The behavior of the graded cellular material under impact

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NANYANG
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**THE BEHAVIOR OF THE GRADED CELLULAR
MATERIAL UNDER IMPACT**

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SCHOOL OF MECHANICAL AND AEROSPACE ENGINEERING

2014

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Abstract

Due to their excellent properties, including low weight, high stiffness and strength and heat insulation, cellular materials are widely used in engineering applications, especially in the aerospace and defense industries, as energy absorption devices. To improve the performance of the cellular material, optimization of the material property has attracted considerable research interest. The present research aims to investigate the influence of the gradient on the behavior of graded cellular material under impact conditions. This research program has four parts.

First, honeycomb composing of regular two dimensional cells is modeled in Finite Element software (ABAQUS/EXPLICIT). A gradient in the quasi-static plateau stress is introduced into the honeycomb. Deformation modes are identified from the observation of the deformation process. Empirical equations for critical velocities, at which the deformation mode changes, are summarized. The analysis for the dynamic stress reveals that the simple shock wave theory is only applicable in the case of positive gradient. Further investigation into the energy absorption capacity demonstrates that the positive gradient enhances the energy absorption capacity of such graded honeycomb structures while the negative one weakens the capacity.

Second, an analytical study is conducted by using the one-dimensional shock theory. A uniform cellular rod with a gradient in the material's yielding stress is studied for two impact scenarios. Two compaction zones appear when the gradient is negative while only one takes place when the gradient is positive. Consequently, Double shock (DS) mode and Single shock (SS) mode are proposed to describe the deformation processes in the

graded cellular rod with negative and positive gradient, respectively. Closed form solutions are obtained, together with finite element results. The analysis reveals that placing the weakest part of the graded cellular material at the proximal end achieves better energy absorption performance.

Third, further analytical investigation focuses on a more practical problem with a gradient in the initial density of the cellular material, which leads to variation of the quasi-static plateau stress and the locking strain along the rod. Similarly, the two basic deformation modes exist. The investigation shows that the SS mode has almost no influence on the energy absorption while the DS mode weakens the energy absorption significantly. Afterwards, the effect of the density gradient is decoupled. Finite element simulations, based on the practical metal foam, are carried out to validate the analytical modeling.

Finally, an experimental study is conducted by using a gas gun, impinging on a stationary foam block with a gradient in its cross-sectional area. The experimental result confirms the existence of the two basic deformation modes in a graded cellular block. Finite element simulations are employed to extend the study to cases with higher impact velocities. The analytical modeling achieves similar conclusions: positive gradient has no influence while the negative weakens the energy absorption capacity; and the transmitted force in the case with negative gradient is smaller.

Keywords: Graded cellular material; Impact loading; Double shock mode; Energy absorption capacity.

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List of symbols

Chapter 2

σ_e, σ_p	Elastic buckling stress and yielding stress of the honeycomb in the X1 direction
E_s, σ_{ys}	Young's modulus and yielding stress of the parent material
h, t	Thickness of the cell wall
l	Length of the cell wall
$\varepsilon_L, \varepsilon_d$	Locking or densification strain
ρ_0, ρ^*	Initial density of the cellular material
ρ_s	Density of the parent material
V_{cr}	Critical velocity
ε_{cr}	The strain where $d\sigma/d\varepsilon = 0$
$c(\varepsilon)$	Speed of stress wave in a solid
E^*, G^*, ν^*	Young's modulus, shear modulus, Poisson ratio of the foam
ϕ	Variable fraction in the cell faces
σ_{pl}^*	Plastic strength of the foam
$C, C', C_1, C_2, C_3, C_4$	Constants determined by experiments
p_0	Pressure of the fluid in the foam
p_{at}	Atmospheric pressure

ε_a	Nominal strain
W	Absorbed energy up to a given nominal strain, ε_a
E	Absorbed energy divided by the corresponding stress
V	Impact velocity
$\dot{\varepsilon}$	Strain rate
σ_d	Crushing stress in cellular material
L_0	Length of the impinged cellular rod
$[\]$	A change in the value across the shock
K	Strength index
m	Strain hardening index
V_{c1}, V_{c2}	Critical velocities
E_1	Plastic hardening modulus
ρ_1, ρ_2	The initial densities at the proximal and distal layers
d, y	Thickness of the layer, the position through the thickness

Chapter 3

σ_p	Plastic yielding stress of the honeycomb in X1 direction
σ_d	Crushing stress in the cellular material
E, σ_{ys}	Young's modulus and yielding stress of the parent material
t, l	Thickness and length of the cell wall

ε_d	Locking strain
ρ_0	The initial density of the cellular material
ρ_s	The density of the parent material
σ_{ave}	Average value of the materials' yielding stress of parent material
σ_0	Average plateau stress of the cellular material
n	The number of the layers
l_0	The initial length of the honeycomb block
g, Θ	Gradient, corresponding non-dimensional gradient
l_{layer}	The length of the layer
$\Delta\sigma_{ys}$	The difference of the yielding stress between each two adjacent layers
C	Constant determined by experiments
c_p	Characteristic velocity
u, μ	The displacement of the impinging plate, corresponding non-dimensional value
E_q, E_d	Absorbed energy under quasi-static and dynamic condition
ψ	Energy ratio
G, m, Γ	Impinging mass, the mass of the honeycomb block, mass ratio
V, ν	The impact velocity, corresponding non-dimensional value

v_d Densification velocity

V_{c1}, V_{c2} Critical velocities

Chapter 4

V_0, v_0 Initial impact velocity, corresponding non-dimensional value

σ_d, Σ_d Crushing stress in the cellular material, corresponding non-dimensional value

E, σ_{ys} Young's modulus and yielding stress of the parent material

t, a Length of the cell wall

ε_d Locking strain

ρ_0 The initial density of the cellular material

ρ_s The density of the parent material

A_0 Cross-section area of the cellular rod

σ_{ave} Average value of the materials' yielding stress of parent material

σ_0 Average plateau stress of the cellular material

n The number of the layers

l_0 The initial length of the cellular rod

g, Θ Gradient, corresponding non-dimensional gradient

l_{layer} The length of the layer

C	Constant determined by experiments
c_p	Characteristic velocity
T, t, τ	Characteristic time, time, non-dimensional time
u, μ	The displacement of the impinging plate, corresponding non-dimensional value
G, m, Γ	Impinging mass, the mass of the honeycomb block, mass ratio
V, v	The impact velocity, corresponding non-dimensional value
V_1, V_2, v_1, v_2	Impinging velocity and the velocity of the undeformed zone, corresponding non-dimensional values
v_d	Non-dimensional densification velocity
v_p	A velocity, beyond which Phase II does not appear
x, ξ	Length of the forward densification zone, corresponding non-dimensional value
l, λ	Length of the undeformed zone, corresponding non-dimensional value
y, η	Length of the backward densification zone, corresponding non-dimensional value
σ_d^f, σ_p^f	Crushing stress and the quasi-static stress at the forward shock front
σ_d^b, σ_p^b	Crushing stress and the quasi-static stress at the backward shock front

E_q, E_d	Absorbed energy under quasi-static and dynamic condition
ψ	Energy ratio
E_d^f, E_d^b	Absorbed energy by the forward, backward densification zone
$\Delta\sigma_{ys}$	The difference of the yielding stress between each two adjacent cells
l_{cell}	The length of a cell
r	Radius of the circular cell
E, ν	Young's modulus, Poisson ratio

Chapter 5

V_0, v_0	Initial impact velocity, corresponding non-dimensional value
σ_d	Crushing stress in the cellular material
E, σ_{ys}	Young's modulus and yielding stress of the parent material
ε_d	Locking strain
ρ_0	The initial density of the cellular material
ρ_s	The density of the parent material
A_0	Cross-section area of the cellular rod

σ_0	Average plateau stress of the cellular material
n	The number of the layers
l_0	The initial length of the cellular rod
Θ_ρ, Θ_s	Density gradient, stress gradient
C	Constant determined by experiments
c_p	Characteristic velocity
T, t, τ	Characteristic time, time, non-dimensional time
u, μ	The displacement of the impinging plate, corresponding non-dimensional value
G, m, Γ	Impinging mass, the mass of the honeycomb block, mass ratio
V, v	The impact velocity, corresponding non-dimensional value
V_1, V_2, v_1, v_2	Impinging velocity and the velocity of the undeformed zone, corresponding non-dimensional values
v_d	Non-dimensional densification velocity
v_p	A velocity, beyond which Phase II does not appear
x, γ	Length of the forward densification zone, corresponding non-dimensional value
X, ξ	Length of the forward deformed zone, corresponding non-dimensional value
l, λ	Length of the undeformed zone, corresponding non-dimensional value

y, ζ	Length of the backward densification zone, corresponding non-dimensional value
Y, η	Length of the forward deformed zone, corresponding non-dimensional value
σ_d^f, σ_p^f	Crushing stress and the quasi-static stress at the forward shock front
σ_d^b, σ_p^b	Crushing stress and the quasi-static stress at the backward shock front
E_q, E_d	Absorbed energy under quasi-static and dynamic condition
ψ	Energy ratio

Chapter 6

V_0, v_0	Initial impact velocity, corresponding non-dimensional value
σ_d	Crushing stress in the cellular material
E, σ_{ys}	Young's modulus and yielding stress of the parent material
ε_d	Locking strain
ρ_0	The initial density of the cellular material
ρ_s	The density of the parent material

A_0	Average cross-section area of the cellular rod
A_{bs}, A_{fs}	Cross-section areas at the backward and forward shock front
A_m, A_f	Average cross-section areas in the undeformed zone and forward densified zone
σ_0	Quasi-static plateau stress of the cellular material
l_0	The initial length of the cellular rod
Θ_A	Gradient in the cross-section
C	Constant determined by experiments
c_p	Characteristic velocity
T, t, τ	Characteristic time, time, non-dimensional time
u, μ	The displacement of the impinging plate, corresponding non-dimensional value
G, m, Γ	Impinging mass, the mass of the honeycomb block, mass ratio
V, v	The impact velocity, corresponding non-dimensional value
V_1, V_2, v_1, v_2	Impinging velocity and the velocity of the undeformed zone, corresponding non-dimensional values
v_d	Non-dimensional densification velocity
v_p	A velocity, beyond which Phase II does not appear
x, ξ	Length of the forward densification zone, corresponding non-dimensional value
l, λ	Length of the undeformed zone, corresponding non-

	dimensional value
y, η	Length of the backward densification zone, corresponding non-dimensional value
σ_d^b	Crushing stress at the backward shock front
E_q, E_d	Absorbed energy under quasi-static and dynamic condition
ψ	Energy ratio
F_{dis}, f_{dis}	Force at the distal end and the corresponding non- dimensional value

Chapter 1 Introduction

1.1 Background

Cellular solids, composed of an interconnected network of solid struts (open cells) and plates (closed cells), are normally categorized into two groups, namely “honeycomb” and “foam”. The “honeycomb” consists of cellular material with essentially two-dimensional microstructures while the “foam” includes those materials with fully three-dimensional microstructures. Figure 1.1a-c shows three examples of cellular solids including a hexagonal honeycomb, an open and a closed-cell foam, respectively.

(a)

(b)

(c)

Figure 1.1 Examples of cellular solids: (a) Aluminum honeycomb; (b) open-cell foam; (c) closed-cell foam (Gibson and Ashby 1997)

Both natural (e.g. bones and wood) and synthetic (e.g. honeycomb, metal foams, *etc.*) cellular solids have attracted numerous research interests due to their outstanding properties, including high relative stiffness and strength, heat insulation and efficiency of energy absorption. As a result of these properties, cellular materials are widely used in aerospace applications and packaging applications. One of their applications is as an energy absorption layer for impact and blast protection due to their ability to deform over a long stroke at an almost constant load.

Over the past decades, the idea of functionally graded materials (FGMs) is introduced to materials sciences research. In a functionally graded material (FGM), both the composition and the structure gradually change over the volume in one direction, resulting in corresponding changes of the material and structure. FGMs are ideal for applications involving severe thermal gradients, ranging from thermal structures in advanced aircraft and aerospace engines to computer circuit boards. Application of these materials tends to reduce stresses resulting from material property mismatch, increases the bonding strength, improves the surface properties and provides protection against adverse thermal and chemical environment. Such FGMs structures can be easily found in nature (e.g. bones and woods). For example, bone possesses a gradient of the density in order to maximize the biological mechanical performance. Nowadays, various methods have been explored in the designing and manufacturing of the artificial FGMs layers, such as adhesive bonding, sintering, thermal spray, and reactive infiltration.

For cellular solids, a gradual variation in the cell size distribution or material strength may significantly influence many properties, such as mechanical shock resistance and

thermal insulation. Therefore, an understanding of mechanical properties of the graded cellular structure is important in applications of these materials.

1.2 Objective

Previous studies on the cellular material under impact merely focused on homogenous cases. Early theoretical work on the geometry and fundamental principles of the mechanics of cellular solids dates back to De Boer (1998). An extensive record on the structure and property of cellular solids is given by Gibson and Ashby (1997). After that, numerous interests were attracted in the understanding of the geometric, mechanical, thermal and electrical characteristic of these solids.

In the past few decades, research interest has been shown in studying various graded cellular structures. It has been confirmed that the introduction of the property gradient will significantly influence the overall response of the graded cellular structure. However, the underlying deformation mechanism of the graded cellular material under dynamic loading remains to be further understood due to the lack of comprehensive theoretical study of the behavior of graded cellular materials.

The present thesis focuses on three main directions: 1) Deformation processes of the graded cellular material under impact; 2) The influence of the gradient on the stress at the proximal and distal ends. 3) The energy absorption capacity of the graded cellular material.

1.3 Scope

This project was designed to understand the mechanical behavior of the graded cellular material under impact loading. The research is carried out from three aspects, namely Finite Element (FE) simulations, analytical modeling and experimental study. Two kinds of structures are investigated in our study, which are the graded honeycomb structure and graded foam material.

At the beginning, Finite Element (FE) simulations of graded honeycomb structures under impact are carried out by using ABAQUS/EXPLICIT. As the first step, only stress gradient in the yielding stress of the parent material is introduced into the honeycomb block. In each layer, the material property is the same except that the yielding stress varies layer by layer. Similar to the work of Ruan et al. (2003), the basic deformation mode is identified and summarized with a classification map. Empirical equations for critical velocities are summarized in terms of the gradient, thickness ratio, average yielding stress and the density of the parent material. Furthermore, one dimensional shock wave theory is employed to predict stresses at the proximal and distal ends. The limitation of the simulation is that the honeycomb block is a two-dimensional structure, resulting in deformations in the X2 direction. As a result, some deviations are expected between the theoretical and FE result.

Before proposing an analytical model, we investigate the response of the one-dimensional hexagonal and circular chains under impact to avoid the deformation in other direction. The observation of the deformation profiles confirms the existence of Double Shock (DS) fronts in a cellular rod with a negative gradient. On the other hand, in the cellular rod

with positive gradient, only one Single Shock (SS) occurs. According to these basic deformation modes, one-dimensional analytical models are proposed by using the simple shock theory. The cellular material is assumed to be continuous with the rigid, perfect-plastic, and locking (R-PP-L) property. The DS mode is proposed to prescribe the deformation process when the gradient was negative while the SS mode is used when the gradient was positive. Comparison between the FE and analytical results shows good agreement, which validates the analytical analysis.

Subsequent work aims to investigate a more realistic case, with the gradient in the initial density of the cellular material. Compared with the stress gradient in previous study, the density gradient not only influences the quasi-static plateau stress but also the dynamic enhancement. General solution is derived to determine the basic deformation mode. Similar to the case with stress gradient, the DS and SS modes exist in the graded cellular rod with initial density gradient. The essential reason of the appearance of the DS mode is the difference in the quasi-static plateau stress between the proximal and distal ends. By using the finite difference method, the influence of the initial density gradient on the dynamic stresses, instant velocity of the impinging mass and the energy absorption capacity are investigated. Furthermore, the effect of the initial density gradient is decoupled into two parts, namely quasi-static plateau stress and the dynamic enhancement effects. FE simulations are carried out with the practical metal foam (Alporas foam). The time histories of the velocity of the impinging mass are compared with those from analytical studies. The FE curves show the same trends but with some deviation in the final displacement, caused by the simplification of the R-PP-L material model.

An experimental study is carried out. Because a graded cellular rod with varying initial density is unavailable, instead a foam block with varying cross-section is used, which is impinged by an impact mass with initial velocities. Gradient in the cross-section is introduced to the foam block. Different velocities are achieved by using a gas gun, varying from 25-41m/s. Observation of experimental results confirms the existence of the DS mode in the graded cellular material with negative gradient. Due to the limitation of the test equipment, FE simulations are employed for the situation with higher velocities, varying from 40 to 110m/s. Analytical work, based on the simple shock theory, is also implemented in this study. Comparison between the FE simulation and analytical results gives a good agreement.

1.4 Outline

This thesis is organized into 7 chapters. The background and objective of the research are introduced in Chapter 1.

Chapter 2 reviews the relevant previous works on the quasi-static and dynamic behavior of honeycomb structures and foams. Basic analytical models are presented. Furthermore, recent studies on the graded cellular materials are also reviewed.

In Chapter 3, a Finite Element (FE) simulation on the graded honeycomb structure under impact is carried out by using ABAQUS/EXPLICIT. A gradient in the material yielding stress is introduced into a uniform honeycomb block. A constant velocity is given to the impact plate, which varies from 3 to 200m/s. The basic deformation mode is identified and a map of classification of the deformation modes is summarized. To further

understand the energy capacity of the graded honeycomb, different initial velocities versus the impinging mass are imposed on the impact plate.

Chapter 4 presents an analytical study of the effect of the gradient in terms of quasi-static plateau stress on dynamic behavior of the graded cellular rod under impact loading. Basic deformation modes are observed from FE simulations of one-dimensional graded hexagonal and circular cells' chains under uniaxial impact loading. Theoretical analysis is carried out following the simple shock theory proposed by Reid and Peng (1997). The cellular material is assumed to be rigid-perfect plastic-locking (R-PP-L). Two impact scenarios are considered: firstly, a stationary cellular rod is subjected to the impact of a rigid mass; secondly, the cellular rod together with a rigid mass impinges onto a rigid target.

In Chapter 5, further investigation focuses on the influence of the density gradient in the graded cellular rod under impact. Different from Chapter 4, the gradient in the initial density not only influences the quasi-static plateau stress but also the locking strain and hence the dynamic enhancement. One impact scenario is considered, in which a stationary cellular rod with graded initial density is impinged by a rigid mass. General analytical derivation is carried out to determine governing equations and deformation modes. The influences of the initial density gradient on the dynamic stresses at the proximal and distal ends, instant velocity of the impinging mass and the energy absorption capacity are investigated. An FE simulation with actual metal foam under impact is simulated by using ABAQUS/EXPLICIT.

Chapter 6 presents the experimental study on the graded cellular foam block. The cross-section of the foam block varies along the loading direction. Different velocities are applied to the impinging mass by using a gas gun. Analytical modeling is carried out by using the simple shock theory. Comparisons between FE and analytical results show a good agreement.

Finally, general conclusions of this research are drawn in Chapter 7. A brief suggestion for future work is also given.

Chapter 2 Literature Review

In the past decades, extensive experimental, numerical and theoretical investigations are attracted to the mechanical behavior of cellular solids, in particular metallic foams and honeycombs. Earlier work are on mechanical properties of the cellular material those of Gent and Thomas (1959, 1963), McFarland (1963, 1964), Shaw and Sata (1996), and Patel and Finnie (1970). The landmark work on the investigation into the cellular material was done by Gibson and Ashby (1988). Most of former results were summarized well in the book “Cellular solids: structure and properties” by Gibson and Ashby (1997). This book provides a general overview of the subject and is an ideal starting point for users.

Equation Chapter 2 Section 1

This chapter only reviews previous works in the fields of cellular solids under quasi-static and dynamic loading conditions, which are relevant to current research. Since the dynamic property of the cellular material is very different from the static one, literature works are divided into two areas, namely static and dynamic properties. Section 2.1 reviews the relevant work of the honeycomb structure. Only in-plane properties of the honeycomb are considered. In Section 2.2, we introduce the behavior of the general foam material. Finally, section 2.3 presents the recent progress on the graded cellular material.

2.1 Two dimensional cellular material: honeycomb structure

Honeycomb structures are a classic species of the cellular material, which are grouped by regular arrays of hexagonal cells. In nature, honeycombs can be found easily, such as beehives, honeycomb weathering in rocks, and bone. Manufacturing techniques have

been developed to make artificial honeycomb structures, such as sandwich panels and aluminum honeycombs. Actually, the honeycomb structure can also be made up with triangular, or square, or rhombic cells. Examples are shown in Figure 2.1.

Figure 2.1 Honeycomb structures: hexagonal, triangular and square cells (Gibson and Ashby 1997)

As a kind of cellular material, honeycomb structures have general properties of high energy absorption capacity, low weight and insulation. Due to these properties, they are widely used as energy absorption devices and package engineering. It is difficult to analyse foam materials. Due to the simple geometry of the honeycomb, two dimensional honeycomb structures are often used to represent foam with more complex three dimensional macro cells. Closed-form expressions for mechanical properties, such as the

Young's modulus and crushing stress, can be expressed as a function of cell properties (i.e. material property and cell shape). In the current work, only in-plane properties are included.

2.1.1 Static in-plane uniaxial compression of honeycombs

The static mechanism of the compressive deformation of honeycomb under loading has been thoroughly studied by previous researchers, which was well summarized by Gibson and Ashby (1997). Figure 2.2 shows typical stress-strain curves for honeycombs under in-plane compression loads. The curves all exhibit broadly similar shapes while slight deviation occurs due to different materials (brittle or elastic). In the present study, we only consider elastic-plastic cases.

The stress-strain curve is classified into three regimes. In the beginning, the cell walls start to bend, undergoing linear elasticity, which is reversible. But when a critical stress reaches, the cells begin to collapse. In the plateau region, the cells continue collapsing with a constant plateau stress. At high strains, the material densifies as the cell walls come into contact.

Figure 2.2 Sketch of stress-strain curves for honeycombs under in-plane loads (Gibson and Ashby 1997)

Equation Section (Next)

Elastic properties of the cellular material (Young's modulus and Poisson ratio) were investigated by using either analytical or finite element approaches (Patel and Finnie 1970, Gibson and Ashby 1982, Warren and Kraynik 1987). McFarland (1963, 1964) carried out the pioneering work on the plastic deformation of the honeycomb structure under axial compression. In this work, the mean stress for the honeycomb was expressed in a semi-empirical formula, related to the ratio of the thickness and side length. Combining both bending and extension, Wierzbicki (1983) proposed an improved model for the crushing of the honeycomb. Focusing on a single cell of the honeycomb structure, Gibson and Ashby (1982) firstly derived the analytical expression for the plastic collapse

stress in closed forms, as well as the linear-elastic modulus for the elastic regime. Master and Evans (1996) extended the work (Gibson et al. 1982) by incorporating the stretching and the bending mechanism.

In the work of Gibson and Ashby (1982), beam bending theory was employed for the elastic analysis and the large deformation theory was used for plastic regions. For honeycombs with regular hexagonal cells (i.e. the side length equal to the face length), elastic buckling and plastic collapse stress are expressed in Eq. (2.1) and Eq. (2.2), respectively (Gibson and Ashby (1982)).

$$\frac{\sigma_e}{E_s} = 0.22 \left(\frac{h}{l} \right)^3 \quad (2.1)$$

$$\frac{\sigma_p}{\sigma_{ys}} = \frac{2}{3} \left(\frac{h}{l} \right)^2 \quad (2.2)$$

where σ_e and σ_p are elastic buckling and yielding stress of the honeycomb in X1 direction, respectively; E_s and σ_{ys} are the Young's modulus and yielding stress of the parent material, respectively; h/l is the ratio of thickness and side length of the cell wall. Comparing with the experimental data, it was found that Eq. (2.2) overestimated the plateau stress. Employing the energy method, Lu and Yu (2003) provided another formula for plateau stress,

$$\frac{\sigma_p}{\sigma_{ys}} = \frac{1}{2} \left(\frac{h}{l} \right)^2 \quad (2.3)$$

Figure 2.3 shows comparison between theory Eqs. (2.2), (2.3) and experiment in terms of non-dimensional yield strength. Eq. (2.3) gives values about 75% of those from Eq. (2.2), which is much closer to experimental results.

Figure 2.3 Comparison between theory Eqs. (2.2), (2.3) and experimental yield strength (Lu and Yu 2003)

Papka and Kyriakide (1994, 1998) firstly employed the Finite element (FE) technique to carry out numerical simulations of the aluminum honeycomb under quasi-static loading. FE results showed that the difference between the limit stress and the local contact minimum stress increased with the increasing of the yield stress of the material. Furthermore, it was found that both the FE and theoretical results (Eq. (2.2)) slightly

overestimated the stress. Their work provided good insight into the crushing process of the honeycomb. Supplementary work was done by Said and Reddy (2002), investigating the response of the honeycomb compressed across faces. Triantafylidis and Schnaidt (1998) investigated the onset of failure in aluminum honeycombs. Chuang and Huang (2002) theoretically analyzed the elastic modulus and collapse strength of hexagonal honeycombs with plateau borders, which was helpful in designing the honeycomb structure.

