Modeling of small-scale structures and the extension to the analysis of micro tool's dynamics

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**MODELING OF SMALL-SCALE STRUCTURES
AND THE EXTENSION TO THE ANALYSIS OF
MICRO TOOL'S DYNAMICS**

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**SCHOOL OF MECHANICAL AND AEROSPACE
ENGINEERING**

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ABSTRACT

Small-scale structures are essential elements in today's search for portable and space-constrained miniaturized systems. From micro-turbines, to micro tools and nano-composites, the idea that "small is the new big thing" is revolutionizing the ways the products of the future are being conceived, designed and built. Due to the extensive applications of miniature systems, the bafflingly complex behavior of their skeletal components, mostly made of micro- and nano-sized beams, plates or shells have to be fully understood. This thesis, therefore, examines the modeling of some small-scale structures based on two recently advanced higher-order theoretical frameworks, namely: Eringen's nonlocal elasticity theory and the modified couple stress theory. Based on these theoretical frameworks, novel higher-order mathematical models of non-uniformly tapered small-scale structures, small-scale structures with pre-twisted deformation and spinning small-scale structures are developed. A new method of analysis, based on the spectral element method, is implemented for parametric analyses of the models. Analyses results revealed that the rate of twist bifurcates the spectrum curve, while the small-scale coefficient improves the dispersion of the traveling wave and generates a group velocity that is more than twice the phase speed for all values of the wavenumbers determined. Under a practical range of the small-scale coefficient, the critical speeds of a spinning, small-scale structure is observed to be at least 20% higher than that predicted by the classical model. The extension of the model to the simulation of the response of a micro end mill revealed that an over-simplification of the model of the micro end mill, in which the micro flute is represented by a model that neglects its rate of twist and length scale, results in a wider margin of difference between the experimental and predicted features.

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In *Codependent No More*, one of those books with a title that almost betrays the tenor of its content, the author Melody Beatie offers this gem:

“Gratitude unlocks the fullness of life. It turns what we have into enough, and more. It turns denial into acceptance, chaos to order, confusion to clarity. It can turn a meal into a feast, a house into a home and a stranger into a friend.”

It is based on the above quote that I would like to express my appreciation to a number of people that have been of help in the course of this PhD journey. First, I express gratitude and heartfelt thanks to my supervisor, Associate Professor Zhong Zhaowei, whose constant encouragement, support and useful directions have been essential ingredients in carrying out this work. His spirit of tolerance, subtle approach to issues and patience are few of the traits I have found motivating in this academic journey. I am also deeply beholden to the management of A*STAR, whose generous scholarship scheme (SINGA) gave me the rare chance of pursuing a PhD in a renown and resourceful University like Nanyang Technological University. My thanks also go to Mr. Kong Seng Ann (Manufacturing Processes Lab 1, School of Mechanical and Aerospace, NTU) and past FYP students (Soh Kian Huat and Gunasekaran Santhosh) for the tremendous assistance rendered during the series of conducted experiments. Some friends fire up the spirit of life-long learning in you anytime you meet them. Others make you swallow the humble pie with the width of their wit and their shining spiritual piety. This doctoral work brought me in contact with friends that combine the above two traits and I say thank you to them all.

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- [1] Mustapha K.B., Zhong Z.W. Free transverse vibration of an axially loaded non-prismatic single-walled carbon nanotube embedded in a two-parameter elastic medium. *Computational Materials Science*, 50(2):742-51, 2010. (*Impact factor: 1.878*) (*Times cited: 20*) (*SCI Q2 Journal*)
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- [3] Mustapha K.B., Zhong Z.W. Stability of single-walled carbon nanotubes and single-walled carbon nanocones under self-weight and an axial tip force. *International Journal of Engineering Science*. 50(1):268-78, 2012. (*Impact factor: 1.691*) (*Times cited: 8*) (*SCI Q1 Journal*)
- [4] Mustapha K.B., Zhong Z.W. Spectral element analysis of a non-classical model of a spinning micro beam embedded in an elastic medium. *Mechanism and Machine Theory*, 53(4): 66-85, 2012. (*Impact factor: 1.24*) (*SCI Q2 Journal*)
- [5] Mustapha K.B., Zhong Z.W. Wave propagation characteristics of a twisted micro scale beam. *International Journal of Engineering Science*, 53(4): 46-57, 2012. (*Impact factor: 1.691*) (*SCI Q1 Journal*)

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- [6] Mustapha K.B., Zhong Z.W. A new modeling approach for the dynamics of a micro end mill in high-speed micro-cutting. *Journal of Vibration and Control*. 19(6): 901-923, 2013 (*Impact factor: 1.966*) (***SCI Q1 Journal***)
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LIST OF SYMBOLS

G	Shear Modulus
E	Young Modulus
$\mathcal{L}$	Lagrangian function
U	Total potential energy
Ω	Angular speed of rotation
I, I_{XX}, I_{YY}	Second moment of area
W	Virtual work done during elastic deformation
ρ	Mass density
A	Area of a cross-section
M_i	Bending moment (where $i = x, y, z$)
k	Shear correction factor
ϵ_b	Taper ratios in principal coordinate directions e.g. x and z
γ	Rate of twist
u, v, w	Displacement components in x, y and z directions
$\bar{W}(y), \bar{U}(y)$	Frequency dependent amplitudes of u, w
r_{1j}, β_{1j}	Amplitude ratios where $j = 1 \dots 8$
N_θ, N_w, N_u , and N_ψ	Frequency-dependent shape functions (uniform beam)
F_θ, F_w, F_u , and F_ψ	Frequency-dependent shape functions (twisted beam)

$\mathbf{S}(\omega)$	Stiffness matrix
λ	Non-dimensional natural frequency
$\boldsymbol{\sigma}$	Cauchy stress tensor
$\boldsymbol{\varepsilon}$	Strain tensor
$\mathbf{m}$	Deviatoric component of the couple stress tensor
$\boldsymbol{\chi}$	Symmetric curvature tensors
∇	Laplacian notation
∂_k, ∂_l	Differential operators
l_s	Material length scale
ϑ	Poisson's ratio
Π_s	Strain energy
ω_{cf}	Cut-off frequency
c_p	Phase speed
c_g	Group speed
K_f	Stiffness of an elastic foundation
K_s	Shear modulus of an elastic foundation
$e_0 a$	Nonlocal scale coefficient
r_g	Radius of gyration
$\mathbf{T}$	Transformation matrix

f_g	Global force vector
S_g	Global stiffness matrix
u_g	Global displacement vector
Q-Q	Quantile-Quantile
f_{pt}	Feed per tooth
φ_u	Cutting angle
c_t	Chip thickness
M_c	Material coefficient
h_β	Tool's helix angle
$\mathbf{C}$	Damping matrix

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CHAPTER 1 INTRODUCTION

1.1 Background

The art of mathematical modeling is almost like the art of programming; it is equal part marvelous and cryptic. However, beyond the layer of its cryptic nature, the elegance of mathematical expressions conveys inspiration and reveals hidden symmetry, and at times awe-inspiring phenomenon. From the time Eratosthenes [1] employed rudimentary mathematical intuition to quantify the circumference of the globe to the discovery of the theory of relativity by Einstein, mathematics has remained the natural language to understand the world around us. Nonetheless, mathematicians and engineers approach mathematics with a different mindset. Mathematical models, in the context of engineering discipline, are not just theoretic conjectures but crucial ingredients in the analyses and designs of many engineering problems. Dealing with engineering problems requires an awareness of the class to which the problem being tackled belongs. That is, is the problem being tackled a direct problem or an inverse problem? Accordingly, knowing the class of problem being confronted makes the selection of apposite analysis tool easier.

The bulk of this study falls majorly under the category of direct problems. Direct problems, as concisely expressed by Gladwell [2], involves analyses and predictions of the behavior of engineering structures or systems from known physical parameters or inputs. Linear and nonlinear stress analyses, forced response prediction, stability and natural frequencies evaluation are all examples of a direct problem. In contrast to direct problems, inverse problems are, most often, ill-posed in formulation and require the use

of the method of quasi-reversibility for solution [2]. The type of direct problem touched on in this study deals with the prediction of the dynamic characteristics of micro-sized structures of varying complexity. In this regard, a set of novel mathematical models, intended to form the prerequisite for direct-type analyses of micro-structures, are developed. A logical question that arises from this preamble is, why the focus on micro-structures? The answer to this question is rooted in the motivation for this research enumerated in the paragraphs that follow.

1.2 Motivation

Based on the projected global market analyses from three independent emerging technology market analysts (iNEMI, NEXUS, and Yole Development), the *Compound Annual Growth Rate (CAGR)* of microsystems, in its various forms, has seen a steady rise since the beginning of this millennium [3-5]. This robust growth in demand for microsystems, as noticed from Figure 1-1, has also generated parallel growth in microsystems application, microsystems materials and microsystems equipments. Micro-scale structures are essential elements in the search for most portable, compact, and space-constrained microsystems or miniature systems. The extensive application of miniature systems has enhanced the systematic integration of complex functionality at a lower manufacturing cost. This, in turn, has ushered in vast possibilities in microsystems and microelectromechanical systems (MEMS) design and capabilities. Miniature systems have diverse applications as demonstrated by their use as microactuators [6], capacitive microsensors [7], and impedance micropump [8], micro-channels for lab-on-chips, stents, micro-fluidic graphite channels for fuel cell applications, fluidic micro-chemical reactors,

micro-molds, sub-miniature actuators and sensors, to mention but a few [9].

Figure 1-1 A projected market analysis of microsystems from year 2002 to year 2008 [3-5]

On a more general level, miniaturization of products have been tipped to provide micro-systems that will enhance quality of life, healthcare and economic growth [10]. Apart from their use as integral parts of microsystems, there are other application areas where micro-scale structures remain invaluable. These areas include micromachining and rotating micro-machinery. In the context of micromachining, micro-drills and micro end

mills have seen increased use as rotating shafts in material removal processes. Similarly, with respect to rotating micro-machinery, the bulk of which constitutes many power MEMS for energy conversion applications, rotating micro-structures have precipitated revolutionary changes with the innovative use of spiral microrotor [11], microgyroscopes [12], micro-rocket engine and micro-compressors [13]. Certain advantages have contributed to the demand for many of the miniaturized components and systems. Some of these advantages include, but not limited to [14]:

- ✚ less power consumption
- ✚ lower thermal emission
- ✚ reduced weight
- ✚ compactness and increased handiness
- ✚ improved strength of the underlying bulk material

The demand for microsystems or miniature systems first started with silicon micro-fabrication technologies which generated a whole lot of MEMS [15] over a wide range of applications. Micro components require features that include but not limited to: high dimensional precision (generally expected to be better than 1 *micron*); accurate geometrical form (with enhanced flatness or roundness); and good surface finishing. These features, in turn, require machine tools to have high static stiffness, low thermal distortion, low motion errors and high damping or dynamic stiffness. Right now, the domain knowledge of micro-fabrication can be said to be well-established. Among the commercially viable and heavily reported micro-fabrication technologies is LIGA (a

photo-lithography technique that makes use of a synchrotron). Others are epitaxy, laser beam fabrication, micro-electro discharge machining, ion beam technique and ultrasonic vibration [16].

With the increasing demand from a whole range of industries for highly accurate, smaller and better components, it becomes clear that the majority of the semiconductor-based methods inhibit the broad commercialization of microsystems by their low productivity. They have also been found to be limited in the breadth of materials they can handle for processing. They are limited in the degree of relative accuracy attainable and are mostly suitable for planar geometries [17]. Consequently, fabrication of micro-components pushes towards robust, flexible, reliable and repeatable methods with accurate analysis tools. The unrelenting quest for mass production and broad commercialization of product miniaturization spurred innovative endeavor in continuous reduction of component feature sizes. It is also responsible for the corresponding reduction in diameter of the cutters employed in micromachining [18]. It is this market pull and the research push that have led to the concept of mechanical micromachining.

Mechanical micromachining is regarded as another choice for an effective method of creating miniature devices and components with features that range from micrometers to a few millimeters in size [19]. Dario et al. [20] described the principle of mechanical micromachining as partly based on systematic shrinking of the size of the well-known mechanical structures and processes in the macro domain down to the micro domain. The result of this systematic shrinking is a cutting process by miniature machine-tools and

micro-cutters. To put it concisely, mechanical micromachining utilizes miniaturized tools to perform cutting operations like turning, drilling and milling at the micro-scale level.

Mechanical micromachining processes have a number of distinctive advantages in the fabrication of micro-sized components. These include the ability to create 3D free-form features and high material removal rate. They are also cost effective (as they do not require the use of expensive mask and expensive set-up of lithographic methods). Mechanical micromachining is also very flexible compared to the other processes on the kind of workpiece material that can be handled [19]. It is a particularly suitable alternative for various metallic micro parts. It has been successfully used to handle metallic alloys, ceramics, and polymer with complex microstructures and functional shapes [21]. Additionally, because the size of the scaled-down machine tool is usually about tenth to several tenths of the size of the conventional machine tool, the effect of power saving is absolutely high [22]. Scaled-down machine tools are easy to maintain, reconfigure and provide atmosphere for rapid response manufacturing processes. Though a scaled-down version of the conventional machining, mechanical micromachining comes with a mirage of critical issues. These issues arise from the miniaturization of the tools, the components and the processes. Investigations of technical issues in cutting fine parts are still very much in the early stages [19]. The performance index of micro-cutting tools in particular has been found to be profoundly influenced by vibrations and unstable cutting force [17]. The depth of cut and feed rates during micromachining are smaller when compared to conventional machining. The depth of cut during micromachining is so small that it can be smaller than the workpiece average grain size [23]. This makes the tool works intra-crystalline and hence cuts through grain boundaries inside the

workpiece. The workpiece material's grains are inherently non-homogenous. As the tool moves from grain to grain and point-to-point inside the material, it passes through irregular terrains. This in turn causes variation in cutting force values and vibration. The cumulative effect of random vibrations and excessive cutting force is obviously detrimental to component's tolerance and longevity of the micro-tools [24].

In reality, there are various types of mechanical micromachining processes that have been widely employed. Micro end milling, according to Filiz et al. [25], is by far the most versatile. The wide acceptability of micromilling has been aided by the easy availability of micro end mills. Most commercial micro end mills are usually made from sintered tungsten carbide and have a meso-scale shank to enable their attachment to the spindle collet. In order to achieve the required cutting speeds and increase productivity, micro milling operations are usually carried out at high spindle speeds (80 *Krpm* – 500 *Krpm*) [26]. High speed machining like the one used for micromilling has the advantages of increased material removal rate, increased dissipation of heat and better surface finishes [27]. Milling has been the fabrication of choice for a wide variety of 3D components, even in the traditional material removal domain. Research studies dealing with cutting force models, tool-interaction models and tool failure in the macro-domain have been immensely reported. Smith and Tlustý [28] provided a detailed overview of the modeling and simulation of the traditional milling process. In this review, the authors classified and evaluated various models earlier developed with respect to mechanistic, empirical and semi-analytical model in the traditional milling operation. There has been recent effort in the same direction of tool modeling in the conventional milling. For example, Balachandran and Zhao [29] proposed a unified mechanics-based model with

multiple degrees of freedom to simulate the tool-workpiece interactions during the milling of ductile workpiece using helical tools. In a different study by Altintas and Engin [30], a generalized model for mechanics and dynamics of helical cutters based on parametric design criteria were presented.

However, the difference between the milling operation at micro-domain and macro-domain is wide. According to Chae and Park [31], problems that generally engender no threat in the macro-level, such as tool run-out, are often amplified in the micro-domain. Further evidences of the difference are provided in Tansel et al. [32]. Tansel et al. revealed that the wear and breakage mechanisms are far more complex in the micro-milling due to complex tool's flexibility and challenging cutting dynamics. When tool's flexibility is coupled with process faults, the resultant effect inevitably leads to large vibration effect, buckling and failure.

There are other challenges facing micro end milling technology arising from the systematic scaling down of the end milling processes. These challenges are summarized below.

1.2.1 Tool dynamics

In his classic book, Shaw [33], suggested that the most common cause of failure of small-size cutting tools (with diameters less than 1.5 mm) is highly related to the tools dynamics. It is also explained that uncontrolled vibration of the tool can easily lead to breakage as a result of either too great a torque or excessive thrust force. According to an internal report [34] by Union Tool Co., one of the largest manufacturer of micro-cutting tools in Europe, tool failure is an imminent threat to machining productivity in

micro-end milling. As opposed to failure mode in the conventional end milling, micro end mills have been found to be prone to failure more by fracture at the tool root, than by edge wear. A number of tests were conducted by Union Tool [34] to provide an understanding of the reasons for micro-cutter breakage. Based on its finding, it is concluded that when a micro end mill breaks, a careful examination of the broken end of the micro-cutting tool can give an indication as to the cause of such a breakage. Researchers at Union Tool Co. determined that micro end mills breakage may occur due to excessive twisting or bending. From mechanics point of view, excessive twisting leads to excessive shear stresses, while excessive bending in turn results in excessive bending stresses. The type of stress that leads to eventual failure is evident by examining the fractured surface of the micro end mills. This holds true for both the static and dynamic situations. Further studies by leading researchers [32, 35] have even shown that examination of the fracture surface of tungsten carbide micro-end mills reveals a failure due to brittle fracture. Based on the above studies and because of the small diameter of the micro-cutting tools, the cutting force must be kept very small in order not to exceed the bending stress fatigue limit of the tool. The comprehensive understanding of tool dynamics, and therefore failure, in micromachining technology is a research area that has left a gap to fill. This is due to the fact that most literature available in this area concentrates only on monitoring tool states that leads to failure, while the mechanics that leads to such a failure are often neglected.

1.2.2 Cutting edge radius

The nature of cutter's edge radius in micromachining has also been a subject of research [36]. Most commercially available micro end mills usually have cutting edge radii of the order of $1.5 - 3 \mu m$. These edge radii are substantially higher than the feedrates. Since the feedrates in micromilling is comparable to the cutter's edge radius, material removal mechanism is dominated by ploughing and rubbing mechanism [37]. This is in stark contrast to the shearing-dominated cutting mechanism in the conventional end milling. This phenomenon also indicates that the effective rake angle of the cutting edge is negative, leading to cutting force coefficients that is much higher than what is experienced in conventional end milling operations [38].

1.2.3 Cutting speed

It is logical that as the cutting tool diameter decreases, the spindle speed must increase. This is to give the optimum peripheral cutting speed for productive material removal rate. For very small diameter tools like micro end mills, spindle speeds of over 300 Krpm are needed for effective 3D micromilling. A tool rotating at such high spindle speeds is prone to experience unbalancing dynamics contributed by the different components of the rotating assembly. Consequently, the unbalancing dynamics coupled with the flexibility of the micro-cutting tool causes excessive deflection of the tool.

1.2.4 Spindle run-out

Spindle run-out remains a fundamental problem in many machining operations where cutters with slender shanks are employed. Micromilling is therefore no exception to this problem. Spindle run-out is a process-based fault that is associated with tool alignment errors. Commercially available micro-milling spindles have been reported to typically have radial run-out at the tool tip on the order of 1 to 2 micrometers or more [39]. While the same run-out creates negligible change at the cutting force profile of traditional end milling operations, its adverse effect has been noticed in micro end milling operations. The magnitude of this radial run-out is larger than the desired chip thickness and inevitably leads to overloading of the tool and premature failure.

With some of the stated challenges above, it is quite clear that a micromachining operation is not simply a rough miniaturization of the conventional material removal technology. It is an operation that is faced with its own unique challenges. A number of works have been done to study the cutting dynamics in micromilling [40-41]. However, the relationship between the complex dynamics arising from the special geometrical designs of micro end mills and some of the cutting characteristics remains largely unclear. The influence of process variables on the dynamics of the micro end mill's flute has not been studied. The prediction of the actual dynamics of these micro cutters is still subject of ongoing research. A large gap in literature therefore exists to be filled.

1.3 Research objectives

The key objective of this research study is the mathematical modeling of small-scale structures through non-classical elasticity theory. However, since the term small-scale structures is wide, the focus of the research is to concentrate on mathematical models that are suitable to provide a window through which the dynamics of a micro end mill (taken as a case study) could be understood. With this premise, the following tasks need to be carried out:

- ✚ Development of analytical models for the evaluation of the dynamic characteristics of small-scale structures with geometries relevant to the model of the micro-cutting tool (such as spinning micro-scale structure with uniform cross-section, spinning tapered micro-scale structures, and spinning twisted micro-scale structures)
- ✚ Development of appropriate solution procedures for the governing equations of the small-scale structures with geometries relevant to the model of the micro-cutting tool

1.4 Research scope

The scope of this research is limited to the study of the dynamic response of micro-sized structures based on elasticity theory (non-classical and classical) and computational analytical and numerical methods. It is a research effort that employed the power of these computational methods to investigate the dynamic behaviors of spinning

micro-sized structures, with micro-cutting tools as a special case. As a result, this research does not cover the issue of chip morphology during high-speed micromachining which is more related to manufacturing and material science. Chip morphology has been found to affect mostly tool wear but not necessarily tool dynamics response. The effect of temperature and thermal stress analysis are not covered in this research. This is because temperature variation is assumed to be constant during most micromachining operations [42].

1.5 Organization of the thesis

The outline of this thesis is as follows. In chapter 2, a comprehensive literature review of the existing works, with some degree of relevance to the current study is presented. It is important to stress that, the review in chapter 2 is meant to provide a window through which the place of the current study is viewed. Further information about other pertinent studies is provided where necessary within the context of each chapter to stress what is available in the literature and what is not.

The objective of chapter 3 is to demonstrate the framework adopted in addressing the objectives highlighted in chapter 2. Hence, in chapter 3, research methodology and the significance of the current research are highlighted.

In chapter 4, the initial phase of the present study involving an analytical study devoted to the mathematical modeling of a micro-cutting tool's dynamics is presented. Chapter 4 also includes the preliminary finite element simulation results conducted with ANSYS.

Chapter 5 is a rigorous demonstration of two higher-order, non-classical elasticity theories suitable for the prediction of response of small-scale structures. In this chapter, novel theoretical models of small-scale structures with non-trivial mechanical behavior and complex geometries are presented.

In chapter 6, detail is provided about the extension of the theoretical models presented in chapter 5 to the study of the forced vibration characteristics of a micro end mill. A number of experiments conducted with respect to the response of micro end mills are also presented to complement the theoretical model.

Chapter 7 is a summary of the conclusions from the numerous analyses carried out in the course of this study and the direction of possible future extension of the work.

CHAPTER 2 LITERATURE REVIEW

2.1 Introduction

Miniaturization is fast becoming a central theme in the contemporary fabrication technology [43]. The capability to fabricate three-dimensional micron features on a wide variety of materials makes micro-cutting operations a realistic and suitable micro-manufacturing platform [44]. However, insufficient understanding of the mechanism responsible for unstable micro tool behaviors, as pointed out by Masuzawa and Tönshoff [45] and Jun et al. [26], hampers progress in micromachining. Tool vibrations, either in micro or macro cutting operations, can lead to a number of disappointingly unacceptable consequences. Prominent among these unwanted consequences are [9, 46-49]: (i) the systematic degradation of the quality of the desired features or machined surfaces, (ii) the impairment of expected tolerances, (iii) the dynamic instability resulting from chip size modification, and (iv) the shortening of tool life and reduction of productivity. Besides, the determination of regions of stability in most machining processes depends on many factors that range from the cutting conditions to the dynamics of the machine spindle-tool system [10, 50].

Shaw [33] demonstrated that the best choice of cutting parameters is prone to the influence of the structural characteristics of the cutting tool. It is a result of the above-mentioned reasons that studies to understand the behavior of the conventional tools have been undertaken to pave way for a more informed behavior of the machine stability for years [30, 50]. Micromachining differs, in concept and precept, from the traditional metal

removal processes as widely reported in the literature [51-53]. As a result of the difference, the tool vibration effect coupled with unique process-induced factors pertaining to the micro domain raises even more concern about process stability and feature accuracy. In micro end milling, which is the focus of this study, the stress distribution on the tiny tip of the micro end mill is disproportionately larger than that of the traditional macro-sized end mill and a high spindle speed is required for a productive operation [9]. Furthermore, the edge radius of the micro end mill and uncut chip thickness approach each other in size, leading to a negative rake angle [41]. Additionally, tool life and productivity show explicit dependence on a cautiously chosen feed-rate to tool radius ratio (f_t/r) [39]. From the unavoidable tool-process interaction enumerated above, it is expected that the geometry of a micro end mill would ultimately contribute to the inputs and outputs of the process.

The mechanics and dynamics of micro-cutting operations have to be understood, empirically or analytically, in order to understand process behaviors, if expensive experimental trials are to be minimized. The stability, vibration and balancing of micro tools are matters of great practical importance that are better understood through system modeling. In their conventional size, cutting tools have been the subject of numerous experimental and theoretical studies [54-56]. However, the geometric modeling of micro tools, in particular micro end mills, remains inchoate [57].

This chapter provides a review of literature in areas that are found somewhat close to the proposed works. The phrase ‘somewhat close’ is used because the proposed research work spans the realm of numerical analysis, structural analysis, as well as the machining processes. However, rather than being about the machining technology itself,

this work is more about the extension of certain structural models to the prediction of the response of an aspect of the micromachining process. Therefore, this literature review will be incomplete without presenting the state-of-the-art of the past and the current endeavors with respect to some aspect of the machining research studies. This is because mechanical micromachining systems are regarded as systematic scaled-down versions of the macro-scale systems. Therefore, it will be logical to review the relevant studies that have been conducted in for instance, conventional end milling processes, to provide the rhythm and flow needed for the present work in micro end milling.

2.2 Design and mechanics of conventional helical cutters

The modeling of macro tools for structural and dynamics analyses has taken on so many directions [58]. Some researchers have favored the definition of the geometry of these tools using the principles of projective geometry. Others have adopted the use of discrete and continuum models. In metal cutting industry, cutters such as conventional end mills play key roles in obtaining the desired size and shape of most complex components. The modeling and simulation of the dynamics behaviors of macro end mills can therefore be said to be crucial steps in the realization of high quality machined parts and features. Besides, for precise simulation of most material removal operations, accurate mathematical and geometrical models of the cutting tools used in the machining processes are of great importance [33]. Wide ranges of milling cutters used in practice are fluted in geometry and a variety of them are used in the manufacturing industry. Helical cylindrical, tapered helical, and helical ball end mills are some of the common end mills used in die machining, aerospace, and automotive industries [59]. Cylindrical helical end

mills are commonly used in peripheral milling of prismatic parts. While straight and helical ball end mills have been reported to be widely used in machining of sculptured die and aerospace part surfaces [30]. The comprehensive geometrical analyses of the tool surfaces together with the cutting forces acting on them play a significant role in the design of the end mills and the quality of the machined part [59].

In 1997, Ehmann et al. [60] presented a summary of the overview of past research studies in the mechanics and dynamics of milling cutters. Although the review paper only focused on the standard end and face milling cutters, the four modeling approaches commonly used for milling cutters were summarized. These approaches include analytical, experimental, mechanistic and numerical techniques. Altintas and Lee [58] and Altintas and Engin [30] were the first set of researchers to present a generalized mechanics and dynamics model suitable to analyze any cutter geometry. In these works, they employed their model to analyze a variety of end mill geometrical shapes used in the manufacturing industry. In 2001, an improved generalized parametric design model for helical end mills was presented by Engin and Altintas [30]. Following the manual and computer-assisted continuous path programs, Engin and Altintas [30] proposed the use of seven geometric parameters ($D, R, R_r, R_z, \alpha, \beta, h$) shown in Figure 2-1, to describe the cutters profile parametrically. In their work, the periphery of the macro end milling cutter is divided into three and each of these zones is described parametrically as[30]:

$$r(z) = \frac{z}{\tan \alpha} \quad (2.1)$$

$$r(z) = \sqrt{R^2 - (R_z - z)^2} + R_r \quad (2.2)$$

$$r(z) = \frac{D}{2} (1 - \tan \alpha \tan \beta) + z \tan \beta \quad (2.3)$$

Eqs. (2.1) to (2.3) are the parametric radial equations for OM , MN and NS (shown in Figure 2-1) respectively. In Eqs. (2.1) – (2.3), $r(z)$ is used to represent the radius of the end milling cutter at an elevation z . α is the angle between the horizontal axis of the reference frame and the clearance face of the cutter, β is the angle between the vertical axis of the reference frame and the clearance face of the cutter, D is the effective diameter of the cutter's base. R_r and R_z are the radii measured along r and z axes. It would be observed that for each of the cutter envelopes OM , MN and NS that Eqs. (2.1) to (2.3) represent, the expression for $r(z)$ depends on the variables like the elevation z , angles α and β , radial distances such as D , R , R_r , and R_z . Among these variables, α is generally in the range $0 < \alpha < 90^\circ$. The value of each of the other variables, as highlighted in Engin and Altintas [30], is a function of the type of helical cutter that is being dealt with (e.g. cylindrical end mill, ball end mill, ball nose end mill, taper end mill, taper ball end mill or general end mill). Hence, according to Engin and Altintas [30], the range of values of the expression for $r(z)$ will depend on the geometry of the end mill that is being studied.

In addition to the above equations for the periphery, an arc with the center at a point C , a radial offset R_r and arc of radius R , intersects the taper lines OL and LS at two points M and N , respectively. However, both points M and N have radial (M_r) and axial (M_z) offsets from the cutter tip and axis. These offset distances for point M are found respectively from [30] as:

$$M_r = \frac{R_z \tan \alpha + R_r + \sqrt{(R^2 - R_r^2) \tan^2 \alpha + 2R_z R_r \tan \alpha - R_z^2 + R^2}}{\tan^2 \alpha + 1} \quad (2.4)$$

$$M_z = M_r \tan \alpha \quad (2.5)$$

The offset distances for N are also found from:

$$N_z = \frac{(R_r - u) \tan \beta + R_z - \sqrt{(R^2 - R_z^2) \tan^2 \beta + 2R_z(R_r - u) \tan \beta - (R_r - u)^2 + R^2}}{\tan^2 \beta + 1} \quad (2.6)$$

$$N_r = u + N_z \tan \beta \quad (2.7)$$

Figure 2-1 General tool geometry for macro end mills [30]

where the angles β and α are all illustrated in Figure 2-1. The other phase of the parametric modeling involves wrapping around the helical cutting edge with the help of a position vector defined as

$$\vec{r}_p = (x_p \vec{i} + y_p \vec{j}) + z_p \vec{k} \quad (2.8a)$$

$$\vec{r}_p = r(\varphi_p)(\sin \varphi_p \vec{i} + \cos \varphi_p \vec{j}) + z(\varphi_p)\vec{k} \quad (2.8b)$$

where φ_p is the depth of immersion of any point P on a particular flute which varies as a function of flute position, local helix angle at the point of consideration and rotation angle. Altintas and co-workers made enormous contributions to the advancement of design and mechanics of milling cutters. Unfortunately most of their available works for specifying the geometry of single and multi-point milling cutters are not suitable for providing downstream technological applications [61].

The developments in the field of computer aided geometric design now provide designers with considerable latitude to specify the cutting tool surfaces as bi-parametric surface patches. Such an approach was recently reported by Tandon et al. [59]. In their studies [59, 61-63], they sought to provide comprehensive three-dimensional definitions of the cutting tools (end mills, fluted cutters and slab mills) suitable for downstream engineering applications like prototyping and finite element analysis. The concepts of their studies involve a blend of surface modeling and computational geometry. Employing a combination of CAD sweeping rules and mapping relations, Tandon and co-workers were able to generate fluted section of end mills with different helical paths (cylindrical, conical and hemispherical). They also presented how the surface model of a cutting tool made from their proposed algorithms can be converted into a solid model and used for finite element analysis and simulation. A procedure similar, but algorithmically different from that of Tandon et al. [59] was employed by Hinds and Treanor [64] for the evaluation of stresses in micro-drills using the method of finite element analysis. In the

same direction, Abele and Fujara [65] also employed the surface generation method for the design and geometry optimization of the conventional twist drill.

2.3 Modeling of cutting forces

The modeling and analysis of the force system is critical to ensuring part quality, process planning, optimizing production rates and for reducing vibration and tool wear [66-67]. Numerous studies, in the realm of conventional machining, have been presented for the study of cutting forces in the conventional end milling process. Pioneering works, such as [68-69] were significant in presenting superior understanding of cutting force. In Smith and Thusty [28], a detailed overview of modeling and simulation of milling process that classifies and evaluates various models earlier developed with respect to cutting force prediction is presented. In essence, a number of approaches have been used for force prediction, but the mechanistic modeling approach remains the most widely employed for force prediction. Altintas and Lee [50] reported a way to estimate cutting forces through the proper transforming of shear stress, shear angle and friction angle to a local oblique coordinate system. Lazoglu and Liang [70] employed angular domain convolution to formulate the generation of total cutting forces in a ball end milling process. In all of the above cited literature, the magnitudes of the instantaneous tangential cutting force, F_t , that acts on the cutting edge is defined in such a way as to contain the axial depth of cut, specific cutting force and the uncut chip thickness. In general the tangential cutting force, F_t can be expressed as [54]:

$$F_t = K_s a h(i, j, k) \quad (2.9)$$

where $h(i, j, k)$ is the chip thickness. The parameter h varies with the feed per tooth (f_t) and the rotation angle β as shown in the equation below:

$$h(i, j, k) = f_t \sin \beta(i, j, k) \quad (2.10)$$

where feed per tooth has a unit of length and (i, j, k) is used to describe the i th axial segment, the j th angular position and the rotation angle of the k th flute cutting edge. The specific cutting force term, $\mathcal{K}_s$, as observed by many investigators [66, 71-72] depends on cutting speed V_c , the cutter's rake angle (α_n) and the chip thickness. It is a term that is often empirically determined based on the relation below [54]:

$$\ln(\mathcal{K}_s) = a_0 + a_1 \ln h + a_2 \ln V + a_3 \ln \alpha_n + a_4 \ln h \ln V \quad (2.11)$$

where the cutting coefficients a_0, a_1, a_2, a_3 , and a_4 are usually determined experimentally. Once they are all determined, the forces are transformed to the machine's XYZ coordinates using an appropriate transformation matrix. Meanwhile, in addition to the mechanistic force prediction models, a number of works on the calculation of the instantaneous cutting forces with cutter run-out have also been reported. Seminal works in this aspect include Armarego and Deshpande [73] and Wang and Liang [74]. Parameter identification of the cutter run-out in end milling was reported by Zheng and Liang [75]. The insight and understanding that the above cited literature have on the processes involved in the conventional end milling domain are invaluable. However, all these studies only based their formulation on the kinematics of the cutter at static or very low rotation. Besides, they usually need quite a number of empirical coefficients to be determined for each combination of machining conditions before the force or chip estimation can be conducted for different cases. These models are not so easily amenable

for simulation and optimization for design and cutter's failure analysis. Therefore, these models are only suitable for the conventional end milling operations and not micro-end milling.

2.4 Development in micro end milling

According to Liu et al. [76], the earlier research on the mechanics of material removal processes and surface generation in the mechanical micromachining were limited to orthogonal cutting and turning. The reason for this limited scope is attributed to the initial erroneous perception that the micro-sized cutter will behave like macro-sized counterparts. However, the roundness of the cutting edge in micro end mills and the existence of a minimum chip thickness below which ploughing occurs pose considerable challenges. Therefore, in micro end milling, the Merchant's assumption [33] that material removal is predominantly by the mechanical shear force of the interaction between the sharp tool and workpiece is no longer valid. Since Merchant's assumption is no longer valid, the development of new models to accurately predict cutting force, chip thickness and tool dynamics at the micro-scale level became necessary.

2.4.1 Specific cutting force and chip formation in micro end milling

Measurement of the specific cutting force is one of the available ways to determine the size effect in micromachining [77-78]. According to Lui and Melkote [79], the size effect in material removal at the micro-scale level is characterized by a non-linear increase in specific cutting energy because of decrease in undeformed chip thickness. The authors further explained that, in most mechanical micromachining, an effective negative

rake angle enhances the ploughing effect. As highlighted in Table 2-1, the mechanism of chip formation in micromilling is more complex than that in the conventional milling process.

Table 2-1 Geometric and process influences on macro- and micro-sized end mills

Micro end mill	Macro end mill	Reference
The run-out of the tool tip greatly affects the accuracy of the milling operations at micro-scale.	Tool run-out is not a great threat.	[39]
Micromilling is associated with sudden tool failure due to its highly unpredictable cutting action.	Higher tool strength prevents sudden failure.	[40]
The chip formation in micromilling depends upon a minimum chip thickness	Chip formation is governed by shear deformation at the tool-workpiece contact points.	[80-81]
Tool deflection in the micromilling greatly affects the chip formation and the accuracy of the desired surface.	Reduced tool deflection.	[82]

A number of other researchers have also established that the ratio of undeformed chip thickness to cutting edge radius plays a significant role on specific cutting force when the undeformed chip thickness is less than the cutting edge radius [83]. Arsecularatne [84] stated that though the ploughing phenomenon during micromilling might be used to explain the size effect, it is really difficult to measure directly. Hence an

indirect way is to observe an increase of the specific cutting force with decrease in the undeformed chip thickness.

Chip formation in micromilling depends on the concept of a minimum chip thickness. Basuray et al. [85] defined the minimum chip thickness as the ratio of the chip thickness to cutting edge radius. The experimental investigation of effect of minimum chip thickness was reported by Yuan et al. [86]. Using two different cutters with edge radii of $0.3\ \mu\text{m}$ and $0.6\ \mu\text{m}$, Yuan et al. found that chips formation does not occur below a critical value of 30 % of the cutter's edge radius. In a separate study, Ikawa et al. [81] also reported the issue of minimum chip thickness and its contribution to surface roughness. Their investigation was based on the result of molecular dynamics simulations (MDS) for both aluminum and copper. In what appears to be another nod to importance of determining the contribution of the minimum chip thickness to the mechanics of micromachining, Wu et al. [87] employed a variant of the plastic strain gradient to proffer another insight into the quantification of minimum chip thickness.

2.4.2 Surface roughness and burr formation in mechanical micromachining

The quality of machined surface and part lies at the heart of the performance index of most micromachining techniques. Weule et al. [88] and other works such as [38, 89] have revealed that material properties like microstructure and hardness significantly impact the achievable surface roughness in micromachining. Some investigators like Yuan et al. [90] presented studies on the measurement of surface roughness of a metal matrix composite such as Silicon Carbide (SiC) machined with the aid of diamond micro-turning operation. In their study, Yuan et al. observed that the amount of SiC whiskers

affects the allowable surface roughness. In another study, Chan et al. [91] carried out a number of cutting experiments with metal matrix composites using different cutting parameters. Their conclusion was that surface roughness has a tendency to increase with increased cutting speed. During micromachining, inhomogeneous multi-phase material of different hardness and ductility consisting of ferrite and pearlite are encountered. Often times, this encounter generates different ratios of cutting radius to grain size. As a result of this encounter with different phase ductility, process mechanism is bound to differ in the micro domain from the conventional machining process [92]. From the available literature, it can be concluded that to achieve a good surface finish for micromachining with a tungsten carbide tool, hard homogeneous material remains the ideal choice. Apart from surface finish, burr formation has been found to be another crucial factor in micro end milling. Burr formation affects the capability to meet tolerance and geometry definition [57]. What makes burr formation such a big threat in micro-cutting operation is that conventional de-burring techniques cannot be applied on micro-components. Therefore, it is imperative to find a way of eliminating burr formation either by developing strategies for minimization or adopting some of the new post-processing technology for burr removal proposed in [93-94].

2.4.3 Process instability and tool condition monitoring in a micro end milling operation

Most micro end milling operations are carried out at high spindle speeds. High-speed machines are associated with new set of challenges. Somewhat chaotic dynamics in cutting processes (e.g. chatter) present problems in micromachining. In conventional

machining, regenerative chatter stability and even occurrence are dependent on cutting conditions, dynamics of machine tool spindle and workpiece material behavior [68]. In micromachining however, chatter stability requires taking into account the ‘spring back’ elastic behavior of the workpiece because of low rate of feed. Efficient estimation of the overall structural dynamics of the micro-machine system is crucial to preventing chatter [31]. Jorgensen et al. [95] reported that at a very high speed, the dynamics of micro-machine tools are affected, resulting in an unbalanced spindle producing both centrifugal and gyroscopic effects. In their studies, a complete bearing load-deflection analysis that includes thermal expansion was derived and coupled with an analysis of spindle dynamic response. They also found steady-state temperature distribution from heat generation at the tool-workpiece contact point from a quasi 3D heat transfer model.

Tool condition monitoring (TCM) has been studied with respect to the conventional machining operations [96]. Based on the available literature, one observes that TCM can be carried out in the form of direct or indirect measurements of the features of interest. However, indirect measurement of the features e.g. vibration, acoustic emission (AE), force signals seems to be mostly favored by the researchers [97]. The reasons why the indirect measurement approaches are favored are because micromachining is carried out at very high speeds and the noise signal for monitoring micromachining is reported to be very high [98]. During or after the indirect measurements, statistical inferential algorithms or intelligent approaches are then applied to estimate the rate or degree of failure. Compared to conventional machining, there has been quite little work in micromachining TCM due to the difficulty encountered in micromachining TCM. Most studies on micro milling are those reported by Tansel and

his research team [39, 99-100]. Using an enhanced vision-based technique, Trujillo et al. [100] monitored the rate of wear of a micro end mill. Trujillo et al. [100] discovered that the intensity of the variation of the reflected laser beam from the micro end mill surface was really very sensitive to the micro end mill wear. Tansel et al. [99] applied cutting force in the monitoring of micro end mill breakage, however, they found that the variations of the static feed force were quite sensitive to tool breakage. This is due to the fact the cutting force in micromachining is highly noisy. In a separate study [99], Tansel et al. attempted to discover the tool wear from the cutting force with the aid of neural network (NN). Motivated by this work, Bao and Tansel [40] presented an analytical cutting force model for micro end milling to predict cutting force and estimate tool wear at the same time using NN. It was later found out that the monitoring systems depend on the network structure and this network cannot be generalized. Other methods that have been proposed for failure monitoring in micro end milling is the tool-workpiece voltage monitoring and the reflective single-beam laser system [18]. Jemielniak and Arrazola [101] employed AE signals in the monitoring of micro end mill wear. The results revealed strong influence of tool wear on AE signal. In a recent study to improve the works of Tansel and his collaborators, Zhu et al. [97] employed the method of hidden Markov models (HMM) for a multi-category monitoring of micro end mill wear. The shortcoming of the HMM is similar to that of NN. HMM is network-dependent.

2.5 On the dynamic of spinning structures

Investigations on the dynamic behavior of spinning structures are related to the vibration and stability of rotating shafts, boring bars, drills, end mills, and spin-stabilized

satellite booms. The stability, vibration and balancing of these elements are matters of great practical importance for rotating machinery. Hence, these structural systems have been the subject of numerous experimental and theoretical studies. In general, the essential feature that distinguishes such rotating systems from their non-rotating counterparts is the complexity of the accelerations associated with them. The accurate prediction of the dynamics of these systems is critical to meet the increasing design requirements in modern applications of high-speed machinery and turbomachinery. Apart from prismatic and tapered rotating beams, rotating pre-twisted beams are widely used as structural elements. According to Madhusudhana et al. [102], a structure is considered pre-twisted if, in the stress-free state, the cross-section of the structure is rotated relative to the end cross-section along the length of the beam. The blades of the aircraft propellers, turbines and fans are usually modeled as pre-twisted beams. Several studies concerning the dynamic aspects of non-rotating and rotating pre-twisted beams and blades can be found in the early review papers by Leissa [103] and Rosen [104]. Recent compact survey of the different finite elements that have been developed to model spinning rotors of various configurations can be found in the works of Nandi [105] and Mackerle [106].

Beyond applications in rotordynamics, spinning structural elements have been employed in many other areas. For instance, in several machining operations, helical fluted cutting tools such as the conventional drill bits, end mills and their micro counterparts like micro-drill bits, and micro end mills are often used. The determination of regions of stability in most operations using these elements depends on many factors. These factors range from the operating condition to the dynamics of the spindle-tool

system [50]. As discussed in the monograph of Shaw [33] and other experimental studies, the choice of cutting parameters is prone to the influence of the structural characteristics of the tool. The structural characteristics of rotating tools in material removal processes therefore play a critical role in determining the quality and productivity of the processes. The problem of spinning beam, because of its paramount technological applications in vast areas of engineering practice, has been a subject of studies by several authors in the past. The first derivation of the characteristics equation of a simply supported spinning Euler-Bernoulli (EB) beam is credited to Dimentberg [107]. Earlier comprehensive treatment of the spinning classical elastic rod of different boundary conditions are presented by Kulla [108] and Meirovitch and Calico [109]. Bauer [110] presented the analytical relationship between the natural frequencies of the rotating EB beam attached to a rigid base and that of a non-spinning beam. Filipich et al. [111] handled the vibration study of a spinning EB beam with orthotropic, elastically restrained ends and an axis of symmetry. In addition to these pioneering works, there have also been quite a number of studies on the free vibration analysis of spinning beams, with diverse solution approach, many of which are reviewed in the studies by Banerjee and Su [112]. While all these studies have offered immense information about the behavior of spinning systems, the shortcomings with most of them is their reliance on the classical elasticity theory in their formulations of the governing equations. In order to be able to come up with a model that can deal with the size effect phenomenon typical in small-scale structures such as the micro-cutting tools, a renewed approach is required.

