Control policy and algorithm analysis for production planning and inventory control

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CONTROL POLICY AND ALGORITHM ANALYSIS FOR
PRODUCTION PLANNING AND INVENTORY CONTROL

FENG YI

2005

CONTROL POLICY AND ALGORITHM ANALYSIS FOR PRODUCTION PLANNING AND INVENTORY CONTROL

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**SCHOOL OF MECHANICAL AND AEROSPACE ENGINEERING
NANYANG TECHNOLOGICAL UNIVERSITY**

2005

Control Policy and Algorithm Analysis for Production Planning and Inventory Control

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ABSTRACT

In this thesis, research literature is reviewed in the field of production planning and inventory control. The scope of the work focuses on the optimal policies and algorithm analysis for stochastic and deterministic models from production planning and inventory control. Specifically, three fundamental mathematical models are investigated in detail. For the single product periodic review problem with setup costs and equal capacity constraint, this study considers the model where inventory cost function is convex, which has not been investigated deeply in the literature earlier. The optimal policy with property of X-Y band is characterized. The average computation time of newly designed algorithm grows more or less doubly with the periodic numbers, which is more efficient than the others for solving such a model so far. This study also considers a particular model in stochastic multiple product flexible manufacturing system, where the respective model parameter is the same for each product. The obtained results particularly characterize the optimal hedging point policy, and also give a chance to observe the order policy of the general model which has been an open problem for about four decades. Further, parameter analysis is conducted for the uncapacitated economic lot-sizing problem where backlogging is allowed, and some new results are achieved. For instance, it is proved that there are critical values for periodic setup and unit production costs respectively. If the setup cost or unit production cost is higher than their respective critical value, the period will be a production period; if it is equal to or lower than the value, the period will be a non-production period. Recommendations for future research in this field are also proposed. Specifically, a hypothesis is suggested to extend the special stochastic multiple-product model into a general one.

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Chapter 1

Introduction

The problem of production planning first appeared when the industry era started. Since that time, the importance of inventory control has been widely recognized. The concept of supply chain planning has been increasingly discussed in academics and industries since mid-1980s. Nowadays, companies not only optimize their internal production planning, but also work cooperatively with their supply chain partners to achieve whole chain optimization.

1.1 Introduction to Production Planning and Inventory Control

Planning problems in production planning and inventory control consider the best use of resources in order to satisfy certain goals over a time horizon. Making the right decisions will directly affect the company performance and productivity, which are important for a firm or chain's capability to compete in the market. The planning problems are normally presented at the company's strategic, tactical and workshop levels, and solved by ERP or MRP systems. Usually, the model of a planning and inventory control problem is a configuration of the following factors:

- (1) Planning horizon. Planning horizon is the time interval on which the planning schedule extends into the future. It may be finite or infinite.
- (2) Echelon numbers. The planning problems may be of single echelon or multiple echelons. In a single echelon system, the raw materials are changed into final products after processed by a single operation. Multiple echelon systems include

serial, assembly, disassembly and general systems mainly decided by the product structure.

- (3) Capacity or resource constraints. The constraints include manpower, equipment, machines, financial budget, and so on.
- (4) Products number. There are two principal types of planning problems in terms of products number: Single-product system and multiple-product system. The complexity of multiple product system is much higher than that of single-product system.
- (5) Demand. Two types of demands are considered: Deterministic and stochastic demands. They mean that the demand is known in advance or based on some probabilities, respectively.
- (6) Setup structure. Setup costs and/or time are usually modeled by introducing zero-one variables in the mathematical model, and thus cause the model to be more complex.
- (7) Inventory cost. Inventory holding or backlogging also complicates the model. Allowing backlogging means current demands will be satisfied in future periods.

Under the roof of planning problems, stochastic and deterministic dynamic models are classified depending on whether there is a random model parameter or not. The objective of planning problem is to solve the stochastic or deterministic dynamic models with multiple products, multiple echelons, different cost functions and different constraints. Such models reflect the real supply chain in the business world.

In fact, in real business world, numerous models in production and supply chain planning are closely related. For example, in consecutive planning time periods with different deterministic periodic demand, and considering the single item production scheduling problem at the manufacturer's site and the same product's inventory control problem at the retailer's site, it is a typical economic lot sizing problem (Van Hoesel and Wagelmans, 1993) that both sites are to be optimized simultaneously in order to achieve supply chain optimization. Another example is that a manufacturer produces two kinds of products which are supplied to two retailers. The retailers forecast their own demands and share the information with the manufacturer. This supply chain optimization problem can be modeled as that in Chen (2004b).

1.2 Dynamic Programming and Production Planning and Inventory Control Problems

Basically production planning is closely related with inventory management. How and when to replenish the inventory also decides how and when to produce or order the products. The problem is often claimed to have started from the Wilson (1934)'s economic order quantity (EOQ) formula. After Bellman (1957) created dynamic programming, researchers found that dynamic programming was an ideal tool for analyzing the multiple period and multiple stage planning problems. Based on the pioneering works of Bellman (1957), and Arrow, Harris, and Marschak (1951) for tackling stochastic models, Wagner and Whitin (1958) also applied dynamic programming in deterministic models and postulated some theorems which are of use even today.

The subsequent research studies have further extended Bellman (1957), Arrow, Harris, and Marschak (1951) and Wagner and Whitin (1958)'s works to incorporate more complex cost functions, capacity constraints, multiple products and multiple echelons. Numerous theories have been established for both deterministic and stochastic dynamic models, which will be reviewed in the next chapter.

All the theories developed and accomplished for production planning and inventory control can be applied not only in manufacturing, but also in service industry such as aircraft seat and hotel room booking, overbooking, cancellation and no-shows management. There is an increasing interest in applying dynamic programming techniques into yield management. Yield management encompasses all practices of discriminatory pricing used to maximize the profit generated from a fixed amount of resources. There are two typical applications of yield management: hotel and airline booking policies. The basic idea behind yield management is that different consumers of the service are willing to pay different amounts for that service. Given a stochastic process for customers calling in reservations prior to a particular booking date of a hotel or airline, the problem is to devise a policy that maximizes the total expected profit of the hotel booking or airline flight (Badinelli, 2000). One sophisticated model in this field is that of Subramanian, Stidham and Lauenbacher (1999).

1.3 Research Motivations

Planning problem is one of the most challenging subjects for the management in many organizations. Making right decisions effectively and efficiently will affect the

company's ability to compete in the market (Drexel and Kimms, 1997; Karimi, Ghomi and Wilson, 2003).

As stated in Section 1.1, the objective of planning problem is to characterize and solve the real supply chain models efficiently and effectively. Nevertheless, the supply chain models and within this area, the class of capacitated multi item lot sizing problems as being of prime importance are still awaiting algorithmic improvements. The objective is still far away from achieved since not many optimal properties have been recognized for many basic models. The general model of inventory control problems is very complex. Florian et al. (1980) have in fact shown that even the single-item model is NP-complete. When several items are involved, no efficient solution methods are known. The one-product system cannot serve as a basic model for multiproduct systems, as it cannot represent interactions among products. Thus, so far, no complete theory has been developed for the inventory systems in general and finding and characterizing the optimal control policies is a very challenging problem. Therefore, by considering the recently developed basic models one by one and extending them, the objective of characterizing the optimal policies can be accomplished. Specifically, this research focuses on three basic models which are also the three different configurations of seven factors listed in section 1.1. All three basic models consider finite and multiple planning horizons, and allow inventory backlogging. These basic models are briefly introduced below.

Chen and Lambrecht (1996) and Chen (2004a) find that for a stochastic capacitated dynamic model with same setup, linear production and convex inventory costs in each period, there exists the X-Y band optimal control policy (A function $f(x)$ is linear if it

satisfies that $f(x + y) = f(x) + f(y)$, and $f(\alpha x) = \alpha f(x)$ for all x and y in the domain, and all scalars α . A function $f(x)$ is convex on an interval $[a, b]$ if for any two points x_1 and x_2 in $[a, b]$, $f(\frac{1}{2}(x_1 + x_2)) \leq \frac{1}{2}(f(x_1) + f(x_2))$ (<http://mathworld.wolfram.com/ConvexFunction.html>). Convex cost function means that the marginal cost is increasing (Silver and Peterson, 1985)). In this thesis, a stochastic capacitated dynamic model means that the planning horizon is multiple periods, the periodic demand is random, and the production or supply capacity is constrained by the upper limit for the model. On the other hand, for the deterministic model with the same model configuration, no optimal property has been characterized since Florian and Klein (1971)'s work. The deterministic model is one of the most frequently used inventory planning models and needs to be solved repeatedly. Thus, there is a motivation to test whether X-Y band optimal control policy developed for stochastic model is applicable to the complementary deterministic capacitated model. If it is applicable, a new algorithm can be designed based on the X-Y band policy. In other words, the problem is to develop an optimal control policy for a more general model in which periodic set-up cost, unit production cost, and the inventory cost function can be different. Furthermore, this research topic is meaningful since applying newly established X-Y band control policy into deterministic convex optimization problems could result into an improved and efficient new algorithm. The theoretical results will also lead to the development of a new method to further explore the optimal policy of deterministic models.

In his recent paper, Chen (2004b) discusses the fundamental result of a hedging point policy for the stochastic capacitated dynamic model considering two products. Extending

his theory to more than two products is not straightforward, and the problem is still an open problem and tough challenge to academia. This research focuses on a particular case of multiple product problem in which stochastic demand distribution, production rate, unit production cost, and periodic expected inventory cost are the same for all products respectively. Since the simplified model is a special case of the general one, if the simplified model can be characterized by some properties such as hedging point optimal policy, it may give clues to the solution for the general model. This will also lead to a major breakthrough and contribution in constrained convex optimization.

Parameter analysis is quite important for saving computational time, especially when the model is NP-hard (A problem belongs to class ξ if, for any instance of the problem, its feasibility or infeasibility can be determined by a polynomial algorithm. It belongs to the class $\tau\xi$ if, for any instance, one can determine in polynomial time whether a given structure affirms its feasibility. Problem P' is said to be reducible to problem P if for any instance of P' an instance of P can be constructed in polynomial time such that solving the instance of P will solve the instance of P' as well. Problem P is called NP-Hard if P' is reducible to P for every P' belonging to $\tau\xi$ (Florian, Lenstra, and Rinnooy Kan, 1980).) and the size is quite large. This is absolutely meaningful for real-time or online planning problems which are very common in today's e-commerce application. Van Hoesel and Wagelmans (1993) conducted sensitivity analysis for an uncapacitated model with setup costs, linear production and inventory costs and without allowing backlogging. However, for a more general model which allows backlogging, how to define the model parameter variation scope is still not reported. This research tries to answer this question by

obtaining new results for the variations of periodic setup cost, total demand variation and periodic unit production cost. Obviously, the purpose of conducting sensitivity analysis is to save computational time. Nowadays, e-commerce and online ERP application is becoming very common; therefore, this research is very helpful especially in large scale economic lot sizing planning (ELSP) problem and online or real time ELSP planning or scheduling problems.

1.4 Research Objectives

This research follows the main research stream in production planning and inventory control. Fundamental models, results and challenges have been reviewed and identified respectively. The following are the objectives of this research:

- (1) The first objective is to prove that for a class of deterministic models with non-increasing setup cost, linear production and convex inventory costs, the optimal result of stochastic model is still true and applicable, and consequently, extend the optimal control policy to a more general model. In other words, the control policy that for every period n , there exists a Y_n , such that it is optimal not to order for any inventory level higher than Y_n , and order up to full capacity for all inventory level lower than $Y_n - Cp \equiv X_n$, where Cp is limit on order size, is extended to a general model. This has also been defined as X-Y band control policy for a stochastic model (Chen, 2004a). After proving the applicability of optimal control policy to deterministic models, a new efficient algorithm is also developed to solve the model.

- (2) This research considers a special case of the multiple product system, in which the stochastic demand distribution, production rate, unit production cost, and periodic expected inventory cost are the same for all products respectively. The second objective is to extend the existing control policy for two products system (Chen, 2004b) to multiple products system, and characterize the hedging point policy for the new model.
- (3) The third objective is to conduct the parameter analysis of setup cost, unit production cost and total demands for the backlogging-allowed Wagner-Whitin model. This research obtains new results and extends the results obtained by Van Hoesel and Wagelmans (1993) from the model where backlogging is not allowed into one where backlogging is allowed.

A fundamental methodology in this research is trying to prove that an optimal control policy exists for the investigated model by using dynamic programming.

1.5 Thesis Organization

This thesis is organized as follows. Chapter 2 provides a literature review on the fundamental problems and research development in production planning and inventory control modeled with stochastic and deterministic dynamic programs. Chapter 3 explains the integration of three different models which are investigated in this thesis. Chapter 4 first extends Chen and Lambrecht (1996) and Chen (2004a)'s results into a more general stochastic model, and subsequently applies the new results into the complementary deterministic model. Chapter 5 is devoted to a special case of multiple product stochastic programming problems. In Chapter 6, the parameter analysis is conducted for a basic

uncapacitated, deterministic, dynamic model where backlogging is allowed. New results are obtained for the model parameters of setup cost, total demands, and unit production cost. Chapter 7 summarizes this research and concludes with recommendations for future research.

Chapter 2

Literature Review

2.1 Introduction

This chapter reviews the research development for both the stochastic and deterministic dynamic models in production planning and inventory control. Numerous papers have been published in both research streams. Kuik, et al. (1994) classify the models by the leadtimes, product structure, planning horizon, service policy and objective function. Drexl and Kimms (1997) focus on the capacitated and deterministic cases. Karimi, et al. (2003) introduce the factors affecting the formulation of the models which are the planning horizon, the number of echelons, the number of products, the capacity or resource constraints, the deterioration of the products, the demand patterns, the setup structures and the service policies. Nevertheless, only the single stage models are reviewed in their paper.

A basic model can be built by considering different factors and cost functions. Numerous mathematical methods have been applied in solving the models. Specifically, Lagrangian relaxation, branch and bound, linear programming, integer programming, dynamic programming and other optimization techniques in operations research such as shortest path, minimum cost network flow are common tools as reviewed in the preceding review articles. In this thesis, dynamic programming is used commonly to prove the control policy and develop algorithms for the investigated models. Actually, dynamic programming has been extensively applied in inventory control modeling. As pointed out

by Smith (1999), one important use of dynamic programming is that it allows the modeler to prove that there is an optimal control policy for the system. All the preceding articles do not review literatures from the viewpoint of control policies and algorithms for the fundamental models in production planning and inventory control. This chapter focuses on above.

2.2 Deterministic Dynamic Models

Problems in this research field are generally referred to as Economic Lot-Sizing Problems (ELSP) (Drexl and Kimms, 1997; Karimi, et al, 2003). The objective of ELSP is to determine the production quantities for each period at a minimum cost or maximum profit in order to satisfy the known demands for one or more than one specific commodity in a number of consecutive periods (the planning horizon). The problem may allow backlogging if it is permitted to produce or order later to satisfy the demand of a prior period. In each period, three costs are considered: fixed setup costs, production and inventory costs. The cost functions for production and inventory are non-decreasing in the amount produced, stored or backlogged, and are usually assumed to be linear, fixed-charge, concave or convex functions. The production can be uncapacitated or capacitated.

Bitran and Yanasse (1982) created the notation of $\alpha/\beta/\gamma/\delta$ to represent an ELSP model, where α , β , γ , and δ specify respectively a special structure for the setup costs, holding costs, production costs and capacities. α , β , γ , and δ can be taken equal to the following letters: G, C, ND, NI, and Z if the parameter under consideration is assumed over time to follow no pre-specified pattern, be constant, non-decreasing, non-increasing, and have a

value zero respectively. Capacity is equal to zero means that the capacity is not a constraint in the model. Bitran and Yanasse (1982)'s notation is extended to include more cost patterns: α , β , γ , and δ can also be CA, CV, PCA, and PCV if the parameter is a concave, convex, piecewise concave and piecewise convex functions over time respectively. Additionally, if the model is for more than one product, the number of products can be expressed at the beginning of the notation. For instance, the symbol 2/G/Z/ND/Ni indicates that the class of models with two products, where the setup costs of the products do not necessarily follow a pre-specified pattern, the holding and unit production costs of both products are respectively equal to zero and non-decreasing, and the capacities are non-increasing over time.

Wagner and Whitin (1958) extended traditional EOQ model into the no-backlogging G/C/C/Z model. They proved that for such a system, there exists an optimal control policy which could be demonstrated as follows.

- (1) it is optimal to produce or order enough products to exactly satisfy the demands for a whole number of time periods, and
- (2) the production or order should only be made when the inventory level is zero.

Interestingly, based on the preceding optimal control policy, Wagner and Whitin (1958) transformed the original problem into the form of a shortest path problem, and proposed an algorithm with time complexity of $O(N^2)$ (N is the length of horizon) for solving the shortest path problem, and also the original one.

Wagner and Whitin (1958)'s work has attracted much research interests in the models with deterministic demands as well as stochastic demands. Many investigations have focused on extensions of the basic Wagner and Whitin model. Some basic extensions are:

- (1) complex cost functions, e.g. the production cost function may be concave for considering learning effect (learning effect means that the worker 'learns' to get used to the new setup, procedures and materials. Scrap costs are likely to be higher than normal while the worker adjusts to the new procedures, gradually reaching full efficiency (Silver and Peterson, 1985)), the inventory cost may be convex for considering Just-In-Time management philosophy or concave for considering a quantity discount (Scarf, 1960; Scarf, Gilford and Shelly, 1963; Wagner, 1972; Chand and Sethi, 1990; Van Hoesel and Wagelmans, 1993),
- (2) different policies with respect to holding inventory are allowed, for instance backlogging is allowed (Scarf, 1960; Veinott, 1966a; Florian and Klein, 1971; Aggarwal and Park, 1993; Lee, et al, 2001),
- (3) incorporation of production and inventory capacities, for example a bound on the number of product that can be produced or held in inventory in a single period (Florian and Klein, 1971; Florian, et al, 1980; Chen and Lambrecht, 1996; Chen, 2004a),
- (4) multiple products are considered in the model (Johnson, 1967; Evans, 1967; Kalin, 1980; Atkin and Sun, 1995; Chen, 2004b), and
- (5) multiple echelons (Clark and Scarf, 1960; Federgruen and Zipkin, 1984; Chen and Zheng, 1994; Chen, 2000).

2.2.1 Optimal Policies and Algorithms

Zangwill (1966, 1969) proved that the control policy in Wagner and Whitin (1958) also holds for the more general model of G/PCA/CA/Z where backlogging could be allowed. He defined the following definition of exact requirements to characterize the policy.

Definition of Exact Requirements: a production schedule is said to satisfy exact requirements if there are integers $0 = s_0 \leq s_1 \leq \dots \leq s_i \leq \dots \leq s_n = N$ such that $x_i = \sum_{h=1+s_{i-1}}^{s_i} r_h$, where x_i is the production quantity in period i , and r_i is the demand for period i .

The control policy, which is referred to as *exact requirements policy*, can be demonstrated as follows.

- (1) The production schedule satisfies the exact requirements,
- (2) The inventory at the end of period i , I_i also satisfies exact requirements. Specifically,

$$I_i = \sum_{h=i+1}^{s_i} r_h, \text{ and}$$

- (3) For an optimal production schedule, the following inequalities will not hold for any period i concurrently: $x_i > 0$, and $I_{i-1} > 0$. This means that a production or an order is never made when the inventory level is positive.

The control policies for the ELSP models with capacity constraint are more complex than the ones without capacity constraint. Florian and Klein (1971) first gave the following definition of capacity constrained production schedule to characterize the control policy for the model class of G/CA/CA/G.

Definition of Capacity Constrained Production Sequence: If the inventory at the end of period t equals to zero, i.e. $I_t = 0$, then the period t is called a *regeneration period*. Further, let S_{uv} represent a subset of a feasible production plan that includes all the periods between the two consecutive regeneration periods u and v , i.e.

$$S_{uv} = \{x_i, i = u+1, \dots, v \mid I_u = 0 = I_v, I_i \neq 0 \text{ for } u < i < v\}.$$

Sequence S_{uv} is called capacity constrained if the production level in at most one period is positive but less than the capacity, and all other production levels are either zero or at their full capacities.

Furthermore, Florian and Klein (1971) proposed the following control policy properties for the models of G/CA/CA/G.

- (1) Inventory decomposition property: If at a period k , its inventory I_k is zero, and the problems for the time horizons from 1 to k and from $k+1$ to the end period N are both feasible, then an optimal solution to the original problem can be found by independently solving the problems for the first k periods and for the last $(N-k)$ periods, and
- (2) There exists an optimal production plan that only consists of capacity constrained sequences.

Florian and Klein (1971) also proved that the preceding properties hold for backlogging-allowed G/CA/CA/G models as well as backlogging-not-allowed ones. Although the properties characterized the optimal solution structure, the authors pointed out that finding an optimal capacity constrained production sequence was, in general, a

combinatorial problem. For the special case where the production capacities were equal for every period, the authors found that it could also be transformed into a shortest path problem, and solved at $O(N^4)$ time. ($O(N^4)$ time means that there exists a constant $c \geq 0$ such that model solving time $\leq c \cdot N^4$ for all $N > 0$). A model's computational complexity is related to the algorithm designed for its solution, and is measured by the running time of the algorithm. If a problem of size n can be solved with running time $O(p(n))$ where p is a polynomial function, then the algorithm may be called 'good' and the problem 'well solved'. On the other hand, many problems require exponential time (Florian, et al, 1980).). In the case that the capacities are not equal, the problem is NP-hard, which was proved by Florian, et al. (1980).

Florian, et al. (1980) and Bitran and Yanasse (1982) did the pioneering study on ELSP from the viewpoint of computational complexity theory. They proved that for a general capacitated ELSP, the problem with equal demands and zero storage costs was still NP-hard in each of the following cases.

- (1) arbitrary cost functions, no setup costs, no capacity limits,
- (2) concave cost functions, no setup costs, arbitrary capacity limits, and
- (3) convex cost functions, unit setup costs, no capacity limits.

This result leaves only a few possibilities for truly polynomial algorithms. Thus, most models of capacitated ELSP are NP-hard.

Bitran and Yanasse (1982) further proved that the following single product ELSP could be solved in polynomial time:

- (1) NI/G/NI/ND can be solved at $O(N^4)$ time,
- (2) C/Z/C/G can be solved at $O(M \log N)$ time, and
- (3) ND/Z/ND/NI can be solved at $O(N)$ time,

and the following problems were all NP-hard:

- (1) C/Z/NI/NI,
- (2) C/Z/ND/ND,
- (3) ND/Z/Z/ND,
- (4) NI/Z/Z/NI,
- (5) C/G/Z/NI, and
- (6) C/C/ND/NI.

Furthermore, for the multiple product ELSP, Bitran and Yanasse (1982) first proved that the following problem classes were NP-hard.

- (1) 2/C/Z/Z/ND,
- (2) 2/C/Z/Z/NI,
- (3) 2/C/Z/NI/C, and
- (4) 3/C/C/Z/C.

2.2.2 Efficient Algorithms and Computational Complexity

After Florian and Klein (1971)'s work, not much breakthrough has been made in finding optimal control policies for the basic deterministic models. Due to the increasing interests in computational complexity, intensive studies have focused on improving algorithm efficiency. Furthermore, new significant progresses were made at the beginning of

1990's. Aggarwal and Park (1993), Federgruen and Tzur (1991), and Wagelmans, et al. (1992) discovered independently that Wagner and Whitin's model could be solved in $O(N \log N)$ and in some special cases even in linear time.

By reformulating the Wagner and Whitin problem into a model without holding cost and expressing the dynamic programming geometrically, Wagelmans, et al. (1992) developed a technique to solve the problem in $O(N \log N)$. Van Hoesel, Wagelmans, and Moerman (1994) proved that similar geometric techniques could also be applied to the model where backlogging was allowed, and the extended model could also be solved in $O(N \log N)$ time. Federgruen and Tzur (1991)'s forward algorithm is based on the key observation that for any pair of periods $k < l$, the difference function $\Delta_{k,l}(t) = F(k,t) - F(l,t)$ ($F(m,t)$ is the minimum cost in the first t periods, if the final setup is performed in period m) is monotone in $t \geq l$. Federgruen and Tzur (1993) extended the result to the model where backlogging was allowed.

Aggarwal and Park (1993) identified the Monge arrays (a $m \times n$ two-dimensional array $A = \{a[i, j]\}$ is said to satisfy the Monge condition if for $1 \leq i < m$ and $1 \leq j < n$, $a[i, j] + a[i+1, j+1] \leq a[i, j+1] + a[i+1, j]$.) that arose in connection with the general model classes of G/CA/CA/Z. It is the properties of Monge arrays that achieve the computation improvement dramatically by applying known array searching techniques.

Lee, et al. (2001) study the uncapacitated dynamic lot-sizing model with demand time windows. Their model differs from the classical dynamic lot-sizing model in the sense

that the demand is not limited to a specific period. There is a grace period for timely delivery or production replenishment. During this grace period which is also called demand time window, the demand is satisfied with no penalty. Lee, et al. (2001) consider two situations where backlogging is allowed in the first case, and not allowed in the second situation, and develop polynomial algorithms at the efficiency of $O(N^3)$ and $O(N^2)$ respectively.

