{(p')^2}{8} \gamma' w_u(f^{\{2,2\}}, p') \right]. \end{aligned} \tag{5.50}$$

where $\gamma = \frac{40p^4}{(4h^2+p^2)(h^2+p^2)}$ and γ' is the same as γ with p replaced by p' .

Proof. Applying Lemma 5.3, we write

$$f - S_\phi f = (f - S_\varphi f) + [S_\varphi(f - S_{\varphi'} f) - (f - S_{\varphi'} f)] + (f - S_{\varphi'} f)$$

which implies

$$\begin{aligned} \|f - S_\phi f\| &\leq \|f - S_\varphi f\| + \|S_\varphi(f - S_{\varphi'} f) - (f - S_{\varphi'} f)\| + \|f - S_{\varphi'} f\| \\ &= \|e_\varphi\| + \|S_\varphi(f - S_{\varphi'} f) - (f - S_{\varphi'} f)\| + \|e_{\varphi'}\| \end{aligned} \quad (5.51)$$

where $e_\varphi(t, u) = f(t, u) - S_\varphi f(t, u)$ and $e_{\varphi'}(t, u) = f(t, u) - S_{\varphi'} f(t, u)$.

Applying (5.20), we get

$$\|e_\varphi\| \leq \frac{p^4}{64}(1 + \gamma)w_t(f^{\{4,0\}}, p) + \frac{p^2}{8}\gamma w_t(f^{\{2,0\}}, p) \quad (5.52)$$

and

$$\|e_{\varphi'}\| \leq \frac{(p')^4}{64}(1 + \gamma')w_u(f^{\{0,4\}}, p') + \frac{(p')^2}{8}\gamma' w_u(f^{\{0,2\}}, p'). \quad (5.53)$$

Further, using (5.20) again gives

$$\begin{aligned} \|S_\varphi(f - S_{\varphi'} f) - (f - S_{\varphi'} f)\| &\leq \frac{p^4}{64}(1 + \gamma)w_t(e_{\varphi'}^{\{4,0\}}, p) + \frac{p^2}{8}\gamma w_t(e_{\varphi'}^{\{2,0\}}, p) \\ &\leq \frac{p^4}{32}(1 + \gamma)\|e_{\varphi'}^{\{4,0\}}\| + \frac{p^2}{4}\gamma\|e_{\varphi'}^{\{2,0\}}\|. \end{aligned} \quad (5.54)$$

Since $(S_{\varphi'} f)^{\{4,0\}} = S_{\varphi'}(f^{\{4,0\}})$, applying (5.20) we obtain

$$\|e_{\varphi'}^{\{4,0\}}\| = \|f^{\{4,0\}} - S_{\varphi'}(f^{\{4,0\}})\| \leq \frac{(p')^4}{64}(1 + \gamma')w_u(f^{\{4,4\}}, p') + \frac{(p')^2}{8}\gamma' w_u(f^{\{4,2\}}, p'). \quad (5.55)$$

Similarly, we find

$$\|e_{\varphi'}^{\{2,0\}}\| = \|f^{\{2,0\}} - S_{\varphi'}(f^{\{2,0\}})\| \leq \frac{(p')^4}{64}(1 + \gamma')w_u(f^{\{2,4\}}, p') + \frac{(p')^2}{8}\gamma' w_u(f^{\{2,2\}}, p'). \quad (5.56)$$

Now, using (5.55) and (5.56) in (5.54) and then together with (5.52) and (5.53) in (5.51), we get (5.50) immediately. $\square$

We shall now present two examples to illustrate the two-variable periodic discrete spline interpolation as well as the error estimates obtained.

In the following numerical examples, let (t_0, u_0) be such that $\|f - S_\phi f\| = |f(t_0, u_0) - S_\phi f(t_0, u_0)|$.

Example 5.2. Consider the function

$$f(t, u) = \frac{1}{100} \left[\sin^2(\pi t) + \frac{19}{20} \cos^2(\pi u) \right], \quad (t, u) \in [0, 1] \times [0, 1].$$

Here, we have $[a, b] = [c, d] = [0, 1]$. Fix $p = p'$ and take $h = \frac{p}{4}$.

To construct $S_\phi f(t, u)$, in view of Remark 5.6 we need only to construct the cardinal splines $s_i(t)$, $0 \leq i \leq n-1$ and $s_j(u)$, $0 \leq j \leq m-1$. To obtain a particular cardinal spline, we solve the systems (5.8) and (5.11) and then the cardinal spline can be written explicitly using (5.3) or (5.4).

We also compute the actual error

$$\|f - S_\phi f\| = \max_{(t,u) \in [0,1]_h \times [0,1]_h} |f(t, u) - S_\phi f(t, u)|$$

and the bound in Theorem 5.4. The results are presented in Table 5.2. To illustrate graphically, we have plotted the original function $f(t, u)$, the spline interpolant $S_\phi f(t, u)$ and the absolute error between the two functions in Figures 5.2 and 5.3.

Table 5.2: Actual error and bound for Example 5.2.

	$p = p' = \frac{1}{8}$	$p = p' = \frac{1}{10}$
$\ f - S_\phi f\ $	$0.56790328e - 03$	$0.28049052e - 03$
Bound	$0.18645590e - 01$	$0.93922244e - 02$
CPU time (s)*	240.2	544.6

* Matlab programme is performed on a laptop of FUJITSU LIFEBOOK S Series.

Relative error $\frac{\ f - S_\phi f\ }{ f(t_0, u_0) }$	$0.42125688e - 01$	$0.18118482e - 01$
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From Table 5.2 we notice that the errors (actual and relative) and the bounds

tend to be smaller when the step sizes become smaller, however, it takes Matlab longer time to get the results with smaller values of the step sizes. With reference to Figure 5.3, we note from the Matlab computations that the maximum error happens at $(t, u) = [13/20, 3/20]$ when $p = p' = 4h = \frac{1}{10}$.

Figure 5.2: f and $S_\phi f$ when $p = p' = 4h = \frac{1}{10}$ (Example 5.2)

Figure 5.3: Absolute error between f and $S_\phi f$ when $p = p' = 4h = \frac{1}{10}$ (Example 5.2)

Example 5.3. Consider the function

$$f(t, u) = \frac{1}{100} [\sin^2(\pi t) \cos^2(\pi u)], \quad (t, u) \in [0, 1] \times [0, 1].$$

Here, we have $[a, b] = [c, d] = [0, 1]$. Fix $p = p'$ and take $h = \frac{p}{4}$.

Table 5.3: Actual errors and error bounds

	$p = p' = \frac{1}{8}$	$p = p' = \frac{1}{10}$
$\ f - S_\phi f\ $	$0.41659718e - 03$	$0.22632005e - 03$
Bound	$0.43015179e - 01$	$0.17140378e - 01$
CPU time (s)*	246.6	565.8

* Matlab programme is performed on a laptop of FUJITSU LIFEBOOK S Series.

Relative error $\frac{\ f - S_\phi f\ }{ f(t_0, u_0) }$	$0.62643452e - 01$	$0.35908684e - 01$
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As the step sizes decrease, the values of $\|f - S_\phi f\|$, the relative errors and the bounds also decrease while it takes Matlab more time to run the programme to obtain the results. With reference to Figure 5.5, we observe from the Matlab computations that the largest error happens at $(t, u) = [13/20, 3/20]$ when $p = p' = 4h = \frac{1}{10}$.

Figure 5.4: f and $S_\phi f$ when $p = p' = 4h = \frac{1}{10}$ (Example 5.3)

Figure 5.5: Absolute error between f and $S_\phi f$ when $p = p' = 4h = \frac{1}{10}$ (Example 5.3)

Chapter 6

Solving Boundary Value Problems

In this chapter, we shall provide numerical solutions to boundary value problems. In section 6.1, we shall discuss a numerical method for second order boundary value problems using cubic discrete splines. Through convergence analysis, the method is shown to be fourth order convergent when the parameter $h = \frac{p}{\sqrt{2}}$, and second order convergent for other values of $h \in (0, p]$. In section 6.2, deficient cubic discrete spline will be employed to obtain a new numerical method for solving a system of second order boundary value problems. This method allows us to approximate the solution as well as its derivatives at every point in the range of integration. Through an application to an obstacle boundary value problem, we see that our method is second-order accurate when the parameter $h = \frac{p}{\sqrt{2}}$, and it outperforms other collocation, finite difference and spline methods in the literature. Finally, in section 6.3, we shall utilize the equations derived from quintic discrete spline to obtain a numerical solution of fourth order boundary value problems. The method is shown to be fourth order convergent when the parameter $h = \frac{p}{\sqrt{3}}$, and second order convergent for other values of $h \in (0, p]$. Numerical examples will be presented in each section to illustrate the efficiency of our results.

6.1 Second Order Boundary Value Problems

This section is based on the work of [31].

6.1.1 Problem Description

We consider the second order boundary value problem

$$\begin{aligned} y''(t) &= f(t)y(t) + g(t), \quad a \leq t \leq b, \\ y(a) &= \bar{\alpha}, \quad y(b) = \bar{\beta} \end{aligned} \tag{6.1}$$

where f and g are continuous functions on $[a, b]$. Such problems arise from many real world situations, for example in the theory which describes the deflection of plates, diffusion occurring in the presence of exothermic chemical reaction, heat conduction associated with radiation effect and a variety of other scientific applications [67]. In general it is difficult to obtain the analytical solution of (6.1) for arbitrary f and g and we usually resort to some numerical methods to obtain an approximate solution of (6.1). In the literature, the finite difference method has been commonly used for the numerical treatment of (6.1) and this method has been discussed by many authors, see, for example [18, 33, 48, 98]. On the other hand, Ahlberg et al. [5] have introduced splines in solving initial as well as boundary value problems. Following this several authors [6, 10, 23, 43, 53] have investigated the use of cubic splines in solving two-point boundary value problems. Other methods that involve quadratic splines as well as collocation methods with splines as basis functions have further been applied to solve various second order boundary value problems, see for example [5, 9, 15, 52, 73, 85, 104] and the references therein.

There is an advantage of a spline method over a finite difference method – once the spline solution is obtained, any information between the mesh points becomes immediately available. Motivated by the research on using splines to solve (6.1), in this section we shall employ *cubic discrete splines* to get an approximate solution of (6.1). It is noted that the methods in the literature that solve (6.1) by *cubic splines* [6, 10, 23, 43, 53] are of order 2 in most cases, except that of Khan [53], which is of order 4 when certain parameters take specific values and is of order 2 otherwise. In comparison, our method is fourth order convergent if a parameter takes a specific value, else it is second order convergent – this convergence is ‘on par’ with the

method of Khan [53] and better than those in [6, 10, 23, 43]. Moreover, computation wise our method is much easier compared to [53]. Indeed, we shall show by two numerical examples that our method outperforms other collocation, finite-difference and spline methods for solving (6.1).

Discrete splines were first introduced by Mangasarian and Schumaker [64] in 1971 as solutions to constrained minimization problems in real Euclidean space, which are discrete analogs of minimization problems in Banach space whose solutions are generalized splines. Subsequent investigations on discrete splines can be found in the work of Schumaker [88], Astor and Duris [19], Lyche [59, 60] and Wong et al [27, 30, 112]. Following [59, 60], the discrete spline we use will involve central differences.