CHAPTER 3 RESEARCH METHODOLOGY

3.1 Introduction

In this chapter a background to the framework adopted in addressing the problems highlighted in chapter two of this thesis is provided. It was revealed in chapter 2 that a number of studies relating to conventional material removal processes that involve cutting tools and micro-cutting operations (such as micro end milling and micro-drilling processes) are available. However, the directions of those existing research, with respect to micro-cutting processes, are generally geared towards two areas of concerns:

- (i) identifying parameters relating to workpiece quality such as chip morphology, surface roughness, and burr formation and
- (ii) estimating process conditions such as instability, cutting force prediction, and machine condition monitoring

It is noticed, too, from the surveyed literature, that some important dynamic problems relating to micro-cutting operations are not yet fully understood. Interestingly, both the parameters relating to workpiece quality and estimations of process conditions are influenced, in one way or the other by machine tool dynamic. In short, the quality and size of miniaturized products depend on the properties of the machine tools employed in their production. A product with a poor surface finish arising from a poor machine tool will not only lose its aesthetic appeal, it will add to the cost of production by being a waste [113]. The demand for improved surface finish and lower sub-surface damage is

even greater for advanced materials like metal matrix composites [114]. The capability of a machine tool is therefore vital to product requirements especially in micromachining operations where micro tools' life become a critical part of the machine tool. A typical machine tool, in the case of a micromachining operation, can be said to consist of three major mechanical sub-systems. In Figure 3-1, the key components of the sub-systems that define machine tool dynamic and affect its characteristics are highlighted. From a mechanical engineering point of view, certain aspects of the machine systems are of utmost interest. These aspects include, but not limited to the mechanical structure, the spindle and drive system, and the tooling and fixture system [115]. It is imperative to point out that, due to the variations in machines functionality and diverse machine configurations, Figure 3-1 serves to provide a conservative summary for the understanding of the machine tool constitution and its performance rather than a generalized machine tool set-up. It also serves to provide the basis for the place of the present work in the scheme of mechanical micromachining research.

Quantifying the contributions of the mechanical sub-systems that make up the machine tool to its performance index is essential. From the sub-systems presented in Figure 3-1, the greatest fear of the machinists ultimately lies in the failure of the micro tooling system. This is because, all things being equal, the catastrophic failures of the other sub-systems are rare occurrences. Besides, the failure of the micro tooling system puts the whole system on hold. The micro tooling system is therefore an essential part of the micromachining system. This system also plays significant roles in the preliminary stages of machine tool design. The deformations of the micro tools in static and dynamic

circumstances will in most cases be transferred to the workpiece surface quality and texture.

Figure 3-1 Subsystems and factors that contribute to machine dynamic

Also, in a flat contrast to the overall dynamic of the machine tool, the dynamic of the micro tools could change significantly, depending on the location of the cutting tool with respect to the workpiece. In essence, micro tooling is a dominant factor that affects the cost, feasibility, and productivity of mechanical micromachining applications.

Under pure commercial conditions, the bottom-line is mostly on the production of parts with satisfactory accuracy under a required time frame and conditions of maximized profit. However, certain factors could prevent the achievement of such profit-oriented objective. Some of these factors are better understood by realizing that the dynamic of micromachining processes is two-pronged. One aspect of the dynamics encompasses the time-dependent mechanics of micromachining that concerns the machining forces generated by the material removal [9, 116]. The other aspect of the machine system dynamic relates to the dynamic of the micro-cutting tool and machine-tool parts. As a result of the limitation of the existing studies, the focus of the present thesis primarily concerns the latter. The dynamic of mechanics of micromachining and the dynamics associated with machine tool-micro tooling system have to be well understood, and possibly controlled, in order to prevent annoying effects such as degradation of machined surface texture, chatter, and accelerated tool breakage.

3.2 Dynamics of micro-sized tooling components

One of the crucial objectives of manufacturing science is to provide a comprehensive framework for effective modeling, realistic simulation, process optimization and control of accurate machined surfaces [76, 117]. The achievement of this basic objective is often based on critical examination of machined surface formation, surface texture, surface integrity and topography, and machine components functionality during production processes. The examination of the dynamics of tooling components for micromachining is even more essential for developing the base knowledge of what the process entails. This is also increasingly important for engaging high speed machining,

ultra-precision machining, and nano-/micro-manufacturing. It is highlighted in chapter two that initial efforts in the field of mechanical micromachining have been devoted to cutting force prediction, chip morphology, process instability, surface roughness prediction, burr formation and tool monitoring. In contrast, the structural modeling of the micro-cutting tools is yet to be fully developed as a result of the inherent complexity of the geometry involved.

In reality, to be able to meet the twin requirements of dimensional accuracy and surface roughness, attention needs to be paid to the structural modeling of the micro-cutting tools. Likewise, to be able to optimize micromachining operations for the maximum tool life, the knowledge of the structural characteristics and deformations of the micro-cutting tools is essential. In the study by Franco-Gasca [118], the cost of the tooling system, for a basic manufacturing system, is said to represent one of the most dominant cost, besides that of raw materials, in the production process of the manufacturing industry. Thus, a rigorous attempt is made in this research endeavor to fill in the gap in the area of micromachining research with respect to the modeling of the complex behavior of micro-sized cutting tools.

From decades of empirical studies, it has been revealed and illustrated that no matter the amount of understanding generated by physics-based mathematical idealization of systems, there are always some degrees of uncertainties that will be associated with the process [119]. These uncertainties, associated with uncontrolled process variables, lead to failures of various kinds [32]. Furthermore, these uncertainties are always, and almost certainly, better captured during the testing and possibly

validation of formulated mathematical model with experimental investigations. Therefore, the basic methodology of the present study is a two-pronged research direction that involves computational mechanics-based formulation of hitherto intractable problem of dynamics of small-scale structures and experimental investigations. The computational mechanics-based aspect of the research involves the non-trivial formulation of the theories of small-scale structures with respect to their mechanical deformation under certain loading conditions. On the other hand, the experimental phase of the study, while not the major phase of the research, is meant to serve as a validation platform. In this aspect, the dynamic of micro-cutting tools used in the micro-cutting processes are critically examined through the mathematical idealization provided by their structural models.

3.3 Structural modeling of macro- and micro-sized tools

The design and modeling of tools for material removal is a central aspect of the machining operation design. For a reliable tool performance, a careful consideration of certain factors is imperative during design and modeling phase. These factors, from mechanics point of view, include but not limited to, the geometry of the tool, the material of the tool and material of the intended workpiece. Another factor that requires consideration, from process modeling point of view, is the type and amount of surface coating applied on the tool. The combination of these factors contributes to the total machining performance of a tool as shown in Figure 3-2.

Figure 3-2 Major factors affecting tool design

Research works on the structural modeling of cutting tools, especially for the determination of their dynamic characteristics in the macro domain have been ongoing for a long while. Pioneering works in this aspect include the long standing studies by Magrab and Gilsinn [120], Telkinap and Ulsoy [121-122], and Ulsoy et al. [123]. With drilling being the most economical manufacturing method in the period in which these pioneering works were done, all of the above works are related to the structural modeling of the conventional drill bits. The similarity between the work by Magrab and Gilsinn [120] and the studies by Telkinap and Ulsoy is the underlying Euler-Bernoulli beam theory employed in their studies. Albeit, Magrab and Gilsinn employed a more representative modified twisted Euler-beam model, while Telkinap and Ulsoy attempted the same problem with an Euler-Bernoulli beam having a variable cross-sectional geometry. As excellent as these works are, they are sprinkled with approximations typical

of any structural element idealization. These approximations include neglecting of the rotary inertia and gyroscopic moment of the rotating system. Some of these shortcomings were later addressed in the study by Rincon and Ulsoy [124]. The inclusion of alignment error in a rather simplified model of the drill bit is reported by Gupta et al. [125]. Arvaje and Ismail [126] considered the bending problem of the drill bits using yet another approximation of a lumped-mass model in the dynamic model that considered the drill bit as a beam with a variable cross-sectional geometry. A number of studies on drill bits are also available in the literature, but they are not directly related to the present research endeavor. While so much can be found on the vibration and buckling of drill bit, the same cannot be said of the conventional end mill. The only known study on the vibration of the conventional end mill, from mechanics point of view, is the work of Liao and Dang [127]. Shortly after this work, Liao collaborated with Tsai in [128] to present the static analysis of the conventional end mill based on the model of the study in Liao and Dang [127]. All these developments are more than a decade ago, after which no study is available in the literature on the structural modeling of the end mills.

With the foray of the machining community into mechanical micromachining where micro drill bits and micro end mills are now employed and extensive breakage of the duo reported, researchers are once again turning to the drawing board for a more accurate structural modeling. Interestingly, just like in the case of the macro-cutting tools (conventional drill bits and end mills), modeling of the micro drill bits has received enormous attention from different researchers as compared to micro end mills. Wijeyewickrema and Keer [129] first presented a study on the determination of the critical loads, critical speeds and natural frequencies of a micro drill bit using the finite

element method. Subsequently, Huang [130], and Gong et al. [131], using different twisted beam models, presented the free transverse vibration of the micro drill bit. The parametric studies of the influence of geometric properties on the stability of micro drill bits were recently reported by Chen [132]. However, micro drill bits are geometrically different from micro end mills and the model of one cannot easily be extrapolated for the other [127, 133]. In the case of micro end mill, the works of Jun et al. [134] and Filiz and Ozdoganlar [25] are the only known works that one could find in which analytical models of a micro end mill is claimed to have been presented. One would also find works by Filiz and Ozdoganlar [92] in which the same model as in [25] is used.

A careful study of the works by Jun et al. [134] and Filiz and Ozdoganlar [25] revealed a degree of similarity in focus. In these two works, it appears that the authors have given more attention to the analysis of alignment error due to process fault than presenting an analytical model suitable for transverse vibration, lateral vibration, stability analysis and optimization of the micro end mills for failure prevention. Correct information about the flute geometry and tapered section of the micro end mill are neglected. The interaction of the tool with the workpiece is not considered. The spindle-micro end mill dynamic have not been taken into account. The impetus for this research is therefore based on the above apparent gap in the research of mechanical micromachining. Experimental studies by Jun et al. [26] and numerical studies by Filiz et al. [135] have shown that, the designs of micro end mills are quite different from that of conventional end mills or drills. Further studies, by Tansel et al. [136] also revealed the difference dynamic behavior, between micro-cutting tools and their macro-sized counterparts. The commercially available micro end mills are designed with a stubby and

long shank that is followed by a tapered neck and then the flute for the cutting edges as shown in Figures 3-2 and Figure 3-3 (Filiz et al. [137]).

Figure 3-3 A collet-tooling system of a micro-milling machine

Figure 3-4 A CAD-generated geometric model of the micro end mill

The most compelling reason for this geometrical combination is to give micro end mills additional rigidity and thwart the issue of self-excited vibrations during machining processes. Unfortunately, the importance of these geometric compensations for rigidity has been neglected in the few micro end mill models reported so far. In addition, Mascardelli et al. [43] and a number of other works like Filiz et al. [137] have elaborately reported the contribution of the fragility of micro end mills to the difficulty in the experimental determination of their dynamic behavior. Analytical modeling therefore remains the viable alternative to comprehend their dynamic characteristics.

The modeling and analyses of practical structures, which are often more complicated in geometry and material constituent, generally calls for: (i) a thoughtful idealization of the physics of the structure, (ii) a careful formulation of the governing equations of the idealized system, and (iii) a rigorous search for the solution of the formulated governing equations. With respect to idealization, the dynamic behaviors of rotating cutting tools (such drill bits and end mills) are analogous to that of rotating shafts under a varying load [25]. As a result of this similarity to rotating shafts, the Euler-Bernoulli (EB) beam theory (twisted or of variable cross section) has been used in the analysis of drills and micro drill bits. The EB beam theory, being a first order beam theory, assumes a set of simplified assumptions in the kinematic description of a deformed elastic solid. These assumptions are [138]:

- a. The cross-section of the deformed solid is infinitely rigid in its own plane*
- b. The cross-section of the deformed solid remains plane after deformation*
- c. The cross-section remains normal to the axis of the beam*

The inevitable limitation of the EB beam theory, arising from the above assumptions, is the non-inclusion of the shear deformation effect. In spite of this limitation, however, the EB beam theory gives reasonably good prediction for bending, vibration and stability analyses of long thin beam. The last statement explains the popularity of the EB beam theory in the analyses of drills and micro-drills [130, 139]. However, in the case of micro end mills, the geometry of the tool is not simple. As a result, the EB beam theory (twisted or of variable cross section) used in the analysis of

drills and micro drill bits becomes inappropriate. This is because the theory does not account for rotary inertia and shear deformation [140]. Hence a more refined theory, based on either the Bresse-Timoshenko or first order shear deformable beam (such as Timoshenko), is required.

The formulation of the principal equations for characterizing the behavior of most solids, according to Wunderlich and Pilkey [141], depends on three important factors: (i) the conditions of equilibrium, (ii) the conditions of geometric-fit (otherwise known as strain-displacement relationship), and (iii) the material law. From the above mentioned conditions, the governing equations suitable for the examination of the static deformation, the stability, and the dynamic of most structural elements (shaft, rod, beam, shell and plate) are derived. These governing equations are mostly and collectively refer to as *linear elastic theory of solid mechanics*, are usually in the form of partial differential equations. In the case of the shear deformable beam, the elastic theory-based governing equations of motion are coupled partial differential equations. These equations are generally of higher complexity to deal with than the EB beam. The complexity in dealing with the Timoshenko beam is further increased if the beam is non-uniform, twisted, rotating, spinning or a combination of any of these.

The linear elastic theory of solid mechanics has an interesting history and has been studied by applied mathematicians and engineers for quite a while [142]. Rigorous efforts to formulate the solutions of equations arising from linear elastic theory have lead to extensive research in applied mathematics. The classical approach, in the quest for their solutions, is to embark on providing analytical exploration. However, the spectrum

of adequate analytical solutions is quite limited, and the problems for which such analytical solutions can be obtained are usually very simple. This, therefore, leads to the reduction of the governing equations to a form that can be handled with modest effort. The art of reducing the governing equations to a form that can be handled with modest effort is commonly referred to as engineering structural analysis. Engineering structural analysis has generated an extensive collection of solutions to practical problems in many areas of engineering science.

As earlier pointed out, finding the exact solutions to the computational mechanics-based governing equations of motion of most structural elements is both formidable and intractable. Consequently, the solutions of these complex governing equations are often sought through numerical techniques. There are extensive research endeavors concerning the solutions of governing equations of beams of various kinds and configurations. In this direction, some researchers have used the Southwell method, the Rayleigh-Ritz method [143], the perturbation technique, the Myklestad method, and the extended Galerkin procedure [144]. Even though their mathematical models have been critiqued to represent an oversimplified model of micro end mills, Jun et al. [134] solved the system of equations for the rotating end mills using the finite element method. In their own study, Filiz et al. [25] used the spectral Chebysev technique. The spectral Chebysev technique method involves the use of the Gauss-Lobatto interpolation function to perform numerical integration of the weak form of the governing equations. To reiterate the points made so far in this chapter, the limitations in the existing works on the structural modeling and dynamic study concerning the behavior of micro end mills are further highlighted below:

- a. The available works on the structural modeling of micro end mill have failed to consider the influence of the tapered section in micro end mills and its contribution to the dynamic of the tool.
- b. The rate of twist, a feature that characterizes the micro end mill flute and distinguishes it from a mere uniform beam has been neglected.
- c. Limited validations of the available model with experimental observations show wider discrepancy.

The above reasons lead to the proposition of the research endeavor that form the bulk of these studies.

3.4 Proposed research

As earlier mentioned, small-scale structures are essential elements in today's search for portable, compact, and space-constrained miniaturized systems. These applications can be seen from micro turbines, to micro tools and nano-composites. In all, these applications reinforce the idea that "small is the new big thing" is influencing the way the products of the future are being conceived, designed and built. It is also necessary to point out that the extensive applications of micro-systems have also enhanced systematic integration of complex functionality at a lower manufacturing cost. However, to understand the dynamic characteristics of micro-systems, the puzzlingly complex behavior of their skeletal components, mostly made of micro- and nano-sized beams, plates or shells have to be fully understood. Therefore, the studies detailed in this thesis,

even though it is tailored, for the reason of practical convenience, to investigate the behavior of micro-cutting tools, have application far beyond micro tools.

In order to address the gap in literature highlighted in the preceding sections, it is necessary to have a theoretical framework through which different mechanical responses of micro-scale structures can be considered. It is also necessary to have some experimental investigation to complement the theoretical model. In Figure 3-5, an overall picture of the proposed research direction is shown.

Figure 3-5 Overview of the underlying tasks of the proposed research

From this figure, it could be gleaned that the philosophy of the present thesis is to provide a novel window that will enhance the modeling and dynamic analyses of small-scale structures of certain and special geometry. The selection of the geometry of interest is based on the relevance of such geometry to the modeling of micro-cutting tools. The adopted approach for the formulation of the theoretical framework will be hinged on the computational mechanics-based *linear theory of elasticity*. This approach leads to two paths. The first path explores the adoption of classical elasticity theory to provide the

basis for initial investigation. The second path examines the shortcoming of the classical elasticity theory and subsequently leads to the formulation of the behavior of small-scale structures based on two novel and advanced higher-order, non-local theoretical frameworks. These two higher-order theoretical frameworks are (i) the Eringen's nonlocal elasticity theory, and (ii) the modified couple stress theory.

The two theoretical routes (classical and non-local theoretical frameworks) yield a complex system of partial differential equations. The computational analyses of the various differential equations, yielded by the two theoretical frameworks, are handled by different numerical methods that include, but not limited to, the spectral finite element method. To validate certain aspect of the study, series of carefully designed experiments needs to be carried out and compared with mathematical model of the micro-cutting tools. Consequent upon this, the target of the present research consists of providing a novel theoretical window through which the predictions of the mechanical deformation of micro-cutting tools are investigated using the elastic theory of solids.

From the information present earlier, it is highlighted that micro-drills and micro end mills are the most common of the micro-cutting tools employed in mechanical micromachining. Now, from the perspective of the theoretical formulation of the micro-cutting tools, it is dimmed appropriate to focus on the modeling of micro end mills. This is because, micro end mills have complex geometries that can be easily reduced to that of micro-drills. With this in mind, the modeling of the micro end mill proceeds with the physical sub-structuring of the micro end mill into three geometrically distinct parts (shown in Figure 3-4) and then treated as a stepped distributed dynamic system. For a

high speed machining operation like micromilling, where there is increased possibility of complex tool's deflection(s), the spinning beam model becomes more appropriate for structural modeling of the micro end mill [145]. Thus the Euler-Lagrange governing equations for each part of the micro end mill will be derived in the framework of a spinning uniform, spinning tapered and spinning twisted Timoshenko beam. The combination of the Euler-Lagrange governing equations for each part will constitute the mathematical model of the micro end mill. These derived equations will then be used to investigate the mechanical deformation of the micro end mill. In this study, the solution of the governing equation of motion for the micro end mill will first be sought for vibration and stability analyses through the spectral finite element (SFEM). With the aid of SFEM, resonance characteristics of the entire micro end mill can then be predicted and other responses like the bending mode along the tool's length will be observed. The reason for this approach of starting with a vibration analysis is to provide the basis for comparison with some existing simplified model. Besides, vibration is a pertinent problem in micromachining because of the high speed of rotation of the micromachine spindle. The presence of this high-spindle speed is bound to cause harmonic force to be applied to the body of the micro end mill. If the frequency of the spindle rotation corresponds to the natural frequency of the body, resonance will occur. Resonance will cause a large deflection which may exceed the elastic limits of the micro end mill and cause the structure to fail. Thus, the results from the initial vibration study will provide the framework for the examination of:

- ✚ The inclusion of the small scale coefficient to investigate the influence of size effect on the dynamic of the micro cutting tools

- ✚ The combined influence of material stiffness, material length scale and gyroscopic moment on the behavior of micro-cutting tool

The methodology of using SFEM in this research endeavor is to provide a unique solution procedure for the study of the dynamic of the micro end mill. This approach enables not only the solution of dynamic equations of motion to investigate free vibration characteristics of micro end mill, but to also predict the behavior of bending waves and dynamic stresses in these complex medium. Additional advantages and limitations of using the SFEM are stated below.

3.4.1 Advantages of SFEM

The spectral finite element method (SFEM), just like any computational method, has its pros and cons. Some of the advantages of the SFEM are enumerated below:

- ✚ Reduced problem size and degree of freedoms (DOFs) compared to the conventional finite element method (CFEM)
- ✚ Lower computational cost compared to the CFEM
- ✚ High accuracy in natural frequency prediction
- ✚ Effective in dealing with frequency-domain and wave propagation problems
- ✚ Free of the shear-locking phenomenon commonly encountered in the CFEM
- ✚ Highly efficient in dealing with digitized data during experimental studies

- ✚ Provides a viable computational platform for solution of inverse problem and joint parameter identification

3.4.2 *Limitations of SFEM*

Although, SFEM is effective in dealing with different kinds of dynamic problems and has been well used in providing validation of many benchmark problems, it has the following limitations.

- ✚ It is a frequency-domain analytical method that depends on the discrete Fourier transform. As a result, it may not be directly applicable to systems for which the principle of superposition does not hold. Examples are non-linear, time-variant systems.
- ✚ SFEM is a bit cumbersome for time-domain analysis. This is because to convert the frequency-domain analytical solution yielded by SFEM, the time-domain solutions must be processed by convolving the frequency transfer functions with external excitations using the inverse fast Fourier transform (IFFT). While the direct problem provides exact solutions in the frequency domain, the inverse problem might be corrupted by aliasing or leakage error.
- ✚ SFEM involves the arduous task of dealing with numerous symbolic representations.

It is important to note that, although SFEM is the preferred numerical method for the most part of this study, switch is made to some other numerical methods including the

conventional FEM where need be for effective comparison. With the aim of providing robust and rigorous analyses of the mechanical deformation of micro-cutting tool, the other complementary phase of the research involve experimental studies that will focus on the forced vibration characteristics of the micro-cutting tool. This aspect will involve statistical quantification of the accuracy of the developed micro-scale tool model when compared with experimental observation. Previous investigations [32, 35] have shown that close to 50% of all cases of tool life of carbide tools expires just because of vibration and excessive stress. Vibration data have been successfully and effectively used in the monitoring of rotating machineries, aerospace and civil engineering application to ensure maximum safety of delicate structures [146]. All experimental works will be carried out based on the philosophy of design of experiment. This will make it easy to find the interaction effects of the different process and geometric variables on the tools behavior.

3.5 Research significance

It is believed that the above research will have significant contribution in:

- Structural optimization of micro-cutting tools (micro end mills and micro drills)
- Process optimization of the micromachining operations for maximum tool life

CHAPTER 4 DEVELOPMENT OF THEORETICAL MODEL FOR THE DYNAMICS OF MICRO-CUTTING TOOLS

4.1 Introduction

Some aspect of engineering sciences involves persistent strive to model processes of diverse complexity occurring in practical situations as precisely as possible. But nature is far too complex to be modeled with impeccable precision. The quest to find a way around nature's complexity leads to the adoption of approximate representations of observed events. These approximate representation act to provide insight, even if imperfect, to understand the underlying law leading to the observed events. The first attempt at most approximations of physical systems is the descriptive illustration and understanding of the system being studied. This, in turn, serves to provide the concepts of logical assumptions necessary to limit the boundary of complexity of the system being considered. With the assumptions in mind, the discovery of how the system being observed depends on some parameters of interest is inferred. The objective of initial observations of the system and identification of potential assumptions is to lay the groundwork for the mathematical framework of the underlying phenomena under investigation. Consequently, the mathematical framework imposes what can be described as an arrow of relationship on certain variables in relation to others.

In chapter three, the limitations of the existing research efforts relating to the structural and dynamic analyses of micro-cutting tools were discussed. The direction of this research endeavor is also presented. The focus of this chapter therefore, is to

systematically provide the details of the initial attempt adopted in the development of alternative mathematical models for the structural modeling and analysis of these micro tools. Micro-cutting tools can in general be idealized structurally as beams. However, not any kind of beams can be used to model micro-cutting tools. A micro-drill for instance, is geometrically made up of two distinct sub-structures namely, the shank and the flute. On the other hand, a micro end mill is made up of three geometrically different parts namely, the shank, the intermediate tapered neck and the flute. In addition, the cross-sectional dimensions of most micro-drills are small compared to their length, thus making the use of the low-order beam theory sufficient. The low-order beam theories include the classical Euler-Bernoulli (EB) and the Rayleigh beam theories. In contrast, the physical geometries of most micro end mills are stubby compared to those of micro drills. This stubbiness inevitably leads to the requirement to employ the shear deformable beam theories.

There are a number of beam types whose mathematical models are based on the shear deformability criterion. Prominent among these beam types are:

- (a) Bresse-Timoshenko beam
- (b) Timoshenko beam
- (c) Reddy beam
- (d) Levinson beam

Based on the geometrical discrepancies and subsequent representation, it is noted that the derivation of the mathematical model for micro end mill dynamic will be more

daunting than that of the micro-drills. However, sequel to the fact the governing equations of the low-order beams, can in most cases, be derived from the shear deformable beams, the derivation of the analytical model of the micro end mills based on the shear deformable beam theory will first be carried out. The governing equation for the micro-drills is then expected to be treated as a special case of that of the micro end mill. The methodology adopted in this research employs mostly differential and integral equations to depict the description of the mechanical deformation of the micro end mill.

This chapter proceeds from the presentation of the background of the analytical model formulation in section 4.2. The governing equations of the sub-structures (that is, the geometrically distinct part of the micro cutting tools) are then presented in section 4.3. A brief introduction of the method of spectral finite element, as well as a brief survey of recent studies based on this method is given in section 4.4. Section 4.5 gives detail of the numerical results and discussions of some of the simulation works on the prediction of the natural frequencies of a two-fluted micro end mill. Finally, the summary of the chapter is presented in section 4.6.

4.2 Analytical model formulation for micro end mills

From mechanics point of view, the analytical model governing the dynamics of most structural members are derivable from the D'Alembert's principle, the 3D elasticity theory or the extended Hamilton's principle. However, due to its ability to yield 'variationally' consistent *Neuman* boundary conditions, the extended Hamilton's principle is adopted in the derivation of all of the governing equations presented in this

chapter and to some extent in this thesis. The schematic of a representative micro tooling system embedded within a workpiece material is shown in Figure 4-1.

Figure 4-1 Schematic of a micro tooling system embedded within a workpiece material

The schematic, shown in Figure 4-1, is divided into three parts. Part A is the spindle/collet part of a micro tool on bearing supports. The bearings represent the stiffness of the machine mechanical structure. Part B is the micro end mill shank and the intermediate tapered neck. Part C represents the micro end mill flute embedded in a workpiece. To be able to capture the response of the micro end mill along its entire length, the micro end mill is modeled as a stepped distributed dynamic system. This modeling approach leads to the spatial sub-structuring of the micro end mill into three different geometrically distinct parts. These parts are: (a) the micro shank, (b) the tapered neck, and (c) the micro flute. The derivation of the governing equations for each of these three parts will follow from the extended Hamilton's principle.

4.2.1 The extended Hamilton's principle

In its most basic form, the extended Hamilton's principle [138] states that “*the motion of a particle acted on by conservative forces between two arbitrary instants time t_1 and t_2 is such that the line integral over the Lagrangian function is an extremum for the path motion.*”

The above statement is written mathematically as:

$$\delta \int_{t_1}^{t_2} \mathcal{L} dt = \delta \int_{t_1}^{t_2} (T - \Pi) dt = 0 \quad (4.1)$$

where $\mathcal{L}$ is the Lagrangian function, T is the total kinetic energy and Π is the total strain energy of the system ($\Pi = U + W$), where U is the total potential energy and W is the total work done by the external forces.

4.2.2 Basic assumptions for the derivation of the Lagrangian function for a spinning micro end mill

As stated in the introduction, establishing the assumptions of a modeling framework makes it easy to impose a boundary on the dimension of the problem being addressed. It is therefore necessary to state before hand the underlying assumptions of the mathematical model to be later presented. The behavior of micro-cutting tools during the machining operation is obviously different from their behavior in a non-rotating machine. In mechanical micromachining especially, the operation is usually carried out at high spindle speeds in order to have any appreciable surface finish [26, 126]. To account for this high spindle speed of the machine, which ultimately leads to a very high rotation of

the micro tools, gyroscopic moments must be taken into account while formulating the governing equations. Hence, the derivations of the governing equations for each of the spatial sub-structures (see Figure 4-2 for the spatial sub-structuring) need to be carried out in the framework of spinning beam elements.

Figure 4-2 Sub-structuring of the micro end mill into three distinct parts

Based on this sub-structuring, a micro-drill can be modeled as a stepped spinning Euler-Bernoulli, while a micro end mill is modeled with spinning Timoshenko beam structural elements. After the sub-structuring, the micro shank is treated as a *spinning uniform Timoshenko beam*, the intermediate part between the shank and the flute is modeled as a *spinning tapered Timoshenko beam*. The micro flute is modeled as a *spinning twisted Timoshenko beam* element experiencing a bending-bending elastic coupling and rate of twist.

Apparently, studies on the dynamic characteristics of spinning Timoshenko beams has been on-going for quite some time as observed in [147-148]. For dynamic analysis of

elastic spinning beams, the following general assumptions are usually made in order to derive the governing equation of the basic spinning Timoshenko beam element:

- The material of the beam is homogenous, elastic and isotropic.
- The deformation of the beam is small compared to its thickness so that the strain-displacement relation can be taken as linear.
- Plane sections of the beam cross-section before the deformation remain plane after the deformation, but no longer remain perpendicular to the neutral surface after bending.
- The displacement of the beam is due to the deflection of the points of the centerline.
- The beam is spinning about an axis that coincides with its longitudinal axis y at a constant speed of Ω .
- The shear and the mass centers of the beam coincide.

Moreover, two coordinate frames are used in order to model the complete micro end mill. The first coordinate xyz runs from the beginning of the shank to the end of the tapered section as shown in Figure 4-2. The second coordinate frame XYZ is fixed to the left end of the spinning flute and moves along the micro flute. The micro end mill is spinning about the y axis at an angular velocity of Ω (rad/s), where both y and Y are coincident.

4.3 The governing equations of the micro end mill's sub-structures

As stated earlier, the governing equations for the entire micro end mill will be based on the governing equations of its sub-structures. These include the micro shank, the tapered neck and the micro flute. In the subsections that follow, the governing equations for each of the sub-structures will be presented. Only a small part of the derivation procedure for each of these sub-structures will be presented for conciseness sake.

4.3.1 Governing equations of the micro shank and the tapered neck

Considering a differential spinning Timoshenko beam element, the total strain energy due to flexural and shear deformation of the spinning uniform Timoshenko beam, and by extension the micro shank, is given by [149]

$$U_{total} = \frac{1}{2} \int_0^{L_1} \left\{ EI_1 \left[\left(\frac{d\theta_1}{dy} \right)^2 + \left(\frac{d\psi_1}{dy} \right)^2 \right] + kA_1 G \left[\left(\frac{du_1}{dy} - \psi_1 \right)^2 + \left(\frac{dw_1}{dy} - \theta_1 \right)^2 \right] \right\} dy \quad (4.2)$$

where E is the Young's modulus and I_1 is the second moment of area of the cross section of the shank. θ_1 and ψ_1 represent section rotation about ox and oz planes, while u_1 and w_1 are the elastic transverse deflections in the x and z directions of the shank. The kinetic energy of the spinning Timoshenko beam element has to be expressed in terms of the velocity components of the point at a distance from the neutral axis and it is given by [149]:

$$T_{total} = \frac{1}{2} \int_0^t \left(\rho I \left[\left(\frac{d\theta_1}{dt} \right)^2 + \left(\frac{d\psi_1}{dt} \right)^2 \right] + \rho A \left[\left(\frac{du_1}{dt} \right)^2 + \left(\frac{dw_1}{dt} \right)^2 \right] \right) dy \quad (4.3)$$

The virtual work due to the gyroscopic moments is expressed as:

$$\delta W = 2\Omega \int_0^{L_1} \left(\rho I \left[\frac{d\theta_1}{dt} \delta\psi_1 - \frac{d\psi_1}{dt} \delta\theta_1 \right] \right) dy \quad (4.4)$$

Substituting Eqs. (4.2) – (4.4) into the expression for the extended Hamilton's principle in Eq. (4.1), we have the following set of coupled partial differential equations:

$$-\mu_1 \left(\frac{\partial^2 w_1}{\partial t^2} \right) + k_1 A_1 G \left(\frac{\partial^2 w_1}{\partial y^2} - \frac{\partial \theta_1}{\partial y} \right) = 0 \quad (4.5a)$$

$$EI_1 \left(\frac{\partial^2 \theta_1}{\partial y^2} \right) + k_1 A_1 G \left(\frac{\partial w_1}{\partial y} - \theta_1 \right) - \rho I \frac{\partial^2 \theta_1}{\partial t^2} - \rho I_1 \Omega \frac{\partial \psi_1}{\partial t} = 0 \quad (4.5b)$$

$$-\mu_1 \left(\frac{\partial^2 u_1}{\partial t^2} \right) + k_1 A_1 G \left(\frac{\partial^2 u_1}{\partial y^2} - \frac{\partial \psi_1}{\partial y} \right) = 0 \quad (4.5c)$$

$$EI_1 \left(\frac{\partial^2 \psi_1}{\partial y^2} \right) + k_1 A_1 G \left(\frac{\partial u_1}{\partial y} - \psi_1 \right) - \rho I \frac{\partial^2 \psi_1}{\partial t^2} + \rho I_1 \Omega \frac{\partial \theta_1}{\partial t} = 0 \quad (4.5d)$$

where μ_1 and k_1 are the mass distribution and shear correction factor for the shank respectively. The following force-displacement relations which represent natural boundary conditions are derived from the above equations:

$$M_{z1} = EI_1 \frac{\partial \theta_1}{\partial y}, M_{x1} = EI_1 \frac{\partial \psi_1}{\partial y}, Q_{z1} = k_1 G A_1 \left(\frac{\partial w_1}{\partial y} - \theta_1 \right), Q_{x1} = k_1 G A_1 \left(\frac{\partial u_1}{\partial y} + \psi_1 \right) \quad (4.6)$$

The equations of motion for the spinning tapered neck of the micro end mill are easily derivable from that of the spinning micro shank. This is done by re-deriving the Lagrangian function in such a way that, the mass distribution (μ), flexural rigidity (EI) and rotary inertia (ρI) are no longer constant coefficients. Hence, in the case of the tapered neck we have a set of variable coefficient partial differential equations given as

$$-\mu_2 \left(\frac{\partial^2 w_2}{\partial t^2} \right) + k_2 G \frac{\partial}{\partial y} \left[A_2 \left(\frac{\partial w_2}{\partial y} - \theta_2 \right) \right] = 0 \quad (4.7a)$$

$$\frac{\partial}{\partial y} \left(EI_2 \frac{\partial \theta_2}{\partial y} \right) + k_2 G \left[\left(\frac{\partial w_2}{\partial y} - \theta_2 \right) \right] - \rho I_2 \frac{\partial^2 \theta_2}{\partial t^2} - \Omega \rho I_2 \frac{\partial \psi_2}{\partial t} = 0 \quad (4.7b)$$

$$-\mu_2 \left(\frac{\partial^2 u_2}{\partial t^2} \right) + k_2 G \frac{\partial}{\partial y} \left[A_2 \left(\frac{\partial u_2}{\partial y} - \psi_2 \right) \right] = 0 \quad (4.7c)$$

$$\frac{\partial}{\partial y} \left(EI_2 \frac{\partial \psi_2}{\partial y} \right) + k_2 G A_2 \left[\left(\frac{\partial u_2}{\partial y} - \psi_2 \right) \right] - \rho I_2 \frac{\partial^2 \psi_2}{\partial t^2} + \Omega \rho I_2 \frac{\partial \theta_2}{\partial t} = 0 \quad (4.7d)$$

where μ_2 and k_2 are the mass distribution and shear correction factor for the tapered segment respectively. The corresponding force-displacement relations gotten from the application of the Hamilton's principle to the Lagrangian function are now of the forms:

$$M_{z2} = EI_2(y) \frac{\partial \theta_2}{\partial y}, M_{x2} = EI_2(y) \frac{\partial \psi_2}{\partial y}, Q_{z2} = k_2 G A_2(y) \left(\frac{\partial w_2}{\partial y} - \theta_2 \right), Q_{x2} = k_2 G A_2(y) \left(\frac{\partial u_2}{\partial y} + \psi_2 \right) \quad (4.8a)$$

The literature on the vibration analysis of beams with tapered geometry revealed that a number of different but complementary techniques have been proposed for handling taper behaviors in structural elements. Wang and Wereley [150] reported the dynamic analysis of rotating blades with uniform taper by using a low degree of freedom model. The authors employed the Frobenius series method to derive the dynamic stiffness matrix. Other works, such as [151] employed polynomial variation to represent the flexural and mass variations along the length of non-uniform beams. In this work, the polynomial taper formulation is employed. Besides, instead of using the Frobenius series method which could complicate the solution procedure for computer implementation, a combination of the SFEM and an approximate scheme based on the well-known variational approach of *Bubnov-Galerkin* is adopted. Additionally, the spinning tapered

neck of the micro end mill is doubly tapered. Therefore, the following taper parameters are defined in the two planes of the deformation:

$$A_{2y} = A_1 \left[\left(1 - \epsilon_z \frac{y}{L_2} \right)^r \left(1 - \epsilon_x \frac{y}{L_2} \right)^r \right] \quad (4.8b)$$

$$I_{2z} = I_1 \left(1 - \epsilon_z \frac{y}{L_2} \right)^{3r}, I_{2x} = I_1 \left(1 - \epsilon_x \frac{y}{L_2} \right)^{3r} \quad (4.8c)$$

Eqs. (4.8b) and (4.8c) show the variation of the area along the length of the beam and the variation of the second moment of inertias I_{2z} and I_{2x} in the z and x directions. The terms ϵ_x and ϵ_z are the taper ratios in both x and z directions. I_1 denote the value second moment of inertia at the left side of the tapered section. This left side coincides with the end of the micro shank. According to [152], the superscript r usually takes the values that depend on the nature of the taper. A linear tapered geometry is assumed linear, thus $r = 1$.

4.3.2 The governing equations of the micro flute

In order to accurately model the micro flute, whose behavior is very critical to the accurate prediction of the dynamic response of most micro cutting tools, a different beam element is needed. In a manner different from the few available works on micro end mills [25, 92, 134], the micro flute is modeled with a spinning shear deformable twisted beam element experiencing a bending-bending elastic coupling and a uniform rate of twist. The Lagrangian of this shear deformable twisted beam element is derived from the following kinetic energy, potential energy and the virtual work done due to the gyroscopic moments:

$$T_3 = \frac{1}{2} \int_0^{L_3} \left[\rho I_{X3} \left(\frac{\partial \theta_3}{\partial t} \right)^2 + \rho I_{Z3} \left(\frac{\partial \psi_3}{\partial t} \right)^2 + \mu_3 \left(\frac{\partial u_3}{\partial t} + \frac{\partial w_3}{\partial t} \right) + \rho I_p \Omega^2 \right. \\ \left. + \mu_3 \Omega^2 (w_3^2 + u_3^2) \right] dY \quad (4.9)$$

$$U_3 = \frac{1}{2} \int_0^{L_3} \left[EI_{X3} \left(\frac{d\theta_3}{dY} + \gamma \frac{d\psi_3}{dY} \right)^2 + EI_{Z3} \left(\frac{d\psi_3}{dY} + \gamma \frac{d\theta_3}{dY} \right)^2 + k_X A_3 G \left(\frac{du_3}{dY} + \psi_3 + \gamma \frac{dw_3}{dY} \right)^2 \right. \\ \left. + k_Z A_3 G \left(\frac{dw_3}{dY} - \theta_3 - \gamma \frac{du_3}{dY} \right)^2 \right] dY \quad (4.10)$$

$$\delta W_3 = 2\Omega \int_0^{L_3} \left[\rho I_{X3} \left(\frac{d\theta_3}{dt} + \gamma \frac{d\psi_3}{dt} \right) \delta \psi_3 + \rho I_{Z3} \left(\frac{d\psi_3}{dt} + \gamma \frac{d\theta_3}{dt} \right) \delta \theta_3 \right] dY \quad (4.11)$$

where T_3 is the kinetic energy, U_3 is the total potential energy and δW_3 is the virtual work done due to the gyroscopic moments of the spinning micro flute. Substituting Eqs. (4.9) – (4.11) into Eq. (4.1), we have the following equations for the micro flute in the local frame XYZ

$$-\mu_3 \left(\frac{\partial^2 w_3}{\partial t^2} \right) + k_{Z3} AG \frac{\partial^2 w_3}{\partial Y^2} - \gamma^2 k_{X3} AG w_3 + \gamma AG (k_{Z3} + k_{X3}) \frac{\partial u_3}{\partial Y} + k_{Z3} AG \frac{\partial \psi_3}{\partial Y} - \\ \gamma k_{X3} AG \theta_3 + \mu_3 \Omega^2 w_3 + 2\mu_3 \Omega \frac{\partial u_3}{\partial t} = 0 \quad (4.12a)$$

$$-J_{Z3} \left(\frac{\partial^2 \theta_3}{\partial t^2} \right) + EI_{Z3} \frac{\partial^2 \theta_3}{\partial Y^2} - \gamma^2 EI_{X3} \theta_3 - k_{X3} AG \theta_3 - \gamma k_{X3} AG w_3 - k_{X3} AG \frac{\partial u_3}{\partial Y} - \\ \gamma (EI_{Z3} + EI_{X3}) \frac{\partial \psi_3}{\partial Y} + J_{X3} \Omega^2 \theta_3 - \Omega (J_{X3} + J_{Z3}) \frac{\partial \psi_3}{\partial t} = 0 \quad (4.12b)$$

$$-\mu_3 \left(\frac{\partial^2 u_3}{\partial t^2} \right) + k_{X3} AG \frac{\partial^2 u_3}{\partial Y^2} - \gamma^2 k_{Z3} AG u_3 - \gamma AG (k_{Z3} + k_{X3}) \frac{\partial w_3}{\partial Y} - k_{X3} AG \frac{\partial \theta_3}{\partial Y} - \\ \gamma k_{Z3} AG \psi_3 + \mu_3 \Omega^2 u_3 - 2\mu_3 \Omega \frac{\partial w_3}{\partial t} = 0 \quad (4.12c)$$

$$\begin{aligned}
& -J_{X3} \left(\frac{\partial^2 \psi_3}{\partial t^2} \right) + EI_{X3} \frac{\partial^2 \psi_3}{\partial y^2} - \gamma^2 EI_{Z3} \psi_3 - k_{Z3} AG \psi_3 - \gamma k_{Z3} AG u_3 + k_{Z3} AG \frac{\partial w_3}{\partial Y} + \\
& \gamma (EI_{Z3} + EI_{X3}) \frac{\partial \theta_3}{\partial Y} + J_{Z3} \Omega^2 \psi_3 + \Omega (J_{X3} + J_{Z3}) \frac{\partial \theta_3}{\partial t} = 0
\end{aligned} \tag{4.12d}$$

For this substructure, the required force-displacement relations evolving from the Hamilton's principle are [153]:

$$\begin{aligned}
M_{Z3} &= -EI_{Z3} \left(\frac{\partial \theta_3}{\partial Y} - \gamma \psi_3 \right), M_{X3} = -EI_{X3} \left(\frac{\partial \psi_3}{\partial Y} + \gamma \theta_3 \right), Q_{X3} = -EI_{Z3} \frac{\partial^2 \theta_3}{\partial Y^2} + \\
& \gamma^2 EI_{X3} \theta_3 + \gamma (EI_{Z3} + EI_{X3}) \frac{\partial \psi_3}{\partial Y} + J_{Z3} \left(\frac{\partial^2 \theta_3}{\partial t^2} \right), Q_{Z3} = EI_{X3} \frac{\partial^2 \psi_3}{\partial Y^2} - \gamma^2 EI_{X3} \psi_3 + \\
& \gamma (EI_{Z3} + EI_{X3}) \frac{\partial \theta_3}{\partial Y} + J_{Z3} \left(\frac{\partial^2 \psi_3}{\partial t^2} \right)
\end{aligned} \tag{4.13}$$

The spinning shear deformable twisted beam element presented here is a new element whose static case is only recently proposed by Banerjee [153]. Unlike the other twisted beam element available in the literature, this element allows the inclusion of shear deformation in the two planes of the twisted beam, a feature that makes it suitable for modeling the micro flute of the micro end mill. There is also an explicit inclusion of the rate of twist, which defines the helix angle of the micro flute. Solving the governing equations presented in sections 4.3.1 and 4.3.2 is definitely not a trivial exercise. An efficient numerical technique of the spectral finite element method (SFEM) is adopted to deal with these governing equations. A brief introduction of the method is given in the next section. The first analysis for which the SFEM is used for is the free transverse vibration analysis of the micro end mill. This analysis involves the prediction of the natural frequencies for a micro end mill of specified geometric and material properties.

4.4 A brief background and survey of recent applications of the SFEM

Most of the computational methods of solid mechanics are born out of the unrelenting quest for solutions of governing equations that are normally encountered in structural analysis. In structural analysis, the determination of the dynamic stress, natural frequencies, and vibration modes, for most practical structures, requires a wide range of computational analytical or numerical methods. One of the most prominent methods of structural analysis is the conventional finite element method (CFEM). In its basic formulation, CFEM needs a substantial number of elements when higher natural frequencies are sought. Besides, the inclusion of gyroscopic terms in the governing equations of flexible spinning structures often leads to quadratic eigenvalue problem (QEP) for an undamped free vibration analysis using the CFEM. However, an alternative powerful computational technique is the SFEM.

