Green's functions and boundary element methods for the analysis of bimaterials with imperfect interfaces

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Green's Functions and
Boundary Element Methods
for the Analysis of Bimaterials
with Imperfect Interfaces

NANYANG
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by

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Abstract

This thesis is concerned with the numerical solution of several important classes of boundary value problems involving bimetals with imperfect interfaces. The use of imperfect interfaces in the analysis of layered materials is in line with the current research trends in engineering science.

Problems in steady state axisymmetric heat conduction and plane elastostatics are considered in this thesis. The imperfect interfaces of the bimetals in the axisymmetric heat conduction analysis are either low or high conducting. For the plane elastostatic analysis, the interfaces are assumed to be either soft or stiff.

For both the axisymmetric heat conduction and plane elastostatic problems considered here, special Green's functions are derived for cases where the imperfect interfaces are flat (planar). The Green's functions are chosen to satisfy the relevant imperfect interfacial conditions and employed to derive boundary integral equations that do not involve integrals over the imperfect interfaces. Boundary element procedures based on the boundary integral equations, which do not require the interfaces to be discretized into elements, are then proposed for solving numerically the boundary value problems for the bimetals.

An alternative boundary element approach based on hypersingular integral formulation of the imperfect interfacial conditions is also proposed for the numerical solution of axisymmetric heat conduction problems involving bima-

terials with low and high conducting interfaces. Unlike the special Green's function boundary element approaches for flat interfaces, the hypersingular boundary integral method may be used to solve problems involving curved interfaces. Together with a corrective-predictor procedure, it is applied too to solve an axisymmetric heat conduction problem involving nonlinear interfacial conditions.

The validity and the accuracy of all the boundary element approaches proposed in this thesis are examined by solving numerically specific test problems that have known analytical solutions. The numerical solutions are found to agree well with the analytical ones. For some problems that may be of practical interest, the effects of the interfacial parameters on the heat conduction or elastic deformation of bimetals with imperfect interfaces are studied using the boundary element procedures. The results obtained appear to be intuitively and qualitatively acceptable.

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Contents

	<i>page</i>
Abstract	i
Acknowledgments	iii
Contents	v
List of Figures	ix
List of Tables	xiii
1 General Introduction	1
1.1 Motivation	1
1.2 Some Prior Works on Imperfect Interfaces	3
1.3 Boundary Integral Equations and Green's Functions	5
1.4 The Present Thesis	7
1.4.1 Research Objective and Scope	7
1.4.2 Overview of Remaining Chapters	8
1.5 Publications	9
2 Mathematical Preliminaries	11
2.1 Axisymmetric Heat Conduction	11

2.1.1	Basic Equations of Heat Conduction	11
2.1.2	Boundary Integral Equations	12
2.2	Plane Elastostatic Deformations	17
2.2.1	Basic Equations of Anisotropic elasticity	17
2.2.2	Boundary Integral Equations	20
2.3	Hypersingular Integrals	22
3	Steady State Axisymmetric Heat Conduction Across Imperfect Planar Interfaces	25
3.1	Introduction	25
3.2	An Axisymmetric Heat Conduction Problem	27
3.3	Three-dimensional Green's Functions	32
3.3.1	Low Conducting Interfaces	33
3.3.2	High Conducting Interfaces	35
3.4	Axisymmetric Green's Functions	37
3.4.1	Low Conducting Interfaces	38
3.4.2	High Conducting Interfaces	40
3.5	Axisymmetric Boundary Integral Equations	41
3.5.1	Low Conducting Interfaces	43
3.5.2	High Conducting Interfaces	43
3.6	Boundary Element Procedures	45
3.6.1	Low Conducting Interfaces	46
3.6.2	High Conducting Interfaces	47
3.7	Specific Problems	47
3.8	Summary	61
4	Steady State Axisymmetric Heat Conduction Across Imperfect Curved Interfaces	63

Contents

4.1	Introduction	63
4.2	Interfacial Conditions for Axisymmetric Imperfect Curved In- terfaces	65
4.2.1	Ideal Interfaces	68
4.2.2	Low Conducting Interfaces	69
4.2.3	High Conducting Interfaces	69
4.3	An Axisymmetric Heat Conduction Problem	73
4.4	Hypersingular Boundary Integral Equation Method	75
4.5	Low Conducting Interfaces	76
4.5.1	Integral Equations	76
4.5.2	Boundary Element Procedure	80
4.6	High Conducting Interfaces	83
4.6.1	Integral Equations	83
4.6.2	Boundary Element Procedure	84
4.7	Nonlinear Interfaces	96
4.8	Specific Problems	100
4.9	Summary	122
5	Plane Elastostatic Deformations of Anisotropic Bimaterials with Soft and Stiff Interfaces	125
5.1	Introduction	125
5.2	Imperfect Soft and Stiff Planar Interfaces	127
5.2.1	Soft Interfacial Conditions	129
5.2.2	Stiff Interfacial Conditions	131
5.3	Green's Functions for Planar Interfaces	132
5.3.1	Perfect Interfaces	133
5.3.2	Soft Interfaces	135

5.3.3	Stiff Interfaces	137
5.4	A Plane Elastostatic Problem	139
5.5	Boundary Integral Equations	140
5.5.1	Soft Interfaces	141
5.5.2	Stiff Interfaces	142
5.6	Boundary Element Procedures	143
5.6.1	Soft Interfaces	145
5.6.2	Stiff Interfaces	146
5.7	Specific Problems	149
5.8	Analytic Solutions for Infinitely Long Bilayered Slabs with Soft and Stiff Interfaces	161
5.8.1	Soft Interfaces	163
5.8.2	Stiff Interfaces	163
5.9	Summary	164
6	Research Contributions and Extensions	165
6.1	Summary of Contributions	165
6.2	Research Extensions	166
	Bibliography	167

List of Figures

2.1	A sketch of an axisymmetric body in the form of two-dimensional region Ω and curve Γ on the rz plane.	14
3.1	The regions Ω_0 , Ω_1 and Ω_2 are rotated by an angle of 360° about the z -axis to form the three-dimensional regions R_0 , R_1 and R_2 respectively.	29
3.2	The layer Ω_0 is replaced by the line Γ_0 as δ tends to zero. Apart from Γ_0 , the curves Γ_1 and Γ_2 make up the remaining boundaries of Ω_1 and Ω_2 respectively.	30
3.3	A geometrical sketch of Problem 3.1 on the rz plane.	49
3.4	Plots of the numerical and exact temperature $T(r, z)$ for $-1 < z < 1.5$ (Problem 3.1).	50
3.5	A geometrical sketch of Problem 3.2 on the rz plane.	50
3.6	A geometrical sketch of Problem 3.3 on the rz plane.	52
3.7	Plots of $\log_{10}(\text{error})$ against $\log_{10}(\Delta x)$ for the numerical solution at several points taken from Table 3.2.	55
3.8	A geometrical sketch of Problem 3.4 on the rz plane.	56
3.9	Plots of the numerical and exact boundary temperature $T(1, z)$ for $-\frac{1}{2} < z < 2$ (Problem 3.4).	57

3.10	Plots of the numerical and exact boundary temperature $T(r, z)$ for $z = 1$ and $z = 2$ ($0 < r < 2$) (Problem 3.4).	58
3.11	A geometrical sketch of Problem 3.5 on the rz plane.	58
3.12	Plots of $\kappa_1(T - T_a)/q_0z_1$ against z/z_1 for a few selected values of $\alpha/z_1\kappa_2$ (for high conducting interfaces).	60
3.13	Plots of $\kappa_1(T - T_a)/q_0z_1$ against z/z_1 for a few selected values of $\lambda z_1/\kappa_2$ (for low conducting interfaces).	61
4.1	A planar sketch of an axisymmetric trilayered body.	65
4.2	An axisymmetric bimaterial with interface Γ_0	66
4.3	A geometrical sketch of the bimaterial on the rz plane.	74
4.4	A geometrical sketch of Problem 4.2 on the rz plane.	103
4.5	Plots of the numerical and exact temperature jump $\Delta T(r, 1)$ for $0.05 \leq r \leq 0.95$	103
4.6	Plots of the numerical and exact boundary temperature $T(1, z)$ for $0 < z < 2$	104
4.7	A geometrical sketch of Problem 4.3 on the rz plane.	105
4.8	Plots of κ_e/κ_1 against $\log_{10}(\lambda r_2/\kappa_1)$ for a few selected values of ℓ_c/r_2 . Horizontal dashed lines give values of κ_e/κ_1 as calculated in Ang, Singh and Tanaka [11] for the case in which the interface between the elastomer and the carbon nanotube is ideal.	106
4.9	A geometrical sketch of Problem 4.4 on the rz plane.	108
4.10	A geometrical sketch of Problem 4.5 on the rz plane.	110
4.11	Plots of $\kappa_1(T - T_a)/q_0r_1$ against r/r_1 for a few selected values of $\alpha/r_1\kappa_2$ (for perfectly conducting and high conducting interfaces).	111
4.12	Plots of $\kappa_1(T - T_a)/q_0r_1$ against r/r_1 for a few selected values of $\lambda r_1/\kappa_2$ (for perfectly conducting and low conducting interfaces).	112

List of Figures

4.13	Plots of $\kappa_1(T - T_a)/q_0z_1$ against z/z_1 for a few selected values of $\alpha/z_1\kappa_2 = 10$ (in comparison to Green's function boundary element method).	113
4.14	Plots of $\kappa_1(T - T_a)/q_0z_1$ against z/z_1 for a few selected values of $\lambda z_1/\kappa_2 = 0.4$ (in comparison to Green's function boundary element method).	114
4.15	A geometrical sketch of Problem 4.7 on the rz plane.	115
4.16	Plots of the numerical and exact temperature $T(r, z)$ for $r = 0.75$ and $r = 0.25$ ($-0.6 \leq z \leq 0.6$).	117
4.17	A geometrical sketch of Problem 4.8 on the rz plane.	118
4.18	Non-dimensionalized temperature T/T_a contour plot for Problem 4.8.	120
4.19	Plots of T/T_a against z/z_1 for $r/r_1 = 1.8$ and $r/r_1 = 1.2$	120
4.20	Plots of T/T_a against z/z_1 for three different interfaces (with different λ).	121
5.1	Two anisotropic elastic half-spaces $x_2 > \delta$ and $x_2 < 0$ with an anisotropic elastic layer of thickness δ sandwiched in between them.	127
5.2	Two dissimilar elastic half-spaces joined along $x_2 = 0$	128
5.3	A sketch of the bimaterial.	139
5.4	A rectangular bilayered slab.	150
5.5	Plots of $u_1(x_1, 0.25)$ against x_1 for $-8 < x_1 < 8$ (for soft interfaces).	152
5.6	Plots of $u_2(x_1, 0.25)$ against x_1 for $-8 < x_1 < 8$ (for soft interfaces).	152

5.7	Plots of $u_1(0.5, x_2)$ against x_2 for $-0.5 < x_2 < 0.5$ (for soft interfaces).	153
5.8	Plots of $u_2(0.5, x_2)$ against x_2 for $-0.5 < x_2 < 0.5$ (for soft interfaces).	153
5.9	Plots of $u_1(x_1, 0.25)$ against x_1 for $-8 < x_1 < 8$ (for stiff interfaces).	154
5.10	Plots of $u_2(x_1, 0.25)$ against x_1 for $-8 < x_1 < 8$ (for stiff interfaces).	155
5.11	Plots of $u_1(0.5, x_2)$ against x_2 for $-0.5 < x_2 < 0.5$ (for stiff interfaces).	155
5.12	Plots of $u_2(0.5, x_2)$ against x_2 for $-0.5 < x_2 < 0.5$ (for stiff interfaces).	156
5.13	Plots of $u_1 E^{(1)}/hT$ on $x_2/h = 0.5$, $-2 < x_1/h < 2$, for selected values of $\lambda_{11}h/E^{(1)}$ (for soft interfaces).	158
5.14	Plots of $u_2 E^{(1)}/hT$ on $x_2/h = 0.5$, $-2 < x_1/h < 2$, for selected values of $\lambda_{11}h/E^{(1)}$ (for soft interfaces).	158
5.15	Plots of $u_2 E^{(1)}/hT$ along $x_1/h = 0$, $-1 < x_2/h < 1$, for selected values of $\lambda_{11}h/E^{(1)}$ (for soft interfaces).	159
5.16	Plots of $u_1 E^{(1)}/hT$ on $x_2/h = 0.5$, $-2 < x_1/h < 2$, for selected values of $\alpha_{11}/E^{(1)}h$ (for stiff interfaces).	159
5.17	Plots of $u_2 E^{(1)}/hT$ on $x_2/h = 0.5$, $-2 < x_1/h < 2$, for selected values of $\alpha_{11}/E^{(1)}h$ (for stiff interfaces).	160
5.18	Plots of $u_2 E^{(1)}/hT$ along $x_1/h = 0$, $-1 < x_2/h < 1$, for selected values of $\alpha_{11}/E^{(1)}h$ (for stiff interfaces).	160

List of Tables

3.1	Numerical and exact values of T at selected interior points for Problem 3.2.	51
3.2	Numerical and exact values of T at selected interior points for Problem 3.3.	54
4.1	A comparison of the numerical values of T with the exact solution at various selected points for Problem 4.1.	101
4.2	Numerical and exact values of T at selected interior points for Problem 4.4.	109
4.3	A comparison of the numerical values of T with the exact solution at various selected points for Problem 4.7.	117

Chapter 1

General Introduction

1.1 Motivation

If two dissimilar materials are joined by a very thin layer material sandwiched in between them, it may be desirable to model the thin interphase layer as an interface in the form of a line (for plane or two-dimensional analysis) or a surface (for three-dimensional analysis). The line or surface interface may help to simplify the mathematics involved in the analysis of multi-layered materials, making the derivation of analytical solutions mathematically more amenable.

Furthermore, the direct application of numerical methods (such as the finite element method or the boundary element method) to solve boundary value problems involving very thin (slender) regions may give rise to solutions of poor accuracy (see, for example, Luo, Liu and Berger [50]). Thus, in the numerical treatment of dissimilar materials separated by a very thin layer of material, the difficulties encountered in obtaining accurate solution may be avoided if the thin interphase layer is modeled as a line or surface interface.

The boundary conditions to impose on the line or surface interface depend on the properties of the material in the thin interphase layer and may be derived by using asymptotic analysis based on Taylor series expansion of the

relevant field quantities. If the interphase layer has material properties that tend to extreme limiting values as the thickness of the layer vanishes to zero, non-classical interfacial conditions involving jumps in certain field quantities (such as temperature or normal heat flux and displacement or normal stress) may be obtained. An interface with such conditions is termed “imperfect” or “non-ideal” in contrast to the classical perfect or ideal interface where the relevant field quantities do not exhibit any jump.

The above mentioned imperfect interfacial conditions, which are based on $O(\delta)$ approximation of the boundary conditions (where δ is the thickness of the thin interphase), may be found in Benveniste [14], Benveniste and Miloh [17], Cheng and Torquato [30], Hashin [43], Miloh and Benveniste [52], Andrianov, Bolshakov, Danishevs'kyi and Weichert [1] and Lipton [49] for conduction-type problems (that is, for analyses in heat conduction, electric conduction, diffusion, magnetic permeability, and so on). For elastic problems, the $O(\delta)$ interface model may be found in Benveniste [15], Benveniste and Miloh [18]-[19], Hashin [41],[42], [44], and Ru [62].

Higher order interface models may also be found in the research literature. For example, Niklasson, Datta and Dunn [53]-[54] constructed an $O(\delta^2)$ model for wave propagation in a thin coating layer and Benveniste [16] derived an interface model for heat conduction in a three-dimensional thin interphase layer accurate to $O(\delta^N)$ (N is an arbitrary integer).

The analysis of imperfect interfaces is currently a subject of considerable interest in engineering, particularly in composites and electronic packaging. A weak imperfect interface may occur when the interface contains microcracks or microscopic gaps due to surface roughness as in Wang, Ang and Fan [72] or when materials are not properly glued along the interface as in Ang and Fan [9] and Fan and Wang [35]. On the other hand, an imperfect interface is regarded

as strong if it is strengthened by micro-inclusions such as carbon nanotubes as described in Desai, Geer and Sammakia [32]. Other applications involving imperfect interfaces are mentioned in the last few paragraphs of Section 1.2.

1.2 Some Prior Works on Imperfect Interfaces

There are many research papers on the solutions of boundary value problems involving imperfect interfaces. Most of them deal with materials with idealized geometries, such as half-spaces that are imperfectly joined along a planar surface.

In the context of heat conduction and other physically analogous problems such as those in electrostatics and magnetostatics, solutions of boundary value problems for imperfect interfaces may be found in Benveniste and Miloh [17], Hashin [43], Cheng and Torquato [30], Quang, Phan and Bonnet [59], Bovik and Olsson [27], Benveniste [15], and Lipton [49]. Boundary value problems involving low and high conducting interfaces were considered in Hashin [43]. A thermally low conducting interface is a weak interface where there is a jump in the temperature across opposite sides of the interface, whereas a thermally high conducting interface is a strong interface where there is a jump in the heat flux. These are explained in greater details in Section 3.1 and Section 3.2. Earlier works seem to be more focused on low conducting or Kapitza type of interface. Only in later years, researchers showed a greater interest in high conducting interfaces, such as in [30] and [59].

For elastic problems, Hashin derived solutions for spherical inclusions with soft interfaces in [42] and later solved both soft and stiff interface problems in [44]. Interaction of dislocations with imperfect interfaces is also of great interest. In particular, Fan and Wang investigated the problem of screw dis-

location interacting with soft interface in [34] and the interaction of screw dislocation with viscoelastic interfaces (both Kelvin and Maxwell models) in [35]. A dislocational solution for a stiff interface may be found in Wang and Pan [74].

Currently, due to increasing applications of piezoelectric materials in engineering, there is a surge in interest in coupled field analyses which include the effects of electric field on the deformations of solids. Exact solutions for multi-layered piezothermoelastic plates with imperfect interfaces were derived in Wang and Pan [73]. Wang and Pan [75] also derived solutions for inclusion of arbitrary shape embedded in one of two imperfectly bonded piezoelectric half-planes. Applications of acoustic wave devices also gave rise to works related to piezoelectric waves in which the shear-lag model is used to describe the imperfect interface (see, Fan, Yang and Xu [36] and [37]). In Yang, Hu, Zeng and Fan [79], the shear-lag model is also used to obtain solutions for thickness-shear vibration modes of a quartz plate.

Time-dependent boundary value problems involving imperfect interfaces are also found in literature. Motivated by the need to analyze an electromagnetic-acoustic transducer system, Berger, Martin and McCaffery [22] had solved the problem of time-harmonic torsional wave propagation in composite elastic cylinders with an imperfect interface. An elastostatic dynamic problem of a triple-layer piezoelectric composite cylinder with imperfect interfaces was solved in Wang [70].

All the works mentioned in the preceding paragraphs are for highly idealized geometries and boundary conditions.

1.3 Boundary Integral Equations and Green's Functions

Solutions in the form of boundary integral equations may be analytically derived for the governing partial differential equations in linear heat conduction and elastostatics. The boundary integral equations may be used to obtain numerical procedures for solving heat conduction and elastostatic boundary value problems. Such numerical procedures known as boundary element methods in the literature require only the boundary of the solution domain of a boundary value problem to be discretized into elements. Details on boundary element methods may be found in, for example, Ang [2], Brebbia, Telles and Wrobel [26], Clements [28], Clements and Rizzo [29], and Sladek and Sladek [65].

The direct boundary integral method (in, for example, [2] and [25]) is a well suited method for the numerical solution of boundary value problems involving multi-domains with non-ideal interfaces. One of its advantages is that all unknown functions in the boundary integral equation are directly related to physical quantities on the boundaries and interfaces of the solution domain. For example, in a heat conduction problem, the unknown quantities in the direct boundary integral formulation are the temperature and the normal heat flux on the boundary and interface. Thus, in dealing with the boundary or interfacial conditions, it is not necessary to approximate the normal heat flux in terms of temperature at selected nodal points. The boundary integral approach can also be used to reduce the number of unknown functions to be determined on imperfect interfaces.

For a wave scattering problem, Martin [51] derived boundary integral equations for various types of imperfect interfaces in the solution domain. For a curved low conducting interface, Ang [5] expressed the interfacial conditions

in terms of a hypersingular integral equation with the interfacial temperature jump as an unknown function. The hypersingular integral formulation of the interfacial conditions are used together with the usual boundary integral equation to develop a simple numerical procedure for solving the plane heat conduction problem of a bimaterial with a low conducting interface of an arbitrary shape. In Ang and Fan [9] and Ang [4], the hypersingular boundary integral approach in [5] was extended to solve antiplane elastic problems involving bimaterials with imperfect viscoelastic interfaces.

Fundamental solutions or so called free space Green's functions of the governing partial differential equations are used to derive the boundary integral equations. As explained in Ang [2], for a boundary value problem, if the boundary integral equation(s) is (are) derived by modifying the fundamental solution in an appropriate way such that it satisfies certain boundary conditions, the integration over some parts of the boundary of the solution domain may be avoided. For example, specially modified fundamental solutions or special Green's functions satisfying particular conditions on planar cracks are used in Ang and Telles [12] and Athanasius and Ang [13] to derive boundary integral equations that do not require integrations over cracks. Also, Green's functions for displacements and stresses expressed in closed form are given in Tonon, Pan and Amadei [69] and Hasabe, Wang and Kondo [40]. More details on the general concepts and the applications of Green's functions may be found in Roach [61] and Cole, Beck, Haji-Sheikh and Litkouhi [31].

Green's functions for perfect interfaces in bimaterials may be found in Berger and Karageorghis [21] for heat conduction problems and in Berger [20], Berger and Tewary [24] and Pavlou [57] for elastostatic problems.

A special Green's function for a low conducting planar interface may be found in Ang, Choo and Fan [7] for a two-dimensional heat conduction prob-

lem. Later on, Ang [3] applied the Green's function in [7] to derive a dual-reciprocity boundary element method for solving time-dependent heat conduction problem involving a low conducting planar interface.

For three-dimensional heat conduction problem, Wang and Sudak [76] derived three-dimensional Green's functions for both thermally low and high conducting planar interfaces. However, the authors in [76] did not apply the Green's functions to obtain boundary element procedures for analyzing bimetals with low and high conducting interfaces. There are very few boundary element solutions for imperfect interfaces in heat conduction problems, especially for high conducting interfaces.

For plane elastostatics, Green's functions for soft interface with specified discontinuities can be found in Berger and Tewary [23]-[24]. Meanwhile, Sudak and Wang [68] had derived Green's functions for a soft planar interface.

1.4 The Present Thesis

1.4.1 Research Objective and Scope

The main objective of the thesis is to extend and develop boundary element techniques for solving numerically several important classes of boundary value problems involving bimetals with imperfect interfaces. Selected problems in steady state axisymmetric heat conduction and plane elastostatics are solved. The imperfect interfaces of the bimetals are either low or high conducting for the axisymmetric heat conduction problem, and are either soft or stiff for the plane elastostatic problems.

To achieve the research objective, special axisymmetric heat conduction and plane elastostatic Green's functions are derived for imperfect planar interfaces. The Green's functions are used to obtain boundary element procedures

that do not require the imperfect interfaces to be discretized into elements. An alternative boundary element method based on hypersingular boundary integral equations, which is applicable to curved interfaces, is also proposed for axisymmetric heat conduction across low and high conducting interfaces.

1.4.2 Overview of Remaining Chapters

The remaining part of the thesis consists of five chapters. The content of each chapters is briefly described below.

Mathematical preliminaries outlining the basic equations of linear heat conduction and elastostatics are given in Chapter 2. These include the governing partial differential equations and boundary integral equations for steady state axisymmetric heat conduction and plane elastostatics for anisotropic materials.

In Chapter 3, special Green's functions are derived for steady state axisymmetric heat conduction across low and high conducting planar interfaces between two dissimilar half-spaces. Boundary element procedures based on the special Green's functions are developed for analyzing the temperature distribution in bimetals with imperfect interfaces. Such boundary element procedures do not require the imperfect interfaces to be discretized into elements.

The problem of steady state axisymmetric heat conduction across low and high conducting curved interfaces in bimetals is considered in Chapter 4. The axisymmetric thermal conditions on the imperfect curved interfaces are derived using asymptotic analysis. An alternative boundary element method based on hypersingular integral formulations of the imperfect interfacial conditions is proposed for the numerical solution of the axisymmetric heat conduction problem. The proposed method is also extended to treat nonlinear thermal conditions on interfaces of bimetals.

Special elastostatic Green's functions are derived for soft and stiff planar interfaces between two dissimilar anisotropic elastic half-spaces in Chapter 5. The Green's functions are used to derive boundary element procedures for analyzing plane elastostatic deformations of bimetals with imperfect interfaces. As in Chapter 3, the boundary element procedures do not require the planar interfaces to be discretized into elements. For the purpose of verifying the boundary element solutions, specific problems involving infinitely long bilayered slabs with imperfect interfaces are also solved analytically in Chapter 5 by using the Fourier transform method.

Lastly, the main contributions of the thesis and some suggestions for extending the research works here are given in Chapter 6.

1.5 Publications

The works reported in the present thesis have been published or accepted for publication as follows.

Chen EL, Ang WT and Fan H, A hypersingular boundary integral analysis of axisymmetric steady-state heat conduction across a non-ideal interface between two dissimilar materials, *Engineering Analysis with Boundary Elements* **35** (2011) 1090-1100.

Chen EL, Boundary element method for axisymmetric heat conduction in bimetals with nonlinear interface conditions, *Proceedings of the 4th International Conference on Computational Methods* (2012) 25-27.

Chen EL and Ang WT, Special Green's function boundary element approach for steady-state axisymmetric heat conduction across low and high conducting planar interfaces, *Applied Mathematical Modelling* **37** (2013) 1948-1956.

Chen EL and Ang WT, Green's functions and boundary element analysis for bimaterials with soft and stiff planar interfaces under plane elastostatic deformations, *Engineering Analysis with Boundary Elements* **40** (2014) 50-61.

Chapter 2

Mathematical Preliminaries

2.1 Axisymmetric Heat Conduction

In Chapter 3 and 4, axisymmetric heat conduction problems involving bimaterials are considered. Thus, a review of basic theory for heat conduction is given here in this section.

2.1.1 Basic Equations of Heat Conduction

If $T(\underline{\mathbf{x}}, t)$ denotes the temperature at the point $\underline{\mathbf{x}}$ in a solid at time t and $\underline{\mathbf{q}}(\underline{\mathbf{x}}, t)$ is the heat flux then the Fourier law for thermally isotropic solids gives

$$\underline{\mathbf{q}}(\underline{\mathbf{x}}, t) = -\kappa \underline{\nabla} T(\underline{\mathbf{x}}, t), \quad (2.1)$$

where κ is the thermal conductivity of the solid. The negative sign implies that the heat flow proceeds from region of high temperature to region of low temperature.

From the law of conservation energy, the differential heat energy equation is given by

$$-\underline{\nabla} \cdot \underline{\mathbf{q}}(\underline{\mathbf{x}}, t) + Q(\underline{\mathbf{x}}, t) = \rho c \frac{\partial}{\partial t} [T(\underline{\mathbf{x}}, t)], \quad (2.2)$$

where ρ is the density, c is the specific heat of the solid and Q is the internal heat generator.

For thermal analysis, the energy equation in (2.2) is to be solved together with the Fourier law of heat conduction in (2.1). Substitution of (2.1) into (2.2) gives the governing partial differential equation of heat conduction for a thermally isotropic solid as

$$\underline{\nabla} \cdot (\kappa \underline{\nabla} T(\underline{\mathbf{x}}, t)) + Q(\underline{\mathbf{x}}, t) = \rho c \frac{\partial}{\partial t} [T(\underline{\mathbf{x}}, t)]. \quad (2.3)$$

The temperature T is governed by (2.3) inside the solid together with given initial and boundary conditions. Once T is known, the heat flux $\underline{\mathbf{q}}$ can be determined using (2.1).

For a homogeneous solid, the coefficients κ , ρ and c are constants and (2.3) reduces to

$$\nabla^2 T(\underline{\mathbf{x}}, t) + \frac{1}{\kappa} Q(\underline{\mathbf{x}}, t) = \frac{\rho c}{\kappa} \frac{\partial}{\partial t} [T(\underline{\mathbf{x}}, t)], \quad (2.4)$$

where ∇^2 denotes the Laplacian operator. We refer to (2.4) as the (classical) heat equation.

For a more comprehensive understanding of the theory of heat conduction, one may refer to Wang, Zhou and Wei [71] and Özışık [55]-[56].

2.1.2 Boundary Integral Equations

The boundary integral equation for axisymmetric steady-state heat equation is used in Chapter 3 and 4 to formulate an axisymmetric steady-state heat conduction problem involving bimaterials with low and high conducting interfaces. A brief account on the derivation of the axisymmetric boundary integral equation is given here.

From (2.4), the partial differential equation governing the steady-state heat conduction in a homogeneous isotropic solid with zero internal heat generation

is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0, \quad (2.5)$$

where x , y and z are the Cartesian coordinates of a point in the solid.

For three-dimensional steady state heat conduction, the boundary integral equation for (2.5) can be written in the form (see, for example, Ang [2])

$$\begin{aligned} \gamma(\xi, \eta, \zeta)T(\xi, \eta, \zeta) &= \iint_S \left\{ T(x, y, z) \frac{\partial}{\partial n} [\tilde{\Phi}(x, y, z; \xi, \eta, \zeta)] \right. \\ &\quad \left. - \tilde{\Phi}(x, y, z; \xi, \eta, \zeta) \frac{\partial}{\partial n} [T(x, y, z)] \right\} ds(x, y, z), \end{aligned} \quad (2.6)$$

where R is the region occupied by the solid, S is the surface bounding R , and $\partial/\partial n$ may be given by $\partial f/\partial n = n_x \partial f/\partial x + n_y \partial f/\partial y + n_z \partial f/\partial z$, where $[n_x, n_y, n_z]$ is the unit normal vector to the surface S pointing out of region R .

Also, in (2.6), $\gamma(\xi, \eta, \zeta)$ is defined by

$$\gamma(\xi, \eta, \zeta) = \begin{cases} 0 & \text{if } (\xi, \eta, \zeta) \notin R \cup S, \\ 1/2 & \text{if } (\xi, \eta, \zeta) \text{ lies on a smooth part of } S, \\ 1 & \text{if } (\xi, \eta, \zeta) \in R, \end{cases} \quad (2.7)$$

and $\tilde{\Phi}(x, y, z; \xi, \eta, \zeta)$ is chosen to be

$$\tilde{\Phi}(x, y, z; \xi, \eta, \zeta) = \Phi_{3D}(x, y, z; \xi, \eta, \zeta) + \Phi^*(x, y, z; \xi, \eta, \zeta), \quad (2.8)$$

where $\Phi_{3D}(x, y, z; \xi, \eta, \zeta)$ is the fundamental solution given by

$$\Phi_{3D}(x, y, z; \xi, \eta, \zeta) = -\frac{1}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}}, \quad (2.9)$$

and $\Phi^*(x, y, z; \xi, \eta, \zeta)$ is any solution of the partial differential equation

$$\frac{\partial^2 \Phi^*}{\partial x^2} + \frac{\partial^2 \Phi^*}{\partial y^2} + \frac{\partial^2 \Phi^*}{\partial z^2} = 0. \quad (2.10)$$

We can always take $\Phi^*(x, y, z; \xi, \eta, \zeta) = 0$ but, for particular problems, it may be advantageous to choose $\Phi^*(x, y, z; \xi, \eta, \zeta)$ in a certain way to derive special Green's functions for the boundary integral equation.

For an axisymmetric body, it is more convenient to describe the points in the region R using cylindrical polar coordinates (r, θ, z) instead of Cartesian coordinates (x, y, z) by applying the relation $x = r \cos \theta$ and $y = r \sin \theta$. The symmetric property of the body about the z -axis allows us to represent the solid R and its surface boundary S in the form of two-dimensional region Ω and curve Γ as shown in Figure 2.1. The axisymmetric body R can be generated simply by rotating the region Ω by an angle of 360° about the z -axis. Note that Γ may be an open curve or a closed curve, depending on the geometry of the axisymmetric body.

Figure 2.1: A sketch of an axisymmetric body in the form of two-dimensional region Ω and curve Γ on the rz plane.

For the case of axisymmetric heat conduction, the temperature is inde-

pendent of θ . Hence, the steady-state heat equation in (2.5) can be written as

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0. \quad (2.11)$$

For axisymmetric heat conduction, we will now briefly explain how the integral over the surface S in (2.6) can be reduced to a line integral over Γ .

For S generated by rotating Γ , the infinitesimal area $ds(x, y, z)$ in (2.6) can be written as

$$ds(x, y, z) = r d\ell d\theta, \quad (2.12)$$

where $d\ell$ is the length of an infinitesimal portion of the curve Γ .

Take a point $(\xi, \eta, \zeta) = (r_0 \cos \theta_0, r_0 \sin \theta_0, z_0)$ on the Oxz plane (where $y = 0$ or $\theta_0 = 0$) and rewrite (2.6) as

$$\begin{aligned} \gamma(r_0, z_0)T(r_0, z_0) &= \iint_S \left\{ T(r, z) \frac{\partial}{\partial n} [\tilde{\Phi}(r \cos \theta, r \sin \theta, z; r_0, 0, z_0)] \right. \\ &\quad \left. - \tilde{\Phi}(r \cos \theta, r \sin \theta, z; r_0, 0, z_0) \frac{\partial}{\partial n} [T(r, z)] \right\} r d\theta d\ell, \end{aligned} \quad (2.13)$$

where $\gamma(r_0, z_0) = 1/2$ if (r_0, z_0) lies on a smooth part of Γ and $\gamma(r_0, z_0) = 1$ if (r_0, z_0) lies in the interior of Ω (Figure 2.1).

Since T and $\partial T / \partial n$ are independent of θ for axisymmetric heat conduction, they can be brought out of the integrands for the integrals over θ (for $0 \leq \theta \leq 2\pi$). Thus, (2.13) can now be rewritten as

$$\begin{aligned} \gamma(r_0, z_0)T(r_0, z_0) &= \int_{\Gamma} \left\{ T(r, z) \tilde{G}_1(r, z; r_0, z_0; n_r, n_z) \right. \\ &\quad \left. - \tilde{G}_0(r, z; r_0, z_0) \frac{\partial}{\partial n} [T(r, z)] \right\} r d\ell(r, z), \end{aligned} \quad (2.14)$$

where $\tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ and $\tilde{G}_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ are given by

$$\begin{aligned}\tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) &= G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0), \\ \tilde{G}_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) &= G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) + G_1^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})),\end{aligned}\quad (2.15)$$

with

$$\begin{aligned}G_0(r, z; r_0, z_0) &= \int_0^{2\pi} \Phi_{3D}(r \cos \theta, r \sin \theta, z; r_0, 0, z_0) d\theta, \\ G_1(r, z; r_0, z_0; n_r, n_z) &= \int_0^{2\pi} \frac{\partial}{\partial n} [\Phi_{3D}(r \cos \theta, r \sin \theta, z; r_0, 0, z_0)] d\theta, \\ G_0^*(r, z; r_0, z_0) &= \int_0^{2\pi} \Phi^*(r \cos \theta, r \sin \theta, z; r_0, 0, z_0) d\theta, \\ G_1^*(r, z; r_0, z_0; n_r, n_z) &= \int_0^{2\pi} \frac{\partial}{\partial n} [\Phi^*(r \cos \theta, r \sin \theta, z; r_0, 0, z_0)] d\theta.\end{aligned}\quad (2.16)$$

The functions G_0 and G_1 in (2.16) can further be expressed explicitly in terms of known special functions as

$$\begin{aligned}G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) &= -\frac{K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))}{\pi \sqrt{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)}}, \\ G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) &= -\frac{1}{\pi \sqrt{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)}} \\ &\quad \times \left\{ \frac{n_r(\underline{\mathbf{x}})}{2r} \left[\frac{r_0^2 - r^2 + (z_0 - z)^2}{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0)} E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \right. \right. \\ &\quad \left. \left. - K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \right] \right. \\ &\quad \left. + n_z(\underline{\mathbf{x}}) \frac{z_0 - z}{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0)} E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \right\},\end{aligned}\quad (2.17)$$

with

$$\begin{aligned} m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) &= \frac{2b(r; r_0)}{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)}, \\ a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) &= r_0^2 + r^2 + (z_0 - z)^2, \quad b(r; r_0) = 2rr_0, \\ K(m) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}, \quad E(m) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 \theta} d\theta. \end{aligned} \quad (2.18)$$

$K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind respectively (see Brebbia, Telles and Wrobel [26]).

To solve steady-state axisymmetric heat conduction problems, a boundary element procedure that requires only the curve Γ in Figure 2.1 to be discretized into elements may be developed by using (2.14). Compared to the surface S , the curve Γ is easier to discretize. Also, in discretizing Γ , the boundary element procedure will have fewer elements and unknowns compared to the full three-dimensional problems.

2.2 Plane Elastostatic Deformations

Bimaterials with soft and stiff planar interfaces under plane elastostatic deformations are discussed in Chapter 5. To facilitate our discussion, a review of the basic equations of anisotropic elasticity is given here. The details on the fundamentals of anisotropic elasticity may be found in Sadd [63] and Shames and Cozzarelli [64].

2.2.1 Basic Equations of Anisotropic elasticity

For an anisotropic elastic body, the generalized Hooke's Law gives

$$\sigma_{ij} = c_{ijkl} \frac{\partial u_k}{\partial x_l}, \quad (2.19)$$

where u_i and σ_{ij} are respectively the displacements and stresses, x_i are the Cartesian coordinates of points in space and the fourth-order tensor c_{ijkl} gives all the elastic moduli of the material. In general, u_i and σ_{ij} are functions of space x_i and time t . Note that the Latin subscripts i and j are the free indices whereas k and l are the dummy indices. The Einsteinian convention of summing over the dummy index (repeated index) is adopted throughout this thesis. Thus, in (2.19), we have a double summation over k and l from 1 to 3.

From physical consideration, elastic moduli c_{ijkl} are required to satisfy

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}, \quad (2.20)$$

and the strict inequality

$$c_{ijkl} \frac{\partial u_k}{\partial x_l} \frac{\partial u_i}{\partial x_j} > 0 \text{ for all non-zero } 3 \times 3 \text{ matrix } \left[\frac{\partial u_i}{\partial x_j} \right]. \quad (2.21)$$

According to (2.21), the strain energy of a deformed elastic system is positive. The conditions in (2.20) and (2.21) reduce the total number of independent elastic components in c_{ijkl} from 81 to 21.

In the absence of body force, the equation of motion gives

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad (2.22)$$

where ρ is the density of the material and t denotes time.

Substitution of (2.19) into (2.22) leads to the governing equation of a homogeneous, anisotropic elastic bodies

$$c_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} = \rho \frac{\partial^2 u_i}{\partial t^2}. \quad (2.23)$$

Note that the elastic moduli c_{ijkl} are constants for homogeneous materials.

Chapter 5 deals with plane elastostatic problems involving homogeneous materials where the displacements u_k are functions of x_1 and x_2 only. For such

a case, (2.23) reduces to the elliptic partial differential equations

$$c_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} = 0, \quad (2.24)$$

where the summation over j and l runs from 1 to 2 only since there is no dependence on x_3 .

The elliptic partial differential equations (2.24) have solutions of the form

$$u_k = A_k f(x_1 + \tau x_2), \quad (2.25)$$

where A_k and τ are constants and f is any analytic function that is at least twice differentiable of $z = x_1 + \tau x_2$. Substituting (2.25) in (2.24), we find that A_k satisfy the system of linear algebraic equation

$$[c_{i1k1} + (c_{i1k2} + c_{i2k1})\tau + c_{i2k2}\tau^2] A_k = 0 \quad (i = 1, 2, 3). \quad (2.26)$$

To ensure that the system (2.26) has non-trivial solution A_k , the determinant of the matrix giving the coefficients of A_k has to be zero. Hence, the constant τ is required to satisfy

$$\det [c_{i1k1} + (c_{i1k2} + c_{i2k1})\tau + c_{i2k2}\tau^2] = 0. \quad (2.27)$$

The characteristic equation (2.27) is a sextic equation in τ . Theorem 1.3.1 in Clements [28] shows that solutions τ of (2.27) are not real because of the ellipticity condition in (2.21). They occur in three complex conjugate pairs since the coefficients of τ in the sextic equation are real.

Assuming that the solutions of (2.27) occur in three distinct conjugate pairs, we denote the three distinct solutions with positive imaginary parts by τ_α ($\alpha = 1, 2, 3$). For $\tau = \tau_\alpha$, the corresponding non-trivial solutions A_k of (2.26) are denoted by $A_{k\alpha}$, that is,

$$[c_{i1k1} + (c_{i1k2} + c_{i2k1})\tau_\alpha + c_{i2k2}\tau_\alpha^2] A_{k\alpha} = 0 \quad (i = 1, 2, 3). \quad (2.28)$$

Naturally, for $\tau = \bar{\tau}_\alpha$, we find that $A_k = \bar{A}_{k\alpha}$.