Chen, et al. (1994a) applied similar geometric techniques of Wagelmans, et al. (1992) into capacitated ELSP with linear cost functions. Although the computation complexity was also exponential theoretically, the authors stated that the algorithm was very efficient for most test problems. In a subsequent paper, Chen, Hearn, and Lee (1994b) extended the findings of their earlier work into the model where cost functions are piecewise linear with finite segments.

Shaw and Wagelmans (1998) consider the model with piecewise linear production and general holding costs and provide a better pseudo-polynomial time algorithm. They express the inventory as a variable in the objective dynamic function and prove that computation can be reduced to $O(N^2qd)$, where q is the average number of pieces required to represent the production cost functions, and d is the average demands over the planning horizon. The authors claim that when the planning periods and q become larger, the algorithm is more efficient than that given by Chen, et al. (1994a, 1994b).

Van Hoesel *et al.* (2002) claim that in the context of supply chain optimization, it is necessary to extend the classical ELSP to consider transportation decisions, as well as the possible inventories held at the different echelons in the supply chain. The authors investigate some models with stationary production capacities and allowing for multiple intermediate level storage and transportation. For the models with concave production costs and linear holding costs, the authors identify some special cases that can be solved in polynomial time, and also give the respective algorithms.

2.2.3 Sensitivity Analysis and Heuristic Algorithms

Sensitivity analysis for the ELSP has also attracted much interests. The research findings in this field are briefly reviewed here. Based on their geometric technique for the G/C/C/Z model in which backlogging is not considered, Van Hoesel and Wagelmans (1993) conducted the sensitivity analysis of the model parameters. They presented the algorithms and computational complexities for computing the lower or upper bounds of every parameter variation scope. Chand and Sethi (1990) studied Wagner and Whitin's model with learning in setup. Their main result is that the minimum holding cost for a given interval of periods declines at a decreasing rate for an increasing number of setups. Chand and Vörös (1992) applied the techniques of Chand and Sethi (1990) to the similar model except that backlogging was allowed, and the setup cost was fixed for all periods. They proved that the minimum cost of holding and backlogging for the T -period problem with n setups ($HB(T, n)$) is a non-increasing convex function of n , where T -period problem means that the planning horizon is T for the same problem. Subsequently, they proved that for any optimal setup number n to the problem, the lower bound of the setup

cost stability is obtained by $HB(T, n) - HB(T, n+1)$, and the upper bound by $HB(T, n-1) - HB(T, n)$. The computational complexity for all these $HB(T, n)$ values was proved to be $O(T^3)$.

Zangwill (1987) also studied the setup cost stability problem for the general uncapacitated ELSP without backlogging. Zangwill's main motivation was to analyze the concepts of the zero inventory philosophy, which states that the inventory level should be as small as possible and that this can be accomplished by reducing the setup costs.

Van Hoesel and Wagelmans (2000) also investigate the effects of reducing all setup costs by the same amount for all production periods in a special case of Wagner and Whitin's model in which there are no speculative motives to hold inventory. When the setup cost reduction is increasing, solution with more production periods becomes relatively attractive since fewer inventories will be carried between periods. They prove that change in production periods is in a very structured way.

Due to the fact that most capacitated ELSP are NP-hard, some investigators try to propose efficient algorithms to obtain a heuristic solution. The works in this field are contributed mainly by Bitran and Matsuo (1986), Gavish and Johnson (1990), and Van Hoesel and Wagelmans (2001). The fundamental differences between the works of Bitran and Matsuo (1986), Gavish and Johnson (1990) and that of Van Hoesel and Wagelmans (2001) can be summarized as follows.

- (1) Van Hoesel and Wagelmans (2001) prove that a full polynomial approximation algorithm exists for the G/CA/CA/G model, and
- (2) The algorithm in Van Hoesel and Wagelmans (2001) can also determine a feasible solution with a value not larger than $(1+\varepsilon)Z^*$ where Z^* is the optimal costs or profits for the problem, and ε can be any positive real value.

Chan *et al.* (2002) investigate the classical ELSP with a special ordering cost function namely the modified all-unit discount function in their research article. They propose an order policy and prove that the policy can be implemented in polynomial time and the result is no more than $4/3$ times the overall optimal cost. There are also some heuristic policies that have been established for multi-echelon system, for instance, Mitchell (1987), Atkin and Sun (1995), and Chen (1998).

2.3 Stochastic Dynamic Models

It is interesting to observe that in the preceding section, the basic control policies are obtained mainly from the models with concave cost functions. This is due to the fact that the model constraints define a closed convex set for the feasible solutions, and therefore the minimum of the concave objective function can be attained at an extreme point of the convex set. Conversely, in this section for the review of stochastic dynamic models, the inventory cost function will be assumed to be convex for most basic models. This is due to the fact that the expected inventory cost in one period is a convex function if the unit holding and backlogging costs are constants (Wagner, 1972).

2.3.1 Models with Single Product

This section only focuses on periodic review system which can be described as follows (Chen and Lambrecht, 1996): At the beginning of each time period, upon a review of the current inventory levels, the decision is made on how much to order; an order, if placed, will be delivered after some time periods which is called lead-time. At each time period, the demand for the product is a random variable, and is independently and identically distributed (i.i.d.) from period to period. The per-period expected holding and backlogging inventory cost function is assumed to be convex. A most fundamental challenge is to find the optimal order policy for such a system, or the special order pattern which characterizes the optimal control policy.

Since Bellman (1957) developed dynamic programming, researchers have found that it is a very powerful tool to tackle the preceding challenge especially for stochastic dynamic models. After Arrow, et al. (1951)'s pioneering work of applying dynamic programming into inventory management, numerous papers have appeared depending on the treatments of the various problem structures with respect to the cost functions, capacity, planning horizon, and so on. A few basic theorems have been established for some essential model structures.

If only one product is considered, the first basic model is that the production cost is linear, setup cost is not considered, and the production capacity or order quantity can be unlimited, then the order policy is described as *base-stock policy*: there exists a single critical number noted as S , when the reviewed inventory level is below S , enough should

be produced or ordered to bring the inventory level to S ; if the reviewed inventory level is equal to or higher than S , then it is optimal to order nothing. This result is described in Scarf (1960), Scarf, et al. (1963), and Veinott (1966a, 1966b).

Based on the first basic model, the second basic model includes the constraint for the capacity. The optimal control policy is found by Federgruen and Zipkin (1986) and named as a *modified base stock policy*. This policy is to follow the base stock policy if necessary, when the capacity constraint is reached, then produce or order up to full capacity.

Also based on the first basic model, the third basic model considers the setup cost. Scarf (1960), Veinott (1966a) independently proved that the optimal policy had the following pattern: when the inventory level is lower than a critical number s , enough should be ordered or produced to bring the inventory level up to another critical number S . Otherwise, nothing is necessary to be produced or ordered. This order pattern is called (s, S) policy. It is worth to point out that Scarf (1960) introduced the notation of K -convex to establish the optimality theorem of (s, S) policy, and Veinott (1966a)'s proof did not use K -convex. This study describes Scarf's notation of K -convex and Chen (2004a)'s notation of (Cp, K) -convex in detail in the next chapter.

The fourth case is to consider the setup cost and capacity constraint simultaneously. Chen and Lambrecht (1996) first find that such a model follows a control pattern of global X-Y band: there exists a pair of global X-Y band for all periods, if the reviewed inventory

level is equal to or lower than a critical number X , it is optimal to order full capacity; if the reviewed inventory level is higher than another critical number Y , then it is optimal to order nothing. Chen (2004a) further characterizes the order pattern and proves that in each time period, there exists a pair of local lower band X and upper band Y . If the reviewed inventory level is equal to or lower than X , it is optimal to order up to full capacity; if the reviewed inventory level is higher than Y , then it is optimal to order nothing. Furthermore, the periodic X - Y band is at most one capacity wide. Chen (2004a) also introduces the notation of (C_p, K) -convex to establish the optimality of the X - Y band policy. Additionally, both Wijngaard (1972) and Chen and Lambrecht (1996) have demonstrated in their findings that modified (s, S) policy is not optimal for the model. The preceding four basic models and their optimal policies can be described in the following Figure 2.1, where K is the setup cost.

Figure 2.1 Four Basic Single Product Deterministic Models and Optimal Policies

2.3.2 Models with Multiple Products or Echelons

Research studies go further to consider more than one product in the model. Johnson (1967) and Kalin (1980) established the optimality of a multi-dimensional (s, S) policy for the multi-product model without considering the capacity constraints. For the capacitated multi-product model, the policy is far more complex. Evans (1967) may be the first one who considers the capacitated multiple-product, periodic review and stochastic demand systems in a finite horizon. Under the assumptions that the one period expected cost function was strictly convex and of second order differentiable, he showed that the optimal policy could be partially characterized by base-stock policy: When all product inventory levels are above the base stock levels, it is optimal to order nothing. When all product inventory levels are below the base stock level and there is sufficient capacity, it is optimal to order up to the base stock levels. Evans (1967) only partially characterized the order policy, and described the existence of various regions in which it was optimal to produce one or more of the products and use all of the available resource in a given period.

After Evans (1967)'s work, the model has been investigated intensively in research. DeCroix and Arreola-Risa (1998) extend Evans (1967)'s results into the infinite horizon case, Pena-Perez and Zipkin (1997) provide a heuristic policy for the system. Ha (1997) proves the optimality of the hedging point policy for the two product system in which the two products have the identical production time. Wein (1992) proposes an approximation for the multiclass queuing control problem. The solution suggests particularly the optimality of a hedging point policy when all products are backlogged. Srivatsan and

Dallery (1998) and de Vericourt, et al. (2000) provide a partial characterization of the optimal hedging point policy by different techniques. Gershwin (1994) provides a systematic review on research works in the area and also suggests the optimality of hedging point policy.

Chen (2004b) fundamentally proves the optimality of the hedging point policy for the system with two products which can be described as follows: there are two curves which intersect at one hedging point, and therefore divide the two dimensional plane into three distinct regions, say Region 0, 1 and 2 in the following Figure 2.1.

Figure 2.2 General Illustration of Hedging Point Policy

According to the hedging point policy of Chen (2004b), it is optimal to order nothing in Region 0, order products 1 and 2 respectively in Regions 1 and 2 only. Chen (2004b) also provides the definitions of the two curves and the hedging point. These are described in detail in later chapters.

When considering setup times and capacity constraints simultaneously in the multiple product models, the systems become much more complex. Federgruen and Katalan (1998), Anupindi and Tayur (1998) and Stadtler (2003) propose some heuristic solutions. It is still a challenge for academics to find the control policy for such a system.

Research is also extended to consider multiple echelons in the model. Most current literatures in this field only consider one product, while the models considering more than one product are far more complex and very rare. One basic model is an N -stage serial supply chain system with deterministic transportation lead time between stages. Stationary random demand occurs at stage 1, which obtains the supply from stage 2, stage 2 obtains supply from stage 3, and so on. Stage N obtains supply from an outsider which has infinite stock. The first pioneer work in this field was done by Clark and Scarf (1960), who showed that an echelon base stock policy was optimal for the finite horizon problem. Federgruen and Zipkin (1984) and Chen and Zheng (1994) extend Clark and Scarf (1960)'s result into the infinite horizon problem. Chen (2000) proves that the base stock policy is still optimal when the order in every stage is in batches. Dong and Lee (2003) further prove that the optimal policy is still true for the model when the demand is time correlated and predicted by a commonly used time evolution model.

2.4 Summary

In this chapter, two main research fields in production planning and inventory control are reviewed mainly from the viewpoint of control policies and algorithms for the basic models. In one research stream for deterministic dynamic models, exact requirement

policy, and capacity constrained production sequence policy are described in detail. In the other stream for stochastic dynamic models, base stock, modified base stock, (s, S) , X-Y band and hedging point policies are reviewed in detail. It can be seen that using dynamic programming, optimal control policies have been discovered and applied to develop efficient algorithms for some basic models. With the incorporation of more than one echelon and more than one product, it is still unknown for many basic models how the dynamic stochastic or deterministic models evolve over time.

Chapter 3

Description and Integration of Three Research Models

3.1 Model Description

As described in Chapters 1 and 2, usually a model of production planning and inventory control problem can be regarded as a configuration of the following seven factors: planning horizon, echelon numbers, capacity or resource constraints, products number, demand type, setup structure, and inventory control policy. Current research studies are focusing on considering multiple echelons, multiple products, different policies with respect to the inventory control, and incorporating complex cost functions and/or production and inventory capacities constraints.

The three models, which are investigated in this research, are also three different configurations of above seven factors. All three models consider finite and multiple planning horizons, and allow inventory backlogging. In the first model, a single product is taken into consideration, and the production capacity for every period is incorporated and assumed to be equal. For the cost functions, it is assumed that the setup cost is non-increasing periodic, production cost is linear, and especially, the inventory cost function is convex. The model first assumes that the demand is stochastic. After Chen (2004a)'s (Cp, K) -convex is proved to be true for the objective function of the model with stochastic demand, the demand can be regarded as deterministic as it is a special case of stochastic demand. Chen (2004a)'s (Cp, K) -convex is easily proved to hold for the model

with deterministic demand. By this theoretical result, a new algorithm is developed to solve the deterministic model.

As reviewed in Chapter 2, not many research studies have investigated the deterministic models with convex cost functions, and identified the control policies of such models. Although Shaw and Wagelmans (1998) propose a pseudo-polynomial time algorithm for the model with piecewise linear production and general holding cost, they do not provide a control policy for their investigated model. On the other hand, Florian, Lenstra and Rinnooy Kan (1980) proved that the above deterministic model is NP-hard even without capacity constraint. Therefore, finding control policies and more efficient algorithms for such a model or more general models are the challenges in research academia.

The second model investigated in this research work considers multiple products with only stochastic demand. Production cost is linear and setup cost is not incorporated. The periodic expected inventory cost function is assumed to be convex. Production capacity is also incorporated. Especially, all model parameters are the same for each product. That is, the stochastic demand distribution, production rate, unit production cost and periodic expected inventory cost function are the same for every product. By proving that between two products the product with lower inventory level will have higher order/production priority compared with the other product, and several other results, hedging point policy is proposed to characterize the second model. Nevertheless, how to extend the above result or Chen (2004b)'s general two-product model into the general one of multiple products is not straightforward, and still a tough challenge in academia. Combining the

results from the special multiple product model with those from Chen (2004b)'s general two-product model would probably give some clues to the optimal policy of general multiple product model.

The third model is actually a typical economic lot-sizing problem where production capacity is not considered. Moreover, the demand is deterministic, the setup cost is incorporated and the production cost is linear. The periodic inventory cost is also linear with different unit holding and backlogging costs. The main objective is to analyze the parameter variation of setup cost, unit production cost and total demand respectively, such that the optimal production schedule remains unchanged. In fact, Van Hoesel and Wagelmans (1993) did the similar work. However, one limitation of their work is that backlogging is not allowed in their model; secondly, most of their results are computational, not theoretical. The research conducted in this thesis relaxes their limitation that backlogging is not allowed, and achieves more theoretical results.

3.2 Integration of the Three Research Models

Obviously, a model of production planning and inventory control usually considers seven factors which are described in the preceding section. Different scenarios of production and inventory control give rise to different configurations of the seven factors, and that in turn, produces different models, some of which have been solved, and the rest remain unsolved. For instance, Wagner and Whitin (1958) studied a single-product and multiple-period model which is very common in the manufacturing environment. The model considers deterministic demand, setup cost, linear production and holding costs, and does

not allow backlogging. They characterized the model with exact requirement policy and proposed an algorithm with $O(N^2)$ time. However, if the holding cost is convex rather than linear, then the new model is only proved to be NP-hard by Florian, Lenstra and Rinnooy Kan (1980), and no optimal control policies based on efficient algorithms have been identified and developed.

Thus, the three different models investigated in this research work are still in the family of production planning and inventory control and are three different configurations of the seven factors of production planning and inventory control model.

Furthermore, a fundamental methodology in this research work is to prove that an optimal control policy exists for the investigated model by using dynamic programming. Further exploiting the properties of obtained optimal control policy could most possibly develop an efficient solution algorithm for the model. Although the three investigated models are different, they all follow the above methodology.

Additionally, in the real business world, many models are with multiple echelons, multiple products, complex cost structures, different inventory strategies and capacity constraints. Such models reflect the real supply chain or supply network in business. How to optimize such models and develop efficient solutions is a tough challenge in academia. However, characterizing the models such as the three investigated in this research and extending the models to more general one can achieve this objective.

3.3 Summary

This chapter describes the three research models investigated in this research, and their closely related results. The integration of the three models is also explained. Basically, the three models belong to the same family of production planning and inventory control problems, and the research methodology for the model solution is the same.

Chapter 4

Single Product Periodic Review Problem with Setup/Ordering Cost, Equal Capacity Constraint and Convex Inventory Cost

4.1 Introduction

The single product, periodic review problem with stochastic demand in production and supply chain planning is a fundamental and classic research area in academia. The subject has been investigated intensively by numerous researchers and theories have been established for fundamental system configurations. As reviewed in Chapter 2, the essential theories are *base stock policy* (Scarf, 1960; Scarf, et al, 1963; Veinott, 1966a, 1966b), *(s, S) policy* (Veinott, 1966a; Veinott and Wagner, 1965; Wagner, 1972), *modified base stock policy* (Federgruen and Zipkin, 1986), and *X-Y band policy* (Chen and Lambrecht, 1996; Chen, 2004a) for the basic models respectively.

On the other hand, the complementary problem with deterministic demands also attracts much interest in research. Some fundamental theories have also been built and reviewed in the preceding chapter. Basically, they are the *exact requirements policy* for the un-capacitated problem (Wagner and Whitin, 1958; Zangwill, 1966), and *capacity constrained policy* for capacitated problem (Florian and Klein, 1971). Most recent results, for instance, Wagelmans, et al. (1992), Federgruen and Tzur (1991, 1993), Aggarwal and Park (1993), Chen, et al. (1994a, 1994b), Shaw and Wagelmans (1998), Lee, et al. (2001), focus on the algorithm efficiency improvement or model variations, and no further fundamental theorems or order policies have been discovered.

For the deterministic dynamic models, the research focuses on concave cost functions. For the convex cost functions, Florian, Lenstra and Rinnooy Kan (1980) and Bitran and Yanasse (1982) prove that despite equal demands, zero storage costs, unit setup costs and without capacity limits, the problem is still NP-hard. Although in Shaw and Wagelmans (1998)'s model, inventory cost can be general, the authors give an algorithm without characterizing the optimal policy for the model.

The main objective of this chapter is to characterize the deterministic dynamic model with single product, production capacity constraint, convex inventory cost, setup cost and backlogging allowed by X-Y band policy of Chen and Lambrecht (1996) and Chen (2004a). Based on this theoretical result, an efficient algorithm can be designed for solving the model. This chapter will first prove that the X-Y band policy holds for a more general stochastic model in which the periodic model parameters may have different values in section 4.4. Next, the policy will be stated that will be true for the complementary deterministic dynamic model by regarding the deterministic model as a special case of the stochastic one in section 4.5.2. Subsequently in section 4.5.3 a new algorithm is developed which is based on the policy for solving the deterministic model. The computational study reported in section 4.5.4 shows that the algorithm performs better than the other methods of model solution.

4.2 The Stochastic Demand Model Description, Notation and Formulation

The model to be investigated has equal capacity constraint for every planning period, linear unit production cost besides setup cost, and convex inventory cost. It is quite

common in real life, especially for inventory holding, which is discouraged in the commonly used JIT philosophy (JIT Philosophy was first introduced by the Toyota Motor Corporation. JIT (Just-In-Time) is a total system encompassing product design, supplier selection, materials management, quality assurance and productivity improvement (Silver and Peterson, 1985).). This section lists the necessary notations with descriptions first, and subsequently formulates the model by a dynamic program.

The following notations are defined for the model with stochastic demands. n is an integer, for $0 < n \leq N$:

N – planning horizon period.

Cp – production capacity, or limit on order size, a positive integer. The model to be investigated in this chapter will have the same capacity constraint for every period.

x_n – inventory level (on hand plus on order) prior to placing any order in period n .

y_n – inventory on hand plus on order subsequent to an order decision but before the demand occurs in period n . Obviously, $x_n \leq y_n$, and $y_n - x_n \leq Cp$ due to the capacity constraint Cp in every period.

D_n – discrete independently and identically distributed demand random variable in period n . The expected value $E(D_n)$ is constant and $E(D_n) < \infty$.

c_n – unit production cost in period n .

α – single period discount factor, $0 < \alpha < 1$.

$K_n \delta(q) + c_n q$ – production or ordering cost for q units ($q \geq 0$), where $K_n (>0)$ is the setup cost in period n . $\delta(x)$ is the binary function, when $x > 0$, $\delta(x) = 1$; $x = 0$, $\delta(x) = 0$.

$l_n(y)$ – one period expected inventory holding and shortage penalty cost function at the end of period n . See, for instance, Wagner (1972), for the calculation of the function. $l_n(y)$ is assumed to be a convex function, polynomially bounded, and $\lim_{y \rightarrow \pm\infty} l_n(y) = \infty$.

$f_n(x)$ – expected optimal discounted cost for periods n through N , given that beginning inventory level in period n is x .

Suppose B denotes the present value at the beginning of period 1, of all costs that are incurred during periods 1, 2, ..., N . c_{N+1} is the salvage value per unit of x_{N+1} if $x_{N+1} \geq 0$. If $x_{N+1} < 0$, c_{N+1} is the penalty cost per unit of backlogged order demands. The following process to obtain the dynamic program (4.4) below is from Heyman and Sobel (1984, section 7.1).

$$B = \sum_{n=1}^N \alpha^{n-1} [K_n \delta(y_n - x_n) + c_n (y_n - x_n) + l_n(y_n)] - \alpha^N c_{N+1} x_{N+1}$$

Substitution of $x_n = y_{n-1} - D_{n-1}$ for $n > 1$ in the preceding equation,

$$B = \sum_{n=1}^N \alpha^{n-1} [K_n \delta(y_n - x_n) + l_n(y_n) + (c_n - \alpha c_{n+1}) y_n] - c_1 x_1 + \sum_{n=1}^N \alpha^n c_{n+1} D_n$$

$$E(B) = E \left[\sum_{n=1}^N \alpha^{n-1} [K_n \delta(y_n - x_n) + l_n(y_n) + (c_n - \alpha c_{n+1}) y_n] \right] - c_1 x_1 + \sum_{n=1}^N \alpha^n c_{n+1} E(D_n)$$

(4.1)

The second and third terms in equation (4.1) will not be affected by the policy for choosing the y_n at period n ; thus, the objective is to minimize the first term which is as follows.

$$E\left[\sum_{n=1}^N \alpha^{n-1} [K_n \delta(y_n - x_n) + l_n(y_n) + (c_n - \alpha c_{n+1}) y_n]\right] \quad (4.2)$$

If the costs of K_n , $l_n(y_n)$ and c_n in equation (4.2) are assumed to be discounted with respect to period n , then the money discount factor can be omitted and equation (4.2) becomes:

$$E\left[\sum_{n=1}^N [K_n \delta(y_n - x_n) + l_n(y_n) + (c_n - c_{n+1}) y_n]\right] \quad (4.3)$$

The equivalent dynamic program of function (4.3) with the beginning inventory of x is as follows. Here, clearly the cost of $-c_1 x_1 + \sum_{n=1}^N \alpha^n c_{n+1} E(D_n)$ is not counted in $f_n(x)$ due to the same reason.

$$f_n(x) = \min_{y \in [x, x+cp]} \{K_n \delta(y - x) + l_n(y) + (c_n - c_{n+1}) y + E[f_{n+1}(y - D_n)]\}$$

$$f_{N+1}(x) = 0.$$

Define $e_n = c_n - c_{n+1}$

Then $f_n(x)$ is equal to the following expression.

$$f_n(x) = \min_{y \in [x, x+cp]} \{K_n \delta(y - x) + g_n(y)\}, \quad (4.4)$$

where

$$g_n(y) = e_n y + l_n(y) + E[f_{n+1}(y - D_n)] \quad n = 1, 2, \dots, N \quad (4.5)$$

From (4.4), $f_n(x)$ is actually the cost comparison of producing something ($y > x$) and producing nothing ($y = x$). This comparison can be described in equation (4.6).

$$f_n(x) = \min \begin{cases} g_n(x) \\ \min_{y \in (x, x+Cp]} g_n(y) + K_n \end{cases} \quad (4.6)$$

Suppose $\min_{y \in (x, x+Cp]} g_n(y) = g_n(y^*)$ where $y^* \in (x, x+Cp]$, then the optimal decision is as follows.

$$y_n^*(x) = \begin{cases} y^* & \text{if } g_n(x) \geq g_n(y^*) + K_n \\ x & \text{if } g_n(x) < g_n(y^*) + K_n \end{cases} \quad (4.7)$$

4.3 Previous Results for Models with Stochastic Demands

This section mainly describes the results obtained by Chen and Lambrecht (1996) and Chen (2004a), especially Chen (2004a)'s (Cp, K) -convex, and global and periodic X-Y band policies.

Chen and Lambrecht (1996) and Chen (2004a) investigated the order policy for the model with discounted costs in the above section when the set up cost K_n , the distribution of stochastic demand D_n , unit production cost c_n , and the inventory cost $l_n(y)$ are the same respectively for every period in the planning horizon. In Chen and Lambrecht (1996), it is proved that the modified (s, S) order policy is not optimal for such a model, and the optimal policy exhibits a special pattern of global X-Y band for every period which can be described as follows. When the inventory level drops below X, the full capacity will be ordered; when the inventory level is higher than Y, then it is optimal to order nothing.