The SFEM belongs to a class of computational methods that is capable of handling an extensive spectrum of dynamic problems [154-155]. The method has proven to have higher efficiency in dealing with dynamics of structural members [156]. Vinod et al. [151] and Banerjee et al. [157] have successfully used both SFEM and its variant, the dynamic stiffness method (DSM) to understand the vibration behaviors of components in aerospace industries. Recently, the works of Wieslaw [158] and Elforjani and Mba [159] have shown how the method can be used for structural health monitoring of delicate structures. One of the major differences between the CFEM and the DSM or the SFEM, is that the CFEM makes use of discrete approximations of the structural members. As a result, its accuracy is mesh-dependent. On the other hand, both the SFEM and the DSM

use continuum element models that often lead to transcendental eigenvalue problems. Consequently, in most cases one element is enough to model an entire member assuming there is no material or significant geometric discontinuity. This feature puts SFEM or DSM in good stead for dynamic analysis of micro end mill. Spectral elements were first advanced by Doyle [160] to study wave propagation in structures with varying degree of complexity. However, the method has since been extended to diverse structural problems [151, 157, 161-162]. In its basic form, the SFEM is a Discrete Fourier Transform-based method in which the solution procedure is formulated in the frequency domain and wavenumbers space [163]. Furthermore, the SFEM, unlike the CFEM, has its shape functions derived from the exact solution of the underlying governing differential equations. With the use of the exact shape functions, the SFEM involves reduced degree of freedom and generates eigenvalue of interest with high degree of accuracy. It is because of the above features that the SFEM is deemed attractive for the problem at hand.

The basic flow chart of the key steps involved in predicting the natural frequencies is shown in Figure 4-3.

4.5 Dynamic stiffness formulation of the micro end mill sub-structures

The beginning of the solution procedure under the SFEM is to assume harmonic variations of the field variables w , u , ψ and θ with circular frequency of ω . By using the discrete Fourier transform (DFT), we can express each of these field variables in the governing equations of motion as:

Figure 4-3 Flowchart of a free vibration analysis of the micro end mill through the SFEM

$$\begin{aligned}
u_i(y, t) &= \sum_{n=1}^N \bar{U}_i(y) e^{i\omega_n t}; w_i(y, t) = \sum_{n=1}^N \bar{W}_i(y) e^{i\omega_n t} \\
\psi_i(y, t) &= \sum_{n=1}^N \bar{\Psi}_i(y) e^{i\omega_n t}; \theta_i(y, t) = \sum_{n=1}^N \bar{\Theta}_i(y) e^{i\omega_n t}
\end{aligned} \tag{4.14}$$

where subscript $i = 1, 2, 3$ is used to represent the three sub-structures of the micro end mill and ω_n is the circular frequency of the n th sampling point. N is the Nyquist frequency used in the Fast Fourier Transform (FFT) of the field variables. The terms $\bar{W}(y)$, $\bar{U}(y)$, $\bar{\Psi}(y)$ and $\bar{\Theta}(y)$ are frequency dependent amplitudes of u, w, ψ and θ respectively. Substituting Eqs. (4.14) into the governing equations earlier defined, we would have a number of equations for each micro end mill sub-structure. Due to the extensive symbolic manipulation involved in this analysis, only the very basic steps are provided in the solution of each part of the micro end mill. The solution procedures are presented in the subsections that follow.

4.5.1 SFEM solution procedure of the micro end mill's shank

In line with the principle of SFEM, the partial differential equations of each sub-structure will first be reduced to ordinary differential equations through spectral analysis. For the shank, this is achieved by substituting Eq. (4.14) in Eq. (4.5a) – (4.5d) to have the spectral forms of the governing equations as:

$$\sum_{n=1}^N \left[\frac{\mu_1 L_1^2 \omega_n^2}{k_1 A_1 G} \bar{W}_1 + \left(\frac{d^2 \bar{W}_1}{d\xi^2} - L_1 \frac{d\bar{\Theta}_1}{d\xi} \right) \right] e^{i\omega_n t} = 0 \tag{4.15a}$$

$$\sum_{n=1}^N \left[\frac{\rho I_1 L_1}{k_1 A_1 G} \omega_n^2 \bar{\Theta}_1 + \frac{EI_1}{k_1 A_1 G L_1} \left(\frac{d^2 \bar{\Theta}_1}{d\xi^2} \right) + \left(\frac{d\bar{W}_1}{d\xi} - L_1 \bar{\Theta}_1 \right) - i \frac{2\Omega \rho I_1 L_1 \omega_n}{k_1 A_1 G} \psi_1 \right] e^{i\omega_n t} = 0 \tag{4.15b}$$

$$\sum_{n=1}^N \left[\frac{\mu_1 L_1^2 \omega_n^2}{k_1 A_1 G} \bar{U}_1 + \left(\frac{d^2 \bar{U}_1}{d\xi^2} - L_1 \frac{d\bar{\Psi}_1}{d\xi} \right) \right] e^{i\omega_n t} = 0 \quad (4.15c)$$

$$\sum_{n=1}^N \left[\frac{\rho I_1 L_1}{k_1 A_1 G} \omega_n^2 \bar{\Psi}_1 + \frac{EI_1}{k_1 A_1 G L_1} \left(\frac{d^2 \bar{\Psi}_1}{d\xi^2} \right) + \left(\frac{d\bar{U}_1}{d\xi} - L_1 \bar{\Psi}_1 \right) + i \frac{2\Omega \rho I_1 L_1 \omega_n}{k_1 A_1 G} \bar{\Theta}_1 \right] e^{i\omega_n t} = 0 \quad (4.15d)$$

where the non-dimensional parameter $\xi = y/L_1$ has been introduced and L_1 is the length of the shank. Without the loss of generality, the spectral analysis has now reduced the partial differential equations of motion of the shank to a set of ordinary differential equations with constant coefficients. The procedure proceeds with the assumption of the general solutions of Eqs. (4.15a) – (4.15d) in the form:

$$\bar{W} = \bar{A}_{1j} e^{i\alpha\xi}, \bar{U} = \bar{C}_{1j} e^{i\alpha\xi}, \bar{\Theta} = \bar{B}_{1j} e^{i\alpha\xi}, \bar{\Psi} = \bar{D}_{1j} e^{i\alpha\xi}, \quad j = 1, \dots, 8 \quad (4.16)$$

where $\bar{A}_{1j}$, $\bar{B}_{1j}$, $\bar{C}_{1j}$, and $\bar{D}_{1j}$ are used to represent the unknown amplitude spectrum of the shank. For any reasonable practical solutions, ω_n cannot be zero for a truly harmonic motion of the shank. Therefore, the term $e^{i\omega_n t}$ cannot be zero in Eqs (4.15a) – (4.15d). Also, knowing that Eqs. (4.15a) – (4.15d) must be satisfied for each value of n and up to the Nyquist frequency, from here on, ω_n will simply be written as ω for brevity sake. In order to be able to obtain the relationship (called the amplitude ratio in literature) between the amplitude spectrums in Eq. (6), a compact form of the resulting equations after the substitution of Eq. (4.16) in Eqs. (4.15) is obtained as:

$$\begin{bmatrix} R_{11} - R_{12} & 0 & 0 \\ R_{13} & R_{14} & 0 & -R_{15} \\ 0 & 0 & R_{11} - R_{12} \\ 0 & R_{15} & R_{13} & R_{14} \end{bmatrix} \begin{Bmatrix} \bar{A}_{1j} \\ \bar{B}_{1j} \\ \bar{C}_{1j} \\ \bar{D}_{1j} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (4.17)$$

where,

$$\begin{aligned} R_{11} &= -\alpha^2 + \left(\frac{\mu_1 L_1^2}{k_1 A_1 G} \omega^2 \right); R_{12} = i L_1 \alpha; R_{13} = i \alpha; R_{14} = -\alpha^2 \frac{E I_1}{k_1 A_1 G L_1} + \left(\frac{\rho I_1 L_1}{k_1 A_1 G} \right) \omega^2 - L_1 \\ R_{15} &= i \left(\frac{2 \Omega \rho I_1 L_1}{k_1 A_1 G} \omega \right); \end{aligned} \quad (4.18)$$

From the matrix Eq. (4.17), the dispersion relation as well as the relations between the four amplitude spectrums is obtained. Through the help of MATHEMATICA[®], the dispersion relation is found to be:

$$\alpha^8 + \alpha^6 \varepsilon_{11} + \alpha^4 \varepsilon_{12} + \alpha^2 \varepsilon_{13} + \varepsilon_{14} = 0 \quad (4.19)$$

The coefficient terms, ε_{11} , ε_{12} , ε_{13} , and ε_{14} in Eq. (9) are given in the Appendix. In addition, the eight roots of the characteristics equation (Eq. 9) are also given in the Appendix. Meanwhile, the eight roots of the characteristics polynomial given in Eq. (4.19), enable us to write the general solutions for $\bar{W}_1$, $\bar{U}_1$, $\bar{\Theta}_1$, and $\bar{\Psi}_1$ as:

$$\begin{aligned} \bar{W}_1 &= \sum_{j=1}^8 A_{1j} e^{i \alpha_j \xi}; \bar{\Theta}_1 = \sum_{j=1}^8 r_{1j} A_{1j} e^{i \alpha_j \xi}; \bar{U}_1 = \sum_{j=1}^8 \beta_{1j} A_{1j} e^{i \alpha_j \xi}; \bar{\Psi}_1 = \\ &\sum_{j=1}^8 r_{1j} \beta_{1j} A_{1j} e^{i \alpha_j \xi} \end{aligned} \quad (4.20)$$

where the amplitude ratios r_{1j} and β_{1j} have been introduced to harmonize the solutions of the nodal degrees of freedom and they are defined below:

$$r_{1j} = \frac{R_{11}}{R_{12}}; \beta_{1j} = \left(\frac{R_{13} + R_{14} r_{1j}}{R_{15}} \right) \quad (4.21)$$

4.5.2 SFEM solution of the micro end mill's tapered sub-structure

A number of complementary techniques have been proposed for handling the analysis of tapered geometries in the literature. Examples of such techniques can be found in Banerjee and Williams [152] and Wang and Wereley [150]. To handle the tapered geometry of the micro end mill, a polynomial power law is combined with the well-known variational approach of Bubnov-Galerkin method in the context of SFEM. For a spinning tapered geometry, the taper parameters must be defined in the two planes of deformation. Thus, the taper parameter has been included in the following relations defining the variation of the area and second moment of inertia of the tapered neck of the micro end mill.

$$A_{2y} = A_1 \left[\left(1 - \epsilon_z \frac{y}{L_2} \right)^c \left(1 - \epsilon_x \frac{y}{L_2} \right)^c \right], \quad I_{2z} = I_1 \left(1 - \epsilon_z \frac{y}{L_2} \right)^{3c}, \quad I_{2x} = I_1 \left(1 - \epsilon_x \frac{y}{L_2} \right)^{3c} \quad (4.22)$$

Eq. (22) represents the variation of the area along the length of the beam and variation of the second moment of area (I_{2z} and I_{2x}) of the tapered neck in the z and x directions. The terms ϵ_x and ϵ_z are constant called the taper ratios in both x and z directions. A_1 and I_1 denote the values of the cross-sectional area and second moment of inertia at the left side of the tapered section. This left side coincides with the end of the shank. According to Banerjee and Williams [152], the superscript c usually takes values that depend on the nature of the taper. For a linear taper assumption, $c = 1$. Bearing in mind the definitions in Eqs.(4.14) and (4.12), the weak form of Eqs. (4.6a) – (4.6d) with the non-dimensional parameter is formed through the application of the Galerkin-type weighted residual formalism on Eqs. (4.6a) – (4.6d) to have:

$$\begin{aligned}
 \delta \mathbf{u}_{d2}^T & \left\{ \int_0^1 \left(EI_{2z}(\xi) \left\{ \frac{dN_\theta}{d\xi} \right\}^T \left\{ \frac{dN_\theta}{d\xi} \right\} + EI_{2x}(\xi) \left\{ \frac{dN_\psi}{d\xi} \right\}^T \left\{ \frac{dN_\psi}{d\xi} \right\} \right) d\xi \right. \\
 & + \int_0^1 \left(k_2 GA_{2z}(\xi) \left\{ \frac{dN_w}{d\xi} \right\}^T \left\{ \frac{dN_w}{d\xi} \right\} + k_2 GA_{2x}(\xi) \left\{ \frac{dN_u}{d\xi} \right\}^T \left\{ \frac{dN_u}{d\xi} \right\} \right) d\xi \\
 & + \int_0^1 \left(k_2 GA_{2z}(\xi) \left\{ \frac{dN_w}{d\xi} \right\}^T \{N_\theta\} + k_2 GA_{2z}(\xi) \{N_\theta\}^T \left\{ \frac{dN_w}{d\xi} \right\} \right) d\xi \\
 & + \int_0^1 \left(k_2 GA_{2x}(\xi) \left\{ \frac{dN_u}{d\xi} \right\}^T \{N_\psi\} + k_2 GA_{2x}(\xi) \{N_\psi\}^T \left\{ \frac{dN_u}{d\xi} \right\} \right) d\xi \\
 & + \int_0^1 \left(k_2 GA_{2z}(\xi) \{N_\theta\}^T \{N_\theta\} + k_2 GA_{2x}(\xi) \{N_\psi\}^T \{N_\psi\} \right) d\xi \\
 & - \int_0^1 (\mu_{2z} \omega^2 \{N_w\}^T \{N_w\} + \mu_{2x} \omega^2 \{N_u\}^T \{N_u\}) d\xi \\
 & - \int_0^1 \rho I_2(\xi) \omega^2 (\{N_\theta\}^T \{N_\theta\} + \{N_\psi\}^T \{N_\psi\}) d\xi \\
 & \left. + \int_0^1 i2\Omega \rho I_2(\xi) \omega (\{N_\theta\}^T \{N_\psi\} - \{N_\psi\}^T \{N_\theta\}) d\xi \right\} \mathbf{u}_{d2} \mathbf{f}_{d2} = 0
 \end{aligned} \tag{4.23a}$$

where N_θ, N_w, N_u , and N_ψ are the frequency-dependent interpolating functions. The interpolating functions are obtained from the exact solution of the governing differential equations of the spinning uniform Timoshenko beam. The full form of these shape functions are defined in the Appendix. Since $\delta \mathbf{u}_d^T$ is arbitrary, we obtain the dynamic stiffness equation at the elemental level of the tapered neck from Eq. (13a) as:

$$\{\mathbf{f}_{d2}\} = [\mathbf{S}_2(\omega)]\{\mathbf{u}_{d2}\} \tag{4.23b}$$

$$\{\mathbf{u}_{d2}\} = \{\bar{\mathbf{W}}_{21} \bar{\boldsymbol{\theta}}_{21} \bar{\mathbf{U}}_{21} \bar{\boldsymbol{\psi}}_{21} \bar{\mathbf{W}}_{22} \bar{\boldsymbol{\theta}}_{22} \bar{\mathbf{U}}_{22} \bar{\boldsymbol{\psi}}_{22}\}^T \tag{4.24a}$$

$$\{f_{d2}\} = \{\bar{Q}_{z21}\bar{M}_{z21}\bar{Q}_{x21}\bar{M}_{x21}\bar{Q}_{z22}\bar{M}_{z22}\bar{Q}_{x22}\bar{M}_{x22}\}^T \quad (4.24b)$$

The dynamic stiffness matrix $S_2(\omega)$ is defined as:

$$\begin{aligned} [S_2(\omega)] = & \left[\int_0^1 \left(EI_{2z}(\xi) \left\{ \frac{dN_\theta}{d\xi} \right\}^T \left\{ \frac{dN_\theta}{d\xi} \right\} + EI_{2x}(\xi) \left\{ \frac{dN_\psi}{d\xi} \right\}^T \left\{ \frac{dN_\psi}{d\xi} \right\} \right) d\xi + \right. \\ & \int_0^1 \left(k_2 GA_{2z}(\xi) \left\{ \frac{dN_w}{d\xi} \right\}^T \left\{ \frac{dN_w}{d\xi} \right\} + k_2 GA_{2x}(\xi) \left\{ \frac{dN_u}{d\xi} \right\}^T \left\{ \frac{dN_u}{d\xi} \right\} \right) d\xi + \\ & \int_0^1 \left(k_2 GA_{2z}(\xi) \left\{ \frac{dN_w}{d\xi} \right\}^T \{N_\theta\} + k_2 GA_{2z}(\xi) \{N_\theta\}^T \left\{ \frac{dN_w}{d\xi} \right\} \right) d\xi + \\ & \left. \int_0^1 \left(k_2 GA_{2x}(\xi) \left\{ \frac{dN_u}{d\xi} \right\}^T \{N_\psi\} + k_2 GA_{2x}(\xi) \{N_\psi\}^T \left\{ \frac{dN_u}{d\xi} \right\} \right) d\xi + Q2 \right] \end{aligned} \quad (4.25a)$$

4.5.3 SFEM solution of the micro end mill's flute

Since it is found that the cross-sections of most micro end mills' flutes are not uniform (usually slightly tapered towards the tip), Eqs. (4.7) are modified during the solution phase to include this non-uniformity. The fluted sub-structure of the micro end mill is therefore characterized by two geometric non-uniformity factors; the rate of twist and the tapered kinematic. Consequently, the SFEM solution procedure of the fluted sub-structure involves more rigorous mathematical manipulations. The first step in the solution procedure for the flute is to obtain the exact solution of the governing equations of the taper-free spinning twisted shear deformable beam (in the manner explained in sub-section 4.5.1). The exact solutions of the taper-free twisted element yield the needed interpolation functions to be combined with the Bubnov-Galerkin method in the context

of the SFEM (as done in sub-section 4.5.2). The local dynamic stiffness matrix for the flute is therefore obtained, after rigorous mathematical simplification, as:

$$\begin{aligned}
 Q2 = & \left[\int_0^1 \left(k_2 G A_{2z}(\xi) \{N_\theta\}^T \{N_\theta\} + k_2 G A_{2x}(\xi) \{N_\psi\}^T \{N_\psi\} \right) d\xi \right. \\
 & - \int_0^1 (\mu_{2z} \omega^2 \{N_w\}^T \{N_w\} + \mu_{2x} \omega^2 \{N_u\}^T \{N_u\}) d\xi \\
 & - \int_0^1 \rho I_2(\xi) \omega^2 \left(\{N_\theta\}^T \{N_\theta\} + \{N_\psi\}^T \{N_\psi\} \right) d\xi \\
 & \left. + \int_0^1 i 2 \Omega \rho I_2(\xi) \omega \left(\{N_\theta\}^T \{N_\psi\} - \{N_\psi\}^T \{N_\theta\} \right) d\xi \right]
 \end{aligned} \tag{4.25b}$$

$$\begin{aligned}
 [S_3(\omega)] = & \left\{ \int_0^1 \left(\left[k_{z3} A_3 G \left\{ \frac{dF_w}{d\xi} \right\}^T \left\{ \frac{dF_w}{d\xi} \right\} - (\gamma^2 k_{x3} A_3 G - \mu_3 \omega^2 - \mu_3 \Omega^2) \{F_w\}^T \{F_w\} + \right. \right. \right. \\
 & \gamma A_3 G (k_{z3} + k_{x3}) \{F_w\}^T \left\{ \frac{dF_u}{d\xi} \right\} + k_{z3} A_3 G \{F_w\}^T \left\{ \frac{dF_\psi}{d\xi} \right\} - \gamma k_{x3} A_3 G \{F_w\}^T \{F_\theta\} + \\
 & \left. \left. i 2 \mu_3 \omega \Omega \{F_w\}^T \{F_u\} \right] + \right. \\
 & \left[E I_{z3} \left\{ \frac{dF_\theta}{d\xi} \right\}^T \left\{ \frac{dF_\theta}{d\xi} \right\} - (\gamma^2 E I_{x3} + k_{x3} A_3 G + J_{z3} \omega^2 + J_{x3} \Omega^2) \{F_\theta\}^T \{F_\theta\} - \right. \\
 & \gamma k_{x3} A_3 G \{F_\theta\}^T \{F_w\} - k_{x3} A_3 G \{F_\theta\}^T \left\{ \frac{dF_u}{d\xi} \right\} - \gamma (E I_{z3} + E I_{x3}) \{F_\theta\}^T \left\{ \frac{dF_\psi}{d\xi} \right\} - i 2 \omega \Omega (J_{x3} + \\
 & J_{z3}) \{F_\theta\}^T \{F_\psi\} \left. \right] + \left[k_{x3} A_3 G \left\{ \frac{dF_u}{d\xi} \right\}^T \left\{ \frac{dF_u}{d\xi} \right\} - (\gamma^2 k_{z3} A_3 - G \mu_3 \omega^2 - \mu_3 \Omega^2) \{F_u\}^T \{F_u\} - \right. \\
 & \gamma A_3 G (k_{z3} + k_{x3}) \{F_u\}^T \left\{ \frac{dF_w}{d\xi} \right\} - k_{x3} A_3 G \{F_u\}^T \left\{ \frac{dF_\theta}{d\xi} \right\} - \gamma k_{z3} A_3 G \{F_u\}^T \{F_\psi\} - \\
 & \left. \left. i 2 \mu_3 \omega \Omega \{F_u\}^T \{F_w\} \right] + \right. \\
 & \left[E I_{x3} \left\{ \frac{dF_\psi}{d\xi} \right\}^T \left\{ \frac{dF_\psi}{d\xi} \right\} - (\gamma^2 E I_{z3} + k_{z3} A_3 - G J_{x3} \omega^2 - J_{z3} \Omega^2) \{F_\psi\}^T \{F_\psi\} - \right.
 \end{aligned}$$

$$\gamma k_{Z3} A_3 G\{F_\psi\}^T \{F_u\} + k_{Z3} A_3 G\{F_\psi\}^T \left\{ \frac{dF_w}{d\xi} \right\} + \gamma (EI_{Z3} + EI_{X3}) \{H_\psi\}^T \left\{ \frac{dF_\theta}{d\xi} \right\} + i2\omega\Omega(J_{X3} + J_{Z3}) \{F_\psi\}^T \{F_\theta\} \Big] d\xi = 0 \Big\} \quad (4.26)$$

where F_θ, F_w, F_u , and F_ψ are frequency-dependent interpolating functions derived from the exact solution of the taper-free twisted shear deformable beam. The full form of these shape functions are also given in the Appendix. It is to be noted that the integration in Eq. (4.26) is over the entire length of the micro flute (L_3). However, this length has been normalized as $\xi = Y/L_3$.

The following parameters that define the taper kinematic of the flute are used in Eq. (4.26) to obtain the local dynamic stiffness matrix.

$$A_{3Y} = A_2 \left(1 - f_Z \frac{Y}{L_3}\right)^c \left(1 - f_X \frac{Y}{L_3}\right)^c, \quad I_{3Z} = I_2 \left(1 - f_Z \frac{Y}{L_3}\right)^{3c}, \quad I_{3X} = I_2 \left(1 - f_X \frac{Y}{L_3}\right)^{3c} \quad (4.27)$$

where A_{3Y} , I_{3Z} , and I_{3X} represent the variations of the area and the second moment of area of the tapered flute in the principal planes and f_Z and f_X are the taper ratios in those planes. The observed geometry of the flute of a standard micro end mill reveals a profile that matches the description of a linear tapered configuration, hence superscript c in Eq. (4.27) is also taken as 1. The dynamic stiffness matrix for the tapered part is numerically obtained through the combination of the SFEM with the Bubnov-Galerkin method. Contrarily, each of the micro shank and the micro flute is represented by a single spectral finite element having just two nodes. The nodal displacements at the nodal ends of the micro shank and the flute are expressed by eight amplitude coefficients as:

$$\{u_{d1}\} = [R_1] \{A_{1k}\} \quad (4.28a)$$

$$\{u_{d3}\} = [R_3]\{A_{3k}\} \quad (4.28b)$$

where $k = 1, 2 \dots 8$. Also, the nodal forces corresponding to the eight nodal displacements are expressed in terms of the eight amplitude coefficients as:

$$\{f_{d1}\} = [P_1]\{A_{1k}\} \quad (4.29a)$$

$$\{f_{d3}\} = [P_3]\{A_{3k}\} \quad (4.29b)$$

The column vector $\{f_d\}$ is the nodal force vector for each spectral finite beam element. Finally, the stiffness equation is obtained by eliminating the coefficients A_{1k} and A_{3k} from Eqs. (4.28a) – (4.28b) and (4.29a) – (4.29b) to have

$$\{f_{d1}\} = [S_1(\omega)]\{u_{d1}\} \quad (4.30a)$$

$$\{f_{d3}\} = [T]'[S_3(\omega)][T]\{u_{d3}\} \quad (4.30b)$$

where $[S(\omega)] = [P][R]^{-1}$. Matrix $\mathbf{S}(\omega)$ in Eqs. (4.30a) – (4.30b) is the non-dimensional 8×8 frequency dependent dynamic stiffness matrix that is used for the determination of each sub-structure's response. The matrix $\mathbf{T}$ and the transpose matrix $\mathbf{T}'$ in Eq. (4.30b) are used to transform the dynamic stiffness matrix of the fluted part from its local coordinates XYZ to the global coordinate xyz . The full forms of the matrices $\mathbf{P}_1, \mathbf{P}_3, \mathbf{R}_1, \mathbf{R}_3$ and $\mathbf{T}$ are listed in the appendix. Since the expression for the tapered section is difficult to obtain in a closed-form solution because of the nature of the governing differential equations being variable coefficients, the equivalent expression for the tapered section dynamic stiffness is given by Eq. (4.31).

$$\begin{aligned}
[S_2(\omega)] = & \delta \mathbf{u}_d^T \left\{ \int_0^{L_2} \left(EI_{2z}(\xi) \left\{ \frac{dN_\theta}{d\xi} \right\}^T \left\{ \frac{dN_\theta}{d\xi} \right\} + EI_{2x}(\xi) \left\{ \frac{dN_\psi}{d\xi} \right\}^T \left\{ \frac{dN_\psi}{d\xi} \right\} \right) d\xi + \right. \\
& \int_0^{L_2} \left(k_2 GA_{2z}(\xi) \left\{ \frac{dN_w}{d\xi} \right\}^T \left\{ \frac{dN_w}{d\xi} \right\} + k_2 GA_{2x}(\xi) \left\{ \frac{dN_u}{d\xi} \right\}^T \left\{ \frac{dN_u}{d\xi} \right\} \right) d\xi + \\
& \int_0^{L_2} \left(k_2 GA_{2z}(\xi) \left\{ \frac{dN_w}{d\xi} \right\}^T \{N_\theta\} + k_2 GA_{2z}(\xi) \{N_\theta\}^T \left\{ \frac{dN_w}{d\xi} \right\} \right) d\xi + \\
& \int_0^{L_2} \left(k_2 GA_{2x}(\xi) \left\{ \frac{dN_u}{d\xi} \right\}^T \{N_\psi\} + k_2 GA_{2x}(\xi) \{N_\psi\}^T \left\{ \frac{dN_u}{d\xi} \right\} \right) d\xi + \\
& \int_0^{L_2} \left(k_2 GA_{2z}(\xi) \{N_\theta\}^T \{N_\theta\} + k_2 GA_{2x}(\xi) \{N_\psi\}^T \{N_\psi\} \right) d\xi - \int_0^{L_2} (\mu_{2z} \omega^2 \{N_w\}^T \{N_w\} + \\
& \mu_{2x} \omega^2 \{N_u\}^T \{N_u\}) d\xi - \int_0^{L_2} \rho I_2(\xi) \omega^2 \left(\{N_\theta\}^T \{N_\theta\} + \{N_\psi\}^T \{N_\psi\} \right) d\xi + \\
& \left. \int_0^{L_2} i 2 \Omega \rho I_2(\xi) \omega \left(\{N_\theta\}^T \{N_\psi\} - \{N_\psi\}^T \{N_\theta\} \right) d\xi = 0 \right\}
\end{aligned} \tag{4.31}$$

where N_θ, N_w, N_u , and N_ψ are the frequency-dependent interpolating functions. The interpolating functions are obtained from the exact solution of the governing differential equations of the spinning uniform Timoshenko beam. The full forms of these shape functions are defined in the appendix.

4.6 Numerical results and discussion

This section presents the results of the numerical analyses. To assess the accuracy of most numerical methods and mathematical models, validation is usually required. The accuracy of the developed micro end mill model is assessed through two simplified models available in the literature. The first attempt is to validate the spectrally formulated elements of the micro end mill sub-structures before combining the elements to get the

overall prediction for the whole micro end mill. This validation step is presented in subsection 4.6.1. The second validation involves the comparison of the proposed method predictions with that of ANSYS simulation. This is in turn presented in subsection 4.6.2. Validation is necessary so as to strengthen the accuracy of the approach.

4.6.1 The validation of the dynamics response of the micro end mill sub-structures

Using the results of free vibration analysis of spinning beams available in the literature, numerical comparison is carried out using the prediction from the proposed method and that based on the CFEM. Tables 4-1, 4-2 and 4-3 where the natural frequencies for the spinning uniform, the spinning taper and the non-spinning twisted shear deformable beams under the clamped-free boundary conditions are presented respectively. In general, since the SFEM uses continuum element, the determinants of the resulting dynamic stiffness matrices when equated to zero are often of infinite roots. It will thus require an efficient iteration method to evaluate each natural frequency. In this work, the Muller root search algorithm detailed in Press [164] is used to generate the natural frequencies. The problem of missing natural frequency is somewhat prevalent while dealing with structures with closely spaced natural frequencies. In such cases, to avoid missing any natural frequencies, the well-known algorithm of Wittrick and Williams [165] becomes quite handy. Each of the natural frequency in the Tables 4-1 and 4-2 is evaluated only up to 4-digit machine precision. The following data taken from Bazoune et al. [166] are used for the spinning uniform and tapered Timoshenko beams:

$$L = 1m, E = 207GPa, G = 77.6GPa, \epsilon_z = 0.15, \epsilon_x = 0.3, \frac{A_1 L^2}{I_1} = 2500, k = \frac{5}{6}, \rho = 7850 kg/m^3$$

When presenting results for Tables 4-1 and 4-2, the natural frequencies (λ_i) and spinning speed (Ω) are given in the non-dimensional forms as

$$\lambda_i = \omega_i/\omega_0, \eta = \Omega \sqrt{\rho AL^4/EI} \quad (4.32a)$$

where the i -th natural frequency is normalized with respect to ω_0 , defined as

$$\omega_0 = \sqrt{\rho A/EI} \quad (4.32b)$$

For the spinning uniform beam, ϵ_z and ϵ_x will be zero. At the same time, the same value of length (L) is used for L_1 and L_2 .

Meanwhile, the natural frequencies of a spinning twisted shear deformable beam are not available in the literature. Therefore, this sub-structure is only validated with the result of a special static case of a twisted beam provided in Lin et al. [167] and Yardimoglu et al. [168]. The validation of the spinning case of the twisted sub-structure is done during the validation of the assembled sub-structures reported in section 4.6.2. The comparison of the predictions from the present study with those of Lin et al. [167] and Yardimoglu et al. [168] is given in Table 4-3, where the following parameters for a twisted beam of dimension ($0.1524 \text{ m} \times 0.0254 \text{ m} \times 0.001727 \text{ m}$) are used:

$$E = 207 \text{ GPa}, G = 82.74 \text{ GPa}, f_z = f_x = 0, k_{x3} = k_{y3} = \frac{5}{6}, \rho = 7850 \frac{\text{kg}}{\text{m}^3}, \gamma = 45^\circ$$

In all of the comparisons made in this first layer of validation, a good agreement between the natural frequencies predicted from the SFEM and previous studies is found. Additionally, it is observed from these Tables that only a few spectrally formulated

elements are needed to predict the natural frequencies of most of the beam models considered; making the SFEM a very powerful computational method. During the validation analyses, the variability in cross-sectional properties of the tapered beam resulted in profound effect on the number of spectral elements required for the prediction of its frequencies. This observation is similar to those reported by Banerjee et al. [169] and Vinod et al. [151]. However, despite this difficulty with geometric nonlinearity, the convergence of the spectral element in the natural frequency or dynamic response prediction is parabolic as the number of element is increased. Apart from showing the agreement of the developed spectrally formulated elements with reported results in the literature, the influence of the spinning speed on the natural frequencies of the beam is also demonstrated in Tables 4-1 and 4-2.

Table 4-1 Prediction of the non-dimensional natural frequencies of a clamped-free spinning uniform Timoshenko beam (rounded up to two decimal places)

η	Number of SFE	Present theory	Ref. [170]
		λ_i	λ_i
3	1	4.80	4.79
		22.75	22.76
		47.83	-- ^a
6	1	7.34	7.43 ^b
		26.26	26.22
		51.62	-

^aDouble dash means the value is not available

^bThe exact value, according to ref. [166] is 7.3319.

4.6.2 Finite element simulations and parametric studies of effects

The geometric properties of the micro end mill used for the analyses are given in Table 4-4. In this preliminary analysis, the study is restricted to a two-fluted micro end mill. Hence, a near rectangular cross-section has been chosen to represent the flute.

Table 4-2 Prediction of the non-dimensional natural frequencies of a clamped-free spinning tapered Timoshenko beam (rounded up to two decimal places)

η	Number of FLE ^a	Number of SFE ^b	Present theory	Ref. [166]
			λ_i	λ_i
3	12	8	5.05	5.05
			21.11	21.11
			52.27	52.86
6	12	8	7.57	7.58
			24.55	24.53
			56.67	56.66

^aFinite line element, ^bSpectrally formulated element

Table 4-3 Validation of natural frequencies of a non-spinning clamped-free twisted Timoshenko beam (rounded up to two decimal places)

Frequency mode	Present theory	Ref. [168]	Lin et al. [167]
1	60.56	61.8	61.7
2	300.11	304.8	300.9
3	936.51	944.5	917.0

Table 4-4 Geometric properties of the micro end mills used in the case studies

Sub-structures	Geometric properties
Shank	$k_1 = 0.90, d_1 = 4.92 \text{ mm}, L_1 = 30.85 \text{ mm}$
Tapered neck	$k_2 = 0.90, \epsilon_z = \epsilon_x = 0.23, L_2 = 10 \text{ mm}, \frac{d_1}{d_2} = 14$
Micro flute	length (L_3) = $2280 \mu\text{m}$, thickness = $160 \mu\text{m}$, width = $300 \mu\text{m}$, $f_z = f_x = 0.48$, $\gamma = 60^\circ$, $k_{x3} = 0.90, k_{x3} = 0.90$

The finite element simulations that would be presented involve the spectral finite element simulations and the CFEM using ANSYS®. Two different modal analyses are performed with ANSYS®. The first analysis (given in Table 4-5) makes use of finite line element (FLE), while the second involves the use of solid volume elements. The natural frequency prediction from the present theory using SFEM and comparison with the ANSYS® simulation results are presented in Table 4-5. The finite element simulation results presented in Table 4-5 are obtained through the use of 1D beam element (BEAM189). A number of beam elements are available in ANSYS®, but to model the micro end mill with spinning Timoshenko beam elements, BEAM188 and BEAM189 seemed to be most suitable. These two beam elements (that is, BEAM188 and BEAM189) are developed based on the thick beam theory and are capable of handling gyroscopic effects and rotary inertia quite efficiently. The first observation from Table 4-5 is the close agreement between the SFEM predictions and the ANSYS® results.

Table 4-5 Prediction of the natural frequencies (in kHz) of the assembled beam elements (rounded up to two decimal places)

Spinning rate (<i>krpm</i>)	Mode	Number FLE ^a	of	Number SFE ^b	of	Present theory	ANSYS [®]
						ω_i	ω_i
0	ω_1	80		10		6.30	6.26
	ω_2	80		10		13.67	13.34
	ω_3	80		10		35.34	35.32
	ω_4	80		10		67.62	67.11
	ω_5	80		10		94.11	88.45
120	ω_1	85		10		4.09	4.07
	ω_2	85		10		12.30	12.01
	ω_3	85		10		41.70	42.38
	ω_4	85		10		81.14	84.55
	ω_5	85		10		126.10	130.34
240	ω_1	85		10		5.53	5.38
	ω_2	85		10		14.76	15.13
	ω_3	85		10		55.88	58.48
	ω_4	85		10		117.66	120.07
	ω_5	85		10		170.23	172.05

^aFinite line element, ^bSpectrally formulated element

In addition, the results of Table 4-5 show that the spectrally formulated elements performed efficiently with just ten elements compared to the total number of finite line elements required for the CFEM. Without the use of modal reduction techniques, the CFEM requires at least 804 DOF to give results close to what the SFEM predicts. On the other hand, only 80 degrees of freedom are used for the spectral finite element

simulations. This great reduction in degrees of freedom helps to reduce computational time for quick check in machining workshops.

Additionally, further prediction of the natural frequency, of the assembled beam elements, from the present theory and comparison with the ANSYS® simulation results are presented in Table 4-6 for three case studies. Case study 1 reveals the natural frequency of a static micro end mill with an untapered flute (i.e. $f_z = f_x = 0$). In case studies 2 and 3, the influence of the taper parameter ($f_z = f_x = 0.48$) on the dynamic response is investigated at the spinning rates of 120 *krpm* and 240 *krpm*, respectively. The predictions, presented in Table 4-6 involving the SFEM and the CFEM are both based on the combination of, uniform, tapered, and twisted beam elements for consistency. An important observation is the reduction in the natural frequency associated with the increasing spin rate. This is generally expected for the backward whirling condition of a spinning system as demonstrated in Genta [171] and Barnerjee [172]. During the backward whirl of the spinning system, a material point on the rotor orbits in opposite direction to the rotational direction. Under the forward whirl, as the spinning speed increases, it is often observed that the gyroscopic moment induced by the spinning make the system behave like a stiff spring. The increased in stiffness tend to increase the natural frequency. However, in the case of the backward whirl, the effect is somewhat reversed. This is because the increasing spin rate often leads to a dynamic reduction in the effective stiffness of the system. With a reduction in the effective stiffness, there is a reduction in the natural frequency. This behavior can also be explained by the skew-symmetric nature of the matrix that the gyroscopic term adds to the equation of motion.

The eventual effect of the nature of this skew-symmetric is stiffening and softening effect.

Table 4-6 Prediction of the natural frequencies (up to two decimal places) of the assembled beam elements

Mode	Number of FLE ^a	Number of SFE ^b	Present theory (kHz)	FEM
			ω_i	ω_i
Case 1: $\Omega = 0$ krpm, $f_z = f_x = 0$				
ω_1	95	16	2.42	2.30
ω_2	95	16	13.93	13.52
ω_3	95	16	23.44	23.21
Case 2: $\Omega = 120$ krpm, $f_z = f_x = 0.48$				
ω_1	104	16	2.37	2.25
ω_2	104	16	13.75	13.34
ω_3	104	16	27.66	27.38
Case 3: $\Omega = 240$ krpm, $f_z = f_x = 0.48$				
ω_1	104	16	3.19	2.98
ω_2	104	16	16.49	16.31
ω_3	104	16	37.07	37.19

^aFinite line element; ^bSpectrally formulated element

4.6.3 Effect of the spinning rate on the natural frequencies of the micro end mill

The spinning rate is one of the process variables of interest in micro milling and in any parametric studies of machining tools. The influence of the spinning rate on the natural frequencies is also included in Table 4-6. Three different values of the spinning rate (0 krpm, 120 krpm, and 240 krpm) are considered. The case of $\Omega = 0$ krpm

reduces the model to a non-spinning micro end mill. Both the FLE of CFEM and the SFE of the SFEM produced very close results, with the SFEM maintaining a stable prediction with minimum degrees of freedom. As earlier stated, for the case of the backward whirling of the assembled beam, a reduction in frequency is also observed. The non-spinning micro end mill is also a good validation option as there would be reduced numbers of terms in the simulations. The CPU time taken for the non-spinning case is about half of the spinning cases. At the beginning of the analyses, it was decided to use both the line element and solid element in the ANSYS library to validate the non-spinning case. Although, the ANSYS simulation with 3D solid elements (shown in Figure 4-4) is more computationally intensive but provides good visual insight of the deformation modes. For the analysis involving the 3D solid element, a tetrahedron solid element was selected for meshing the geometric micro end mill model due to its ability to handle geometrical discontinuity in CAD models accurately. The element is defined by ten nodes with three degrees of freedom at each node and the mesh is shown in Figure 4-4. The number of elements required to mesh the geometric model for the 3D simulation is around one thousand two hundred. The corresponding deformation modes that correspond to each forward natural frequency for the first four frequency modes of the micro end mill resulting from this volume element modal analysis in ANSYS are presented in Figures 4-6.

Beneath each of the deformation modes in Figure 4-6 is a bar expressing the stresses (in N/mm^3) that are associated with each deformation modes. These stresses obtained from a modal analysis do not represent actual stresses in the micro end mill, but provide an idea of the relative stress distributions for each deformation mode.

Importantly, since each of the natural frequencies predicted is associated with distinct deformation mode, avoiding them is necessary in avoiding resonance.

Figure 4-4 Finite element meshes using quadratic tetrahedron element

Otherwise, if a disturbance's frequency coincides with any of these natural frequencies, the deformation arising from the effect of the resonance would become excessive. This in turn will impart on the life of the tool as the deformation may exceed the material's elastic or strength limit. The natural frequencies and mode shapes are principal parameters for dynamic loading conditions. Their extraction offers the benefit of observing the different deformation modes of any structural elements without the influence of an external excitation force. Further inclusion of the spinning rate results in significant change of the natural frequencies. By using the relation $(S_{120} - S_0) \times 100/$

S_0 , it was discovered that at the spinning speed of 120 krpm, the first natural frequency dropped by about 2.17 %, while the second natural frequency dropped by around 1.33%. The third natural frequency increased by 1.8 %. This phenomenon is very common with spinning beams. In his work on drill bit, Hsu [139] observed that in between this drop and the increase in the natural frequencies lies a critical speed of the micro end mill. The significance of the critical speed, according to Wijeyewickrema and Keer [129] and Gong et al. [131], is in its association with micro tool's instability when a buckling load of the lowest mode is reduced to zero. Generally, a spinning structural element with equal flexural rigidity is stable for any angular speed of spinning except for certain critical values.

4.6.4 Influence of the helix angle on the natural frequencies of the micro end mill

The micro flutes of most commercially available micro end milling cutters are mostly helical. A cutter with a straight flute usually experiences high vibration because the whole cutter would impact the workpiece at once, causing poor material removal and reduced accuracy. In the study by Boucher et al. [173], the helix angle was discovered to affect both the cutting forces and tool's life. In the design of micro end mills, the choice of suitable helix angle is usually arrived at after a number of recursive optimization trials. Three different values of helix angle are chosen to observe the influence of this design variable on the micro end mill natural frequencies. The results of the variation are given in Table 4-7.

Figure 4-5 (a) First mode shape of a micro end mill.

Figure 4-5 (b) Second mode shape of a micro end mill.

Figure 4-5 Frequency mode shapes (a) First mode shape (b) Second mode shape

Table 4-7 Variation of the natural frequencies (up to two decimal places) with respect to helix angle

Helix Angle (<i>deg</i>)	Natural frequencies (<i>kHz</i>)	
	$\Omega = 120 \text{ krpm}$	$\Omega = 240 \text{ krpm}$
5	7.72	9.31
	7.76	10.42
	34.99	46.88
	67.89	98.44
	112.22	151.49
30	5.65	7.62
	12.52	15.03
	39.50	52.93
	95.64	138.68
	139.98	188.98
45	4.21	5.51
	16.78	18.04
	47.48	59.42
	109.02	154.89
	151.76	202.28

With a helix angle of 5° , the natural frequency exhibits a fluctuation that resembles that of using a micro end mill with a straight cutter. By keeping the helix angle at this value, the rotational speed is varied between 120 krpm and 240 krpm . In both cases, the first and the second natural frequencies become extremely close, the third natural frequency gets to about half the fourth and the fourth also becomes about half the fifth. At the higher values of 30° and 45° , the natural frequencies at the different modes begin to get distinct from each other. These two values of helix angle are widely used in micro end mill design and can be said to be optimum depending on the values of other geometrical parameters of the tool. From mechanics point of view, the helix angle introduced a coupling between the flexural planes of the structural element resulting in

strengthening of the weaker plane. Too much helix angle however may impact negatively on the lives of the tool by reducing its bending stiffness. Especially, since micro end mills are usually coated with cemented carbide, a very high helix angle could induce a crack formation.

4.6.5 Shear deformation and rotary inertia effects on the natural frequencies

Shear deformation is an effect that is not included in the classical beam theory used in pioneering works such as that of Magrab and Gilsinn [120] and Telkinap and Ulsoy [121] that dealt with the modeling of drill bits. The awfully small machining zone, process forces of various forms, the possibility of excessive tool's deflection from bending stresses and shearing stress deformation necessitate the need to carry out parametric studies of shear deformation and rotary inertia variables on the micro end mill dynamics.

To investigate the effect of the shear and rotary inertia on the natural frequencies, the parameters relating to shear strain and rotary inertia are eliminated from the global dynamic matrix. The influence of the elimination of these terms on the natural frequencies of a spinning micro end mill is presented in Table 4-8. The spinning speed of 120 *krpm* and a helix angle of 45° are employed, along with the other geometrical and material properties previously defined in Table 4-4. The percentage difference for the first five natural frequencies reveals that at the first natural frequency, a difference of around 2.61% is found with the predicted result. However, the effect is quite significant

for the remaining values. The spinning classical beams, also known as the Euler-Bernoulli beam is suitable for modeling micro-drills.

