Distributed formation for large swarm of agents

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Distributed Formation For Large Swarm Of Agents

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Doctor of Philosophy

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Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

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Date

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Abstract

Recent advances in sensor technology have led to the development of complex networked control systems consisting of large number of interacting agents, thus opening up new opportunities and challenges in analysis and control of such complex systems. Recently, formation control of a large swarm of networked multi-agent system has received considerable attention. Due to the large size of the system and the communication limitations of the complex networks, traditional formation control methods fail to demonstrate satisfactory performance. This thesis presents several methodologies to solve distributed formation control problem for a large swarm of networked multi-agent system.

Formation control of network of agents with single-integrator dynamic can be simplified as a consensus problem. This thesis firstly addresses the distributed consensusability problem in complex networked agent systems based on the structural observability of network systems. A notion of unobservable node is introduced which leads to structural unobservability, and the relationship between the existence of unobservable nodes and the distributed consensusability of network systems is explored. Subsequently, a simple tool is presented to identify the unobservable nodes of a network system which is equivalent to examining the distributed consensusability of the network system. Moreover, a simple mathematical method is proposed to modify a distributed consensusable network into a weight-balanced one.

To obtain formation control of large-scale multi-agent systems with general Euler-Lagrange dynamic equations, the shape formation technique has been presented. However, in existing shape formation control problem, having access to the desired reference of the shape by all agents is a requirement. Due to limitations in communication bandwidth and range when operating in a large group, the agents are not able to have access to the desired reference through broadcasting. To overcome these problems, an estimator is presented to construct the desired reference for each agent based on local information exchange by its neighboring agents. In addition, two distributed shape formation control methodology is proposed to deal with bidirectional and unidirectional interactions among agents.

The existing methods on shape formation control are limited to simple formation problems by single group of agents. The main obstacles that prevent the achievement of the complex shape formation are difficulties in representing complex formations. This problem is overcome by introducing a multi-group coordination methodology. The multi-group coordination problem is formulated into two parts: inter-group formation where groups form a desired formation and intra-group formation such that members of each group form a specified shape. To guarantee the stability of multi-group system, adaptive interactive forces are proposed to cope with inter-group interactions.

Finally, a distributed dynamic multi-group methodology is presented for complex shape formation control of a large swarm of agents. To obtain the distributed dynamic multi-group formation, a global virtual leader is assigned to perform the overall planning and local virtual leaders are assigned to obtain their desired information from the global virtual leader in a distributed way. Therefore, the problem is formulated into two parts: global group behavior and local group behavior. The proposed method is effective in achieving

distributed complex formations, distributed multi-group reconfiguration, and distributed time-varying shape formations for large swarm of agents.

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Chapter 1

Introduction

Rapid advances in sensing, computing and communication technologies have led to the development of networked multi-agent functioning in indoor and outdoor environments. A network of multi-agent system is composed of multiple interacting units which perform specified tasks. In many applications, a given task is too complex to be achieved by a single agent acting alone. As a result, multi-agent systems working cooperatively are required to complete the task. In addition, a team of small agents is preferred to one complex agent for several reasons. First, utilizing a team of small agents will increase the efficiency. From the economic aspect, producing large numbers of identical agents would reduce the cost. Therefore, it is more effective to implement the same type of small agents with different functions to work together than to have a single agent with many functions. Moreover, multi-agent system is more robust as compared to the single agent system because a team of agents can provide some amount of redundancy which is useful in the case when one agent fails.

1.1 Literature Review

Multi-agent systems may appear in various types of networked systems such as sensor networks [1],[2], multiple spacecraft [3]-[5], multiple unmanned aerial vehicles [6],[7], multiple autonomous underwater vehicles [8], and multiple unmanned ground vehicles [9]. During the past decades, there has been an increasing interest in building more and more sophisticated multi-agent systems due to their wide applications in various disciplines of science and engineering including search and rescue [10], target detection, localization, and tracking [11], traffic and transportation systems [12], forest-fire detection [13], supply chain management [14], cooperative coordination and formation flying [15].

Due to development of technologies of agent based systems, there have been a growth in designing large-scale multi-agent systems. Systems of this kind have been employed in power grid systems, communication networks, swarm robotics, traffic networks, etc. The control of large-scale systems presents new challenges due to the extensive size of such systems. Typically, control of large-scale multi-agent systems is required some form of decentralized architectures. The goal to be achieved with the control of this kind of systems, is to obtain a control methodology with a reasonable effort in implementation. This thesis focuses on the formation control of large-scale multi-agent systems.

In the following, previous works in multi-agent systems are reviewed in two main topics including structural behavior of multi-agent systems and formation control of multi-agent systems.

1.1.1 Structural Property of Multi-Agent Systems

A large-scale multi-agent system is characterized by having a large number of components that cooperatively work together. To examine a large-scale system, instead of investigating components separately it is more desirable to consider the interconnections among components. In linear multivariable control systems, the idea of graph-theoretic approach has been developed, which has proved to be suitable for large-scale systems [16]. A pioneer work in graph-theoretic representation of large-scale dynamical systems was done by Lin [17]. He put forward the concept of structured matrix which represents the connectivity among elements. Moreover, the concept of structural property has been introduced which concerns the property of structured matrix with associated a system. Structural properties include structural stabilizability, controllability, and observability. In particular, structural controllability and observability play a critical role in large-scale systems, since these approaches determine the minimum number of sub-systems to be observed or controlled such that the overall controllability or observability of the systems are obtained. A survey on structural properties of linear systems was carried out by Dion et al. [18].

The concept of structural controllability has been studied since the classical work by Lin [17]. In [17] structural controllability of SISO linear systems was explored by introducing a notion of structured matrix whose elements are either fixed zeros or independent free parameters. Moreover, a graph is defined associated with the structured system matrix, and some graph theoretic conditions were obtained to examine the structural controllability of the system. Shields and Pearson [19] extended the results of [17] to structural controllability of multi-input linear systems. Glover and Silverman [20] presented a recursive algorithm to examine the structural controllability based on Boolean operations. Davison [21] defined the notion of connectability for interconnected systems and examined its relationship with structural controllability. Hosoe and Matsumoto [22] derived some prop-

erties on the irreducibility condition to provide easy proof to the structural controllability theorem. Later, Mayeda [23] derived a graph theoretic condition for full generic rank to simplify the proof of the structural controllability. A structurally controllable system can be uncontrollable due to dependencies among system parameters. Hence, Mayeda and Yamada [24] put forward the concept of strong structural controllability such that a system is strongly structurally controllable if the system is controllable for all the non-zero values of parameters of the system. Another result on strong structural controllability of linear systems was presented in [25]. Yamada and Luenberger [26] studied some generic properties of column-structured matrices where entities are columnwise independent, and presented theorems on controllability of such matrices.

Structural observability has been studied in literature based on the duality between controllability and observability [27]. Rech and Perret [28] established a sufficient condition for structural observability of interconnected systems. Sueur and Dauphin-Tanguy [29],[30] employed the bond-graph methodology to examine the structural controllability and observability of linear systems. Chan and Shachter [31] examined structural controllability and observability in influence diagrams by introducing controllable and observable nodes. Boukhobza and Hamelin [32] explored the structural observability of bilinear systems.

In recent years, controllability and observability of multi-agent systems have received considerable attention. Feddema et al. [33] studied input/output reachability, structural observability and controllability of multi-vehicle systems. Tanner [34] addressed the controllability of multi-agent systems with leader-follower structures, and presented a necessary and sufficient condition for controllability of such systems based on the eigenvector of the associated graph laplacian matrix. Rahmani and Mesbahi [35] studied a sufficient condition for the uncontrollability of a network. Ji and Egerstedt [36] introduced network equitable partitions to present a necessary condition for the controllability of a multi-agent networked system. Later, Ji and Egerstedt [37] addressed the observability of a networked

system based on the notion of equitable partitions of a graph. Inspired by the above works, Rahmani et al. [38] purposed the controllability of multi-agent systems with multiple leaders. The observability of networks with ring and chain topologies is discussed in [39]. Zelazo and Mesbahi [40] investigated observability for homogeneous and heterogeneous networks of decoupled linear dynamical systems with coupled output. Parlangeli and Notarstefano [41] provided necessary and sufficient conditions for the observability of path and cycle graphs. An application of network observability in intelligent transportation systems was examined by Castillo et al. [42]. Jafari et al. [43] considered structural controllability of leader-follower multi-agent systems. Sundaram et al. [44] examined the structural controllability and observability of multi-agent systems when the state-space of the agents is constrained to be finite.

However, as networks become more complex, above methods do not provide satisfactory performance due to computational issues. To alleviate the problem, considerable investigations have focused on the understanding of the characteristics of complex networks [45]-[47] recently. A notable work in this area is carried out by Liu et al. [48] which addressed the controllability of complex networks. Several works have been devoted to investigating the pinning control of complex networks [49]-[51]. Song and Cao [52] presented some pinning criteria for synchronization of both directed and undirected complex networks. Porfiri and Bernardo [53] studied pinning-controllability of complex networks.

1.1.2 Formation Control Of Multi-Agent Systems

The study of multi-agent systems brings up new research problems that do not appear in the single agent case [54]. An important problem in multi-agent systems is the coordination control of agents in the team [55]. Formation control is one of the research areas of the coordination control for multi-agent systems, which has become an active research area. The main source of inspiration for the formation control has come from collective

behaviors in nature [56]. Formations can be observed in various animal species as a result of cooperative behaviors among its members. Such fascinating group formations in nature include bird flocks [57],[58], fish schools [59],[60], mammal herds, ant colonies, pattern formation in bacteria [61], and self-organization in cell biology [62]. Animals choose to stay in a group because they can get benefits from the group. For example, birds flying in V formation to save energy for a long migration. The first work that tried to monitor these group formation was done by Reynolds [63], such that a computer simulation was built to model the motion of a flock of birds, called boids. Reynolds believes the motion of the boids, as a whole, is the result of the actions of each individual member that follow some simple rules.

In many applications, a group of autonomous agents are required to follow a predefined trajectory while maintaining a desired pattern. Many approaches have been proposed by researchers in which each of them has their own advantages and disadvantages based on the three major concepts: scalability, asynchronously and anonymous. Formation controller must not be restricted to limited number of agents and must be robust to the dynamic deletion or addition of new agents. Agents should operate asynchronously and rely as much as possible on local communication. Furthermore, agents must be anonymous due to difficulties in uniquely identifying individual members [64].

Formation control of multi-agent systems has a wide range of applications which can be categorized as follows:

1. *Formation control of multiple autonomous ground vehicles:* Multiple autonomous ground vehicles can be employed for the tasks like guarding an object, patrolling an area [65], detect an invader, search and rescue. Moreover, they can be used for data gathering such as environmental monitoring and surveillance.

2. *Formation control of multiple unmanned aerial vehicles (UAVs):* multi-UAV systems represent mobile flying networks which can carry a variety of sensors [66],[67]. UAVs are widely used in military applications such as reconnaissance, target search mission, chemical and biological attack detection. Multi-UAV systems can be developed for civilian applications, such as urban collaborative surveillance [68], forest fires detection and monitoring [69], and geographic mapping.
3. *Formation control of multiple autonomous underwater vehicles (AUVs):* Multi-AUV systems can provide significant benefits to a number of applications including underwater mapping, ocean sampling [70], marine mine-sweeping [71], and deep-sea oil-well maintenance.

A variety of different approaches have been proposed in the literature to solve formation control of multi-agent systems, which can generally be classified into behavior-based approach, leader-follower approach, virtual-structure strategy, artificial potential based method, consensus based method and shape formation approach.

In the behavior-based approach [72]-[74], the overall behavior for each agent is derived by weighting the importance of several desired behaviors. Possible behaviors include goal tracking, collision and obstacle avoidance, and formation maintenance. However, mathematical analysis of this method is difficult in general; Consequently, it is difficult to study the convergence of such collective behavior.

In the leader-follower approach [75],[76], some agents of a group are designated as the leaders while the remaining agents are required to maintain certain relative distances with respect to the specified leaders such that a desired formation is achieved. This approach has advantage of simplicity in theoretical analysis. The disadvantage of this approach is that each follower must have the information access to the specified leader which leads

to very high implementation cost. A reasonable strategy for using the advantages of this approach is to combine the leader-follower control with the decentralized control, which means that each agent only needs the information exchange with its local neighbors. Other disadvantages of leader-follower method are poor disturbance rejection [77], and that a leader has no feedback from the followers. Thus, the group can show such incoherent behaviors as run-away of a fast leader with slower followers left behind. Moreover, the leader-following approach is not suitable for controlling a large group of agents because as the number of agents in the group increases, the constraint relationships among the agents become more complicated.

In the virtual structure strategy [78]-[80], a virtual dynamical system is used to abstract the group maneuver. A virtual structure is a collection of agents, which maintain a (semi-) rigid geometric relationship to each other and to a frame of reference. In this approach, the entire formation is treated as a single rigid body, so desired trajectories are assigned to the entire formation. The virtual structure can evolve as a whole in a given direction with some given orientation while the rigid geometric relationship among agents is maintained during movement. The advantage of the virtual structure strategy is that it is fairly easy to prescribe a coordinated behavior of the group. Also it allows attaining a certain robustness of the formation to perturbations on agents. The main disadvantage of this method is that it is centralized. Similar to the leader-following methods, the virtual structure approaches are not suitable for controlling a large group of agents, since the constraint relationships among the agents become more complicated.

Potential field methods have been studied extensively for path planning of autonomous mobile robot [81]-[83]. Recently this approach has gained increased popularity among researchers in the field of group maneuvering such as formation, migration and obstacle

avoidance. The idea of imaginary forces acting on an agent has been suggested by Khatib [84] and Reif and Wang [85]. The basic idea is to create energy like function in terms of the distance constraints between the agents. The negative gradient of the potential function is used as a restoring force on each agent to achieve coordination. In these approaches obstacles exert repulsive forces onto the agent, while the target applies an attractive force to the agent. The sum of all forces, the resultant force, determines the subsequent direction and speed of movement. One of the reasons for the popularity of this method is its simplicity. Artificial potential based control has been frequently used for formation control of multi-agent systems [86],[87]. However, artificial potential methods can only be used in keeping the agents together as a cluster but no specific shape can be formed.

Consensus control is one of the important problems in networked control systems that has gained increasing attention, owing to its wide potential applications [88],[89]. The consensus problem for a network of agents with undirected communication topology was introduced in [90]-[92]. Later, Moreau [93], Ren and Beard [94] extended the results to a network of agents with directed communication topology. In a recent work, Ma and Zhang [95] introduced the notion of consensusability, and presented the necessary and sufficient conditions for consensusability of multi-agent systems. A special class of the consensus problem is called distributed average consensus, where the consensus value is the average of the initial values [96],[97]. Average consensus is achievable for networks with undirected information flow if and only if the networks are connected. Olfati-Saber and Murray [91] studied average consensus problem for networks with directed graph, and showed that the balanced and strong connectivity are necessary and sufficient conditions for reaching the average consensus. Hence, the balanced condition is a crucial factor in reaching average consensus. Lee and Spong [98] used balanced information graph for flocking problem. Hu and Hong [99] examined the leader-following consensus problem in switching topology when all interconnection graphs associated to followers are balanced.

Cortes [100] proposed a distributed consensus algorithm for weight-balanced networks. Li et al. [101] addressed average consensus for discrete-time first order agents. Ghare-sifard and Cortes [102] proposed a distributed algorithm for generating weight-balanced digraphs.

Consensus algorithms have been frequently used for formation control of multi-agent systems. Recently, consensus based formation control strategies have gained attention. Moshtagh et al. [103] utilized consensus approach to obtain circular formation for a group of nonholonomic ground robots. Xiao et al. [104] proposed a finite-time formation control framework for multi-agent systems using finite-time consensus protocols. Cao et al. [105] proposed decentralized finite-time sliding mode estimators based on finite-time consensus algorithms to obtain finite-time formation control. Zhao et al. [106] addressed the finite-time formation problem based on the finite-time consensus tracking.

As the number of robots in a team increases, accurate positioning of each agent in the formation becomes more complicated. Hence, for formation control of large swarm of agents, methods that take advantage of the anonymity property, are more suitable and gain more attention. A solution to this problem is considering shape formation approaches. In the shape formation method, a desired shape is defined based on a pre-planned desired reference that is available to all members of the group. The methods do not require specific orders or positions of the agents within the specified region. Several methods have been proposed to deal with shape formation problem. Belta and Kumar [107] defined abstraction manifold to coordinate large number of agents by designing control law on small dimensional manifold. Ekanayake and Pathirana [108] by using the Cauchy integral theorem in complex plane introduced a robust and stable algorithm for multi-agent systems aggregation in 2D space. Zhang and Leonard [109] proposed a method for steering control of a system of unit speed particles and used the relative arc-length between particles instead of phase angle differences to measure the relative position between agents on a closed curve. Zhang [110] designed particle formation shape control based on

Jacobi shape theory, such that by using shape theoretical approach each agent modeled as a particle free to choose a coordinate frame to describe observed formation shape dynamics. Kalantar and Zimmer [111] by using active contours presented a controller for compound shape formation of swarms of autonomous underwater vehicles. It was also shown that the problem of shape formation can be simplified if the aggregate is treated as being composed of two components, its boundary and its interior, with the boundary performing shape formation and the interior part maintaining a uniform distribution. Hsieh et al. [112] proposed a feedback controller based on artificial potential function, such that in the proposed decentralized controller two problem was considered: first desired curve forming and second is collision avoidance between members. Cheah et al. [113] proposed a region band control method to deal with shape formation for a swarm of agents. This method can generate shapes formed by intersection of several regions. To specify shape that is formed by union of regions, multiplication potential functions are proposed [114]. In formation navigation, two types of formation structure are possible: fixed and dynamic formation. Fixed formation is more suitable for obstacle-free environments in which formation can be preplanned. Formation control with fixed structure has been studied extensively in the literature Hendrickx et al. [115]. Some applications of rigid formation can be listed as: cooperative object transportation [116], automated highway systems [117] and satellite formation flying [80], [5]. For large number of agents, rigid formation can be interpreted as fixed shape formation, such that group of agents maintains a consistent shape during movement.

Despite of the above mentioned applications of the fixed formation, it is too restricted in utilization due to limitations in operating versatility. In real implementation, the group formation must have the ability to adapt to the environment and avoid obstacles. For small number of robots, dynamic formation is achievable by reconfiguration of robots' positions. Desai [118] studied the formation reconfiguration problem based on the transition from one control graph to another. Defoort et al. [119] addressed time-varying leader-follower

formations. For large number of robots, dynamic formation can be obtained by changing the shape of the group. Hou et al. [120] addressed the dynamic region formation problem based on shape transformation. Sun et al. [121] considered generalized superellipse formation with time-varying parameters. Varghese and Mckee [122] addressed pattern transformation for swarm of robots based on Moebius transformation.

Recently, there has been considerable study on improving formation control techniques to make it more distributed. As a result, distributed leader-follower approach has been proposed which rely on information exchange between neighboring agents. Hong et al. [123] proposed a distributed leader-follower method for an undirected network of multi-agent system. Ren and Sorensen [124] proposed a distributed formation architecture based on the virtual structure scheme. Su et al. [125] extended the multi-agent flocking algorithm proposed by Olfati-Saber [126] to the case that only a fraction of agents are informed of the virtual leader with time-varying velocity. Bai et al. [127] studied the coordination problem of a group of agents where the reference velocity is only available for the leader, while the other agents reconstruct this information with an adaptive design based on local information exchange. Choi et al. [128] addressed a leader-follower formation control in the absence of leader's velocity information. An adaptive law was developed for each agent to construct leader's velocity based on its distance and the bearing angle to the leader. Dong [129] discussed the formation control of multiple mobile agents under the condition that the desired trajectory is available to only portion of agents. Su and Huang [130] considered the cooperative output regulation for linear multi-agent systems which is a generalization of distributed leader-follower problem. However, the proposed methods in distributed formation control are based on leader-follower and virtual structure approaches which are therefore limited to groups with small number of agents.

1.2 Motivation

This section presents the major problems that arise from formation control of large-scale multi-agent systems which is the main motivation of this thesis.

Distributed consensus as an interesting topic of formation control of networked agents, has wide range of applications. It was shown that the distributed consensus is achievable in a network of integrator agents under balanced and strong connectivity conditions. However, the results on balanced graphs are limited to the properties of balanced graphs and how the average consensus is obtained in the balanced graphs. Moreover, a mathematical tool that identify strong connected digraph has not been explored thoroughly. This problem emerges as a crucially complicated issue when dealing with complex networks with more than thousands of nodes. Therefore, a rigorous mathematical formulation on distributed consensusability and weight-balancing of large-scale networked systems are required.

In recent years, there has been a considerable interest in using distributed techniques for formation control problems [131]. This is mostly due to the limitations of communication bandwidth and range in various applications, which lead to a requirement for distributed algorithms based on local information exchange. In this regard, consensus algorithms have been frequently used in the distributed coordination. In shape formation control, having access to the desired reference of the shape is a requirement. The desired reference can be pre-planned to all members, but this reduces the flexibility of the group during movement. Another way is peer-to-peer broadcasting of the desired reference which increases uncertainties due to data lost. Moreover, limitations in communication bandwidth hinder the direct broadcasting of the states of the desired reference. Therefore, a consensus protocol is needed to construct the desired reference for each agent based on local information exchange by its neighboring agents.

The next issue related to shape formation control of multi-agent systems is difficulty in representing complex formations. Therefore existing methods are only limited to simple shape formation problems. The main obstacles that prevent the achievement of the shape formation problem for swarm agents are the large size of the community and complexity of the shape. Therefore, a methodology in cooperative coordination to achieve more complex formations for large-scale multi-agent systems is required.

In the existing methods, dynamic formation is obtained by the overall synchronous shape transformation, in the sense that overall shape of the group is deformed simultaneously which leads to repositioning of all agents. In most applications repositioning of all agents is unnecessary and inefficient especially for a large group of agents. To eliminate the superfluous movements to obtain more intelligent dynamic formation, other issues like formation reconfiguration and local formation transformation must be taken into consideration.

1.3 Contributions

The main contributions of this thesis are summarized as follows :

1. The notion of unobservable node has been introduced based on the structural observability concept. It has been shown that the existence of unobservable nodes leads to loss of distributed consensusability. In the previous results on distributed consensus problem, mathematically examining the strongly connectivity in a network has not been worked out thoroughly. Therefore for the large number of nodes it would be difficult to identify distributed consensusability of the network. A mathematical method for identifying the unobservable nodes is presented. Consequently, the nodes that lead to structural unbalanceability can be easily identified. In addition, globally reachable nodes can be specifically identified in large-scale

networks by using the proposed method which is based on the principal minors of the graph laplacian matrix.

2. A mathematical method is presented based on principal minors of the in-degree laplacian matrix to identify the distributed consensusability of large-scale multi-agent systems and modify the structurally balanceable network into a weight-balanced one.
3. In shape formation control, the movement of the entire group is defined by a reference trajectory and having access to the desired reference of the shape is a requirement. However, due to limitations in communication, it is difficult to access the desired reference for a large group of agents. Therefore, a distributed shape formation control is proposed based on the estimation of the desired reference by group members. The estimation of the desired reference is obtained for each agent based on the local information exchange by its neighboring agents.
4. The other issue in shape formation control involves the difficulty in representing complex formations. Thus, the existing methods have only considered simple formation problems for a single group of agents. The main obstacles that hinder the development of complex shape formation problem for swarm of agents are the large size of the community and the difficulty in mathematically representing a complex shape. To overcome these obstacles, a multi-group coordination methodology is proposed to obtain the complex shape formation for a large swarm of agents. To achieve multi-group formation, the problem is formulated into two parts: inter-group formation where groups form a desired formation, and intra-group formation such that members of each group form a specified shape. Moreover, adaptive inter-active forces are introduced to cope with inter-group interactions and guarantee the stability of the system with multi-group interactions..
5. In some applications, fixed formation problem may be too restricted due to limita-

tions in operating versatility. Moreover, existing methods on dynamic formation are based on synchronous shape transformation, in the sense that overall shape of the group is deformed simultaneously which leads to superfluous repositioning of all agents in the group. To eliminate unnecessary movements of agents in a large group of agents, a distributed dynamic multi-group formation control is presented. The proposed method enhances the previous results in two aspects: (i) more efficient dynamic formations can be obtained, including local transformations and multi-group reconfiguration. (ii) complex formations can be achieved in a distributed way.

1.4 Organization

This dissertation is organized as follows. Chapter 2 provides some preliminaries in graph theory, and reviews the consensus problem in network of agents. The concept of structural observability in network systems is studied, and its relationship with distributed consensus problem is explored. Chapter 3 addresses the distributed shape formation problem for networks of multi-agent systems. A distributed shape formation control is presented for two cases, bidirectional and unidirectional interactions. Chapter 4 considers complex shape formation problem and introduces a multi-group coordination control methodology to obtain complex formations in large swarm of agents. Chapter 5 extends the previous results and presents a distributed dynamic multi-group formation control. Chapter 6 summarizes this work and gives some directions for future research.

Chapter 2

Distributed Consensus in Complex Networks

Rapid growing of network technologies has led to a great demand for development of networked control systems, thus opening up new opportunities and challenges in understanding such control systems. Information consensus is one of the important problems in networked control systems that has gained increasing attention, owing to its wide potential applications. For example distributed formation control for integrator agents is a consensus problem. The consensus protocols are typically introduced by distributed strategies based on neighbor-to-neighbor information exchange. A special case of the consensus problem is called distributed average consensus, where the consensus value is the average of the initial values. Average consensus is achievable for networks with undirected information flow if and only if the networks are strongly connected. For directed graphs, balanced and strong connectivity are necessary and sufficient conditions to obtain average consensus. However, the balanced and strong connectivity have not been explored sufficiently, and rigorous mathematical formulation for weight-balancing of networks is lacking.

This chapter addresses distributed formation problem for integrator agents, and extended results on distributed formation control for agents with general nonlinear dynamic equations will be presented in the next chapters. Distributed formation problem for integrator agents can be simplified to distributed consensus problem. To study distributed consensus problem, a notion of distributed consensusability is introduced which represents the ability of a complex network system in solving distributed consensus problem. To examine distributed consensusability, the structural observability concept is considered in complex network systems. A notion of *unobservable node* is introduced, which leads to structural unobservability. Consequently, it is shown that existence of *unobservable node* yields loss of distributed consensusability. Hence, a simple tool is presented to identify the *unobservable nodes* in complex networks. Moreover, a simple mathematical method is proposed to modify a distributed consensusable network into a weight-balanced one.

2.1 Preliminaries

The communication between nodes can be expressed by a weighted directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$, such that $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$ represents the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, $\mathcal{A} = [a_{ij}]$ is the weighted adjacency matrix where $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. A graph $\mathcal{H}$ is said to be a subgraph of a graph $\mathcal{G}$ if $\mathcal{V}(\mathcal{H}) \subseteq \mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{H}) \subseteq \mathcal{E}(\mathcal{G})$. A directed path in a digraph is an ordered sequence of nodes so that any two consecutive nodes in the sequence are an edge of the digraph. An undirected graph is a tree if and only if for any two nodes, there is a unique path connecting them. A directed spanning tree or arborescence is a digraph such that there is a unique directed path from a designated root node to every other node. The (in-degree) graph laplacian matrix of a digraph $\mathcal{G}$, denoted by $\mathcal{L}$, is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \text{diag}\{d_1, d_2, \dots, d_N\}$ is the degree matrix such that $d_i = a_{ii} = \sum_{\substack{j=1 \\ j \neq i}}^N a_{ij}$. If a digraph contains a directed spanning tree, then the cor-

responding laplacian matrix has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}_N$, and all other eigenvalues have positive real parts [132].

Definition 2.1. A digraph $\mathcal{G}$ is called an *arborescence diverging from node u* , if there is only one directed path between root u and any other node of $\mathcal{G}$. If $\mathcal{G}$ is an arborescence diverging from u , then its reverse digraph is called an *arborescence converging to u* . [133].

If there is an arborescence sub-digraph diverging from an arbitrary node u , then u is called a globally reachable node.

Lemma 2.1. Consider a digraph $\mathcal{G}$ with the associate laplacian matrix $\mathcal{L}$. Let $\mathcal{G}'$ be a sub-digraph of $\mathcal{G}$ which contains an arborescence, the associate laplacian matrix $\mathcal{L}'$ of $\mathcal{G}'$ has rank $N - 1$.

Proof. Since $\mathcal{G}'$ contains an arborescence, therefore $\mathcal{L}'$ has only one zero eigenvalue. Using the rank-multiplicity theorem expressed in Appendix A (see theorem A.1), one can conclude that $\text{rank}(\mathcal{L}') = N - 1$. $\square$

Definition 2.2. A matrix A_s is said to be a *structured matrix* if its elements are either fixed zeros or independent free parameters [134].

Definition 2.3. For a matrix A , $\text{max_rank}(A) \triangleq \max_{A^*} (\text{rank}(A^*))$ where A^* s have the same structure as A with the consideration of the dependency among the parameters.

Consequently, the following inequality is obtained:

$$\text{rank}(A) \leq \text{max_rank}(A) \leq \text{s_rank}(A), \quad (2.1)$$

where $s_rank(A)$ is the structural rank of the matrix A . For example

$$A = \begin{bmatrix} a_1 & a_2 \\ 2a_1 & 2a_2 \end{bmatrix} \Rightarrow \begin{cases} max_rank(A) = 1 \\ s_rank(A) = 2 \end{cases}, \quad (2.2)$$

where a_1 and a_2 are scalars.

2.2 Consensus Problem

Consider a network consisting of N nodes with single integrator dynamics given by

$$\dot{x}_i = u_i, \quad (2.3)$$

where $x_i \in \mathbb{R}$ is state variable of node i which can be information about position or measured quantity; $u_i \in \mathbb{R}$ are control inputs. A protocol is said to solve the consensus problem, if $x_i - x_j \rightarrow 0$ as $t \rightarrow \infty$, for every $i, j \in \{1, 2, \dots, N\}$. A consensus protocol can be expressed as follows [135]:

$$u_i = - \sum_{j=1}^N a_{ij} (x_i - x_j), \quad (2.4)$$

where a_{ij} is the $(i, j)^{th}$ entry of the weighted adjacency matrix $\mathcal{A}$. The closed-loop equation can be written compactly in a matrix form as follows:

$$\dot{x} = -\mathcal{L}x, \quad (2.5)$$

where $x = [x_1, x_2, \dots, x_N]^T$. Let $e = [x_2 - x_1, x_3 - x_1, \dots, x_N - x_1]^T$, therefore (2.5) can be rewritten as follows:

$$\begin{aligned} \dot{e} &= -\mathcal{L}_e e \\ \mathcal{L}_e &= \begin{bmatrix} a_{22} + a_{12} & -a_{23} + a_{13} & \dots & -a_{2N} + a_{1N} \\ -a_{32} + a_{12} & a_{33} + a_{13} & \dots & -a_{3N} + a_{1N} \\ \vdots & \vdots & \dots & \vdots \\ -a_{N2} + a_{12} & -a_{N3} + a_{13} & \dots & a_{NN} + a_{1N} \end{bmatrix}. \end{aligned} \quad (2.6)$$

If the graph contains an arborescence, by similarity transformation, $\mathcal{L}$ can be represented as a block upper-triangular form as follows:

$$\mathcal{P}\mathcal{L}\mathcal{P}^{-1} = \begin{bmatrix} 0 & -\mathcal{A}(1, 2 : N) \\ \mathbf{0} & \mathcal{L}_e \end{bmatrix}, \quad (2.7)$$

where $\mathcal{P} = \begin{bmatrix} 1 & \mathbf{0}_{N-1}^T \\ -\mathbf{1}_{N-1} & \mathbf{I}_{N-1} \end{bmatrix}$ such that $\mathbf{1}_N = \underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T}_N$, $\mathbf{0}_N = \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}^T}_N$

and $\mathbf{I}_{N-1}$ is an identity matrix. Since similarity transformation does not change the eigenvalues, hence, the eigenvalues of $\mathcal{L}_e$ are greater than zero. Therefore, e is asymptotically converging to zero, which implies that the network reach to a consensus.