An analytical method for finding a pair of the global X and Y bands is also given in Chen and Lambrecht (1996).

Chen (2004a) further characterizes the order policy by defining the notation of the (Cp, K) convexity. The definition of (Cp, K) -convex is as follows (Chen, 2004a).

Definition of (Cp, K) -Convex: a function $F(x)$ is defined as (Cp, K) -convex if $F(x)$ satisfies:

$$\text{I. } K + F(x + a) \geq F(x) + \frac{a}{b} \{F(y) - F(y - b)\}, \quad 0 < a \leq Cp, 0 < b \leq Cp, y \leq x;$$

$$\text{II. } \frac{F(y - Cp) - F(y) - K}{Cp} \geq \frac{F(x) - F(x + a) - K}{a}, \quad y \leq x, 0 < a \leq Cp.$$

By proving that the model's system cost function for every period is (Cp, K) -convex, Chen (2004a) further characterizes the order policy as follows. For every period n , there exists a Y_n , such that it is optimal not to order for any inventory level $x > Y_n$, and order up to full capacity for all inventory level x if $x \leq Y_n - Cp \equiv X_n$. Thus, for each period, there exists a periodic $X_n - Y_n$ band, and such a band is at most Cp units in width.

Chen (2004a) also gives the following properties for (Cp, K) -convex functions.

(Cp, K) -Convex Properties: If $F(x)$ and $G(x)$ are (Cp, K) -convex, then

- I. $aF(x)$ is (Cp, K) -convex, $a > 0$,
- II. $F(x - a)$ is (Cp, K) -convex, for any a ,
- III. $F(x)$ is $(Cp, K + a)$ -convex, $a > 0$, and

IV. $\alpha F(x) + \beta G(x)$ is $(Cp, \alpha K + \beta K)$ -convex.

It is worth pointing out that the (s, S) policy can be characterized by Scarf (1960)'s notation of K -convex, which is described as follows.

K -Convex Definition: a function $F(x)$ is a K -convex function if $F(x)$ satisfies:

$$K + F(x+a) \geq F(x) + \frac{a}{b} \{F(x) - F(x-b)\}, \quad a \geq 0, \text{ and } b > 0.$$

Clearly Chen's (Cp, K) -convex is more general than Scarf's K -convex, and it also clearly characterizes the solution to the capacitated model which has not been solved for decades.

4.4 Application Extensions to Global X-Y Band and (Cp, K) -Convex

The objective of this section is to test whether the global X-Y band and (Cp, K) -convex which are described in Chen and Lambrecht (1996) and Chen (2004a) respectively can also hold for the model of dynamic program (4.4). It will be proved that with slightly different or additional assumptions, the global X-Y band and (Cp, K) -convex still hold.

4.4.1 Global X Band

This section proves that results similar to Lemma 1 in Chen and Lambrecht (1996) hold for dynamic program (4.4), and therefore global X band can be defined accordingly.

Define:

x_{li} : the value(s) at which $l_i(y)$ reaches its minimum, $i \in \{1, 2, \dots, N\}$

$$x_l = \min \{x_{li} \mid 1 \leq i \leq N\},$$

$$x_l' = \max \{x_{li} \mid 1 \leq i \leq N\} ,$$

x_{mi} : the value(s) at which $e_i y + l_i(y)$ reaches its minimum, $i \in \{1, 2, \dots, N\}$

$$x_m = \min \{x_{mi} \mid 1 \leq i \leq N\} ,$$

$$x_m' = \max \{x_{mi} \mid 1 \leq i \leq N\} ,$$

x_{si} : the maximum value of x that satisfies $e_i(x - Cp) + l_i(x - Cp) \geq e_i x + l_i(x) + K_i$,

and $x \leq x_{mi}$. This value is assumed to exist.

$$x_s = \min \{x_{si} \mid 1 \leq i \leq N\} ,$$

$$x_s' = \max \{x_{si} \mid 1 \leq i \leq N\} .$$

It will be proved that $x_s - Cp$ is a global X-band.

Lemma 4.1 The following results are true for dynamic program (4.4) for all $n \geq 1$,

- a. $f_{n+1}(x)$ is a non-increasing function for $x \leq x_m$,
- b. $g_n(y)$ is a non-increasing function for $y \leq x_m$,
- c. $g_n(y) \geq g_n(y + Cp) + K_n$ for $y \leq x_s - Cp \equiv X$,
- d. $y_n^*(x) = x + Cp$ for $x \leq x_s - Cp \equiv X$.

Proof: The proof is similar to Lemma 1 in Chen and Lambrecht (1996) for similar results.

It is conducted by backward induction and summarized as follows.

For $n = N$, $f_{N+1}(x) = 0$, so (a) is true. $g_N(y) = e_N y + l_N(y)$ is non-increasing function for $y \leq x_{mN}$. Because $y \leq x_m \leq x_{mN}$, (b) is true. As $g_N(y)$ is convex, thus for $y \leq x_s - Cp \leq x_{sN} - Cp$, $g_N(y) - g_N(y + Cp) \geq g_N(x_{sN} - Cp) - g_N(x_{sN}) \geq K_N$, (c) is true. Because $x_s \leq x_m$, so (d) can be drawn directly from (b), (c), (4.7).

Now, suppose for $n = i$, the lemma is true.

Next for $n = i - 1$.

For (a), define $F_i(x) = \min_{y \in (x, x+Cp]} g_i(y)$. By the same arguments in Lemma 1 of Chen and

Lambrecht (1996), $F_i(x)$ is a non-increasing function of x for $x \leq x_m$. By equation (4.6)

$$f_i(x) = \min \begin{cases} g_i(x) \\ F_i(x) + K_i \end{cases}$$

since both $g_i(x)$ and $F_i(x) + K_i$ are non-increasing functions for $x \leq x_m$, so (a) is true.

For (b), by (4.5) $g_{i-1}(y) = e_{i-1}y + l_{i-1}(y) + E[f_i(y - D_{i-1})]$. If $y \leq x_m$ and by (a), $E[f_i(y - D_{i-1})]$ is non-increasing, $e_{i-1}y + l_{i-1}(y)$ is also non-increasing for $y \leq x_m \leq x_{m(i-1)}$. Thus, (b) is proved.

For (c), if $y \leq x_s - Cp$, then by (a),

$$E[f_i(y - D_{i-1})] \geq E[f_i(y + Cp - D_{i-1})]. \quad (4.8)$$

By the definition of $x_{s(i-1)}$, $y \leq x_s - Cp \leq x_{s(i-1)} - Cp$ and the convexity of $e_{i-1}y + l_{i-1}(y)$,

$$[e_{i-1}y + l_{i-1}(y)] - [e_{i-1}(y + Cp) + l_{i-1}(y + Cp)] \geq$$

$$[e_{i-1}(x_{s(i-1)} - Cp) + l_{i-1}(x_{s(i-1)} - Cp)] - [e_{i-1}(x_{s(i-1)}) + l_{i-1}(x_{s(i-1)})] \geq K_{i-1}$$

$$\text{or } [e_{i-1}y + l_{i-1}(y)] \geq [e_{i-1}(y + Cp) + l_{i-1}(y + Cp)] + K_{i-1} \quad (4.9)$$

Adding equations (4.8) and (4.9), and by the definition of $g_i(y)$, the following holds.

$$g_{i-1}(y) \geq g_{i-1}(y + Cp) + K_{i-1}, \text{ so (c) is proved.}$$

Due to $x_s \leq x_m$, (d) follows directly from (b), (c) and equation (4.7). This concludes the proof. $\square$

The following proposition derives from Lemma 4.1.

Proposition 4.1 Under the assumption that x_{si} exists for every period $1 \leq i \leq N$, there exists a global lower band $X \equiv x_s - Cp$, such that for all $x \leq X$ and all $n \geq 1$, it is optimal to order or produce at the full capacity.

4.4.2 Global Y-Band

This section proves that the global Y-band exists for dynamic program (4.4) with additional assumptions. Define D_{max} as the maximum demand in all periods. Similar to that in Chen and Lambrecht (1996), the following two scenarios are considered.

Scenario 1: $D_{max} \leq Cp$;

Scenario 2: $D_{max} > Cp$.

Scenario 1 is analyzed first.

Similar to Proposition 1 in Chen and Lambrecht (1996), the following lemma for dynamic program (4.4) is true.

Lemma 4.2 For any $a \in [0, Cp]$, there exists a' , $a' \in [0, Cp]$, such that for any $n \geq 1$,

$$f_n(x) - f_n(x+a) \leq \max\{K_n, g_n(y') - g_n(y'+a')\} \text{ where } y' = x + Cp.$$

Proof: The proof is following the idea of Proposition 1 in Chen and Lambrecht (1996) and summarized as follows.

If at $x+a$, it is optimal not to order, then $f_n(x+a) = g_n(x+a)$. From (4.6)

$$f_n(x) \leq g_n(x+a) + K_n, \text{ thus } f_n(x) - f_n(x+a) \leq K_n.$$

If at $x+a$, it is optimal to order, say b units, $0 < b \leq Cp$, then $f_n(x+a) = g_n(x+a+b) + K_n$. If $x+a+b \leq x+Cp$, by (4.6) $f_n(x) \leq g_n(x+a+b) + K_n$, and hence $f_n(x) - f_n(x+a) \leq 0$. If $x+Cp+Cp \geq x+a+b > x+Cp$, define $a' = a+b-Cp$, then $f_n(x) - f_n(x+a) \leq g_n(x+Cp) + K_n - g_n(x+a+b) - K_n = g_n(y') - g_n(y'+a)$. Clearly, $a' \in [0, Cp]$. This proves the lemma. $\square$

As stated in Chen and Lambrecht (1996), the Y band under the scenario of $D_{\max} \leq Cp$ is quite tight. The following two assumptions are necessary for applying Chen and Lambrecht (1996)'s results into dynamic program (4.4).

Assumption 1: $c_1 \geq c_2 \geq \dots \geq c_{N-1} \geq c_N \geq 0$,

Assumption 2: $K_1 \geq K_2 \geq \dots \geq K_{N-1} \geq K_N \geq 0$.

Under the preceding Assumption 1, the following inequality is true due to the respective definitions of x_{li} , x_{mi} , x_{si} , x_l' , x_m' , and x_s' .

for any $i \in \{1, 2, \dots, N\}$ $x_{li} \geq x_{mi} \geq x_{si}$, and therefore $x_l' \geq x_m' \geq x_s'$.

The following lemma will prove that x_l' is a global upper bound.

Lemma 4.3 If $D_{\max} \leq Cp$, then under the preceding Assumptions 1 and 2, the following results are true.

- $f_{n+1}(x) \leq f_{n+1}(x+a) + K_{n+1}$ for all $x \geq x_l' - Cp$, and $0 \leq a \leq Cp$;
- $g_n(y) < g_n(y+a) + K_n$ for all $y \geq x_l' \equiv Y$, and $0 \leq a \leq Cp$;
- $y_n^*(x) = x$ for all $x \geq x_l' \equiv Y$.

Proof: The proof is conducted by backward induction.

For $n = N$, $f_{N+1}(x) = 0$, so (a) is true. For $y \geq x_l' \geq x_m' \geq x_{mN}$, $g_N(y) = e_N y + l_N(y) \leq e_N(y+a) + l_N(y+a) = g_N(y+a)$, so (b) is true. (c) follows directly from (b) and (4.7).

Now suppose for $n = i$, the lemma is true.

Next for $n = i - 1$. By Lemma 4.2 and (b) in supposition, (a) is true.

For (b), since $y \geq x_l' \geq x_m' \geq x_{m(i-1)}$, $K_{i-1} \geq K_i$, $D_{\max} \leq Cp$ and by (a):

$$\begin{aligned} g_{i-1}(y) &= e_{i-1}y + l_{i-1}(y) + E[f_i(y - D_{i-1})] \leq e_{i-1}(y+a) + l_{i-1}(y+a) + E[f_i(y - D_{i-1})] \\ &\leq e_{i-1}(y+a) + l_{i-1}(y+a) + E[f_i(y+a - D_{i-1})] + K_i \leq g_{i-1}(y+a) + K_{i-1}, \text{ so (b) is true.} \end{aligned}$$

(c) is directly from (b) and (4.7). This concludes the lemma. $\square$

Next for scenario 2: $D_{\max} > Cp$

The following assumption is slightly different from those in Chen and Lambrecht (1996):

Assumption: $l_i(y) - l_i(y + D_{\max}) \leq \alpha^i (A + B |y - x_l|^\tau)$ for some non-negative integer τ , and for some positive constants A and B . As explained in section 4.2, the inventory cost function $l_i(y)$ is already discounted.

Define $M = \sum_{i=1}^{\infty} \alpha^i (A + B(iD_{\max})^\tau)$, Chen and Lambrecht (1996) prove that M is a finite

value, and U is the smallest integer which satisfies $\alpha^U M \leq \min\{K_i \mid i = 1, 2, \dots, N\}$.

The result that will be proved is that $Y = U * D_{\max}$ is one global upper bound.

The proof is following the similar process of Chen and Lambrecht (1996) and summarized as follows. Suppose at period j , the beginning inventory x is greater than or

equal to the preceding $Y = U * D_{\max}$, and there are N periods to go. Clearly if $N \leq U$, then nothing needs to be ordered and thus the result holds. Now suppose $N > U$, and there are only two possible policies to be considered.

Policy (1): order nothing,

Policy (2): order some units, say b units, $0 < b \leq Cp$.

It will be proved that Policy (1) is always better than Policy (2). The following strategy is used to analyze the cost of preceding two policies.

- I. Assume an optimal ordering path is to be followed by the system under policy (2) over the remaining $N-1$ periods;
- II. In each future period, the system under policy (1) will order exactly the same order quantity as that implied in I.

Define $f_n(x)$ and $f_n(x+b)$ as the expected costs under policies (1) and (2) for the rest of the planning horizon respectively, and y_i and y_i' as the respective inventory levels subsequent to an ordering decision but before the demand occurs in the period i under policies (1) and (2). By the preceding strategy, the following equations hold.

$$y_i' = y_i + b, \quad j \leq i \leq j + N$$

$$f_n(x) - f_n(x+b) = \sum_{i=j}^{j+N-1} [l_i(y_i) - l_i(y_i + b)] - K_j - c_j b$$

Because the ordering cost is the same for both policies, and the first shortage can only occur after U periods, $l_i(y_i) - l_i(y_i') \leq 0$ $i = j, j+1, \dots, j+U-1$, for the first U periods.

Thus

$$f_n(x) - f_n(x+b) \leq \sum_{i=j+U}^{j+N-1} [l_i(y_i) - l_i(y_i + b)] - K_j - c_j b$$

$$\leq \sum_{i=0}^{N-U-1} \alpha^{j+U+i} (A + B |y - x_l'|^\tau) - K_j - c_j b \leq \alpha^{j+U} M - K_j - c_j b < 0$$

This means that Policy (1) of ordering nothing is always better than Policy (2), so Y is a global upper bound. Clearly, the Y -band under the scenario of $D_{\max} > Cp$ is also applicable to the scenario of $D_{\max} \leq Cp$.

Concluding the results for the preceding two scenarios, the following result can be drawn.

Proposition 4.2

1. With the assumptions of non-increasing unit production cost and setup cost and $D_{\max} \leq Cp$, x_l' is a global upper bound, and is defined in section 4.4.1.

2. With the assumption of $l_i(y) - l_i(y + D_{\max}) \leq \alpha^i (A + B |y - x_l'|^\tau)$ for some non-negative integer τ , and for some positive constants A and B , define

$$M = \sum_{i=1}^{\infty} \alpha^i (A + B (i D_{\max})^\tau), \text{ and } U \text{ as the smallest integer that satisfies}$$

$$\alpha^U M \leq \min\{K_i \mid i = 1, 2, \dots, N\}, \text{ then } Y = U * D_{\max} \text{ is a global upper bound.}$$

4.4.3 (Cp, K) -Convex and Order Policy

The purpose of this section is to prove that dynamic program (4.4) is also (Cp, K) -Convex. In this section the following assumption is necessary to prove that (Cp, K) -convex holds for dynamic program (4.4).

$$K_1 \geq K_2 \geq \dots \geq K_{N-1} \geq K_N \geq 0 \tag{4.10}$$

In section 4.2, it is known that all K_n are the discounted setup costs.

Chen (2004a) proves the following order policy which will also be proved to be true for dynamic program (4.4) by proving Proposition (4.3) below.

(Chen (2004a), Lemma 1) If $g_n(y)$ is (Cp, K_n) -convex, then there exists Y_n , such that it is optimal not to order for any $x > Y_n$, and order up to full capacity for all $x \leq Y_n - Cp = X_n$. Thus, the X_n - Y_n band is at most Cp units wide.

Lemma 4.4 Suppose $g(x)$ is (Cp, K) -convex, then $f(x)$ is (Cp, V) -convex where

$$f(x) = \min_{y \in [x, x+Cp]} \{V\delta(y-x) + g(y)\} \quad V \geq K$$

Proof: Because (Cp, K) -convex means (Cp, V) -convex for any $V \geq K$, thus $g(\cdot)$ is (Cp, K) -convex means that $g(x)$ is (Cp, V) -convex. Theorem 3 of Chen (2004a) proves that if $g(x)$ is (Cp, V) -convex, then $f(x) = \min_{y \in [x, x+Cp]} \{V\delta(y-x) + g(y)\}$ is also (Cp, V) -convex.

This proves the lemma. $\square$

Proposition 4.3 $f_n(x)$ and $g_n(x)$ defined in dynamic program (4.4) are (Cp, K_n) -convex under the assumption that the setup cost is non-increasing.

Proof: The proof is conducted by induction. First for $n = N$, $f_{N+1}(x) = 0$ yields $g_N(y) = e_N y + l_N(y)$, both are convex and thus (Cp, K_N) -convex. Next for $n = i$, suppose $g_i(x)$ is (Cp, K_i) -convex, from Lemma 4.4, $f_i(x)$ is also (Cp, K_i) -convex. Lastly for $n = i - 1$, because $g_{i-1}(y) = e_{i-1}y + l_{i-1}(y) + E[f_i(y - D_{i-1})]$, $E[f_i(y - D_{i-1})]$ is (Cp, K_i) -convex due to the properties of (Cp, K) -convex, thus $g_{i-1}(x)$ is (Cp, K_i) -convex; since

$K_{i-1} \geq K_i$, therefore $g_{i-1}(x)$ is (Cp, K_{i-1}) -convex. From Lemma 4.4, $g_{i-1}(x)$ is (Cp, K_{i-1}) -convex means that $f_{i-1}(x)$ is also (Cp, K_{i-1}) -convex. This completes the proof. $\square$

Proposition 4.3 states that if the discounted setup costs for the time horizon is non-increasing, the order policy proved in Lemma 1 of Chen (2004a) will also hold for the model described in dynamic program (4.4).

The X-Y band policy is very meaningful for production or purchasing managers in the stochastic environment. Since every period has two critical X-Y values, when the inventory level is higher than Y, nothing is required to be done; if the inventory level drops below X, then producing or purchasing at the full capacity is always optimal.

4.5 The Complementary Deterministic Model

In this section, (Cp, K) -convex and X-Y band order policy described in section 4.4 will be applied to the complementary deterministic model which is a single item multiple period planning problem with equal capacity, deterministic demands, convex inventory cost function, and linear production cost. As reviewed in section 2.2 of Chapter 2, most fundamental results have been built for the cost functions, for instance, inventory or production cost functions are concave. Most current findings focus on algorithm complexity improvement and model variations, and not many production or order policies have been reported. This section will first point out that the results obtained in section 4.4 can be applied to the complementary model with deterministic demands,

subsequently an algorithm will be developed based on the results, and finally some computational issues will be discussed.

4.5.1 Previous Results

Although not many production or order policies have been found for the model with general or convex cost function, the model is still solvable by algorithms with higher computational complexity or more computational time. In this sub-section, a general model and its algorithm are described first (Florian, Lenstra and Rinnooy Kan (1980)), and subsequently a specific model and its model solving technique studied by Shaw and Wagelmans (1998) are explained.

The following are the additional notations for describing the general model.

d_i : the known deterministic demand at period i ($i = 1, \dots, N$),

CP_i : the production capacity limit at period i ($i = 1, \dots, N$),

q_i : the production or order quantity at period i ($i = 1, \dots, N$),

I_i : the inventory level at the end of period i ($i = 1, \dots, N$), if $I_i < 0$, it means that there is backlogging,

$h_i(I_i)$: the inventory cost at the end of period i ($i = 1, \dots, N$), it can be the holding cost or backlogging cost depending on $I_i > 0$ or $I_i < 0$. It is continuous and non-decreasing with the amounts of the inventory held or backlogged.

$p_i(q_i)$: the production cost at period i ($i = 1, \dots, N$), it can be any continuous and non-decreasing function. Setup cost is included if $q_i > 0$.

$F_i(s)$: the cost of an optimal production plan over periods $i, \dots, N$ when the starting inventory in period i is equal to s . It is defined in equation (4.11) below.

$$TQ_i = \sum_{j=1}^i q_j, \quad TD_i = \sum_{j=1}^i d_j, \quad TDI_i = \sum_{j=i}^N d_j, \quad TCP_i = \sum_{j=1}^i CP_j$$

Florian, Lenstra and Rinnooy Kan (1980) first give a general dynamic program for the preceding general model without backlogging as follows.

$$G_0(Q) = \begin{cases} 0 & Q = 0 \\ \infty & Q \neq 0 \end{cases}$$

$$G_i(Q) = \begin{cases} \min_{q \in \varphi_i(Q)} \{G_{i-1}(Q - q) + p_i(q)\} + h_i(Q - TD_i) & Q \in \Psi_i \\ \infty & Q \notin \Psi_i \end{cases}$$

Where $G_i(Q)$ is the cost of an optimal production plan over periods 1, ..., i subject to $TQ_i = Q$. Ψ_i is the set of feasible cumulative production levels at period i ($i = 1, \dots, N$). $\varphi_i(Q)$ is the set of feasible production quantity at period i subject to $TQ_i = Q$. Clearly, the value of Ψ_i is from the set of $\Psi_i \subseteq \{TD_i, TD_i + 1, \dots, TD_N\}$ and the value of $\varphi_i(Q)$ is from the set of $\varphi_i(Q) \subseteq \{0, 1, \dots, CP_i\}$. Florian, Lenstra and Rinnooy Kan (1980) give the computational complexity of the preceding dynamic program as $O(TD_N TCP_N)$.

Florian, Lenstra and Rinnooy Kan (1980, Proposition 2) and Bitran and Yanasse (1982, Proposition 4.1) also prove that for the arbitrary cost functions, even very simple planning problems are *NP*-hard. In the case that $h_i(I_i)$ and $p_i(q_i)$ are convex functions, Florian, Lenstra and Rinnooy Kan (1980) point out the problem is *NP*-hard even all setup cost $K_i = 1$ ($i = 1, \dots, N$) and there is no capacity constraint.

Shaw and Wagelmans (1998) interpret the algorithm complexity $O(TD_N TCP_N)$ of Florian, Lenstra and Rinnooy Kan (1980) as $O(N^2 \bar{c} \bar{d})$ where $\bar{c}$ and $\bar{d}$ are the average

production capacity and demand over the planning horizon N periods respectively. They consider a specific model that the production cost is linear and the inventory cost can be any general function. The model can be described as follows.

$$F_i(s) = \min \sum_{t=i}^N (c_t q_t + K_t \delta_t(q_t) + h_t(I_t)) \quad (4.11)$$

$$\text{subject to: } I_t = I_{t-1} + q_t - d_t \quad t = i, i+1, \dots, N.$$

$$q_t \leq CP_t \delta_t \quad t = i, i+1, \dots, N.$$

$$I_{i-1} = s, \quad I_N = 0,$$

$$q_t \geq 0, \quad \delta_t \in \{0,1\}, \quad t = i, i+1, \dots, N.$$

The initial conditions at period N are:

$$F_N(d_N) = 0,$$

$$F_N(s) = K_N + c_N(d_N - s) \quad \text{if } d_N - CP_N \leq s \leq d_N - 1$$

$$F_N(s) = \infty \quad \text{if } s \leq d_N - CP_N - 1$$

and $F_i(s) = \infty$ for any infeasible solution.

The above integer program can be transformed into a backward dynamic program as follows for all $i \in \{1, 2, \dots, N-1\}$ and all $s \in \{-TD_i, -TD_i + 1, \dots, -1, 0, 1, \dots, TD_i\}$.

$$F_i(s) = \min \{h_i(s - d_i) + F_{i+1}(s - d_i), \min_{q_i \in [0, CP_i]} \{K_i + c_i q_i + h_i(s - d_i + q_i) + F_{i+1}(s - d_i + q_i)\}\} \quad (4.12)$$

Shaw and Wagelmans (1998) re-write the second part of the right hand side in (4.12) as follows.

$$\min_{q_i \in [0, CP_i]} \{K_i + c_i q_i + h_i(s - d_i + q_i) + F_{i+1}(s - d_i + q_i)\} =$$

$$K_i - c_i(s - d_i) + \min_{s-d_i+1 \leq \tau \leq s-d_i+CP_i} \{w_i(\tau)\}$$

where $w_i(\tau) = c_i\tau + h_i(\tau) + F_{i+1}(\tau)$.