Table 4-8 Effects of rotary inertia and shear deformation parameters on the natural frequencies (kHz) (rounded up to two decimal places)

Frequency mode	Natural frequencies (without shear and rotary inertia)	Natural frequencies (shear and rotary inertia included)
ω_1	2.43	2.37
ω_2	16.14	13.75
ω_3	33.17	27.66

4.7 Summary

In this chapter, a detailed analytical study devoted to the mathematical modeling of a micro end mill's dynamics in high-speed micromilling operation is presented. Dealing with the micro end mill as a special case of the micro cutting tool, the micro end mill is sub-structured into three parts. The governing dynamic equations of motion for each of these parts are derived. The SFEM is then employed to solve the derived coupled partial differential equations. The adoption of the SFEM provides a framework to effectively deal with the complex mathematical model of motion that characterizes each sub-structured part of the micro end mill. The SFEM also contributes significantly to the reduction of degrees of freedom needed to deal with such a complex structural element.

The influence of the spinning rate on the natural frequencies of the micro end mill is found to be significant. It was discovered that at the spinning speed of 120 krpm, the first natural frequency dropped by about 2.17 %, while the second natural frequency dropped by around 1.33%. The third natural frequency increased by 1.8 %. The study of the influence of the helix angle indicates that at the values of 30° and 45° helix angles, the natural frequencies at the different modes begin to get distinct from each other, representing a very stable behavior of the micro tool. The presence of the shear deformation and rotary inertia in the mathematical model adds flexibility to the dynamic response of the micro tool.

CHAPTER 5 NON-CLASSICAL CONTINUUM THEORY OF SMALL-SCALE STRUCTURES

5.1 Introduction

In this chapter, novel theoretical models of micro-scale structures, with non-trivial mechanical behavior and complex geometries, will be presented. In chapter 4, a model of a micro end mill (spatially sub-structured) was presented. It would be observed that, even though, the structures of the micro tools (micro-drills and micro end mill) have micro scale dimensions, the analytical model presented in chapter four is based purely on the classical theory of elasticity. The inadequacy of the classical theory would be addressed in this chapter. This is done by detailing, in this chapter, the novel theoretical formulations of some structural elements whose geometries and mechanical behaviors bear relevance to the structural elements employed in the model of the micro-cutting tool provided in chapter four.

This chapter proceeds from the presentation of the need for the non-classical elasticity theory of small-scale structures in section 5.2. In section 5.3, the detail of one of the higher-order, non-classical elasticity theories of solid, *the modified couple stress theory*, is presented. Section 5.4 contains the presentation of another higher-order elasticity theory called the *Eringen's nonlocal elasticity theory* and its brief extension to the modeling of non-prismatic embedded nano-sized beam. Key conclusions from the analyses contained in the chapter are summarized in section 5.5.

5.2 On the need for continuum theory of small-scale structures

Beyond micromachining, there are extensive applications of micro-scale structures due to their enhancement of systematic integration of complex functionality at a lower manufacturing cost. In general, micro-scale structures have become essential elements in the unrelenting search for portable, compact, and space-constrained miniature systems. In most miniature systems, the skeletal idealizations of the essential components of the systems are vibrating micro-sized beams, plates or shells. Often, the mechanical response of these rotating micro-sized beams, plates or shells shows a size effect phenomenon that the classical beam, plate or shell theories have failed to demonstrate [174]. From extensive experimentation, it has been proven for decades that size dependency exists in metals, hardened copper polycrystals, glassy polymers, anisotropic materials like human bones, and elastic-perfectly plastic materials [175-179].

Apparently, the classical theory of linear elasticity and plasticity has been used for years in explaining many bafflingly complex phenomena. However, the scale upon which the classical theory of linear elasticity is based lacks the spatial resolution required to grasp the microstructural effect in micro-sized and nano-sized structures. As a result of this inadequacy, a number of investigators have questioned the legitimacy of using the classical elasticity theory of structures to rationalize the intricate microscopic phenomenon (generally referred to as size effect) in micro- and nano-scale structures [180-183]. Due to the significance of the size effect, researchers have pursued, with verve and vigor, the refinement and reformulation of the classical size-free elasticity theory to address the issue of size effect. In this respect, a number of elasticity theories with

higher-order constitutive laws have emerged to predict the size dependency of micro- and nano-size structures by the inclusion of the material length scale in their formulations. Of these higher-order elasticity theories, Eringen's nonlocal elasticity theory [184-186], the grade-2 strain gradient theory, the couple stress theory and the micropolar elasticity have gained considerable attention among interdisciplinary scientists. The distinguishing feature between the divergent theories, upon which the inclusion of material length scale is based, is the number of material constants (apart from the Lamé constants) present in the reformulated governing equations. For instance, the Eringen's nonlocal elasticity theory [186-187], generates two material constants and has been widely employed in a number of recent studies to investigate the impact of size effect on the deformational response of carbon nanotubes [188-191]. The strain gradient theory, based on the works of Mindlin and Eshel [192] and further developed by Aifantis [193], Aifantis and Willis [194], and Papargyri-Beskou et al. [195], generates four material constants, the determinations of all of which are non-trivial. On account of the non-triviality of the determination of the microstructure-dependent length parameters, as observed by Lam et al. [196], Maranganti and Sharma [197] and Reddy [198], reformulated models of structures involving tractable number of length scale parameter offer mathematical convenience and advantages. It is evidenced from the above studies that, compared to the classical theory of elasticity, the governing equations of the refined higher-order elasticity theories are better equipped with further spatial gradients of pertinent variables [199]. In fact, most of these studies revealed that by including these relevant additional gradient parameters in the formulations of the new models, events (such as stress concentrations

around crack tips) that are engendered by microstructural effects can be realistically captured.

In the sections that follow, two of the new, nonlocal, non-classical higher-order theoretical frameworks will be presented for some geometries of interest to the present research objective. These two higher-order theoretical frameworks are based on (i) the modified couple stress theory, and (ii) the Eringen's nonlocal elasticity theory. The selection of these two theories is predicated on two reasons. The first reason is the relevance of their underlying theory to the objectives of the proposed study. The second reason has to do with the mathematical convenience the two theories offered.

5.3 The modified couple stress theory

The modified couple stress theory (MCST), which would be rigorously treated in this chapter, is one of the higher-order elasticity theories in which the characterization of size-effect is done with just one material length scale parameter [195, 200]. The MCST originates from the classical couple stress theory. It is well known that, the classical couple stress theory is itself a variant of the micropolar elasticity theory[180]. The formulation of the couple stress theory can be traced back to the pioneering works of Mindlin and Tiersten [201], Toupin [202] Fleck and Hutchinson [203]. Under the MCST, the analysis of the applied force on a Hookean material particle does not only include the force to drive the material particle to translate but also a couple to drive it to rotate [204]. As a result, the macro-rotation of the material particle is equivalent to the micro-rotation of its inner structural configuration. Based on the MCST, a number of

recent studies have presented refined models of the classical beams and plates to account for the size effect. For instance, Park and Gao [205] presented a size-dependent model for the static analysis of the Euler-Bernoulli (EB) beam based on the MCST. By employing the properties of epoxy polymeric micro beam, Park and Gao [205] revealed the over-prediction of the deflection of a micro beam by the classical EB beam theory. Kong et al. [206] extended the model of Park and Gao [207] to analyze the free transverse vibration of the non-rotating EB beam. In another study, Ma et al. [208] presented a variationally consistent formulations for the static deformation and free vibration analyses of a microstructure-dependent Timoshenko beam based on the MCST. The versatility of the MCST has also been adopted by Simsek [209] to quantify the influence of size effect on an embedded micro-beam with a moving micro-particle. Some extension of the MCST to other practical applications include Kahrobaiyan et al. [210] and Fu and Zhang [211], where the theory is used to decipher the size dependence dynamics of Atomic Force micro-cantilever and stability of protein micro-tubules, respectively. Furthermore, the MCST is being utilized to provide deeper understanding of the mechanical response of functionally graded beams [198, 212]. From the above studies, the results of the static, vibration and stability analyses obtained by the classical beam theory and those predicted by the MCST differs when the characteristic size of a small-scale beam becomes comparable to its internal material length scale parameter.

Meanwhile, from the survey of the literature it is noticed that the static bending and free vibration analysis of non-rotating micro beams have been treated. However, the size-dependent mechanical deformation of spinning micro beams (with uniform cross-section and tapered cross-section) and twisted micro beams (non-spinning and spinning) based

on all of the higher-order elasticity theory is nonexistent. As a result of their relevance to the objective of modeling micro-cutting tools, the theoretical formulations of these structural elements (that is, spinning micro beams with uniform and tapered cross-sections and twisted micro beams) will be the major focus of the remaining sections of this chapter.

5.4 Microstructure-dependent model of spinning micro beams

The problem of spinning beam, as a result of its paramount technological applications in vast areas of engineering practice such as turbomachinery, space technology, and machining, has been the subject of studies by a number of authors in the past. The first derivation of the characteristics equation of a simply supported spinning Euler-Bernoulli (EB) beam is credited to Dimentberg [107]. Earlier comprehensive treatment of the spinning classical elastic rod of different boundary conditions are presented by Kulla [108] and Meirovitch and Calico [109]. Bauer [110] presented the analytical relationship between the natural frequencies of the rotating EB beam attached to a rigid base and that of a non-spinning beam. Filipich et al. [213] handled the vibration study of a spinning EB beam with orthotropic, elastically restrained ends and an axis of symmetry. In addition to these pioneering works, there have also been quite a number of studies on the free vibration analysis of spinning beams, with diverse solution approach, many of which are reviewed in the studies by Banerjee and Su [112].

However, the models presented in this section, which is to an extent driven by the use of micro-scale beams as rotating shafts in micromachining processes and micro-

turbomachinery, are different from all of the referenced studies that are based on the size-free elasticity theory. In the current mathematical derivation, a combination of the hypothesis of the extended Hamilton's principle, and the modified couple stress theory is used to formulate the size-dependent coupled equations (in two orthogonal planes) of a spinning micro beam satisfying the shear deformable condition. The shear deformable beam presented in this sub-subsection follows the assumption of the Timoshenko beam theory (TBT). The derivation starts out by employing the kinematic description of the TBT in formulating the higher-order strain energy of deformation of the spinning micro beams. The higher-order strain energy of deformation is then combined with the kinetic energy through the use of the extended Hamilton's principle to derive the governing equation of motion.

5.4.1 Derivation of the governing equations

To establish the equations that govern the dynamics of the spinning micro beam, the strain energy and the kinetic energy of the micro beam are required. For the classical macro-sized beam, the constitutive relations upon which the strain energy depends, are available in the literature [110, 112]. However, the inclusion of the small-scale effect in the governing equation calls for a non-classical constitutive law. Based on the modified coupled stress formulation, the strain energy density of an isotropic Hookean material (with a volume V) in a domain $\mathcal{V}$ is given by [200, 207, 214]

$$\Pi_s = \frac{1}{2} \int_{\mathcal{V}} (\boldsymbol{\sigma} : \boldsymbol{\varepsilon} + \mathbf{m} : \boldsymbol{\chi}) dV \quad (5.1)$$

where $\boldsymbol{\varepsilon}$ is the dilatation strain tensor, $\boldsymbol{\sigma}$ is the Cauchy stress tensors, $\boldsymbol{m}$ is the deviatoric component of the couple stress and $\boldsymbol{\chi}$ is the symmetric curvature tensors. If the displacement and rotation vectors are both represented by $\boldsymbol{u}$ and $\boldsymbol{\theta}$, respectively, then the tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$ in Eq. (5.1) satisfy the following geometric relations:

$$\boldsymbol{\varepsilon} = \frac{1}{2} [\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T] \quad (5.2)$$

$$\boldsymbol{\chi} = \frac{1}{2} [\nabla \boldsymbol{\theta} + (\nabla \boldsymbol{\theta})^T] \quad (5.3)$$

where ∇ is the Laplacian notation. The rotation vector $\boldsymbol{\theta}$ is related to the displacement vector in the form:

$$\boldsymbol{\theta} = \frac{1}{2} \text{curl } \boldsymbol{u} \quad (5.4)$$

Eqs. (5.2) – (5.4) can be re-written in indicial notation as:

$$\varepsilon_{kl} = \frac{1}{2} [\partial_l u_k + \partial_k u_l] \quad (k, l = 1, 2, 3) \quad (5.5)$$

$$\chi_{kl} = \frac{1}{2} [\partial_l \theta_k + \partial_k \theta_l] \quad (5.6)$$

$$\theta_k = \frac{1}{2} \text{curl}(u_k) \quad (5.7)$$

where ∂_k and ∂_l are differential operators and u_k is the displacement vector. The dilatation strain and the symmetric curvature tensors in Eqs. (5.2) and (5.3) are related to the Cauchy stress and the deviatoric component of the couple stress to form the constitutive relations as:

$$\boldsymbol{\sigma} = \lambda \text{tr}(\boldsymbol{\varepsilon}) \mathbf{I} + 2G \boldsymbol{\varepsilon} \quad (5.8)$$

$$\boldsymbol{m} = 2(l_s)^2 G \boldsymbol{\chi} \quad (5.9)$$

Where λ and G are the bulk and shear modulus respectively. An additional material constant in the form of l_s would be noticed in Eq. (5.9). This parameter is used to denote the material length scale and is physically a property measuring the effect of the couple stress [206, 215]. In terms of its mathematical equivalent, the material length scale l_s , according to Park and Gao [205] and Ma et al. [208], is found to be analogous to the square root of the ratio of the modulus of curvature of a deformed body to the modulus of shear. Experimental determination of the value l_s have been carried out through the bending tests (as shown in e.g. Stölken and Evans [216]) of thin beams or torsion tests of slim cylinders [217]. In subsequent derivations, λ and G are written in terms of the conventional Poisson's ratio (ϑ) and the Young's modulus (E). That is, $\lambda = E\vartheta/(1 + \vartheta)(1 - 2\vartheta)$ and $G = E/2(1 + \vartheta)$.

To derive the strain energy expression, the schematic, shown in Figure 5-1, of a micro-scale beam of length L , area A and mass density ρ is considered. The micro beam is subjected to a spinning speed Ω along axis Z , where Z is the longitudinal axis of a right-handed inertia Cartesian coordinate system XYZ . If it is assumed, for convenience, a second set of axes xyz as a body-fixed coordinate system with its z axis coinciding with the undeformed neutral surface of the beam, then the equations of the micro beam's motion using the higher-order constitutive relations presented above can be derived. Let S (in Figure 5-1) be an arbitrary material point at a cross-section of the micro beam whose position vector from the neutral axis can be written as [112]

$$\mathbf{R} = d_3 \mathbf{e}_x + d_2 \mathbf{e}_y + d_1 \mathbf{e}_z \quad (5.10)$$

Figure 5-1 Schematic of a spinning micro-scale beam with representative coordinate axes

where $\mathbf{e}_x, \mathbf{e}_y$ and $\mathbf{e}_z$ are the dextral orthogonal projections of the relative displacements of point S in the undeformed state about x, y and z respectively, while d_1, d_2 and d_3 represent the components of the displacement field defined as

$$d_1 = w(z, t) - y\varphi_x(z, t) - x\varphi_y(z, t), d_2 = v(z, t), d_3 = u(z, t) \quad (5.11)$$

where d_1, d_2 , and d_3 are the components of the displacement along z, y and x axis respectively. u, v and w are the corresponding displacement components of the points on the centroidal axis along x, y and z axes respectively. φ_x and φ_y are the rotation angles of the principal axes (x, y) of the spinning micro beam. In the absence of axial strain and by eliminating the nonlinear derivative terms, after substituting Eq. (5.11) in Eqs. (5.5) – (5.9), bearing in mind the definition of Cauchy strain [218], one arrives at the following non-zero components of the strain and stress tensors:

$$\varepsilon_{zz} = -y \frac{\partial \varphi_x(z, t)}{\partial z} - x \frac{\partial \varphi_y(z, t)}{\partial z}, \varepsilon_{yz} = \frac{1}{2} \left(-\varphi_x + \frac{\partial v(z, t)}{\partial z} \right), \varepsilon_{xz} = \frac{1}{2} \left(\varphi_y + \frac{\partial u(z, t)}{\partial z} \right) \quad (5.12)$$

$$\theta_x = \frac{1}{2} \left(\frac{\partial v(z,t)}{\partial z} + \varphi_x \right), \quad \theta_y = -\frac{1}{2} \left(\varphi_y + \frac{\partial u(z,t)}{\partial z} \right), \quad \theta_z = 0 \quad (5.13)$$

$$\chi_{xz} = \frac{1}{4} \left(\frac{\partial \varphi_x}{\partial z} + \frac{\partial^2 v(z,t)}{\partial z^2} \right), \quad \chi_{yz} = -\frac{1}{4} \left(\frac{\partial \varphi_y}{\partial z} + \frac{\partial^2 u(z,t)}{\partial z^2} \right) \quad (5.14)$$

With Eqs.(5.12) – (5.14), the total strain energy given in Eq. (5.1) can now be re-written as:

$$\Pi_s = \int_V (\sigma_{kl} \delta \varepsilon_{kl} + m_{kl} \delta \chi_{kl}) dV \quad (5.15)$$

The indicial notations in Eq. (5.15) are replaced with the non-zero tensors components to have:

$$\Pi_s = \int_V (\sigma_{zz} \delta \varepsilon_{zz} + 2\sigma_{yz} \delta \varepsilon_{yz} + 2\sigma_{xz} \delta \varepsilon_{xz} + 2m_{xz} \delta \chi_{xz} + 2m_{yz} \delta \chi_{yz}) dV \quad (5.16)$$

With $dV = dA dz$ and knowing that in general for points along the centroid of the micro beam $\int_A z dA = 0$, the strain energy due to bending U_b is given by:

$$U_b = \frac{1}{2} \frac{E(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} \left[\int_0^L I_{yy} \left(\frac{\partial \varphi_x}{\partial z} \right)^2 + \int_0^L I_{xx} \left(\frac{\partial \varphi_y}{\partial z} \right)^2 \right] dz dA \quad (5.17)$$

where the definitions $\int_A y^2 dA = I_{xx}$ and $\int_A x^2 dA = I_{yy}$ have been introduced in Eq. (5.17) to represent the principal moments of area of the cross section of the micro-scale beam. From the shear stress definitions given in Eq. (5.12) and Eq. (5.14), the strain energy due to shear (U_s) and the deviatoric components of the couple stress(U_c) can also be written as:

$$U_s = \frac{GkA}{2} \left[\int_0^L \left(\frac{\partial v(z,t)}{\partial z} + \varphi_x \right)^2 + \int_0^L \left(\varphi_y + \frac{\partial u(z,t)}{\partial z} \right)^2 \right] dz \quad (5.18)$$

$$U_c = \frac{G(l_s)^2 A}{4} \left[\int_0^L \left(\frac{\partial \varphi_x}{\partial z} + \frac{\partial^2 v(z,t)}{\partial z^2} \right)^2 + \int_0^L \left(\frac{\partial \varphi_y}{\partial z} + \frac{\partial^2 u(z,t)}{\partial z^2} \right)^2 \right] dz \quad (5.19)$$

where k is the shear correction factor that is determined by the cross-section of the micro-scale beam being considered. The strain energy terms are combined to form the total strain energy of the spinning micro-scale beam as:

$$\Pi_s = U_b + U_s + U_c \quad (5.20)$$

$$\Pi_s = \frac{1}{2} \left[\int_0^L \left\{ \frac{E(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{yy} \left(\frac{\partial \varphi_x}{\partial z} \right)^2 + \frac{E(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{xx} \left(\frac{\partial \varphi_y}{\partial z} \right)^2 + kGA \left(\frac{\partial v}{\partial z} + \varphi_x \right)^2 \right. \right. \\ \left. \left. + kGA \left(\varphi_y + \frac{\partial u}{\partial z} \right)^2 + \frac{G(l_s)^2 A}{2} \left(\frac{\partial \varphi_x}{\partial z} + \frac{\partial^2 v}{\partial z^2} \right)^2 + \frac{G(l_s)^2 A}{2} \left(\frac{\partial \varphi_y}{\partial z} + \frac{\partial^2 u}{\partial z^2} \right)^2 \right\} dz \right] \quad (5.21)$$

The kinetic energy of the spinning shear deformable beam element in terms of the velocity components of the material point (S) at a distance is given by [112, 219]

$$\mathbf{V} = \dot{\mathbf{R}} + \boldsymbol{\Omega} \times \mathbf{R} \quad (5.22)$$

where $\boldsymbol{\Omega} = \Omega \mathbf{e}_k$ and $\mathbf{e}_k$ is the unit vector in the z direction. Thus, the kinetic energy of the system can now be written as

$$T = \frac{1}{2} \int_0^t \left(\left[\rho I_{xx} \left(\frac{d\varphi_y}{dt} \right)^2 + \rho I_{yy} \left(\frac{d\varphi_x}{dt} \right)^2 \right] + \rho A \left[\left(\frac{du}{dt} \right)^2 + \left(\frac{dv}{dt} \right)^2 \right] \right) dz \quad (5.23)$$

The first variation of the virtual work due to the gyroscopic moments is expressed as:

$$\delta W = 2\Omega \int_0^{L_1} \left(\left[\rho I_{xx} \frac{d\varphi_y}{dt} \delta \varphi_x - \rho I_{yy} \frac{d\varphi_x}{dt} \delta \varphi_y \right] \right) dz \quad (5.24)$$

The principle of extended Hamilton holds that the true dynamic trajectory of a point under motion renders the integral $\delta \int_0^T (T - \Pi_s + W) dt$ stationary with respect to any variation of the trajectory between the on the time interval $[0, T] \subseteq \mathbb{R}_+$ [138]. By employing the postulates of the extended Hamilton principles, Eqs. (5.21), (5.23) and (5.24) are combined to lead to:

$$\delta \int_0^T (T - \Pi_s + W) dt \quad (5.25)$$

By employing the fundamental lemma of variational calculus [138] after integrating by parts of each term of the integrals resulting from Eq. (5.25), one arrives at the size-dependent, coupled elastodynamics governing equations of the shear deformable, spinning micro-scale beam given by

$$-\rho A \left(\frac{\partial^2 u}{\partial t^2} \right) + kAG \left(\frac{\partial^2 u}{\partial z^2} - \frac{\partial \varphi_y}{\partial z} \right) - \frac{1}{4} G(l_s)^2 A \left(\frac{\partial^3 \varphi_y}{\partial z^3} + \frac{\partial^4 u}{\partial z^4} \right) = 0 \quad (5.26)$$

$$\begin{aligned} & \frac{E(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{xx} \left(\frac{\partial^2 \varphi_y}{\partial z^2} \right) + kAG \left(\frac{\partial u}{\partial z} - \varphi_y \right) - \rho I_{xx} \frac{\partial^2 \varphi_y}{\partial t^2} - \rho I_{yy} \Omega \frac{\partial \varphi_x}{\partial t} \\ & + \frac{1}{4} G(l_s)^2 A \left(\frac{\partial^2 \varphi_y}{\partial z^2} + \frac{\partial^3 u}{\partial z^3} \right) = 0 \end{aligned} \quad (5.27)$$

$$-\rho A \left(\frac{\partial^2 v}{\partial t^2} \right) + kAG \left(\frac{\partial^2 v}{\partial z^2} - \frac{\partial \varphi_x}{\partial z} \right) + \frac{1}{4} G(l_s)^2 A \left(\frac{\partial^3 \varphi_x}{\partial z^3} + \frac{\partial^4 v}{\partial z^4} \right) = 0 \quad (5.28)$$

$$\begin{aligned} & \frac{E(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{yy} \left(\frac{\partial^2 \varphi_x}{\partial z^2} \right) + kAG \left(\frac{\partial v}{\partial z} - \varphi_x \right) - \rho I_{yy} \frac{\partial^2 \varphi_x}{\partial t^2} + \rho I_{xx} \Omega \frac{\partial \varphi_y}{\partial t} \\ & + \frac{1}{4} G(l_s)^2 A \left(\frac{\partial^3 \varphi_x}{\partial z^3} + \frac{\partial^4 v}{\partial z^4} \right) = 0 \end{aligned} \quad (5.29)$$

A number of differences will be noticed in the appearance of Eqs. (5.26) - (5.29), which are the governing equations of a spinning micro-scale beam, and those presented in chapter 4, given in Eqs. (4.5a) to (4.5d). The first difference is the inclusion of the material length scale (l_s) in Eqs. (5.26) to (5.29). The second is the presence of Poisson's ratio (ϑ) in the flexural terms. For preliminary analysis and validation, the shear and rotary terms in Eqs. (5.26) – (5.29) will be temporarily eliminated so as to reduce the governing equations to those of a spinning Euler-Bernoulli (EB) micro-scale beam. The elimination is done by simply allowing the terms k , ρI_{yy} and ρI_{xx} to be zero. This elimination straightforwardly leads to:

$$E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{yy} \frac{\partial^4 u}{\partial z^4} + (l_s)^2 A \mu \frac{\partial^4 u}{\partial z^4} - \rho A \Omega^2 u - 2\rho A \Omega \frac{\partial v}{\partial t} + \rho A \frac{\partial^2 u}{\partial t^2} = 0 \quad (5.30)$$

$$E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{xx} \frac{\partial^4 v}{\partial z^4} + (l_s)^2 A \mu \frac{\partial^4 v}{\partial z^4} - \rho A \Omega^2 v + 2\rho A \Omega \frac{\partial u}{\partial t} + \rho A \frac{\partial^2 v}{\partial t^2} = 0 \quad (5.31)$$

With the Hamiltonian formulation, the following shear forces (Q_y and Q_x) and bending moments (M_x and M_y) relations are yielded as:

$$M_x(z, t) = E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{xx} \frac{\partial^2 v}{\partial z^2}, \quad M_y(z, t) = -E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{yy} \frac{\partial^2 u}{\partial z^2}, \quad Q_y(z, t) = E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{xx} \frac{\partial^3 v}{\partial z^3}, \quad Q_x(z, t) = E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{yy} \frac{\partial^3 u}{\partial z^3} \quad (5.32)$$

Before moving on, it is necessary to contrast Eqs. 5.30 and 5.31 with some of the equations available in the literature. For instance, if the material length scales parameters (l_s), the Poisson's ratio (ϑ) vanish from Eqs. (5.30) and (5.31), the governing equations of the spinning classical EB beam [112] are retrieved. Also, if the gyroscopic moment caused by the spinning speed (Ω) is eliminated from Eqs. (5.30) and (5.31), the decoupled, non-classical, size-dependent governing equations of the micro beam

presented in [214] are obtained. Furthermore, if the gyroscopic moment due to the spinning speed (Ω) is eliminated, then the non-classical equation of an embedded micro beam presented by Simsek [209] is also obtained. The comparison made for the governing equation of the EB micro-scale beam can easily be extended to that of the shear deformable, spinning micro-scale beam given in Eqs. (5.26) – (5.29).

5.4.2 SEM Procedure for the spinning micro-scale beam embedded in an elastic foundation

As highlighted in chapter 4, the SEM-based solution procedure begins by seeking a plane wave type representation of the field variables u and v (in Eqs. 5.30 and 5.31). This representation is achieved via the discrete Fourier transform (DFT) algorithm which allows the field variables in Eqs (5.30) and (5.31) to be expressed in the spectral forms as [220-221]

$$u(z, t) = \frac{1}{N} \sum_{n=1}^N U_n(z; \omega_n) \cos \omega_n t \quad (5.33)$$

$$v(z, t) = \frac{1}{N} \sum_{n=1}^N V_n(z; \omega_n) \sin \omega_n t \quad (5.34)$$

where $\omega_n (n = 1, 2, 3, \dots, N)$ is the circular frequency of the n th sampling point and N is the Nyquist frequency in the fast Fourier transform (FFT) employed for the computer implementation of DFT. The terms U_n and V_n are frequency dependent amplitudes of u and v . When Eqs. (5.33) and (5.34) are substituted in the governing equations, a set of frequency-domain ordinary differential equations are obtained as

$$E \frac{(1 - \vartheta)}{(1 + \vartheta)(1 - 2\vartheta)} I_{yy} \frac{d^4 U}{dz^4} + (l_s)^2 A \mu \frac{d^4 U}{dz^4} + e_x U - \rho A \Omega^2 U + 2\rho A \Omega \omega_n V - \rho A \omega_n^2 U = 0$$

(5.35)

$$E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{xx} \frac{d^4 V}{dz^4} + (l_s)^2 A \mu \frac{d^4 V}{dz^4} + e_y V - \rho A \Omega^2 V + 2\rho A \Omega \omega_n U - \rho A \omega_n^2 V = 0 \quad (5.36)$$

where e_u and e_v in Eqs. 5.35 – 5.36 represents the foundation stiffness in the orthogonal planes. Similarly, the shear forces (Q_y and Q_x) and the bending moments (M_x and M_y) are spectrally transformed as

$$Q_x = \frac{1}{N} \sum_{n=1}^N Q_{Un}(z) \cos \omega_n t, \quad Q_y = \frac{1}{N} \sum_{n=1}^N Q_{Vn}(z) \sin \omega_n t \quad (5.37)$$

$$M_x = \frac{1}{N} \sum_{n=1}^N M_{Un}(z) \cos \omega_n t, \quad M_y = \frac{1}{N} \sum_{n=1}^N M_{Vn}(z) \sin \omega_n t \quad (5.38)$$

Since n represents the n th Fourier components of the transformed variables, the subscript n in subsequent derivations is henceforth dropped without any loss of clarity. With the help of the following non-dimensional variables

$$\vartheta_r = \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)}, \quad \alpha_{zx} = \frac{(l_s)^2 A \mu}{E \vartheta_r I_{yy}}, \quad \alpha_{zy} = \frac{(l_s)^2 A \mu}{E \vartheta_r I_{xx}}, \quad \eta = \frac{\Omega}{\omega_n}, \quad \lambda_x = \left(\frac{\rho A \omega_n^2 L^4}{E \vartheta_r I_{xx}} \right)^{1/2}, \quad \lambda_y = \left(\frac{\rho A \omega_n^2 L^4}{E \vartheta_r I_{yy}} \right)^{1/2}, \quad \xi = z/L, \quad k_u = \frac{e_x L^4}{E \vartheta_r I_{yy}}, \quad k_v = \frac{e_y L^4}{E \vartheta_r I_{xx}} \quad (5.39)$$

Eqs. (5.35) - (5.36) are cast into the non-dimensional ordinary differential forms below

$$\frac{d^4 U}{d\xi^4} + \alpha_{zx} \frac{d^4 U}{d\xi^4} - \lambda_y^2 \eta^2 U + k_u U + 2\lambda_y^2 \eta V - \lambda_y^2 U = 0 \quad (5.40)$$

$$\frac{d^4 V}{d\xi^4} + \alpha_{zy} \frac{d^4 V}{d\xi^4} - \lambda_x^2 \eta^2 V + k_v V + 2\lambda_x^2 \eta U - \lambda_x^2 V = 0 \quad (5.41)$$

With the reduction of the equations to ordinary differential equations, the spatial solutions of the transformed field variables could then be assumed as

$$U = \bar{U} e^{-i\beta\xi}, \quad V = \bar{V} e^{-i\beta\xi} \quad (5.42a)$$

where β is called the wavenumber [160]. Substitution of Eq. (5.45) in Eqs. (5.40) – (5.41) yields

$$\begin{bmatrix} (\beta^4 + \alpha_{zx}\beta^4 + k_u - 2\lambda_y^2\eta^2 - \lambda_y^2) & 2\lambda_y^2\eta \\ 2\lambda_x^2\eta & (\beta^4 + \alpha_{zy}\beta^4 + k_v - 2\lambda_x^2\eta^2 - \lambda_x^2) \end{bmatrix} \begin{Bmatrix} \bar{U} \\ \bar{V} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (5.42b)$$

For a non-trivial solution, the determinant of the square matrix in Eq. (5.42) is set to zero to obtain the dispersion relation as

$$H_1\beta^8 + H_2\beta^4 + H_3 = 0 \quad (5.43)$$

The solution of Eq. (5.43) yields four real and four imaginary roots (or wavenumbers) as follows

$$\beta_{1,3} = \pm \left(\frac{1}{2}\right)^{1/4} \left[\frac{H_2 + \sqrt{H_2^2 - 4H_1H_3}}{H_1} \right]^{1/4}, \quad \beta_{2,4} = \pm \left(\frac{1}{2}\right)^{1/4} \left[\frac{H_2 - \sqrt{H_2^2 - 4H_1H_3}}{H_1} \right]^{1/4} \quad (5.44)$$

$$\beta_{5,7} = \pm i \left(\frac{1}{2}\right)^{1/4} \left[\frac{H_2 + \sqrt{H_2^2 - 4H_1H_3}}{H_1} \right]^{1/4}, \quad \beta_{6,8} = \pm i \left(\frac{1}{2}\right)^{1/4} \left[\frac{H_2 - \sqrt{H_2^2 - 4H_1H_3}}{H_1} \right]^{1/4} \quad (5.45)$$

where

$$\begin{aligned} H_1 &= (1 + \alpha_{zx})(1 + \alpha_{zy}) \\ H_2 &= [(1 + \alpha_{zy})k_u + (1 + \alpha_{zx})k_v - (1 + 2\eta^2)[(1 + \alpha_{zx})\lambda_x^2 + (1 + \alpha_{zy})\lambda_y^2]] \\ H_3 &= k_u k_v - k_u(1 + 2\eta^2)\lambda_x^2 - k_v(1 + 2\eta^2)\lambda_y^2 + \lambda_x^2\lambda_y^2 + 4\eta^4\lambda_x^2\lambda_y^2 \end{aligned} \quad (5.46)$$

With the help of the eight wavenumbers, the general solutions of the flexible deformations in the orthogonal planes could then be written as a superposition of eight waves in the form:

$$U(\xi) = \bar{U}_1 \sin \beta_1 \xi + \bar{U}_2 \cos \beta_2 \xi + \bar{U}_3 \sinh \beta_3 \xi + \bar{U}_4 \cosh \beta_4 \xi + \bar{U}_5 \sin \beta_5 \xi + \bar{U}_6 \cos \beta_6 \xi + \bar{U}_7 \sinh \beta_7 \xi + \bar{U}_8 \cosh \beta_8 \xi \quad (5.47)$$

$$V(\xi) = r_1 \bar{U}_1 \sin \beta_1 \xi + r_2 \bar{U}_2 \cos \beta_2 \xi + r_3 \bar{U}_3 \sinh \beta_3 \xi + r_4 \bar{U}_4 \cosh \beta_4 \xi + r_5 \bar{U}_5 \sin \beta_5 \xi + r_6 \bar{U}_6 \cos \beta_6 \xi + r_7 \bar{U}_7 \sinh \beta_7 \xi + r_8 \bar{U}_8 \cosh \beta_8 \xi \quad (5.48)$$

where

$$r_j = \frac{2\lambda_x^2 \eta}{(\beta_j^4 + \alpha_{zy} \beta_j^4 + k_v - 2\lambda_x^2 \eta^2 - \lambda_x^2)} \quad (j = 1, 2, 3, \dots, 8) \quad (5.49)$$

r_j is an amplitude ratio derived from Eq. (5.49) to harmonize the solutions of the orthogonal nodal displacements. For convenience, Eqs. (5.47) - (5.48) are put in the following forms

$$U(\xi) = \Phi(\xi) \bar{\mathbf{U}}, V(\xi) = \Phi(\xi) [\text{diag}(r_j)] \bar{\mathbf{U}}, \quad (5.50)$$

where

$$\Phi(\xi) = [\sin \beta_1 \quad \cos \beta_2 \quad \sinh \beta_3 \cosh \beta_4 \quad \sin \beta_5 \quad \cos \beta_6 \sinh \beta_7 \quad \cosh \beta_8] \quad (5.51)$$

$$\bar{\mathbf{U}} = \{\bar{U}_1 \quad \bar{U}_2 \quad \bar{U}_3 \bar{U}_4 \quad \bar{U}_5 \quad \bar{U}_6 \bar{U}_7 \quad \bar{U}_8\}^T \quad (5.52)$$

$\bar{U}_j (j = 1, 2, 3, \dots, 8)$ are constants to be determined from the boundary conditions of a finite section of the beam. Figure 5-2 shows the depiction of the nodal force and nodal bending moments of a two-noded EB beam elements. From Eqs. (5.47) – (5.48), the nodal rotations (φ_x, φ_y) , the shear forces (Q_x, Q_y) and the bending moments (M_x, M_y) are equally defined as

Figure 5-2 Schematic of the nodal forces and nodal bending moments of a spinning micro-scale beam

$$\begin{aligned}
 \varphi_x &= -\frac{1}{L} \left(\frac{dV}{d\xi} \right), & \varphi_y &= -\frac{1}{L} \left(\frac{dU}{d\xi} \right), & Q_x &= \frac{E\vartheta_r I_{yy}}{L^3} \left(\frac{d^3 U}{d\xi^3} \right), \\
 Q_y &= \frac{E\vartheta_r I_{xx}}{L^3} \left(\frac{d^3 V}{d\xi^3} \right), & M_x &= \frac{E\vartheta_r I_{xx}}{L^2} \left(\frac{d^2 V}{d\xi^2} \right), & M_y &= -\frac{E\vartheta_r I_{yy}}{L^2} \left(\frac{d^2 U}{d\xi^2} \right)
 \end{aligned} \quad (5.53)$$

For a two-noded spectral element, the transformed field variables in Eqs. (5.47) – (5.48), the shear forces and the bending moments in Eq. (5.53) are evaluated at the two ends of the finite micro beam. That is, $\xi = 0$ and $\xi = 1$ as

$$U(0), \quad V(0), \quad \varphi_U(0), \quad \varphi_V(0), \quad U(1), \quad V(1), \quad \varphi_U(1), \quad \varphi_V(1) \quad (5.54)$$

$$Q_x(0), \quad Q_y(0), \quad M_x(0), \quad M_y(0), \quad Q_x(1), \quad Q_y(1), \quad M_x(1), \quad M_y(1) \quad (5.55)$$

With the above evaluation, the Fourier components of the spatial degrees of freedom, the nodal forces, and the moments for a two-noded spectral element are given by

$$\mathbf{d} = \{U_1 \quad V_1 \quad \varphi_{U1} \varphi_{V1} \quad U_2 \quad V_2 \varphi_{U2} \quad \varphi_{V2}\} \quad (5.56)$$

$$\mathbf{F} = \{Q_{x1} \quad Q_{y1} \quad M_{x1}M_{y1} \quad Q_{x2} \quad Q_{y2}M_{x2} \quad M_{y2}\} \quad (5.57)$$

Evaluation of Eqs. (5.54) and (5.55) through the definitions in Eqs. (5.47) – (5.48), leads to:

$$\mathbf{d} = \mathbf{Z}(\omega)\bar{\mathbf{U}} \quad (5.58)$$

where

$$\mathbf{Z}(\omega) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & r_2 & 0 & r_4 & 0 & r_6 & 0 & r_8 \\ \frac{\beta_1^3}{L} & 0 & -\frac{\beta_3^3}{L} & 0 & \frac{\beta_5^3}{L} & 0 & -\frac{\beta_7^3}{L} & 0 \\ -\frac{r_1\beta_1^3}{L} & 0 & \frac{r_3\beta_3^3}{L} & 0 & -\frac{r_5\beta_5^3}{L} & 0 & \frac{r_7\beta_7^3}{L} & 0 \\ S(\beta_1) & C(\beta_2) & Sh(\beta_3) & Ch(\beta_4) & S(\beta_5) & C(\beta_6) & Sh(\beta_7) & Ch(\beta_8) \\ S(\beta_1)r_1 & C(\beta_2)r_2 & Sh(\beta_3)r_3 & Ch(\beta_4)r_4 & S(\beta_5)r_5 & C(\beta_6)r_6 & Sh(\beta_7)r_7 & Ch(\beta_8)r_8 \\ \frac{C(\beta_1)\beta_1^3}{L} & -\frac{S(\beta_2)\beta_2^3}{L} & -\frac{Ch(\beta_3)\beta_3^3}{L} & -\frac{Sh(\beta_4)\beta_4^3}{L} & \frac{C(\beta_5)\beta_5^3}{L} & -\frac{S(\beta_6)\beta_6^3}{L} & -\frac{Ch(\beta_7)\beta_7^3}{L} & -\frac{Sh(\beta_8)\beta_8^3}{L} \\ -\frac{C(\beta_1)r_1\beta_1^3}{L} & \frac{S(\beta_2)r_2\beta_2^3}{L} & \frac{Ch(\beta_3)r_3\beta_3^3}{L} & \frac{Sh(\beta_4)r_4\beta_4^3}{L} & -\frac{C(\beta_5)r_5\beta_5^3}{L} & \frac{S(\beta_6)r_6\beta_6^3}{L} & \frac{Ch(\beta_7)r_7\beta_7^3}{L} & \frac{Sh(\beta_8)r_8\beta_8^3}{L} \end{bmatrix} \quad (5.59)$$

where,

$$S = \sin, C = \cos, Sh = \sinh, Ch = \cosh \quad (5.60)$$

By employing Eq. (5.58), Eq. (5.50) can be re-expressed (by eliminating $\bar{\mathbf{U}}$) in terms of the dynamic shape functions as

$$U(\xi) = \mathbf{N}_u(\xi; \omega)\mathbf{d}, \quad V(\xi) = \mathbf{N}_v(\xi; \omega)\mathbf{d} \quad (5.61)$$

where N_u and N_v are the dynamic shape functions defined by

$$\mathbf{N}_u = \Phi(\xi)\mathbf{Z}^{-1}, \mathbf{N}_v = \Phi(\xi)[diag(r_j)]\mathbf{Z}^{-1} \quad (5.62)$$

To formulate the characteristics matrix of the spectral element, the so-called weak form (or integro-differential form) of the governing equation is obtained as

$$\int_0^1 \left[\begin{array}{c} E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{yy} \left(\frac{d^2 U}{d\xi^2} \right) \delta \left(\frac{d^2 U}{d\xi^2} \right) + (l_s)^2 A \mu \left(\frac{d^2 U}{d\xi^2} \right) \delta \left(\frac{d^2 U}{d\xi^2} \right) \\ + k_1 U \delta U - \rho A \Omega^2 U \delta U + 2\rho A \Omega \omega U \delta V - \rho A \omega^2 U \delta U + \\ E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{xx} \left(\frac{d^2 V}{d\xi^2} \right) \delta \left(\frac{d^2 V}{d\xi^2} \right) + \\ (l_s)^2 A \mu \left(\frac{d^2 V}{d\xi^2} \right) \delta \left(\frac{d^2 V}{d\xi^2} \right) + k_2 V \delta V - \rho A \Omega^2 V \delta V + 2\rho A \Omega \omega V \delta U - \rho A \omega^2 V \delta V \end{array} \right] d\xi =$$

$$\{Q_{x1} \quad Q_{y1} \quad M_{x1} M_{y1} \quad Q_{x2} \quad Q_{y2} M_{x2} \quad M_{y2}\}^T \quad (5.63)$$

Substitution of the dynamic shape functions, derived in Eq. 5.62, into Eq. (5.63) leads to the spectral element equation

$$\delta \mathbf{d}^T \left\{ \int_0^1 \left[E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{yy} \left[\frac{d^2 \mathbf{N}_u}{d\xi^2} \right]^T \left[\frac{d^2 \mathbf{N}_u}{d\xi^2} \right] + (l_s)^2 A \mu \left[\frac{d^2 \mathbf{N}_u}{d\xi^2} \right]^T \left[\frac{d^2 \mathbf{N}_u}{d\xi^2} \right] + k_1 [\mathbf{N}_u]^T [\mathbf{N}_u] - \right. \right.$$

$$\rho A \Omega^2 [\mathbf{N}_u]^T [\mathbf{N}_u] + 2\rho A \Omega \omega [\mathbf{N}_u]^T [\mathbf{N}_v] - \rho A \omega^2 [\mathbf{N}_u]^T [\mathbf{N}_u] +$$

$$E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{xx} \left[\frac{d^2 \mathbf{N}_v}{d\xi^2} \right]^T \left[\frac{d^2 \mathbf{N}_v}{d\xi^2} \right] + (l_s)^2 A \mu \left[\frac{d^2 \mathbf{N}_v}{d\xi^2} \right]^T \left[\frac{d^2 \mathbf{N}_v}{d\xi^2} \right] + k_2 [\mathbf{N}_v]^T [\mathbf{N}_v] -$$

$$\left. \rho A \Omega^2 [\mathbf{N}_v]^T [\mathbf{N}_v] + 2\rho A \Omega \omega [\mathbf{N}_v]^T [\mathbf{N}_u] - \rho A \omega^2 [\mathbf{N}_v]^T [\mathbf{N}_v] \right] d\xi \Big\} \mathbf{d} = \delta \mathbf{d}^T \{\mathbf{F}\} \quad (5.64)$$

Bearing in mind that the functional term $\delta \mathbf{d}^T$ is arbitrary and hence cannot be zero, its coefficients on both sides of the equality signs are equated to form the frequency-dependent dynamic stiffness relation symbolically written as

$$[\mathbf{K}]\{\mathbf{U}\} = \{\mathbf{F}\} \quad (5.65)$$

where

$$\mathbf{K} = \int_0^1 \left[\begin{array}{l} E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{yy} \left[\frac{d^2 \mathbf{N}_u}{d\xi^2} \right]^T \left[\frac{d^2 \mathbf{N}_u}{d\xi^2} \right] + (l_s)^2 A \mu \left[\frac{d^2 \mathbf{N}_u}{d\xi^2} \right]^T \left[\frac{d^2 \mathbf{N}_u}{d\xi^2} \right] \\ + k_1 [\mathbf{N}_u]^T [\mathbf{N}_u] - \rho A \Omega^2 [\mathbf{N}_u]^T [\mathbf{N}_u] \\ + 2\rho A \Omega \omega [\mathbf{N}_u]^T [\mathbf{N}_v] - \rho A \omega^2 [\mathbf{N}_u]^T [\mathbf{N}_u] \\ + E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{xx} \left[\frac{d^2 \mathbf{N}_v}{d\xi^2} \right]^T \left[\frac{d^2 \mathbf{N}_v}{d\xi^2} \right] \\ + (l_s)^2 A \mu \left[\frac{d^2 \mathbf{N}_v}{d\xi^2} \right]^T \left[\frac{d^2 \mathbf{N}_v}{d\xi^2} \right] + k_2 [\mathbf{N}_v]^T [\mathbf{N}_v] - \rho A \Omega^2 [\mathbf{N}_v]^T [\mathbf{N}_v] \\ + 2\rho A \Omega \omega [\mathbf{N}_v]^T [\mathbf{N}_u] - \rho A \omega^2 [\mathbf{N}_v]^T [\mathbf{N}_v] \end{array} \right] d\xi \quad (5.66)$$

The matrices $\mathbf{K}$, $\mathbf{U}$ and $\mathbf{F}$ are the 8×8 dynamic stiffness matrix, the 8×1 nodal degrees of freedom and the 8×1 nodal forces, in frequency domain, respectively. The dynamic stiffness matrix obtained is symmetric. Similar to an eigenvalue problem, the vanishing determinant of the dynamic stiffness matrix indicates resonances and hence the natural frequencies. As earlier stated in chapter 4, in the absence of geometric discontinuities, the compactness of the dynamic stiffness matrix, yielded by the SEM, provides the chance to resort to the usual assemblage procedures of the traditional FEM. For a uniform structural element, such as the spinning uniform micro-scale beam treated in this section, the natural frequencies can be computed to a higher degree of accuracy from Eq. (5.66) after the application of the appropriate boundary conditions. Again, the Muller root search algorithm [164] is employed for the computation of the natural frequencies from the characteristics dynamic stiffness matrix.