In the following section, the network system (2.5) with an arbitrary node identified as output node is considered, such that the weights of the links between the nodes are adjustable. It would be interesting to find out under what conditions the specified network system loses the observability. Consequently, the relationship between the unobservability condition and distributed consensus problem will be presented.

2.3 Network Observability

According to classical control theory, a dynamical system is observable if the knowledge of the input and output over a finite time is sufficient to determine the state of the system at any time. Even though a system with a pair of (A, C) might be unobservable, it can be observable for another pair of (A_1, C_1) with the same structure [17]. Consequently, a system is said to be *structurally observable* if there exists at least one admissible realization which is observable [30]. As stated in the classical observability theories, a system is unobservable if it has unobservable states, therefore the following lemma can be expressed for structured systems:

Lemma 2.2. *A structured system is structurally unobservable if there exists a state x^* which is unobservable for all parameter values.*

Hence, to show that a dynamic system is *structurally unobservable*, it is sufficient to find an unobservable state for its structured system. In the following, the *structural observability* of network systems is examined, and the condition that leads to *structural unobservability* of network systems is expressed.

Consider the network system (2.5) with an arbitrary node identified as output node, as follows:

$$\begin{cases} \dot{x} = -\mathcal{L}x \\ y = x_i \end{cases}, \quad (2.8)$$

where $x = [x_1, x_2, \dots, x_N]^T$, and y represents the output. To examine the observability, the zero-input response of the network system is considered, as follows:

$$y_z = C_i e^{-\mathcal{L}t} x_0, \quad (2.9)$$

where $C_i = \begin{bmatrix} 0, 0, \dots, 0, 1, 0, \dots, 0 \end{bmatrix}$. Therefore, the term $C_i e^{-\mathcal{L}t}$ refers to the i^{th} row of the matrix $e^{-\mathcal{L}t}$. Using Cayley-Hamilton theorem, $e^{-\mathcal{L}t}$ can be expanded as follows:

$$e^{-\mathcal{L}t} = \sum_{i=0}^{N-1} \alpha_i(t) \mathcal{L}^i, \quad (2.10)$$

where $\alpha_i(t)$ are scalar functions of time. According to lemma A.1 in Appendix A, the $(i, j)^{th}$ element of the matrix series (2.10) is zero, if there is no path from node j to node i . In this case, the j^{th} element of the $C_i e^{-\mathcal{L}t}$ is zero, and remains zero for all values of the network link weights, therefore at least $x^* = \begin{bmatrix} 0, 0, \dots, 0, \alpha, 0, \dots, 0 \end{bmatrix}^T$ is an unobservable state of the system. Hence, the following lemma can be stated:

Lemma 2.3. *Consider the following structured network system:*

$$\begin{cases} \dot{x} = -\mathcal{L}_s x \\ y = x_i \end{cases}, \quad (2.11)$$

where $\mathcal{L}_s$ is the structured laplacian matrix. If there is no path from node j to the output node i , then $x^* = \begin{bmatrix} 0, 0, \dots, 0, \alpha, 0, \dots, 0 \end{bmatrix}^T$ is an unobservable state of the system, and hence the network system is structurally unobservable.

It has been shown that the existence of a path from all nodes to the output node is a necessary condition for the *structural observability*. In the following, the *unobservable node* is defined based on the above results on the existence of the unobservable state in network systems.

Definition 2.4. *Consider the network system (2.8) with any arbitrary output node $i = 1, 2, \dots, N$. Node j is said to be an unobservable node of the network, if there is no path from node j to at least one arbitrary output node.*

In fact a node is an *unobservable node* if there is at least one node in the system that cannot receive information from that node directly or indirectly. Hence, one can deduce that there is no arborescence sub-digraph diverging from an *unobservable node*, and consequently it is not a globally reachable node. For better understanding, a network consisting of 4 nodes as shown in figure 2.1 is illustrated. The laplacian matrix corresponding to the

Figure 2.1: An example of a network with 4 nodes

4-node graph is expressed as follows:

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}. \quad (2.12)$$

If node 1 is the output node, equation (2.8) is expressed as follows:

$$\begin{cases} \dot{x} = -\mathcal{L}x \\ y = C_1 x \end{cases}, \quad (2.13)$$

where $C_1 = [1 \ 0 \ 0 \ 0]$. In order to check the observability of the system (2.13) based on the classical method, the observability matrix for the system (2.13) is constructed, as

follows:

$$\mathcal{O}_1 = \begin{bmatrix} C_1 \\ -C_1\mathcal{L} \\ C_1\mathcal{L}^2 \\ -C_1\mathcal{L}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & -1 \\ 4 & -2 & 0 & -2 \\ 8 & -4 & 0 & -4 \end{bmatrix}. \quad (2.14)$$

The rank of the observability matrix is, $\text{rank}(\mathcal{O}_1) = 2$. From the observability matrix one can conclude that node 3 is an *unobservable node*. Another factor leading to the rank deficiency of the observability matrix is the symmetricity of nodes 2 and 4 with respect to node 1. By changing the weights, a_{24} and a_{42} in such a way that $a_{24} \neq a_{42}$, the rank of the observability matrix can increase to 3, i.e. $\text{max_rank}(\mathcal{O}_1) = 3$. If other nodes are set as the output, the following results are obtained by constructing the observability matrices:

$$\begin{aligned} \text{max_rank}(\mathcal{O}_2) &= 2, & \text{Unobservable nodes : } 1, 3 \\ \text{max_rank}(\mathcal{O}_3) &= 3, & \text{Unobservable nodes : } 1 \\ \text{max_rank}(\mathcal{O}_4) &= 2, & \text{Unobservable nodes : } 1, 3 \end{aligned} \quad (2.15)$$

Hence, one can conclude that nodes 1 and 3 are *unobservable nodes*. It has been shown that how *unobservable nodes* of a network system can be identified by using the observability matrices. However, the procedure is tedious and time consuming, especially for a complex network, since it requires to check N observability matrices.

In network systems, identifying *unobservable nodes* are useful, since it provides the knowledge about the information flow in the network. Moreover, it shows which nodes cannot be assigned as the leader node. Therefore, it is desirable to find a simple and easy way to identify *unobservable nodes*. In the following, it is shown how principal minors of the laplacian matrix can be used to identify the *unobservable nodes*.

Lemma 2.4. *Consider a network system with the associate digraph $\mathcal{G}$ and the corre-*

sponding laplacian matrix $\mathcal{L}$. The principal minor of $\mathcal{L}$ corresponding to an arbitrary node i is zero if and only if there is no arborescence sub-digraph diverging from node i .

Proof. Necessary condition: assume that there is a sub-digraph which is an arborescence diverging from node i (without loss of generality, it is assumed that $i = 1$), then it must be shown that the principal minor corresponding to node i is not zero. Consider the sub-digraph $\mathcal{G}'$ which is obtained by removing the incoming links to node i . Since removing the incoming links to node i has no effect on the arborescence sub-digraph diverging from node i , hence $\mathcal{G}'$ contains an arborescence. Using lemma 2.1, the associate laplacian matrix $\mathcal{L}'$ of $\mathcal{G}'$ has rank $N - 1$. Therefore,

$$\text{rank}(\mathcal{L}'(G)) = \text{rank} \left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & & & \\ \vdots & & \mathcal{L}(G/1) & \\ -a_{N1} & & & \end{bmatrix} \right) = N - 1. \quad (2.16)$$

Since elementary row/column operations do not change the rank of a matrix, adding columns 2,...,N to the column 1, yields

$$\text{rank}(\mathcal{L}'(G)) = \text{rank} \left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathcal{L}(G/1) & \\ 0 & & & \end{bmatrix} \right) = N - 1. \quad (2.17)$$

The matrix $\mathcal{L}'(G)$ has rank $N - 1$ while first row and column are dependent, hence one can conclude that the matrix $\mathcal{L}(G/1)$ is full rank. Moreover, since $\mathcal{L}(G/1)$ is a diagonally dominant matrix with positive diagonal entries, $\det(\mathcal{L}(G/1)) > 0$.

Sufficient condition: Let $\mathcal{G}$ be a digraph which has no arborescence sub-digraph diverging from node i . Therefore $\mathcal{G}$ can be decomposed into two subdigraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ as illustrated

in figure 2.2, where $\mathcal{G}_1$ contains r nodes which includes node i plus those nodes that have access to information from node i , and $\mathcal{G}_2$ contains $N - r$ nodes which have no access to the information from node i . By renumbering of the nodes of $\mathcal{G}$, the laplacian matrix of

Figure 2.2: An example of the partitioning $\mathcal{G}$ into two subdigraphs $\mathcal{G}_1$ and $\mathcal{G}_2$

the directed graph $\mathcal{G}$ can be expressed in the following decomposed form:

$$\mathcal{L}(\mathcal{G}) = \begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_{12} \\ \mathbf{0} & \mathcal{L}_2 \end{bmatrix}, \quad (2.18)$$

where $\mathcal{L}_1$ is the $r \times r$ matrix associated to the sub-digraph $\mathcal{G}_1$, $\mathcal{L}_2$ is the $(N - r) \times (N - r)$ matrix associated to the sub-digraph $\mathcal{G}_2$, $\mathcal{L}_{12}$ is the $r \times (N - r)$ matrix associated to the information transfer from $\mathcal{G}_2$ to $\mathcal{G}_1$, and $\mathbf{0}$ is $(N - r) \times r$ zero matrix. Since information is only transferred from $\mathcal{G}_2$ to $\mathcal{G}_1$, and $\mathcal{L}$ has zero row sums, $\mathcal{L}_2$ has also zero row sums. The principal minor corresponding to node i can be expressed as follows:

$$\det(\mathcal{L}(\mathcal{G}/1)) = \begin{vmatrix} \mathcal{L}'_1 & \mathcal{L}'_{12} \\ \mathbf{0}' & \mathcal{L}_2 \end{vmatrix}, \quad (2.19)$$

where $\mathcal{L}'_1$, $\mathcal{L}'_2$, $\mathcal{L}'_{12}$ and $\mathbf{0}'$ are obtained by removing the row and column corresponding to node i from $\mathcal{L}$. Using the determinant of the partitioned matrices, $\det(\mathcal{L}(\mathcal{G}/1))$ is expressed as follows:

$$\det(\mathcal{L}(\mathcal{G}/1)) = \det(\mathcal{L}'_1) \cdot \det(\mathcal{L}_2). \quad (2.20)$$

Since matrix with zero row sums has zero determinant, $\det(\mathcal{L}_2) = 0$ which leads to $\det(\mathcal{L}(\mathcal{G}/1)) = 0$. $\square$

Remark 2.1. *The renumbering of the nodes is similar to the row/column switching, and since the row/column switching has no effect on the value of principal minors, the renumbering does not change the absolute value of the principal minors corresponding to the nodes.*

A direct conclusion of the lemma 2.4 is expressed in the following corollary:

Corollary 2.1. *A node is globally reachable node, if and only if the corresponding principal minor of the laplacian matrix is greater than zero.*

It has been shown in [136] that the directed graph $\mathcal{G}$ has a globally reachable node if and only if laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ has a simple zero eigenvalue. However, the result only shows the existence of a globally reachable node and the problem of identifying the locations of globally reachable nodes has not been formerly solved. In lemma 2.4, a simple method is presented to specifically identify the globally reachable nodes by using principal minors of $\mathcal{L}$.

Theorem 2.1. *Consider a network of nodes, such that the topology information between nodes is represented by the corresponding digraph $\mathcal{G}$. Node i is an unobservable node if and only if, the i^{th} principal minor of the laplacian matrix is zero.*

Proof. The proof can be obtained by using definition 2.4 and lemma 2.4. $\square$

As an example for the proposed node observability theorems, again the network in figure 2.1 is considered. To find the unobservable nodes, the principal minors of the laplacian

matrix (2.12) are calculated.

$$\begin{cases} \mathcal{M}_{22} = \mathcal{M}_{44} = 4 \\ \mathcal{M}_{11} = \mathcal{M}_{33} = 0 \end{cases}, \quad (2.21)$$

where $\mathcal{M}_{ii}$ is the i^{th} principal minor. (2.21) shows that nodes 1 and 3 are *unobservable nodes*, which is a similar result to the one found by calculating the observability matrices. However, computing principal minors requires much less effort to identify the *unobservable nodes*. Corollary 2.1 has wide applications in network systems, such as in sensor networks to identify the suitable nodes to be chosen as anchor nodes, or in multi-agent systems to determine the agents that should be assigned as the leaders.

In theorem 2.1, a method to find *unobservable nodes* is proposed. In the following, an alternative theorem to find *unobservable nodes* for different output nodes is presented.

Theorem 2.2. *Consider the network system (2.8) with an arbitrary output node i . A node j is an unobservable node of the system, if the $(j, i)^{th}$ minor of the matrix $\mathbf{I} - \mathcal{A}t$ is zero, where $\mathbf{I}$ is an identity matrix, $\mathcal{A}$ is the associate adjacency matrix, and t is a positive constant such that the spectral radius of $\mathcal{A}t$ is less than 1.*

Proof. To prove this theorem, at first the Taylor series expansion of the matrix inverse (see lemma A.3 in Appendix A) is used, as follows:

$$(\mathbf{I} - \mathcal{A}t)^{-1} = \sum_{i=0}^{\infty} t^i \mathcal{A}^i. \quad (2.22)$$

Using lemma A.2, $(\mathcal{A}^k)_{ij}$ for $k = 1, 2, \dots, \infty$ is zero if and only if there is no path from node j to node i . Hence, the zeroness of the $(i, j)^{th}$ entry of the $(\mathbf{I} - \mathcal{A}t)^{-1}$ which is equivalent to the zeroness of the $(j, i)^{th}$ minor of the $\mathbf{I} - \mathcal{A}t$ represents that there is no path from node j to the output node i , which implies that node j is an *unobservable node*. □

Using Gershgorin's theorem [137], the suitable t which satisfies the condition in theorem 2.2 is obtained as follows:

$$t = \left(\max_i a_{ii} \right)^{-1} - \varepsilon, \quad (2.23)$$

where ε is a small number.

To illustrate the result in theorem 2.2, the network in figure 2.1 is considered. From (2.23) it is obtained that, $t = 0.4$. Therefore, the matrix of minors of $(\mathbf{I} - \mathcal{A}t)^{-1}$ is obtained as follows:

$$\mathcal{M}(\mathbf{I} - \mathcal{A}t) = \begin{bmatrix} 0.84 & 0.00 & 0.00 & 0.00 \\ 0.24 & 1.00 & 0.24 & -0.40 \\ 0.00 & 0.00 & 0.84 & 0.00 \\ 0.24 & -0.40 & 0.24 & 1.00 \end{bmatrix}, \quad (2.24)$$

such that, $\mathcal{M}(1, 4)$ is equal to zero, which means that node 1 is an *unobservable node* of the network system.

After presenting the method for identifying *unobservable nodes* of a network system, the relationship between the distributed consensus problem and the existence of *unobservable nodes* is shown in the next section.

2.4 Distributed Consensus

It is known that consensus is reachable if the corresponding graph of the network contains an arborescence. However, in many applications such as distributed computation [138], distributed filtering [139], and distributed sensor fusion [140],[141], it is desirable to reach the average or distributed consensus.

The distributed consensus is achieved in a network, if

$$\lim_{t \rightarrow \infty} \left(x_i - \frac{1}{N} \sum_{j=1}^N x_j(0) \right) = 0, \quad (2.25)$$

for every $i \in \{1, 2, \dots, N\}$, where $x_j(0)$ refer to initial value of state variables. Consequently, the following definitions are obtained:

Definition 2.5. *A network is said to be distributed consensusable if there is a set of links weights, such that the consensus protocol (2.4) solves the distributed consensus problem.*

Definition 2.6. *A laplacian matrix, $\mathcal{L}$, is said to be structurally balanceable if there exists a balanced laplacian matrix $\mathcal{L}^*$ with the same structure as $\mathcal{L}$.*

For undirected graphs, existence of spanning tree guarantees that the network can reach to average consensus; however, for directed graph existence of directed spanning tree is a necessary but not sufficient condition. It has been shown [124],[142], that the balanced and strong connectivity are necessary and sufficient conditions for reaching the average consensus in networks with directed topology. The balanced condition is described as follows:

Consider a network of nodes with the following closed loop equation:

$$\dot{x} = -\mathcal{L}x. \quad (2.26)$$

The average value of the nodes is obtained as follows:

$$x_{ave} = \frac{1}{N} \sum_{j=1}^N x_j = \frac{1}{N} \mathbf{1}_N x. \quad (2.27)$$

Differentiating equation (2.27), yields

$$\begin{aligned}\dot{x}_{ave} &= \frac{1}{N}\mathbf{1}_N\dot{x} = \frac{1}{N}\mathbf{1}_N(-\mathcal{L}x) \\ &= \frac{1}{N} \begin{bmatrix} c_1 & c_2 & \cdots & c_N \end{bmatrix} x,\end{aligned}\tag{2.28}$$

where $c_i = \sum_{\substack{j=1 \\ j \neq i}}^N a_{ji} - \sum_{\substack{j=1 \\ j \neq i}}^N a_{ij}$. When $\dot{x}_{ave} = 0$ the average of nodes values will not change during the time, which implies that the nodes values will converge to initial average value of them. Since zeroness of c_i where $i = 1, 2, \dots, N$, leads to zeroness of $\dot{x}_{ave}$, then, if signals reach to a consensus, the consensus value is the average of the initial values of the nodes. Therefore, for the consensusable networks the *distributed consensusability* is equivalent to the *structural balanceability*. Though it has been shown that the distributed consensus is achievable in strongly connected networks, a mathematical representation of strong connectivity has not been investigated. In this section, it is shown how the existence of *unobservable nodes* leads to loss of *distributed consensusability*. In addition, a rigorous mathematical formulation is provided to check the *distributed consensusability* of complex networks and modifying a distributed consensusable network into a weight-balanced one.

The following theorem presents the relationship between *distributed consensusability* and the results on the *structural observability* of network systems.

Theorem 2.3. *A network of nodes is distributed consensusable if and only if it has no unobservable nodes. Hence, the following statements are equivalent:*

1. *the network is distributed consensusable.*
2. *the consensusable network is structurally balanceable.*

3. the network has no unobservable node.

4. all the principal minors of the associate laplacian matrix are greater than zero.

Proof. To prove the necessary condition, it must be shown that if the network has no unobservable nodes, then it is distributed consensusable. Using theorem 2.1, since the network has no unobservable nodes, therefore all the principal minors are greater than zero. To show the distributed consensusability of the network, it must be shown that the network is structurally balanceable. To do so, the determinant of the laplacian matrix is calculated by expanding along the first column, as follows:

$$\begin{aligned}
 \det(\mathcal{L}) &= a_{11} \begin{vmatrix} a_{22} & -a_{23} & \cdots & -a_{2N} \\ -a_{32} & a_{33} & \cdots & -a_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N2} & -a_{N3} & \cdots & a_{NN} \end{vmatrix} \\
 &+ (-1)^{2+1}(-a_{21}) \begin{vmatrix} -a_{12} & -a_{13} & \cdots & -a_{1N} \\ -a_{32} & a_{33} & \cdots & -a_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N2} & -a_{N3} & \cdots & a_{NN} \end{vmatrix} \\
 &+ \cdots \\
 &+ (-1)^{N+1}(-a_{N1}) \begin{vmatrix} -a_{12} & -a_{13} & \cdots & -a_{1N} \\ a_{22} & -a_{23} & \cdots & -a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{(N-1)2} & -a_{(N-1)3} & \cdots & -a_{(N-1)N} \end{vmatrix}.
 \end{aligned} \tag{2.29}$$

Since, the sum over each row is zero, $a_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^N -a_{ij} = 0$, therefore by elementary operations, the second cofactor can be rewritten as follows:

$$\begin{aligned}
 & \begin{vmatrix} -a_{12} & -a_{13} & \cdots & -a_{1N} \\ -a_{32} & a_{33} & \cdots & -a_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N2} & -a_{N3} & \cdots & a_{NN} \end{vmatrix} \\
 &= \begin{vmatrix} -\sum_{\substack{j=2 \\ j \neq 3}}^N a_{1j} & -a_{13} & \cdots & -a_{1N} \\ a_{33} - \sum_{\substack{j=2 \\ j \neq 3}}^N a_{3j} & a_{33} & \cdots & -a_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{NN} - \sum_{\substack{j=2 \\ j \neq N}}^N a_{Nj} & -a_{N3} & \cdots & a_{NN} \end{vmatrix} \quad (2.30) \\
 &= - \begin{vmatrix} a_{11} & -a_{13} & \cdots & -a_{1N} \\ -a_{31} & a_{33} & \cdots & -a_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N1} & -a_{N3} & \cdots & a_{NN} \end{vmatrix} = -\mathcal{M}_{22},
 \end{aligned}$$

where $\mathcal{M}_{22}$ represents the second principal minor. Similarly, all other cofactors in (2.29) can be expressed as the corresponding principal minors. Hence, the determinant of the laplacian matrix can be represented as the following linear combination of the principal minors:

$$\det(\mathcal{L}) = a_{11}\mathcal{M}_{11} - a_{21}\mathcal{M}_{22} - \cdots - a_{N1}\mathcal{M}_{NN}, \quad (2.31)$$

where $\mathcal{M}_{ii}$ represents the i^{th} principal minor of $\mathcal{L}$. Next, by expanding the determinant

of the laplacian matrix over the second column, the following equation can be achieved:

$$\det(\mathcal{L}) = -a_{12}\mathcal{M}_{11} + a_{22}\mathcal{M}_{22} - \cdots - a_{N2}\mathcal{M}_{NN}. \quad (2.32)$$

Similarly, the determinant of the laplacian matrix can be expanded over other columns. Since the determinant of the laplacian matrix is zero, the following set of linear combinations of the principal minors can be obtained:

$$\begin{bmatrix} a_{11} & -a_{21} & \cdots & -a_{N1} \\ -a_{12} & a_{22} & \cdots & -a_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1N} & -a_{2N} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} \mathcal{M}_{11} \\ \mathcal{M}_{22} \\ \vdots \\ \mathcal{M}_{NN} \end{bmatrix} = 0. \quad (2.33)$$

Since all the principal minors are greater than zero, equation (2.33) can be expressed as follows:

$$\begin{bmatrix} \mathcal{M}_{11}a_{11} & -\mathcal{M}_{22}a_{21} & \cdots & -\mathcal{M}_{NN}a_{N1} \\ -\mathcal{M}_{11}a_{12} & \mathcal{M}_{22}a_{22} & \cdots & -\mathcal{M}_{NN}a_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathcal{M}_{11}a_{1N} & -\mathcal{M}_{22}a_{2N} & \cdots & \mathcal{M}_{NN}a_{NN} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0, \quad (2.34)$$

which implies that, if all the principal minors are greater than zero, then by multiplying each row of the laplacian matrix by the corresponding principal minor, a balanced laplacian matrix can be achieved, i.e. the following laplacian matrix satisfies the balanced condition:

$$\mathcal{L}' = \begin{bmatrix} a'_{11} & -a'_{12} & \cdots & -a'_{1N} \\ -a'_{21} & a'_{22} & \cdots & -a'_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -a'_{N1} & -a'_{N2} & \cdots & a'_{NN} \end{bmatrix}, \quad (2.35)$$

where $a'_{ij} = \mathcal{M}_{ii}a_{ij}$, and $\sum_{\substack{j=1 \\ j \neq i}}^N a'_{ij} = \sum_{\substack{j=1 \\ j \neq i}}^N a'_{ji}$. Since the network is both consensusable and balanceable, then it is distributed consensusable.

To prove the sufficient condition, it must be shown a distributed consensusable network has no unobservable nodes. From theorem 2.1, a node is unobservable if and only if the corresponding principal minor of the laplacian matrix is equal to zero. The proof proceeds to show that in network systems all the principal minors of the laplacian matrix associated with a distributed consensusable network are nonzero. Since the network is distributed consensusable, without loss of generality, it is assumed that the laplacian matrix $\mathcal{L}$ associated with the network is balanced. Multiplying the laplacian matrix with nonsingular matrices, yields

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & & & \\ \vdots & & \mathbf{I}_{N-1} & \\ 0 & & & \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & -a_{1N} \\ \vdots & \ddots & \vdots \\ -a_{N1} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ \vdots & & \mathbf{I}_{N-1} & \\ 1 & & & \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & -a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -a_{N2} & \cdots & a_{NN} \end{bmatrix}, \quad (2.36)$$

where $\mathbf{I}_{N-1}$ is an identity matrix with dimension $N - 1$. Since the network is distributed consensusable, the rank of the laplacian matrix is $N - 1$ (see lemma A.1 in Appendix A). Since the rank of a matrix is invariant under multiplication by a non-singular matrix, the resultant matrix of (2.36) has rank $N - 1$. The first row and column are dependent, hence rows 2 to N and columns 2 to N are independent. Therefore, $\mathcal{M}_{11}$ which is the first principal minor, is nonzero. Similarly, it can be shown that the other principal minors are nonzero. $\square$

The following result follows directly from equations (2.33) and (2.34):

Corollary 2.2. *Consider a network with corresponding directed graph $\mathcal{G}$. If the network*

has no unobservable nodes, then the network can be balanced by multiplying each row of the laplacian matrix of $\mathcal{G}$ by the corresponding principal minor.

Remark 2.2. Distributed consensus is achievable for a network of nodes, if and only if, the corresponding laplacian matrix has no zero principal minors.

The following example (see figure 2.3), shows a 6-node network expressed by the following laplacian matrix,

$$\mathcal{L} = \begin{bmatrix} 1 & -0.5 & 0 & 0 & 0 & -0.5 \\ -0.5 & 1 & 0 & 0 & 0 & -0.5 \\ 0 & -0.5 & 1.5 & -0.5 & -0.5 & 0 \\ -0.5 & -0.5 & -0.5 & 2 & -0.5 & 0 \\ 0 & 0 & -0.5 & -0.5 & 1.5 & -0.5 \\ -0.5 & -0.5 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2.37)$$

Calculating the principal minors of the laplacian matrix, yields

Figure 2.3: Examples for unbalanceable networks

$$\begin{cases} \mathcal{M}_{11} = \mathcal{M}_{22} = \mathcal{M}_{66} = 2.25 \\ \mathcal{M}_{33} = \mathcal{M}_{44} = \mathcal{M}_{55} = 0.00 \end{cases} \quad (2.38)$$

which means that nodes 3, 4 and 5 are *unobservable nodes*. Hence, the network is *structurally unbalanceable*.

The other example is a network depicted in figure 2.4 with the following laplacian matrix,

$$\mathcal{L} = \begin{bmatrix} 1.5 & -0.5 & -0.5 & 0 & 0 & -0.5 \\ -0.5 & 1.5 & 0 & 0 & -0.5 & -0.5 \\ 0 & -0.5 & 1.5 & -0.5 & -0.5 & 0 \\ -0.5 & -0.5 & -0.5 & 2 & -0.5 & 0 \\ 0 & 0 & -0.5 & -0.5 & 1.5 & -0.5 \\ -0.5 & -0.5 & -0.5 & 0 & 0 & 1.5 \end{bmatrix}. \quad (2.39)$$

which has no *unobservable node*, therefore based on theorem 2.3 it is *structurally bal-*

Figure 2.4: Examples for balanceable networks

anceable. The principal minors of the laplacian matrix are, $\mathcal{M}_{11} = 3.72$, $\mathcal{M}_{22} = 4.88$, $\mathcal{M}_{33} = 4.63$, $\mathcal{M}_{44} = 2.13$, $\mathcal{M}_{55} = 3.88$, and $\mathcal{M}_{66} = 4.16$. After applying the balancing method proposed in theorem 2.3 (see also corollary 2.2), the balanced laplacian matrix corresponding to network 3(b) is expressed as follows:

$$\mathcal{L} = \begin{bmatrix} 5.58 & -1.86 & -1.86 & 0 & 0 & -1.86 \\ -2.44 & 7.32 & 0 & 0 & -2.44 & -2.44 \\ 0 & -2.31 & 6.94 & -2.32 & -2.31 & 0 \\ -1.06 & -1.07 & -1.06 & 4.26 & -1.07 & 0 \\ 0 & 0 & -1.94 & -1.94 & 5.82 & -1.94 \\ -2.08 & -2.08 & -2.08 & 0 & 0 & 6.24 \end{bmatrix}. \quad (2.40)$$

2.5 Simulations

In this section, simulation results are presented to illustrate the performance of the proposed methods for complex networks. For the numerical calculations and simulations MATLAB software was used. For illustration purpose, at first a network with 15 nodes with the following laplacian matrix is considered (as depicted in figure 2.6), Calculating

$$\mathcal{L} = \begin{pmatrix} 2.18 & -0.13 & 0 & 0 & -1.02 & 0 & 0 & 0 & 0 & 0 & 0 & -1.03 & 0 & 0 & 0 \\ 0 & 1.01 & 0 & -0.15 & 0 & 0 & 0 & 0 & -0.62 & 0 & 0 & -0.24 & 0 & 0 & 0 \\ 0 & 0 & 0.99 & -0.99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.39 & 0 & -1.03 & 0 & -0.39 & 0 & 0 & 0 & 0 & 0 & -0.97 & 0 \\ 0 & 0 & 0 & 0 & 0.79 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.79 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -0.94 & 0 & 0 & 1.29 & 0 & -0.35 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.07 & 2.18 & 0 & -0.15 & 0 & 0 & 0 & -0.96 & 0 \\ 0 & 0 & 0 & 0 & -0.7 & 0 & -0.67 & 0 & 1.94 & 0 & -0.58 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.89 & 0 & 0 & 0 & -1.02 & -0.59 & 0 & 3.51 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -0.29 & 0 & 0 & 0 & 0 & -0.76 & -0.88 & 1.93 & 0 & 0 & 0 & 0 \\ 0 & -1.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.04 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.64 & 0 & 0 & 0.64 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.95 & 0 & 0 & 0 & 0.95 & 0 \\ -0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.32 & 0 & 0 & 0.92 \end{pmatrix}$$

Figure 2.5: Laplacian matrix of a randomly generated network with 15 nodes

Figure 2.6: An example of a network consists of 15 nodes

the principal minors of the laplacian matrix, gives

$$\mathcal{M} = \left\{ \begin{array}{l} 0.00, 0.00, 2.93, 12.83, 3.29, 13.20, 18.58, 6.23, \\ 3.76, 14.47, 1.13, 0.00, 24.61, 34.81, 0.00 \end{array} \right\} \quad (2.41)$$

such that $\mathcal{M}_{ii} = 0$ for $i = 1, 2, 12, 15$ and the other principal minors are greater than zero. Therefore one can conclude that nodes 1, 2, 12 and 15 are *unobservable nodes*. The consensus protocol was applied, and the result is depicted in figure 2.7. The initial values are randomly selected from $[0,10]$, and have an average 4.62. The consensus value is 4.02.