For plane elastostatic problems, we can construct a general solution for (2.24) in the form (Stroh [67])

$$u_k = \operatorname{Re}\left\{\sum_{\alpha=1}^3 A_{k\alpha} f_\alpha(z_\alpha)\right\}, \quad (2.29)$$

where Re denotes the real part of a complex number and f_α are holomorphic functions of complex variable $z_\alpha = x_1 + \tau_\alpha x_2$.

From (2.19) and (2.29), the stresses σ_{ij} are given by

$$\sigma_{ij} = \operatorname{Re}\left\{\sum_{\alpha=1}^3 L_{ij\alpha} f'_\alpha(z_\alpha)\right\}, \quad (2.30)$$

where $L_{ij\alpha} = [c_{ijk1} + \tau_\alpha c_{ijk2}] A_{k\alpha}$.

A useful matrix $M_{\alpha p}$, which appears in Chapter 5, is related to $L_{k2\alpha}$ by

$$\sum_{\alpha=1}^3 L_{k2\alpha} M_{\alpha p} = \delta_{kp}. \quad (2.31)$$

2.2.2 Boundary Integral Equations

The boundary integral equations for plane elastostatic deformations are used in Chapter 5 for solving plane elastostatic problems involving bimaterials with soft and stiff planar interfaces. A brief account on boundary integral equation for plane elastostatic deformations is given here. Details on the boundary integral equations for plane elastostatics can be found in Clements [28].

If the system (2.24) holds in a two-dimensional region Π bounded by a simple closed curve C , the boundary integral equations may be derived using

the reciprocal relation in Clements and Rizzo [29] as

$$\begin{aligned} \gamma(\xi_1, \xi_2)u_k(\xi_1, \xi_2) &= \int_C [u_k(x_1, x_2)\Gamma_{km}(x_1, x_2; \xi_1, \xi_2) \\ &\quad - t_k(x_1, x_2)\Phi_{km}(x_1, x_2; \xi_1, \xi_2)]ds(x_1, x_2) \\ &\quad \text{for } (\xi_1, \xi_2) \text{ in the interior of } \Pi, \end{aligned} \quad (2.32)$$

where $\Phi_{km}(x_1, x_2; \xi_1, \xi_2)$ is given by

$$\begin{aligned} &\Phi_{km}(x_1, x_2; \xi_1, \xi_2) \\ &= \Phi_{km}^{(\text{fund})}(x_1, x_2; \xi_1, \xi_2) + \Phi_{km}^*(x_1, x_2; \xi_1, \xi_2). \end{aligned} \quad (2.33)$$

In (2.33), $\Phi_{km}^{(\text{fund})}(x_1, x_2; \xi_1, \xi_2)$ is the plane elastostatic fundamental solutions of the system (2.24), given by

$$\begin{aligned} &\Phi_{km}^{(\text{fund})}(x_1, x_2; \xi_1, \xi_2) \\ &= \frac{1}{2\pi} \text{Re}\left\{\sum_{\alpha=1}^3 A_{k\alpha}[N_{\alpha p} \ln(x_1 - \xi_1 + \tau_{\alpha}(x_2 - \xi_2))]\right\}d_{pm}, \end{aligned} \quad (2.34)$$

and $\Phi_{km}^*(x_1, x_2; \xi_1, \xi_2)$ is any solution of the partial differential equation

$$c_{ijk\ell} \frac{\partial^2}{\partial x_j \partial x_{\ell}} [\Phi_{km}^*(x_1, x_2; \xi_1, \xi_2)] = 0 \text{ in } \Pi. \quad (2.35)$$

The corresponding tractions of $\Phi_{km}^{(\text{fund})}(x_1, x_2; \xi_1, \xi_2)$ are

$$\begin{aligned} &\Gamma_{km}^{(\text{fund})}(x_1, x_2; \xi_1, \xi_2) \\ &= \frac{1}{2\pi} \text{Re}\left\{\sum_{\alpha=1}^3 L_{kj\alpha} \frac{N_{\alpha p} n_j(x_1, x_2)}{(x_1 - \xi_1 + \tau_{\alpha}(x_2 - \xi_2))}\right\}d_{pm}, \end{aligned} \quad (2.36)$$

with $n_j(x_1, x_2)$ being components of the outward unit normal vector to C at (x_1, x_2) .

The constants $N_{\alpha p}$ and d_{pm} in (2.34) are defined by

$$\begin{aligned} \sum_{\alpha=1}^3 A_{k\alpha} N_{\alpha p} &= \delta_{kp}, \\ \text{Im}\left\{\sum_{\alpha=1}^3 L_{k2\alpha} N_{\alpha p}\right\} d_{pm} &= \delta_{kp}, \end{aligned} \quad (2.37)$$

where δ_{kp} is the Krocnecker-delta and Im denotes the imaginary part of a complex number. Also,

$$\gamma(\xi_1, \xi_2) = \begin{cases} 1 & \text{if } (\xi_1, \xi_2) \text{ lies in the interior of } \Pi, \\ 1/2 & \text{if } (\xi_1, \xi_2) \text{ lies on a smooth part of } C, \\ 0 & \text{if } (\xi_1, \xi_2) \text{ lies outside of } \Pi. \end{cases} \quad (2.38)$$

2.3 Hypersingular Integrals

In Chapter 4, the conditions on the imperfect interfaces are formulated in terms of hypersingular integral equations. Hypersingular integrals are essentially the Hadamard finite-part integrals introduced in Hadamard [39].

The Hadamard finite-part integrals arise out of the differentiation of certain Cauchy principal integrals or more specifically Hilbert transforms as follows:

$$\frac{d}{dx} \left[\mathcal{C} \int_a^b \frac{f(t)dt}{t-x} \right] = \mathcal{H} \int_a^b \frac{f(t)dt}{(t-x)^2} \quad \text{for } a < x < b, \quad (2.39)$$

where $\mathcal{C}$ and $\mathcal{H}$ denote respectively the Cauchy principal and Hadamard finite-part integrals defined by

$$\left. \begin{aligned} \mathcal{C} \int_a^b \frac{f(t)dt}{t-x} &= \lim_{\epsilon \rightarrow 0^+} \left\{ \int_a^{x-\epsilon} \frac{f(t)dt}{t-x} + \int_{x+\epsilon}^b \frac{f(t)dt}{t-x} \right\} \\ \mathcal{H} \int_a^b \frac{f(t)dt}{(t-x)^2} &= \lim_{\epsilon \rightarrow 0^+} \left\{ \int_a^{x-\epsilon} \frac{f(t)dt}{(t-x)^2} + \int_{x+\epsilon}^b \frac{f(t)dt}{(t-x)^2} - \frac{2f(x)}{\epsilon} \right\} \end{aligned} \right\} \quad \text{for } a < x < b. \quad (2.40)$$

Ang and Clements [8] gave the following alternative but equivalent definition for the Hadamard finite-part integrals:

$$\mathcal{H} \int_a^b \frac{f(t)dt}{(t-x)^2} \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0^+} \left\{ \int_a^b \frac{(t-x)^2 f(t)dt}{[(t-x)^2 + \epsilon^2]^2} - \frac{\pi}{2\epsilon} f(x) \right\} \quad \text{for } a < x < b. \quad (2.41)$$

The limit in the alternative definition above appears directly in Chapter 4 in the derivation of the conditions for the imperfect interfaces.

Some details on hypersingular integrals with applications to fracture analysis may be found in Ang [6].

Chapter 3

Steady State Axisymmetric Heat Conduction Across Imperfect Planar Interfaces

3.1 Introduction

Axisymmetric heat conduction in multi-layered cylindrical solids has attracted the attention of researchers in recent years. For example, Desai, Geer and Sammakia [32] derived an analytical solution for the steady state axisymmetric heat conduction in perfectly bonded dissimilar co-axial cylindrical solids to investigate the performance of thermal management systems in electronic packaging, and Ang, Singh and Tanaka [11] proposed a model for steady state axisymmetric heat conduction in a multi-material cylindrical system containing a thermal superconductor and applied the model to analyze the thermal behaviors of carbon nanotube based composites.

In this chapter, boundary integral solutions are derived for a class of steady state axisymmetric problems involving bimetals with imperfect planar interfaces. The imperfect interfaces are either low or high conducting, described by the interfacial conditions given in Benveniste [14].

For a thermally low conducting interface, the temperature is discontinuous across opposite sides of the interface and the interfacial temperature jump is linearly related to the normal heat flux which is continuous on the interface. The interface between two imperfectly joined solids may be modeled as thermally low conducting if it contains microscopic gaps filled with air or a material of extremely low thermal conductivity.

The temperature is continuous on a thermally high conducting interface, but the normal heat flux exhibits a jump across the interface. The jump on the normal heat flux is proportional in magnitude to the Laplacian of the interfacial temperature. If two dissimilar materials are joined by an extremely thin layer of superconductor, such as carbon nanotubes as described in Desai, Geer and Sammakia [32], the thin interphase layer may be modeled as a thermally high conducting interface.

Three-dimensional Green's functions for thermally low and high conducting planar interfaces between two thermally isotropic half-spaces are given in Wang and Sudak [76], for the case where the singular internal heat point source lies on the interface between the half-spaces. The analysis in [76] is generalized here to include the case where the singular heat point source is located at an arbitrary point in space. The three-dimensional Green's functions are then integrated axially to obtain the corresponding axisymmetric Green's functions for the low and high conducting interfaces.

The axisymmetric Green's functions are employed to obtain boundary integral equations for steady state axisymmetric heat conduction across the low or high conducting planar interface of a bimaterial of finite extent. As the Green's functions satisfy the interfacial conditions, the boundary integral equations do not contain any integral over the imperfect interface. The boundary element procedures for solving numerically the boundary integral equations do not re-

quire the interface of the bimaterial to be discretized into elements. Thus, the resulting system of linear algebraic equations to be solved contains a smaller number of unknowns. To check the validity and accuracy of the Green's function boundary element procedures, some specific problems are solved.

There are some earlier works on the use of special Green's functions for developing numerical methods for analyzing heat conduction in bimaterials. Berger and Karageorghis [21] used the Green's function for steady state two-dimensional heat conduction across a perfect planar interface between two thermally anisotropic half-spaces to develop a meshless method for computing the temperature distribution in a bimaterial. Ang, Choo and Fan [7] derived a Green's function for two-dimensional heat conduction across a low conducting planar interface between two thermally isotropic half-spaces and employed the Green's function to derive an interface free boundary element procedure for solving two-dimensional heat conduction problems involving bimaterials of finite extent. The Green's function in [7] was also used in Ang [3] to develop a dual-reciprocity boundary element method for solving non-steady two-dimensional heat conduction problems.

3.2 An Axisymmetric Heat Conduction Problem

Consider two dissimilar materials bonded together with a thin layer of material sandwiched in between them. The regions occupied by the layer and the two dissimilar materials are denoted by R_0 , R_1 and R_2 respectively. With reference to a Cartesian coordinate system denoted by $Oxyz$, R_0 occupies part of the space $-\delta/2 < z < \delta/2$, where δ is a given positive number. The regions R_1 and R_2 are subsets of the half-spaces $z < -\delta/2$ and $z > \delta/2$ respectively. The

regions R_0 , R_1 and R_2 are axisymmetric, obtained by rotating respectively two-dimensional regions Ω_0 , Ω_1 and Ω_2 on the Orz (axisymmetric coordinate) plane (as sketched in Figure 3.1) by an angle of 360° about the z -axis. Note that r is the distance of a point from the z -axis.

It is assumed that the temperature in the bimetals is steady and varies spatially with the axisymmetric coordinates r and z . Denoted by $T(\underline{\mathbf{x}})$ (where $\underline{\mathbf{x}} = (r, z)$), the temperature is governed by the partial differential equation (2.11) in Chapter 2. Thus,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \quad \text{for } \underline{\mathbf{x}} \in \Omega_0 \cup \Omega_1 \cup \Omega_2. \quad (3.1)$$

The thermal conductivities of the materials in Ω_0 , Ω_1 and Ω_2 are positive constants κ_0 , κ_1 and κ_2 respectively. We are interested in modeling the sandwiched layer Ω_0 as a line interface on the z -axis of the Orz plane for the limiting case in which the thickness δ tends to zero. A geometrical sketch of the body with the line interface $0 < r < b$, $z = 0$ denoted by Γ_0 is shown in Figure 3.2. In general, depending on the geometries of Ω_0 , Ω_1 and Ω_2 , the interface Γ_0 lies on $a < r < b$, $z = 0$, where a is not necessarily zero.

For the planar interface Γ_0 where $a < r < b$, $z = 0$, the low and high conducting interfacial conditions can be directly extracted from the asymptotic analysis in Benveniste [14]. They can also be recovered as special cases of axisymmetric interfacial conditions for imperfect interfaces of arbitrary shapes. The derivations of the more general axisymmetric interfacial conditions are given in Section 4.2. Thus, we will only briefly explain here how the imperfect conditions for the planar interface $a < r < b$, $z = 0$, come about.

The line interface Γ_0 between Ω_1 and Ω_2 in Figure 3.2 is said to be ideal or perfectly conducting if the steady state axisymmetric temperature $T(r, z)$ and the corresponding normal heat flux are continuous on Γ_0 , that is, if the

Figure 3.1: The regions Ω_0 , Ω_1 and Ω_2 are rotated by an angle of 360° about the z -axis to form the three-dimensional regions R_0 , R_1 and R_2 respectively.

thermal conditions on the interface Γ_0 are given by

$$\left. \begin{aligned} T(r, 0^+) &= T(r, 0^-) \\ \kappa_2 \frac{\partial T}{\partial z} \Big|_{z=0^+} &= \kappa_1 \frac{\partial T}{\partial z} \Big|_{z=0^-} \end{aligned} \right\} \text{for } a < r < b. \quad (3.2)$$

If the thermal conductivity κ_0 in the layer Ω_0 is such that

$$\frac{\kappa_0}{\delta} \rightarrow \lambda \text{ (a finite positive constant) as } \delta \rightarrow 0^+, \quad (3.3)$$

then the interfacial conditions given in (4.9) for the curved interface can be written here for the planar interface as

$$\left. \begin{aligned} \kappa_2 \frac{\partial T}{\partial z} \Big|_{z=0^+} &= \kappa_1 \frac{\partial T}{\partial z} \Big|_{z=0^-} \\ \lambda [T(r, 0^+) - T(r, 0^-)] &= \kappa_2 \frac{\partial T}{\partial z} \Big|_{z=0^+} \end{aligned} \right\} \text{for } a < r < b. \quad (3.4)$$

Figure 3.2: The layer Ω_0 is replaced by the line Γ_0 as δ tends to zero. Apart from Γ_0 , the curves Γ_1 and Γ_2 make up the remaining boundaries of Ω_1 and Ω_2 respectively.

Note that (3.3) implies that κ_0 approaches zero as δ tends to zero. Thus, (3.4) gives the thermal conditions on a layer with thickness and thermal conductivity that tend to zero in such a way that there is a temperature jump across opposite sides of the layer of vanishing thickness. An imperfect interface with such thermal conditions is said to be low conducting. The line interface Γ_0 between two materials as sketched in Figure 3.2 may be modeled as low conducting if the interface contains microscopic gaps filled with air.

For another imperfect interface, if the thermal conductivity of the layer Ω_0 is given by

$$\kappa_0 \delta \rightarrow \alpha \text{ (a finite positive constant) as } \delta \rightarrow 0^+, \quad (3.5)$$

then the thermal conditions in (4.10) and (4.24) for the curved interface can be written here for the planar interface as

$$\left. \begin{aligned} T(r, 0^+) &= T(r, 0^-) \\ \kappa_2 \frac{\partial T}{\partial z} \Big|_{z=0^+} - \kappa_1 \frac{\partial T}{\partial z} \Big|_{z=0^-} &= \alpha \frac{\partial^2 T}{\partial z^2} \Big|_{z=0} \end{aligned} \right\} \text{ for } a < r < b. \quad (3.6)$$

In view of the first condition in (3.6) and the governing partial differential equation in (3.1), note that

$$\begin{aligned} \frac{\partial^2 T}{\partial z^2} \Big|_{z=0^+} &= - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \Big|_{z=0^+} \\ &= - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \Big|_{z=0^-} = \frac{\partial^2 T}{\partial z^2} \Big|_{z=0^-}. \end{aligned} \quad (3.7)$$

As implied by (3.5), the thermal conductivity κ_0 in the layer Ω_0 tends to infinity as δ vanishes. Thus, an imperfect interface with the thermal conditions (3.6) in which the normal heat flux is discontinuous across the vanishing interphase layer is said to be high conducting. For a practical example, if the two materials in Figure 3.2 are joined together by an extremely thin layer of carbon nanotubes, the line interface may be modeled as high conducting.

The problem of interest here is to solve (3.1) for the axisymmetric steady-state temperature in the bimaterial sketched in Figure 3.2, that is, in $\Omega_1 \cup \Omega_2$, subject to either (3.4) or (3.6) (as the thermal conditions on Γ_0) and suitably prescribed temperature or flux at each point on the exterior boundary $\Gamma_1 \cup \Gamma_2$ of the bimaterial. Specifically, the boundary conditions on $\Gamma_1 \cup \Gamma_2$ are given by

$$\begin{aligned} T(\underline{\mathbf{x}}) &= f_0(\underline{\mathbf{x}}) \text{ for } \underline{\mathbf{x}} \in \Xi_1, \\ P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}})) &= f_1(\underline{\mathbf{x}}) + f_2(\underline{\mathbf{x}})T(\underline{\mathbf{x}}) \text{ for } \underline{\mathbf{x}} \in \Xi_2, \end{aligned} \quad (3.8)$$

where $f_0(\underline{\mathbf{x}})$, $f_1(\underline{\mathbf{x}})$ and $f_2(\underline{\mathbf{x}})$ are suitably prescribed functions, Ξ_1 and Ξ_2

are non-intersecting curves (for the different boundary conditions) such that $\Xi_1 \cup \Xi_2 = \Gamma_1 \cup \Gamma_2$.

3.3 Three-dimensional Green's Functions

With reference to a Cartesian coordinate system $Oxyz$, consider the half-spaces $z < 0$ and $z > 0$ being occupied by two dissimilar homogeneous materials having thermal conductivities κ_1 and κ_2 respectively. We present here Green's functions for three-dimensional steady-state heat conduction across low and high conducting interfaces at $z = 0$.

The required Green's functions denoted by $\tilde{\Phi}(x, y, z; \xi, \eta, \zeta)$ are solutions of the partial differential equation

$$\frac{\partial^2 \tilde{\Phi}}{\partial x^2} + \frac{\partial^2 \tilde{\Phi}}{\partial y^2} + \frac{\partial^2 \tilde{\Phi}}{\partial z^2} = \delta(x - \xi)\delta(y - \eta)\delta(z - \zeta), \quad (3.9)$$

where $\delta(x)$ denotes the Dirac-delta function and (ξ, η, ζ) denotes the position of singular heat source.

Equation (3.9) admits solutions of the form

$$\begin{aligned} \tilde{\Phi}(x, y, z; \xi, \eta, \zeta) &= [H(z)H(\zeta) + H(-z)H(-\zeta)]\Phi_{3D}(x, y, z; \xi, \eta, \zeta) \\ &\quad + \Phi^*(x, y, z; \xi, \eta, \zeta). \end{aligned} \quad (3.10)$$

$\tilde{\Phi}(x, y, z; \xi, \eta, \zeta)$ has already been shown in (2.8), together with $\Phi_{3D}(x, y, z; \xi, \eta, \zeta)$ and $\Phi^*(x, y, z; \xi, \eta, \zeta)$ in Subsection 2.1.2. Note that $H(x)$ denotes the unit-step Heaviside function.

The function $\Phi^*(x, y, z; \xi, \eta, \zeta)$ satisfying the imperfect interfacial conditions are given in Wang and Sudak [76] for $\zeta = 0$, that is, for the case where the singular point heat source lies on the imperfect interface. Here, we extend the analysis in [76] to include the case where $\zeta \neq 0$.

We take $\Phi^*(x, y, z; \xi, \eta, \zeta)$ to be of the form

$$\begin{aligned}
 & \Phi^*(x, y, z; \xi, \eta, \zeta) \\
 = & H(-\zeta) \left\{ H(-z) \left[\frac{a_0}{4\pi \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2}} \right. \right. \\
 & + a_1 \int_0^\infty \frac{\exp(-a_3 u) du}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta-u)^2}} \left. \right] \\
 & + H(z) a_2 \int_0^\infty \frac{\exp(-a_3 u) du}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta+u)^2}} \left. \right\} \\
 & + H(\zeta) \left\{ H(z) \left[\frac{b_0}{4\pi \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2}} \right. \right. \\
 & + b_1 \int_0^\infty \frac{\exp(-b_3 u) du}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta+u)^2}} \left. \right] \\
 & + H(-z) b_2 \int_0^\infty \frac{\exp(-b_3 u) du}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta-u)^2}} \left. \right\}, \quad (3.11)
 \end{aligned}$$

where $a_0, a_1, a_2, a_3, b_0, b_1, b_2$ and b_3 are constants. We assume a priori that a_3 and b_3 are positive constants so that the improper integrals over $[0, \infty)$ in (3.11) exist.

It may be easily verified that (3.11) is a solution of (2.35) at all points (x, y, z) in space. The constants $a_0, a_1, a_2, a_3, b_0, b_1, b_2$ and b_3 are chosen to satisfy the conditions on the interface.

3.3.1 Low Conducting Interfaces

For the case in which the interface $z = 0$ is low conducting, $\tilde{\Phi}(x, y, z; \xi, \eta, \zeta)$ is required to satisfy the interfacial conditions

$$\kappa_1 \frac{\partial}{\partial z} [\tilde{\Phi}(x, y, z; \xi, \eta, \zeta)] \Big|_{z=0^-} = \kappa_2 \frac{\partial}{\partial z} [\tilde{\Phi}(x, y, z; \xi, \eta, \zeta)] \Big|_{z=0^+},$$

$$\lambda[\tilde{\Phi}(x, y, 0^+; \xi, \eta, \zeta) - \tilde{\Phi}(x, y, 0^-; \xi, \eta, \zeta)] = \kappa_2 \frac{\partial}{\partial z} [\tilde{\Phi}(x, y, z; \xi, \eta, \zeta)] \Big|_{z=0^+}. \quad (3.12)$$

If we take $a_0 = b_0 = -1$, the condition on the first line of (3.12) is satisfied if

$$\begin{aligned} -\kappa_1 a_1 &= \kappa_2 a_2, \\ -\kappa_1 b_2 &= \kappa_2 b_1. \end{aligned} \quad (3.13)$$

For $\zeta < 0$, the condition on the second line of (3.12) is satisfied if

$$\begin{aligned} & \lambda(a_2 - a_1) \int_0^\infty \frac{\exp(-a_3 u) du}{\sqrt{(x - \xi)^2 + (y - \eta)^2 + (u - \zeta)^2}} \\ & + \frac{\lambda}{2\pi \sqrt{(x - \xi)^2 + (y - \eta)^2 + \zeta^2}} \\ & = -\kappa_2 a_2 \int_0^\infty \frac{(u - \zeta) \exp(-a_3 u) du}{[(x - \xi)^2 + (y - \eta)^2 + (u - \zeta)^2]^{3/2}}. \end{aligned} \quad (3.14)$$

Using the integration by parts, we obtain

$$\begin{aligned} & \int \frac{(u - \zeta) \exp(-a_3 u) du}{[(x - \xi)^2 + (y - \eta)^2 + (u - \zeta)^2]^{3/2}} \\ & = -\frac{\exp(-a_3 u)}{[(x - \xi)^2 + (y - \eta)^2 + (u - \zeta)^2]^{1/2}} \\ & - a_3 \int \frac{\exp(-a_3 u) du}{\sqrt{(x - \xi)^2 + (y - \eta)^2 + (u - \zeta)^2}}. \end{aligned} \quad (3.15)$$

From (3.14) and (3.15), it follows that

$$\begin{aligned} -2\pi\kappa_2 a_2 &= \lambda, \\ \kappa_2 a_2 b_3 &= \lambda(a_2 - a_1). \end{aligned} \quad (3.16)$$

Similarly, for $\zeta > 0$, we obtain

$$\begin{aligned} 2\pi\kappa_2 b_1 &= \lambda, \\ -\kappa_2 b_1 b_3 &= \lambda(-b_1 + b_2). \end{aligned} \quad (3.17)$$

Solving (3.13), (3.16) and (3.17) gives

$$\begin{aligned} a_1 &= -b_2 = \frac{\lambda}{2\pi\kappa_1}, \\ a_2 &= -b_1 = -\frac{\lambda}{2\pi\kappa_2}, \\ a_3 &= b_3 = \frac{\lambda}{\kappa_2}\left(1 + \frac{\kappa_2}{\kappa_1}\right). \end{aligned} \quad (3.18)$$

Note that a_3 and b_3 are positive (as assumed in (3.11)).

Thus, the required three-dimensional Green's function for the case in which the interface $z = 0$ is low conducting is given by (3.10), (3.11) and (3.18).

3.3.2 High Conducting Interfaces

For the case in which the interface $z = 0$ is high conducting, $\tilde{\Phi}(x, y, z; \xi, \eta, \zeta)$ is required to satisfy the interfacial conditions

$$\begin{aligned} \tilde{\Phi}(x, y, 0^+; \xi, \eta, \zeta) &= \tilde{\Phi}(x, y, 0^-; \xi, \eta, \zeta), \\ \kappa_2 \frac{\partial}{\partial z} [\tilde{\Phi}(x, y, z; \xi, \eta, \zeta)] \Big|_{z=0^+} &- \kappa_1 \frac{\partial}{\partial z} [\tilde{\Phi}(x, y, z; \xi, \eta, \zeta)] \Big|_{z=0^-} \\ &= \alpha \frac{\partial^2}{\partial z^2} [\tilde{\Phi}(x, y, z; \xi, \eta, \zeta)] \Big|_{z=0}. \end{aligned} \quad (3.19)$$

Taking $a_0 = b_0 = 1$, the condition on the first line of (3.19) is satisfied if

$$\begin{aligned} a_1 &= a_2, \\ b_1 &= b_2. \end{aligned} \quad (3.20)$$

For $\zeta < 0$, the condition on the second line of (3.19) is satisfied if

$$\begin{aligned}
 & -(\kappa_1 a_1 + \kappa_2 a_2) \int_0^\infty \frac{(u - \zeta) \exp(-a_3 u) du}{[(x - \xi)^2 + (y - \eta)^2 + (u - \zeta)^2]^{3/2}} \\
 & \quad + \frac{\kappa_1 \zeta}{2\pi[(x - \xi)^2 + (y - \eta)^2 + \zeta^2]^{3/2}} \\
 = & \alpha a_2 \int_0^\infty \exp(-a_3 u) \frac{\partial^2}{\partial z^2} \left[\frac{1}{\sqrt{(x - \xi)^2 + (y - \eta)^2 + (u - \zeta)^2}} \right] \Big|_{z=0} du.
 \end{aligned} \tag{3.21}$$

Using integration by parts and the relation

$$\begin{aligned}
 & \frac{\partial^2}{\partial z^2} \left[\frac{1}{\sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta + u)^2}} \right] \Big|_{z=0} \\
 = & \frac{\partial^2}{\partial u^2} \left[\frac{1}{\sqrt{(x - \xi)^2 + (y - \eta)^2 + (u - \zeta)^2}} \right],
 \end{aligned} \tag{3.22}$$

we obtain

$$\begin{aligned}
 & \frac{\kappa_1 \zeta}{2\pi[(x - \xi)^2 + (y - \eta)^2 + \zeta^2]^{3/2}} - \frac{(\kappa_1 a_1 + \kappa_2 a_2)}{\sqrt{(x - \xi)^2 + (y - \eta)^2 + \zeta^2}} \\
 & + a_3(\kappa_1 a_1 + \kappa_2 a_2) \int_0^\infty \frac{\exp(-a_3 u) du}{\sqrt{(x - \xi)^2 + (y - \eta)^2 + (u - \zeta)^2}} \\
 = & \alpha a_2 \left\{ -\frac{\zeta}{[(x - \xi)^2 + (y - \eta)^2 + \zeta^2]^{3/2}} - \frac{a_3}{\sqrt{(x - \xi)^2 + (y - \eta)^2 + \zeta^2}} \right. \\
 & \left. + a_3^2 \int_0^\infty \frac{\exp(-a_3 u) du}{\sqrt{(x - \xi)^2 + (y - \eta)^2 + (u - \zeta)^2}} \right\}.
 \end{aligned} \tag{3.23}$$

From (3.23), it follows that

$$\begin{aligned}
 \kappa_1 & = -2\pi\alpha a_2, \\
 (\kappa_1 a_1 + \kappa_2 a_2) & = \alpha a_2 a_3.
 \end{aligned} \tag{3.24}$$

Similarly, for $\zeta > 0$, we obtain

$$\begin{aligned}
 \kappa_2 & = -2\pi\alpha b_2, \\
 (\kappa_1 b_2 + \kappa_2 b_1) & = \alpha b_2 b_3.
 \end{aligned} \tag{3.25}$$

Solving (3.20), (3.24) and (3.25) gives

$$\begin{aligned} a_1 &= a_2 = -\frac{\kappa_1}{2\pi\alpha}, \\ b_1 &= b_2 = -\frac{\kappa_2}{2\pi\alpha}, \\ a_3 &= b_3 = \frac{1}{\alpha}(\kappa_1 + \kappa_2). \end{aligned} \tag{3.26}$$

Note that a_3 and b_3 are positive (as assumed in (3.11)).

Thus, the required three-dimensional Green's function for the case in which the interface $z = 0$ is high conducting is given by (3.10), (3.11) and (3.26).

3.4 Axisymmetric Green's Functions

For the bimaterial in Figure 3.2, according to (2.15), the functions $\tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ and $\tilde{G}_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ in the boundary integral equation in (2.14) can be written as

$$\begin{aligned} \tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) &= [H(z)H(z_0) + H(-z)H(-z_0)]G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0), \\ \tilde{G}_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) &= [H(z)H(z_0) + H(-z)H(-z_0)]G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \\ &\quad + G_1^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})), \end{aligned} \tag{3.27}$$

with the functions $G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ and $G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ as given in (2.17), the function $G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ being any solution of

$$\frac{\partial^2}{\partial r^2}[G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)] + \frac{1}{r} \frac{\partial}{\partial r}[G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)] + \frac{\partial^2}{\partial z^2}[G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)] = 0 \text{ for } \underline{\mathbf{x}} \in \Omega_1 \cup \Omega_2, \tag{3.28}$$

and the function $G_1^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ defined by

$$G_1^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) = n_r(\underline{\mathbf{x}}) \frac{\partial}{\partial r} [G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)] + n_z(\underline{\mathbf{x}}) \frac{\partial}{\partial z} [G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)]. \quad (3.29)$$

In general, we may take $G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ in (3.27) to be $G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) = 0$ (so that $\tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) = G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$). Nevertheless, we may find it advantageous to solve (3.28) for $G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ that satisfies the imperfect conditions on the interface Γ_0 of the bimaterial.

3.4.1 Low Conducting Interfaces

For the case in which the interface Γ_0 of the bimaterial is low conducting, $G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ is chosen in such a way that $\tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ in (3.27) satisfies the interfacial conditions

$$\left. \begin{aligned} \kappa_2 \frac{\partial \tilde{G}_0}{\partial z} \Big|_{z=0^+} &= \kappa_1 \frac{\partial \tilde{G}_0}{\partial z} \Big|_{z=0^-} \\ \lambda [\tilde{G}_0(r, 0^+; \underline{\mathbf{x}}_0) - \tilde{G}_0(r, 0^-; \underline{\mathbf{x}}_0)] &= \kappa_2 \frac{\partial \tilde{G}_0}{\partial z} \Big|_{z=0^+} \end{aligned} \right\} \text{for } 0 < r < \infty. \quad (3.30)$$

The Green's function $\tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ satisfying (3.30) can be obtained by performing an axial integration on the corresponding Green's function $\tilde{\Phi}(x, y, z; \xi, \eta, \zeta)$ for three-dimensional heat conduction (given by (3.10) together with (3.11) and (3.18)). We find that the required axisymmetric Green's function $\tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ for

low conducting interface Γ_0 is given by (3.27) with

$$\begin{aligned}
 & G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) \\
 = & H(-z_0) \left\{ H(-z) [G_0(\underline{\mathbf{x}}; r_0, -z_0) \right. \\
 & - \frac{2\lambda}{\kappa_1} \int_0^\infty G_0(\underline{\mathbf{x}}; r_0, u - z_0) \exp\left(-\frac{\lambda}{\kappa_2} \left(1 + \frac{\kappa_2}{\kappa_1}\right) u\right) du] \\
 & + H(z) \frac{2\lambda}{\kappa_2} \int_0^\infty G_0(\underline{\mathbf{x}}; r_0, z_0 - u) \exp\left(-\frac{\lambda}{\kappa_2} \left(1 + \frac{\kappa_2}{\kappa_1}\right) u\right) du \} \\
 & + H(z_0) \left\{ H(z) [G_0(\underline{\mathbf{x}}; r_0, -z_0) \right. \\
 & - \frac{2\lambda}{\kappa_2} \int_0^\infty G_0(\underline{\mathbf{x}}; r_0, -z_0 - u) \exp\left(-\frac{\lambda}{\kappa_2} \left(1 + \frac{\kappa_2}{\kappa_1}\right) u\right) du] \\
 & + H(-z) \frac{2\lambda}{\kappa_1} \int_0^\infty G_0(\underline{\mathbf{x}}; r_0, z_0 + u) \exp\left(-\frac{\lambda}{\kappa_2} \left(1 + \frac{\kappa_2}{\kappa_1}\right) u\right) du \}.
 \end{aligned} \tag{3.31}$$

Note that $G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ is obtained by integrating axially $\Phi^*(x, y, z; \xi, \eta, \zeta)$ given by (3.11) and (3.18).

The function $G_1^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ which corresponds to $G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ in (3.31) is given by

$$\begin{aligned}
 & G_1^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \\
 = & H(-z_0) \left\{ H(-z) [G_1(\underline{\mathbf{x}}; r_0, -z_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \right. \\
 & - \frac{2\lambda}{\kappa_1} \int_0^\infty G_1(\underline{\mathbf{x}}; r_0, u - z_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \exp\left(-\frac{\lambda}{\kappa_2} \left(1 + \frac{\kappa_2}{\kappa_1}\right) u\right) du] \\
 & + H(z) \frac{2\lambda}{\kappa_2} \int_0^\infty G_1(\underline{\mathbf{x}}; r_0, z_0 - u; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \exp\left(-\frac{\lambda}{\kappa_2} \left(1 + \frac{\kappa_2}{\kappa_1}\right) u\right) du \} \\
 & + H(z_0) \left\{ H(z) [G_1(\underline{\mathbf{x}}; r_0, -z_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \right. \\
 & - \frac{2\lambda}{\kappa_2} \int_0^\infty G_1(\underline{\mathbf{x}}; r_0, -z_0 - u; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \exp\left(-\frac{\lambda}{\kappa_2} \left(1 + \frac{\kappa_2}{\kappa_1}\right) u\right) du] \\
 & + H(-z) \frac{2\lambda}{\kappa_1} \int_0^\infty G_1(\underline{\mathbf{x}}; r_0, z_0 + u; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \exp\left(-\frac{\lambda}{\kappa_2} \left(1 + \frac{\kappa_2}{\kappa_1}\right) u\right) du \}.
 \end{aligned} \tag{3.32}$$

3.4.2 High Conducting Interfaces

For the case in which the interface Γ_0 of the bimaterial is high conducting, $G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ is chosen in such a way that $\tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ in (3.27) satisfies the interfacial conditions

$$\left. \begin{aligned} \tilde{G}_0(r, 0^+; \underline{\mathbf{x}}_0) &= \tilde{G}_0(r, 0^-; \underline{\mathbf{x}}_0) \\ \kappa_2 \frac{\partial \tilde{G}_0}{\partial z} \Big|_{z=0^+} - \kappa_1 \frac{\partial \tilde{G}_0}{\partial z} \Big|_{z=0^-} &= \alpha \frac{\partial^2 \tilde{G}_0}{\partial z^2} \Big|_{z=0} \end{aligned} \right\} \text{ for } 0 < r < \infty. \quad (3.33)$$

The function $G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ such that (3.33) holds is obtained by integrating axially $\Phi^*(x, y, z; \xi, \eta, \zeta)$ given by (3.11) and (3.26) (for three-dimensional heat conduction across a high conducting planar interface at $z = 0$), that is,

$$\begin{aligned} & G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) \\ = & H(-z_0) \left\{ H(-z) [-G_0(\underline{\mathbf{x}}; r_0, -z_0) \right. \\ & + \frac{2\kappa_1}{\alpha} \int_0^\infty G_0(\underline{\mathbf{x}}; r_0, u - z_0) \exp(-\frac{1}{\alpha}(\kappa_1 + \kappa_2)u) du] \\ & + H(z) \frac{2\kappa_1}{\alpha} \int_0^\infty G_0(\underline{\mathbf{x}}; r_0, z_0 - u) \exp(-\frac{1}{\alpha}(\kappa_1 + \kappa_2)u) du \} \\ & + H(z_0) \left\{ H(z) [-G_0(\underline{\mathbf{x}}; r_0, -z_0) \right. \\ & + \frac{2\kappa_2}{\alpha} \int_0^\infty G_0(\underline{\mathbf{x}}; r_0, -z_0 - u) \exp(-\frac{1}{\alpha}(\kappa_1 + \kappa_2)u) du] \\ & + H(-z) \frac{2\kappa_2}{\alpha} \int_0^\infty G_0(\underline{\mathbf{x}}; r_0, z_0 + u) \exp(-\frac{1}{\alpha}(\kappa_1 + \kappa_2)u) du \}. \end{aligned} \quad (3.34)$$

The function $G_1^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ which corresponds to $G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ in (3.34) is

given by

$$\begin{aligned}
 & G_1^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \\
 = & H(-z_0) \left\{ H(-z) [-G_1(\underline{\mathbf{x}}; r_0, -z_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \right. \\
 & + \frac{2\kappa_1}{\alpha} \int_0^\infty G_1(\underline{\mathbf{x}}; r_0, u - z_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \exp(-\frac{1}{\alpha}(\kappa_1 + \kappa_2)u) du] \\
 & + H(z) \frac{2\kappa_1}{\alpha} \int_0^\infty G_1(\underline{\mathbf{x}}; r_0, z_0 - u; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \exp(-\frac{1}{\alpha}(\kappa_1 + \kappa_2)u) du \} \\
 & + H(z_0) \left\{ H(z) [-G_1(\underline{\mathbf{x}}; r_0, -z_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \right. \\
 & + \frac{2\kappa_2}{\alpha} \int_0^\infty G_1(\underline{\mathbf{x}}; r_0, -z_0 - u; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \exp(-\frac{1}{\alpha}(\kappa_1 + \kappa_2)u) du] \\
 & + H(-z) \frac{2\kappa_2}{\alpha} \int_0^\infty G_1(\underline{\mathbf{x}}; r_0, z_0 + u; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \exp(-\frac{1}{\alpha}(\kappa_1 + \kappa_2)u) du \}.
 \end{aligned} \tag{3.35}$$

3.5 Axisymmetric Boundary Integral Equations

The boundary integral equations for axisymmetric heat conduction are given by (2.14) in Chapter 2. Here, for the bimaterial sketched in Figure 3.2, the boundary integral equations for axisymmetric heat conduction governed by (3.1) can be rewritten as

$$\begin{aligned}
 & \gamma_1(\underline{\mathbf{x}}_0) T(\underline{\mathbf{x}}_0) \\
 = & \int_{\Gamma_1} (T(\underline{\mathbf{x}}) \tilde{G}_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) - \tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))) r ds(\underline{\mathbf{x}}) \\
 & + \int_a^b (T(r, 0^-) \tilde{G}_1(r, 0^-; \underline{\mathbf{x}}_0; 0, 1) - \tilde{G}_0(r, 0^-; \underline{\mathbf{x}}_0) \frac{\partial}{\partial z} [T(\underline{\mathbf{x}})] \Big|_{z=0^-}) r dr \\
 & \text{for } \underline{\mathbf{x}}_0 = (r_0, z_0) \in \Omega_1 \cup \Gamma_0 \cup \Gamma_1,
 \end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
 & \gamma_2(\underline{\mathbf{x}}_0)T(\underline{\mathbf{x}}_0) \\
 &= \int_{\Gamma_2} (T(\underline{\mathbf{x}})\tilde{G}_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) - \tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}})))rds(\underline{\mathbf{x}}) \\
 & - \int_a^b (T(r, 0^+)\tilde{G}_1(r, 0^+; \underline{\mathbf{x}}_0; 0, 1) - \tilde{G}_0(r, 0^+; \underline{\mathbf{x}}_0) \frac{\partial}{\partial z}[T(\underline{\mathbf{x}})] \Big|_{z=0^+})rdr \\
 & \quad \text{for } \underline{\mathbf{x}}_0 \in \Omega_2 \cup \Gamma_0 \cup \Gamma_2,
 \end{aligned} \tag{3.37}$$

where $\gamma_i(\underline{\mathbf{x}}_0) = 1$ if $\underline{\mathbf{x}}_0$ lies in the interior of Ω_i , $\gamma_1(\underline{\mathbf{x}}_0)$ and $\gamma_2(\underline{\mathbf{x}}_0)$ are defined by

$$\begin{aligned}
 \gamma_1(\underline{\mathbf{x}}_0) &= \int_{\Gamma_1} \tilde{G}_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))rds(\underline{\mathbf{x}}) + \int_a^b \tilde{G}_1(r, 0^-; \underline{\mathbf{x}}_0; 0, 1)rdr \\
 & \quad \text{for } \underline{\mathbf{x}}_0 = (r_0, z_0) \in \Omega_1 \cup \Gamma_0 \cup \Gamma_1,
 \end{aligned} \tag{3.38}$$

and

$$\begin{aligned}
 \gamma_2(\underline{\mathbf{x}}_0) &= \int_{\Gamma_2} \tilde{G}_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))rds(\underline{\mathbf{x}}) - \int_a^b \tilde{G}_1(r, 0^+; \underline{\mathbf{x}}_0; 0, 1)rdr \\
 & \quad \text{for } \underline{\mathbf{x}}_0 \in \Omega_2 \cup \Gamma_0 \cup \Gamma_2,
 \end{aligned} \tag{3.39}$$

$ds(\underline{\mathbf{x}})$ denotes the length of an infinitesimal part of the curve $\Gamma_0 \cup \Gamma_i$, $\underline{\mathbf{n}}(\underline{\mathbf{x}}) = [n_r(\underline{\mathbf{x}}), n_z(\underline{\mathbf{x}})] = n_r(\underline{\mathbf{x}})\underline{\mathbf{e}}_r + n_z(\underline{\mathbf{x}})\underline{\mathbf{e}}_z$ ($\underline{\mathbf{e}}_r$ and $\underline{\mathbf{e}}_z$ are the unit base vectors along the r and z axes respectively) is the unit normal vector to $\Gamma_1 \cup \Gamma_2$ (at the point $\underline{\mathbf{x}}$) pointing out of $\Omega_1 \cup \Omega_2$, $P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ is the directional rate of change of the axisymmetric temperature along the vector $\underline{\mathbf{n}}(\underline{\mathbf{x}})$ as defined by

$$P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}})) = n_r(\underline{\mathbf{x}})\frac{\partial}{\partial r}[T(\underline{\mathbf{x}})] + n_z(\underline{\mathbf{x}})\frac{\partial}{\partial z}[T(\underline{\mathbf{x}})]. \tag{3.40}$$

$\tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ and $\tilde{G}_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ are the axisymmetric Green's functions derived in Section 3.4.