5.4.3 Discussions on the analysis of the embedded spinning micro-scale beam

Here, numerical experiments are discussed out to demonstrate the accuracy and efficiency of the theory and method presented. First, a simplified case of the governing equation presented is used for validation with available results in the literature. This is

done by eliminating the material length scale parameter (l_s), the Poisson's ratio (ν) and the foundation parameters (e_x and e_y) from Eqs. (5.30) and (5.31). This then leads to the free vibration analysis of the classical spinning EB beam. Using the following non-dimensional parameters

$$\lambda_n = \omega_n / \omega_0, \quad \eta_n = \Omega / \omega_0, \quad \omega_0 = \left(\frac{\sqrt{EI_{xx}I_{yy}}}{\rho AL^4} \right)^{\frac{1}{2}} \quad (5.67)$$

where η is the spinning speed parameter and ω_n is the natural frequencies of the n th mode of vibration of the spinning beam, the predictions from the present analysis are contrasted with the analytical work of Likins et al. [222] and the DSM-based computations of Banerjee and Su [112] in Table 5-1. From Table 5-1, it is noticed that there are negligible differences between the SEM predictions and the computations from the other alternative methods. A peculiar characteristic of spinning systems is the existence of synchronous forward and backward circular motions in the presence of the spinning-induced gyroscopic effect. In essence, the gyroscopic effect of spinning splits each of the natural frequencies into two: one lower, the other higher. This is further depicted in Figure 5-3. From Figure 5-3, it is noticed that the lower of the split natural frequency corresponds to the backward precessional mode, while the higher of the split natural frequency, mostly caused by unbalanced mass effect during spinning, leads to forward precessional modes. The forward precessional modes have increased natural frequencies. In Table 5-1, this splitting of the natural frequencies is noticeable. The attendant effect of this phenomenon is the introduction of critical speeds. The determination of the critical speed is of utmost interest when dealing with spinning beams. The value of the spin rate at which the natural frequency equals to zero

corresponds to the critical speed of the spinning beam. For the classical EB beam, the first critical speed occurs at $\eta = 3.5$ as shown in Table 5-1. Identifying the critical speed is strategic to the isolation of the onset of divergence-type instability [171].

Table 5-1 Validation of natural frequencies (rounded up to two decimal places) of a clamped-free classical spinning beam

η	Mode 1		Mode 2		Mode 3	
	λ_B	λ_F	λ_B	λ_F	λ_B	λ_F
0	3.52(3.52) ^a [3.51] ^b	3.52(3.52) [3.51]	22.03(22.03) [22.03]	22.03(22.03) [22.03]	61.73(61.69) [61.69]	61.73(61.69) [61.69]
2	1.52(1.52) [1.52]	5.52 (5.52) [5.52]	20.03 (20.03) [20.03]	24.03 (24.03) [24.13]	59.73 (59.69) [59.69]	63.73 (63.69) [63.71]
3.5	0 (0)	7.03 (7.01) [7.11]	18.52 (18.53) [18.55]	25.55 (25.53) [25.61]	58.22 (58.19) [58.19]	65.25 (65.19) [65.19]
4	— ^d	7.52 (7.51) 7.52	18.035 (18.03) [18.02]	26.03 (26.03) [26.03]	57.73 (57.69) [57.71]	65.73 (65.69) [65.71]
8	—	11.51 (11.52) [11.52]	14.03 (14.03) [14.04]	30.03 (30.03) [30.04]	53.74 (53.69) [53.72]	69.74 (69.69) [69.71]

^dThe dash indicates not available in references

With the validation carried out, the other sets of parameters (l_s, k_x, k_y, ϑ) in the governing equations are eventually restored to investigate their influence on the dynamic response of the micro-scale beam. It is worth mentioning that the primary focus of this particular analysis is concentrated on the synchronous forward motion of the spinning micro beam. The reason for focusing on the synchronous forward motion is because of the possibility of the asynchronous backward motion to generally lead to chaotic behavior, a response that is well beyond the scope of the research study.

Figure 5-3 Splitting of the frequency values of a clamped-free circular beam at different spin rates

The material parameters used in the analyzes that follow involve material properties of tungsten-carbide, for which $E = 568 \text{ GPa}$, $\rho = 14,450 \text{ kg/m}^3$, $\nu = 0.38$. Additionally, a micro-scale beam of $800 \text{ }\mu\text{m}$ and $60 \text{ }\mu\text{m}$ in length (L) and diameter(d), respectively, is considered. Besides, the assumption of a circular axis-symmetric micro beam is made for the analyses carried out in this sub-subsection. Hence the material length scale is assumed to be equal in the two orthogonal planes (i.e. $l_s = l$). By starting with the influence of the higher-order material length scale, the variation of the first and

the second natural frequency with the size effect parameter (d/l) is shown in Figures 5-4 and 5-5. For these two figures, the Poisson's ratio and the Winkler's foundation stiffness are still temporarily kept out of the equation, however. The first noticeable influence from Figures 5-4 and 5-5 is the fact that, analogous to the behavior in the classical case, the spin rate (η) still contributes to the increase of the natural frequencies in the synchronous forward motion. This is expected since the spin rate is an integral part of the kinetic energy term rather than the strain energy term. Another observation from Figures 5-4 and 5-5 is the significant influence of the size effect parameter (d/l) in the range $1 \leq d/l \leq 6$. Furthermore, the size effect parameter happens to be more slightly pronounced on the second mode (Figure 5-5) of the free vibration than it does on the fundamental mode (Figure 5-4).

Qualitatively, for instance, a 40% and 42% increase is observed on the first and second natural frequencies, respectively, of the micro beam when the size effect parameter (d/l) is 4 and at the non-dimensional spin rate (η) is 3. Similar increase, due to the size effect, is observed under different values of the spin rate. Meanwhile, for beams with thicker diameters, ($d/l \geq 12$), it is noted from Figures 5-4 and 5-5 that the size effect parameter has only marginal influence on the dynamic response of the structure. This agrees with the observations in refs. [206, 214]. Still at the size effect parameter (d/l) of 4, the critical speed is observed to have more than a 30% increase.

Figure 5-4 Size effect on the fundamental frequency of a clamped-free circular micro beam with different spin rates

Figure 5-5 Size effect on the second natural frequency of a clamped-free circular micro beam with different spin rates

This increase in critical speed means the micro-scale beams can be used at a much higher speed range than the equivalent macro-sized beam before their first critical speeds are reached. While this increase in critical speed is beneficial, other in-service process parameters could militate against attaining such doubling effect in reality, which is why breaking micro tools remains a bottleneck to achieving the full benefit of micromachining. In Figure 5-6, the phenomenon of frequency splitting, in the presence of the size effect parameter, is revealed through the variation of the natural frequencies of the micro beam with the imposed angular spin rate. As noted earlier, the first critical speed of the beam with the higher-order material length scale is increased compared to the 3.51 of the classical beam theory.

In Figures 5-7 and 5-8, the effects of the Winkler-type foundation modulus and the Poisson's ratio are considered on the fundamental natural frequency of the micro beam. Specifically, in Figure 5-7, the Poisson's ratio is maintained at zero, while different values of the non-dimensional stiffness (k_r) of an elastic foundation parameter is employed. For a qualitative description of the effect of both the foundation modulus and the Poisson's ratio on the fundamental natural frequency, a frequency ratio is defined as

$$\lambda_r = \lambda_{embedded} / \lambda_0 \quad (5.68)$$

where $\lambda_{embedded}$ is the computed fundamental frequency of the embedded micro beam and λ_0 is the fundamental natural frequency of the non-spinning, non-embedded EB beam. This ratio provides a quick visual assessment of the influence of the foundation

parameter and the Poisson's ratio. As one moves from left to right on the abscissa of Figure 5-7, the upward trend in the value of the natural frequency becomes noticeable.

Figure 5-6 Variation of the natural frequencies with the external spin rate

This observation of increase in frequency due to foundation stiffness is generally true, even in the case of classical beam, since foundation stiffness is expected to have a stabilizing effect. Contrarily, however, the upward trend of the natural frequency is observed not to be true in the presence of spinning. At higher spinning rate, the effect of the foundation on the natural frequency becomes marginal as the spin rate is numerically raised upward. In Figure 5-8, the Poisson's ratio effect on the frequency ratio is studied for different values of the spin rate and the elastic foundation stiffness.

Figure 5-7 Variation of the fundamental natural frequency ratio with the non-dimensional Winkler-type elastic modulus and a zero Poisson's ratio

Now, since the addition of the Poisson's ratio arises from the modified couple stress theory, it is discerned from this figure that neglecting the Poisson's ratio (which is obviously the case in the classical theory) could potentially lead to predictive computational error. Thus, the existence of the Poisson's ratio in the mathematical model is bound to play a major role on the dynamic response of micro-sized beam. The most perceptible and fascinating aspect of the analyzes carried out so far is the observation that the four parameters of interest (l, k_r, ϑ and η) contained in the mathematical model mostly contribute stiffening effect, albeit of different degrees, to the dynamic response of the micro beam. Which of these parameters therefore has the most profound stiffening effect? Furthermore, is there any degree of dependency between these parameters under nonlinear variations of each of the parameters that could affect the dynamic response of the micro beam?

Figure 5-8 Variation of the fundamental natural frequency ratio with the non-dimensional Winkler-type elastic modulus and a Poisson's ratio of 0.3

To answer these questions, an approach that is far removed from the traditional single-input-single-output (SISO) method of analysis (that yields Figures 5-4 to 5-8) is needed. In this wise, an applied statistical tool of theoretical Design of Experiment (DOE) is employed [223]. As expected, the term experiment, in this context, is used in the sense of numerical experiment, where the numerical experiment is the numerical predictions from the SEM analysis. A major advantage of DOE is its provision of a convenient framework for the investigation of the main effect, which in this study represent the size effect (d/l), the elastic foundation's stiffness (k_r), the Poisson's ratio (ϑ), the spin rate (η), and any interaction between them.

For ease of representation in the implementation of the DOE, the four parameters of interest in the mathematical model (i.e. spin rate, size effect, foundation stiffness and Poisson's ratio) are denoted by dummy variables A, B, C and D respectively. The

interaction effects, on the other hand, are denoted by a combination of the dummy variables (e.g. AB, AC, BD, ABC). In the words of the applied statisticians, interaction effects exist when variations of the response due to one of the main factors at a particular level manifests dependency on another factor at a different level. It is crucial to point out, though, that an interaction is between the main effects and not levels. When an interaction effect occurs, then the response being monitored is no longer equal to the simple sum of the main effects. Sometimes, the interaction effect of two or more variables becomes more significant than the individual effects. Moreover, we limit ourselves to two seemingly extreme values for each of the parameters under consideration. The values for each of the parameters, under two levels of variation, are shown in Table 5-2. The design of experiments involving four main factors at two levels (2^y), where the superscript y is the number of factors of interest, involves 2^4 runs of the theoretical experiment [224].

Based on the random combination of the values of the parameters given in Table 5-2, the corresponding values of the first three natural frequencies are computed through the SEM. The computed values of the natural frequencies and the values of the parameters, at each level of the combination, are then fed into a commercial statistical package MINITAB®, for the generation of the Pareto charts. Figures 5-9 to 5-11 represent the Pareto charts of the impact of the different factors on the first, second and third natural frequencies.

Table 5-2 Uncoded and coded values of input parameters for the numerical experiments

Input parameter for numerical experiment	Levels	
	<i>Low</i>	<i>High</i>
Non-dimensional spin rate (A)	2	8
Size effect (B)	4	12
Non-dimensional foundation stiffness (C)	10	100
Poisson's ratio (D)	0.2	0.3

Figure 5-9 Pareto chart for the influence of the spin rate, size effect, foundation parameter and Poisson's ratio on the first natural frequency

The Pareto charts, show not only the main effects, but also the interaction effect generated by the main effects. In this numerical experiment, the interaction effects are limited to the order two, because the higher levels of interactions are found to have no influence on the dynamic response of the micro-scale beam. The magnitude of influence

of the size effect is then followed by the spin rate and then the Poisson's ratio, while that of the elastic foundation parameter has diminished considerably. All these observations point to a non-negligible contribution of the size effect in the mathematical model. The observations all reveal why it is important to put the size effect parameter in consideration for detailed analysis of micro-sized structures.

Figure 5-10 Pareto chart for the influence of the spin rate, size effect, foundation parameter and Poisson's ration on the second natural frequency

The observations from Figure 5-9 show the order of influence of the main parameters as $\eta > k_r > d/l > \vartheta$. At the same time, the interaction effect of the foundation modulus and the size effect (BC) is found to have more impact on the first natural frequency than the Poisson's ratio, a fact that may not be obvious from the SISO-type analysis approach. With respect to the second and third natural frequencies, in

Figures 5-10 and 5-11 respectively, the situation changes. The size effect is observed to be far more dominant than the other parameters.

Figure 5-11 Pareto chart for the influence of the spin rate, size effect, foundation parameter and Poisson's ration on the third natural frequency

5.5 Microstructure-dependent model of tapered micro beams

In sub-subsection 5.3.1, the mathematical expressions of a microstructure-dependent model of a spinning micro-scale beam with a uniform cross-section are presented. The derivation is first worked out for a shear deformable beam. The governing equations of a spinning micro-scale beam with a uniform cross-section based on the EB beam theory are then derived from Eqs. (5.26) – (5.29). In reality, the equations of

motion for a spinning tapered micro-scale beam are easily derivable from that of the spinning micro-scale beam with uniform cross-section. In order to achieve this, the Lagrangian function is re-derived in such a way that, the mass distribution (ρA), the flexural rigidity terms (EI_{xx} and EI_{yy}) and the rotary inertia terms (ρI_{xx} and ρI_{yy}) are no longer constant coefficients. With the variation of the mass distribution, the flexural rigidity, and the rotary inertia, a set of variable coefficient partial differential equations governing the transverse vibration of the spinning micro-scale beam with a taper cross-section is given as:

$$-\rho A \left(\frac{\partial^2 u}{\partial t^2} \right) + kG \frac{\partial}{\partial z} \left[A \left(\frac{\partial u}{\partial z} - \varphi_y \right) \right] - \frac{1}{4} G(l_s)^2 \frac{\partial}{\partial z} \left[A \left(\frac{\partial^2 \varphi_y}{\partial z^2} + \frac{\partial^3 u}{\partial z^3} \right) \right] = 0 \quad (5.69)$$

$$\begin{aligned} & \frac{E(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} \frac{\partial}{\partial z} \left[I_{xx} \left(\frac{\partial \varphi_y}{\partial z} \right) \right] + kAG \left(\frac{\partial u}{\partial z} - \varphi_y \right) - \rho I_{xx} \frac{\partial^2 \varphi_y}{\partial t^2} - \rho I_{yy} \Omega \frac{\partial \varphi_x}{\partial t} + \\ & \frac{1}{4} G(l_s)^2 \frac{\partial}{\partial z} \left[A \left(\frac{\partial \varphi_y}{\partial z} + \frac{\partial^2 u}{\partial z^2} \right) \right] = 0 \end{aligned} \quad (5.70)$$

$$-\rho A \left(\frac{\partial^2 v}{\partial t^2} \right) + kG \frac{\partial}{\partial z} \left[A \left(\frac{\partial v}{\partial z} - \varphi_x \right) \right] + \frac{1}{4} G(l_s)^2 \frac{\partial}{\partial z} \left[A \left(\frac{\partial^2 \varphi_x}{\partial z^2} + \frac{\partial^3 v}{\partial z^3} \right) \right] = 0 \quad (5.71)$$

$$\begin{aligned} & \frac{E(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} \frac{\partial}{\partial z} \left[I_{yy} \left(\frac{\partial \varphi_x}{\partial z} \right) \right] + kAG \left(\frac{\partial v}{\partial z} - \varphi_x \right) - \rho I_{yy} \frac{\partial^2 \varphi_x}{\partial t^2} + \rho I_{xx} \Omega \frac{\partial \varphi_y}{\partial t} + \\ & \frac{1}{4} G(l_s)^2 \frac{\partial}{\partial z} \left[A \left(\frac{\partial \varphi_x}{\partial z} + \frac{\partial^2 v}{\partial z^2} \right) \right] = 0 \end{aligned} \quad (5.72)$$

5.6 Microstructure-dependent model of pre-twisted micro beams

Pre-twisted structural elements are naturally complex. Due to their extensive use in engineering systems, a deeper understanding of their mechanical response is required. In this sub-subsection, based on the modified couple stress theory, the elastodynamics

governing partial differential equations of motion for the transverse dynamics of a twisted micro-scale beam will be derived. In general, micro-scale beams are characterized with different cross-sections. However, in this derivation, a rectangular cross-section, for which the rate of twist introduced a bending-bending coupling effect, will be considered. For analysis, the derived governing equations of motion will be used to examine the effects of the rate of twist and the material length scale on the bending wave propagation characteristics of the micro-scale beam. Before presenting the governing equations and subsequent analysis, it is necessary to give some background on the rigorous efforts of earlier researchers in this realm.

From the meticulous survey, presented by Rosen [104], of the works done by researchers up to the early nineties, it is evident that the static and dynamic analyses of twisted beams have long been a subject of great interest to researchers. Almost two decades after Rosen's review, that interest has not waned. A carefully selected list of work on the subjects include the studies by Subrahmanyam and Rao [225] on the use of Reissner method for the vibration study of a twisted beam and Leung [226] on the development of exact dynamic stiffness of a pre-twisted beam. Liao and Dang [127] analyzed the transverse vibration and stability of an orthotropic twisted beam. Banerjee [227] employed the dynamic stiffness method for the free vibration analysis of an Euler-Bernoulli (EB) beam. Yardimoglu and Yildirim [168] developed a finite element model of a twisted Timoshenko beam. Leung and Fan [228] studied the effect of multiple initial stresses on the natural frequencies of a pre-twisted Timoshenko beam. During the course of this study, there is no available study on the modeling of twisted micro-scale beams based on higher-order elasticity theories with the capability to quantify the size effect

phenomenon. Besides, as a result of its relevance to the stated research objectives, effort is made to present a size-dependent governing equation for the dynamics of a twisted micro-scale beam on the basis of the MCST. Propagation of elastic waves in structures is crucial for most of the well-known non-destructive tests. For fragile micro-scale structures, especially those that require non-invasive modal analysis, the presence of the small-scale effect is expected to affect the wave propagation characteristics [229].

5.6.1 Fundamental relation of the modified couple stress theory of the twisted micro beam

The generalized constitutive model based on the nonlocal MCST is the same as those for the isotropic Hookean material presented in sub-subsection 5.3.1 and will therefore not be repeated here again. Instead, from the higher-order constitutive model (defined in sub-subsection 5.3.1), the coupled bending-bending deformation energy of the twisted micro-scale beam will be derived. To derive the coupled bending-bending deformation energy of the twisted micro beam, the schematic shown in Figure 5-12, of a twisted micro-scale beam of total length L , thickness h and width b is considered. In deriving the governing equation of the twisted micro-scale beam, two fundamental assumptions are made. The first is based on the postulates of the Euler-Bernoulli (EB) beam theory, while the second is the assumption of a uniform rate of twist along the length of the micro-scale beam. With these assumptions, a set of kinematically admissible flexible displacements and rotations is employed to form the system of direct and shearing strains.

Figure 5-12 Schematic of a uniformly twisted micro scale beam and coordinate systems

In Figure 5-13, the two sets of coordinate systems needed (inertia frame of reference and body coordinates) to derive the governing equation of this geometry are shown. Along the neutral surface of the undeformed micro beam cross-section is the inertial frame of reference XYZ located at origin O . CG taken to be the centroid at a distance z from the origin of the inertial frame, while CGx and CGy are the principal axes in bending of the cross section.

Since the local coordinates derivation in Banerjee [227] is adopted, the $CG(x, y)$ axis system has a right-handed rotation about OZ . This right-handed rotation ensures the angle of rotation (φ) between CGx and OZ and CGz and OZ to be equal. At the centroidal point CG , φ is the angle of twist, therefore the rate of twist along the length of the micro scale beam is $d\varphi/dz$.

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$$W(z, t) = -y\beta_x + x\beta_y \quad (5.73)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \quad (5.74)$$

$$\begin{bmatrix} U(z, t) \\ V(z, t) \end{bmatrix} = \begin{bmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{bmatrix} \begin{bmatrix} u(z) \\ v(z) \end{bmatrix} \quad (5.75)$$

where

$$\beta_x = \frac{dv(z)}{dz} - \frac{d\varphi}{dz} u(z), \beta_y = -\frac{du(z)}{dz} - \frac{d\varphi}{dz} v(z) \quad (5.76)$$

Putting Eq. (5.76) into Eq. (5.73) result in:

$$W = -y \left(\frac{dv}{dz} - \frac{d\varphi}{dz} u \right) + x \left(\frac{du}{dz} - \frac{d\varphi}{dz} v \right) \quad (5.77)$$

From Eq. (5.74), the expressions for y and x are substituted in Eq. (5.77) to have:

$$W = -(X \sin \varphi + Y \cos \varphi) \left(\frac{dv}{dz} - \frac{d\varphi}{dz} u \right) + (X \cos \varphi - Y \sin \varphi) \left(\frac{du}{dz} - \frac{d\varphi}{dz} v \right) \quad (5.78)$$

With the displacement functions defined, the normal component of the Cauchy strain [218] is defined as:

$$\begin{aligned} \varepsilon_{zz} = \frac{\partial W}{\partial z} = \frac{d\varphi}{dz} (Y \cos \varphi + X \sin \varphi) \left(\frac{\partial \varphi}{\partial z} w + \frac{\partial u}{\partial z} \right) + \frac{\partial \varphi}{\partial z} (X \cos \varphi - Y \sin \varphi) \left(\frac{\partial \varphi}{\partial z} u - \frac{\partial v}{\partial z} \right) - \\ (X \cos \varphi - Y \sin \varphi) \left(\frac{\partial \varphi}{\partial z} \frac{\partial v}{\partial z} + \frac{\partial^2 u}{\partial z^2} \right) + (Y \cos \varphi + X \sin \varphi) \left(\frac{\partial \varphi}{\partial z} \frac{\partial u}{\partial z} - \frac{\partial^2 v}{\partial z^2} \right) \end{aligned} \quad (5.79)$$

By taking note of the transformation matrix in Eq. (5.74), the expression for the normal strain (ε_{zz}) is further simplified as:

$$\varepsilon_{zz} = x \left[\left(\frac{\partial \varphi}{\partial z} \right)^2 u - 2 \frac{\partial \varphi}{\partial z} \frac{\partial v}{\partial z} - \frac{\partial^2 u}{\partial z^2} \right] + z \left[\left(\frac{\partial \varphi}{\partial z} \right)^2 w + 2 \frac{\partial \varphi}{\partial z} \frac{\partial u}{\partial z} - \frac{\partial^2 v}{\partial z^2} \right] \quad (5.80)$$

$$\varepsilon_{yy} = \varepsilon_{xx} = \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{xz} = 0 \quad (5.81)$$

From the constitutive relation, the components of the symmetric curvature tensor are obtained as:

$$\chi_{ij} = \begin{bmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial z} \left(\frac{\partial v}{\partial z} - \frac{\partial \varphi}{\partial z} u \right) \\ 0 & 0 & -\frac{\partial}{\partial z} \left(-\frac{\partial u}{\partial z} - \frac{\partial \varphi}{\partial z} v \right) \\ 0 & 0 & 0 \end{bmatrix} \quad (5.82)$$

The non-zero components of the Cauchy strain and the symmetric curvature tensors in Eqs. (5.80) and (5.82) are appropriately substituted in the higher-order constitutive relations, to arrive at the following non-zero components of the Cauchy stress and the deviatoric component of the couple stress:

$$\sigma_{zz} = \frac{\partial}{\partial z} E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} \left(x \left[\left(\frac{\partial \varphi}{\partial z} \right)^2 u - 2 \frac{\partial \varphi}{\partial z} \frac{\partial v}{\partial z} - \frac{\partial^2 u}{\partial z^2} \right] + y \left[\left(\frac{\partial \varphi}{\partial z} \right)^2 w + 2 \frac{\partial \varphi}{\partial z} \frac{\partial u}{\partial z} - \frac{\partial^2 v}{\partial z^2} \right] \right) \quad (5.83)$$

$$m_{zy} = -l^2 G \frac{\partial}{\partial z} \left(-\frac{\partial u}{\partial z} - \frac{\partial \varphi}{\partial z} v \right), \quad m_{zx} = l^2 G \frac{\partial}{\partial z} \left(\frac{\partial v}{\partial z} - \frac{\partial \varphi}{\partial z} u \right) \quad (5.84)$$

As earlier pointed out $\frac{\partial \varphi}{\partial z}$ is the rate of twist along the length of the micro-scale beam. For convenience this term $\left(\frac{\partial \varphi}{\partial z} \right)$ is replaced by γ from now on. From the derived tensor components, the total strain energy (in bending) of the twisted micro-scale beam is obtained as follows:

$$\begin{aligned}
 \Pi_s = \frac{1}{2} \int_0^L \left[E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{YY} \left[\gamma^2 u - 2\gamma \frac{\partial v}{\partial z} - \frac{\partial^2 u}{\partial z^2} \right]^2 \right. \\
 \left. + E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{XX} \left[\gamma^2 v + 2\gamma \frac{\partial u}{\partial z} - \frac{\partial^2 v}{\partial z^2} \right]^2 - l^2 AG \left[-\frac{\partial^2 u}{\partial z^2} - \gamma \frac{\partial v}{\partial z} - v \frac{\partial \gamma}{\partial z} \right]^2 \right. \\
 \left. + l^2 AG \left[\frac{\partial^2 v}{\partial z^2} - \gamma \frac{\partial u}{\partial z} - u \frac{\partial \gamma}{\partial z} \right]^2 \right] dz
 \end{aligned} \quad (5.85)$$

where the definitions $\int_A y^2 dA = I_{XX}$ and $\int_A x^2 dA = I_{YY}$ have been introduced in Eq. (5.85) to represent the principal moment of area of the cross section of the micro scale beam. Using the allowable displacement components, u and v , the total kinetic energy (Π_t) of the micro-scale beam is given as:

$$\Pi_t = \frac{1}{2} \int_0^L \rho A \left[\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 \right] dz \quad (5.86)$$

where ρ is the density of the beam material and A is the area. The governing equations of the twisted micro scale beam are now obtained with the help of the extended Hamilton's principle as:

$$\delta \int_0^T (\Pi_s - \Pi_t + \Pi_w) dt = 0 \quad (5.87)$$

Neglecting the external work done Π_w , the expressions for Π_s from Eq. (5.85) and Π_t from Eq. (5.87) are put into Eq. (5.87) and the criterion of variational calculus is applied to obtain:

$$\begin{aligned}
 \int_0^T \int_0^L \left(E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{YY} \left[\gamma^2 u - 2\gamma \frac{\partial v}{\partial z} - \frac{\partial^2 u}{\partial z^2} \right] \left[\gamma^2 \delta u - 2\gamma \frac{\partial \delta v}{\partial z} - \delta \frac{\partial^2 u}{\partial z^2} \right] + \right. \\
 E \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{XX} \left[\gamma^2 v + 2\gamma \frac{\partial u}{\partial z} - \frac{\partial^2 v}{\partial z^2} \right] \left[\gamma^2 \delta v + 2\gamma \frac{\partial \delta u}{\partial z} - \delta \frac{\partial^2 v}{\partial z^2} \right] + l^2 AG \left[-\frac{\partial^2 u}{\partial z^2} - \gamma \frac{\partial v}{\partial z} - \right. \\
 \left. v \frac{\partial \gamma}{\partial z} \right] \left[-\delta \frac{\partial^2 u}{\partial z^2} - \gamma \frac{\partial \delta v}{\partial z} - \delta v \frac{\partial \gamma}{\partial z} \right] + l^2 AG \left[\frac{\partial^2 v}{\partial z^2} - \gamma \frac{\partial u}{\partial z} - u \frac{\partial \gamma}{\partial z} \right] \left[\delta \frac{\partial^2 v}{\partial z^2} - \gamma \frac{\partial \delta u}{\partial z} - \delta u \frac{\partial \gamma}{\partial z} \right] - \\
 \left. \rho A \frac{\partial \delta}{\partial t} \frac{\partial u}{\partial t} - \rho A \frac{\partial \delta}{\partial t} \frac{\partial v}{\partial t} \right) dz dt = 0
 \end{aligned} \quad (5.88)$$

The above equation is integrated by part to give:

$$\int_0^T \int_0^L \left\{ \begin{aligned} & E \vartheta_r I_{YY} \left[\frac{\partial^4 u}{\partial z^4} + 2\gamma \frac{\partial^3 v}{\partial z^3} - 2\gamma^2 \frac{\partial^2 u}{\partial z^2} - 2\gamma^3 \frac{\partial v}{\partial z} + \gamma^4 u \right] - \\ & 2\gamma E \vartheta_r I_{XX} \left[\gamma^2 \frac{\partial v}{\partial z} + 2\gamma \frac{\partial^2 u}{\partial z^2} - \frac{\partial^3 v}{\partial z^3} \right] + \\ & l^2 AG \left[\frac{\partial^4 u}{\partial z^4} + 2\gamma \frac{\partial^3 v}{\partial z^3} + 2 \frac{\partial \gamma}{\partial z} \frac{\partial^2 v}{\partial z^2} + 3 \frac{\partial^2 \gamma}{\partial z^2} \frac{\partial v}{\partial z} \right. \\ & \left. + v \frac{\partial^3 \gamma}{\partial z^3} - \gamma^2 \frac{\partial^2 u}{\partial z^2} - \gamma \frac{\partial \gamma}{\partial z} \frac{\partial u}{\partial z} - u \gamma \frac{\partial^2 \gamma}{\partial z^2} + u \left(\frac{\partial \gamma}{\partial z} \right)^2 \right] + \rho A \frac{\partial^2 u}{\partial t^2} \end{aligned} \right\} \delta u \\ + \left\{ \begin{aligned} & E \vartheta_r I_{XX} \left[\frac{\partial^4 v}{\partial z^4} - 2\gamma \frac{\partial^3 u}{\partial z^3} - 2\gamma^2 \frac{\partial^2 v}{\partial z^2} + 2\gamma^3 \frac{\partial u}{\partial z} + \gamma^4 v \right] \\ & + 2\gamma E \vartheta_r I_{YY} \left[\gamma^2 \frac{\partial u}{\partial z} - 2\gamma \frac{\partial^2 v}{\partial z^2} - \frac{\partial^3 u}{\partial z^3} \right] \\ & + l^2 AG \left[\frac{\partial^4 v}{\partial z^4} - 2\gamma \frac{\partial^3 u}{\partial z^3} - 2 \frac{\partial \gamma}{\partial z} \frac{\partial^2 u}{\partial z^2} - 3 \frac{\partial^2 \gamma}{\partial z^2} \frac{\partial u}{\partial z} - u \frac{\partial^3 \gamma}{\partial z^3} - \right. \\ & \left. \gamma \frac{\partial \gamma}{\partial z} \frac{\partial v}{\partial z} - \gamma^2 \frac{\partial^2 v}{\partial z^2} - v \frac{\partial^2 \gamma}{\partial z^2} + v \left(\frac{\partial \gamma}{\partial z} \right)^2 \right] + \rho A \frac{\partial^2 v}{\partial t^2} \end{aligned} \right\} \delta v \Bigg\} dz dt = 0$$

+Q5

(5.89a)

$$Q5 = \int_0^T \left[E \vartheta_r I_{XX} \left[\gamma^2 \frac{\partial v}{\partial z} + 2\gamma \frac{\partial^2 u}{\partial z^2} - \frac{\partial^3 v}{\partial z^3} \right] \delta v - E \vartheta_r I_{XX} \left[\gamma^2 v + 2\gamma \frac{\partial u}{\partial z} - \frac{\partial^2 v}{\partial z^2} \right] \frac{\partial \delta v}{\partial z} \right. \\ + E \vartheta_r I_{XX} 2\gamma \left[\gamma^2 v + 2\gamma \frac{\partial u}{\partial z} - \frac{\partial^2 v}{\partial z^2} \right] \delta u \\ + E \vartheta_r I_{YY} \left[\gamma^2 \frac{\partial u}{\partial z} - 2\gamma \frac{\partial^2 v}{\partial z^2} - \frac{\partial^3 u}{\partial z^3} \right] \delta u \\ - E \vartheta_r I_{YY} \left[\gamma^2 u - 2\gamma \frac{\partial v}{\partial z} - \frac{\partial^2 u}{\partial z^2} \right] \frac{\partial \delta u}{\partial z} \\ \left. - 2\gamma E \vartheta_r I_{YY} \left[\gamma^2 u - 2\gamma \frac{\partial v}{\partial z} - \frac{\partial^2 u}{\partial z^2} \right] \delta v \right]_0^L$$

(5.89b)

where ϑ_r is now used to represent $\frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)}$. From Eq. (5.89), the fundamental lemma of variational calculus is invoked to arrive at the elastodynamics governing equation of a twisted micro-scale beam as:

$$\begin{aligned}
 & E\vartheta_r I_{YY} \left[\frac{\partial^4 u}{\partial z^4} + 2\gamma \frac{\partial^3 v}{\partial z^3} - 2\gamma^2 \frac{\partial^2 u}{\partial z^2} - 2\gamma^3 \frac{\partial v}{\partial z} + \gamma^4 u \right] - 2\gamma E\vartheta_r I_{XX} \left[\gamma^2 \frac{\partial v}{\partial z} + 2\gamma \frac{\partial^2 u}{\partial z^2} - \frac{\partial^3 v}{\partial z^3} \right] + \\
 & l^2 AG \left[\frac{\partial^4 u}{\partial z^4} + 2\gamma \frac{\partial^3 v}{\partial z^3} + 2 \frac{\partial \gamma}{\partial z} \frac{\partial^2 v}{\partial z^2} + 3 \frac{\partial^2 \gamma}{\partial z^2} \frac{\partial v}{\partial z} + v \frac{\partial^3 \gamma}{\partial z^3} - \gamma^2 \frac{\partial^2 u}{\partial z^2} - \gamma \frac{\partial \gamma}{\partial z} \frac{\partial u}{\partial z} - u \gamma \frac{\partial^2 \gamma}{\partial z^2} \right] + \rho A \frac{\partial^2 u}{\partial t^2} \\
 & \quad + u \left(\frac{\partial \gamma}{\partial z} \right)^2
 \end{aligned} \tag{5.90}$$

$$\begin{aligned}
 & E\vartheta_r I_{XX} \left[\frac{\partial^4 v}{\partial z^4} - 2\gamma \frac{\partial^3 u}{\partial z^3} - 2\gamma^2 \frac{\partial^2 v}{\partial z^2} + 2\gamma^3 \frac{\partial u}{\partial z} + \gamma^4 v \right] + 2\gamma E\vartheta_r I_{YY} \left[\gamma^2 \frac{\partial u}{\partial z} - 2\gamma \frac{\partial^2 v}{\partial z^2} - \frac{\partial^3 u}{\partial z^3} \right] + \\
 & l^2 AG \left[\frac{\partial^4 v}{\partial z^4} - 2\gamma \frac{\partial^3 u}{\partial z^3} - 2 \frac{\partial \gamma}{\partial z} \frac{\partial^2 u}{\partial z^2} - 3 \frac{\partial^2 \gamma}{\partial z^2} \frac{\partial u}{\partial z} - u \frac{\partial^3 \gamma}{\partial z^3} - \gamma \frac{\partial \gamma}{\partial z} \frac{\partial v}{\partial z} - \gamma^2 \frac{\partial^2 v}{\partial z^2} - v \gamma \frac{\partial^2 \gamma}{\partial z^2} \right] + \rho A \frac{\partial^2 v}{\partial t^2} \\
 & \quad + v \left(\frac{\partial \gamma}{\partial z} \right)^2
 \end{aligned} \tag{5.91}$$

Again, before moving on to the analysis of the bending waves in the twisted micro-scale beam, a couple remarks about the derived governing equations are in order. First, if the rate of twist γ is set to zero, $I_{XX} = I_{YY} = I$ and $G = \mu$, the size-dependent governing equation of the micro scale beam presented by Kong et al. [206] is obtained. Secondly, if the material length scale parameter l and Poisson's ratio are set to zero, then the governing equation of the classical twisted EB beam presented by Banerjee [231] is retrieved. Thirdly, it would be observed that the modified couple stress theory has introduced higher derivatives of the rate of twist γ . The higher derivatives of the rate of twist needs to be considered for a non-uniformly twisted micro-scale beam. However,

due to the assumption of a constant rate of twist, the higher derivatives of the rate of twist introduced by the couple stress are neglected in analyses that follow.

5.6.2 Discussion of the bending wave analysis in the twisted micro-scale beam

To determine the phase speed and phase velocity of a wave guided through a medium, the dispersion relation, which provides the relationship between the frequency of the travelling wave and the wavenumbers, is needed [160]. Also, in order to investigate the propagation of the bending elastic wave in the twisted micro-scale beam the dispersion relation is needed. The dispersion relation is derived by writing the general solution of Eqs. (5.90) – (5.91) in the form

$$u(z, t) = Q_1(z - c_s t) + Q_2(z + c_s t) \quad (5.92)$$

$$v(z, t) = Q_3(z - c_s t) + Q_4(z + c_s t) \quad (5.93)$$

where $Q_i (i = 1 \dots 4)$ are used to represent wave profiles. $Q_1(z - c_s t)$ and $Q_3(z - c_s t)$ represent waves moving in the positive z direction with a velocity of c_s , while $Q_2(z + c_s t)$ and $Q_4(z + c_s t)$ are the waves moving in the opposite direction. Considering an oscillatory motion of the wave, which is typical in dynamics analyses [232], the following travelling wave profile in the xz and yz planes are considered:

$$u(z, t) = U(z, \omega)e^{-j(kz - \omega t)}, v(z, t) = V(z, \omega)e^{-j(kz - \omega t)} \quad (5.94)$$

where $U(z, \omega)$ and $V(z, \omega)$ are the frequency domain amplitudes of the flexural deformation of the twisted micro scale beam in the xz and yz respectively. k represents the wavenumbers (which is equal to the inverse of the wave's wavelength in the positive

z direction). ω denotes the frequency of the motion of the wave. The substitution of Eq. (5.94) into Eqs. (5.92) and (5.93) leads to two homogenous algebraic equations in terms of U and V as

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} U \\ V \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (5.95)$$

where

$$\begin{aligned} M_{11} &= GA l^2 (k^4 + k^2 \gamma^2) + 4E \vartheta_r I_{XX} k^2 \gamma^2 + E \vartheta_r I_{YY} (k^4 + 2k^2 \gamma^2 + \gamma^4) - \rho A \omega^2 \\ M_{12} &= (i2GA l^2 k^3 \gamma - 2E \vartheta_r I_{XX} \gamma (-ik^3 - ik\gamma^2) + E \vartheta_r I_{YY} (2ik^3 \gamma + 2ik\gamma^3)) \\ M_{21} &= -i2GA l^2 k^3 \gamma + 2E \vartheta_r I_{YY} \gamma (-ik^3 - ik\gamma^2) + E \vartheta_r I_{XX} (-2ik^3 \gamma - 2ik\gamma^3) \\ M_{22} &= GA l^2 (k^4 + k^2 \gamma^2) + 4E \vartheta_r I_{YY} k^2 \gamma^2 + E \vartheta_r I_{XX} (k^4 + 2k^2 \gamma^2 + \gamma^4) - \rho A \omega^2 \end{aligned}$$

The dispersion relation of the bending wave is now determined by the condition for the non-trivial solution of Eq. (5.95) given as the following form:

$$\begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix} = 0 \quad (5.96)$$

The resulting dispersion relation from Eq. (5.96) is found as:

$$k^8 S_1 + k^6 S_2 + k^4 S_3 + k^2 S_4 + S_5 \quad (5.97)$$

where

$$S_1 = G^2 A^2 l^4 + E \vartheta_r I_{XX} GA l^2 + E \vartheta_r I_{YY} GA l^2 + E^2 \vartheta_r^2 I_{XX} I_{YY} \quad (5.98)$$

$$S_2 = -2G^2 A^2 l^4 \gamma^2 - E \vartheta_r I_{XX} GA l^2 \gamma^2 - E \vartheta_r I_{YY} GA l^2 \gamma^2 - 4E^2 \vartheta_r^2 I_{XX} I_{YY} \gamma^2 \quad (5.99)$$

$$S_3 = G^2 A^2 l^4 \gamma^4 - E \vartheta_r I_{XX} G A l^2 \gamma^4 - E \vartheta_r I_{YY} G A l^2 \gamma^4 + 6 E^2 \vartheta_r^2 I_{XX} I_{YY} \gamma^4 - 2 G A^2 l^2 \rho \omega^2 - E \vartheta_r I_{XX} \rho A \omega^2 - E \vartheta_r I_{YY} \rho A \omega^2 \quad (5.100)$$

$$S_4 = E \vartheta_r I_{XX} G A l^2 \gamma^6 + E \vartheta_r I_{YY} G A l^2 \gamma^6 - 4 E^2 \vartheta_r^2 I_{XX} I_{YY} \gamma^6 - 2 G A^2 l^2 \gamma^2 \rho \omega^2 - 6 E \vartheta_r I_{XX} \gamma^2 \rho A \omega^2 - 6 E \vartheta_r I_{YY} \gamma^2 \rho A \omega^2 \quad (5.101)$$

$$S_5 = E^2 \vartheta_r^2 I_{XX} I_{YY} \gamma^8 - E \vartheta_r I_{XX} \gamma^4 \rho A \omega^2 - E \vartheta_r I_{YY} \gamma^4 \rho A \omega^2 + \rho A^2 \omega^4 \quad (5.102)$$

The zeros of Eq. (5.97) are the wavenumbers of the travelling bending wave in the twisted micro-scale beam. From Eqs. (5.97), bearing in mind Eqs. (5.98) – (5.102), the dependence of the dispersion on the material length scale parameter l and the rate of twist γ is easily noticed. The zeros of the quartic form of Eq. (5.97) are then obtained by applying the normal root-finding technique to find the wavenumbers in the form:

$$k_{1,2} = \pm \sqrt{q_1}, k_{3,4} = \pm \sqrt{q_2}, k_{5,6} = \pm \sqrt{q_3}, k_{7,8} = \pm \sqrt{q_4} \quad (5.103)$$

where $q(i = 1, 2 \dots 4)$ are the roots of the quartic form of Eq. (5.97). However, it is worth pointing out that, four of the eight wavenumbers are real, while the other four are imaginary. The effects of the material length scale and the rate of twist on the fundamental wave propagation behavior of a micro-scale beam are reported from here on. For the purpose of numerical analysis and illustration of the effects of the material length scale on the wave propagation characteristics, the micro-scale beam is taken to have the following material properties [233]: $E = 1.44 \text{ GPa}$, $\vartheta = 0.38$, $\rho = 1.22 \times 10^3 \text{ kg/m}^3$. Furthermore, the cross section of the beam is such that $L = 20h$, $b = 2h$, $17.6 \mu\text{m} \leq h \leq 100 \mu\text{m}$, and a uniform rate of twist $\gamma = 45^\circ = \pi/4$.

In Figure 5-14, the spectrum curve of a micro-scale beam with a zero rate of twist is shown, where *CEB* in the figure denotes classical Euler-Bernoulli beam. The wavenumber-frequency relationship is almost linear in the absence of the rate of twist. Besides, as the thickness h decreases from $6l$ to around $2l$ (l being the material length scale parameter), there is a gradual decrease in the propagating space of the bending wave. The pattern of the dispersion curve of the propagating wave mode, in the case of the *CEB*, agrees with those predicted by Doyle [160].

Figure 5-14 Effect of material length scale on the dispersion curve of a micro scale beam with zero twist

The combined effect of the rate of twist and the material length scale is shown in Figure 5-15, where *TCEB* in the figure represents twisted classical Euler-Bernoulli beam. Here, the rate of twist is kept at a value of $\gamma = 45^\circ = \pi/4$, while the material length

scale is varied for the spectrum curve. Similar to the case of the untwisted beam, the increase of the stiffness of the micro scale beam due to the presence of the material length scale is observed to narrow the spectrum curve. In addition to the aforementioned effect, bifurcations of the spectrum curves for the *TCEB*, $h = 4l$, and $h = 4l$ is also noticed.