The outline of this section is as follows. In section 6.1.2 we shall derive the consistency relations and develop the new cubic discrete spline method for solving (6.1). Section 6.1.3 is devoted to the convergence analysis of the method. Finally, in section 6.1.4 we present the numerical experiments and comparison with other methods.

6.1.2 Cubic Discrete Spline Method

Recall that φ is the uniform partition of $[a, b]$ with $p = (b - a)/n$. Based on central differences defined in Chapter 5, we shall now introduce the cubic discrete spline.

Definition 6.1. Let $S(t; h)$ be a piecewise continuous function defined over $[a, b]$ (with mesh φ) and $S_i(t)$ be its restriction in $[t_{i-1}, t_i]$, $1 \leq i \leq n$ passing through the points (t_{i-1}, s_{i-1}) and (t_i, s_i) . We say $S(t; h)$ is a *cubic discrete spline* if $S_i(t)$, $1 \leq i \leq n$ is a polynomial of degree 3 or less and

$$(S_{i+1} - S_i)(t_i + jh) = 0, \quad j = -1, 0, 1, \quad 1 \leq i \leq n - 1. \quad (6.2)$$

Indeed, the condition (6.2) has the following equivalent form

$$D_h^{\{j\}} S_i(t_i) = D_h^{\{j\}} S_{i+1}(t_i), \quad j = 0, 1, 2, \quad 1 \leq i \leq n-1. \quad (6.3)$$

Note that [60] has the same definition for discrete splines.

Throughout, in the context of (6.1) we shall use the notations

$$y^{(k)}(t_i) = y_i^{(k)}, \quad f(t_i) = f_i, \quad g(t_i) = g_i, \quad 0 \leq i \leq n.$$

We shall approximate a solution $y(t)$ of (6.1) by the cubic discrete spline $S(t; h)$, i.e., $y(t)$ will be approximated by $S_i(t)$ over the subinterval $[t_{i-1}, t_i]$, $1 \leq i \leq n$. Indeed, for any $t \in [a, b]$ (t may be between mesh points), we propose the following approximates:

$$y(t) \cong S(t; h), \quad y'(t) \cong D_h^{\{1\}} S(t; h), \quad y''(t) \cong f(t)S(t; h) + g(t). \quad (6.4)$$

In particular, at the mesh points we have

$$y_i \cong s_i \equiv S_i(t_i), \quad y'_i \cong s'_i \equiv D_h^{\{1\}} S_i(t_i), \quad y''_i \cong s''_i \equiv f_i S_i(t_i) + g_i, \quad 0 \leq i \leq n. \quad (6.5)$$

From the boundary conditions, we note that $s_0 = y_0 = \bar{\alpha}$ and $s_n = y_n = \bar{\beta}$, and s_i is an approximate of y_i , $1 \leq i \leq n-1$.

Our task now is to obtain an explicit expression of $S_i(t)$. Clearly, $S_i(t)$ should pass through the points (t_{i-1}, s_{i-1}) and (t_i, s_i) . Let $c_i = D_h^{\{2\}} S(t_i; h)$, $0 \leq i \leq n$ denote the ‘discrete moments’. Since $D_h^{\{2\}} S(t; h)$ is piecewise linear, we have for $t \in [t_{i-1}, t_i]$ and $1 \leq i \leq n$,

$$D_h^{\{2\}} S(t; h) = D_h^{\{2\}} S_i(t) = \frac{t_i - t}{p} c_{i-1} + \frac{t - t_{i-1}}{p} c_i. \quad (6.6)$$

It follows that

$$S_i(t) = \frac{(t_i - t)^{\{3\}}}{6p} c_{i-1} + \frac{(t - t_{i-1})^{\{3\}}}{6p} c_i + \frac{t_i - t}{p} u_i + \frac{t - t_{i-1}}{p} w_i, \quad (6.7)$$

$$t \in [t_{i-1}, t_i], \quad 1 \leq i \leq n$$

where u_i and w_i are arbitrary constants. The interpolation conditions $S_i(t_j) = s_j$, $j = i - 1, i$ then lead to

$$u_i = s_{i-1} - \frac{p^2 - h^2}{6} c_{i-1}, \quad w_i = s_i - \frac{p^2 - h^2}{6} c_i, \quad 1 \leq i \leq n. \quad (6.8)$$

Now, substituting (6.8) into (6.7) and taking central difference gives

$$D_h^{\{1\}} S_i(t) = -\frac{(t_i - t)^2}{2p} c_{i-1} - \frac{(p^2 - h^2)(c_i - c_{i-1})}{6p} + \frac{(t - t_{i-1})^2}{2p} c_i + \frac{s_i - s_{i-1}}{p},$$

$$t \in [t_{i-1}, t_i], \quad 1 \leq i \leq n. \quad (6.9)$$

The ‘continuity’ requirement $D_h^{\{1\}} S_i(t_i) = D_h^{\{1\}} S_{i+1}(t_i)$ (see (6.3)) then leads to the system of equations

$$\frac{(p^2 - h^2)}{6} c_{i-1} + \frac{2(2p^2 + h^2)}{6} c_i + \frac{(p^2 - h^2)}{6} c_{i+1} = s_{i-1} - 2s_i + s_{i+1}, \quad 1 \leq i \leq n - 1. \quad (6.10)$$

In view of the fact that we approximate $y(t)$ by $S(t; h)$ and (6.5), we set $c_i = s_i''$ or

$$c_i = f_i s_i + g_i, \quad 0 \leq i \leq n. \quad (6.11)$$

Upon substituting (6.11) into (6.10), the system (6.10) becomes

$$\left[\frac{(p^2 - h^2)}{6} f_{i-1} - 1 \right] s_{i-1} + \left[\frac{2(2p^2 + h^2)}{6} f_i + 2 \right] s_i + \left[\frac{(p^2 - h^2)}{6} f_{i+1} - 1 \right] s_{i+1}$$

$$= -\frac{(p^2 - h^2)}{6} (g_{i-1} + g_{i+1}) - \frac{2(2p^2 + h^2)}{6} g_i, \quad 1 \leq i \leq n - 1. \quad (6.12)$$

Together with the boundary conditions $s_0 = \bar{\alpha}$, $s_n = \bar{\beta}$, we may solve (6.12) to get

s_i , $1 \leq i \leq n-1$. The solvability of the system (6.12) will be tackled in section 6.1.3 where we shall establish the existence of a unique solution for (6.12).

Finally, c_i is computed via (6.11) and then substituted into (6.8) as well as the expression (6.7) of $S_i(t)$. This completes the task of finding the cubic discrete spline solution of (6.1).

6.1.3 Convergence Analysis

In this section, we shall establish the existence of a unique cubic discrete spline solution for (6.1) (i.e., (6.12) has a unique solution) and also conduct a convergence analysis for the method presented in section 6.1.2.

Let $e_i = y_i - s_i$, $1 \leq i \leq n-1$ be the errors. Let $y = [y_i]$, $s = [s_i]$, $r = [r_i]$, $v = [v_i]$ and $e = [e_i]$ be $(n-1)$ -dimensional column vectors. To recall the definition for vector norm, $|e|_0 = \max_{1 \leq i \leq n-1} |e_i|$, $|v|_0 = \max_{1 \leq i \leq n-1} |v_i|$. The system (6.12) can be written as

$$As = r \quad (6.13)$$

where

$$A = A_0 + Q, \quad Q = BF, \quad F = \text{diag}(f_i), \quad i = 1, 2, \dots, n-1, \quad (6.14)$$

$B = [b_{ij}]$ and $A_0 = [a_{ij}]$ are $(n-1) \times (n-1)$ tridiagonal matrices given by

$$b_{ij} = \begin{cases} \frac{2(2p^2 + h^2)}{6}, & i = j = 1, 2, \dots, n-1, \\ \frac{(p^2 - h^2)}{6}, & |i - j| = 1, \\ 0, & \text{otherwise,} \end{cases} \quad (6.15)$$

$$a_{ij} = \begin{cases} 2, & i = j = 1, 2, \dots, n-1, \\ -1, & |i - j| = 1, \\ 0, & \text{otherwise,} \end{cases} \quad (6.16)$$

and

$$r_i = \begin{cases} \bar{\alpha} - \frac{1}{6}[(p^2 - h^2)f_0\bar{\alpha} + (p^2 - h^2)g_0 + 2(2p^2 + h^2)g_1 + (p^2 - h^2)g_2], & i = 1, \\ -\frac{1}{6}[(p^2 - h^2)g_{i-1} + 2(2p^2 + h^2)g_i + (p^2 - h^2)g_{i+1}], & 2 \leq i \leq n-2, \\ \bar{\beta} - \frac{1}{6}[(p^2 - h^2)f_n\bar{\beta} + (p^2 - h^2)g_{n-2} + 2(2p^2 + h^2)g_{n-1} + (p^2 - h^2)g_n], & i = n-1. \end{cases} \quad (6.17)$$

From (6.13) we have $A(y - e) = r$ or

$$Ay = r + v \quad (6.18)$$

where

$$Ae = v. \quad (6.19)$$

The i -th equation of the system (6.18) is

$$-y_{i-1} + 2y_i - y_{i+1} = -\frac{1}{6}[(p^2 - h^2)y''_{i-1} + 2(2p^2 + h^2)y''_i + (p^2 - h^2)y''_{i+1}] + v_i$$

where v_i , $1 \leq i \leq n-1$ are the local truncation errors given by

$$v_i = \frac{p^2(p^2 - 2h^2)}{12}y_i^{(4)} + \frac{p^4(4p^2 - 5h^2)}{360}y_i^{(6)} + O(p^8). \quad (6.20)$$

Remark 6.1. For the special case $h = \frac{p}{\sqrt{2}}$, it is clear from (6.20) that

$$v_i = \frac{1}{240}p^6 y_i^{(6)} + O(p^8).$$

Thus,

$$|v|_0 = \frac{1}{240}p^6 M_6 \quad (6.21)$$

where $M_6 = \max_t |y^{(6)}(t)|$.

Lemma 6.1. [98] *The explicit inverse of A_0 , namely $A_0^{-1} = [\eta_{ij}]$, is given by*

$$\eta_{ij} = \begin{cases} \frac{j(n-i)}{n}, & i \geq j \\ \frac{i(n-j)}{n}, & i \leq j. \end{cases}$$

Note that $A_0^{-1} > 0$, i.e., all the entries of A_0^{-1} are positive. Moreover,

$$\|A_0^{-1}\| \leq \frac{n^2}{8}. \quad (6.22)$$

Our first result guarantees the existence of a unique cubic discrete spline solution for (6.1).

Theorem 6.1. *The system (6.13) has a unique solution if*

$$K\hat{f} < 1 \quad (6.23)$$

where $K = \frac{1}{8}(b-a)^2$ and $\hat{f} = \max_{1 \leq i \leq n-1} |f_i|$.

Proof. If (6.13) has a unique solution, then it can be written as

$$s = A^{-1}r = (A_0 + Q)^{-1}r = [A_0(I + A_0^{-1}Q)]^{-1}r = (I + A_0^{-1}Q)^{-1}A_0^{-1}r. \quad (6.24)$$

From Lemma 6.1 the inverse A_0^{-1} exists, hence it remains to show that $(I + A_0^{-1}Q)$

is nonsingular.