We may multiply κ_1 and κ_2 to (3.36) and (3.37) respectively and add up the two equations to obtain boundary integral equation

$$\begin{aligned}
 & \gamma_1(\underline{\mathbf{x}}_0)\kappa_1 T(\underline{\mathbf{x}}_0) + \gamma_2(\underline{\mathbf{x}}_0)\kappa_2 T(\underline{\mathbf{x}}_0) \\
 &= \sum_{i=1}^2 \int_{\Gamma_i} \kappa_i (T(\underline{\mathbf{x}}) \tilde{G}_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) - \tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))) r ds(\underline{\mathbf{x}}) \\
 &+ \int_a^b \kappa_1 (T(r, 0^-) \tilde{G}_1(r, 0^-; \underline{\mathbf{x}}_0; 0, 1) - \tilde{G}_0(r, 0^-; \underline{\mathbf{x}}_0) \left. \frac{\partial}{\partial z} [T(\underline{\mathbf{x}})] \right|_{z=0^-}) r dr \\
 &- \int_a^b \kappa_2 (T(r, 0^+) \tilde{G}_1(r, 0^+; \underline{\mathbf{x}}_0; 0, 1) - \tilde{G}_0(r, 0^+; \underline{\mathbf{x}}_0) \left. \frac{\partial}{\partial z} [T(\underline{\mathbf{x}})] \right|_{z=0^+}) r dr \\
 &\quad \text{for } \underline{\mathbf{x}}_0 \in \Omega_1 \cup \Omega_2 \cup \Gamma_1 \cup \Gamma_2.
 \end{aligned} \tag{3.41}$$

As we shall see, the integrals over the imperfect interface may be eliminated in (3.41) if $\tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ and $\tilde{G}_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ are appropriately selected.

3.5.1 Low Conducting Interfaces

Using (3.4) and (3.30) for low conducting interface Γ_0 , we find that (3.41) can be reduced to

$$\begin{aligned}
 & \{\gamma_1(\underline{\mathbf{x}}_0)\kappa_1 + \gamma_2(\underline{\mathbf{x}}_0)\kappa_2\} T(\underline{\mathbf{x}}_0) \\
 &= \sum_{i=1}^2 \int_{\Gamma_i} \kappa_i (T(\underline{\mathbf{x}}) \tilde{G}_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) - \tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))) r ds(\underline{\mathbf{x}}) \\
 &\quad \text{for } \underline{\mathbf{x}}_0 \in \Omega_1 \cup \Omega_2 \cup \Gamma_1 \cup \Gamma_2,
 \end{aligned} \tag{3.42}$$

if $\tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ and $\tilde{G}_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ are given by (3.27), together with (3.31) and (3.32).

Note that (3.42) does not contain any integral over the interface.

3.5.2 High Conducting Interfaces

For high conducting interface Γ_0 , using (3.6), (3.7) and (3.33) and noting that

$$\left. \frac{\partial^2 \tilde{G}_0}{\partial z^2} \right|_{z=0} = -\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} [\tilde{G}_0(r, 0; \mathbf{x}_0)] \right), \quad (3.43)$$

we find that (3.41) can be rewritten as

$$\begin{aligned} & \gamma_1(\mathbf{x}_0) \kappa_1 T(\mathbf{x}_0) + \gamma_2(\mathbf{x}_0) \kappa_2 T(\mathbf{x}_0) \\ &= \sum_{i=1}^2 \int_{\Gamma_i} \kappa_i (T(\mathbf{x}) \tilde{G}_1(\mathbf{x}; \mathbf{x}_0; \mathbf{n}(\mathbf{x})) - \tilde{G}_0(\mathbf{x}; \mathbf{x}_0) P(\mathbf{x}; \mathbf{n}(\mathbf{x}))) r ds(\mathbf{x}) \\ &+ \int_a^b \alpha \left\{ -\tilde{G}_0(r, 0; \mathbf{x}_0) \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \right\}_{z=0^+} + T(r, 0) \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} [\tilde{G}_0(r, 0; \mathbf{x}_0)] \right) \} r dr \\ & \quad \text{for } \mathbf{x}_0 \in \Omega_1 \cup \Omega_2 \cup \Gamma_1 \cup \Gamma_2. \end{aligned} \quad (3.44)$$

Using integration by parts, we find that

$$\begin{aligned} & \int_a^b \alpha \left\{ -\tilde{G}_0(r, 0; \mathbf{x}_0) \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \right\}_{z=0^+} \\ &+ T(r, 0) \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} [\tilde{G}_0(r, 0; \mathbf{x}_0)] \right) \} r dr \\ &= -b \tilde{G}_0(b, 0; \mathbf{x}_0) \alpha \frac{\partial T}{\partial r} \Big|_{(r,z)=(b,0)} + a \tilde{G}_0(a, 0; \mathbf{x}_0) \alpha \frac{\partial T}{\partial r} \Big|_{(r,z)=(a,0)} \\ &+ b T(b, 0) \alpha \frac{\partial}{\partial r} [\tilde{G}_0(\mathbf{x}; \mathbf{x}_0)] \Big|_{(r,z)=(b,0)} - a T(a, 0) \alpha \frac{\partial}{\partial r} [\tilde{G}_0(\mathbf{x}; \mathbf{x}_0)] \Big|_{(r,z)=(a,0)}. \end{aligned} \quad (3.45)$$

It follows that (3.44) reduces to

$$\begin{aligned}
 & \{\gamma_1(\underline{\mathbf{x}}_0)\kappa_1 + \gamma_2(\underline{\mathbf{x}}_0)\kappa_2\}T(\underline{\mathbf{x}}_0) \\
 & - bT(b, 0)\alpha \frac{\partial}{\partial r}[\tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)] \Big|_{(r,z)=(b,0)} + aT(a, 0)\alpha \frac{\partial}{\partial r}[\tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)] \Big|_{(r,z)=(a,0)} \\
 & + b\tilde{G}_0(b, 0; \underline{\mathbf{x}}_0)\alpha \frac{\partial T}{\partial r} \Big|_{(r,z)=(b,0)} - a\tilde{G}_0(a, 0; \underline{\mathbf{x}}_0)\alpha \frac{\partial T}{\partial r} \Big|_{(r,z)=(a,0)} \\
 & = \sum_{i=1}^2 \int_{\Gamma_i} \kappa_i (T(\underline{\mathbf{x}})\tilde{G}_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) - \tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))) r ds(\underline{\mathbf{x}}) \\
 & \quad \text{for } \underline{\mathbf{x}}_0 \in \Omega_1 \cup \Omega_2 \cup \Gamma_1 \cup \Gamma_2.
 \end{aligned} \tag{3.46}$$

Thus, the integral over the high conducting interface Γ_0 vanishes if the Green's function $\tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ is given by (3.27) with $G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ in (3.34).

3.6 Boundary Element Procedures

In this section, we describe boundary element procedures for determining $T(\underline{\mathbf{x}})$ and $P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ (whichever is not known) on $\Gamma_1 \cup \Gamma_2$. Once $T(\underline{\mathbf{x}})$ and $P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ are completely known on $\Gamma_1 \cup \Gamma_2$, we can obtain the temperature at any point $\underline{\mathbf{x}}_0$ in the interior of the domains by using $\gamma_1(\underline{\mathbf{x}}_0) = 1$ and $\gamma_2(\underline{\mathbf{x}}_0) = 0$ for $\underline{\mathbf{x}}_0$ in the interior of Ω_1 or $\gamma_1(\underline{\mathbf{x}}_0) = 0$ and $\gamma_2(\underline{\mathbf{x}}_0) = 1$ for $\underline{\mathbf{x}}_0$ in the interior of Ω_2 in (3.42) (for low conducting interface) or in (3.46) (for high conducting interface).

We discretize the boundary $\Gamma_1 \cup \Gamma_2$ into N straight line elements denoted by $B^{(1)}, B^{(2)}, \dots, B^{(N-1)}$ and $B^{(N)}$. As (3.42) or (3.46) does not contain any integral over the interface Γ_0 (because of the use of the special Green's function), we do not need to discretize the interface Γ_0 .

For a simple approximation, T and P are taken to be constants over an

element of $\Gamma_1 \cup \Gamma_2$, specifically

$$\left. \begin{aligned} T(\underline{\mathbf{x}}) &\simeq T^{(m)} \\ P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}})) &\simeq P^{(m)} \end{aligned} \right\} \text{ for } \underline{\mathbf{x}} \in B^{(m)} \ (m = 1, 2, \dots, N), \quad (3.47)$$

where $T^{(m)}$ and $P^{(m)}$ are constants.

Each boundary element is associated with only one unknown constant. Specifically, if T is specified over the element $B^{(m)}$ according to the first line of (3.8) then $P^{(m)}$ is the unknown over $B^{(m)}$. On the other hand, if P is given by the second line of (3.8) over $B^{(m)}$, we can express $P^{(m)}$ in terms of $T^{(m)}$ and regard $T^{(m)}$ as the unknown constant over $B^{(m)}$.

3.6.1 Low Conducting Interfaces

For the case where Γ_0 is a low conducting interface, we let $\underline{\mathbf{x}}_0$ in (3.42) be given in turn by the midpoints of $B^{(i)}$ ($i = 1, 2, \dots, N$), together with (3.8) to obtain

$$\begin{aligned} &\{\gamma_1(\widehat{\underline{\mathbf{x}}}^{(i)})\kappa_1 + \gamma_2(\widehat{\underline{\mathbf{x}}}^{(i)})\kappa_2\}[d^{(i)}T^{(i)} + (1 - d^{(i)})f_0(\widehat{\underline{\mathbf{x}}}^{(i)})] \\ &= \sum_{m=1}^N \kappa^{(m)} \{[d^{(m)}T^{(m)} + (1 - d^{(m)})f_0(\widehat{\underline{\mathbf{x}}}^{(m)})] \int_{B^{(m)}} \widetilde{G}_1(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{x}}}^{(i)}; \underline{\mathbf{n}}^{(m)}) r ds(\underline{\mathbf{x}}) \\ &\quad - [d^{(m)}(f_1(\widehat{\underline{\mathbf{x}}}^{(m)}) + f_2(\widehat{\underline{\mathbf{x}}}^{(m)})T^{(m)}) + (1 - d^{(m)})P^{(m)}] \int_{B^{(m)}} \widetilde{G}_0(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{x}}}^{(i)}) r ds(\underline{\mathbf{x}})\} \\ &\quad \text{for } i = 1, 2, \dots, N, \end{aligned} \quad (3.48)$$

where $\widehat{\underline{\mathbf{x}}}^{(i)}$ is the midpoint of $B^{(i)}$, $d^{(m)} = 0$ if T is specified on the m -th element $B^{(m)}$ as given by the first line of (3.8), $d^{(m)} = 1$ if the boundary condition given by the second line of (3.8) is applicable on $B^{(m)}$, $\underline{\mathbf{n}}^{(m)}$ is the unit normal vector to $B^{(m)}$ pointing away from the solution domain $\Omega_1 \cup \Omega_2$, $\kappa^{(m)} = \kappa_1$ if $B^{(m)}$ is an element on the boundary of Ω_1 and $\kappa^{(m)} = \kappa_2$ if $B^{(m)}$ is an element on the boundary of Ω_2 .

In (3.48), the integrals over $B^{(m)}$ are to be interpreted in the Cauchy principal sense if $\hat{\mathbf{x}}^{(i)}$ is the midpoint of $B^{(m)}$ (that is, if $m = i$). The Cauchy principal integrals can be accurately evaluated by using a highly accurate Gaussian quadrature. In the numerical problems below, 10-point Gaussian quadrature is used. For problems that have analytical solutions, the numerical solutions show reasonable good accuracy, indicating that the Gaussian quadrature used is sufficiently accurate.

Now (3.48) gives a system of N linear algebraic equations containing N unknowns given by either $T^{(k)}$ or $P^{(k)}$ ($k = 1, 2, \dots, N$). Once the unknowns on the boundary are determined, the temperature at the interior point of the domain $\Omega_1 \cup \Omega_2$ can be obtained as explained above.

3.6.2 High Conducting Interfaces

For the case where Γ_0 is a high conducting interface, if we proceed as before by collocating (3.46) at the midpoint of each boundary element, we obtain

$$\begin{aligned}
 & \{\gamma_1(\hat{\mathbf{x}}^{(i)})\kappa_1 + \gamma_2(\hat{\mathbf{x}}^{(i)})\kappa_2\} [d^{(i)}T^{(i)} + (1 - d^{(i)})f_0(\hat{\mathbf{x}}^{(i)})] \\
 & - bT(b, 0)\alpha \frac{\partial}{\partial r} [\tilde{G}_0(\mathbf{x}; \hat{\mathbf{x}}^{(i)})] \Big|_{(r,z)=(b,0)} + aT(a, 0)\alpha \frac{\partial}{\partial r} [\tilde{G}_0(\mathbf{x}; \hat{\mathbf{x}}^{(i)})] \Big|_{(r,z)=(a,0)} \\
 & + b\tilde{G}_0(b, 0; \hat{\mathbf{x}}^{(i)})\alpha \frac{\partial T}{\partial r} \Big|_{(r,z)=(b,0)} - a\tilde{G}_0(a, 0; \hat{\mathbf{x}}^{(i)})\alpha \frac{\partial T}{\partial r} \Big|_{(r,z)=(a,0)} \\
 & = \sum_{m=1}^N \kappa^{(m)} \{ [d^{(m)}T^{(m)} + (1 - d^{(m)})f_0(\hat{\mathbf{x}}^{(m)})] \int_{B^{(m)}} \tilde{G}_1(\mathbf{x}; \hat{\mathbf{x}}^{(i)}; \mathbf{n}^{(m)}) r ds(\mathbf{x}) \\
 & - [d^{(m)}(f_1(\hat{\mathbf{x}}^{(m)}) + f_2(\hat{\mathbf{x}}^{(m)})T^{(m)}) + (1 - d^{(m)})P^{(m)}] \\
 & \times \int_{B^{(m)}} \tilde{G}_0(\mathbf{x}; \hat{\mathbf{x}}^{(i)}) r ds(\mathbf{x}) \} \\
 & \text{for } i = 1, 2, \dots, N, \quad (3.49)
 \end{aligned}$$

where $\kappa^{(m)}$ is as defined below (3.48).

The terms $T(a, 0)$, $T(b, 0)$, $\partial T/\partial r|_{(r,z)=(a,0)}$ and $\partial T/\partial r|_{(r,z)=(b,0)}$ in (3.49) are unknown constants. They can, however, be approximated in terms of T and P on boundary elements near $(a, 0)$ and/or $(b, 0)$. How the required approximations may be made depends on the geometries of the solution domains – see, for example, Problems 3.3 and 3.4 in Section 3.7 below. Thus, (3.49) can be solved as a system of N linear algebraic equations for N unknowns given by either $T^{(k)}$ or $P^{(k)}$ ($k = 1, 2, \dots, N$).

3.7 Specific Problems

In all the numerical problems below as well as those in the subsequent chapters, the setting up of relevant system of linear algebraic equation is coded in FORTRAN 77. The linear algebraic equations are solved using the *LU* decomposition technique, specifically by using subroutines listed in [58].

Problem 3.1

To test the boundary element procedure for Γ_0 that is low conducting, consider the regions Ω_1 and Ω_2 as sketched in Figure 3.3. Note that Ω_1 and Ω_2 are defined by the curves $r^2 + z^2 = 4$ and $r^2 + z^2 = 1$ and the lines $r = 0$, $r = 1$, $r = 2$ and $z = -1$ on the rz plane. For a particular problem take $\kappa_1 = 1$, $\kappa_2 = 2$ and $\lambda = 1$. The exterior boundary of $\Omega_1 \cup \Omega_2$ is approximated using N straight line elements.

The boundary conditions on the exterior boundary of $\Omega_1 \cup \Omega_2$ are given by

$$\left. \begin{aligned} P(2, z; 1, 0) &= 4z \\ P(1, z; -1, 0) &= -2z \end{aligned} \right\} \text{ for } -1 < z < 0,$$

Figure 3.3: A geometrical sketch of Problem 3.1 on the rz plane.

$$\begin{aligned}
 T(r, -1) &= -r^2 + \frac{2}{3} \text{ for } 1 < r < 2, \\
 T(r, z) &= \frac{1}{2}r^2z - \frac{1}{3}z^3 + r^2 - 2z^2 \\
 &\quad \text{for } r^2 + z^2 = 1, \ 0 < r < 1, \\
 P(r, z; \frac{1}{2}r, \frac{1}{2}z) &= -\frac{1}{2}z^3 - 2z^2 + \frac{3}{4}r^2z + r^2 \\
 &\quad \text{for } r^2 + z^2 = 4, \ 0 < r < 2.
 \end{aligned}$$

It can be easily verified that the exact solution for the problem here is given by

$$T(r, z) = \begin{cases} r^2z - \frac{2}{3}z^3 & \text{for } (r, z) \in \Omega_1, \\ \frac{1}{2}r^2z - \frac{1}{3}z^3 + r^2 - 2z^2 & \text{for } (r, z) \in \Omega_2. \end{cases}$$

Numerical values are obtained for the temperature at the interior of the regions by solving equation (3.48) using $N = 60$ and $N = 120$. Through the use of (3.42) with $\gamma_1(\underline{\mathbf{x}}_0) = 1, \gamma_2(\underline{\mathbf{x}}_0) = 0$ for $\underline{\mathbf{x}}_0$ in the interior of Ω_1 and $\gamma_1(\underline{\mathbf{x}}_0) = 0, \gamma_2(\underline{\mathbf{x}}_0) = 1$ for $\underline{\mathbf{x}}_0$ in the interior of Ω_2 , numerical values of the temperature at $r = 1.5$ and $r = 1.75$ for $-1 < z < 0$ and at $r^2 + z^2 = (1.75)^2$

Figure 3.4: Plots of the numerical and exact temperature $T(r, z)$ for $-1 < z < 1.5$ (Problem 3.1).

and $r^2 + z^2 = (1.5)^2$ for $0 < z < 1.5$ are obtained and compared graphically with the exact temperature in Figure 3.4. On the whole, the numerical and exact temperature agree well with each other. Note that the gap in the graph is due to the temperature jump across the interface Γ_0 at $z = 0$.

Problem 3.2

Consider now the case in which Ω_1 and Ω_2 are given by

$$\begin{aligned}\Omega_1 &= \{(r, z) : 0 \leq r < 1, -1 < z < 0\}, \\ \Omega_2 &= \{(r, z) : 0 \leq r < \frac{3}{2}, 0 < z < \frac{3}{2}\},\end{aligned}$$

as sketched in Figure 3.5. As in Problem 3.1, the interface Γ_0 between Ω_1 and Ω_2 is taken to be low conducting. Note that for this particular case the

Figure 3.5: A geometrical sketch of Problem 3.2 on the rz plane.

exterior boundary of the bimaterial lies on part of the $z = 0$ plane (that is, $1 < r < 3/2$, $z = 0$).

For a particular problem, we take $\kappa_1 = 1$, $\kappa_2 = 1/2$ and $\lambda = 1$ and the boundary conditions as

$$\begin{aligned}
 T(1, z) &= 4 - 2z^2 + 2z \text{ for } -1 < z < 0, \\
 P(r, -1; 0, -1) &= -6 \text{ for } 0 < r < 1, \\
 T(r, \frac{3}{2}) &= r^2 + \frac{13}{2} \text{ for } 0 < r < \frac{3}{2}, \\
 P(\frac{3}{2}, z; 1, 0) &= 3 \text{ for } 0 < z < \frac{3}{2}, \\
 T(r, 0) &= r^2 + 5 \text{ for } 1 < r < \frac{3}{2}.
 \end{aligned}$$

The exact solution for the particular problem here is given by

$$T(r, z) = \begin{cases} r^2 - 2z^2 + 2z + 3 & \text{for } (r, z) \in \Omega_1, \\ r^2 - 2z^2 + 4z + 5 & \text{for } (r, z) \in \Omega_2. \end{cases}$$

Table 3.1: Numerical and exact values of T at selected interior points for Problem 3.2.

Point (r, z)	$N = 25$	$N = 50$	$N = 100$	$N = 200$	Exact
$(0.30, -0.80)$	0.21635	0.21094	0.21005	0.20996	0.21000
$(0.75, -0.35)$	2.61420	2.61655	2.61724	2.61743	2.61750
$(0.40, -0.40)$	2.03921	2.03947	2.03977	2.03992	2.04000
$(1.40, 0.20)$	7.60894	7.66118	7.67354	7.67800	7.68000
$(0.50, 1.25)$	7.11813	7.12282	7.12431	7.12478	7.12500
$(0.20, 0.80)$	6.94401	6.95498	6.95848	6.95954	6.96000

The exterior boundary of the bimaterial is discretized into N straight line elements. The numerical values of T at various selected points in $\Omega_1 \cup \Omega_2$ are computed using (3.42) with $\gamma_1(\mathbf{x}_0) = 1$ and $\gamma_2(\mathbf{x}_0) = 0$ for $\mathbf{x}_0$ in the interior of Ω_1 and $\gamma_1(\mathbf{x}_0) = 0$ and $\gamma_2(\mathbf{x}_0) = 1$ for $\mathbf{x}_0$ in the interior of Ω_2 . They are compared with the exact values in Table 3.1 for $N = 25, 50, 100$ and 200 . The numerical values are reasonably accurate and they converge to the exact solution when the calculation is refined by reducing the sizes of the boundary elements used (that is, when N is increased from 25 to 200). All percentage errors of the numerical values for $N = 200$ are less than 0.05%.

Problem 3.3

To check the boundary element procedure for a bimaterial with a high conducting interface, take

$$\begin{aligned}\Omega_1 &= \{(r, z) : 0 \leq r < 1, -1 < z < 0\}, \\ \Omega_2 &= \{(r, z) : 0 \leq r < 1, 0 < z < 1\},\end{aligned}$$

as illustrated in Figure 3.6.

Figure 3.6: A geometrical sketch of Problem 3.3 on the rz plane.

We take $\kappa_1 = 6$, $\kappa_2 = 2$, and $\alpha = 7/4$ and the boundary conditions as

$$\begin{aligned}
 P(1, z; 1, 0) &= 4 + z \text{ for } -1 < z < 0, \\
 T(1, z) &= 2 + \frac{1}{2}z - 4z^2 - z^3 \text{ for } 0 < z < 1, \\
 \\
 \left. \begin{aligned}
 T(r, -1) &= \frac{3}{2}r^2 - \frac{17}{3} \\
 P(r, 1; 0, 1) &= \frac{3}{2}r^2 - 12
 \end{aligned} \right\} \text{ for } 0 < r < 1.
 \end{aligned}$$

The exact solution of the particular problem here is given by

$$T(r, z) = \begin{cases} 2r^2 - 4z^2 + \frac{1}{2}r^2z - \frac{1}{3}z^3 + 2z & \text{for } (r, z) \in \Omega_1, \\ 2r^2 - 4z^2 + \frac{3}{2}r^2z - z^3 - z & \text{for } (r, z) \in \Omega_2. \end{cases}$$

The exterior boundary of the bimaterial is discretized into N straight line elements. To solve (3.49) as a system of N linear algebraic equations for N

Table 3.2: Numerical and exact values of T at selected interior points for Problem 3.3.

Point (r, z)	$N = 40$	$N = 80$	$N = 160$	$N = 320$	Exact
$(0.500, -0.500)$	-1.52954	-1.52386	-1.52198	-1.52130	-1.52083
$(0.200, -0.100)$	-0.17288	-0.16598	-0.16350	-0.16249	-0.16167
$(0.700, -0.950)$	-4.47882	-4.47771	-4.47724	-4.47707	-4.47696
$(0.900, 0.900)$	-2.14050	-2.15158	-2.15451	-2.15525	-2.15550
$(0.100, 0.500)$	-1.59994	-1.59895	-1.59828	-1.59790	-1.59750
$(0.750, 0.001)$	1.10860	1.11827	1.12191	1.12347	1.12484

unknowns, we have to approximate $T(1, 0)$ and $\partial T / \partial r|_{(r,z)=(1,0)}$ in terms of T and P on the boundary elements. (Note that for this particular problem, $a = 0$ and $b = 1$.) If the exterior boundary is discretized in such a way that the first and the last elements ($B^{(1)}$ and $B^{(N)}$ respectively) are of equal length, lie on $r = 1$ and have $(1, 0)$ as one of their endpoints, then we can make the approximations

$$\begin{aligned}
T(1, 0) &\simeq \frac{1}{2}(T^{(1)} + T^{(N)}), \\
\left. \frac{\partial T}{\partial r} \right|_{(r,z)=(1,0)} &\simeq \frac{1}{2}(P^{(1)} + P^{(N)}).
\end{aligned}$$

Numerical values of the temperature at selected interior points are compared with the exact values in Table 3.2. On the whole, the numerical values at the selected interior points are in good agreement with the exact solution and there is improve in accuracy when N is increased from 40 to 320 (again, calculation is refined by reducing the sizes of the boundary elements used). The numerical results here also justify the above approximations for the terms $T(1, 0)$ and $\partial T / \partial r|_{(r,z)=(1,0)}$ in (3.49).

Constant elements are used in the calculations in all the numerical examples of this chapter. If the Δx is the length of a typical element, the order of

Figure 3.7: Plots of $\log_{10}(|\text{error}|)$ against $\log_{10}(\Delta x)$ for the numerical solution at several points taken from Table 3.2.

accuracy is expected to be $O(\Delta x)$. To illustrate this, we plot $\log_{10}(|\text{error}|)$ (where $|\text{error}|$ denotes the magnitude of the error of the numerical solution) against $\log_{10}(\Delta x)$ for the numerical solution at several points (from Table 3.2) in Figure 3.7. The slopes of the lines of best fit are found to be between 1 and 2, mainly closer to 1, as may be expected.

Problem 3.4

Problem 3.3 deals with relatively simple rectangular domains on the rz -plane. For a more general test problem involving a high conducting interface, we take here Ω_1 with a slanted boundary $r = z + 3/2$ and Ω_2 with part of $z = 0$ as its exterior boundary. A sketch of $\Omega_1 \cup \Omega_2$ is given in Figure 3.8.

We take $\kappa_1 = 1$, $\kappa_2 = 1/2$ and $\alpha = 1/8$ and the boundary conditions on

 Figure 3.8: A geometrical sketch of Problem 3.4 on the rz plane.

the exterior boundary of $\Omega_1 \cup \Omega_2$ as

$$\begin{aligned}
 T(r, z) &= r^2 - 2z^2 + \frac{1}{2}r^2z - \frac{1}{3}z^3 \\
 \text{for } r &= z + \frac{3}{2}, -1 < z < 0, \\
 P(r, -1; 0, -1) &= -3 - \frac{1}{2}r^2 \text{ for } 0 < r < \frac{1}{2}, \\
 P(r, 2; 0, 1) &= -17 + r^2 \text{ for } 0 < r < 2, \\
 P(2, z; 1, 0) &= 4(1 + z) \text{ for } 0 < z < 2, \\
 T(r, 0) &= r^2 \text{ for } \frac{3}{2} < r < 2.
 \end{aligned}$$

The exact solution of the problem here is given by

$$T(r, z) = \begin{cases} r^2 - 2z^2 + \frac{1}{2}r^2z - \frac{1}{3}z^3 & \text{for } (r, z) \in \Omega_1, \\ r^2 - 2z^2 + r^2z - \frac{2}{3}z^3 - z & \text{for } (r, z) \in \Omega_2. \end{cases}$$

To solve (3.49), we have to approximate $T(3/2, 0)$ and $\partial T / \partial r|_{(r,z)=(3/2,0)}$ in terms of T and P on the boundary elements which approximate the exterior

Figure 3.9: Plots of the numerical and exact boundary temperature $T(1, z)$ for $-\frac{1}{2} < z < 2$ (Problem 3.4).

boundary of $\Omega_1 \cup \Omega_2$. If there is a small horizontal boundary element at the intersection between the slanted boundary $r = z + 3/2$ and the vertical boundary $z = 0$ then $\partial T / \partial r|_{(r,z)=(3/2,0)}$ can be approximated as P on that horizontal element. For this purpose, we approximate the line segment $r = z + 3/2$, $-d < z < 0$, where d is a very small number, by using a small horizontal line element of length d (see Figure 3.8). If the small horizontal line element is taken to be the first element $B^{(1)}$ then

$$\begin{aligned} T(3/2, 0) &\simeq T^{(1)}, \\ \left. \frac{\partial T}{\partial r} \right|_{(r,z)=(3/2,0)} &\simeq P^{(1)}, \end{aligned}$$

where $T^{(1)}$ can be easily worked out from the given boundary conditions (since T is specified on $r = z + 3/2$) and $P^{(1)}$ is an unknown to be determined.

Numerical values are obtained for T by using $N = 141$ and $N = 281$

Figure 3.10: Plots of the numerical and exact boundary temperature $T(r, z)$ for $z = 1$ and $z = 2$ ($0 < r < 2$) (Problem 3.4).

with $d = 0.0001$. The numerical results are compared graphically with the exact solution as shown in Figure 3.9 and Figure 3.10. Figure 3.9 shows the temperature along $r = 1$ ($-\frac{1}{2} < z < 2$) while Figure 3.10 captures the variation of the temperature at $z = 1$ and $z = 2$ ($0 < r < 2$). On the whole, the numerical and exact temperature values agree well with each other.

Problem 3.5

Here we consider a thermal management system modeled by two homogeneous cylindrical solids as sketched in Figure 3.11. The regions Ω_1 and Ω_2 model the computer chip and the heat sink respectively while the interface Γ_0 (line $z = 0$, $0 < r < r_2$) represents a thin layer of carbon nanotubes or nanocylinders of high thermal conductivity. We model the interface Γ_0 as high conducting.

A constant heat flux q_0 flows into the system through $z = -z_1$, $0 < r < r_1$.

Figure 3.11: A geometrical sketch of Problem 3.5 on the rz plane.

There is a uniform convective cooling at the end $z = z_2$, $0 < r < r_2$. Elsewhere, the exterior boundary of the thermal system is thermally insulated. More precisely, the boundary conditions on the sides that are not thermally insulated are as follows:

$$\begin{aligned} -\kappa_1 P(r, -z_1; 0, -1) &= q_0 \text{ for } 0 < r < r_1, \\ -\kappa_2 P(r, z_2; 0, 1) &= h[T(r, z_2) - T_a] \text{ for } 0 < r < r_2, \end{aligned}$$

where q_0 is the magnitude of the specified heat flux, h is the heat convection coefficient and T_a is the ambient temperature of the system.

We study the effect of the interfacial parameter α (assumed to be constant) on the thermal performance of the heat dissipation system. For this purpose, we take the radii r_1 and r_2 and the lengths z_1 and z_2 to be $r_2/r_1 = 2$, $z_2/r_1 = 2$ and $z_1/r_1 = 1$. The numerical results are obtained by employing a total of 140 elements on the exterior boundary of $\Omega_1 \cup \Omega_2$. To capture the thermal behaviors more accurately near the region of heating, it may be necessary to employ

Figure 3.12: Plots of $\kappa_1(T - T_a)/q_0 z_1$ against z/z_1 for a few selected values of $\alpha/z_1 \kappa_2$ (for high conducting interfaces).

more elements on the side $z = -z_1$, especially near the point $(r, z) = (r_1, -z_1)$ where the boundary heat flux is discontinuous. Using $h z_1 / \kappa_2 = 5 \times 10^{-3}$ and $\kappa_1 / \kappa_2 = 0.5$, the non-dimensionalized temperature $\kappa_1(T - T_a)/q_0 z_1$ along the z -axis are plotted against z/z_1 for selected values of the non-dimensionalized parameter $\alpha/z_1 \kappa_2$.

In Figure 3.12, the solid line ($\alpha/z_1 \kappa_2 = 0.1$) approximates the plot of the non-dimensionalized temperature profile for the case in which the interface between the chip and heat sink is nearly perfectly bonded (for perfectly bonded or ideal interface, $\alpha/z_1 \kappa_2 = 0$). As anticipated, at a given point on the z -axis, the non-dimensionalized temperature in both the computer chip and heat sink decreases as $\alpha/z_1 \kappa_2$ increases. Hence, the thin layer of carbon nanotubes or nanocylinders of high thermal conductivity enhances the heat dissipation performance of the system.

Figure 3.13: Plots of $\kappa_1(T - T_a)/q_0z_1$ against z/z_1 for a few selected values of $\lambda z_1/\kappa_2$ (for low conducting interfaces).

Still with $r_2/r_1 = 2$, $z_2/r_1 = 2$, $z_1/r_1 = 1$, $hz_1/\kappa_2 = 5 \times 10^{-3}$ and $\kappa_1/\kappa_2 = 0.5$, we will now investigate the case whereby the interface between the chip and the sink is filled with microscopic voids. We regard this interface as low conducting. Again, we plot $\kappa_1(T - T_a)/q_0z_1$ against z/z_1 for selected values of the non-dimensionalized parameter $\lambda z_1/\kappa_2$ as shown in Figure 3.13. The solid line ($\lambda z_1/\kappa_2 = 100$) gives the temperature profile for the case of a nearly ideal interface as there is negligible temperature jump across the interface at $z = 0$. Note that the solid lines in both Figure 3.12 and Figure 3.13 give the temperature profile for a nearly perfect interface. The temperature in the chip in Figure 3.13 is higher than that in Figure 3.12. Thus, the effect of the low conducting interface on heat flow is opposite to that of the high conducting one, that is, the low conducting interface obstructs rather than enhance the heat flow from the chip into the sink. As expected, when the obstruction of the heat

flow is higher (that is, when $\lambda z_1/\kappa_2$ has a lower value), the temperature jump across the interface at $z/z_1 = 0$ is bigger. Also, the differences between the temperature distributions for the different values of $\lambda z_1/\kappa_2$ are much smaller in the sink compared to those in the chip.

On the whole, Figures 3.12 and 3.13 summarize the effects of the three types of interfaces – low conducting, perfectly conducting and high conducting ones – on the thermal performance of the heat dissipation system in Figure 3.11.

3.8 Summary

Special steady state axisymmetric Green's functions are derived for low and high conducting interfaces between two thermally isotropic half-spaces. Boundary element procedures based on these special Green's functions are proposed for analyzing the steady state axisymmetric temperature distribution in bimetals with low or high conducting planar interfaces. As the Green's functions satisfy the relevant interfacial conditions, the boundary element procedures do not require the interfaces to be discretized into elements, giving rise to smaller systems of linear algebraic equations to be solved. The procedures are applied to solve particular problems with known exact solutions. The numerical solutions obtained confirm the validity of the Green's functions and the proposed Green's function boundary element procedures.

Chapter 4

Steady State Axisymmetric Heat Conduction Across Imperfect Curved Interfaces

4.1 Introduction

The steady state axisymmetric heat conduction in bimetals with curved interfaces is considered in this chapter. Low and high conducting interfacial conditions are derived in axisymmetric coordinates for the curved interfaces. As Green's functions satisfying the interfacial conditions are difficult (if not impossible) to derive analytically in explicit forms for curved interfaces, a hypersingular boundary integral equation method is proposed here for analyzing the temperature distribution in the bimetals.

In the approach proposed here, the functions $\tilde{G}_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ and $\tilde{G}_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ in the axisymmetric boundary integral equation (2.14) are respectively given by $G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ and $G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ in (2.17) (that is, the functions $G_0^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ and $G_1^*(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ in (2.15) are set to zero). Since the interfacial conditions are not satisfied by the fundamental solution $G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$, the boundary integral equation for the bimaterial under consideration inevitably contains integrals

over opposite sides of the curved interface of the bimaterial. Nevertheless, the boundary integral equation can be written in such a way that only the temperature jump across opposite sides of the interface appears as unknown function on a low conducting interface and only the directional rate of change of temperature functions on opposite sides of the interface appear as unknown functions on a high conducting interface. The boundary integral equation is then used to derive a hypersingular boundary integral equation for the imperfect interfacial condition.

The hypersingular integral solution approach mentioned above was used in Ang and Fan [9] to analyze the quasi-state antiplane deformations of an elastic bimaterial with a weak visco-elastic planar interface. It was also used in Ang [5] to formulate a steady state two-dimensional heat conduction problem for a bimaterial with a low conducting curved interface. Prior to the work in this chapter, there is apparently no published work on the extension of the hypersingular boundary integral approach to high conducting interfaces or to axisymmetric heat conduction across imperfect interfaces.

In recent years, the analysis of interfaces with nonlinear conditions prescribed in accordance with the Stefan-Boltzmann law for radiations has attracted the attention of some researchers. For example, Yang, Yamamoto and Cheng [78] studied a problem involving one-dimensional heat conduction in multi-layered materials with nonlinear interfaces, while Hu, Xu and Chen [46] investigated the same problem for three-dimensional heat conductions. Those nonlinear interfaces may be regarded as low conducting interfaces with temperature dependent properties. The axisymmetric hypersingular boundary integral approach here for low conducting interfaces is also applied together with a corrective-predictor (iterative) procedure to analyze the steady state axisymmetric heat conduction in a bimaterial with the nonlinear Stefan-Boltzmann

Figure 4.1: A planar sketch of an axisymmetric trilayered body.

conditions.

4.2 Interfacial Conditions for Axisymmetric Imperfect Curved Interfaces

On the Orz axisymmetric coordinate plane, consider the regions Ω_1 and Ω_2 separated by a thin layer Ω_0 of uniform thickness δ as sketched in Figure 4.1. The boundary separating Ω_0 and Ω_1 is the curve Γ_{01} while that separating Ω_0 and Ω_2 is Γ_{02} . If $\underline{n}$ is the unit normal vector to the curve Γ_{02} pointing towards Ω_1 , then the curve Γ_{01} is obtained by shifting each and every point on Γ_{02} by a distance of δ in the direction of $\underline{n}$.

A multi-layered body is obtained by rotating Ω_0 , Ω_1 and Ω_2 by an angle of 360° about the z axis.