Figure 5-15 Spectrum curves of a micro scale beam at a rate of twist of $\pi/4$ and different material length scale

The bifurcation of the spectrum curve is observed to occur in the frequency range $0.2 \text{ MHz} \leq \omega \leq 0.4 \text{ MHz}$. To probe the bifurcation point further, a plot of the real and imaginary wavenumbers is given in Figure 5-16. Each of the plots in the positive region of Figure 5-16 is the real component of the dispersion curve at the different values

of the material length scale, while those in the negative region are the imaginary equivalent. Figure 5-16 confirms two important points: (i) that the bifurcation of the spectrum curve does occur in the frequency range $0.2 \text{ MHz} \leq \omega \leq 0.4 \text{ MHz}$ and (ii) that the wave profile of the couple stress theory travels at a faster frequency than that of the classical theory. Furthermore, it is observed that the imaginary parts of the wave profile are spatially decaying waves and hence can actually not be used to effectively transport energy from point to point within the material of the beam.

Figure 5-16 Real and imaginary components of the dispersion curve at a rate of twist of $\pi/4$ and different material length scale

5.6.3 Cut-off frequency and wave speeds

The speed of propagation of a wave pulse in a medium is given by its phase velocity. Besides, when a wave packet or a wave pulse comprising harmonic waves of

different wavenumbers travels through such a dispersive medium as an elastic micro beam, the speed of the wave packet or wave pulse is bound to be determined by the group velocity of the waves. Additionally, the magnitude of the frequency of a wave pulse traveling within an elastic dispersive medium depends on the magnitude of the cut-off frequency. It is therefore of great interest to investigate the degree to which the material and the geometric parameter of the twisted micro-scale beam contributes to its cut-off frequency and the wave speeds (phase and group velocity).

The cut-off frequency of the twisted micro scale beam is obtained by setting $k = 0$ in the dispersion relation earlier defined to lead to:

$$\omega_{cf} = \frac{\sqrt{Eh^2\gamma^4\vartheta_r + Eh^2\alpha^2\gamma^4\vartheta_r - \frac{Eh\gamma^4\vartheta_r\sqrt{-48\alpha^2 + h^2\rho A + 2h^2\alpha^2\rho A + h^2\alpha^4\rho A}}{\sqrt{\rho A}}}}{2\sqrt{6}} \quad (5.104)$$

From Eq. (5.104), it is obvious that the cut-off frequency is independent of the material length scale. However, it is a function of the rate of twist and the second moment of inertia in the zx and zy planes. In Eq. (5.104), a division by I_{XX} has been made and the term $\alpha^2 = (I_{YY}/I_{XX})^{0.5}$ is introduced, where α^2 now represent the thickness-to-width ratio of the micro-scale beam. The rate of change of the cut-off frequency with the rate of twist is shown in Figure 5-17 for different values of the thickness-to-width ratio. The rate of twist considered is in the range $0 \leq \gamma \leq 2\pi$. It is interesting to note when the value of the thickness-to-width ratio (α) of the twisted micro scale beam is 1, the relationship between the cut-off frequency and the rate of twist disappears. This is because when the thickness-to-width ratio is 1, the rectangular cross section is essentially reduced to a

square cross section. Furthermore, the cut-off frequency increases with higher value of the thickness-to-width ratio.

Figure 5-17 Variation of the rate of twist and the cut-off frequency of the twisted micro-scale beam

Figures 5-18 and 5-19 show the trend of the group velocity and the phase speed for different values of the rate of twist. Both the phase speed and the group velocity of the bending wave in the twisted micro-scale beam are found to be a function of the material length scale and other geometric parameters. By definition, the phase speed c_p is a ratio of the propagating frequency and the wavenumbers (that is, $c_p = \omega/k$), while the group velocity is $c_g = \partial\omega/\partial k$.

Figure 5-18 Variation of the phase speed and the propagating frequency of the wave profile at $h = 12$

Figure 5-19 Variation of the group velocity and the propagating frequency of the wave profile at $h = 2l$

The two expressions for the phase velocity and the group speed were numerically obtained from Eq. (5.103). An important observation in the analysis of the wave speed for the twisted micro scale beam is that, unlike in the classical EB beam, the group velocity is more than twice the phase speed for all values of the wavenumbers obtained. Furthermore, it is noticed that both the phase speed and the group velocity show high propagating frequency at lower values of the rate of twist. Increasing the rate of twist therefore leads to a decrease in the magnitude of the propagating frequency. It is noteworthy to point out that the model developed and presented in this sub-subsection is easily extended to study the transverse vibration of a twisted micro-scale beam. Furthermore, the analyses served as a platform to provide useful insight into the dynamics of twisted micro-scale structure based on the (MCST).

5.7 Dynamics of an embedded, axially loaded, non-prismatic small-scale beam based on Eringen's nonlocal elasticity theory

In the previous sub-subsections, a nonlocal, non-classical elasticity theory (so-called modified couple stress theory) is employed to formulate the governing equations of micro-sized beams of different cross-sections and geometries. This sub-subsection aims to provide a rather brief exploration of another higher-order elasticity theory (Eringen's nonlocal elasticity theory [186]). (Eringen's nonlocal elasticity theory is also endowed with the capacity to quantify size-effect phenomenon in small-scale structures. In terms of mathematical implementation, the Eringen's nonlocal theory [186-187], is characterized by two material constants. This theory has been widely employed to examine and explore the degree to which the size-effect phenomenon contributes to the

structural response of carbon nanotubes (CNT) [183, 190-191]. Discrete atomistic methods, such as the density functional theory and the molecular dynamics are generally the theoretical routes for the simulation of the responses of CNTs [234]. However, because of the computational cost, the structural mechanics-based continuum theory provided a viable alternative theoretical technique to simulate the behavior of CNTs. This explains why, in recent years, several types of continuum models (such as beam and elastic shell models) have also been used in analytical investigations of the properties of CNTs.

The primary aim of this sub-subsection is to therefore present the formulation of the governing equation of nano-sized structures (nano-scale beam-like structure) through the Eringen's nonlocal elasticity theory. The results from the use of the Eringen's nonlocal elasticity theory will then be used to highlights the difference between the two theories. To derive the governing dynamics equation of a nano-sized beam, the von Kármán nonlinear strain-displacement relationship along with the constitutive equation for the nonlocal elasticity theory is employed. Recent refinement of the nonlocal continuum theory is found in Refs. [189, 235]. Vibration analyses are of primary importance in the design of novel nanodevices and Nanoelectromechanical systems (NEMS). A vital preliminary analysis involves the determination of the natural frequencies and the associated mode shape. The vibration response of CNTs combined with the effect of the surrounding medium is also of practical importance. For instance, beams supported on elastic foundations are commonly incorporated in the design of aircraft structures where CNTs are finding immense applications. Another practical example is the mechanical characteristics of CNTs embedded in polymer or metal matrix

[236]. In what follows, a mathematical model is presented for the vibration analysis of a geometrically tapered nano-beam. The nano-beam is idealized as a nonlocal Rayleigh (NLR) beam that is axially loaded and embedded in a deformable surrounding medium. A polynomial power law that defines the range of tapers along the longitudinal axis of the continuum will also be proposed. Additionally, the variable coefficient governing differential equation of the tapered nano-scale beam is solved with the *Bubnov-Galerkin* method. The current investigation is focused on a different problem of the free vibration of a nano-sized beam embedded in an elastic foundation of the Pasternak-type. The choice of modeling the nano-sized beam with a Rayleigh beam arises from the fact that it gives a relatively better prediction of the dynamic behavior of a long structure as a result of its inclusion of the rotary inertia. The effect of the rotary inertia effect is neglected in the Euler-Bernoulli beam. At the same time, the use of the Pasternak model is justified based on the argument of Kerr [237] that the behavior of a large class of foundation materials occurring in practice cannot be described by the Winkler models. This argument is also supported by the work of Rades [238] and De Rosa [239], who pointed out that the influence of the shear modulus in the Pasternak foundation model is most pronounced over the range of low forcing frequencies, where the Winkler's hypothesis leads to great errors in the frequency range of practical interest. Apart from the above argument of Kerr [240], another reason for modeling the surrounding as a Pasternak model is because the model is more appropriate when CNTs are used as a secondary reinforcing element in multi-scale composites. Besides, the Pasternak model is more consistent with the nonlocal continuum theory used in our analysis because it has been classified by Di Paola et al. [241] as a nonlocal foundation. This is actually the first study

in which the axial force, Pasternak elastic foundation, small scale effect (nonlocal parameter) and non-uniformity in the cross-section of the nano-sized beam modeled as a nonlocal continuum beam is dealt with.

Based on the principle of virtual displacement, the following integro-differential equation holds for the axially loaded non-local Rayleigh (NLR) beam embedded in an elastic medium of the Pasternak-type:

$$\int_{t_1}^{t_2} \int_{L_1}^{L_2} \left[N \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} - N \delta \varepsilon_{xx} - M \delta \kappa + \rho A \left(\frac{\partial w}{\partial t} \frac{\partial \delta w}{\partial t} + \frac{\partial w}{\partial t} \frac{\partial \delta w}{\partial t} \right) + \rho I \left(\frac{\partial^2 w}{\partial x \partial t} \frac{\partial^2 \delta w}{\partial x \partial t} \right) + K_f \delta w - K_s \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} + \eta_{th} \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \right] dx dt = 0 \quad (5.105)$$

where x is the axial coordinate, w is the transverse displacement, κ is the slope of the transverse deflection of the beam, N is the axial compressive force due to the pre-stressed state, and M is the bending stress resultant. η_{th} is the axial force arising from the thermal effect. ρA and ρI are the mass per unit length and the rotary inertia of the nano-scale beam-like structure, respectively. The stiffness and the shear modulus parameters of the deformable medium are represented by K_f and K_s . Given that $t_1 = 0$, $t_2 = t$, $L_1 = 0$ and $L_2 = L$, the Euler-Lagrange equations of the nano-scale beam is obtained as

$$\frac{\partial N}{\partial x} - \rho A \frac{\partial^2 w}{\partial t^2} = 0 \quad (5.106)$$

$$\frac{\partial^2 M}{\partial x^2} - \frac{\partial}{\partial x} \left(N \frac{\partial w}{\partial x} \right) + K_f w - K_s \frac{\partial^2 w}{\partial x^2} - \rho A \frac{\partial^2 w}{\partial t^2} + \rho I \left(\frac{\partial^4 w}{\partial x^2 \partial t^2} \right) + \frac{\partial}{\partial x} \left(\eta_{th} \frac{\partial w}{\partial x} \right) = 0 \quad (5.107)$$

The second order derivative of the moment of inertia is obtained from Eq. (5.107) and then substituted into the nonlocal constitutive relation (given in e.g. Reddy and Pang [189]) to obtain the nonlocal moment of inertia M^N as

$$M^N = -EI\kappa + (e_0a)^2 \left[\frac{\partial}{\partial x} \left(N \frac{\partial w}{\partial x} \right) - K_f w + K_s \frac{\partial^2 w}{\partial x^2} + \rho A \frac{\partial^2 w}{\partial t^2} \right] - \rho I \left(\frac{\partial^4 w}{\partial x^2 \partial t^2} \right) + \frac{\partial}{\partial x} \left(\eta_{th} \frac{\partial w}{\partial x} \right) \quad (5.108)$$

where EI is the flexural rigidity of the nano-scale beam-like structure. (e_0a) is the nonlocal scale coefficient parameter describing the internal characteristics length of the nano-scale beam-like structure. By replacing M of the local model with the M^N of the nonlocal model in Eq. (5.107), the elastodynamics governing equation of the axially loaded nano-scale beam-like structure embedded in an elastic medium of the Pasternak-type is arrived at as:

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 w}{\partial x^2} \right) + \frac{\partial}{\partial x} \left(\eta_{th} \frac{\partial w}{\partial x} \right) + (e_0a)^2 \frac{\partial^2}{\partial x^2} \left[-\frac{\partial}{\partial x} \left(N \frac{\partial w}{\partial x} \right) - K_f w + K_s \frac{\partial^2 w}{\partial x^2} - \rho A \frac{\partial^2 w}{\partial t^2} + \right. \\ \left. \rho I \left(\frac{\partial^4 w}{\partial x^2 \partial t^2} \right) - \frac{\partial}{\partial x} \left(\eta_{th} \frac{\partial w}{\partial x} \right) \right] + \frac{\partial}{\partial x} \left(N \frac{\partial w}{\partial x} \right) + K_f w - K_s \frac{\partial^2 w}{\partial x^2} = \rho A \frac{\partial^2 w}{\partial t^2} - \rho I \left(\frac{\partial^4 w}{\partial x^2 \partial t^2} \right) \end{aligned} \quad (5.109a)$$

For a geometrically non-uniform NLR beam model, clearly the mass distribution (ρA) , the rotary inertia (ρI) , and the flexural stiffness (EI) of the SWCNT cannot be constant. We therefore assume a polynomial variations with a power law of these properties, which is consistent with the works of Wang and Wereley [150]. Based on the proposed polynomial variation, the variable coefficients of the governing equation are defined as:

$$\rho A(\xi) = \rho A_0(1 - \varepsilon\xi)^n \quad (5.109b)$$

$$\rho I(\xi) = \rho I_0(1 - \varepsilon\xi)^{n+2} \quad (5.109c)$$

$$EI(\xi) = EI_0(1 - \varepsilon\xi)^{n+2} \quad (5.109d)$$

where A_0 and I_0 are the area and the mass moment of area of the continuum at the left end respectively. The integer n takes values of 1 for the first type of tapered beam (Figure 5-20a) and 2 for the second type of tapered beam (Figure 5-20b). As pointed out by Banerjee et al. [169] (see also Ref. [152]), a large number of cross-sections can be constructed by employing these two values. In general, the taper ratio ε that defines the degree of inhomogeneity of the cross-section is such that $0 < \varepsilon < 1$.

A number of cases derivable from Eq. (5.109) are considered and the solutions are sought using the Bubnov-Galerkin method [138]. Some of the numerical result from the application of the Galerkin formalism to Eq. (5.109) is presented next. In the succeeding paragraphs, the following non-dimensional parameters are used

$$\bar{p} = \frac{NL^2}{EI_0}; \mu = \frac{e_0 a}{L}; \lambda^2 = \omega^2 \frac{\rho A_0 L^4}{EI_0}; \bar{K}_f = \frac{K_f L^4}{EI_0}; \bar{K}_s = \frac{K_s L^4}{EI_0}; F_{th} = \frac{\eta_{th} L^2}{EI} \quad (5.110)$$

Throughout the numerical case studies presented in this subsection, the effective material properties of the nano-scale beam structure will be those of the single-walled carbon nanotube (SWCNT) taken from Murmu and Pradhan [242]. The modulus of elasticity of the SWCNT is taken as $E = 1 \text{ TPa}$, its mass density (ρ) is 2.3 g/cm^3 , and Poisson's ratio (ν) of 0.19 is used. In the same vein, the geometrical property employed includes a diameter of 1 nm . A conservative value of the scale coefficient ($e_0 a$) is said to be less than 2 nm .

Figure 5-20 Schematic diagram of different tapered cross-sectional geometry of a nano-sized beam, (a) geometry for which the taper integer $n = 1$, (b) geometry for which the taper integer $n = 2$

Hence, the scale coefficient in this study varies in the range $(0 \leq \mu \leq 2 \text{ nm})$. Furthermore, for most of the analyses, the non-dimensional variables are all varied in the ranges that are believed to be of engineering interest. The variation of all the other parameters will be explicitly stated in the appropriate cases. The coefficient of thermal expansion for CNTs is taken as $-1.6 \times 10^{-6} K^{-1}$.

For validation purpose, three different special cases of the above developed mathematical model are used to assess the accuracy of the solution procedure. All calculated values are obtained for up to four-figure accuracy.

Case study 1: *Classical Rayleigh beam of uniform cross-section without elastic foundation*

The vibration of classical Rayleigh beam of uniform cross-section without elastic foundation is first considered. The mathematical model for this case is obtained when $\varepsilon = 0$, $\bar{p} = 0$, $\bar{K}_s = 0$, $\bar{K}_w = 0$, $F_{th} = 0$ and $\mu = 0$. With this specified condition, the model in Eq. (5.109a) reduces to that of the free vibration of the classical Rayleigh beam with a uniform cross-sectional geometry. Table 5-3 shows the values of the first three non-dimensional natural frequencies of a uniform clamped-free Rayleigh beam for various values of the rotary inertia (r_g) from the present analysis and the exact results presented in Bottega [243]. It is apparent from Table 5-3 that, the calculated results with minimal effort match closely with that of the exact approach. The natural frequencies decrease with increase in the values of the normalized radius of gyration.

Case study 2: *Vibration of a SWCNT based on the nonlocal EB beam without elastic foundation*

The mathematical representation for this problem will simply be obtained by allowing ε , r_g , $\bar{p}$, $\bar{K}_s$ and $\bar{K}_w$ to vanish from the governing equation (i.e. $\varepsilon = 0$, $r_g = 0$, $\bar{p} = 0$, $\bar{K}_s = 0$ and $\bar{K}_w = 0$). When this is done, the value of the small scale coefficient is then varied appropriately. The values of the non-dimensional frequencies for the boundary conditions of pinned-pinned and clamped-pinned obtained from the present analysis and that of Wang et al. [244] are presented in Table 5-4. The decrease in the values of the natural frequencies as the nonlocal parameter is varied is evident of its importance contribution to the mathematical model.

Case study 3: *Natural frequencies of a pinned-pinned local EB beam on a Pasternak foundation*

Another chosen validation model is that of the classical EB beam embedded on elastic foundation of the Pasternak-type. Table 5-5 shows the comparison between the values in Yokoyama's work based on a finite elements analysis and the present *Bubnov-Galerkin* method. Figure 5-21 shows the schematics of the boundary conditions employed in the discussions that follow.

Figure 5-21 Schematic of the nano beam, (a) boundary configurations, (b) the reference coordinate and the loading condition of the embedded nano beam

Table 5-6 shows the variation of the first two natural frequencies of a pinned-pinned NLR beam for $\varepsilon = 0.2$, $\mu = 0.1$, $\bar{p} = 0$, $\bar{K}_s = 0$ and $\bar{K}_w = 0$. The rotary inertia is what differentiates the NLR beam from the purely flexible nonlocal Euler-Bernoulli (NLEB) beam. The influence of the rotary inertia is investigated in terms of the radius of gyration (r_g) of the beam.

Table 5-3 The first three non-dimensional natural frequencies of a uniform cantilevered Rayleigh beam

Non-dimensional natural frequencies (λ)						
r_g	Mode 1		Mode 2		Mode 3	
	Present	Exact [243]	Present	Exact [243]	Present	Exact [243]
0.01	3.51	3.52	22.00	22.00	61.53	61.46
0.05	3.49	3.49	21.19	21.19	56.50	56.48
0.10	3.44	3.47	19.14	19.14	46.60	46.49
0.15	3.34	3.34	16.76	16.75	38.04	37.81

Table 5-4 The first two non-dimensional frequencies for a pinned-pinned and clamped-pinned SWCNT

Non-dimensional natural frequencies ($\sqrt{\lambda}$)								
Mode	$\mu = 0$		$\mu = 0.3$		$\mu = 0.5$		$\mu = 0.8$	
	Present	Ref. [244]	Present	Ref. [244]	Present	Ref. [244]	Present	Ref. [244]
pinned-pinned								
1	3.14	3.14	2.68	2.68	2.30	2.30	1.91	—
2	6.31	6.28	4.32	4.30	3.47	3.46	2.78	—
clamped-pinned								
1	3.93	3.93	3.28	3.28	2.79	2.78	2.30	—
2	7.1	7.06	4.79	4.77	3.84	3.83	3.15	—

Table 5-5 Non-dimensional natural frequencies of a classical EB beam embedded in a Pasternak foundation

Non-dimensional natural frequencies (λ)				
Mode	$\bar{K}_s = 25$ and $\bar{K}_w = 25$		$\bar{K}_s = 36$ and $\bar{K}_w = 36$	
	Present	Ref. [245]	Present	Ref. [245]
1	19.22	19.21	22.11	—
2	50.78	50.71	55.19	—

All values in Tables 5-3 to 5-5 have been rounded up to two decimal places. The results in Table 5-6 are given for two types of tapered SWCNTs whose profiles are uniquely defined by the integer n . This integer takes values of 1 and 2 respectively. It is evident from Figure 5-20, that the case of $n = 1$ will represent a scenario in which one of the dimensions of the SWCNT, say, the thickness has a linear variation and the other dimension, say the width is kept constant. However if the It is also worthy to point out that the taper ratio (ε) in this study is different from the term height ratio mentioned in [246]. Instead, the taper ratio (ε) of the cross-section of the non-uniform SWCNT is such that $0 < \varepsilon < 1$. It is also interesting to know that, when $\varepsilon = 0$, the mathematical model presented in this study reduces to that of the uniform NLR beam. However, the situation when $\varepsilon = 1$, is a theoretical limit that may never be reached in practice. This is because, the beam tapers to a point which makes the lowest elastic buckling load of the continuum structure to be zero [247]. In Table 5-6, an inverse relationship is observed between the natural frequency parameters and the normalized radius of gyration of the SWCNT for both cases of $n = 1$ and $n = 2$. That is, the frequency parameter decreases as the normalized radius of gyration increases. However, the behavior of the SWCNT for both cases of the integer n differs. For instance, the natural frequency for the first mode of the SWCNT when $n = 1$ is higher than its corresponding natural frequency for the first mode when $n = 2$. Contrarily, the second natural frequency of the case of $n = 1$ is less than the corresponding second natural frequency of the case when $n = 2$, for all values of the radius of gyration considered. From the design engineer's perspective, the conclusion that can be drawn here is that, if the desire is to have a SWCNT with a slightly lower fundamental frequency compared to a uniform SWCNT, then the case of $n = 2$ becomes

a suitable configuration. On the other hand, if the desire is to slightly ‘minimize’ the second natural frequency, the case of $n = 1$ will be a good choice. In the presence of the nonlocal parameter, it is observed that the natural frequencies for the two tapered geometries seem to be approaching the value of the natural frequencies of a uniform NLR beam for the high value of the normalized radius of gyration ($r_g = 0.10$).

Table 5-6 The first two natural frequencies of a pinned-pinned non-uniform SWCNT

Non-dimensional natural frequencies (λ) ($\varepsilon = 0.2, \mu = 0.1, \bar{p} = 0, \bar{K}_s = 0$ and $\bar{K}_w = 0$)			
Mode	r_g	$n = 1$	$n = 2$
First mode			
	0.00	10.33	10.32
	0.02	10.31	10.29
	0.04	10.25	10.24
	0.06	10.15	10.14
	0.08	10.02	10.00
	0.10	9.89	9.84
Second mode			
	0.00	37.21	37.32
	0.02	36.91	37.03
	0.04	36.08	36.19
	0.06	34.80	34.91
	0.08	33.22	33.32
	0.10	31.47	31.57

This is because the normalized radius of gyration of any geometry is related to its slenderness ratio. It is therefore only logical that the effect of taperedness on the natural frequencies of a tapered geometry with a large value of the normalized radius of gyration will be very minimal. It is also worthy to point out that the taper ratio (ε) in this study is different from the term height ratio mentioned in [246]. Instead, the taper ratio (ε) of the

cross-section of the non-uniform SWCNT is such that $0 < \varepsilon < 1$. It is also interesting to know that, when $\varepsilon = 0$, the mathematical model presented in this study reduces to that of the uniform NLR beam. However, the situation when $\varepsilon = 1$, is a theoretical limit that may never be reached in practice. This is because, the beam tapers to a point which makes the lowest elastic buckling load of the continuum structure to be zero [247].

Figure 5-22 shows the variation of the first natural frequencies of a pinned-pinned NLR beam for $r_g = 0.05$, $\bar{p} = 0$, $\bar{K}_s = 0$ and $\bar{K}_w = 0$ and $n = 1$. The values of μ is varied in the range $0 \leq \mu \leq 0.5 \text{ nm}$ and ε is varied in the range $0 \leq \varepsilon \leq 0.5$. The influence of the variation of these two variables on the first and second natural frequencies of the non-uniform SWCNT is depicted in Figures 5-22(a) and 5-22(b) respectively. Although, both figures show the drop in the frequency values with the increase in the values of the taper ratio and the small scale coefficient, the drop is more pronounced on the second mode than on the first mode of vibration. With respect to the drop in the frequency value as the taper ratio is increased, it can be explained that, it is due to the reduction in the flexural stiffness of the SWCNT. Of course, the taper ratio defines the degree of loss of stiffness of the non-uniform NLR beam model; therefore increasing the value of the taper ratio renders the SWCNT more flexible. In addition, for the case of the pinned-pinned boundary condition considered here, the vibration behavior of the nonlocal model of the SWCNT is also found to be greatly influenced by the small scale coefficient. The small scale coefficient reduces the frequency of the nonlocal model. Without this parameter, the frequency of the beam model will be over-predicted.

(a)

(b)

Figure 5-22 Variation of the natural frequency ratio with the taper ratio for different values of the small scale coefficient, (a) the first mode shape, (b) the second mode shape

It is therefore very important to include this parameter for nano-scale mathematical models. It will also be observed that the clamped-free configuration of the NLEB beam treated in [246] exhibits almost the same trend of frequency parameter increase with increase in the height ratio. However, the taper ratio in this study is conditioned to be less than unity for practical reasons.

In Figure 5-23, the Pareto chart of the effects of the variations in the values of the nonlocal parameter and the taper ratio is presented. In the present study, the prediction of the natural frequency is treated as a ‘theoretical experiment’ and the response, for all the Pareto chart presented is the first natural frequency. Also, in the different sub-sections, the factors that are varied to obtain the chart will be stated. The essence of the charts is to show the factor that has the most significant effect on the first natural frequency when two or more factors are varied. It is observed from Figure 5-23 that when, both the taper ratio and the nonlocal parameter are varied, the nonlocal parameter has the most pronounced effect on the changes in the first natural frequency value. The term AB indicated in the chart is, in the parlance of the industrial experimentalists, called an interaction effect between the factors A and B . Sometimes, the interaction effect can be more significant than the individual effect, in which case, both individual factors must be used for the optimization of the response when it is desired. The presence of the axial force is bound to influence the vibration behavior of a non-uniform NLR beam model of the SWCNT. To investigate how the vibration behavior is affected, some of the parameters in the mathematical model are varied, while some are kept at values that are believed to be valid for practical purposes.

Figure 5-23 Pareto chart of the effects of variation of the taper ratio and the nonlocal parameter

Thus, the following values are used: $r_g = 0.08$, $\bar{K}_s = 0$, $\bar{K}_w = 0$, $\varepsilon = 0.3$ and $n = 2$. The values of the non-dimensional axial force parameter and the small scale coefficient are varied for this analysis. In essence, this situation is basically dealing with an axially loaded non-uniform NLR beam without elastic foundation. Figure 5-24 shows the plots of the axial force against the natural frequency of the nonlocal model of the SWCNT. For this analysis, when the value of the axial force parameter is above 20, no real eigenvalue is gotten for the first vibration mode of the nonlocal model. It is worthy to know that, in the presence of the axial force and non-uniformity in the cross-section of the nonlocal model, the small scale coefficient tends to increase the natural frequency to

the value of the same beam in which the effects of the axial force and the taper ratio are absent.

The plot on the left side of Figure 5-24 is the corresponding Pareto chart for the variations in the values of the non-dimensional axial force parameter and the nonlocal parameter. The interaction effect AB of factors A and B here is more significant than the individual effects of factors A and B . The next significant influence on the response is the nonlocal parameter.

Next, the effects of the Pasternak shear modulus and the nonlocal factor on the natural frequencies are treated. This phase of the analysis contains all the variables presented in the mathematical model. Albeit, some of the parameters are constant, while two are varied to be able to observe the changes in the fundamental frequency of the non-uniform NLR beam model of the SWCNT with respect to these variations. Thus, the parameters are set in such a way that $\varepsilon = 0.3$, $r_g = 0.06$, $\bar{p} = 10$. The present analysis is done for both values of the integer $n = 1, 2$. The value of μ is varied in the range $0.1 \leq \mu \leq 0.5$, $\bar{K}_w$ is varied in the range $5 \leq \bar{K}_w \leq 100$, and $\bar{K}_s$ is varied in the range $1 \leq \bar{K}_s \leq 10$ for a pinned-pinned boundary condition. Figures 5-25(a) and 5-25(b) show the variation of the natural frequency ratio with the Pasternak modulus. The relationship between the frequency ratio and the Pasternak modulus appears to be linearly increasing at a lower value of the nonlocal parameter, but gradually become quadratic as the nonlocal parameter is increased further.

(a)

(b)

Figure 5-24 Effect of the axial force and the nonlocal parameter on the natural frequencies, (a) Variation of the axial force against the natural frequency, and (b) Pareto chart for the main and interaction effects of the axial and the nonlocal parameter

It is obvious that the use of the Pasternak foundation model is very important, as the behavior of the model cannot be easily extrapolated from the simple Winkler modulus

in which the shear modulus of the foundation is neglected. It is also noticed that the frequencies of the NLR beam model of the SWCNT embedded in a Pasternak elastic medium will in most cases be higher than those in a medium of the Winkler-type. Next, the combined effect of three variables on the natural frequencies is considered. This is used to demonstrate how a higher level of variation in more than two factors affects the fundamental natural frequency of the NLR beam model of the SWCNT. The first consideration is the case in which there are variations in the values of the rotary inertia, Winkler modulus and the nonlocal factor for the case when $\varepsilon = 0.3$, $\bar{K}_s = 2$, $\bar{p} = 5$ and $n = 1$. The chart generated is shown in Figure 5-26.

The significance of the influence of the rotary inertia, Winkler modulus and the nonlocal parameter on the response is very evident in Figure 5-26. Terms B , C , A and BC (the interaction effects of terms B and C) are all observed to pass the critical vertical red line. The term AB (the interaction effects of A and B) is also exactly on the line. This indicates that terms B , C , and A , which represent the nonlocal factor, the Winkler modulus and the rotary inertia respectively, have noticeable effects on the fundamental frequency of the SWCNT. With respect to the order of the degree of influence of the main factors, the Winkler modulus has the most significant influence on the response, followed by the nonlocal factor and then the rotary inertia. However, the interaction effects of B and C has more influence on the response than does the rotary inertia.

(a)

(b)

Figure 5-25 The variation of the first natural frequencies with the Pasternak shear modulus, (a) a non-uniform NLR beam model with $n = 1$, (b) a non-uniform NLR beam model with $n = 2$

Figure 5-26 The Pareto chart of the variations in values of the rotary inertia, the Winkler modulus and the nonlocal factor

Further, the variation of the Winkler modulus, the Pasternak shear modulus and the nonlocal factor for the case when ($\varepsilon = 0.3, r_g = 0.04, \bar{p} = 5$ and $n = 1$) is examined. The corresponding Pareto chart in Figure 5-27 shows again that the nonlocal parameter which defines the size of the nanostructures has the most significant influence. In addition, the importance of the use of the advanced elastic foundation model of the Pasternak-type is also further revealed in Figure 5-27, in which the term AC (the interaction effects of terms A and C) has more significant influence on the response than either B or BC . This is in spite of the low values of the Pasternak shear modulus ($1 \leq \bar{K}_s \leq 10$) chosen compared to the values of the Winkler modulus ($10 \leq \bar{K}_s \leq 100$)

considered.

Figure 5-27 The Pareto chart of the variations in values of the Winkler modulus, Pasternak shears modulus and the nonlocal factor

5.8 Summary

This chapter is a presentation of the theoretical modeling and analysis of small-scale structures based on higher-order, non-classical theoretical framework. The chapter detailed novel theoretical models of small-scale structures, with non-trivial mechanical behavior and complex geometries. Based on two non-classical higher-order elasticity theories, the inadequacy of the classical elasticity theory was addressed in this chapter. Specifically, in this chapter, the novel theoretical formulations of some structural

elements whose geometries and mechanical behaviors bear relevance to the structural elements employed in the model of the micro-cutting tool are presented.

A new dynamic model of a spinning doubly-symmetric circular micro beam is developed. The mathematical model, which contains the effect of the material length scale, is formulated on the framework of the modified couple stress theory and the Euler-Bernoulli beam theory. Numerical results from the SEM analysis of the size-dependent governing equations show that the classical equations of the spinning beam underestimate the natural frequencies. However, the effect of the size-dependency diminishes with increase in the diameter of the beam. Furthermore, the coupled elastodynamics governing equation of a twisted micro scale beam based on the modified couple stress theory is presented. Based on the derived equation, the propagating characteristics of a monochromatic bending elastic wave are studied. It is observed that the rate of twist bifurcates the spectrum curve of the micro scale beam within a given frequency range, while the material length scale improves the dispersion of the traveling wave. The cut-off frequency is found to be independent of the material length scale, but proportional to the fourth power of the rate of twist. Increasing the material length scale is further observed to increase the group velocity of the wave, while a high rate of twist lowers the wave speeds. It was also observed that for the micro-scale geometry, unlike in the classical EB beam, the group velocity is more than twice the phase speed for all values of the wavenumbers determined.

Furthermore, the free transverse vibration problem of an axially loaded non-uniform nano-sized beam is investigated based on the Eringen's nonlocal elasticity

theory. A polynomial variation of the mass distribution, flexural stiffness and the rotary inertia is assumed along with a power law that determines the nature of the non-uniformity. The effect of an elastic surrounding medium modeled as an elastic foundation of the Pasternak-type is included in the mathematical model. With the polynomial power law, an inverse relationship is observed between the natural frequency parameters and the radius of gyration of the nano-sized beam. The natural frequency for the first mode of the nano-sized beam when $n = 1$ is higher than its corresponding natural frequency for the first mode when $n = 2$, but the two frequencies are higher than that of the corresponding uniform nano-sized beam. Based on the observation from the numerous analyses in this chapter, it is confirmed that the classical theory of elasticity theory needs refinement for it to be applicable to the prediction of small-scale structures.

CHAPTER 6 MODELS OF SMALL-SCALE STRUCTURES AND THE DYNAMICS OF MICRO END MILLS

6.1 Introduction

This chapter discusses the extension of the theoretical models presented in chapter 5 to the study of the vibration characteristics of micro-cutting tools (micro end mill in particular). As a result of its detrimental effect on tool life and product quality, vibration analyses are crucial if the full potential of micro-milling operation is to be attained. In this chapter, a hybrid analytical model (HAM) for estimating the transverse response of a micro end mill will be presented. This so-called HAM will be a combination of discrete and distributed structural elements. On the one hand, the discrete elements will account for the stiffness and the damping coefficients of the machining system. On the other hand, the distributed elements will, as done in chapter 4, idealize the geometrical representation of the micro end mill with some novel mathematical models.

The interest in providing a framework for predicting the mechanical behavior of micro end mill is, as highlighted in chapters 2 and 3, a result of the fact that miniaturization is fast becoming a central theme in the contemporary fabrication technology [43, 248]. The capability to fabricate three-dimensional micron features on a wide variety of materials makes micro end milling a realistic and suitable micro-manufacturing platform [44]. However, insufficient understanding of the mechanism responsible for unstable micro tool behaviors, as pointed out by Masuzawa and Tönshoff [45] and Jun et al. [26], hampers progress in micromachining. Tool vibrations, either in

micro or macro cutting operations, can lead to a number of disappointingly unacceptable consequences. Prominent among these unwanted consequences are [9, 46-49]:

- (i) The systematic degradation of the quality of the desired features or machined surfaces,
- (ii) Impairment of expected tolerances
- (iii) Dynamic instability resulting from chip size modification, and
- (iv) Shortening of tool life and deterioration of productivity

Besides, the determination of regions of stability in most machining processes depends on many factors that range from the cutting condition to the dynamics of the machine spindle-tool system [10, 50]. As elaborately discussed by Shaw [33], the best choice of cutting parameters is prone to the influence of the structural characteristics of the cutting tool. It is a result of the above-mentioned reasons that studies to understand the behavior of the conventional tools have been undertaken to pave way for a more informed behavior of the machine stability for years [30, 50]. Micromachining differs, in concept and precept, from the traditional metal removal processes as widely reported in the literature [51-53]. As a result of the difference, the tool vibration effect coupled with unique process-induced factors pertaining to the micro domain raises even more concern about process stability and feature accuracy. In micro end milling, which is the focus of this study, the stress distribution on the tiny tip of the micro end mill is disproportionately larger than that of the traditional macro-sized end mill and a high speed spindle is required for a productive operation [9]. Furthermore, the edge radius of the micro end mill and uncut chip thickness approach each other in size, leading to a negative rake

angle [41]. Additionally, tool life and productivity show explicit dependence on a cautiously chosen ratio of feed rate to tool radius (f_t/r) [39-40]. From the unavoidable tool-process interaction enumerated above, it is expected that the geometry of a micro end mill would ultimately contribute to the inputs and outputs of the micro end milling process. In this chapter, a re-formulated novel hybrid analytical model is presented to study the vibration response of a micro end milling system, with the micro end mill as the main focus. As already stated and done in chapter 4, the proposed model treats a micro end mill as a stepped distributed dynamic system.

6.2 Re-visiting of the analytical model of micro end mill

As it would be recalled, chapter 5 involves the use of nonlocal, non-classical elastic theory of solids to formulate the governing equations of some structural elements with non-trivial geometries and cross-sections. The structural analyses detailed in chapter 5 were focused on as a result of their specific relevance to the modeling of micro-scale cutting tool. Of course, those structural elements have applications beyond the modeling of micro-scale cutting tool. However, for the purpose of the defined research objectives, the developed models, based on the nonlocal, non-classical elastic theory of solids will be re-visited here and extended to the modeling of the micro end mill (a versatile micro-cutting tool). A collection of micro end mills along with the geometric sub-structuring of the micro end mill is shown in Figure 6-1. One of the micro end mills (pointed to by a red arrow) in preparation for a routine SEM observation is shown in Figure 6-1(b).

Figure 6-1 Depiction of micro-scale cutting tool, (a) a collection of tungsten carbide micro end mills, (b) a micro end mill in pre-SEM measurement procedure, (c) physical sub-structuring of the micro end mill

As earlier stated, to be able to capture the response of the micro end mill along its entire length, the micro end mill is modeled as a stepped distributed dynamic system. This modeling approach leads to the spatial sub-structuring of the micro end mill into three different geometrically distinct parts as shown in Figure 6-1(c). The sub-structured parts are: (a) the shank, (b) the tapered neck, and (c) the micro flute. With the spatial sub-structuring of the micro end mill into the geometrically distinct parts, the governing equations for the transverse vibration of each of these parts are re-visited and presented in the framework of the non-local elasticity theory in the next sub-subsection.