At the final stage, the deformation of the cell reaches the limitation and the structure becomes densified at the densification strain. Usually, the densification strain is also known as the locking strain, ε_d . As for the regular honeycomb structure, the theoretical densification strain is expressed as (Gibson and Ashby 1997),

$$\varepsilon_d = 1 - \frac{\rho_0}{\rho_s} \quad (2.4)$$

where $\rho_0 = \frac{2}{\sqrt{3}} \frac{h}{l} \rho_s$ is the initial density of the hexagonal honeycomb and ρ_s is the density of the parent material. However, in the practical engineering, the honeycomb structure cannot be fully densified in a compressive process. Hence, Eq. (2.4) only gives the maximum value of the densification strain. Hu and Yu (2010) proposed another analytical equation for the densification strain based on the deformation profile from the FE simulation,

$$\varepsilon_d = 1 - \frac{4}{3} \frac{\rho_0}{\rho_s} \quad (2.5)$$

For natural or man-made honeycomb structures, imperfection and inhomogeneity (e.g. missing cells, fractured or misaligned cell wall thickness) always exist in these structures. Warren and Kraynik (1987) and Kraynik et al. (1991) found that the elastic response of honeycombs was little affected by the non-uniform thickness of the cell wall. Employing the FE method, elastic properties of the non-periodic honeycomb with Voronoi cells were studied by Silva et al. (1995). Afterwards, Silva and Gibson (1997) also investigated the compressive strength of the Voronoi honeycomb. In this work, defects were introduced to the structure by randomly removing cell edges of the honeycomb. The result stated that the stiffness and the plateau stress were significantly influenced by the defect. Simone and Gibson (1998a) studied the effect of the non-uniform thickness on hexagonal honeycombs and the idealized cellular foam with closed tetrakaidecahedral cells. Their results suggested that the distribution of the material had little influence on the stiffness, as well as the strength of the hexagonal honeycomb. Afterwards, they (Simone and Gibson 1998b) also investigated the influence of the cell edge curvature and corrugations. They found that the Young's modulus and the plastic plateau stress significantly decreased due to the wavy imperfection. The size effect on mechanical properties of the hexagonal honeycomb was studied in the work of Onck et al. (2001) and Andrews et al. (2001). For low relative density honeycombs, Chuang and Huang (2002) employed a semi-analytical integral-equation approach to analyze the effect of the solid distribution. The result showed the influence was significant. Chen et al. (1999, 2001) systematically studied the influence of different types of imperfections in the honeycomb, including the waviness, non-uniform cell wall thickness, cell size variation, fractured cell walls, cell-wall misalignments, missing cells, rigid inclusions and holes. Guo and Gibson (1999)

studied the behavior of intact and damaged honeycombs using ABAQUS. Results were consistent with those of previous analyses. The single, isolated defects reduced the modulus and strength. The localization behavior was also studied and it was found that the localization strain decreased with the increment of the ratio of thickness and length of the cell wall.

2.1.2 Dynamic in-plane crushing of honeycombs

Under quasi-static compression, the collapse is dominated by the distribution of the material and imperfections so that it always initiates from the weakest row or layer of the cell. However, the response of the honeycomb under dynamic loading is very different from the static one. Under dynamic loading, the deformation mode and the crushing strength are much more complex than the quasi-static case. The dynamic behavior of the honeycomb is strongly influenced by the inertial and strain rate effect.

As for the strain rate effect, it influences the parent material of the cell wall. However, common metal hexagonal honeycombs (e.g. aluminum) are not sensitive to the strain rate. Usually, they are regarded as rate-independent honeycombs. The inertia of the rotation and lateral motion affect the deformation of the honeycomb when the cell collapses.

The inertia effect was clarified in the work of Stronge (1987) and Stronge and Shim (1988). They carried out both the static and dynamic crushing tests with hexagonal and square packed of thin-walled ductile metal tubes. In their work, softening within the cell and localization of the deformation were observed. Due to this feature, the honeycomb collapsed layer by layer. In their following work (Shim et al. 1990, Shim et al. 1992),

one-dimensional mass-chain models were proposed to account for such softening behavior. Later, this analytical model was employed in the work of Li and Meng (2002) to analysis the dynamic effect in the foam. Klintworth (1988) correlated the increase of the peak stress with the inertia effect in the hexagonal honeycomb. Zhao and Gary (1998) employed Hopkinson bar to carry out the dynamic experiment of the honeycomb structure under in-plane compression.

Honig and Stronge (2002a, 2002b) employed the FE simulation to analyze crushing bands and the wave trapping in the honeycomb subjected to in-plane impact. Their results showed that the elastic wave propagation was very important in localized band initiation and the stress enhancement was mainly caused by the inertia effect. Moreover, they found a critical velocity. For the impact speeds larger than the critical velocity, the localized band initiated from the loading surface; for those smaller than the critical velocity, collapse took place from cells with imperfections. The critical velocity was expressed as follows,

$$V_{cr} = \int_0^{\varepsilon_{cr}} c(\varepsilon) d\varepsilon \quad (2.6)$$

where $c(\varepsilon) = \sqrt{(d\sigma/d\varepsilon)/\rho_0}$ is the speed of stress wave in a solid with a uniaxial stress-strain curve $\sigma = \sigma(\varepsilon)$, and ε_{cr} is the strain where $d\sigma/d\varepsilon = 0$.

Ruan et al. (2003) studied the crushing behavior of the regular hexagonal honeycomb by using FE simulations. Based on deformation profiles of the honeycomb structure under impact, three deformation modes were identified for different impact velocities, namely “X”, “V” and “I” shown in Figure 2.4. “X” mode is a quasi-static mode, which occurs

under low impact velocities; “I” mode is a shock-type mode, appearing when the velocity is extremely high; and “V” mode is a transitional mode, which initiates at the moderate velocity. In the work of Ruan et al. (2003), a mode classification map was summarized, shown in Figure 2.5 and critical velocities were expressed in terms of the ratio of the thickness and the length of cell walls of the honeycomb (h/l),

$$v_{c1} = 14\text{m/s} \quad (2.7)$$

and,

$$v_{c2} = 277\sqrt{h/l} \quad (2.8)$$

Figure 2.4 Sketch of the three types of deformation modes (Ruan et al. 2003)

Figure 2.5 A deformation mode map for regular hexagonal honeycombs (Ruan et al. 2003)

Zheng et al. (2005) investigated the crushing behavior of the honeycomb with Voronoi cells and obtained the similar conclusion. To minimize the boundary effect, Zou et al. (2009) employed the long honeycomb with hexagonal cells to analyze the crushing behavior of the honeycomb structure. Similar deformation modes were observed from their FE simulation. The dynamic stress was well predicted by the one-dimensional shock theory (Reid and Peng 1997), considering the honeycomb as continuum. Moreover, their results showed that the densification strain of the honeycomb structure increased with the impinging velocity. Hu and Yu (2010) obtained similar analytical formula of the dynamic plateau stress from a structural point of view.

2.1.3 Honeycombs with other different cells

The above analysis mainly focused on the hexagonal honeycomb, which is the bending-dominated honeycomb. According to the collapse mechanism, the honeycomb structure may be classified into two groups, which are membrane-dominated (triangular, square) and bending dominated (hexagonal, rhombus) honeycombs. Torquato et al. (1998) investigated the effective linear elastic property of square, hexagonal and triangular cell honeycombs by two different approaches, obtaining the bound of the cross-sectional property relating to the bulk modulus and other elastic moduli to the thermal conductivity. Christensen (1986) derived the effective elastic moduli for low density cellular materials using composite micromechanical approaches. Hunt (1993) studied the mechanical property of simple triangular cell honeycombs with ceramic materials, for catalysis application. Their work focused the elastic property of honeycomb made with brittle material. Santosa and Wierzbicki (1999) contributed to understanding the plastic collapse in bending of thin-walled prismatic columns, relevant to out of plane loading conditions. The kinematical criteria of cellular solids was investigated by Deshpande et al. (2001). In this work, they also reviewed the bending and stretching mechanism for the periodic honeycomb. Wang and McDowell (2004) theoretically investigated in-plane elastic and plastic mechanical properties (including the stiffness, Poisson ratio and yielding stress) of the periodic honeycomb with various shaped cells. Table 2.1 represents in-plane mechanical properties of various periodic honeycomb cell structures, derived by Wang and McDowell. Wang and McDowell (2003) also investigated the influence of imperfections with square/triangular cells on the in-plane mechanical property of the honeycomb. Their results indicated that the effect stiffness and the yielding stress of the

triangular cell honeycombs were little influenced by the imperfections. The post-collapse behavior of a circular honeycomb under in-plane compression was analyzed by Karagiozova and Yu (2005). They employed the concept of an equivalent structure and the limit analysis to describe the large deformation during the collapse. Hu et al. (2008) observed the deformation modes in the circular honeycomb from the FE simulation. Lin et al. (2012) theoretically analyzed the in-plane elastic and plastic behavior of the honeycomb with hexagonally packed circular cells.

Table 2.1 In-plane mechanical properties of various periodic honeycomb cell (Wang and McDowell 2004)

Relative density $r = \rho^* / \rho_s$	E_1^* / E_s or E_2^* / E_s	G_{12}^* / E_s	ν_{12}^*	σ_{el}^* / E_s	$\sigma_{pl}^* / \sigma_{ys}$	$\tau_{pl}^* / \sigma_{ys}$	E_{dia}^* / E_s	$\frac{(\sigma_{pl}^*)_{dia}}{\sigma_{ys}}$	r_{crit}
Square Cell	$2 t/l$	$0.5 r$	$0.0625 r^3$	$0.5 \nu_s r$	$0.103 r^3$	$0.5 r$	$0.125 r^2$	$0.25 r^3$	$0.25 r^2$	0.0698
Hexagonal Cell	$(2/\sqrt{3}) t/l$	$1.50 r^3$	$0.375 r^3$	1.0	$0.143 r^3$	$0.5 r^2$	$0.217 r^2$	--	--	0.0035
Triangular Cell	$2\sqrt{3} t/l$	$0.333 r$	$0.125 r$	0.333	$0.0914 r^3$	$\frac{0.5 r}{0.333 r}$	$0.289 r$	--	--	0.0604
Mixed Cell	$(2 + \sqrt{2}) t/l$	$0.369 r$	$0.104 r$	0.146	$0.0292 r^3$	$0.369 r$	$0.207 r$	$0.293 r$	$0.293 r$	0.0594
Kagome Cell	$\sqrt{3} t/l$	$0.333 r$	$0.125 r$	0.333	$0.366 r^3$	$\frac{0.5 r}{0.333 r}$	$0.289 r$	--	--	0.0301
Diamond Cell	$(5/\sqrt{3}) t/l$	$\frac{0.4 r}{0.2 r}$	$0.15 r$	0.333	$0.0789 r^3$	$\frac{0.6 r}{0.2 r}$	$0.346 r$	--	--	0.0502
Rectangular Cell	$\frac{at_2 + bt_1}{ab}$	$\frac{t_1/a}{t_2/b}$	$\frac{t_1^3 t_2^3}{ab(at_1^3 + bt_2^3)}$	$\nu_{12} = \nu_s \frac{t_1}{a}$ $\nu_{21} = \nu_s \frac{t_2}{b}$	$\frac{\pi^2 t_1^3}{12 ab^2}$ $\frac{\pi^2 t_2^3}{12 a^2 b}$	t_1/a t_2/b	$\frac{1}{2} \frac{t_2^2}{ab}$	$\frac{(a^2 + b^2)^2 t_2^3 t_1^3}{a^3 b^3 (at_1^3 + bt_2^3)}$	$\frac{t_2^2 (a^2 + b^2)}{2a^2 b^2}$	--

From the FE simulation, it was found that dynamic stresses of the membrane-dominated honeycombs were not so high as those from the theoretical prediction. Employing FE simulations, Liu and Zhang (2009) summarized an empirical formulas to predict the

dynamic stress for triangular and square honeycombs under impact. Qiu et al. (2009) considered the elastic and large deformation effects in the honeycomb with five kinds of periodic cells (i.e. hexagonal, rhombus, square, triangular and Kagome) under quasi-static uniaxial compression. In their following work (Qiu et al. 2009), they proposed empirical formulas for the dynamic crushing strength of the honeycomb with five periodic cells based on the result from the FE simulation. Ma et al. (2009) employed the numerical model based on Voronoi tessellation to investigate the loading rate effect on the crushing stress of the cellular material. The numerical result showed that the crushing stress at the loading side was mainly related to the impact velocity, regardless of the specimen size; and the increase of the crushing stress was caused by inertial effect. Similar conclusion was obtained in the numerical study of Liu et al. (2009).

2.2 General cellular material: foam

Foams are the cellular material with three dimensional cells, e.g. woods, metal foams. According to the micro-structure of the cell, the foam can be classified into two types, namely open-cell foam and closed-cell foam. For the open-cell foam, cells are connected by beam-type edges only, in which fluids can flow. On the other hand, the closed-cell foam is made up by cell walls and there is no space for a fluid to flow. Similar to the review for honeycombs, the following reviews are catalogued into two groups: static and dynamic properties.

2.2.1 Static compression of the foam

Due to the different parent material (e.g. rubber liked material, metal and brittle foam), the stress-strain relationship may exhibit a little difference. In this chapter, only the foam made from materials which have a plastic yielding point (e.g. metal) is reviewed. For such material, the stress-strain relationship of the foam under compression is similar to that of the honeycomb structure, possessing three stages: elastic stage, long plastic region with almost constant plateau stress, and densification. Elastic properties are determined by foam types (i.e. open and closed-cell foam). In the linearly elastic range, the deformation mechanism of the open-cell foam is controlled by the bending of the cell walls while it is caused by the cell face stretching in the closed-cell foam (Gibson and Ashby 1997). Similar to honeycombs, the mechanical property of the foam is largely determined by the parent material of the cell wall. Researchers tried to measure the cell-wall's moduli by using tensile and bending tests directly (Ryan and Williams 1989, Choi et al. 1990). However, the measured result underestimated the true modulus due to the local deformation or slip at the loading points. The ultrasound technique was used to obtain elastic properties of the foam by Ashman and Rho (1988, 1998). Gibson and Ashby (1997) summarized experimental results and proposed the following two equations for the open and closed-cell foam, respectively.

$$\frac{E^*}{E_s} \approx \left(\frac{\rho^*}{\rho_s} \right)^2, \frac{G^*}{E_s} \approx \frac{3}{8} \left(\frac{\rho^*}{\rho_s} \right)^2, \nu^* \approx \frac{1}{3} \quad (2.9)$$

$$\begin{aligned}\frac{E^*}{E_s} &\approx \phi^2 \left(\frac{\rho^*}{\rho_s} \right)^2 + (1-\phi) \frac{\rho^*}{\rho_s} + \frac{\rho_0(1-2\nu^*)}{E_s(1-\rho^*/\rho_s)} \\ \frac{G^*}{E_s} &\approx \frac{3}{8} \left\{ \phi^2 \left(\frac{\rho^*}{\rho_s} \right)^2 + (1-\phi) \frac{\rho^*}{\rho_s} \right\}, \nu^* \approx \frac{1}{3}\end{aligned}\tag{2.10}$$

where ρ^*/ρ_s is the relative density, E^* is the Young's modulus, G^* is the shear modulus, ν^* is the Poisson ratio of the foam and ϕ is the variable fraction in the cell faces, respectively.

When the load exceeds the elastic limit, σ_p , plastic collapse occurs due to the buckling or fracture of the cell wall or edge. Sometimes, softening behavior exhibits, leading to a stress drop after the plastic yielding. As the deformation growing, other cells or bands begin to collapse, leading to the plateau stage in the stress-strain curve. The stress in this stage can be approximately regarded as a constant, although a small slope was found in many experimental works.

To measure the response of cellular metals, Bastawros and McManuis (1998, 2000) employed the technique of the digital photography and image correlation. Also, the X-ray was used to measure the plastic response (Bart-Smith et al. 1998). Their results confirmed that the collapse in the foam initiated from the weaker cell or the one with concentrated stress due to heterogeneities (e.g. variation in density, cell shape, orientation, cell wall curvature) and propagated to adjacent cells. The stress-strain relationship of the foam was expressed with the continuum constitutive law (Gibson et al. 1989, Deshpande and Fleck 2000). For low-density foams, the Poisson ratio (Plastic Poisson ratio) is regarded as zero.

For the open-cell foam, the deformation is primarily caused by the elastic bending of the cell edge. Gibson and Ashby (1997) empirically proposed the plastic strength σ_{pl} in terms of the relative density as,

$$\frac{\sigma_{pl}^*}{\sigma_{ys}} = C \left(\frac{\rho^*}{\rho_s} \right)^2 \quad (2.11)$$

where σ_{pl}^* and ρ^* are the plastic strength and density of the foam; σ_{ys} and ρ_s are those of the parent material of the cell wall; and C is a constant, which is determined by the experimental result. Simone and Gibson (1998a) compared the equation with the experimental result and suggested that $C = 0.3$ for the open-cell foam. Gibson and Ashby (1982) also proposed a modified equation to include the density correction,

$$\frac{\sigma_{pl}^*}{\sigma_{ys}} = C' \left(\frac{\rho^*}{\rho_s} \right)^{3/2} \left(1 + \left(\frac{\rho^*}{\rho_s} \right)^{1/2} \right) \quad (2.12)$$

Comparing with the experimental result, the value of the constant C' was determined to be 0.23.

For the closed-cell foam, collapse (e.g. yielding) is the result of the coupling of the stretching and bending of the cell faces. Hence, the stiffness and strength are higher than those of open-cell foam. The stretching of the cell wall contributes to the strength of the foam as a term linear with the relative density. Therefore, the equation for the plastic collapse stress is,

$$\frac{\sigma_{pl}^*}{\sigma_{ys}} = C_1 \left(\phi \frac{\rho^*}{\rho_s} \right)^{3/2} + C_2 (1 - \phi) \left(\frac{\rho^*}{\rho_s} \right) + \frac{p_0 - p_{at}}{\sigma_{ys}} \quad (2.13)$$

where ϕ is the volume fraction in cell edges of the foam, p_0 is the gas pressure and p_{at} is the atmospheric pressure. Usually, the contribution of the gas pressure $(p_0 - p_{at})/\sigma_{ys}$ is neglected because $\sigma_{ys} \gg p_{at}$. However, in some condition, if the cell contains fluid a pressure p_0 much larger than atmospheric pressure, the item must be considered. Comparison with the experimental data gives the constants $C_1 = 0.3$, and $C_2 = 0.4$ for Eq. (2.13). Comparing with FE results, Simone and Gibson (1998a) obtained an equation for a tetrakaidecahedral closed-cell foam with flat faces, which was expressed as,

$$\frac{\sigma_{pl}^*}{\sigma_{ys}} = 0.33 \left(\frac{\rho^*}{\rho_s} \right)^{3/2} + 0.44 \left(\frac{\rho^*}{\rho_s} \right) \quad (2.14)$$

Santosa and Wierzbicki (1998) employed a truncated cube to capture the mechanical properties of the closed foam. Theoretical analysis predicted the crushing strength of the closed foam,

$$\frac{\sigma_{pl}^*}{\sigma_{ys}} = 0.63 \left(\frac{\rho^*}{\rho_s} \right)^{2/3} + 0.07 \left(\frac{\rho^*}{\rho_s} \right) + 0.8 \left(\frac{\rho^*}{\rho_s} \right)^2 \quad (2.15)$$

Their FE result gave an approximate result,

$$\frac{\sigma_{pl}^*}{\sigma_{ys}} = 1.05 \left(\frac{\rho^*}{\rho_s} \right)^{1.52} \quad (2.16)$$

Similar to the honeycomb, the densification strain is defined as the nominal strain, at which cell walls of the foam come into contact and crush together. In the book of Gibson and Ashby (1997), it was defined as follows,

$$\varepsilon_d = 1 - C_3 \left(\frac{\rho^*}{\rho_s} \right) \quad (2.17)$$

where C_3 is a constant, which can be obtained from the experiment.

An “efficiency” technique was employed to determine the densification strain from the stress-strain relationship (Tan et al. 2002, Tan et al. 2005a). In this method, a coefficient E is defined as the energy absorbed up to a given nominal strain, ε_a , divided by the corresponding stress value,

$$E(\varepsilon_a) = \frac{W}{\sigma(\varepsilon_a)} \quad (2.18)$$

where $W = \int_0^{\varepsilon_a} \sigma(\varepsilon) d\varepsilon$ is the absorbed energy. The value of the locking strain is obtained from the condition, at which the efficiency E reaches the maximum value. Therefore, it is determined by the equation,

$$\frac{dE(\varepsilon)}{d\varepsilon} = 0 \quad (2.19)$$

Consequently, the quasi-static plateau stress can be determined,

$$\sigma_{pl}^* = \frac{W}{\varepsilon_d} \quad (2.20)$$

Fitting the empirical expression for the densification strain by Ashby et al. (2000), the form was expressed as follows,

$$\varepsilon_d = 0.7 \left[1 - 1.5 \left(\frac{\rho^*}{\rho_s} \right) - 143.14 \left(\frac{\rho^*}{\rho_s} \right)^3 \right] \quad (2.21)$$

Irregularities have been identified by microstructural observation by many researchers. For example, the Alporas foam contains some cells with initial curvature (Simone and Gibson 1998b); some area in the Alcan foams have lower relative density (Sugimura et al. 1997). However, these irregularities were not taken into account for the analytical model. As a result, experimental results of the closed foam (Gibson and Ashby 1997, Sugimura et al. 1997, Andrews et al. 1999, Ramamurty and Paul 2004) were lower than the theoretical prediction. Simone and Gibson (1998a) employed the FE simulation to investigate the influence of the curvature and corrugation on the yielding strength of the foam. They found that the curvature and corrugation significantly reduced the strength of the aluminum foam compared to that of the ideal foam. The effect of distribution of the material was also investigated by Simone and Gibson (1998a) by using the FE method. Due to the defect in the closed foam, the contribution of the stretching decreased so that the property of the closed foam was nearly the same as those of the open-cell foam (McCullough et al. 1999, Olurin et al. 2000). Hence, the plastic collapse strength σ_{pl}^* of either open or closed-cell foams can be described by a power law function,

$$\frac{\sigma_{pl}^*}{\sigma_{ys}} = C_4 \left(\frac{\rho^*}{\rho_s} \right)^n \quad (2.22)$$

where C_4 and n are constants, determined by the experimental result.

2.2.2 Dynamic crushing behavior of the foam

Due to the influence of the strain rate and the effect of the inertia, the dynamic crushing behavior of the foam differs from that of the static loading condition. Hence, the following reviews focused on the two aspects, namely the strain rate effect and the inertial effect.

2.2.2.1 Strain rate effect

The definition of the strain rate is the ratio between the impact velocity, V and the length of the impinged object, L_0 (i.e. $\dot{\varepsilon} \equiv V/L_0$). The literature on the influence of the strain rate on the crushing strength of the foam contains apparent conflict. The conflict is mainly caused by several factors, including the definition of key parameters (e.g. plateau stress and densification strain), choice of the experimental technique, the size of the specimen, and the material of the foam.

Prakash et al. (1995) reported that crushing behavior of the closed-cell foam Al-SiC foam is insensitive to the strain rate in the range of 10^{-3} to 10^{-1} s^{-1} . Kenny (1996) found the similar point that the strain rate has little effect on mechanical properties of the closed-cell foam Alcan aluminum foam over the strain rate range 10^{-4} to 100 s^{-1} . However, they did not measure the stress-strain curve under dynamic and quasi-static condition. Similar conclusion was obtained by Lankford and Dannemann (1998) for the Duocel aluminum foam (open-cell foam). Deshpande and Fleck (2000) investigated the high strain rate influence on the crushing behavior of Alulight (open-cell foam) and Duocel (closed-cell foam) foams by using split Hopkinson bar (SHPB) and direct impact tests. Their

experimental results showed that the plateau stress remained insensitive to the strain rate over the range strain rate from 10^{-3} to 5000 s^{-1} . Benouali et al. (2002) also reported no strain rate effect on the compression of the closed-cell Hydro Al alloy foam in the velocity range from 5 to 250mm/min. Tan et al. (2005a) performed the impact test on the closed-cell Hydro/Cymat aluminum foam. Their results verified that the foam did not show significant strain rate sensitivity. Peroni et al. (2013) also used the SHPB to carry the behavior of Hydro aluminum foam (closed-cell foam) over the strain rate range from 100 to 300 s^{-1} . No obvious strain rate effect was observed in their study.

The contradicted result about the strain rate dependence was also reported by some researchers. Mukai et al. (1999) reported that the plateau strength of the closed-cell Alporas foam was enhanced 50-100% at the strain rate of 10^3 s^{-1} . The study by Paul and Ramamurty (2000) showed the plateau strength increased by 31% with increasing strain rate at the strain rate of $1.6 \times 10^{-1} \text{ s}^{-1}$. Ramachandra et al. (2003) studied the response of a closed-cell Alporas aluminum foam with relative density 8% under the impact velocity of 3-30m/s. Their results showed that the absorbed energy gradually increased in the quasi-static regime; significant increase appeared at the velocity larger than 10m/s due to shockwave effects. Mukai et al. (2006) also used the SHPB to test Alporas aluminum foam with different relative densities. Some degree of strain rate sensitivity was observed, especially on the elastic limit. Similar conclusion was obtained by Itraquin et al. (2013) after correcting the dispersion of the stress wave. Wang et al. (2011) carried out the investigation into the compressive behavior of the closed-cell foam at the medium strain rate, showing that the plateau stress and absorbed energy were remarkable dependent on the strain rate over the range from 10^{-3} to 450 s^{-1} .

Kanahashi et al. (2000) employed a SHPB to study the compressive behavior of an open-cell aluminum foam with the relative density of 0.03-0.06. Comparing with the quasi-static stress-strain curve, the sensitivity of the yielding stress was shown at the strain rate of $1.4 \times 10^3 \text{ s}^{-1}$.

2.2.2.2 Inertia effect and plastic shock wave propagation

Various experiments, numerical and analytical studies have confirmed that the inertia effect leads to plastic shock propagation in the foam structure, resulting in an enhancement of the crushing stress under dynamic loading.