The temperature distribution T inside the multi-layered solid is a function

Figure 4.2: An axisymmetric bimaterial with interface Γ_0 .

of r and z only and it satisfies the axisymmetric steady-state heat equation in (3.1). The materials in Ω_0 , Ω_1 and Ω_2 are assumed perfectly bonded along the common boundary between any two materials, that is,

$$\left. \begin{aligned} T^{(0)}(\underline{\mathbf{x}}) &= T^{(1)}(\underline{\mathbf{x}}) \\ \kappa^{(0)} P^{(0)}(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}})) &= \kappa^{(1)} P^{(1)}(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \end{aligned} \right\} \text{ for } \underline{\mathbf{x}} \in \Gamma_{01},$$

$$\left. \begin{aligned} T^{(0)}(\underline{\mathbf{x}}) &= T^{(2)}(\underline{\mathbf{x}}) \\ \kappa^{(0)} P^{(0)}(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}})) &= \kappa^{(2)} P^{(2)}(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}})) \end{aligned} \right\} \text{ for } \underline{\mathbf{x}} \in \Gamma_{02},$$
(4.1)

where $\underline{\mathbf{x}} = (r, z)$, $\kappa^{(i)}$ is the thermal conductivity of the material in Ω_i , $T^{(i)}$ is the temperature in Ω_i and $P^{(i)}$ (see, (3.40)) is defined by

$$P^{(i)}(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}})) = n_r(\underline{\mathbf{x}}) \frac{\partial T^{(i)}(\underline{\mathbf{x}})}{\partial r} + n_z(\underline{\mathbf{x}}) \frac{\partial T^{(i)}(\underline{\mathbf{x}})}{\partial z}, \text{ for } i = 0, 1, 2, \quad (4.2)$$

with $\underline{\mathbf{n}}(\underline{\mathbf{x}}) = [n_r(\underline{\mathbf{x}}), n_z(\underline{\mathbf{x}})]$ being the unit normal vector to the curves Γ_{01} and Γ_{02} pointing into the region Ω_1 .

If the thickness δ of the interphase layer Ω_0 is sufficiently small, we may consider simplifying the task of determining the temperature distribution in Ω_1 and Ω_2 by replacing the layer with the line Γ_0 (where $\Gamma_{01} \rightarrow \Gamma_0$ and $\Gamma_{02} \rightarrow \Gamma_0$ when $\delta \rightarrow 0^+$), that is, by replacing the trilayered axisymmetric body in Figure 4.1 with the bimaterial in Figure 4.2.

Since the layer Ω_0 is reduced to a line, we are interested in conditions involving only $T^{(1)}$ and $T^{(2)}$ on Γ_0 . Following closely the analysis in Benveniste [14], we derive the conditions on Γ_0 in axisymmetric coordinates. The derivation is given below.

Take a fixed point $\underline{\mathbf{x}}_0 = (r_0, z_0)$ on a smooth part of the curve Γ_{02} in Figure 4.1 and define

$$\Theta(s) = T^{(0)}(\underline{\mathbf{x}}_0 + \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)s) \text{ for } 0 \leq s \leq \delta, \quad (4.3)$$

where $\underline{\mathbf{n}}(\underline{\mathbf{x}})$ is the unit normal vector to Γ_{02} on $\underline{\mathbf{x}}$ pointing towards the region Ω_1 . The function $\Theta(s)$ defined in (4.3) gives the temperature in the layer Ω_0 along the line perpendicular to Γ_{02} at $\underline{\mathbf{x}}_0$.

The Taylor series of $\Theta(s)$ about $s = 0$ gives

$$\Theta(s) = \Theta(0) + s\Theta'(0) + O(s^2). \quad (4.4)$$

If we let $s = \delta$ in (4.4), we obtain

$$\Theta(\delta) = \Theta(0) + \delta\Theta'(0) + O(\delta^2),$$

which can be rewritten as

$$\begin{aligned} & T^{(0)}(\underline{\mathbf{x}}_0 + \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)\delta) - T^{(0)}(\underline{\mathbf{x}}_0) \\ &= \delta P^{(0)}(\underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)) + O(\delta^2). \end{aligned} \quad (4.5)$$

Use of (4.1) in (4.5) leads to

$$\begin{aligned} & T^{(1)}(\underline{\mathbf{x}}_0 + \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)\delta) - T^{(2)}(\underline{\mathbf{x}}_0) \\ &= \frac{\delta}{\kappa^{(0)}} \kappa^{(2)} P^{(2)}(\underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)) + O(\delta^2), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} & T^{(1)}(\underline{\mathbf{x}}_0) - T^{(2)}(\underline{\mathbf{x}}_0) + \delta P^{(1)}(\underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)) \\ &= \frac{\delta}{\kappa^{(0)}} \kappa^{(2)} P^{(2)}(\underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)) + O(\delta^2). \end{aligned} \quad (4.6)$$

In a similar way, we may start from

$$\Theta'(\delta) = \Theta'(0) + \delta \Theta''(0) + O(\delta^2),$$

to obtain

$$\begin{aligned} & \kappa^{(1)} P^{(1)}(\underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)) - \kappa^{(2)} P^{(2)}(\underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)) \\ & + \delta \kappa^{(1)} [n_r(r, z) \frac{\partial}{\partial r} (P^{(1)}(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))) \\ & + n_z(r, z) \frac{\partial}{\partial z} (P^{(1)}(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}})))] \Big|_{\underline{\mathbf{x}}=\underline{\mathbf{x}}_0} \\ &= \delta k^{(0)} \Theta''(0) + O(\delta^2). \end{aligned} \quad (4.7)$$

4.2.1 Ideal Interfaces

If the thermal conductivity $\kappa^{(0)}$ is a fixed number (that is, its magnitude is $O(1)$ with reference to the thickness δ of the layer Ω_0) and if terms of magnitude $O(\delta)$ are neglected in (4.6) and (4.7) then

$$\begin{aligned} T^{(2)}(\underline{\mathbf{x}}_0) &= T^{(1)}(\underline{\mathbf{x}}_0), \\ \kappa^{(1)} P^{(1)}(\underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)) &= \kappa^{(2)} P^{(2)}(\underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)), \end{aligned} \quad (4.8)$$

that is, for sufficiently small δ , the interphase layer Ω_0 can be modeled as an ideal line interface Γ_0 on which the temperature and the normal heat flux are continuous.

4.2.2 Low Conducting Interfaces

Consider now the case in which interphase layer Ω_0 is occupied by a material of extremely low thermal conductivity such that there is a non-vanishing temperature jump across the layer as $\delta \rightarrow 0^+$. From (4.6), it is clear that the thermal conductivity $\kappa^{(0)}$ should take the form $\kappa^{(0)} = \lambda\delta$, where λ is a positive coefficient. With $\kappa^{(0)} = \lambda\delta$, if we let $\delta \rightarrow 0^+$ in (4.6) and (4.7), we obtain

$$\begin{aligned} & \lambda[T^{(1)}(\underline{\mathbf{x}}_0) - T^{(2)}(\underline{\mathbf{x}}_0)] \\ = & \kappa^{(1)}P^{(1)}(\underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)) = \kappa^{(2)}P^{(2)}(\underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)). \end{aligned} \quad (4.9)$$

If the conditions in (4.9) are applicable on the interface Γ_0 of the bimaterial in Figure 4.2, then Γ_0 is a low conducting interface. We may regard (4.9) as an $O(\delta)$ approximation for a low conducting interface — it is obtained by neglecting $O(\delta)$ terms in (4.6) and (4.7).

4.2.3 High Conducting Interfaces

Consider now the other extreme case in which interphase layer Ω_0 is occupied by a material of extremely high thermal conductivity. For such a case, (4.6) gives

$$T^{(1)}(\underline{\mathbf{x}}_0) = T^{(2)}(\underline{\mathbf{x}}_0), \quad (4.10)$$

that is, the temperature is continuous on the interface Γ_0 which models Ω_0 as a very thin layer of extremely high thermal conductivity. This is physically expected since Ω_0 conducts heat extremely well between the regions Ω_1 and Ω_2 .

If the thermal conductivity $\kappa^{(0)}$ is extremely high such that the normal heat flux has a finite jump across the layer Ω_0 as $\delta \rightarrow 0^+$, then $\kappa^{(0)}$ should take the form $\kappa^{(0)} = \alpha/\delta$, where α is a positive coefficient. With $\kappa^{(0)} = \alpha/\delta$, if we let

$\delta \rightarrow 0^+$ in (4.7), we obtain

$$\kappa^{(1)} P^{(1)}(\underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)) - \kappa^{(2)} P^{(2)}(\underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)) = \alpha \Theta''(0). \quad (4.11)$$

From (4.3), we find that $\Theta''(0)$ can be written in the form

$$\begin{aligned} \Theta''(0) = & [n_r(\underline{\mathbf{x}}_0)]^2 \frac{\partial^2 T^{(0)}}{\partial r^2} \Big|_{\underline{\mathbf{x}}=\underline{\mathbf{x}}_0} + 2n_r(\underline{\mathbf{x}}_0)n_z(\underline{\mathbf{x}}_0) \frac{\partial^2 T^{(0)}}{\partial r \partial z} \Big|_{\underline{\mathbf{x}}=\underline{\mathbf{x}}_0} \\ & + [n_z(\underline{\mathbf{x}}_0)]^2 \frac{\partial^2 T^{(0)}}{\partial z^2} \Big|_{\underline{\mathbf{x}}=\underline{\mathbf{x}}_0} \\ & + \left\{ n_r(\underline{\mathbf{x}}_0) \frac{\partial n_r(\underline{\mathbf{x}}_0)}{\partial r} + n_z(\underline{\mathbf{x}}_0) \frac{\partial n_r(\underline{\mathbf{x}}_0)}{\partial z} \right\} \frac{\partial T^{(0)}}{\partial r} \Big|_{\underline{\mathbf{x}}=\underline{\mathbf{x}}_0} \\ & + \left\{ n_r(\underline{\mathbf{x}}_0) \frac{\partial n_z(\underline{\mathbf{x}}_0)}{\partial r} + n_z(\underline{\mathbf{x}}_0) \frac{\partial n_z(\underline{\mathbf{x}}_0)}{\partial z} \right\} \frac{\partial T^{(0)}}{\partial z} \Big|_{\underline{\mathbf{x}}=\underline{\mathbf{x}}_0}. \end{aligned} \quad (4.12)$$

Using the governing partial differential equation in (3.1) for axisymmetric heat conduction, we obtain

$$\begin{aligned} \Theta''(0) = & -[n_r(\underline{\mathbf{x}}_0)]^2 \frac{\partial^2 T^{(0)}}{\partial z^2} \Big|_{\underline{\mathbf{x}}=\underline{\mathbf{x}}_0} + 2n_r(\underline{\mathbf{x}}_0)n_z(\underline{\mathbf{x}}_0) \frac{\partial^2 T^{(0)}}{\partial r \partial z} \Big|_{\underline{\mathbf{x}}=\underline{\mathbf{x}}_0} \\ & - [n_z(\underline{\mathbf{x}}_0)]^2 \frac{\partial^2 T^{(0)}}{\partial r^2} \Big|_{\underline{\mathbf{x}}=\underline{\mathbf{x}}_0} \\ & + \left\{ n_r(\underline{\mathbf{x}}_0) \frac{\partial n_r(\underline{\mathbf{x}}_0)}{\partial r} + n_z(\underline{\mathbf{x}}_0) \frac{\partial n_r(\underline{\mathbf{x}}_0)}{\partial z} - \frac{1}{r} \right\} \frac{\partial T^{(0)}}{\partial r} \Big|_{\underline{\mathbf{x}}=\underline{\mathbf{x}}_0} \\ & + \left\{ n_r(\underline{\mathbf{x}}_0) \frac{\partial n_z(\underline{\mathbf{x}}_0)}{\partial r} + n_z(\underline{\mathbf{x}}_0) \frac{\partial n_z(\underline{\mathbf{x}}_0)}{\partial z} \right\} \frac{\partial T^{(0)}}{\partial z} \Big|_{\underline{\mathbf{x}}=\underline{\mathbf{x}}_0}. \end{aligned} \quad (4.13)$$

Although $T^{(0)} = T^{(2)}$ on Γ_0 , we cannot replace $T^{(0)}$ in (4.13) by $T^{(1)}$ unless we can show that the right hand side of (4.13) describes the change of $T^{(0)}$ along the curve Γ_0 on the Orz plane.

Now the rate of change of $T^{(0)}$ per unit distance along the curve Γ_0 at the point $\underline{\mathbf{x}}$ is given by

$$[n_z(\underline{\mathbf{x}}), -n_r(\underline{\mathbf{x}})] \cdot \left[\frac{\partial T^{(0)}}{\partial r}, \frac{\partial T^{(0)}}{\partial z} \right] = n_z(\underline{\mathbf{x}}) \frac{\partial T^{(0)}}{\partial r} - n_r(\underline{\mathbf{x}}) \frac{\partial T^{(0)}}{\partial z}, \quad (4.14)$$

where $[n_z(\underline{\mathbf{x}}), -n_r(\underline{\mathbf{x}})]$ is a unit tangential vector to Γ_2 at the point $\underline{\mathbf{x}}$.

It follows that the second order rate of change of $T^{(0)}$ along the middle layer with respect to distance along the curve Γ_0 at the point $\underline{\mathbf{x}}$ is given by

$$\begin{aligned}
 & n_z(\underline{\mathbf{x}}) \frac{\partial}{\partial r} \left[n_z(\underline{\mathbf{x}}) \frac{\partial T^{(0)}}{\partial r} - n_r(\underline{\mathbf{x}}) \frac{\partial T^{(0)}}{\partial z} \right] \\
 & - n_r(\underline{\mathbf{x}}) \frac{\partial}{\partial r} \left[n_z(\underline{\mathbf{x}}) \frac{\partial T^{(0)}}{\partial r} - n_r(\underline{\mathbf{x}}) \frac{\partial T^{(0)}}{\partial z} \right] \\
 = & [n_z(\underline{\mathbf{x}})]^2 \frac{\partial^2 T^{(0)}}{\partial r^2} - 2n_r(\underline{\mathbf{x}})n_z(\underline{\mathbf{x}}) \frac{\partial^2 T^{(0)}}{\partial r \partial z} + [n_r(\underline{\mathbf{x}})]^2 \frac{\partial^2 T^{(0)}}{\partial z^2} \\
 & + \left\{ n_z(\underline{\mathbf{x}}) \frac{\partial n_z(\underline{\mathbf{x}})}{\partial r} - n_r(\underline{\mathbf{x}}) \frac{\partial n_z(\underline{\mathbf{x}})}{\partial z} \right\} \frac{\partial T^{(0)}}{\partial r} \\
 & + \left\{ -n_z(\underline{\mathbf{x}}) \frac{\partial n_r(\underline{\mathbf{x}})}{\partial r} + n_r(\underline{\mathbf{x}}) \frac{\partial n_r(\underline{\mathbf{x}})}{\partial z} \right\} \frac{\partial T^{(0)}}{\partial z}. \tag{4.15}
 \end{aligned}$$

Since $T^{(0)}$, $T^{(1)}$ and $T^{(2)}$ are equal along the interface as $\delta \rightarrow 0^+$, we can apply (4.15) in (4.13) to obtain

$$\begin{aligned}
 \Theta''(0) = & -n_z(\underline{\mathbf{x}}) \frac{\partial}{\partial r} \left[n_z(\underline{\mathbf{x}}) \frac{\partial T^{(1)}}{\partial r} - n_r(\underline{\mathbf{x}}) \frac{\partial T^{(1)}}{\partial z} \right] \\
 & + n_r(\underline{\mathbf{x}}) \frac{\partial}{\partial r} \left[n_z(\underline{\mathbf{x}}) \frac{\partial T^{(1)}}{\partial r} - n_r(\underline{\mathbf{x}}) \frac{\partial T^{(1)}}{\partial z} \right] \\
 & + \left[A(\underline{\mathbf{x}}) - \frac{1}{r} \right] \frac{\partial T^{(0)}}{\partial r} + B(\underline{\mathbf{x}}) \frac{\partial T^{(0)}}{\partial z}, \tag{4.16}
 \end{aligned}$$

as $\delta \rightarrow 0^+$. The functions $A(\underline{\mathbf{x}})$ and $B(\underline{\mathbf{x}})$ are given as

$$\begin{aligned}
 A(\underline{\mathbf{x}}) = & n_z(\underline{\mathbf{x}}) \frac{\partial n_z(\underline{\mathbf{x}})}{\partial r} - n_r(\underline{\mathbf{x}}) \frac{\partial n_z(\underline{\mathbf{x}})}{\partial z} \\
 & + n_r(\underline{\mathbf{x}}) \frac{\partial n_r(\underline{\mathbf{x}})}{\partial r} + n_z(\underline{\mathbf{x}}) \frac{\partial n_r(\underline{\mathbf{x}})}{\partial z}, \\
 B(\underline{\mathbf{x}}) = & -n_z(\underline{\mathbf{x}}) \frac{\partial n_r(\underline{\mathbf{x}})}{\partial r} + n_r(\underline{\mathbf{x}}) \frac{\partial n_r(\underline{\mathbf{x}})}{\partial z} \\
 & + n_r(\underline{\mathbf{x}}) \frac{\partial n_z(\underline{\mathbf{x}})}{\partial r} + n_z(\underline{\mathbf{x}}) \frac{\partial n_z(\underline{\mathbf{x}})}{\partial z}. \tag{4.17}
 \end{aligned}$$

Note that $A(\underline{\mathbf{x}}) = 0$ and $B(\underline{\mathbf{x}}) = 0$ if Γ_0 is a flat (planar) interface as in Chapter 3.

To deal with the terms $\partial T^{(0)}/\partial r$ and $\partial T^{(0)}/\partial z$, we note that

$$n_r \frac{\partial T^{(0)}}{\partial r} + n_z \frac{\partial T^{(0)}}{\partial z} \rightarrow 0^+ \text{ as } \delta \rightarrow 0^+, \quad (4.18)$$

since the normal heat flux from Ω_0 into Ω_1 or Ω_2 should physically be bounded as $\delta \rightarrow 0^+$ (hence $\kappa^{(0)} \rightarrow \infty$), that is, we require

$$\kappa^{(0)} \left[n_r \frac{\partial T^{(0)}}{\partial r} + n_z \frac{\partial T^{(0)}}{\partial z} \right] \rightarrow \text{finite number as } \delta \rightarrow 0^+. \quad (4.19)$$

Since $T^{(0)} = T^{(1)}$ along the interface as $\delta \rightarrow 0^+$, we may write

$$-n_z \frac{\partial T^{(0)}}{\partial r} + n_r \frac{\partial T^{(0)}}{\partial z} \rightarrow -n_z \frac{\partial T^{(1)}}{\partial r} + n_r \frac{\partial T^{(1)}}{\partial z} \text{ as } \delta \rightarrow 0^+. \quad (4.20)$$

From (4.18) and (4.20), we obtain

$$\begin{aligned} \frac{\partial T^{(0)}}{\partial r} &\rightarrow n_z^2 \frac{\partial T^{(1)}}{\partial r} - n_r n_z \frac{\partial T^{(1)}}{\partial z}, \\ \frac{\partial T^{(0)}}{\partial z} &\rightarrow -n_r n_z \frac{\partial T^{(1)}}{\partial r} + n_r^2 \frac{\partial T^{(1)}}{\partial z}. \end{aligned} \quad (4.21)$$

Using (4.21), we can now rewrite (4.16) as

$$\begin{aligned} \Theta''(0) &= -n_z(\underline{\mathbf{x}}) \frac{\partial}{\partial r} \left[n_z(\underline{\mathbf{x}}) \frac{\partial T^{(1)}}{\partial r} - n_r(\underline{\mathbf{x}}) \frac{\partial T^{(1)}}{\partial z} \right] \\ &\quad + n_r(\underline{\mathbf{x}}) \frac{\partial}{\partial r} \left[n_z(\underline{\mathbf{x}}) \frac{\partial T^{(1)}}{\partial r} - n_r(\underline{\mathbf{x}}) \frac{\partial T^{(1)}}{\partial z} \right] \\ &\quad + C(\underline{\mathbf{x}}) \frac{\partial T^{(1)}}{\partial r} + D(\underline{\mathbf{x}}) \frac{\partial T^{(1)}}{\partial z}, \end{aligned} \quad (4.22)$$

where

$$\begin{aligned} C(\underline{\mathbf{x}}) &= \left[A(\underline{\mathbf{x}}) - \frac{1}{r} \right] n_z^2 - B(\underline{\mathbf{x}}) n_r n_z, \\ D(\underline{\mathbf{x}}) &= -\left[A(\underline{\mathbf{x}}) - \frac{1}{r} \right] n_r n_z + B(\underline{\mathbf{x}}) n_r^2. \end{aligned} \quad (4.23)$$

It follows that the boundary condition in (4.11) for the interface Γ_0 which models the extremely thin interphase layer Ω_0 with thermal conductivity $\kappa^{(0)} =$

α/δ is given by

$$\begin{aligned} & \kappa^{(1)} P^{(1)}(\underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)) - \kappa^{(2)} P^{(2)}(\underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}_0)) \\ &= -\alpha \mathcal{S}(T(\underline{\mathbf{x}}_0)), \end{aligned} \quad (4.24)$$

where $\mathcal{S}(T(\underline{\mathbf{x}}))$ is the differential operator defined by

$$\begin{aligned} \mathcal{S}(T(\underline{\mathbf{x}})) &= n_z(\underline{\mathbf{x}}) \frac{\partial}{\partial r} \left[n_z(\underline{\mathbf{x}}) \frac{\partial T^{(1)}}{\partial r} - n_r(\underline{\mathbf{x}}) \frac{\partial T^{(1)}}{\partial z} \right] \\ &\quad - n_r(\underline{\mathbf{x}}) \frac{\partial}{\partial r} \left[n_z(\underline{\mathbf{x}}) \frac{\partial T^{(1)}}{\partial r} - n_r(\underline{\mathbf{x}}) \frac{\partial T^{(1)}}{\partial z} \right] \\ &\quad - C(\underline{\mathbf{x}}) \frac{\partial T^{(1)}}{\partial r} - D(\underline{\mathbf{x}}) \frac{\partial T^{(1)}}{\partial z}. \end{aligned} \quad (4.25)$$

Note that $\mathcal{S}(T(\underline{\mathbf{x}}))$ is the Laplacian of the interface temperature.

If the conditions in (4.10) and (4.24) are applicable on the interface Γ_0 of the bimaterial in Figure 4.2, then Γ_0 is a high conducting interface. We may regard (4.10) and (4.24) as an $O(\delta)$ model for a high conducting interface. If $\alpha = 0$, (4.10) and (4.24) give the conditions for the ideal interface.

4.3 An Axisymmetric Heat Conduction Problem

Consider a bimaterial comprising regions R_1 and R_2 which have thermal conductivities κ_1 and κ_2 respectively. With reference to a Cartesian coordinate system denoted by $Oxyz$, the regions R_1 and R_2 are symmetrical about the z -axis, that is, the regions R_1 and R_2 can be respectively obtained by rotating two-dimensional regions Ω_1 and Ω_2 on the rz (axisymmetric coordinate) plane by an angle of 360° about the z -axis. (Note that $r = \sqrt{x^2 + y^2}$.) The bimaterial is similar to the one sketched in Figure 3.2 in Chapter 3 except here the interface Γ_0 is a general curve as shown in Figure 4.3. In Figure 4.3, Γ_0 ,

Figure 4.3: A geometrical sketch of the bimaterial on the rz plane.

Γ_1 and Γ_2 are sketched as open curves, each with an endpoint on the z -axis. In general, they do not have to be always open curves with endpoints on the z -axis. For example, Γ_0 may be a horizontal straight line segment above the z -axis, that is, Γ_0 may be parallel to the z -axis. Furthermore, unlike in Figure 4.3, $\Gamma_1 \cup \Gamma_2$ may possibly form a simple closed curve as in, for example, the case in which $R_1 \cup R_2$ is a hollow cylinder.

The problem of interest here is to determine the steady-state axisymmetric temperature distribution $T(\underline{\mathbf{x}})$ by solving the governing partial differential equation (3.1), that is,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \quad \text{for } \underline{\mathbf{x}} = (r, z) \in \Omega_1 \cup \Omega_2, \quad (4.26)$$

subject to the boundary conditions (3.8), which is

$$\begin{aligned} T(\underline{\mathbf{x}}) &= f_0(\underline{\mathbf{x}}) \quad \text{for } \underline{\mathbf{x}} \in \Xi_1, \\ P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}})) &= f_1(\underline{\mathbf{x}}) + f_2(\underline{\mathbf{x}})T(\underline{\mathbf{x}}) \quad \text{for } \underline{\mathbf{x}} \in \Xi_2, \end{aligned} \quad (4.27)$$

and the interfacial conditions given by either

$$\kappa_1 Q_1(\underline{\mathbf{x}}) = \kappa_2 Q_2(\underline{\mathbf{x}}) = \lambda \Delta T(\underline{\mathbf{x}}) \text{ for } \underline{\mathbf{x}} \in \Gamma_0, \quad (4.28)$$

or

$$\left. \begin{aligned} \Delta T(\underline{\mathbf{x}}) &= 0 \\ \kappa_1 Q_1(\underline{\mathbf{x}}) - \kappa_2 Q_2(\underline{\mathbf{x}}) &= -\alpha \mathcal{S}(T(\underline{\mathbf{x}})) \end{aligned} \right\} \text{ for } \underline{\mathbf{x}} \in \Gamma_0. \quad (4.29)$$

The notations $P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$, $\underline{\mathbf{n}}(\underline{\mathbf{x}})$, $f_0(\underline{\mathbf{x}})$, $f_1(\underline{\mathbf{x}})$, $f_2(\underline{\mathbf{x}})$, Ξ_1 and Ξ_2 are as described in Chapter 3. $\underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}})$ is the unit normal vector to Γ_0 (at $\underline{\mathbf{x}}$) pointing into Ω_1 , $\underline{\mathbf{t}}^{\text{int}}(\underline{\mathbf{x}}) = [t_r^{\text{int}}(\underline{\mathbf{x}}), t_z^{\text{int}}(\underline{\mathbf{x}})]$ is a unit tangential vector to Γ_0 (at $\underline{\mathbf{x}}$), $Q_i(\underline{\mathbf{x}})$ (for $\underline{\mathbf{x}} \in \Gamma_0$) is the function $P(\underline{\mathbf{x}}; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}}))$ calculated using the temperature field $T(\underline{\mathbf{x}})$ in Ω_i , $\Delta T(\underline{\mathbf{x}})$ is the temperature jump across Γ_0 (at $\underline{\mathbf{x}}$) as defined by

$$\Delta T(\underline{\mathbf{x}}) = \lim_{\varepsilon \rightarrow 0^+} [T(\underline{\mathbf{x}} + \varepsilon \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}})) - T(\underline{\mathbf{x}} - \varepsilon \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}}))] \text{ for } \underline{\mathbf{x}} \in \Gamma_0, \quad (4.30)$$

and $\mathcal{S}$ is the differential operator defined in (4.25).

Note that (4.28) and (4.29) are the interfacial conditions on low conducting and high conducting interfaces derived in Section 4.2.

4.4 Hypersingular Boundary Integral Equation Method

The boundary integral equations for axisymmetric heat conduction are given by (2.14) in Chapter 2. Applying (2.14) to the regions Ω_1 and Ω_2 of the bimaterial in Figure 4.3, we obtain

$$\begin{aligned} \gamma_1(\underline{\mathbf{x}}_0) T(\underline{\mathbf{x}}_0) &= \int_{\Gamma_1} (T(\underline{\mathbf{x}}) G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) - G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))) r ds(\underline{\mathbf{x}}) \\ &\quad - \int_{\Gamma_0} (T(\underline{\mathbf{x}}) G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}})) - G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) Q_1(\underline{\mathbf{x}})) r ds(\underline{\mathbf{x}}) \\ &\quad \text{for } \underline{\mathbf{x}}_0 = (r_0, z_0) \in \Omega_1 \cup \Gamma_0 \cup \Gamma_1, \end{aligned} \quad (4.31)$$

and

$$\begin{aligned}
 \gamma_2(\underline{\mathbf{x}}_0)T(\underline{\mathbf{x}}_0) &= \int_{\Gamma_2} (T(\underline{\mathbf{x}})G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) - G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}})))rds(\underline{\mathbf{x}}) \\
 &\quad + \int_{\Gamma_0} (T(\underline{\mathbf{x}})G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}})) - G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)Q_2(\underline{\mathbf{x}}))rds(\underline{\mathbf{x}}) \\
 &\quad \text{for } \underline{\mathbf{x}}_0 = (r_0, z_0) \in \Omega_2 \cup \Gamma_0 \cup \Gamma_2, \quad (4.32)
 \end{aligned}$$

where $\gamma_i(\underline{\mathbf{x}}_0) = 1$ if $\underline{\mathbf{x}}_0$ lies in the interior of Ω_i , $\gamma_i(\underline{\mathbf{x}}_0) = 1/2$ if $\underline{\mathbf{x}}_0$ lies on a smooth part of $\Gamma_0 \cup \Gamma_i$, $ds(\underline{\mathbf{x}})$ denotes the length of an infinitesimal part of the curve $\Gamma_0 \cup \Gamma_i$, $G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ and $G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ are as defined in (2.17). The boundary integrals in (4.31) and (4.32) are improper and are to be interpreted in the Cauchy principal sense if $\underline{\mathbf{x}}_0$ lies on Γ_0 or Γ_1 or Γ_2 .

4.5 Low Conducting Interfaces

4.5.1 Integral Equations

For low conducting interfaces, addition of (4.31) and (4.32) together with the use of (4.28) yields

$$\begin{aligned}
 \gamma(\underline{\mathbf{x}}_0)T(\underline{\mathbf{x}}_0) &= \int_{\Gamma_1 \cup \Gamma_2} [T(\underline{\mathbf{x}})G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) - G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))]rds(\underline{\mathbf{x}}) \\
 &\quad - \int_{\Gamma_0} \Delta T(\underline{\mathbf{x}})[G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}})) - \frac{(\kappa_2 - \kappa_1)}{\kappa_1 \kappa_2} \lambda(\underline{\mathbf{x}})G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)]rds(\underline{\mathbf{x}}) \\
 &\quad \text{for } \underline{\mathbf{x}}_0 \in \Omega_1 \cup \Omega_2 \cup \Gamma_1 \cup \Gamma_2, \quad (4.33)
 \end{aligned}$$

where $\gamma(\underline{\mathbf{x}}_0) = 1$ if $\underline{\mathbf{x}}_0$ lies in the interior of $\Omega_1 \cup \Omega_2$ or on Γ_0 and $\gamma(\underline{\mathbf{x}}_0) = 1/2$ if $\underline{\mathbf{x}}_0$ lies on a smooth part of $\Gamma_1 \cup \Gamma_2$.

From (4.33), we may write

$$\begin{aligned}
 & n_r^{\text{int}}(\underline{\mathbf{y}}) \frac{\partial}{\partial r_0} [T(\underline{\mathbf{x}}_0)] + n_z^{\text{int}}(\underline{\mathbf{y}}) \frac{\partial}{\partial z_0} [T(\underline{\mathbf{x}}_0)] \\
 = & \int_{\Gamma_1 \cup \Gamma_2} [T(\underline{\mathbf{x}}) \Phi_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}); \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}})) \\
 & - \Phi_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}})) P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))] r ds(\underline{\mathbf{x}}) \\
 & - \int_{\Gamma_0} \Delta T(\underline{\mathbf{x}}) [\Phi_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}}); \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}})) \\
 & - \frac{(\kappa_2 - \kappa_1)}{\kappa_1 \kappa_2} \lambda(\underline{\mathbf{x}}) \Phi_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}}))] r ds(\underline{\mathbf{x}}) \\
 & \text{for } \underline{\mathbf{x}}_0 \text{ in the interior of } \Omega_1 \text{ or } \Omega_2, \quad (4.34)
 \end{aligned}$$

where $\underline{\mathbf{y}}$ is a point on the interface Γ_0 and the functions $\Phi_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}}))$ and $\Phi_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}); \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}}))$ are given by

$$\begin{aligned}
 & \Phi_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}})) = G_1(\underline{\mathbf{x}}_0; \underline{\mathbf{x}}; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}})), \\
 & \Phi_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}); \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}})) \\
 = & \frac{n_r^{\text{int}}(\underline{\mathbf{y}}) \Theta(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) + n_z^{\text{int}}(\underline{\mathbf{y}}) \Psi(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}))}{\pi \sqrt{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)} (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0))^2}, \quad (4.35)
 \end{aligned}$$

with

$$\begin{aligned}
 \Theta(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) &= (r + r_0)(1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \left\{ \frac{n_r(\underline{\mathbf{x}})}{2r} \right. \\
 &\quad \times [(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r^2) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 &\quad - (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0)) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 &\quad - n_z(\underline{\mathbf{x}})(z - z_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \} \\
 &\quad - \frac{n_r(\underline{\mathbf{x}})}{2r_0} [(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2) (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 &\quad \times [(r - r_0) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - (r + r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 &\quad - 2r_0 ((r - r_0)^2 - (z - z_0)^2) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 &\quad + \frac{n_z(\underline{\mathbf{x}})}{2r_0} (z - z_0) [4r_0(r - r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 &\quad + (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2) \\
 &\quad \times (E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)))] , \\
 \\
 \Psi(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) &= -\{(z - z_0) (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 &\quad \times [\frac{n_r(\underline{\mathbf{x}})}{2r} [(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r^2) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 &\quad - (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0)) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 &\quad - n_z(\underline{\mathbf{x}})(z - z_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 &\quad + n_r(\underline{\mathbf{x}})(z - z_0) [(1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 &\quad \times [(r - r_0) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - (r + r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 &\quad - 2(r - r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 &\quad - n_z(\underline{\mathbf{x}}) [(2 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) (z - z_0)^2 \\
 &\quad - (r - r_0)^2) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 &\quad - (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) (z - z_0)^2 K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \}. \quad (4.36)
 \end{aligned}$$

Noting that $K(m)$ can be expanded about $m = 1$ as (see, for example, Whittaker and Watson [77])

$$\begin{aligned} K(m) = & -\frac{1}{2}\ln(1-m) \cdot \left[1 - \frac{1}{4}(m-1) + \frac{9}{64}(m-1)^2 + \dots\right] \\ & + \ln(4) + \frac{1}{4}(1 - \ln(4))(m-1) + \frac{3}{128}(6\ln(4) - 7)(1-m)^2 + \dots, \end{aligned} \quad (4.37)$$

and letting $\underline{\mathbf{x}}_0$ tend to $\underline{\mathbf{y}} = (\rho_0, \zeta_0)$ on Γ_0 (from within the region Ω_1), we find that the interfacial conditions in (4.28) give rise to

$$\begin{aligned} & \left[\frac{1}{\kappa_1} - \frac{(\kappa_2 - \kappa_1)}{2\kappa_1\kappa_2}\right]\lambda(\underline{\mathbf{y}})\Delta T(\underline{\mathbf{y}}) \\ = & \mathcal{H} \int_{\Gamma_0} \frac{\Delta T(\underline{\mathbf{x}})}{\sqrt{a(\underline{\mathbf{x}}; \underline{\mathbf{y}}) + b(r; \rho_0)}} \Phi_3(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}}); \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}})) r ds(\underline{\mathbf{x}}) \\ & - \mathcal{C} \int_{\Gamma_0} \Delta T(\underline{\mathbf{x}}) [\Phi_2(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}}); \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}})) \\ & - \frac{(\kappa_2 - \kappa_1)}{\kappa_1\kappa_2} \lambda(\underline{\mathbf{x}}) \Phi_0(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}}))] r ds(\underline{\mathbf{x}}) \\ & + \int_{\Gamma_1 \cup \Gamma_2} [T(\underline{\mathbf{x}}) \Phi_1(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}); \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}})) \\ & - \Phi_0(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}})) P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))] r ds(\underline{\mathbf{x}}) \\ & \text{for } \underline{\mathbf{y}} = (\rho_0, \zeta_0) \in \Gamma_0 \text{ (smooth part)}, \end{aligned} \quad (4.38)$$

where $\mathcal{C}$ and $\mathcal{H}$ denotes that the integral over Γ_0 is to be interpreted in the Cauchy principal and the Hadamard finite-part sense respectively and

$$\begin{aligned} \Phi_2(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}}); \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}})) = & \Phi_1(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}}); \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}})) \\ & + \frac{\Phi_3(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}}); \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}}))}{\sqrt{a(\underline{\mathbf{x}}; \underline{\mathbf{y}}) + b(r; \rho_0)}}, \end{aligned}$$

$$\begin{aligned}
 & \Phi_3(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}}); \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}})) \\
 = & -\{(n_r^{\text{int}}(\underline{\mathbf{x}})n_r^{\text{int}}(\underline{\mathbf{y}}) - n_z^{\text{int}}(\underline{\mathbf{x}})n_z^{\text{int}}(\underline{\mathbf{y}}))([r - \rho_0]^2 - [z - \zeta_0]^2) \\
 & + 2[n_r^{\text{int}}(\underline{\mathbf{y}})n_z^{\text{int}}(\underline{\mathbf{x}}) + n_r^{\text{int}}(\underline{\mathbf{x}})n_z^{\text{int}}(\underline{\mathbf{y}})] \\
 & \times (r - \rho_0)(z - \zeta_0)\} \frac{1}{\pi(a(\underline{\mathbf{x}}; \underline{\mathbf{y}}) - b(r; \rho_0))^2}.
 \end{aligned} \tag{4.39}$$

The derivation of (4.38) is given in Ang [5] for two-dimensional heat conduction across a low conducting interface. The fundamental solution of the governing partial differential equation for the two-dimensional heat conduction has a relatively simple form.

4.5.2 Boundary Element Procedure

We outline here a boundary element procedure for solving (4.33) and (4.38). As in Section 3.6, the exterior boundary $\Gamma_1 \cup \Gamma_2$ of the bimaterial is discretized into N straight line elements denoted by $B^{(1)}, B^{(2)}, \dots, B^{(N-1)}$ and $B^{(N)}$. The interface Γ_0 is discretized into M straight line elements $I^{(1)}, I^{(2)}, \dots, I^{(M-1)}$ and $I^{(M)}$.

The temperature and the directional rate of change of temperature on the boundary elements $B^{(1)}, B^{(2)}, \dots, B^{(N-1)}$ and $B^{(N)}$ are approximated as given in (3.47), and the temperature jump ΔT as a constant $\Delta T^{(j)}$ over the element $I^{(j)}$, that is,

$$\Delta T(\underline{\mathbf{x}}) \simeq \Delta T^{(j)} \quad \text{for } \underline{\mathbf{x}} \in I^{(j)} \quad (j = 1, 2, \dots, M). \tag{4.40}$$

If we let $\underline{\mathbf{x}}_0$ in (4.33) to be given in turn by each of the midpoints of $B^{(1)}$, $B^{(2)}$, $\dots$, $B^{(N-1)}$ and $B^{(N)}$, then the use of (4.27) and (4.40) gives

$$\begin{aligned}
 & \frac{1}{2}[d^{(i)}T^{(i)} + (1 - d^{(i)})f_0(\widehat{\underline{\mathbf{x}}}^{(i)})] \\
 = & \sum_{k=1}^N \{[d^{(k)}T^{(k)} + (1 - d^{(k)})f_0(\widehat{\underline{\mathbf{x}}}^{(k)})] \int_{B^{(k)}} G_1(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{x}}}^{(i)}; \underline{\mathbf{n}}^{(k)}) r ds(\underline{\mathbf{x}}) \\
 & - [d^{(k)}(f_1(\widehat{\underline{\mathbf{x}}}^{(k)}) + f_2(\widehat{\underline{\mathbf{x}}}^{(k)})T^{(k)}) + (1 - d^{(k)})P^{(k)}] \\
 & \times \int_{B^{(k)}} G_0(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{x}}}^{(i)}) r ds(\underline{\mathbf{x}})\} \\
 & - \sum_{j=1}^M \Delta T^{(j)} \int_{I^{(j)}} [G_1(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{x}}}^{(i)}; \underline{\mathbf{m}}^{(j)}) - \frac{(\kappa_2 - \kappa_1)}{\kappa_1 \kappa_2} \lambda(\underline{\mathbf{x}}) G_0(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{x}}}^{(i)})] r ds(\underline{\mathbf{x}}) \\
 & \text{for } i = 1, 2, \dots, N,
 \end{aligned} \tag{4.41}$$

where $d^{(k)}$ and $\underline{\mathbf{n}}^{(k)}$ are as defined in Subsection 3.6.1, $\underline{\mathbf{m}}^{(j)}$ is the unit normal vector to $I^{(j)}$ pointing into Ω_1 and $\widehat{\underline{\mathbf{x}}}^{(i)}$ is the midpoint of the element $B^{(i)}$. In (4.41), note that the integrals over $B^{(k)}$ are Cauchy principal if $\widehat{\underline{\mathbf{x}}}^{(i)}$ is the midpoint of $B^{(k)}$ (that is, if $k = i$). The Cauchy principal integrals can be accurately evaluated by using a highly accurate Gaussian quadrature.