The network topology was changed by adding links in such a way that the network has no unobservable node. Therefore, the network is distributed consensusable based on theorem 2.3. The simulation result for the unbalanced distributed consensusable network is shown in figure 2.8. In this case the consensus value, which is 4.35, gets closer to the average value; however since the network is not balanced the consensus value did not converge to average of initial values. the weigh-balancing technique proposed in corollary 2.2, was applied to obtain a weight-balanced network. The simulation result for the balance network is depicted in figure 2.9. It is shown that, the distributed consensus is obtained after applying the balancing method.

In the next simulation, a network of 1000 nodes is considered which are distributed randomly within a square region as shown in figure 2.10. The communication link is generated between neighboring nodes with the probability of 0.65. The algebraic connectivity of the graph $\lambda_2(\mathcal{L}) = 0.36$. The simulation is done for two cases: I. In the first case, the consensus protocol was applied for unbalance network, depicted in figure 2.11. The initial values were randomly selected from $[0,100]$. It is shown that since the network is not balanced, the consensus value, which is 41.67, has considerable difference with average of initial values which is 51.10. II. In the second case, by using the proposed method

Figure 2.7: Applying consensus protocol on an unobservable network (dashed line represents the average of the initial values of the states)

(see corollary 2.2), the balanced network was obtained. The result of applied consensus protocol for the balanced network is illustrated in figure 2.12. It is shown that since the network is balanced the distributed consensus is obtained.

Figure 2.8: Applying consensus protocol on an unbalanced network (dashed line represents the average of the initial values of the states)

Figure 2.9: Applying consensus protocol on a balanced network (dashed line represents the average of the initial values of the states)

Figure 2.10: Network consists of 1000 nodes

Figure 2.11: Applying consensus protocol before balancing the network (dashed line represents the average of the initial values of the states)

Figure 2.12: Applying consensus protocol after balancing the network (dashed line represents the average of the initial values of the states)

Chapter 3

Distributed Shape Formation of Multi-Agent Systems

In the previous chapter, consensus control as a formation control of integrator agents was examined. However, the results is limited to agents with linear and one-dimensional dynamics. This chapter extends the previous chapter's results to formation control of agents with nonlinear Euler-Lagrange dynamics in n -dimensional space.

In formation control of mobile agents, a group of agents are required to follow a desired trajectory while maintaining a desired pattern. In order for all agents in the group to have access to the desired trajectory, several techniques can be used, depending on the control and communication structure of the group. Here, a distributed approach is proposed to obtain the distributed shape formation for multi-agent systems. In shape formation control, having access to the desired reference of the shape is a requirement. The desired reference can be pre-planned to all members, but this reduces the flexibility of the group during movement. Another way is peer-to-peer broadcasting of the desired reference which increases communication cost and uncertainties due to data lost. Moreover, limitations in communication bandwidth hinder the direct broadcasting of the states of the

desired reference. To overcome these problems, an estimator is presented to construct the desired reference for each agent based on local information exchange by its neighboring agents.

3.1 Dynamic Equations of Multi-Agent Systems

Consider a group of N agents, which have the following dynamic equations:

$$\begin{cases} M_1(x_1)\ddot{x}_1 + C_1(x_1, \dot{x}_1)\dot{x}_1 + D_1(x_1)\dot{x}_1 + g_1(x_1) = u_1 \\ M_2(x_2)\ddot{x}_2 + C_2(x_2, \dot{x}_2)\dot{x}_2 + D_2(x_2)\dot{x}_2 + g_2(x_1) = u_2 \\ \vdots \\ M_N(x_N)\ddot{x}_N + C_N(x_N, \dot{x}_N)\dot{x}_N + D_N(x_N)\dot{x}_N + g_N(x_N) = u_N \end{cases} \quad (3.1)$$

where $x_i \in \mathbb{R}^n$ is a generalized coordinate, $M_i(x_i) \in \mathbb{R}^{n \times n}$ are inertia matrices, $C_i(x_i, \dot{x}_i) \in \mathbb{R}^{n \times n}$ are matrices of Coriolis and centripetal terms which together with inertia matrices satisfy skew-symmetric property, $D_i(x_i) \in \mathbb{R}^{n \times n}$ represent the damping forces, $g_i(x_i) \in \mathbb{R}^n$ denote gravitational force vectors, and $u_i \in \mathbb{R}^n$ denote the control inputs. Some properties of equation (3.1) are as follows [143]:

Property 3.1. *The inertia matrices $M_i(x_i)$ are symmetric and positive definite for all $x_i \in \mathbb{R}^n$.*

Property 3.2. *Matrices $\dot{M}_i(x_i) - 2C_i(x_i, \dot{x}_i)$ are skew-symmetric for all $x_i \in \mathbb{R}^n$.*

Property 3.3. *The damping matrices $D_{ki}(x_{ki})$ are positive definite for all $x_i \in \mathbb{R}^n$.*

By linear parameterization, the dynamic equation of each agent is written as follows [144]:

$$M_i(x_i)\ddot{x}_i + C_i(x_i, \dot{x}_i)\dot{x}_i + D_i(x_i)\dot{x}_i + g_i(x_i) = Y_i(x_i, \dot{x}_i, \ddot{x}_i)\theta_i \quad (3.2)$$

where $Y_i(x_i, \dot{x}_i, \ddot{x}_i)$ are known regressor matrices and θ_i are unknown parameter vectors.

3.2 Distributed Shape Forces

In shape formation problem, the shape functions are defined which specify the desired region by intersection of m subregions, as follows [113]:

$$f_i(\Delta x_{io}) = [f_i^1(\Delta x_{io}), f_i^2(\Delta x_{io}), \dots, f_i^m(\Delta x_{io})]^T \leq 0 \quad (3.3)$$

where $\Delta x_{io} = x_i - x_o$, such that x_o is the desired reference. For the agent i the shape function, $f_i(\Delta x_{io})$, is less than or equal to zero when the agent i is located inside or on the boundary of the desired shape, otherwise $f_i(\Delta x_{io})$ is greater than zero. The shape functions have the following properties:

Property 3.4. $f_i(\Delta x_{io})$ are continuous scalar functions with continuous partial derivatives that satisfy $|f_i(\Delta x_{io})| \rightarrow \infty$ as $||\Delta x_{io}|| \rightarrow \infty$.

Property 3.5. The partial differentiation of the shape functions with respect to the Δx_{io} , i.e. $\frac{\partial f_i(\Delta x_{io})}{\partial \Delta x_{io}}$ represent the vectors towards the desired region.

Property 3.6. Boundedness of $f_i(\Delta x_{io})$ represents the boundedness of $\frac{\partial f_i(\Delta x_{io})}{\partial \Delta x_{io}}$ and $\frac{\partial^2 f_i(\Delta x_{io})}{\partial \Delta x_{io}^2}$.

An example of the shape function in 2D space is

$$f_i(\Delta x_{io}) = (\Delta x_{io1})^2 + (\Delta x_{io2})^2 - r^2 \leq 0 \quad (3.4)$$

for circle shape formation with radius r . This function is continuous and its derivative

with respect to Δx_{io} is

$$\frac{\partial f_i(\Delta x_{io})}{\partial \Delta x_{io}} = [2\Delta x_{io1}, 2\Delta x_{io2}]^T \quad (3.5)$$

The boundedness of $f_i(\Delta x_{io})$ represents the boundedness of Δx_{io1} and Δx_{io2} , consequently this leads to the boundedness of $\frac{\partial f_i(\Delta x_{io})}{\partial \Delta x_{io}}$. Some other examples of the shape functions in 2D space are illustrated in figure 3.1.

In [113], it is assumed that the desired reference is available for all agents, however due

Figure 3.1: Examples of the shape functions in 2D space, where a , b , a_i , b_i and d_i are constants. The solid circles represent the agents.

to limitations in communication bandwidth and range, it is reasonable to assume that the desired reference, x_o , is only available for some agents. Therefore, for agents that have no access to x_o , an estimator is proposed to construct the desired reference based on the local information exchange with their neighboring agents. Consider that communication topology among agents is represented by a directed graph $\mathcal{G}$ with the associate laplacian matrix $\mathcal{L} = [l_{ij}]$, where

$$l_{ij} = \begin{cases} -a_{ij} & i \neq j \\ \sum_{j=1}^N a_{ij} & i = j \end{cases} \quad (3.6)$$

such that, if the agent i can obtain information from the agent j , $a_{ij} > 0$, otherwise

$a_{ij} = 0$. The proposed estimator to construct the desired reference is expressed as follows:

$$\dot{\hat{x}}_o^i = \frac{1}{c_i} \sum_{j=1}^N a_{ij} \dot{\hat{x}}_o^j + \frac{1}{c_i} b_{io} \dot{x}_o - \frac{k_e}{c_i} \left(\sum_{j=1}^N a_{ij} (\hat{x}_o^i - \hat{x}_o^j) + b_{io} (\hat{x}_o^i - x_o) \right) \quad (3.7)$$

where $\hat{x}_o^i$ is an estimation of the desired reference, x_o , by the agent i . k_e is positive constant. b_{io} is equal to 1 if agent i has an access to the desired reference and 0 otherwise; $c_i = \sum_{\substack{j=1 \\ j \neq i}}^N a_{ij} + b_{io}$. Since, it is assumed that each agent receives information from at least one agent, therefore c_i is always greater than zero. To obtain the distributed shape formation, the shape functions are defined with respect to $\Delta \hat{x}_{io}$ as follows:

$$f_i(\Delta \hat{x}_{io}) = [f_i^1(\Delta \hat{x}_{io}), f_i^2(\Delta \hat{x}_{io}), \dots, f_i^m(\Delta \hat{x}_{io})]^T \leq 0 \quad (3.8)$$

where $\Delta \hat{x}_{io} = x_i - \hat{x}_o^i$. The potential energy shape function can be defined as follows:

$$P_i(\Delta \hat{x}_{io}) = \sum_{l=1}^m P_i^l(\Delta \hat{x}_{io}) \quad (3.9)$$

such that $P_i^l(\Delta \hat{x}_{io})$ is the potential function for the agent i which is defined for each subregion as follows:

$$\begin{aligned} P_i^l(\Delta \hat{x}_{io}) &= \frac{k_l}{n} [\max(0, f_i^l(\Delta \hat{x}_{io}))]^n \\ &= \begin{cases} 0 & f_i^l(\Delta \hat{x}_{io}) \leq 0 \\ \frac{k_l}{n} (f_i^l(\Delta \hat{x}_{io}))^n & f_i^l(\Delta \hat{x}_{io}) > 0 \end{cases} \end{aligned} \quad (3.10)$$

where k_l are positive constants, n is defined in such a way that the formation potential functions at least belong to class $\mathcal{C}^2$. The distributed shape error with respect to the estimated desired reference, $\hat{x}_o^i$, is obtained by the partial differentiating of the potential

energy shape function with respect to $\Delta\hat{x}_{io}$ as follows:

$$\Delta\hat{\xi}_i = \sum_{l=1}^m k_l [\max(0, f_i^l(\Delta\hat{x}_{io}))]^{n-1} \left(\frac{\partial f_i^l(\Delta\hat{x}_{io})}{\partial \Delta\hat{x}_{io}} \right)^T \quad (3.11)$$

It should be noted that the distributed shape error, equation (3.11), for n greater than two is a smooth function. Moreover, the following relationship holds between the shape functions and the potential shape functions:

$$\begin{cases} \text{if } f_i^l(\Delta\hat{x}_{io}) \rightarrow +\infty \Rightarrow \max(0, f_i^l(\Delta\hat{x}_{io})) \rightarrow +\infty \\ \text{if } f_i^l(\Delta\hat{x}_{io}) \rightarrow -\infty \Rightarrow \max(0, f_i^l(\Delta\hat{x}_{io})) = 0 \end{cases} \quad (3.12)$$

Therefore, if $\max(0, f_i^l(\Delta\hat{x}_{io}))$ is bounded, $f_i^l(\Delta\hat{x}_{io})$ is either bounded or negatively unbounded. If $f_i^l(\Delta\hat{x}_{io})$ is bounded, using the property 3.6 yields the boundedness of $\Delta\hat{\xi}_i$. If $f_i^l(\Delta\hat{x}_{io})$ is negatively unbounded, therefore, $\max(0, f_i^l(\Delta\hat{x}_{io}))$ is equal to zero, and consequently, $\Delta\hat{\xi}_i$ is equal to zero. Hence, the following property is obtained:

Property 3.7. *The boundedness of the potential energy functions (3.9) leads to the boundedness of the distributed shape error (3.11).*

For an example, consider the circle shape formation with radius a , such that $f_i(\Delta\hat{x}_{io}) = (\Delta\hat{x}_{io1})^2 + (\Delta\hat{x}_{io2})^2 - a^2 \leq 0$. An illustration of the potential energy shape function, $P_i^l(\Delta\hat{x}_{io})$, and distributed shape force, $\Delta\hat{\xi}_i$, are shown in figure 3.2.

3.3 Interactive Forces

After defining the control force for the shape formation, the interactive forces, $\Delta\rho_{i-j}$, are identified between agents to assure that the agents maintain the minimum distance with

Figure 3.2: Potential shape function, $P_i^l(\Delta \hat{x}_{io})$, and shape control force $\Delta \hat{\xi}_i$ (is shown by vectors) for circle shape formation, $f_i(\Delta \hat{x}_{io}) = (\Delta \hat{x}_{io1})^2 + (\Delta \hat{x}_{io2})^2 - a^2 \leq 0$

respect to each other. Hence, the interactive potential functions are defined as follows:

$$Q(\Delta x_{ij}) = \sum_{j=1}^N \frac{a_{ij} k_{ij}}{n} [\max(0, g(\Delta x_{ij}))]^n \quad (3.13)$$

where Δx_{ij} refers to the relative distance between the agent i and the agent j , k_{ij} are positive constants, n is defined in such a way that the interactive potential functions at least belong to class $\mathcal{C}^2$, and

$$g(\Delta x_{ij}) = r^2 - \|\Delta x_{ij}\|^2 \leq 0 \quad (3.14)$$

where r is positive constant which specifies the minimum distance between agents. An illustration of the interactive potential function in 2D space is depicted in figure 3.3. From (3.14), the following equalities are obtained:

$$g(\Delta x_{ij}) = g(\Delta x_{ji}) \quad (3.15)$$

Figure 3.3: An illustration of the interactive potential function in 2D space for $n = 3$.

and

$$\frac{\partial g(\Delta x_{ij})}{\partial \Delta x_{ij}} = -\frac{\partial g(\Delta x_{ji})}{\partial \Delta x_{ji}} \quad (3.16)$$

The interactive forces are obtained by partial differentiation of the potential functions with respect to Δx_{ij} as follows:

$$\Delta \rho_{i-j} = \sum_{j=1}^N a_{ij} k_{ij} \underbrace{[\max(0, g(\Delta x_{ij}))]^{n-1} \left(\frac{\partial g(\Delta x_{ij})}{\partial \Delta x_{ij}} \right)^T}_{\Delta \bar{\rho}_{i-j}} \quad (3.17)$$

It should be noted that the interactive force only exerts to agents inside the neighborhood range. In other words, for each two agents i and j , the interactive force between them is zero when $\|\Delta x_{ij}\|$ is greater than r .

Property 3.8. *The distributed shape forces and interactive forces, equation (3.11) and (3.17) are smooth functions.*

Since, $[\max(0, g(\Delta x_{ij}))]^{n-1}$ is zero when $\|\Delta x_{ij}\| \geq r$, and for $\|\Delta x_{ij}\| < r$, $[\max(0, g(\Delta x_{ij}))]^{n-1}$ and $\left(\frac{\partial g(\Delta x_{ij})}{\partial \Delta x_{ij}} \right)^T$ are bounded, therefore, the following properties are obtained:

Property 3.9. *The interactive forces are upper bounded, i.e. $\|\Delta \rho_{i-j}\| < \sigma$, where σ is positive constant.*

Property 3.10. *There is positive constant $\bar{\sigma}$, such that $\|\Delta\dot{\rho}_{i-j}\| < \bar{\sigma} \sum_{j=1}^N a_{ij} \|\Delta\dot{x}_{ij}\|$.*

3.4 Distributed Shape Formation Control Based On Bidirectional Interactions

To design the distributed shape formation control, a reference vector is defined for each agent, as follows:

$$\dot{\hat{x}}_{ri} = \dot{\hat{x}}_o^i - \alpha_i \Delta \hat{\xi}_i - \gamma \Delta \rho_{i-j} \quad (3.18)$$

such that $\hat{x}_o^i$ is the estimation of the desired reference by the agent i , which can be computed by the proposed estimator (3.7). α_i are positive constants, and $\Delta \hat{\xi}_i$ are computed by (3.11); γ is a positive constant, and $\Delta \rho_{i-j}$ are computed by (3.17). Using the reference vector (3.18), a sliding vector is defined for the agent i as follows:

$$\hat{s}_i = \dot{x}_i - \dot{\hat{x}}_{ri} = \Delta \dot{\hat{x}}_{io} + \alpha_i \Delta \hat{\xi}_i + \gamma \Delta \rho_{i-j} \quad (3.19)$$

where $\Delta \dot{\hat{x}}_{io} = \dot{x}_i - \dot{\hat{x}}_o^i$. By differentiating equation (3.19) with respect to time, $\dot{\hat{s}}_i$ is obtained as follows:

$$\dot{\hat{s}}_i = \ddot{x}_i - \ddot{\hat{x}}_o^i = \Delta \ddot{\hat{x}}_{io} + \alpha_i \Delta \dot{\hat{\xi}}_i + \gamma \Delta \dot{\rho}_{i-j} \quad (3.20)$$

Substituting equations (3.19) and (3.20) into (3.1), the following equation is obtained:

$$M_i(x_i) \dot{\hat{s}}_i + C_i(x_i, \dot{x}_i) \hat{s}_i + D_i(x_i) \hat{s}_i + Y_i(x_i, \dot{x}_i, \dot{\hat{x}}_{ri}, \ddot{\hat{x}}_{ri}) \theta_i = u_i \quad (3.21)$$

where $Y_i(x_i, \dot{x}_i, \dot{\hat{x}}_{ri}, \ddot{\hat{x}}_{ri})$ are known regressor matrices and θ_i are unknown parameter vectors, which are defined as follows:

$$Y_i(x_i, \dot{x}_i, \dot{\hat{x}}_{ri}, \ddot{\hat{x}}_{ri})\theta_i = M_i(x_i)\ddot{\hat{x}}_{ri} + C_i(x_i, \dot{x}_i)\dot{\hat{x}}_{ri} + D_i(x_i)\dot{\hat{x}}_{ri} + g_i(x_i) \quad (3.22)$$

The distributed feedback law for the shape formation is expressed as follows:

$$u_i = -K_{si}\hat{s}_i - k_p(\alpha_i\Delta\hat{\xi}_i + \gamma\Delta\rho_{i-j}) + Y_i(x_i, \dot{x}_i, \dot{\hat{x}}_{ri}, \ddot{\hat{x}}_{ri})\hat{\theta}_i \quad (3.23)$$

where parameters k_p is positive constant, K_{si} are positive definite matrices, and $\hat{\theta}_i$ is the vector of estimated parameters of the agent i , which is computed based on the following parameter update laws:

$$\dot{\hat{\theta}}_i = -L_i Y_i^T(x_i, \dot{x}_i, \dot{\hat{x}}_{ri}, \ddot{\hat{x}}_{ri})\hat{s}_i \quad (3.24)$$

where L_i are positive definite matrices.

In the stability analysis, firstly the closed-loop equations of the multi-agent system and the estimator are obtained. The closed-loop equations for the multi-agent system are obtained by substituting the distributed feedback law (3.23) into the dynamic equations (3.21), as follows:

$$\begin{aligned} M_i(x_i)\dot{\hat{s}}_i + C_i(x_i, \dot{x}_i)\hat{s}_i + D_i(x_i)\hat{s}_i + K_{si}\hat{s}_i \\ + k_p(\alpha_i\Delta\hat{\xi}_i + \gamma\Delta\rho_{i-j}) + Y_i(x_i, \dot{x}_i, \dot{\hat{x}}_{ri}, \ddot{\hat{x}}_{ri})\Delta\theta_i = 0 \end{aligned} \quad (3.25)$$

where $\Delta\theta_i = \theta_i - \hat{\theta}_i$. Next, equation (3.7) is rewritten as follows:

$$\dot{\varepsilon}_i = -k_e\varepsilon_i \quad (3.26)$$

such that

$$\begin{aligned}\varepsilon_i &= \sum_{j=1}^N a_{ij} (\hat{x}_o^i - \hat{x}_o^j) + b_{io} (\hat{x}_o^i - x_o) \\ &= \sum_{j=1}^N a_{ij} (e_i - e_j) - b_{io} e_i\end{aligned}\quad (3.27)$$

where $e_i = \hat{x}_o^i - x_o$. Equation (3.26) can be written in the following closed form:

$$\dot{\varepsilon} = -k_e \varepsilon \quad (3.28)$$

where $\varepsilon = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N]^T$. From equation (3.27), equation (3.28) can be written as follows:

$$\{(\mathcal{L} + \mathcal{B}) \otimes I_n\} \dot{e} = -k_e \{(\mathcal{L} + \mathcal{B}) \otimes I_n\} e \quad (3.29)$$

where $\mathcal{L}$ is the laplacian matrix defined by (3.6), $\otimes$ is the Kronecker product, I_n is the identity matrix with the dimension of $n \times n$, and

$$\begin{cases} e = \begin{bmatrix} e_1 & e_2 & \cdots & e_N \end{bmatrix}^T \\ B = \text{diag}\{b_{1o} & b_{2o} & \cdots & b_{No}\}^T\end{cases} \quad (3.30)$$

If $\mathcal{L} + \mathcal{B}$ satisfies the condition in lemma A.4 in Appendix A, then $\mathcal{L} + \mathcal{B}$ is invertible.

Therefore, multiplying both sides of (3.29) by $\{(\mathcal{L} + \mathcal{B})^{-1} \otimes I_n\}$, yields

$$\{(\mathcal{L} + \mathcal{B})^{-1} \otimes I_n\} \{(\mathcal{L} + \mathcal{B}) \otimes I_n\} \dot{e} = -k_e \{(\mathcal{L} + \mathcal{B})^{-1} \otimes I_n\} \{(\mathcal{L} + \mathcal{B}) \otimes I_n\} e \quad (3.31)$$

Using properties of Kronecker product, it gives

$$\{(\mathcal{L} + \mathcal{B})^{-1} (\mathcal{L} + \mathcal{B})\} \otimes \{I_n I_n\} \dot{e} = -k_e \{(\mathcal{L} + \mathcal{B})^{-1} (\mathcal{L} + \mathcal{B})\} \otimes \{I_n I_n\} e \quad (3.32)$$

Hence, (3.32) can be simplified as follows:

$$\dot{e} = -k_e e \Rightarrow \dot{e}_i = -k_e e_i \text{ for } i = 1, 2, \dots, N \quad (3.33)$$

Hence, for each i and j , the following equation is obtained:

$$\begin{cases} \dot{e}_i = -k_e e_i \\ \dot{e}_j = -k_e e_j \end{cases} \Rightarrow (\dot{e}_i - \dot{e}_j) = -k_e (e_i - e_j) \Rightarrow \Delta \dot{\hat{x}}_o^{ij} = -k_e \Delta \hat{x}_o^{ij} \quad (3.34)$$

where $\Delta \hat{x}_o^{ij} = \hat{x}_o^i - \hat{x}_o^j$. Solution of (3.34) is expressed as follows:

$$\Delta \hat{x}_o^{ij}(t) = \Delta \hat{x}_o^{ij}(0) e^{-k_e t} \quad (3.35)$$

where t represents the time.

To examine the stability of the overall system described by the equations (3.25) and (3.26), the following Lyapunov-like candidate is proposed:

$$\begin{aligned} V_T = & \sum_{i=1}^N \frac{1}{2} \hat{s}_i^T M_i \hat{s}_i + \sum_{i=1}^N \frac{1}{2} \Delta \theta_i^T L_i^{-1} \Delta \theta_i + \sum_{i=1}^N \frac{1}{2} e_i^T e_i \\ & + \sum_{i=1}^N \alpha_i k_p \sum_{l=1}^m \frac{k_l}{n} [\max(0, f_i^l(\Delta x_{io}))]^n \\ & + \frac{1}{2} \sum_{i=1}^N \gamma k_p \sum_{j=1}^N \frac{a_{ij} k_{ij}}{n} [\max(0, g(\Delta x_{ij}))]^n \end{aligned} \quad (3.36)$$

By differentiating the Lyapunov-like candidate with respect to time, substituting the closed-loop equations (3.25) and (3.26) into it, and considering the skew-symmetry of the

matrix $\dot{M}_i(x_i) - 2C_i(x_i, \dot{x}_i)$, $\dot{V}_T$ is obtained as follows:

$$\begin{aligned}
 \dot{V}_T &= - \sum_{i=1}^N \hat{s}_i^T K_{si} \hat{s}_i - \sum_{i=1}^N \hat{s}_i^T D_i(x_i) \hat{s}_i - \sum_{i=1}^N \hat{s}_i^T k_p (\alpha_i \Delta \hat{\xi}_i + \gamma \Delta \rho_{i-j}) \\
 &\quad - k_e \sum_{i=1}^N e_i^T e_i + \sum_{i=1}^N \alpha_i k_p \Delta \hat{x}_{io}^T \Delta \hat{\xi}_i + \frac{1}{2} \sum_{i=1}^N \gamma k_p \sum_{j=1}^N a_{ij} \Delta \dot{x}_{ij} \Delta \bar{\rho}_{ij} \\
 &= - \sum_{i=1}^N \hat{s}_i^T K_{si} \hat{s}_i - \sum_{i=1}^N \hat{s}_i^T D_i(x_i) \hat{s}_i - \sum_{i=1}^N \Delta \omega_i^T k_p \Delta \omega_i - k_e \sum_{i=1}^N e_i^T e_i \\
 &\quad - \sum_{i=1}^N \gamma k_p \Delta \hat{x}_{io}^T \Delta \rho_{i-j} + \frac{1}{2} \sum_{i=1}^N \gamma k_p \sum_{j=1}^N a_{ij} \Delta \dot{x}_{ij} \Delta \bar{\rho}_{i-j} \tag{3.37}
 \end{aligned}$$

where $\Delta \omega_i = \alpha_i \Delta \hat{\xi}_i + \gamma \Delta \rho_{i-j}$, and $\Delta \dot{x}_{ij}$ is expressed as follows:

$$\Delta \dot{x}_{ij} = \Delta \dot{x}_{io} - \Delta \dot{x}_{jo} + \Delta \dot{x}_o^{ij} \tag{3.38}$$

Therefore, the following equation is obtained:

$$\begin{aligned}
 \sum_{i=1}^N \gamma k_p \sum_{j=1}^N a_{ij} \Delta \dot{x}_{ij} \Delta \bar{\rho}_{i-j} &= \sum_{i=1}^N \gamma k_p \sum_{j=1}^N a_{ij} \Delta \dot{x}_{io} \Delta \bar{\rho}_{i-j} - \sum_{i=1}^N \gamma k_p \sum_{j=1}^N a_{ij} \Delta \dot{x}_{jo} \Delta \bar{\rho}_{i-j} \\
 &\quad + \sum_{i=1}^N \gamma k_p \sum_{j=1}^N a_{ij} \Delta \dot{x}_o^{ij} \Delta \bar{\rho}_{i-j} \tag{3.39}
 \end{aligned}$$

According to equations (3.15), (3.16) and bidirectionality of the interactive forces, $a_{ji} \Delta \bar{\rho}_{j-i} = -a_{ij} \Delta \bar{\rho}_{i-j}$. Hence, it yields

$$\begin{aligned}
 \sum_{i=1}^N \gamma k_p \sum_{j=1}^N a_{ij} \Delta \dot{x}_{jo} \Delta \bar{\rho}_{i-j} &= \sum_{j=1}^N \gamma k_p \sum_{i=1}^N a_{ji} \Delta \dot{x}_{io} \Delta \bar{\rho}_{j-i} \\
 &= - \sum_{i=1}^N \gamma k_p \sum_{j=1}^N a_{ij} \Delta \dot{x}_{io} \Delta \bar{\rho}_{i-j} \\
 &= - \sum_{i=1}^N \gamma k_p \Delta \dot{x}_{io} \Delta \rho_{i-j} \tag{3.40}
 \end{aligned}$$

Using (3.39) and (3.40), yields

$$\begin{aligned} \sum_{i=1}^N \gamma k_p \sum_{j=1}^N a_{ij} \Delta \dot{x}_{ij} \Delta \bar{\rho}_{i-j} &= 2 \sum_{i=1}^N \gamma k_p \Delta \dot{x}_{io} \Delta \rho_{i-j} \\ &+ \sum_{i=1}^N \gamma k_p \sum_{j=1}^N a_{ij} \Delta \hat{x}_o^{ij} \Delta \bar{\rho}_{i-j} \end{aligned} \quad (3.41)$$

Substituting (3.41) into (3.37) yields

$$\dot{V}_T = -U + \frac{1}{2} \sum_{i=1}^N \gamma k_p \sum_{j=1}^N a_{ij} \Delta \hat{x}_o^{ij} \Delta \bar{\rho}_{i-j} \quad (3.42)$$

such that

$$U = \sum_{i=1}^N \hat{s}_i^T K_{si} \hat{s}_i + \sum_{i=1}^N \hat{s}_i^T D_i(x_i) \hat{s}_i + \sum_{i=1}^N \Delta \omega_i^T k_p \Delta \omega_i + k_e \sum_{i=1}^N e_i^T e_i \quad (3.43)$$

Now, the following theorem can be expressed on the convergence of the signals:

Theorem 3.1. *The multi-agent system, represented by equation (3.1) with the bidirectional interactions, the distributed shape control scheme (3.23) and the parameter update law (3.24), result in the convergence of $\Delta \dot{x}_{io}$ and $\Delta \omega_i$ to zero, for all $i, j = 1, 2, \dots, N$, as $t \rightarrow \infty$*

Proof. Since $M_i(x_i)$, $D_i(x_i)$ and L_i are positive definite, from (3.36), V_T is positive definite in $\hat{s}_i$ and $\Delta \theta_i$. Similarly, since K_{si} are positive definite, from (3.43), U is positive definite in $\hat{s}_i$ and $\Delta \omega_i$. Integrating both sides of (3.42), gives

$$V_T(t) + \int_0^t U(\tau) d\tau = V(0) + \frac{1}{2} \int_0^t \sum_{i=1}^N \gamma k_p \sum_{j=1}^N a_{ij} \Delta \hat{x}_o^{ij} \Delta \bar{\rho}_{i-j} dt \quad (3.44)$$

Using property 3.9 and equation (3.35), the following inequality is obtained:

$$\begin{aligned} \sum_{i=1}^N \gamma k_p \sum_{j=1}^N a_{ij} \Delta \dot{\hat{x}}_o^{ij} \Delta \bar{\rho}_{ij} &\leq \gamma k_p \left| \sum_{i=1}^N \sum_{j=1}^N a_{ij} \Delta \dot{\hat{x}}_o^{ij} \right| \left| \sum_{i=1}^N \sum_{j=1}^N a_{ij} \Delta \bar{\rho}_{ij} \right| \\ &\leq \gamma k_p N \sigma \left| \sum_{i=1}^N \sum_{j=1}^N a_{ij} \Delta \hat{x}_o^{ij}(0) e^{-k_e t} \right| \end{aligned} \quad (3.45)$$