Some researchers employed the honeycomb structure for simplification of the analysis to investigate the dynamic response of the cellular material, which has been reviewed in Section 2.1. In FE simulations, the foam is always considered as a continuum material for simplification, which can be found in a lot of research work (Hanssen et al. 2009, Langdon et al. 2010, Zeng et al. 2010).

A number of analytical methods have been proposed to explain the crushing enhancement and the shock wave propagation, including the one-dimensional shock theory (Reid and Peng 1997), spring mass model (Shim et al. 1990, Li and Meng 2002), and the energy balance method (Fleck and Deshpande 2004, Seitz and Skews 2007, Main and Gazonas 2008). Some debates (Tan et al., 2005; Fleck et al., 2005; Li and Reid, 2006) have been lead due to the application of the different methods. Clarity has been provided by Harrigan et al. (2010) by comparison these methods for analysis of two impact scenarios, which shows that the applicability of the simple shock theory. Most of analytical models

for the dynamic behavior of the cellular material were reviewed by Zhu et al. (2011). The one-dimensional shock theory well predicted the shock front of the plastic wave, as well as the enhancement of the crushing stress while the other models are regarded as less accurate and potentially flawed. Hence, the following reviews concentrates on the one-dimensional shock theory.

a. One-dimensional shock theory

Jahsman (1968) proposed a one-dimensional non-linear theory of syntactic foam. Reid et al. (1983, 1984) employed the one-dimensional ring-system to carry out the impact test. Analytical results were also presented to explain the plastic shock wave by neglecting the elastic property of the ring. Later, Reid and Peng (1997) proposed a one-dimensional shock model to explain the crushing behavior of the wood under impact. Afterwards, Deshpande et al. (2000) employed this model in their experiment to investigate the strain rate effect of the foam. Further application of this theory can be found in the analysis of the honeycomb (Harrigan et al. 1999, Zou et al. 2009), aluminum foam (Tan et al. 2005a, Tan et al. 2005b), and the crushing of a foam under blast (Hanssen et al. 2002, Main and Gazonas 2008).

In the one dimensional shock theory, the stress-strain curve of the foam material (e.g. wood) is simplified into rigid-perfect plastic-locking densification (R-PP-L), which is shown in Figure 2.6. Two key parameters are involved in this model, namely the plateau stress, σ_{pl}^* and the locking strain ε_L . The locking strain is usually regarded as the densification strain of the cellular material.

Figure 2.6 Stress-strain relationship for the cellular material and the R-PP-L material model

In this theory, the densified material is assumed to be fully densified at the locking strain while no deformation occurs in the undeformed zone (see Figure 2.7).

Figure 2.7 States of the foam at the shock front

By analyzing the stress discontinuity at the shock front and employing mass and momentum conservation, the crushing stress, σ_d is found to be enhanced by the inertia, expressed as follows,

$$\sigma_d = \sigma_{pl}^* + \rho^* \frac{v^2}{\varepsilon_L} \quad (2.23)$$

where ρ^* is the initial density of the foam and v is the velocity of the material behind the shock front.

Developments have been applied to this simple shock theory in order to give better predictions. Retaining the basic characteristic, Tan et al. (2005a, 2005b) took accounts the elastic property of the material; Harrigan et al. (1998, 2005) proposed an elastic-plastic model with hardening. The elasticity property was also considered in the work of Lopatnikov et al. (2003, 2004, 2007) with the different definition of the locking strain. In their work, the locking strain was defined as the maximum value (i.e. $\varepsilon_{\max} = 1 - \rho^* / \rho_s$) while it was determined by using the “efficiency” method in the research work (Tan et al. 2005a, Tan et al. 2005b).

The analytical models mentioned above, all used a constant densification strain during the whole crushing process. Researchers found that the locking strain in R-PP-L material model significantly influences the prediction of the shock front speed and the dynamic stress (Pattofatto et al. 2007). In the work of Zou et al. (2009), the densification strain was found to increase with the increase of the crushing velocity, approaching to a maximum value (15% higher than the static value, see Figure 2.8). The simple shock

theory was used together with the Rankine-Hugoniot jump conditions at the cross-section of the shock front; consequently, the evolution of Eq. (2.23) is expressed as follows,

$$\sigma_d = \sigma_{pl}^* + \rho^* \frac{[v]^2}{[\varepsilon]} \quad (2.24)$$

where $[]$ denote a change in the value across the shock.

Figure 2.8 Densification strain of the honeycomb under different crushing velocities (Zou et al. 2009)

Recently, Tan et al.(2012) carried out experimental studies on the open cell metal foam under dynamic loading condition. Comparing with the experimental results, they proposed an equation to determine the locking strain,

$$\varepsilon_d = \rho^* v^2 / (\sigma_d - \sigma_{pl}^*) \quad (2.25)$$

where v is the instant velocity of the particle behind the shock front; σ_d is the dynamic stress measured from the experiment; and σ_{pl}^* is the quasi-static plateau stress.

Pattofatto et al. (2007) proposed a power law densification to make more accurate prediction, expressed by Eq. (2.26).

$$\sigma(\varepsilon) = \sigma_{pl}^* + K\varepsilon^m \quad (2.26)$$

where K is the strength index and m is the strain hardening index. Combining with Eq. (2.23), the densification strain is determined as,

$$\varepsilon_d = (\rho^* v^2 / K)^{1/(m+1)} \quad (2.27)$$

Instead of using the concept of the locking densification, Karagiozova et al. (2012) proposed analytical models based on actual experimental stress-strain curves. Also, conservation of the mass and momentum was employed for analysis the proposed impact scenarios. Comparing with the results by using R-PP-L material model, they found that the R-PP-L material model with only two parameters, one quasi-static plateau stress and locking strain, could lead to inaccurate predictions, especially for the impact problems with really high densities.

b. Deformation modes and critical velocity

The experimental and theoretical analysis showed that only when the impact velocity reaches a critical value, plastic compaction shock occurs. For honeycomb structure, three deformation modes, namely “X” mode (quasi-static mode), “V” mode (transitional mode), and “I” (shock-type mode), were identified by Ruan et al. (2003) and Zou et al. (2009).

Similar phenomenon was observed in the work of Zheng et al. (2005) and Liu et al. (2009). Within different velocity region, different waves initiate from the proximal end of the foam, leading to different deformation modes, namely homogenous mode, transitional mode and shock mode. For the material without plastic hardening, the transitional mode does not exist.

It should be noted that the application of the shock simple shock is only valid when the plastic shock occurs. Apparently, R-PP-L material cannot capture these deformation modes. Hence, the E-PP-L material model in the work of Lopatnikov et al. (2004, 2007) was employed to describe the deformation process under low impact velocities. Based on the wave trap theory (see E. (2.6)), the first critical velocity was determined for the E-PP-L material model,

$$V_{cr} = \frac{\sigma_y}{\sqrt{E\rho^*}} \quad (2.28)$$

where E and σ_y are the Young's modulus and the yielding stress, respectively. When the Young's modulus approaches infinite, the E-PP-L model degenerates to the R-PP-L model and the critical velocity decreases to zero.

The plastic hardening property was considered by Zheng et al. (2012). They proposed a more realistic material model with a rigid-linear hardening plastic-locking (R-LHP-L) idealization. The second critical velocity, from which the deformation mode transits from the transitional mode to the shock mode, was determined,

$$V_{c2} = \varepsilon_d \sqrt{E_1/\rho^*} \quad (2.29)$$

where E_1 is the plastic hardening modulus. Since they did not consider the elastic property of the foam, the stress uniformity index was employed to determine the first critical velocity, showing as follows,

$$V_{c1} = \sigma_{pl}^* / 9\rho_0 \sqrt{E_1 / \rho^*} \quad (2.30)$$

More recently, Zheng et al. (2013) employed the power law densification and determined the second critical velocity,

$$V_{c2} = \varepsilon_d^{(n+1)/2} \sqrt{K / \rho^*} \quad (2.31)$$

2.3 Literature on graded cellular material

The concept of the functionally graded material is to optimize the global response of the structure by introducing a variation of the material property, e.g. density and yielding stress. Previous studies on the inhomogeneity and imperfection of the cellular material in Section 2.1 and 2.2 give insight that introducing a gradual variation in one direction of the cellular material significantly influences the mechanical property of the cellular material. Such graded cellular material can be easily found in nature. For example, human bone processes a density gradient, maximizing the biological mechanical performance. Hence, it is worthwhile to further explore mechanical properties of graded cellular structures under dynamic loading.

2.3.1 Numerical simulations on the graded cellular material

Up to now, most research works in the literature were based on the FE simulation. By using the FE modeling, Ali et al. (2008) investigated the response of the hexagonal honeycomb block with different wall thickness under low velocity impact (up to 20m/s). In their work, the regular honeycomb block was divided into 6 layers and each layer was assigned with different wall thickness. They observed the deformation process and investigated the energy absorption capacity of the graded honeycomb. However, the impact velocity in their study was so low that no obvious influence on the deformation process was found.

The study by Ajdari et al. (2009) focused on the quasi-static compressive behavior of functionally graded Voronoi structures by using FE simulations. It was found that the effective modulus and yielding strength of the structure increased after introducing the density gradient. In their later study (Ajdari et al. 2011), a gradient in the cell wall thickness was introduced into the regular hexagonal honeycomb. Results showed that the decrease of the relative density along the direction of crushing enhanced the energy absorption of honeycombs at early stages of crushing. Similar work on the graded Voronoi honeycomb was done by Wang et al. (2011). However, these studies mentioned above focused on the final energy absorption of the structure under low impact velocity, neglecting the deformation process under high impact velocities.

Cui et al. (2009) proposed a graded foam model to design the graded foam by using FE simulation. Based on the model proposed by Schraad and Harlow (2006), they determined the stress-strain curves for all layers of the foam due to different initial density. The density gradient is expressed as,

$$\rho(y) = \rho_1 + (\rho_2 - \rho_1) \left(\frac{y}{d} \right)^n, 0 \leq y \leq d \quad (2.32)$$

where ρ_1 and ρ_2 are the initial densities at the proximal and distal layers, respectively; d is the thickness and y is the position through the thickness; and n is the power number. Their study focused on the type of the density gradient (i.e. the value of n), neglecting the influence of the gradient on the deformation mode.

Bruck (2000) proposed a one-dimensional model to describe the stress wave propagation in the functionally graded material. Li et al. (2001) investigated the behavior of the graded structure under impulsive loading. However, these works are only applicable for the graded continuum material, instead of the graded cellular material. The propagation of the stress wave in the graded cellular material was investigated by Kiernan et al. (2009). They adopted same approaches in the work of Cui et al. (2009) to introduce the density gradient. Similarly, their work did not pay attention on the deformation mode. Employing energy balance method, Zhu et al. (2009) studied of the effect of the length and density on the maximum deflection of a foam sandwich plate.

2.3.2 Experimental work on the graded cellular material

Only a few experimental works in the literature are related to the graded cellular material. A graded structure for hollow particle filled syntactic foams material was studied in the work of Gupta (2007). Comparing to the homogenous structure, no significant loss in the strength or failure took place when the graded foam was under 60-75% compression.

Brother and Dunand (2008) measured the compressive mechanical properties of the Al-6061 foam with density gradient under quasi-static loading. Their results showed that the plateau stress of the density-graded foam was higher than the uniform one.

Zeng et al. (2010) investigated the influence of the density gradient on the mechanical response of graded polymeric hollow sphere agglomerates under impact loading. Focused on the early stage, the force at the distal end was found much higher when the strongest layers were placed near the proximal end. Their study showed that placing the strongest layer as the first impacted layer and weakest layer as the last had some benefit to minimize the force level transmitting to the protected structure.

2.4 Concluding remarks

The literature review provides an overview of the quasi-static and dynamic behavior of cellular solids, including the honeycomb and foam structure. Under a uniaxial compression, static or dynamic, the stress-strain relationship contains three regimes, i.e. a short elastic stage, a long plateau stage and the densification.

Under quasi-static loading, mechanical properties, such as the elastic modulus and plateau stress, are strongly influenced by defects and imperfections of the cellular material. The dynamic behavior of the foam is very different from the quasi-static one. When the velocity is low, the compression with the homogenous material is in a uniform manner; for the material with defects, the crush imitates from the cells or layer with defects, similar to that under quasi-static load. Under high velocity impact, the plastic shock front takes places from the proximal end and propagates to the distal end due to the

inertia effect, leading to an enhancement of the crushing strength. Between these two velocities, the crushing process is in a transitional mode. The one-dimensional shock theory successfully described the plastic shock wave propagation and predicted the enhancement of the crushing strength.

To optimize the global mechanical performance, a gradient in a property of the cellular material (i.e. strength, density and cell size) is incorporated into the structure. Previous studies focused on the energy absorption capacity of the graded cellular material and the stress transmitted to the protected structure, regardless of the deformation process.

Few researches on the deformation process and its mechanism of the graded cellular material under impact were reported. Furthermore, previous work mainly employed the FE simulation to investigate response of the graded cellular material. Current study attempts to present a basic understanding of the influence of the gradient in the graded cellular material under impact, including the deformation modes, energy absorption capacity and the crushing strength.

Chapter 3 Dynamic behavior of graded honeycombs —— a finite element study

3.1 Introduction

In this chapter, we investigate the in-plane dynamic mechanical behavior of the functionally graded honeycomb. Finite element simulations are carried out using ABAQUS/EXPLICIT. As a first step of the whole research, an artificial case was proposed. The material's yielding stress of a honeycomb varies layer by layer, introducing a stress gradient into the uniform honeycomb. In practical engineering, the manufacturing of such kind of honeycomb can be the same with the homogenous honeycomb except that different materials with varying strength are used as parent material for each layer. In the FE simulation, a constant velocity is applied to the impact plate which then crushed the honeycomb. From trial tests and the observation of the deformation modes in the crushing process, the range of the constant impact velocity is determined from 5 to 100m/s. From the observation of the deformation profiles, three deformation modes have been identified when the strongest layer is placed at the impact end while two deformation modes have been found when the weaker layer is placed at the impact end. The critical velocities, at which the deformation modes transits are determined with a classification map. Dynamic stress and quasi-static plateau stress are obtained from the loading and the supporting plate, respectively. The stress-displacement curves are compared with the analytical prediction based on the shock wave theory. Then, the absorbed energy under constant velocity loading is investigated. To further investigate the influence of the stress gradient on the behavior of the graded honeycomb,

different initial velocities versus the impinging mass are imposed on the impact plate. By considering a critical case, in which the velocity reduces to zero when the honeycomb structure is just fully crushed, the densification velocity is obtained. The results confirm that the positive gradient will enhance the energy absorption capacity of honeycombs.

Equation Chapter 3 Section 1

3.2 Analytical consideration

In general, the stress-strain curve of cellular material possesses the feature sketched by the fine line in Figure 2.6 and it is characterized by three stages, viz. a short elastic stage, a long plateau stage and finally a densification stage (Gibson and Ashby 1997). Reid and Peng (1997) firstly proposed a rigid-perfectly-plastic-locking (R-P-P-L) model (see Figure 2.6) for the cellular material and derived the equation (see Eq. (2.23)) for the dynamically-enhanced stress of the cellular material under dynamic loading based on shock wave theory.

From Eq. (2.23), it is noted that two kinds of gradients can be introduced to a honeycomb block, i.e. the gradient in the density ρ_0 , and that in the quasi-static plateau stress σ_p .

In Chapter 2, it has been reviewed that, Gibson and Ashby (1997) employed the large deformation bending theory, theoretically gave the equations for the quasi-static plateau stress σ_p (Eq. (2.2)), the density of the undeformed cell $\rho_0 = \frac{2}{\sqrt{3}} \left(\frac{t}{l} \right) \rho_s$ and the

locking strain $\varepsilon_d = 1 - C \frac{\rho_0}{\rho_s}$, where t and l are the thickness and the length of the cell edge, respectively; σ_{ys} and ρ_s are the yielding stress and density of the parent material,

respectively; and C is a constant, which is ideally equal to 1. It should be noted the cell of the honeycomb block cannot be wholly densified, and so C is always greater than 1.

Hence, the dynamic stress at the shock front for the honeycomb structure is,

$$\sigma_d = \frac{2}{3} \left(\frac{t}{l} \right)^2 \sigma_{ys} + \frac{\rho_s}{\frac{\sqrt{3}}{2} \left(\frac{l}{t} \right) - C} V^2 \quad (3.1)$$

Although the above equations are determined for homogenous honeycombs, they provide good insight into the graded honeycomb structure (GHS). Eq. (3.1) shows that the gradient can be introduced into the structure by merely changing the yielding stress or the density of the parent material; the thickness ratio t/l not only influences the quasi-static plateau stress but also the density of the cellular material.

In current study, we start this problem as the simplest case of gradient cellular materials, trying to decouple the two most dominant factors (i.e. quasi-static plateau stress and density). In reality, changing thickness is the easiest way to introduce the gradient. However, it would involve gradient both in the quasi-static plateau stress and density (see Eq. (3.1)), which is hard to see the effect of changing one of them alone. Hence, an artificial case is invented, in which a gradient only in the materials' yielding stress is introduced to the honeycomb structure, in order to reveal any new phenomena and define some parameters first, such as gradient and some dimensionless parameters.

As a first step, gradient in stress is introduced by assigning different values of yielding stress to the parent material of each layer of a regular hexagonal honeycomb. The deformation modes are observed under a constant impact velocity. Deformation profiles

are obtained for each deformation mode. The critical velocities, at which the deformation mode changes, are obtained empirically in the terms of the impact velocity, stress gradient and the thickness ratio of cells. To further investigate the influence of the gradient on the energy absorption capacity, the densification velocity of a given impact mass is obtained, at which the instant velocity of the impinging plate reduces to zero when the honeycomb block is just fully crushed.

3.3 Finite Element model

The Finite Element (FE) model is similar to that previously proposed by Ruan et al. (2003), but a gradient in the material's yielding stress is introduced to represent the graded hexagonal honeycomb structure (GHS) (Figure 3.1a). The structure is composed of 16 cells in the X1 direction and 15 in the X2 direction. The material property of all the layers is the same except that the yielding stress varies layer by layer. It is noted that the neighboring layers have cells sharing the same side. Within a cell, five edges are assigned the same material while the left one edge has a different material property (Figure 3.1b). The parent material is assumed to be elastic, perfectly plastic with $E = 69\text{GPa}$ and $\nu = 0.3$, where E and ν are Young's modulus and Poisson ratio of the material, respectively. The cell is regular with the same value of edge length, which is $l = 2.7\text{mm}$, and the corresponding angle between the edges is 120° . The wall-thickness of the cells t and the density of the parent material ρ_s are 0.2mm and 2700kg/m^3 , respectively. Hence, the density of the honeycomb is $\rho_0 = 2/\sqrt{3}(t/l)\rho_s = 230\text{kg/m}^3$. The out-of-plane width of the honeycomb is $b = 0.9\text{mm}$. Hourglass controlled, four nodes, reduced integration shell

elements (type S4R) are used to mesh the structure. Each cell wall is meshed into eight shell elements with one-layer elements along the out-of-plane direction. Both plates A and B are defined as rigid body.

As shown in Figure 3.1, plate A is fixed by constraining all the degrees of freedom at the reference point; plate B only can move along the X_1 direction. Interaction properties are imposed using general contact conditions and surface to surface kinematic contact conditions between the honeycomb block and the two rigid plates without considering friction. Furthermore, the out-of-plane displacement of nodes within the X_1 - X_2 plane is constrained so as to prevent the honeycomb block from out-of-plane bulking. Five integration points through the thickness of the shell elements are adopted.

Figure 3.1 (a) Finite element model composed by 16 hexagonal cells in X_1 direction and 15 cells in X_2 direction. (b) Within a cell, five edges have the same material property while the left one is assigned with different yielding stress. Note: plate A is fixed and plate B is the impinging plate

For clarity, it is defined that the stress gradient is negative when the strongest layer is placed at the impact end while it is positive when the weakest layer is placed at the impact end. The stress gradient of GHS is defined as follows,

$$g = \frac{\Delta\sigma_{ys}}{\sigma_{ave}l_{layer}} \quad (3.2)$$

where $\Delta\sigma_{ys}$ is the difference of the material's yielding stress between two layers; σ_{ave} is the average value of the materials' yielding stress of parent material and l_{layer} is the length of each cell in X_1 direction. Non-dimensional parameters are selected to investigate the dynamic behavior of the honeycomb under impact loading. In our study, the average materials' yielding stress of the parent material is $\sigma_{ave} = 77$ MPa. Using Eq. (2.2), the average plateau stress of the cellular material is obtained as $\sigma_0 = 0.28$ MPa.

Non-dimensionalization of the parameters is implemented in current study. The displacement of the impinging plate is non-dimensionalized respected to the initial length of the GHS,

$$\mu \equiv u/l_0 \quad (3.3)$$

where l_0 is the total length of the graded honeycomb block.

A characteristic velocity of the GHS is defined as,

$$c_p \equiv \sqrt{\sigma_0/\rho_0} \quad (3.4)$$

In terms of the thickness ratio, t/l the density, ρ_s and the average stress of the parent material, Eq. (3.4) is written as,

$$c_p \equiv \sqrt{\frac{1}{\sqrt{3}} \left(\frac{t}{l} \right) \frac{\sigma_{ave}}{\rho_s}} \quad (3.5)$$

Substituting the values of the physical parameters, the characteristic velocity of the GHS is $c_p = 34.92$ m/s.

Hence, the non-dimensional velocity can be determined as follows,

$$v \equiv V/c_p \equiv V / \sqrt{\frac{1}{\sqrt{3}} \left(\frac{t}{l} \right) \frac{\sigma_{ave}}{\rho_s}} \quad (3.6)$$

The non-dimensional gradient is defined as follows,

$$\Theta \equiv gl_0 = n \frac{\Delta\sigma_{ys}}{\sigma_{ave}} \quad (3.7)$$

where n is the number of the layers. The non-dimensional gradient can be regarded as the ratio between the difference of the yielding stress at impinged and distal ends and the average yielding stress.

3.4 Finite Element Results

3.4.1 Deformation modes

In this section, attention is put on the effect of the velocity on the deformation modes. Constant velocities are applied to the impinging plate (i.e. plate B). The deformation profiles are obtained to investigate the deformation modes of the GHS under dynamic loading. It should be noted that applying a constant velocity to a rigid plate is not the same as an impact event. Compared to the constant velocity loading, the impact condition is much more complex, where the velocity decreases to zero at the end of the crushing.

3.4.1.1 Deformation modes - positive gradient

When the gradient Θ is positive, the weakest layer is placed at the impact end. Under low velocity impact, the deformation is mainly dominated by the distribution of the quasi-static plateau stress of the layers. Since the weakest layer is placed at the proximal end, the initial collapse occurs at the impinging end regardless of the impact velocity. Figure 3.2-Figure 3.4 show deformation modes of a GHS with the gradient $\Theta = 0.42$ under various impact velocities. As shown in Figure 3.2, when $V = 5$ m/s deformation localization occurs initially at the loading end. It is found that some cells in the other layers with stronger material collapse while the cells in the 15th layer remains undeformed, which produces a ‘V’ shaped band (Figure 3.2a). Afterwards, the “V” shaped band continues to propagate to the supporting end (Figure 3.2b-c). With the increase of the displacement of the impinging plate, the cells ahead of the “V” shaped band begin to collapse (Figure 3.2d). At a higher impact velocity of 20m/s (Figure 3.3), the “V” shaped band appears later than the case depicted in Figure 3.2. Besides, no deformation is found near the supporting end during the whole crushing. When the impact velocity increases to $V = 35$ m/s (Figure 3.4), only a localized transverse band in the shape of “I” is observed near the loading end and it continues to propagate, layer by layer to the supporting end, which is like the shock wave propagation.

The deformation profile for the GHS with a higher stress gradient of $\Theta = 1.66$ is shown in Figure 3.5. The impact velocity is the same as for Figure 3.3 (i.e. $V = 20$ m/s). Comparison between Figure 3.3 Figure 3.5 shows that the “V” shaped band appears in the honeycomb with lower stress gradient (Figure 3.3) while only “I” shaped band appears during the whole crushing process in the cases of higher stress gradients (Figure

3.5). This indicates that the stress gradient influences the deformation mode under the dynamic loading.

Figure 3.2 Deformation profile of the graded honeycomb in the case of $\Theta = 0.42$, $V = 5$ m/s. (a) $u = 7$ mm; (b) $u = 21$ mm; (c) $u = 35$ mm; (d) $u = 49$ mm. u is the displacement of the impinging plate

Figure 3.3 Deformation profile of the graded honeycomb in the case of $\Theta = 0.42$, $V = 20$ m/s. (a) $u = 7$ mm; (b) $u = 21$ mm; (c) $u = 35$ mm; (d) $u = 49$ mm. u is the displacement of the impinging plate

Figure 3.4 Deformation profile of the graded honeycomb in the case of $\Theta = 0.42$, $V = 35$ m/s. (a) $u = 7$ mm; (b) $u = 21$ mm; (c) $u = 35$ mm; (d) $u = 49$ mm. u is the displacement of the impinging plate

3.4.1.2 Deformation modes - negative gradient

When the gradient is negative, the strongest layer is placed at the impact end while the weakest layer is at the supporting end. The deformation modes are more complex than the cases with positive gradient. Figure 3.6-Figure 3.8 show the deformation modes for the honeycomb with negative gradient of $\Theta = -0.42$ under various impact velocities. Figure 3.6 depicts the response under the velocity $V = 5$ m/s. The collapse of cells initiates from the supporting end while no deformation occurs at the loading end. Afterwards, a “V” shaped band appears near the supporting end and it becomes more obvious with the cells’ collapse near the shaped band. Near the end of the crushing, some cells adjacent to the impact plate (i.e. plate B) begin to collapse. The deformation process seems a reversal of that shown in Figure 3.2, as expected. Under low impact loading condition, the deformation process is determined by the distribution of the quasi-static plateau stress no matter where the load is applied—the weakest part deforms first.