Based on the fact that when computing $F_i(s+1)$ instead of $F_i(s)$, the minimization of $w_i(\cdot)$ is over almost the same set of integer values, namely $\{s - d_k + 2, s - d_k + 3, \dots, s - d_k + CP_i + 1\}$. Shaw and Wagelmans (1998) also propose the following Problem (CI) and prove that the time complexity to solve Problem (CI) is $O(D)$.

Problem (CI) Given positive integer c and D where $c \leq D$, and function values $w(\tau)$ over integer argument values $\tau = 0, 1, \dots, D$, for every $s = 0, \dots, D-c$, determine the value $\min\{w(\tau) \mid s+1 \leq \tau \leq s+c\}$.

Based on the above result, Shaw and Wagelmans (1998) prove that the original model of dynamic program (4.12) can be solved in $O(N^2 \bar{d})$ time.

4.5.2 Deterministic Demand Model Description

This section formulates a dynamic program for the complementary deterministic model of dynamic program (4.4). It will also be stated that the deterministic model is in fact a special case of the stochastic one.

The model has the following assumptions.

- The demand for every period is deterministic and known,
- The capacity is the same for every period, $CP_i = Cp$ for all i ($i = 1, \dots, N$),
- The inventory cost function $h_i(I_i)$ is a continuous convex function,

- d. The production cost function is linear,
- e. The setup cost is non-increasing, i.e. $K_1 \geq K_2 \geq \dots \geq K_{N-1} \geq K_N \geq 0$.

For simplification, this model is referred to as Model I, and the model described in section 4.2 and expressed as dynamic program (4.4) is referred to as Model II.

There is a close relationship between Models I and II, and it will be demonstrated that Model I is actually a particular case of Model II. First, the known deterministic demand d_i at period i can be regarded as the random variable D_i in Model II by assigning $D_i = d_i$ with probability 1, and $D_i \neq d_i$ with probability 0. Next, as to the inventory cost function, due to the preceding arrangement of the stochastic demand in Model II, the notations used in Models I and II descriptions have the following relationships: $l_i(y) = h_i(I_i)$, $x_i = I_{i-1}$, $y_i = I_{i-1} + q_i$, and $I_i = y_i - d_i = I_{i-1} + q_i - d_i$. Last, for the ending inventory level condition at period N , $I_N = 0$ is often assumed in the literature (for instance, Florian, Lenstra and Rinnooy Kan (1980), Shaw and Wagelmans (1998)) due to the following arguments from Zangwill (1966): Since the demands $(d_1, d_2, \dots, d_N)$ are known in advance in Model I, it is clear that producing more than the demand values is wasteful, therefore the ending inventory level must be non-positive, i.e. $I_N \leq 0$. By appending an additional artificial $N+1$ st period with non-negative production quantity, no demands in this $N+1$ st period, and all costs occurred in this $N+1$ st period are zero, then setting $I_{N+1} = 0$ leads to the following equation.

$$0 = I_{N+1} = q_{N+1} - d_{N+1} + I_N = q_{N+1} + I_N.$$

So the problem of N -periods with $I_N \leq 0$ is transformed into the similar one of $N+1$ periods with $I_{N+1} = 0$ without incurring any extra cost. Thus for the problem description, Zangwill (1966) states that there is no loss in generality between assuming $I_N \leq 0$ and $I_N = 0$.

As stated in section 4.2, if $x_{N+1} = I_N \leq 0$, c_{N+1} is the unit backlogging penalty cost. From the preceding analysis, it will be assumed that $c_{N+1} = 0$ and $x_{N+1} = I_N \leq 0$ for Model I in the rest of this chapter.

Clearly, by assigning a large enough unit backlogging penalty cost to c_{N+1} that the system will not afford, I_N can be forced to be zero. Such a large enough c_{N+1} exists, for instance,

$$c_{N+1} = \max\{c_i \mid 1 \leq i \leq N\} * TD_N + \sum_{j=1}^N K_j + N * \max\{h_i(TD_N) \mid 1 \leq i \leq N\}$$

This value means that if there is one unit backlogged at the end of period N , the cost will be more than producing only once at the first period and satisfying all the demands in the planning horizon. Clearly, the system will not allow backlogging due to such a high unit backlogging cost at the end of planning horizon.

Summarizing the above assumptions and analysis, and following the similar process in section 4.2 with the deterministic demands, Model I can be expressed as:

$$f_n(x) = \min_{y \in [x, x+CP]} \{K_n \delta(y-x) + g_n(y)\} \quad (4.13)$$

$$\begin{aligned} g_n(y) &= e_n y + l_n(y) + f_{n+1}(y - d_n) \\ &= e_n y + h_n(y - d_n) + f_{n+1}(y - d_n) \quad n = 1, 2, \dots, N \end{aligned} \quad (4.14)$$

$$f_{N+1}(x) = 0.$$

where $e_n = c_n - c_{n+1}$, $c_{N+1} = 0$

Clearly, $f_n(x)$ is not the expected optimal cost but the optimal cost from period n to N with the inventory level x at the beginning of the period n .

4.5.3 Algorithm Description and Computational Complexity

From the analysis in the previous section, deterministic dynamic program (4.13) is in fact a particular case of stochastic dynamic program (4.4). With the same assumption for the setup costs, it is clear that Proposition 4.3 also holds for dynamic program (4.13), and therefore the order policy for (Cp, K) -convex is also true for (4.13). This fact will be used to develop an efficient algorithm for solving (4.13). A lemma on which the algorithm is based is proposed first, the algorithm and its explanation are given next, followed by the analysis of the algorithm's computational complexity.

If the periodic X_n - Y_n band for period n is known, due to the fact that the order policy for $x \leq X_n$ and $x > Y_n$ is already known, the computation of dynamic program (4.13) could be narrowed down to the minimization problem for $X_n < x \leq Y_n$. So the first step is to compute the periodic X_n - Y_n band efficiently. In order to introduce the algorithm, the following lemma is necessary.

Lemma 4.5 For any period n , suppose the beginning inventory level is x_n , the periodic highest X band is X'_n and the lowest Y-band is Y'_n then

a. if $g_n(x_n + Cp) + K_n > g_n(x_n)$, and $g_n(x_n) + K_n > g_n(x_n - Cp)$ then

$$x_n > X'_n + Cp \geq Y'_n;$$

b. if $g_n(x_n + Cp) + K_n > g_n(x_n)$, and $g_n(x_n) + K_n \leq g_n(x_n - Cp)$ then

$$X'_n < x_n \leq Y'_n + Cp \leq X'_n + 2Cp;$$

c. if $g_n(x_n + Cp) + K_n \leq g_n(x_n)$, and $g_n(x_n + 2Cp) + K_n \leq g_n(x_n + Cp)$ then

$$x_n \leq Y'_n - Cp \leq X'_n;$$

d. if $g_n(x_n + Cp) + K_n \leq g_n(x_n)$, and $g_n(x_n + 2Cp) + K_n > g_n(x_n + Cp)$ then

$$Y'_n - 2Cp \leq X'_n - Cp < x_n \leq Y'_n.$$

Proof: For the inequality $g_n(x_n + Cp) + K_n > g_n(x_n)$, it means that at point x_n , not full capacity or even nothing will be ordered, thus $x_n > X'_n$. At the same time if $g_n(x_n) + K_n > g_n(x_n - Cp)$ holds, for the same reason, $x_n - Cp > X'_n$ holds. Furthermore, because $Y'_n - X'_n \leq Cp$, therefore $x_n > X'_n + Cp \geq Y'_n$, and this proves (a). For (b), $g_n(x_n) + K_n \leq g_n(x_n - Cp)$ means that at point $x_n - Cp$, something can be ordered, so $x_n - Cp \leq Y'_n$, or $x_n \leq Y'_n + Cp \leq X'_n + 2Cp$. By combining with $x_n > X'_n$, the inequality of $X'_n < x_n \leq Y'_n + Cp \leq X'_n + 2Cp$ can be obtained. This proves (b).

For the inequality $g_n(x_n + Cp) + K_n \leq g_n(x_n)$, it means that at point x_n , something will be ordered, thus $x_n \leq Y'_n$. If at the same time $g_n(x_n + 2Cp) + K_n \leq g_n(x_n + Cp)$, then for

the same reason, $x_n + Cp \leq Y'_n$, and because $Y'_n - X'_n \leq Cp$, so $x_n \leq Y'_n - Cp \leq X'_n$, this proves (c). On the other hand if $g_n(x_n + 2Cp) + K_n > g_n(x_n + Cp)$, then at the point of $x_n + Cp$, not full or even nothing will be ordered, so $x_n + Cp > X'_n$. Because $Y'_n - X'_n \leq Cp$, thus $x_n + Cp > X'_n \geq Y'_n - Cp$, or $x_n > X'_n - Cp \geq Y'_n - 2Cp$. By combining with $x_n \leq Y'_n$, $Y'_n - 2Cp \leq X'_n - Cp < x_n \leq Y'_n$ can be obtained. This proves (d) and also concludes the whole lemma. $\square$

Based on Lemma 4.5, the following Algorithm 1 can be developed to compute a pair of periodic X-Y band which are defined as X''_n and Y''_n .

Algorithm 1 Initialize x_n as any value, here arbitrarily set $x_n = 0$.

Step 1: If $g_n(x_n + Cp) + K_n > g_n(x_n)$ is true,

Step 1.1: If $g_n(x_n) + K_n \leq g_n(x_n - Cp)$ is true

Set $X''_n = x_n - 2Cp$, and $Y''_n = x_n + Cp$

Stop the algorithm.

Else

Step 1.2: If $g_n(x_n - Cp) + K_n > g_n(x_n - 2Cp)$, then set $x_n = x_n - Cp$,

Go to Step 1.1,

Step 1.3: Else set $X''_n = x_n - 3Cp$, and $Y''_n = x_n$,

Stop the algorithm,

EndIf

EndIf

Else

Step 2: If $g_n(x_n + 2Cp) + K_n > g_n(x_n + Cp)$ is true

Set $X''_n = x_n - Cp$, and $Y''_n = x_n + 2Cp$,

Stop the algorithm.

Else

Step 2.1: If $g_n(x_n + 3Cp) + K_n \leq g_n(x_n + 2Cp)$ is true, then set $x_n = x_n + Cp$,

Go to Step 2,

Step 2.2: Else set $X_n'' = x_n$, and $Y_n'' = x_n + 3Cp$,

Stop the algorithm.

EndIf

EndIf

EndIf

Algorithm Explanation: Clearly in Step 1, if the condition test is true, and the condition in Step 1.1 is also true, then from (b) of Lemma 4.5, $X_n' < x_n \leq Y_n' + Cp \leq X_n' + 2Cp$. So $X_n'' = x_n - 2Cp \leq X_n'$ is a lower X band, and $Y_n'' = x_n + Cp > X_n' + Cp \geq Y_n'$ is an upper Y band, and such a pair of periodic X-Y band is $3 * Cp$ in width. If the condition in Step 1.1 is not true, it means that $g_n(x_n) + K_n > g_n(x_n - Cp)$. Additionally, if the condition $g_n(x_n - Cp) + K_n \leq g_n(x_n - 2Cp)$, then also by (b) of Lemma 4.5, $X_n' < x_n - Cp \leq Y_n' + Cp \leq X_n' + 2Cp$. As described in Step 1.3, $X_n'' = x_n - 3Cp \leq X_n'$, and $Y_n'' = x_n > X_n' + Cp \geq Y_n'$. Therefore, such a pair of periodic X-Y band is also $3 * Cp$ wide. If the condition $g_n(x_n - Cp) + K_n > g_n(x_n - 2Cp)$ is true, by (a) of Lemma 4.5, $x_n - Cp > X_n' + Cp \geq Y_n'$, or $x_n > X_n' + 2Cp \geq Y_n' + Cp$. As demonstrated in Step 1.2, x_n is decreased by one Cp . Since $x_n - Cp > Y_n'$, thus the condition in Step 1 will still hold, and the program goes to Step 1.1 again.

In Step 2, $g_n(x_n + Cp) + K_n \leq g_n(x_n)$ is considered. If $g_n(x_n + 2Cp) + K_n > g_n(x_n + Cp)$ in Step 2 is true, by (d) of Lemma 4.5, $Y_n' - 2Cp \leq X_n' - Cp < x_n \leq Y_n'$. So $X_n'' = x_n - Cp \leq Y_n' - Cp \leq X_n'$ is a lower X band, and $Y_n'' = x_n + 2Cp > Y_n'$ is an upper Y

band, and this pair of X-Y band is also $3 * Cp$ wide. The condition of $g_n(x_n + 2Cp) + K_n > g_n(x_n + Cp)$ does not hold implies that $g_n(x_n + 2Cp) + K_n \leq g_n(x_n + Cp)$. Further if $g_n(x_n + 3Cp) + K_n > g_n(x_n + 2Cp)$, by (d) of Lemma 4.5, $Y'_n - 2Cp \leq X'_n - Cp < x_n + Cp \leq Y'_n$. As described in Step 2.2, $X''_n = x_n \leq Y'_n - Cp \leq X'_n$, and $Y''_n = x_n + 3Cp > Y'_n$, and such a pair of periodic X-Y band is also $3 * Cp$ in width. If $g_n(x_n + 3Cp) + K_n \leq g_n(x_n + 2Cp)$, by (c) of Lemma 4.5, $x_n + Cp \leq Y'_n - Cp \leq X'_n$, or $x_n \leq Y'_n - 2Cp \leq X'_n - Cp$. As demonstrated in Step 2.1, x_n will be increased by one Cp . Since the new value is still less than X'_n , $g_n(x_n + Cp) + K_n \leq g_n(x_n)$ still holds, and thus the algorithm goes to Step 2 again. The loops in Steps 1.2 and 2.2 cannot be endless since the existence of periodic X-Y band.

In Algorithm 1, the two loops in Steps 1.2 and 2.1 determine the computation time. For Step 1.2, a periodic X band can be reached by at most $Integer[|X''_n - x_n| / Cp] = Integer[|X''_n| / Cp]$ steps, and so Step 1.2 can be completed in $O(|X''_n|)$ time. Similarly for Step 2.1, the algorithm will stop at most $Integer[|Y''_n - x_n| / Cp] = Integer[|Y''_n| / Cp]$ steps, where Y''_n is a periodic Y band. So computational complexity of Algorithm 1 is $O(\max(|X''_n|, |Y''_n|))$.

After Algorithm 1 is completed, the dynamic program (4.13) can be narrowed down to the minimization problem for $X''_n \leq x \leq Y''_n$ and $Y''_n - X''_n$ is $3 * Cp$ wide. Next, Shaw and

Wagelmans (1998)'s technique can be applied to compute $f_n(x)$ for $X_n'' \leq x \leq Y_n''$. In order to apply the technique, (4.13) needs to be reformulated as follows.

$$\begin{aligned} f_n(x) &= \min_{y \in [x, x+CP]} \{K_n \delta(y-x) + e_n y + h_n(y-d_n) + f_{n+1}(y-d_n)\} \\ &= \min \{e_n x + h_n(x-d_n) + f_{n+1}(x-d_n), K_n + \min_{x+1 \leq y \leq x+CP} \{e_n y + h_n(y-d_n) + f_{n+1}(y-d_n)\}\} \\ &= \min \{e_n x + h_n(x-d_n) + f_{n+1}(x-d_n), K_n + e_n d_n + \min_{x+1 \leq y \leq x+CP} \{e_n(y-d_n) + h_n(y-d_n) + f_{n+1}(y-d_n)\}\} \end{aligned}$$

Define $w_n(\tau) = e_n(\tau) + h_n(\tau) + f_{n+1}(\tau)$, then the preceding $f_n(x)$ becomes as follows for

$$X_n'' \leq x \leq Y_n''.$$

$$f_n(x) = \min \{e_n x + h_n(x-d_n) + f_{n+1}(x-d_n), K_n + e_n d_n + \min_{x+1-d_n \leq \tau \leq x+CP-d_n} \{w_n(\tau)\} \} \quad (4.15)$$

Since the value of x is from the set of $\{-TD_{n-1}, -TD_{n-1} + 1, \dots, -1, 0, 1, \dots, TDI_n\}$, the other value which is not in this set will not have a better solution for the model, so the values of x form the set of $\max(X_n'', -TD_{n-1}) \leq x \leq \min(Y_n'', TDI_n)$.

Dynamic program (4.15) is the same problem which is described as Problem (CI) in section 4.5.1. Applying Shaw and Wagelmans (1998)'s result, such a problem (4.15) can be solved in $O(\min(Y_n'', TDI_n) - \max(X_n'', -TD_{n-1}))$ time.

Define $X = \min\{X_n'' \mid 1 \leq n \leq N\}$, $Y = \max\{Y_n'' \mid 1 \leq n \leq N\}$. The preceding analysis which is based on one period n ($0 < n \leq N$) only leads to the following conclusion.

Proposition 4.4 Dynamic program (4.13) can be solved at $O(N(\min(Y, TD_N) - \max(X, -TD_N)))$ time.

4.5.4 Issues for Computational Study

In order to test the effectiveness of the algorithm described in section 4.5.3, the algorithm is implemented in JAVA, and computational experiments are conducted on a commonly used computer. The computer's CPU processor is Intel P4 1.7GHz, memory is 128Mb, and hard disk is 20Gb. In the rest of this section, the computational experiments are explained first followed by the computational results and comparison.

The experiments are very similar to those described in Shaw and Wagelmans (1998), which is described below in detail. Demand in period t is randomly generated from the following formula.

$$d_t = \text{int}[u + \delta z_t + a \sin(\frac{2\pi t}{b} + \frac{\pi}{2}) + 0.5]$$

where,

u : the mean demand,

δ : the standard error of the demand,

a : the amplitude of the seasonality component,

b : the number of periods in one seasonal cycle,

z_t : i.i.d. standard normal random variables.

The following four demand patterns are investigated.

Pattern 1: $\delta = 0.335u$, $a = 0$,

Pattern 2: $\delta = 1.185u$, $a = 0$,

Pattern 3: $\delta = 0.335u$, $a = 0.625u$, $b = n$, and

Pattern 4: $\delta = 0.335u$, $a = 0.625u$, $b = 12$.

The unit production cost for every period is generated from the uniform distribution in $[200/u, 600/u]$. The capacity is drawn from the uniform distribution in $[0.8u, 1.2u]$. Unit holding cost is generated from uniform distribution $[0.8, 1.2]$, and unit backlogging cost is from uniform distribution $[9, 12]$. Therefore the inventory cost function is different in every period. The setup cost for period t is generated from the uniform distribution in $[0.5f, 1.5f]$, and the following three cases for setup cost f are investigated: $f = 40$, $f = 80$, and $f = 160$. Since the model requires that the setup cost is non-increasing, a simple test is conducted to assure the requirement.

First observation is that the demand patterns and different setup cost do not affect the computational time much, which can be observed in Tables 4.1 and 4.2 respectively.

Table 4.1 CPU Times (in Milliseconds) for Different Demand Patterns

Pattern	Average Times
1	31.3
2	32.5
3	30.9
4	32.0

Average demand = 20, Setup Cost = 40, Period No. = 48

Table 4.2 CPU Times (in Milliseconds) for Different Setup Cost

Setup Cost	Average Times
40	31.3
80	33.1

160	32.5
-----	------

Average demand = 20, Period No. = 48

It is also observed that when the number of periods is increased while keeping all other parameters the same, the average time grows more or less doubly. This is illustrated in Table 4.3. While in this case, Shaw and Wagelmans' algorithm grows more or less in quadruple times.

Table 4.3 CPU Times (in Milliseconds) for Different Numbers of Periods

Period Numbers	Average Times
24	20
48	31.3
96	57.6
192	110.2

Average demand is 20

Period Numbers	Average Times
24	40.7
48	55
96	130.3

Average demand is 100

The experiments also indicate that CPU times do not grow so fast as the average demand.

This phenomenon is demonstrated in Table 4.4.

Table 4.4 CPU Times (in Milliseconds) for Different Average Demands

Average Demands	Average Times
-----------------	---------------

20	31.3
50	49.4
100	55
200	87.2

Planning Period Number is 48

It is worth pointing out that although the computational time for the model is decreased by more and more efficient algorithms, the memory space required by the algorithms is not reduced. It is clear to see that the algorithm discussed in this chapter and some other algorithms such as those in Florian, Lenstra and Rinnooy Kan (1980) and Shaw and Wagelmans (1998) need the same memory space.

For the models with single product, when the average demands or planning period number is quite large, for instance, when the planning period number is 384 for Table 4.1, or the average demands are 300 for Table 4.2, the memory requirements could be a problem for many computers. If the models for multiple product planning problems are developed, computer memory requirements could be a big issue for computation. For example, suppose there is a planning problem where the average demand is 20 for these different products respectively, the planning periods are 24, and backlogging is allowed, then in order to trace all feasible solution, normally the size of required array will be $(20 * 24)^3 * 24 * 32$ byte ≈ 79 Gb, assuming the array is in Float type which requires 32 bytes each. Thus, developing efficient algorithms to reduce space requirement for the dynamic program models could be an interesting question.

4.6 Summary

This chapter proves that the X-Y band policy of Chen and Lambrecht (1996) and Chen (2004a) holds for a more general model in which periodic set-up cost, stochastic demand, unit production cost, and the inventory cost function can be different. A new efficient algorithm based on this theoretical result is designed for the counterpart deterministic dynamic model. Computational studies show that the algorithm performs better than the other methods of model solution.

It also implies that although the model is proved NP-hard, it is still possible to find optimal policy and develop efficient algorithms for solving the model. However, how to characterize a model is a tough challenge. For instance, how to characterize the same model without the constraint of equal periodic capacity is still unsolved.

Chapter 5

Order Policy Characterization for the Stochastic Multi-Product Flexible

Manufacturing Systems: Analysis and Hypothesis

5.1 Introduction

Research efforts have been put in finding the optimal production/order policies for stochastic dynamic models in production and supply chain planning for almost half a century. The challenges in this area have attracted much attention of many researchers. Many theories have been developed especially for single-product system and the main contributions are made by Scarf (1960), Iglehart (1963), Veinott (1966*b*), Federgruen and Zipkin (1986), Chen and Lambrecht (1996), and Chen (2004a).

However, very few of the theories have been established for the capacitated multiple-product multiple-period system. Evans (1967) first investigates such a system without considering setup. Further research (for example, Wein (1992), Gershwin (1994), de Vericourt, Karaesmen and Dallery (2000), Srivatsan and Dallery (1998), etc.) has suggested the optimality of hedging point order policy for the model. Chen (2004b) fundamentally characterizes the optimal hedging point policy for the two-product flexible manufacturing system.

However, extending Chen (2004b)'s results into more than two-product system is not straightforward, and still a tough challenge in academia. This chapter focuses on a simplified multiple-product system where stochastic demand distribution, production

rate, unit production cost, and periodic expected inventory cost are the same for all products respectively. The objective is trying to characterize the system with hedging point policy. In this chapter, the model of general multiple-product system will be introduced first; subsequently, the main results will be demonstrated, especially the hedging point policy for two-product system. Next a simplified model will be introduced and the production/order policy will be illustrated and proved. Finally, a hypothesis for the general multiple-product system will be proposed.

5.2 Model Description, Notation and Dynamic Program Formulation

The general model of a capacitated multiple-product and multiple-period system investigated here considers a production system with a single, flexible machine that produces multiple distinct products in a make-to-stock mode. Here the machine is flexible means that the machine is able to perform operations on a random sequence of parts with little or no time or other expenditure for changeover from one part to the next (Gershwin (1994)). Demands for each item are stochastic, independently and identically distributed (i.i.d.). Unsatisfied demands are backlogged with a penalty cost. The machine is assumed to be perfectly maintained and reliable. The objective is to minimize the total expected discounted cost over the planning time horizon.

A general model for the m -product system can be expressed as the following dynamic program based on Chen (2004b).

$$f_n(x_1, x_2, \dots, x_m) = \min_{\substack{\frac{x_1 - x_1}{u_1} + \frac{x_2 - x_2}{u_2} + \dots + \frac{x_m - x_m}{u_m} \leq 1; x_1 \leq X_1, x_2 \leq X_2, \dots, x_m \leq X_m}} \{L_1(X_1) + L(X_2) + \dots + L(X_m) \\ + c_1 * (X_1 - x_1) + c_2 * (X_2 - x_2) + \dots + c_m * (X_m - x_m) \\ + \alpha * E(f_{n-1}(X_1 - d_1, X_2 - d_2, \dots, X_m - d_m))\} \quad (5.1)$$

Where;

$x_1(X_1), x_2(X_2), \dots, x_m(X_m)$: the inventory levels before (after) the order is placed for

products 1, 2, ... and m respectively at the beginning of a period.

$L_1(X_1), L_2(X_2), \dots, L_m(X_m)$: the one period expected inventory cost (including holding and

backlogging costs), assuming convex, for products 1, 2, ..., and m

respectively. For example, in Wagner (1972), for the calculation of

$L_i(y)$ under different postulations, $L_i(x) \rightarrow \infty$ when $|x| \rightarrow \infty$, $i=1,$

2, ..., m .

u_i, c_i : production rate and unit production cost for product i respectively.

d_i : the one-period demand for product i , and is an i.i.d. random variable.

$f_n(x_1, x_2, \dots, x_m)$: n -period minimum expected discounted cost function.