6.2.1 Updated model of the shank

The new mathematical model of the shank, modeled as a spinning uniform shear deformable beam, based on the modified couple stress (MCST) theoretical framework; a nonlocal, non-classical and higher-order elasticity theory are given as:

$$-\rho A \left(\frac{\partial^2 u}{\partial t^2} \right) + kAG \left(\frac{\partial^2 u}{\partial z^2} - \frac{\partial \varphi_y}{\partial z} \right) - \frac{1}{4} G(l_s)^2 A \left(\frac{\partial^3 \varphi_y}{\partial z^3} + \frac{\partial^4 u}{\partial z^4} \right) = 0 \quad (6.1)$$

$$\begin{aligned} & \frac{E(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{xx} \left(\frac{\partial^2 \varphi_y}{\partial z^2} \right) + kAG \left(\frac{\partial u}{\partial z} - \varphi_y \right) - \rho I_{xx} \frac{\partial^2 \varphi_y}{\partial t^2} - \rho I_{yy} \Omega \frac{\partial \varphi_x}{\partial t} + \\ & \frac{1}{4} G(l_s)^2 A \left(\frac{\partial^2 \varphi_y}{\partial z^2} + \frac{\partial^3 u}{\partial z^3} \right) = 0 \end{aligned} \quad (6.2)$$

$$-\rho A \left(\frac{\partial^2 v}{\partial t^2} \right) + kAG \left(\frac{\partial^2 v}{\partial z^2} - \frac{\partial \varphi_x}{\partial z} \right) + \frac{1}{4} G(l_s)^2 A \left(\frac{\partial^3 \varphi_x}{\partial z^3} + \frac{\partial^4 v}{\partial z^4} \right) = 0 \quad (6.3)$$

$$\begin{aligned} & \frac{E(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} I_{yy} \left(\frac{\partial^2 \varphi_x}{\partial z^2} \right) + kAG \left(\frac{\partial v}{\partial z} - \varphi_x \right) - \rho I_{yy} \frac{\partial^2 \varphi_x}{\partial t^2} + \rho I_{xx} \Omega \frac{\partial \varphi_y}{\partial t} + \\ & \frac{1}{4} G(l_s)^2 A \left(\frac{\partial^2 \varphi_x}{\partial z^2} + \frac{\partial^3 v}{\partial z^3} \right) = 0 \end{aligned} \quad (6.4)$$

6.2.2 Updated model of the tapered neck of the micro end mill

The tapered neck of the micro end mill is idealized as a spinning beam with a taper cross-section. The governing equation of a microstructure-dependent model of a spinning micro-scale beam with a taper cross-section is a set of variable coefficient partial differential equations given as:

$$-\rho A \left(\frac{\partial^2 u}{\partial t^2} \right) + kG \frac{\partial}{\partial z} \left[A \left(\frac{\partial u}{\partial z} - \varphi_y \right) \right] - \frac{1}{4} G(l_s)^2 \frac{\partial}{\partial z} \left[A \left(\frac{\partial^2 \varphi_y}{\partial z^2} + \frac{\partial^3 u}{\partial z^3} \right) \right] = 0 \quad (6.5)$$

$$\begin{aligned} & \frac{E(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} \frac{\partial}{\partial z} \left[I_{xx} \left(\frac{\partial \varphi_y}{\partial z} \right) \right] + kAG \left(\frac{\partial u}{\partial z} - \varphi_y \right) - \rho I_{xx} \frac{\partial^2 \varphi_y}{\partial t^2} - \rho I_{yy} \Omega \frac{\partial \varphi_x}{\partial t} + \\ & \frac{1}{4} G(l_s)^2 \frac{\partial}{\partial z} \left[A \left(\frac{\partial \varphi_y}{\partial z} + \frac{\partial^2 u}{\partial z^2} \right) \right] = 0 \end{aligned} \quad (6.6)$$

$$-\rho A \left(\frac{\partial^2 v}{\partial t^2} \right) + kG \frac{\partial}{\partial z} \left[A \left(\frac{\partial v}{\partial z} - \varphi_x \right) \right] + \frac{1}{4} G(l_s)^2 \frac{\partial}{\partial z} \left[A \left(\frac{\partial^2 \varphi_x}{\partial z^2} + \frac{\partial^3 v}{\partial z^3} \right) \right] = 0 \quad (6.7)$$

$$\begin{aligned} & \frac{E(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)} \frac{\partial}{\partial z} \left[I_{yy} \left(\frac{\partial \varphi_x}{\partial z} \right) \right] + kAG \left(\frac{\partial v}{\partial z} - \varphi_x \right) - \rho I_{yy} \frac{\partial^2 \varphi_x}{\partial t^2} + \rho I_{xx} \Omega \frac{\partial \varphi_y}{\partial t} + \\ & \frac{1}{4} G(l_s)^2 \frac{\partial}{\partial z} \left[A \left(\frac{\partial \varphi_x}{\partial z} + \frac{\partial^2 v}{\partial z^2} \right) \right] = 0 \end{aligned} \quad (6.8)$$

6.2.3 Updated model of the micro flute

A re-derived mathematical model of the micro flute based on the framework of a microstructure-dependent model of a twisted micro-scale beam, satisfying the Timoshenko shear deformable criterion is given here. Again, the re-derived governing equations, given by below, are based on the local coordinates system:

$$-\mu_3 \left(\frac{\partial^2 w_3}{\partial t^2} \right) + k_{Z3} A_3 G \frac{\partial^2 w_3}{\partial Y^2} - \gamma^2 k_{X3} A_3 G w_3 + \gamma A_3 G (k_{Z3} + k_{X3}) \frac{\partial u_3}{\partial Y} + k_{Z3} A_3 G \frac{\partial \psi_3}{\partial Y} - \gamma k_{X3} A_3 G \theta_3 + \mu_3 \Omega^2 w_3 + 2\mu_3 \Omega \frac{\partial u_3}{\partial t} + Y_1 = 0 \quad (6.9)$$

$$-J_{Z3} \left(\frac{\partial^2 \theta_3}{\partial t^2} \right) + E \vartheta_r I_{Z3} \frac{\partial^2 \theta_3}{\partial Y^2} - \gamma^2 E \vartheta_r I_{X3} \theta_3 - k_{X3} A_3 G \theta_3 - \gamma k_{X3} A_3 G w_3 - k_{X3} A_3 G \frac{\partial u_3}{\partial Y} - \gamma \vartheta_r (E I_{Z3} + E I_{X3}) \frac{\partial \psi_3}{\partial Y} + J_{X3} \Omega^2 \theta_3 - \Omega (J_{X3} + J_{Z3}) \frac{\partial \psi_3}{\partial t} + Y_2 = 0 \quad (6.10)$$

$$-\mu_3 \left(\frac{\partial^2 u_3}{\partial t^2} \right) + k_{X3} A_3 G \frac{\partial^2 u_3}{\partial Y^2} - \gamma^2 k_{Z3} A_3 G u_3 - \gamma A_3 G (k_{Z3} + k_{X3}) \frac{\partial w_3}{\partial Y} - k_{X3} A_3 G \frac{\partial \theta_3}{\partial Y} - \gamma k_{Z3} A_3 G \psi_3 + \mu_3 \Omega^2 u_3 - 2\mu_3 \Omega \frac{\partial w_3}{\partial t} + Y_3 = 0 \quad (6.11)$$

$$-J_{X3} \left(\frac{\partial^2 \psi_3}{\partial t^2} \right) + E \vartheta_r I_{X3} \frac{\partial^2 \psi_3}{\partial Y^2} - \gamma^2 E \vartheta_r I_{Z3} \psi_3 - k_{Z3} A_3 G \psi_3 - \gamma k_{Z3} A_3 G u_3 + k_{Z3} A_3 G \frac{\partial w_3}{\partial Y} + \gamma \vartheta_r (E I_{Z3} + E I_{X3}) \frac{\partial \theta_3}{\partial Y} + J_{Z3} \Omega^2 \psi_3 + \Omega (J_{X3} + J_{Z3}) \frac{\partial \theta_3}{\partial t} + Y_4 = 0 \quad (6.12)$$

where the terms ϑ_r and $Y_i (i = 1 \dots 4)$ in Eqs. (6.9) – (6.12) are defined below:

$$\vartheta_r = \frac{E(1 - \vartheta)}{(1 + \vartheta)(1 - 2\vartheta)}$$

$$Y_1 = \frac{GA(l_s)^2}{4} \left[-\frac{\partial^3 \psi_3}{\partial Y^3} - \frac{\partial^4 w_3}{\partial Y^4} - \gamma \frac{\partial^2 u_3}{\partial Y^2} - \gamma \frac{\partial^2 \theta_3}{\partial Y^2} + \gamma^2 \frac{\partial^2 w_3}{\partial Y^2} - \gamma \frac{\partial^3 u_3}{\partial Y^3} \right]$$

$$Y_2 = \frac{GA(l_s)^2}{4} \left[\frac{\partial^2 \theta_3}{\partial Y^2} + \frac{\partial^3 w_3}{\partial Y^3} + \gamma \frac{\partial^2 u_3}{\partial Y^2} \right]$$

$$Y_3 = \frac{GA(l_s)^2}{4} \left[-\frac{\partial^3 \theta_3}{\partial Y^3} - \frac{\partial^4 u_3}{\partial Y^4} + \gamma \frac{\partial^2 w_3}{\partial Y^2} - \gamma \frac{\partial^2 \psi_3}{\partial Y^2} + \gamma^2 \frac{\partial^2 u_3}{\partial Y^2} - \gamma \frac{\partial^3 w_3}{\partial Y^3} \right]$$

$$Y_4 = \frac{GA(l_s)^2}{4} \left[\frac{\partial^2 \psi_3}{\partial Y^2} + \frac{\partial^3 u_3}{\partial Y^3} - \gamma \frac{\partial^2 w_3}{\partial Y^2} \right]$$

In order to be able to simulate the transverse vibration of the micro end milling operation, three steps are required. The first step is to have a reasonably accurate model of the cutting force(s) experience by the micro end mill in the course of micro end milling operation. The second step is determining the mass, the stiffness and the damping coefficient of the machine employed for the micro end milling operation. Due to their

nature, the mass, stiffness and damping coefficient of the machine would have to be represented by discrete elements. The third step is to combine the model of cutting force(s), the discrete elements of the mass, stiffness and damping coefficient of the machine, and the continuum model of the micro end mill to establish the basis of the vibration analysis. The flow chart of the steps involved in the simulation is shown in Figure 6-2. From Figure 6-2, it would be observed the spectral element method (SEM) is the basic solution methodology adopted in solving the hybrid dynamics model required to simulate the vibration response of the micro end milling operation. Without repeating the elaborate steps involved in the solution phase of the hybrid dynamic model, the dynamic stiffness matrices $S_1(\omega)$, $S_2(\omega)$ and $S_3(\omega)$ representing the dynamic stiffness matrix of the shank, the tapered neck and the micro flute of the micro end mill are given below:

$$\begin{aligned}
S_1(\omega) = & \int_0^1 E \vartheta_r I_1 \left(\left\{ \frac{dN_\theta}{d\xi} \right\}^T \left\{ \frac{dN_\theta}{d\xi} \right\} + \left\{ \frac{dN_\psi}{d\xi} \right\}^T \left\{ \frac{dN_\psi}{d\xi} \right\} \right) d\xi \\
& + \int_0^1 k_1 G A_1 \left(\left\{ \frac{dN_v}{d\xi} \right\}^T \left\{ \frac{dN_v}{d\xi} \right\} + \left\{ \frac{dN_u}{d\xi} \right\}^T \left\{ \frac{dN_u}{d\xi} \right\} \right) d\xi \\
& + \int_0^1 k_1 G A_1 \left(\left\{ \frac{dN_v}{d\xi} \right\}^T \{N_\theta\} + \{N_\theta\}^T \left\{ \frac{dN_v}{d\xi} \right\} \right) d\xi \\
& + \int_0^1 k_1 G A_1 \left(\left\{ \frac{dN_u}{d\xi} \right\}^T \{N_\psi\} + \{N_\psi\}^T \left\{ \frac{dN_u}{d\xi} \right\} \right) d\xi \\
& + \int_0^1 k_1 G A_1 \left(\{N_\theta\}^T \{N_\theta\} + \{N_\psi\}^T \{N_\psi\} \right) d\xi \\
& - \int_0^1 \mu_1 \omega^2 (\{N_v\}^T \{N_v\} + \{N_u\}^T \{N_u\}) d\xi \\
& - \int_0^1 \rho I_1 \omega^2 (\{N_\theta\}^T \{N_\theta\} + \{N_\psi\}^T \{N_\psi\}) d\xi \\
& + \int_0^1 i 2 \Omega \rho I_1 \omega (\{N_\theta\}^T \{N_\psi\} - \{N_\psi\}^T \{N_\theta\}) d\xi + P_2
\end{aligned} \tag{6.13a}$$

Figure 6-2 Flow chart of the solution approach for the hybrid analytical model

where the expression for $P2$ in Eq. (6.13) is given below as:

$$\begin{aligned}
 P2 = 1/4(l_s)^2 AG \left[\int_0^1 \left(\left\{ \frac{d^2 N_v}{d\xi^2} \right\}^T \left\{ \frac{d^2 N_v}{d\xi^2} \right\} + \left\{ \frac{d^2 N_u}{d\xi^2} \right\}^T \left\{ \frac{d^2 N_u}{d\xi^2} \right\} \right) d\xi \right. \\
 - \int_0^1 \left(\left\{ \frac{d^2 N_\theta}{d\xi^2} \right\}^T \left\{ \frac{dN_\theta}{d\xi} \right\} + \left\{ \frac{d^2 N_\psi}{d\xi^2} \right\}^T \left\{ \frac{dN_\psi}{d\xi} \right\} \right) d\xi \\
 + \int_0^1 \left(\left\{ \frac{dN_\theta}{d\xi} \right\}^T \left\{ \frac{dN_\theta}{d\xi} \right\} + \left\{ \frac{dN_\psi}{d\xi} \right\}^T \left\{ \frac{dN_\psi}{d\xi} \right\} \right) d\xi \\
 \left. - \int_0^1 \left(\left\{ \frac{d^2 N_u}{d\xi^2} \right\}^T \left\{ \frac{dN_u}{d\xi} \right\} + \left\{ \frac{d^2 N_v}{d\xi^2} \right\}^T \left\{ \frac{dN_v}{d\xi} \right\} \right) d\xi \right]
 \end{aligned} \tag{6.13b}$$

$S_2(\omega) =$

$$\begin{aligned}
 \int_0^1 \left(E \vartheta_r I_{2z}(\xi) \left\{ \frac{dN_\theta}{d\xi} \right\}^T \left\{ \frac{dN_\theta}{d\xi} \right\} + E \vartheta_r I_{2x}(\xi) \left\{ \frac{dN_\psi}{d\xi} \right\}^T \left\{ \frac{dN_\psi}{d\xi} \right\} \right) d\xi \\
 + \int_0^1 k_2 G A_{2z}(\xi) \left(\left\{ \frac{dN_v}{d\xi} \right\}^T \left\{ \frac{dN_v}{d\xi} \right\} + k_2 G A_{2x}(\xi) \left\{ \frac{dN_u}{d\xi} \right\}^T \left\{ \frac{dN_u}{d\xi} \right\} \right) d\xi \\
 + \int_0^1 \left(k_2 G A_{2z}(\xi) \left\{ \frac{dN_v}{d\xi} \right\}^T \{N_\theta\} + k_2 G A_{2z}(\xi) \{N_\theta\}^T \left\{ \frac{dN_v}{d\xi} \right\} \right) d\xi \\
 + \int_0^1 \left(k_2 G A_{2x}(\xi) \left\{ \frac{dN_u}{d\xi} \right\}^T \{N_\psi\} + k_2 G A_{2x}(\xi) \{N_\psi\}^T \left\{ \frac{dN_u}{d\xi} \right\} \right) d\xi + P3
 \end{aligned} \tag{6.14a}$$

where the expression for $P3$ is given below as:

$$\begin{aligned}
 P3 = & \int_0^1 \left(k_2 G A_{2z} \{N_\theta\}^T \{N_\theta\} + k_2 G A_{2x} \{N_\psi\}^T \{N_\psi\} \right) d\xi \\
 & - \int_0^1 (\mu_{2z} \omega^2 \{N_v\}^T \{N_v\} + \mu_{2x} \omega^2 \{N_u\}^T \{N_u\}) d\xi \\
 & - \int_0^1 \rho I_2(\xi) \omega^2 \left(\{N_\theta\}^T \{N_\theta\} + \{N_\psi\}^T \{N_\psi\} \right) d\xi \\
 & + \int_0^1 i 2 \Omega \rho I_2(\xi) \omega \left(\{N_\theta\}^T \{N_\psi\} - \{N_\psi\}^T \{N_\theta\} \right) d\xi \\
 & + \frac{(l_s)^2 A(\xi) G}{4} \left[\int_0^1 \left(\left\{ \frac{d^2 N_v}{d\xi^2} \right\}^T \left\{ \frac{d^2 N_v}{d\xi^2} \right\} + \left\{ \frac{d^2 N_u}{d\xi^2} \right\}^T \left\{ \frac{d^2 N_u}{d\xi^2} \right\} \right) d\xi \right. \\
 & - \int_0^1 \left(\left\{ \frac{d^2 N_\theta}{d\xi^2} \right\}^T \left\{ \frac{d N_\theta}{d\xi} \right\} + \left\{ \frac{d^2 N_\psi}{d\xi^2} \right\}^T \left\{ \frac{d N_\psi}{d\xi} \right\} \right) d\xi \\
 & + \int_0^1 \left(\left\{ \frac{d N_\theta}{d\xi} \right\}^T \left\{ \frac{d N_\theta}{d\xi} \right\} + \left\{ \frac{d N_\psi}{d\xi} \right\}^T \left\{ \frac{d N_\psi}{d\xi} \right\} \right) d\xi \\
 & \left. - \int_0^1 \left(\left\{ \frac{d^2 N_u}{d\xi^2} \right\}^T \left\{ \frac{d N_u}{d\xi} \right\} + \left\{ \frac{d^2 N_v}{d\xi^2} \right\}^T \left\{ \frac{d N_v}{d\xi} \right\} \right) d\xi \right]
 \end{aligned} \tag{6.14b}$$

$$\text{where } \vartheta_r = \frac{(1-\vartheta)}{(1+\vartheta)(1-2\vartheta)}$$

$$\begin{aligned}
 S_3(\omega) = & \int_0^1 \left(\left[k_{Z3} A_3 G \left\{ \frac{dF_w}{d\xi} \right\}^T \left\{ \frac{dF_w}{d\xi} \right\} - (\gamma^2 k_{X3} A_3 G - \mu_3 \omega^2 - \mu_3 \Omega^2) \{F_w\}^T \{F_w\} + \right. \right. \\
 & \gamma A_3 G (k_{Z3} + k_{X3}) \{F_w\}^T \left\{ \frac{dF_u}{d\xi} \right\} + k_{Z3} A_3 G \{F_w\}^T \left\{ \frac{dF_\psi}{d\xi} \right\} - \gamma k_{X3} A_3 G \{F_w\}^T \{F_\theta\} + \\
 & \left. i 2 \mu_3 \omega \Omega \{F_w\}^T \{F_u\} \right] +
 \end{aligned}$$

$$\begin{aligned}
& \left[EI_{Z3} \left\{ \frac{dF_\theta}{d\xi} \right\}^T \left\{ \frac{dF_\theta}{d\xi} \right\} - (\gamma^2 EI_{X3} + k_{X3} A_3 G + J_{Z3} \omega^2 + J_{X3} \Omega^2) \{F_\theta\}^T \{F_\theta\} - \right. \\
& \gamma k_{X3} A_3 G \{F_\theta\}^T \{F_w\} - k_{X3} A_3 G \{F_\theta\}^T \left\{ \frac{dF_u}{d\xi} \right\} - \gamma (EI_{Z3} + EI_{X3}) \{F_\theta\}^T \left\{ \frac{dF_\psi}{d\xi} \right\} - i2\omega\Omega (J_{X3} + \\
& J_{Z3}) \{F_\theta\}^T \{F_\psi\} \left. \right] + \left[k_{X3} A_3 G \left\{ \frac{dF_u}{d\xi} \right\}^T \left\{ \frac{dF_u}{d\xi} \right\} - (\gamma^2 k_{Z3} A_3 - G\mu_3 \omega^2 - \mu_3 \Omega^2) \{F_u\}^T \{F_u\} - \right. \\
& \gamma A_3 G (k_{Z3} + k_{X3}) \{F_u\}^T \left\{ \frac{dF_w}{d\xi} \right\} - k_{X3} A_3 G \{F_u\}^T \left\{ \frac{dF_\theta}{d\xi} \right\} - \gamma k_{Z3} A_3 G \{F_u\}^T \{F_\psi\} - \\
& i2\mu_3 \omega \Omega \{F_u\}^T \{F_w\} \left. \right] + \\
& \left[EI_{X3} \left\{ \frac{dF_\psi}{d\xi} \right\}^T \left\{ \frac{dF_\psi}{d\xi} \right\} - (\gamma^2 EI_{Z3} + k_{Z3} A_3 - GJ_{X3} \omega^2 - J_{Z3} \Omega^2) \{F_\psi\}^T \{F_\psi\} - \right. \\
& \gamma k_{Z3} A_3 G \{F_\psi\}^T \{F_u\} + k_{Z3} A_3 G \{F_\psi\}^T \left\{ \frac{dF_w}{d\xi} \right\} + \gamma (EI_{Z3} + EI_{X3}) \{F_\psi\}^T \left\{ \frac{dF_\theta}{d\xi} \right\} + i2\omega\Omega (J_{X3} + \\
& J_{Z3}) \{F_\psi\}^T \{F_\theta\} \left. \right] \quad (6.15)
\end{aligned}$$

where N_θ, N_w, N_u , and N_ψ in Eqs. (6.13) and (6.15) are, as earlier explained, the frequency-dependent interpolating functions. These interpolating functions are obtained from the exact solution of the governing differential equations of the spinning uniform Timoshenko beam. Also, F_θ, F_w, F_u , and F_ψ are the frequency-dependent shape functions derived from the exact solution of the taper-free twisted shear deformable beam. Now, for the three dynamic matrices to be easily added, the dynamic stiffness matrix of the micro flute $S_3(\omega)$ (obtained in the principal coordinates) has to be transformed into the inertia frame as:

$$[S_3^{tr}(\omega)] = [T]'[S_3(\omega)][T] \quad (6.16)$$

where $S_3^{tr}(\omega)$ is now the transformed local dynamic matrix of the flute. For all the three basic sub-structures, the dynamic stiffness matrix is an 8×8 frequency dependent matrix

that is useful for the determination of each sub-structure's response. The full form of the transformation matrix $\mathbf{T}$ is provided in the Appendix. Before the integration of the equations is carried out, the taper parameters have to be quantified for the tapered neck of the micro end mill. Besides, from the careful examination of the micro end mill, it is noticed that the micro flute is also slightly tapered along its length. Therefore, the micro flute is eventually idealized as tapered, twisted Timoshenko beam. The taperedness in these two geometries is handled with the polynomial power law defined in Eqs. (6.17) and (6.18) below.

$$A_{2Y} = A_1 \left(1 - \epsilon_z \frac{y}{L_2}\right)^c \left(1 - \epsilon_x \frac{y}{L_2}\right)^c, \quad I_{2Z} = I_1 \left(1 - \epsilon_z \frac{y}{L_2}\right)^{3c}, \quad I_{2X} = I_1 \left(1 - \epsilon_x \frac{y}{L_2}\right)^{3c} \quad (6.17)$$

$$A_{3Y} = A_2 \left(1 - f_Z \frac{Y}{L_3}\right)^c \left(1 - f_X \frac{Y}{L_3}\right)^c, \quad I_{3Z} = I_2 \left(1 - f_Z \frac{Y}{L_3}\right)^{3c}, \quad I_{3X} = I_2 \left(1 - f_X \frac{Y}{L_3}\right)^{3c} \quad (6.18)$$

where A_{2Y} is the variation of the area of the tapered neck, while I_{2Z} and I_{2X} are the second moment of inertias. ϵ_x and ϵ_z are the taper ratios in both x and z directions. A_1 and I_1 denote the values of the cross-sectional area and second moment of inertia at the left side of the tapered section. It is important to point out that the left side of the tapered section side coincides with the end of the shank. Also, I_{3Z} and I_{3X} are the variations of the second moment of area of the tapered flute in the principal planes and f_Z and f_X are the taper ratios in those planes. According to [152], the term c usually takes the values that depend on the nature of the taper. The observed taper geometry of the micro end mill reveals a profile that matches that of a linear tapered configuration, hence superscript c in Eq. (6.17) and (6.18) is taken as 1. The compactness of the dynamic

stiffness matrix, yielded by the SEM, provides the chance to resort to the usual assemblage procedures of the traditional FEM. As a result, the different independent spectral characteristics matrix for each segment needs to be added to form the global spectral characteristics matrix of the system as:

$$\{f_g\} = [S_g(\omega)]\{u_g\} \quad (6.19)$$

where Eq. (6.19) is now the global dynamic stiffness equation. f_g , S_g and u_g are the global force vector, the global dynamic stiffness matrix and the global displacement vector respectively. In general, the formation of the global dynamic stiffness matrix is not just a simple addition of the three dynamic stiffness matrices for the three segments, the appropriate continuity conditions must be taken into account. The global spectral characteristics matrix helps to determine the response of the entire multi-step beam. With the addition done, the response of the micro end mill and the supporting bearings can be obtained with the necessary conversion of the frequency domain response to the time domain response, after the application of the appropriate boundary conditions and the cutting force.

6.3 Vibration response during experimental micro-milling processes

It is almost a well-established consensus that the mechanics and dynamics of micro-cutting operations have to be understood, empirically or analytically, in order to understand process behaviors in advance of expensive experimental trials. The stability, vibration and balancing of micro-scale tools are matters of great practical importance that are better understood through system modeling. In their conventional size, cutting tools

have been the subject of numerous experimental and theoretical studies [54-56]. However, the geometric modeling of micro tools, in particular micro end mills, remains inchoate [57, 249-250].

Although details of the many variations of the experimental tests carried out are reported in refs. [251-252], attempt is made to in this section to provide some explanation for a selected number of the slot micro end milling experiments carried out. These slot micro end milling experiments are used to acquire the signal data set needed for the validation of the forced response of the micro end mill. The machining experiments were carried out at the NTU's Manufacturing Processes Laboratory 1. The set-up of the experiment for the slot micro end milling operation comprises two parts, namely the hardware and the software parts. The hardware set-up involves (i) The machine tool (ii) a miniature tri-axial accelerometer (iii) DAQ board (NI PCI-6289 M series) and (iv) micro end mills. The machine tool employed in the experiment is the highly-rigid Mazak Variaxis 500-5X series shown in Figure 6-3. The *Mazak Variaxis* machine tool is a CNC machine with a capability for a simultaneous 5-axis control. The accelerometer, with the manufacturer's specified sensitivity of 200 mv/g and a good linearity over a wide dynamic range, is used. The accelerometer is attached to the polished workpiece' surface. The polishing ensures a flat clean surface for better sensor performance. The software setup also consists of two parts. The first part is the coding of the CNC machine for each of the experiments. The coding of the CNC machine is done via an interface that allows the supply of the experimental tool path as well as the cutting variables listed in Table 6-3. The second aspect of the software set-up is the data acquisition. The software used for the data capturing on the desktop PC is DASyLab. DASyLab, just like LabVIEW

provides a basic graphical programming environment that can be easily set-up to interface conveniently with the NI-DAQ board. It is also packed with a number of signal filtering algorithms. The signal data are stored in both binary and text file formats for post-processing in MATLAB. The reported machining experimental tests here are all dry machining. Dry machining is adopted to eliminate, among other side effects, the potential contribution of the noise from the coolant supply nozzle to the desired signal to be captured.

The slot micro end milling experiments are all carefully designed with the Taguchi method of design of experiments to examine the accuracy of the model. The Taguchi method helps to streamline the experiments by providing a framework that facilitates the design of fewer experimental to provide a good a much better understanding of the tool dynamics due to the varying nature of the cutting parameters. After each run, the machined slot is thoroughly sprayed with an air gun before the next experimental run is conducted to ensure the chips are cleared and debris is not clogged onto the workpiece which may affect the tool behavior.

The workpiece material involved in all of the presented experiments is Aluminum 6061-T6 with its physical properties given in Table 6-1. The machining experiments involved a number of set-ups, but one of the basic experimental set-ups is shown in Figure 6-4, while the schematic of the data acquisition system is depicted in Figure 6-5. The cutting conditions that were subjected to variations during the experiments are depth of cut, spindle speed and the feed rate per tooth of the micro end mill.

Figure 6-3 Experimental machine tool (*Mazak Variaxis 5X series*)

Figure 6-4 Set-up of workpiece material, machine fixture and sensors

Figure 6-5 A schematic of the experimental and data acquisition set-up

Table 6-1 Physical properties of the workpiece material

Physical properties	
Density	$2.7 \text{ (kg/dm}^3\text{)}$
Melting point	650°C
Coefficient of thermal expansion	$23.3 \mu\text{m}^{-1}\text{K}^{-1}$
Thermal conductivity	$166 \text{ Wm}^{-1}\text{K}^{-1}$
Specific heat capacity	$895 \text{ Jkg}^{-1}\text{K}^{-1}$
Electrical conductivity	$43\% \text{ IACS}$

Figure 6-5 shows the workpiece-micro end mill-spindle assembly. The model of the machine is simplified by representing the spindle assembly with a set of isotropic bearings. Two pairs of isotropic identical bearings are involved in the HAM. The experimentally determined stiffness of the isotropic bearing in the xy plane is $1.856 \times 10^7 \text{ N/m}$ and that of the bearing in the zy plane is $1.418 \times 10^7 \text{ N/m}$. The damping coefficient in the two planes, xy and xz , are $354.45 \times 10^{-2} \text{ Ns/cm}$ and $386.12 \times 10^{-2} \text{ Ns/m}$ respectively. The geometric properties of the micro end mill (with two flutes)

used are shown in Table 6-2. The magnitude of the small-scale coefficient used is based on the definition given in chapter 5. In addition, the spindle is made of a material with Young's modulus of 207 GPa , mass density of $7.9 \times 10^3 \text{ kg/cm}^3$.

Table 6-2 Geometric properties of the micro end mill sub-structures

Sub-structures	Geometric properties
Shank	$k_1 = 0.90, d_1 = 4.92 \text{ mm}, L_1 = 30.85 \text{ mm}$
Tapered neck	$k_2 = 0.90, \epsilon_z = \epsilon_x = 0.23, L_2 = 10 \text{ mm}, d_1/d_2 = 14$
Micro flute	length (L_3) = $2280 \text{ }\mu\text{m}$, thickness = $160 \text{ }\mu\text{m}$, width = $300 \text{ }\mu\text{m}$, $f_z = f_x = 0.48$, $\gamma = 60^\circ$, $k_{x3} = 0.90, k_{y3} = 0.90$

Various cutting conditions were explored through a number of experimental runs designed with the methodology of design of experiment (DOE) [224], with the stated material and geometric properties. Table 6-3 shows just 18 representative sets of experiments conducted on Aluminum 6061-T6. The DOE methodology helps to eliminate linear dependency of the experimental factors, which are the depth of cut, the spindle speed and the feed rate shown in Table 6-3. In order to have reliable signal data set, each of the experimental runs in Table 6-3 is conducted twice, thereby resulting in a total of 36 experimental runs. For each of the experiment the vibration digital signal are independently acquired. In order to remove transient effect, the first and the last 50 points of each acquired signal data set are removed. The removal of the first and the last 50 points of each acquired signal data set is based on the observation from Figure 6-6 where the Q-Q plots (quantile-quantile plots) of the signal from the first nine experimental runs revealed slight deviation mostly from the first few points and the last few points.

Table 6-3 Design of experiment for the slot micro end milling of aluminum 6061-T6

Experimental runs	depth of cut (mm)	spindle speed (rpm)	feedrate/per tooth (mm)
1	0.100	20000	0.01
2	0.200	20000	0.03
3	0.300	20000	0.05
4	0.100	22000	0.03
5	0.200	22000	0.05
6	0.300	22000	0.01
7	0.10	25000	0.05
8	0.200	25000	0.01
9	0.300	25000	0.03
10	0.048	20000	0.30
11	0.060	22000	0.30
12	0.096	24000	0.30
13	0.048	22000	0.50
14	0.060	24000	0.50
15	0.096	20000	0.50
16	0.048	24000	0.70
17	0.060	20000	0.70
18	0.096	22000	0.70

The sub-captions $a - i$ in Figures 6-6 represent experiment numbers 1 – 9. In each of the Q-Q plot, the red dotted and the blue starred lines depict the distribution of experiment A and B , respectively, for each of the experimental runs. The Q-Q plot, a versatile tool in exploratory data analysis, was initially advanced by Wilk and Gnanadesikan [253] to help in the visual comparison of two distributions. The comparison is generally done by having the quantiles of one sample plotted against the quantiles of the other.

(a) Quantile-Quantile plot of vectors of signal data of the first experiment (Vector 1: *****, Vector 2: - - - -)[‡]

Figure 6-6 Quantile-Quantile plots of the nine experimental data set (a)

[‡] For Figures 6-6(a) – 6.6(i) the stars (**) not lying on the straight lines represent signal points that deviate from the distribution of the measuring sample population.

(b) Quantile-Quantile plot of vectors of signal data of the second experiment (Vector 1: *****, Vector 2: - - -)

(c) Quantile-Quantile plot of vectors of the signal data of the third experiment (Vector 1: *****, Vector 2: - - -)

Figure 6-6 Quantile-Quantile plots of the nine experimental data set (b)-(c)

(d) Quantile-Quantile plot of vectors of the signal data of the fourth experiment (Vector 1: *****, Vector 2: - - - -)[§]

(e) Quantile-Quantile plot of vectors of the signal data of the fifth experiment (Vector 1: *****, Vector 2: - - - -)

Figure 6-6 Quantile-Quantile plots of the nine experimental data set (d)-(e)

[§] For Figures 6-6(a) – 6.6(i) the stars (**) not lying on the straight lines represent signal points that deviate from the distribution of the measuring sample population.

(f) Quantile-Quantile plot of vectors of signal data of the sixth experiment (Vector 1: *****, Vector 2: - - -)

(g) Quantile-Quantile plot of vectors of signal data of the seventh experiment (Vector 1: *****, Vector 2: - - -)

Figure 6-6 Quantile-Quantile plots of the nine experimental data set (f)-(g)

(h) Quantile-Quantile plot of vectors of signal data of the eighth experiment (Vector 1: *****, Vector 2: - - - -)

(i) Quantile-Quantile plot of vectors of signal data of the ninth experiment (Vector 1: *****, Vector 2: - - - -)**

Figure 6-6 Quantile-Quantile plots of the nine experimental data set (h)-(i)

** For Figures 6-6(a) – 6.6(i) the stars (**) not lying on the straight lines represent signal points that deviate from the distribution of the measuring sample population.

In statistical analysis, one of the other principal uses of the Q-Q plot is in establishing if the vector of features in a particular sample, say A , is drawn from a specific continuous or discrete distribution. It is employed to determine if two samples, say A or B or Y and X happens to be from the same distribution. In situations where the samples are derived from the same statistical distribution, even if it does happen that one distribution is a shifted and re-scaled form of the other, the Q-Q plot is generally found to be linear. Of course, it should be stated that the flexibility of the Q-Q plot allows that both or any one of the distributions between which comparisons is being made to be empirically or theoretically formulated.

Although the Q-Q plots for the repeated measures of the signal data for the nine experiments appear to support the assessment that the signal data at each experimental run are statistically likely to originate from the same distribution, it is possible to explore this further through the use of the conventional hypothesis test. In order to do this an appropriate statistic has to be chosen. A common statistic for analyzing repeated measures of this kind is the *t*-statistic of the t-test [254]. In order to employ t-tests, however, a variable has to be chosen. In this particular study, the aim is to show that two vectors of signal data acquired from an experiment performed under a similar cutting condition are statistically similar. Therefore, an appropriate variable around which the hypothesis test could be framed would be the mean of the difference between the signals for the respective experiment. Based on this premise, both the null and alternative hypotheses are defined as given below:

Null hypothesis: $H_0: \mu_D = 0$

Alternative hypothesis:

$$H_1: \mu_D \neq 0$$

where μ_D is the mean of the difference for each set of signal data for the respective experiment. The null hypothesis defined above states that there is an absence of a systematic difference between the two vectors of the signals of the repeated experiments for each level of the experimental run. That is, a systematic difference does not exist between, say, E_{11} (the first run of experiment number 1) and E_{12} (the second run of experiment number 1) etc. Of course, the hypothesis does not stipulate that every data point will match and thus have a zero difference, but rather that the mean of the differences should be zero or very close to zero. In Table 6-4, the results of the t-test on the samples are shown for the nine experiments. The tests were all run within the statistical package R.

In Table 6-4, the terms diffExp1, diffExp2, ..., diffExp9 represent the difference of the vectors of acquired signals for experiment number 1 to experiment number 9 respectively. The interpretation of the result of the hypothesis test is as follows. In each of the row corresponding to the experiment number, the hypothesis test result shows the values of (i) *p – value* (ii) *t* statistic (iii) 95% confidence interval (iv) degree of freedom (*df*) and (v) sample estimate of the mean. In each row, the magnitude of the *p – value* is observed to be large enough, at a 95% confidence interval, to make the null hypothesis acceptable. In line with this conclusion, it is also observed that the estimated mean (the last floating number in each row) of the difference of the vectors of signal data in each of the row is close to zero. Hence, the hypothesis test agrees with the analysis of the Q-Q plots earlier presented.

Table 6-4 Results of the hypothesis for the first four experiments (A)

Experiment	Hypothesis test result
1	One Sample t-test Data: <u>diffExp1</u> $t = -0.489$, $df = 1999$, $p\text{-value} = 0.8249$ alternative hypothesis: true mean is not equal to 0 95 percent confidence interval: -0.07787186 0.04679009 sample estimates: mean of <u>diffExp1</u> -0.001554088
2	One Sample t-test Data: <u>diffExp2</u> $t = 0.5229$, $df = 1999$, $p\text{-value} = 0.6011$ alternative hypothesis: true mean is not equal to 0 95 percent confidence interval: -0.04609788 0.07961408 sample estimates: mean of <u>diffExp2</u> 0.0167581
3	One Sample t-test Data: <u>diffExp3</u> $t = -0.2373$, $df = 1999$, $p\text{-value} = 0.8124$ alternative hypothesis: true mean is not equal to 0 95 percent confidence interval: -0.06962418 0.05459234 sample estimates: mean of <u>diffExp3</u> -0.00751592
4	One Sample t-test Data: <u>diffExp4</u> $t = -0.1779$, $df = 1999$, $p\text{-value} = 0.8588$ alternative hypothesis: true mean is not equal to 0 95 percent confidence interval: -0.06969937 0.05810416 sample estimates: mean of <u>diffExp4</u> -0.005797604

Table 6-4 Results of the hypothesis for the experiments (B)

Experiment	Hypothesis test result
5	One Sample t-test Data: <u>diffExp5</u> $t = -0.0655$, $df = 1999$, $p\text{-value} = 0.9478$ alternative hypothesis: true mean is not equal to 0: 95 percent confidence interval : -0.06409621 0.05995582 sample estimates : mean of <u>diffExp5</u> -0.002070194
6	One Sample t-test Data: <u>diffExp6</u> $t = -0.1741$, $df = 1999$, $p\text{-value} = 0.8618$ alternative hypothesis: true mean is not equal to 0: 95 percent confidence interval : -0.06781004 0.05675168 sample estimates : mean of <u>diffExp6</u> -0.005529178
7	One Sample t-test Data: <u>diffExp7</u> $t = 0.2299$, $df = 1999$, $p\text{-value} = 0.8182$ alternative hypothesis: true mean is not equal to 0: 95 percent confidence interval : -0.05322729 0.06736658 sample estimates : mean of <u>diffExp7</u> 0.007069647
8	One Sample t-test Data: <u>diffExp8</u> $t = -0.1993$, $df = 1999$, $p\text{-value} = 0.842$ alternative hypothesis: true mean is not equal to 0: 95 percent confidence interval : -0.06714505 0.05475681 sample estimates : mean of <u>diffExp8</u> -0.00619412
9	One Sample t-test Data: <u>diffExp9</u> $t = -0.177$, $df = 1999$, $p\text{-value} = 0.8595$ alternative hypothesis: true mean is not equal to 0: 95 percent confidence interval : -0.06836401 0.05704706 sample estimates : mean of <u>diffExp9</u> -0.005658478

Among the typical test of variation of samples in statistical analyses, the use of the concept of analysis of variance is common. However, it would be noted that an implicit assumption is made in the statistical hypothesis testing presented in Table 6-4, which does not rely explicitly on the traditional concept of variance. This assumption, which arises from the nature of repeated measurements analysis, is based on the premise that the statistical t-test could be used to evaluate the variation in two samples of a repeated measurement. With the t-test, a quantifiable hypothesis that bears semblance to the variation between (or among) the samples has to be defined. In the t-test analysis carried out and presented in Table 6-4, the hypothesis testing is defined through the mean of the difference between the samples.

6.3.1 The adopted cutting force model

The prediction of the HAM depends on the nature of the cutting force generated by the micro-milling operation. In this study, the cutting force model employed to supplement the analytical model of the micro end mill is based on the study of Bao and Tansel [40]. In a nutshell, the derivation of the e cutting force model is premised on the assumptions that: (i) the tangential cutting force is proportional to the cutting area, (ii) the radial cutting force is proportional to the tangential cutting force, and (iii) the cutting area is a product of the chip thickness and the radial depth-of-cut or width of the chip. The cutting force in most machining operation involving helical tools (like the micro end mill) is a function of chip load. Thus, the development of the adopted cutting force model follows from the coordinate system depicted in Figure 6-7. To derive the cutting force model, it is necessary, for the purpose of clarity to start from the classical definition of

the chip thickness (c_t) with respect to a cutting angle (φ_u) and feed per tooth (f_{pt}) given in [40]:

$$c_t = f_{pt} \sin \varphi_u \quad (6.20)$$

Figure 6-7 Schematic of the coordinate system for micro-milling process

Bearing in mind that for certain values of φ_u , the chip thickness would not be a constant, the expressions for the differential instantaneous tangential and radial cutting forces can be written as:

$$df_t = 2 \frac{M_c r f_{pt}}{\tan(h_\beta/2)} c_t(\varphi_u) d\varphi_u \quad (6.21)$$

$$df_r = 2 \frac{M_c r f_{pt}}{\tan(h_\beta/2)} p_f c_t(\varphi_u) d\varphi_u \quad (6.22)$$

where M_c is the material coefficient, r is the tool radius, h_β is the tool's helix angle and p_f is the proportional factor. From the expressions of the differential instantaneous tangential and radial cutting forces given above, the differential instantaneous cutting forces in the normal and feed directions are then expressed as:

$$dF_x = -df_t \cos \varphi_u - df_r \sin \varphi_u = -2 \left(\frac{M_c r f_{pt}}{\tan(h_\beta/2)} \right) c_t \varphi_u [\cos \varphi_u d\varphi_u + p \sin \varphi_u d\varphi_u] \quad (6.23)$$

$$dF_z = df_t \sin \varphi_u - df_r \cos \varphi_u = 2 \left(\frac{M_c r f_{pt}}{\tan(h_\beta/2)} \right) c_t \varphi_u [\sin \varphi_u d\varphi_u + p \cos \varphi_u d\varphi_u] \quad (6.24)$$

By making use of the definition of the chip thickness (c_t) in Eqn. (6.20), the instantaneous cutting forces in the normal and feed directions after the integration of Eqns. (6.23) and (6.24) now take the form shown below:

$$F_x = -2 \left(\frac{M_c r f_{pt}}{\tan(h_\beta/2)} \right) [p(\varphi_{u2} - \varphi_{u1}) + (\sin^2 \varphi_{u2} - \sin^2 \varphi_{u1}) - 0.5p(\sin 2\varphi_{u2} - \sin 2\varphi_{u1})] \quad (6.25)$$

$$F_z = 2 \left(\frac{M_c r f_{pt}}{\tan(h_\beta/2)} \right) [(\varphi_{u2} - \varphi_{u1}) + p(\sin^2 \varphi_{u2} - \sin^2 \varphi_{u1}) - 0.5(\sin 2\varphi_{u2} - \sin 2\varphi_{u1})] \quad (6.26)$$

where φ_{u1} and φ_{u2} are the starting and end integration angles respectively. Before moving on, it is necessary to declare that both the starting and end integrating angles have an acceptable range that are well illustrated in [40]. From a practical point of view, Eqs. (6.25) and (6.26) are perfect for many applications, but there is a catch. The underlying principle for the definition of the chip thickness (given in Eq. 6.20) assumes that the trajectory of the tool in the conventional end milling operation is analogous to that of the micro-milling operation with a micro end mill. However, a key empirical difference between these two operations is the realization that micro-milling operations are often characterized with a disproportionately large feed per tooth per tool radius ratio. This therefore calls for a re-evaluation of the definition of the chip thickness. An approximate definition that rectifies the weakness in Eq. (6.20) expresses the chip thickness as [40]:

$$c_t = f_{pt} \sin \varphi_u - \frac{N_f}{2\pi r} (f_{pt})^2 \sin \varphi_u \cos \varphi_u + \frac{1}{2\pi r} (f_{pt})^2 \cos^2 \varphi_u \quad (6.27)$$

where $N_f = 2$ for a micro end mill with two flutes and $N_f = 4$ for a micro end mill with four flutes. The substitution of this new definition of the chip thickness in Eqs. 6.21 and 6.22 and subsequent integration leads to new expressions for the cutting forces in the normal and feed directions as:

$$F_x = Q \left[C_1 \frac{f_{pt}}{r} \sin^3 \varphi_u + C_2 \frac{f_{pt}}{r} \cos^3 \varphi_u - \sin^2 \varphi_u + 0.5p \sin 2\varphi_u - \frac{f_{pt}}{r} \sin \varphi_u - p\varphi_u \right] \quad (6.28)$$

$$F_z = Q \left[C_2 \frac{f_{pt}}{r} \sin^3 \varphi_u - C_1 \frac{f_{pt}}{r} \cos^3 \varphi_u - \sin^2 \varphi_u - 0.5 \sin 2\varphi_u - \frac{p f_{pt}}{r} \sin \varphi_u + \varphi_u \right] \quad (6.29)$$

where

$$Q = \frac{M_c r f_{pt}}{\tan(h\beta/2)}; C_1 = \frac{1}{3} \left(1 + \frac{p N_f}{\pi} \right); C_2 = \frac{1}{3} \left(p - \frac{N_f}{\pi} \right); p = 0.3 \quad (6.30)$$

6.3.2 Forced vibration response and signal processing

A crucial aspect of transient dynamic analysis is the evaluation of the time-varying flexible or non-flexible displacements, different types of strains, and stresses in a system in response to the singular effect or mixture of static, transient, or harmonic loads. There are two basic methods of obtaining the dynamic response (used here to refer to time-history response) from the analytical model of a system based on the SEM formulation. As it would be remembered, it has been stated that the SEM is based on the Discrete Fourier Transform (DFT). Therefore, one way of obtaining the dynamic response, in the context of the present study, is to obtain the response of the assembled system in frequency domain and then convert the frequency domain response to the time domain through the use of the inverse discrete Fourier transform (IDFT). The second approach is based on the use precise time integration first proposed by Zhong and William [255]. From the global dynamic matrix (given in Eq. 6.19), and the cutting force model in the preceding sub-subsection, the basic equation of motion for the forced vibration analysis of the system can be written as:

$$\mathbf{S}_g(\omega)\mathbf{u} + \mathbf{C}\dot{\mathbf{u}} = \mathbf{F}(t) \quad (6.31)$$

where the dot represents a differentiation with respect to time, $\mathbf{u}$ is a vector of generalized displacement, $\mathbf{C}$ is the damping matrix, and $\mathbf{F}$ is the generalized vector of the cutting

force acting on the tip of the micro end mill. It would be noticed that Eqn. (6.31) is quite different in form from the way the basic equation of motion is often expressed for a typical transient dynamic analysis which is generally of the form

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}(t) \quad (6.32)$$

where $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix *but not the dynamic stiffness matrix*. In order to be able to carry out the time integration, it is desired to have Eqn. 6.31 re-expressed in the form of Eq. 6.32. A significant breakthrough in the development of the SEM (and its cousin the dynamic stiffness method), is the discovery by Richards and Leung [256] that the overall mass matrix and stiffness matrix of a system, whose solution of governing equations is sought in the framework of the dynamic stiffness matrix, can be expressed as:

$$\mathbf{M} = \frac{\partial \mathbf{S}_g(\omega)}{\partial \omega^2}, \mathbf{K} = \mathbf{S}_g(\omega) + \omega^2 \mathbf{M} \quad (6.33)$$

With the aid of the above formula, Eqn. (6.31) is now expressed as:

$$\left[\frac{\partial \mathbf{S}_g}{\partial \omega^2} \right] \ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + [\mathbf{S}_g + \omega^2 \mathbf{M}] \mathbf{u} = \mathbf{F}(t) \quad (6.34)$$

The method of precise time integration (PTI), which is characterized with an explicit integral scheme and unconditional stability, is now used on Eq. 6.34. The PTI method has an interesting conceptual beginning. It starts with the transformation of the highest-order differential in the equation of interest to a lower-order differential. Under the present condition, the differential transformation is done from the second-order to the first-order as:

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C}/2 & \mathbf{M}^{-1} \\ \frac{\mathbf{CM}^{-1}\mathbf{C}}{4} - \mathbf{K} & -\mathbf{CM}^{-1}/2 \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} + \{0 \quad F(t)^T\}^T \quad (6.35)$$

where $v = \mathbf{M}\dot{u} + \mathbf{C}u/2$. For the sake of simplification, let

$$\mathbf{Q} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C}/2 & \mathbf{M}^{-1} \\ \frac{\mathbf{CM}^{-1}\mathbf{C}}{4} - \mathbf{K} & -\mathbf{CM}^{-1}/2 \end{bmatrix}, \mathbf{x} = \begin{Bmatrix} u \\ v \end{Bmatrix}, \mathbf{f} = \{0 \quad F(t)^T\}^T \quad (6.36)$$

With the above symbols, the procedure of PTI allows the representation of $\mathbf{x}$ at different time step (τ) as:

$$\mathbf{x}_{n+1} = \mathbf{e}^{(\mathbf{Q}\tau)}\mathbf{x}_n + \int_{t_n}^{t_{n+1}} \mathbf{e}^{\mathbf{Q}(t_{n+1}-\mu)} \mathbf{f}(\mu) d\mu \quad (6.37)$$

Using the derived expression, forced vibration response of the system is obtained and compared with experimental observations. The plots of the raw signal of the transverse vibration characteristics of a micro-milling system, captured with the data acquisition system shown in Figure 6-5, are shown in Figures 6-8 to 6-10. Most of the details about the experimental set up are contained in Refs. [251-252]. The raw vibratory responses shown are for different cutting conditions in the dominant cutting direction (yz plane) for experiments 1-9. The results presented here are mostly of the experimental runs involving the end milling of a single slot per micro end mill. From Figures 6-8 to 6-10, it would be noticed that each of the experimentally obtained data sets is dense and can hardly be interpreted without further analyses. Therefore, to be able to make valid comparisons between the experimentally obtained signals and the signals generated by the HAM, the experimentally obtained signals are decomposed with the wavelet transformation method. The time domain equivalents of the wavelet-transformed signals

are then obtained through the inverse wavelet transformation. Before moving on, it is imperative to offer a justification for the choice of the wavelet transformation technique with respect to the present study.