The deformation profile of the honeycomb under a higher impact velocity of $V = 20\text{m/s}$ is given in Figure 3.7. The initial collapse occurs at the loading end (though stronger) while no obvious deformation occurs at the supporting end. Afterwards, some cells near the proximal end remain undeformed while others in the middle of the specimen begin to collapse, producing a “V” shaped band. Subsequently, the cells near the supporting end begin to collapse, forming a new “V” shaped band (Figure 3.7c). With the increase of the displacement, the “V” shaped band near the proximal end almost remains unchanged and further deformation is concentrated near the supporting end. Finally, the bands interact with each other until the whole structure is crushed.

For $V = 50\text{m/s}$ the deformation profile is depicted in Figure 3.8. Similar to the cases with positive gradient under high impact velocities, only a localized transverse band is observed at the loading end, which then propagates to the supporting end. For the cells adjacent to the supporting end, some small deformation still can be found.

The deformation modes of the honeycomb with larger magnitudes of the stress gradient, -1.02 and -1.66, are shown in Figure 3.9 and Figure 3.10, respectively. The impact velocity is the same as for Figure 3.6 (i.e. $V = 5\text{m/s}$). The initial deformation at the supporting end in Figure 3.6 is also present in Figure 3.9 and Figure 3.10 while no deformation occurs at the loading end. Comparing these figures, it is noted that the “V” shaped band is more obvious in the honeycomb with a lower magnitude of the stress gradient, indicating that the deformation mode is significantly influenced by the local quasi-static plateau stress under low velocity impact.

Figure 3.5 Deformation profile of the graded honeycomb in the case of $\Theta = 1.66$, $V = 20$ m/s. (a) $u = 7$ mm; (b) $u = 21$ mm; (c) $u = 35$ mm; (d) $u = 49$ mm. u is the displacement of the impinging plate

Figure 3.6 Deformation profile of the graded honeycomb in the case of $\Theta = -0.42$, $V = 5$ m/s. (a) $u = 7$ mm; (b) $u = 21$ mm; (c) $u = 35$ mm; (d) $u = 49$ mm. u is the displacement of the impinging plate

Figure 3.7 Deformation profile of the graded honeycomb in the case of $\Theta = -0.42$, $V = 20$ m/s. (a) $u = 7$ mm; (b) $u = 21$ mm; (c) $u = 35$ mm; (d) $u = 49$ mm. u is the displacement of the impinging plate

Figure 3.8 Deformation profile of the graded honeycomb in the case of $\Theta = -0.42$, $V = 50$ m/s. (a) $u = 7$ mm; (b) $u = 21$ mm; (c) $u = 35$ mm; (d) $u = 49$ mm. u is the displacement of the impinging plate

Figure 3.9 Deformation profile of the graded honeycomb in the case of $\Theta = -1.02$, $V = 5$ m/s. (a) $u = 7$ mm; (b) $u = 21$ mm; (c) $u = 35$ mm; (d) $u = 49$ mm. u is the displacement of the impinging plate

Figure 3.10 Deformation profile of the graded honeycomb in the case of $\Theta = -1.66$, $V = 5$ m/s. (a) $u = 7$ mm; (b) $u = 21$ mm; (c) $u = 35$ mm; (d) $u = 49$ mm. u is the displacement of the impinging plate

3.4.1.3 Mode classification map

The above deformation characteristics can be summarized as follows. Two deformation modes are found for the GHS with positive gradient while three deformation modes in the GHS with negative gradient.

When the gradient is positive, the first deformation mode is “V” mode, which takes place when the impact velocity is low (Figure 3.2) and is characterized by oblique localized bands initiating from the loading end. The “V” shaped band occurs at the loading end and then extends to the other layers. With the increase of the impact velocity, localized bands tend to be orientated in the transversal direction (Figure 3.3 and Figure 3.4), which is the second, known as “I” mode.

When the gradient is negative, the deformation processes are quite complex. Hence, the names of the modes are taken according to the initial mode. Three deformation modes exist: two involve a “V” mode, either initiating from the supporting end or from the impacting side and one for “I” mode. The first is “V_S” mode, as shown in Figure 3.6. In this mode, deformation occurs from the “supporting” end, resulting in a “V” shaped band. With the increase of the impact velocity, the inertial effect gradually influences the deformation mode so that more and more deformation takes places at the loading end. The second is “V_L” mode, in which the initial deformation occurs at the “loading” end, resulting in a “V” shaped band. “V_L” mode is a mixed mode, in which further deformation may be concentrated at the supporting end in the later deformation process (Figure 3.7). The third is “I” mode, which appears under high velocity impact (Figure 3.8). This “I” mode is similar to the case for graded honeycomb with positive gradient under high velocity impact. Compared with the deformation modes for the homogenous

honeycomb under impact loading [20], the “V_s” mode is the quasi-static deformation mode while the “I” mode represents the deformation mode under high velocity impact.

Figure 3.11 Non-dimensional critical velocities versus the gradient

Figure 3.11 plots the non-dimensional critical velocities for the GHS against the stress gradient, based on FE simulations of a number of cases. In terms of the stress gradient Θ , the thickness ratio t/l , the average quasi-static plateau stress σ_{ave} and the density of the parent material ρ_s , two empirical equations are proposed for the critical velocity as,

$$V_{cr1} = (-0.27\Theta + 0.43) \sqrt{\frac{1}{\sqrt{3}} \left(\frac{t}{l} \right) \frac{\sigma_{ave}}{\rho_s}} \quad (3.8)$$

$$V_{cr2} = (-0.51\Theta + 1.31) \sqrt{\frac{1}{\sqrt{3}} \left(\frac{t}{l} \right) \frac{\sigma_{ave}}{\rho_s}} \quad (3.9)$$

It should be noted that when the gradient is positive (i.e. $\Theta > 0$), the first critical velocity does not exist. Also, the empirical equations are suitable for the analyzed cell block geometry only, since the strain localization near the stationary end is size dependent.

3.4.1.4 Dynamic stresses in the graded honeycombs

In this section, the dynamic stress in graded honeycomb is investigated by analyzing the reaction stresses of the two plates (i.e. plates A and B). For the low velocity impact, the stresses of the two plates are nearly the same as the quasi-static plateau stress, neglecting the inertia effect. On the other hand, when the impact velocity is sufficiently high, the deformation mode is a propagation of a plane plastic wave (i.e. “T” mode) regardless of the stress gradient, indicating that the shock wave theory may be applicable in this case. The locking strain ε_d is assumed to be a constant at $\varepsilon_d = 0.82$, which is determined from the FE simulations.

Figure 3.12 The stress-displacement relationship for the GHS with positive gradient. (a) $\Theta = 0.42$, $V = 5 \text{ m/s}$ and $V = 35 \text{ m/s}$; (b) $\Theta = 1.04$, $V = 5 \text{ m/s}$ and $V = 35 \text{ m/s}$; (c) $\Theta = 1.66$, $V = 5 \text{ m/s}$ and $V = 30 \text{ m/s}$

Note: the displacement is the non-dimensional one respect to the initial length

Figure 3.12 shows the reaction stresses versus the non-dimensional displacement of the impact plate (i.e. plate B) when the gradient is positive. The main reason for the oscillation in the FE results (shown in Figure 3.12) is due to the softening behavior in the collapse of the hexagonal cell. Under the low velocity impact, the stresses at the supporting and loading ends are almost the same. Compared with the analytical prediction some deviations are found at the finishing stage of crushing. The stresses are much smaller than the analytical prediction. The reason can be found from the deformation profiles. For example, it is found that the cells ahead of the compaction band are collapsed in Figure 3.2d. Under high velocity impact, the dynamic stress is obtained from the loading end while the stress on the supporting end is equal to the quasi-static plateau stress at the compaction front. The analytical prediction shows a good agreement with the FE results, indicating that the shock wave theory can be employed for the GHS with positive gradient.

Figure 3.13 The stress-displacement relationship for the GHS with negative gradient. (a) $\Theta = -0.42$, $V = 5$ m/s and $V = 50$ m/s; (b) $\Theta = -1.04$, $V = 5$ m/s and $V = 80$ m/s; (c) $\Theta = -1.66$, $V = 5$ m/s and $V = 100$ m/s;

Note: the displacement is the non-dimensional one respect to the initial length

Figure 3.13 represents the reaction stresses versus the non-dimensional displacement of the impinging plate when the gradient is negative and the deformation mode is either “V_S” or “I”. For the low velocity impact, similar to the cases in Figure 3.12, the reaction stresses of the loading and supporting plates are almost the same, which can be regarded as quasi-static behavior. Under the high velocity impact, the stress at the loading plate is much higher than that at the supporting plate due to the inertial effect. The stresses decrease with the increase of the displacement because the collapse occurs from the loading end with strong material to the supporting end with weak material. However, it is found that there is a sudden drop for the stress after the initial peak force and the value is much smaller than those predicted by the shock wave theory. The reason is that the collapse of the cells is not layer by layer as assumed analytically. For example, in Figure 3.8b the cells behind the compaction front are not fully densified while some cells in the other layers are already densified, leading to the deviation from the analytical prediction.

Figure 3.14 (a) The stress-displacement relationship for the transitional mode "V_L" in the GHS with negative gradient. $\Theta = -1.04$, $V = 20$ m/s ; (b) the corresponding deformation profile at the instants $\mu=0.078$, $\mu=0.267$, $\mu=0.457$ and $\mu=0.647$.

Under an intermediate velocity impact, the transitional deformation mode takes places in the GHS with negative gradient. There seem two compaction fronts appearing in this mode and it is difficult to give a good analytical prediction by using shock wave theory. Figure 3.14a represents the stress-displacement relationship of such a case of $V = 20$ m/s and $\Theta = -1.02$. The corresponding deformation profiles are shown in Figure 3.14b. The whole crushing process can be divided into two phases. In Phase I, the stress of the loading end is higher than that at the supporting end, indicating that most deformation occurs adjacent to the loading end; the compaction band adjacent to the loading end becomes more obvious with the collapse of the cells ahead of the compaction band. At the beginning of Phase II, there is a sudden increase in the stress at the supporting end. The reason is that no further deformation occurs at the loading end at that moment, but at the supporting end instead, forming a new compaction band. Afterwards, the stress at the supporting plate increases due to the inertial effect. Meanwhile, the stress at the loading end suddenly drops to a very low value because the compaction band adjacent to the loading plate stops and the stress becomes close to the quasi-static plateau stress at the new compaction band adjacent to the supporting end.

3.4.2 Energy absorption capacity of the graded honeycomb

In this section, the energy absorbed by the GHS under the dynamic loading conditions is investigated, including the constant velocity loading and impact with initial velocity.

3.4.2.1 Energy absorbed under constant impact velocity

Figure 3.15 depicts the absorbed energy by the GHS versus the displacement of the impinging plate. For the low velocity impact (Figure 3.15a), at the early stage the energy absorbed by the homogenous honeycomb is larger than that for GHS but they approach nearly the same value when the displacement reaches the maximum. The reason is that the inertia effect can be neglected so that the deformation always initiates from the weakest layer and then transmits to the stronger layers regardless of the loading direction.

Figure 3.15 Absorbed energy versus the displacement of the impinging plate. a) $V_0 = 5$ m/s; b) $V_0 = 20$ m/s; c) $V_0 = 60$ m/s; d) $V_0 = 100$ m/s

With the increase of the impact velocity, the inertia influences the deformation mode. For the GHS with negative gradient, the deformation may initiate from the loading end (i.e. strongest layer), resulting in slightly larger amount of energy absorbed than the homogenous structure at the early stage (Figure 3.15b-d). For the GHS with positive gradient, the deformation always initiates from the loading end (i.e. weakest layer), thus the absorbed energy is always smaller than the homogenous structure at the early stage.

It is found that the energy absorbed by the GHS with positive gradient approaches to nearly the same value as the homogenous honeycomb block regardless of the velocity. The reason is that only one compaction band propagates during the whole crushing, so that the average dynamic stress is the same as that of the homogenous honeycomb. On the other hand, under high velocity impact, the energy absorbed by the GHS with a negative gradient is much smaller (Figure 3.15c-d). This is because the deformation mode for the GHS with negative gradient is very different from that of the homogenous honeycomb; that is there are two compaction bands for the GHS with a negative gradient under an intermediate impact velocity. Even under higher velocity impact, small deformations can still be found at the cells adjacent to the supporting end.

3.4.2.2 Energy absorbed under a mass impact with initial velocity

When the honeycomb is subjected to impact of a mass with initial velocity, the velocity of the impinging plate cannot remain constant and it decreases with the displacement of the impinging plate. Consequently, the deformation modes may change from one to another during a response process. Hence, the absorbed energy under a constant velocity

loading cannot truly represent the energy-absorbing capacity of the GHS. In this section, an initial velocity is applied to the impinging plate (i.e. plate B). Depending on the magnitude of the initial input energy from the impinging mass, there are two possible outcomes. When the initial kinetic energy is sufficiently high, the structure will be fully crushed with a residual velocity of the impinging plate. On the other hand, if the initial kinetic energy is relatively low the honeycomb block will not be fully densified when the velocity of the plate reaches zero. Hence, a special initial velocity, called *densification velocity* v_d , can be defined by considering a critical case, in which the GHS is fully crushed when the input kinetic energy is just exhausted. In this case, when the displacement of the rigid plate reaches the maximum, the velocity decreases to zero. Since there is no heat transfer or energy loss in the whole crushing process, the energy absorbed by the GHS can be determined by the conservation of the energy.

Under the quasi-static loading the honeycomb collapses layer by layer and the energy absorption capacity of the GHS is calculated as,

$$E_q = \int A_0 \sigma_p du = A_0 \varepsilon_d l_0 \sigma_0 = \frac{\sigma_0}{\rho_0} m \varepsilon_d \quad (3.10)$$

where σ_p is the plateau stress, σ_0 is the average plateau stress, and $m = \rho_0 l_0 A_0$ is the total mass of the GHS. Since the average plateau stress is defined as a constant, the quasi-static energy-absorbing capacity of graded honeycomb blocks with different gradient is the same.

The conservation of the energy requires the energy-absorbed by the graded honeycomb under a mass impact loading as,

$$E_d = \frac{1}{2} \Gamma \frac{\sigma_0}{\rho_0} m v_d^2 \quad (3.11)$$

where $\Gamma = G/m$, G is the mass of the impinging plate. Hence, the dynamic enhancement of the energy absorption can be defined as,

$$\psi \equiv \frac{E_d}{E_q} = \frac{\Gamma v_d^2}{2\varepsilon_d} \quad (3.12)$$

where v_d is the non-dimensional densification velocity.

Harrigan et al. (2010) theoretically gave the densification velocity for the homogenous cellular rod as,

$$v_d = \frac{1}{\Gamma} \sqrt{(2\Gamma + 1)\varepsilon_d} \quad (3.13)$$

Eq. (3.13) shows that the densification velocity is only related to the mass ratio Γ .

Figure 3.16a depicts the densification velocity v_d against the mass ratio Γ from FE simulation. It shows that the densification velocity v_d decreases with the increase of the mass ratio Γ . Compared with the homogenous honeycomb, it is found that the densification velocity is higher when the gradient is positive while it is lower when the gradient is negative. Similar to the homogenous honeycombs, the densification velocity is significantly influenced by mass ratio Γ . Using Eq. (3.12), the corresponding energy ratio ψ is plotted against the mass ratio Γ in Figure 3.16b. The positive gradient enhances the energy ratio while the negative gradient weakens the energy ratio.

Figure 3.16 (a) Densification velocity versus the mass ratio; (b) Energy ratio versus the mass ratio

Most recently, the authors conducted theoretical derivation of the one-dimensional graded cellular rod under impact by using R-P-P-L material model (see the next Chapter). Assuming only a single shock compaction front, the densification velocity v_d is given in a closed form,

$$v_d = \frac{1}{\Gamma} \sqrt{\left(2\Gamma + 1 + \frac{1}{6}\Theta\right) \varepsilon_d} \quad (3.14)$$

Compared with Eq. (3.13), the gradient Θ is now incorporated into the equation. Combining with Eq. (3.12), the energy ratio for the graded cellular rod is,

$$\psi = 1 + \frac{1}{2\Gamma} + \frac{1}{12\Gamma} \Theta \quad (3.15)$$

It is easily seen that placing the weakest material at the proximal end leads to higher energy absorption, which is consistent with the FE results.

3.5 Conclusion

This chapter employs FE simulation to investigate the dynamic of crushing band and wave trapping of the graded honeycomb structure under in-plane dynamic loading. A gradient in the material's yielding stress has been introduced to a regular honeycomb structure. Constant velocity loading is used to analyze the deformation modes, critical velocities and the dynamic stress. To investigate the energy absorbing capacity of the GHS, both the cases of constant velocity loading and a mass impact with initial velocity are employed.

Two types of deformation modes have been observed when the gradient is positive while three types of deformation modes exist when the gradient is negative. The deformation modes are nearly the same when the velocity is either small or sufficiently high, whether the gradient is positive or negative. “V” shaped band initiates from the weakest layer at the low velocity impact while “I” shaped band occurs from the loading end at the high velocity impact. A transitional mode is present when the velocity is intermediate and the gradient is negative. More deformation has been found at the loading end with the increase of the impact velocity. A map has been constructed showing the critical velocity.

The shock wave theory was proved to be applicable for the homogenous honeycomb structure. Unsurprisingly, the analysis of the stress of the graded honeycomb structure indicates that it is applicable only when one compaction front appears. The key reason is that when the gradient is negative, two compaction zones appear in the honeycomb, which is very different from the homogenous structure. The reaction stress at the supporting end is equal to the quasi-static plateau stress of the layer under collapse; the stress at the loading end equals the dynamic stress at the compaction front.

Densification velocity of the GHS is not only related to the mass ratio but also the stress gradient. The investigation into the energy absorption shows that the positive gradient enhances the energy absorbing capacity of the GHS.

Chapter 4 Double shock mode in graded cellular rod under impact

4.1 Introduction

In Chapter 3, the response of the honeycomb structure with stress gradient under impact has been investigated by the FE simulation. The FE results presented the influence of the gradient on the deformation mode, crushing stress and the energy absorption capacity of the cellular material. However, the underlying mechanism remains to be understood since no theoretical study has been carried out. Equation Chapter 4 Section 1

This chapter presents an analytical study of the effect of the gradient in quasi-static plateau stress on the dynamic behavior of graded cellular rods under impact loading. To avoid the Poisson ratio and boundary effect, FE simulations of graded hexagonal and circular cells' chains (one-dimension) under uniaxial impact loading are first carried out by using ABAQUS/EXPLICIT.

From the basic deformation modes from the FE simulation, an analytical model is built. A graded cellular rod, whose quasi-static plateau stress varies along the axial direction, is supposed to be impinged by a rigid mass, G with an initial velocity V_0 . Only one shock front appears when the gradient is positive, while two shock fronts appear in the rod with negative gradient. Analytical expressions of the dynamic response parameters for the graded cellular rod are theoretically derived by using the one-dimensional shock theory originally proposed by Reid and Peng (1997). Closed form solution is found for the single shock (SS) mode, while finite difference method is employed to obtain solutions for the double shock (DS) mode. Densification velocity, at which the graded rod is just fully crushed at the end of its dynamic response, is determined; and accordingly, the

maximum energy-absorbing capacity of the graded cellular rod is determined. The weakest part of the graded rod is suggested to be placed at the impact end to achieve higher energy absorption. The theoretical models are then extended to the second scenario in which the graded cellular rod together with attached mass G impinges onto a rigid target. Similar to Scenario I, the gradient significantly influences the capacity of the graded cellular rod when the ratio of mass, G to the mass of the rod, m is relatively small. The comparison between the FE simulation results and analytical predictions shows good agreement, which validates the theoretical model.

4.2 Observation from the FE simulations

In this section, FE simulations of two kinds of 1D graded cellular chain under impact loadings are carried out. The deformation modes are qualitatively described here while more quantitative analysis will be given in section 4.5. The graded cellular chain contains 25 cells of identical dimensions. Figure 4.1a and b represent the graded hexagonal and circular cells' chain, respectively. A constant gradient in the materials' yielding stress is introduced to the cells. The details of the FE model are given in the Appendix 4A. As shown in Figure 4.1, plate A is the impinging plate while plate B is fixed by constraining all the degrees of freedom. A range of initial impact velocities are given to the impact plate in order to observe the response of the graded cellular chain. The strongest cell is placed at the proximal end while the weakest cell is placed at the fixed end. It should be noted that the yield stress is the elastic limit stress of the parent material of the cellular

material while the plateau stress is the stress in the plateau stage in the compression for the cellular material.

Figure 4.1 1D cellular chains composed of 25 cells. (a) Hexagonal cellular chain; (b) circular cellular chain

Figure 4.2a shows the deformation profile of the graded hexagonal cells' chain under the initial impact velocity of $V_0=10\text{m/s}$. The mass of the impact plate is 6mg. After the deformation initiated from both ends, further deformation occurs near the fixed end while the proximal end remains undeformed. Figure 4.2b depicts three snap-shots of the deformation process of the same chain but under the initial impact velocity of $V_0=100\text{m/s}$ with the mass 0.06mg. Initially, a compaction front is found at the proximal end while no obvious deformation occurs at the fixed end. Afterwards, another compaction front gradually appears at the fixed end while the deformation at the proximal end keeps expanding. Later, it is found that the deformation at the proximal end stops expanding while the deformation at the fixed end keeps increasing. Similar deformation processes are found in the graded circular cells' chain, as shown in Figure 4.3.

Figure 4.4 displays the deformation profiles of the graded hexagonal cells' chain under impact when the weakest cell is placed at the proximal end. It is found that deformation always starts from the proximal end and then propagates to the fixed end regardless of the value of the initial impact velocity.

Figure 4.2 Deformation profiles for graded hexagonal cells' chain, the strongest cell at the proximal end. (a) $V_0 = 10 \text{ m/s}$, $G = 6 \text{ mg}$ ($t = 0.68 \text{ ms}$, 3.60 ms , 6.67 ms , 10.43 ms); (b) $V_0 = 100 \text{ m/s}$, $G = 0.06 \text{ mg}$ ($t = 0.14 \text{ ms}$, 0.74 ms , 1.34 ms , 2.09 ms)

Figure 4.3 Deformation profiles for graded circular cells' chain, the strongest cell at the proximal end. (a) $V_0 = 10 \text{ m/s}$, $G = 6 \text{ mg}$ ($t = 1.8 \text{ ms}$, 4.80 ms , 7.8 ms , 10.8 ms); (b) $V_0 = 100 \text{ m/s}$, $G = 0.06 \text{ mg}$ ($t = 0.18 \text{ ms}$, 0.98 ms , 1.78 ms , 2.78 ms). Note: the rigid plates placed between each pair of cells are not displayed.

Figure 4.4 Deformation profiles for graded hexagonal cells' chain, the weakest cell at the proximal end. (a) $V_0=10\text{m/s}$, $G=6\text{mg}$ ($t=0.60\text{ms}$, 3.6ms , 6.6ms , 9.6ms); (b) $V_0=100\text{m/s}$, $G=0.06\text{mg}$ ($t=0.06\text{ms}$, 0.66ms , 1.26ms , 1.86ms)

Based on the observation of the FE simulations, it is remarked that two compaction fronts appear when the strongest cell is placed at the proximal end while only one compaction front propagates if the weakest cell is placed at the proximal end. In the subsequent sections, an analytical model will be established for a graded continuum rod by using the shock theory. Two impact scenarios are considered. Scenario I is the same as the above FE simulations, in which a stationary rod is impinged by a rigid mass. In Scenario II, the rod together with an attached mass impinges onto a rigid target.

4.3 Deformation modes in Scenario I

As shown in Figure 4.5, a stationary cellular rod with initial length l_0 and initial density ρ_0 is impinged by a rigid mass G with initial impact velocity V_0 . The distal end of the rod is fixed on the support. A constant gradient in the quasi-static plateau stress, g , is introduced to the cellular rod and it is defined as,

$$g = \frac{\Delta\sigma}{\sigma_0 l_0} \quad (4.1)$$

where $\Delta\sigma$ is the difference in the quasi-static plateau stress between the two ends of the cellular rod with length l_0 and σ_0 the average plateau stress over the rod length. For clarity, it is defined that $g > 0$ when the weakest part is at the proximal end (i.e. the strength increases from the proximal end along the rod); and $g < 0$ when the strongest part is at the proximal end. They are sketched in Figure 4.6.

Figure 4.5 Scenario I: a stationary cellular rod directly impinged by a rigid mass G

Figure 4.6 Quasi-static plateau stress distribution in the graded cellular rods

Let ρ_s denote the density of the fully densified material and σ_d the dynamic crushing stress. The cross-sectional area A_0 is assumed to remain unchanged during the whole crushing process. Similar to the previous studies, e.g. (Reid and Peng 1997, Tan et al. 2005b), the R-PP-L model and one-dimensional shock theory are employed for the graded cellular rod under concern.