α : discount factor.

Define: $g_i(X) = L_i(X) + c_i * X$, and

$$G_n(X_1, X_2, \dots, X_m) = g_1(X_1) + g_2(X_2) + \dots + g_m(X_m) \\ + \alpha * E(f_{n-1}(X_1 - d_1, X_2 - d_2, \dots, X_m - d_m)) \quad (5.2)$$

then equation (5.1) can be simplified as follows.

$$f_n(x_1, x_2, \dots, x_m) = \min_{\substack{\frac{x_1 - x_1}{u_1} + \frac{x_2 - x_2}{u_2} + \dots + \frac{x_m - x_m}{u_m} \leq 1; x_1 \leq X_1, x_2 \leq X_2, \dots, x_m \leq X_m}} \{G_n(X_1, X_2, \dots, X_m)\} \\ - c_1 * x_1 - c_2 * x_2 - \dots - c_m * x_m \quad (5.3)$$

The initial condition is: $f_0(x_1, x_2, \dots, x_m) = 0$

For example, for a three-product model, it can be expressed as follows directly from equations (5.2) and (5.3). Define:

$$G_n(X, Y, Z) = g_1(X) + g_2(Y) + g_3(Z) + \alpha * E(f_{n-1}(X - d_1, Y - d_2, Z - d_3)) \quad (5.4)$$

The cost function for the three-product model is:

$$f_n(x, y, z) = \min_{\substack{\frac{X-x}{u_1} + \frac{Y-y}{u_2} + \frac{Z-z}{u_3} \leq 1; x \leq X; y \leq Y; z \leq Z}} \{G_n(X, Y, Z)\} - c_1 * x - c_2 * y - c_3 * z \quad (5.5)$$

$$f_0(x, y, z) = 0$$

$x(X)$, $y(Y)$, and $z(Z)$ represent the inventory levels before (after) the order is placed for products 1, 2, and 3 respectively at the beginning of a period.

5.3 Previous Results

This section reviews the results of multiple product system related to the model investigated in this chapter, especially the hedging point policy in Chen (2004b). Some useful results are also explained here, which are Theorem 23.1 for one-sided directional partial derivatives in Rockafellar (1970), and a higher order priority means $u_1 G'_{nx^+}(x, y) \leq u_2 G'_{ny^+}(x, y)$.

First, the directional partial derivatives are introduced. The one-sided directional partial derivatives of the convex functions $G_n(X, Y, Z)$ and $f_n(X, Y, Z)$ exist by Theorem 23.1 of Rockafellar (1970) and $G'_{nX^+}(X, Y, Z)$ is defined by:

$$G'_{nX^+}(X, Y, Z) = \lim_{t \rightarrow 0^+} \frac{G_n(X + t, Y, Z) - G_n(X, Y, Z)}{t}. \quad (5.6)$$

$t \rightarrow 0^+$ means that t approaches zero from the side which is greater than zero. The other partial derivatives are defined similarly.

Evans (1967) partially characterizes the order policy of the above multiple product system, and describes the existence of various regions in which it is optimal to produce one or more of the products and use all of the available resource in a given period. Additionally, in Evans (1967) the function $f_n(x_1, x_2, \dots, x_m)$ in equation (5.3) is proved to be convex for each n . By Proposition B-2 of Heyman and Sobel (1984), clearly $G_n(X_1, X_2, \dots, X_m)$ of (5.2) is also convex for each n .

Chen (2004b) finds that the two-product system has the properties named μ -difference monotone. By using geometrical techniques, Chen (2004b) proves that $G_n(x, y)$ and $f_n(x, y)$ are both μ -difference monotone for all period n . Exploiting the properties of μ -difference monotone, Chen (2004b) further proves the optimality of hedging point which is described by Figure 5.1. $X^*(y)$, $Y^*(x)$ and $\bar{y}(x)$ are the boundaries of a hedging point policy, intersecting at a hedging point $S(s_x, s_y)$ which is the global minimum point of $G_n(x, y)$. $X^*(y)$ is the minimum point of $G_n(x, y)$ as a function of x for a given y , and $Y^*(x)$ is defined similarly. $\bar{y}(x)$ is defined as:

$$\bar{y}(x) = \min \{y \mid u_1 G'_{x^+}(x, y) \leq u_2 G'_{y^+}(x, y), \text{ given } x \leq s_x\}.$$

In Figure 5.1, the whole two dimensional space is divided into three distinct regions, say regions 0, 1 and 2. In Region 0, nothing is needed to be ordered, items 1 and 2 will be ordered only in regions 1 and 2 respectively.

Figure 5.1 Hedging Point Policy Regions (Chen (2004b))

In the analysis of Chen (2004b), suppose item 1 has a higher priority to be ordered, then it means that the marginal cost reduction by producing item 1 in Δt time units is no less than that by item 2. It thus means that $u_1 G'_{nx^+}(x, y) \leq u_2 G'_{ny^+}(x, y)$. This result will also be used in this chapter.

5.4 Analytical Results for a Simplified Three-Product System

In this section, model (5.5) is simplified and its order policy is analyzed, then, the results are extended to multiple-product system with the similar simplifications in the next section. Based on the general model descriptions made in section 5.2, the simplified model assumes that the system has the same i.i.d. stochastic demands, unit production costs and production rates for the three products respectively.

Therefore, define

$L(X)$: the one period expected inventory cost (including holding and backlogging costs) assumed convex, is the same for products 1, 2 and 3.

u, c : production rate and unit production cost are the same for products 1, 2 and 3.

d : the one-period demand is the same for products 1, 2 and 3. d is a random variable.

and $g(X) = L(X) + c * X$,

$$G_n(X, Y, Z) = g(X) + g(Y) + g(Z) + \alpha * E(f_{n-1}(X - d, Y - d, Z - d)) \quad (5.9)$$

Equation (5.5) can be simplified as:

$$f_n(x, y, z) = \min_{\substack{\frac{X-x}{u} + \frac{Y-y}{u} + \frac{Z-z}{u} \leq 1; x \leq X; y \leq Y; z \leq Z}} \{G_n(X, Y, Z)\} - c * (x + y + z) \quad (5.10)$$

Due to the symmetry, it is intuitive to imagine that products 1, 2 and 3 divide the three dimensional space into three equal parts and hedging point policy holds for such a system. The question for this intuition is how the inventory levels of three items divide the three-dimensional space. The objective of this section is to answer this question.

The main theoretical results obtained in this section are that products 1, 2, and 3 divide the three dimensional space equally with three planes of $X=Y$, $Y=Z$, and $Z=X$. Further, the three planes intersect at one global minimum point $p(X_1, Y_1, Z_1)$ with equal coordinate values, i.e., $X_1=Y_1=Z_1$. Lemma 5.2 proves that such a global minimum point exists. Proposition 5.1 provides the three planes in the solution framework, and Proposition 5.2 proves that the three planes divide the three dimensional space. Finally Proposition 5.3 characterizes an optimal policy for the simplified model.

Lemma 5.1 is obvious.

Lemma 5.1 For all n , $G_n(X, Y, Z) = G_n(Y, X, Z)$, and $f_n(x, y, z) = f_n(y, x, z)$,

$$G_n(X, Y, Z) = G_n(X, Z, Y), \text{ and } f_n(x, y, z) = f_n(x, z, y), \text{ and}$$

$$G_n(X, Y, Z) = G_n(Z, Y, X), \text{ and } f_n(x, y, z) = f_n(z, y, x).$$

Lemma 5.2 There exists at least one global minimum point $P(x_1, y_1, z_1)$ for each of $G_n(x, y, z)$ and $f_n(x, y, z)$ where its coordinate values are equal, i.e. $x_1 = y_1 = z_1$.

Proof: Suppose one global optimum point is $P_1(x_0, y_0, z_0)$, without losing the generality, let $x_0 \leq y_0 \leq z_0$. If two equalities hold, then the lemma is proved. Suppose at least one equality does not hold, by Lemma 5.1, $G_n(x_0, y_0, z_0) = G_n(z_0, x_0, y_0)$, and $G_n(x_0, y_0, z_0) = G_n(y_0, z_0, x_0)$. Because $P_1(x_0, y_0, z_0)$ is the global minimum point, thus $P_2(z_0, x_0, y_0)$ and $P_3(y_0, z_0, x_0)$ are also global minimum points. Due to the convexity of $G_n(x, y, z)$, define $P_4 = \frac{1}{2}P_2 + \frac{1}{2}P_3 = (\frac{z_0 + y_0}{2}, \frac{x_0 + z_0}{2}, \frac{y_0 + x_0}{2})$, then P_4 is also a global minimum point. By the same logic, define $P = \frac{1}{3}P_1 + \frac{2}{3}P_4 = (\frac{x_0 + y_0 + z_0}{3}, \frac{x_0 + y_0 + z_0}{3}, \frac{x_0 + y_0 + z_0}{3})$, clearly P is a global optimum point with equal coordinate values. This completes the proof. $\square$

Proposition 5.1

- a. if $x = y$, then $G'_{nx^+}(x, y, z) = G'_{ny^+}(x, y, z)$, and $f'_{nx^+}(x, y, z) = f'_{ny^+}(x, y, z)$,
- b. if $y = z$, then $G'_{ny^+}(x, y, z) = G'_{nz^+}(x, y, z)$, and $f'_{ny^+}(x, y, z) = f'_{nz^+}(x, y, z)$, and

c. if $z = x$, then $G'_{nz^+}(x, y, z) = G'_{nx^+}(x, y, z)$, and $f'_{nz^+}(x, y, z) = f'_{nx^+}(x, y, z)$.

Proof: The proof for (a) will be given, and (b) and (c) can be obtained similarly.

From equation (5.6):

$$\begin{aligned}
 G'_{nx^+}(x, y, z) &= \lim_{t \rightarrow 0^+} \frac{G_n(x+t, y, z) - G_n(x, y, z)}{t} \\
 &= \lim_{t \rightarrow 0^+} \frac{G_n(y, x+t, z) - G_n(y, x, z)}{t} \quad (\text{based on Lemma 5.1}) \\
 &= \lim_{t \rightarrow 0^+} \frac{G_n(x, y+t, z) - G_n(x, y, z)}{t} \quad (\text{as } x = y) \\
 &= G'_{ny^+}(x, y, z) \quad (\text{based on the definition of } G'_{ny^+}(x, y, z))
 \end{aligned}$$

Similar process is applicable to $f'_{nx^+}(x, y, z)$. This completes the proof. $\square$

Proposition 5.1 states that when two items have equal inventory levels, the order priorities for both items are also equal.

Proposition 5.2

a. if $x < y$, then $G'_{nx^+}(x, y, z) \leq G'_{ny^+}(x, y, z)$, and $f'_{nx^+}(x, y, z) \leq f'_{ny^+}(x, y, z)$,

b. if $x > y$, then $G'_{nx^+}(x, y, z) \geq G'_{ny^+}(x, y, z)$ and $f'_{nx^+}(x, y, z) \geq f'_{ny^+}(x, y, z)$.

Proof: The proof is conducted by induction.

First $n = 1$, from equation (5.9):

$$G_1(x+t, y, z) = g(x+t) + g(y) + g(z), \text{ and}$$

$$G_1(x, y+t, z) = g(x) + g(y+t) + g(z).$$

Since $g(\cdot)$ is convex, if $x < y$, then

$$g(x+t) - g(x) \leq g(y+t) - g(y)$$

$$\text{or } g(x+t) + g(y) \leq g(x) + g(y+t)$$

$$\text{or } g(x+t) + g(y) + g(z) \leq g(x) + g(y+t) + g(z)$$

$$\text{or } G_1(x+t, y, z) \leq G_1(x, y+t, z)$$

$$\text{or } G_1(x+t, y, z) - G_1(x, y, z) \leq G_1(x, y+t, z) - G_1(x, y, z)$$

$$G'_{1x^+}(x, y, z) = \lim_{t \rightarrow 0^+} \frac{G_1(x+t, y, z) - G_1(x, y, z)}{t} \leq \lim_{t \rightarrow 0^+} \frac{G_1(x, y+t, z) - G_1(x, y, z)}{t} = G'_{1y^+}(x, y, z)$$

Clearly the above process can be applicable to prove that if $x > y$, then

$$G'_{1x^+}(x, y, z) \geq G'_{1y^+}(x, y, z).$$

Assuming that for $n = i$, if $x < y$, then $G'_{ix^+}(x, y, z) \leq G'_{iy^+}(x, y, z)$, and if $x > y$, then

$$G'_{ix^+}(x, y, z) \geq G'_{iy^+}(x, y, z). \text{ First, some further explanations are necessary for the}$$

assumption and are described as follows. The assumption that if $x < y$, $G'_{ix^+}(x, y, z) \leq$

$G'_{iy^+}(x, y, z)$ implies:

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{G_i(x+t, y, z) - G_i(x, y, z)}{t} &\leq \lim_{t \rightarrow 0^+} \frac{G_i(x, y+t, z) - G_i(x, y, z)}{t} \\ \text{or } \lim_{t \rightarrow 0^+} \frac{G_i(x+t, y, z) - G_i(x, y+t, z)}{t} &\leq 0 \end{aligned} \quad (5.11)$$

Further, the following statements are from the assumptions and Proposition 5.1.

$$\text{if } x < y, \text{ then } G'_{ix^+}(x, y, z) \leq G'_{iy^+}(x, y, z),$$

$$\text{if } x > y, \text{ then } G'_{ix^+}(x, y, z) \geq G'_{iy^+}(x, y, z), \text{ and}$$

$$\text{if } x = y, \text{ then } G'_{ix^+}(x, y, z) = G'_{iy^+}(x, y, z).$$

Under the condition that there is a need to order x or y , the first order policy is obvious: when $x < y$, then ordering item 1 has higher priority compared with ordering item 2. Thus, ordering item 1 is an optimal policy when $x < y$. The second specified order policy is: when $x = y$, then ordering items 1 and 2 simultaneously along $x = y$ is always an optimal policy. This is due to the following reasons. When $x > y$, $G'_{ix^+}(x, y, z) \geq G'_{iy^+}(x, y, z)$ and ordering item 2 will have the higher priority comparing with ordering item 1. So to the left hand side of $x = y$ item 1 is preferred to item 2, and to the right hand side of $x = y$ item 2 is preferred; thus, one optimal order policy is to order items 1 and 2 simultaneously along $x = y$. Although there is a possibility that Chattering Phenomena which is described in Chen (2004b) exists for the system, $x = y$ is included in the chattering area. Therefore, ordering along $x = y$ is always an optimal policy.

Summarizing the preceding analysis, and following the policies specified above, the post ordering inventory levels of X and Y will still have the relationship of $X \leq Y$ if the beginning inventory levels x and y have the relationship of $x < y$. Obviously, if the beginning inventory level has the relationship of $x = y$, the ending inventory level will have the same relationship by the second order policy. Following the similar analysis, the post ordering inventory level will still have the relationship of $X \geq Y$ if the beginning inventory level is $x > y$.

Next, it will be proved that if $x < y$, $f'_{ix^+}(x, y, z) \leq f'_{iy^+}(x, y, z)$, and $G'_{(i+1)x^+}(x, y, z) \leq G'_{(i+1)y^+}(x, y, z)$ and if $x > y$, $f'_{ix^+}(x, y, z) \geq f'_{iy^+}(x, y, z)$, and $G'_{(i+1)x^+}(x, y, z) \geq G'_{(i+1)y^+}(x, y, z)$. The result that if $x < y$, $f'_{ix^+}(x, y, z) \leq f'_{iy^+}(x, y, z)$ is proved first.

From equation (5.6),

$$f'_{ix^+}(x, y, z) = \lim_{t \rightarrow 0^+} \frac{f_i(x+t, y, z) - f_i(x, y, z)}{t}, \text{ and}$$

$$f'_{iy^+}(x, y, z) = \lim_{t \rightarrow 0^+} \frac{f_i(x, y+t, z) - f_i(x, y, z)}{t}.$$

From equation (5.10),

$$f_i(x+t, y, z) = \min_{\frac{x-(x+t)}{u} + \frac{y-y}{u} + \frac{z-z}{u} \leq 1; x+t \leq X; y \leq Y; z \leq Z} \{G_i(X, Y, Z)\} - c^*(x+y+z+t) \quad (5.12)$$

$$f_i(x, y+t, z) = \min_{\frac{x-x}{u} + \frac{y-(y+t)}{u} + \frac{z-z}{u} \leq 1; x \leq X; y+t \leq Y; z \leq Z} \{G_i(X, Y, Z)\} - c^*(x+y+z+t) \quad (5.13)$$

Subtracting equation (5.13) from (5.12) results into:

$$\begin{aligned} f_i(x+t, y, z) - f_i(x, y+t, z) &= \min_{\frac{x-(x+t)}{u} + \frac{y-y}{u} + \frac{z-z}{u} \leq 1; x+t \leq X; y \leq Y; z \leq Z} \{G_i(X, Y, Z)\} \\ &\quad - \min_{\frac{x-x}{u} + \frac{y-(y+t)}{u} + \frac{z-z}{u} \leq 1; x \leq X; y+t \leq Y; z \leq Z} \{G_i(X, Y, Z)\} \end{aligned} \quad (5.14)$$

Assign $X' = X - t$, then $X = X' + t$ and

$$\begin{aligned} \min_{\frac{x-(x+t)}{u} + \frac{y-y}{u} + \frac{z-z}{u} \leq 1; x+t \leq X; y \leq Y; z \leq Z} \{G_i(X, Y, Z)\} &= \min_{\frac{X'-x}{u} + \frac{y-y}{u} + \frac{z-z}{u} \leq 1; x \leq X'; y \leq Y; z \leq Z} \{G_i(X'+t, Y, Z)\} \\ &= \min_{\frac{X-x}{u} + \frac{y-y}{u} + \frac{z-z}{u} \leq 1; x \leq X; y \leq Y; z \leq Z} \{G_i(X+t, Y, Z)\} \end{aligned} \quad (5.15)$$

Again, assign $Y' = Y - t$, then $Y = Y' + t$ and

$$\min_{\frac{x-x}{u} + \frac{y-(y+t)}{u} + \frac{z-z}{u} \leq 1; x \leq X; y+t \leq Y; z \leq Z} \{G_i(X, Y, Z)\} = \min_{\frac{x-x}{u} + \frac{Y'-y}{u} + \frac{z-z}{u} \leq 1; x \leq X; y \leq Y'; z \leq Z} \{G_i(X, Y'+t, Z)\}$$

$$= \min_{\frac{X-x}{u} + \frac{Y-y}{u} + \frac{Z-z}{u} \leq 1; x \leq X; y \leq Y; z \leq Z} \{G_i(X, Y+t, Z)\} \quad (5.16)$$

Clearly, the right hand sides of equations (5.15) and (5.16) have the same constraints for their respective minimum problems; thus, they have the same set of feasible solutions.

Equation (5.14) can be written as:

$$\begin{aligned} f_i(x+t, y, z) - f_i(x, y+t, z) &= \min_{\frac{X-x}{u} + \frac{Y-y}{u} + \frac{Z-z}{u} \leq 1; x \leq X; y \leq Y; z \leq Z} \{G_i(X+t, Y, Z)\} \\ &\quad - \min_{\frac{X-x}{u} + \frac{Y-y}{u} + \frac{Z-z}{u} \leq 1; x \leq X; y \leq Y; z \leq Z} \{G_i(X, Y+t, Z)\} \end{aligned} \quad (5.17)$$

Since $x < y$, for a small enough positive t , $x+t \leq y$, for point $(x+t, y, z)$ in equation (5.12), suppose its optimal point after ordering is (X_0, Y_0, Z_0) . From the assumption and the specified order policies, clearly $X_0 \leq Y_0$. For point $(x, y+t, z)$ in equation (5.13), suppose its optimal point after ordering is (X_1, Y_1, Z_1) , and since $x < y < y+t$, thus $X_1 \leq Y_1$.

The following is to prove that for $X \leq Y$, and a very small enough positive t ,

$$G_i(X_0, Y_0, Z_0) \leq \min_{\frac{X-x}{u} + \frac{Y-y}{u} + \frac{Z-z}{u} \leq 1; x \leq X; y \leq Y; z \leq Z} \{G_i(X, Y+t, Z)\},$$

and then to prove that for $X_1 \leq Y_1$, $G_i(X_0, Y_0, Z_0) \leq G_i(X_1, Y_1, Z_1)$.

Equation (5.11) states that for an infinitesimally small positive t , and $X < Y$, $G_i(X+t, Y, Z) - G_i(X, Y+t, Z) \leq 0$. Thus, for a feasible solution (X, Y, Z) of the minimum problem at the right hand side of equation (5.15), if $X < Y$, then there is a

small enough positive t to let the following inequality hold: $G_i(X+t, Y, Z) \leq G_i(X, Y+t, Z)$. Since equations (5.15) and (5.16) have the same set of feasible solutions, and for every feasible solution (X, Y, Z) with $X < Y$, $G_i(X+t, Y, Z) \leq G_i(X, Y+t, Z)$ holds. Therefore, for a very small positive t , and $X < Y$,

$$\min_{\frac{X-x}{u} + \frac{Y-y}{u} + \frac{Z-z}{u} \leq 1; x \leq X; y \leq Y; z \leq Z} \{G_i(X+t, Y, Z)\} \leq \min_{\frac{X-x}{u} + \frac{Y-y}{u} + \frac{Z-z}{u} \leq 1; x \leq X; y \leq Y; z \leq Z} \{G_i(X, Y+t, Z)\}$$

Since the minimum of equation (5.15) is attained at (X_0, Y_0, Z_0) with $X_0 \leq Y_0$, thus for $X < Y$,

$$G_i(X_0, Y_0, Z_0) \leq \min_{\frac{X-x}{u} + \frac{Y-y}{u} + \frac{Z-z}{u} \leq 1; x \leq X; y \leq Y; z \leq Z} \{G_i(X, Y+t, Z)\}$$

Since for $X = Y$, and an infinitesimally small positive t , $G_i(X+t, Y, Z) = G_i(X, Y+t, Z)$. Thus, for $X \leq Y$, and a very small enough positive t ,

$$G_i(X_0, Y_0, Z_0) \leq \min_{\frac{X-x}{u} + \frac{Y-y}{u} + \frac{Z-z}{u} \leq 1; x \leq X; y \leq Y; z \leq Z} \{G_i(X, Y+t, Z)\} \quad (5.18)$$

Next is to prove that for $X_1 \leq Y_1$, $G_i(X_0, Y_0, Z_0) \leq G_i(X_1, Y_1, Z_1)$. Since the optimum value of the right hand side of equation (5.16) is attained at (X_1, Y_1, Z_1) with the relationship $X_1 \leq Y_1$, for $X_1 < Y_1$, it means that to right hand side of equation (5.16), there exists X and Y such that following inequality holds.

$$X_1 = X < Y + t = Y_1$$

Since $t \rightarrow 0^+$, $X \leq Y$; this means that the minimum to the right hand side of the inequality is attained at $X \leq Y$. Thus, from equation (5.18):

$$G_i(X_0, Y_0, Z_0) \leq G_i(X_1, Y_1, Z_1).$$

If $X_1 = Y_1$, it means that for the minimum problem to the left hand side of (5.16), (X_1, Y_1, Z_1) is the minimum point and satisfies the following inequalities.

$$\frac{X_1 - x}{u} + \frac{Y_1 - (y+t)}{u} + \frac{Z_1 - z}{u} \leq 1, \quad x \leq X_1, \quad y+t \leq Y_1, \quad z \leq Z_1.$$

Since $X_1 = Y_1$, and $x < y$, thus $X_1 = Y_1 \geq y+t > x+t$, and $Y_1 \geq y+t > y$, $z \leq Z_1$. This means (X_1, Y_1, Z_1) also satisfies the following inequalities which are the constraints for the minimum problem at the left hand side of equation (5.15).

$$\frac{X_1 - (x+t)}{u} + \frac{Y_1 - y}{u} + \frac{Z_1 - z}{u} \leq 1, \quad x+t \leq X_1, \quad y \leq Y_1, \quad z \leq Z_1.$$

This implies that (X_1, Y_1, Z_1) is the feasible solution of the minimum problem at the left hand side of (5.15), thus

$$G_i(X_0, Y_0, Z_0) \leq G_i(X_1, Y_1, Z_1).$$

Concluding the above analysis, the following inequality is true if the order policies specified before are followed, and $t \rightarrow 0^+$.

$$G_i(X_0, Y_0, Z_0) \leq G_i(X_1, Y_1, Z_1) \quad (5.19)$$

By equation (5.14), for $t \rightarrow 0^+$ $f_i(x+t, y, z) - f_i(x, y+t, z) \leq 0$

Since $f'_{ix^+}(x, y, z)$ and $f'_{iy^+}(x, y, z)$ exist, thus the following inequality holds.

$$f'_{ix^+}(x, y, z) - f'_{iy^+}(x, y, z) = \lim_{t \rightarrow 0^+} \frac{f_i(x+t, y, z) - f_i(x, y+t, z)}{t} \leq 0. \quad (5.20)$$

From equation (5.9),

$$G_{i+1}(X, Y, Z) = g(X) + g(Y) + g(Z) + \alpha * E(f_i(X-d, Y-d, Z-d))$$

$$G'_{(i+1)X^+}(X, Y, Z) - G'_{(i+1)Y^+}(X, Y, Z)$$

$$= g'(X) + \alpha * E(f'_{ix^+}(X-d, Y-d, Z-d)) - g'(Y) - \alpha * E(f'_{iy^+}(X-d, Y-d, Z-d))$$

$$= g'(X) - g'(Y) + \alpha * E(f'_{iX^+}(X-d, Y-d, Z-d) - f'_{iY^+}(X-d, Y-d, Z-d)) \quad (5.21)$$

$$\text{Since } X < Y, \quad g'(X) - g'(Y) \leq 0, \quad (5.22)$$

By equation (5.20), for every possible value of demand random variable d ,

$$\begin{aligned} & f'_{iX^+}(X-d, Y-d, Z-d) - f'_{iY^+}(X-d, Y-d, Z-d) \leq 0, \text{ and} \\ & \alpha * E(f'_{iX^+}(X-d, Y-d, Z-d) - f'_{iY^+}(X-d, Y-d, Z-d)) \leq 0 \end{aligned} \quad (5.23)$$

Summing up equations (5.22) and (5.23), the following result can be drawn.

$$G'_{(i+1)X^+}(X, Y, Z) - G'_{(i+1)Y^+}(X, Y, Z) \leq 0$$

This completes the first part of the proposition.