It is clear from (6.15) that $\|B\| = p^2$. Since $Q = BF$, we find

$$\|Q\| \leq \|B\| \|F\| \leq p^2 \hat{f}. \quad (6.25)$$

It follows from (6.22), (6.25) and the fact $n = \frac{b-a}{p}$ that

$$\|A_0^{-1}Q\| \leq \|A_0^{-1}\| \|Q\| \leq \frac{(b-a)^2}{8p^2} (p^2 \hat{f}) = K \hat{f} < 1 \quad (6.26)$$

where we have used (6.23) in the last inequality. Since $\|A_0^{-1}Q\| < 1$, we conclude from Lemma 3.3 that $(I + A_0^{-1}Q)$ is nonsingular. Hence, (6.13) has a unique solution given by (6.24). $\square$

The condition (6.23) can be relaxed when $f(t)$ is a positive constant.

Theorem 6.2. *The system (6.13) has a unique solution if $f(t) \equiv f_0 > 0$.*

Proof. We shall show that the coefficient matrix A in (6.13) is strictly diagonally dominant, then A^{-1} exists and the conclusion is immediate.

In fact, from (6.12) we see that A is tridiagonal and is given by

$$A = [a_{ij}] = \begin{bmatrix} \frac{2(2p^2+h^2)}{6}f_0 + 2 & \frac{(p^2-h^2)}{6}f_0 - 1 & & & \\ \frac{(p^2-h^2)}{6}f_0 - 1 & \frac{2(2p^2+h^2)}{6}f_0 + 2 & \frac{(p^2-h^2)}{6}f_0 - 1 & & \\ & \ddots & & \ddots & \\ & & \frac{(p^2-h^2)}{6}f_0 - 1 & \frac{2(2p^2+h^2)}{6}f_0 + 2 & \frac{(p^2-h^2)}{6}f_0 - 1 \\ & & & \frac{(p^2-h^2)}{6}f_0 - 1 & \frac{2(2p^2+h^2)}{6}f_0 + 2 \end{bmatrix}. \quad (6.27)$$

It can be easily checked that

$$|a_{ii}| - \sum_{j \neq i} |a_{ij}| = \begin{cases} \frac{2(p^2+2h^2)}{6}f_0 + 4, & \frac{(p^2-h^2)}{6}f_0 - 1 \geq 0 \\ p^2 f_0, & \frac{(p^2-h^2)}{6}f_0 - 1 \leq 0. \end{cases} \quad (6.28)$$

Hence, the matrix A given in (6.27) is indeed strictly diagonally dominant, this completes the proof. $\square$

The next result gives the order of convergence for the cubic discrete spline method.

Theorem 6.3. *Suppose $K\hat{f} < 1$ or $f(t) \equiv f_0 > 0$. Then,*

$$|e|_0 \cong O(p^4) \quad \text{if} \quad h = \frac{p}{\sqrt{2}}$$

and $|e|_0 \cong O(p^2)$ for other values of $h \in (0, p]$, i.e., the cubic discrete spline method (6.13) is fourth order convergent if $h = \frac{p}{\sqrt{2}}$ and is second order convergent otherwise.

Proof. First, suppose $K\hat{f} < 1$. We consider the special case when $h = \frac{p}{\sqrt{2}}$. From (6.19) we have

$$e = A^{-1}v = (A_0 + Q)^{-1}v = (I + A_0^{-1}Q)^{-1}A_0^{-1}v.$$

Noting (6.26) we apply Lemma 3.3, and together with (6.22), (6.21) and the fact $n = \frac{b-a}{p}$, we find

$$\begin{aligned} |e|_0 &\leq \|(I + A_0^{-1}Q)^{-1}\| \|A_0^{-1}\| |v|_0 \\ &\leq \frac{\|A_0^{-1}\| |v|_0}{1 - \|A_0^{-1}Q\|} \\ &\leq \frac{(b-a)^2}{8p^2} \left(\frac{1}{240} p^6 M_6 \right) \left(\frac{1}{1 - K\hat{f}} \right) \\ &= \frac{KM_6 p^4}{240(1 - K\hat{f})} \cong O(p^4). \end{aligned} \tag{6.29}$$

This inequality shows that (6.13) is a fourth order convergence method when $h = \frac{p}{\sqrt{2}}$. Using a similar argument, for other values of $h \in (0, p]$, from (6.20) we have $|v|_0 \cong O(p^4)$ and it follows that (6.13) is second order convergent.

Next, suppose $f(t) \equiv f_0 > 0$. Here, the matrix A (see (6.27)) is strictly diago-

nally dominant. It is well known that for a strictly diagonally dominant matrix [106],

$$\|A^{-1}\| \leq \left\{ \min_i \left[|a_{ii}| - \sum_{j \neq i} |a_{ij}| \right] \right\}^{-1}.$$

If $\frac{(p^2-h^2)}{6}f_0 - 1 \geq 0$, then using (6.28) we find

$$\|A\|^{-1} \leq \left[\frac{2(p^2 + 2h^2)}{6}f_0 + 4 \right]^{-1} \leq \frac{1}{\frac{p^2}{3}f_0 + 4} \leq \frac{3}{p^2f_0}. \quad (6.30)$$

If $\frac{(p^2-h^2)}{6}f_0 - 1 \leq 0$, then it follows from (6.28) that

$$\|A^{-1}\| \leq \frac{1}{p^2f_0}. \quad (6.31)$$

Combining (6.30) and (6.31), we get

$$\|A^{-1}\| \leq \max \left\{ \frac{3}{p^2f_0}, \frac{1}{p^2f_0} \right\} = \frac{3}{p^2f_0}. \quad (6.32)$$

Now for the special case $h = \frac{p}{\sqrt{2}}$, from (6.19), (6.21) and (6.32) we get

$$|e|_0 \leq \|A^{-1}\| |v|_0 \leq \frac{3}{p^2f_0} \left(\frac{1}{240}p^6M_6 \right) = \frac{M_6p^4}{80f_0} \cong O(p^4). \quad (6.33)$$

Hence, (6.13) is a fourth order convergence method when $h = \frac{p}{\sqrt{2}}$. For other values of $h \in (0, p]$, from (6.20) we have $|e|_0 \cong O(p^4)$ and it follows that (6.13) is second order convergent. $\square$

6.1.4 Numerical Examples

In this section, we present two numerical examples to demonstrate the cubic discrete spline method proposed in section 6.1.2 as well as to illustrate the comparative performance with some well known numerical methods for solving (6.1).

Example 6.1. Consider the boundary value problem

$$y'' = \frac{2}{t^2}y - \frac{1}{t}, \quad y(2) = y(3) = 0. \quad (6.34)$$

The analytical solution is given by $y(t) = \frac{1}{38}(-5t^2 + 19t - \frac{36}{t})$.

Example 6.2. Consider the boundary value problem

$$y'' = 100y, \quad y(0) = y(1) = 1. \quad (6.35)$$

The exact solution is given by $y(t) = \frac{\cosh(10t-5)}{\cosh 5}$.

In Example 6.1, we have $a = 2$, $b = 3$, $f(t) = \frac{2}{t^2}$ and so $K = \frac{1}{8}(b-a)^2 = \frac{1}{8}$ and $\hat{f} < \frac{1}{2}$. Thus, $K\hat{f} < 1$ and it follows from Theorem 6.1 that (6.34) has a unique cubic discrete spline solution. On the other hand, in Example 6.2 we have $f(t) = 100 > 0$ and so Theorem 6.2 ensures that (6.35) has a unique cubic discrete spline solution. The steps of computing the cubic discrete spline solutions of (6.34) and (6.35) are listed as follows:

- (i) Fix the mesh φ (and hence the mesh size p) and choose a value for h ($\in (0, p]$).
- (ii) Solve (6.12) to get s_i , $1 \leq i \leq n-1$, which approximates y_i .
- (iii) Calculate c_i by using $c_i = f_i s_i + g_i$, $0 \leq i \leq n$. Noting (6.5), $s_i'' = c_i$ serves as an approximate to y_i'' .
- (iv) Construct the cubic discrete spline solution $S_i(t)$ over the subinterval $[t_{i-1}, t_i]$ using (6.7).
- (v) Compute $D_h^{\{1\}} S_i(t_i)$ from (6.9). Noting (6.5), $s_i' = D_h^{\{1\}} S_i(t_i)$ serves as an approximate to y_i' .
- (vi) Since the exact solutions of (6.34) and (6.35) are known, we are able to compute $\max_i |y_i^{(k)} - s_i^{(k)}|$, $k = 0, 1, 2$ to get an indication of the accuracy of our method, see Tables 6.1, 6.2, 6.4 and 6.5. We shall also plot the graphs of the cubic discrete spline solutions and the exact solutions for comparison, see Figures 6.1 and 6.2.

First, we take $h = \frac{p}{\sqrt{2}}$. The maximum absolute errors ($\max_i |y_i - s_i|$) for different mesh sizes p are given in Table 6.1. We note that if the mesh size p is reduced by a factor of $\frac{1}{2}$, then the maximum absolute errors are approximately reduced by $(\frac{1}{2})^4 = \frac{1}{16}$. Thus, the numerical results confirm that our method is fourth order convergent when $h = \frac{p}{\sqrt{2}}$ (Theorem 6.3). Moreover, the maximum absolute errors recorded in Table 6.1 *coincide* with those obtained by Khan [53] using the parametric cubic spline method with the parameters $(\alpha, \beta) = (\frac{1}{12}, \frac{5}{12})$ in which case the method is also of order 4. We remark that the expression of the spline given by our method is much easier to obtain and the approximate values s_i are easy to compute (see (6.7), (6.12), (6.11), (6.8), while in [53] the expression of the parametric cubic spline contains two parameters which makes the computation complicated.

Table 6.1: Maximum absolute errors using the present method with $h = \frac{p}{\sqrt{2}}$

p	BVP (6.34)	BVP (6.35)
1/8	$1.74e - 07$	$1.74e - 03$
1/16	$1.10e - 08$	$1.13e - 04$
1/32	$6.85e - 10$	$7.28e - 06$

Next, for the sake of comparison with other second order methods we take $h = \frac{3}{4}p$. The maximum absolute errors ($\max_i |y_i - s_i|$) obtained by various methods for the boundary value problem (6.34) are given in Table 6.2. The numerical experiment confirms that our method is second order convergent when $h = \frac{3}{4}p$ (Theorem 6.3), and our results are notably *better* than others'. A brief description of the various methods presented: (i) Khan [53] uses parametric cubic splines, the parameters (α, β) included in Table 6.2 give second order convergence; (ii) in Al-Said's papers [9, 10], cubic splines and quadratic splines have been employed respectively in a non-traditional manner – here the author first obtains the approximate solution at the mid point of two nodes, then more approximations are needed to get the explicit expression of the spline, in comparison our method is quite straightforward in getting the expression of the spline; (iii) Sakai and Usmani [85] use a collocation method

with quadratic splines as basis functions; (iv) Albasiny and Hoskins [6] introduce the traditional cubic spline method to solve boundary value problems (the numerical computation is provided in [10]); (v) a second order centered-difference method is applied and the numerical computation is provided in [10].