The R-PP-L material model, which is adopted here, is a special case of *E-PP-L* material with the Young's modulus, E approaching infinite. Thus, the elastic wave speed is infinite and the wave front reaches the distal end immediately after impact. Thus, the elastic wave speed is infinite and the wave front reaches the distal end immediately after impact. Also, the stress of every material point reaches yielding stress after impact. This assumption also can be found at the paper of Reid et al.(1983). When the gradient g is positive, the weakest part is at the proximal end and it deforms immediately after impact. However, at the distal end the reaction stress is still below its quasi-static plateau stress so that the distal end remains undeformed during the entire crushing process. When the gradient is negative, the weakest part is placed at the distal end, and the reaction stress there could easily exceed its quasi-static plateau stress. On the other hand, for the R-PP-L model, the critical velocity given by Eq. (2.28) becomes zero as the Young's Modulus approaches infinite. Consequently, shock fronts at both the proximal end and the distal end may co-exist from the very beginning. Therefore, a single shock (SS) mode and a double shock (DS) mode have to be respectively proposed to describe these two possible crushing processes. It should be noted that the exact initiation or formation of a shock front has been critical and needs further study. The present study is mainly based on the postulated mechanism of two shock fronts, very much similar to the analysis of upper-

bound (though equilibrium equations are also used here) or low bound analysis in plastic analysis, and hence may not be exact. The initiation for the second shock in the E-PP-L material model has been described in the work of Harrigan et al. (2005). For R-PP-L material model, the stress wave is reflected on the far end of the rod and could build up to form a compaction wave, although it takes place in an infinitely small time interval.

Figure 4.7a shows an SS mode when the gradient is positive. Deformation in the form of shock wave occurs from the proximal end and then gradually propagates to the distal end. During the whole crushing process, only single forward shock front occurs starting from the proximal end. The initial length of rod is l_0 , the densified portion is x and the undeformed length is l . The rigid block displacement is u .

Figure 4.7 (a) The SS mode in the graded cellular rod impinged by a rigid mass (positive gradient, single shock front); (b) corresponding stress distribution in the graded cellular rod at instant t

Figure 4.8 (a) The DS mode in the graded cellular rod impinged by a rigid mass (negative gradient, double shock fronts); (b) corresponding stress distribution in the graded cellular rod at instant t

Figure 4.8a shows a DS mode which occurs when the gradient is negative. Crushing occurs initially from both the distal and proximal end. In this DS mode, a forward shock and a backward shock propagate simultaneously before they finally meet, if the input energy is sufficiently large.

Non-dimensionalization of the parameters is first implemented for the analytical derivations. For the two crushing modes, the physical parameters involved are the same, which are the locking strain ε_d , the average plateau stress σ_0 , the gradient g , the initial length l_0 , the initial density ρ_0 , the cross-section area A_0 , and the impinging mass G . The process variables involved are the displacement of the impinging mass u , the length of the densified zone adjacent to the proximal end x , the length of the densified zone adjacent to the distal (fixed) end y , the length of the undeformed zone l , and the instantaneous velocity of the impinging mass V .

The length parameters are all non-dimensionalized with respect to rod initial length as,

$$\frac{x}{l_0} = \xi, \quad \frac{u}{l_0} = \mu, \quad \frac{l}{l_0} = \lambda, \quad \frac{y}{l_0} = \eta \quad (4.2)$$

A characteristic velocity of the cellular rod is,

$$c_p \equiv \sqrt{\sigma_0 / \rho_0} \quad (4.3)$$

Hence, the non-dimensional velocity can be defined as

$$\nu \equiv V / c_p \quad (4.4)$$

while the characteristic time is,

$$T \equiv l_0/c_p \quad (4.5)$$

Accordingly, the non-dimensional time is defined as,

$$\tau \equiv tc_p/l_0 \quad (4.6)$$

4.3.1 Positive gradient – Single Shock (SS) mode

The dynamic deformation process of a graded cellular rod is sketched in Figure 4.7a when the gradient g is positive. Since the weakest part is at the proximal (impinged) end, the crushing starts from the proximal end and then propagates to the distal end. The analysis of such a case is similar to that for a homogenous rod given by Reid and Peng (1997).

From the geometry, the displacement of the attached rigid mass is easily found as,

$$u = (x + u)\varepsilon_d = (l_0 - l)\varepsilon_d \quad (4.7)$$

where ε_d is the locking strain and l is the length of the undeformed zone in the cellular rod.

Ahead of the shock front the cellular rod is undeformed and the stress is equal to the quasi-static plateau stress at that cross section. Since it is assumed that the quasi-static plateau stress of the material linearly increases along the position, the plateau stress ahead the shock front is obtained in terms of the gradient g ,

$$\sigma_p = \sigma_0 - \left(\frac{l_0}{2} - \frac{x}{1 - \varepsilon_d} \right) g \sigma_0 \quad (4.8)$$

where σ_0 is the average plateau stress of the cellular rod. Since the gradient g is a constant, σ_0 is also the plateau stress at the central section of the cellular rod.

For a small increment of time dt , consider a small element ahead of the shock front. The velocity of the element increases from zero to V . The length of the element after densification is dx and so the mass of the element is

$$dm = \frac{\rho_0 A_0 dx}{1 - \varepsilon_d} \quad (4.9)$$

Furthermore, from the geometry and the kinematic considerations,

$$\frac{dx}{dt} = \frac{1 - \varepsilon_d}{\varepsilon_d} V \quad (4.10)$$

The conservation of the momentum gives,

$$(\sigma_d - \sigma_p) A_0 dt = V dm \quad (4.11)$$

where σ_d is the dynamic crushing stress and σ_p is the quasi-static plateau stress.

Substituting Eqs. (4.9) and (4.10) into Eq. (4.11) gives

$$\sigma_d = \sigma_p + \frac{\rho_0}{\varepsilon_d} V^2 \quad (4.12)$$

The derivation of the dynamic equation is the same as that first given by Reid and Peng (1997). That is because in the present case the only difference is that the plateau stress

σ_p varies with the position rather than being a constant, while all the other parameters remain the same. For this scenario, the stress distribution along the graded cellular rod with a positive gradient at the instant t is depicted in Figure 4.7b. For the stationary unformed portion, the stress is uniform, which is smaller than the plateau stress. For the fully densified zone which moves as one body with the same velocity and acceleration, the internal force and hence the stress varies linearly, as a result of Newton's second law.

We are now interested in the relationships among the instantaneous velocity, displacement and time. The conservation of momentum for the whole system consisting of the cellular rod and the impinging mass shown in Figure 4.7a over the period from t to $(t+dt)$ leads to the governing equation,

$$\frac{dV}{dt} = - \frac{\sigma_p + \frac{\rho_0}{\varepsilon_d} V^2}{\frac{G}{A_0} + \frac{\rho_0}{\varepsilon_d} u} \quad (4.13)$$

Combining Eqs.(4.8) and (4.13) gives,

$$\frac{dV}{dt} = - \frac{\sigma_0 \left(1 - \frac{l_0}{2} g \right) + \frac{g \sigma_0}{\varepsilon_d} u + \frac{\rho_0}{\varepsilon_d} V^2}{\frac{G}{A_0} + \frac{\rho_0}{\varepsilon_d} u} \quad (4.14)$$

Non-dimensionalization of Eq. (4.14) gives,

$$\frac{dv}{d\tau} = \left[\left(\frac{1}{2} - \frac{\mu}{\varepsilon_d} \right) \Theta - 1 - \frac{v^2}{\varepsilon_d} \right] / \left(\Gamma + \frac{\mu}{\varepsilon_d} \right) \quad (4.15)$$

where $\Theta = gl_0 = \Delta\sigma/\sigma_0$ is the non-dimensional gradient, $\Gamma = G/m$ is the mass ratio, and $m = \rho_0 l_0 A_0$ is the total mass of the cellular rod.

From Eq. (4.15), the non-dimensional velocity is obtained in terms of the non-dimensional displacement μ in a closed form as follows, and the derivation is shown in Appendix 4B.1.

$$v = \frac{v_0}{\Gamma + \frac{1}{\varepsilon_d} \mu} \sqrt{\Gamma^2 - \frac{1}{v_0^2} \left[(2 - \Theta) \Gamma \mu + \frac{2\Theta\Gamma + 2 - \Theta}{2\varepsilon_d} \mu^2 + \frac{2\Theta}{3\varepsilon_d^2} \mu^3 \right]} \quad (4.16)$$

Based on Eq. (4.12), the non-dimensional dynamic crushing stress is obtained as a function of the displacement, μ ,

$$\Sigma_d = 1 - \frac{1}{2} \Theta + \frac{1}{\varepsilon_d} \Theta \mu + \frac{(\Gamma v_0)^2 - (2 - \Theta) \Gamma \mu - \frac{2\Theta\Gamma + 2 - \Theta}{2\varepsilon_d} \mu^2 - \frac{2\Theta}{3\varepsilon_d^2} \mu^3}{\left(\Gamma + \frac{1}{\varepsilon_d} \mu \right)^2} \quad (4.17)$$

where $\Sigma_d \equiv \sigma_d/\sigma_0$ is the non-dimensional dynamic crushing stress just behind the shock front.

Depending on the value of initial input energy from the impinging mass, there exist two possible deformation outcomes. When the initial energy is sufficiently small, the velocity of the impacting mass decreases to zero before the graded cellular rod is fully crushed and so there is an undeformed portion left. The other possible outcome occurs when the initial energy is sufficiently large. The impacting mass still has residual velocity even when the whole rod is fully densified.

To characterize the dynamic energy-absorbing capacity of the graded cellular rod, a critical case is considered, in which the cellular rod is just fully crushed when the input kinetic energy is exhausted. The initial impact velocity, in the case, is called as *densification velocity*, v_d . In this case, the displacement of the impinging mass, u , reaches the maximum value $l_0 \varepsilon_d$ and the instant velocity of the impinging mass decreases to zero, where l_0 is the initial length of the cellular rod. The non-dimensionalisation displacement is $\mu = u/l_0$, Hence, the non-dimensionalisation displacement $\mu = l_0 \varepsilon_d / l_0 = \varepsilon_d$. By applying the boundary condition, $\mu = \varepsilon_d$ and $v = 0$, the initial velocity $v_0 = v_d$ and the densification velocity is obtained from Eq. (4.16) as,

$$v_d = \frac{1}{\Gamma} \sqrt{\left(2\Gamma + 1 + \frac{1}{6}\Theta\right)} \varepsilon_d \quad (4.18)$$

It is seen that the densification velocity relates to the non-dimensionalisation parameter $\mu = \varepsilon_d$.

Consequently, the energy absorbed by the cellular rod is equal to the initial kinetic energy of the rigid mass,

$$E_d = \frac{1}{2} \Gamma \frac{\sigma_0}{\rho_0} m v_d^2 \quad (4.19)$$

where E_d denotes the maximum dynamic energy-absorbing capacity and m is the total mass of the cellular rod.

On the other hand, the maximum energy absorption of the rod under quasi-static loading is,

$$E_q = \int A_0 \sigma_p du = A_0 \varepsilon_d l_0 \sigma_0 = \frac{\sigma_0}{\rho_0} m \varepsilon_d \quad (4.20)$$

Comparing Eqs. (4.19) and (4.20) gives an energy ratio,

$$\psi = \frac{E_d}{E_q} = \frac{\Gamma v_d^2}{2 \varepsilon_d} \quad (4.21)$$

Substituting Eq. (4.18) into (4.21) gives,

$$\psi = 1 + \frac{1}{2\Gamma} + \frac{1}{12\Gamma} \Theta \quad (4.22)$$

Hence, the maximum dynamic energy-absorbing enhancement ratio ψ is determined by two non-dimensional parameters, i.e. the mass ratio Γ and the non-dimensional gradient Θ . The range of the non-dimensional gradient is $\Theta \in [0, 2)$ as $g > 0$. The solid lines in Figure 4.9 represent the dynamic energy-absorbing enhancement ratio ψ versus the mass ratio Γ , indicating that the maximum dynamic energy-absorption decreases with Γ . It is evident that the enhancement is significant when the impinging mass is small but the impinging velocity is high. Compared with the homogenous cellular rod ($\Theta = 0$), the energy-absorbing capacity of the graded cellular rod is enhanced slightly by the positive gradient ($\Theta > 0$).

Figure 4.9 Energy ratio $\psi = E_d/E_q$ versus mass ratio $\Gamma = G/m$ for direct impact, Scenario I

4.3.2 Negative gradient – Double Shock (DS) mode

When the gradient is negative, the strongest part of the graded cellular rod is at the proximal end. Dynamic crushing will take place from both the distal and proximal ends, resulting in a double shock mode. The deformation process is shown in Figure 4.8a. V_{s1} is the speed of the forward shock front, while V_{s2} is the speed of the shock front travelling backward. V_1 is the velocity of the impinging mass and V_2 is the velocity of the undeformed zone between the two shock fronts. The dynamic crushing stress in the graded cellular rod for the DS mode is depicted in Figure 4.8b. σ_d^b and σ_p^b represent the dynamic crushing and quasi-static plateau stresses at two sides of the backward shock front, respectively; and σ_d^f and σ_p^f are the stresses at two sides of the forward shock front. The stress in densified zone 2 is the same as the dynamic stress at the backward shock front, because the material in this zone is in the equilibrium state.

From the geometry, the following relationship can be easily obtained and the details of the derivation is shown in Appendix 4B.2,

$$\frac{du}{dt} = V_1 \quad (4.23)$$

$$\frac{dy}{dt} = V_{s2} \quad (4.24)$$

$$\frac{dl}{dt} = -\frac{1}{\varepsilon_d} V_1 \quad (4.25)$$

$$\frac{dy}{dt} = \frac{1 - \varepsilon_d}{\varepsilon_d} V_2 \quad (4.26)$$

$$\frac{dx}{dt} = \frac{1 - \varepsilon_d}{\varepsilon_d} (V_1 - V_2) \quad (4.27)$$

By considering the small element ahead of the forward shock front over a small time increment dt , the velocity of the element increases from V_2 to V_1 after densification. The length of the element after densification is dx and the mass of the element can be obtained by Eq. (4.9). Thus, conservation of momentum for this small element leads to,

$$(\sigma_d^f - \sigma_p^f) A_0 dt = (V_1 - V_2) dm \quad (4.28)$$

where σ_d^f is the dynamic crushing stress behind the forward shock front and σ_p^f is the relevant quasi-static plateau stress.

Substituting Eqs. (4.9), (4.27) and (4.28) gives the dynamic crushing stress behind the forward shock front as,

$$\sigma_d^f = \sigma_p^f + \frac{\rho_0}{\varepsilon_d} (V_1 - V_2)^2 \quad (4.29)$$

Similarly, taking a small element ahead of the backward shock front over a small time interval dt , the velocity of the element decreases from V_2 to zero. The length of element after compression is dy . The conservation of momentum for the element gives,

$$(\sigma_d^b - \sigma_p^b) A_0 dt = -V_2 dm \quad (4.30)$$

where σ_d^b is the dynamic crushing stress behind the forward shock front and σ_p^b is the relevant quasi-static plateau stress.

Replacing dx with dy in Eq. (4.9) for backward shock front and substituting together with Eq. (4.26) into (4.30) gives the dynamic crushing stress behind the backward shock front,

$$\sigma_d^b = \sigma_p^b + \frac{\rho_0}{\varepsilon_d} V_2^2 \quad (4.31)$$

Typical stress distribution in the graded cellular rod in the DS mode is depicted in Figure 4.8b at an instant $t = t$. Stress discontinuities exist at the two shock fronts, which are caused by the R-PP-L material model.

Since the undeformed zone has velocity V_2 , which varies with time, by taking the first shock front as the reference frame and considering the first shock front over the period from t to $(t+dt)$, the conservation of momentum gives,

$$-\sigma_p^f dt = \left[\frac{G}{A_0} + \rho_s (x + dx) \right] (V_1 + dV_1) - \left(\frac{G}{A_0} + \rho_s x \right) V_1 - \frac{\rho_0}{1 - \varepsilon_d} V_2 dx \quad (4.32)$$

where $\sigma_p^f = \sigma_0 - \left(\frac{1}{2} - \frac{x}{1 - \varepsilon_d} \right) g \sigma_0$ is the quasi-static plateau stress ahead of the forward shock front and $\rho_s = \rho_0 / (1 - \varepsilon_d)$ is the density of the densified material. The densified zone 1 has a length of x ; hence, the mass of this zone is the density of the parent material, ρ_s , multiplied by the cross-section area and the length, x .

For the backward shock front, the reference frame can be fixed at the distal end. The conservation of the momentum gives,

$$-\sigma_d^b dt = d \left[\left(\frac{G}{A_0} + \rho_s x \right) V_1 \right] + d(\rho_0 l V_2) \quad (4.33)$$

where σ_d^b is the reaction stress at the distal end. Since the densified zone adjacent to the distal end is in equilibrium, the reaction stress σ_d^b equals the dynamic plateau stress behind the backward shock front.

$$\sigma_d^b = \sigma_0 + \left(\frac{l_0}{2} - \frac{y}{1 - \varepsilon_d} \right) g \sigma_0 + \frac{\rho_0}{\varepsilon_d} V_2^2 \quad (4.34)$$

Subtracting Eq. (4.32) from Eq. (4.33) gives,

$$\rho_0 l \frac{dV_2}{dt} + \rho_0 V_2 \frac{dl}{dt} + \frac{\rho_0 V_2}{1 - \varepsilon_d} \frac{dx}{dt} = - \left(l_0 - \frac{x + y}{1 - \varepsilon_d} \right) g \sigma_0 - \frac{\rho_0}{\varepsilon_d} V_2^2 \quad (4.35)$$

Here $l = l_0 - \frac{x + y}{1 - \varepsilon_d}$ is used based on geometry.

Then, substituting Eqs. (4.25) and (4.27) into Eq. (4.35) gives,

$$\frac{dV_2}{dt} = - \frac{g \sigma_0}{\rho_0} \quad (4.36)$$

Initially, $V_2 = 0$ at $t = 0$. From Eq. (4.36), it is found that dV_2/dt varies linearly with the gradient g . It is evident that when the gradient $g > 0$ (i.e. the impact occurs from the weakest part), V_2 cannot decrease so that $V_2 \equiv 0$, implying that only a single shock occurs from the proximal end. On the other hand, when the gradient $g < 0$ (i.e. the impact occurs from the strongest part), $dV_2/dt > 0$, indicating that V_2 increases with time. In the latter case, double shocks initiate and propagate from both the distal and proximal ends. However, Eq. (4.27) also requires $V_1 > V_2$ to ensure $dx/dt > 0$. Thus, two

successive phases should exist with the DS mode. In Phase I, V_2 increases with time and V_1 decreases until they reach the same value. In Phase II, immediately following Phase I, $V_1 = V_2$ and the length of the densified zone near the proximal end, x , remains unchanged (i.e. $dx/dt = 0$), while only the densified zone adjacent to the distal end continues to expand.

Non-dimensionalization of Eqs. (4.23), (4.25), (4.26) and (4.27) leads to,

$$\frac{d\mu}{d\tau} = v_1, \quad \frac{d\lambda}{d\tau} = -\frac{v_1}{\varepsilon_d}, \quad \frac{d\eta}{d\tau} = -\frac{1-\varepsilon_d}{\varepsilon_d} v_2, \quad \frac{d\xi}{d\tau} = \frac{1-\varepsilon_d}{\varepsilon_d} (v_1 - v_2) \quad (4.37)$$

In Phase I, the deformation is governed by Eqs. (4.32) and (4.36). Non-dimensionalization of Eqs. (4.32) and (4.36) gives the following equations,

$$\frac{dv_1}{d\tau} = \left[-1 + \left(\frac{1}{2} - \frac{\xi}{1-\varepsilon_d} \right) \Theta - \frac{1}{\varepsilon_d} (v_1 - v_2)^2 \right] / \left(\Gamma + \frac{\xi}{1-\varepsilon_d} \right) \quad (4.38)$$

$$\frac{dv_2}{d\tau} = -\Theta \quad (4.39)$$

Phase II starts from the instant $\tau = \tau^*$ at which $v_1 = v_2$. Thus, the initial conditions of Phase II are,

$$\tau = \tau^*, \quad v_1 = v_2 = \nu, \quad \xi = \xi^* = \text{known constant}, \quad \eta^* \text{ and } \lambda^* \text{ are known} \quad (4.40)$$

where ν , ξ^* , η^* and λ^* are the non-dimensional velocity, densified length near the proximal end, densified length near the distal end and the undeformed length at the end of Phase I, respectively.

In Phase II, Eq. (4.37) can be re-written as,

$$\frac{d\mu}{d\tau} = \nu, \quad \frac{d\lambda}{d\tau} = -\frac{\nu}{\varepsilon_d}, \quad \frac{d\eta}{d\tau} = -\frac{1-\varepsilon_d}{\varepsilon_d}\nu, \quad \frac{d\xi}{d\tau} = 0 \quad (4.41)$$

Non-dimensionalization of Eq. (4.33) and applying the initial conditions give,

$$\frac{d\nu}{d\tau} = -\left[1 + \left(\frac{1}{2} - \frac{\eta}{1-\varepsilon_d}\right)\Theta\right] \bigg/ \left(\Gamma + 1 - \frac{\eta}{1-\varepsilon_d}\right) \quad (4.42)$$

Further variations of ν , ξ , η and λ with time τ in Phase II can be determined from Eqs. (4.41) and (4.42). With the initial conditions and governing equations, the time histories of the velocities (ν_1 , ν_2) and the lengths l can be determined numerically, although closed form solutions cannot be obtained. Finite difference method is employed, therefore, to solve the equations for the dynamic response of the cellular rod with negative gradient under impact.

Figure 4.10 (a) Typical time histories of v_1 and v_2 ; (b) typical time histories of the position of the shock front. $\Gamma = 0.5$, $\Theta = -1.0$ and $v_0 = 2$. Scenario I

Figure 4.11 (a) Time histories of v_1 and v_2 ; (b) time histories of the shock front. $\Gamma = 0.5$, $\Theta = -1.0$ and $v_0 = 3$. Scenario I

Typical time histories of v_1 and v_2 are plotted in Figure 4.10a with mass ratio $\Gamma = 0.5$, gradient $\Theta = -1.0$ and initial velocity $v_0 = 2$. Correspondingly, the time histories of the positions of the two shock fronts are shown Figure 4.10b. Figure 4.10a shows that v_1 decreases, while v_2 increases in Phase I; Phase II starts at the end of Phase I (i.e. at instant $\tau = \tau^*$) with the condition of $v_1 = v_2$. The whole response ends either when v decreases to zero or when the undeformed length λ decreases to zero. Note that at the phase transition, $dv/d\tau$ exhibits discontinuity, deviating from $dv_1/d\tau$ and $dv_2/d\tau$, which is caused by a sudden drop of the stress at the forward shock front. When v_1 just decreases to v_2 , no further deformation occurs at the forward shock front. In fact, in the R-PP-L material model, the stress at the forward shock front suddenly drops to a value lower than σ_p^f , which can be determined by the Newton's second law.

In the case that the initial kinetic energy is so large that the two shock fronts meet each other (i.e. the length of the undeformed zone λ decreases to zero) before v_1 decreases to v_2 , Phase II does not appear. The corresponding velocity, v_p , can be determined by applying the end condition of $v_1 = v_2$ and $\lambda = 0$ to the Eqs. (4.37)-(4.39). Figure 4.11a shows the time histories of v_1 and v_2 in such a case, in which the mass ratio is $\Gamma = 0.5$, the gradient $\Theta = -1.0$ and the initial velocity $v_0 = 3.5$. Figure 4.11b shows the corresponding histories of the positions of the shock fronts.

Figure 4.12 (a) Densification velocities v_d versus gradient Θ ; (b) energy ratio $\psi = E_d/E_q$ versus gradient Θ . Scenario I

Figure 4.13 (a) The energy ratio E_d^f/E_q versus gradient Θ ($\Gamma = 0.2$); (b) energy ratio E_d^f/E_d versus gradient Θ . Scenario I

Similar to the SS mode, the capacity of the energy absorption for the DS mode is investigated for the range of the negative gradient $\Theta \in (-2, 0)$. Consider the same critical case that ν decreases to zero exactly at the moment when the rod is just fully crushed. Under this condition, the end condition of the crushing is $\nu = 0$ and $\lambda = 0$. By using the finite difference method, the densification velocities, ν_d , can be determined for the DS mode. On the other hand, a closed form solution of the densification velocity ν_d is obtained by Eq. (4.18) for the SS mode. Figure 4.12a plots densification velocities against the gradient Θ . It is found that the densification velocity ν_d increases with the value of the gradient.

The broken lines in Figure 4.9 represent the energy ratio ψ versus the mass ratio Γ . It is also found that the energy ratio ψ increases when the mass ratio Γ decreases, which is similar to the conclusion for homogenous cellular rods (Reid and Peng 1997, Harrigan et al. 2010). The enhancement of the energy absorption by the small mass ratio (e.g. $\Gamma = 0.5$) is significant, no matter what the deformation mode is.

Figure 4.12b plots the energy ratio ψ versus the gradient Θ for both the SS and DS modes. For the SS mode, the energy ratio ψ increases linearly with the gradient Θ . For the DS mode it still increases with the gradient Θ , but less significantly when the gradient Θ approaches -2. Continuities of the densification velocity and energy ratio are found for the DS and SS modes at $\Theta = 0$. It is seen that the capacity of the energy absorption is higher when the weaker part of cellular rod is placed at the proximal end. The energy absorbed by the forward shock E_d^f versus the gradient when $\Gamma = 0.2$ is plotted in Figure 4.13a, indicating that the percentage of the energy absorbed by the

backward shock decreases when the magnitude of the gradient decreases. When the gradient approaches zero, the backward shock disappears so that all the input energy is absorbed by the forward shock. The ratios of the energy absorbed by the forward and backward shock fronts over the total dynamically absorbed energy are plotted in Figure 4.13b. It is evident that the energy ratio E_d^b/E_q increases with the mass ratio Γ , implying that the densification zone adjacent to the proximal end is longer with a higher mass ratio Γ .