The similar process is applicable to prove that if $x > y$, $f'_{iX^+}(x, y, z) \geq f'_{iY^+}(x, y, z)$, and

then $G'_{(i+1)X^+}(x, y, z) \geq G'_{(i+1)Y^+}(x, y, z)$. This completes the proof. $\square$

Due to the symmetry of $x(X)$, $y(Y)$, and $z(Z)$, the following results can be obtained.

if $y < z$, then $G'_{ny^+}(x, y, z) \leq G'_{nz^+}(x, y, z)$, and $f'_{ny^+}(x, y, z) \leq f'_{nz^+}(x, y, z)$;

if $y > z$, then $G'_{ny^+}(x, y, z) \geq G'_{nz^+}(x, y, z)$ and $f'_{ny^+}(x, y, z) \geq f'_{nz^+}(x, y, z)$.

if $z < x$, then $G'_{nz^+}(x, y, z) \leq G'_{nx^+}(x, y, z)$, and $f'_{nz^+}(x, y, z) \leq f'_{nx^+}(x, y, z)$;

if $z > x$, then $G'_{nz^+}(x, y, z) \geq G'_{nx^+}(x, y, z)$ and $f'_{nz^+}(x, y, z) \geq f'_{nx^+}(x, y, z)$.

From Proposition 5.1, for every period n , $G'_{nx^+}(x, y, z) = G'_{ny^+}(x, y, z)$ if $x = y$, and this equality is for any value of z . Thus, the plane $x = y$ will divide the whole three dimensional space into two parts: $x < y$ and $x > y$. In the first part with $x < y$, by

Proposition 5.2, $G'_{nx^+}(x, y, z) \leq G'_{ny^+}(x, y, z)$; in the other part with $x > y$,
 $G'_{nx^+}(x, y, z) \geq G'_{ny^+}(x, y, z)$. Similar arguments are applicable to the planes of $y = z$
and $z = x$. Plane $y = z$ divides the space into two parts where one part has the properties
of $G'_{ny^+}(x, y, z) \leq G'_{nz^+}(x, y, z)$ and $y < z$, and the other part with $G'_{ny^+}(x, y, z) \geq$
 $G'_{nz^+}(x, y, z)$ and $y > z$. Plane $z = x$ divides the space into two parts where one part has
the properties of $G'_{nz^+}(x, y, z) \leq G'_{nx^+}(x, y, z)$ and $z < x$, and the other part with
 $G'_{nz^+}(x, y, z) \geq G'_{nx^+}(x, y, z)$ and $z > x$. The three planes of $x = y$, $y = z$, and $z = x$
have an intersection curve which actually is a line of $x = y = z$. This line is illustrated as
line L_1 in Figure 5.2. From Lemma 5.2, this line passes through a global minimum point

Figure 5.2 Solution Framework for the Simplified Three-Product System

$S(S_x, S_y, S_z)$. On the other hand, the three planes also divide the space into six different sub-spaces which share the line L_1 as the common boundary. Figure 5.2 also describes the space divisions and the detailed explanation is as follows.

The plane $x = y$ divides the space into the following two parts:

In sub-spaces ①, ⑤ and ⑥, $G'_{nx^+}(x, y, z) \leq G'_{ny^+}(x, y, z)$

In sub-spaces ②, ③ and ④, $G'_{nx^+}(x, y, z) \geq G'_{ny^+}(x, y, z)$,

Similarly for plane $y = z$:

In sub-spaces ①, ② and ③, $G'_{ny^+}(x, y, z) \leq G'_{nz^+}(x, y, z)$

In sub-spaces ④, ⑤ and ⑥, $G'_{ny^+}(x, y, z) \geq G'_{nz^+}(x, y, z)$,

for plane $z = x$:

In sub-spaces ①, ② and ⑥, $G'_{nx^+}(x, y, z) \leq G'_{nz^+}(x, y, z)$

In sub-spaces ③, ④ and ⑤, $G'_{nx^+}(x, y, z) \geq G'_{nz^+}(x, y, z)$.

Therefore, in sub-space ①: $G'_{nx^+}(x, y, z) \leq G'_{ny^+}(x, y, z) \leq G'_{nz^+}(x, y, z)$,

in sub-space ②: $G'_{ny^+}(x, y, z) \leq G'_{nx^+}(x, y, z) \leq G'_{nz^+}(x, y, z)$,

in sub-space ③: $G'_{ny^+}(x, y, z) \leq G'_{nz^+}(x, y, z) \leq G'_{nx^+}(x, y, z)$,

in sub-space ④: $G'_{nz^+}(x, y, z) \leq G'_{ny^+}(x, y, z) \leq G'_{nx^+}(x, y, z)$,

in sub-space ⑤: $G'_{nz^+}(x, y, z) \leq G'_{nx^+}(x, y, z) \leq G'_{ny^+}(x, y, z)$, and

in sub-space ⑥: $G'_{nx^+}(x, y, z) \leq G'_{nz^+}(x, y, z) \leq G'_{ny^+}(x, y, z)$.

In sub-spaces ① and ⑥, $G'_{nx^+}(x, y, z)$ has the smallest value compared with $G'_{ny^+}(x, y, z)$ and $G'_{nz^+}(x, y, z)$. Because $G_n(x, y, z)$ is convex, $G'_{nx^+}(x, y, z)$ is a non-decreasing function of x . Therefore, the boundary between ordering something and nothing in sub-spaces ① and ⑥ is the curve surface $X^*(y, z)$ which is defined as:

$$G'_{nx^+}(x, y, z) \begin{cases} \geq 0, & \text{for } x \geq X^*(y, z) \\ < 0, & \text{for } x < X^*(y, z) \end{cases}$$

Similarly, in sub-spaces ② and ③, $G'_{ny^+}(x, y, z)$ has the smallest value, and the boundary of ordering something and nothing is the curve surface $Y^*(z, x)$ which is defined as:

$$G'_{ny^+}(x, y, z) \begin{cases} \geq 0, & \text{for } y \geq Y^*(z, x) \\ < 0, & \text{for } y < Y^*(z, x) \end{cases}$$

In sub-spaces ④ and ⑤, $G'_{nz^+}(x, y, z)$ is the smallest, and the boundary of ordering something and nothing is the curve surface $Z^*(x, y)$ which is defined as:

$$G'_{nz^+}(x, y, z) \begin{cases} \geq 0, & \text{for } z \geq Z^*(x, y) \\ < 0, & \text{for } z < Z^*(x, y) \end{cases}$$

From the above definitions, $X^*(y, z)$, $Y^*(z, x)$ and $Z^*(x, y)$ are clearly the minimum point of $G_n(x, y, z)$ as a function of y and z , z and x , and x and y respectively. $X^*(y, z)$, $Y^*(z, x)$ and $Z^*(x, y)$ intersect at the global minimum point with equal coordinate values defined in Lemma 5.2.

The following proposition describes one optimal order policy for the system.

Proposition 5.3

- a. For any point $P(x, y, z)$ in sub-space ① and ⑥ with its x value greater than or equal to $X^*(y, z)$, nothing needs to be ordered;
- b. For any point $P(x, y, z)$ in sub-space ① (⑥) with its x value less than $X^*(y, z)$, order item 1 first until $X^*(y, z)$ or a global minimum point is reached, or plane $x = y$ ($z = x$) is reached if capacity allows. If the plane $x = y$ ($z = x$) is reached, then order items 1 and 2 (items 1 and 3) along curve $x = y$ ($z = x$) at the given $z(y)$ value until $X^*(y, z)$ or a global minimum point is reached, or line L_1 is reached provided capacity allows. If line L_1 is reached, then order items 1, 2 and 3 along the line until a global minimum point is reached provided there is any capacity left.

Similar results hold for sub-spaces ② and ③, and ④ and ⑤ respectively.

Proof: According to the definition of $X^*(y, z)$, for a point $P(x, y, z)$ in sub-space ① and ⑥, if $x \geq X^*(y, z)$, then $G'_{nx^+}(x, y, z) \geq 0$. It is already known that in sub-space ①

$G'_{nx^+}(x, y, z) \leq G'_{ny^+}(x, y, z) \leq G'_{nz^+}(x, y, z)$. Thus

$$0 \leq G'_{nx^+}(x, y, z) \leq G'_{ny^+}(x, y, z) \leq G'_{nz^+}(x, y, z).$$

Similarly, if the point is in sub-space ⑥, then:

$$0 \leq G'_{nx^+}(x, y, z) \leq G'_{nz^+}(x, y, z) \leq G'_{ny^+}(x, y, z).$$

Both mean that ordering any product will not decrease the cost. Thus, nothing is required to be ordered. This proves (a).

For the point $P(x, y, z)$ within sub-spaces ①, if $x < X^*(y, z)$, then $G'_{nx^+}(x, y, z) < 0$.

Further, as $G'_{nx^+}(x, y, z) \leq G'_{ny^+}(x, y, z) \leq G'_{nz^+}(x, y, z)$ for any point in sub-space ①,

ordering item 1 has more advantage. The following optimal order policies specified in the proof of Proposition 5.2 are applied here.

Order Policy 1: when $x < y$, order item 1 only if necessary,

Order Policy 2: when $x = y$, order items 1 and 2 at the same time along $x = y$ if necessary.

So in the sub-space ① when $x < X^*(y, z)$, according to Order Policy 1, the inventory level x of item 1 will be increased. There are the following scenarios while increasing x .

Scenario 1: $X^*(y, z)$ is reached. Due to part (a) of this proposition, further cost reduction is impossible, so the local optimum is reached.

Scenario 2: A global minimum point is reached. Obviously, further cost reduction is also impossible.

Scenario 3: $x = y$ is reached. This is actually the case in Order Policy 2, and items 1 and 2 will be increased at the same time along $x = y$ if necessary. While this increasing x and y , preceding scenarios 1 and 2 could also occur and their respective analysis above applies.

Scenario 4: the line L_1 is reached. Since L_1 is the line with $x = y = z$, according to Order Policy 2, the optimal ordering is to increase the inventory levels of items 1, 2 and 3 at the same time along $x = y$, $y = z$, and $z = x$ which actually is the line L_1 itself, until the global minimum point is reached provided the capacity allows.

The preceding analysis is applicable to sub-space ⑥. This concludes part (b) of the proposition. For sub-spaces ② and ③ where item 2 has the highest order priority, and ④ and ⑤ where item 3 has the highest order priority, the above analysis is applicable. This completes the proof. $\square$

5.5 Extension to Simplified m -Product System

This section focuses on extending the results in the previous section for the simplified three-product system into m -product system with similar simplifications. Similarly, the simplified m -product model assumes that the system has the same i.i.d. stochastic demands, unit production costs and production rates for each product. The organization of this section is similar to that in the previous section.

Therefore, define

$L(X)$: the per period expected inventory cost (including holding and backlogging costs), assumed to be convex, is the same for all items $1, 2, \dots, m$.

u, c : production rate and unit production cost are the same for all items $1, 2, \dots, m$.

d : the per-period demand is the same for all items $1, 2, \dots, m$ and d is a random variable.

and $g(X) = L(X) + c * X$,

Define:

$$G_n(X_1, X_2, \dots, X_m) = g(X_1) + g(X_2) + \dots + g(X_m) + \alpha * E(f_{n-1}(X_1 - d, X_2 - d, \dots, X_m - d)) \quad (5.24)$$

then the simplified m -product system can be expressed as the following dynamic program.

$$f_n(x_1, x_2, \dots, x_m) = \min_{\substack{\frac{x_1 - x_1}{u} + \frac{x_2 - x_2}{u} + \dots + \frac{x_m - x_m}{u} \leq 1; x_1 \leq X_1, x_2 \leq X_2, \dots, x_m \leq X_m}} \{G_n(X_1, X_2, \dots, X_m)\} - c^*(x_1 + x_2 + \dots + x_m) \quad (5.25)$$

The following Lemma 5.3 has a result similar to Lemma 5.1 for simplified m -product system.

Lemma 5.3 For all n , $G_n(X_1, \dots, X_i, \dots, X_j, \dots, X_m) = G_n(X_1, \dots, X_j, \dots, X_i, \dots, X_m)$,

$$f_n(x_1, \dots, x_i, \dots, x_j, \dots, x_m) = f_n(x_1, \dots, x_j, \dots, x_i, \dots, x_m).$$

$$1 \leq i, j \leq m.$$

The following Lemma 5.4 is the complement of Lemma 5.2 for the simplified m -product system.

Lemma 5.4 There exists at least one global minimum point $P(x_1, x_2, \dots, x_m)$ for $G_n(x_1, x_2, \dots, x_m)$ and $f_n(x_1, x_2, \dots, x_m)$ respectively that its coordinate values are equal, i.e. $x_1 = x_2 = \dots = x_m$.

Proof: Suppose a global minimum point is $P_1(x_1, x_2, \dots, x_m)$ for $G_n(x_1, x_2, \dots, x_m)$. If $x_1 = x_2 = \dots = x_m$, then the lemma is proved. Suppose at least one equality does not hold, the following m points are obtained by putting the i th element of the previous sequence as the $(i-1)$ th and the first one as the last of the new sequence. Here the sequence consists of the m coordinate values of point P_1 . The first sequence is $(x_1, x_2, \dots, x_m)$.

$$P_1(x_1, x_2, \dots, x_m), P_2(x_2, x_3, x_4, \dots, x_m, x_1), P_3(x_3, x_4, \dots, x_1, x_2), \dots, P_m(x_m, x_1, \dots, x_{m-2}, x_{m-1})$$

By Lemma 5.3, if $P_1(x_1, x_2, \dots, x_m)$ is the global minimum point, the above m points are all global minimum points.

$$\text{Define } P_{12} = \frac{1}{2}P_1 + \frac{1}{2}P_2 = \left(\frac{x_1 + x_2}{2}, \frac{x_2 + x_3}{2}, \dots, \frac{x_m + x_1}{2}\right),$$

$$P_{123} = \frac{2}{3}P_{12} + \frac{1}{3}P_3 = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{x_2 + x_3 + x_4}{3}, \dots, \frac{x_m + x_1 + x_2}{3}\right)$$

Because of the convexity of $G_n(x_1, x_2, \dots, x_m)$, P_{12} and P_{123} are also global minimum points.

Suppose $P_{12\dots(i-1)}$ is a global minimum point, and

$$P_{12\dots i} = \frac{i-1}{i}P_{12\dots(i-1)} + \frac{1}{i}P_i = \left(\frac{x_1 + x_2 + \dots + x_i}{i}, \frac{x_2 + x_3 + \dots + x_{i+1}}{i}, \dots, \frac{x_m + x_1 + \dots + x_{i-1}}{i}\right),$$

then $P_{12\dots i}$ is a global minimum point, and the following equality holds.

$$P_{12\dots i(i+1)} = \frac{i}{i+1}P_{12\dots i} + \frac{1}{i+1}P_{i+1} = \left(\frac{x_1 + x_2 + \dots + x_i + x_{i+1}}{i+1}, \frac{x_2 + x_3 + \dots + x_{i+2}}{i+1}, \dots, \frac{x_m + x_1 + \dots + x_i}{i+1}\right)$$

The preceding result proves the following equality,

$$P_{12\dots m} = \frac{m-1}{m}P_{12\dots m-1} + \frac{1}{m}P_m = \left(\frac{x_1 + x_2 + \dots + x_{m-1} + x_m}{m}, \frac{x_2 + x_3 + \dots + x_m + x_1}{m}, \dots, \frac{x_m + x_1 + \dots + x_{m-1}}{m}\right)$$

and $P_{12\dots m}$ is also a global minimum point.

The above analysis is applicable to $f_n(x_1, x_2, \dots, x_m)$. This completes the lemma. $\square$

Proposition 5.4

If $x_i = x_j$, then $G'_{nx_i^+}(x_1, \dots, x_i, \dots, x_j, \dots, x_m) = G'_{nx_j^+}(x_1, \dots, x_i, \dots, x_j, \dots, x_m)$, and

$$f'_{nx_i^+}(x_1, \dots, x_i, \dots, x_j, \dots, x_m) = f'_{nx_j^+}(x_1, \dots, x_i, \dots, x_j, \dots, x_m).$$

Proof: The proof is the straightforward extension of that for Proposition 5.1. $\square$

Proposition 5.5

If $x_i < x_j$, then $G'_{nx_i^+}(x_1, \dots, x_i, \dots, x_j, \dots, x_m) \leq G'_{nx_j^+}(x_1, \dots, x_i, \dots, x_j, \dots, x_m)$, and

$$f'_{nx_i^+}(x_1, \dots, x_i, \dots, x_j, \dots, x_m) \leq f'_{nx_j^+}(x_1, \dots, x_i, \dots, x_j, \dots, x_m).$$

If $x_i > x_j$, then $G'_{nx_i^+}(x_1, \dots, x_i, \dots, x_j, \dots, x_m) \geq G'_{nx_j^+}(x_1, \dots, x_i, \dots, x_j, \dots, x_m)$, and

$$f'_{nx_i^+}(x_1, \dots, x_i, \dots, x_j, \dots, x_m) \geq f'_{nx_j^+}(x_1, \dots, x_i, \dots, x_j, \dots, x_m).$$

Proof: The proof is the straightforward extension of that for Proposition 5.2. It needs to be pointed out that for the simplified m -product system, the two optimal order policies specified in the proof of Proposition 5.2 becomes as follows. Under the condition that there is a need to order item i or j , the first order policy is when $x_i < x_j$, then item i has higher order priority. The second order policy is when $x_i = x_j$, then ordering i and j at the same time along $x_i = x_j$. $\square$

Solution framework can be constructed based on the above results. The framework description will be based on the example for $m = 4$. When $m = 4$, every following equality in the 3-dimensional space will divide the 4-dimensional space into two parts.

$$x_i = x_j, \quad i \neq j, \quad 1 \leq i, j \leq 4$$

The divided two parts in 4-dimensional space are:

Part 1: $x_i < x_j$, and $G'_{nx_i^+}(x_1, x_2, x_3, x_4) \leq G'_{nx_j^+}(x_1, x_2, x_3, x_4)$,

Part 2: $x_i > x_j$, and $G'_{nx_i^+}(x_1, x_2, x_3, x_4) \geq G'_{nx_j^+}(x_1, x_2, x_3, x_4)$.

All the above $x_i = x_j$ passes through the line of $x_1 = x_2 = x_3 = x_4$ in the 4-dimensional space. There are $C_4^1 * C_3^1 * C_2^1 * C_1^1 = 24$ possibilities for the relationships among (x_1, x_2, x_3, x_4) . By Proposition 5.5, the relationship of the (x_1, x_2, x_3, x_4) is also the relationship of their respective partial derivatives. Thus, the 24 relationships among (x_1, x_2, x_3, x_4) and their partial derivatives are:

Sub-spaces 1-6: $x_1 < x_j < x_k < x_l, j \neq k \neq l, j, k, l \in \{2, 3, 4\}$,

$$G'_{nx_1^+}(x_1, x_j, x_k, x_l) \leq G'_{nx_j^+}(x_1, x_j, x_k, x_l) \leq G'_{nx_k^+}(x_1, x_j, x_k, x_l) \leq G'_{nx_l^+}(x_1, x_j, x_k, x_l)$$

Sub-spaces 7-12: $x_2 < x_j < x_k < x_l, j \neq k \neq l, j, k, l \in \{1, 3, 4\}$,

$$G'_{nx_2^+}(x_2, x_j, x_k, x_l) \leq G'_{nx_j^+}(x_2, x_j, x_k, x_l) \leq G'_{nx_k^+}(x_2, x_j, x_k, x_l) \leq G'_{nx_l^+}(x_2, x_j, x_k, x_l)$$

Sub-spaces 13-18: $x_3 < x_j < x_k < x_l, j \neq k \neq l, j, k, l \in \{1, 2, 4\}$,

$$G'_{nx_3^+}(x_3, x_j, x_k, x_l) \leq G'_{nx_j^+}(x_3, x_j, x_k, x_l) \leq G'_{nx_k^+}(x_3, x_j, x_k, x_l) \leq G'_{nx_l^+}(x_3, x_j, x_k, x_l)$$

Sub-spaces 19-24: $x_4 < x_j < x_k < x_l, j \neq k \neq l, j, k, l \in \{1, 2, 3\}$,

$$G'_{nx_4^+}(x_4, x_j, x_k, x_l) \leq G'_{nx_j^+}(x_4, x_j, x_k, x_l) \leq G'_{nx_k^+}(x_4, x_j, x_k, x_l) \leq G'_{nx_l^+}(x_4, x_j, x_k, x_l)$$

In sub-spaces 1, 2, 3, 4, 5 and 6 (denoted as 1-6), $G'_{nx_1^+}(x_1, x_2, x_3, x_4)$ has the smallest value compared with $G'_{nx_2^+}(x_1, x_2, x_3, x_4)$, $G'_{nx_3^+}(x_1, x_2, x_3, x_4)$ and $G'_{nx_4^+}(x_1, x_2, x_3, x_4)$.

Thus, item 1 has the highest order priority. Because $G_n(x_1, x_2, x_3, x_4)$ is convex,

$G'_{nx_1^+}(x_1, x_2, x_3, x_4)$ is a non-decreasing function of x_1 ; therefore, the boundary between

ordering something and nothing for sub-spaces 1-6 is $X_1^*(x_2, x_3, x_4)$ which is defined as follows.

$$G'_{nx_1^+}(x_1, x_2, x_3, x_4) \begin{cases} \geq 0, & \text{for } x_1 \geq X_1^*(x_2, x_3, x_4); \\ < 0, & \text{for } x_1 < X_1^*(x_2, x_3, x_4). \end{cases}$$

Similarly, in sub-spaces 7-12, item 2 has the highest order priority; in sub-spaces 13-18, item 3 has the highest order priority; and in sub-spaces 19-24, item 4 has the highest order priority. Also, similar definitions can be made for $X_2^*(x_3, x_4, x_1)$, $X_3^*(x_4, x_1, x_2)$, and $X_4^*(x_1, x_2, x_3)$ for sub-spaces 7-12, 13-18, and 19-24 respectively.

The preceding analysis can be applied to m -dimension. For the simplified m -product system, every $x_i = x_j$, $i \neq j, 1 \leq i, j \leq m$ divides the m -dimensional space into two parts.

One part is with $x_i < x_j$, and $G'_{nx_i^+}(x_1, \dots, x_i, \dots, x_j, \dots, x_m) \leq G'_{nx_j^+}(x_1, \dots, x_i, \dots, x_j, \dots, x_m)$,

the other part is with $x_i > x_j$, and $G'_{nx_i^+}(x_1, \dots, x_i, \dots, x_j, \dots, x_m) \geq G'_{nx_j^+}(x_1, \dots, x_i, \dots,$

$x_j, \dots, x_m)$. All these parts have a common boundary of $x_1 = x_2 = \dots = x_i = \dots = x_m$, and

construct the sub-spaces in number of $C_m^1 * C_{m-1}^1 * \dots * C_2^1 * C_1^1 = m!$. For any x_i ,

$1 \leq i \leq m$, there are $C_{m-1}^1 * \dots * C_2^1 * C_1^1 = (m-1)!$ sub-spaces in which item i has the

highest order priority. Due to convexity of $G_n(x_1, \dots, x_i, \dots, x_m)$, $G'_{nx_i^+}(x_1, \dots, x_i, \dots, x_m)$ is a

non-decreasing function of x_i . Therefore, the boundary between ordering something and

nothing for the sub-spaces where x_i has the highest order priority is the following

$$X_i^*(\dots, x_{i-1}, x_{i+1}, \dots).$$

$$G'_{nx_i^+}(x_1, \dots, x_i, \dots, x_m) \begin{cases} \geq 0, & \text{for } x_i \geq X_i^*(\dots, x_{i-1}, x_{i+1}, \dots); \\ < 0, & \text{for } x_i < X_i^*(\dots, x_{i-1}, x_{i+1}, \dots). \end{cases}$$

For a point $P(x_1, \dots, x_i, \dots, x_m)$ in the sub-spaces where x_i has the highest order priority, ordering is possible clearly means that $x_i < X_i^*(\dots, x_{i-1}, x_{i+1}, \dots)$, capacity is not used up, and the point is not a global minimum point. By Proposition 5.5, for a point $P(x_1, \dots, x_i, \dots, x_m)$ in the sub-spaces where x_i is the smallest among $\{x_1, \dots, x_i, \dots, x_m\}$, it also means that item i has the highest order priority. By the policies defined in the proof of Proposition 5.5, x_i will be increased to the next smallest, say x_{i1} , among $\{x_1, \dots, x_i, \dots, x_m\}$. Since $x_i = x_{i1}$, if ordering is possible, then according to the order policies, x_i and x_{i1} will be increased at the same time; also by the order policy, x_i and x_{i1} will be increased to the next smallest value among $\{x_1, \dots, x_i, \dots, x_m\}$. This process can continue until ordering is impossible. Concluding the above analysis, the following proposition can be stated.