Table 6.2: Maximum absolute errors for BVP (6.34)

Methods	$p = 1/4$	$p = 1/8$	$p = 1/16$
Our method with $h = (3/4)p$	$1.77e - 05$	$5.00e - 06$	$1.29e - 06$
Khan [53]			
$(\alpha, \beta) = (1/14, 3/7)$	$2.05e - 05$	$5.74e - 06$	$1.47e - 06$
$(\alpha, \beta) = (1/10, 2/5)$	$3.50e - 05$	$8.46e - 06$	$2.09e - 06$
$(\alpha, \beta) = (1/18, 4/9)$	$5.14e - 05$	$1.36e - 06$	$3.46e - 06$
Al-Said [10]	$5.49e - 05$	$1.87e - 05$	$5.07e - 06$
Al-Said [9]	$1.60e - 04$	$2.66e - 05$	$5.58e - 06$
Sakai and Usmani [85]	$7.93e - 05$	$2.06e - 05$	$5.20e - 06$
Albasiny and Hoskins [6]	$1.65e - 04$	$4.17e - 05$	$1.04e - 05$
Second order centered-difference	$2.79e - 04$	$5.42e - 05$	$1.19e - 05$

Now, we shall compare the performance of the ‘non-traditional’ cubic *continuous* spline method of [10] (which is superior to traditional cubic spline method) with our cubic *discrete* spline method. The values $\max_i |y_i^{(k)} - s_i^{(k)}|$, $k = 0, 1, 2$ obtained for the boundary value problem (6.34) by using the method in [10] and our method with $h = \frac{2}{3}p$ (second order convergent) are respectively presented in Tables 6.3 and 6.4. From the two tables we see that the actual error $\max_i |y_i - s_i|$ when using our method is much smaller, whereas $\max_i |y'_i - s'_i|$ is slightly worse, but $\max_i |y''_i - s''_i|$ is again much smaller – this indicates that our cubic discrete spline method gives a better approximation of $y(t_i)$ and $y''(t_i)$ for the boundary value problem (6.34).

Table 6.3: Maximum absolute errors for BVP (6.34) using cubic continuous spline method [10]

p	$\max_i y_i - s_i $	$\max_i y'_i - s'_i $	$\max_i y''_i - s''_i $
1/10	$1.247e - 05$	$7.818e - 05$	$8.734e - 04$
1/20	$3.286e - 06$	$1.931e - 05$	$2.211e - 04$
1/40	$8.466e - 07$	$4.812e - 06$	$5.546e - 05$

Table 6.4: Maximum absolute errors for BVP (6.34) using the present method with $h = \frac{2}{3}p$

p	$\max_i y_i - s_i $	$\max_i y'_i - s'_i $	$\max_i y''_i - s''_i $
1/10	$3.038e - 06$	$2.797e - 04$	$1.082e - 06$
1/20	$7.461e - 07$	$7.003e - 05$	$2.655e - 07$
1/40	$1.858e - 07$	$1.751e - 05$	$6.630e - 08$

Finally, in Table 6.5 we present the maximum absolute errors ($\max_i |y_i - s_i|$) for the boundary value problem (6.35) obtained by various second order methods. Once again we observe that our method is second order convergent and offers *better* results than other methods. Here the various methods considered include those of Table 6.2 and also that of Khalifa and Eilbeck [52] which uses collocation with quadratic splines. While doing the numerical experiments with different $h \in (0, p]$, we observe that as $h \rightarrow 0$, the result reduces to that of the cubic continuous spline [6]; when h approaches $\frac{p}{\sqrt{2}}$ from 0 or from p , the maximum absolute errors become smaller, this is in agreement with our theoretical results given in Theorem 6.3.

Table 6.5: Maximum absolute errors for BVP (6.35)

Methods	$p = 1/16$	$p = 1/32$	$p = 1/20$	$p = 1/40$
Our method with $h = (2/3)p$	$7.64e - 04$	$1.74e - 04$	$4.75e - 04$	$1.10e - 04$
Our method with $h = (3/4)p$	$6.18e - 04$	$1.80e - 04$	$4.32e - 04$	$1.17e - 04$
Khan [53] $(\alpha, \beta) = (1/10, 2/5)$	$1.28e - 03$	$3.07e - 04$	$8.17e - 04$	$1.95e - 04$
$(\alpha, \beta) = (1/14, 3/7)$	$7.22e - 04$	$2.06e - 04$	$5.00e - 04$	$1.34e - 04$
$(\alpha, \beta) = (1/18, 4/9)$	$1.83e - 03$	$4.91e - 04$	$1.22e - 03$	$3.16e - 04$
Al-Said [10]	$2.27e - 03$	$6.84e - 04$	$1.57e - 03$	$4.53e - 04$
Sakai and Usmani [85]	$3.06e - 03$	$7.58e - 04$		
Albasiny and Hoskins [6]	$6.05e - 03$	1.51×10^{-3}	$3.93e - 03$	$9.66e - 04$
Khalifa and Eilbeck [52]			$1.8e - 03$	$4.7e - 04$

Figure 6.1: Example 6.1 when $p = \frac{1}{8}$, $h = \frac{p}{\sqrt{2}}$

Figure 6.2: Example 6.2 when $p = \frac{1}{8}$, $h = \frac{p}{\sqrt{2}}$

6.2 Second Order Obstacle Boundary Value Problems

This section is based on the work of [32].

6.2.1 Introduction

We consider a system of second order boundary value problems of the type

$$y'' = \begin{cases} f(t), & a \leq t \leq c \\ g(t)y(t) + f(t) + r, & c \leq t \leq d \\ f(t), & d \leq t \leq b \end{cases} \quad (6.36)$$

$$y(a) = \bar{\alpha}, \quad y(b) = \bar{\beta}$$

with continuity conditions of y and y' at c and d . Here, f and g are continuous functions on $[a, b]$ and $[c, d]$ respectively, r , $\bar{\alpha}$ and $\bar{\beta}$ are real finite constants. Such type of systems arise in the study of obstacle, unilateral, moving and free boundary value problems and have important applications in other branches of pure and applied sciences, see [21, 34, 35, 54, 71, 72, 84].

The literature on the numerical treatment of (6.36) is abundant. Noor and Khalifa [73] have solved (6.36) using a collocation method with *first-order* accuracy, adopting cubic B-splines as basis functions. Also, Noor and Tirmizi [75] have used finite difference schemes based on the central difference and the well known Numerov method to solve (6.36), all these give *first-order* accurate approximations to the solution of (6.36). In [15], the authors use spline and finite difference methods to obtain numerical solutions of (6.36) – it is shown that the numerical solutions derived using both spline and finite difference techniques are *first-order* accurate approximations regardless of the order of the methods used, and the authors illustrate this idea further with the second-order cubic spline method of Albasiny and Hoskins [6] and the fourth-order quintic spline method of Usmani and Warsi [103]. For methods of *second-order* accuracy, we note that a modified Numerov method has been discussed in [13]. Splines have also been employed in solving (6.36), for example in the papers of Al-Said [8, 11, 12], quadratic and cubic spline methods have been developed and analyzed, these methods are of *second order*. On the other hand, quintic spline is used in [20] to solve (6.36), the method developed is *second-order* accurate.

Motivated by all the above research especially the use of splines in solving (6.36), in this section we shall employ a *deficient cubic discrete spline* to get a numerical solution of (6.36). Following [60, 78], the deficient discrete spline we use will involve central differences. It has been observed by [78] that deficient splines are more applicable than usual splines as they need less continuity requirement at the mesh points. In our proposed method, we shall relax the continuity of y' at c and d , and instead impose the continuity of the *first central difference* of y at c and d . Our proposed method is second-order convergent, and through a well know numerical example on obstacle boundary value problem, we illustrate that

our method outperforms other collocation, finite difference and spline methods for solving (6.36) in the literature [8, 11–13, 15, 20, 73, 75].

The plan of this section is as follows. In section 6.2.2 we shall derive the consistency relations and develop the new discrete spline method for solving (6.36). Section 6.2.3 is devoted to the convergence analysis of the method. Finally, in section 6.2.4 we present a well known example and compare the performance of our method with other methods in the literature.

6.2.2 Deficient cubic discrete Spline Method

With reference to the uniform partition φ of $[a, b]$, without loss of generality, we shall take

$$c = \frac{3a + b}{4} = t_{n/4} \quad \text{and} \quad d = \frac{a + 3b}{4} = t_{3n/4},$$

and require the positive integer n (≥ 8) to be divisible by 4. Based on the central differences defined in Chapter 5, we shall now introduce the deficient cubic spline.

Definition 6.2. Let $S(t; h)$ be a piecewise continuous function defined over $[a, b]$ (with mesh φ) and $S_i(t)$ be its restriction in $[t_{i-1}, t_i]$, $1 \leq i \leq n$ passing through the points (t_{i-1}, s_{i-1}) and (t_i, s_i) . We say $S(t; h)$ is a *deficient cubic discrete spline* if $S_i(t)$, $1 \leq i \leq n$ is a polynomial of degree 3 or less and

$$(S_{i+1} - S_i)(x_i + jh) = 0, \quad j = -1, 0, 1, \quad i \in I$$

$$(S_{i+1} - S_i)(x_i) = 0, \quad (S_{i+1} - S_i)(x_i + h) = (S_{i+1} - S_i)(x_i - h), \quad i = \frac{n}{4}, \frac{3n}{4} \quad (6.37)$$

where $I = \{i \mid 1 \leq i \leq n-1, i \neq \frac{n}{4}, \frac{3n}{4}\}$.

The above definition of deficient cubic discrete spline coincides with that given in [78], indeed the condition (6.37) has the following equivalent form

$$\begin{aligned} D_h^{\{j\}} S_i(x_i) &= D_h^{\{j\}} S_{i+1}(x_i), \quad j = 0, 1, \quad 1 \leq i \leq n-1 \\ D_h^{\{2\}} S_i(x_i) &= D_h^{\{2\}} S_{i+1}(x_i), \quad i \in I. \end{aligned} \quad (6.38)$$

Throughout, in the context of (6.36) we shall use the notations

$$y^{(k)}(x_i) = y_i^{(k)}, \quad f(x_i) = f_i, \quad g(x_i) = g_i, \quad 0 \leq i \leq n.$$

We shall approximate a solution $y(t)$ of (6.36) by the deficient cubic discrete spline $S(t; h)$, i.e., $y(t)$ will be approximated by $S_i(t)$ over the subinterval $[t_{i-1}, t_i]$, $1 \leq i \leq n$. Indeed, for any $t \in [a, b]$ (t may be between mesh points), we propose the following approximates:

$$\begin{aligned} y(t) &\cong S(t; h), \quad y'(t) \cong D_h^{\{1\}} S(t; h), \quad t \in [a, b] \\ y''(t) &\cong \begin{cases} f(t), & t \in [a, c) \cup (d, b] \\ g(t)S(t; h) + f(t) + r, & t \in (c, d). \end{cases} \end{aligned} \quad (6.39)$$

In particular, at the mesh points we have

$$\begin{aligned} y_i &\cong s_i \equiv S_i(t_i), \quad y'_i \cong s'_i \equiv D_h^{\{1\}} S_i(t_i), \quad 0 \leq i \leq n \\ y''_i &\cong \begin{cases} f_i, & 0 \leq i \leq \frac{n}{4} - 1 \text{ and } \frac{3n}{4} + 1 \leq i \leq n \\ g_i s_i + f_i + r, & \frac{n}{4} + 1 \leq i \leq \frac{3n}{4} - 1. \end{cases} \end{aligned} \quad (6.40)$$

Moreover, since the left and right second derivatives y''_- and y''_+ exist at both c and d , we propose

$$\begin{aligned} \text{when } i = \frac{n}{4}, \quad y''_{i-} &\cong f_i, \quad y''_{i+} \cong g_i s_i + f_i + r, \\ \text{when } i = \frac{3n}{4}, \quad y''_{i-} &\cong g_i s_i + f_i + r, \quad y''_{i+} \cong f_i. \end{aligned} \quad (6.41)$$

From the boundary conditions, we note that $s_0 = y_0 = \bar{\alpha}$ and $s_n = y_n = \bar{\beta}$, and s_i is an approximate of y_i , $1 \leq i \leq n - 1$.