In fact, the SS mode can be regarded as a reduced DS mode. As expected, no backward shock front appears for SS mode, resulting in that the velocity of the undeformed zone V_2 remains to be zero. When $V_2 = 0$, all the governing equations for the DS mode are degenerated to those for the SS mode.

4.4 Deformation modes in Scenario II

In this section, the theoretical analysis is extended to another impact scenario of graded cellular rods. Figure 4.14 shows this Scenario II in which the graded cellular rod together with an attached mass impinges on a rigid target. Since the impact occurs now at the right end, the distribution of the quasi-static plateau stress of the graded cellular rod is reversed from Figure 4.6. When the gradient is positive, the weakest part is at the proximal end. In this case, the reaction stress at the distal end is not large enough to cause the densification, resulting in only one shock front from the proximal end. When the gradient is negative, the strongest part of the rod is placed at the proximal end and two shock fronts from both the proximal and distal ends may co-exist.

Figure 4.14 Scenario II: a graded cellular rod with an attached mass G impinges onto a rigid target

4.4.1 Positive gradient – Single Shock (SS) mode

Figure 4.15a shows the SS mode in this scenario; and the corresponding dynamic distribution at instant t is depicted in Figure 4.15b. When the gradient is positive, only one shock wave appears from the proximal end during the whole crushing process, which is equivalent to the case of $V_1 = V_2 = V$ in the DS mode. The initial condition is that $V_1 = V_2 = V_0$, so there is no densified zone adjacent to the distal end (i.e. $x = 0$). Since the material in the second densified zone is in the equilibrium state, the stress in this zone is the same as the dynamic stress at the shock front. As for the undeformed part, it has the same velocity with the attached mass. Therefore, the stress can be determined by using Newton's second law and the stress is expressed $\sigma = Ma/A_0$, resulting that the stress decreases with the mass.

Figure 4.15 (a) The SS mode in the graded cellular rod with attached mass impinges onto a rigid target (positive gradient, single shock front); (b) corresponding stress distribution in the graded cellular rod at instant t

Applying the conditions that $V_1 = V_2 = V$ and $x = 0$, the DS mode degenerates to SS mode and the governing equation is,

$$-\sigma_d^b dt = \frac{G}{A_0} dV + d(\rho_0 l V) \quad (4.43)$$

where σ_d^b is the reaction force at the proximal end.

Because the gradient is reversed, Eq. (4.34) changes to,

$$\sigma_d^b = \sigma_0 - \left(\frac{l_0}{2} - \frac{y}{1 - \varepsilon_d} \right) g \sigma_0 + \frac{\rho_0}{\varepsilon_d} V^2 \quad (4.44)$$

Substituting Eq. (4.44) into Eq. (4.43) and simplifying it give the governing equation,

$$\left(\frac{G}{A_0} + \rho_0 l \right) \frac{dV}{dt} = -\sigma_0 + \left(\frac{l_0}{2} - \frac{u}{\varepsilon_d} \right) g \sigma_0 \quad (4.45)$$

Combining Eqs. (4B.10), (4B.11) (see Appendix. 4B) and the condition of $x = 0$, Eq. (4.45) can be re-written in terms of the displacement of the attached mass u , as,

$$\left(\frac{G + m}{A_0} - \frac{\rho_0}{\varepsilon_d} u \right) \frac{dV}{dt} = -\sigma_0 + \left(\frac{l_0}{2} - \frac{y}{1 - \varepsilon_d} \right) g \sigma_0 \quad (4.46)$$

Non-dimensionalization of Eq. (4.46) gives

$$\frac{dv}{d\tau} = - \frac{1 - \left(\frac{1}{2} - \frac{\mu}{\varepsilon_d} \right) \Theta}{\Gamma + 1 - \frac{\mu}{\varepsilon_d}} \quad (4.47)$$

Hence, we obtain the non-dimensional velocity in terms of the non-dimensional displacement, μ ,

$$\nu = \sqrt{\nu_0^2 + 2\varepsilon_d \left[1 + \left(\frac{1}{2} + \Gamma \right) \Theta \right] \ln \left[1 - \frac{\mu}{\varepsilon_d (\Gamma + 1)} \right] + 2\Theta \mu} \quad (4.48)$$

The detail of the derivation is shown in Appendix 4B. 3.

The non-dimensional dynamic crushing stress behind the shock front, Σ_d , is obtained from Eq. (4.44) with respect to the average plateau stress, σ_0 ,

$$\Sigma_d \equiv \sigma_d / \sigma_0 = 1 - \left(\frac{1}{2} - \frac{\mu}{\varepsilon_d} \right) \Theta + \frac{\nu_0^2}{\varepsilon_d} + 2 \left[1 + \left(\frac{1}{2} + \Gamma \right) \Theta \right] \ln \left[1 - \frac{\mu}{\varepsilon_d (\Gamma + 1)} \right] + 2\Theta \frac{\mu}{\varepsilon_d} \quad (4.49)$$

The shock front will continue to propagate until either the velocity of the attached mass decreases to zero or the shock front reaches the distal end. Similar to Scenario I, the densification velocity ν_d , at which the graded cellular rod is just fully crushed, is obtained from Eq. (4.48) by setting the initial velocity $\nu_0 = \nu_d$ and the final velocity as $\nu = 0$ when the displacement of the attached mass reaches the maximum value of $\mu = \varepsilon_d$.

Thus,

$$\nu_d = \sqrt{-2\varepsilon_d \left[1 + \left(\frac{1}{2} + \Gamma \right) \Theta \right] \ln \frac{\Gamma}{\Gamma + 1} - 2\varepsilon_d \Theta} \quad (4.50)$$

From Eq. (4.50), the energy ratio $\psi = E_d / E_q$ is obtained, which is the ratio of the energy absorbed under the densification velocity ν_d to that absorbed up to the full densification under quasi-static compression,

$$\psi = \frac{E_d}{E_q} = (\Gamma + 1) \left[\left(1 + \frac{\Theta}{2} + \Gamma \Theta \right) \ln \left(1 + \frac{1}{\Gamma} \right) - \Theta \right] \quad (4.51)$$

4.4.2 Negative gradient – Double Shock (DS) mode

When the gradient is negative, the strongest part is placed at the proximal end. In this condition, two shock fronts will appear during the crushing process. The deformation profile of the DS mode in this scenario is the same as in Figure 4.8a. The dynamic crushing stress in the graded cellular material for the DS mode at the instant t in this scenario is depicted in Figure 4.16. Also, the stress at the densification zone 1 can be determined by using Newton's second law it decreases with the mass.

Figure 4.16 Stress distribution in the graded cellular rod at instant t (negative gradient, double shock front, Scenario II)

The governing equations are obtained as follows by substituting a reversed gradient $-\Theta$ into Eqs. (4.38) and (4.39),

$$\frac{dv_1}{d\tau} = \left[-1 - \left(\frac{1}{2} - \frac{\xi}{1-\varepsilon_d} \right) \Theta - \frac{1}{\varepsilon_d} (\nu_1 - \nu_2)^2 \right] / \left(\Gamma + \frac{\xi}{1-\varepsilon_d} \right) \quad (4.52)$$

$$\frac{dv_2}{d\tau} = \Theta \quad (4.53)$$

From Eqs. (4.52) and (4.53), it is found that ν_1 and ν_2 are always decreasing. To ensure the existence of double shock mode, another condition $dv_1/d\tau > dv_2/d\tau$ is required. Using Eqs. (4.52) and (4.53), we obtain,

$$\frac{dv_1}{d\tau} - \frac{dv_2}{d\tau} = \left(-1 - \left(\frac{1}{2} + \Gamma \right) \Theta - \frac{1}{\varepsilon_d} (\nu_1 - \nu_2)^2 \right) / \left(\Gamma + \frac{\xi}{1-\varepsilon_d} \right) > 0 \quad (4.54)$$

Substituting the initial condition $\nu_1 = \nu_2$ into Eq. (4.54) and simplifying it give,

$$\Theta < -\frac{2}{2\Gamma+1} \quad (4.55)$$

Eq. (4.55) provides a necessary condition, under which a DS mode appears for the graded cellular rod with negative gradient. It is seen that the necessary condition only depends on the mass ratio Γ and the gradient Θ ; in other words, the initial velocity has no influence on the existence of the DS mode. When $2 > \Theta \geq -\frac{2}{2\Gamma+1}$, only a single shock appears during the whole crushing process; when $-2 < \Theta < -\frac{2}{2\Gamma+1}$, double shock fronts appear. $|\Theta| < 2$ holds because the plateau stress of the cellular material cannot be negative.

Since $dv_1/d\tau - dv_2/d\tau > 0$ is true for the DS mode, it is obvious that v_2 decreases faster than v_1 . However, Eq. (4.55) holds only when the velocity v_2 is positive. Hence, two deformation phases should exist. The deformation in Phase I is governed by Eqs. (4.52) and (4.53). Phase II starts immediately after the velocity v_2 decreases to zero. When this happens, no more deformation takes place near the proximal end, indicating $v_2 \equiv 0$ and $\eta = \eta^*$. In Phase II, only v_1 exists, which can be determined by Eq. (4.52) together with the condition $v_2 \equiv 0$.

$$\frac{dv_1}{d\tau} = \left[-1 - \left(\frac{1}{2} - \frac{\xi}{1 - \varepsilon_d} \right) \Theta - \frac{1}{\varepsilon_d} v_1^2 \right] / \left(\Gamma + \frac{\xi}{1 - \varepsilon_d} \right) \quad (4.56)$$

Finite different method is again employed to solve the above equations. Figure 4.17 plots typical time histories of non-dimensional velocities v_1 , v_2 and positions of the shock fronts when $\Theta = -1.5$, $\Gamma = 1$ and $v_0 = 0.8$. If the initial energy is so large (i.e. $v > v_p$) that the cellular rod has been fully crushed before v_2 decreases to zero, only Phase I appears during the whole crushing process.

Figure 4.17 (a) Typical time histories of non-dimensional velocities v_1 and v_2 ; (b) typical time histories of the position of the shock front $\Theta = -1.5$, $\Gamma = 1$ and $v_0 = 0.8$. Scenario II

The same case is considered to determine the densification velocity v_d for the DS mode under impact Scenario II, that is, the velocity of the attached mass v_1 decreases to zero just at the instant when the graded cellular rod is fully crushed. Combining with Eq. (4.50), the densification velocity v_d is plotted against the gradient in Figure 4.18a. The DS mode dominates in the regime on the left of the broken line, while the SS mode appears on the right regime. It is found that the gradient Θ has little influence on the densification velocity for the mass ratio Γ larger than 0.5 for the SS regime. For the DS regime, the influence decreases when the gradient Θ approaches to -2, while it is significant when it is closer to the border of the regime. When the mass ratio Γ is small (e.g. $\Gamma = 0.2$), the densification velocity significantly increases with the gradient whether the SS or DS mode occurs.

To exhibit the energy-absorbing capacity of the graded cellular rod, the energy ratio ψ is plotted in Figure 4.18b. When the gradient Θ is positive, the energy ratio is higher, suggesting that the weakest part of graded cellular rod be the proximal end. Moreover, the gradient has significant effect on the energy ratio when the mass ratio is small (e.g. $\Gamma = 0.2$), but much less when it is sufficiently large (e.g. $\Gamma = 0.5$).

Figure 4.18 (a) Densification velocities v_d versus gradient Θ ; (b) energy ratio $\psi = E_d/E_q$ versus gradient Θ . Scenario II

4.5 Comparison with FE results and discussion

In this section, the theoretical prediction is compared with the FE simulation result of cellular chains. Two kinds of 1D graded chain, composed of the hexagonal and circular cells, are impinged by a rigid mass with various initial impact velocities. The FE models with hexagonal and circular cells are shown previously in Figure 4.1a and Figure 4.1b, respectively. Generally, for cellular solids, the densification strain, ε_d , is the strain at which the material has the same density as parent material. However, in reality, the chain cannot be completely densified; it is hard to determine the locking strain ε_d theoretically. In our study, the method called as “efficiency” method (Tan et al. 2005b) is employed to obtain the value of the densification strain. The efficiency E of the foam in absorbing energy is defined as the energy absorbed up to a given nominal strain, ε_a , divided by the corresponding stress value,

$$E(\varepsilon_a) = \frac{W}{\sigma(\varepsilon_a)} \quad (4.57)$$

where $W = \int_0^{\varepsilon_a} \sigma(\varepsilon) d\varepsilon$ is the absorbed energy.

The value of the densification strain is obtained from the condition, at which the efficiency E reaches the maximum value. Therefore, it is determined by the equation as follows,

$$\frac{dE(\varepsilon)}{d\varepsilon} = 0 \quad (4.58)$$

Figure 4.19 Uniaxial compression stress-strain relationship of a single cell. (a) Hexagonal cell; (b) circular cell.

Hence, the densification strains, ε_d predicted by the “efficiency” method for hexagonal and circular chains, are 0.94 and 0.78, respectively (see Figure 4.19). From the final non-dimensional displacement (see Figure 4.22b), the locking strain is modified by multiplying a factor 1.05 for circular chain (i.e. $\varepsilon_d = 0.82$) to improve the fit.

The quasi-static plateau stress σ_p is obtained from the quasi-static uniaxial compression of a single cell in FE simulations. Figure 4.19 shows the stress-strain relationship of a single hexagonal and circular cell. The quasi-static plateau stress of the chain is defined as,

$$\sigma_p = \frac{\int_0^{\varepsilon_d} \sigma(\varepsilon) d\varepsilon}{\varepsilon_d} \quad (4.59)$$

Because the quasi-static plateau stress is linearly related to the material’s yielding stress (Gibson and Ashby, 1997), the gradient in the graded chain is expressed as,

$$g = \frac{\Delta\sigma_{ys}}{l_{cell}\sigma_{ys}} \quad (4.60)$$

where l_{cell} and $\Delta\sigma_{ys}$ are the length of a cell and the difference in material’s yielding stress between two adjacent cells, respectively; σ_{ys} is the corresponding average yielding stress of the material. The non-dimensional gradient in graded cellular chain is obtained,

$$\Theta = \frac{25\Delta\sigma_{ys}}{\sigma_{ys}} \quad (4.61)$$

Table 4.1 lists the values of physical parameters of the two cellular chains. The non-dimensional velocity of the impinging plate versus the non-dimensional displacement is compared with their corresponding FE results for both the SS and DS modes under direct impact (i.e. Scenario I). For the FE result, v_1 is obtained by the velocity-time history of the impinging mass while v_2 is obtained by the velocity-time history of a gauge in the undeformed zone. For the SS mode, the variation of the velocity of the impinging plate with its displacement is calculated from Eq. (4.16), while it is obtained by using finite difference method for the DS mode. The position of the shock fronts can be easily obtained by simple integration. Also, the position of the shock front can be obtained by a close look at the deformation profiles, because the compaction zones can be easily identified (Figure 4.2-Figure 4.4).

Table 4.1: Values of the physical parameters in the graded cellular chain

Chain	Θ	ρ_0 (kg/m ³)	σ_0 (MPa)	c_p (m/s)	l_0 (m)	T (ms)	m (mg)
Hexagonal	± 1.62	230.94	0.29	35.9	0.074	3.26	0.165
Circular		400	0.35	26.65	0.135	5.07	0.206

Note: the densities of the cellular chains are shown in Appendix. 4A.

Figure 4.20 The non-dimensional velocity versus displacement of the graded hexagonal cells' chain, the SS mode. (a) $\Gamma = 36$, $v_0 = 0.279$ corresponds $G = 6\text{mg}$ and $V_0 = 10\text{ m/s}$; (b) $\Gamma = 0.36$, $v_0 = 2.79$ corresponds $G = 0.06\text{mg}$ and $V_0 = 100\text{ m/s}$

Figure 4.20 depicts the non-dimensional velocity of the impinging plate against the displacement for the hexagonal cells' chain when the gradient is positive. Two initial impact velocities are given to the impinging plate (i.e. 10m/s and 100m/s). To keep the initial kinetic energy constant, the mass of the impinging plate are chosen as 6mg and 0.06mg, respectively. Thus, the non-dimensional initial velocities v_0 are 0.279 and 2.79; and the mass ratios Γ are 36 and 0.36. The analytical predictions all show good agreement with the FE results. Moreover, it is found that the chain is not fully crushed when a smaller mass but higher velocity are employed, implying that the energy-absorbing capacity is significantly influenced by the mass ratio Γ .

Figure 4.21 Non-dimensional velocities versus displacement of the graded hexagonal cells' chain, DS mode. (a) $\Gamma = 36$, $\nu_0 = 0.279$; (b) $\Gamma = 0.36$, $\nu_0 = 2.79$; (c) $\Gamma = 1$, $\nu_0 = 2.79$

Figure 4.21 depicts the velocity against the displacement of the impinging plate for the DS mode in the graded hexagonal cells' chain. Three cases are considered: $V_0=10\text{m/s}$ and $G = 6\text{mg}$, $V_0=100\text{m/s}$ and $G = 0.06\text{mg}$, and $V_0=10\text{m/s}$ and $G = 0.145\text{mg}$. Non-dimensionalization gives that $\Gamma = 36$, $v_0 = 0.279$; $\Gamma = 0.36$, $v_0 = 2.79$; and $\Gamma = 1$, $v_0 = 2.79$, respectively. The analytical predictions for the DS mode also agree well with the FE results. Two phases exist in Figure 4.21a and b while only Phase I appears in the third case (Figure 4.21c).

The non-dimensional velocity versus the displacement of the impinging plate for the DS mode in the circular cells' chain is depicted in Figure 4.22, similar to those cases considered in Figure 4.20. It is found that the velocity of the undeformed zone v_2 is almost zero when the displacement $\mu < 0.2$; afterwards, the velocity begins to pick up, which deviated from the analytical prediction. This lag phenomenon is also found in the hexagonal cells' chains. The velocity begins to increase when $\mu > 0.07$ in Figure 4.21b; and it begins to increase when $\mu > 0.1$ in Figure 4.21c. This deviation is mainly caused by the elasticity of the material, which is included in FE simulations but neglected in the analytical derivation with the R-PP-L material model. It will take a little time for the stress wave to travel from the proximal end to the distal end. Meanwhile, if the stress in the distal end is smaller than the quasi-static plateau stress, only small elastic deformation takes place. The stress will be enhanced by reflection of the stress wave at the two ends and plastic deformation only begins to occur when the stress exceeds the quasi-static plateau stress.

Figure 4.22 Non-dimensional velocities versus displacement of the graded circular cells' chain, the DS mode. (a) $\Gamma = 29$, $\nu_0 = 0.375$; (b) $\Gamma = 0.29$, $\nu_0 = 3.75$

Figure 4.23 and Figure 4.24 show the comparison of the forces at the proximal end between the predicted (analytical) and those from finite element simulations for the graded hexagonal and circular chains, respectively. From the comparison, it is seen that both the analytical predictions for the graded hexagonal and circular show good agreement against the FE result, which also validates the analytical modeling.

Figure 4.23 Comparison of the force at proximal end for the graded hexagonal chain between FE result and analytical prediction

Figure 4.24 Comparison of the force at proximal end for the graded circular chain between FE result and analytical prediction

4.6 Conclusion

In this chapter, the dynamic response of graded cellular rods under impact is investigated using analytical approaches. FE simulations of hexagonal and circular cells' chain under uniaxial dynamic compression are carried out by using ABAQUS/EXPLICIT. From the observation of the deformation profiles of cellular chains, it is found that two shock fronts appear when the strongest cell is placed at the proximal end while only one shock front appears when the weakest cell is placed at the proximal end. Hence, a gradient in the plateau stress of the cellular material is introduced to a continuum cellular rod. Two impact scenarios are considered in the analysis. In Scenario I, the graded cellular rod is impinged by a rigid mass, while in Scenario II the rod together with an attached mass impinges onto a rigid target. Both the Single Shock (SS) and Double Shock (DS) modes are found in these two scenarios. Based on the shock theory originally proposed by Reid and Peng (1997), theoretical derivations lead to analytical expressions of the dynamic response parameters associated with the SS and DS modes in the graded cellular rods.

Closed form solutions are obtained for the SS mode, while finite difference method is employed to obtain the solution for the DS mode.

Figure 4.25 summarizes the deformation modes and phases for the graded cellular rods under the impact Scenario I. The DS mode occurs when the gradient is negative, while SS mode takes place when the gradient is positive. If the initial velocity is lower than the densification velocity, the initial kinetic energy will be exhausted before the cellular rod is fully densified. The DS mode contains two deformation phases, which are characterized by double shock fronts and a single shock front, respectively. If the initial velocity is higher than the densification velocity, not the entire initial kinetic energy is dissipated at the end of the crushing of the rod.

Similarly, Figure 4.26 depicts the deformation modes and phases for the graded cellular rod under the impact Scenario II. For Scenario II, a critical gradient is obtained in terms of the mass ratio Γ . Beyond the critical gradient, the DS mode appears. The densification velocities are also found for this impact scenario. Similar to Scenario I, the DS mode may also contain two deformation phases.

The analysis for both the scenarios shows that the gradient significantly influences the dynamic energy-absorbing capacity of the graded cellular rod when the mass ratio is small. To achieve higher energy absorption, the weakest part of the graded cellular material is suggested to be placed at the proximal end.

As for the experiments in the literature, most of them employed the density gradient in the specimen, while the present analysis adopts a gradient for the yield stress in the parent material. Therefore, the current results cannot be directly compared with published

experimental results. However, the favorable agreement between the FE results and theoretical predictions has verified the analytical solutions. A lag phenomenon is found in FE simulations when the strongest cell is placed at the proximal end, as a result of the elastic deformation in the FE models.

In the current study, only the gradient in the materials' yielding stress is investigated. If the gradient in the density of the material exists, the deformation mechanism is similar to the current work. Because the density of the material affects the quasi-static plateau stress, the locking strain and the inertia, the governing equation will be more complex. However, the SS mode and the DS mode would still be applicable, and a full analysis will be conducted in future.

Scenario I	$-2 < \Theta < 0$ (DS mode)	$0 < \Theta < 2$ (SS mode)
$v > v_p$ <ul style="list-style-type: none"> When wholly crushed, $v_1 > v_2$; No Phase II in the DS mode
$v_d < v < v_p$ <ul style="list-style-type: none"> When wholly crushed, $v > 0$; Phase II appears in the DS mode.
$v < v_d$ <ul style="list-style-type: none"> Not wholly crushed; Phase II appears in the DS mode.

Figure 4.25 A map showing various deformation histories in Scenario I. v_p denotes the velocity, beyond which Phase II does not appear for the DS mode while does not exist in the SS mode.

Scenario II	$-2 < \Theta < -2/(2\Gamma + 1)$ (DS mode)	$-2/(2\Gamma + 1) < \Theta < 2$ (SS mode)
$\nu > \nu_p$ <ul style="list-style-type: none"> When wholly crushed, $\nu_1 > \nu_2$; No Phase II in the DS mode
$\nu_d < \nu < \nu_p$ <ul style="list-style-type: none"> When wholly crushed, $\nu_1 > 0$; Phase II appears in the DS mode.
$\nu < \nu_d$ <ul style="list-style-type: none"> Not wholly crushed; Phase II appears in the DS mode.

Figure 4.26 A map showing various deformation histories in Scenario II.

Appendix 4A – FE Model

A chain is composed of 25 uniform hexagonal or circular cells along the axial loading direction, as shown in Figure 4.1. For the hexagonal cells' chain, the cell is regular with the same edge length $a = 2.7 \text{ mm}$ and the corresponding angle 120° between edges. For the circular cells' chain, the cell is a circle with radius $r = 2.7 \text{ mm}$. The wall-thickness of the cells and the density of the material are 0.2 mm and 2700 kg/m^3 , respectively. Thus, the density of the hexagonal cells' chain is $\rho_0 = \frac{2}{\sqrt{3}} \left(\frac{t}{a} \right) \rho_s = 230.94 \text{ kg/m}^3$ while for the circular cells' chain it is $\rho_0 = \frac{2t}{r} \rho_s = 400 \text{ kg/m}^3$. The out-of-plane width of the honeycomb is $b = 0.9 \text{ mm}$. The material of the cell wall is assumed to be elastic, perfectly plastic with Young's modulus $E = 69 \text{ GPa}$, and Poisson's $\nu = 0.3$. The total masses of the hexagonal and circular models are 0.165 mg and 0.206 mg , respectively. Actually the material employs is the aluminum; and the main reason why the mass is so small is that the width is very small, only 0.9 mm . To prevent the out-plane buckling of the cellular chain, the whole FE model is constrained with no movement in out-plane direction. Hence, to promote the computing efficiency and saving time, only 0.9 mm thickness is taken as the width. Three elements are used in the width direction and 10 elements for the cell edge in order to ensure the accuracy of the FE result. The gradient in the plateau stress is achieved by assigning a yielding stress of parent material which varies cell by cell. In our study, the difference of material's yield stress between each pair of adjacent cells is $\Delta\sigma_{ys} = \pm 5 \text{ MPa}$. The average materials' yielding stress is 77 MPa . For the circular cells' chain, a rigid plate is placed in-between two adjacent cells, which is to provide stiffness so that a more regular deformation could be obtained for each cell with the

constraint provided by the plate. The assigned mass of these plates is so small that the inertia effect can be neglected. After the trial simulation, it is found that if there is no plate between rings, the ring's deformation would show irregular interaction between the two adjacent ones (e.g. wrapping onto each other), resulting that the shock wave theory cannot predict well against the FE results. The reason is that the mode of the deformation of each ring is very different from the rigid-plate compression mode in the ring-plate system. This phenomenon is also found in Reid and Reddy's work (1983). To prevent the wrapping, massless rigid plates are inserted. It is confirmed that when the wrapping is prevented the distribution of deformation is more regular. Furthermore, trial simulations show that the rigid massless plate is also helpful in developing the "shock-type" deformation.