Consequently, it yields

$$V_T(t) + \int_0^t U(\tau) d\tau \leq V(0) + \frac{\gamma k_p N \sigma}{2k_e} \left| \sum_{i=1}^N \sum_{j=1}^N a_{ij} \Delta \hat{x}_o^{ij}(0) \right| \quad (3.46)$$

Since the RHS is bounded, and $V_T(t)$ and $\int_0^t U(\tau) d\tau$ are both positive, then $V_T(t)$ and $\int_0^t U(\tau) d\tau$ are bounded, which yields the boundedness of U . According to (3.36) and the definition of ultimately bounded [145], $\hat{s}_i$ and potential shape functions are bounded. Using property 3.7, one can conclude $\Delta \hat{\xi}_i$ is bounded. Since $\hat{x}_o^i$ exponentially approaches to x_o , therefore, boundedness of x_o and its derivatives lead to the boundedness of $\hat{x}_o^i$ and its derivatives. Hence, from (3.19), $\dot{x}_i$ is bounded. From the closed-loop equations (3.25), the boundedness of $\dot{\hat{s}}_i$ is obtained. Since $\dot{x}_i$ and $\Delta \hat{\xi}_i$ are bounded, $\Delta \dot{\hat{\xi}}_i$ is bounded. Moreover, using property 3.10, $\Delta \dot{\rho}_{i-j}$ is bounded. From (3.20), $\ddot{x}_i$ is bounded. Therefore, $\dot{U}$ is bounded. Consequently, by invoking Extended Barbalat's Lemma (see lemma A.5 in Appendix A), one concludes that U goes to zero as $t \rightarrow \infty$, which leads to $\hat{s}_i \rightarrow 0$ and $\Delta \omega_i \rightarrow 0$ as $t \rightarrow \infty$. Hence, $\Delta \dot{\hat{x}}_{io} \rightarrow 0$ as $t \rightarrow \infty$. Since $\hat{x}_o^i$ goes to x_o , one can conclude $\Delta \dot{\hat{x}}_{io}$ goes to zero as $t \rightarrow \infty$. $\square$

3.5 Distributed Shape Formation Control Based On Unidirectional Interactions

In the previous section, it was assumed that the interactions between agents are bidirectional. In this section, the unidirectional interactions between agents are considered. A reference vector for each agent is defined as follows:

$$\dot{\hat{x}}_{ri} = \dot{\hat{x}}_o^i - \alpha_i \Delta \hat{\xi}_i \quad (3.47)$$

such that $\dot{\hat{x}}_o^i$ is the estimation of the velocity of the desired reference which is computed by the proposed estimator (3.7). α_i are positive constants, and $\Delta \hat{\xi}_i$ are computed by (3.11). Using the reference vector (3.47), a sliding vector is defined for the agent i as follows:

$$\hat{s}_i = \dot{x}_i - \dot{\hat{x}}_{ri} = \Delta \dot{\hat{x}}_{io} + \alpha_i \Delta \hat{\xi}_i \quad (3.48)$$

where $\Delta \dot{\hat{x}}_{io} = \dot{x}_i - \dot{\hat{x}}_o^i$. By differentiating equation (3.48) with respect to time, $\dot{\hat{s}}_i$ is obtained as follows:

$$\dot{\hat{s}}_i = \ddot{x}_i - \ddot{\hat{x}}_o^i = \Delta \ddot{\hat{x}}_{io} + \alpha_i \Delta \dot{\hat{\xi}}_i \quad (3.49)$$

Substituting equations (3.48) and (3.49) into (3.1) gives

$$M_i(x_i) \dot{\hat{s}}_i + C_i(x_i, \dot{x}_i) \hat{s}_i + D_i(x_i) \hat{s}_i + Y_i(x_i, \dot{x}_i, \dot{\hat{x}}_{ri}, \ddot{\hat{x}}_{ri}) \theta_i = u_i \quad (3.50)$$

where $Y_i(x_i, \dot{x}_i, \dot{\hat{x}}_{ri}, \ddot{\hat{x}}_{ri})$ are known regressor matrices and θ_i are unknown parameter vectors, which are defined by (3.22). The distributed feedback law for the shape formation

is expressed as follows:

$$\begin{aligned}
 u_i = & -k_p \left(\alpha_i \Delta \hat{\xi}_i + k_a \sum_{j=1}^N a_{ij} (\hat{s}_i - \hat{s}_j) + \gamma \sum_{j=1}^N a_{ij} \Delta \bar{\rho}_{i-j} \right) \\
 & -K_{si} \text{sign}(\hat{s}_i) + Y_i(x_i, \dot{x}_i, \dot{\hat{x}}_{ri}, \ddot{\hat{x}}_{ri}) \hat{\theta}_i
 \end{aligned} \tag{3.51}$$

where k_p and k_a are positive constants; K_{si} are positive definite matrices, and $\hat{\theta}_i$ is estimated parameter vector of the agent i which is computed based on the following parameter update laws:

$$\dot{\hat{\theta}}_i = -L_i Y_i^T(x_i, \dot{x}_i, \dot{\hat{x}}_{ri}, \ddot{\hat{x}}_{ri}) \hat{s}_i \tag{3.52}$$

where L_i are positive definite matrices. The closed-loop equations are obtained by substituting the distributed feedback law (3.51) into the dynamic equations (3.50), as follows:

$$\begin{aligned}
 & M_i(x_i) \dot{\hat{s}}_i + C_i(x_i, \dot{x}_i) \hat{s}_i + D_i(x_i) \hat{s}_i + K_{si} \text{sign}(\hat{s}_i) \\
 & + k_p \left(\alpha_i \Delta \hat{\xi}_i + k_a \sum_{j=1}^N a_{ij} (\hat{s}_i - \hat{s}_j) + \gamma \sum_{j=1}^N a_{ij} \Delta \bar{\rho}_{i-j} \right) + Y_i(x_i, \dot{x}_i, \dot{\hat{x}}_{ri}, \ddot{\hat{x}}_{ri}) \Delta \theta_i = 0
 \end{aligned} \tag{3.53}$$

where $\Delta \theta_i = \theta_i - \hat{\theta}_i$. To examine the stability of the overall system described by equations (3.53) and (3.26), the following Lyapunov-like candidate is proposed:

$$\begin{aligned}
 V_T = & \sum_{i=1}^N \frac{1}{2} \mathcal{M}_i \hat{s}_i^T M_i \hat{s}_i + \sum_{i=1}^N \frac{1}{2} \mathcal{M}_i \Delta \theta_i^T L_i^{-1} \Delta \theta_i + \sum_{i=1}^N \frac{1}{2} e_i^T e_i \\
 & + \sum_{i=1}^N \mathcal{M}_i \alpha_i k_p \sum_{l=1}^m \frac{k_l}{n} [\max(0, f_i^l(\Delta x_{io}))]^n
 \end{aligned} \tag{3.54}$$

where $\mathcal{M}_i$ is the i^{th} principal minor of the laplacian matrix. If the network is structurally observable $\mathcal{M}_i > 0$ (see theorem 2.3). By differentiating the Lyapunov-like candidate with respect to time, substituting the closed-loop equations (3.53) and (3.26) into it, and considering the skew-symmetry of the matrix $\dot{M}_i(x_i) - 2C_i(x_i, \dot{x}_i)$, $\dot{V}_T$ is obtained as

follows:

$$\begin{aligned}
 \dot{V}_T &= -\sum_{i=1}^N \mathcal{M}_i \hat{s}_i^T D_i(x_i) \hat{s}_i - \sum_{i=1}^N \mathcal{M}_i \hat{s}_i^T K_{si} \text{sign}(\hat{s}_i) - k_e \sum_{i=1}^N e_i^T e_i \\
 &\quad - \sum_{i=1}^N \mathcal{M}_i \hat{s}_i^T k_p \left(\alpha_i \Delta \hat{\xi}_i + k_a \sum_{j=1}^N a_{ij} (\hat{s}_i - \hat{s}_j) + \gamma \sum_{j=1}^N a_{ij} \Delta \bar{\rho}_{i-j} \right) \\
 &\quad + \sum_{i=1}^N \mathcal{M}_i \alpha_i k_p \Delta \hat{x}_{io}^T \Delta \hat{\xi}_i \\
 &= -\sum_{i=1}^N \mathcal{M}_i \hat{s}_i^T D_i(x_i) \hat{s}_i - \sum_{i=1}^N \mathcal{M}_i \hat{s}_i^T K_{si} \text{sign}(\hat{s}_i) - \sum_{i=1}^N \mathcal{M}_i \Delta \hat{\xi}_i^T k_p \Delta \hat{\xi}_i \\
 &\quad - k_e \sum_{i=1}^N e_i^T e_i - k_p k_a \sum_{i=1}^N \mathcal{M}_i \hat{s}_i^T \sum_{j=1}^N a_{ij} (\hat{s}_i - \hat{s}_j) \\
 &\quad + k_p \sum_{i=1}^N \gamma \mathcal{M}_i \sum_{j=1}^N a_{ij} \hat{s}_i^T \Delta \bar{\rho}_{i-j}
 \end{aligned} \tag{3.55}$$

Since the network is structurally balanceable (see theorem 2.3), therefore, $\sum_{j=1}^N a'_{ij} = \sum_{j=1}^N a'_{ji}$, where $a'_{ij} = \mathcal{M}_i a_{ij}$. Hence,

$$\sum_{j=1}^N \mathcal{M}_i a_{ij} = \sum_{j=1}^N \mathcal{M}_j a_{ji} \tag{3.56}$$

Consequently,

$$\begin{aligned}
 \sum_{i=1}^N \hat{s}_i^T \sum_{j=1}^N \mathcal{M}_i a_{ij} (\hat{s}_i - \hat{s}_j) &= \sum_{i=1}^N \hat{s}_i^T \sum_{j=1}^N \mathcal{M}_j a_{ji} (\hat{s}_i - \hat{s}_j) \\
 &= \sum_{j=1}^N \hat{s}_j^T \sum_{i=1}^N \mathcal{M}_i a_{ij} (\hat{s}_j - \hat{s}_i) \\
 &= -\sum_{i=1}^N \sum_{j=1}^N \mathcal{M}_i a_{ij} \hat{s}_j^T (\hat{s}_i - \hat{s}_j)
 \end{aligned} \tag{3.57}$$

Therefore,

$$\begin{aligned}
 \sum_{i=1}^N \hat{s}_i^T \sum_{j=1}^N \mathcal{M}_i a_{ij} (\hat{s}_i - \hat{s}_j) &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \mathcal{M}_i a_{ij} \hat{s}_i^T (\hat{s}_i - \hat{s}_j) \\
 &\quad - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \mathcal{M}_i a_{ij} \hat{s}_j^T (\hat{s}_i - \hat{s}_j) \\
 &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \mathcal{M}_i a_{ij} (\hat{s}_i - \hat{s}_j)^T (\hat{s}_i - \hat{s}_j)
 \end{aligned} \tag{3.58}$$

Moreover, the following inequality holds:

$$\hat{s}_i^T K_{si} \text{sign}(\hat{s}_i) \geq \lambda_{\min}(K_{si}) \|\hat{s}_i\| \tag{3.59}$$

Using property 3.9, the following inequality is obtained:

$$\sum_{j=1}^N a_{ij} \hat{s}_i^T \Delta \bar{\rho}_{i-j} \leq \sigma \|\hat{s}_i\| \tag{3.60}$$

Hence, (3.55) is written as follows:

$$\begin{aligned}
 \dot{V}_T &\leq - \sum_{i=1}^N \mathcal{M}_i \hat{s}_i^T D_i(x_i) \hat{s}_i - \sum_{i=1}^N \mathcal{M}_i \{ \lambda_{\min}(K_{si}) - k_p \gamma \sigma \} \|\hat{s}_i\| \\
 &\quad - \sum_{i=1}^N \mathcal{M}_i \Delta \hat{\xi}_i^T k_p \Delta \hat{\xi}_i - k_e \sum_{i=1}^N e_i^T e_i \\
 &\quad - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \mathcal{M}_i a_{ij} (\hat{s}_i - \hat{s}_j)^T (\hat{s}_i - \hat{s}_j)
 \end{aligned} \tag{3.61}$$

Therefore, one can guarantee the negativeness of $\dot{V}_T$ by choosing K_{si} in such a way that $\lambda_{\min}(K_{si}) > k_p \gamma \sigma$. Now, the following theorem can be expressed on the convergence of the signals:

Theorem 3.2. *The multi-agent system, represented by equation (3.1) with the unidirectional*

tional interactions, the distributed shape control scheme (3.51) and the parameter update law (3.52), result in the convergence of $\Delta \dot{x}_{io}$, $\Delta \xi_i$ and the synchronization term, $\hat{s}_i - \hat{s}_j$ to zero, for all $i, j = 1, 2, \dots, N$, as $t \rightarrow \infty$; where $\Delta \xi_i$ is the shape error with respect to the actual desired reference, x_o , and it is obtained as follows:

$$\Delta \xi_i = \sum_{l=1}^m k_l [\max(0, f_i^l(\Delta x_{io}))]^{n-1} \left(\frac{\partial f_i^l(\Delta x_{io})}{\partial \Delta x_{io}} \right)^T \quad (3.62)$$

Proof. Since $M_i(x_i)$, $D_i(x_i)$ and L_i are positive definite, from (3.54), V_T is positive definite in $\hat{s}_i$ and $\Delta \theta_i$. Therefore $\hat{s}_i$ and potential shape functions are bounded. Using property 3.7, $\Delta \hat{\xi}_i$ is bounded. Since $\hat{x}_o^i$ exponentially approaches to x_o , therefore, boundedness of x_o and its derivatives lead to the boundedness of $\hat{x}_o^i$ and its derivatives. Equations (3.47) and (3.48) yields the boundedness of $\dot{x}_i$. From the closed-loop equations (3.53), the boundedness of $\dot{\hat{s}}_i$ is obtained. Since $\dot{x}_i$ and $\Delta \hat{\xi}_i$ are bounded, $\Delta \dot{\hat{\xi}}_i$ is bounded. Moreover, using property 3.10, $\Delta \dot{\rho}_{i-j}$ is bounded. From (3.49), $\ddot{x}_i$ is bounded. Therefore, $\ddot{V}_T$ is bounded, and hence $\dot{V}_T$ is uniformly continuous. By invoking Barbalat's Lemma [144], it then follows $\dot{V}_T$ goes to zero as $t \rightarrow \infty$, so it is concluded that $\hat{s}_i$, $\Delta \hat{\xi}_i$ and $\hat{s}_i - \hat{s}_j$ goes to zero as $t \rightarrow \infty$. From convergence of e_i one can reach to the convergence of $\Delta \dot{x}_{io}$. Moreover, since e_i and $\Delta \hat{\xi}_i$ goes to zero, one can deduce that $\Delta \xi_i$ goes to zero as t goes to infinity. $\square$

Remark 3.1. In this chapter, a switching controller is proposed to deal with nonuniform interactions. In next chapters, a concept of adaptive interactive forces will be introduced to overcome this problem.

Remark 3.2. It was shown that the estimator (3.7) yields exponential convergence of the $\hat{x}_o^i$ to x_o . To obtain the finite-time convergence of the $\hat{x}_o^i$ to x_o , the following estimator is

proposed:

$$\begin{aligned} \dot{\hat{x}}_o^i = & \frac{1}{c_i} \sum_{j=1}^N a_{ij} \dot{\hat{x}}_o^j + \frac{1}{c_i} b_{io} \dot{x}_o - \frac{k_e}{c_i} \text{sign} \left(\sum_{j=1}^N a_{ij} (\hat{x}_o^i - \hat{x}_o^j) + b_{io} (\hat{x}_o^i - x_o) \right) \\ & \times \left| \sum_{j=1}^N a_{ij} (\hat{x}_o^i - \hat{x}_o^j) + b_{io} (\hat{x}_o^i - x_o) \right|^\delta \end{aligned} \quad (3.63)$$

where $0 < \delta < 1$. To show the finite-time convergence of the proposed estimator, equation (3.63) is rewritten as follows:

$$\dot{\varepsilon}_i = -k_e \text{sign}(\varepsilon_i) |\varepsilon_i|^\delta \quad (3.64)$$

where ε_i is defined by (3.27). Consider the following Lyapunov-like candidate:

$$V = \frac{1}{2} \sum_{i=1}^N \varepsilon_i^T \varepsilon_i \quad (3.65)$$

By differentiating the Lyapunov-like candidate with respect to time, and substituting equation (3.64) into it, $\dot{V}$ is obtained as follows:

$$\dot{V} = -k_e \sum_{i=1}^N |\varepsilon_i|^{1+\delta} = -k_e \sum_{i=1}^N (\varepsilon_i^T \varepsilon_i)^{\frac{1+\delta}{2}} \quad (3.66)$$

Using the property A.1 expressed in Appendix A, (3.66) is written as follows:

$$\dot{V} \leq -2^{\frac{1+\delta}{2}} k_e \left(\frac{1}{2} \sum_{i=1}^N \varepsilon_i^T \varepsilon_i \right)^{\frac{1+\delta}{2}} = -2^{\frac{1+\delta}{2}} k_e V^{\frac{1+\delta}{2}} \quad (3.67)$$

Using the lemma A.6 in Appendix A, one can obtain $\varepsilon_i = 0$ for $t \geq T$, such that T is a constant with the following upper bound:

$$T \leq \frac{(2V(0))^{\frac{1-\delta}{2}}}{k_e(1-\delta)} \quad (3.68)$$

Remark 3.3. *The potential shape function defined by (3.10) belongs to $\mathcal{C}^2$, if n greater than two. However, it may lead to large initial values of the shape errors when the agents are initially located far away from the desired shape. To alleviate this problem, the following potential function is proposed:*

$$P_i^l(\Delta \hat{x}_{io}) = \frac{k_l}{n} \ln \left(\cosh \left(\left[\max \left(0, f_i^l(\Delta \hat{x}_{io}) \right) \right]^n \right) \right) \quad (3.69)$$

where "ln" is the natural logarithm, and "cosh" is the hyperbolic cosine function. Therefore, the distributed shape error is obtained as follows:

$$\Delta \hat{\xi}_i = \sum_{l=1}^m k_l \left[\max \left(0, f_i^l(\Delta \hat{x}_{io}) \right) \right]^{n-1} \tanh \left(\left[\max \left(0, f_i^l(\Delta \hat{x}_{io}) \right) \right]^n \right) \left(\frac{\partial f_i^l(\Delta \hat{x}_{io})}{\partial \Delta \hat{x}_{io}} \right)^T \quad (3.70)$$

such that $\Delta \hat{\xi}_i$ is a smooth function for n greater than one.

3.6 Simulation

A numerical simulation was performed to illustrate the effectiveness of the proposed distributed shape formation control. For the simulation, Virtual Reality Toolbox (VRLT) in Matlab software was employed. A group of omnidirectional mobile agents consists of 30 agents was considered, which have the following dynamic equations:

$$M_i \ddot{x}_i + C_i \dot{x}_i = u_i \quad (3.71)$$

where M_i and C_i are mass and damping constants, respectively. The actual mass of the agents are taken to be $M_i = 1kg$, and the actual damping coefficients are taken to be $C_i = 0.5$. The radius of each agent is $r_i = 0.3m$. To obtain the parameter update law,

equation (3.22) is expressed as follows:

$$\begin{aligned} M_i \ddot{x}_{ri} + C_i \dot{x}_{ri} &= \begin{bmatrix} \ddot{\hat{x}}_{ri} & \dot{\hat{x}}_{ri} \end{bmatrix} \begin{bmatrix} M_i \\ C_i \end{bmatrix} \\ &= Y_i \theta_i \end{aligned} \quad (3.72)$$

The initial estimate of the physical parameters for the parameter update law was set as $\hat{\theta}_i = [0.8 \ 0.3]^T$. Any two agents with the probability of 0.6 were been able to directly communicate with each other if their Euclidean distance was smaller than the given threshold $r_0 = 10m$. The control gains and estimator gain used in the simulation are listed in table 3.1.

Two simulations were conducted to examine the proposed distributed shape formation

Table 3.1: Control gains

k_e	2
α_i	30
k_p	1
K_{si}	diag {20, 20}
k_{ij}	1
γ	500
L_i	diag {0.05, 0.05}

control in presence of bidirectional and unidirectional interactions.

i) In the first simulation, bidirectional interactions were assumed among the agents. The agents were assigned to form a circle shape with radius $R_s = 2.5m$.

Initially the agents were distributed randomly within a square region of side length $10m$ as it is depicted in figure 3.4., such that the desired shape is shown by a transparent blue region. The desired reference was chosen as follows:

Figure 3.4: The initial position of agents from two different views

$$\begin{cases} x_{o1} = \cos(\pi/4) \left\{ 0.5t - 4 \sin(t/2) \right\} - 4 \\ x_{o2} = \sin(\pi/4) \left\{ 0.5t + 4 \sin(t/2) \right\} - 4 \end{cases} \quad (3.73)$$

The simulation was run for $t = 35s$ with a simulation time step of $10ms$. The agents' motions at different time instances of the simulation from two different views are illustrated in figure 3.5 and figure 3.6. The initial values of $\hat{x}_o^i$ were randomly chosen. The

Figure 3.5: Distributed shape formation of multi-agent system (view1).

Figure 3.6: Distributed shape formation of multi-agent system (view2).

Euclidean norm of the desired reference $\|x_o\|$ and the Euclidean norm of the estimations of the desired reference $\|\hat{x}_o^i\|$ are depicted in figure 3.7.

ii) In the second simulation, unidirectional interactions were considered among the agents. The agents were assigned to form an astroid-like shape obtained by intersection of five circle regions as depicted in figure 3.8.

Initially the agents were distributed randomly within a square region as shown in figure 3.9. The desired reference was chosen as follows:

$$\begin{cases} x_{o1} = 4. \sin(t/4) \\ x_{o2} = 0.6t \end{cases} \quad (3.74)$$

The simulation was conducted for $t = 35s$ with a simulation time step of $10ms$. The agents' motions at different time instances of the simulation from two different views are illustrated in figure 3.10 and figure 3.11. The initial values of $\hat{x}_o^i$ were randomly chosen. The Euclidean norm of the desired reference $\|x_o\|$ and the Euclidean norm of the

Figure 3.7: The Euclidean norm of the desired reference $\|x_o\|$ and the Euclidean norm of the estimations of the desired reference $\|\hat{x}_o^i\|$.

estimations of the desired reference $\|\hat{x}_o^i\|$ are depicted in figure 3.12.

Figure 3.8: Astroid-like shape obtained by intersection of five circle regions

Figure 3.9: The initial position of agents from two different views

Figure 3.10: Distributed shape formation of multi-agent system (view1).

Figure 3.11: Distributed shape formation of multi-agent system (view2).

Figure 3.12: The Euclidean norm of the desired reference $\|x_o\|$ and the Euclidean norm of the estimations of the desired reference $\|\hat{x}_o^i\|$.

Chapter 4

Multi-Group Coordination Problem

In the previous chapter, the region formation control was presented as a technique to solve the shape formation problem for multi-agent systems. However, due to difficulties in representing complex formations, the existing methods have only considered simple formation problems for a single group of agents. The main obstacles that hinder the development of complex shape formation problem for swarm of agents are the large size of the community and the difficulty in mathematically representing a complex shape. Both these obstacles are overcome by introducing a multi-group coordination methodology to obtain the complex shape formation for a large swarm of agents. Complexity is dealt with by decomposing the swarms into multiple groups of agents which constitute the basic building blocks of the complex formation. The problem is thus formulated as coordination control of multiple groups of agents. The multi-group coordination control concept has many advantages in large swarm formation, complex formation transformation and formation reconfiguration. To achieve multi-group formation, group architectures must be defined, between and within these groups. Therefore, the problem is formulated into two parts: inter-group formation where groups form a desired formation, and intra-group formation such that members of each group form a specified shape. Compared with the

mutual interactions among individuals in single groups, social interactions in multi-group of agents are more complicated. Generally, there are two kinds of interactions: intra-group interactions, which can occur among members of a group, and inter-group interactions, which will occur between members of one group and members of adjacent groups. Despite the intra-group interactions, inter-group interactions have more intricacy, and interactive forces that are used for single groups cannot guarantee the stability of the system with multi-group interactions. Hence, an adaptive interactive force is proposed to cope with inter-group interactions.

4.1 Problem Description

Two issues which must be considered in the coordination of multi-group systems are: (1) Inter-group architecture between groups; and (2) Intra-group architecture within each group. For the inter-group architecture, the linear dominance hierarchy [146] is considered. The main structure of this architecture is composed of several levels of dominancy (as illustrated in figure 4.1) where each level consists of several follower groups such that groups in level k of dominancy dominate over groups in the $(k + 1)^{th}$ level of dominancy and is dominated by groups in the $(k - 1)^{th}$ level of dominancy. This dominancy can be implemented as interaction and following. The number of levels can be defined on the basis of the desired task and leader communication pattern. For intra-group architecture, a leader-follower architecture is considered, such that the group leader is leading all of the members.

Existing methods in leader-follower and hierarchical formations [147],[148] are limited to a single group structure and thus only simple formations can be formed. To achieve complex formations, a general multi-group coordination method is proposed such that the

Figure 4.1: Linear dominance hierarchy

inter-group and intra-group architectures can be specified as hierarchical, leader-follower, virtual structure or any other architectures on the basis of the desired task and application. In region based shape control, a controller is designed such that all agents move inside a specified region. The restriction of this method is the difficulty in representing complex shapes. By formulating a group coordination problem, one can decompose every complex shape into several simple regions. In this case, complex shapes can be formed by using some basic building blocks of regions such as circles, ellipses and rectangles. An illustration of multi-group coordination control is shown in figure 4.2. In this example, a swarm of agents are assigned to form a Chinese character. To achieve this goal, eleven groups in six levels of dominance are defined. As is shown, in the primary stage groups form simple shapes and in the secondary stage, by coordination of these groups, complex formation is achieved.

Considering the architecture described, the dynamic equations of the multi-group systems are expressed by (4.1), where: for the leading group $i = 1, 2, \dots, n_{Lg}$ such that n_{Lg} is the number of members of the leading group; for follower groups $i = 1, 2, \dots, n_{Fg_j^k}$ where $k = 1, 2, \dots, N$ (N denotes the number of levels) and $j = 1, 2, \dots, N_k$ (N_k denotes the number of groups in each level), such that $n_{Fg_j^k}$ is the number of members of the follower

Figure 4.2: An example of multi-group coordination control

group Fg_j^k ; $x_L, x_{Lg_i}, x_{Fl_j^k}, x_{Fg_{ji}^k} \in \mathbb{R}^n$ are generalized coordinates of the global leader, leading group members, local group leaders, and following group members, respectively; $M_L(x_L), M_{Lg_i}(x_{Lg_i}), M_{Fl_j^k}(x_{Fl_j^k}), M_{Fg_{ji}^k}(x_{Fg_{ji}^k}) \in \mathbb{R}^{n \times n}$ are symmetric and positive definite inertia matrices, $C_L(x_L, \dot{x}_L), C_{Lg_i}(x_{Lg_i}, \dot{x}_{Lg_i}), C_{Fl_j^k}(x_{Fl_j^k}, \dot{x}_{Fl_j^k}), C_{Fg_{ji}^k}(x_{Fg_{ji}^k}, \dot{x}_{Fg_{ji}^k}) \in \mathbb{R}^{n \times n}$ are matrices of Coriolis and centripetal terms which together with inertia matrices satisfy the skew-symmetric property, $D_L(x_L), D_{Lg_i}(x_{Lg_i}), D_{Fl_j^k}(x_{Fl_j^k}), D_{Fg_{ji}^k}(x_{Fg_{ji}^k}) \in \mathbb{R}^{n \times n}$ represent the positive definite damping matrices, $g_L(x_L), g_{Lg_i}(x_{Lg_i}), g_{Fl_j^k}(x_{Fl_j^k}), g_{Fg_{ji}^k}(x_{Fg_{ji}^k}) \in \mathbb{R}^n$ denote gravitational force vectors, and $u_L, u_{Lg_i}, u_{Fl_j^k}, u_{Fg_{ji}^k} \in \mathbb{R}^n$ denote the control inputs.

$$\left\{ \begin{array}{l} \text{Leading group} \left\{ \begin{array}{l} L : M_L(x_L)\ddot{x}_L + C_L(x_L, \dot{x}_L)\dot{x}_L + D_L(x_L)\dot{x}_L + g_L(x_L) = u_L \\ Lg : M_{Lg_i}(x_{Lg_i})\ddot{x}_{Lg_i} + C_{Lg_i}(x_{Lg_i}, \dot{x}_{Lg_i})\dot{x}_{Lg_i} + D_{Lg_i}(x_{Lg_i})\dot{x}_{Lg_i} + g_{Lg_i}(x_{Lg_i}) = u_{Lg_i} \end{array} \right. \\ \\ \text{Level 1 of dominance} \left\{ \begin{array}{l} \text{Follower group 1} \left\{ \begin{array}{l} Fl_1^1 : M_{Fl_1^1}(x_{Fl_1^1})\ddot{x}_{Fl_1^1} + C_{Fl_1^1}(x_{Fl_1^1}, \dot{x}_{Fl_1^1})\dot{x}_{Fl_1^1} \\ \quad + D_{Fl_1^1}(x_{Fl_1^1})\dot{x}_{Fl_1^1} + g_{Fl_1^1}(x_{Fl_1^1}) = u_{Fl_1^1} \\ Fg_1^1 : M_{Fg_1^1}(x_{Fg_1^1})\ddot{x}_{Fg_1^1} + C_{Fg_1^1}(x_{Fg_1^1}, \dot{x}_{Fg_1^1})\dot{x}_{Fg_1^1} \\ \quad + D_{Fg_1^1}(x_{Fg_1^1})\dot{x}_{Fg_1^1} + g_{Fg_1^1}(x_{Fg_1^1}) = u_{Fg_1^1} \end{array} \right. \\ \vdots \\ \text{Follower group } N_1 \left\{ \begin{array}{l} Fl_{N_1}^1 : M_{Fl_{N_1}^1}(x_{Fl_{N_1}^1})\ddot{x}_{Fl_{N_1}^1} + C_{Fl_{N_1}^1}(x_{Fl_{N_1}^1}, \dot{x}_{Fl_{N_1}^1})\dot{x}_{Fl_{N_1}^1} \\ \quad + D_{Fl_{N_1}^1}(x_{Fl_{N_1}^1})\dot{x}_{Fl_{N_1}^1} + g_{Fl_{N_1}^1}(x_{Fl_{N_1}^1}) = u_{Fl_{N_1}^1} \\ Fg_{N_1}^1 : M_{Fg_{N_1}^1}(x_{Fg_{N_1}^1})\ddot{x}_{Fg_{N_1}^1} + C_{Fg_{N_1}^1}(x_{Fg_{N_1}^1}, \dot{x}_{Fg_{N_1}^1})\dot{x}_{Fg_{N_1}^1} \\ \quad + D_{Fg_{N_1}^1}(x_{Fg_{N_1}^1})\dot{x}_{Fg_{N_1}^1} + g_{Fg_{N_1}^1}(x_{Fg_{N_1}^1}) = u_{Fg_{N_1}^1} \end{array} \right. \end{array} \right. \\ \vdots \\ \text{Level N of dominance} \left\{ \begin{array}{l} \text{Follower group 1} \left\{ \begin{array}{l} Fl_1^N : M_{Fl_1^N}(x_{Fl_1^N})\ddot{x}_{Fl_1^N} + C_{Fl_1^N}(x_{Fl_1^N}, \dot{x}_{Fl_1^N})\dot{x}_{Fl_1^N} \\ \quad + D_{Fl_1^N}(x_{Fl_1^N})\dot{x}_{Fl_1^N} + g_{Fl_1^N}(x_{Fl_1^N}) = u_{Fl_1^N} \\ Fg_1^N : M_{Fg_1^N}(x_{Fg_1^N})\ddot{x}_{Fg_1^N} + C_{Fg_1^N}(x_{Fg_1^N}, \dot{x}_{Fg_1^N})\dot{x}_{Fg_1^N} \\ \quad + D_{Fg_1^N}(x_{Fg_1^N})\dot{x}_{Fg_1^N} + g_{Fg_1^N}(x_{Fg_1^N}) = u_{Fg_1^N} \end{array} \right. \\ \vdots \\ \text{Follower group } N_N \left\{ \begin{array}{l} Fl_{N_N}^N : M_{Fl_{N_N}^N}(x_{Fl_{N_N}^N})\ddot{x}_{Fl_{N_N}^N} + C_{Fl_{N_N}^N}(x_{Fl_{N_N}^N}, \dot{x}_{Fl_{N_N}^N})\dot{x}_{Fl_{N_N}^N} \\ \quad + D_{Fl_{N_N}^N}(x_{Fl_{N_N}^N})\dot{x}_{Fl_{N_N}^N} + g_{Fl_{N_N}^N}(x_{Fl_{N_N}^N}) = u_{Fl_{N_N}^N} \\ Fg_{N_N}^N : M_{Fg_{N_N}^N}(x_{Fg_{N_N}^N})\ddot{x}_{Fg_{N_N}^N} + C_{Fg_{N_N}^N}(x_{Fg_{N_N}^N}, \dot{x}_{Fg_{N_N}^N})\dot{x}_{Fg_{N_N}^N} \\ \quad + D_{Fg_{N_N}^N}(x_{Fg_{N_N}^N})\dot{x}_{Fg_{N_N}^N} + g_{Fg_{N_N}^N}(x_{Fg_{N_N}^N}) = u_{Fg_{N_N}^N} \end{array} \right. \end{array} \right. \end{array} \right. \quad (4.1)$$

4.2 Multi-Group Formation

In order to achieve multi-group formation, the problem is formulated into two parts: inter-group formation and intra-group formation. In the inter-group formation, formation of groups will be specified. Formation of groups is achievable based on group leaders formation. In the intra-group formation, formation of members of groups will be determined. This stage can be accomplished by determining formation of the group members with respect to the group leader.