Proposition 5.6

- a. For any point $P(x_1, \dots, x_i, \dots, x_m)$ in the sub-spaces where x_i is the smallest among $\{x_1, \dots, x_i, \dots, x_m\}$, if x_i is greater than or equal to $X_i^*(\dots, x_{i-1}, x_{i+1}, \dots)$, nothing needs to be ordered;
- b. For any point $P(x_1, \dots, x_i, \dots, x_m)$ in the sub-spaces where x_i is the smallest among $\{x_1, \dots, x_i, \dots, x_m\}$, if x_i is less than $X_i^*(\dots, x_{i-1}, x_{i+1}, \dots)$, then order item i first until $X_i^*(\dots, x_{i-1}, x_{i+1}, \dots)$ is reached, or a global minimum point is reached, or $x_i = x_{i1}$ (x_{i1} is the second smallest value among $\{x_1, \dots, x_i, \dots, x_m\}$) is reached if

capacity allows. If the $x_i = x_{i1}$ is reached, then order items i and $i1$ simultaneously along $x_i = x_{i1}$ until $X_i^*(..., x_{i-1}, x_{i+1}, ...)$ is reached, or a global minimum point is reached, or the next smallest value say x_{i2} is reached if capacity allows, then $x_i = x_{i1} = x_{i2}$. This process can continue if ordering is possible until the common boundary line $x_1 = x_2 = ... = x_i = ... = x_m$ is reached. If this common boundary line is reached, then order items 1, 2, ..., i , ... and m simultaneously along the line until the global minimum point is reached if there is any capacity left.

For the simplified system, production or purchasing manager can obtain an optimal implementation easily. For instance, if ordering or production is necessary, ordering or producing the product(s) with the lowest inventory is always optimal.

5.6 Hypothesis for General m -Product System

Based on the results for simplified multiple product system, and Chen (2004b)'s general two-product system, this section tries to propose a hypothesis for the general multiple product system.

Chen (2004b) proves that $X^*(y)$, and $Y^*(x)$ have the following properties which characterize the shapes of the two curves for the two-product system described in Figure 5.1.

$$(1) \quad X^*(S_y) = S_x, \quad Y^*(S_x) = S_y;$$

- (2) $X^*(y) \downarrow x$, and $Y^*(x) \downarrow y$;
- (3) $X^*(y)$ and $Y^*(x)$ are right-sided continuous;
- (4) The slope of the curve $X^*(y)$ is less than or equal to $(-u_2/u_1)$, while that of $Y^*(x)$ is greater than or equal to $(-u_2/u_1)$.

Where $\uparrow(\downarrow)x$ means non-decreasing (non-increasing) in x , and the others are defined similarly.

Although sections 5.4 and 5.5 partially describe the order policy characterization for the simplified m -product system, the characteristics of $X_i^*(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_m)$ are still unknown. The reason is that the μ -difference monotone defined in section 5.3 still cannot be proved to be true even for the simplified n -product system. If it is true, using similar characterizations for the order region boundaries can be easily obtained by using similar processes as in Chen (2004b).

Therefore, for the simplified or general m -product system one challenge is to test whether the following similar μ -difference monotone for the m -product system holds or not.

Possible Definition 1 of μ -Difference Monotone for m -Product System: a convex function $G_n(x_1, x_2, \dots, x_i, \dots, x_m)$ is said to be μ -difference monotone if and only if:

- (1) $G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m) \uparrow x_1, \uparrow x_2, \dots, \uparrow x_m, i = 1, 2, \dots, m$;
- (2) $u_i G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m) - u_j G_{nx_j^+}'(x_1, x_2, \dots, x_j, \dots, x_m) \uparrow x_i, \downarrow x_j, i, j = 1, 2, \dots, m, i \neq j$.

For the simplified m -product system, the Proposition 5.3 is true due to symmetry of the items, and it states that the inventory level of the third item x_k does not affect

$G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m) = G_{nx_j^+}'(x_1, x_2, \dots, x_j, \dots, x_m)$ when $x_i = x_j$, $i, j, k = 1, 2, \dots, m$, $i \neq j \neq k$. However, for the general m -product system, how x_k affects $u_i G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m) - u_j G_{nx_j^+}'(x_1, x_2, \dots, x_j, \dots, x_m)$ is still not clear. From the order policy for the simplified m -product system, it is known that when x_i has the highest priority to be increased, e.g. $u_i G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m) < u_j G_{nx_j^+}'(x_1, x_2, \dots, x_j, \dots, x_m)$, $i \neq j, j = 1, 2, \dots, m$, then x_i will be increased first until it reaches the boundary of ordering and non-ordering regions, or it reaches the global minimum point, or it reaches $x_i = x_j$, then x_i and x_j will be increased at the same time until the next minimum x_k is reached, or the boundary of ordering and non-ordering regions is reached, or the global minimum point is reached. It can be seen that $G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m)$ is increased to reach the next minimum derivative, say $G_{nx_j^+}'(x_1, x_2, \dots, x_i, \dots, x_m)$, subsequently both are increased to reach the next minimum derivative, and this process can continue if possible. Furthermore, as described in Chen (2004b), inventory could be regarded as stored capacity; if the inventory level of a product is higher, then it will reduce the chance and intensity that the product competes for the limited capacity with the others in future. So from the preceding observation and arguments, basically, there is following possible relationship for how the third item inventory level x_k affects $u_i G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m) - u_j G_{nx_j^+}'(x_1, x_2, \dots, x_j, \dots, x_m)$.

If $u_i G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m) < u_j G_{nx_j^+}'(x_1, x_2, \dots, x_j, \dots, x_m)$, then

$$u_i G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m) - u_j G_{nx_j^+}'(x_1, x_2, \dots, x_j, \dots, x_m) \uparrow x_k, \text{ and}$$

if $u_i G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m) > u_j G_{nx_j^+}'(x_1, x_2, \dots, x_j, \dots, x_m)$, then

$$u_i G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m) - u_j G_{nx_j^+}'(x_1, x_2, \dots, x_j, \dots, x_m) \downarrow x_k.$$

The above two cases can be summarized in the following formula:

$$|u_i G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m) - u_j G_{nx_j^+}'(x_1, x_2, \dots, x_j, \dots, x_m)| \downarrow x_k.$$

In summary, the possible properties for the general m -product system could be defined as follows.

Possible Definition 2 of μ -Difference Monotone for m -Product System: A convex function $G_n(x_1, x_2, \dots, x_i, \dots, x_m)$ is said to be μ -difference monotone if and only if:

- (1) $G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m) \uparrow x_1, \uparrow x_2, \dots, \uparrow x_m, i = 1, 2, \dots, m$;
- (2) $u_i G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m) - u_j G_{nx_j^+}'(x_1, x_2, \dots, x_j, \dots, x_m) \uparrow x_i, \downarrow x_j, i, j = 1, 2, \dots, m, i \neq j$.
- (3) $|u_i G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m) - u_j G_{nx_j^+}'(x_1, x_2, \dots, x_j, \dots, x_m)| \downarrow x_k, i, j, k = 1, 2, \dots, m, i \neq j \neq k$.

In the following analysis, $u_i G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m) = u_j G_{nx_j^+}'(x_1, x_2, \dots, x_j, \dots, x_m)$ is assumed to have solution(s) for simplification. If $G_n(x_1, x_2, \dots, x_i, \dots, x_m)$ and $f_n(x_1, x_2, \dots, x_i, \dots, x_m)$ defined in equations (5.2) and (5.3) are μ -difference monotone of the preceding Possible Definition 2, then the solution framework for the general m -product system will be similar to that for the simplified m -product system described in sections 5.4 and 5.5. Clearly, if $u_i G_{nx_i^+}'(x_1, x_2, \dots, x_i, \dots, x_m) = u_j G_{nx_j^+}'(x_1, x_2, \dots, x_j, \dots, x_m)$, then for any x_k value, $i \neq j \neq k$, the equality still holds due to

$|u_i G_{nx_i} (x_1, x_2, \dots, x_i, \dots, x_m) - u_j G_{nx_j} (x_1, x_2, \dots, x_j, \dots, x_m)| \downarrow x_k$. Thus, the whole m -dimensional space will be divided by the $(m-1)$ dimensional spaces with $u_i G_{nx_i} (x_1, x_2, \dots, x_i, \dots, x_m) = u_j G_{nx_j} (x_1, x_2, \dots, x_j, \dots, x_m)$ $i, j = 1, 2, \dots, m, i \neq j$, and the whole space will be separated into different sub-spaces with the exact relationships among $u_1 G_{nx_1} (x_1, x_2, \dots, x_m)$, $u_2 G_{nx_2} (x_1, x_2, \dots, x_m)$, $\dots$, $u_n G_{nx_n} (x_1, x_2, \dots, x_m)$.

Therefore, a similar solution framework can be constructed. On the other hand, because of conditions (1), and (2) of the Possible Definition 2, the similar properties described at the beginning of this section for the ordering boundary can also be obtained.

5.7 Summary

This chapter discusses a basic problem of multiple-product flexible manufacturing system. The focus is on the simplified model where the stochastic demands, unit production costs and production rates are the same for each product. For the simplified model, hedging point policy is proved to be optimal; the ordering and not-ordering regions for every product are defined. These new results not only characterize the optimal policy for the simplified model, but also give clues to the general multiple-product system.

Chapter 6

Parameter Analysis for Economic Lot Size Problem with Backlogging

6.1 Introduction

Economic lot size models study the production and inventory system. Products are produced in batches and placed into inventory. When the inventory depletes sufficiently, another batch is produced. This procedure is repeated as often as necessary during the planning time horizon. Here a single product is assumed throughout the chapter. In real life, numerous problems in production planning and supply chain optimization can be modeled as an economic lot size problem (ELSP). For example, in consecutive planning time periods with different deterministic periodic demand, and considering the single item production scheduling problem at the manufacturer's site and the same product's inventory control problem at the retailer's site, it is a typical ELSP that both sites are to be optimized simultaneously in order to achieve supply chain optimization.

ELSP is a classical problem which was first proposed by Wagner and Whitin (1958). After Wagner and Whitin (1958) solved the basic economic lot size model without backlogging by an $O(N^2)$ algorithm (N is the length of time planning horizon), considerable research efforts have been made to extend the basic model. Two important extensions are the consideration of backlogging, and the improvement of computational complexity. In this chapter the algorithms proposed by Zangwill (1969), and Van Hoesel, Wagelmans and Moerman (1994) are re-interpreted, and the forward and backward

algorithms are applied for conducting parameter analysis on setup cost, total demands, and unit production cost.

6.2 Previous Results

This section reviews the theoretical results and sensitivity analysis for ELSP, especially the geometric technique in Van Hoesel, Wagelmans and Moerman (1994). Zangwill (1966) considers inventory backlogging in the basic economic lot size model, and gives the definition of exact requirements which is described in Chapter 2. The fundamental result in Zangwill (1966) is that there exists an optimal production schedule that satisfies exact requirements. In the same paper Zangwill also developed an $O(N^3)$ algorithm to solve the model. In another paper, Zangwill (1969) expresses the economic lot size problem with and without backlogging as a network that has one entry with the demand summations and multiple exits with the demand of every period. He also develops an efficient $O(N^2)$ algorithm for the model where the production cost is fixed. It is worth noting that the cost functions of every period in Zangwill (1969)'s work can be linear or nonlinear and the value should be computed in constant time.

In the rest of this chapter, ELSP specifically refers to the model having linear production and inventory costs in addition to setup cost. Wagelmans, Van Hoesel and Kolen (1992) reformulate Wagner-Whitin's model into a new model without considering holding cost, and employ a geometric method to reduce the computational complexity from $O(N^2)$ to $O(N \log N)$. Furthermore, Van Hoesel, Wagelmans and Moerman (1994) generalize the

results of Wagelmans, Van Hoesel and Kolen (1992). They consider the following dynamic program.

$$G(0) = 0,$$

$$G(i) = A_i + \min_{0 \leq j < i} \{G(j) + B_j + C_i D_j\}, 1 \leq i \leq N$$

Here A_i, B_j, C_i, D_j ($1 \leq i \leq N, 1 \leq j \leq N-1$) are constants that only depend on the index.

First they define the line m_j ($0 \leq j < i$) as:

$$m_j(x) = G(j) + B_j + D_j x$$

Actually, m_j is the line passing through the point $(0, G(j) + B_j)$ and has the slope D_j , where the x -axis of (x, y) represents the index i , and y -axis represents a possible value on the index. Therefore, finding the minimum of $\{G(j) + B_j + C_i D_j \mid 0 \leq j < i\}$ is same as finding the minimum of the values $m_j(C_i)$ over $j, 0 \leq j < i$. Furthermore, the dynamic program recursion can be viewed as constructing, maintaining and finding the lower concave envelope of a set of lines. By this basic technique, Van Hoesel, Wagelmans and Moerman (1994) first prove that the preceding dynamic program can be solved in the time of $O(N \log N)$. Subsequently, they prove that Wagner-Whitin model, ELSP with backlogging, and some other extension models can be expressed as the same dynamic program and thus all of these models have the same computational complexity.

Based on the geometric interpretation of the ELSP developed in Wagelmans, Van Hoesel and Kolen (1992), Van Hoesel and Wagelmans (1993) study the variation scopes of setup, production, holding costs and demand respectively such that the optimal production schedule remains unchanged when the parameter varies. The results obtained

are all computation related. They give the computational complexity and detailed data structure for every parameter about how to compute the variation scope. If the setup cost for every period is constant and the unit production cost is also fixed, Chand and Vörös (1992) prove that the total cost of holding and backlogging is a non-increasing convex function of the number of setups for the ELSP with and without backlogging, and develop an $O(N^2)$ forward algorithm to compute the stability region of the setup cost.

6.3 Model Description and Notations

Mathematically, economic lot sizing model with backlogging is to satisfy the known demands in a planning horizon for a single commodity at minimum cost. ELSP with backlogging permits to produce later to satisfy the demand of a prior period. The inventory levels at the beginning and ending periods of the planning horizon are zero. The difference between this research and Van Hoesel and Wagelmans (1993)'s work is that backlogging is considered in this study. Furthermore, the results obtained in this chapter are not all computation related. The following notations will be used in the rest of this chapter.

N : the length of the planning horizon,

d_i : the demand in period $i \in \{1, \dots, N\}$,

p_i : the unit production cost in period $i \in \{1, \dots, N\}$,

f_i : the setup cost in period $i \in \{1, \dots, N\}$,

h_i^+ : the unit inventory cost in period $i \in \{1, \dots, N\}$,

h_i^- : the unit backlogging cost in period $i \in \{1, \dots, N\}$,

$$d_{i,j} = \sum_{t=i}^j d_t, \quad 1 \leq i \leq j \leq N,$$

$$h_{i,j}^+ = \sum_{t=i}^j h_t^+, 1 \leq i \leq j \leq N,$$

$$h_{i,j}^- = \sum_{t=i}^j h_t^-, 1 \leq i \leq j \leq N,$$

$F(i)$: the minimum cost from the beginning of period 1 to the end of period i , the demands are satisfied from the productions in these periods. The inventory at the end of period i is zero (such period is defined as regeneration period). $F(i)$ is actually the optimal cost for the problem restricted to the planning horizon from period 1 to period i .

$F'(j)$: the minimum cost from the beginning of period 1 to the end of period j , and the demands are satisfied by the productions in these periods. Period j is a production period which produces some units of the product in this period. The cost is actually the optimum of the restricted problem for planning horizon from period 1 to j with the special additional requirement that period j is a production period.

$B(i)$: the minimum cost from the beginning of period i to the end of period N , and the demands are satisfied by the productions in these periods. The inventories at the beginning of period i and at the end of period N are zero. Similarly, it is the optimal cost of the problem restricted to the planning horizon from period i to N .

$B'(j)$: the minimum cost from the beginning of period j to the end of period N , and the demands are satisfied by the productions in these periods, the end inventory is zero. Period j is a production period. Similarly, the cost is actually the optimum of the restricted problem for planning horizon from period j to N with the special additional requirement that period j is a production period.

6.4 Forward and Backward Algorithms

Zangwill (1969) gives a two-step expression to demonstrate ELSP model with backlogging. Van Hoesel, Wagelmans and Moerman (1994) reformulate the formulas into the form described in section 6.2. Based on both the preceding works, the backward and forward dynamic program expressions are re-formulated into similar dynamic program forms but only with the parameters which are within the studied period scope. By this method, some results in the following sections for parameter analysis can easily be observed. The following sections 6.4.1 and 6.4.2 describe the procedures to obtain the backward and forward dynamic programs respectively.

6.4.1 Backward Algorithm

This section demonstrates that ELSP model can be expressed in terms of form only with the parameters which are within the studied period scope backwardly. In Zangwill (1969) and Van Hoesel, Wagelmans and Moerman (1994), the backward algorithm of ELSP model with backlogging can be expressed as the following equations:

$$B(s) = \min_{s \leq t \leq N+1} \{p_t d_{s,t-1} + \sum_{\tau=s}^{t-1} h_{\tau}^{-} d_{s,\tau} + B'(t)\} \quad (6.1)$$

$$B'(t) = \min_{t < u \leq N+1} \{f_t + p_t d_{t,u-1} + \sum_{\tau=t}^{u-2} h_{\tau}^{+} d_{\tau+1,u-1} + B(u)\} \quad (6.2)$$

$$B(N+1) = 0, \quad B'(N+1) = 0.$$

$$\text{Define } M(s, t) = \sum_{\tau=s}^{t-1} h_{\tau}^{-} d_{s,\tau} \text{ and } M'(t, u) = \sum_{\tau=t}^{u-2} h_{\tau}^{+} d_{\tau+1,u-1}$$

The following equalities hold:

$$M(s, t) = M(s, N) - M(t, N) - h_{t,N-1}^{-} d_{s,t-1},$$

$$M'(t, u) = M'(t, N) - M'(u, N) - h_{t,u-1}^+ d_{u,N-1}.$$

Therefore equations (6.1) and (6.2) can be re-written as:

$$\begin{aligned} B(s) &= \min_{s \leq t \leq N+1} \{p_t d_{s,t-1} + M(s, N) - M(t, N) - h_{t,N-1}^- d_{s,t-1} + B'(t)\} \\ &= M(s, N) + \min_{s \leq t \leq N+1} \{B'(t) - M(t, N) - (p_t - h_{t,N-1}^-) d_{t,N} + (p_t - h_{t,N-1}^-) d_{s,N}\} \end{aligned} \quad (6.1a)$$

$$\begin{aligned} B'(t) &= \min_{t < u \leq N+1} \{f_t + p_t d_{t,u-1} + M'(t, N) - M'(u, N) - h_{t,u-1}^+ d_{u,N-1} + B(u)\} \\ &= M'(t, N) + f_t - h_{t,N}^+ d_{t,N-1} + \min_{t < u \leq N+1} \{B(u) - M'(u, N) + h_{u,N}^+ d_{u,N-1} + (p_t + h_{t,N}^+) d_{t,u-1}\} \end{aligned} \quad (6.2a)$$

The geometric meaning of equalities (6.1a) and (6.2a) can be explained as follows:

The lines passing through $(0, B'(t) - M(t, N) - (p_t - h_{t,N-1}^-) d_{t,N})$ with slopes $(p_t - h_{t,N-1}^-)$ construct the concave lower envelope, and the $B(s)$ can be calculated from $B'(t)$ ($t = s + 1, \dots, N$) by maintaining and updating this concave lower envelope. The points of $(d_{u,N}, B(u) - M'(u, N) + h_{u,N}^+ d_{u,N-1})$ construct a convex lower envelope, and $B'(t)$ can be obtained from $B(u)$ ($u = t + 1, \dots, N$) by searching the point that the line passing through with slope $p_t + h_{t,N}^+$ is tangent to the envelope. The detailed proof and illustration are in Van Hoesel, Wagelmans and Moerman (1994).

6.4.2 Forward Algorithm

This section shows that form similar to the one shown in section 6.4.1 can be obtained forwardly. The forward algorithm can be expressed as the following formulas:

$$F(s) = \min_{1 \leq t \leq s} \{p_t d_{t+1,s} + \sum_{\tau=t}^{s-1} h_{\tau}^{+} d_{\tau+1,s} + F'(t)\} \quad (6.3)$$

$$F'(t) = \min_{1 \leq u < t} \{f_t + p_t d_{u+1,t} + \sum_{\tau=u+1}^{t-1} h_{\tau}^{-} d_{u+1,\tau} + F(u)\} \quad (6.4)$$

and $F(0) = 0, \quad F'(0) = 0$.

Similarly, define $m(t, s) = \sum_{\tau=t}^{s-1} h_{\tau}^{+} d_{\tau+1,s}$, and $m'(u, t) = \sum_{\tau=u+1}^{t-1} h_{\tau}^{-} d_{u+1,\tau}$.

It is trivial to prove the following two equalities.

$$m(t, s) = m(1, s) - m(1, t) - h_{1,t-1}^{+} d_{t+1,s},$$

$$m'(u, t) = m'(1, t) - m'(1, u) - h_{u,t-1}^{-} d_{2,u}.$$

By applying the above equations, equalities (6.3) and (6.4) can be rewritten as:

$$\begin{aligned} F(s) &= \min_{1 \leq t \leq s} \{p_t d_{t+1,s} + m(1, s) - m(1, t) - h_{1,t-1}^{+} d_{t+1,s} + F'(t)\} \\ &= m(1, s) + \min_{1 \leq t \leq s} \{F'(t) - m(1, t) - (p_t - h_{1,t-1}^{+}) d_{1,t} + (p_t - h_{1,t-1}^{+}) d_{1,s}\} \end{aligned} \quad (6.3a)$$

$$\begin{aligned} F'(t) &= \min_{1 \leq u < t} \{f_t + p_t d_{u+1,t} + m'(1, t) - m'(1, u) - h_{u,t-1}^{-} d_{2,u} + F(u)\} \\ &= f_t + m'(1, t) - h_{1,t-1}^{-} d_{2,t} + \min_{1 \leq u < t} \{F(u) - m'(1, u) + h_{1,u-1}^{-} d_{2,u} + (p_t + h_{1,t-1}^{-}) d_{u+1,t}\} \end{aligned} \quad (6.4a)$$

The similar geometric meaning of equalities (6.3a) and (6.4a) can be explained as:

The lines passing through $(0, F'(t) - m(1, t) - (p_t - h_{1,t-1}^{+}) d_{1,t})$ with slopes $(p_t - h_{1,t-1}^{+})$ construct the concave lower envelope, and the $F(s)$ can be computed by maintaining and updating this concave lower envelope. The points of $(d_{1,u}, F(u) - m'(1, u) + h_{1,u-1}^{-} d_{2,u})$ construct a convex lower envelope, and $F'(t)$ can be obtained by searching the point that

the line passing through with slope $p_t + h_{1,t-1}^-$ is tangent to the envelope. Van Hoesel, Wagelmans and Moerman (1994) prove that the time complexity of solving (6.1a) or (6.3a) for the planning horizon is $O(N \log N)$.

Now, the backward and forward dynamic algorithms are expressed with the parameters within the study periods. In the following sections, the algorithms are used for parameter analysis of setup cost, total demands and unit production cost.

6.5 Parameter Analysis of Setup Cost

In this section, two scenarios have been considered: setup cost in a period is decreased, or increased by an amount δ . The variation range of δ is analyzed such that the production schedule remains unchanged. First, it is obvious to have the following lemma directly from equalities (6.2a) and (6.4a).

Lemma 6.1 If in period i , the setup cost is changed from f_i to $f_i \pm \delta$, then

- I. From period 1 to $i-1$, $F'(\cdot)$ remains unchanged, and $F'(i)$ changes by $\pm \delta$,
- II. From period $i+1$ to N , $B'(\cdot)$ remains unchanged, and $B'(i)$ changes by $\pm \delta$.

Scenario (i) Setup cost is decreased from f_i to $f_i - \delta$.

Two situations are considered: period i is a production period, and again is not. For the first situation, it is intuitive that period i remains production period if setup cost is decreased, which is concluded in Proposition 6.2. For the second situation, it will be proved in Proposition 6.3 that there is a bound for the setup cost decreasing.

First, consider period i is a production period in the original production schedule. The following Proposition 6.1 is obvious.

Proposition 6.1 If period i is a production period in ELSP with backlogging, it will still be a production period if its setup cost is decreased.

Lemma 6.2 If period i is a production period in the final optimal production schedule, then $B(1) = F(N) = F'(i) + B'(i) - f_i - p_i d_i$.