Hourglass controlled, four nodes, reduced integration shell elements (type S4R) are used to mesh the structures. The plates at both the ends are defined as rigid body. As shown in Figure 4.1, plate B is fixed by constraining the degrees of freedom of the reference point; plate A can only move along the axial loading direction. Interaction properties are imposed using general contact conditions and surface to surface kinematic contact conditions between the cellular chain and the two rigid plates with no friction. Furthermore, the out-of-plane displacement of the nodes is constrained so as to prevent the chain from out-of-plane buckling. Five integration points through the thickness of the shell elements are adopted. An initial impact velocity is given to the impinging plate.

To check that the explicit simulation has been set-up properly, a finite element model in ABAQUS/EXPLICIT is established to model a homogenous circular chain with rigid

plates by using the data in the work (Reid and Reddy 1983). Comparison between FE and experimental results shows good agreement.

Appendix 4B – Detail Derivations

4B.1 Solution of the Single Shock (SS) mode in Scenario I

For convenience, let $A = 1 - \frac{1}{2}\Theta$, $B = \frac{1}{\varepsilon_d}\Theta$, $C = \frac{1}{\varepsilon_d}$, $D = \Gamma$. Then Eq. (4.15) can be re-

written as, Equation Section 2

$$\frac{dv}{d\tau} = \frac{dv}{d\mu} \frac{d\mu}{d\tau} = v \frac{dv}{d\mu} = -\frac{A + B\mu + Cv^2}{D + C\mu} \quad (4B.1)$$

To solve Eq. (4B.1), the solution is supposed to be in the form of,

$$v = v_0 \sqrt{f(\mu)} \quad (4B.2)$$

where v_0 is the non-dimensional initial impacting velocity.

Substituting the initial condition $v|_{t=0} = v_0$, $\mu|_{t=0} = 0$, we find $f(0) = 1$. Then, substituting

Eq. (4B.2) into Eq. (4B.1) gives,

$$f'(\mu) + \frac{2C}{D + C\mu} f(\mu) = -\frac{2(A + B\mu)}{v_0^2 (D + C\mu)} \quad (4B.3)$$

which has a solution,

$$f(\mu) = e^{-\int P(\mu) d\mu} \left(\int Q(\mu) e^{-\int P(\mu) d\mu} d\mu + E \right) \quad (4B.4)$$

where $P(\mu) = \frac{2C}{D + C\mu}$, $Q(\mu) = -\frac{2(A + B\mu)}{v_0^2 (D + C\mu)}$, E is a constant to be determined from

the initial conditions. Since

$$e^{-\int P(\mu)d\mu} = \left(\frac{1}{D+C\mu} \right)^2, \int Q(\mu)e^{-\int P(\mu)d\mu} d\mu = -\frac{2}{v_0^2} \left(AD\mu + \frac{BD+AC}{2}\mu^2 + \frac{BC}{3}\mu^3 \right) \quad (4B.5)$$

by applying the initial condition $f(0)=1$ to Eq. (4B.4), it is found that

$$E = D^2 = \Gamma^2 \quad (4B.6)$$

Substituting Eqs. (4B.2)-(4B.6) into Eq. (4B.1) leads to the solution,

$$v = \frac{v_0}{D+C\mu} \sqrt{D^2 - \frac{2}{v_0^2} \left(AD\mu + \frac{BD+AC}{2}\mu^2 + \frac{BC}{3}\mu^3 \right)} \quad (4B.7)$$

Finally, by substituting constants A , B , C and D into Eq. (4B.7), we obtain an expression for velocity in terms of the displacement of the impinging mass, μ .

4B.2 Geometry relationship in Double Shock (DS) mode

In geometry, the following relationships are easily obtained,

$$\frac{du}{dt} = V_1 \quad (4B.8)$$

$$\frac{dy}{dt} = V_{s2} \quad (4B.9)$$

$$u + x + l + y = l_0 \quad (4B.10)$$

$$x + y = (l_0 - l)(1 - \varepsilon_d) \quad (4B.11)$$

where x is the length of the densified zone adjacent to the proximal end, y is the length of the densified zone adjacent to the distal end, u is the displacement of the impinging rigid mass, l is the length of the undeformed zone in the rod, and l_0 is the initial length.

Combining Eqs. (4B.10) and (4B.11),

$$u = (l_0 - l) \varepsilon_d \quad (4B.12)$$

Differentiation of Eq. (4B.12) gives,

$$\frac{dl}{dt} = -\frac{1}{\varepsilon_d} \frac{du}{dt} = -\frac{1}{\varepsilon_d} V_1 \quad (4B.13)$$

Considering a small element at the backward shock front, conservation of mass gives,

$$\rho_0 (V_2 + V_{s2}) = \rho_d V_{s2} \quad (4B.14)$$

so that we have,

$$\frac{dy}{dt} = V_{s2} = \frac{1 - \varepsilon_d}{\varepsilon_d} V_2 \quad (4B.15)$$

Since l_0 is a constant, differentiating Eq. (4B.11) gives,

$$\frac{dx}{dt} + \frac{dy}{dt} + \frac{dl}{dt} + \frac{du}{dt} = 0 \quad (4B.16)$$

Substituting Eqs. (4B.8), (4B.13) and (4B.15) into (4B.16) gives,

$$\frac{dx}{dt} = \frac{1 - \varepsilon_d}{\varepsilon_d} (V_1 - V_2) \quad (4B.17)$$

4B.3 Solution of the Single Shock (SS) mode in Scenario II

For convenience, let $A' = 1 - \frac{1}{2}\Theta$, $B' = \frac{\Theta}{\varepsilon_d}$, $C' = \Gamma + 1$, $D' = -\frac{1}{\varepsilon_d}$; consequently, Eq. (4.47)

can be re-written as,

$$\frac{dv}{dt} = v \frac{dv}{du} = -\frac{A' + B'u}{C' + D'u} \quad (4B.18)$$

By integrating Eq. (4B.18) and using the initial condition, $v|_{\mu=0} = v_0$ gives,

$$v = \sqrt{v_0^2 - 2\left(\frac{A'D' - B'C'}{D'^2}\right)\ln\left(1 + \frac{D'}{C'}\mu\right) - \frac{2B'}{D'}\mu} \quad (4B.19)$$

Substituting the constants A' , B' , C' and D' into Eq. (4B.19) results in the final solution for non-dimensional velocity in term of the displacement of the attached mass, μ .

Chapter 5 Investigation into the behavior of a graded cellular rod under impact

5.1 Introduction

Previous work (Chapter 4) has presented basic deformation patterns in a simple case, in which only the quasi-static plateau stress gradually decreases or increases along the rod. In the current chapter, further investigations are carried out into the effect of the gradient in the initial density, which inherently leads to variation in the quasi-static plateau stress. A simple impact scenario is considered, in which a rigid mass strikes a stationary cellular rod with variation in the initial density of the material. Similar to the previous studies, the rigid-perfectly plastic-locking (R-PP-L) material model and the simple shock theory are employed to carry out the analysis. Current study identified that the basic deformation modes (i.e. double shock (DS) and single shock (SS) modes) still exist in the cellular rod with density gradient. The analytical results indicate that the negative gradient weakens the capacity of energy absorption while positive gradient has little influence. Then, the effect of the initial density gradient is further investigated by considering two artificial cases with the gradient in the quasi-static stress or initial density only. Finally, finite element (FE) simulations are carried out with practical metal foam in order to verify the analytical modeling.

Equation Chapter 5 Section 1

5.2 Governing equations for a graded cellular rod with density gradient

Consider a stationary cellular rod impinged by a rigid mass, G with an initial velocity, V_0 (see Figure 5.1 and Figure 5.2). The material properties of the rod are uniform except for the initial density of the cellular material, which varies linearly along the rod. The initial length of the rod is l_0 and initial average density is ρ_0 . The gradient in the initial density, Θ_ρ , is defined as follows,

$$\Theta_\rho = \frac{\Delta\rho_0}{\rho_0} \quad (5.1)$$

where $\Delta\rho_0$ is the difference in the initial density of the undeformed cellular material between the two ends of the cellular rod. The gradient is defined as positive if the initial density increases from the proximal end to the distal end.

Introducing the density gradient not only influences the inertia item but also the quasi-static plateau stress. Therefore, the double shock mode (DS) and single shock (SS) modes, which were obtained in our previous work, are also employed here for the present analytical model.

Figure 5.1 Double Shock (DS) mode. A stationary cellular rod impinged by a rigid mass G with initial velocity V_0 . Shock waves are generated from both distal and proximal ends. The velocity of the forward shock wave is V_{s1} while the velocity of backward is V_{s2} . The shock fronts separate the rod into three parts, namely forward densified zone, undeformed zone and backward densified zone. In densified zones, the material is fully densified (i.e. $\rho = \rho_s, \varepsilon = \varepsilon_d$); in undeformed zone, the material density is $\rho = \rho_0$ and it has a rigid body velocity V_2 . The velocity of forward shock zone and rigid mass have velocity V_1 . During the impact, the displacement of the rigid mass is u , the length of the forward densified zone is x , the length of backward densified zone is y , the length of undeformed zone is l . Correspondingly, the length of the forward deformation zone is X and the length of backward deformation zone is Y .

Figure 5.2 Single Shock (SS) mode. The initial density of the material at the proximal end is the lowest; only one shock front appears from the proximal end; the material ahead of the shock front remains undeformed.

5.2.1 Double Shock (DS) mode

In the present study, the initial lengths of the two deformed zones are employed to characterize the deformation process. Figure 5.3 shows the typical initial density distribution. The initial density at the two shock fronts, defined with X and Y , are as follows,

$$\begin{aligned}\frac{\rho_0(X)}{\rho_0} &= 1 - \Theta_\rho \left(\frac{1}{2} - \frac{X}{l_0} \right) \\ \frac{\rho_0(Y)}{\rho_0} &= 1 - \Theta_\rho \left(\frac{Y}{l_0} - \frac{1}{2} \right)\end{aligned}\tag{5.2}$$

where X is the length of the forward deformation zone and Y is the length of the backward deformation zone. From geometry, we have $l = l_0 - X - Y$. Thus, the following relationship is held,

$$\rho_0(X) - \rho_0(Y) = -\rho_0 \frac{\Theta_\rho}{l_0} l\tag{5.3}$$

where l is the length of the remaining undeformed zone.

Figure 5.3 Initial density of the cellular rod in Scenario I. A constant gradient Θ is introduced to the cellular rod. The average initial density is ρ_0 . It is defined that $\Theta_\rho > 0$ when the weakest part of the cellular rod is placed at the proximal end; $\Theta_\rho < 0$ when the strongest part is at the proximal end.

Referring to equation for cellular material (Gibson and Ashby 1997), i.e. Eq. (2.17) and for simplicity taking the constant C_3 as 1, the locking strain at the shock fronts is expressed as,

$$\begin{aligned}\varepsilon_d(X) &= 1 - \frac{\rho_0(X)}{\rho_s} \\ \varepsilon_d(Y) &= 1 - \frac{\rho_0(Y)}{\rho_s}\end{aligned}\tag{5.4}$$

Hence, the lengths of the densified zones (x, y) are obtained as follow,

$$\begin{aligned}x &= \int_0^X (1 - \varepsilon_d(X)) dX \\ y &= \int_0^Y (1 - \varepsilon_d(Y)) dY\end{aligned}\tag{5.5}$$

On the other hand, the quasi-static plateau stress of the cellular material is related to the initial density. In the current study, the relationship $\sigma_p/\sigma_{ys} = C_4(\rho_0/\rho_s)^{3/2}$ (see ref. (Gibson and Ashby 1997)) is employed to carry out the analytical derivation. Hence they may be expressed as,

$$\begin{aligned}\sigma_p(X) &= C_4 \left(\frac{\rho_0(X)}{\rho_s} \right)^{3/2} \sigma_{ys} \\ \sigma_p(Y) &= C_4 \left(\frac{\rho_0(Y)}{\rho_s} \right)^{3/2} \sigma_{ys}\end{aligned}\tag{5.6}$$

where C_4 is a constant, determined by experimental results.

The cross-section A_0 is assumed to be unchanged during the crushing process. The cellular material in the densified zone is ideally compressed to its locking strain $\varepsilon_d(X)$.

From kinematic consideration, the following relationships are found,

$$\frac{du}{dt} = V_1, \quad \frac{dx}{dt} = \frac{1 - \varepsilon_d(X)}{\varepsilon_d(X)} (V_1 - V_2), \quad \frac{dy}{dt} = V_{s2} = \frac{1 - \varepsilon_d(Y)}{\varepsilon_d(Y)} V_2 \quad (5.7)$$

$$\frac{dX}{dt} = \frac{V_1 - V_2}{\varepsilon_d(X)}, \quad \frac{dY}{dt} = \frac{V_2}{\varepsilon_d(Y)}, \quad \frac{dl}{dt} = -\frac{V_1 - V_2}{\varepsilon_d(X)} - \frac{V_2}{\varepsilon_d(Y)} \quad (5.8)$$

Details of the above derivation and the physical meaning of the parameters are elaborated Appendix. 5A.

For the backward shock front, conservation of momentum for the whole system shown in Figure 5.1 gives,

$$-A_0 \sigma_d(Y) dt = d \left[(G + \rho_s x A_0) V_1 \right] + d \left(\frac{\rho_0(X) + \rho_0(Y)}{2} l V_2 A_0 \right) \quad (5.9)$$

where $\sigma_d(Y)$ is the dynamic stress at the backward shock front and it can be obtained by using shock wave theory $\sigma_d(Y) = \sigma_p(Y) + \rho_0(Y) V_2^2 / \varepsilon_d(Y)$. The derivation of the dynamic stress is given in Appendix. 5B.

Conservation of momentum at the forward shock front gives,

$$-A_0 \sigma_p(X) dt = d \left[(G + \rho_s x A_0) V_1 \right] - \frac{\rho_0(X)}{1 - \varepsilon_d(X)} V_2 A_0 dx \quad (5.10)$$

Subtracting by Eq. (5.10) from Eq. (5.9) leads to,

$$-A_0\sigma_d(Y)dt + A_0\sigma_p(X)dt = d\left(\frac{\rho_0(X) + \rho_0(Y)}{2}lW_2A_0\right) + \frac{\rho_0(X)}{1-\varepsilon_d(X)}V_2A_0dx \quad (5.11)$$

Considering the geometry and kinetic relationships, we obtain a differential equation from Eq.(5.11),

$$\frac{dV_2}{dt} = -2 \frac{\sigma_p(X) - \sigma_p(Y)}{\rho_0^2(X) - \rho_0^2(Y)} \frac{\rho_0\Theta_\rho}{l_0} \quad (5.12)$$

Consequently, we have,

$$\frac{dV_1}{dt} = - \frac{\sigma_p(X) + \frac{\rho_0(X)}{\varepsilon_d(X)}(V_1 - V_2)^2}{G/A_0 + \rho_s x} \quad (5.13)$$

Note that the quasi-static plateau stress is also a function of the initial density. V_2 is only related to the initial density distribution, regardless of the initial velocity. The initial condition is that $V_2 = 0$ at $t = 0$. Obviously, V_2 cannot be less than zero, implying that $dV_2/dt \geq 0$. Hence, when the impact occurs at the weakest part (i.e. $\Theta_\rho > 0$), $dV_2/dt < 0$, indicating $V_2 \equiv 0$ since V_2 cannot decrease. Only a single shock initiates from the proximal end. On the other hand, when the impact takes place at the strongest part (i.e. $\Theta_\rho < 0$), $dV_2/dt > 0$, double shocks propagates from both distal and proximal ends, simultaneously. Furthermore, Eq. (5.12) reveals the necessary condition for the DS mode that is the difference of the quasi-static plateau stress at the two ends. The homogenous condition is the same as the case with positive gradient. Since $\Theta_\rho \equiv 0$, $dV_2/dt \equiv 0$, indicating only one shock front imitates from the impinged end of the rod.

Non-dimensionalisation is implemented for the analytical derivation and the details are shown in Appendix. 5C. The governing differential equations are expressed as follows,

$$\frac{d\nu_2}{d\tau} = -2\Theta_\rho \frac{\left(1 - \Theta_\rho \left(\frac{1}{2} - \xi\right)\right)^{1.5} - \left(1 - \Theta_\rho \left(\eta - \frac{1}{2}\right)\right)^{1.5}}{\left(1 - \Theta_\rho \left(\frac{1}{2} - \xi\right)\right)^2 - \left(1 - \Theta_\rho \left(\eta - \frac{1}{2}\right)\right)^2} \quad (5.14)$$

$$\frac{d\nu_1}{d\tau} = - \frac{\left(1 - \Theta_\rho \left(\frac{1}{2} - \xi\right)\right)^{1.5} + \frac{(\nu_1 - \nu_2)^2 \left(1 - \Theta_\rho \left(\frac{1}{2} - \xi\right)\right)}{1 - \left(1 - \Theta_\rho \left(\frac{1}{2} - \xi\right)\right)(1 - \varepsilon_d)}}{\Gamma + \xi - \frac{1}{2}\xi\Theta_\rho + \frac{1}{2}\Theta_\rho\xi^2} \quad (5.15)$$

As proposed in the previous work (Chapter 4), there are two successive phases in the DS mode. In Phase I, ν_2 increases and ν_1 decreases until they reach the same value, when Phase II starts. In Phase II, only backward densified zone continues to propagate and the length of the forward densified zone, x , remains unchanged (i.e. $dx/dt = 0$). Thus, the initial conditions of Phase II are,

$$\tau = \tau^*, \quad \nu_1 \equiv \nu_2 = \nu, \quad \xi = \xi^* \quad (5.16)$$

where τ^* is the instant when the Phase II starts; and ξ^* is a constant in Phase II.

Applying the initial conditions (i.e. Eq.(5.16)) to Eq. (5.9) and non-dimensionalization lead to the following governing equation in Phase II,

$$\frac{d\nu}{d\tau} = - \frac{\left(1 - \Theta_\rho \left(\eta - \frac{1}{2}\right)\right)^{1.5}}{\Gamma + \left(1 - \frac{1}{2}\Theta_\rho\right)\xi^* + \frac{1}{2}\Theta_\rho(\xi^*)^2 + \frac{\lambda}{2}(2 - \Theta_\rho(\eta - \xi^*))} \quad (5.17)$$

The general solutions of the variations of the velocities (ν_1, ν_2), length parameters ($\mu, \xi, \lambda, \eta, \gamma, \zeta$) and the stress (σ/σ_0), although implicit, can be determined numerically with the boundary conditions. Therefore, finite difference method is applied.

The deformation processes are similar to those depicted in the previous work for the yield stress gradient (Chapter 4). By assuming that the average locking strain $\varepsilon_d = 0.8$, a typical deformation process of the DS mode with two phases is shown in Figure 5.4 with the mass ratio $\Gamma = 0.5$, the gradient $\Theta_\rho = -0.5$ and the initial velocity $\nu_0 = 2$. Figure 5.4a shows the time histories of the velocities (ν_1 and ν_2) while the corresponding positions of the shock fronts is presented in Figure 5.4b. In Phase I, immediately after impact ν_1 decreases and ν_2 increases; after ν_1 and ν_2 reaches a same value, Phase II starts (i.e. at instant $\tau = \tau^*$), in which further deformation concentrates at the backward densification zone. When the velocity of the impinging mass, ν , or the undeformed length, λ decreases to zero, the whole response completes. When the shock ceases, a sudden drop in the stress at the shock front takes place. The discontinuity of $d\nu/d\tau$ appears at the instant of phase transition ($\tau = \tau^*$), deviating from $d\nu_1/d\tau$ and $d\nu_2/d\tau$. The reason is that the forward shock propagation ceases when Phase II takes place.

Figure 5.4 Typical deformations process of the Double Shock (DS) mode, $\Gamma = 0.5$, $v_0 = 2$, $\Theta_p = -0.5$, (a) Time histories of the velocities; (b) the corresponding time histories position of shock fronts. It is assumed that the average locking strain $\varepsilon_d = 0.8$.

Figure 5.5 Typical deformations process of the Double Shock (DS) mode, $\Gamma = 0.5, v_0 = 3, \Theta_p = -0.5$. (a) Time histories of the velocities; (b) the corresponding time histories position of shock fronts. It is assumed that the average locking strain $\varepsilon_d = 0.8$.

If the initial kinetic energy is large enough, the two shock fronts may collide with each other before ν_1 decreases to ν_2 . In such a case, Phase II does not exist. We may determine the lower bound of the velocity ν_p from Eq. (5.15) with the condition that $\nu_1 = \nu_2$ and $\lambda = 0$. Figure 5.5a shows the time histories of ν_1 and ν_2 in such a case, where the mass ratio is $\Gamma = 0.5$, the gradient $\Theta_\rho = -0.5$ and the initial velocity $\nu_0 = 3$. The corresponding histories of the positions of the shock fronts are plotted in Figure 5.5b. Compared with Figure 5.4b, the forward densification zone is larger when the initial velocity increases.

Under the case with higher impact velocity, the deformation is the same as the Figure 5.5. Figure 5.6a shows curves, representing the non-dimensional lengths of the forward deformation zone (ξ) and backward deformation zone (η) against the impact velocity. It shows that the forward deformation zone (ξ) always increases with the impact velocity. As for the backward deformation zone (η), it increases before the velocity reaches the densification velocity and it reaches the maximum value under the densification velocity; afterwards, it decreases. Figure 5.6b plots the maximum backward deformed length (η) against the density gradient (Θ_ρ) with different mass ratio. It is shown that the backward deformed length increases with the density gradient. With the mass ratio increase, the backward deformed length also increases. From the Figure 5.6a, it has been verified that the maximum deformation at the backward zone occurs when the velocity reaches the densification velocity. Also, we may see from Figure 5.6b that if the mass ratio is too large, most of the deformation is concentrated at the distal end of the rod. Therefore, to

better evaluate the forward and backward deformation zone, we need choose proper parameters.

Figure 5.6 (a) Non-dimensional length versus the impact velocity ($\Gamma = 0.5$ and $\Theta_p = -0.5$) (b) Maximum backward deformed length (η) versus density gradient (Θ_p). It is assumed that the average locking strain $\varepsilon_d = 0.8$.

5.2.2 Single Shock (SS) mode

For the SS mode, only one shock front appears starting from the proximal end of the cellular rod (see Figure 5.2). Actually, SS mode is a degenerated DS mode with the condition that $V_2 \equiv 0$ and $Y \equiv 0$. Hence, the governing equation for the SS mode can be obtained from Eq. (5.9) by substituting the condition, $V_2 \equiv 0$,

$$\frac{dV}{dt} = - \frac{\sigma_p(X) + \frac{\rho_0(X)}{\varepsilon_d(X)} V^2}{G/A_0 + \rho_s x} \quad (5.18)$$

Considering the geometrical and kinetic relationships and non-dimensionalization leads to the following differential equation,

$$\frac{dv}{d\tau} = - \frac{\left(1 - \Theta_\rho\left(\frac{1}{2} - \xi\right)\right)^{1.5} + \frac{\left(1 - \Theta_\rho\left(\frac{1}{2} - \xi\right)\right)}{1 - \left(1 - \Theta_\rho\left(\frac{1}{2} - \xi\right)\right)(1 - \varepsilon_d)} v^2}{\Gamma + \xi - \frac{1}{2} \xi \Theta_\rho + \frac{1}{2} \Theta_\rho \xi^2} \quad (5.19)$$

Similar to the DS mode, one can numerically obtain the values of all the parameters, including the length parameters and velocities. However, the deformation process of the SS mode is similar to that of a homogenous cellular rod and is much simpler than the DS mode. Hence, it is not presented here.

5.3 Comparison and Discussions

In this section, the influences of the gradient are investigated in terms of the energy absorption capacity, the velocity of the impinging mass and the dynamic stress. Then, FE simulations are carried out with a typical metal foam (Alporas foam). Comparison with the analytical result demonstrates that the analytical modelling is suitable for the practical foam materials.

5.3.1 Energy absorption capacity

From the previous studies (Ruan et al. 2003, Ajdari et al. 2009, Zou et al. 2009, Ajdari et al. 2011), it has been studied that the deformation mode of the cellular structure is significantly influenced by velocity, leading to that the energy absorption capacity is velocity dependent. Therefore, in this manuscript we consider a critical case for all the deformation process, under which the rod is just fully crushed at the very end of the crushing.

First of all, the maximum energy absorbed by the rod under quasi-static loading is,

$$\begin{aligned} E_q &= \int A_0 \sigma_p(X) \varepsilon_d(X) dX \\ &= m \frac{\sigma_0}{\rho_0} \int_0^1 \left(1 - \Theta_\rho \left(\frac{1}{2} - \xi \right) \right)^{3/2} \left(1 - (1 - \varepsilon_d) \left(1 - \Theta_\rho \left(\frac{1}{2} - \xi \right) \right) \right) d\xi \end{aligned} \quad (5.20)$$

For a homogenous cellular rod without any gradient (i.e. $\Theta_\rho = 0$), the absorbed energy under quasi-static loading is,

$$E_q^h = m \frac{\sigma_0}{\rho_0} \varepsilon_d \quad (5.21)$$

By using Eqs. (5.20) and (5.21), we plot the energy ratio E_q/E_q^h versus the density gradient Θ_ρ under quasi-static conditions in Figure 5.7. The gradient has little influence on the energy absorption under the quasi-static loading.