We shall now obtain an explicit expression of $S_i(t)$. Let $c_i = D_h^{\{2\}} S(t_i; h)$ denote the ‘discrete moments’. Taking into account the fact that we approximate $y(t)$ by

$S(t; h)$ as well as (6.40) and (6.41), we set

$$c_i = \begin{cases} f_i, & 0 \leq i \leq \frac{n}{4} - 1 \text{ and } \frac{3n}{4} + 1 \leq i \leq n, \\ g_i s_i + f_i + r, & \frac{n}{4} + 1 \leq i \leq \frac{3n}{4} - 1, \end{cases} \quad (6.42)$$

$$\begin{cases} \text{when } i = \frac{n}{4}, & c_{i-} = f_i, \quad c_{i+} = g_i s_i + f_i + r, \\ \text{when } i = \frac{3n}{4}, & c_{i-} = g_i s_i + f_i + r, \quad c_{i+} = f_i. \end{cases}$$

Since $D_h^{\{2\}} S(t; h)$ is piecewise linear, we have for $t \in [t_{i-1}, t_i]$ and $1 \leq i \leq n$,

$$D_h^{\{2\}} S(t; h) = D_h^{\{2\}} S_i(t) = \frac{t_i - t}{p} c_{i-1} + \frac{t - t_{i-1}}{p} c_i. \quad (6.43)$$

It follows from (6.43) that

$$S_i(t) = \frac{(t_i - t)^{\{3\}}}{6p} c_{i-1} + \frac{(t - t_{i-1})^{\{3\}}}{6p} c_i + \frac{t_i - t}{p} u_i + \frac{t - t_{i-1}}{p} w_i, \quad (6.44)$$

$$t \in [t_{i-1}, t_i], \quad 1 \leq i \leq n$$

where u_i and w_i are arbitrary constants. The interpolation conditions $S_i(t_j) = s_j$, $j = i - 1, i$ then lead to

$$u_i = s_{i-1} - \frac{p^2 - h^2}{6} c_{i-1}, \quad w_i = s_i - \frac{p^2 - h^2}{6} c_i, \quad 1 \leq i \leq n. \quad (6.45)$$

Now, substituting (6.45) into (6.44) and taking central difference gives

$$D_h^{\{1\}} S_i(t) = -\frac{(t_i - t)^2}{2p} c_{i-1} - \frac{(p^2 - h^2)(c_i - c_{i-1})}{6p} + \frac{(t - t_{i-1})^2}{2p} c_i + \frac{s_i - s_{i-1}}{p}, \quad (6.46)$$

$$t \in [t_{i-1}, t_i], \quad 1 \leq i \leq n.$$

Note that in (6.43)–(6.46), when $i = \frac{n}{4}, \frac{3n}{4}$, we take $c_i = c_{i,-}$; when $i = \frac{n}{4} + 1, \frac{3n}{4} + 1$, we take $c_{i-1} = c_{i-1,+}$ (see (6.42) for the definitions).

For $i \in I$, the ‘continuity’ requirement $D_h^{\{1\}} S_i(t_i) = D_h^{\{1\}} S_{i+1}(t_i)$ (see (6.38))

leads to the system of equations

$$\frac{(p^2 - h^2)}{6}c_{i-1} + \frac{2(2p^2 + h^2)}{6}c_i + \frac{(p^2 - h^2)}{6}c_{i+1} = s_{i-1} - 2s_i + s_{i+1}, \quad i \in I. \quad (6.47)$$

Note that in (6.47), when $i = \frac{n}{4} - 1$, $\frac{3n}{4} - 1$, we take $c_{i+1} = c_{i+1,-}$; when $i = \frac{n}{4} + 1$, $\frac{3n}{4} + 1$, we take $c_{i-1} = c_{i-1,+}$ (see (6.42) for the definitions).

When $i = \frac{n}{4}, \frac{3n}{4}$, from (6.46) we have the following

$$D_h^{\{1\}}S_i(t_i) = \frac{p}{2}c_{i-} + \frac{s_i - s_{i-1}}{p} - \frac{(p^2 - h^2)}{6p}(c_{i-} - c_{i-1}),$$

$$D_h^{\{1\}}S_{i+1}(t_i) = -\frac{p}{2}c_{i+} + \frac{s_{i+1} - s_i}{p} - \frac{(p^2 - h^2)}{6p}(c_{i+1} - c_{i+}),$$

and $D_h^{\{1\}}S_i(t_i) = D_h^{\{1\}}S_{i+1}(t_i)$ yields

$$\frac{(p^2 - h^2)}{6}c_{i-1} + \frac{(h^2 + 2p^2)}{6}(c_{i+} + c_{i-}) + \frac{(p^2 - h^2)}{6}c_{i+1} = s_{i-1} - 2s_i + s_{i+1}, \quad i = \frac{n}{4}, \frac{3n}{4}. \quad (6.48)$$

Upon substituting (6.42) into (6.47) and (6.48), we obtain for $1 \leq i \leq \frac{n}{4} - 1$ and $\frac{3n}{4} + 1 \leq i \leq n - 1$,

$$-s_{i-1} + 2s_i - s_{i+1} = -\frac{(p^2 - h^2)}{6}f_{i-1} - \frac{2(2p^2 + h^2)}{6}f_i - \frac{(p^2 - h^2)}{6}f_{i+1}, \quad (6.49)$$

for $\frac{n}{4} + 1 \leq i \leq \frac{3n}{4} - 1$,

$$\begin{aligned} & \left[-1 + \frac{(p^2 - h^2)}{6}g_{i-1} \right] s_{i-1} + \left[2 + \frac{2(2p^2 + h^2)}{6}g_i \right] s_i + \left[-1 + \frac{(p^2 - h^2)}{6}g_{i+1} \right] s_{i+1} \\ &= -\frac{(p^2 - h^2)}{6}f_{i-1} - \frac{2(2p^2 + h^2)}{6}f_i - \frac{(p^2 - h^2)}{6}f_{i+1} - p^2r, \end{aligned} \quad (6.50)$$

and for $i = \frac{n}{4}, \frac{3n}{4}$,

$$\begin{aligned} & -s_{i-1} + \left[2 + \frac{(2p^2 + h^2)}{6}g_i \right] s_i + \left[-1 + \frac{(p^2 - h^2)}{6}g_{i+1} \right] s_{i+1} \\ &= -\frac{(p^2 - h^2)}{6}f_{i-1} - \frac{2(2p^2 + h^2)}{6}f_i - \frac{(p^2 - h^2)}{6}f_{i+1} - \frac{p^2}{2}r. \end{aligned} \quad (6.51)$$

Having established (6.49)–(6.51), the values of s_i , $1 \leq i \leq n-1$ can be determined by solving the system of linear equations, the solvability of which will be discussed in section 6.2.3. Then, c_i 's are computed via (6.42) and then substituted into (6.45) and finally the expression of $S_i(t)$ in (6.44). This completes the task of finding the deficient cubic discrete spline solution of (6.36).

6.2.3 Convergence Analysis

In this section, we shall establish the existence of a unique deficient cubic discrete spline solution for (6.36) (i.e., (6.49)–(6.51) give a unique solution) and also conduct a convergence analysis for the method presented in section 6.2.2.

Let $e_i = y_i - s_i$, $1 \leq i \leq n-1$ be the errors. Let $y = [y_i]$, $s = [s_i]$, $r = [r_i]$, $v = [v_i]$ and $e = [e_i]$ be $(n-1)$ -dimensional column vectors. The system (6.49)–(6.51) can be written as

$$As = r \quad (6.52)$$

where

$$A = A_0 + B, \quad (6.53)$$

$A_0 = [a_{ij}]$ and $B = [b_{ij}]$ are $(n-1) \times (n-1)$ matrices given by

$$a_{ij} = \begin{cases} 2, & i = j \\ -1, & |i - j| = 1 \\ 0, & \text{otherwise} \end{cases} \quad (6.54)$$

$$b_{ij} = \begin{cases} \frac{2(2p^2 + h^2)g_i}{6}, & i = j, \quad \frac{n}{4} + 1 \leq i \leq \frac{3n}{4} - 1 \\ \frac{(p^2 - h^2)g_{i-1}}{6}, & i - j = 1, \quad \frac{n}{4} + 1 \leq i \leq \frac{3n}{4} - 1 \\ \frac{(p^2 - h^2)g_{i+1}}{6}, & j - i = 1, \quad \frac{n}{4} + 1 \leq i \leq \frac{3n}{4} - 1 \\ \frac{(2p^2 + h^2)g_i}{6}, & i = j, \quad i = \frac{n}{4}, \frac{3n}{4} \\ \frac{(p^2 - h^2)g_{i+1}}{6}, & j - i = 1, \quad i = \frac{n}{4}, \frac{3n}{4} \\ 0, & \text{otherwise} \end{cases} \quad (6.55)$$

and

$$r_i = \begin{cases} \bar{\alpha} - \frac{1}{6} [(p^2 - h^2)f_{i-1} + 2(2p^2 + h^2)f_i + (p^2 - h^2)f_{i+1}], & i = 1 \\ -\frac{1}{6} [(p^2 - h^2)f_{i-1} + 2(2p^2 + h^2)f_i + (p^2 - h^2)f_{i+1}], & 2 \leq i \leq \frac{n}{4} - 1 \text{ and } \frac{3n}{4} + 1 \leq i \leq n - 2 \\ -\frac{1}{6} [(p^2 - h^2)f_{i-1} + 2(2p^2 + h^2)f_i + (p^2 - h^2)f_{i+1}] - \frac{p^2}{2}r, & i = \frac{n}{4}, \frac{3n}{4} \\ -\frac{1}{6} [(p^2 - h^2)f_{i-1} + 2(2p^2 + h^2)f_i + (p^2 - h^2)f_{i+1} + 6p^2r], & \frac{n}{4} + 1 \leq i \leq \frac{3n}{4} - 1 \\ \bar{\beta} - \frac{1}{6} [(p^2 - h^2)f_{i-1} + 2(2p^2 + h^2)f_i + (p^2 - h^2)f_{i+1}], & i = n - 1. \end{cases} \quad (6.56)$$

From (6.52) we have $A(y - e) = r$ or

$$Ay = r + v \quad (6.57)$$

where

$$Ae = v. \quad (6.58)$$

For $i \in I$, the i -th equation of system (6.57) is

$$-y_{i-1} + 2y_i - y_{i+1} = -\frac{1}{6}[(p^2 - h^2)y''_{i-1} + 2(2p^2 + h^2)y''_i + (p^2 - h^2)y''_{i+1}] + v_i$$

while for $i = \frac{n}{4}, \frac{3n}{4}$, we have

$$-y_{i-1} + 2y_i - y_{i+1} = -\frac{(p^2 - h^2)}{6}(y''_{i-1} + y''_{i+1}) - \frac{(h^2 + 2p^2)}{6}(y''_{i+} + y''_{i-}) + v_i.$$

By Taylor series expansion, we obtain the truncation error v_i as

$$v_i = \frac{p^2(p^2 - 2h^2)}{12}y_i^{(4)} + \frac{p^4(4p^2 - 5h^2)}{360}y_i^{(6)} + O(p^8), \quad 1 \leq i \leq n-1. \quad (6.59)$$

Remark 6.2. For the special case $h = \frac{p}{\sqrt{2}}$, it is clear from (6.59) that

$$|v|_0 = \frac{1}{240}p^6 M_6 \quad (6.60)$$

where $M_6 = \max_t |y(t)^{(6)}|$.