4.2.1 Inter-group Formation

In inter-group formation, the formation of groups with respect to each other will be determined. This can be accomplished by defining formation among leaders such that for each local group leader, formation can be defined with respect to its dominant local group leader and other local group leaders in that level of dominancy. An example of the formation of group leaders is illustrated in figure 4.3. A global leader is assigned to follow the desired path while local group leaders in first level of dominancy maintain the desired formation with respect to global leader and each other. Similarly, local group leaders in the second level form the desired formation with respect to the local group leaders in the first level and each other.

Figure 4.3: An example of inter-group formation

In order to achieve the desired formation, a formation function is defined for the j^{th} local group leader in the k^{th} level of dominancy with respect to the q^{th} local group leader in the $(k-1)^{th}$ level and other local group leaders $\bar{j}$ in the k^{th} level of dominancy (which are in

communication with F_l^k) as

$$\mathfrak{f}_{F_l^k} = \begin{cases} \left[\mathfrak{f}_{F_l^k}^1 \left(\Delta x_{F_l^k F_l^{k-1}} \right), \dots, \mathfrak{f}_{F_l^k}^{z_{F_l^k}} \left(\Delta x_{F_l^k F_l^{k-1}} \right) \right] \leq 0 \\ \left[\mathfrak{f}_{F_l^k}^{z_{F_l^k}+1} \left(\Delta x_{F_l^k F_l^k} \right), \dots, \mathfrak{f}_{F_l^k}^{Z_{F_l^k}} \left(\Delta x_{F_l^k F_l^k} \right) \right] \leq 0 \end{cases} \quad (4.2)$$

such that $\Delta x_{F_l^k F_l^{k-1}} = x_{F_l^k} - x_{F_l^{k-1}}$ and $\Delta x_{F_l^k F_l^k} = x_{F_l^k} - x_{F_l^k}$; $z_{F_l^k}$ are the number of objective functions defined for F_l^k . $\mathfrak{f}_{F_l^k}^{\ell_j^k}(\Delta x_{F_l^k F_l^{k-1}})$ for $\ell_j^k = 1, 2, \dots, z_{F_l^k}$ are objective functions which are defined for F_l^k with respect to F_l^{k-1} , and $\mathfrak{f}_{F_l^k}^{\ell_j^k}(\Delta x_{F_l^k F_l^k})$ for $\ell_j^k = z_{F_l^k} + 1, \dots, Z_{F_l^k}$ are objective functions which are defined for F_l^k with respect to F_l^k . For $k = 1$, F_l^{k-1} is substituted by L . Therefore, the following potential functions are defined:

$$\begin{aligned} P_{F_l^k} = & \sum_{\ell_j^k=1}^{z_{F_l^k}} \frac{k_{F_l^k}}{n} \left[\max \left(0, \mathfrak{f}_{F_l^k}^{\ell_j^k} \left(\Delta x_{F_l^k F_l^{k-1}} \right) \right) \right]^n \\ & + \sum_{\ell_j^k=z_{F_l^k}+1}^{Z_{F_l^k}} \frac{k_{F_l^k}}{n} \left[\max \left(0, \mathfrak{f}_{F_l^k}^{\ell_j^k} \left(\Delta x_{F_l^k F_l^k} \right) \right) \right]^n \end{aligned} \quad (4.3)$$

where $k_{F_l^k}$ are positive constants and n is defined in such a way that the formation potential functions at least belong to class $\mathcal{C}^2$. Formation errors are obtained by partial differentiation of potential functions with respect to their independent variables as:

$$\begin{aligned} \Delta \zeta_{F_l^k} = & \underbrace{\sum_{\ell_j^k=1}^{z_{F_l^k}} k_{F_l^k} \left[\max \left(0, \mathfrak{f}_{F_l^k}^{\ell_j^k} \left(\Delta x_{F_l^k F_l^{k-1}} \right) \right) \right]^{n-1} \left(\frac{\partial \mathfrak{f}_{F_l^k}^{\ell_j^k} \left(\Delta x_{F_l^k F_l^{k-1}} \right)}{\partial \Delta x_{F_l^k F_l^{k-1}}} \right)^T}_{\Delta \zeta_{F_l^k} \left(\Delta x_{F_l^k F_l^{k-1}} \right)} \\ & + \underbrace{\sum_{\ell_j^k=z_{F_l^k}+1}^{Z_{F_l^k}} k_{F_l^k} \left[\max \left(0, \mathfrak{f}_{F_l^k}^{\ell_j^k} \left(\Delta x_{F_l^k F_l^k} \right) \right) \right]^{n-1} \left(\frac{\partial \mathfrak{f}_{F_l^k}^{\ell_j^k} \left(\Delta x_{F_l^k F_l^k} \right)}{\partial \Delta x_{F_l^k F_l^k}} \right)^T}_{\Delta \zeta_{F_l^k} \left(\Delta x_{F_l^k F_l^k} \right)} \end{aligned} \quad (4.4)$$

Remark 4.1. *Since the formation potential functions (4.3) at least belong to class $\mathcal{C}^2$, therefore, the formation errors, equation (4.4), are smooth functions.*

4.2.2 Intra-group Formation

After the coordination of leaders, shape formation is defined for the members of each group. An illustration is shown in figure 4.2, where the shape of each group is determined by the corresponding group leader. The shape functions (see (3.3)) for the members of the leading group with respect to the global leader are defined as follows:

$$f_{Lg}(\Delta x_{Lg_iL}) = \left[f_{Lg}^1, f_{Lg}^2, \dots, f_{Lg}^{Z_{Lg}} \right] \leq 0 \quad (4.6)$$

where $\Delta x_{Lg_iL} = x_{Lg_i} - x_L$; $f_{Lg}^m(\Delta x_{Lg_iL})$ are objective functions which for $m = 1, 2, \dots, Z_{Lg}$ are representing the desired subregion, such that Z_{Lg} is the number of objective functions defined for the members of the leading group. Similarly, for follower groups, the shape functions can be defined with respect to the corresponding local group leaders,

$$f_{Fg_j^k}(\Delta x_{Fg_{ji}^kFl_j^k}) = \left[f_{Fg_j^k}^1, f_{Fg_j^k}^2, \dots, f_{Fg_j^k}^{Z_{Fg_j^k}} \right] \leq 0 \quad (4.7)$$

where $\Delta x_{Fg_{ji}^kFl_j^k} = x_{Fg_{ji}^k} - x_{Fl_j^k}$; $f_{Fg_j^k}^{m_j^k}(\Delta x_{Fg_{ji}^kFl_j^k})$ are objective functions which for $m_j^k = 1, 2, \dots, Z_{Fg_j^k}$ are representing the desired subregion for each follower group, such that $Z_{Fg_j^k}$ is the number of objective functions defined for Fg_j^k . For example, a circle shape formation is defined for the follower group as illustrated in figure 4.4.

$$f_{Fg_j^k}(\Delta x_{Fg_{ji}^k Fl_j^k}) = \left(x_{Fg_{ji}^k} - x_{Fl_j^k} \right)^T \left(x_{Fg_{ji}^k} - x_{Fl_j^k} \right) - r^2 \leq 0$$

Figure 4.4: Circle formation of follower group Fg_j^k with respect to the local leader Fl_j^k

Therefore, the potential energies corresponding to the shape functions for the leading group and follower groups are defined as follows:

$$P_{Lg} = \sum_{m=1}^{Z_{Lg_i}} \frac{k_{Lg}}{n} \left[\max(0, f_{Lg}^m(\Delta x_{Lg_i L})) \right]^n \quad (4.8)$$

$$P_{Fg_j^k} = \sum_{m_j^k=1}^{Z_{Fg_j^k}} \frac{k_{Fg_j^k}}{n} \left[\max\left(0, f_{Fg_j^k}^{m_j^k}(\Delta x_{Fg_{ji}^k Fl_j^k})\right) \right]^n \quad (4.9)$$

where k_{Lg} and $k_{Fg_{ji}^k}$ are positive constants. Shape errors are obtained for the members of the leading group and follower groups by partial differentiation of the potential energy functions with respect to $\Delta x_{Lg_i L}$ and $\Delta x_{Fg_{ji}^k Fl_j^k}$, respectively, as:

$$\begin{aligned} \Delta \xi_{Lg_i} &= \sum_{m=1}^{Z_{Lg_i}} k_{Lg} \left[\max(0, f_{Lg}^m(\Delta x_{Lg_i L})) \right]^{n-1} \left(\frac{\partial f_{Lg}^m(\Delta x_{Lg_i L})}{\partial \Delta x_{Lg_i L}} \right)^T \\ \Delta \xi_{Fg_{ji}^k} &= \sum_{m_j^k=1}^{Z_{Fg_j^k}} k_{Fg_j^k} \left[\max\left(0, f_{Fg_j^k}^{m_j^k}(\Delta x_{Fg_{ji}^k Fl_j^k})\right) \right]^{n-1} \left(\frac{\partial f_{Fg_j^k}^{m_j^k}(\Delta x_{Fg_{ji}^k Fl_j^k})}{\partial \Delta x_{Fg_{ji}^k Fl_j^k}} \right)^T \end{aligned} \quad (4.10)$$

4.3 Social Interactions

The coexistence of any large community is a result of the balance between inter-species and intra-species interactions. Gradient forces are common tools for representing these social interactions [149]. In this regard attraction-repulsion models are frequently used in literature to explain these biological collective behaviors [150],[151].

Investigations on interference among conspecific individuals reveal that the surrounding environment for each individual can be divided to four distinct zones [152]. First is stress zone which individuals repelled by those who enter their personal space. Second is neutral zone which individuals do not respond to their neighbors. Third is attraction zone which is the result of predation avoidance, and the last zone is rear zone. On the basis of these natural findings, potential functions are proposed for intra-group interactions (which take place among group members), and decompose environment around each agent to four areas (figure 4.5). Separation area, in order to keep minimum distance between agents;

Figure 4.5: Surrounding environment of each agent shown by four distinct areas

Neutral area, which agents can choose a desired range with respect to each other in order to increase the flexibility of movement; Attractive area in order to keep the group unity during movement, also group aggregation after passing obstacles; Inactive area, which appears by increasing distance among agents and vanishing in attractive force.

The intra-group interactions take place between each agent identified by $I_{G_{ji}}$ with other in-group members which can be followers or local group leaders and are identified by I_{G_j} , are defined by the proposed interactive potential function as follows:

$$Q = \sum_{\substack{\nu \in \mathbb{N}_{I_{G_{ji}}} \\ \nu \in I_{G_j}}} \frac{k_{I_{G_{ji}}I_{G_j}}}{n} [\max(0, g(\Delta x_{I_{G_{ji}}\nu}))]^n \quad (4.11)$$

where $k_{I_{G_{ji}}I_{G_j}}$ are positive constants, $\mathbb{N}_{I_{G_{ji}}}$ refers to neighbors of agent $I_{G_{ji}}$, and $\Delta x_{I_{G_{ji}}\nu} = x_{I_{G_{ji}}} - x_\nu$; and the function $g(\cdot)$ is defined as follows

$$g(\Delta x_{I_{G_{ji}}\nu}) = \left(1 - e^{-a_1(\|\Delta x_{I_{G_{ji}}\nu}\|^2 - d_1^2)}\right) \left(1 - e^{-a_2(\|\Delta x_{I_{G_{ji}}\nu}\|^2 - d_2^2)}\right) \quad (4.12)$$

where a_1 , a_2 , d_1 and d_2 are positive constants. The function $[\max(0, g(\Delta x_{I_{G_{ji}}\nu}))]^n$ and potential function Q are illustrated in figure 4.6. The value of d_1 determines short range

Figure 4.6: Function $[\max(0, g(\Delta x_{I_{G_{ji}}\nu}))]^n$ and potential function Q .

repulsion and neutral area width can be determined by both d_1 and d_2 . The values of a_1 , a_2 and d_2 determine long range attraction. Parameters d_1 and d_2 can be defined on the basis of the physical properties of the agent sensors, such as communication or vision

range. From equation (4.12), it is obtained:

$$g(\Delta x_{I_{G_{ji}}\nu}) = g(\Delta x_{\nu I_{G_{ji}}}) \quad (4.13)$$

$$\frac{\partial g(\Delta x_{I_{G_{ji}}\nu})}{\partial \Delta x_{I_{G_{ji}}\nu}} = - \frac{\partial g(\Delta x_{\nu I_{G_{ji}}})}{\partial \Delta x_{\nu I_{G_{ji}}}} \quad (4.14)$$

The interaction force can be obtained by partial differentiation of the potential function as follows:

$$\Delta \rho_{I_{G_{ji}}-I_{G_j}} = \sum_{\substack{\nu \in \mathbb{N}_{I_{G_{ji}}} \\ \nu \in I_{G_j}}} k_{I_{G_{ji}}I_{G_j}} \underbrace{\left[\max \left(0, g(\Delta x_{I_{G_{ji}}\nu}) \right) \right]^{n-1} \left(\frac{\partial g(\Delta x_{I_{G_{ji}}\nu})}{\partial \Delta x_{I_{G_{ji}}\nu}} \right)^T}_{\Delta \bar{\rho}_{I_{G_{ji}}-\nu}} \quad (4.15)$$

Since $\frac{\partial g(\Delta x_{I_{G_{ji}}\nu})}{\partial \Delta x_{I_{G_{ji}}\nu}}$ go to zero as $\Delta x_{I_{G_{ji}}\nu}$ go to infinity, therefore, from (4.15), the intra-group interactive forces vanish as agents become far from each other. Moreover, as illustrated in figure 4.6, $\left[\max \left(0, g(\Delta x_{I_{G_{ji}}\nu}) \right) \right]^n$ is bounded. Hence, the following properties are obtained:

Property 4.1. *The intra-group interactive forces, $\left\| \Delta \rho_{I_{G_{ji}}-I_{G_j}} \right\|$, go to zero as $\left\| \Delta x_{I_{G_{ji}}\nu} \right\|$ go to infinity.*

Property 4.2. *The intra-group interactive forces, $\Delta \rho_{I_{G_{ji}}-I_{G_j}}$, are upper bounded, i.e. $\left\| \Delta \rho_{I_{G_{ji}}-I_{G_j}} \right\| < \sigma_1$, where the value of σ_1 depends on the parameters a_1 , a_2 , d_1 , and d_2 .*

On the other hand, another common social interaction which can be seen between heterospecifics is agonistic behaviors, such that individuals from different groups repel each other. To demonstrate the inter-group interactions, an inter-group interactive potential functions is proposed such that for each agent identified by $I_{G_{ji}}$ which belongs to group G_j and is interacting with members of another group $G_{\bar{j}}$ identified by $I_{G_{\bar{j}}}$, it is defined as

follows:

$$H = \sum_{\substack{\nu \in \mathbb{N}_{I_{G_{ji}}} \\ \nu \in I_{G_{ji}^-}}} \frac{k_{I_{G_{ji}} I_{G_{ji}^-}}}{n} [\max(0, h(\Delta x_{I_{G_{ji}} \nu}))]^n \quad (4.16)$$

where function $h(\cdot)$ is defined as follows:

$$h(\Delta x_{I_{G_{ji}} \nu}) = - \left(1 - e^{-a(\|\Delta x_{I_{G_{ji}} \nu}\|^2 - d^2)} \right) \quad (4.17)$$

where a and d are positive constants. The function $\left[\max(0, h(\Delta x_{I_{G_{ji}} \nu})) \right]^n$ and potential function H are illustrated in figure 4.7. In fact, the interactive potential energy function,

Figure 4.7: Function $\left[\max(0, h(\Delta x_{I_{G_{ji}} \nu})) \right]^n$ and potential function H .

H , maintains the minimum distance, d , between the agent $I_{G_{ji}}$ and members of the group G_{ji}^- . From equation (4.17), it is obtained:

$$h(\Delta x_{I_{G_{ji}} \nu}) = h(\Delta x_{\nu I_{G_{ji}}}) \quad (4.18)$$

$$\frac{\partial h(\Delta x_{I_{G_{ji}} \nu})}{\partial \Delta x_{I_{G_{ji}} \nu}} = - \frac{\partial h(\Delta x_{\nu I_{G_{ji}}})}{\partial \Delta x_{\nu I_{G_{ji}}}} \quad (4.19)$$

Therefore, the inter-group interactive force can be obtained similarly as follows:

$$\Delta\psi_{I_{G_{ji}}-I_{G_{\bar{j}}}} = \sum_{\substack{\nu \in \mathbb{N}_{I_{G_{ji}}} \\ \nu \in I_{G_{\bar{j}}}}} \underbrace{k_{I_{G_{ji}}I_{G_{\bar{j}}}} \left[\max \left(0, h(\Delta x_{I_{G_{ji}}\nu}) \right) \right]^{n-1} \left(\frac{\partial h(\Delta x_{I_{G_{ji}}\nu})}{\partial \Delta x_{I_{G_{ji}}\nu}} \right)^T}_{\Delta\bar{\psi}_{I_{G_{ji}}-\nu}} \quad (4.20)$$

As illustrated in figure 4.7, $\left[\max \left(0, h(\Delta x_{I_{G_{ji}}\nu}) \right) \right]^n$ is bounded, therefore, the following property is obtained:

Property 4.3. *The inter-group interactive forces, $\Delta\psi_{I_{G_{ji}}-I_{G_{\bar{j}}}}$, are upper bounded, i.e. $\left\| \Delta\psi_{I_{G_{ji}}-I_{G_{\bar{j}}}} \right\| < \sigma_2$, where the value of σ_2 depends on the parameters a and d .*

Remark 4.2. *The interactive forces (4.15) and (4.20) are smooth functions and at least belong to class $\mathcal{C}^1$.*

In order to deal with the stability of groups with different desired speeds, an adaptive interactive force is proposed. In our scenario this condition will happen during the transient movement of agents. In fact, there exists a condition where control forces align with interactive forces but with the opposite direction, and this causes agents to get stuck in a local minima; therefore the agents are not able to follow the desired speeds. A solution to this problem is updating the direction of interactive force in such a way that it releases the agent from local minima (as illustrated in figure 4.8). Hence, adaptive interactive force,

Figure 4.8: Adaptive interactive force

$\Delta\eta_{I_{G_{ji}}-I_{G_{\bar{j}}}}$, for each $I_{G_{ji}}$ and $I_{G_{\bar{j}}}$, is considered as the interactive force which can update

Formation control for local group leaders (Fl_1^1, Fl_2^1) to keep a formation with respect to the global leader and each other. 4. Shape formation for members of the follower groups (Fg_{1i}^1, Fg_{2i}^1) with respect to their corresponding local group leaders. In the following, the general formulation for multi-group coordination control is expressed in four steps.

4.4.1 Proposed controller for global leader

At first, a controller is introduced for the global leader in order to have it follow a specified path. Therefore, a reference vector $\dot{x}_{r_L}$ is defined as follows:

$$\dot{x}_{r_L} = \dot{x}_o - k\Delta x_L \quad (4.22)$$

where k is a positive constant, $\dot{x}_o$ is the velocity of the desired path of the global leader, and $\Delta x_L = x_L - x_o$. A sliding vector can be defined for the global leader as follows:

$$s_L = \dot{x}_L - \dot{x}_{r_L} = \Delta \dot{x}_L + k\Delta x_L \quad (4.23)$$

Substituting the sliding vector into the dynamic equation for the global leader described by (4.1) and applying linear parameterization, yields

$$\begin{aligned} M_L(x_L)\dot{s}_L + C_L(x_L, \dot{x}_L)s_L + D_L(x_L)s_L \\ + Y_L(x_L, \dot{x}_L, \dot{x}_{r_L}, \ddot{x}_{r_L})\theta_L = u_L \end{aligned} \quad (4.24)$$

where $Y_L(x_L, \dot{x}_L, \dot{x}_{r_L}, \ddot{x}_{r_L})$ is a known regressor matrix and θ_L is an unknown parameter vector, which is obtained from

$$\begin{aligned} M_L(x_L)\ddot{x}_{r_L} + C_L(x_L, \dot{x}_L)\dot{x}_{r_L} + D_L(x_L)\dot{x}_{r_L} \\ + g_L(x_L) = Y_L(x_L, \dot{x}_L, \dot{x}_{r_L}, \ddot{x}_{r_L})\theta_L \end{aligned} \quad (4.25)$$

The feedback control law for the global leader is expressed as follows:

$$u_L = -K_L s_L - k_{pL} \Delta x_L + Y_L(x_L, \dot{x}_L, \dot{x}_{rL}, \ddot{x}_{rL}) \hat{\theta}_L \quad (4.26)$$

where K_L is a positive definite matrix, k_{pL} is a positive constant and $\hat{\theta}_L$ is an estimation of the unknown θ_L . Moreover, the parameter update law for the global leader is introduced as

$$\dot{\hat{\theta}}_L = -L_L Y_L^T(x_L, \dot{x}_L, \dot{x}_{rL}, \ddot{x}_{rL}) s_L \quad (4.27)$$

where L_L is a positive definite matrix.

4.4.2 Proposed controller for members of leading group

After designation of the global leader, reference vectors are defined for the leading group members as follows:

$$\dot{x}_{rLg_i} = \dot{x}_L - \alpha_{Lg_i} \Delta \xi_{Lg_i} - \gamma_{Lg_i} \Delta Q_{Lg_i} \quad (4.28)$$

where α_{Lg_i} and γ_{Lg_i} are positive constants and $\Delta \xi_{Lg_i}$ are defined by equation (4.10); $\Delta Q_{Lg_i} = \Delta \rho_{Lg_i-L} + \Delta \rho_{Lg_i-Lg} + \Delta \psi_{Lg_i-F_{\mathfrak{N}_{Lg}}^1}$, where $F_{\mathfrak{N}_{Lg}}^1$ refers to those local leaders in first level of dominancy which are in the neighborhood of the leading group; therefore $\Delta \rho_{Lg_i-L}$ and $\Delta \rho_{Lg_i-Lg}$ are forces of interaction between leading group members and the global leader, and among leading group members, respectively, that can be computed by equations (4.15). $\Delta \psi_{Lg_i-F_{\mathfrak{N}_{Lg}}^1}$ are forces of interaction between leading group members and local group leaders in first level of dominancy, and can be computed by equations (4.20). Sliding vectors for the leading group members are defined as follows:

$$s_{Lg_i} = \dot{x}_{Lg_i} - \dot{x}_{rLg_i} = \Delta \dot{x}_{Lg_iL} + \Delta w_{Lg_i} \quad (4.29)$$

such that $\Delta w_{Lg_i} = \alpha_{Lg_i} \Delta \xi_{Lg_i} + \gamma_{Lg_i} \Delta Q_{Lg_i}$. Substituting the sliding vectors into the dynamic equations of leading group members described by (4.1) and applying linear parameterization, yields

$$\begin{aligned} M_{Lg_i}(x_{Lg_i})\dot{s}_{Lg_i} + C_{Lg_i}(x_{Lg_i}, \dot{x}_{Lg_i})s_{Lg_i} + D_{Lg_i}(x_{Lg_i})s_{Lg_i} \\ + Y_{Lg_i}(x_{Lg_i}, \dot{x}_{Lg_i}, \dot{x}_{r_{Lg_i}}, \ddot{x}_{r_{Lg_i}})\theta_{Lg_i} = u_{Lg_i} \end{aligned} \quad (4.30)$$

where $Y_{Lg_i}(x_{Lg_i}, \dot{x}_{Lg_i}, \dot{x}_{r_{Lg_i}}, \ddot{x}_{r_{Lg_i}})$ are known regressor matrices and θ_{Lg_i} are unknown parameter vectors, which can be defined as follows:

$$\begin{aligned} M_{Lg_i}(x_{Lg_i})\ddot{x}_{r_{Lg_i}} + C_{Lg_i}(x_{Lg_i}, \dot{x}_{Lg_i})\dot{x}_{r_{Lg_i}} + D_{Lg_i}(x_{Lg_i})\dot{x}_{r_{Lg_i}} \\ + g_{Lg_i}(x_{Lg_i}) = Y_{Lg_i}(x_{Lg_i}, \dot{x}_{Lg_i}, \dot{x}_{r_{Lg_i}}, \ddot{x}_{r_{Lg_i}})\theta_{Lg_i} \end{aligned} \quad (4.31)$$

The feedback control laws for the leading group members are defined as follows:

$$\begin{aligned} u_{Lg_i} = & -K_{Lg_i}s_{Lg_i} - k_{p_{Lg_i}} \left(\Delta w_{Lg_i} + \Delta \eta_{Lg_i - \mathcal{N}_{Fg}^{Lg}} \right) \\ & + Y_{Lg_i}(x_{Lg_i}, \dot{x}_{Lg_i}, \dot{x}_{r_{Lg_i}}, \ddot{x}_{r_{Lg_i}})\hat{\theta}_{Lg_i} \end{aligned} \quad (4.32)$$

where K_{Lg_i} are positive definite matrices, $k_{p_{Lg_i}}$ are positive constants and $\hat{\theta}_{Lg_i}$ are estimations of θ_{Lg_i} . $\mathcal{N}_{Fg}^{Lg}$ refers to follower groups which are in the neighborhood of the leading group, therefore $\Delta \eta_{Lg_i - \mathcal{N}_{Fg}^{Lg}}$ refers to interactions between the members of the leading group and members of neighboring follower groups, and can be computed by equation (4.21). The parameter update laws for the members of the leading group are introduced as

$$\dot{\hat{\theta}}_{Lg_i} = -L_{Lg_i} Y_{Lg_i}^T(x_{Lg_i}, \dot{x}_{Lg_i}, \dot{x}_{r_{Lg_i}}, \ddot{x}_{r_{Lg_i}})s_{Lg_i} \quad (4.33)$$

where L_{Lg_i} are positive definite matrices.

4.4.3 Proposed controller for local group leaders

For the local group leaders, reference vectors are proposed as follows:

$$\dot{x}_{r_{Fl_j^k}} = \dot{x}_{Fl_q^{k-1}} - \alpha_{Fl_j^k} \Delta \zeta_{Fl_j^k} - \gamma_{Fl_j^k} \Delta \psi_{Fl_j^k - Fl_q^{k-1}} \quad (4.34)$$

where $\alpha_{Fl_j^k}$ and $\gamma_{Fl_j^k}$ are positive constants; $\Delta \psi_{Fl_j^k - Fl_q^{k-1}}$ are forces of interaction between local group leaders in k^{th} level of dominancy and follower groups in the $(k-1)^{th}$ level of dominancy. For $k=1$, $\Delta \psi_{Fl_j^k - Fl_q^{k-1}}$ is substituted by $\Delta \psi_{Fl_j^k - Lg}$. Consequently, sliding vectors for the local group leaders are defined as

$$s_{Fl_j^k} = \dot{x}_{Fl_j^k} - \dot{x}_{r_{Fl_j^k}} = \Delta \dot{x}_{Fl_j^k Fl_q^{k-1}} + \Delta w_{Fl_j^k} \quad (4.35)$$

such that $\Delta w_{Fl_j^k} = \alpha_{Fl_j^k} \Delta \zeta_{Fl_j^k} + \gamma_{Fl_j^k} \Delta \psi_{Fl_j^k - Fl_q^{k-1}}$. Substituting the sliding vectors into the dynamic equations of the local group leaders described by (4.1) and applying linear parameterization, yields

$$\begin{aligned} M_{Fl_j^k}(x_{Fl_j^k}) \dot{s}_{Fl_j^k} + C_{Fl_j^k}(x_{Fl_j^k}, \dot{x}_{Fl_j^k}) s_{Fl_j^k} + D_{Fl_j^k}(x_{Fl_j^k}) s_{Fl_j^k} \\ + Y_{Fl_j^k}(x_{Fl_j^k}, \dot{x}_{Fl_j^k}, \dot{x}_{r_{Fl_j^k}}, \ddot{x}_{r_{Fl_j^k}}) \theta_{Fl_j^k} = u_{Fl_j^k} \end{aligned} \quad (4.36)$$

where $Y_{Fl_j^k}(x_{Fl_j^k}, \dot{x}_{Fl_j^k}, \dot{x}_{r_{Fl_j^k}}, \ddot{x}_{r_{Fl_j^k}})$ are known regressor matrices and $\theta_{Fl_j^k}$ are estimations of unknown parameter vectors, and can be represented as follows:

$$\begin{aligned} M_{Fl_j^k}(x_{Fl_j^k}) \ddot{x}_{r_{Fl_j^k}} + C_{Fl_j^k}(x_{Fl_j^k}, \dot{x}_{Fl_j^k}) \dot{x}_{r_{Fl_j^k}} + D_{Fl_j^k}(x_{Fl_j^k}) \dot{x}_{r_{Fl_j^k}} \\ + g_{Fl_j^k}(x_{Fl_j^k}) = Y_{Fl_j^k}(x_{Fl_j^k}, \dot{x}_{Fl_j^k}, \dot{x}_{r_{Fl_j^k}}, \ddot{x}_{r_{Fl_j^k}}) \theta_{Fl_j^k} \end{aligned} \quad (4.37)$$

The feedback control laws for the local group leaders are expressed as follows:

$$u_{Fl_j^k} = -K_{Fl_j^k} s_{Fl_j^k} - k_{p_{Fl_j^k}} \Delta w_{Fl_j^k} + Y_{Fl_j^k}(x_{Fl_j^k}, \dot{x}_{Fl_j^k}, \dot{x}_{r_{Fl_j^k}}, \ddot{x}_{r_{Fl_j^k}}) \hat{\theta}_{Fl_j^k} \quad (4.38)$$

such that $K_{Fl_j^k}$ are positive definite matrices, $k_{p_{Fl_j^k}}$ are positive constants, $Y_{Fl_j^k}(\cdot)$ are known regressor matrices and $\hat{\theta}_{Fl_j^k}$ are estimations of unknown parameter vectors $\theta_{Fl_j^k}$. The parameter update laws for the local group leaders are proposed as

$$\dot{\hat{\theta}}_{Fl_j^k} = -L_{Fl_j^k} Y_{Fl_j^k}^T (x_{Fl_j^k}, \dot{x}_{Fl_j^k}, \dot{x}_{r_{Fl_j^k}}, \ddot{x}_{r_{Fl_j^k}}) s_{Fl_j^k} \quad (4.39)$$

where $L_{Fl_j^k}$ are positive definite matrices.