Figure 6-8 Raw vibratory signal for the parameters of experiments 1, 2 and 3

Figure 6-9 Raw vibratory signal for the parameters of experiments 4, 5 and 6

Figure 6-10 Raw vibratory signal for the parameters of experiments 7, 8 and 9

A Chebyshev bandpass filter is employed during the acquisition of the raw signals during the micro-milling process. As expected, the aim of the Chebyshev bandpass filter is essentially to eliminate the spectrum of noise below and above the frequency range of interest. However, due to the complex dynamics of most micro-milling processes, it is believed that certain types of noise could still manage to fall in-between the pass-band range specified for the signal. In the context of micro end milling for example, one of such types of noise could be in the form of the air cutting noise signal that is generated as a result of micro end mill's intermittent disengagement from the workpiece. In micro end milling, the small edge radius of the tool often results in larger negative rake angles. This large negative rake angle causes a rather non-uniform cyclical movement of the air around the cutter's helical segment when it disengages the workpiece. However, during engagement with the workpiece in the course of the cutting operation, an elastic-plastic deformation of the workpiece occurs, thereby generating signal with a different frequency band compare to what occurs during the tool's disengagement. Another important factor that could potentially contribute to the air cutting noise is the tool run-out. Additionally, the presence of the signal generated by the ploughing phenomenon, that has been associated with the mechanics of micromachining, especially micro end milling [257], could be another source of noise. The ploughing phenomenon is the situation that occurs when rubbing takes place between the micro-tool and the workpiece instead of actual material removal. Ploughing results from the size effect engendered by the small ratio of chip to the tip of the micro tool [257].

Even though these types of noise falling in-between the pass-band frequency range may still be removed, as illustrated in Diniz et al. [258], by employing, for

instance, the so-called moving averages filter, some challenges still remain. Prominent among these challenges is the adverse effect of such moving average filter on the amplitude of the frequency of the signal of interest. The other challenge is the choice of the filter width, which could lead to the distortion in the original signal. Besides, there is always the compromise that has to be made with regard to the loss of the first n and the last n points, when designing and evaluating a moving average filter. Again, it is also necessary to point out that although, the elimination of such distortion may be achievable by employing a filtering method like bi-directional filtering techniques [107, 259], such filtering further involves time-consuming correction that could impact on the vibration data gathered. In order to retain as much of the gathered signal without much loss to the original signature of the data, the wavelet-based transformation procedure offers a higher benefit as recently demonstrated in different studies. In theoretical comparison, the wavelet transform is somewhat analogous to the short-time Fourier transform, but the wavelet transform has a time-localization characteristic that renders it exceptional for analyzing non-stationary phenomena [199]. By projecting an observed signal onto a space of a set of basis functions, the wavelet-based transformation technique essentially decomposes a signal into a set of frequency bands that are called scales. In reality, the mapping of the observed signal onto distinct scales is almost equivalent to bandpass filtering with a bank of constant-Q filters, where Q refers to the fidelity factor of a filter [260]. Therefore, one of the advantages of the wavelet transformation is in the identification and localization of singularity or discontinuity in the transformed signals in the time-scale plane.

Since the mathematical theorem for wavelet transformation is well-covered in the literature (for instance [261]), only the application is demonstrated here for the sake of brevity. In Figures 6-11 to 6-18, the decomposition of the three raw signals in Figures 6-8 to 6-10 by using a 4-level wavelet transformation (Daubechies-8) is shown. The selection of Daubechies-8 is based on the correlation between the original signal and the de-noised signal and the cumulative energy distribution. To de-noise the signals, it is necessary to set an appropriate threshold. In our analysis, it is found that 3.8 times of the standard deviation of the decomposed signal at each level of the decomposition produces the results that match the criteria of higher correlation between the original signal and the de-noised signal and also better cumulative energy distribution.

The produced figures represent the plots of the wavelet coefficients for each of the nine experiments. In Figures 6-11 to 6-19, the dashed lines represent the upper and lower threshold limit for the decomposed signal at each level of the wavelet decomposition. During the de-noising phase, the hard thresholding method is employed to remove the noises in the signals. The overall de-noised signal at each level is shown at the top-right corner of each of Figures 6-11 to 6-18. Furthermore, in evaluating the appropriateness of the wavelet type and decomposition level, Eqs. (6.38) and (6.39) are further used to examine:

- (i) the signal-to-noise ratio (SNR), and
- (ii) the peak error (PE) of the de-noised signal based on the different levels of decomposition

Figure 6-11 Decomposition and de-noising of the vibratory signal for experiment 1

Figure 6-12 Decomposition and de-noising of the vibratory signal for experiment 2

Figure 6-13 Decomposition and de-noising of the vibratory signal for experiment 3

Figure 6-14 Decomposition and de-noising of the vibratory signal for experiment 4

Figure 6-15 Decomposition and de-noising of the vibratory signal for experiment 5

Figure 6-16 Decomposition and de-noising of the vibratory signal for experiment 6

Figure 6-17 Decomposition and de-noising of the vibratory signal for experiment 7

Figure 6-18 Decomposition and de-noising of the vibratory signal for experiment 8

Figure 6-19 Decomposition and de-noising of the vibratory signal for experiment 9

The level-4 of Daubechies-8 wavelet decomposition gives the highest value of the *SNR* (9.50) and the lowest value of the peak error (0.86).

$$SNR = 10 \log_{10} \left[\frac{\sum_{k=1}^N x^2(k)}{\sum_{k=1}^N (x^2(k) - x_d^2(k))} \right] \quad (6.38)$$

$$PE = \max_{i \leq k \leq N} (|x^2(k) - x_d^2(k)|) \quad (6.39)$$

where $\mathbf{x}$ is a vector of the original signal and $\mathbf{x}_d$ is a vector of the de-noised signal.

Figures 6-20 to 6-22 show the comparison of the processed experimental signal and the theoretical prediction of the vibratory response from the HAM, again for nine of the experiments. As earlier noted, the HAM involves the analytical model developed for the micro end mill, the machine elements and the presented force model. It is important to

point that, since the experiment is carried out on a machine that has the maximum attainable spindle speed of 25000 *rpm*, attempt is made to utilize this maximum spindle speed of the machine. However, this maximum spindle speed is lower than the required speed needed for a tool of diameter of micro-sized diameter.

Based on the above fact, the HAM neglects the torsional–axial chatter as a result of the limitation imposed by the available spindle speed employed in the experiment. For all of the experimental runs, the aliasing effect in the digital signal processing of the signal is avoided by having the signals sampled at the rate of 10 *kHz* after the application of a Chebyshev bandpass filter. After the application of the direct and inverse wavelet transformation to the experimental signals, comparison is made between the simulated and experimental signals as shown in Figures 6-20 to 6-22. Figure 6-20 represents the comparison of the experimental signal and the theoretical prediction for experiments 1 to 3. Figure 6-21 shows the comparison of the experimental signal and theoretical prediction for experiments 4 to 6, while Figure 6-22 reveals the comparison of the decomposed vibratory signal and theoretical prediction for experiments 7 to 9 shown in Table 6-3. It would be noticed that the plots in Figures 6-20 to 6-22 reveal how close the theoretical predictions are to the experimental observations. Of course, there are some slight discrepancies in the comparisons. This slight discrepancy between the experimental observations and the theoretical predictions can be attributed to some of the underlying assumptions made in the derivation of the analytical model. One of such assumptions is the non-inclusion of the magnitude of tool run-out in the theoretical model. Furthermore, a number of factors that cannot be easily captured by analytical models exist such as, sensor sensitivity, human error etc.

Figure 6-20 Comparison of the decomposed vibratory signal and theoretical predictions for experiments 1, 2 and 3

Figure 6-21 Comparison of the decomposed vibratory signal and theoretical predictions for experiments 4, 5 and 6

Figure 6-22 Comparison of the decomposed vibratory signal and theoretical predictions for experiments 7, 8 and 9

In Table 6-4, the principle of forward causal inference is employed to examine the percentage difference between the root mean square (RMS) of the experimental signals and that of the predictions from the HAM. In the parlance of statistician, the principle of forward causation is a very well-established concept, and it simply means what happens to is likely to happen if certain changes are made to an empirical or non-empirical model. In the present case, the question of what happens to the response of the micro end mill when the rate of twist (helix angle) and the taper ratio are eliminated from the model of the micro-end mill is examined. In order to answer this question, there has to be a modification to the model of the micro end mill through the simplification made to Eqn. 6.15. This simplification involves the elimination of the rate of twist (γ) and the taper parameter ϵ .

Table 6-5 Percentage difference between the RMS of the experimental and predicted signals from the HAM

Experimental runs	% Difference between the RMS of the vibration signals along the x direction		% Difference between the RMS of the vibration signals along the y direction	
Number	S_{Exp} and S_{HAMt}	S_{Exp} and S_{HAMu}	S_{Exp} and S_{HAMt}	S_{Exp} and S_{HAMu}
1	5.2	13.3	9.8	15.1
2	5.3	15.2	7.9	18.2
3	5.9	21.8	9.7	25.8
4	6.4	16.5	9.2	20.2
5	3.9	13.5	10.7	15.8
6	4.6	23.2	9.6	20.9
7	8.1	16.8	10.1	21.4
8	3.6	18.7	8.9	19.3
9	8.7	14.9	8.9	17.6
10	7.6	20.1	10.2	16.5
11	6.2	14.3	7.7	19.3
12	6.7	15.6	7.9	17.5
13	4.8	16.2	8.1	20.5
14	6.2	19.1	8.7	18.9
15	8.9	14.6	10.2	25.3
16	6.6	19.4	10.2	24.9
17	6.5	14.3	7.7	16.0
18	4.8	18.9	8.8	23.1

The reason for the elimination of these key variables is predicated on the suggestions in the literature that the mathematical model of the shank (a uniform spinning beam) can be employed to model the helical flute due to its geometrical complexity. With the simplification suggested, the kinematic of the micro end mill reduces to just the simple addition of a collection of three uniform spinning beam.

In the current study, however, it is observed that the elimination of the parameters (that is, the rate of twist and the taper ratio) results in a higher deviation of the predicted

signal from the experimental observation as seen in the third and fifth columns of Table 6-4. The values of the percentage difference in Table 6-4 are rounded up to the first decimal place, where S_{Exp} represents experimental signal. S_{HAMt} represents the predicted vibration signal from the HAM based on the twisted element representation of the micro flute, while S_{HAMu} represents the predicted vibration signal from the HAM based on the uniform element representation of the micro flute. From Table 6-5, it is noticed that the average of the percentage differences between S_{Exp} and S_{HAMt} is well below 9 % in the two orthogonal planes, xy and zy . On the contrary, the average of the percentage differences between S_{Exp} and S_{HAMu} is well above 15 % in the two orthogonal planes, xy and zy . This observation shows that, the accurate representation of the micro flute plays a very crucial role in the analysis of the vibration response of the micro end mill.

6.4 Summary

The presentation in this chapter comprises two major segments. The first involves the presentation of updated models of the spinning distributed systems, based on the higher-order, non-classical modified couple stress theory. The spinning distributed systems focused on composed of (i) a spinning shear deformable beam with a uniform cross-section, (ii) a spinning shear deformable beam with a taper cross-section, and (iii) a spinning, twisted shear deformable beam. These three spinning distributed systems are employed in the formulation of a hybrid analytical model (HAM) for a micro-milling process. The HAM, whose basis is formulated with the effect of micro end mill geometry in mind, involves the combination of spinning distributed systems and discrete elements.

The spinning distributed systems, on the one hand, are employed to model the distinct parts of the micro end mill spinning about its longitudinal axis. The discrete elements, on the other hand, are used to represent the isotropic bearing supports of the spindle. The stiffness of each of the isotropic bearings is experimentally determined.

A number of machining experiments were carried out to validate the transverse response of the micro end mill predicted by the HAM. From the analyses, it is observed that the simulated prediction of the vibration of the micro end mill is close to the experimental observation. Additionally, it is noted that an over-simplification of the model of the micro end mill, in which the micro flute is represented with a model that neglects its rate of twist, results in a wider margin of difference between the RMS of the experimental and predicted signals. The contribution of the geometric parameters contained in the present model of the micro flute should therefore not be neglected if accurate response simulation is to be obtained.

CHAPTER 7 CONCLUSION AND FUTURE WORK

7.1 Introduction

The synthesis of most systems or structures, no matter how deceptively simple they appear in functionality, shape, and form, is connected to a number of factors. These factors, depending on the system under consideration, range from a whole gamut of issues: material properties and availability, economic factor, interaction with external loads, dynamic stress, temperature effect and economic factors. The pursuit of synthesis of structures, though not entirely restricted to mechanical consideration alone, is achievable through a superior blend of theoretical analyses and well-planned experimental studies. The physical as well as mathematical modeling of mechanical structures therefore plays a crucial role in the mechanical system design and analyses. During the early phase of mechanical system analysis, a structural model is formulated in an attempt to describe the domain of the problem being studied. Often times, as situations are understood and event advances, earlier versions of the developed model are subjected to occasional refinement and periodic iteration. The iterations are done so as to establish a new or an updated model appropriate for the analysis process. The ability to specify, evaluate and possibly generate structural elements' behavior from physical model has always being an integral aspect of systematic engineering design.

A well-established branch of structural mechanics responsible for the determination of the association that exists between the characteristics of a structure with the nature of external influences is, as elaborated in chapter 3, known as structural analysis. Structural

vibration analyses, which can be said to broadly comprise: (i) the determination of natural frequencies and mode shapes from free vibration analysis, and (ii) the evaluation of transient dynamic response from a forced vibration analysis, are constantly employed for a number of reasons in system analyses. In general, free vibration analysis, *ipso facto*, occurs without any form of long term, outside forces. As Schmitz and Smith [262] succinctly put it, free vibration arises out of some initial conditions that a particular system is subjected to and generates motion in one of the system's natural frequencies. The initial conditions can be, for instance, a displacement, an applied torque or a moment from the system's equilibrium position. In contrast to free vibration, a forced vibration is induced by the presence of external force(s). Structural vibration analyses offer many benefits. One of the prime usages of detailed vibration analyses is in the adjustment, control and regulation of the mechanical response of structures. From vibration analyses, a deeper understanding could be generated for the off-line quantification of the dynamics of mechanical systems. Likewise, an improved insight into the identification of the conditions that could potentially lead to the failure of mechanical systems can also be developed. One of the key pre-requisites of mechanical failure analysis is structural vibration analysis. The bulk of the analyses carried out in this study would be noticed to fall under the category of structural analysis. These analyses have in turn fulfilled most of the tasks spelt out to achieve the tasks surrounding the objective of the present research.

7.2 Review of objectives and conclusions

The aim of this sub-section is to address how the research objective, stated in chapter 1, is met. To re-iterate, the major objective of the study is the mathematical modeling of

small-scale structures through the framework of the non-classical elasticity theory. As also stated, the focus of the work is to concentrate on providing understanding of the mechanical behavior of small-scale structures with geometries of relevant to the analysis of a micro end mill. Based on this premise, the achievement of the research objective could be said to have been met as a result of the following research studies that have been carried out:

- Initial assessment and evaluation of the model of a micro end mill through the classical elasticity theory and the development of a solution approach based on the spectral element method (SEM)
- Development of the improved model and SEM analysis of a spinning doubly-symmetric micro-scale beam to represent the shank of the micro end mill
- Development of the improved model and investigation of the dynamic characteristics of a twisted micro-scale beam suitable for the modeling of the flute of the micro end mill
- Development of the model and SEM analysis of a geometrically tapered micro-scale beam suitable for the modeling of the intermediate section of the micro end mill
- Development of a revised model of the micro end mill based on the higher-order non-classical elasticity theory (the modified couple stress theory)
- Development of the solution search for the revised model through the SEM

Further elaborations on some of the above highlighted points are detailed in sub-sections 7.2.1 to 7.2.5 that follow. These sub-sections (7.2.1 to 7.2.5) also contain summary of the conclusions that were derived from the many analyses carried out.

7.2.1 Initial model of the micro end mill

At the initial phase of the present study, a detailed analytical study devoted to the mathematical modeling of micro-cutting tool's dynamics is first tackled. In this phase, a micro end mill is selected as a special case of the micro-cutting tool and studies relating to its transverse vibration analysis are carried out. The model presented for this initial modeling phase is based purely on the classical theory of continuum mechanics. Nonetheless, the free transverse vibration analyses were carried out to determine the influence of helix angle, spindle rotation, rotary inertia and shear deformation on its natural frequencies. These factors are found to have varying degree of influence on the natural frequencies of the micro end mill. The high-speed spindle rotation is found to have the most significant effect on the natural frequencies of the micro end mill. This is then followed by the absolute value of the helix angle. The study of the influence of the helix angle indicates that at the values of 30° and 45° helix angles, the natural frequencies at the different modes begin to get distinct from each other, representing a very stable behavior of the micro tool with these helix angles. The presence of the shear deformation and rotary inertia in the mathematical model adds flexibility to the dynamic response of the micro tool. However, the rotary inertia has the least effect on the dynamic response of the micro end mill.

Since it is recognized that micro-cutting tools are themselves a subset of micro-scale structures, a tremendous effort is made to dig deeper into the dynamics of small-scale structures. Furthermore, in order to compensate for the inadequacy of the classical theory of continuum mechanics employed in the initial model of the micro end mill, novel theoretical models of micro-scale structures, with non-trivial mechanical behavior and complex geometries, are explored and presented. These new models of micro-scale structures are based on two of the new, nonlocal, non-classical higher-order theoretical frameworks of the modified couple stress theory (MCST) and the Eringen's nonlocal elasticity theory. As explained in chapter 5, the selection of these two theories is predicated on two reasons. The reasons have to do both with the relevance of their underlying theory to the objectives of the proposed study and the mathematical convenience the two theories offered. A number of rigorous analyses were carried out based on these two higher-order theoretical frameworks of non-classical elasticity theory.

In particular, after the presentation of the initial modeling phase of the micro end mill, attention is devoted to the evaluation of the study of the behavior of a number of small-scale structures highlighted below.

7.2.2 Analysis of a spinning doubly-symmetric micro-scale beam

Under this analysis, a new dynamic model of a spinning doubly-symmetric circular micro-scale beam is developed. The mathematical model, which contains the effect of the material length scale, is formulated on the framework of the modified couple stress theory. The model is developed for a shear deformable micro-scale beam and then

reduced to that of an Euler-Bernoulli beam for rigorous analysis. Apart from the material length scale, another intrinsic property of the micro-scale beam included in the model is the Poisson's ratio effect. Furthermore, to take into account the effect of a surrounding elastic medium, a Winkler-type reactive pressure is included in the higher-order strain energy expression of the beam. Consequently, size-dependent, coupled partial differential governing equations emerge in two orthogonal plane of spinning of the beam. Through an efficient computational tool of the spectral element method (SEM), the developed elastodynamics equations are employed in the free vibration analysis of the spinning axisymmetric circular micro beam under a clamped-free boundary condition. The accuracy of the SEM predictions is numerically ascertained by contrasting with exact solutions of the classical model. Numerical results from the SEM analysis of the size-dependent governing equations show that the classical equations of the spinning beam underestimate the natural frequencies by more than 30%. However, the effect of the size-dependency diminishes with increase in the diameter of the beam. Furthermore, detailed analysis with an applied statistical tool of DOE demonstrates that both the size effect and the spin rate have domineering role in influencing the dynamic response of the beam than does the elastic foundation.

7.2.3 Wave propagation characteristics of a twisted micro-scale beam

Under this study, the elastodynamics governing partial differential equations of motion for the transverse dynamics of a twisted micro-scale beam is also presented. A micro-scale beam of rectangular cross-section, for which the rate of twist introduced a bending-bending coupling effect, is then considered. The presented governing equation of

motion is used to address the effects of the rate of twist and the material length scale on the bending wave propagation characteristics of the micro-scale beam. Results are presented for the spectrum curve, the cut-off frequency, the phase speed and the group velocity of a propagating harmonic wave profile in the twisted micro-scale beam. Based on the derived equation, the propagating characteristics of a monochromatic bending elastic wave are studied. The following points are deduced from this particular analysis:

- [i] The rate of twist of the micro-scale beam is observed to cause the bifurcation of the spectrum curve of the wave profile.
- [ii] The wave motion in the micro-scale beam based on the modified couple stress theory is found to travel at a set of faster frequency values than that of the classical theory.
- [iii] It is confirmed that the evanescent components of the wave profile decays exponentially in the spatial framework and can thus not be used for effective energy transport.
- [iv] Both the phase speed and the group velocity of the bending wave in the twisted micro-scale beam are found to be a function of the material length scale.
- [v] The cut-off frequency is found to be independent of the material length scale but it is greatly affected by the rate of twist.
- [vi] The analysis of the wave speed for the geometry also reveals that, unlike in the classical EB beam, the group velocity is more than twice the phase speed for all values of the wavenumbers determined.

7.2.4 Analysis of an axially loaded non-prismatic nano-scale beam

Here, attempt is made to present a mathematical model for the vibration analysis of a geometrically tapered nano-scale beam based on the Eringen's nonlocal elasticity theory. The nano-scale beam is idealized as a nonlocal Rayleigh (NLR) beam that is axially loaded and embedded in a deformable surrounding medium. A polynomial variation of the mass distribution, flexural stiffness and the rotary inertia is assumed along with a power law that determines the nature of the non-uniformity. The effect of an elastic surrounding medium modeled as an elastic foundation of the Pasternak-type is included in the mathematical model. The *Bubnov-Galerkin* method is used to solve the variable coefficient governing differential equation of the NLR beam model. The presence and nature of the geometric non-uniformity is observed to have tremendous influence on the natural frequencies of the NLR beam model of the nano-scale beam. With the polynomial power law, an inverse relationship is observed between the natural frequency parameters and the radius of gyration of the nano-scale beam. The natural frequency for the first mode of the nano-scale beam when the taper index parameter $n = 1$ (the taper index is defined in chapter 5) is higher than its corresponding natural frequency for the first mode when $n = 2$, but the two frequencies are higher than that of the corresponding uniform beam model. The combined influence of the Pasternak shear modulus and the nonlocal factor is more pronounced than the case of a model in which these factors are neglected. In all of the studies conducted with the Eringen's nonlocal elasticity theory, it is observed that the scale coefficient parameter strongly influences the natural frequencies of the structures. However, the influence of the scale coefficient parameter on the frequency values is different from those based on the modified couple

stress theory. Furthermore, it is noticed that the results from the use of modified couple stress theory is more compatible with the experimental observations of the behavior of micro-scale structures.

7.2.5 Revised model of the micro end mill

Finally, the updated models of the spinning distributed systems, based on the higher-order, non-classical MCST are used to examine the forced vibration behavior of a micro end mill. The spinning distributed systems focused on composed of (i) a spinning shear deformable beam with a uniform cross-section, (ii) a spinning shear deformable beam with a taper cross-section, and (iii) a spinning, twisted shear deformable beam. These three spinning distributed systems are employed in the formulation of a hybrid analytical model (HAM) for a micro-milling process. The HAM, is especially formulated with the effect of micro end mill geometry in mind and therefore involves the combination of the mentioned spinning distributed systems and discrete elements. Basically, it is demonstrated that the spinning distributed systems are employed to model the distinct parts of the micro end mill spinning about its longitudinal axis, while the discrete elements are used to represent the discrete elements of the machine. A number of machining experiments were carried out to validate the transverse response of the micro end mill predicted by the HAM. From the analyses, it is observed that the simulated prediction of the vibration of the micro end mill is close to the experimental observation. Additionally, it is noted that an over-simplification of the model of the micro end mill, in which the micro flute is represented with a model that neglects its rate of twist, results in a wider margin of difference between the RMS of the experimental and predicted signals.

As a result of the observation from this analysis, it is concluded that the contribution of the geometric parameters contained in the model of the micro flute should therefore not be neglected if accurate response simulation is to be obtained.

7.2.6 The merits of the developed models

It is important to examine the logic behind the models developed in chapter 5 and then employed in chapter 6. Especially, it is important to highlight the extra benefit that could be derived from these seemingly complex models when compared to a simple model like, say, a beam based on the classical Euler-Bernoulli beam (EB) theory. First, the classical EB beam theory, being a first order theory, assumes a simplified assumption that a plane section before deformation remain plane after deformation in the kinematic description of a deformed elastic material [138]. This assumption automatically prevents the classical EB beam theory from allowing the inclusion of the effect of shear deformation that becomes important for a structure like that of a micro end mill with a considerable cross-section [140]. This is because the theory does not account for rotary inertia and shear deformation. Hence a more refined theory, based on either the Bresse-Timoshenko or first order shear deformable beam (such as Timoshenko), is required for a micro end mill. However, the EB beam can be used to analyze the behavior of micro drills as reported in chapter 3. There is therefore an advantage in selecting the micro end mill as a case study for validating the mathematical models. This advantage has to do with the fact that, by eliminating the shear deformation and rotary terms from the model of a micro end mill, the reduced mathematical model can be employed to analyze the behavior of a micro drill.

Furthermore, the formulation of the models in chapters 5 and 6 is premised on the on the higher-order constitutive model of the modified couple stress theory. Based on the higher-order constitutive model, the extended Hamilton's principle is applied to the size-dependent energy expressions to arrive at the coupled elastodynamics governing equations. This is important because the classical theory of linear elasticity, even though, has been used for years in explaining many complex phenomena, the scale upon which the theory is based lacks the spatial resolution required to grasp the microstructural effect in micro-sized structures. Consequently, as reported in chapter 5, a number of investigations have questioned the legitimacy of using the classical models of beam to rationalize the intricate microscopic phenomenon in micro- and nano-scale structures. Thus, the models based on the modified couple stress theory, though appears more complex, offers the possibility to include a material length scale parameter to characterize the size effect in micro-scale structures such as the micro end mill. As also enumerated in chapter 6, the elimination of this important parameters leads to a wider difference between the prediction of the response of the micro end mill by the model and the experimental observation.

7.3 Major contributions

From the above presentation, the contribution of the work undertaken by the author can be summarized as follows:

- The establishment of a novel unified model of a micro end mill with a number of non-classical kinematic (with the inclusion of the tool's tip material's length scale)
- Development of a new and advanced nonlocal model of a spinning micro-scale beam (with a uniform cross-section conforming with both Euler beam theory and the first-order shear deformable beam theory) with the inclusion of material's small-scale coefficient
- Development of novel nonlocal^{***} model of a pre-twisted micro-scale beam (conforming with both Euler beam theory and the first-order shear deformable beam theory) with the inclusion of material's small-scale coefficient
- Provision of a unique insight into the use of power-law for enhancing the modeling of a geometrically tapered micro- and nano-structure and the investigation of the combined effect of the geometric taperedness and small-scale coefficient
- Development of advanced spectral element models of a spinning micro-scale beam, a spinning uniformly tapered shear deformable beam and a twisted micro-scale beam

^{***} In the context of computational mechanics, a nonlocal model is a model in which the stress at a point depends on the strain in a domain near the point of interest (in a way, the stress terms are treated in a cohesive sense).

- Provision of novel insight into the difference between the wave propagation characteristics in a twisted micro-scale beam with and without the combined effect of the material's small-scale coefficient and rate of twist
- Novel application of the SEM of analysis to the investigation of the non-local partial differential equations with constant and variable coefficient
- Provision of insight into the stability of conical small-scale structures in the form of nano-scale beam under self-weight and tip force

7.4 Future work

As early pointed out, the physical as well as mathematical modeling of mechanical structures plays an important role in the mechanical system design and analyses. With respect to the work done so far, it would be noticed that a considerable amount of the numerical computations, in relation to the theoretical modeling and understanding of the dynamics of small-scale structures has been carried out. With the adoption of a micro end mill as a case study, experimental work is carried out to validate the appropriateness of the mathematical model. In spite of this validation procedure, further experimental work is required to strengthen the accuracy of predictions. The future direction of the present work is likely to be directed towards the following tasks:

- Development of a failure monitoring system for the spinning micro-scale systems and life distribution of the failure mode of the micro-scale structures (especially micro tools)

- Determination of the unbalance response of the high-speed spinning micro-scale systems under heterogeneous elastic foundation

7.4.1 Failure monitoring system for the spinning micro-scale systems and life distribution of the failure mode of the micro-cutting tools

While physics-based modeling of structural behavior offers tremendous help in comprehending systems of various levels of complexity and functionality, not all behavior can be completely understood from purely analytical structural models. This is also true in the case of understanding and possibly predicting the behavior of micro tools or for that matter, micro-scale structures. Therefore, in order to complement the understanding from the developed analytical model, identification of the behavior of micro-scale structures and micro-cutting tools under different experimental conditions (not yet explored) is necessary. Specifically, detail experimental work with different micro-cutting tools from different manufacturer is likely to guarantee the ability to build an empirical model, not only for the life distribution of the micro end mill, but for comprehensive collection of data to test the robustness of the developed model under varying conditions.

7.4.2 Determination of the unbalance response of the high-speed spinning micro-scale systems under heterogeneous elastic foundation

Another future direction of the research is the investigation of the behavior of the micro-cutting tools under a number of heterogeneous materials, especially, multi-phase material like the CNT-reinforced composite. The reason for choosing CNT-reinforced

composites is because these materials are gaining ground and are in good stead to becoming materials of choice for future products. The evaluation of the possibility of using micromachining techniques for the fabrication of these products is therefore very important. It will also be of interest to observe the unbalance response of micro-scale tools in the course of the micromachining of these materials. These two possible research directions are expected to be able to come up with an emphatic assessment of the feasibility and economic cost of using micro tools for the machining different materials, including multi-phase materials.

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APPENDIX

$$\varepsilon_{11} = (-2H_1 + 2H_2 + 2\zeta L_1) \quad (A1)$$

$$\varepsilon_{12} = (H_1^2 - 4H_1H_2 + H_2^2 - H_3^2 - 2\zeta H_1L_1 + 2\zeta H_2L_1 + \zeta^2 L_1^2) \quad (A2)$$

$$\varepsilon_{13} = (2H_1^2H_2 - 2H_1H_2^2 + 2H_1H_3^2 - 2\zeta H_1H_2L_1) \quad (A3)$$

$$\varepsilon_{14} = (H_1^2H_2^2 - H_1^2H_3^2) \quad (A4)$$

$$H_1 = \frac{\mu_1 L_1}{k_1 A_1 G} \omega^2 \quad (A5)$$

$$H_2 = \frac{J_1 L_1^2}{EI_1} \omega^2 - \left(\frac{k_1 A_1 G L^2}{EI_1} \right) \quad (A6)$$

$$\zeta = \frac{k_1 A_1 G L_1}{EI_1} \quad (A7)$$

$$H_3 = \frac{\Omega J_{y1} L_1^2 \omega}{EI_1} \quad (A8)$$

The roots of the characteristic equation referenced in chapter 4 are:

$$\alpha_{1,2} = \pm \frac{\sqrt{H_1 - H_2 + H_3 - \zeta L_1 - \sqrt{4H_1(H_2 - H_3) + (H_1 - H_2 + H_3 - \zeta L_1)^2}}}{\sqrt{2}} \quad (A9)$$

$$\alpha_{3,4} = \pm \frac{\sqrt{H_1 - H_2 + H_3 - \zeta L_1 + \sqrt{4H_1(H_2 - H_3) + (H_1 - H_2 + H_3 - \zeta L_1)^2}}}{\sqrt{2}} \quad (A10)$$

$$\alpha_{5,6} = \pm \frac{\sqrt{H_1 - H_2 - H_3 - \zeta L_1 - \sqrt{4H_1(H_2 + H_3) + (-H_1 + H_2 + H_3 + \zeta L_1)^2}}}{\sqrt{2}} \quad (A11)$$

$$\alpha_{7,8} = \pm \frac{\sqrt{H_1 - H_2 - H_3 - \zeta L_1 + \sqrt{4H_1(H_2 + H_3) + (-H_1 + H_2 + H_3 + \zeta L_1)^2}}}{\sqrt{2}} \quad (A12)$$

$$R_1 =$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ r_{11} & r_{12} & r_{13} & r_{14} & r_{15} & r_{16} & r_{17} & r_{18} \\ t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16} & t_{17} & t_{18} \\ r_{11}t_{11} & r_{12}t_{12} & r_{13}t_{13} & r_{14}t_{14} & r_{15}t_{15} & r_{16}t_{16} & r_{17}t_{17} & r_{18}t_{18} \\ e^{ik_1L_1} & e^{-ik_1L_1} & e^{ik_3L_1} & e^{-ik_3L_1} & e^{ik_5L_1} & e^{-ik_5L_1} & e^{ik_7L_1} & e^{-ik_7L_1} \\ e^{ik_1L_1}r_{11} & e^{-ik_1L_1}r_{12} & e^{ik_3L_1}r_{13} & e^{-ik_3L_1}r_{14} & e^{ik_5L_1}r_{15} & e^{-ik_5L_1}r_{16} & e^{ik_7L_1}r_{17} & e^{-ik_7L_1}r_{18} \\ e^{ik_1L_1}t_{11} & e^{-ik_1L_1}t_{12} & e^{ik_3L_1}t_{13} & e^{-ik_3L_1}t_{14} & e^{ik_5L_1}t_{15} & e^{-ik_5L_1}t_{16} & e^{ik_7L_1}t_{17} & e^{-ik_7L_1}t_{18} \\ e^{ik_1L_1}r_{11}t_{11} & e^{-ik_1L_1}r_{12}t_{12} & e^{ik_3L_1}r_{13}t_{13} & e^{-ik_3L_1}r_{14}t_{14} & e^{ik_5L_1}r_{15}t_{15} & e^{-ik_5L_1}r_{16}t_{16} & e^{ik_7L_1}r_{17}t_{17} & e^{-ik_7L_1}r_{18}t_{18} \end{bmatrix} \quad (A13)$$

$$P_1 = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} & P_{17} & P_{18} \\ P_{21} & P_{22} & P_{23} & P_{24} & P_{25} & P_{26} & P_{27} & P_{28} \\ P_{31} & P_{32} & P_{33} & P_{34} & P_{35} & P_{36} & P_{37} & P_{38} \\ P_{41} & P_{42} & P_{43} & P_{44} & P_{45} & P_{46} & P_{47} & P_{48} \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} & P_{56} & P_{57} & P_{58} \\ P_{61} & P_{62} & P_{63} & P_{64} & P_{65} & P_{66} & P_{67} & P_{68} \\ P_{71} & P_{72} & P_{73} & P_{74} & P_{75} & P_{76} & P_{77} & P_{78} \\ P_{81} & P_{82} & P_{83} & P_{84} & P_{85} & P_{86} & P_{87} & P_{88} \end{bmatrix} \quad (A14)$$

where the elements of matrix P_1 are given by:

$$\begin{aligned} P_{1j} &= AGk(\alpha_j + iL_1r_{1j}); P_{2j} = \frac{iEl\alpha_j r_{1j}}{L_1}; P_{3j} = iAGk(\alpha_j + iL_1r_{1j})t_{1j}; P_{4j} = \frac{iEl\alpha_j r_{1j}t_{1j}}{L_1} \\ P_{5j} &= iAe^{i\alpha_j L_1}Gk(\alpha_j + iL_1r_{1j}); P_{6j} = \frac{iEl e^{i\alpha_j L_1} \alpha_j r_{1j}}{L_1}; P_{7j} = iAGke^{i\alpha_j L_1}(\alpha_j + iL_1r_{1j})t_{1j}; \\ P_{8j} &= \frac{iEl e^{i\alpha_j L_1} \alpha_j r_{1j}t_{1j}}{L_1}; j = 1, \dots, 8 \end{aligned} \quad (A15)$$

The interpolating functions used for the generation of the dynamic stiffness for the micro end mill's taper are:

$$N_w = \{e^{ib_1L_3}, e^{-ik_1\xi}, e^{-ik_3(L_2-\xi)}, e^{-ik_3\xi}, e^{-ik_5(L_2-\xi)}, e^{-ik_5\xi}, e^{-ik_7(L_2-\xi)}, e^{-ik_7\xi}\}$$

$$\begin{aligned}
N_\theta &= \{e^{ik_1(L_2-\xi)}r_{11}, e^{-i\xi k_1}r_{12}, e^{ik_3(L_2-\xi)}r_{13}, e^{-i\xi k_3}r_{14}, e^{ik_5(L_2-\xi)}r_{15}, e^{-i\xi k_5}r_{16}, e^{ik_7(L_2-\xi)}r_{17}, e^{-i\xi k_7}r_{18}\} \\
N_u &= \{e^{ik_1(L_2-\xi)}t_{11}, e^{-i\xi k_1}t_{12}, e^{ik_3(L_2-\xi)}t_{13}, e^{-i\xi k_3}t_{14}, e^{ik_5(L_2-\xi)}t_{15}, e^{-i\xi k_5}t_{16}, e^{ik_7(L_2-\xi)}t_{17}, e^{-i\xi k_7}t_{18}\} \\
N_\psi &= \{e^{ik_1(L_2-\xi)}r_1t_1, e^{-i\xi k_1}r_2t_2, e^{ik_3(L_2-\xi)}r_3t_3, e^{-i\xi k_3}r_4t_4, e^{ik_5(L_2-\xi)}r_5t_5, e^{-i\xi k_5}r_6t_6, e^{ik_7(L_2-\xi)}r_7t_7, e^{-i\xi k_7}r_8t_8\}
\end{aligned} \tag{A16}$$

The interpolating functions used for the generation of the dynamic stiffness for the micro end mill's flute are:

$$\begin{aligned}
F_w &= \{e^{ib_1(L_3-\xi)}, e^{-ib_1\xi}, e^{ib_3(L_3-\xi)}, e^{-ib_3\xi}, e^{ib_5(L_3-\xi)}, e^{-ib_5\xi}, e^{ib_7(L_3-\xi)}, e^{-ib_7\xi}\} \\
F_\theta &= \{e^{ib_1(L_3-\xi)}r_{31}, e^{-i\xi b_1}r_{32}, e^{ib_3(L_3-\xi)}r_{33}, e^{-i\xi b_3}r_{34}, e^{ib_5(L_3-\xi)}r_{35}, e^{-i\xi b_5}r_{36}, e^{ib_7(L_3-\xi)}r_{37}, e^{-i\xi b_7}r_{38}\} \\
F_u &= \{e^{ib_1(L_3-\xi)}\beta_1, e^{-i\xi b_1}\beta_2, e^{ib_3(L_3-\xi)}\beta_3, e^{-i\xi b_3}\beta_4, e^{ib_5(L_3-\xi)}\beta_5, e^{-i\xi b_5}\beta_6, e^{ib_7(L_3-\xi)}\beta_7, e^{-i\xi b_7}\beta_8\} \\
F_\psi &= \{e^{ib_1(L_3-\xi)}q_1, e^{-i\xi b_1}q_2, e^{ib_3(L_3-\xi)}q_3, e^{-i\xi b_3}q_4, e^{ib_5(L_3-\xi)}q_5, e^{-i\xi b_5}q_6, e^{ib_7(L_3-\xi)}q_7, e^{-i\xi b_7}q_8\}
\end{aligned}$$

Where b_j (for $j = 1, 2 \dots 8$) are the roots of the dispersion relation for the taper-free twisted beam, while r_{3j} , β_j and q_j are the corresponding amplitude-ratios to harmonize the four amplitude contained in the general solution of the taper-free twisted beam. These terms are defined below:

$$r_{3j} = -\frac{R_{33}}{\underbrace{R_{32}R_{37}-R_{33}R_{36}}_{B_1}} \left\{ \underbrace{\frac{(R_{34}R_{37}-R_{38}R_{33})D_1}{R_{33}}}_{B_1} + \underbrace{\frac{R_{31}R_{37}-R_{35}R_{33}}{R_{33}}}_{B_3} \right\} \tag{A18}$$

$$q_j = \overbrace{\left(\frac{R_{34}R_{31} + B_1B_3(R_{31}^* + R_{32}R_{34}) - R_{32}^*}{R_{34}^2 - B_1B_2(-R_{31}^* + R_{32}R_{34}) - R_{33}^*} \right)}^{D_1} \quad (A19)$$

$$\beta_j = (R_{31} + R_{32}B_1B_3 + D_1(R_{34} - R_{32}B_1B_2)) \quad (A20)$$

The R terms in Eqs. (A18) – (A20) are given below:

$$\begin{aligned} R_{31} &= -b_j^2 + \left(-\frac{\gamma^2 k_{X3} L_3^2}{k_{Z3}} + \frac{\mu_3 L_3^2}{k_{Z3AG}} \Omega^2 + \frac{\mu_3 L_3^2}{k_{Z3AG}} \omega_n^2 \right); R_{31}^* = -b_j^2 + \left(-\frac{\gamma^2 k_{Z3} L_3^2}{k_{X3}} + \right. \\ &\left. \frac{\mu_3 L_3^2}{k_{X3AG}} \omega_n^2 + \frac{\mu_3 L_3^2}{k_{X3AG}} \Omega^2 \right); R_{32} = i \left(\frac{\gamma(k_{Z3} + k_{X3})L_3}{k_{Z3}} b_j + 2 \frac{\mu_3 L_3^2}{k_{Z3AG}} \Omega \omega_n \right); \\ R_{32}^* &= i \left(\frac{\gamma(k_{Z3} + k_{X3})L_3}{k_{X3}} b_j + 2 \frac{\mu_3 L_3^2}{k_{X3AG}} \Omega \omega_n \right) \\ R_{33} &= \frac{\gamma k_{X3} L_3^2}{k_{Z3}}; R_{33}^* = \frac{\gamma k_{Z3} L_3^2}{k_{X3}}; R_{34} = i L_3 b_j; R_{35} = \frac{\gamma k_{X3} A G L_3^2}{E I_{Z3}}; -R_{35}^* = \frac{\gamma k_{Z3} A G L_3^2}{E I_{X3}}; \\ R_{36} &= i \frac{k_{X3} A G L_3}{E I_{Z3}} b_j; R_{36}^* = i \frac{k_{Z3} A G L_3}{E I_{X3}} b_j; R_{37} = -b_j^2 + \left(-\frac{\gamma^2 E I_{X3} L_3^2}{E I_{Z3}} - \frac{k_{X3} A G L_3^2}{E I_{Z3}} + \right. \\ &\left. \frac{J_{Z3} L_3^2}{E I_{Z3}} \omega_n^2 + \frac{J_{X3} L_3^2 \Omega^2}{E I_{Z3}} \right); R_{37}^* = -b_j^2 + \left(-\frac{\gamma^2 E I_{Z3} L_3^2}{E I_{X3}} - \frac{k_{Z3} A G L_3^2}{E I_{X3}} + \frac{J_{X3} L_3^2}{E I_{X3}} \omega_n^2 + \frac{J_{Z3} L_3^2 \Omega^2}{E I_{X3}} \right); \\ R_{38} &= i \left(\frac{\gamma(E I_{Z3} + E I_{X3})L_3}{E I_{Z3}} b_j + \frac{\Omega(J_{X3} + J_{Z3})L_3^2 \omega_n}{E I_{Z3}} \right); \\ R_{38}^* &= i \left(\frac{\gamma(E I_{Z3} + E I_{X3})L_3}{E I_{X3}} b_j + \frac{\Omega(J_{X3} + J_{Z3})L_3^2 \omega_n}{E I_{X3}} \right) \end{aligned} \quad (A21)$$

The matrix T and its transpose needed to transform the dynamic stiffness matrix of the fluted section from local XYZ coordinates to the global coordinates xyz are given below:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos \gamma & -\sin \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin \gamma & \cos \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos \gamma & -\sin \gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin \gamma & \cos \gamma \end{bmatrix} \quad (A22)$$

The total potential and kinetic energy terms used for the derivation of the governing equation for the spinning shank and the tapered neck of the micro end mill are given by:

$$U_{total} = \frac{1}{2} \int_0^L \left\{ EI_1 \left[\left(\frac{d\theta_1}{dy} \right)^2 + \left(\frac{d\psi_1}{dy} \right)^2 \right] + kAG \left[\left(\frac{du_1}{dy} - \psi_1 \right)^2 + \left(\frac{dw_1}{dy} - \theta_1 \right)^2 \right] \right\} dy \quad (A23)$$

$$T_{total} = \frac{1}{2} \int_0^t \left(\rho I \left[\left(\frac{d\theta_1}{dt} \right)^2 + \left(\frac{d\psi_1}{dt} \right)^2 \right] + \rho A \left[\left(\frac{du_1}{dt} \right)^2 + \left(\frac{dw_1}{dt} \right)^2 \right] \right) dy \quad (A24)$$

The virtual work due to the gyroscopic moments for both elements is:

$$\delta W = 2\Omega \int_0^L \left(\rho I \left[\frac{d\theta_1}{dt} \delta \psi_1 - \frac{d\psi_1}{dt} \delta \theta_1 \right] \right) dy \quad (A25)$$

The total energy expressions and the virtual work due to the gyroscopic moments used for the derivation of the governing equation for the spinning flute of the micro end mill are given by:

$$T_3 = \frac{1}{2} \int_0^{L_3} \left[\rho I_{X3} \left(\frac{\partial \theta_3}{\partial t} \right)^2 + \rho I_{Z3} \left(\frac{\partial \psi_3}{\partial t} \right)^2 + \mu_3 \left(\frac{\partial u_3}{\partial t} + \frac{\partial w_3}{\partial t} \right) + \rho I_p \Omega^2 + \mu_3 \Omega^2 (w_3^2 + u_3^2) \right] dY \quad (A26)$$

$$U_3 = \frac{1}{2} \int_0^{L_3} \left[EI_{X3} \left(\frac{d\theta_3}{dY} + \gamma \frac{d\psi_3}{dY} \right)^2 + EI_{Z3} \left(\frac{d\psi_3}{dY} + \gamma \frac{d\theta_3}{dY} \right)^2 + k_X A_3 G \left(\frac{du_3}{dY} + \psi_3 + \gamma \frac{dw_3}{dY} \right)^2 + k_Z A_3 G \left(\frac{dw_3}{dY} - \theta_3 - \gamma \frac{du_3}{dY} \right)^2 \right] dY \quad (A27)$$

$$\delta W_3 = 2\Omega \int_0^{L_3} \left[\rho I_{X3} \left(\frac{d\theta_3}{dt} + \gamma \frac{d\psi_3}{dt} \right) \delta\psi_3 + \rho I_{Z3} \left(\frac{d\psi_3}{dt} + \gamma \frac{d\theta_3}{dt} \right) \delta\theta_3 \right] dY \quad (\text{A28})$$