Similarly, we can let $\underline{\mathbf{y}}$ in (4.38) be given in turn by each of the midpoints of $I^{(1)}$, $I^{(2)}$, $\dots$, $I^{(M-1)}$ and $I^{(M)}$ to approximately obtain

$$\begin{aligned}
 & [(\frac{1}{\kappa_1} - \frac{(\kappa_2 - \kappa_1)}{2\kappa_1 \kappa_2}) \lambda(\widehat{\underline{\mathbf{y}}}^{(p)}) + \frac{2}{\pi \ell^{(p)}}] \Delta T^{(p)} \\
 = & \sum_{\substack{j=1 \\ j \neq p}}^M \Delta T^{(j)} \int_{I^{(j)}} \frac{\Phi_3(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{y}}}^{(p)}; \underline{\mathbf{m}}^{(j)}; \underline{\mathbf{m}}^{(p)})}{\sqrt{a(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{y}}}^{(p)}) + b(r; \widehat{\rho}_0^{(p)})}} r ds(\underline{\mathbf{x}})
 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{j=1}^M \Delta T^{(j)} \int_{I^{(j)}} [\Phi_2(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)}; \underline{\mathbf{m}}^{(j)}; \underline{\mathbf{m}}^{(p)}) \\
 & - \frac{(\kappa_2 - \kappa_1)}{\kappa_1 \kappa_2} \lambda(\underline{\mathbf{x}}) \Phi_0(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)}; \underline{\mathbf{m}}^{(p)})] r ds(\underline{\mathbf{x}}) \\
 & + \sum_{k=1}^N \{ [d^{(k)} T^{(k)} + (1 - d^{(k)}) f_0(\underline{\hat{\mathbf{x}}}^{(k)})] \int_{B^{(k)}} \Phi_1(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)}; \underline{\mathbf{n}}^{(k)}; \underline{\mathbf{m}}^{(p)}) r ds(\underline{\mathbf{x}}) \\
 & - [d^{(k)} (f_1(\underline{\hat{\mathbf{x}}}^{(k)}) + f_2(\underline{\hat{\mathbf{x}}}^{(k)}) T^{(k)}) + (1 - d^{(k)}) P^{(k)}] \\
 & \times \int_{B^{(k)}} \Phi_0(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)}; \underline{\mathbf{m}}^{(p)}) r ds(\underline{\mathbf{x}}) \} \\
 & \text{for } p = 1, 2, \dots, M,
 \end{aligned} \tag{4.42}$$

where $\ell^{(p)}$ and $\underline{\hat{\mathbf{y}}}^{(p)} = (\hat{\rho}_0^{(p)}, \hat{\zeta}_0^{(p)})$ are respectively the length and the midpoint of $I^{(p)}$.

Note that in deriving (4.42) the Hadamard finite-part integral in (4.38) is evaluated by using the approximation

$$\begin{aligned}
 & \mathcal{H} \int_{\Gamma_0} \frac{\Delta T(\underline{\mathbf{x}})}{\sqrt{a(\underline{\mathbf{x}}; \underline{\mathbf{y}}) + b(r; \rho_0)}} \Phi_3(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}}); \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}})) r ds(\underline{\mathbf{x}}) \\
 & \simeq \Delta T^{(p)} \mathcal{H} \int_{I^{(p)}} \frac{\Phi_3(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)}; \underline{\mathbf{m}}^{(p)}; \underline{\mathbf{m}}^{(p)})}{\sqrt{a(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)}) + b(r; \hat{\rho}_0^{(p)})}} r ds(\underline{\mathbf{x}}) \\
 & + \sum_{\substack{j=1 \\ j \neq p}}^M \Delta T^{(j)} \int_{I^{(j)}} \frac{\Phi_3(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)}; \underline{\mathbf{m}}^{(j)}; \underline{\mathbf{m}}^{(p)})}{\sqrt{a(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)}) + b(r; \hat{\rho}_0^{(p)})}} r ds(\underline{\mathbf{x}}).
 \end{aligned} \tag{4.43}$$

From (4.43), if the expression $r/\sqrt{a(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)}) + b(r; \hat{\rho}_0^{(p)})}$ is approximated as a constant given by its value at the midpoint of $I^{(p)}$, the Hadamard finite-part integral on the right hand side of (4.43) is approximately given by $-2/(\pi \ell^{(p)})$.

The integrals over $I^{(j)}$ in (4.42) which have the functions Φ_0 and Φ_2 in their integrands approximate the Cauchy principal integral in (4.38). Hence

the integrals over $I^{(j)}$ must be interpreted in the Cauchy principal sense if $\hat{\underline{\mathbf{y}}}^{(p)}$ is the midpoint of $I^{(j)}$ (that is, if $j = p$). The integrals over $I^{(j)}$ in (4.42) are proper if $j \neq p$. The integrals over $B^{(k)}$ are also proper.

The equations in (4.41) and (4.42) give a system of $N + M$ linear algebraic equations containing $N + M$ unknowns given by either $T^{(k)}$ or $P^{(k)}$ (not both) for $k = 1, 2, \dots, N$ and by $\Delta T^{(j)}$ for $j = 1, 2, \dots, M$. Once the unknowns are determined, the temperature at any interior point $\underline{\mathbf{x}}_0$ in the bimaterial can be determined from (4.33) with $\gamma(\underline{\mathbf{x}}_0) = 1$ by computing approximately the line integrals over Γ_0 and $\Gamma_1 \cup \Gamma_2$.

4.6 High Conducting Interfaces

4.6.1 Integral Equations

From (4.31) and (4.32) and the interfacial condition on the first line of (4.29), we obtain

$$\begin{aligned} \gamma(\underline{\mathbf{x}}_0)T(\underline{\mathbf{x}}_0) &= \int_{\Gamma_1 \cup \Gamma_2} [T(\underline{\mathbf{x}})G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}})) - G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))]rds(\underline{\mathbf{x}}) \\ &\quad + \int_{\Gamma_0} G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)[Q_1(\underline{\mathbf{x}}) - Q_2(\underline{\mathbf{x}})]rds(\underline{\mathbf{x}}). \end{aligned} \quad (4.44)$$

From (4.44), we may write

$$\begin{aligned} &n_r^{\text{int}}(\underline{\mathbf{y}})\frac{\partial}{\partial r_0}[T(\underline{\mathbf{x}}_0)] + n_z^{\text{int}}(\underline{\mathbf{y}})\frac{\partial}{\partial z_0}[T(\underline{\mathbf{x}}_0)] \\ &= \int_{\Gamma_1 \cup \Gamma_2} [T(\underline{\mathbf{x}})\Phi_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}(\underline{\mathbf{x}}); \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}})) \\ &\quad - \Phi_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}}))P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))]rds(\underline{\mathbf{x}}) \\ &\quad + \int_{\Gamma_0} \Phi_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}}))[Q_1(\underline{\mathbf{x}}) - Q_2(\underline{\mathbf{x}})]rds(\underline{\mathbf{x}}) \end{aligned}$$

for $\underline{\mathbf{x}}_0$ in the interior of Ω_1 or Ω_2 , (4.45)

where $\underline{\mathbf{y}}$ is a point on the interface Γ_0 .

If we let $\underline{\mathbf{x}}_0$ tends to the point $\underline{\mathbf{y}}$ on Γ_0 (from within the region Ω_1), we obtain

$$\begin{aligned}
 & \frac{1}{2}[Q_1(\underline{\mathbf{y}}) + Q_2(\underline{\mathbf{y}})] \\
 = & \int_{\Gamma_1 \cup \Gamma_2} [T(\underline{\mathbf{x}})\Phi_1(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}); \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}})) \\
 & - \Phi_0(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}}))P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))]rds(\underline{\mathbf{x}}) \\
 & + \mathcal{C} \int_{\Gamma_0} \Phi_0(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{y}}))[Q_1(\underline{\mathbf{x}}) - Q_2(\underline{\mathbf{x}})]rds(\underline{\mathbf{x}}) \\
 & \text{for } \underline{\mathbf{y}} \in \Gamma_0 \text{ (smooth part)}. \tag{4.46}
 \end{aligned}$$

4.6.2 Boundary Element Procedure

The exterior boundary $\Gamma_1 \cup \Gamma_2$ and the interface Γ_0 are discretized into elements as described in Section 4.5.2. The temperature $T(\underline{\mathbf{x}})$ and the directional rate of change of temperature $P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))$ are approximated as constants $T^{(k)}$ and $P^{(k)}$ respectively on $B^{(k)}$ as in (3.47). The interfacial directional rate of change of temperature function $Q_i(\underline{\mathbf{x}})$ is approximated using

$$Q_i(\underline{\mathbf{x}}) \simeq Q_i^{(m)} \text{ for } \underline{\mathbf{x}} \in I^{(m)} \text{ } (m = 1, 2, \dots, M), \tag{4.47}$$

where $Q_i^{(m)}$ are constants.

From (4.44), we obtain

$$\begin{aligned}
 \gamma(\underline{\mathbf{x}}_0)T(\underline{\mathbf{x}}_0) = & \sum_{k=1}^N \left\{ T^{(k)} \int_{B^{(k)}} G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)})rds(\underline{\mathbf{x}}) \right. \\
 & \left. - P^{(k)} \int_{B^{(k)}} G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)rds(\underline{\mathbf{x}}) \right\} \\
 & + \sum_{m=1}^M (Q_1^{(m)} - Q_2^{(m)}) \int_{I^{(m)}} G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)rds(\underline{\mathbf{x}}). \tag{4.48}
 \end{aligned}$$

Collocating (4.48) at the midpoint of each boundary element and using the boundary conditions in (4.27) give

$$\begin{aligned}
 & \frac{1}{2}[d^{(i)}T^{(i)} + (1 - d^{(i)})f_0(\widehat{\mathbf{x}}^{(i)})] \\
 = & \sum_{k=1}^N \{[d^{(k)}T^{(k)} + (1 - d^{(k)})f_0(\widehat{\mathbf{x}}^{(k)})] \int_{B^{(k)}} G_1(\mathbf{x}; \widehat{\mathbf{x}}^{(i)}; \mathbf{n}^{(k)}) rds(\mathbf{x}) \\
 & - [d^{(k)}(f_1(\widehat{\mathbf{x}}^{(k)}) + f_2(\widehat{\mathbf{x}}^{(k)})T^{(k)}) + (1 - d^{(k)})P^{(k)}] \\
 & \times \int_{B^{(k)}} G_0(\mathbf{x}; \widehat{\mathbf{x}}^{(i)}) rds(\mathbf{x})\} \\
 & + \sum_{m=1}^M (Q_1^{(m)} - Q_2^{(m)}) \int_{I^{(m)}} G_0(\mathbf{x}; \widehat{\mathbf{x}}^{(i)}) rds(\mathbf{x}) \\
 & \text{for } i = 1, 2, \dots, N.
 \end{aligned} \tag{4.49}$$

Similarly, from (4.46), we obtain

$$\begin{aligned}
 & \frac{1}{2}[Q_1^{(i)} + Q_2^{(i)}] \\
 = & \sum_{k=1}^N \{[d^{(k)}T^{(k)} + (1 - d^{(k)})f_0(\widehat{\mathbf{x}}^{(k)})] \int_{B^{(k)}} \Phi_1(\mathbf{x}; \widehat{\mathbf{y}}^{(i)}; \mathbf{n}^{(k)}; \mathbf{m}^{(i)}) rds(\mathbf{x}) \\
 & - [d^{(k)}(f_1(\widehat{\mathbf{x}}^{(k)}) + f_2(\widehat{\mathbf{x}}^{(k)})T^{(k)}) + (1 - d^{(k)})P^{(k)}] \\
 & \times \int_{B^{(k)}} \Phi_0(\mathbf{x}; \widehat{\mathbf{y}}^{(i)}; \mathbf{m}^{(i)}) rds(\mathbf{x})\} \\
 & + \sum_{m=1}^M (Q_1^{(m)} - Q_2^{(m)}) \int_{I^{(m)}} \Phi_0(\mathbf{x}; \widehat{\mathbf{y}}^{(i)}; \mathbf{m}^{(i)}) rds(\mathbf{x}) \\
 & \text{for } i = 1, 2, \dots, M.
 \end{aligned} \tag{4.50}$$

To deal with the interfacial condition over $I^{(i)}$, as given in the second line of (4.29), we work out a formula for $\mathcal{S}(T(\mathbf{x}))$ (given in (4.25)) at $\mathbf{x} \in I^{(i)}$ as follows.

From (4.48), we find that

$$\begin{aligned}
 & t_r^{(i)} \frac{\partial}{\partial r_0} (T(\mathbf{x}_0)) + t_z^{(i)} \frac{\partial}{\partial z_0} (T(\mathbf{x}_0)) \\
 = & \sum_{k=1}^N \left\{ T^{(k)} \int_{B^{(k)}} \Phi_1(\mathbf{x}; \mathbf{x}_0; \mathbf{n}^{(k)}; \mathbf{t}^{(i)}) r ds(\mathbf{x}) \right. \\
 & \left. - P^{(k)} \int_{B^{(k)}} \Phi_0(\mathbf{x}; \mathbf{x}_0; \mathbf{t}^{(i)}) r ds(\mathbf{x}) \right\} \\
 & + \sum_{m=1}^M (Q_1^{(m)} - Q_2^{(m)}) \int_{I^{(m)}} \Phi_0(\mathbf{x}; \mathbf{x}_0; \mathbf{t}^{(i)}) r ds(\mathbf{x}) \\
 & \text{for } \mathbf{x}_0 \text{ in the interior of } \Omega_1 \text{ or } \Omega_2, \tag{4.51}
 \end{aligned}$$

where $\mathbf{t}^{(i)} = [t_r^{(i)}, t_z^{(i)}]$ is a unit tangential vector to the i -th interface element $I^{(i)}$. Note that $[t_r^{(i)}, t_z^{(i)}] = [n_z^{(i)}, -n_r^{(i)}]$.

It follows that

$$\begin{aligned}
 & t_r^{(i)} \frac{\partial}{\partial r_0} [t_r^{(i)} \frac{\partial}{\partial r_0} (T(\mathbf{x}_0)) + t_z^{(i)} \frac{\partial}{\partial z_0} (T(\mathbf{x}_0))] \\
 & + t_z^{(i)} \frac{\partial}{\partial z_0} [t_r^{(i)} \frac{\partial}{\partial r_0} (T(\mathbf{x}_0)) + t_z^{(i)} \frac{\partial}{\partial z_0} (T(\mathbf{x}_0))] \\
 = & \sum_{k=1}^N \left\{ T^{(k)} \int_{B^{(k)}} \Lambda_1(\mathbf{x}; \mathbf{x}_0; \mathbf{n}^{(k)}; \mathbf{t}^{(i)}) r ds(\mathbf{x}) \right. \\
 & \left. - P^{(k)} \int_{B^{(k)}} \Lambda_0(\mathbf{x}; \mathbf{x}_0; \mathbf{t}^{(i)}) r ds(\mathbf{x}) \right\} \\
 & + \sum_{m=1}^M (Q_1^{(m)} - Q_2^{(m)}) \int_{I^{(m)}} \Lambda_0(\mathbf{x}; \mathbf{x}_0; \mathbf{t}^{(i)}) r ds(\mathbf{x}) \\
 & \text{for } \mathbf{x}_0 \text{ in the interior of } \Omega_1 \text{ or } \Omega_2, \tag{4.52}
 \end{aligned}$$

where

$$\begin{aligned}
 \Lambda_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{t}}^{(i)}) &= t_r^{(i)} \frac{\partial}{\partial r_0} [\Phi_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{t}}^{(i)})] + t_z^{(i)} \frac{\partial}{\partial z_0} [\Phi_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{t}}^{(i)})], \\
 \Lambda_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)}; \underline{\mathbf{t}}^{(i)}) &= t_r^{(i)} \frac{\partial}{\partial r_0} [\Phi_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)}; \underline{\mathbf{t}}^{(i)})] \\
 &\quad + t_z^{(i)} \frac{\partial}{\partial z_0} [\Phi_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)}; \underline{\mathbf{t}}^{(i)})].
 \end{aligned} \tag{4.53}$$

More explicitly, the function $\Lambda_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{t}}^{(i)})$ in (4.53) is given by

$$\begin{aligned}
 &\Lambda_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{t}}^{(i)}) \\
 &= \frac{1}{\pi \sqrt{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)} (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0))^2} \\
 &\quad \times \{t_r^{(i)} Y_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{t}}^{(i)}) + t_z^{(i)} Y_2(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{t}}^{(i)})\},
 \end{aligned} \tag{4.54}$$

where

$$\begin{aligned}
 &Y_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{t}}^{(i)}) \\
 &= (r + r_0)(1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \left[\frac{t_r^{(i)}}{2r_0} [(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \right. \\
 &\quad \left. - (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0)) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] + t_z^{(i)} (z - z_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \right] \\
 &\quad - \frac{t_r^{(i)}}{2r_0^2} [(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0))^2 K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2) \\
 &\quad \times (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0)) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - r_0 (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2) \\
 &\quad \times (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) [(r + r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) + (r - r_0) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 &\quad + 2r_0 E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) [r(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0)) - 2r_0 (z - z_0)^2] \\
 &\quad - t_z^{(i)} (z - z_0) \left[\frac{1}{2r_0} (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2) \right. \\
 &\quad \left. \times (E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))) + 2(r - r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \right],
 \end{aligned} \tag{4.55}$$

and

$$\begin{aligned}
 & Y_2(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{t}}^{(i)}) \\
 = & -\{(z - z_0)(1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \left[\frac{t_r^{(i)}}{2r_0} [(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \right. \\
 & \left. - (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0)) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] + t_z^{(i)}(z - z_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \right] \\
 & + t_r^{(i)}(z - z_0)[2(r - r_0)E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & \times [(r + r_0)E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) + (r - r_0)K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))]] \\
 & + t_z^{(i)}[(z - z_0)^2 [(2 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & \left. - (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] - (r - r_0)^2 E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \}. \quad (4.56)
 \end{aligned}$$

The partial derivatives defining the function $\Lambda_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)}; \underline{\mathbf{t}}^{(i)})$ are explicitly given by

$$\begin{aligned}
 & \frac{\partial}{\partial r_0} [\Phi_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)}; \underline{\mathbf{t}}^{(i)})] \\
 = & \frac{1}{\pi \sqrt{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)} (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0))^2} \\
 & \times \left\{ \frac{4(r - r_0)(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)) - (r + r_0)(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0))}{(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0))(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0))} \right. \\
 & \times [t_r^{(i)} \Theta(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)}) + t_z^{(i)} \Psi(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)})] \\
 & \left. + [t_r^{(i)} Y_3(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)}) + t_z^{(i)} Y_4(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)})] \right\}, \quad (4.57)
 \end{aligned}$$

and

$$\begin{aligned}
 & \frac{\partial}{\partial z_0} [\Phi_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)}; \underline{\mathbf{t}}^{(i)})] \\
 = & \frac{1}{\pi \sqrt{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)} (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0))^2} \\
 & \times \left\{ \frac{(z - z_0)[5(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)) - 4rr_0]}{(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0))(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0))} \right. \\
 & \times [t_r^{(i)} \Theta(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)}) + t_z^{(i)} \Psi(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)})] \\
 & \left. + [t_r^{(i)} Y_5(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)}) + t_z^{(i)} Y_6(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)})] \right\}, \quad (4.58)
 \end{aligned}$$

where

$$\begin{aligned}
 Y_3(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)}) = & [(1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - \frac{m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)(r + r_0)(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2)}{r_0(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0))}] \\
 & \times \left\{ \frac{n_r^{(k)}}{2r} [(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r^2) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0)) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \right. \\
 & - n_z^{(k)}(z - z_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \left. \right\} + (r + r_0)(1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & \times \left\{ \frac{n_r^{(k)}}{r} [r_0 E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) + (r - r_0) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \right. \\
 & + \frac{(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2)}{2r_0(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0))} \\
 & \times (n_r^{(k)}[(r - r_0) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - (r + r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 & - n_z^{(k)}(z - z_0)(E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)))) \left. \right\} \\
 & - n_r^{(k)} \left\{ - \frac{(1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)}{2r_0^2} [(r - r_0) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \right. \\
 & - (r + r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] + 2(r - r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & + \frac{(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2)^2}{4r_0^2(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0))} \\
 & \times [(3m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 1)(r + r_0) + (r - r_0)] E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & + 2(r_0 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)r) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \left. \right\} - \frac{(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2)}{r_0(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0))} \\
 & \times [(r - r_0)^2 E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) + (z - z_0)^2 K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \left. \right\} \\
 & + n_z^{(k)}(z - z_0) \left\{ \frac{(E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)))(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2)(r - r_0)}{r_0(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0))} \right. \\
 & - 2E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) + \frac{(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2)^2}{4r_0^2(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0))} \\
 & \times [m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)(2K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - 3E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)))] \\
 & - \frac{(1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)}{2r_0^2} (E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))) \left. \right\}, \tag{4.59}
 \end{aligned}$$

$$\begin{aligned}
 & Y_4(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)}) \\
 = & -(z - z_0) \left\{ (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \frac{n_r^{(k)}}{r} \right. \\
 & \times [r_0 E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) + (r - r_0) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 & + \frac{(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2)}{2r_0(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0))} \\
 & \times \{ (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) [n_r^{(k)} ((r - r_0) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & - (r + r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))) \\
 & - n_z^{(k)} (z - z_0) (E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)))] \\
 & - 2m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) \left[\frac{n_r^{(k)}}{2r} [(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r^2) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \right. \\
 & - (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0)) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 & \left. - n_z^{(k)} (z - z_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \} \} \\
 & - n_r^{(k)} (z - z_0) \left\{ \frac{(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2)}{2r_0(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0))} \right. \\
 & \times [E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) (3m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) (r + r_0) - 2r) \\
 & + K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) (2r (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)))] \\
 & - (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) (E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & + K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) + 2E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))) \} \\
 & + n_z^{(k)} \left\{ \frac{(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2)}{2r_0(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0))} \right. \\
 & \times [(z - z_0)^2 [(1 - 3m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & + (2m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 1) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] - (r - r_0)^2 (E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & - K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)))] + 2(r - r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \}, \tag{4.60}
 \end{aligned}$$

$$\begin{aligned}
 & Y_5(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)}) \\
 = & \left[-2m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)(r + r_0) \frac{(z - z_0)}{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)} \right] \\
 & \times \left\{ \frac{n_r^{(k)}}{2r} [(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r^2) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \right. \\
 & - (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0)) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 & - n_z^{(k)}(z - z_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 & + (r + r_0) (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & \times \left\{ \frac{n_r^{(k)}}{r} (z - z_0) (K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))) \right. \\
 & + \frac{1}{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)} \\
 & \times [n_r^{(k)}(z - z_0) [(r - r_0) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - (r + r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 & + n_z^{(k)} [(r + r_0)^2 E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) + (z - z_0)^2 K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \} \\
 & - \frac{n_r^{(k)}(z - z_0)}{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)} \left\{ \left(-\frac{1}{r_0} \right) [(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0)) \right. \\
 & + m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2)] \\
 & \times [(r - r_0) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - (r + r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 & + \frac{(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2)}{2r_0} [(r - r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & \times ((r + r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - 2r_0 K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)))] - (E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & - K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))) ((r - r_0)^2 + (z - z_0)^2) \\
 & - 2E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)) \} \\
 & + \frac{n_z^{(k)}}{2r_0} \{ -4r_0(r - r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & - (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2) \\
 & \times (E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))) \\
 & + \frac{(z - z_0)^2}{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)}
 \end{aligned}$$

$$\begin{aligned}
 & \times [(4r_0(r - r_0) - 2(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0)))(E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))) \\
 & + m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r_0^2) \\
 & \times (2K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - 3E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)))]], \tag{4.61}
 \end{aligned}$$

and

$$\begin{aligned}
 & Y_6(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)}) \\
 = & [(1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) + \frac{2m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)(z - z_0)^2}{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)}] \\
 & \times \left\{ \frac{n_r^{(k)}}{2r} [(a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 2r^2) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \right. \\
 & - (a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - b(r; r_0)) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] - n_z^{(k)}(z - z_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \} \\
 & - (z - z_0) (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \left\{ \frac{n_r^{(k)}}{r} (z - z_0) (K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) - E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))) \right. \\
 & + \frac{1}{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)} (n_r^{(k)}(z - z_0) [(r - r_0) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & - (r + r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 & + n_z^{(k)} [(r + r_0)^2 E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) + (z - z_0)^2 K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \} \\
 & - n_r^{(k)} \{ 2E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))(r - r_0) - (1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) [(r - r_0) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & - (r + r_0) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \\
 & + \frac{(z - z_0)^2}{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)} [(3m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)(r + r_0) - 2r) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & + (2r(1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] \} \\
 & + n_z^{(k)}(z - z_0) \left\{ \frac{(z - z_0)^2}{a(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) + b(r; r_0)} [(2 - 3m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \right. \\
 & - 2(1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0))] + (3m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0) - 5) E(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \\
 & \left. + 3(1 - m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) K(m(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) \right\}. \tag{4.62}
 \end{aligned}$$

Thus, taking (4.52) and (4.48) (with $\gamma(\underline{\mathbf{x}}_0) = 1$) and apply them in (4.25) gives the formula for $\mathcal{S}(T(\underline{\mathbf{x}}))$ as

$$\begin{aligned}
 \mathcal{S}(T(\underline{\mathbf{x}}))|_{\underline{\mathbf{x}}=\underline{\mathbf{x}}_0} &\simeq \sum_{k=1}^N \{ T^{(k)} [\int_{B^{(k)}} \Lambda_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)}; \underline{\mathbf{t}}^{(i)}) r ds(\underline{\mathbf{x}}) \\
 &\quad - C(\underline{\mathbf{x}}_0) \int_{B^{(k)}} \frac{\partial}{\partial r_0} (G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)})) r ds(\underline{\mathbf{x}}) \\
 &\quad - D(\underline{\mathbf{x}}_0) \int_{B^{(k)}} \frac{\partial}{\partial z_0} (G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)})) r ds(\underline{\mathbf{x}})] \\
 &\quad - P^{(k)} [\int_{B^{(k)}} \Lambda_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{t}}^{(i)}) r ds(\underline{\mathbf{x}}) \} \\
 &\quad - C(\underline{\mathbf{x}}_0) \int_{B^{(k)}} \frac{\partial}{\partial r_0} (G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) r ds(\underline{\mathbf{x}}) \\
 &\quad - D(\underline{\mathbf{x}}_0) \int_{B^{(k)}} \frac{\partial}{\partial z_0} (G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) r ds(\underline{\mathbf{x}})] \\
 &\quad + \sum_{m=1}^M (Q_1^{(m)} - Q_2^{(m)}) [\int_{I^{(m)}} \Lambda_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{t}}^{(i)}) r ds(\underline{\mathbf{x}}) \\
 &\quad - C(\underline{\mathbf{x}}_0) \int_{I^{(m)}} \frac{\partial}{\partial r_0} (G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) r ds(\underline{\mathbf{x}}) \\
 &\quad - D(\underline{\mathbf{x}}_0) \int_{I^{(m)}} \frac{\partial}{\partial z_0} (G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)) r ds(\underline{\mathbf{x}})] \\
 &\quad \text{for } \underline{\mathbf{x}}_0 \text{ in the interior of } \Omega_1 \text{ or } \Omega_2, \tag{4.63}
 \end{aligned}$$

where $C(\underline{\mathbf{x}}_0)$ and $D(\underline{\mathbf{x}}_0)$ are as given in (4.23).

If we let $\underline{\mathbf{x}}_0$ in (4.63) be given by $\widehat{\underline{\mathbf{y}}}^{(i)}$ (midpoint of $I^{(i)}$), the integral over $I^{(m)}$ is proper if $i \neq m$. For $i = m$, the limit of the integral as $\underline{\mathbf{x}}_0$ approaches $\widehat{\underline{\mathbf{y}}}^{(m)}$ can be written as the sum of a Hadamard finite-part integral and a Cauchy principal integral, that is,

$$\begin{aligned} \int_{I^{(m)}} \Lambda_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{t}}^{(m)}) r ds(\underline{\mathbf{x}}) &= \mathcal{H} \int_{I^{(m)}} \frac{\Phi_3(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{y}}}^{(m)}; \underline{\mathbf{t}}^{(m)}; \underline{\mathbf{t}}^{(m)})}{\sqrt{a(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{y}}}^{(m)}) + b(r; \widehat{\rho}_0^{(m)})}} r ds(\underline{\mathbf{x}}) \\ &+ \mathcal{C} \int_{I^{(m)}} \Phi_4(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{y}}}^{(m)}; \underline{\mathbf{t}}^{(m)}; \underline{\mathbf{t}}^{(m)}) r ds(\underline{\mathbf{x}}), \end{aligned} \quad (4.64)$$

where

$$\begin{aligned} &\Phi_4(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{y}}}^{(m)}; \underline{\mathbf{t}}^{(m)}; \underline{\mathbf{t}}^{(m)}) \\ &= \Lambda_0(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{y}}}^{(m)}; \underline{\mathbf{t}}^{(m)}) - \frac{\Phi_3(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{y}}}^{(m)}; \underline{\mathbf{t}}^{(m)}; \underline{\mathbf{t}}^{(m)})}{\sqrt{a(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{y}}}^{(m)}) + b(r; \widehat{\rho}_0^{(m)})}}. \end{aligned} \quad (4.65)$$

The Hadamard finite-part integral above which contains the function Φ_3 in its integrand has also appeared in the boundary element formulation for the low conducting interface. As explained below (4.43), it is approximately given by $-2/(\pi \ell^{(m)})$.

Thus, the interfacial condition in the second line of (4.29) becomes

$$\begin{aligned} &\kappa_1 Q_1^{(i)} - \kappa_2 Q_2^{(i)} \\ &= -\alpha \sum_{k=1}^N \{ [d^{(k)} T^{(k)} + (1 - d^{(k)}) f_0(\widehat{\underline{\mathbf{x}}}^{(k)})] \\ &\quad \times [\int_{B^{(k)}} \Lambda_1(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{y}}}^{(i)}; \underline{\mathbf{n}}^{(k)}; \underline{\mathbf{t}}^{(i)}) r ds(\underline{\mathbf{x}}) \\ &\quad - C(\widehat{\underline{\mathbf{y}}}^{(i)}) \int_{B^{(k)}} \frac{\partial}{\partial \rho_0} (G_1(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{y}}}^{(i)}; \underline{\mathbf{n}}^{(k)})) r ds(\underline{\mathbf{x}}) \\ &\quad - D(\widehat{\underline{\mathbf{y}}}^{(i)}) \int_{B^{(k)}} \frac{\partial}{\partial \zeta_0} (G_1(\underline{\mathbf{x}}; \widehat{\underline{\mathbf{y}}}^{(i)}; \underline{\mathbf{n}}^{(k)})) r ds(\underline{\mathbf{x}})] \end{aligned}$$

$$\begin{aligned}
 & -[d^{(k)}(f_1(\hat{\mathbf{x}}^{(k)}) + f_2(\hat{\mathbf{x}}^{(k)})T^{(k)}) + (1 - d^{(k)})P^{(k)}] \\
 & \times \left[\int_{B^{(k)}} \Lambda_0(\mathbf{x}; \hat{\mathbf{y}}^{(i)}; \mathbf{t}^{(i)}) r ds(\mathbf{x}) \right. \\
 & - C(\hat{\mathbf{y}}^{(i)}) \int_{B^{(k)}} \frac{\partial}{\partial \rho_0} (G_0(\mathbf{x}; \hat{\mathbf{y}}^{(i)})) r ds(\mathbf{x}) \\
 & \left. - D(\hat{\mathbf{y}}^{(i)}) \int_{B^{(k)}} \frac{\partial}{\partial \zeta_0} (G_0(\mathbf{x}; \hat{\mathbf{y}}^{(i)})) r ds(\mathbf{x}) \right] \} \\
 & - \alpha \sum_{m=1}^M \{ (Q_1^{(m)} - Q_2^{(m)}) \\
 & \times [L^{(im)} - C(\hat{\mathbf{y}}^{(i)}) \int_{I^{(m)}} \frac{\partial}{\partial \rho_0} (G_0(\mathbf{x}; \hat{\mathbf{y}}^{(i)})) r ds(\mathbf{x}) \\
 & - D(\hat{\mathbf{y}}^{(i)}) \int_{I^{(m)}} \frac{\partial}{\partial \zeta_0} (G_0(\mathbf{x}; \hat{\mathbf{y}}^{(i)})) r ds(\mathbf{x})] \} \\
 & \text{for } i = 1, 2, \dots, M,
 \end{aligned} \tag{4.66}$$

where

$$\begin{aligned}
 L^{(im)} &= \int_{I^{(m)}} \Lambda_0(\mathbf{x}; \hat{\mathbf{y}}^{(i)}; \mathbf{t}^{(i)}) r ds(\mathbf{x}) \text{ for } i \neq m, \\
 L^{(mm)} &\simeq \frac{2}{\pi \ell^{(m)}} + \mathcal{C} \int_{I^{(m)}} \Phi_4(\mathbf{x}; \hat{\mathbf{y}}^{(m)}; \mathbf{t}^{(m)}; \mathbf{t}^{(m)}) r ds(\mathbf{x}).
 \end{aligned} \tag{4.67}$$

Equations (4.49), (4.50) and (4.66) can be solved as a system of $N + 2M$ linear algebraic equations for $N + 2M$ unknowns given by either $T^{(k)}$ or $P^{(k)}$ ($k = 1, 2, \dots, N$) and $Q_i^{(m)}$ ($i = 1, 2$ and $m = 1, 2, \dots, M$).

4.7 Nonlinear Interfaces

In this section, we extend the hypersingular boundary integral method to nonlinear interfacial conditions prescribed in accordance with the Stefan-Boltzmann law (see Yang, Yamamoto and Cheng [78] and Hu, Xu and Chen [46]). More specifically, the nonlinear interfacial conditions are given by

$$\begin{aligned} \kappa_1 Q_1(\underline{\mathbf{x}}) &= \kappa_2 Q_2(\underline{\mathbf{x}}) = \sigma(\underline{\mathbf{x}}) \lim_{\varepsilon \rightarrow 0^+} [T^4(\underline{\mathbf{x}} + \varepsilon \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}})) - T^4(\underline{\mathbf{x}} - \varepsilon \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}}))] \\ &\quad \text{for } \underline{\mathbf{x}} \in \Gamma_0, \end{aligned} \quad (4.68)$$

where $\sigma(\underline{\mathbf{x}})$ is a positive function.

If we define the interfacial temperature functions $T_1(\underline{\mathbf{x}})$ and $T_2(\underline{\mathbf{x}})$ by

$$\left. \begin{aligned} T_1(\underline{\mathbf{x}}) &= \lim_{\varepsilon \rightarrow 0^+} T(\underline{\mathbf{x}} + \varepsilon \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}})) \\ T_2(\underline{\mathbf{x}}) &= \lim_{\varepsilon \rightarrow 0^+} T(\underline{\mathbf{x}} - \varepsilon \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}})) \end{aligned} \right\} \text{ for } \underline{\mathbf{x}} \in \Gamma_0, \quad (4.69)$$

we may rewrite (4.68) as

$$\begin{aligned} \kappa_1 Q_1(\underline{\mathbf{x}}) &= \kappa_2 Q_2(\underline{\mathbf{x}}) = \lambda(\underline{\mathbf{x}}, T_1(\underline{\mathbf{x}}), T_2(\underline{\mathbf{x}})) \Delta T(\underline{\mathbf{x}}) \\ &\quad \text{for } \underline{\mathbf{x}} \in \Gamma_0, \end{aligned} \quad (4.70)$$

where

$$\lambda(\underline{\mathbf{x}}, T_1(\underline{\mathbf{x}}), T_2(\underline{\mathbf{x}})) = \sigma(\underline{\mathbf{x}})[T_1^2(\underline{\mathbf{x}}) + T_2^2(\underline{\mathbf{x}})][T_1(\underline{\mathbf{x}}) + T_2(\underline{\mathbf{x}})]. \quad (4.71)$$

Comparing with (4.28), we may regard nonlinear interfaces with interfacial conditions (4.70) as low conducting. Thus, the boundary element procedure for low conducting interfaces, as outlined in Subsection 4.5.2, may be adopted to analyze steady state axisymmetric heat conduction problems involving bimetals with the nonlinear interfacial conditions in (4.70), as described below.

Now, replacing $\lambda(\underline{\mathbf{x}})$ in (4.33) by $\lambda(\underline{\mathbf{x}}, T_1(\underline{\mathbf{x}}), T_2(\underline{\mathbf{x}}))$ and letting $\underline{\mathbf{x}}_0$ tend to $\underline{\mathbf{y}} = (\rho_0, \zeta_0)$ on Γ_0 (from within the region Ω_1 and Ω_2), we find that the

interfacial temperature functions $T_1(\underline{\mathbf{x}})$ and $T_2(\underline{\mathbf{x}})$ are given by

$$\begin{aligned}
 T_1(\underline{\mathbf{y}}) = & \frac{1}{2}\Delta T(\underline{\mathbf{y}}) + \int_{\Gamma_1 \cup \Gamma_2} [T(\underline{\mathbf{x}})G_1(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}(\underline{\mathbf{x}})) - G_0(\underline{\mathbf{x}}; \underline{\mathbf{y}})P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))]rds(\underline{\mathbf{x}}) \\
 & - \int_{\Gamma_0} \Delta T(\underline{\mathbf{x}})[G_1(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}})) \\
 & - \frac{(\kappa_2 - \kappa_1)}{\kappa_1 \kappa_2} \lambda(\underline{\mathbf{x}}, T_1(\underline{\mathbf{x}}), T_2(\underline{\mathbf{x}}))G_0(\underline{\mathbf{x}}; \underline{\mathbf{y}})]rds(\underline{\mathbf{x}}) \\
 & \text{for } \underline{\mathbf{y}} \in \Gamma_0, \tag{4.72}
 \end{aligned}$$

and

$$\begin{aligned}
 T_2(\underline{\mathbf{y}}) = & -\frac{1}{2}\Delta T(\underline{\mathbf{y}}) + \int_{\Gamma_1 \cup \Gamma_2} [T(\underline{\mathbf{x}})G_1(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}(\underline{\mathbf{x}})) - G_0(\underline{\mathbf{x}}; \underline{\mathbf{y}})P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}}))]rds(\underline{\mathbf{x}}) \\
 & - \int_{\Gamma_0} \Delta T(\underline{\mathbf{x}})[G_1(\underline{\mathbf{x}}; \underline{\mathbf{y}}; \underline{\mathbf{n}}^{\text{int}}(\underline{\mathbf{x}})) \\
 & - \frac{(\kappa_2 - \kappa_1)}{\kappa_1 \kappa_2} \lambda(\underline{\mathbf{x}}, T_1(\underline{\mathbf{x}}), T_2(\underline{\mathbf{x}}))G_0(\underline{\mathbf{x}}; \underline{\mathbf{y}})]rds(\underline{\mathbf{x}}) \\
 & \text{for } \underline{\mathbf{y}} \in \Gamma_0. \tag{4.73}
 \end{aligned}$$

To allow for the possibility of nonlinear conditions on the exterior boundary of the bimaterial, we modify the boundary conditions in (4.27) to become

$$\begin{aligned}
 T(\underline{\mathbf{x}}) &= f_0(\underline{\mathbf{x}}) \text{ for } \underline{\mathbf{x}} \in \Xi_1, \\
 P(\underline{\mathbf{x}}; \underline{\mathbf{n}}(\underline{\mathbf{x}})) &= f_1(\underline{\mathbf{x}}, T(\underline{\mathbf{x}})) \text{ for } \underline{\mathbf{x}} \in \Xi_2, \tag{4.74}
 \end{aligned}$$

where f_1 is a function of $\underline{\mathbf{x}}$ and T .

The boundary element procedure follows that of Subsection 4.5.2. In addition to the approximations in (3.47) and (4.40), we have the following approximations over an element on the interface:

$$\left. \begin{aligned} T_1(\underline{\mathbf{x}}) &\simeq T_1^{(j)} \\ T_2(\underline{\mathbf{x}}) &\simeq T_2^{(j)} \\ \lambda(\underline{\mathbf{x}}, T_1(\underline{\mathbf{x}}), T_2(\underline{\mathbf{x}})) &\simeq \lambda^{(j)} \end{aligned} \right\} \text{ for } \underline{\mathbf{x}} \in I^{(j)} (j = 1, 2, \dots, M), \tag{4.75}$$

where $T_1^{(j)}$, $T_2^{(j)}$ and $\lambda^{(j)}$ are constants.