4.4.4 Proposed controller for members of follower groups

Finally, reference vectors are defined for the members of the follower groups as follows:

$$\dot{x}_{r_{Fg_{ji}^k}} = \dot{x}_{Fl_j^k} - \alpha_{Fg_{ji}^k} \Delta \xi_{Fg_{ji}^k} - \gamma_{Fg_{ji}^k} \Delta Q_{Fg_{ji}^k} \quad (4.40)$$

where $\alpha_{Fg_{ji}^k}$ and $\gamma_{Fg_{ji}^k}$ are positive constants; $\Delta Q_{Fg_{ji}^k} = \Delta \rho_{Fg_{ji}^k - Fl_j^k} + \Delta \rho_{Fg_{ji}^k - Fg_j^k} + \Delta \psi_{Fg_{ji}^k - Fl_{\mathfrak{N}_{Fg_j^k}^{k+1}}}$; $Fl_{\mathfrak{N}_{Fg_j^k}^{k+1}}$ for $k = 1, 2, \dots, N - 1$ refer to those local leaders that follow local leader Fl_j^k and are in the neighborhood of the follower group Fg_j^k . Sliding vectors for the members of the follower groups are given as follows:

$$s_{Fg_{ji}^k} = \dot{x}_{Fg_{ji}^k} - \dot{x}_{r_{Fg_{ji}^k}} = \Delta \dot{x}_{Fg_{ji}^k Fl_j^k} + \Delta w_{Fg_{ji}^k} \quad (4.41)$$

such that $\Delta w_{Fg_{ji}^k} = \alpha_{Fg_{ji}^k} \Delta \xi_{Fg_{ji}^k} + \gamma_{Fg_{ji}^k} \Delta Q_{Fg_{ji}^k}$. Substituting the sliding vectors into the dynamic equation of the members of the follower groups described by (4.1) and applying linear parameterization, yields

$$\begin{aligned} M_{Fg_{ji}^k}(x_{Fg_{ji}^k}) \dot{s}_{Fg_{ji}^k} + C_{Fg_{ji}^k}(x_{Fg_{ji}^k}, \dot{x}_{Fg_{ji}^k}) s_{Fg_{ji}^k} + D_{Fg_{ji}^k}(x_{Fg_{ji}^k}) s_{Fg_{ji}^k} \\ + Y_{Fg_{ji}^k}(x_{Fg_{ji}^k}, \dot{x}_{Fg_{ji}^k}, \dot{x}_{r_{Fg_{ji}^k}}, \ddot{x}_{r_{Fg_{ji}^k}}) \theta_{Fg_{ji}^k} = u_{Fg_{ji}^k} \end{aligned} \quad (4.42)$$

where $Y_{Fg_{ji}^k}(x_{Fg_{ji}^k}, \dot{x}_{Fg_{ji}^k}, \dot{x}_{r_{Fg_{ji}^k}}, \ddot{x}_{r_{Fg_{ji}^k}})$ are known regressor matrices and $\theta_{Fg_{ji}^k}$ are unknown parameter vectors, which can be defined as follows:

$$\begin{aligned} M_{Fg_{ji}^k}(x_{Fg_{ji}^k})\ddot{x}_{r_{Fg_{ji}^k}} + C_{Fg_{ji}^k}(x_{Fg_{ji}^k}, \dot{x}_{Fg_{ji}^k})\dot{x}_{r_{Fg_{ji}^k}} + D_{Fg_{ji}^k}(x_{Fg_{ji}^k})\dot{x}_{r_{Fg_{ji}^k}} \\ + g_{Fg_{ji}^k}(x_{Fg_{ji}^k}) = Y_{Fg_{ji}^k}(x_{Fg_{ji}^k}, \dot{x}_{Fg_{ji}^k}, \dot{x}_{r_{Fg_{ji}^k}}, \ddot{x}_{r_{Fg_{ji}^k}})\theta_{Fg_{ji}^k} \end{aligned} \quad (4.43)$$

The feedback control laws for the members of the follower groups are expressed as follows:

$$\begin{aligned} u_{Fg_{ji}^k} = & -K_{Fg_{ji}^k} s_{Fg_{ji}^k} - k_{p_{Fg_{ji}^k}} \left(\Delta w_{Fg_{ji}^k} + \Delta T_{Fg_{ji}^k} \right) \\ & + Y_{Fg_{ji}^k}(x_{Fg_{ji}^k}, \dot{x}_{Fg_{ji}^k}, \dot{x}_{r_{Fg_{ji}^k}}, \ddot{x}_{r_{Fg_{ji}^k}}) \hat{\theta}_{Fg_{ji}^k} \end{aligned} \quad (4.44)$$

such that $K_{Fg_{ji}^k}$ are positive matrices and $k_{p_{Fg_{ji}^k}}$ are positive constants; and

$$\Delta T_{Fg_{ji}^k} = \gamma_{Fg_{ji}^k} \left(\Delta \eta_{Fg_{ji}^k - \mathcal{N}_{Fl}^{Fg_j^k}} + \Delta \eta_{Fg_{ji}^k - \mathcal{N}_{Fg}^{Fg_j^k}} \right) \quad (4.45)$$

where $\mathcal{N}_{Fl}^{Fg_j^k}$ and $\mathcal{N}_{Fg}^{Fg_j^k}$ refer to those local leaders (except $Fl_{\mathfrak{N}_{Fg_j^k}}^{k+1}$) and follower groups which are in the neighborhood of follower group Fg_j^k , respectively. Therefore $\Delta \eta_{Fg_{ji}^k - \mathcal{N}_{Fl}^{Fg_j^k}}$ refers to interactions between members of group Fg_j^k and adjacent local group leaders, and $\Delta \eta_{Fg_{ji}^k - \mathcal{N}_{Fg}^{Fg_j^k}}$ refers to interactions between members of group Fg_j^k and members of adjacent follower groups. The estimated parameters $\hat{\theta}_{Fg_{ji}^k}$ are updated as follows:

$$\dot{\hat{\theta}}_{Fg_{ji}^k} = -L_{Fg_{ji}^k} Y_{Fg_{ji}^k}^T (x_{Fg_{ji}^k}, \dot{x}_{Fg_{ji}^k}, \dot{x}_{r_{Fg_{ji}^k}}, \ddot{x}_{r_{Fg_{ji}^k}}) s_{Fg_{ji}^k} \quad (4.46)$$

where $L_{Fg_{ji}^k}$ are positive matrices. Substituting the feedback control laws described by (4.26), (4.32), (4.38) and (4.44) into (4.24), (4.30), (4.36) and (4.42), respectively, the closed-loop equations are obtained as

$$\left\{ \begin{array}{l} \text{Leading group} \\ \text{Follower group } j \text{ of } k^{th} \text{ level} \end{array} \right\} \left\{ \begin{array}{l} L : M_L(x_L)\dot{s}_L + C_L(x_L, \dot{x}_L)s_L + D_L(x_L)s_L + K_Ls_L + k_{p_L}\Delta x_L \\ \quad + Y_L(x_L, \dot{x}_L, \dot{x}_{r_L}, \ddot{x}_{r_L})\Delta\theta_L = 0 \\ Lg : M_{Lg_i}(x_{Lg_i})\dot{s}_{Lg_i} + C_{Lg_i}(x_{Lg_i}, \dot{x}_{Lg_i})s_{Lg_i} + D_{Lg_i}(x_{Lg_i})s_{Lg_i} + K_{Lg_i}s_{Lg_i} \\ \quad + k_{p_{Lg_i}} \left(\Delta w_{Lg_i} + \Delta \eta_{Lg_i - \mathcal{N}_{Fg}^{Lg}} \right) + Y_{Lg_i}(x_{Lg_i}, \dot{x}_{Lg_i}, \dot{x}_{r_{Lg_i}}, \ddot{x}_{r_{Lg_i}})\Delta\theta_{Lg_i} = 0 \\ Fl_j^k : M_{Fl_j^k}(x_{Fl_j^k})\dot{s}_{Fl_j^k} + C_{Fl_j^k}(x_{Fl_j^k}, \dot{x}_{Fl_j^k})s_{Fl_j^k} + D_{Fl_j^k}(x_{Fl_j^k})s_{Fl_j^k} + K_{Fl_j^k}s_{Fl_j^k} \\ \quad + k_{p_{Fl_j^k}} \Delta w_{Fl_j^k} + Y_{Fl_j^k}(x_{Fl_j^k}, \dot{x}_{Fl_j^k}, \dot{x}_{r_{Fl_j^k}}, \ddot{x}_{r_{Fl_j^k}})\Delta\theta_{Fl_j^k} = 0 \\ Fg_j^k : M_{Fg_j^k}(x_{Fg_j^k})\dot{s}_{Fg_j^k} + C_{Fg_j^k}(x_{Fg_j^k}, \dot{x}_{Fg_j^k})s_{Fg_j^k} + D_{Fg_j^k}(x_{Fg_j^k})s_{Fg_j^k} + K_{Fg_j^k}s_{Fg_j^k} \\ \quad + k_{p_{Fg_j^k}} \left(\Delta w_{Fg_j^k} + \Delta T_{Fg_j^k} \right) + Y_{Fg_j^k}(x_{Fg_j^k}, \dot{x}_{Fg_j^k}, \dot{x}_{r_{Fg_j^k}}, \ddot{x}_{r_{Fg_j^k}})\Delta\theta_{Fg_j^k} = 0 \end{array} \right. \quad (4.47)$$

where $\Delta\hat{\theta}_L = \theta_L - \hat{\theta}_L$, $\Delta\hat{\theta}_{Lg_i} = \theta_{Lg_i} - \hat{\theta}_{Lg_i}$, $\Delta\hat{\theta}_{Fl_j^k} = \theta_{Fl_j^k} - \hat{\theta}_{Fl_j^k}$, $\Delta\hat{\theta}_{Fg_j^k} = \theta_{Fg_j^k} - \hat{\theta}_{Fg_j^k}$.

For stability analysis the following Lyapunov-like candidate is proposed:

$$V_T = V + U \quad (4.48)$$

where V and U are defined as follows:

$$V = V_L + \sum_{i=1}^{n_{Lg}} V_{Lg_i} + \sum_{k=1}^N \sum_{j=1}^{N_k} \left\{ V_{Fl_j^k} + \sum_{i=1}^{n_{Fg_j^k}} V_{Fg_{ji}^k} \right\} \quad (4.49)$$

$$U = \sum_{i=1}^{n_{Lg}} U_{Lg_i} + \sum_{k=1}^N \sum_{j=1}^{N_k} \left\{ U_{Fl_j^k} + \sum_{i=1}^{n_{Fg_j^k}} U_{Fg_{ji}^k} \right\} \quad (4.50)$$

such that

$$\begin{aligned}
V_L &= \frac{1}{2} s_L^T M_L(x_L) s_L + \frac{1}{2} \Delta \theta_L^T L_L^{-1} \Delta \theta_L + \frac{1}{2} k_{pL} \Delta x_L^T \Delta x_L \\
V_{Lg_i} &= \frac{1}{2} s_{Lg_i}^T M_{Lg_i}(x_{Lg_i}) s_{Lg_i} + \frac{1}{2} \Delta \theta_{Lg_i}^T L_{Lg_i}^{-1} \Delta \theta_{Lg_i} \\
V_{Fl_j^k} &= \frac{1}{2} s_{Fl_j^k}^T M_{Fl_j^k}(x_{Fl_j^k}) s_{Fl_j^k} + \frac{1}{2} \Delta \theta_{Fl_j^k}^T L_{Fl_j^k}^{-1} \Delta \theta_{Fl_j^k} \\
V_{Fg_{ji}^k} &= \frac{1}{2} s_{Fg_{ji}^k}^T M_{Fg_{ji}^k}(x_{Fg_{ji}^k}) s_{Fg_{ji}^k} + \frac{1}{2} \Delta \theta_{Fg_{ji}^k}^T L_{Fg_{ji}^k}^{-1} \Delta \theta_{Fg_{ji}^k}
\end{aligned} \tag{4.51}$$

where V_L denotes the energy function for the global leader, V_{Lg_i} denote the energy functions for the members of the leading group, $V_{Fl_j^k}$ denote the energy functions for the local leaders, and $V_{Fg_{ji}^k}$ denote the energy functions for the members of the follower groups. Moreover, U_{Lg_i} , $U_{Fl_j^k}$ and $U_{Fg_{ji}^k}$ are related to formation potentials and interaction potentials, and are defined as follows:

$$\begin{aligned}
U_{Lg_i} &= \alpha_{Lg_i} k_{pLg_i} \sum_{m=1}^{Z_{Lg}} \frac{k_{Lg_i}}{n} [\max(0, f_{Lg}^m(\Delta x_{Lg_iL}))]^n \\
&\quad + \frac{\gamma_{Lg_i} k_{pLg_i} k_{Lg_iL}}{n} [\max(0, g(\Delta x_{Lg_iL}))]^n \\
&\quad + \frac{1}{2} \gamma_{Lg_i} k_{pLg_i} \sum_{\substack{\nu \in \mathbb{N}_{Lg_i} \\ \nu \in Lg}} \frac{1}{n} k_{Lg_iLg} [\max(0, g(\Delta x_{Lg_i\nu}))]^n \\
&\quad + \frac{1}{2} \gamma_{Lg_i} k_{pLg_i} \sum_{\substack{\nu \in \mathbb{N}_{Lg_i} \\ \nu \in Fl_{\mathfrak{N}_{Lg}}^1}} \frac{1}{n} k_{Lg_iFl_{\mathfrak{N}_{Lg}}^1} [\max(0, h(\Delta x_{Lg_i\nu}))]^n
\end{aligned} \tag{4.52}$$

where the first term refers to the shape formation potential energy, and the other terms represent interactive potential energies;

$$\begin{aligned}
U_{Fl_j^k} = & \alpha_{Fl_j^k} k_{p_{Fl_j^k}} \sum_{\ell_j^k=1}^{z_{Fl_j^k}} \frac{k_{Fl_j^k}}{n} \left[\max \left(0, f_{Fl_j^k}^{\ell_j^k} \left(\Delta x_{Fl_j^k Fl_q^{k-1}} \right) \right) \right]^n \\
& + \frac{1}{2} \alpha_{Fl_j^k} k_{p_{Fl_j^k}} \sum_{\ell_j^k=z_{Fl_j^k}+1}^{z_{Fl_j^k}} \frac{k_{Fl_j^k}}{n} \left[\max \left(0, f_{Fl_j^k}^{\ell_j^k} \left(\Delta x_{Fl_j^k Fl_j^k} \right) \right) \right]^n \\
& + \frac{1}{2} \gamma_{Fl_j^k} k_{p_{Fl_j^k}} \sum_{\substack{\nu \in \mathbb{N}_{Fl_j^k} \\ \nu \in Fl_q^{k-1}}} \frac{1}{n} k_{Fl_j^k Fl_q^{k-1}} \left[\max \left(0, h(\Delta x_{Fl_j^k \nu}) \right) \right]^n
\end{aligned} \tag{4.53}$$

such that the first and second terms stand for formation potential energies, and the third term represents the interactive potential energy;

$$\begin{aligned}
U_{Fg_{ji}^k} = & \alpha_{Fg_{ji}^k} k_{p_{Fg_{ji}^k}} \sum_{m_j^k=1}^{z_{Fg_{ji}^k}} \frac{k_{Fg_{ji}^k}}{n} \left[\max \left(0, f_{Fg_{ji}^k}^{m_j^k} \left(\Delta x_{Fg_{ji}^k Fl_j^k} \right) \right) \right]^n \\
& + \frac{1}{n} \gamma_{Fg_{ji}^k} k_{p_{Fg_{ji}^k}} k_{Fg_{ji}^k Fl_j^k} \left[\max \left(0, g(\Delta x_{Fg_{ji}^k Fl_j^k}) \right) \right]^n \\
& + \frac{1}{2} \gamma_{Fg_{ji}^k} k_{p_{Fg_{ji}^k}} \sum_{\substack{\nu \in \mathbb{N}_{Fg_{ji}^k} \\ \nu \in Fl_j^k}} \frac{1}{n} k_{Fg_{ji}^k Fl_j^k} \left[\max \left(0, g(\Delta x_{Fg_{ji}^k \nu}) \right) \right]^n \\
& + \frac{1}{2} \gamma_{Fg_{ji}^k} k_{p_{Fg_{ji}^k}} \sum_{\substack{\nu \in \mathbb{N}_{Fg_{ji}^k} \\ \nu \in Fl_{\mathfrak{N}_{Fg_{ji}^k}}^{k+1}}} \frac{1}{n} k_{Fg_{ji}^k Fl_{\mathfrak{N}_{Fg_{ji}^k}}^{k+1}} \left[\max \left(0, h(\Delta x_{Fg_{ji}^k \nu}) \right) \right]^n
\end{aligned} \tag{4.54}$$

such that the first term stands for potential energy related to the shape formation, and the other terms represent interactive potential energies. By differentiating the Lyapunov-like candidate with respect to time, substituting the closed-loop equations described by (4.47) and update laws (4.27), (4.33), (4.39) and (4.46) into $\dot{V}$, and also using the bidirectional

property of the interaction forces, $\dot{V}_T$ are obtained as follows:

$$\begin{aligned}
\dot{V}_T = & -s_L^T K_L s_L - s_L^T D_L(x_L) s_L - \Delta x_L^T k_{p_L} \Delta x_L \\
& - \sum_{i=1}^{n_{Lg}} \left\{ s_{Lg_i}^T K_{Lg_i} s_{Lg_i} + s_{Lg_i}^T D_{Lg_i}(x_{Lg_i}) s_{Lg_i} + \Delta w_{Lg_i}^T k_{p_{Lg_i}} \Delta w_{Lg_i} \right. \\
& + s_{Lg_i}^T k_{p_{Lg_i}} \Delta \eta_{Lg_i - \mathcal{N}_{Fg}^{Lg}} \left. \right\} - \sum_{k=1}^N \sum_{j=1}^{N_k} \left\{ s_{Fl_j^k}^T K_{Fl_j^k} s_{Fl_j^k} \right. \\
& + s_{Fl_j^k}^T D_{Fl_j^k}(x_{Fl_j^k}) s_{Fl_j^k} + \Delta w_{Fl_j^k}^T k_{p_{Fl_j^k}} \Delta w_{Fl_j^k} \\
& + \sum_{i=1}^{n_{Fg_{ji}^k}} \left\{ s_{Fg_{ji}^k}^T K_{Fg_{ji}^k} s_{Fg_{ji}^k} + s_{Fg_{ji}^k}^T D_{Fg_{ji}^k}(x_{Fg_{ji}^k}) s_{Fg_{ji}^k} \right. \\
& \left. \left. + \Delta w_{Fg_{ji}^k}^T k_{p_{Fg_{ji}^k}} \Delta w_{Fg_{ji}^k} + s_{Fg_{ji}^k}^T k_{p_{Fg_{ji}^k}} \Delta T_{Fg_{ji}^k} \right\} \right\} \quad (4.55)
\end{aligned}$$

Next, the inter-group interaction terms are defined, $\Delta \eta_{Lg_i - \mathcal{N}_{Fg}^{Lg}}$ and $\Delta T_{Fg_{ji}^k}$, so that

$$\sum_{i=1}^{n_{Lg}} s_{Lg_i}^T k_{p_{Lg_i}} \Delta \eta_{Lg_i - \mathcal{N}_{Fg}^{Lg}} + \sum_{k=1}^N \sum_{j=1}^{N_k} \sum_{i=1}^{n_{Fg_{ji}^k}} s_{Fg_{ji}^k}^T k_{p_{Fg_{ji}^k}} \Delta T_{Fg_{ji}^k} = 0 \quad (4.56)$$

In this regard, the adaptive interactive forces (see (4.21)) are updated between each member of the leading groups, Lg_i , and its out-group neighbors ν as follows:

$$\Delta \eta_{Lg_i - \mathcal{N}_{Fg}^{Lg}} = \sum_{\substack{\nu \in \mathbb{N}_{Lg_i} \\ \nu \in \mathcal{N}_{Fg}^{Lg}}} \mathcal{R}(\varphi_{Lg_i \nu}) \Delta \bar{\psi}_{Lg_i - \nu} \quad (4.57)$$

where $\mathcal{R}(\varphi_{Lg_i \nu})$ are rotation matrices and $\Delta \bar{\psi}_{Lg_i - \nu}$ are introduced by (4.20). The rotation

matrices for the members of the leading group are defined such that

$$\begin{aligned} & \sum_{i=1}^{n_{Lg}} s_{Lg_i}^T k_{p_{Lg_i}} \sum_{\substack{\nu \in \mathbb{N}_{Lg_i} \\ \nu \in \mathcal{N}_{Fg}^{Lg}}} \mathcal{R}(\varphi_{Lg_i \nu}) \Delta \bar{\psi}_{Lg_i - \nu} \\ & + \sum_{k=1}^N \sum_{j=1}^{N_k} \sum_{i=1}^{n_{Fg_{ji}^k}} s_{Fg_{ji}^k}^T k_{p_{Fg_{ji}^k}} \sum_{\substack{\nu \in \mathbb{N}_{Fg_{ji}^k} \\ \nu \in Lg}} \mathcal{R}(\varphi_{Fg_{ji}^k \nu}) \Delta \bar{\psi}_{Fg_{ji}^k - \nu} = 0 \quad (4.58) \end{aligned}$$

For each member of the leading group Lg_i in a neighborhood of a member of the follower group Fg_{ji}^k , (4.58) can be simplified to the following equation:

$$y^T \mathcal{R}(\varphi_{Lg_i Fg_{ji}^k}) \Delta \bar{\psi}_{Lg_i - Fg_{ji}^k} = 0 \quad (4.59)$$

where $y = k_{p_{Lg_i}} s_{Lg_i} - k_{p_{Fg_{ji}^k}} s_{Fg_{ji}^k}$. Rotation matrices are defined such that equation (4.59) is satisfied. In fact, the rotation matrices are updated so that the interactive forces that maintain the minimum distance between agents are perpendicular to the sliding vectors. When y is zero, no updates in the rotation matrices are needed since equation (4.59) is satisfied for any $\mathcal{R}(\varphi_{Lg_i Fg_{ji}^k})$; hence previous value of $\varphi_{Lg_i Fg_{ji}^k}$ can be used.

Now, the following theorem on the convergence of the signals is stated:

Theorem 4.1. *The leader follower groups system represented by (4.1) with the adaptive control schemes (4.26), (4.32), (4.38) and (4.44) and the parameter update laws (4.27), (4.33), (4.39) and (4.46) results in the convergence of $\Delta x_L \rightarrow 0$, $\Delta \dot{x}_L \rightarrow 0$, $\Delta \dot{x}_{Lg_i L} \rightarrow 0$, $\Delta w_{Lg_i} \rightarrow 0$, $\Delta \dot{x}_{Fl_j^k Fl_q^{k-1}} \rightarrow 0$, $\Delta w_{Fl_j^k} \rightarrow 0$, $\Delta \dot{x}_{Fg_{ji}^k Fl_j^k} \rightarrow 0$ and $\Delta w_{Fg_{ji}^k} \rightarrow 0$ for the global leader, leading group, local group leaders and members of follower groups, respectively, as $t \rightarrow \infty$.*

Proof. Since inertia matrices $M_L(x_L)$, $M_{Lg_i}(x_{Lg_i})$, $M_{Fl_j^k}(x_{Fl_j^k})$ and $M_{Fg_{ji}^k}(x_{Fg_{ji}^k})$ are positive definite, and moreover, L_L , L_{Lg_i} , $L_{Fl_j^k}$ and $L_{Fg_{ji}^k}$ are positive definite matrices, V_T is positive definite in s_L , $\Delta \theta_L$, s_{Lg_i} , $\Delta \theta_{Lg_i}$, $s_{Fl_j^k}$, $\Delta \theta_{Fl_j^k}$, $s_{Fg_{ji}^k}$ and $\Delta \theta_{Fg_{ji}^k}$. Hence, V_T is

bounded, which asserts the boundedness of all sliding vectors, formation functions, shape functions and interaction potentials. Therefore, one can conclude that Δw_{Lg_i} , $\Delta w_{Fl_j^k}$ and $\Delta w_{Fg_{ji}^k}$ are bounded. By considering reference vector equations, sliding vector equations and the boundedness of $\dot{x}_o$, one can reach to the boundedness of $\dot{x}_{r_L}$ and $\Delta \dot{x}_L$, and consequently, it is concluded that $\dot{x}_{r_{Lg_i}}$, $\Delta \dot{x}_{Lg_i}$, $\dot{x}_{r_{Fl_j^k}}$, $\Delta \dot{x}_{Fl_j^k Fl_q^{k-1}}$, $\dot{x}_{r_{Fg_{ji}^k}}$ and $\Delta \dot{x}_{Fg_{ji}^k Fl_j^k}$ are bounded. Hence, equation (4.47) shows the boundedness of all $\dot{s}_L$, $\dot{s}_{Lg_i}$, $\dot{s}_{Fl_j^k}$ and $\dot{s}_{Fg_{ji}^k}$. Therefore $\ddot{V}_T$ is bounded and this ensures that $\ddot{V}_T$ is uniformly continuous. By invoking Barbalat's lemma, it is obtained that, $\dot{V}_T \rightarrow 0$ as $t \rightarrow \infty$. Subsequently, one can conclude that $\Delta x_L \rightarrow 0$, $\Delta \dot{x}_L \rightarrow 0$, $\Delta w_{Lg_i} \rightarrow 0$, $\Delta \dot{x}_{Lg_i L} \rightarrow 0$, $\Delta \dot{x}_{Fl_j^k Fl_q^{k-1}} \rightarrow 0$, $\Delta w_{Fl_j^k} \rightarrow 0$, $\Delta \dot{x}_{Fg_{ji}^k Fl_j^k} \rightarrow 0$ and $\Delta w_{Fg_{ji}^k} \rightarrow 0$ as $t \rightarrow \infty$. $\square$

Summing the signals Δw_{Lg_i} , $\Delta w_{Fl_j^k}$ and $\Delta w_{Fg_{ji}^k}$ over all agents and considering the bidirectionality of the interaction forces, yields

$$\sum_{i=1}^{n_{Lg}} \alpha_{Lg_i} \Delta \xi_{Lg_i} + \sum_{k=1}^N \sum_{j=1}^{N_k} \left\{ \alpha_{Fl_j^k} \Delta \zeta_{Fl_j^k} + \sum_{i=1}^{n_{Fg_{ji}^k}} \alpha_{Fg_{ji}^k} \Delta \xi_{Fg_{ji}^k} \right\} = 0 \quad (4.60)$$

Equation (4.60) represents the overall group behavior of the multi-group system in the steady state. One solution of (4.60) is that formation/region errors are equal to zero. This can be easily guaranteed if the agents are initially located inside the regions or at one side of the regions. In the case where the agents are located at the opposite side of the regions, the control gains should be adjusted such that a higher weighting is given to the region errors with respect to the interaction forces. That is, the overall group behavior can be obtained by adjusting the control gains.

4.5 SIMULATION

In this section, two examples are presented to illustrate the performance of the proposed method. The simulations are conducted by Matlab. Large numbers of agents are chosen for the simulations to represent the scalability of the proposed method. Initially, the agents are randomly scattered within the environment. The snapshots of the simulations are represented in figures 4.10 and 4.11, which illustrate initial positions, transient response and steady state movement of the agents. The agents are shown by solid circles. It should be noted that figures have different axes limits with respect to each other in order to better display the agents positions. The interspace agents are considered as a coefficient of the size of agents. The control gain constants used in the numerical simulation are $k = 20$, $K_L = K_{Fl_j^k} = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$, $K_{Lg_i} = K_{Fg_{ji}^k} = \begin{bmatrix} 60 & 0 \\ 0 & 60 \end{bmatrix}$, $k_{vv} = 1$, $\gamma_L = \gamma_{Lg_i} = \gamma_{Fl_j^k} = \gamma_{Fg_{ji}^k} = 500$, $k_{pL} = k_{p_{Fl_j^k}} = 1$ and $k_{p_{Lg_i}} = k_{p_{Fg_{ji}^k}} = 10$.

In the first simulation, a six-group agent system consists of 140 agents is considered. Initial positions were generated randomly for all agents as shown in figure 4.10(a). The swarm is decomposed to a leading group and 5 follower groups assigned to form a plane-like shape. The follower groups is divided into two level of dominancy which the first level contains three groups and the second level contains two groups. The global leader is represented by L , the local group leaders in the first level of dominancy are represented by Fl_1^1 , Fl_2^1 and Fl_3^1 , and the local group leaders in the second level of dominancy are represented by Fl_1^2 and Fl_2^2 . Figures 4.10(b)-4.10(d) depict snapshots of the swarm movement within a time frame of 99 simulation seconds, such that the corresponding time instant is given below each figure. The desired path for the global leader is represented by dash-dot line. The solid lines denote the desired shape of each group, and the dashed lines denote the linkages among group leaders. The desired path for the global leader is $x_{o1} = 1.75\sin(0.25t)$, $x_{o2} = 0.25t$ where t represents time in second.

In the second simulation, an eleven-group agent system consists of 11 leaders and 300 followers is considered, which decomposes to a leading group and 10 follower groups. The swarm is divided into seven level of dominancy which the first level contains one group, the second level contains three groups, the third level contains one group, the fourth level contains two groups and each of the rest level contain one group. To illustrate the performance of the proposed method in compound pattern formation, the swarm forms a Chinese character with calligraphic strokes. Initial positions were generated randomly for all agents as shown in figure 4.11(a). Figures 4.11(b)-4.11(f) depict snapshots of the swarm movement within a time frame of 81 simulation seconds, such that the corresponding time instant is given below each figure. The desired path for the global leader is represented by dash-dot line. The solid lines denote the desired shape of each group, and the dashed lines denote the linkages among group leaders. The desired path for the global leader is $x_{o1} = 1.75\sin(0.25t)$, $x_{o2} = 0.25t$ where t represents time in second.

It is observable that any complex pattern formation is achievable by coordinating local group leaders as well as properly choosing simple shapes for leader and follower groups.

Figure 4.10: A six-group system forms a plane-shape pattern.

Figure 4.11: An eleven-group system forms a Chinese character with calligraphic strokes.