Theorem 6.4. Suppose $K = \frac{(b-a)^2}{8}\hat{g} < 1$ where $\hat{g} = \max_{0 \leq i \leq n} |g_i|$. Then, the system (6.52) has a unique solution and

$$|e|_0 \cong O(p^2) \quad \text{if} \quad h = \frac{p}{\sqrt{2}},$$

i.e., the deficient cubic discrete spline method (6.52) is second-order convergent if $h = \frac{p}{\sqrt{2}}$.

Proof. If (6.52) has a unique solution, then it can be written as

$$s = A^{-1}r = (A_0 + B)^{-1}r = [A_0(I + A_0^{-1}B)]^{-1}r = (I + A_0^{-1}B)^{-1}A_0^{-1}r. \quad (6.61)$$

From Lemma 6.1 the inverse A_0^{-1} exists, hence it remains to show that $(I + A_0^{-1}B)$

is nonsingular.

From (6.55), we find $\|B\| \leq p^2 \hat{g}$. Then, using (6.22) gives

$$\|A_0^{-1}B\| \leq \|A_0^{-1}\| \|B\| \leq \frac{(b-a)^2}{8p^2} (p^2 \hat{g}) = K < 1. \quad (6.62)$$

Hence, we conclude from Lemma 3.3 that $(I + A_0^{-1}B)$ is nonsingular, and (6.52) has a unique solution given by (6.61).

Next, we consider the special case when $h = \frac{p}{\sqrt{2}}$. From (6.58), we have

$$e = A^{-1}v = (A_0 + B)^{-1}v = (I + A_0^{-1}B)^{-1}A_0^{-1}v. \quad (6.63)$$

Applying (6.22), (6.60), (6.62) and Lemma 3.3, it follows from (6.63) that

$$\begin{aligned} |e|_0 &\leq \|(I + A_0^{-1}B)^{-1}\| \|A_0^{-1}\| |v|_0 \\ &\leq \frac{\|A_0^{-1}\| |v|_0}{1 - \|A_0^{-1}B\|} \\ &\leq \frac{(b-a)^2}{8p^2} \left(\frac{1}{240} p^6 M_6 \right) \left(\frac{1}{1-K} \right) \\ &= \frac{(b-a)^2 M_6 p^4}{1920(1-K)}. \end{aligned}$$

This shows that (6.52) is a fourth-order convergence method when $h = \frac{p}{\sqrt{2}}$. As in [13, 15], due to the fact that the continuity of the function exists only up to the second derivative, the overall accuracy of this method is only second order. $\square$

Remark 6.3. Theorem 6.4 gives a *sufficient* condition for the existence and uniqueness of deficient cubic discrete spline solution and the order of convergence. Actually, the weakest condition is just to have the matrix A *invertible*. Then, the system (6.52) has a unique solution $s = A^{-1}r$. Moreover, the deficient cubic discrete spline method (6.52) is convergent if $h = \frac{p}{\sqrt{2}}$, since from (6.58) we have $e = A^{-1}v$ which in view of (6.60) leads to

$$|e|_0 \leq \|A^{-1}\| |v|_0 \leq \frac{1}{240} p^6 M_6 \|A^{-1}\| < \infty.$$

6.2.4 Application to Obstacle Boundary Value Problems

To illustrate the application of the deficient cubic discrete spline method developed in section 6.2.2, we consider the obstacle boundary value problem

$$\begin{aligned}
 -y''(t) &\geq f(t), & \text{on } \Omega = [0, \pi] \\
 y(t) &\geq \psi(t), & \text{on } \Omega = [0, \pi] \\
 (y''(t) - f(t))(y(t) - \psi(t)) &= 0, & \text{on } \Omega = [0, \pi] \\
 y(0) &= y(\pi) = 0,
 \end{aligned} \tag{6.64}$$

where $f(t)$ is a given force acting on the string and $\psi(t)$ is the elastic obstacle.

By studying problem (6.64) in the framework of variational inequality approach, it can be shown that, see [21,34,54,74], problem (6.64) is equivalent to the variational inequality problem

$$a(y, v - y) \geq \langle f, v - y \rangle, \quad \text{for all } v \in K \tag{6.65}$$

where $a(\cdot, \cdot)$ is a coercive continuous bilinear form and K is the closed convex set given by $K = \{v \in H_0^1(\Omega) \mid v \geq \psi \text{ on } \Omega\}$ ($H_0^1(\Omega)$ is a Sobolev space).

Following the idea and technique of Lewy and Stampacchia [58], the variational inequality (6.65) can be written as

$$\begin{aligned}
 y'' - [\mu(y - \psi)](y - \psi) &= f, \quad 0 < x < \pi \\
 y(0) &= y(\pi) = 0
 \end{aligned} \tag{6.66}$$

where $\mu(t)$, known as the penalty function, is the discontinuous function defined by

$$\mu(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{6.67}$$

and ψ is the given obstacle function defined by

$$\psi(t) = \begin{cases} -1, & 0 \leq t \leq \frac{\pi}{4} \\ 1, & \frac{\pi}{4} \leq t \leq \frac{3\pi}{4} \\ -1, & \frac{3\pi}{4} \leq t \leq \pi. \end{cases} \quad (6.68)$$

Equation (6.66) describes the equilibrium configuration of an obstacle string pulled at the ends and lying over elastic step of constant height 1 and unit rigidity. Since the obstacle function ψ is known, it is possible to find the solution of the problem in the interval $[0, \pi]$.

From equations (6.66)–(6.68), we obtain the following system of boundary value problem

$$y'' = \begin{cases} f, & 0 \leq t \leq \frac{\pi}{4} \text{ and } \frac{3\pi}{4} \leq t \leq \pi \\ y + f - 1, & \frac{\pi}{4} \leq t \leq \frac{3\pi}{4} \end{cases} \quad (6.69)$$

$$y(0) = y(\pi) = 0$$

and the condition of the continuity of y and y' at $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

Example 6.3. We consider the system (6.69) when $f = 0$, i.e.,

$$y'' = \begin{cases} 0, & 0 \leq t \leq \frac{\pi}{4} \text{ and } \frac{3\pi}{4} \leq t \leq \pi \\ y - 1, & \frac{\pi}{4} \leq t \leq \frac{3\pi}{4} \end{cases} \quad (6.70)$$

$$y(0) = y(\pi) = 0.$$

The analytical solution for this problem is given by

$$y(t) = \begin{cases} \frac{4}{\gamma_1}t, & 0 \leq t \leq \frac{\pi}{4} \\ 1 - \frac{4}{\gamma_2} \cosh\left(\frac{\pi}{2} - t\right), & \frac{\pi}{4} \leq t \leq \frac{3\pi}{4} \\ \frac{4}{\gamma_1}(\pi - t), & \frac{3\pi}{4} \leq t \leq \pi \end{cases} \quad (6.71)$$

where $\gamma_1 = \pi + 4 \coth \frac{\pi}{4}$ and $\gamma_2 = \pi \sinh \frac{\pi}{4} + 4 \cosh \frac{\pi}{4}$.

We observe from the analytical solution that y and y' are continuous at $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, but y'' is not continuous at these two points, so the overall accuracy of our method is only second order. This is also verified from the numerical evidence in Table 6.6.

In this example, $f = 0$, $g = 1$, $r = -1$, and we take $h = \frac{p}{\sqrt{2}}$. The system of linear equations (6.49)–(6.51) is explicitly given as

$$\begin{aligned} -s_{i-1} + 2s_i - s_{i+1} &= 0, & 1 \leq i \leq \frac{n}{4} - 1 \text{ and } \frac{3n}{4} + 1 \leq i \leq n - 1 \\ \left(-1 + \frac{p^2}{12}\right) s_{i-1} + \left(2 + \frac{5p^2}{6}\right) s_i + \left(-1 + \frac{p^2}{12}\right) s_{i+1} &= p^2, & \frac{n}{4} + 1 \leq i \leq \frac{3n}{4} - 1 \\ -s_{i-1} + \left(2 + \frac{5p^2}{12}\right) s_i + \left(-1 + \frac{p^2}{12}\right) s_{i+1} &= \frac{p^2}{2}, & i = \frac{n}{4}, \frac{3n}{4}. \end{aligned}$$

For different values of p , we can solve the unknowns s_i , $1 \leq i \leq n - 1$ from the above system. Then, we can get c_i using (6.42) and finally obtain the deficient discrete spline $S_i(t)$ as well as $D_h^{\{1\}} S_i(t)$ in (6.44) and (6.46) respectively. In Tables 6.6 and 6.7 respectively, we present the maximum absolute errors $\max_i |y_i - s_i|$ and $\max_i |y'_i - s'_i|$ obtained from our method as well as from other methods in the literature. It is noted that our method outperforms other methods tabulated, moreover the numerical results from Table 6.6 confirm that our method is of second order.

Further, to compare graphically, in Figure 6.2 we plot the exact solution y and

the deficient cubic discrete spline solution $S(t; h)$; in Figure 6.3 we plot the exact y' , $D_h^{\{1\}}S(t; h)$ and the first derivative of the cubic spline solution obtained in [11]; in Figure 6.4 we plot the exact y'' , $D_h^{\{2\}}S(t; h)$ and the second derivative of the cubic spline solution obtained in [11]. It is seen from the figures that our method gives better approximations for y , y' and y'' .

A brief description of the methods listed in Tables 6.6 and 6.7:

- (i) In [8, 11, 12], quadratic and cubic spline methods have been developed for solving (4.7), these methods are of *second order*. In the papers [11, 12], Al-said has used $c_{i+\frac{1}{2}}$ instead of c_i in the cubic spline, this technically avoids the points $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, but more approximations are needed which results in more computations.
- (ii) In [13], a *second-order* modified Numerov method is proposed. First, a finite difference scheme is developed to obtain approximations at the midknots. Using a second-order interpolation, these approximations are then used to compute the numerical solutions at the knots.
- (iii) In [20], quintic spline is used to solve (4.7). Here, at the two points of discontinuity $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, the mean of the left- and right-hand limits of g and r has been used. The method developed is *second-order* accurate.
- (iv) In [73], a collocation method is employed with cubic B-splines as basis functions. The method has *first-order* accuracy.
- (v) In [15], the authors have produced *first-order* accurate approximations to the solution of (6.70) and its first derivative using numerical methods based on the cubic spline method of Albasiny and Hoskins [6] and the quintic spline method of Usmani and Warsi [103]. Here, they have chosen

$$c_i = \begin{cases} 0, & 0 \leq i \leq \frac{n}{4} \text{ and } \frac{3n}{4} < i \leq n \\ s_i - 1, & \frac{n}{4} < i \leq \frac{3n}{4}, \end{cases}$$

i.e., at $t_i = \frac{\pi}{4}$, the authors take $c_i = 0$, while at $t_i = \frac{3\pi}{4}$, they take $c_i = s_i - 1$.