It follows that (4.41) can be modified to become

$$\begin{aligned}
 & \frac{1}{2}[d^{(i)}T^{(i)} + (1 - d^{(i)})f_0(\widehat{\mathbf{x}}^{(i)})] \\
 = & \sum_{k=1}^N \{[d^{(k)}T^{(k)} + (1 - d^{(k)})f_0(\widehat{\mathbf{x}}^{(k)})] \int_{B^{(k)}} G_1(\mathbf{x}; \widehat{\mathbf{x}}^{(i)}; \mathbf{n}^{(k)}) r ds(\mathbf{x}) \\
 & - [d^{(k)}f_1(\widehat{\mathbf{x}}^{(k)}, T^{(k)}) + (1 - d^{(k)})P^{(k)}] \int_{B^{(k)}} G_0(\mathbf{x}; \widehat{\mathbf{x}}^{(i)}) r ds(\mathbf{x})\} \\
 & - \sum_{j=1}^M \Delta T^{(j)} \int_{I^{(j)}} [G_1(\mathbf{x}; \widehat{\mathbf{x}}^{(i)}; \mathbf{m}^{(j)}) - \frac{(\kappa_2 - \kappa_1)}{\kappa_1 \kappa_2} \lambda^{(j)} G_0(\mathbf{x}; \widehat{\mathbf{x}}^{(i)})] r ds(\mathbf{x}) \\
 & \text{for } i = 1, 2, \dots, N.
 \end{aligned} \tag{4.76}$$

Similarly, (4.42) becomes

$$\begin{aligned}
 & [(\frac{1}{\kappa_1} - \frac{(\kappa_2 - \kappa_1)}{2\kappa_1 \kappa_2})\lambda^{(p)} + \frac{2}{\pi \ell^{(p)}}]\Delta T^{(p)} \\
 = & \sum_{\substack{j=1 \\ j \neq p}}^M \Delta T^{(j)} \int_{I^{(j)}} \frac{\Phi_3(\mathbf{x}; \widehat{\mathbf{y}}^{(p)}; \mathbf{m}^{(j)}; \mathbf{m}^{(p)})}{\sqrt{a(\mathbf{x}; \widehat{\mathbf{y}}^{(p)}) + b(r; \widehat{\rho}_0^{(p)})}} r ds(\mathbf{x}) \\
 & - \sum_{j=1}^M \Delta T^{(j)} \int_{I^{(j)}} [\Phi_2(\mathbf{x}; \widehat{\mathbf{y}}^{(p)}; \mathbf{m}^{(j)}; \mathbf{m}^{(p)}) \\
 & - \frac{(\kappa_2 - \kappa_1)}{\kappa_1 \kappa_2} \lambda^{(j)} \Phi_0(\mathbf{x}; \widehat{\mathbf{y}}^{(p)}; \mathbf{m}^{(p)})] r ds(\mathbf{x}) \\
 & + \sum_{k=1}^N \{[d^{(k)}T^{(k)} + (1 - d^{(k)})f_0(\widehat{\mathbf{x}}^{(k)})] \int_{B^{(k)}} \Phi_1(\mathbf{x}; \widehat{\mathbf{y}}^{(p)}; \mathbf{n}^{(k)}; \mathbf{m}^{(p)}) r ds(\mathbf{x}) \\
 & - [d^{(k)}f_1(\widehat{\mathbf{x}}^{(k)}, T^{(k)}) + (1 - d^{(k)})P^{(k)}] \int_{B^{(k)}} \Phi_0(\mathbf{x}; \widehat{\mathbf{y}}^{(p)}; \mathbf{m}^{(p)}) r ds(\mathbf{x})\} \\
 & \text{for } p = 1, 2, \dots, M.
 \end{aligned} \tag{4.77}$$

From (4.72) and (4.73), we obtain

$$\begin{aligned}
 T_1^{(p)} &= \frac{1}{2}\Delta T^{(p)} + \sum_{k=1}^N \int_{B^{(k)}} [T^{(k)} G_1(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)}; \underline{\mathbf{n}}^{(k)}) - G_0(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)}) P^{(k)}] r ds(\underline{\mathbf{x}}) \\
 &\quad - \sum_{j=1}^M \Delta T^{(j)} \int_{I^{(j)}} [G_1(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)}; \underline{\mathbf{m}}^{(j)}) - \frac{(\kappa_2 - \kappa_1)}{\kappa_1 \kappa_2} \lambda^{(j)} G_0(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)})] r ds(\underline{\mathbf{x}}) \\
 &\quad \text{for } p = 1, 2, \dots, M,
 \end{aligned} \tag{4.78}$$

and

$$\begin{aligned}
 T_2^{(p)} &= -\frac{1}{2}\Delta T^{(p)} + \sum_{k=1}^N \int_{B^{(k)}} [T^{(k)} G_1(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)}; \underline{\mathbf{n}}^{(k)}) - G_0(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)}) P^{(k)}] r ds(\underline{\mathbf{x}}) \\
 &\quad - \sum_{j=1}^M \Delta T^{(j)} \int_{I^{(j)}} [G_1(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)}; \underline{\mathbf{m}}^{(j)}) - \frac{(\kappa_2 - \kappa_1)}{\kappa_1 \kappa_2} \lambda^{(j)} G_0(\underline{\mathbf{x}}; \underline{\hat{\mathbf{y}}}^{(p)})] r ds(\underline{\mathbf{x}}) \\
 &\quad \text{for } p = 1, 2, \dots, M.
 \end{aligned} \tag{4.79}$$

We can solve for the unknown values in (4.76), (4.77), (4.78) and (4.79) as follows.

1. Make an initial guess of the temperature $T(\underline{\mathbf{x}})$ in the bimaterial to compute $\lambda^{(j)}$ and $f_1(\underline{\hat{\mathbf{x}}}^{(k)}, T^{(k)})$ approximately. Go to Step 2
2. Solve (4.76) and (4.77) for the unknown boundary values and the interface temperature jump using the latest known values of $\lambda^{(j)}$ and $f_1(\underline{\hat{\mathbf{x}}}^{(k)}, T^{(k)})$. Go to Step 3.
3. With the latest known values of $\lambda^{(j)}$, $T^{(k)}$, $P^{(k)}$ and $\Delta T^{(j)}$, use (4.78) and (4.79) to compute $T_1^{(j)}$ and $T_2^{(j)}$ and update the values of $\lambda^{(j)}$ accordingly. $f_1(\underline{\hat{\mathbf{x}}}^{(k)}, T^{(k)})$ is also updated using the latest $T^{(k)}$. If each updated values of $\lambda^{(j)}$ and $f_1(\underline{\hat{\mathbf{x}}}^{(k)}, T^{(k)})$ changes by a very small prescribed absolute value, stop the iteration. Otherwise, go to Step 2 to continue the loop with the latest values of $\lambda^{(j)}$ and $f_1(\underline{\hat{\mathbf{x}}}^{(k)}, T^{(k)})$.

4.8 Specific Problems

Problem 4.1

To test the boundary element procedure for low conducting interfaces, take

$$\begin{aligned}\Omega_1 &= \{(r, z) : r_{\text{int}} < r < r_{\text{outer}}, 0 < z < d\}, \\ \Omega_2 &= \{(r, z) : r_{\text{inner}} < r < r_{\text{int}}, 0 < z < d\},\end{aligned}$$

where r_{int} , r_{inner} and r_{outer} are constants such that $0 < r_{\text{inner}} < r_{\text{int}} < r_{\text{outer}}$ and d is a given positive constant. Note that the low conducting interface Γ_0 between Ω_1 and Ω_2 lies in the region $r = r_{\text{int}}$, $0 < z < d$.

The boundary conditions on the exterior boundary of $\Omega_1 \cup \Omega_2$ are given by

$$\begin{aligned}T(r_{\text{inner}}, z) &= T_c \text{ for } 0 < z < d, \\ P(r_{\text{outer}}, z; 1, 0) &= -\frac{c}{\kappa_1}(T(r_{\text{outer}}, z) - T_a) \text{ for } 0 < z < d, \\ P(r, 0; 0, -1) &= P(r, h; 0, 1) = 0 \text{ for } r_{\text{inner}} < r < r_{\text{outer}},\end{aligned}$$

where c , T_c and T_a are given constants. Note that T_a is the outside ambient temperature surrounding the body.

It is assumed that (4.28) is applicable with $\lambda(r, z) = \lambda_0$ (a constant), that is, the low conducting interface Γ_0 is homogeneous.

The exact solution of this specific problem is

$$T(r, z) = \sigma_i + \tau_i \ln(r) \text{ for } (r, z) \in \Omega_i \ (i = 1, 2),$$

where

$$\begin{aligned}\sigma_1 &= T_a - \tau_1 \left[\frac{\kappa_1}{cr_{\text{outer}}} + \ln(r_{\text{outer}}) \right], \quad \sigma_2 = T_c - \tau_2 \ln(r_{\text{inner}}), \\ \tau_1 &= \frac{\kappa_2}{\kappa_1} \tau_2, \quad \tau_2 = \frac{\lambda_0}{\chi} (T_a - T_c), \\ \chi &= \frac{\kappa_2}{r_{\text{int}}} - \lambda_0 \left[\ln(r_{\text{inner}}) - \frac{\kappa_2}{\kappa_1} (\ln(r_{\text{outer}}) + \frac{\kappa_1}{cr_{\text{outer}}}) - (1 - \frac{\kappa_2}{\kappa_1}) \ln(r_{\text{int}}) \right].\end{aligned}$$

Table 4.1: A comparison of the numerical values of T with the exact solution at various selected points for Problem 4.1.

Point (r, z)	$N_0 = 5$	$N_0 = 10$	$N_0 = 20$	$N_0 = 30$	Exact
(0.600,0.100)	4.616703	4.613403	4.612124	4.611786	4.611326
(0.750,0.500)	4.139119	4.137097	4.136263	4.136022	4.135628
(0.900,0.900)	3.754166	3.750142	3.748390	3.747864	3.746954
(1.250,0.750)	2.652760	2.650462	2.649568	2.649317	2.648913
(1.100,0.100)	3.061885	3.059512	3.058493	3.058192	3.057687
(1.490,0.100)	2.094195	2.089170	2.087941	2.087660	2.087292

For the purpose of testing the boundary element procedure in Subsection 4.5.2, take $d = 1$, $r_{\text{outer}} = 3/2$, $r_{\text{int}} = 1$, $r_{\text{inner}} = 1/2$, $\kappa_1 = 1/2$, $\kappa_2 = 3/4$, $\lambda_0 = 10$, $c = 1$, $T_a = 1$ and $T_c = 5$. The exterior boundary of $\Omega_1 \cup \Omega_2$ and the interface Γ_0 are approximated as straight lines with $2N_0$ sides (so that $N = 8N_0$ and $M = 2N_0$).

Equations (4.41) and (4.42) are solved using $N_0 = 5, 10, 20$ and 30 and the numerical values of T at various selected points in $\Omega_1 \cup \Omega_2$ as computed by using (4.33) with $\gamma(\underline{\mathbf{x}}_0) = 1$ are compared with the exact values in Table 4.1. The numerical values are in good agreement with the exact ones and they converge to the exact solution when N_0 is increased from 5 to 30 (that is, when the calculation is refined by reducing the sizes of the boundary elements used).

As expected, for a given N_0 , the numerical value of the temperature jump ΔT is found to have approximately the same value on all the elements of the interface Γ_0 . Moreover, the percentage errors in the numerical values of ΔT are around 0.92%, 0.45%, 0.22% and 0.15% for N_0 given by 5, 10, 20 and 30 respectively.

Problem 4.2

Problem 4.1 deals with one-dimensional heat conduction across a homogeneous interface. For a more general test problem involving the low conducting interfacial conditions (4.28) with coefficient λ which varies in space, take

$$\begin{aligned}\Omega_1 &= \{(r, z) : 0 \leq r < 1, 0 < z < 1\}, \\ \Omega_2 &= \{(r, z) : 0 \leq r < 1, 1 < z < 2\},\end{aligned}$$

together with $\lambda = 1/(1 + r^2)$, $\kappa_1 = 1/4$, $\kappa_2 = 1$ and the boundary conditions

$$\begin{aligned}\left. \begin{aligned} P(r, 0; 0, -1) &= 0 \\ T(r, 2) &= -4 \end{aligned} \right\} \text{ for } 0 < r < 1, \\ P(1, z; 1, 0) &= \begin{cases} 2 & \text{for } 0 < z < 1, \\ 0 & \text{for } 1 < z < 2. \end{cases}\end{aligned}$$

Note that the interface between Ω_1 and Ω_2 is given by $0 \leq r < 1$, $z = 1$. Refer to Figure 4.4.

It may be easily verified that the exact solution for the test problem here is given by

$$T(r, z) = \begin{cases} r^2 - 2z^2 & \text{for } (r, z) \in \Omega_1, \\ -2 - z & \text{for } (r, z) \in \Omega_2. \end{cases}$$

For the problem here, each of Γ_1 and Γ_2 comprises two straight line segments of unit length on the rz plane. The interface Γ_0 is a vertical line segment of unit length. To obtain some numerical results, each of the unit length line segments is discretized into N_0 equal length boundary elements (so that $N = 4N_0$ and $M = N_0$). Three sets of numerical values are obtained for $\Delta T(r, 1)$ across the interface Γ_0 by solving the equations (4.41) and (4.42) using $N_0 = 50, 150$ and 450 . Figure 4.5 compares the numerical $\Delta T(r, 1)$ with the values obtained from the exact solution for $0.05 \leq r \leq 0.95$. On the whole, there is a good agreement between the numerical and exact temperature jump, except for points which are very close to $r = 0$ due to their close proximity to

Figure 4.4: A geometrical sketch of Problem 4.2 on the rz plane.

Figure 4.5: Plots of the numerical and exact temperature jump $\Delta T(r, 1)$ for $0.05 \leq r \leq 0.95$.

Figure 4.6: Plots of the numerical and exact boundary temperature $T(1, z)$ for $0 < z < 2$.

the sharp corners. Nevertheless, as clearly shown in Figure 4.5, the errors for the temperature jump for r close to zero are significantly reduced when the number of boundary and interface elements is increased.

As the temperature is not known a priori on the boundary $r = 1$ ($0 < z < 2$), the numerical values of the boundary temperature $T(1, z)$ are compared graphically with the exact temperature in Figure 4.6. The numerical and exact temperature agree well with each other. Note that the gap in the graph is due to the temperature jump across the interface Γ_0 at $z = 1$.

Problem 4.3

Consider now a homogeneous cylindrical representative volume element containing a centrally located cylindrical carbon nanotube as in Ang, Singh and Tanaka [11]. The regions Ω_1 and Ω_2 are as sketched in Figure 4.7. As the

Figure 4.7: A geometrical sketch of Problem 4.3 on the rz plane.

carbon nanotube is centrally located in the composite, the lengths ℓ_1 , ℓ_2 and ℓ_3 are such that $\ell_1 + \ell_2 = \ell_3$. Here Ω_1 is taken to be occupied by elastomer S160 with thermal conductivity $\kappa_1 = 0.56 \text{ Wm}^{-1}\text{K}^{-1}$ and Ω_2 is occupied by the carbon nanotube whose thermal conductivity κ_2 is taken to be given by $6000 \text{ Wm}^{-1}\text{K}^{-1}$.

The boundary conditions on the exterior of Ω_1 are given by

$$\left. \begin{aligned} T(r, 0) &= 200 \text{ K} \\ T(r, \ell_3) &= 100 \text{ K} \end{aligned} \right\} \text{ for } 0 < r < r_2,$$

$$P(r_2, z; 1, 0) = 0 \text{ for } 0 < z < \ell_3.$$

Of interest here is to examine the effect of the interfacial parameter λ (assumed to be a constant) on the equivalent (effective) thermal conductivity κ_e of the carbon nanotube based composite along the z direction. In Ang, Singh and Tanaka [11], the equivalent thermal conductivity κ_e is calculated for the limiting case $\lambda \rightarrow \infty$ (that is, the case in which the interface between

the elastomer and the carbon nanotube is perfectly conducting) by modeling the carbon nanotube as a thermal superconductor.

The equivalent thermal conductivity κ_e is given by

$$\kappa_e = -\frac{q_{\text{ave}}\ell_3}{T(r, \ell_3) - T(r, 0)},$$

where q_{ave} is the average heat flux across $0 \leq r < r_2$, $z = \ell_3$. The average heat flux can be calculated approximately from the numerical solution in Section 4.5 by using

$$q_{\text{ave}} \simeq -\frac{2\kappa_1}{r_2^2} \sum_{k=1}^J P^{(k)} \left\{ r^{(k)} \ell^{(k)} + \frac{1}{2} (\ell^{(k)})^2 \right\},$$

if the side $z = \ell_3$ for $0 \leq r \leq r_2$ (on the rz plane) is discretized into J elements denoted by $C^{(1)}$, $C^{(2)}$, $\dots$, $C^{(J-1)}$ and $C^{(J)}$.

Figure 4.8: Plots of κ_e/κ_1 against $\log_{10}(\lambda r_2/\kappa_1)$ for a few selected values of ℓ_c/r_2 . Horizontal dashed lines give values of κ_e/κ_1 as calculated in Ang, Singh and Tanaka [11] for the case in which the interface between the elastomer and the carbon nanotube is ideal.

The radii r_1 and r_2 and the lengths ℓ_1 , ℓ_2 and ℓ_3 are taken to be such that $r_1/r_2 = 1/2$ and $(\ell_1 + \ell_2)/r_2 = \ell_3/r_2 = 10$. To calculate the non-dimensionalized equivalent thermal conductivity κ_e/κ_1 , as many as 3600 elements are employed on the exterior boundary and interface of the carbon nanotube composite. Figure 4.8 gives plots of κ_e/κ_1 against $\log_{10}(\lambda r_2/\kappa_1)$ for some selected values of the non-dimensionalized length ℓ_c/r_2 of the carbon nanotube. The non-dimensionalized equivalent thermal conductivity κ_e/κ_1 is found to be less than 1 for $\lambda r_2/\kappa_1$ which is very close to zero. This is expected, because if the interface between the elastomer and the carbon nanotube is highly damaged, the carbon nanotube behaves as a thermal insulator which obstructs the flow of heat. From Figure 4.8, it is obvious that κ_e/κ_1 tends to a lower value as $\lambda r_2/\kappa_1$ approaches 0 (from above) for larger ℓ_c/r_2 (that is, for a longer carbon nanotube). As $\lambda r_2/\kappa_1$ increases in magnitude, the carbon nanotube serves to enhance the flow of heat through the composite. Thus, κ_e/κ_1 is greater than 1 for larger $\lambda r_2/\kappa_1$. In Figure 4.8, the values of κ_e/κ_1 calculated in [11] for the case in which the interface between the elastomer and the carbon nanotube is perfectly conducting are shown using horizontal dashed lines. For a given ℓ_c/r_2 , it appears that κ_e/κ_1 becomes closer but is less than the value given by dashed line, as $\lambda r_2/\kappa_1$ increases. As may be expected, for a larger value of ℓ_c/r_2 , the difference between the lower and the upper bounds of κ_e/κ_1 is bigger.

Problem 4.4

To check the boundary element procedure for the high conducting interfacial conditions (4.29), consider the regions Ω_1 and Ω_2 as sketched in Figure 4.9. Note that Ω_1 and Ω_2 are defined by the curves $r^2 + z^2 = 9/4$, $r^2 + z^2 = 1$ and $r^2 + z^2 = 1/4$ and the lines $r = 0$ and $r = (\sqrt{3}/3)z$ on the rz plane.

Figure 4.9: A geometrical sketch of Problem 4.4 on the rz plane.

For a particular problem, take $\kappa_1 = 3/4$ and $\kappa_2 = 1/2$. The interface Γ_0 between the two regions is high conducting with

$$\alpha = \frac{r^2 - 2z^2}{4(2r^2 - z^2)}.$$

Note that α is taken to be a function of $\underline{\mathbf{x}}$ merely for the purpose of constructing a test problem that has a simple analytic solution.

The boundary conditions on the exterior boundary of $\Omega_1 \cup \Omega_2$ are given by

$$\begin{aligned} P(r, z; \frac{2}{3}r, \frac{2}{3}z) &= \frac{4}{3}(r^2 - 2z^2) \text{ for } r^2 + z^2 = \frac{9}{4}, 0 < r < \frac{\sqrt{3}}{3}z, \\ T(r, z) &= 3r^2 - \frac{1}{2} \text{ for } r^2 + z^2 = \frac{1}{4}, 0 < r < \frac{\sqrt{3}}{3}z, \\ T(r, z) &= -\frac{5}{3}z^2 \text{ for } r = \frac{\sqrt{3}}{3}z, \frac{1}{4} < r^2 + z^2 < 1, \\ T(r, z) &= -\frac{5}{3}z^2 \text{ for } r = \frac{\sqrt{3}}{3}z, 1 < r^2 + z^2 < \frac{9}{4}. \end{aligned}$$

The interface Γ_0 is discretized into $2N_0$ elements and the exterior boundary into $5N_0$ elements (hence $N = 5N_0$ and $M = 2N_0$). Equations (4.49), (4.50)

Table 4.2: Numerical and exact values of T at selected interior points for Problem 4.4.

Point (r, z)	$N_0 = 5$	$N_0 = 10$	$N_0 = 20$	$N_0 = 40$	Exact
(0.10, 0.60)	-0.709956	-0.709985	-0.709982	-0.709995	-0.71000
(0.20, 0.70)	-0.940185	-0.940049	-0.939990	-0.939999	-0.94000
(0.45, 0.80)	-1.098310	-1.080885	-1.076857	-1.077520	-1.07750
(0.50, 0.90)	-1.382515	-1.368887	-1.370157	-1.370010	-1.37000
(0.20, 1.10)	-2.378008	-2.379544	-2.379830	-2.380000	-2.38000
(0.10, 1.40)	-3.904657	-3.908711	-3.909656	-3.909920	-3.91000

and (4.66) are then solved as a system of $9N_0$ linear algebraic equations with $N_0 = 5, 10, 20$ and 40 . The largest elements for $N_0 = 5, 10, 20$ and 40 have magnitudes of $0.10, 0.05, 0.025$ and 0.0125 units respectively. The numerically computed temperature at various selected points in $\Omega_1 \cup \Omega_2$ are then compared with the exact solution given by

$$T(r, z) = r^2 - 2z^2 \text{ for } (r, z) \in \Omega_1 \cup \Omega_2.$$

As shown in Table 4.2, the numerical values for T are reasonably accurate and they converge to the exact solution when the calculation is refined by increasing the number of elements used. All percentage errors of the numerical values for $N_0 = 40$ are less than 0.01% .

Problem 4.5

Consider now a thermal management system modeled by two homogeneous hollow cylindrical solids. The system is assembled such that one hollow cylindrical solid residing inside the other hollow cylindrical solid. In addition, the cylindrical solids are joined together by a thin layer of carbon nanotubes or nanocylinders of high thermal conductivity. Such an interface may be modeled as high conducting. As sketched in Figure 4.10, Region Ω_1 gives the inner

Figure 4.10: A geometrical sketch of Problem 4.5 on the rz plane.

hollow cylinder while region Ω_2 gives the outer cylinder. The line $r = r_2$, $0 < z < z_1$, denoted by Γ_0 , is the high conducting interface.

A constant heat flux q_0 flows into the system from the hollow space of the cylinder solids at $r = r_1$, $0 < z < z_1$ and there is a uniform convective cooling at the outer surface of the assembled cylinders at $r = r_3$, $0 < z < z_1$. Elsewhere, the bimaterial thermal system is thermally insulated. More specifically, the boundary conditions on the sides that are not thermally insulated are as follows:

$$\begin{aligned} -\kappa_1 P(r_1, z; -1, 0) &= q_0 \text{ for } 0 < z < z_1, \\ -\kappa_2 P(r_3, z; 1, 0) &= h[T(r_3, z) - T_a] \text{ for } 0 < z < z_1, \end{aligned}$$

where h is the heat convection coefficient, q_0 is the magnitude of the specified heat flux and T_a is the ambient temperature of the system.

Figure 4.11: Plots of $\kappa_1(T - T_a)/q_0r_1$ against r/r_1 for a few selected values of $\alpha/r_1\kappa_2$ (for perfectly conducting and high conducting interfaces).

We are interested in analyzing the effect of the interfacial parameter α (assumed to be constant) on the thermal performance of the heat dissipation system. The radii r_1, r_2 and r_3 and the lengths z_1 and z_2 are chosen to be such that $r_2/r_1 = 2$, $r_3/r_1 = 3$ and $z_1/r_1 = 2$. To obtain some numerical results, a total of 1920 elements are employed on the exterior boundary of $\Omega_1 \cup \Omega_2$ and the interface Γ_0 . For $hr_1/\kappa_2 = 2.5 \times 10^{-2}$ and $\kappa_1/\kappa_2 = 2.50$, the non-dimensionalized temperature $\kappa_1(T - T_a)/q_0r_1$ along $z/r_1 = 0.5$ are plotted against r/r_1 for selected values of the non-dimensionalized parameter $\alpha/r_1\kappa_2$ in Figure 4.11.

In Figure 4.11, the solid line gives the plot of the non-dimensionalized temperature profile for the case in which the interface between the solids is perfectly bonded (that is, for the case $\alpha/r_1\kappa_2 = 0$). As anticipated, at a given point along $z/r_1 = 0.5$, the non-dimensionalized temperature in both regions

Figure 4.12: Plots of $\kappa_1(T - T_a)/q_0r_1$ against r/r_1 for a few selected values of $\lambda r_1/\kappa_2$ (for perfectly conducting and low conducting interfaces).

decreases as $\alpha/r_1\kappa_2$ increases. Hence, high conducting interfaces enhance the heat dissipation performance of the system.

Still with $r_2/r_1 = 2$, $r_3/r_1 = 3$, $z_1/r_1 = 2$, $hr_1/\kappa_2 = 2.5 \times 10^{-2}$ and $\kappa_1/\kappa_2 = 2.50$, we plot $\kappa_1(T - T_a)/q_0r_1$ against r/r_1 for the case in which the interface between the solids is a low conducting one. The plots for selected values of the non-dimensionalized parameter $\lambda r_1/\kappa_2$ are given in Figure 4.12. As the low conducting interface tends to obstruct rather than enhance heat flow from the chip into the sink, the temperature profiles in the chip in Figure 4.12 are higher compared to those in 4.11. As expected, for a lower value of $\lambda r_1/\kappa_2$, there is a bigger temperature jump across the interface at $r/r_1 = 2$. The differences between the temperature distributions for the different values of $\lambda r_1/\kappa_2$ are more pronounced in region Ω_1 compared to those in region Ω_2 .

The effects of the three types of interfaces – low conducting, perfectly

Figure 4.13: Plots of $\kappa_1(T - T_a)/q_0z_1$ against z/z_1 for a few selected values of $\alpha/z_1\kappa_2 = 10$ (in comparison to Green's function boundary element method).

conducting and high conducting ones – on the thermal performance of the heat dissipation system in Figure 4.10 are clearly shown by the temperature profiles in Figures 4.11 and 4.12.

Problem 4.6

In order to compare the hypersingular boundary integral analysis to the special Green's function boundary element method as explained in Chapter 3, we reproduce the results for the Problem 3.5, whereby the interface is vertical. Specifically, we examine non-dimensionalized interface parameters $\alpha/z_1\kappa_2 = 10$ and $\lambda z_1/\kappa_2 = 0.4$ for high conducting interface and low conducting interface respectively. To obtain some numerical results, the exterior boundary and the interface are discretized into $N = 21N_0$ and $M = 2N_0$ elements respectively (so that a total of $23N_0$ elements are used).

Figure 4.14: Plots of $\kappa_1(T - T_a)/q_0z_1$ against z/z_1 for a few selected values of $\lambda z_1/\kappa_2 = 0.4$ (in comparison to Green's function boundary element method).

In Figure 4.13 and 4.14, non-dimensionalized temperature $\kappa_1(T - T_a)/q_0z_1$ along the z -axis are plotted against z/z_1 for non-dimensionalized interface parameters $\alpha/z_1\kappa_2 = 10$ (for high conducting interface) and $\lambda z_1/\kappa_2 = 0.4$ (for low conducting interface). For this specific problem, it can be observed that the hypersingular boundary integral analysis method requires a very high number of elements to converge to the results of the special Green's function boundary element method. At least 2760 number of elements ($N_0 = 120$) are used in the hypersingular boundary integral analysis method, whereas only 140 elements are used in the Green's function boundary element method. This is expected since the hypersingular boundary integral analysis method requires multiple differentiation of the fundamental solutions $G_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0)$ and $G_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)})$ (see, (2.17)) in the boundary integral equations. The effect is especially apparent for the functions $\Lambda_0(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{t}}^{(i)})$ and $\Lambda_1(\underline{\mathbf{x}}; \underline{\mathbf{x}}_0; \underline{\mathbf{n}}^{(k)}; \underline{\mathbf{t}}^{(i)})$ for the high conducting

interface case as shown in (4.53). As a result, more elements are needed at the interface for better accuracy. For example, in Figure 4.13, when $N_0 = 30$, the results we obtained using hypersingular boundary integral analysis method deviate a lot from the results obtained from the Green's function method as the number of elements used at the interface is not sufficient. A straight forward solution to this problem is to use higher order elements for discretization. On the other hand, we have incorporated the interfacial conditions into the special Green's function. Thus, only the exterior boundary of the bimaterial needs to be discretized for the special Green's function method. Note that even though special Green's function method gives a smaller linear algebraic system, the computation time incurred and the numerical complexity involved for the Fourier integrals in the Green's functions have to be taken into consideration.

Problem 4.7

Figure 4.15: A geometrical sketch of Problem 4.7 on the rz plane.

To devise a test problem for the numerical procedure for the nonlinear interface in Section 4.7, we take the solution domain to be as sketched in Figure 4.15 and the analytical solution of the test problem to be

$$T(r, z) = \begin{cases} \frac{1}{2}r^2 - z^2 + 3z + 6 & \text{for } (r, z) \in \Omega_1, \\ r^2 - 2z^2 + z + 4 & \text{for } (r, z) \in \Omega_2. \end{cases}$$

Note that Γ_1 and Γ_2 are defined by $r^2 + z^2 = 1$ for $z > 0$ and $z < 0$ respectively.

If we take $\kappa_1 = 1$ and $\kappa_2 = 3$, the conditions on the exterior boundary of the bimaterial are given by

$$P(r, z; r, z) = \frac{2(T^2 + 10)}{[(\frac{1}{2}r^2 - z^2 + 3z + 6)^2 + 10]} + r^2 - 2z^2 + 3z - 2$$

for $(r, z) \in \Gamma_1$,

$$T(r, z) = r^2 - 2z^2 + z + 4$$

for $(r, z) \in \Gamma_2$,

and the nonlinear interfacial conditions are given by (4.68) with

$$\sigma = \frac{3}{(\frac{5}{4}r^4 + 14r^2 + 52)(\frac{3}{2}r^2 + 10)(2 - \frac{1}{2}r^2)}.$$

The exterior boundary Γ_1 and Γ_2 are discretized into $2N_0$ elements each and the interface Γ_0 is discretized into N_0 elements (a total of $5N_0$ elements are used). The first guess for T_1 and T_2 are 0.5 while T (for P in the boundary conditions) is set to be 2. The stopping criteria for the iterating scheme is that each updated values of $\lambda^{(j)}$ and $f_1(\hat{\mathbf{x}}^{(k)}, T^{(k)})$ changes not more than 0.1%.

To check for accuracy and convergence, we start with $N_0 = 10$ and doubling the number each time until we reach $N_0 = 80$. The values of the numerical temperature at various points in the interior of the domains are tabulated into Table 4.3 where they are compared to the exact solution. Indeed, the results clearly show that the numerical results agree well and converge to the

Table 4.3: A comparison of the numerical values of T with the exact solution at various selected points for Problem 4.7.

Point (r, z)	$N_0 = 10$	$N_0 = 20$	$N_0 = 40$	$N_0 = 80$	Exact
(0.50, 0.20)	3.10875	3.12616	3.13150	3.13309	3.13400
(0.10, 0.40)	2.82931	2.86701	2.87857	2.88201	2.88400
(0.90, 0.75)	3.89550	3.95302	3.97057	3.97582	3.97875
(1.10, 0.30)	4.31641	4.33610	4.34217	4.34401	4.34500
(1.30, 0.90)	4.96250	5.01247	5.02782	5.03248	5.03500
(1.80, 0.50)	5.48010	5.51890	5.53100	5.53472	5.53667

Figure 4.16: Plots of the numerical and exact temperature $T(r, z)$ for $r = 0.75$ and $r = 0.25$ ($-0.6 \leq z \leq 0.6$).

Figure 4.17: A geometrical sketch of Problem 4.8 on the rz plane.

exact solution as the elements are refined. For illustration purposes, the values of the numerical temperature $T(r, z)$ along $r = 0.25$ and $r = 0.75$ (for $-0.6 \leq z \leq 0.6$) are compared graphically with the exact ones in Figure 4.16 using $N_0 = 40$ and 80. The gap in the graph shows temperature jump across the nonlinear interface Γ_0 which emphasizes the low conducting nature of the nonlinear interface. Again, the results are reasonably accurate.

Problem 4.8

Consider now the bimaterial in Figure 4.17 with nonlinear interfacial and boundary conditions given by

$$\left. \begin{aligned} \kappa_1 Q_1(r, z_1) &= \kappa_2 Q_2(r, z_1) \\ &= \sigma [T_1^4(r, z_1) - T_2^4(r, z_1)] \end{aligned} \right\} \text{ for } r_1 < r < r_2,$$

$$\begin{aligned}
 -\kappa_1 P(r, z; n_r, n_z) &= \sigma(T^4 - T_a^4) \text{ for } \begin{cases} r = r_2, 0 < z < z_1 \\ z = 0, r_1 < r < r_2 \end{cases}, \\
 -\kappa_2 P(r, z; n_r, n_z) &= \sigma(T^4 - T_a^4) \text{ for } \begin{cases} r = r_2, z_1 < z < z_2 \\ z = z_2, 0 < r < r_2 \end{cases}, \\
 T(r, z) &= T_b \text{ for } \begin{cases} r = r_1, 0 < z < z_1 \\ z = z_1, 0 < r < r_1 \end{cases},
 \end{aligned}$$

where σ is the Stefan–Boltzmann constant given by $5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$, T_a is the ambient temperature and T_b is the fixed temperature at the specified boundaries.

For the purpose of studying this problem, we take the radii r_1 and r_2 and the lengths z_1 and z_2 to be $r_2/r_1 = 2$, $z_2/r_1 = 3$ and $z_1/r_1 = 2$. The numerical results are obtained by employing a total of 1600 elements on the exterior boundary of $\Omega_1 \cup \Omega_2$ and the interface Γ_0 . Initial values of T_1/T_a , T_2/T_a and T/T_a (for the heat flux boundary conditions) are set to be 5. Once each iterated values of $\lambda^{(j)}$ and $f_1(\widehat{\mathbf{x}}^{(k)}, T^{(k)})$ changes by less than 0.1%, the boundary element procedure proceeds with the latest known $\lambda^{(j)}$ and $f_1(\widehat{\mathbf{x}}^{(k)}, T^{(k)})$. Using $\kappa_1/\kappa_2 = 55/18$, $T_b/T_a = 170/33$ and non-dimensionalized Stefan–Boltzmann constant $\sigma T_a^3 r_1/\kappa_2 = 0.01132$, a non-dimensionalized temperature T/T_a contour plot is produced as shown in Figure 4.18. On the whole, we can see that the radiation boundary conditions contribute to the cooling of the domain and there is an abrupt change to the colour contour across the interface. To further examine the temperature profile, non-dimensionalized temperature is plotted along $r/r_1 = 1.8$ and $r/r_1 = 1.2$ across the domain in Figure 4.19. Here, the gap at $z/r_1 = 2$ confirms that there is a jump in the temperature profile across the nonlinear interface. Also, the magnitude of the temperature jump is higher for $r/r_1 = 1.8$ because the region is of lower temperature (which means lower λ and thus suggests larger ΔT , see (4.70)).

Still with the same conditions (that is, $r_2/r_1 = 2$, $z_2/r_1 = 3$, $z_1/r_1 = 2$,

Figure 4.18: Non-dimensionalized temperature T/T_a contour plot for Problem 4.8.

Figure 4.19: Plots of T/T_a against z/z_1 for $r/r_1 = 1.8$ and $r/r_1 = 1.2$.

Figure 4.20: Plots of T/T_a against z/z_1 for three different interfaces (with different λ).

$\kappa_1/\kappa_2 = 55/18$, $T_b/T_a = 170/33$, and $\sigma T_a^3 r_1/\kappa_2 = 0.01132$), we explore and compare this specific nonlinear interface to two other types of interface, namely, the perfect and low conducting interface. A perfect interface is on which the temperature and the normal heat flux are continuous and this type of interface may be recovered if $\lambda \rightarrow \infty$. To make a close approximation of a perfect interface, we choose $\lambda r_1/\kappa_2 = 10000$. Meanwhile, the interface can be regarded as low conducting interface if the interface is filled with microscopic voids ($\lambda r_1/\kappa_2 = 2$ is chosen for this study). The comparison of the three interfaces is captured in Figure 4.20. The solid line in Figure 4.20 indicates perfect contact at the interface as the temperature is continuous across it. On the other hand, there is temperature jump across the interface for the remaining two interfaces. From here we can see that the nonlinear interface (given by the Stefan-Boltzmann law) is also a low conducting interface. This

can be supported by the fact that the nonlinear interface studied here is totally void in nature and the only means of heat transfer across the interface is by radiation.

4.9 Summary

Hypersingular boundary integral formulations for axisymmetric heat conduction across low conducting and high conducting interfaces between two dissimilar materials are derived. Simple boundary element procedures based on these formulations are proposed for solving numerically axisymmetric heat conduction problems. To assess the validity and accuracy of the numerical procedures, they are used to solve some specific test problems which have analytical (exact) solutions. The numerical solutions obtained suggest that the boundary integral formulations are correctly derived and the proposed numerical procedure can be used as an accurate and reliable tool for analyzing heat flow across the imperfect interfaces.

The boundary element procedures are also applied to solve some problems of practical interest. One of the problems requires the computation of the equivalent thermal conductivity of a homogeneous cylindrical representative volume element containing a centrally located cylindrical carbon nanotube. The interface between the constituent parts of the carbon nanotube based composite is assumed to be microscopically damaged and is modeled as low conducting. In another problem, the thermal performance of a heat dissipation system comprising the assembly of two hollow cylindrical solids is simulated. The effects of both low and high conducting interfaces (between the chip and the sink) on the temperature distribution in the thermal system are examined. Also, radiation heat transfer problem involving nonlinear interface is solved.

The numerical results obtained for the equivalent thermal conductivity of the carbon nanotube composite, the temperature profiles in the thermal system and the radiation heat transfer problem appear to be intuitively and qualitatively acceptable. Heat conduction is shown to be impeded across a low conducting interface and enhanced across a high conducting interface.

Chapter 5

Plane Elastostatic Deformations of Anisotropic Bimaterials with Soft and Stiff Interfaces

5.1 Introduction

In many studies on the elastic deformations of multi-layered materials, such as Ang and Park [10], Fenner [38], Kattis and Mavroyannis [47] and Yu, Sanday, Rath and Chang [80], the interface between two dissimilar materials is taken to be perfect, that is, the materials are assumed to be bonded in such a way that the displacements and the traction stresses are continuous on the interface. Nevertheless, if the interface contains micro-cracks or rigid micro-inclusions, or if it contains microscopic gaps filled with materials that have extreme elastic properties, the perfect interface model may not be suitable for analyzing the deformation of the bimaterial.

Most studies on imperfect interfaces deal with compliant or soft interfaces which are modeled as distributions of springs. In the spring model, the displacements may jump across opposite sides of a soft interface but the traction stresses are continuous on the interface, and the traction stresses on the in-

terface are linearly related to the displacement jumps. For stiff interfaces considered in Benveniste and Miloh [18] and Hashin [44], the displacements are continuous on a stiff interface but the traction stresses may exhibit a jump across opposite sides of the interface. Relatively few papers on the analysis of stiff interfaces may be found in literature.

In this chapter, plane elastostatic Green's functions satisfying the relevant conditions on imperfect soft and stiff planar interfaces between two dissimilar anisotropic half-spaces under elastic deformations are explicitly derived with the aid of Fourier integral transformation technique. The Green's functions are applied to obtain special boundary integral equations for the deformation of a bimaterial with an imperfect planar interface that is either soft or stiff. The boundary integral equations do not contain any integral over the imperfect interface. They are used to obtain a boundary element procedure for determining the displacements and stresses in the bimaterial. The numerical procedure does not require the interface to be discretized into elements.

Earlier works on plane elastostatic Green's functions for imperfect planar interfaces in bimaterials may be found in Berger and Tewary [23] and Sudak and Wang [68]. In [23], the displacement jumps across opposite sides of the imperfect soft interface are assumed known a priori and the Green's function is chosen to have explicitly prescribed displacement jumps. The Green's function in [68] satisfies a specific case of the interfacial conditions in the spring model for imperfect soft interfaces. A more general form of the soft interfacial conditions is considered in this chapter.

5.2 Imperfect Soft and Stiff Planar Interfaces

Referring to a Cartesian coordinate system $Ox_1x_2x_3$, consider two anisotropic elastic half-spaces $x_2 > \delta$ and $x_2 < 0$ with an anisotropic elastic layer of thickness δ sandwiched in between them. The elastic layer $0 < x_2 < \delta$ is denoted by $\mathcal{R}^{(0)}$ and the elastic half-spaces $x_2 > \delta$ and $x_2 < 0$ by $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ respectively. The elastic moduli of the region $\mathcal{R}^{(p)}$ are given by $c_{ijkl}^{(p)}$ ($p = 0, 1, 2$). Refer to Figure 5.1 for a sketch of the multi-layered elastic space on the Ox_1x_2 plane.

Figure 5.1: Two anisotropic elastic half-spaces $x_2 > \delta$ and $x_2 < 0$ with an anisotropic elastic layer of thickness δ sandwiched in between them.

The multi-layered elastic space in Figure 5.1 undergoes a plane elastostatic deformation such that the elastic displacement and stress fields are functions of x_1 and x_2 only. The deformation is governed by the elliptic system of partial differential equations in (2.24). If the displacements and stresses in $\mathcal{R}^{(p)}$ are denoted by $u_i^{(p)}$ and $\sigma_{ij}^{(p)}$ respectively then the continuity conditions imposed

Figure 5.2: Two dissimilar elastic half-spaces joined along $x_2 = 0$.

on the planes $x_2 = 0$ and $x_2 = \delta$ are given by

$$\left. \begin{aligned} u_i^{(0)}(x_1, \delta^-) &= u_i^{(1)}(x_1, \delta^+) \\ \sigma_{i2}^{(0)}(x_1, \delta^-) &= \sigma_{i2}^{(1)}(x_1, \delta^+) \\ u_i^{(0)}(x_1, 0^+) &= u_i^{(2)}(x_1, 0^-) \\ \sigma_{i2}^{(0)}(x_1, 0^+) &= \sigma_{i2}^{(2)}(x_1, 0^-) \end{aligned} \right\} \text{ for } -\infty < x_1 < \infty. \quad (5.1)$$

We are interested in modeling the sandwiched layer $\mathcal{R}^{(0)}$ as a line interface on the x_1 axis of the Ox_1x_2 plane for the limiting case in which the thickness δ tends to zero. A geometrical sketch of the layered elastic space for the vanishing layer replaced by the planar interface on $x_2 = 0$ is shown in Figure 5.2.