Chapter 5

Distributed Dynamic Multi-Group Formation of Multi-Agent Systems

In the previous chapter, the idea of multi-group coordination was proposed. However, the results are limited to fixed formations. In some applications, a dynamic formation is required for the multi-agent system to change dynamically to suit the environment. Moreover, communications among agents were not taken into consideration. This chapter presents a distributed dynamic multi-group formation control to extend the results in the previous chapter from two aspects: (i) dynamic complex formations can be obtained for large-scale multi agent systems. (ii) distributed estimators are presented to consider the information exchange among agents. To obtain the distributed dynamic multi-group formation, a global virtual leader that defines the overall movement, and local virtual leaders that define the local group movement, are introduced. Therefore, the distributed dynamic multi-group formation problem is formulated into two parts: (i) specifying the global group behavior, and (ii) specifying the local group behavior. In the global group behavior, the global virtual leader and the local virtual leaders form a dynamic formation based on the directed communication topology. Hence, the multi-group reconfiguration can be

obtained. In the local group behavior, the members of each group form a dynamic shape formation with respect to their local virtual leader based on the directed communication topology. Hence, asynchronous shape transformation can be obtained. In this chapter, the asynchronous shape transformation is defined as an independent shape formation control of each group, therefore local transformation can be achieved.

5.1 Dynamic Equations of the Multi-Group Agent System

Consider N groups of agents which each group has n_k members. The dynamic equations of the system can be represented as follows:

$$\begin{cases} G_1 : M_{1i}(x_{1i})\ddot{x}_{1i} + C_{1i}(x_{1i}, \dot{x}_{1i})\dot{x}_{1i} + D_{1i}(x_{1i})\dot{x}_{1i} + g_{1i}(x_{1i}) = u_{1i} \\ G_2 : M_{2i}(x_{2i})\ddot{x}_{2i} + C_{2i}(x_{2i}, \dot{x}_{2i})\dot{x}_{2i} + D_{2i}(x_{2i})\dot{x}_{2i} + g_{2i}(x_{1i}) = u_{2i} \\ \vdots \\ G_N : M_{Ni}(x_{Ni})\ddot{x}_{Ni} + C_{Ni}(x_{Ni}, \dot{x}_{Ni})\dot{x}_{Ni} + D_{Ni}(x_{Ni})\dot{x}_{Ni} + g_{Ni}(x_{Ni}) = u_{Ni} \end{cases} \quad (5.1)$$

where G_k represent group k , the subscript ki refers to member i of group k , $k = 1, 2, \dots, N$, $i = 1, 2, \dots, n_k$, $x_{ki} \in \mathbb{R}^n$ are generalized coordinates, $M_{ki}(x_{ki}) \in \mathbb{R}^{n \times n}$ are symmetric and positive definite inertia matrices, $C_{ki}(x_{ki}, \dot{x}_{ki}) \in \mathbb{R}^{n \times n}$ are matrices of Coriolis and centripetal terms which together with inertia matrices satisfy skew-symmetric property, $D_{ki}(x_{ki}) \in \mathbb{R}^{n \times n}$ represent the positive definite damping matrices, $g_{ki}(x_{ki}) \in \mathbb{R}^n$ denote gravitational force vectors, and $u_{ki} \in \mathbb{R}^n$ denote the control inputs.

By linear parameterization, the dynamic equation of each agent can be written as [144]

$$M_{ki}(x_{ki})\ddot{x}_{ki} + C_{ki}(x_{ki}, \dot{x}_{ki})\dot{x}_{ki} + D_{ki}(x_{ki})\dot{x}_{ki} + g_{ki}(x_{ki})$$

$$= Y_{ki}(x_{ki}, \dot{x}_{ki}, \ddot{x}_{ki})\theta_{ki} \quad (5.2)$$

where $Y_{ki}(x_{ki}, \dot{x}_{ki}, \ddot{x}_{ki})$ are known regressor matrices and θ_{ki} are unknown parameter vectors.

5.2 Problem Description

Dynamic multi-group formation has several advantages including group reconfiguration and local transformation which enhance the flexibility of swarm during movement in uncertain environments. An application of dynamic multi-group formation is illustrated in figure 5.1, where a swarm of agents by reconfiguration of overall formation and local transformation for middle groups can easily pass the obstacles. Moreover, dynamic complex formation can be obtained as illustrated in figure 5.2.

However, due to limitations in communication bandwidth and range it is reasonable to assume local information exchange among agents. Hence, the distributed techniques are applied to achieve distributed dynamic multi-group formation. To obtain the distributed dynamic multi-group formation, a global virtual leader and local virtual leaders are introduced. The global virtual leader performs the overall planning and local virtual leaders obtain their desired information from the global virtual leader in a distributed way. Therefore, the problem is formulated into two parts: global group behavior and local group behavior. In the global group behavior, overall configuration of swarm is defined. In the local group behavior, time-varying shape formation of each group is defined. An example of a distributed three-group formation is illustrated in figure 5.3.

The virtual leaders [153] are imaginary agents who define the global desired path and the desired reference of each group. The virtual leaders can be defined in various ways, for example a member of a group can be assigned to have information about the desired

Figure 5.1: An application of dynamic multi-group formation in dealing with obstacle-based environment.

reference. Hence, the specified agent which is called semi-leader agent plays a dual role as a leader and member. As illustrated in figure 5.4, the semi-leader agent shown in different appearance, has information about the desired reference. Therefore, at the same time, it transfers information about itself and the desired reference to the other agents. Another way is using aerial agent that has an overview of the formation, to send the desired information to the ground agents in a distributed way, as illustrated in figure 5.5.

Figure 5.2: An example of dynamic complex formation.

Figure 5.3: An example of a distributed three-group formation. The big circle is the global virtual leader and the small circles are the local virtual leaders. The arrows illustrate the information transfer among agents.

In the following development, the global group behavior and the local group behavior are examined.

5.3 Global Group Behavior

In the proposed methodology, the global virtual leader performs the overall planning and have the knowledge of the overall formation. Local virtual leaders obtain their desired in-

Figure 5.4: Virtual leader is specified by semi-virtual agent. The arrows illustrate the information transfer among agents.

Figure 5.5: Virtual leader is specified by aerial agent. The arrows illustrate the information transfer among agents.

formation in a distributed way, such that the communication topology among local virtual leaders is represented by a directed graph $\mathcal{G}$ with the associate laplacian matrix $\mathcal{L} = [l_{ij}]$, where

$$l_{ij} = \begin{cases} -a_{k\bar{k}} & k \neq \bar{k} \\ \sum_{\bar{k}=1}^N a_{k\bar{k}} & k = \bar{k} \end{cases} \quad (5.3)$$

such that, $a_{k\bar{k}}$ is greater than zero if the local virtual leader of group k obtain information from the local virtual leader of group $\bar{k}$, otherwise $a_{k\bar{k}} = 0$. Hence, the proposed algorithm for local virtual leaders to obtain their desired information from the global virtual

leader is expressed as follows:

$$\dot{\hat{\mathcal{X}}}_k = \frac{1}{c_k} \sum_{\bar{k}=1}^N a_{k\bar{k}} \dot{\hat{\mathcal{X}}}_{\bar{k}} + \frac{b_{ko}}{c_k} \dot{\mathcal{X}}_o - \frac{k_{\mathcal{X}_k}}{c_k} \left(\sum_{\bar{k}=1}^N a_{k\bar{k}} (\hat{\mathcal{X}}_k - \hat{\mathcal{X}}_{\bar{k}}) + b_{ko} (\hat{\mathcal{X}}_k - \mathcal{X}_o) \right) \quad (5.4)$$

where $\mathcal{X}_o = [x_{1o}, x_{2o}, \dots, x_{No}]^T \in \mathbb{R}^{nN}$ such that x_{ko} is the actual reference position for local virtual leader of group k determined by the global virtual leader. $\hat{\mathcal{X}}_k = [\hat{x}_{1o}^k, \hat{x}_{2o}^k, \dots, \hat{x}_{No}^k]^T$ such that $\hat{x}_{ko}^k$ is an estimation of x_{ko} , estimated by local virtual leader k . b_{ko} is greater than zero if the local virtual leader of group k obtain information from the global virtual leader and 0 otherwise. $k_{\mathcal{X}_k}$ are positive constants and $c_k = \sum_{\substack{\bar{k}=1 \\ \bar{k} \neq k}}^N a_{k\bar{k}} + b_{ko}$.

To better understand the problem, a simple example is illustrated in figure 5.6. In the depicted figure, an aerial agent is considered as the global virtual leader who determines the overall movement including the reference positions of local virtual leaders. In each group an agent, represented in different appearance, is assigned as semi-leader. The semi-leader agent of each group obtains the information of the local virtual leader of that group by computing the proposed algorithm (5.4). Equation (5.4) can be rewritten as follows:

$$\dot{\epsilon}_{\mathcal{X}_k} = -k_{\mathcal{X}_k} \epsilon_{\mathcal{X}_k} \quad (5.5)$$

where

$$\epsilon_{\mathcal{X}_k} = \sum_{\bar{k}=1}^N a_{k\bar{k}} (\hat{\mathcal{X}}_k - \hat{\mathcal{X}}_{\bar{k}}) + b_{ko} (\hat{\mathcal{X}}_k - \mathcal{X}_o) \quad (5.6)$$

In addition, equation (5.6) is written in the following closed-form:

$$\epsilon_{\mathcal{X}} = \{(\mathcal{L} + \mathcal{B}) \otimes I_{nN}\} \Delta \mathcal{X} \quad (5.7)$$

Figure 5.6: An example of three-group reconfiguration obtained by global group behavior. The black filled circles are virtual leaders.

where $\epsilon_{\mathcal{X}} = [\epsilon_{\mathcal{X}_1}, \epsilon_{\mathcal{X}_2}, \dots, \epsilon_{\mathcal{X}_N}]^T$, $\mathcal{B} = \text{diag}\{b_{1o}, b_{2o}, \dots, b_{No}\}$, $\otimes$ is the Kronecker product, $I_{\mathfrak{m}N}$ is the identity matrix with the dimension of $\mathfrak{m}N \times \mathfrak{m}N$, and

$$\Delta \mathcal{X} = \begin{bmatrix} \hat{\mathcal{X}}_1 - \mathcal{X}_o \\ \hat{\mathcal{X}}_2 - \mathcal{X}_o \\ \vdots \\ \hat{\mathcal{X}}_N - \mathcal{X}_o \end{bmatrix} \quad (5.8)$$

The global virtual leader, not only determines the positions of local virtual leaders, but it also determines the size and deformation of the shape of each group. To do so, scaling vectors are defined for groups $k = 1, 2, \dots, N$ as follows :

$$\mathcal{S}_k = [\mathfrak{s}_k^1, \mathfrak{s}_k^2, \dots, \mathfrak{s}_k^{\mathfrak{n}}]^T \in \mathbb{R}^{\mathfrak{n}} \quad (5.9)$$

where s_k^i are positive scalars for $k = 1, 2, \dots, N$ and $i = 1, 2, \dots, \mathfrak{n}$. S_k represents the shape transformation for group k such that for different values of s_k^i , formation expansion, contraction and deformation can be obtained. Local virtual leaders obtain their desired scaling vectors by the proposed estimators as follows:

$$\dot{\hat{S}}_k = \frac{1}{c_k} \sum_{\bar{k}=1}^N a_{k\bar{k}} \dot{\hat{S}}_{\bar{k}} + \frac{b_{ko}}{c_k} \dot{S}_o - \frac{k_{S_k}}{c_k} \left(\sum_{\bar{k}=1}^N a_{k\bar{k}} (\hat{S}_k - \hat{S}_{\bar{k}}) + b_{ko} (\hat{S}_k - S_o) \right) \quad (5.10)$$

where k_{S_k} are positive constants, $S_o = [S_1, S_2, \dots, S_N]^T \in \mathbb{R}^{\mathfrak{n}N}$, and $\hat{S}_k = [\hat{S}_1^k, \hat{S}_2^k, \dots, \hat{S}_N^k]^T$ such that $\hat{S}_{\bar{k}}^k$ is an estimation of $S_{\bar{k}}$, estimated by local virtual leader k . Equation (5.10) can be rewritten as follows:

$$\dot{\epsilon}_{S_k} = -k_{S_k} \epsilon_{S_k} \quad (5.11)$$

where

$$\epsilon_{S_k} = \sum_{\bar{k}=1}^N a_{k\bar{k}} (\hat{S}_k - \hat{S}_{\bar{k}}) + b_{ko} (\hat{S}_k - S_o) \quad (5.12)$$

In addition, equation (5.12) is written in the following closed-form:

$$\epsilon_S = \{(\mathcal{L} + \mathcal{B}) \otimes I_{\mathfrak{n}N}\} \Delta S \quad (5.13)$$

where $\epsilon_S = [\epsilon_{S_1}, \epsilon_{S_2}, \dots, \epsilon_{S_N}]^T$, $\mathcal{B} = \text{diag}\{b_{1o}, b_{2o}, \dots, b_{No}\}$ and

$$\Delta S = \begin{bmatrix} \hat{S}_1 - S_o \\ \hat{S}_2 - S_o \\ \vdots \\ \hat{S}_N - S_o \end{bmatrix} \quad (5.14)$$

5.4 Local Group Behavior

In the local group behavior, a distributed shape formation for each group is obtained based on the estimation of the local virtual leader of that group. Due to limitations in communication bandwidth and range, it is reasonable to assume that the estimation of local virtual leader of group k , is only available for some agents, and for the other agents, an estimator is proposed to construct the estimation of local virtual leader. Consider that communication topology among agents is represented by a directed graph $\mathcal{G}_k$ with the associate laplacian matrix $\mathcal{L}_k = [l_{ij}^k]$, where

$$l_{ij}^k = \begin{cases} -a_{ij}^k & i \neq j \\ \sum_{j=1}^{n_k} a_{ij}^k & i = j \end{cases} \quad (5.15)$$

such that, if the agent i of group k can obtain information from the agent j of group k , $a_{ij}^k > 0$, otherwise $a_{ij}^k = 0$. The proposed estimator to construct the desired reference of group k is expressed as follows:

$$\dot{\hat{x}}_{ko}^{ki} = \frac{1}{c_{ki}} \sum_{j=1}^{n_k} a_{ij}^k \dot{\hat{x}}_{ko}^{kj} + \frac{b_{io}^k}{c_{ki}} \dot{\hat{x}}_{ko}^k - \frac{k_{x_{ki}}}{c_{ki}} \left(\sum_{j=1}^{n_k} a_{ij}^k (\hat{x}_{ko}^{ki} - \hat{x}_{ko}^{kj}) + b_{io}^k (\hat{x}_{ko}^{ki} - \hat{x}_{ko}^k) \right) \quad (5.16)$$

where b_{io}^k is greater than zero if agent i of group k has an access to the estimated desired reference of group k and 0 otherwise; $k_{x_{ki}}$ are positive constants and $c_{ki} = \sum_{\substack{j=1 \\ j \neq i}}^{n_k} a_{ij}^k + b_{io}^k$.

Equation (5.16) can be rewritten as follows:

$$\dot{\epsilon}_{x_{ki}} = -k_{x_{ki}} \epsilon_{x_{ki}} \quad (5.17)$$

where

$$\epsilon_{x_{ki}} = \sum_{j=1}^{n_k} a_{ij}^k (\hat{x}_{ko}^{ki} - \hat{x}_{ko}^{kj}) + b_{io}^k (\hat{x}_{ko}^{ki} - \hat{x}_{ko}^k) \quad (5.18)$$

In addition, equation (5.18) is written in the following closed-form:

$$\epsilon_{x_k} = \{(\mathcal{L}_k + \mathcal{B}_k) \otimes I_n\} \Delta \hat{x}_{ko}^k \quad (5.19)$$

where $\epsilon_x = [\epsilon_{x_{k1}}, \epsilon_{x_{k2}}, \dots, \epsilon_{x_{kn_k}}]^T$, $\mathcal{B}_k = \text{diag}\{b_{1o}^k, b_{2o}^k, \dots, b_{No}^k\}$ and

$$\Delta \hat{x}_{ko}^k = \begin{bmatrix} \hat{x}_{ko}^{k1} - \hat{x}_{ko}^k \\ \hat{x}_{ko}^{k2} - \hat{x}_{ko}^k \\ \vdots \\ \hat{x}_{ko}^{kN} - \hat{x}_{ko}^k \end{bmatrix} \quad (5.20)$$

Moreover, it is assumed that the estimation of time-varying scaling vector for each group is only available for some agents, and for the other agents, an estimator is proposed to construct the estimation of time-varying scaling vector, as follows:

$$\dot{\hat{S}}_k^{ki} = \frac{1}{c_{ki}} \sum_{j=1}^{n_k} a_{ij}^k \dot{\hat{S}}_k^{kj} + \frac{b_{io}^k}{c_{ki}} \dot{\hat{S}}_k^k - \frac{k_{s_{ki}}}{c_{ki}} \left(\sum_{j=1}^{n_k} a_{ij}^k (\hat{S}_k^{ki} - \hat{S}_k^{kj}) + b_{io}^k (\hat{S}_k^{ki} - \hat{S}_k^k) \right) \quad (5.21)$$

where $\hat{S}_k^{ki}$ is the estimation of $\hat{S}_k^k$ by agent ki , and $k_{s_{ki}}$ are positive constants. Equation (5.21) can be written in the following closed-form:

$$\dot{\epsilon}_{s_{ki}} = -k_{s_{ki}} \epsilon_{s_{ki}} \quad (5.22)$$

where

$$\epsilon_{s_{ki}} = \sum_{j=1}^{n_k} a_{ij}^k (\hat{S}_k^{ki} - \hat{S}_k^{kj}) + b_{io}^k (\hat{S}_k^{ki} - \hat{S}_k^k) \quad (5.23)$$

In addition, (5.23) are written in the following closed-form:

$$\epsilon_{s_k} = \{(\mathcal{L}_k + \mathcal{B}_k) \otimes I_n\} \Delta \hat{S}_k \quad (5.24)$$

where $\epsilon_{\mathcal{S}_k} = [\epsilon_{\mathcal{S}_{k1}}, \epsilon_{\mathcal{S}_{k2}}, \dots, \epsilon_{\mathcal{S}_{kn_k}}]^T$, and

$$\Delta \hat{\mathcal{S}}_k = \begin{bmatrix} \hat{\mathcal{S}}_k^{k1} - \hat{\mathcal{S}}_k^k \\ \hat{\mathcal{S}}_k^{k2} - \hat{\mathcal{S}}_k^k \\ \vdots \\ \hat{\mathcal{S}}_k^{kN} - \hat{\mathcal{S}}_k^k \end{bmatrix} \quad (5.25)$$

After presenting the distributed estimators (5.16) and (5.21) for group members to obtain the information about the position of local virtual leaders and time-varying scaling vectors, the distributed time-varying shape formation for each group is considered. Hence, the following shape functions are presented:

$$\begin{cases} f_{G_1} = [f_{G_{11}}(\Delta \hat{X}_{1io}), f_{G_{12}}(\Delta \hat{X}_{1io}), \dots, f_{G_{1m_1}}(\Delta \hat{X}_{1io})]^T \leq 0 \\ f_{G_2} = [f_{G_{21}}(\Delta \hat{X}_{2io}), f_{G_{22}}(\Delta \hat{X}_{2io}), \dots, f_{G_{2m_2}}(\Delta \hat{X}_{2io})]^T \leq 0 \\ \vdots \\ f_{G_N} = [f_{G_{N1}}(\Delta \hat{X}_{Nio}), f_{G_{N2}}(\Delta \hat{X}_{Nio}), \dots, f_{G_{Nm_N}}(\Delta \hat{X}_{Nio})]^T \leq 0 \end{cases} \quad (5.26)$$

where $f_{G_k}(\cdot)$ are shape functions defined by (3.3), $\Delta \hat{X}_{kio} = \hat{\mathcal{S}}_k^{ki} \Delta \hat{x}_{kio}$, such that $\hat{\mathcal{S}}_k^{ki}$ is the diagonal matrix form of vector $\hat{\mathcal{S}}_k^{ki}$, defined by (5.21), and $\Delta \hat{x}_{kio} = x_{ki} - \hat{x}_{ko}^{ki}$, where $\hat{x}_{ko}^{ki}$ is given by (5.16).

The potential energy shape functions for each group are defined as follows:

$$\begin{cases} P_{G_1}(\Delta \hat{X}_{1io}) = \sum_{l=1}^{m_1} P_{G_{1l}}(\Delta \hat{X}_{1io}) \\ P_{G_2}(\Delta \hat{X}_{2io}) = \sum_{l=1}^{m_2} P_{G_{2l}}(\Delta \hat{X}_{2io}) \\ \vdots \\ P_{G_N}(\Delta \hat{X}_{Nio}) = \sum_{l=1}^{m_N} P_{G_{Nl}}(\Delta \hat{X}_{Nio}) \end{cases} \quad (5.27)$$

where $P_{G_k}(\Delta \hat{X}_{kio})$ is the potential function for agent i of group k . The potential functions

$P_{G_{kl}}(\Delta \hat{X}_{kio})$ are defined as follows:

$$\begin{aligned} P_{G_{kl}}(\Delta \hat{X}_{kio}) &= \frac{1}{n} k_{kl} \left[\max \left(0, f_{G_{kl}}(\Delta \hat{X}_{kio}) \right) \right]^n \\ &= \begin{cases} 0, & f_{G_{kl}}(\Delta \hat{X}_{kio}) \leq 0 \\ \frac{k_{kl}}{n} f_{G_{kl}}^n(\Delta \hat{X}_{kio}), & f_{G_{kl}}(\Delta \hat{X}_{kio}) > 0 \end{cases} \end{aligned} \quad (5.28)$$

where k_{kl} are positive constants. n is positive constant, and is defined in such a way that the potential functions at least belong to class $\mathcal{C}^2$. The distribute time-varying shape forces for whole swarm to form the specified shape are obtained by the partial differentiation of the potential functions with respect to $\Delta \hat{X}_{kio}$ as

$$\begin{cases} \Delta \hat{\xi}_{1i} = \sum_{l=1}^{m_1} k_{kl} \left[\max \left(0, f_{G_{1l}}(\Delta \hat{X}_{1io}) \right) \right]^{n-1} \left(\frac{\partial f_{G_{1l}}(\Delta \hat{X}_{1io})}{\partial \Delta \hat{X}_{1io}} \right)^T \\ \Delta \hat{\xi}_{2i} = \sum_{l=1}^{m_2} k_{kl} \left[\max \left(0, f_{G_{2l}}(\Delta \hat{X}_{2io}) \right) \right]^{n-1} \left(\frac{\partial f_{G_{2l}}(\Delta \hat{X}_{2io})}{\partial \Delta \hat{X}_{2io}} \right)^T \\ \vdots \\ \Delta \hat{\xi}_{Ni} = \sum_{l=1}^{m_N} k_{kl} \left[\max \left(0, f_{G_{Nl}}(\Delta \hat{X}_{Nio}) \right) \right]^{n-1} \left(\frac{\partial f_{G_{Nl}}(\Delta \hat{X}_{Nio})}{\partial \Delta \hat{X}_{Nio}} \right)^T \end{cases} \quad (5.29)$$

5.5 Social Interactions

After defining the control force for shape formation, the interactive forces between agents are defined. In this regard two potential functions are defined; the first one is related to intra-group interactions which occur between members of a group to maintain minimum distance between them as well as keeping group unity. The second potential function is related to inter-group interactions which occur between members of one group with other groups and only need to ensure minimum distance between them. The first proposed potential interaction functions for members of group k are defined as follows:

$$Q(\Delta \hat{X}_{kikj}) = \sum_{kj \in \mathbb{N}_{ki}} \frac{k_{kikj}}{n} \left[\max \left(0, g(\Delta \hat{X}_{kikj}) \right) \right]^n \quad (5.30)$$

where $\Delta\hat{X}_{kikj} = \hat{S}_k^i \Delta x_{kikj}$ such that $\Delta x_{kikj} = x_{ki} - x_{kj}$, k_{kikj} are positive constants, and $g(\cdot)$ was defined by (4.12). $\mathbb{N}_{ki}$ refers to neighbors of agent ki . The interactive forces between members of the group k are obtained by the partial differentiation of the potential interaction functions with respect to $\Delta\hat{X}_{kikj}$ as follows:

$$\Delta\hat{\rho}_{ki-kj} = \underbrace{\sum_{kj \in \mathbb{N}_{ki}} k_{kikj} \left[\max \left(0, g(\Delta\hat{X}_{kikj}) \right) \right]^{n-1} \left(\frac{\partial g(\Delta\hat{X}_{kikj})}{\partial \Delta\hat{X}_{kikj}} \right)^T}_{\Delta\hat{\rho}_{ki-kj}} \quad (5.31)$$

To maintain a minimum distance between the members of the neighboring groups, the following potential energy function are defined:

$$H(\Delta\hat{X}_{ki\bar{k}j}) = \sum_{\bar{k}j \in \mathbb{N}_{ki}} \frac{k_{ki\bar{k}j}}{n} \left[\max \left(0, h(\Delta\hat{X}_{ki\bar{k}j}) \right) \right]^n \quad (5.32)$$

where $k_{ki\bar{k}j}$ are positive constants, $\Delta\hat{X}_{ki\bar{k}j} = \hat{S}_k^i \Delta x_{ki\bar{k}j}$ such that $\Delta x_{ki\bar{k}j} = x_{ki} - x_{\bar{k}j}$ and subscript $\bar{k}j$ refers to those members which are neighbors of the agent i of the group k but don't belong to group k . $h(\Delta\hat{X}_{ki\bar{k}j})$ was defined by (4.17). The inter-group interaction force can be defined by the partial differentiation of (5.32) with respect to $\Delta\hat{X}_{ki\bar{k}j}$, as

$$\Delta\hat{\psi}_{ki-\bar{k}j} = \underbrace{\sum_{\bar{k}j \in \mathbb{N}_{ki}} k_{ki\bar{k}j} \left[\max \left(0, h(\Delta\hat{X}_{ki\bar{k}j}) \right) \right]^{n-1} \left(\frac{\partial h(\Delta\hat{X}_{ki\bar{k}j})}{\partial \Delta\hat{X}_{ki\bar{k}j}} \right)^T}_{\Delta\hat{\psi}_{ki-\bar{k}j}} \quad (5.33)$$

Finally to ensure the stability of distributed multi-group system with intra-group and inter-group interactions, two adaptive interactive forces are introduced. To avoid members of a group being get stuck in local minima in transient response due to difference in estimations of scaling matrix by each agent, the intra-group adaptive interactive forces,

$\Delta \hat{\eta}_{\rho_{ki-\bar{k}j}}$, are defined as follows:

$$\Delta \hat{\eta}_{\rho_{ki-kj}} = \sum_{kj \in \mathbb{N}_{ki}} \mathcal{R}(\varphi_{kikj}) \Delta \hat{\rho}_{ki-kj} \quad (5.34)$$

where $\Delta \hat{\rho}_{ki-kj}$ are defined by (5.31), and $\mathcal{R}(\varphi_{kikj})$ are rotation matrices and φ_{kikj} are rotation angles that shall be defined in a way to guarantee the stability of system. Similarly the inter-group interactive forces, $\Delta \hat{\eta}_{\psi_{ki-\bar{k}j}}$, are introduced as follows:

$$\Delta \hat{\eta}_{\psi_{ki-\bar{k}j}} = \sum_{\bar{k}j \in \mathbb{N}_{ki}} \mathcal{R}(\varphi_{ki\bar{k}j}) \Delta \hat{\psi}_{ki-\bar{k}j} \quad (5.35)$$

where $\Delta \hat{\psi}_{ki-\bar{k}j}$ can be computed by (5.33). The details will be given in later development.