Proof: From the definitions of $B(i)$ and $F(i)$, $B(1) = F(N)$ is clear. Suppose the final production periods are indexed by $1 \leq i_1 < i_2 < \dots < i_j < i < i_r < \dots < i_k \leq N$. If the production schedule before and after period i in this final production schedule is also the optimal production schedule for $F'(i)$ and $B'(i)$ respectively, then the total cost can be counted separately by $F'(i)$ and $B'(i)$, and thus the lemma will be proved. This will be proved by contradiction. Suppose there is a better production schedule $i_1, i_2, \dots, i_j, i$ other than $i_1, i_2, \dots, i_j, i$ for which the total cost for $F'(i)$ is smaller, then obviously $i_1, i_2, \dots, i_j, i, i_r, \dots, i_k$ will be a better production schedule than $i_1, i_2, \dots, i_j, i, i_r, \dots, i_k$. This is a contradiction to our supposition. In a similar way, $i, i_r, \dots, i_k$ can be proved as the optimal production schedule for $B'(i)$. Because $f_i + p_i d_i$ is counted both in $F'(i)$ and $B'(i)$, therefore $B(1) = F(N) = F'(i) + B'(i) - f_i - p_i d_i$. $\square$

From Proposition 6.1 and Lemma 6.1, the following conclusion can be obtained.

Proposition 6.2 If period i is a production period in ELSP model with backlogging, its setup cost is decreased from f_i to $f_i - \delta$, and the production schedule remains unchanged, then the optimal cost decreases by δ .

Next, the situation that period i is not a production period in the original problem is analyzed. Suppose period i is not a production period in the original optimal production schedule and it becomes a production period in the new problem where its setup cost decreases from f_i to $f_i - \delta$, then by Lemma 6.2, the new optimal cost will be $F'(i) + B'(i) - f_i - p_i d_i - \delta$, which should be less than or equal to the original optimal cost $B(1)$ for the model in which period i is not a production period. So:

$$F'(i) + B'(i) - f_i - p_i d_i - \delta < B(1),$$

$$\text{or} \quad \delta > F'(i) + B'(i) - f_i - p_i d_i - B(1).$$

Thus, if $\delta \leq F'(i) + B'(i) - f_i - p_i d_i - B(1)$, then period i will not become a production period in the new optimal production schedule. So δ is bounded by $\min\{f_i, F'(i) + B'(i) - f_i - p_i d_i - B(1)\}$. Hence, the following conclusion can be drawn.

Proposition 6.3 If period i is not a production period in ELSP with backlogging, and its setup cost is decreased from f_i to $f_i - \delta$, then the variation of δ is bounded by $\text{Min}\{f_i, F'(i) + B'(i) - f_i - p_i d_i - B(1)\}$.

Scenario (ii) Setup cost is increased from f_i to $f_i + \delta$.

It is trivial to prove that if a period is not a production period in the final production schedule, it is impossible for the period to become a production period if its setup cost is increased. Thus, only the period, which is a production period in the final production schedule, is needed to be considered. By following Lemmas 6.3 and 6.4, it is clear to see that there is a critical value for the periodic setup cost. When setup cost is higher than the critical value, the period is not a production period; otherwise, it is a production period. The method to compute the critical value is also described.

Lemma 6.3 If period i is still a production period after its setup cost is increased from f_i to $f_i + \delta$, then the final optimal production schedule will remain the same.

Proof: From the proof of Lemma 6.2, it is clear that if period i is a production period, the optimal production schedules is still optimal for $F'(i)$ and $B'(i)$. Now suppose the optimal production periods for $F'(i)$ are indexed by $1 \leq i_1 < i_2 < \dots < i_j < i$, and the final optimal production schedule for the planning horizon is different for the first i periods, i.e. $1 \leq i_1' < i_2' < \dots < i_j' < i < i_r < \dots < i_k \leq N$, then obviously following the production schedule of $1 \leq i_1 < i_2 < \dots < i_j < i < i_r < \dots < i_k \leq N$ will have a smaller cost. So if period i is a production period in the final optimal schedule, optimal production schedule for $F'(i)$ will also be in the final production schedule. The same result also holds for $B'(i)$ by applying the same analysis. From equations (6.2) and (6.4), it is also clear that the different value of setup cost f_i does not affect the choice of the production schedule backward and forward respectively. Thus, the optimal production schedule is the same for $F'(i)$ and $B'(i)$ irrespective of the value of f_i , thus the lemma is proved. $\square$

Based on the preceding analysis, the basic idea here is to compute the optimal minimum value for the solution that has no production in period i in the final production schedule, and the obtained optimal minimum value is clearly an upper bound for the problem where the setup cost of period i is increased and period i remains a production period.

Mathematically, the optimal minimum cost when period i is not in the production period can be expressed as follows.

$$G = \min_{1 \leq j < i < t \leq N} \{F'(j) + g(j, t) + B'(t)\}, \quad (6.5)$$

where

$$g(j, t) = \min_{j \leq l \leq t} \{p_j d_{j+1,l} + p_t d_{l+1,t-1} + \sum_{\tau=j}^{l-1} h_{\tau}^{+} d_{\tau+1,l} + \sum_{\tau=l+1}^{t-1} h_{\tau}^{-} d_{l+1,\tau}\}$$

In order to get the value of G in equality (6.5), the following lemma is necessary.

Lemma 6.4 If a period is not a production period in the final production schedule, and its setup cost is increased, this period will still not be a production period in the final production schedule; and the final production schedule and optimal cost will remain unchanged.

Proof: Similar to Proposition 6.1, the first part is obvious. The second part is proved as follows: Suppose setup cost at period i is $f_i + \delta_1$ and $f_i + \delta_2$ respectively, and period i is not a production period in the final production schedule for both problems with different setup cost. The final optimal total cost will only count the setup costs of final production periods, holding and backlogging costs between the consecutive production periods. All parameter values except the setup cost at period i are the same for both problems.

Because period i is not a production period in the final schedule for both problems, so a feasible production schedule for the problem with the setup cost $f_i + \delta_1$ at period i is also feasible for the other one with the setup cost $f_i + \delta_2$ at period i , and vice versa. Thus, the final production schedule and optimal cost can be the same for both problems. This proves the lemma. $\square$

Lemma 6.4 is used in the following idea to compute the value of G :

Due to increasing setup cost, it is possible that a production period i will be out of the final production schedule at a certain value. So there is a break value of setup cost that period i is a production period in the final production schedule if its setup cost is less than this value (Proposition 6.1), and period i is not a production period in the final production schedule if its setup cost is larger than this value (Lemma 6.4). Also from Lemma 6.4, if setup cost is larger than the setup break point value, the optimal final cost is the same no matter how big the setup cost is. So value G in equality (6.5) can be determined if a big enough setup cost is assigned to period i such that period i will not be a production period in the final optimal solution.

Now, it is clear that the value of G can be determined at time $O(N \log N)$ if the setup cost for period i is assigned a big enough value so that the period i will not be a production period in the final optimal schedule. Such big enough setup value for period i exists, for example, this new setup cost can be:

$$f_i = \sum_{j=1, j \neq i}^N f_j + d_{1,N} [N * \text{Max}(h_1^-, h_2^-, \dots, h_N^-) + N * \text{Max}(h_1^+, h_2^+, \dots, h_N^+) + \text{Max}(p_1, p_2, \dots, p_N)]$$

If production occurs at such a period, the cost will be more than that for setup at every period, carrying and backlogging the total demands from the beginning to the end of the planning horizon, and producing all demands at the most expensive unit cost. Any production schedule other than that including this period will definitely be better.

After value of G is computed, the period i will remain the production period if $F(N) + \delta \leq G$. So the upper bound for the value of δ is $G - F(n)$. So the following result holds:

Proposition 6.4

- a. There is a critical value V for the setup cost f_i , such that if $f_i > V$, then period i will not be a production period; otherwise, period i will be a production period in the final optimal schedule.
- b. The maximal allowable increase of f_i can be calculated at the time complexity of $O(N \log N)$.

6.6 Total Demand Variation Analysis

In this section, the following scenario will be analyzed:

During the planning horizon, there is a total demand variation, the increased or decreased demand is produced or subtracted at only one period in the planning horizon. The objective is to find the variation scope such that production schedule remains unchanged. Two equations are formulated first for the situations where the total demand is increased and decreased respectively. The equations are then expressed geometrically. Based on the geometric meaning, the variation scope can be obtained.

By the following analysis, this problem will be modeled as a minimization problem where the geometric technique described in section 6.2 can be applied. First, it is obvious to get the following lemma directly from equalities (6.2) and (6.4):

Lemma 6.5 If in period i , the production is changed by $\pm \delta$, then

- I. From period 1 to $i-1$, $F'(\cdot)$ remains unchanged, $F'(i)$ changes by $\pm p_i \delta$, and
- II. From period $i+1$ to N , $B'(\cdot)$ remains unchanged, $B'(i)$ changes by $\pm p_i \delta$.

Suppose the total demand variation $\pm \delta$ units are produced at or subtracted from period i , the objective is still to let the whole costs be minimum. Such minimum value can be expressed mathematically for the cases where the total demands in planning horizon is increased and decreased respectively:

$$VI(\delta) = \min_{1 \leq i \leq N} \{F'(i) + B'(i) - f_i - p_i d_i + p_i \delta\} \quad (6.6)$$

$$VD(\delta) = \min_{1 \leq i \leq N} \{F'(i) + B'(i) - f_i - p_i d_i - p_i \delta\} \quad (6.7)$$

Consider (6.6) first. Define l_i as the line which passes through point $(0, F'(i) + B'(i) - f_i - p_i d_i)$ with slope value of p_i . So lines of l_i ($1 \leq i \leq N$) construct a concave lower envelop for the parameter of δ . Because the values of $F'(i) + B'(i) - f_i - p_i d_i$ are the same for all production periods, so only the period with the lowest p_i can be in the concave lower envelop for all production periods. Define such period as period A . The line l_A has the minimum value of $F'(i) + B'(i) - f_i - p_i d_i$ due to A being a production period.

From Figure 6.1, it is clear that if the increased δ is within $[0, C_1]$, where C_1 is the first break point of the concave lower envelop, the production period A will produce the increased units. Otherwise, another period, which is not a production period at the original optimal solution will produce the increased units, and therefore becomes a production period in the new solution. So if the increased δ is greater than C_1 , the original optimal production schedule will not remain the same.

Figure 6.1 Illustration for the Case of Total Demand Increase

Same logic can be applied to (6.7) except that l_i is the line which passes through point $(0, F'(i) + B'(i) - f_i - p_i d_i)$ with slope of $-p_i$; and only the period with the biggest p_i can be in the concave lower envelop for all production periods due to the line gradient

Figure 6.2 Illustration for the Case of Total Demand Decrease

being negative, such period is defined as period B . Figure 6.2 describes this case. From Figure 6.2, it is also clear that if the decreased δ is within $[0, C_1]$, where C_1 is the first break point of the concave lower envelop, the production period B will decrease its production by δ . Otherwise, another period, which is not a production period at the original optimal solution will decrease its production, and therefore becomes a production period in the new solution. So if the decreased δ is greater than C_1 , the original optimal production schedule will not be optimal.

In Van Hoesel, Wagelmans and Moerman (1994), it is proved that time complexity of constructing such a concave lower envelop in Figures 6.1 and 6.2 will be $O(\log N)$. They also propose that balanced tree such as 2-3 tree (Aho, Hopcroft and Ullman (1983)) is the efficient data structure for supporting such a procedure.

The following result can conclude the preceding analysis:

Proposition 6.5

- a. If the solution remains the same, the increased total demand is produced at the period having lowest unit production cost, and the decreased total demand is subtracted from the period having highest unit production cost among all production periods.
- b. The maximum increased and decreased total demand variations are the values of C_1 which are described in Figures 6.1 and 6.2 respectively.
- c. The value of C_1 in Figures 6.1 and 6.2 can be determined at time complexity of $O(\log N)$.

6.7 Parameter Analysis of Unit Production Cost

This section analyzes the variation scope of unit production cost p_i . Similar to the analysis for setup cost, by following Proposition 6.6 and Lemma 6.7, it can also be seen that there is a critical value for unit production cost, which is concluded in Proposition 6.7. However, it will be demonstrated that computing the variation scope is different from that for the setup cost.

Lemma 6.6 If period i is a regeneration period in the final optimal production schedule, then $B(1) = F(N) = F(i) + B(i + 1)$.

Proof: First of all, the following result is a special case of Theorem 1' at Florian and Klein (1971): If period i is a regeneration period in the final optimal production schedule, then period i decomposes the original problem and the optimal solution to the original problem can be found by independently solving the problem for the first i periods and for the last $(N - i)$ periods. Clearly, the optimal cost for the first i periods is $F(i)$ and the optimal cost for the last $(N - i)$ periods is $B(i + 1)$ from their respective definitions. Thus, the lemma is proved. $\square$

As pointed out in the proof, the regeneration period decomposes the original problem into two sub-problems. It is called inventory decomposition property in Florian and Klein (1971) and this property also holds for the Wagner-Whitin model [Theorem 4, Wagner and Whitin (1958)].

From the following analysis, it will be seen that the unit production cost has many similar properties as period setup cost. First, similar to Proposition 6.1, the following proposition is true for the unit production cost.

Proposition 6.6 If period i is a production period in ELSP with backlogging, then period i will still be a production period if its unit production cost is decreased.

Proof: The proof for Proposition 6.1 is applicable here. $\square$

Furthermore, similar to Lemma 6.4, the following lemma holds.

Lemma 6.7 If a period is not a production period in the final production schedule, and its unit production cost is increased, this period will not become a production period in the final production schedule. Additionally, the final production schedule and optimal cost will remain unchanged.

Proof: Suppose the unit production cost is increased from p_i to $p_i + \delta$ and period i is not a production period in the final solution when the unit production cost is p_i . Further, suppose its production schedule is $1 \leq k_1 < k_2 < \dots < k_j < \dots < k_r \leq N$, $i \neq k_1, k_2, \dots, k_j, \dots, k_r$.

The first part of the lemma is proved by contradiction. Suppose period i becomes a production period in the final solution when the unit production cost is increased to $p_i + \delta$, and its production schedule is $1 \leq i_1 < i_2 < \dots < i_j < i < i_r < \dots < i_k \leq N$. Clearly:

Optimal Cost (following production schedule $1 \leq i_1 < i_2 < \dots < i_j < i < i_r < \dots < i_k \leq N$ with unit production cost $p_i + \delta$) $>$ Cost (following production schedule $1 \leq i_1 < i_2 < \dots < i_j < i < i_r < \dots < i_k \leq N$ with unit production cost p_i) $>$ Optimal Cost

(following production schedule $1 \leq k_1 < k_2 < \dots < k_j < \dots < k_r \leq N, i \neq k_1, k_2, \dots, k_j, \dots, k_r$ with unit production cost p_i).

Thus, for the model with unit production cost $p_i + \delta$ at period i , following the production schedule of $1 \leq k_1 < k_2 < \dots < k_j < \dots < k_r \leq N, i \neq k_1, k_2, \dots, k_j, \dots, k_r$ will definitely have a smaller cost, therefore period i can not be a production period. This proves the first part.

Similar to the proof for the second part of Lemma 6.4, the second part of this lemma is proved as follows: Suppose unit production cost at period i is $p_i + \delta_1$ and $p_i + \delta_2$ respectively, and period i is not a production period in the final production schedule for both problems with different unit production cost. The final optimal cost will only count the setup costs and production costs of final production periods, holding and backloging costs between the consecutive production periods. All parameter values except the unit production cost at period i are the same for both problems. Because period i is not a production period, a feasible solution for the problem with the unit production cost $p_i + \delta_1$ is also feasible for the other one with the setup cost $p_i + \delta_2$ at period i , and vice versa. Therefore, the final production schedule and optimal cost can be the same for both problems. This proves the lemma. $\square$

From Proposition 6.6 and Lemma 6.7, it is clear that for the unit production cost at a given period, there is also a break value that period i is a production period in the final production schedule if its unit production cost is less than this value (Proposition 6.6), otherwise it is not a production period (Lemma 6.7). Also from Lemma 6.7, if production

cost is larger than the break value, the optimal final cost is the same no matter how big the unit production cost is. Concluding the preceding analysis and results, the following conclusion can be drawn.

Proposition 6.7 There is a critical value V for the setup cost p_i , such that if $p_i > V$, then period i will not be a production period; otherwise, period i will be a production period in the final optimal schedule.

From the above analysis, it is clear that the technique in section 6.5 is applicable here. The value of G computed in section 6.5 is also the upper bound of the optimal cost for the model where period i remains the production period, and unit production cost at period i is increased by δ . The justification is from the following analysis: The value of G is obtained by assigning a big enough setup cost for period i such that period i is not a production period, and this model is termed as Model 1 for short. From Lemma 6.7, there also exists a big enough unit production cost for period i such that period i is not a production period in the final schedule. This model is termed as Model 2 for short. All the parameters for Models 1 and 2 are the same except the setup and unit production costs at period i . Because period i is not a production period, the setup and unit production costs at period i will not affect the final cost. Thus, any production schedule for Model 1 has the same cost for Model 2, and vice versa. Therefore, Models 1 and 2 have the same optimal cost.

For unit production cost, the counterpart of Lemma 6.3 for setup cost does not hold since the unit production cost p_i will affect the optimal production schedule of $F'(i)$ and

$B'(i)$. Due to increased unit production cost, the production amount at period i may change. Only one special scenario is considered here: when the unit production cost is increased, the production schedule remains the same. In the final production solution, define the nearest regeneration period prior to period i as rp_1 , the nearest regeneration period after period i as rp_2 . Thus, by Lemma 6.6, the optimal cost for the model where period i remains the production period, and unit production cost at period i is increased by δ , can be expressed as follows.

$$F(rp_1) + B(rp_2 + 1) + p_i d_{rp_1+1, rp_2} + \sum_{\tau=rp_1+1}^{t-1} h_{\tau}^{-} d_{rp_1+1, \tau} + \sum_{\tau=t}^{rp_2-1} h_{\tau}^{+} d_{\tau+1, rp_2} + f_t + \delta * d_{rp_1+1, rp_2}$$

This value should be less than or equal to G , thus, the upper bound for δ is

$$\delta \leq \{G(.) - [F(rp_1) + B(rp_2 + 1) + p_i d_{rp_1+1, rp_2} + \sum_{\tau=rp_1+1}^{t-1} h_{\tau}^{-} d_{rp_1+1, \tau} + \sum_{\tau=t}^{rp_2-1} h_{\tau}^{+} d_{\tau+1, rp_2} + f_t]\} / d_{rp_1+1, rp_2}$$

Clearly for this special case, the computational complexity of the maximal allowable increase in unit production cost for a period is same as that of computing the maximal allowable increase of set up cost for a period; thus, the maximal allowable increase of p_i can also be determined in $O(N \log N)$ time.

6.8 Summary

This chapter describes the backward and forward algorithms for the ELSP with backlogging, and subsequently applies the algorithms to conduct parameter analysis. Some new results have been obtained for the variations of periodic setup cost, total demand variation and periodic unit production cost. In summary, there are critical values for periodic setup and unit production costs respectively, such that if the periodic setup or unit production cost is higher than their critical value, then the period will not be a

production period; otherwise, the period will be a production period in the final optimal schedule. Total demand variation scope can be determined in time of $O(\log N)$. The preceding results are novel even compared with that in Van Hoesel and Wagelmans (1993)'s model where backlogging is not allowed. The variation studies for the inventory costs and periodic demand are more complex than the other parameters, and no new results have been achieved.

It can be observed that the complexity of solving the problem itself is not augmented by parameter analysis since these two can be implemented concurrently. Similar to the sensitivity analysis of linear programming, parameter analysis discussed in this chapter is also very useful for large scale ELSP and online or real time ELSP planning or scheduling problems which are very common in today's e_commerce applications (Shapiro, 2001).

Chapter 7

Conclusions and Future Research Recommendations

7.1 Conclusions

The research conducted in this thesis focuses on three fundamental models. The first one is a stochastic model with linear production and convex inventory costs, equal capacity and non-increasing setup cost. The research findings demonstrate that Chen (2004a)'s X-Y band control policy still holds for such a model. Furthermore, based on the new results, a fresh efficient algorithm is developed for finding the solution of the complementary deterministic dynamic model. It is also stated that X-Y band control policy holds for the deterministic model. Computational studies demonstrate that the new algorithm performs much better than the others for solving the deterministic model.

Although many research studies have suggested the optimality of the hedging point policy, it is still a tough challenge to academia to prove the hypothesis. The second model investigated in this research is a simplified case of the multiple-product system. In the model, the stochastic demand distribution, production rate, unit production cost and periodic expected inventory cost are the same for all products respectively. It is proved in this research that the hedging point policy is optimal for this particular model; the ordering and not-ordering regions for every product are also defined. These novel results not only characterize the optimal policy for the simplified model, but also propose clues for the general model.

The parameter variation scope is very meaningful in obtaining the solution more quickly when only one variable changes. Motivated by Van Hoesel and Wagelmans (1993), the parameter analysis for the backordering-allowed Wagner-Whitin model is conducted, and some novel results have been obtained. Some novel results have been obtained for the variation analysis of setup cost, total demands and unit production cost. There are critical values for periodic setup cost and unit production cost respectively, such that if the periodic setup or unit production cost is higher than their critical value, then the period will not be a production period; otherwise, the period will be a production period in the final optimal schedule. Total demand variation scope can be determined in time of $O(\log N)$.

For the above three fundamental models in production and supply chain planning, the essential challenge is to characterize the properties of the objective function under the constraints. Based on these properties, order policies and efficient model-solving algorithms could be developed. By this study, the problem for the first model is narrowed down to characterize the local X-Y band. The X-Y band is also applied to design an algorithm to solve the deterministic model. Additionally, the efficiency of the designed algorithm is better than Shaw and Wagelmans (1998)'s and is the latest development. The limitation of the algorithm in this study is that the capacity must be the same for every period.

For the second model, this study proves that a hedging point policy is optimal. This result also provides the opportunity to build a solution framework in multiple dimensions. For

the third model, two new results have been achieved: there is a break value for periodic setup and unit production cost that decides if the period is a production or non-production period, and total demand variation scope can be determined in time of $O(\log N)$.

In summary, this research follows the main research stream in production and supply chain planning, tries to tackle some fundamental problems, and achieves the following contributions:

- (1) X-Y band control policy is proved to be true for the stochastic model with linear production and convex inventory costs, equal capacity and non-increasing setup cost.
- (2) A better algorithm is designed for solving the deterministic model with linear production and convex inventory costs, equal capacity and non-increasing setup cost.
- (3) The multiple product dynamic model where the stochastic demand distribution, production rate, unit production cost and periodic expected inventory cost are the same for all products respectively is characterized by hedging point policy.
- (4) For the backordering-allowed Wagner-Whitin model, it is proved that there are critical values for periodic setup and unit production costs that decide if the period is a production or non-production period, and total demand variation scope can be determined in time of $O(\log N)$.

7.2 Limitations of the Three Research Models

As described earlier, a usual research methodology is to extend the three models to become more general and approach the real supply chain or network. Therefore, identifying the limitations of the three models investigated in this research work is absolutely necessary for further model extension.

One main limitation of the first model is that the periodic production capacity constraint must be equal. How to relax this constraint and let the periodic capacity be different is still unknown. In fact, removing this constraint is closely related with the definition of (C_p, K) -convex. Generalizing the (C_p, K) -convex definition or investigating the interrelationship of the two inequalities of (C_p, K) -convex are the two probable methods for understanding the function of production capacity in the system.

If the production cost is convex or concave rather than linear, clearly, (C_p, K) -convex cannot characterize the objective function. How to identify the optimal policy for such a model is still unknown. Another limitation is that the periodic setup cost must be non-increasing. How to relax the above limitations and characterize the models' optimal policy is a tough challenge in research.

Clearly, for the second model, the limitation is that the stochastic demand distribution, production rate, unit production cost and periodic expected inventory cost must be the same for each product respectively. Secondly, the setup cost is not incorporated in the models. Actually, considering setup cost in two-product system is still an unsolved

problem. Thirdly, if the production cost is convex or concave rather than linear, whether hedging point policy still holds true is still not answered.

Although the third model extends Van Hoesel and Wagelmans (1993)'s model from the backlogging not allowed to backlogging allowed, due to the model complexity, the parameter variation ranges of unit holding cost, unit backlogging cost and periodic demand are still not obtained. Additionally, production capacity is not incorporated into the model.

7.3 Future Research Recommendations

Numerous investigations have focused on computational efficiency improvement in computing time leading to the development of more and more efficient algorithms by different techniques. Unfortunately, the memory space required by many dynamic programming algorithms has not been reduced since the feasible solutions need to be kept during the dynamic programming. This is called the curse of dimensionality (Nahmias, (1978)). As explained in Section 4.5.4, this issue will become very complex for the models with multiple products and periods. Thus, more research on the method of dynamic programming itself is absolutely necessary.

Secondly, there are many fundamental challenges for the models with multiple products. In Section 5.6, a hypothesis is proposed for the general model with multiple products, and it is suggested that if this hypothesis holds, then most probably the optimality of the

hedging point policy for the general model will be true. However, proving the hypothesis or the optimality of hedging point policy is still a fundamental challenge.

Furthermore, when the setup costs were considered in a multiple product system, only heuristic solutions were proposed. Another research area could be incorporating multiple echelons in the model, this area matches the current intensive research studies in supply chain management. As reviewed earlier, single product, multiple-echelon capacitated model has been very rarely investigated in the literature. How to characterize these systems is still the open problem in academia.

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