If the elastic moduli $c_{ijkl}^{(0)}$ in the layer $\mathcal{R}^{(0)}$ either vanish or become unbounded as δ tends to zero, then the perfect interfacial conditions

$$\left. \begin{aligned} u_i^{(1)}(x_1, 0^+) &= u_i^{(2)}(x_1, 0^-) \\ \sigma_{i2}^{(1)}(x_1, 0^+) &= \sigma_{i2}^{(2)}(x_1, 0^-) \end{aligned} \right\} \text{ for } -\infty < x_1 < \infty, \quad (5.2)$$

may not necessarily hold for the bimaterial in Figure 5.2.

The asymptotic analysis in Benveniste and Miloh [18] may be applied to derive conditions on $x_2 = 0$ (in the limit as δ tends to zero) for soft and stiff interfaces as explained in Subsections 5.2.1 and 5.2.2 below. For the asymptotic analysis, the displacement and the stress fields in $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ as well as the partial derivatives of those fields with respect to x_i are assumed to be $O(1)$ (for small δ) on the interface $x_2 = 0$ and the Taylor series is used to write

$$u_i^{(0)}(x_1, \delta^-) - u_i^{(0)}(x_1, 0^+) = \delta \left. \frac{\partial u_i^{(0)}}{\partial x_2} \right|_{x_2=0^+} + \sum_{m=2}^{\infty} \frac{\delta^m}{m!} \left. \frac{\partial^m u_i^{(0)}}{\partial x_2^m} \right|_{x_2=0^+}, \quad (5.3)$$

and

$$\sigma_{i2}^{(0)}(x_1, \delta^-) - \sigma_{i2}^{(0)}(x_1, 0^+) = \delta \left. \frac{\partial \sigma_{i2}^{(0)}}{\partial x_2} \right|_{x_2=0^+} + \sum_{m=2}^{\infty} \frac{\delta^m}{m!} \left. \frac{\partial^m \sigma_{i2}^{(0)}}{\partial x_2^m} \right|_{x_2=0^+}. \quad (5.4)$$

5.2.1 Soft Interfacial Conditions

The soft interfacial conditions are obtained if the elastic moduli $c_{ijkl}^{(0)}$ in the layer $R^{(0)}$ decreases to zero as δ tends to zero in accordance with

$$c_{ijkl}^{(0)} = \delta c_{ijkl}^{(\text{soft})}, \quad (5.5)$$

where $c_{ijkl}^{(\text{soft})}$ are constants independent of δ .

From the equilibrium equations (see (2.24) in Chapter 2)

$$\frac{\partial \sigma_{ij}^{(0)}}{\partial x_j} = 0 \text{ in } R^{(0)}, \quad (5.6)$$

and the generalized Hooke's law (taken from (2.19))

$$\sigma_{ij}^{(0)} = c_{ijkl}^{(0)} \frac{\partial u_k^{(0)}}{\partial x_\ell}, \quad (5.7)$$

we may write

$$\delta \left. \frac{\partial \sigma_{i2}^{(0)}}{\partial x_2} \right|_{x_2=0^+} = -\delta \left[c_{i1k1}^{(0)} \frac{\partial^2 u_k^{(0)}}{\partial x_1^2} + c_{i1k2}^{(0)} \frac{\partial^2 u_k^{(0)}}{\partial x_1 \partial x_2} \right] \Big|_{x_2=0^+}. \quad (5.8)$$

Again, note that the Einsteinian convention of summing over a repeated index is assumed here for lowercase Latin subscripts which take the values 1 and 2.

From $u_i^{(0)}(x_1, 0^+) = u_i^{(2)}(x_1, 0^-)$ (in the third line of (5.1)), (5.8) may be rewritten as

$$\delta \left. \frac{\partial \sigma_{i2}^{(0)}}{\partial x_2} \right|_{x_2=0^+} = -\delta c_{i1k1}^{(0)} \left. \frac{\partial^2 u_k^{(2)}}{\partial x_1^2} \right|_{x_2=0^-} - \delta c_{i1k2}^{(0)} \left. \frac{\partial^2 u_k^{(0)}}{\partial x_1 \partial x_2} \right|_{x_2=0^+}. \quad (5.9)$$

Using $\sigma_{i2}^{(0)}(x_1, 0^+) = \sigma_{i2}^{(2)}(x_1, 0^-)$ (in the fourth line of (5.1)) and (5.7), we find that

$$c_{i2k2}^{(0)} \left. \frac{\partial u_k^{(0)}}{\partial x_2} \right|_{x_2=0^+} = -c_{i2k1}^{(0)} \left. \frac{\partial u_k^{(2)}}{\partial x_1} \right|_{x_2=0^-} + \sigma_{i2}^{(2)}(x_1, 0^-), \quad (5.10)$$

and hence

$$c_{i2k2}^{(0)} \left. \frac{\partial^2 u_k^{(0)}}{\partial x_1 \partial x_2} \right|_{x_2=0^+} = -c_{i2k1}^{(0)} \left. \frac{\partial^2 u_k^{(2)}}{\partial x_1^2} \right|_{x_2=0^-} + \left. \frac{\partial \sigma_{i2}^{(2)}}{\partial x_1} \right|_{x_2=0^-}. \quad (5.11)$$

Since $u^{(2)}$ and $\sigma_{i2}^{(2)}$ are assumed to be $O(1)$ (for small δ) and $c_{ijkl}^{(0)}$ is given by (5.5), the right hand side of (5.11) is $O(1)$ and (5.9) implies that the first order partial derivative of $\sigma_{i2}^{(0)}$ with respect to x_2 at $x_2 = 0^+$ is $O(1)$. It follows that (5.4) reduces to

$$\sigma_{i2}^{(1)}(x_1, 0^+) = \sigma_{i2}^{(2)}(x_1, 0^-), \quad (5.12)$$

as δ tends to zero.

Similarly, from (5.5) and (5.10), as δ approaches zero, (5.3) reduces to

$$\lambda_{ik} [u_k^{(1)}(x_1, 0^+) - u_k^{(2)}(x_1, 0^-)] = \sigma_{i2}^{(2)}(x_1, 0^-), \quad (5.13)$$

where $\lambda_{ik} = c_{i2k2}^{(\text{soft})}$.

To summarize, the soft interfacial conditions are given by (5.12) and (5.13). The conditions may also be used to describe an interface that is damaged by a distribution of micro-cracks or micro-voids.

5.2.2 Stiff Interfacial Conditions

For the stiff interfacial conditions, the elastic moduli $c_{ijkl}^{(0)}$ of the vanishing layer $\mathcal{R}^{(0)}$ are taken to be given by

$$c_{ijkl}^{(0)} = \frac{1}{\delta} c_{ijkl}^{(\text{stiff})}, \quad (5.14)$$

where $c_{ijkl}^{(\text{stiff})}$ are constants independent of δ .

Since $u^{(2)}$ and $\sigma_{i2}^{(2)}$ are assumed to be $O(1)$ (for small δ) and $c_{ijkl}^{(0)}$ is given by (5.14), the right hand side of (5.10) is $O(\delta^{-1})$ and hence the first order partial derivative of $u_k^{(0)}$ with respect to x_2 at $x_2 = 0^+$ is $O(1)$. It follows that (5.3) reduces to

$$u_k^{(2)}(x_1, 0^-) = u_k^{(1)}(x_1, 0^+), \quad (5.15)$$

as δ tends to zero.

Similarly, (5.4) together with (5.1), (5.8) and (5.11) gives

$$\sigma_{i2}^{(1)}(x_1, 0^+) - \sigma_{i2}^{(2)}(x_1, 0^-) = -\alpha_{ik} \left. \frac{\partial^2 u_k^{(1)}}{\partial x_1^2} \right|_{x_2=0^+}, \quad (5.16)$$

where

$$\alpha_{ik} = -c_{i1k1}^{(\text{stiff})} + c_{i1p2}^{(\text{stiff})} b_{pj} c_{j2k1}^{(\text{stiff})}, \quad (5.17)$$

with b_{pi} defined by

$$b_{pi} c_{i2k2}^{(\text{stiff})} = \delta_{pk}. \quad (5.18)$$

Note that δ_{pk} is the Kronecker-delta.

To summarize, the stiff interfacial conditions are given by (5.15) and (5.16). The conditions may also be used to model an interface containing a distribution of rigid micro-inclusions.

5.3 Green's Functions for Planar Interfaces

In this section, we derive plane elastostatic Green's functions for imperfect soft and stiff planar interfaces between the anisotropic elastic half-spaces in Figure 5.2.

As explained in Subsection 2.2.2, the functions $\Phi_{km}(x_1, x_2; \xi_1, \xi_2)$ and $\Gamma_{km}(x_1, x_2; \xi_1, \xi_2)$ in the boundary integral equations (2.32) for anisotropic elasticity are not uniquely determined. Hence, they are chosen to satisfy the conditions on the imperfect interfaces $\mathcal{I}$.

The task of deriving the Green's functions for the interfaces requires solving the systems of partial differential equations

$$\begin{aligned} c_{ijkl}^{(p)} \frac{\partial^2}{\partial x_j \partial x_\ell} [\Phi_{km}(x_1, x_2; \xi_1, \xi_2)] &= \delta_{im} \delta(x_1 - \xi_1, x_2 - \xi_2) \\ \text{for } (x_1, x_2) &\in \mathcal{R}^{(p)} \quad (p = 1, 2), \end{aligned} \quad (5.19)$$

subject to the relevant far-field and interfacial conditions. Note that $\delta(x_1, x_2)$ is the Dirac delta function.

For the far-field conditions, the stresses $S_{ijm}(x_1, x_2; \xi_1, \xi_2)$ defined by

$$\begin{aligned} S_{ijm}(x_1, x_2; \xi_1, \xi_2) &= c_{ijkl}^{(p)} \frac{\partial}{\partial x_\ell} [\Phi_{km}(x_1, x_2; \xi_1, \xi_2)] \\ \text{for } (x_1, x_2) &\in \mathcal{R}^{(p)} \quad (p = 1, 2), \end{aligned} \quad (5.20)$$

are required to tend to zero as $x_1^2 + x_2^2 \rightarrow \infty$.

5.3.1 Perfect Interfaces

If the interface between the anisotropic elastic half-spaces in Figure 5.2 is perfect, the interfacial conditions for the Green's function are given by

$$\left. \begin{aligned} \Phi_{km}(x_1, 0^+; \xi_1, \xi_2) &= \Phi_{km}(x_1, 0^-; \xi_1, \xi_2) \\ S_{k2m}(x_1, 0^+; \xi_1, \xi_2) &= S_{k2m}(x_1, 0^-; \xi_1, \xi_2) \end{aligned} \right\} \text{ for } -\infty < x_1 < \infty, \quad (5.21)$$

The solution of (5.19) satisfying (5.20) and (5.21), which is given in Berger and Tewary [24], may be written as

$$\begin{aligned} & \Phi_{km}^{(\text{perfect})}(x_1, x_2; \xi_1, \xi_2) \\ &= \frac{1}{2\pi} \operatorname{Re} \left\{ \sum_{\alpha=1}^2 (H(x_2)H(\xi_2)A_{k\alpha}^{(1)}[N_{\alpha p}^{(1)} \ln(x_1 - \xi_1 + \tau_{\alpha}^{(1)}(x_2 - \xi_2)) \right. \\ & \quad + \sum_{\beta=1}^2 Q_{\alpha\beta j}^{(1)} \ln(x_1 - \xi_1 + \tau_{\alpha}^{(1)}x_2 - \bar{\tau}_{\beta}^{(1)}\xi_2)]d_{pm}^{(1)} \\ & \quad + H(-x_2)H(\xi_2) \sum_{\beta=1}^2 A_{k\alpha}^{(2)}Q_{\alpha\beta p}^{(2)}d_{pm}^{(1)} \ln(x_1 - \xi_1 + \tau_{\alpha}^{(2)}x_2 - \tau_{\beta}^{(1)}\xi_2) \\ & \quad + H(x_2)H(-\xi_2) \sum_{\beta=1}^2 A_{k\alpha}^{(1)}T_{\alpha\beta p}^{(1)}d_{pm}^{(2)} \ln(x_1 - \xi_1 + \tau_{\alpha}^{(1)}x_2 - \tau_{\beta}^{(2)}\xi_2) \\ & \quad + H(-x_2)H(-\xi_2)A_{k\alpha}^{(2)}[N_{\alpha p}^{(2)} \ln(x_1 - \xi_1 + \tau_{\alpha}^{(2)}(x_2 - \xi_2)) \\ & \quad \left. + \sum_{\beta=1}^2 T_{\alpha\beta p}^{(2)} \ln(x_1 - \xi_1 + \tau_{\alpha}^{(2)}x_2 - \bar{\tau}_{\beta}^{(2)}\xi_2)]d_{pm}^{(2)}) \right\}, \end{aligned} \quad (5.22)$$

where the overhead bar denotes the complex conjugate of a complex number, $H(x)$ is the unit-step Heaviside function, the constants $Q_{\alpha\beta j}^{(p)}$ are implicitly given by

$$\begin{aligned} N_{\gamma k}^{(2)}(A_{k\beta}^{(1)}N_{\beta p}^{(1)} + \sum_{\alpha=1}^2 \bar{A}_{k\alpha}^{(1)}\bar{Q}_{\alpha\beta j}^{(1)}) &= Q_{\gamma\beta p}^{(2)}, \\ M_{\gamma k}^{(2)}(L_{k2\beta}^{(1)}N_{\beta p}^{(1)} + \sum_{\alpha=1}^2 \bar{L}_{k2\alpha}^{(1)}\bar{Q}_{\alpha\beta p}^{(1)}) &= Q_{\gamma\beta p}^{(2)}, \end{aligned} \quad (5.23)$$

and the constants $T_{\alpha\beta j}^{(p)}$ by

$$\begin{aligned} N_{\gamma k}^{(1)}(A_{k\beta}^{(2)}N_{\beta p}^{(2)} + \sum_{\alpha=1}^2 \overline{A}_{k\alpha}^{(2)}\overline{T}_{\alpha\beta p}^{(2)}) &= T_{\gamma\beta p}^{(1)}, \\ M_{\gamma k}^{(1)}(L_{k2\beta}^{(2)}N_{\beta p}^{(2)} + \sum_{\alpha=1}^2 \overline{L}_{k2\alpha}^{(2)}\overline{T}_{\alpha\beta p}^{(2)}) &= T_{\gamma\beta p}^{(1)}, \end{aligned} \quad (5.24)$$

where $\tau_{\alpha}^{(p)}$, $A_{k\alpha}^{(p)}$, $N_{\alpha j}^{(p)}$, $L_{kj\alpha}^{(p)}$, $M_{\gamma k}^{(p)}$ and $d_{jm}^{(p)}$ are respectively the constants τ_{α} , $A_{k\alpha}$, $N_{\alpha j}$, $L_{kj\alpha}$, $M_{\gamma k}$ and d_{jm} in Chapter 2 computed using $c_{ijkl} = c_{ijkl}^{(p)}$.

From (5.20), the stresses $S_{kjm}^{(\text{perfect})}(x_1, x_2; \xi_1, \xi_2)$ which correspond to the displacements $\Phi_{km}^{(\text{perfect})}(x_1, x_2; \xi_1, \xi_2)$ in (5.22) are given by

$$\begin{aligned} &S_{kjm}^{(\text{perfect})}(x_1, x_2; \xi_1, \xi_2) \\ &= \frac{1}{2\pi} \operatorname{Re}\left\{ \sum_{\alpha=1}^2 (H(x_2)H(\xi_2)L_{kj\alpha}^{(1)} \left[\frac{N_{\alpha p}^{(1)}}{(x_1 - \xi_1 + \tau_{\alpha}^{(1)}(x_2 - \xi_2))} \right. \right. \right. \\ &\quad + \sum_{\beta=1}^2 \frac{Q_{\alpha\beta p}^{(1)}}{(x_1 - \xi_1 + \tau_{\alpha}^{(1)}x_2 - \overline{\tau}_{\beta}^{(1)}\xi_2)} \left. \right] d_{pm}^{(1)} \\ &\quad + H(-x_2)H(\xi_2) \sum_{\beta=1}^2 \frac{L_{kj\alpha}^{(2)}Q_{\alpha\beta p}^{(2)}d_{pm}^{(1)}}{(x_1 - \xi_1 + \tau_{\alpha}^{(2)}x_2 - \tau_{\beta}^{(1)}\xi_2)} \\ &\quad + H(x_2)H(-\xi_2) \sum_{\beta=1}^2 \frac{L_{kj\alpha}^{(1)}T_{\alpha\beta p}^{(1)}d_{pm}^{(2)}}{(x_1 - \xi_1 + \tau_{\alpha}^{(1)}x_2 - \tau_{\beta}^{(2)}\xi_2)} \\ &\quad + H(-x_2)H(-\xi_2)L_{kj\alpha}^{(2)} \left[\frac{N_{\alpha p}^{(2)}}{(x_1 - \xi_1 + \tau_{\alpha}^{(2)}(x_2 - \xi_2))} \right. \\ &\quad \left. \left. + \sum_{\beta=1}^2 \frac{T_{\alpha\beta p}^{(2)}}{(x_1 - \xi_1 + \tau_{\alpha}^{(2)}x_2 - \overline{\tau}_{\beta}^{(2)}\xi_2)} \right] d_{pm}^{(2)} \right\}. \end{aligned} \quad (5.25)$$

5.3.2 Soft Interfaces

If the interface is soft, the interfacial conditions for the Green's function are given by

$$\left. \begin{aligned} S_{k2m}(x_1, 0^+; \xi_1, \xi_2) &= S_{k2m}(x_1, 0^-; \xi_1, \xi_2) \\ S_{k2m}(x_1, 0^\pm; \xi_1, \xi_2) &= \lambda_{ks} [\Phi_{sm}(x_1, 0^+; \xi_1, \xi_2) - \Phi_{sm}(x_1, 0^-; \xi_1, \xi_2)] \end{aligned} \right\} \quad \text{for } -\infty < x_1 < \infty. \quad (5.26)$$

To derive $\Phi_{km} = \Phi_{km}^{(\text{soft})}$ satisfying (5.19), (5.20) and (5.26), we take

$$\begin{aligned} \Phi_{km}^{(\text{soft})}(x_1, x_2; \xi_1, \xi_2) &= \Phi_{km}^{(\text{perfect})}(x_1, x_2; \xi_1, \xi_2) \\ &+ \text{Re} \left\{ \sum_{\alpha=1}^2 \int_0^\infty [H(x_2) A_{k\alpha}^{(1)} E_{\alpha m}^{(1)}(p; \xi_1, \xi_2) \right. \\ &\quad \times \exp(ip(x_1 + \tau_\alpha^{(1)} x_2)) \\ &\quad + H(-x_2) A_{k\alpha}^{(2)} E_{\alpha m}^{(2)}(p; \xi_1, \xi_2) \\ &\quad \times \exp(-ip(x_1 + \tau_\alpha^{(2)} x_2))] dp \Big\}, \end{aligned} \quad (5.27)$$

and hence

$$\begin{aligned} S_{kjm}^{(\text{soft})}(x_1, x_2; \xi_1, \xi_2) &= S_{kjm}^{(\text{perfect})}(x_1, x_2; \xi_1, \xi_2) \\ &+ \text{Re} \left\{ \sum_{\alpha=1}^2 \int_0^\infty [H(x_2) L_{kj\alpha}^{(1)} E_{\alpha m}^{(1)}(p; \xi_1, \xi_2) \right. \\ &\quad \times \exp(ip(x_1 + \tau_\alpha^{(1)} x_2)) \\ &\quad - H(-x_2) L_{kj\alpha}^{(2)} E_{\alpha m}^{(2)}(p; \xi_1, \xi_2) \\ &\quad \times \exp(-ip(x_1 + \tau_\alpha^{(2)} x_2))] ip dp \Big\}, \end{aligned} \quad (5.28)$$

where $E_{\alpha m}^{(1)}(p; \xi_1, \xi_2)$ and $E_{\alpha m}^{(2)}(p; \xi_1, \xi_2)$ are functions to be determined.

The conditions in the first line of (5.26) are satisfied if

$$\begin{aligned} E_{\alpha m}^{(1)}(p; \xi_1, \xi_2) &= M_{\alpha r}^{(1)} \Psi_{rm}(p; \xi_1, \xi_2), \\ E_{\alpha m}^{(2)}(p; \xi_1, \xi_2) &= M_{\alpha r}^{(2)} \overline{\Psi}_{rm}(p; \xi_1, \xi_2), \end{aligned} \quad (5.29)$$

where $\Psi_{rm}(p; \xi_1, \xi_2)$ are to be determined as explained below.

The conditions in the second line of (5.26) are rewritten as

$$\begin{aligned} & \lambda_{ks} [\Phi_{sm}^{(\text{soft})}(x_1, 0^+; \xi_1, \xi_2) - \Phi_{sm}^{(\text{soft})}(x_1, 0^-; \xi_1, \xi_2)] \\ &= H(\xi_2) S_{k2}^{(\text{soft})}(x_1, 0^-; \xi_1, \xi_2) + H(-\xi_2) S_{k2}^{(\text{soft})}(x_1, 0^+; \xi_1, \xi_2) \\ & \quad \text{for } -\infty < x_1 < \infty. \end{aligned} \quad (5.30)$$

If we apply the Fourier exponential transformation on both sides of (5.30) and use (5.22), (5.25), (5.27), (5.28) and (5.29) together with

$$\int_{-\infty}^{\infty} \frac{\exp(-ivx)dx}{(a - ib - x)} = H(b) 2\pi i \exp(-iv(a - ib)), \quad (5.31)$$

where a , b and v are real numbers, we obtain

$$\begin{aligned} & \pi [\lambda_{ik} \sum_{\alpha=1}^2 (A_{k\alpha}^{(1)} M_{\alpha r}^{(1)} - \overline{A}_{k\alpha}^{(2)} \overline{M}_{\alpha r}^{(2)}) - ip\delta_{ir}] \Psi_{rm}(p; \xi_1, \xi_2) \\ &= -\frac{1}{2} H(\xi_2) \sum_{\alpha=1}^2 \overline{L}_{i2\alpha}^{(2)} \sum_{\beta=1}^2 \overline{Q}_{\alpha\beta p}^{(2)} i \exp(-ip(\xi_1 + \overline{\tau}_{\beta}^{(1)} \xi_2)) d_{pm}^{(1)} \\ & \quad - \frac{1}{2} H(-\xi_2) \sum_{\alpha=1}^2 L_{i2\alpha}^{(1)} \sum_{\beta=1}^2 T_{\alpha\beta p}^{(1)} i \exp(-ip(\xi_1 + \tau_{\beta}^{(2)} \xi_2)) d_{pm}^{(2)}. \end{aligned} \quad (5.32)$$

Note that (5.31) is extracted from Erdélyi, Magnus, Oberhettinger and Tricomi [33].

For a fixed value of m , the functions $\Psi_{rm}(p; \xi_1, \xi_2)$ may be obtained by inverting (5.32) as a system of linear algebraic equations. Once $\Psi_{rm}(p; \xi_1, \xi_2)$ are determined, the integral in (5.27) and (5.28) may be easily evaluated by using a numerical integration procedure.

Note that the Green's function for the imperfect soft interface in a less general form, that is, for the special case where $\lambda_{12} = \lambda_{21} = 0$, is given in Sudak and Wang [68].

5.3.3 Stiff Interfaces

If the interface is stiff, the interfacial conditions for the Green's function are given by

$$\left. \begin{aligned} \Phi_{km}(x_1, 0^+; \xi_1, \xi_2) &= \Phi_{km}(x_1, 0^-; \xi_1, \xi_2) \\ -\alpha_{kp} \frac{\partial^2 \Phi_{pm}}{\partial x_1^2} \Big|_{x_2=0^\pm} &= S_{k2m}(x_1, 0^+; \xi_1, \xi_2) - S_{k2m}(x_1, 0^-; \xi_1, \xi_2) \end{aligned} \right\} \quad \text{for } -\infty < x_1 < \infty. \quad (5.33)$$

To derive $\Phi_{km} = \Phi_{km}^{(\text{stiff})}$ satisfying (5.19), (5.20) and (5.33), we take

$$\begin{aligned} \Phi_{km}^{(\text{stiff})}(x_1, x_2; \xi_1, \xi_2) &= \Phi_{km}^{(\text{perfect})}(x_1, x_2; \xi_1, \xi_2) \\ &+ \text{Re} \left\{ \sum_{\alpha=1}^2 \int_0^\infty [H(x_2) A_{k\alpha}^{(1)} F_{\alpha m}^{(1)}(p; \xi_1, \xi_2) \right. \\ &\quad \times \exp(ip(x_1 + \tau_\alpha^{(1)} x_2)) \\ &\quad + H(-x_2) A_{k\alpha}^{(2)} F_{\alpha m}^{(2)}(p; \xi_1, \xi_2) \\ &\quad \times \exp(-ip(x_1 + \tau_\alpha^{(2)} x_2))] dp \Big\}, \end{aligned} \quad (5.34)$$

and hence

$$\begin{aligned} S_{kjm}^{(\text{stiff})}(x_1, x_2; \xi_1, \xi_2) &= S_{kjm}^{(\text{perfect})}(x_1, x_2; \xi_1, \xi_2) \\ &+ \text{Re} \left\{ \sum_{\alpha=1}^2 \int_0^\infty [H(x_2) L_{kj\alpha}^{(1)} F_{\alpha m}^{(1)}(p; \xi_1, \xi_2) \right. \\ &\quad \times \exp(ip(x_1 + \tau_\alpha^{(1)} x_2)) \\ &\quad - H(-x_2) L_{kj\alpha}^{(2)} F_{\alpha m}^{(2)}(p; \xi_1, \xi_2) \\ &\quad \times \exp(-ip(x_1 + \tau_\alpha^{(2)} x_2))] ip dp \Big\}, \end{aligned} \quad (5.35)$$

where $F_{\alpha m}^{(1)}(p; \xi_1, \xi_2)$ and $F_{\alpha m}^{(2)}(p; \xi_1, \xi_2)$ are functions to be determined.

The conditions in the first line of (5.33) are satisfied if $F_{\alpha m}^{(1)}(p; \xi_1, \xi_2)$ and

$F_{\alpha m}^{(2)}(p; \xi_1, \xi_2)$ are given by

$$\begin{aligned} F_{\alpha m}^{(1)}(p; \xi_1, \xi_2) &= N_{\alpha r}^{(1)} \Upsilon_{rm}(p; \xi_1, \xi_2), \\ F_{\alpha m}^{(2)}(p; \xi_1, \xi_2) &= N_{\alpha r}^{(2)} \overline{\Upsilon}_{rm}(p; \xi_1, \xi_2), \end{aligned} \quad (5.36)$$

where $\Upsilon_{rm}(p; \xi_1, \xi_2)$ are to be determined as explained below.

The conditions in the second line of (5.33) are rewritten as

$$\begin{aligned} &S_{k2m}(x_1, 0^+; \xi_1, \xi_2) - S_{k2m}(x_1, 0^-; \xi_1, \xi_2) \\ &= -H(\xi_2) \alpha_{kp} \frac{\partial^2 \Phi_{pm}}{\partial x_1^2} \Big|_{x_2=0^-} - H(-\xi_2) \alpha_{kp} \frac{\partial^2 \Phi_{pm}}{\partial x_1^2} \Big|_{x_2=0^+} \\ &\quad \text{for } -\infty < x_1 < \infty. \end{aligned} \quad (5.37)$$

If we apply the Fourier exponential transformation on both sides of (5.37) and use (5.22), (5.25), (5.34), (5.35) and (5.36) together with (5.31) and

$$\int_{-\infty}^{\infty} \frac{\exp(-ivx)dx}{(a - ib - x)^2} = -H(b)2\pi v \exp(-iv(a - ib)), \quad (5.38)$$

where a , b and v are real numbers, we obtain

$$\begin{aligned} &\pi \left[i \sum_{\alpha=1}^2 (L_{s2\alpha}^{(1)} N_{\alpha r}^{(1)} - \overline{L}_{s2\alpha}^{(2)} \overline{N}_{\alpha r}^{(2)}) - \alpha_{sr} p \right] \Upsilon_{rm}(p; \xi_1, \xi_2) \\ &= -\frac{1}{2} \alpha_{sk} H(\xi_2) \sum_{\alpha=1}^2 \overline{A}_{k\alpha}^{(2)} \sum_{\beta=1}^2 \overline{Q}_{\alpha\beta p}^{(2)} \exp(-ip(\xi_1 + \overline{\tau}_{\beta}^{(1)} \xi_2)) d_{pm}^{(1)} \\ &\quad - \frac{1}{2} \alpha_{sk} H(-\xi_2) \sum_{\alpha=1}^2 A_{k\alpha}^{(1)} \sum_{\beta=1}^2 T_{\alpha\beta p}^{(1)} \exp(-ip(\xi_1 + \tau_{\beta}^{(2)} \xi_2)) d_{pm}^{(2)}. \end{aligned} \quad (5.39)$$

Note that (5.38) may be obtained by partially differentiating both sides of (5.31) with respect to either b or a .

For a fixed value of m , the functions $\Upsilon_{rm}(p; \xi_1, \xi_2)$ may be obtained by inverting (5.39) as a system of linear algebraic equations. Once $\Upsilon_{rm}(p; \xi_1, \xi_2)$ are determined, the integral in (5.34) and (5.35) may be easily evaluated by using a numerical integration procedure.

Figure 5.3: A sketch of the bimaterial.

5.4 A Plane Elastostatic Problem

With reference to a Cartesian coordinate system $Ox_1x_2x_3$, consider an elastic body with a geometry that does not change along the x_3 axis. The body is made up of two dissimilar anisotropic materials occupying the regions $\Pi^{(1)}$ and $\Pi^{(2)}$. The interface between $\Pi^{(1)}$ and $\Pi^{(2)}$, denoted by $\mathcal{I}$, lies on part of the $x_2 = 0$ plane. A geometrical sketch of the bimaterial on the Ox_1x_2 plane is given in Figure 5.3. As shown in Figure 5.3, $C^{(1)} \cup C^{(2)}$ forms the exterior boundary of the bimaterial and the regions $\Pi^{(1)}$ and $\Pi^{(2)}$ are bounded by $\mathcal{I} \cup C^{(1)}$ and $\mathcal{I} \cup C^{(2)}$ respectively. The elastic moduli of the material in $\Pi^{(p)}$ are denoted by $c_{ijkl}^{(p)}$.

Either the displacements u_k or the tractions t_k are suitably prescribed at each point on the exterior boundary $C^{(1)} \cup C^{(2)}$ of the bimaterial. The interface $\mathcal{I}$ is imperfect such that either the soft or the stiff interfacial conditions hold.

The problem of interest is to determine the displacement and the stress fields throughout the bimaterial $\Pi^{(1)} \cup \Pi^{(2)}$.

5.5 Boundary Integral Equations

Boundary integral equation for plane elastostatic deformations of two dimensional region Π bounded by simple closed curve C is given by (2.32) in Chapter 2. For a bimaterial, if the interface $\mathcal{I}$ is given by $a < x_1 < b$, $x_2 = 0$, then the boundary integral equations for elastostatic deformations of the materials in $\Pi^{(1)}$ and $\Pi^{(2)}$ are given by

$$\begin{aligned} \gamma^{(1)}(\xi_1, \xi_2)u_m(\xi_1, \xi_2) &= \int_{C^{(1)}} [u_k(x_1, x_2)\Gamma_{km}(x_1, x_2; \xi_1, \xi_2) \\ &\quad - t_k(x_1, x_2)\Phi_{km}(x_1, x_2; \xi_1, \xi_2)]ds(x_1, x_2) \\ &\quad + \int_a^b [-u_k(x_1, 0^+)S_{k2m}(x_1, 0^+; \xi_1, \xi_2) \\ &\quad + \sigma_{k2}(x_1, 0^+)\Phi_{km}(x_1, 0^+; \xi_1, \xi_2)]dx_1, \end{aligned} \quad (5.40)$$

and

$$\begin{aligned} \gamma^{(2)}(\xi_1, \xi_2)u_m(\xi_1, \xi_2) &= \int_{C^{(2)}} [u_k(x_1, x_2)\Gamma_{km}(x_1, x_2; \xi_1, \xi_2) \\ &\quad - t_k(x_1, x_2)\Phi_{km}(x_1, x_2; \xi_1, \xi_2)]ds(x_1, x_2) \\ &\quad + \int_a^b [u_k(x_1, 0^-)S_{k2m}(x_1, 0^-; \xi_1, \xi_2) \\ &\quad - \sigma_{k2}(x_1, 0^-)\Phi_{km}(x_1, 0^-; \xi_1, \xi_2)]dx_1, \end{aligned} \quad (5.41)$$

where Φ_{km} are any displacements satisfying the partial differential equations in (5.19), Γ_{km} are the traction stresses defined by

$$\Gamma_{km}(x_1, x_2; \xi_1, \xi_2) = S_{kjm}(x_1, x_2; \xi_1, \xi_2)n_j(x_1, x_2), \quad (5.42)$$

S_{kjm} are defined in (5.20), $n_j(x_1, x_2)$ are components of the unit outward normal vector to $C^{(1)} \cup C^{(2)}$ at the point (x_1, x_2) , and $\gamma^{(p)}(\xi_1, \xi_2)$ ($p = 1, 2$) is defined by

$$\gamma^{(p)}(\xi_1, \xi_2) = \begin{cases} 1 & \text{if } (\xi_1, \xi_2) \text{ lies in the interior of } \Pi^{(p)}, \\ 1/2 & \text{if } (\xi_1, \xi_2) \text{ lies on a smooth part of } \mathcal{I} \cup C^{(p)}, \\ 0 & \text{if } (\xi_1, \xi_2) \text{ lies outside } \mathcal{I} \cup C^{(p)} \cup \Pi^{(p)}. \end{cases} \quad (5.43)$$

As we shall see below, the integration over the imperfect interface $\mathcal{I}$ in (5.40) and (5.41) may be eliminated, if Φ_{km} are chosen to be given by $\Phi_{km}^{(\text{soft})}$ and $\Phi_{km}^{(\text{stiff})}$ for soft and stiff interfaces respectively.

5.5.1 Soft Interfaces

For the case where the interface is assumed to be soft, we take

$$\Phi_{km}(x_1, x_2; \xi_1, \xi_2) = \Phi_{km}^{(\text{soft})}(x_1, x_2; \xi_1, \xi_2), \quad (5.44)$$

where $\Phi_{km}^{(\text{soft})}$ are as given in Subsection 5.3.2.

If we add (5.40) and (5.41) and apply (5.12), (5.13) and (5.26), we obtain the boundary integral equations

$$\begin{aligned} \gamma(\xi_1, \xi_2) u_m(\xi_1, \xi_2) &= \int_{C^{(1)} \cup C^{(2)}} [u_k(x_1, x_2) \Gamma_{km}^{(\text{soft})}(x_1, x_2; \xi_1, \xi_2) \\ &\quad - t_k(x_1, x_2) \Phi_{km}^{(\text{soft})}(x_1, x_2; \xi_1, \xi_2)] ds(x_1, x_2) \\ &\quad \text{for } (\xi_1, \xi_2) \in \Pi^{(1)} \cup \Pi^{(2)} \cup C^{(1)} \cup C^{(2)}, \end{aligned} \quad (5.45)$$

where $\Gamma_{km}^{(\text{soft})}$ are given by

$$\Gamma_{km}^{(\text{soft})}(x_1, x_2; \xi_1, \xi_2) = S_{kjm}^{(\text{soft})}(x_1, x_2; \xi_1, \xi_2) n_j(x_1, x_2), \quad (5.46)$$

and $\gamma(\xi_1, \xi_2)$ by

$$\gamma(\xi_1, \xi_2) = \begin{cases} 1 & \text{if } (\xi_1, \xi_2) \text{ lies in the interior of } \Pi^{(1)} \cup \Pi^{(2)}, \\ 1/2 & \text{if } (\xi_1, \xi_2) \text{ lies on a smooth part of } C^{(1)} \cup C^{(2)}, \\ 0 & \text{if } (\xi_1, \xi_2) \text{ lies outside the bimaterial.} \end{cases} \quad (5.47)$$

Note that the boundary integral equations in (5.45) do not contain any integral over the imperfect soft interface $\mathcal{I}$. If (5.45) is used to derive a boundary element procedure for solving the plane elastostatic problem stated in Section 5.4, only the exterior boundary $C^{(1)} \cup C^{(2)}$ of the bimaterial has to be discretized into elements.

5.5.2 Stiff Interfaces

For the case where the interface is assumed to be stiff, we take

$$\Phi_{km}(x_1, x_2; \xi_1, \xi_2) = \Phi_{km}^{(\text{stiff})}(x_1, x_2; \xi_1, \xi_2), \quad (5.48)$$

where $\Phi_{km}^{(\text{stiff})}$ are as given in Subsection 5.3.3.

If we add (5.40) and (5.41) and apply (5.15), (5.16) and (5.33), we obtain the boundary integral equations

$$\begin{aligned} \gamma(\xi_1, \xi_2)u_m(\xi_1, \xi_2) &= \int_{C^{(1)} \cup C^{(2)}} [u_k(x_1, x_2)\Gamma_{km}^{(\text{stiff})}(x_1, x_2; \xi_1, \xi_2) \\ &\quad - t_k(x_1, x_2)\Phi_{km}^{(\text{stiff})}(x_1, x_2; \xi_1, \xi_2)]ds(x_1, x_2) \\ &\quad + \int_a^b \alpha_{kp}[u_k(x_1, 0)\frac{\partial^2}{\partial x_1^2}(\Phi_{pm}^{(\text{stiff})}(x_1, 0; \xi_1, \xi_2)) \\ &\quad - \Phi_{pm}^{(\text{stiff})}(x_1, x_2; \xi_1, \xi_2)\frac{\partial^2}{\partial x_1^2}(u_k(x_1, 0))]dx_1(x_1, x_2) \\ &\quad \text{for } (\xi_1, \xi_2) \in \Pi^{(1)} \cup \Pi^{(2)} \cup C^{(1)} \cup C^{(2)}, \end{aligned} \quad (5.49)$$

where $\Gamma_{km}^{(\text{stiff})}$ are given by

$$\Gamma_{km}^{(\text{stiff})}(x_1, x_2; \xi_1, \xi_2) = S_{kjm}^{(\text{stiff})}(x_1, x_2; \xi_1, \xi_2)n_j(x_1, x_2). \quad (5.50)$$

Performing integration by parts on the integral over the imperfect stiff

interface, we find that (5.49) reduces to

$$\begin{aligned}
 \gamma(\xi_1, \xi_2)u_m(\xi_1, \xi_2) = & \int_{C^{(1)} \cup C^{(2)}} [u_k(x_1, x_2)\Gamma_{km}^{(\text{stiff})}(x_1, x_2; \xi_1, \xi_2) \\
 & - t_k(x_1, x_2)\Phi_{km}^{(\text{stiff})}(x_1, x_2; \xi_1, \xi_2)] ds(x_1, x_2) \\
 & + \alpha_{kp} \left\{ -u_k(a, 0) \frac{\partial \Phi_{pm}^{(\text{stiff})}}{\partial x_1} \right|_{(x_1, x_2)=(a, 0)} \\
 & + u_k(b, 0) \frac{\partial \Phi_{pm}^{(\text{stiff})}}{\partial x_1} \right|_{(x_1, x_2)=(b, 0)} \\
 & + \Phi_{pm}^{(\text{stiff})}(a, 0; \xi_1, \xi_2) \frac{\partial u_k}{\partial x_1} \bigg|_{(x_1, x_2)=(a, 0)} \\
 & - \Phi_{pm}^{(\text{stiff})}(b, 0; \xi_1, \xi_2) \frac{\partial u_k}{\partial x_1} \bigg|_{(x_1, x_2)=(b, 0)} \} \\
 & \text{for } (\xi_1, \xi_2) \in \Pi^{(1)} \cup \Pi^{(2)} \cup C^{(1)} \cup C^{(2)}. \quad (5.51)
 \end{aligned}$$

Note that the boundary integral equations in (5.51) do not contain any integral over the stiff interface. A boundary element procedure based on (5.51) for solving the plane elastostatic problem in Section 5.4 does not require the imperfect stiff interface to be discretized into elements. Nevertheless, the values of u_k and $\partial u_k / \partial x_1$ at the interface tips $(a, 0)$ and $(b, 0)$, which appear on the right hand side of (5.51), must be properly treated in the boundary element procedure.

5.6 Boundary Element Procedures

Boundary element procedures based on (5.45) or (5.51) are proposed here for the numerical solution of the plane elastostatic problem in Section 5.4 for soft and stiff interfaces.