5.6 Distributed Dynamic Shape Control Methodology

In this section, the multi-group shape controller is presented to obtain complex dynamic formation. First, a reference vector $\dot{x}_{rki}$ is defined for member i of group k as follows:

$$\dot{x}_{rki} = \dot{x}_{kio}^{ki} - \left(\hat{S}_k^{ki} \right)^{-1} \dot{\hat{S}}_k^{ki} \Delta \hat{x}_{kio} - \alpha_{ki} \left(\hat{S}_k^{ki} \right)^{-1} \Delta \hat{\xi}_{ki} \quad (5.36)$$

where α_{ki} are positive constants, and $\Delta \hat{\xi}_{ki}$ are defined by (5.29). Using the reference vector, a sliding vector are defined as follows:

$$\hat{s}_{ki} = \dot{x}_{ki} - \dot{x}_{kio}^{ki} = \Delta \dot{\hat{x}}_{kio} + \left(\hat{S}_k^{ki} \right)^{-1} \dot{\hat{S}}_k^{ki} \Delta \hat{x}_{kio} + \alpha_{ki} \left(\hat{S}_k^{ki} \right)^{-1} \Delta \hat{\xi}_{ki} \quad (5.37)$$

By differentiating equation (5.37) with respect to time, $\dot{\hat{s}}_{ki}$ are obtained as follows:

$$\begin{aligned} \dot{\hat{s}}_{ki} = \ddot{x}_{ki} - \ddot{\hat{x}}_{ko}^{ki} = & \Delta \ddot{\hat{x}}_{kio} + \frac{d \left[\left(\hat{S}_k^{ki} \right)^{-1} \right]}{dt} \dot{\hat{S}}_k^{ki} \Delta \hat{x}_{kio} + \left(\hat{S}_k^{ki} \right)^{-1} \ddot{\hat{S}}_k^{ki} \Delta \hat{x}_{kio} \\ & + \left(\hat{S}_k^{ki} \right)^{-1} \dot{\hat{S}}_k^{ki} \Delta \dot{\hat{x}}_{kio} + \alpha_{ki} \frac{d \left[\left(\hat{S}_k^{ki} \right)^{-1} \right]}{dt} \Delta \hat{\xi}_{ki} + \alpha_{ki} \left(\hat{S}_k^{ki} \right)^{-1} \Delta \dot{\hat{\xi}}_{ki} \end{aligned} \quad (5.38)$$

Substituting equations (5.37) and (5.38) into (5.1), the following equations are obtained:

$$\begin{cases} G_1 : M_{1i}(x_{1i})\dot{\hat{s}}_{1i} + C_{1i}(x_{1i}, \dot{x}_{1i})\hat{s}_{1i} + D_{1i}(x_{1i})\hat{s}_{1i} + Y_{1i}(x_{1i}, \dot{x}_{1i}, \dot{\hat{x}}_{r1i}, \ddot{\hat{x}}_{r1i})\theta_{1i} = u_{1i} \\ G_2 : M_{2i}(x_{2i})\dot{\hat{s}}_{2i} + C_{2i}(x_{2i}, \dot{x}_{2i})\hat{s}_{2i} + D_{2i}(x_{2i})\hat{s}_{2i} + Y_{2i}(x_{2i}, \dot{x}_{2i}, \dot{\hat{x}}_{r2i}, \ddot{\hat{x}}_{r2i})\theta_{2i} = u_{2i} \\ \vdots \\ G_N : M_{Ni}(x_{Ni})\dot{\hat{s}}_{Ni} + C_{Ni}(x_{Ni}, \dot{x}_{Ni})\hat{s}_{Ni} + D_{Ni}(x_{Ni})\hat{s}_{Ni} \\ \quad + Y_{Ni}(x_{Ni}, \dot{x}_{Ni}, \dot{\hat{x}}_{rNi}, \ddot{\hat{x}}_{rNi})\theta_{Ni} = u_{Ni} \end{cases} \quad (5.39)$$

where $Y_{ki}(x_{ki}, \dot{x}_{ki}, \dot{\hat{x}}_{rki}, \ddot{\hat{x}}_{rki})$ and θ_{ki} are known regressor matrices and unknown parameter vectors, respectively. The control laws for group formation are expressed as

$$\begin{aligned} u_{ki} = & -K_{ski}\hat{s}_{ki} - \left(\hat{S}_k^{ki} \right)^T k_{kp} \left\{ \alpha_{ki} \Delta \hat{\xi}_{ki} + \gamma_{ki} \Delta \hat{\eta}_{\rho_{ki}-k_j} + \lambda_k \Delta \hat{\eta}_{\psi_{ki}-\bar{k}_j} \right\} \\ & + Y_{ki}(x_{ki}, \dot{x}_{ki}, \dot{\hat{x}}_{rki}, \ddot{\hat{x}}_{rki})\hat{\theta}_{ki} \end{aligned} \quad (5.40)$$

where K_{ski} are positive matrices, k_{kp} , γ_{ki} and λ_k are positive constants. $\hat{\theta}_{ki}$ are estimated parameters vectors. The update laws are expressed as follows:

$$\dot{\hat{\theta}}_{ki} = -L_{ki} Y_{ki}^T(x_{ki}, \dot{x}_{ki}, \dot{\hat{x}}_{rki}, \ddot{\hat{x}}_{rki}) \hat{s}_{ki} \quad (5.41)$$

where L_{ki} are positive constants. The closed-loop equations for the multi-agent system are obtained by substituting the feedback laws (5.40) into the dynamic equations (5.39),

as follows:

$$\left\{ \begin{array}{l} G_1 : M_{1i}(x_{1i})\dot{\hat{s}}_{1i} + C_{1i}(x_{1i}, \dot{x}_{1i})\hat{s}_{1i} + D_{1i}(x_{1i})\hat{s}_{1i} + K_{s1i}\hat{s}_{1i} + Y_{1i}(x_{1i}, \dot{x}_{1i}, \dot{\hat{x}}_{r1i}, \ddot{\hat{x}}_{r1i})\theta_{1i} \\ \quad + \left(\hat{S}_1^{1i}\right)^T k_{p1} \left\{ \alpha_{1i}\Delta\hat{\xi}_{1i} + \gamma_{1i}\Delta\hat{\eta}_{\rho_{1i-1j}} + \lambda_1\Delta\hat{\eta}_{\psi_{1i-1j}} \right\} = 0 \\ G_2 : M_{2i}(x_{2i})\dot{\hat{s}}_{2i} + C_{2i}(x_{2i}, \dot{x}_{2i})\hat{s}_{2i} + D_{2i}(x_{2i})\hat{s}_{2i} + K_{s2i}\hat{s}_{2i} + Y_{2i}(x_{2i}, \dot{x}_{2i}, \dot{\hat{x}}_{r2i}, \ddot{\hat{x}}_{r2i})\theta_{2i} \\ \quad + \left(\hat{S}_2^{2i}\right)^T k_{p2} \left\{ \alpha_{2i}\Delta\hat{\xi}_{2i} + \gamma_{2i}\Delta\hat{\eta}_{\rho_{2i-2j}} + \lambda_2\Delta\hat{\eta}_{\psi_{2i-2j}} \right\} = 0 \\ \vdots \\ G_N : M_{Ni}(x_{Ni})\dot{\hat{s}}_{Ni} + C_{Ni}(x_{Ni}, \dot{x}_{Ni})\hat{s}_{Ni} + D_{Ni}(x_{Ni})\hat{s}_{Ni} + K_{sNi}\hat{s}_{Ni} + Y_{Ni}(x_{Ni}, \dot{x}_{Ni}, \dot{\hat{x}}_{rNi}, \ddot{\hat{x}}_{rNi})\theta_{Ni} \\ \quad + \left(\hat{S}_N^{Ni}\right)^T k_{pN} \left\{ \alpha_{Ni}\Delta\hat{\xi}_{Ni} + \gamma_{Ni}\Delta\hat{\eta}_{\rho_{Ni-Nj}} + \lambda_N\Delta\hat{\eta}_{\psi_{Ni-\bar{N}j}} \right\} = 0 \end{array} \right. \quad (5.42)$$

where $\Delta\theta_{ki} = \theta_{ki} - \hat{\theta}_{ki}$. For stability analysis the following Lyapunov-like candidate is proposed:

$$\begin{aligned} V = & \sum_{k=1}^N \frac{1}{2} \epsilon_{\mathcal{X}_k}^T \epsilon_{\mathcal{X}_k} + \sum_{k=1}^N \frac{1}{2} \epsilon_{\mathcal{S}_k}^T \epsilon_{\mathcal{S}_k} + \sum_{k=1}^N \left[\sum_{i=1}^{n_k} \frac{1}{2} \hat{s}_{ki}^T M_{ki}(x_{ki}) \hat{s}_{ki} \right. \\ & + \sum_{i=1}^{n_k} \frac{1}{2} \Delta\theta_{ki}^T L_{ki}^{-1} \Delta\theta_{ki} + \sum_{i=1}^{n_k} \frac{1}{2} \epsilon_{x_{ki}}^T \epsilon_{x_{ki}} + \sum_{i=1}^{n_k} \frac{1}{2} \epsilon_{\mathcal{S}_{ki}}^T \epsilon_{\mathcal{S}_{ki}} \\ & \left. + \sum_{i=1}^{n_k} \alpha_{ki} k_{p_k} \sum_{l=1}^{m_k} \frac{k_{kl}}{n} [\max(0, f_{G_{kl}}(\Delta\hat{X}_{kio}))]^n \right] \end{aligned} \quad (5.43)$$

By differentiating the Lyapunov-like candidate with respect to time, substituting the closed-loop equations and considering the skew-symmetry of the matrix $\dot{M}_{ki}(x_{ki}) - 2C_{ki}(x_{ki}, \dot{x}_{ki})$, $\dot{V}$ is obtained as follows:

$$\begin{aligned} \dot{V} = & - \sum_{k=1}^N k_{\mathcal{X}_k} \epsilon_{\mathcal{X}_k}^T \epsilon_{\mathcal{X}_k} - \sum_{k=1}^N k_{\mathcal{S}_k} \epsilon_{\mathcal{S}_k}^T \epsilon_{\mathcal{S}_k} - \sum_{k=1}^N \left[\sum_{i=1}^{n_k} \hat{s}_{ki}^T K_{ski} \hat{s}_{ki} + \sum_{i=1}^{n_k} \hat{s}_{ki}^T D_{ki}(x_{ki}) \hat{s}_{ki} \right. \\ & + \sum_{i=1}^{n_k} \hat{s}_{ki}^T \left(\hat{S}_k^{ki} \right)^T k_{p_k} \left\{ \alpha_{ki}\Delta\hat{\xi}_{ki} + \gamma_{ki}\Delta\hat{\eta}_{\rho_{ki-kj}} + \lambda_k\Delta\hat{\eta}_{\psi_{ki-\bar{k}j}} \right\} \\ & \left. + \sum_{i=1}^{n_k} k_{x_{ki}} \epsilon_{x_{ki}}^T \epsilon_{x_{ki}} + \sum_{i=1}^{n_k} k_{\mathcal{S}_{ki}} \epsilon_{\mathcal{S}_{ki}}^T \epsilon_{\mathcal{S}_{ki}} - \sum_{i=1}^{n_k} \alpha_{ki} k_{p_k} \Delta\dot{X}_{kio}^T \Delta\hat{\xi}_{ki} \right] \end{aligned} \quad (5.44)$$

The following equation can be obtained:

$$\begin{aligned}\hat{s}_{ki}^T \left(\hat{S}_k^{ki} \right)^T k_{p_k} \alpha_{ki} \Delta \hat{\xi}_{ki} &= \left\{ \hat{S}_k^{ki} \Delta \dot{\hat{x}}_{kio} + \dot{\hat{S}}_k^{ki} \Delta \hat{x}_{kio} + \alpha_{ki} \Delta \hat{\xi}_{ki} \right\}^T k_{p_k} \alpha_{ki} \Delta \hat{\xi}_{ki} \\ &= k_{p_k} \alpha_{ki} \Delta \dot{\hat{X}}_{kio}^T \Delta \hat{\xi}_{ki} + k_{p_k} \alpha_{ki}^2 \Delta \hat{\xi}_{ki}^T \Delta \hat{\xi}_{ki}\end{aligned}\quad (5.45)$$

Therefore the derivative of Lyapunov-like candidate can be rewritten as

$$\begin{aligned}\dot{V} &= - \sum_{k=1}^N k_{\mathcal{X}_k} \epsilon_{\mathcal{X}_k}^T \epsilon_{\mathcal{X}_k} - \sum_{k=1}^N k_{\mathcal{S}_k} \epsilon_{\mathcal{S}_k}^T \epsilon_{\mathcal{S}_k} - \sum_{k=1}^N \left[\sum_{i=1}^{n_k} \hat{s}_{ki}^T K_{ski} \hat{s}_{ki} + \sum_{i=1}^{n_k} \hat{s}_{ki}^T D_{ki}(x_{ki}) \hat{s}_{ki} \right. \\ &\quad + \sum_{i=1}^{n_k} k_{p_k} \alpha_{ki}^2 \Delta \hat{\xi}_{ki}^T \Delta \hat{\xi}_{ki} + \sum_{i=1}^{n_k} k_{x_{ki}} \epsilon_{x_{ki}}^T \epsilon_{x_{ki}} + \sum_{i=1}^{n_k} k_{s_{ki}} \epsilon_{s_{ki}}^T \epsilon_{s_{ki}} \\ &\quad \left. + \sum_{i=1}^{n_k} \hat{s}_{ki}^T \left(\hat{S}_k^{ki} \right)^T k_{p_k} \left\{ \gamma_{ki} \Delta \hat{\eta}_{\rho_{ki-kj}} + \lambda_k \Delta \hat{\eta}_{\psi_{ki-kj}} \right\} \right]\end{aligned}\quad (5.46)$$

In order to ensure the stability, the intra-group adaptive interactive forces, are defined, so that

$$\sum_{k=1}^N \sum_{i=1}^{n_k} \hat{s}_{ki}^T \left(\hat{S}_k^{ki} \right)^T k_{p_k} \gamma_{ki} \Delta \hat{\eta}_{\rho_{ki-kj}} = 0 \quad (5.47)$$

Substituting (5.34) into (5.47), yields

$$\sum_{k=1}^N \sum_{i=1}^{n_k} \hat{s}_{ki}^T \left(\hat{S}_k^{ki} \right)^T k_{p_k} \gamma_{ki} \sum_{kj \in \mathfrak{n}_{ki}} \mathcal{R}(\varphi_{kikj}) \Delta \hat{\rho}_{ki-kj} = 0 \quad (5.48)$$

Using (4.13) and (4.14), and simplifying, (5.48) is written as

$$y_{\rho_{ki}}^T \mathcal{R}(\varphi_{kikj}) \Delta \hat{\rho}_{ki-kj} = 0 \quad (5.49)$$

where $y_{\rho_{ki}} = \gamma_{ki} \hat{S}_k^i s_{ki} - \gamma_{kj} \hat{S}_k^j s_{kj}$. The rotation matrices are defined such that equation (5.49) is satisfied; that is rotating the intra-group interactive forces such that become perpendicular to $y_{\rho_{ki}}$. In fact, the estimation of scaling matrices causes different desired speed during transient response. In this regard, $\mathcal{R}(\varphi_{kikj})$ is responsible to rotate intra-

group interactive forces in such a way that agents will not get stuck into local minima. An example of a rotation matrix in 2D is given as follows:

$$\mathcal{R}(\varphi_{kikj}) = \begin{bmatrix} \cos(\varphi_{kikj}) & -\sin(\varphi_{kikj}) \\ \sin(\varphi_{kikj}) & \cos(\varphi_{kikj}) \end{bmatrix} \quad (5.50)$$

where the rotation angle φ_{kikj} is defined as

$$\varphi_{kikj} = \tan^{-1} \left\{ -\frac{y_{\rho_{ki}}^T \Delta \hat{\rho}_{ki-kj}}{y_{\rho_{ki}}^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Delta \hat{\rho}_{ki-kj}} \right\} \quad (5.51)$$

Similarly, $\varphi_{ki\bar{k}j}$ is defined such that

$$\sum_{k=1}^N \sum_{i=1}^{n_k} \hat{s}_{ki}^T \left(\hat{S}_k^{ki} \right)^T k_{p_k} \lambda_k \Delta \hat{\eta}_{\psi_{ki-\bar{k}j}} = 0 \quad (5.52)$$

Now, the following theorem can be stated on the convergence of the signals:

Theorem 5.1. *The multi-group system represented by equation (5.1) with the estimator (5.4) for the overall configuration and adaptive control schemes (5.40) and the parameter update laws (5.41), result in the convergence of $\hat{x}_{ko}^k - x_{ko} \rightarrow 0$, $\hat{s}_{ki} \rightarrow 0$ and $\Delta \hat{\xi}_{ki} \rightarrow 0$ for all $k = 1, 2, \dots, N$, and $i = 1, 2, \dots, n_k$, as $t \rightarrow \infty$.*

Proof. Since $M_{ki}(x_{ki})$ and L_{ki} are positive definite, so V is positive definite in $\epsilon_{\mathcal{X}_k}$, $\epsilon_{\mathcal{S}_k}$, $\hat{s}_{ki}$, $\Delta \theta_{ki}$, $\epsilon_{x_{ki}}$ and $\epsilon_{\mathcal{S}_{ki}}$. Therefore, $\epsilon_{\mathcal{X}_k}$, $\epsilon_{\mathcal{S}_k}$, $\hat{s}_{ki}$, $\epsilon_{x_{ki}}$, $\epsilon_{\mathcal{S}_{ki}}$ and potential shape functions are bounded. Using property 3.7, one can conclude $\Delta \hat{\xi}_{ki}$ are bounded. Since $\epsilon_{\mathcal{X}_k}$, $\epsilon_{\mathcal{S}_k}$, $\epsilon_{x_{ki}}$ and $\epsilon_{\mathcal{S}_{ki}}$ are bounded, using (5.5), (5.11), (5.17) and (5.22) lead to the boundedness of $\dot{\epsilon}_{\mathcal{X}_k}$, $\dot{\epsilon}_{\mathcal{S}_k}$, $\dot{\epsilon}_{x_{ki}}$ and $\dot{\epsilon}_{\mathcal{S}_{ki}}$. Therefore, boundedness of x_{ko} and its derivatives lead to the

boundedness of $\hat{x}_{ko}^{ki}$ and its derivatives. Similarly, boundedness of S_k and its derivatives lead to the boundedness of $\hat{S}_k^{ki}$ and its derivatives. From (5.36) the boundedness of $\dot{\hat{x}}_{kri}$ is concluded, and (5.37) yields the boundedness of $\dot{x}_{ki}$. Therefore $\Delta\hat{\xi}_{ki}$ are bounded, yields the boundedness of $\ddot{x}_{tki}$. From closed-loop equations (5.42), the boundedness of $\hat{s}_{ki}$ are obtained. Therefore, $\ddot{V}$ is bounded, which implies that $\dot{V}$ is uniformly continuous. By invoking Barbalat's Lemma, it follows $\dot{V}$ goes to zero as $t \rightarrow \infty$, yields $\epsilon_{\mathcal{X}_k}, \epsilon_{\mathcal{S}_k}, \hat{s}_{ki}, \Delta\hat{\xi}_{ki}, \epsilon_{x_{ki}}$ and $\epsilon_{S_{ki}}$ go to zero as $t \rightarrow \infty$.

If $\mathcal{L} + \mathcal{B}$ and $\mathcal{L}_k + \mathcal{B}_k$ satisfy the condition in lemma A.4 in Appendix A, then $\mathcal{L} + \mathcal{B}$ and $\mathcal{L}_k + \mathcal{B}_k$ are invertible. Using properties of Kronecker product, it gives $(\mathcal{L} + \mathcal{B}) \otimes I_{nN}$ and $(\mathcal{L}_k + \mathcal{B}_k) \otimes I_n$ are invertible. Therefore, equations (5.7), (5.13), (5.19) and (5.24), leads to $\Delta\mathcal{X} \rightarrow 0$, $\Delta\mathcal{S} \rightarrow 0$, $\Delta\hat{x}_{ko}^k \rightarrow 0$ and $\Delta\hat{S}_k \rightarrow 0$ as $t \rightarrow \infty$.

Since, $\hat{S}_k^{ki} \rightarrow S_k$ and $\Delta\hat{x}_{kio} \rightarrow \Delta x_{kio}$ as $t \rightarrow \infty$, $\Delta\xi_{ki} \rightarrow 0$, where

$$\Delta\xi_{ki} = \sum_{l=1}^{m_k} k_{kl} [\max(0, f_{G_{kl}}(\Delta X_{kio}))]^{n-1} \left(\frac{\partial f_{G_{kl}}(\Delta X_{kio})}{\partial \Delta X_{kio}} \right)^T \quad (5.53)$$

such that $\Delta X_{kio} = S_k \Delta x_{kio}$. In addition, (5.37) gives

$$\Delta\dot{\hat{x}}_{kio} + \left(\hat{S}_k^{ki}\right)^{-1} \dot{\hat{S}}_k^{ki} \Delta\hat{x}_{kio} + \alpha_{ki} \left(\hat{S}_k^{ki}\right)^{-1} \Delta\hat{\xi}_{ki} \rightarrow \Delta\dot{x}_{kio} + (S_k)^{-1} \dot{S}_k \Delta x_{kio} \rightarrow 0 \quad (5.54)$$

which imply $\Delta\dot{X}_{kio} \rightarrow 0$ as $t \rightarrow \infty$. □

5.6.1 Simulation

A numerical simulation was performed to illustrate the effectiveness of the proposed distributed dynamic multi-group formation control methodology. For the simulation, Virtual Reality Toolbox (VRLT) in Matlab software was employed. A group of omnidirectional

mobile agents was considered, which have the following dynamic equations:

$$M_i \ddot{x}_i + C_i \dot{x}_i = u_i \quad (5.55)$$

where M_i and C_i are mass and damping constants, respectively. The actual mass of the agents are taken to be $M_i = 1kg$, and the actual damping coefficients are taken to be $C_i = 0.5$. The radius of each agent is $r_i = 0.3m$. To obtain the parameter update law, the following equation is expressed :

$$\begin{aligned} M_i \ddot{x}_{ri} + C_i \dot{x}_{ri} &= \begin{bmatrix} \ddot{x}_{ri} & \dot{x}_{ri} \end{bmatrix} \begin{bmatrix} M_i \\ C_i \end{bmatrix} \\ &= Y_i(\dot{x}_{ri}, \ddot{x}_{ri}) \theta_i \end{aligned} \quad (5.56)$$

The initial estimate of the physical parameters for the parameter update law was set as $\hat{\theta}_i = [0.8 \ 0.3]^T$. Any two agents with the probability of 0.6 were been able to directly communicate with each other if their Euclidean distance was smaller than the given threshold $r_0 = 10m$. The control gains and estimator gain used in the simulation are listed in table 5.1.

Table 5.1: Control gains

$k_{\mathcal{X}_k}$	5	$k_{\mathcal{S}_k}$	5
$k_{x_{ki}}$	2	$k_{\mathcal{S}_{ki}}$	2
α_{ki}	30	k_{kp}	1
K_{ski}	diag {20, 20}	k_{ij}	1
γ_{ki}	500	λ_k	500
L_{ki}	diag {0.05, 0.05}		

Two simulations were conducted to examine the proposed distributed dynamic multi-group formation control.

i) In the first simulation, the agents were assigned to move in an obstacle based envi-

ronment. Initially the agents were distributed randomly within a rectangular region as shown in figure 5.7. The simulation was conducted for $t = 55s$ with a simulation time

Figure 5.7: The initial position of agents from two different views

step of $10ms$. The agents' motions at different time instances of the simulation from two different views are illustrated in figure 5.8 and figure 5.9. It is illustrated that the multiple groups of agents by global reconfiguration and local deformation can increase the flexibility of swarm in moving through obstacle based environment.

Figure 5.8: Distributed multi-group formation of multi-agent system (view1).

Figure 5.9: Distributed multi-group formation of multi-agent system (view2).

ii) In the second simulation, the performance of the proposed method in dynamic complex formation is illustrated. A swarm of 150 agents are divided to 10 groups which each group forms a time-varying elliptical formation. The overall group are assigned to form different alphabetical shapes. Initially the agents were distributed randomly as shown in figures 5.10 and 5.11. The simulation was conducted for $t = 25s$ with a simulation time step of $10ms$. Firstly, the swarm form an A-like shape as illustrated in figure 5.13, where figure 5.12 shows the transient movement of agents to form the A-like shape. Figure 5.14 shows the reconfiguration from A-like shape to B-like shape. Figures 5.15 and 5.16 show the B-like shape formation by agents.

Figure 5.10: The initial position of agents view 1.

Figure 5.11: The initial position of agents view 2.

Figure 5.12: Transient movement of agents to form an A-like shape ($t=4s$).

Figure 5.13: A-like shape formation ($t=10s$).

Figure 5.14: Transient from A-like shape formation to B-like shape formation ($t=15s$).

Figure 5.15: B-like shape formation view 1 ($t=25s$).

Figure 5.16: B-like shape formation view 2 ($t=25s$).

Chapter 6

Conclusion and Future Work

6.1 Conclusion

In the past decade, considerable efforts have been devoted to developing formation control techniques for multi-agent systems. Recently, large-scale systems have gained increasing attentions due to various applications in power grid, chemical processes, transportation systems, and communication networks. Large-scale systems present new challenges for both system analysts and control designers. The size and complexity of these systems make it difficult to examine them based on conventional methods. In particular, there is a great interest in analysis and control of large-scale multi-agent systems, where a large number of agents are equipped with communication and control capabilities.

This thesis has devoted to the development of formation control methodologies for large-scale multi-agent systems. Some conclusions of this thesis are listed as follows:

Formation control of network of integrator agents can be simplified to consensus control problem. Meanwhile distributed consensus problem has gained increasing interest owing to its various applications. This thesis firstly has focused on distributed consensus

problem in large-scale networks. By using the concept of structural observability, an important problem in distributed consensus of large-scale networks which is balanceability, has been examined. Structural balanceability, a missing concept in distributed consensus problem, has been introduced. It has been shown that structural unobservability yields structural unbalanceability. A simple mathematical method has been proposed to modify a distributed consensusable network into a weight-balanced one.

Formation control of large-scale multi-agent systems with general Euler-Lagrange dynamic equations is another interesting problem considered in this thesis. In existing shape formation control problem, having access to the desired reference of the shape by all agents is a requirement. Due to limitations in communication bandwidth and range when operating in a large group, the agents are not able to have access to the desired reference through broadcasting. To overcome this problem, an estimator has been presented to construct the desired reference for each agent based on local information exchange by its neighboring agents. Consequently, a distributed shape formation control methodology has been proposed.

Since the existing methods on shape formation control are limited to simple formation problems by single group of agents, this thesis has been also devoted to complex shape formation of large swarm of agents. The main obstacles that prevent the achievement of the complex shape formation are difficulties in representing complex formations and coordination of the large number of agents. Hence, a multi-group coordination control concept has been introduced as a novel methodology to deal with complex formations that is not obtainable using existing methods. A rigorous theoretical formulation and analysis of the stability of the multi-group control system has been presented. The concept of adaptive interactive forces is introduced to guarantee the stability of the system in

the presence of inter-group interactions.

The other issue that has been explored in this thesis is dynamic formation problem for large-scale multi-agent systems. Existing methods on dynamic formation is based on synchronous shape transformation, in the sense that overall shape of the group is deformed simultaneously which leads to repositioning of all agents. This issue emerges as a serious problem when dealing with large-scale multi-agent systems. Hence, a distributed dynamic multi-group methodology has been presented which enhances the existing results from several aspects: distributed complex formation can be obtained. Multi-group dynamic formation for large-scale multi-agent systems can be achieved which increases the flexibility of group movement in obstacle-based environments.

6.2 Future Work

In the following, some research problems related to formation control of large-scale multi-agent systems are expressed for future developments.

1. In this thesis, the concept of unobservable nodes which lead to structural unobservability, has been introduced. By using the duality principal, the concept of uncontrollable nodes could be introduced based on examining the principal minors of associated graph laplacian matrix. Future work would be devoted to examining the controllability problem in large-scale systems.
2. In the proposed distributed estimators to construct the desired reference, it is assumed that the communication topology is fixed and there is no delay in the communication. Various reasons lead to dynamically changing communication topologies such as link failures or creations in networks with mobile agents. In further

development, dynamic communication topology, uncertainty and delay in the communication could be taken into consideration in the development of the distributed shape controller.

3. Fault detection in large-scale systems is another interesting research challenges. Future research could be devoted to distributed multi-group fault detection, in such a way that each local leader is responsible for monitoring its group members to detect any malicious behavior in the group. By applying distributed techniques, each local leader can have knowledge about the behavior of each group member.
4. In this thesis, agents motion are represented by fully actuated Euler-Lagrange systems. However, in some practical applications, wheeled agents are characterized by non-holonomic constraints. In future development, the non-holonomic constraints could be taken into consideration in the development of the controller.
5. The proposed shape formation control laws obtain shape formation at steady state. In some applications, it is desirable that the steady-state condition is obtained in a finite time interval. Recently, there is a great interest in the finite-time stabilization problem via finite-time controllers. The problem of finite-time stabilization of a dynamic system has been studied by discontinuous and continuous feedback laws. Future research could be devoted to the development of finite-time shape formation techniques, such that agents form the desired shape in a finite-time.

Appendix A

Theorem A.1. [154] *For every $N \times N$ matrix A and every number ω we have*

$$\text{rank}(A - \omega \mathbf{I}) \geq N - m_\omega(A) \quad (\text{A.1})$$

where $m_\omega(A)$ stands for the multiplicity of ω as eigenvalue of A . For $\omega = 0$ we obtain

$$\begin{aligned} \text{rank}(A) \geq \\ N - (\text{the number of zero eigenvalues of } A) \end{aligned} \quad (\text{A.2})$$

Since for any arbitrary digraph $\mathcal{G}$, the rank of the associate laplacian matrix $\mathcal{L}$ is less than $N - 1$, therefore, for a digraph which contains a spanning tree, we have

$$\text{rank}(\mathcal{L}) = N - 1 \quad (\text{A.3})$$

where $\mathcal{L}$ is the laplacian matrix of $\mathcal{G}$.

Lemma A.1. *Let $\mathcal{L}$ be the (in-degree) laplacian matrix of a digraph $\mathcal{G}$. Consider the following matrix:*

$$\mathcal{P} = \sum_{i=1}^{N-1} \beta_i \mathcal{L}^i \quad (\text{A.4})$$

where β_i are scalars. $(\mathcal{P})_{ij}$ is zero for any arbitrary values of β_i , if there is no path of any length from node j to node i .

Proof. To proof the lemma, we show that the $(i, j)^{th}$ element of all matrices $\mathcal{L}^i$ where $i = 1, 2, \dots, N$ are zero, if there is no path of any length from node j to node i . Since there is no adjacent path from node j to i , then $a_{ij} = 0$. Therefore, the $(i, j)^{th}$ element of the $\mathcal{L}^2$ can be expressed as follows:

$$(\mathcal{L}^2)_{ij} = \sum_{\substack{k_1=1 \\ k_1 \neq j}}^N a_{ik_1} a_{k_1j} = (\mathcal{A}^2)_{ij} \quad (\text{A.5})$$

Using lemma A.2, we obtain $(\mathcal{L}^2)_{ij} = 0$. Therefore, the $(i, j)^{th}$ element of the $\mathcal{L}^3$ can be expressed as follows:

$$(\mathcal{L}^3)_{ij} = \sum_{\substack{k_1=1 \\ k_1 \neq i,j}}^N \sum_{\substack{k_2=1 \\ k_2 \neq i,j}}^N a_{ik_1} a_{k_1k_2} a_{k_2j} = (\mathcal{A}^3)_{ij} \quad (\text{A.6})$$

Using lemma A.2, we obtain $(\mathcal{L}^3)_{ij} = 0$. Similarly, we can proceed for $\mathcal{L}^4, \mathcal{L}^5, \dots, \mathcal{L}^{N-1}$, and show that $(\mathcal{L}^k)_{ij} = 0$ for $k = 1, 2, \dots, N - 1$. $\square$

Lemma A.2. [155] *Let $\mathcal{A}$ be the adjacency matrix of a digraph $\mathcal{G}$, then $(\mathcal{A}^k)_{ij}$ is greater than zero if and only if there is a path of length k from node j to node i .*

Lemma A.3. *For two arbitrary matrices A and B , the Taylor series expansion of the matrix inverse is expressed as follows:*

$$(A + B)^{-1} = A^{-1} \sum_{i=0}^{\infty} (-1)^i (BA^{-1})^i \quad (\text{A.7})$$

where the spectral radius of BA^{-1} is less than 1.

Lemma A.4. [99] *Consider a digraph $\mathcal{G}$ with the corresponding laplacian matrix $\mathcal{L}$, which has a directed spanning tree; and a diagonal matrix $\mathcal{B}$ with nonnegative diagonal entries. The matrix $\mathcal{H} = \mathcal{L} + \mathcal{B}$ is nonsingular, if and only if, at least one of the diagonal entries of $\mathcal{B}$ is positive.*

Lemma A.5. *Extended Barbalat's Lemma [156]: If a differentiable function $f(t) \in \mathbb{R}$ has a finite limit as t goes to infinity, and its time derivative can be written as follows:*

$$\dot{f}(t) = g_1(t) + g_2(t) \quad (\text{A.8})$$

where $g_1(t)$ is a uniformly continuous function and

$$\lim_{t \rightarrow \infty} g_2(t) = 0 \quad (\text{A.9})$$

then

$$\lim_{t \rightarrow \infty} \dot{f}(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} g_1(t) = 0 \quad (\text{A.10})$$

Property A.1. *For any real numbers a_i , $i = 1, 2, \dots, N$ and $0 < p \leq 1$, the following inequality holds:*

$$\sum_{i=1}^N |a_i|^p \geq \left(\sum_{i=1}^N |a_i| \right)^p \quad (\text{A.11})$$

Lemma A.6. *Finite-time convergence theorem [157]: Consider the following non-Lipschitz continuous autonomous system*

$$\begin{cases} \dot{x} = f(x) & x \in \mathbb{R}^n \\ f(0) = 0 \end{cases} \quad (\text{A.12})$$

Suppose there exists a continuous function V defined on a neighborhood $\mathcal{D} \in \mathbb{R}^n$, such that the following conditions hold:

- (i) V is positive definite.
- (ii) There exist real numbers $c > 0$ and $\tau \in (0, 1)$ and an open neighborhood $\mathcal{V} \subseteq \mathcal{D}$

of the origin such that

$$\dot{V}(x) + c(V(x))^\tau \leq 0, \quad x \in \mathcal{V} \setminus \{0\} \quad (\text{A.13})$$

Then the origin is locally finite-time stable equilibrium of (A.12). The settling time, depending on the initial state $x(0) = x_0$, satisfies

$$T(x_0) \leq \frac{V(x_0)^{1-\tau}}{c(1-\tau)} \quad (\text{A.14})$$

for all x_0 in some open neighborhood of the origin. if $\mathcal{D} = \mathbb{R}^n$ and V is also proper (or radially unbounded), the origin of system (A.12) is globally finite-time stable.

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