(vi) In [75], the authors have employed the well known Numerov method as well as finite difference schemes based on the central difference

$$p^2 y_i'' = y_{i-1} - 2y_i + y_{i+1} - \frac{p^4}{12} y_i^{(4)}. \quad (4.9)$$

Convergence analysis in this section shows that both methods are *first-order* accurate.

Table 6.6: Maximum absolute errors $\max_i |y_i - s_i|$

Methods	$p = \pi/20$	$p = \pi/40$	$p = \pi/80$
Deficient cubic discrete spline	$1.19e - 03$	$3.04e - 04$	$7.68e - 06$
Cubic spline [12]	$1.26e - 03$	$3.29e - 04$	$8.43e - 05$
Modified Numerov method [13]	$1.65e - 03$	$4.33e - 04$	$1.11e - 05$
Cubic spline [11]	$1.94e - 03$	$4.99e - 04$	$1.27e - 04$
Quadratic spline [8]	$2.20e - 03$	$5.87e - 04$	$1.51e - 04$
Quintic spline [20]	$2.57e - 03$	$7.31e - 04$	$1.94e - 04$
Collocation-cubic [73]	$1.40e - 02$	$7.71e - 03$	$4.04e - 03$
Cubic spline [15]	$1.80e - 02$	$9.13e - 03$	$4.60e - 03$
Quintic spline [15]	$1.82e - 02$	$9.17e - 03$	$4.61e - 03$
Numerov [75]	$2.32e - 02$	$1.21e - 02$	$6.17e - 03$
Scheme (4.9) [75]	$2.50e - 02$	$1.29e - 02$	$6.58e - 03$

Table 6.7: Maximum absolute errors $\max_i |y'_i - s'_i|$

Methods	$p = \pi/20$	$p = \pi/40$	$p = \pi/80$
Deficient cubic discrete spline	$7.58e - 04$	$1.91e - 04$	$4.79e - 05$
Cubic spline [12]	$8.32e - 04$	$2.09e - 04$	$5.22e - 05$
Cubic spline [15]	$2.75e - 02$	$1.39e - 02$	$7.02e - 03$
Quintic spline [15]	$9.05e - 02$	$4.70e - 02$	$2.44e - 02$

Figure 6.3: Exact solution vs deficient discrete spline solution ($p = \frac{\pi}{20}$)

Figure 6.4: The first derivative of exact solution vs the first central difference/derivative of approximate solutions ($p = \frac{\pi}{20}$)

Figure 6.5: The second derivative of exact solution vs the second central difference/derivative of approximate solutions ($p = \frac{\pi}{20}$)

6.3 Fourth Order Boundary Value Problems

This section is based on the work of [29].

6.3.1 Problem Description

In this section, we consider the fourth order Lidstone boundary value problem

$$\begin{aligned} y^{(4)}(t) &= f(t)y(t) + g(t), \quad a \leq t \leq b \\ y(a) &= A_1, \quad y(b) = B_1, \quad y^{(2)}(a) = A_2, \quad y^{(2)}(b) = B_2 \end{aligned} \tag{6.72}$$

where $f(t)$ and $g(t)$ are continuous on $[a, b]$ and $A_i, B_i, i = 1, 2$ are arbitrary real finite constants.

Lidstone boundary value problems have received a lot of attention in the literature, notably on the existence of positive solutions, see for example [1, 38, 110] and the references cited therein. The fourth order Lidstone boundary value problem (6.72) considered arises from the physical problem of bending a rectangular simply supported beam resting on an elastic foundation [83, 97], here y is the vertical deflection of the plate. The use of polynomial splines in the numerical treatment of (6.72) has gathered substantial interests over the years. Usmani and Warsi [102] have used quintic and sextic splines respectively to develop second and fourth order convergent methods for (6.72). Thereafter, quartic splines are employed by Usmani [101] to formulate second order convergent method. Also, during their investigation on fourth order obstacle boundary value problems, Al-Said and Noor [14] and Al-Said et al [16] have respectively used cubic and quartic splines to obtain second order convergent methods for (6.72). Recently, nonpolynomial spline functions have been proposed by Ramadan et al [77] to obtain second and fourth order convergent methods for (6.72), these methods reduced to those of [14, 101, 102] when certain parameters take certain values. A related problem to (6.72) arises from the bending of a long uniformly loaded rectangular plate supported over the entire surface by an elastic foundation and rigidly supported along the edges [83, 97], here the boundary con-

ditions are the conjugate type $y(a) - A_1 = y(b) - B_1 = y'(a) - A_2 = y' - B_2 = 0$. For this problem, second order convergent methods based on quintic splines have been established in [82,92,99], while fourth order convergent method based on sextic splines has been discussed in [99]. The general observation from all these research is that spline methods usually give better (or comparable) approximation than finite difference methods and shooting type methods.

Motivated by all the above research especially the use of splines in solving (6.72), in this section we shall employ a *quintic discrete spline* to get a numerical solution of (6.72). Our proposed method is fourth-order convergent when a parameter takes certain value, else it is second-order convergent. Through a well known numerical example, we illustrate that our method outperforms other spline methods for solving (6.72) in the literature [14,16,77,101,102].

The plan of this section is as follows. In section 6.3.2, we shall derive our method. The matrix form of the method is presented in section 6.3.3 and its convergence analysis is performed. In section 6.3.4, we present a well known example and compare the performance of our method with other methods in the literature.

6.3.2 Numerical Method for (6.72)

In this section, we shall obtain a numerical solution of (6.72) based on the results of the quintic discrete spline obtained in Chapter 5.

To recall the notations used in Chapter 5, throughout, we shall use the notations

$$y_i^{(k)} = y^{(k)}(x_i), \quad f_i = f(x_i), \quad g_i = g(x_i), \quad s_i = S_i(x_i),$$

$$M_i = D_h^{\{2\}} S_i(x_i), \quad F_i = D_h^{\{4\}} S_i(x_i), \quad 0 \leq i \leq n.$$

We propose s_i 's to be the numerical solution of (6.72) at the mesh points, i.e.,

$$y_i \cong s_i, \quad 0 \leq i \leq n. \tag{6.39}$$

Discretizing (6.72) and noting the Lidstone boundary conditions, we set

$$s_0 = y_0 = A_1, \quad s_n = y_n = B_1, \quad M_0 = y_0'' = A_2, \quad M_n = y_n'' = B_2, \quad F_i = f_i s_i + g_i, \\ 0 \leq i \leq n. \quad (6.73)$$

From Chapter 5 (see (5.11)), we have the following equation

$$a_1 F_{i-2} + a_2 F_{i-1} + a_3 F_i + a_2 F_{i+1} + a_1 F_{i+2} = s_{i-2} - 4s_{i-1} + 6s_i - 4s_{i+1} + s_{i+2}, \\ 2 \leq i \leq n-2 \quad (6.74)$$

where

$$a_1 = \frac{(p^2 - h^2)(p^2 - 4h^2)}{120}, \quad a_2 = \frac{2(p^2 - h^2)(8h^2 + 13p^2)}{120}, \quad a_3 = \frac{6(4h^4 + 5h^2p^2 + 11p^4)}{120}. \quad (6.75)$$

Upon substituting $F_j = f_j s_j + g_j$ into (6.74), we see that (6.74) gives $(n-3)$ equations with $(n-1)$ unknowns s_i , $1 \leq i \leq n-1$.

In order to solve for the unknown s_i 's, we need two more equations which we write as

$$b_1 F_0 + b_2 F_1 + b_3 F_2 + b_4 F_3 = p^2 M_0 + b_5 s_0 + b_6 s_1 + b_7 s_2 + b_8 s_3 \quad (6.76)$$

and

$$c_1 F_{n-3} + c_2 F_{n-2} + c_3 F_{n-1} + c_4 F_n = p^2 M_n + c_5 s_{n-3} + c_6 s_{n-2} + c_7 s_{n-1} + c_8 s_n \quad (6.77)$$

where b_i and c_i , $1 \leq i \leq 8$ are real numbers. We require the local truncation errors in both (6.76) and (6.77) to be $O(p^8)$ (the reason will be clear when we perform the convergence analysis in section 6.3.3). To fulfill this, we carry out Taylor series expansion in (6.76) about x_0 and set the coefficients of $s_0^{(k)}$, $0 \leq k \leq 7$ to zeros. This yields 8 equations which we can solve to get b_i , $1 \leq i \leq 8$. Similarly, in (6.77) we expand about x_n and set the coefficients of $s_n^{(k)}$, $0 \leq k \leq 7$ to zeros, then we

solve 8 equations to get c_i , $1 \leq i \leq 8$. The resulting (6.76) and (6.77) are given as follows

$$\frac{p^4}{360}(28F_0 + 245F_1 + 56F_2 + F_3) - p^2M_0 = -2s_0 + 5s_1 - 4s_2 + s_3, \quad (6.78)$$

$$\frac{p^4}{360}(F_{n-3} + 56F_{n-2} + 245F_{n-1} + 28F_n) - p^2M_n = s_{n-3} - 4s_{n-2} + 5s_{n-1} - 2s_n. \quad (6.79)$$

Once again, we substitute $F_j = f_j s_j + g_j$ into (6.78) and (6.79) to give two equations in s_i , $i = 1, 2, 3, n-3, n-2, n-1$.

Noting (6.73) the values of s_0 , s_n , M_0 and M_n are already known, hence we can now solve (6.74), (6.78), (6.79) to obtain the values of s_i , $1 \leq i \leq n-1$. The solvability of the linear system will be discussed in section 6.3.3.

6.3.3 Convergence Analysis

In this section, we shall establish the existence of a unique solution for (6.74), (6.78), (6.79) and also conduct a convergence analysis for the method presented in section 6.3.2.