The exterior boundary $C^{(1)} \cup C^{(2)}$ of the bimaterial in Figure 5.3 is discretized into N straight line elements denoted by $B^{(1)}, B^{(2)}, \dots, B^{(N-1)}$ and

$B^{(N)}$, that is, $C^{(1)} \cup C^{(2)}$ is approximated as

$$C^{(1)} \cup C^{(2)} \simeq B^{(1)} \cup B^{(2)} \cup \dots \cup B^{(N-1)} \cup B^{(N)}. \quad (5.52)$$

Each element is assumed to lie completely in the region given by either $x_2 > 0$ or $x_2 < 0$. The two endpoints of the elements $B^{(n)}$ are denoted by $(x_1^{(n)}, x_2^{(n)})$ and $(y_1^{(n)}, y_2^{(n)})$.

On $B^{(n)}$, we define the points $(\xi_1^{(n)}, \xi_2^{(n)})$ and $(\xi_1^{(N+n)}, \xi_2^{(N+n)})$ by

$$\begin{aligned} (\xi_1^{(n)}, \xi_2^{(n)}) &= (x_1^{(n)}, x_2^{(n)}) + \omega[(y_1^{(n)}, y_2^{(n)}) - (x_1^{(n)}, x_2^{(n)})], \\ (\xi_1^{(N+n)}, \xi_2^{(N+n)}) &= (x_1^{(n)}, x_2^{(n)}) + (1 - \omega)[(y_1^{(n)}, y_2^{(n)}) - (x_1^{(n)}, x_2^{(n)})], \end{aligned} \quad (5.53)$$

where ω is a selected positive number such that $0 < \omega < 1/2$. For the numerical calculations in Section 5.7, we take $\omega = 1/4$.

For the displacements u_k and the tractions t_k on the elements, we make the discontinuous linear approximations

$$\left. \begin{aligned} u_k &\simeq \frac{[s^{(n)}(x_1, x_2) - (1 - \omega)\ell^{(n)}]\hat{u}_k^{(n)} - [s^{(n)}(x_1, x_2) - \omega\ell^{(n)}]\hat{u}_k^{(N+n)}}{(2\omega - 1)\ell^{(n)}} \\ t_k &\simeq \frac{[s^{(n)}(x_1, x_2) - (1 - \omega)\ell^{(n)}]\hat{t}_k^{(n)} - [s^{(n)}(x_1, x_2) - \omega\ell^{(n)}]\hat{t}_k^{(N+n)}}{(2\omega - 1)\ell^{(n)}} \end{aligned} \right\} \quad \text{for } (x_1, x_2) \in B^{(n)}, \quad (5.54)$$

where $\hat{u}_k^{(p)}$ and $\hat{t}_k^{(p)}$ are the values of u_k and t_k at the point $(\xi_1^{(p)}, \xi_2^{(p)})$ ($p = 1, 2, \dots, 2N$), $\ell^{(n)}$ is the length of $B^{(n)}$ and

$$s^{(n)}(x_1, x_2) = \sqrt{(x_1 - x_1^{(n)})^2 + (x_2 - x_2^{(n)})^2}. \quad (5.55)$$

If the displacements u_k are specified on $B^{(n)}$, then $\hat{u}_k^{(n)}$ and $\hat{u}_k^{(n+N)}$ are known values and $\hat{t}_k^{(n)}$ and $\hat{t}_k^{(n+N)}$ are unknown values to be determined. Likewise, if the tractions t_k are prescribed on $B^{(n)}$, then $\hat{t}_k^{(n)}$ and $\hat{t}_k^{(n+N)}$ are known values and $\hat{u}_k^{(n)}$ and $\hat{u}_k^{(n+N)}$ are unknown values to be determined.

5.6.1 Soft Interfaces

From (5.52) and (5.54), the boundary integral equations in (5.45) for the soft interface may be approximated as

$$\begin{aligned}
 \gamma(\xi_1^{(i)}, \xi_2^{(i)}) \hat{u}_m^{(i)} &= \sum_{n=1}^N \frac{1}{(2\omega - 1)\ell^{(n)}} \\
 &\times \{ \hat{u}_k^{(n)} [-(1 - \omega)\ell^{(n)} G_{2km}^{(n)}(\xi_1^{(i)}, \xi_2^{(i)}) + G_{4km}^{(n)}(\xi_1^{(i)}, \xi_2^{(i)})] \\
 &+ \hat{u}_k^{(N+n)} [\omega\ell^{(n)} G_{2km}^{(n)}(\xi_1^{(i)}, \xi_2^{(i)}) - G_{4km}^{(n)}(\xi_1^{(i)}, \xi_2^{(i)})] \\
 &- \hat{t}_k^{(n)} [-(1 - \omega)\ell^{(n)} G_{1km}^{(n)}(\xi_1^{(i)}, \xi_2^{(i)}) + G_{3km}^{(n)}(\xi_1^{(i)}, \xi_2^{(i)})] \\
 &- \hat{t}_k^{(N+n)} [\omega\ell^{(n)} G_{1km}^{(n)}(\xi_1^{(i)}, \xi_2^{(i)}) - G_{3km}^{(n)}(\xi_1^{(i)}, \xi_2^{(i)})] \} \\
 &\quad \text{for } i = 1, 2, \dots, 2N, \quad (5.56)
 \end{aligned}$$

where

$$\begin{aligned}
 G_{1km}^{(n)}(\xi_1, \xi_2) &= \int_{B^{(n)}} \Phi_{km}^{(\text{soft})}(x_1, x_2; \xi_1, \xi_2) ds(x_1, x_2), \\
 G_{2km}^{(n)}(\xi_1, \xi_2) &= \int_{B^{(n)}} \Gamma_{km}^{(\text{soft})}(x_1, x_2; \xi_1, \xi_2) ds(x_1, x_2), \\
 G_{3km}^{(n)}(\xi_1, \xi_2) &= \int_{B^{(n)}} s^{(n)}(x_1, x_2) \Phi_{km}^{(\text{soft})}(x_1, x_2; \xi_1, \xi_2) ds(x_1, x_2), \\
 G_{4km}^{(n)}(\xi_1, \xi_2) &= \int_{B^{(n)}} s^{(n)}(x_1, x_2) \Gamma_{km}^{(\text{soft})}(x_1, x_2; \xi_1, \xi_2) ds(x_1, x_2). \quad (5.57)
 \end{aligned}$$

The integrals in (5.57) may be evaluated numerically. We may solve (5.56) as a system of linear algebraic equations for the unknowns values of the displacements or tractions on the boundary elements. Once the values of the displacements and tractions are known on all the elements, the displacements

at any point (ξ_1, ξ_2) in the interior of the bimaterial may be calculated using

$$\begin{aligned}
 u_m(\xi_1, \xi_2) = & \sum_{n=1}^N \frac{1}{(2\omega - 1)\ell^{(n)}} \\
 & \times \{ \widehat{u}_k^{(n)} [-(1 - \omega)\ell^{(n)} G_{2km}^{(n)}(\xi_1, \xi_2) + G_{4km}^{(n)}(\xi_1, \xi_2)] \\
 & + \widehat{u}_k^{(N+n)} [\omega\ell^{(n)} G_{2km}^{(n)}(\xi_1, \xi_2) - G_{4km}^{(n)}(\xi_1, \xi_2)] \\
 & - \widehat{t}_k^{(n)} [-(1 - \omega)\ell^{(n)} G_{1km}^{(n)}(\xi_1, \xi_2) + G_{3km}^{(n)}(\xi_1, \xi_2)] \\
 & - \widehat{t}_k^{(N+n)} [\omega\ell^{(n)} G_{1km}^{(n)}(\xi_1, \xi_2) - G_{3km}^{(n)}(\xi_1, \xi_2)] \}. \quad (5.58)
 \end{aligned}$$

5.6.2 Stiff Interfaces

If we proceed as before, the boundary integral equations in (5.51) for the stiff interface may be approximated as

$$\begin{aligned}
 \gamma(\xi_1^{(i)}, \xi_2^{(i)}) \widehat{u}_m^{(i)} = & \sum_{n=1}^N \frac{1}{(2\omega - 1)\ell^{(n)}} \\
 & \times \{ \widehat{u}_k^{(n)} [-(1 - \omega)\ell^{(n)} K_{2km}^{(n)}(\xi_1^{(i)}, \xi_2^{(i)}) + K_{4km}^{(n)}(\xi_1^{(i)}, \xi_2^{(i)})] \\
 & + \widehat{u}_k^{(N+n)} [\omega\ell^{(n)} K_{2km}^{(n)}(\xi_1^{(i)}, \xi_2^{(i)}) - K_{4km}^{(n)}(\xi_1^{(i)}, \xi_2^{(i)})] \\
 & - \widehat{t}_k^{(n)} [-(1 - \omega)\ell^{(n)} K_{1km}^{(n)}(\xi_1^{(i)}, \xi_2^{(i)}) + K_{3km}^{(n)}(\xi_1^{(i)}, \xi_2^{(i)})] \\
 & - \widehat{t}_k^{(N+n)} [\omega\ell^{(n)} K_{1km}^{(n)}(\xi_1^{(i)}, \xi_2^{(i)}) - K_{3km}^{(n)}(\xi_1^{(i)}, \xi_2^{(i)})] \} \\
 & + \alpha_{kp} \left\{ -u_k(a, 0) \frac{\partial}{\partial x_1} (\Phi_{pm}^{(\text{stiff})}(x_1, x_2; \xi_1^{(i)}, \xi_2^{(i)})) \right\} \Big|_{(x_1, x_2)=(a, 0)} \\
 & + u_k(b, 0) \frac{\partial}{\partial x_1} (\Phi_{pm}^{(\text{stiff})}(x_1, x_2; \xi_1^{(i)}, \xi_2^{(i)})) \Big|_{(x_1, x_2)=(b, 0)} \\
 & + \Phi_{pm}^{(\text{stiff})}(a, 0; \xi_1^{(i)}, \xi_2^{(i)}) \frac{\partial}{\partial x_1} [u_k(x_1, x_2)] \Big|_{(x_1, x_2)=(a, 0)} \\
 & - \Phi_{pm}^{(\text{stiff})}(b, 0; \xi_1^{(i)}, \xi_2^{(i)}) \frac{\partial}{\partial x_1} [u_k(x_1, x_2)] \Big|_{(x_1, x_2)=(b, 0)} \} \\
 & \text{for } i = 1, 2, \dots, 2N, \quad (5.59)
 \end{aligned}$$

where

$$\begin{aligned}
 K_{1km}^{(n)}(\xi_1, \xi_2) &= \int_{B^{(n)}} \Phi_{km}^{(\text{stiff})}(x_1, x_2; \xi_1, \xi_2) ds(x_1, x_2), \\
 K_{2km}^{(n)}(\xi_1, \xi_2) &= \int_{B^{(n)}} \Gamma_{km}^{(\text{stiff})}(x_1, x_2; \xi_1, \xi_2) ds(x_1, x_2), \\
 K_{3km}^{(n)}(\xi_1, \xi_2) &= \int_{B^{(n)}} s^{(n)}(x_1, x_2) \Phi_{km}^{(\text{stiff})}(x_1, x_2; \xi_1, \xi_2) ds(x_1, x_2), \\
 K_{4km}^{(n)}(\xi_1, \xi_2) &= \int_{B^{(n)}} s^{(n)}(x_1, x_2) \Gamma_{km}^{(\text{stiff})}(x_1, x_2; \xi_1, \xi_2) ds(x_1, x_2). \quad (5.60)
 \end{aligned}$$

In (5.59), the unknowns on the boundary elements are as in (5.56). As values of u_k and $\partial u_k / \partial x_1$ at the interface tips $(a, 0)$ and $(b, 0)$ are possibly unknown, more equations are needed to complement (5.59). The additional equations may be set up as explained below.

We assume that the interface tip $(a, 0)$ is an endpoint of the first element $B^{(1)}$ and the other interface tip $(b, 0)$ is an endpoint of the last element $B^{(N)}$. Furthermore, $B^{(1)}$ and $B^{(N)}$ are assumed to lie in the regions $x_2 > 0$ and $x_2 < 0$ respectively.

From the first line of (5.54), we obtain approximately the following formulae for the displacements at the interface tips $(a, 0)$ and $(b, 0)$:

$$\begin{aligned}
 u_k(a, 0) &= \frac{[s^{(1)}(a, 0) - (1 - \omega)\ell^{(1)}]\hat{u}_k^{(1)} - [s^{(1)}(a, 0) - \omega\ell^{(1)}]\hat{u}_k^{(N+1)}}{(2\omega - 1)\ell^{(1)}}, \\
 u_k(b, 0) &= \frac{[s^{(N)}(b, 0) - (1 - \omega)\ell^{(N)}]\hat{u}_k^{(N)} - [s^{(N)}(b, 0) - \omega\ell^{(N)}]\hat{u}_k^{(2N)}}{(2\omega - 1)\ell^{(N)}}. \quad (5.61)
 \end{aligned}$$

For $(x_1, x_2) \in B^{(n)}$ ($n = 1$ and $n = N$), if we differentiate the approximation

of u_k in (5.54) with respect to the distance along $B^{(n)}$, we obtain

$$\begin{aligned} -n_2^{(1)} \frac{\partial u_k}{\partial x_1} \Big|_{(x_1, x_2)=(a, 0)} + n_1^{(1)} \frac{\partial u_k}{\partial x_2} \Big|_{(x_1, x_2)=(a, 0^+)} &= \frac{(\hat{u}_k^{(1)} - \hat{u}_k^{(N+1)})}{(2\omega - 1)\ell^{(1)}}, \\ -n_2^{(N)} \frac{\partial u_k}{\partial x_1} \Big|_{(x_1, x_2)=(b, 0)} + n_1^{(N)} \frac{\partial u_k}{\partial x_2} \Big|_{(x_1, x_2)=(b, 0^-)} &= \frac{(\hat{u}_k^{(N)} - \hat{u}_k^{(2N)})}{(2\omega - 1)\ell^{(N)}}, \end{aligned} \quad (5.62)$$

where $[n_1^{(k)}, n_2^{(k)}]$ is the unit outward normal vector to $B^{(k)}$.

Note that

$$\begin{aligned} \frac{\partial u_k}{\partial x_1} \Big|_{(x_1, x_2)=(a, 0^+)} &= \frac{\partial u_k}{\partial x_1} \Big|_{(x_1, x_2)=(a, 0^-)}, \\ \frac{\partial u_k}{\partial x_1} \Big|_{(x_1, x_2)=(b, 0^+)} &= \frac{\partial u_k}{\partial x_1} \Big|_{(x_1, x_2)=(b, 0^-)}, \end{aligned} \quad (5.63)$$

since $u_k(x_1, 0^+) = u_k(x_1, 0^-)$ for points $(x_1, 0)$ on the imperfect stiff interface.

From the generalized Hooke's law and the second line of (5.54), we approximately obtain

$$\begin{aligned} &c_{ijk1}^{(1)} n_j^{(1)} \frac{\partial u_k}{\partial x_1} \Big|_{(x_1, x_2)=(a, 0)} + c_{ijk2}^{(1)} n_{j1}^{(1)} \frac{\partial u_k}{\partial x_2} \Big|_{(x_1, x_2)=(a, 0^+)} \\ &= \frac{[s^{(1)}(a, 0) - (1 - \omega)\ell^{(1)}] \hat{t}_i^{(1)} - [s^{(1)}(a, 0) - \omega\ell^{(1)}] \hat{t}_i^{(N+1)}}{(2\omega - 1)\ell^{(1)}}, \\ &c_{ijk1}^{(2)} n_j^{(N)} \frac{\partial u_k}{\partial x_1} \Big|_{(x_1, x_2)=(b, 0)} + c_{ijk2}^{(2)} n_{j1}^{(N)} \frac{\partial u_k}{\partial x_2} \Big|_{(x_1, x_2)=(b, 0^-)} \\ &= \frac{[s^{(N)}(b, 0) - (1 - \omega)\ell^{(N)}] \hat{t}_i^{(N)} - [s^{(N)}(b, 0) - \omega\ell^{(N)}] \hat{t}_i^{(2N)}}{(2\omega - 1)\ell^{(N)}}. \end{aligned} \quad (5.64)$$

We may solve (5.59) together with (5.61), (5.62) and (5.64) for unknown values of u_k or t_k on the boundary elements and unknown values of u_k and $\partial u_k / \partial x_\ell$ at the interface tips $(a, 0)$ and $(b, 0)$. Once the unknown values are determined, the displacements at any point (ξ_1, ξ_2) in the interior of the bima-

terial may be calculated using

$$\begin{aligned}
 u_m(\xi_1, \xi_2) = & \sum_{n=1}^N \frac{1}{(2\omega - 1)\ell^{(n)}} \\
 & \times \{ \widehat{u}_k^{(n)} [-(1 - \omega)\ell^{(n)} K_{2km}^{(n)}(\xi_1, \xi_2) + K_{4km}^{(n)}(\xi_1, \xi_2)] \\
 & + \widehat{u}_k^{(N+n)} [\omega\ell^{(n)} K_{2km}^{(n)}(\xi_1, \xi_2) - K_{4km}^{(n)}(\xi_1, \xi_2)] \\
 & - \widehat{t}_k^{(n)} [-(1 - \omega)\ell^{(n)} K_{1km}^{(n)}(\xi_1, \xi_2) + K_{3km}^{(n)}(\xi_1, \xi_2)] \\
 & - \widehat{t}_k^{(N+n)} [\omega\ell^{(n)} K_{1km}^{(n)}(\xi_1, \xi_2) - K_{3km}^{(n)}(\xi_1, \xi_2)] \} \\
 & + \alpha_{kp} \left\{ -u_k(a, 0) \frac{\partial}{\partial x_1} (\Phi_{pm}^{(\text{stiff})}(x_1, x_2; \xi_1, \xi_2)) \right\} \Big|_{(x_1, x_2)=(a, 0)} \\
 & + u_k(b, 0) \frac{\partial}{\partial x_1} (\Phi_{pm}^{(\text{stiff})}(x_1, x_2; \xi_1, \xi_2)) \Big|_{(x_1, x_2)=(b, 0)} \\
 & + \Phi_{pm}^{(\text{stiff})}(a, 0; \xi_1, \xi_2) \frac{\partial}{\partial x_1} [u_k(x_1, x_2)] \Big|_{(x_1, x_2)=(a, 0)} \\
 & - \Phi_{pm}^{(\text{stiff})}(b, 0; \xi_1, \xi_2) \frac{\partial}{\partial x_1} [u_k(x_1, x_2)] \Big|_{(x_1, x_2)=(b, 0)} \}. \quad (5.65)
 \end{aligned}$$

5.7 Specific Problems

Consider the rectangular bilayered slab in Figure 5.4. The vertices of the rectangular slab are (w, h) , $(w, -h)$, $(-w, -h)$ and $(-w, h)$, where w and h are given positive real numbers. The regions $\Pi^{(1)}$ and $\Pi^{(2)}$ are occupied by particular transversely isotropic materials. Specifically, the only non-zero elastic moduli $c_{ijkl}^{(p)}$ in $\Pi^{(p)}$ are given by

$$\begin{aligned}
 c_{1111}^{(p)} &= C^{(p)}, \quad c_{2222}^{(p)} = A^{(p)}, \\
 c_{1212}^{(p)} &= c_{2112}^{(p)} = c_{1221}^{(p)} = c_{2121}^{(p)} = L^{(p)}, \\
 c_{1122}^{(p)} &= c_{2211}^{(p)} = F^{(p)}, \quad (5.66)
 \end{aligned}$$

where $A^{(p)}$, $F^{(p)}$, $C^{(p)}$ and $L^{(p)}$ are independent elastic coefficients.

Figure 5.4: A rectangular bilayered slab.

The boundary conditions on the exterior boundary of the rectangular slab are given by

$$\left. \begin{aligned} u_k(x_1, -h) &= 0 \\ \sigma_{k2}(x_1, h) &= P_k(x_1) \end{aligned} \right\} \text{ for } -w < x_1 < w, \\ \sigma_{k1}(\pm w, x_2) &= 0 \text{ for } -h < x_2 < h, \quad (5.67)$$

where $P_k(x_1)$ are suitably prescribed tractions.

The interface between the materials in $\Pi^{(1)}$ and $\Pi^{(2)}$ is imperfect, satisfying either the soft or stiff interfacial conditions.

Problem 5.1

For the mere purpose of checking the validity of the boundary element procedures, that is, for a test problem, we take $h = 0.5$, $w = 8$, $P_k(x_1) = \exp(-|x_1|)$, $A^{(1)} = 5$, $C^{(1)} = 4$, $F^{(1)} = 1$, $L^{(1)} = 1$, $A^{(2)} = 1$, $C^{(2)} = 1$, $F^{(2)} = 1/2$ and $L^{(2)} = 1/5$.

The exterior boundary of the bimaterial is discretized into 132 boundary elements and the boundary element method together with the relevant Green's function is used to compute the displacements $u_k(x_1, 0.25)$ ($-8 < x_1 < 8$) and $u_k(0.5, x_2)$ ($-0.5 < x_2 < 0.5$) for soft and stiff interfaces. For the parameters in the interfacial conditions, we take $\lambda_{11} = 2$, $\lambda_{12} = \lambda_{21} = 0$ and $\lambda_{22} = 5$ for the soft interface and $\alpha_{11} = 1/10$ and $\alpha_{12} = \alpha_{21} = \alpha_{22} = 0$ for the stiff interface.

As the width $2w$ of the rectangular slab is relatively large compared to the height $2h$, that is, $w/h = 16$, we compare the computed displacements with the corresponding explicit solutions for an infinitely long slab where $w \rightarrow \infty$. The explicit solutions are derived analytically in Section 5.8 – they are expressed in terms of Fourier integrals which may be evaluated numerically.

For the soft interface, Figure 5.5 and 5.6 compare graphically the values of the displacements $u_k(x_1, 0.25)$ (for $-8 < x_1 < 8$) computed using the Green's function boundary element method and those calculated using the explicit analytical solution in the next section. On the whole, the two sets of value agree well. Observing more closely, we find that the displacements computed using the boundary element method actually deviate slightly from the analytical solution at both ends of the slab. This is to be expected since the analytical solution is derived for an infinitely long slab instead of the finite width slab in Figure 5.4.

Plots of the displacements $u_k(0.5, x_2)$ along the direction perpendicular to the interface $\mathcal{I}$ are given in Figure 5.7 and 5.8 for $-0.5 < x_2 < 0.5$. Again, the values computed using the boundary element method match well the values calculated using the analytical solution. Figure 5.7 and 5.8 show clearly the displacement jumps Δu_k across the soft interface $\mathcal{I}$.

For the stiff interface, we have also observed a good agreement between the boundary element and the analytical solutions for the displacements $u_k(x_1, 0.25)$

Figure 5.5: Plots of $u_1(x_1, 0.25)$ against x_1 for $-8 < x_1 < 8$ (for soft interfaces).

Figure 5.6: Plots of $u_2(x_1, 0.25)$ against x_1 for $-8 < x_1 < 8$ (for soft interfaces).

Figure 5.7: Plots of $u_1(0.5, x_2)$ against x_2 for $-0.5 < x_2 < 0.5$ (for soft interfaces).

Figure 5.8: Plots of $u_2(0.5, x_2)$ against x_2 for $-0.5 < x_2 < 0.5$ (for soft interfaces).

$(-8 < x_1 < 8)$ in Figure 5.9 and 5.10. The slight differences between the solutions at points closer to $x_1 = \pm 8$ are as explained above for the soft interface. Figure 5.11 and 5.12 show the displacements along $(0.5, x_2)$ for $-0.5 < x_2 < 0.5$ for both numerical and analytical model. As expected, we observe continuity of displacements u_k across the stiff interface $\mathcal{I}$.

Figure 5.9: Plots of $u_1(x_1, 0.25)$ against x_1 for $-8 < x_1 < 8$ (for stiff interfaces).

Problem 5.2

Consider now the case where $\Pi^{(1)}$ and $\Pi^{(2)}$ are occupied by isotropic materials. The Young's modulus and the Poisson ratio of the isotropic material in $\Pi^{(p)}$ ($p = 1, 2$) are denoted by $E^{(p)}$ and $\nu^{(p)}$ respectively. The boundary element analysis here may be recovered for isotropic materials by taking the constants

Figure 5.10: Plots of $u_2(x_1, 0.25)$ against x_1 for $-8 < x_1 < 8$ (for stiff interfaces).

Figure 5.11: Plots of $u_1(0.5, x_2)$ against x_2 for $-0.5 < x_2 < 0.5$ (for stiff interfaces).

Figure 5.12: Plots of $u_2(0.5, x_2)$ against x_2 for $-0.5 < x_2 < 0.5$ (for stiff interfaces).

$A^{(p)}$, $F^{(p)}$, $C^{(p)}$ and $L^{(p)}$ in (5.66) to be approximately given by

$$\begin{aligned} A^{(p)} &= C^{(p)} = \frac{E^{(p)}(1 - \nu^{(p)})}{(1 + \nu^{(p)})(1 - 2\nu^{(p)})}, \\ F^{(p)} &= \frac{E^{(p)}\nu^{(p)}(1 - \epsilon)}{(1 + \nu^{(p)})(1 - 2\nu^{(p)})}, \quad L^{(p)} = \frac{E^{(p)}}{2(1 + \nu^{(p)})}, \end{aligned}$$

where ϵ is a positive real number with a very small magnitude.

The boundary conditions are as given in (5.67) with

$$P_k(x_1) = \begin{cases} \delta_{k2}T & \text{for } |x_1| < a, \\ 0 & \text{for } a < |x_1| < w, \end{cases}$$

where T is a given positive real constant and a is a given positive real number such that $a < w$.

For the purpose of our calculation here, we take $w/h = 2$, $a/h = 1$, $\epsilon = 10^{-6}$, $\nu^{(1)} = \nu^{(2)} = 0.30$ and $E^{(1)}/E^{(2)} = 2$ (the material in $\Pi^{(2)}$ is “softer”

than that in $\Pi^{(1)}$). The exterior boundary $C_1 \cup C_2$ of the rectangular slab is discretized into 60 elements.

For the case where the interface is soft, we take $\lambda_{11}/\lambda_{22} = 0.2$, $\lambda_{12}h/E^{(1)} = \lambda_{21}h/E^{(1)} = 0$ and examine the effects of altering the parameter $\lambda_{11}h/E^{(1)}$ on the deformation of the rectangular slab. Plots of the non-dimensionalized displacements $u_k E^{(1)}/hT$ along $x_2/h = 0.5$ for $-2 < x_1/h < 2$ are given in Figures 5.13 and 5.14 for selected values of $\lambda_{11}h/E^{(1)}$. The solid line plots in both figures give the non-dimensionalized displacements $u_k E^{(1)}/hT$ for the case in which the interface is perfectly bonded, that is, for the limiting case where $\lambda_{11}h/E^{(1)} \rightarrow \infty$. In Figures 5.13 and 5.14, $u_1 E^{(1)}/hT$ approaches the solid line as the non-dimensionalized parameter $\lambda_{11}h/E^{(1)}$ increases. Hence, a lower value of $\lambda_{11}h/E^{(1)}$ indicates a weaker interface. This may also be seen in the plots of the displacement $u_2 E^{(1)}/hT$ along $x_1/h = 0$ for $-1 < x_2/h < 1$ in Figure 5.15, where we observe a larger deformation for a smaller value of $\lambda_{11}h/E^{(1)}$. A weaker interface is less able to distribute out the stress in $\Pi^{(1)}$ to $\Pi^{(2)}$ across the interface. As may be expected, Figure 5.15 shows that a smaller value of $\lambda_{11}h/E^{(1)}$ gives a bigger jump in the displacement $u_2 E^{(1)}/hT$ across the interface $\mathcal{I}$. Note that the displacement $u_2 E^{(1)}/hT$ in Figure 5.15 does not show any jump across the ideal interface.

For the case where the interface is stiff, we take α_{ij} to be such that only α_{11} is not zero. Figures 5.16, 5.17 and 5.18 show plots of the non-dimensionalized displacements $u_k E^{(1)}/hT$ for selected values of $\alpha_{11}/E^{(1)}h$. As before, the solid line plots give the non-dimensionalized displacements $u_k E^{(1)}/hT$ for the special case where the interface is ideal.

We observe that the effect of the stiff interface on the displacement field is opposite to that of the soft interface. The presence of the stiff interface appears to reduce the magnitudes of the displacements. In Figure 5.16, the

Figure 5.13: Plots of $u_1 E^{(1)} / hT$ on $x_2/h = 0.5$, $-2 < x_1/h < 2$, for selected values of $\lambda_{11} h / E^{(1)}$ (for soft interfaces).

Figure 5.14: Plots of $u_2 E^{(1)} / hT$ on $x_2/h = 0.5$, $-2 < x_1/h < 2$, for selected values of $\lambda_{11} h / E^{(1)}$ (for soft interfaces).

Figure 5.15: Plots of $u_2 E^{(1)} / hT$ along $x_1/h = 0$, $-1 < x_2/h < 1$, for selected values of $\lambda_{11} h / E^{(1)}$ (for soft interfaces).

Figure 5.16: Plots of $u_1 E^{(1)} / hT$ on $x_2/h = 0.5$, $-2 < x_1/h < 2$, for selected values of $\alpha_{11} / E^{(1)} h$ (for stiff interfaces).

Figure 5.17: Plots of $u_2 E^{(1)}/hT$ on $x_2/h = 0.5$, $-2 < x_1/h < 2$, for selected values of $\alpha_{11}/E^{(1)}h$ (for stiff interfaces).

Figure 5.18: Plots of $u_2 E^{(1)}/hT$ along $x_1/h = 0$, $-1 < x_2/h < 1$, for selected values of $\alpha_{11}/E^{(1)}h$ (for stiff interfaces).

displacement $u_1 E^{(1)}/hT$ approaches the solid line as $\alpha_{11}/E^{(1)}h$ decreases, that is, as the stiffness of the interface decreases. Figures 5.17 and 5.18 also show that the stiff interface is able to resist deformation. Moreover, in Figure 5.18, the gradient of the non-dimensionalized displacement $u_2 E^{(1)}/hT$ appears to decrease as non-dimensionalized stiff interface parameter $\alpha_{11}/E^{(1)}h$ increases, that is, the overall strength of the bimaterial apparently increases with increasing $\alpha_{11}/E^{(1)}h$. As expected, the displacement $u_2 E^{(1)}/hT$ is continuous across the stiff interface.

5.8 Analytic Solutions for Infinitely Long Bilayered Slabs with Soft and Stiff Interfaces

For the purpose of comparing the numerical solutions of Problem 5.1, the displacements and stresses in the rectangular bilayered slab in Figure 5.4, which satisfy the exterior boundary conditions (5.67) and the imperfect interfacial conditions, are given here explicitly in terms of Fourier integrals for the special case where $w \rightarrow \infty$, that is, for an infinitely long bilayered slab.

Guided by the analysis in Clements [28], we take the displacements to be in the form

$$\begin{aligned}
 & u_k(x_1, x_2) \\
 = & H(x_2) \operatorname{Re} \left\{ \sum_{\alpha=1}^2 A_{k\alpha}^{(1)} \int_0^\infty [V_\alpha^{(1)}(p) \exp(ip(x_1 + \tau_\alpha^{(1)}x_2)) \right. \\
 & \left. + W_\alpha^{(1)}(p) \exp(-ip(x_1 + \tau_\alpha^{(1)}x_2))] dp \right\} \\
 & + H(-x_2) \operatorname{Re} \left\{ \sum_{\alpha=1}^2 A_{k\alpha}^{(2)} \int_0^\infty [V_\alpha^{(2)}(p) \exp(ip(x_1 + \tau_\alpha^{(2)}x_2)) \right. \\
 & \left. + W_\alpha^{(2)}(p) \exp(-ip(x_1 + \tau_\alpha^{(2)}x_2))] dp \right\}, \tag{5.68}
 \end{aligned}$$

where $V_\alpha^{(k)}(p)$ and $W_\alpha^{(k)}(p)$ ($k = 1, 2$) are functions yet to be determined.

The stresses corresponding to the displacements in (5.68) are given by

$$\begin{aligned}
 & \sigma_{kj}(x_1, x_2) \\
 = & H(x_2) \operatorname{Re} \left\{ \sum_{\alpha=1}^2 L_{kj\alpha}^{(1)} \int_0^\infty [ipV_\alpha^{(1)}(p) \exp(ip(x_1 + \tau_\alpha^{(1)}x_2)) \right. \\
 & \left. - ipW_\alpha^{(1)}(p) \exp(-ip(x_1 + \tau_\alpha^{(1)}x_2))] dp \right\} \\
 & + H(-x_2) \operatorname{Re} \left\{ \sum_{\alpha=1}^2 L_{kj\alpha}^{(2)} \int_0^\infty [ipV_\alpha^{(2)}(p) \exp(ip(x_1 + \tau_\alpha^{(2)}x_2)) \right. \\
 & \left. - ipW_\alpha^{(2)}(p) \exp(-ip(x_1 + \tau_\alpha^{(2)}x_2))] dp \right\}. \tag{5.69}
 \end{aligned}$$

If the integrals in (5.69) exist, the stresses σ_{kj} decay to zero as $|x_1|$ tends to infinity (Sneddon [66]).

From (5.68), we find that the conditions on the lower edge $x_2 = -h$ of the infinitely long slab, that is,

$$u_k(x_1, -h) = 0 \text{ for } -\infty < x_1 < \infty, \tag{5.70}$$

are satisfied if $V_\alpha^{(2)}(p)$ and $W_\alpha^{(2)}(p)$ are such that

$$\sum_{\alpha=1}^2 [A_{k\alpha}^{(2)} V_\alpha^{(2)}(p) \exp(-ip\tau_\alpha^{(2)}h) + \overline{A}_{k\alpha}^{(2)} \overline{W}_\alpha^{(2)}(p) \exp(-ip\overline{\tau}_\alpha^{(2)}h)] = 0. \tag{5.71}$$

The conditions on the upper edge $x_2 = h$ of the infinitely long slab are given by

$$\sigma_{k2}(x_1, h) = P_k(x_1) \text{ for } -\infty < x_1 < \infty. \tag{5.72}$$

If we apply the Fourier exponential transformation on both sides of (5.72), that is, if we rewrite (5.72) as

$$\int_{-\infty}^\infty \sigma_{k2}(x_1, h) \exp(-i\gamma x_1) dx_1 = \int_{-\infty}^\infty P_k(x_1) \exp(-i\gamma x_1) dx_1, \tag{5.73}$$

and use (5.69), we obtain

$$\begin{aligned} \pi i p \sum_{\alpha=1}^2 [L_{k2\alpha}^{(1)} V_{\alpha}^{(1)}(p) \exp(ip\tau_{\alpha}^{(1)} h) + \bar{L}_{k2\alpha}^{(1)} \bar{W}_{\alpha}^{(1)}(p) \exp(ip\bar{\tau}_{\alpha}^{(1)} h)] \\ = \int_{-\infty}^{\infty} P_k(x_1) \exp(-ipx_1) dx_1. \end{aligned} \quad (5.74)$$

5.8.1 Soft Interfaces

The conditions in (5.12) are satisfied if

$$\sum_{\alpha=1}^2 [L_{i2\alpha}^{(1)} V_{\alpha}^{(1)}(p) + \bar{L}_{i2\alpha}^{(1)} \bar{W}_{\alpha}^{(1)}(p)] = \sum_{\alpha=1}^2 [L_{i2\alpha}^{(2)} V_{\alpha}^{(2)}(p) + \bar{L}_{i2\alpha}^{(2)} \bar{W}_{\alpha}^{(2)}(p)]. \quad (5.75)$$

The conditions in (5.13) give

$$\begin{aligned} \lambda_{ik} \left\{ \sum_{\alpha=1}^2 [A_{k\alpha}^{(1)} V_{\alpha}^{(1)}(p) - A_{k\alpha}^{(2)} V_{\alpha}^{(2)}(p) + \bar{A}_{k\alpha}^{(1)} \bar{W}_{\alpha}^{(1)}(p) - \bar{A}_{k\alpha}^{(2)} \bar{W}_{\alpha}^{(2)}(p)] \right\} \\ = ip \sum_{\alpha=1}^2 [L_{i2\alpha}^{(1)} V_{\alpha}^{(1)}(p) + \bar{L}_{i2\alpha}^{(1)} \bar{W}_{\alpha}^{(1)}(p)]. \end{aligned} \quad (5.76)$$

Thus, if the interface $\mathcal{I}$ is soft, the functions $V_{\alpha}^{(1)}$, $W_{\alpha}^{(1)}(p)$, $V_{\alpha}^{(2)}(p)$, and $W_{\alpha}^{(2)}(p)$ are obtained by solving (5.71), (5.74), (5.75) and (5.76).

5.8.2 Stiff Interfaces

The conditions in (5.15) hold if

$$\sum_{\alpha=1}^2 [A_{k\alpha}^{(1)} V_{\alpha}^{(1)}(p) + \bar{A}_{k\alpha}^{(1)} \bar{W}_{\alpha}^{(1)}(p)] = \sum_{\alpha=1}^2 [A_{k\alpha}^{(2)} V_{\alpha}^{(2)}(p) + \bar{A}_{k\alpha}^{(2)} \bar{W}_{\alpha}^{(2)}(p)]. \quad (5.77)$$

The conditions in (5.16) are satisfied if

$$\begin{aligned} i \left\{ \sum_{\alpha=1}^2 [L_{i2\alpha}^{(1)} V_{\alpha}^{(1)}(p) + \bar{L}_{i2\alpha}^{(1)} \bar{W}_{\alpha}^{(1)}(p) - L_{i2\alpha}^{(2)} V_{\alpha}^{(2)}(p) - \bar{L}_{i2\alpha}^{(2)} \bar{W}_{\alpha}^{(2)}(p)] \right\} \\ = p\alpha_{ik} \sum_{\alpha=1}^2 [A_{k\alpha}^{(1)} V_{\alpha}^{(1)}(p) + \bar{A}_{k\alpha}^{(1)} \bar{W}_{\alpha}^{(1)}(p)]. \end{aligned} \quad (5.78)$$

Thus, if the interface $\mathcal{I}$ is stiff, the functions $V_\alpha^{(1)}$, $W_\alpha^{(1)}(p)$, $V_\alpha^{(2)}(p)$, and $W_\alpha^{(2)}(p)$ are obtained by solving (5.71), (5.74), (5.77) and (5.78).

5.9 Summary

Green's functions are derived for imperfect soft and stiff planar interfaces between two dissimilar anisotropic elastic half-spaces and applied to obtain boundary element procedures for determining the elastic fields in bimetals of finite extent. As the derived Green's functions satisfy the relevant interfacial conditions, the boundary element formulations do not require the interfaces to be discretized into elements, giving rise to smaller systems of linear algebraic equations.

To access the validity and accuracy of the proposed boundary element methods for the bimetals with soft and stiff interfaces, specific test problems involving relatively long rectangular bilayered slabs are solved numerically. The boundary element solutions obtained compare favorably with the analytical solutions for the corresponding case of an infinitely long bilayered slab. The analytical solutions for the infinitely long bilayered slab are derived in explicit forms using exponential Fourier integral transforms.

The boundary element procedures are also applied to study the effects of interface parameters on the displacement fields in bilayered slabs with soft and stiff interfaces. The bilayered slabs are shown to be weakened by soft interfaces and strengthened by stiff interfaces.

Chapter 6

Research Contributions and Extensions

6.1 Summary of Contributions

The main contributions of the thesis are summarized below.

- Special Green's functions for axisymmetric heat conduction are derived for low conducting and high conducting flat interfaces. Boundary element procedures based on the special Green's functions are developed for analyzing the temperature distribution in the bimetals. Such boundary element procedures do not require the imperfect interfaces to be discretized into elements.
- Hypersingular boundary integral formulations are derived for the numerical solution of the axisymmetric steady-state heat conduction problem across low and high conducting curved interfaces in bimetals. Earlier works have been mainly confined to two-dimensional problems involving low conducting interfaces. The hypersingular boundary integral method is also extended to treat steady state axisymmetric heat conduction in bimetals with the nonlinear Stefan-Boltzmann conditions.

- Special elastostatic Green's functions are derived for soft and stiff planar interfaces between two dissimilar anisotropic half-spaces. The Green's functions are used to derive boundary element procedures for analyzing plane elastostatic deformations of bimaterials with imperfect interfaces. Similarly, the boundary element procedures do not require the planar interfaces to be discretized into elements.

6.2 Research Extensions

Some possibilities for extending the works in this thesis are as follows.

- The works in Chapter 3 and 4 are considered in the context of the classical theory of heat conduction. Extension to the non-classical dual-phase-lag heat flux model such as in the works of Ramadan [60] and Ho, Kuo and Jiaung [45] may be considered.
- Extension of the boundary element approaches here to higher order interfaces may be considered. Higher order interfacial conditions are given in Benveniste [16] and Niklasson, Datta and Dunn [53]-[54].
- Problems concerning the interactions between cracks and the imperfect interfaces may be found in Li and Lee [48] and Zhong, Li and Lee [81]. The elastostatic Green's function boundary element method for soft and stiff interfaces in Chapter 5 may be further developed for solving problems involving interaction of cracks and the soft and stiff interfaces.

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