Let $e_i = y_i - s_i$, $1 \leq i \leq n-1$ be the errors. Let $y = [y_i]$, $s = [s_i]$, $r = [r_i]$, $v = [v_i]$ and $e = [e_i]$ be $(n-1)$ -dimensional column vectors. The system (6.74), (6.78), (6.79) can be written as

$$As = r \quad (6.80)$$

where

$$A = A_0 + Q, \quad Q = BF, \quad F = \text{diag}(f_i), \quad i = 1, 2, \dots, n-1, \quad (6.81)$$

A_0 and B are $(n-1) \times (n-1)$ five-band symmetric matrices given by

$$A_0 = \begin{pmatrix} 5 & -4 & 1 & & & \\ -4 & 6 & -4 & 1 & & \\ 1 & -4 & 6 & -4 & 1 & \\ & & & \ddots & & \\ & & & 1 & -4 & 6 & -4 & 1 \\ & & & & 1 & -4 & 6 & -4 \\ & & & & & 1 & -4 & 5 \end{pmatrix}, \quad (6.82)$$

$$B = \begin{pmatrix} -\frac{245p^4}{360} & -\frac{56p^4}{360} & -\frac{p^4}{360} & & & \\ -a_2 & -a_3 & -a_2 & -a_1 & & \\ -a_1 & -a_2 & -a_3 & -a_2 & -a_1 & \\ & & & \ddots & & \\ & & & -a_1 & -a_2 & -a_3 & -a_2 & -a_1 \\ & & & & -a_1 & -a_2 & -a_3 & -a_2 \\ & & & & & -\frac{p^4}{360} & -\frac{56p^4}{360} & -\frac{245p^4}{360} \end{pmatrix} \quad (6.83)$$

and for the vector $r = [r_i]$, we have

$$r_i = \begin{cases} 2s_0 - p^2 M_0 + \frac{p^4}{360} [28f_0 s_0 + 28g_0 + 245g_1 + 56g_2 + g_3], & i = 1 \\ -s_0 + a_1 f_0 s_0 + a_1 g_0 + a_2 g_1 + a_3 g_2 + a_2 g_3 + a_1 g_4, & i = 2 \\ a_1 g_{i-2} + a_2 g_{i-1} + a_3 g_i + a_2 g_{i+1} + a_1 g_{i+2}, & 3 \leq i \leq n-3 \\ -s_n + a_1 g_{n-4} + a_2 g_{n-3} + a_3 g_{n-2} + a_2 g_{n-1} + a_1 g_n + a_1 f_n s_n, & i = n-2 \\ 2s_n - p^2 M_n + \frac{p^4}{360} [g_{n-3} + 56g_{n-2} + 245g_{n-1} + 28g_n + 28f_n s_n], & i = n-1. \end{cases} \quad (6.84)$$

From (6.80) we have $A(y - e) = r$ or

$$Ay = r + v \quad (6.85)$$

where

$$Ae = v. \quad (6.86)$$

For $2 \leq i \leq n-2$, the i -th equation of the linear system (6.85) is

$$y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2} = a_1 y_{i-2}^{(4)} + a_2 y_{i-1}^{(4)} + a_3 y_i^{(4)} + a_2 y_{i+1}^{(4)} + a_1 y_{i+2}^{(4)} + v_i,$$

where v_i 's are the local truncation errors given by

$$v_i = \frac{p^4(p^2 - 3h^2)}{12} y_i^{(6)} + \frac{p^4(4p^4 - 15p^2 h^2 + 8h^4)}{240} y_i^{(8)} + O(p^9). \quad (6.87)$$

For $i = 1, n-1$, the i -th equations of the linear system (6.85) are respectively

$$-2y_0 + 5y_1 - 4y_2 + y_3 = \frac{p^4}{360} \left(28y_0^{(4)} + 245y_1^{(4)} + 56y_2^{(4)} + y_3^{(4)} \right) - p^2 y_0'' + v_1$$

and

$$y_{n-3} - 4y_{n-2} + 5y_{n-1} - 2y_n = \frac{p^4}{360} \left(y_{n-3}^{(4)} + 56y_{n-2}^{(4)} + 245y_{n-1}^{(4)} + 28y_n^{(4)} \right) - p^2 y_n'' + v_{n-1}$$

where v_1 and v_{n-1} are the local truncation errors given by

$$v_1 = v_{n-1} = -\frac{241}{60480}p^8 y_i^{(8)} + O(p^9). \quad (6.88)$$

Now it is clear that the requirement of the parameters in (6.76) and (6.77) is to ensure that order of the local errors v_1 and v_{n-1} is consistent with v_i , $2 \leq i \leq n-2$.

Remark 6.4. For the special case $h = \frac{p}{\sqrt{3}}$, it is clear from (6.87) that

$$v_i = -\frac{p^8}{2160}y_i^{(8)} + O(p^9), \quad 2 \leq i \leq n-2.$$

Thus, taking (6.88) into consideration, we have

$$|v|_0 = \frac{241}{60480}p^8 L, \quad (6.89)$$

where $L = \max_t |y^{(8)}(t)|$.

Lemma 6.2. [14] *The matrix A_0 is invertible and*

$$\|A_0^{-1}\| \leq \frac{5n^4 + 4n^2}{384} = \frac{5(b-a)^4 + 4(b-a)^2 p^2}{384p^4}. \quad (6.90)$$

Our first result guarantees the existence of a unique solution for (6.74), (6.78), (6.79).

Theorem 6.5. *The system (6.80) has a unique solution if*

$$\frac{489}{480}K\hat{f} < 1 \quad (6.91)$$

where $K = \frac{5(b-a)^4 + 4(b-a)^2 p^2}{384}$ and $\hat{f} = \max_{1 \leq i \leq n-1} |f_i|$.

Proof. If (6.80) has a unique solution, then it can be written as

$$s = A^{-1}r = (A_0 + Q)^{-1}r = [A_0(I + A_0^{-1}Q)]^{-1}r = (I + A_0^{-1}Q)^{-1}A_0^{-1}r. \quad (6.92)$$

From Lemma 6.2 the inverse A_0^{-1} exists, hence it remains to show that $(I + A_0^{-1}Q)$ is nonsingular.

From (6.83), a direct computation gives $\|B\| \leq \frac{489}{480}p^4$. Since $Q = BF$, we find

$$\|Q\| \leq \|B\| \|F\| \leq \frac{489}{480}p^4\hat{f}. \quad (6.93)$$

It follows from (6.90) and (6.93) that

$$\|A_0^{-1}Q\| \leq \|A_0^{-1}\| \|Q\| \leq \frac{5(b-a)^4 + 4(b-a)^2p^2}{384p^4} \left(\frac{489}{480}p^4\hat{f} \right) = \frac{489}{480}K\hat{f} < 1 \quad (6.94)$$

where we have used (6.91) in the last inequality. Since $\|A_0^{-1}Q\| < 1$, we conclude from Lemma 3.3 that $(I + A_0^{-1}Q)$ is nonsingular. Hence, (6.80) has a unique solution given by (6.92). $\square$

The next result gives the order of convergence of our method.

Theorem 6.6. *Suppose $\frac{489}{480}K\hat{f} < 1$ Then,*

$$|e|_0 \cong O(p^4) \quad \text{if} \quad h = \frac{p}{\sqrt{3}}$$

and $|e|_0 \cong O(p^2)$ for other values of $h \in (0, p]$, i.e., the method (6.80) is fourth order convergent if $h = \frac{p}{\sqrt{3}}$ and is second order convergent otherwise.

Proof. First, we consider the special case when $h = \frac{p}{\sqrt{3}}$. From equation (6.86) we have

$$e = A^{-1}v = (A_0 + Q)^{-1}v = (I + A_0^{-1}Q)^{-1}A_0^{-1}v$$

Noting (6.94) we apply Lemma 3.3, and together with (6.90) and (6.89), we find

$$\begin{aligned}
|e|_0 &\leq \|(I + A_0^{-1}Q)^{-1}\| \|A_0^{-1}\| |v|_0 \\
&\leq \frac{\|A_0^{-1}\| |v|_0}{1 - \|A_0^{-1}Q\|} \\
&\leq \frac{5(b-a)^4 + 4(b-a)^2 p^2}{384p^4} \left(\frac{241}{60480} p^8 L \right) \left(\frac{1}{1 - \frac{489}{480} K \hat{f}} \right) \\
&= \frac{241KLp^4}{60480 \left(1 - \frac{489}{480} K \hat{f} \right)} \cong O(p^4).
\end{aligned}$$

This inequality shows that (6.80) is a fourth order convergence method when $h = \frac{p}{\sqrt{3}}$.

For other values of $h \in [0, p]$, from (6.87) and (6.88) we have $|v|_0 \cong O(p^6)$. Using a similar argument as above, we see that (6.80) is second order convergent.

□

6.3.4 Numerical Example

In this section, we present a numerical example to demonstrate our proposed method as well as to illustrate the comparative performance with some well known numerical methods for solving (6.72).

Example 6.4. Consider the Lidstone boundary value problem

$$\begin{aligned}
y^{(4)} + ty &= -(8 + 7t + t^3)e^t, \quad 0 \leq t \leq 1 \\
y(0) = y(1) &= 0, \quad y''(0) = 0, \quad y''(1) = -4e.
\end{aligned} \tag{6.95}$$

The analytical solution of (6.95) is

$$y(t) = t(1-t)e^t.$$

In this example, we have $a = 0$, $b = 1$, $f(t) = -t$ and $g(t) = -(8 + 7t + t^3)e^t$.

So $K = \frac{5+4p^2}{384}$ and $\hat{f} < 1$. For any $p \in (0, 1)$, we have $\frac{489}{480}K\hat{f} < 1$ and hence it follows from Theorem 6.5 that our method gives a unique numerical solution for (6.95) .

To compute the numerical solution of (6.95) , first we fix the mesh φ (and hence the step size p) and choose $h = \frac{p}{\sqrt{3}}$. Then, we solve the system (6.74) , (6.78) , (6.79) to get s_i , $1 \leq i \leq n - 1$, which approximates y_i .

The maximum absolute errors ($\max_i |y_i - s_i|$) obtained by our method as well as by other methods in the literature are presented in Table 6.8. From the table we can see that our method is fourth-order convergent when $h = \frac{p}{\sqrt{3}}$. Moreover, a clear comparison shows that our method outperforms continuous polynomial spline (cubic, quartic, quintic, sextic) and nonpolynomial spline (quintic) methods.

A brief description of the methods listed in Table 6.8:

- (i) In [77], second and fourth order convergent methods are derived using a non-polynomial spline function that has a polynomial part and a trigonometric part. The methods of [14, 101, 102] are special cases of nonpolynomial spline methods when certain parameters take certain values.
- (ii) In [102], quintic and sextic splines are employed respectively to establish second and fourth order convergent methods.
- (iii) In [101], a second order convergent method is formulated using quartic splines. Here, the consistency relations are obtained at the *midknots*, this approach is different from other spline methods where consistency relations are usually obtained at the uniformly spaced knots.
- (iv) In [16], cubic splines are used to develop a second order convergent method.
- (v) In [14], a second order convergent method is proposed based on quartic splines.

Table 6.8: Maximum absolute errors $\max_i |y_i - s_i|$

Methods	$p = 1/8$	$p = 1/16$	$p = 1/32$
Our method	$7.48e - 08$	$5.30e - 09$	$4.91e - 10$
Quintic nonpolynomial spline (4th order) [77]	$2.09e - 07$	$7.92e - 09$	$1.27e - 09$
Sextic spline [102]	$1.26e - 06$	$7.87e - 08$	$4.91e - 09$
Quintic nonpolynomial spline (2nd order) [77]	$9.42e - 05$	$6.17e - 06$	$3.95e - 07$
Quartic spline [101]	$4.24e - 04$	$1.08e - 04$	$2.70e - 05$
Cubic spline [16]	$5.69e - 04$	$1.47e - 04$	$3.71e - 05$
Quintic spline [102]	$8.67e - 04$	$2.16e - 04$	$5.40e - 05$
Quartic spline [14]	$1.62e - 03$	$6.39e - 04$	$5.88e - 05$

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Author's Publications

Journal papers:

1. F. Chen and P. J. Y. Wong, Error inequalities for quintic and biquintic discrete Hermite interpolation, *J. Comput. Appl. Math.* 235(2011), 4589-4600.
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