Figure 5.7 The energy ratio versus the gradient Θ_ρ in the general case

To better evaluate the impact energy absorption capacity of the graded cellular rod under impact, a critical case is considered, in which the cellular rod is just fully crushed when

the velocity of the impinging mass (i.e. V_1) decreases to zero. The initial velocity in such a case is called densification velocity, V_d . In Section 2, the differential equations are theoretically obtained for the DS mode (Eqs. (5.14)-(5.17)) and the SS mode (Eq. (5.19)), respectively. By using the finite difference method, the densification velocity can be obtained. Since no heat transfer is considered, conservation of the energy gives the initial input kinetic energy as follows,

$$E_d = \frac{1}{2} M V_d^2 \quad (5.22)$$

Hence, the energy ratio E_d/E_q^h is obtained as,

$$\frac{E_d}{E_q^h} = \frac{\Gamma v_d^2}{2\varepsilon_d} \quad (5.23)$$

where v_d is the non-dimensional densification velocity. In Figure 5.7, the ratio E_d/E_q^h is also plotted versus the density gradient Θ_ρ for different values of mass ratio Γ . Similar to the homogenous case, the influence becomes more evident when the mass ratio is smaller. It is noted that when the mass ratio takes a small value (e.g. $\Gamma = 0.2$), the positive gradient causes a small decrease in the energy ratio as it increases; when the gradient is smaller than -1, the energy ratio increases as it becomes more negative. When the mass ratio becomes larger, it is shown that the density gradient Θ_ρ nearly has little influence on the energy absorption capacity of the graded cellular rod when the gradient is positive while the energy absorption capacity is significantly weakened by the negative gradient in density.

Under dynamic conditions, the curves of the non-dimensional velocity versus the displacement of the rigid mass for different gradient are plotted in Figure 5.8. Since the cellular rod can be fully or partially densified, two different initial velocities are adopted to carry out the analysis, as shown in Figure 5.8a and Figure 5.8b, respectively. Clearly, it is seen that in the SS mode the velocity initially decreases faster for a small value of the gradient; but towards the end the velocity decreases almost to the same value as it varies in the DS mode. When the gradient is negative with a larger magnitude, the residual velocity is higher. We may conclude, therefore that in a fully densified condition, the energy absorption capacity is not influenced by the positive gradient while it is weakened by the negative gradient. Figure 5.8b shows a partially densified condition. Similarly, the velocity drops much faster initially but less so afterwards when the gradient is smaller. Discontinuities are found at the instant when the deformation phase transfers. Comparing the final displacement of the rigid mass, it is found that the uniform case is the smallest, which may benefit the protected structures.

Figure 5.8 Non-dimensional velocity versus the displacement of the rigid mass with the mass ratio $\Gamma = 0.5$. (a) Fully densified condition with the initial velocity $v_0 = 4$; (b) partially densified condition with the initial velocity $v_0 = 2$. It is assumed that the average locking strain $\varepsilon_d = 0.8$.

5.3.2 Dynamic stress at the proximal and distal ends

In practical applications, the cellular materials are widely used as sacrificial material to protect the equipment or structures from impact loadings. Hence, it is very necessary to investigate the stresses at the proximal and distal ends. In this section, the dynamic stresses at proximal and distal ends in the crushing process are discussed. The details of the derivation are shown in Appendix. 5B.

For the DS mode, the dynamic stress at the forward shock front is expressed,

$$\sigma_d(X) = \sigma_p(X) + \frac{\rho_0(X)}{\varepsilon_d(X)} (V_1 - V_2)^2 \quad (5.24)$$

On the other hand, the dynamic stress at the backward shock front as,

$$\sigma_d(Y) = \sigma_p(Y) + \frac{\rho_0(Y)}{\varepsilon_d(Y)} V_2^2 \quad (5.25)$$

As for the stress at the proximal end, it may be derived from Newton's second law,

$$\sigma_d^p = M \frac{dV_1}{dt} \quad (5.26)$$

In the SS mode, the stress at the forward shock front is obtained by applying to Eq. (5.24) the condition that $V_2 \equiv 0$; the stress at the distal end σ_d^d equals the quasi-static plateau stress at the forward shock front; and Eq. (5.26) still holds for the stress at the proximal end in the SS mode.

Figure 5.9a shows non-dimensional dynamic stresses (σ/σ_0) at the proximal and distal ends versus the non-dimensional displacement of the rigid mass μ in a partially densified case. The gradients have the same magnitude but opposite sign. The initial value of the stress at the proximal end in the DS mode is much higher than the stress in the SS mode. Afterwards, it decreases rapidly with the displacement. A sudden drop in the stress occurs, caused by the transition of the deformation phase, described in Section 5.2.1. Compared with the stress at the distal end, it is found that the stress initially has the same value, which is determined by weakest end of the foam; afterwards, it becomes smaller in the DS mode than that in the SS mode. This is the same as the result reported by Zeng et al. (2010). Figure 5.9b shows the non-dimensional dynamic stresses versus the displacement of the rigid mass in a fully crushed case. The trends of the curves are similar to Figure 5.9a, except that there is no sudden drop at the proximal end in the DS mode, because only Phase I exists under such a condition.

Figure 5.9 Dynamic stresses at proximal and distal ends. (a) Partially densified case with $\Gamma = 0.5$, $|\Theta| = 1$ and $\nu_0 = 1.5$ (b) fully densified case with $\Gamma = 0.5$, $|\Theta| = 1$ and $\nu_0 = 3$. Note: SS denotes the single shock mode (i.e. positive gradient); DS denotes the double shock mode (i.e. negative gradient)

5.3.3 Two artificial special cases

Since the gradient in the initial density influences the quasi-static plateau stress, inertia part and locking strain, which may all affect the energy absorption capacity of the graded cellular rod, two artificial cases are considered in this section in an attempt to understand the coupled effect more clearly. The case with the gradient in its initial density described in Section 5.2 is called as “general” case.

5.3.3.1 Gradient in quasi-static plateau stress only

In this case, the variation of the quasi-static plateau stress is the same with the general case in Section 2 while the density and locking strain are assumed to be constants, which are the average value (i.e. $\rho_0(X) = \rho_0(Y) = \rho_0$, $\varepsilon_d(X) = \varepsilon_d(Y) = \varepsilon_d$ and $\sigma_0(X) = f(\Theta, X)$). The non-dimensional governing equations in Phase I are then expressed as follows,

$$\frac{dv_2}{d\tau} = \frac{\left[1 - \Theta_\rho \left(\frac{1}{2} - \xi\right)\right]^{3/2} - \left[1 - \Theta_\rho \left(\eta - \frac{1}{2}\right)\right]^{3/2}}{\lambda} \quad (5.27)$$

$$\frac{dv_1}{d\tau} = - \frac{\left[1 - \Theta_\rho \left(\frac{1}{2} - \xi\right)\right]^{1.5} + \frac{(v_1 - v_2)^2}{\varepsilon_d}}{\Gamma + \xi} \quad (5.28)$$

In Phase II, the non-dimensional governing equation is,

$$\frac{dv}{d\tau} = - \frac{\left[1 - \Theta_\rho \left(\eta - \frac{1}{2}\right)\right]^{3/2}}{\Gamma + 1 - \eta} \quad (5.29)$$

Applying the condition that $v_2 \equiv 0$ to Eq. (5.28) gives the governing equation for the SS mode,

$$\frac{dv}{d\tau} = - \frac{\left[1 - \Theta_\rho \left(\frac{1}{2} - \xi \right) \right]^{1.5} + \frac{v^2}{\varepsilon_d}}{\Gamma + \xi} \quad (5.30)$$

5.3.3.2 Gradient in density only

For the investigation into the effect of the dynamic enhancement, the quasi-static plateau stress and locking strains are assumed to be the average value while the variation of the dynamic enhancement remains the same with the general case (i.e. $\sigma_0(X) = \sigma_0(Y) = \sigma_0$ and $\frac{\rho_0(X)}{\varepsilon_d(X)} = g(\Theta, X)$). In Section 5.2, it has been confirmed that only single shock (SS)

mode occurs if there is no variation in the quasi-static plateau stress. With such condition, the non-dimensional governing equation is,

$$\frac{dv}{d\tau} = - \frac{1 + \frac{1 - \Theta_\rho \left(\frac{1}{2} - \xi \right)}{1 - \left(1 - \Theta_\rho \left(\frac{1}{2} - \xi \right) \right) (1 - \varepsilon_d)} v^2}{\Gamma + \xi - \frac{1}{2} \xi \Theta_\rho + \frac{1}{2} \Theta_\rho \xi^2} \quad (5.31)$$

With all these differential equations the densification velocity for each case can be obtained by using the finite difference method.

Figure 5.10 The energy ratio versus the gradient Θ_ρ in the artificial cases, $\Gamma = 0.5, \Gamma = 1$

Figure 5.10 shows the energy ratio versus the gradient Θ_ρ in the two artificial cases. It shows that the energy absorption capacity in the SS mode is enhanced by the quasi-static plateau stress but weakened by the term of density gradient. In the DS mode, the energy ratio is largely determined by the quasi-static plateau stress while the density gradient has little effect. It should be noted that the yielding stress and density are inherently related. The initial intention was to study the effect of gradient in stress or density alone, so that the coupled effect may be understood more clearly. It was not intended to decouple the two and then superimpose the two. Rather than decoupling the effects, more attention is put on the comparison of the energy ratio for the two artificial cases and the general case.

5.3.4 Numerical simulations with metal foams

Previous researchers employed the initial density gradient to investigate the dynamic response of the graded cellular material. However, it is hard to compare the result with experiments. The reason is that the quasi-static plateau stress and the locking strain do not follow the relationship described in Eqs. (5.4) and (5.6). For example, Zeng et al. (2010) investigated the influence of the density gradient on the impact behaviour of hollow sphere agglomerates. However, only 4 layers of the agglomerates were used in their experiment, which seems not be sufficient to fully understand the influence of the gradient.

The finite element software ABAQUS/EXPLICIT is employed to carry out the numerical simulations. A FE model is shown in Figure 5.11, with 9 layers. The dimension of each layer is the same (2.5cm*5cm*5cm), as shown in Figure 5.11b. Constrains are assigned

to the mass in the transitions and rotations except that it can only move along the rod. As for the impinged rod, it is supported by a rigid support and the rotations of all the elements are constrained so that no buckling will occur in the impact. The mass is allowed to rebound when the rod is fully crushed. The initial density, the quasi-static plateau stress as well as the locking strain all vary along the loading directions while the material properties in each layer are the same. The constitutive law “crushable foam” available in the ABAQUS/EXPLICIT is chosen to present the cellular properties, which has been validated in the work by Zeng et al. (2010).

Figure 5.11 (a) FE model in ABAQUS/EXPLICIT (9 layers) (b) dimension of one layer, $a = b = 5\text{cm}$, $c = 2.5\text{cm}$

The material properties of the middle layer (i.e. the average values) are obtained from the quasi-static stress-strain curve of Alporas foam, as reported in Langdon et al. (2010), which is shown in Figure 5.12. The Young's modulus is of a constant value for all the layers, 1.1GPa. The Poisson ratio is set to be zero so that the deformation in the lateral direction is negligible. Isotropic hardening property is employed in the FE model. The compression yield stress ratio is defined as 1.1. Both the elastic and plastic Poisson ratios are set to be zero so that the deformation in the lateral direction is negligible.

The densification strain for the middle layer, ε_d predicted by the “efficiency” method for the Alporas foam is 0.6 (see Figure 5.12); and the quasi-static plateau stress $\sigma_0 = 2.07$ MPa. The corresponding characteristic velocity $c_p = 95$ m/s.

Given that the gradient of the initial density is $\Theta_p = 1$ and employing Eqs. (5.4) and (5.6) result in the stress-strain curves for the foam materials in the other layers. . Referring to Eq. (5.6), the stress ratio over the average value σ_0 is expressed as follows,

$$\frac{\sigma_0(n)}{\sigma_0} = \left(\frac{\rho_0(n)}{\rho_0} \right)^{3/2}. \text{ As for the corresponding strain, linear scale over its locking strain}$$

is used and can be expressed as follows, $\frac{\varepsilon(n)}{\varepsilon} = \frac{\varepsilon_d(n)}{\varepsilon_d}$, where $\varepsilon(n)$ and ε are the strain

in the n^{th} layer and the middle layer, respectively. As for the relationship between the locking strain, it is obtained by referring to Eq. (5.4) and the expression is

$$\frac{1 - \varepsilon_d(n)}{1 - \varepsilon_d} = \frac{\rho_0(n)}{\rho_0}.$$

Figure 5.12 (a) Stress strain curve of Alporas foam with initial density $\rho_0 = 230 \text{ kg/m}^3$ (layer 5); (b) scaling stress strain curves to the other layers

Table 1 lists the material properties of each layer and the scaling for the curves are presented in Figure 5.12b. The total mass of the foam, $m = 0.129$ kg; the impinging mass, G is taken as 0.09kg in the simulation. Hence, the mass ratio is $\Gamma = 0.7$.

Table 5.1 Summary of properties of the metal foam in each layer

Layer	Initial density $((\rho_0(n)/\rho_0))$	Quasi-static plateau stress $(\sigma_0(n)/\sigma_0)$	Locking strain
1	0.5	0.354	0.8
2	0.625	0.494	0.75
3	0.75	0.650	0.7
4	0.875	0.818	0.65
5	1	1	0.6
6	1.125	1.193	0.55
7	1.25	1.398	0.5
8	1.375	1.612	0.45
9	1.5	1.837	0.4

Note: In the FE simulation, the average initial density $\rho_0 = 230$ kg/m³; the corresponding quasi-static plateau stress $\sigma_0 = 2.07$ MPa.

Since the material properties are not continuous, trial FE simulations are employed to determine how many layers are needed in order to represent the continuous graded cellular rod. Figure 5.13a shows the time-histories of the impinging mass in a DS case with different layers. The non-dimensional parameters are the same, which are $\Gamma = 0.7$, $\nu_0 = 0.78$ and $\Theta_\rho = -1$. Comparison shows that current model (9 layers) is appropriate. To avoid the strain inhomogeneity in the FE model, the ratio of the artificial energy over the internal energy is checked for each case. It is found that it is so small that it can be neglected, indicating that the current model does not introduce strain inhomogeneity. To verify the mesh size, 2.5mm and 1.25mm wide elements are used to mesh the foam

structure. Figure 5.13b shows the velocities of the nodes located at the neighborhood of the layers against the time for three kinds of mesh size. It is noted that current mesh size (5mm size) can give results with sufficient accuracy.

Figure 5.13 (a) Time-histories of the velocity of the impinging mass in DS mode ($\Theta_p = -1$, $\Gamma = 0.7$, $\nu_0 = 0.78$); (b) verification of the mesh size using three mesh size: 5mm, 2.5mm and 1.25mm.

Figure 5.14 shows the average strain distribution of each layer for two instants in the crushing case of $\nu_0 = 1.56$, $\Gamma = 0.7$, $\Theta_\rho = -1$. From the two instants, we may see that two plastic compaction zones appear both at the impinged and distal ends. At the early stage (Figure 5.14a), most of the deformation is concentrated at the impinged end; the FE strain at the distal end is smaller than predicted one but later it increases to the analytical result. The main reason is that the stress takes time to travel to the distal end while no time is consumed in the analytical model due to the assumption of the “Rigid” elastic stage.

Figure 5.14 The average strain in each layer at two instants of the crushing case $\nu_0 = 1.56$, $\Gamma = 0.7$, $\Theta_\rho = -1$.

Figure 5.15 shows the comparison of velocities versus displacement between the FE and analytical results. As predicted, double shock (DS) mode occurs in the cellular rod with negative gradient while only single shock (SS) mode appears when the gradient is positive. The trends of the theoretical and FE curves are similar in all the cases. However, differences are also observed in these cases. For the final displacement of the impinging mass, the analytical results are smaller than those obtained from FE. The main reason for

the difference is the idealisation of the metal foam to R-PP-L material model. It is really true that any positive impact velocity is expected to cause a shock wave for the R-PP-L material model. The essential reason is that R-PP-L material model employs “rigid” for the elastic range, leading to the critical velocity approaches zero. However, for actual metal foam, Young’s modulus is a constant value, instead of an infinite value, resulting in that the critical velocity is larger than zero. As a result, the discrepancy appears between the numerical and the theoretical model.

Figure 5.15 Comparison between FE and analytical results, mass ratio $\Gamma = 0.7$. (a) SS mode, density gradient $\Theta_\rho = 1$, initial velocity $v_0 = 0.78$; (b) SS mode, density gradient $\Theta_\rho = 1$, initial velocity $v_0 = 1.56$; (c) the DS mode, density gradient $\Theta_\rho = -1$, initial velocity $v_0 = 0.78$; (d) the DS mode, density gradient $\Theta_\rho = -1$, initial velocity $v_0 = 1.56$

However, differences are also observed in these cases. For the final displacement of the impinging mass, the analytical results are smaller than those obtained from FE. The main reason for the difference is the idealisation of the metal foam to R-PP-L material model. The R-PP-L material model assumes that the maximum strain of the compressed material is the locking strain while further deformation still takes place in the FE simulations, leading to inaccurate predictions.

Figure 5.16 Locking strain determined by efficiency method and maximum displacement in FE simulation

Figure 5.16 shows the maximum strain of each layer in a fully crushed FE case ($\nu_0 = 2$, $\Gamma = 0.7$ and $\Theta_\rho = -1$) as well as the predicted locking strain by using efficiency method. The predicted locking strain is smaller than the FE results. Another reason is that the R-

PP-L material model cannot capture the behaviour of the cellular material under low velocity (smaller than the critical velocity), resulting in the disappearance of the plastic shock front. Similar observations are reported by Karagiozova et al. (2012).

Hence, current analytical model is helpful in understanding the basic response of the graded cellular rod under impact conditions. However, to obtain more accurate predictions for the actual metal foams, modifications on the R-PP-L material model and on the definition of locking strain are necessary.

5.4 Conclusion

A cellular rod with a gradient in the initial density is proposed. With this gradient, both the quasi-static plateau stress and the locking strain vary along the axial direction. By considering a simple impact scenario, in which a graded cellular rod is impinged by a rigid mass, a complete theoretical analysis has been carried out.

The theoretical analysis reveals two possible deformation processes in the impact due to variation of the gradient: the SS mode occurs when the weakest part is impinged while the DS mode takes place when the strongest part is placed at the impinged end. Also, it reveals the necessary condition for the existence of the DS mode, which is the difference in the quasi-static plateau stress within the cellular rod. The two deformation modes in the impact scenario are similar to those depicted in our previous work.

Consequently, the energy absorption capacity of the gradient cellular rod and the transmitted force on the protected structure are investigated. The investigation shows that,

for the velocity range considered, the SS mode has almost no influence on the energy absorption while the DS mode weakens the energy absorption significantly. In view of the dynamic stress imposed on the protected structure, however, the stress induced by the DS mode is lower than that by the SS mode. This suggests that in practice the weakest end should be placed as the impinged end in order to maximize the total energy absorbed by the cellular rod. However, if possible excessive force at the support or transmitting to the protected structure is of main concern, the strongest end should then be the impinged end, at a small sacrifice of energy absorption.

Then, the effect of the gradient in the initial density is further investigated by considering two artificial cases, i.e. with only quasi-static plateau stress and density gradient only. Comparing with the general case, it shows that the energy absorption capacity in the SS mode is enhanced by the quasi-static plateau stress but weakened by the density gradient. For the DS mode, the energy ratio is largely determined by the quasi-static plateau stress while the density gradient has little effect.

Finally, FE results show the same trends as those revealed by the analytical approach but with difference in terms of the final displacement.

Appendix. 5A – Geometry and kinetic relationships

Equation Chapter 5 Section 1A typical deformation process of the Double Shock (DS) mode is depicted in Figure 5.1. The displacement of the rigid mass is u . The forward shock front moves at velocity V_{s1} while the backward shock front moves at velocity V_{s2} . The material ahead of the shock wave (undeformed) is treated as rigid that moves at velocity V_2 . The material behind the shock front is fully densified (i.e. $\varepsilon = \varepsilon_d$) and its density has been raised from ρ_0 to a constant value ρ_s . The length of the forward densified zone is x and the backward densified zone is y . Correspondingly, the initial lengths of the deformation zone are X and Y , respectively. The initial density distribution is shown in Figure 5.3.

From kinetic consideration, it is clear that,

$$\frac{du}{dt} = V_1 \quad (5A.1)$$

The mass conservation at forward shock front is,

$$\rho_s (V_1 - V_{s1}) = \rho_0 (X) (V_2 - V_{s1}) \quad (5A.2)$$

The velocity of the forward shock front V_{s1} is,

$$V_{s1} = \frac{d(x+u)}{dt} = \frac{dx}{dt} + V_1 \quad (5A.3)$$

Substituting Eq. (5A.3) into Eq. (5A.2) gives and then simplifying it give,

$$\frac{dx}{dt} = (V_1 - V_2) \frac{\rho_0(X)/\rho_s}{(1 - \rho_0(X)/\rho_s)} \quad (5A.4)$$

Referring to Eq. (5.4) we find,

$$\frac{dx}{dt} = \frac{1 - \varepsilon_d(X)}{\varepsilon_d(X)} (V_1 - V_2) \quad (5A.5)$$

Similarly, the mass conservation on the backward shock front is,

$$\rho_s(0 + V_{s2}) = \rho_0(Y)(V_2 + V_{s2}) \quad (5A.6)$$

The velocity of the backward shock front V_{s2} is,

$$\frac{dy}{dt} = V_{s2} \quad (5A.7)$$

Hence, substituting Eq. (5A.7) into Eq. (5A.6) and simplifying give,

$$\frac{dy}{dt} = V_{s2} = \frac{\rho_0(Y)/\rho_s}{1 - \rho_0(Y)/\rho_s} V_2 \quad (5A.8)$$

Referring to Eq. (5.4) we find,

$$\frac{dy}{dt} = V_{s2} = \frac{1 - \varepsilon_d(Y)}{\varepsilon_d(Y)} V_2 \quad (5A.9)$$

In the present study, the lengths of the deformation zone X and Y are employed to carry out the derivation. Referring to Eq. (5.5) we have,

$$\frac{dX}{dt} = \frac{V_1 - V_2}{\varepsilon_d(X)} \quad (5A.10)$$

$$\frac{dY}{dt} = \frac{V_2}{\varepsilon_d(Y)} \quad (5A.11)$$

Consequently, the differential equation of the undeformed length l is obtained as,

$$\frac{dl}{dt} = -\frac{V_1 - V_2}{\varepsilon_d(X)} - \frac{V_2}{\varepsilon_d(Y)} \quad (5A.12)$$

For the Single Shock (SS) mode, only forward shock front appears in the cellular rod.

Hence, the geometry and kinetic relationship can be obtained by simply applying the boundary conditions $V_2 \equiv 0$, $Y \equiv 0$,

$$\frac{du}{dt} = V, \quad \frac{dx}{dt} = \frac{1 - \varepsilon_d(X)}{\varepsilon_d(X)} V, \quad \frac{dX}{dt} = \frac{V}{\varepsilon_d(X)}, \quad \frac{dl}{dt} = -\frac{V}{\varepsilon_d(X)} \quad (5A.13)$$

Appendix. 5B – Dynamic stresses

Equation Section (Next) The dynamic stresses at the shock fronts can be determined by using the simple shock theory. By considering the small element ahead of the forward front over a small time increment dt in the DS mode, the velocity of the element increases from V_2 to V_1 after densification. The deformed length of the element after densification is dX and the mass of the element is determined as,

$$dm = \rho_0(X) A_0 dX \quad (5B.1)$$

Thus, conservation of momentum for this small element leads to,

$$(\sigma_d(X) - \sigma_p(X)) A_0 dt = (V_1 - V_2) dm \quad (5B.2)$$

where $\sigma_d(X)$, $\sigma_p(X)$ are the dynamic stress and quasi-static plateau stress at the forward shock front, respectively.

Substituting Eqs. (5A. 10) and (5B. 1) into Eq. (5B. 2) gives,

$$\sigma_d(X) = \sigma_p(X) + \frac{\rho_0(X)}{\varepsilon_d(X)} (V_1 - V_2)^2 \quad (5B.3)$$

Similar approach is employed for a small element at the backward shock fronts. Over a small time increment dt , the velocity of the element decreases from V_2 to zero. The mass of the small element is,

$$dm = \rho_0(Y) A_0 dY \quad (5B.4)$$

Conservation of momentum at the backward shock front leads to,

$$\left(\sigma_d(Y) - \sigma_p(Y)\right) A_0 dt = V_2 dm \quad (5B.5)$$

Substituting Eqs. (5A. 11) and (5B. 4) into Eq. (5B. 5) gives,

$$\sigma_d(Y) = \sigma_p(Y) + \frac{\rho_0(Y)}{\varepsilon_d(Y)} V_2^2 \quad (5B.6)$$

The reaction stress at the distal end is equal to the dynamic stress at the backward shock front since the backward densification zone is in equilibrium.

Non-dimensionalization of these stresses is carried out in respect to the quasi-static plateau stress σ_0 at average initial density ρ_0 , which can be found in Appendix 5C.

Equation Section (Next)

Appendix. 5C – Non-dimensionalization

The non-dimensional length parameters are obtained with respect to the initial length of the rod,

$$\frac{X}{l_0} = \xi, \frac{u}{l_0} = \mu, \frac{l}{l_0} = \lambda, \frac{Y}{l_0} = \eta, \frac{x}{l_0} = \gamma, \frac{y}{l_0} = \zeta \quad (5C.1)$$

The non-dimensional velocity is determined with respect to the characteristic velocity,

$$v \equiv V/c_p \quad (5C.2)$$

where c_p is a characteristic velocity, defined by

$$c_p \equiv \sqrt{\sigma_0/\rho_0} \quad (5C.3)$$

where ρ_0 is the average initial density of the cellular material and σ_0 is the corresponding quasi-static plateau stress.

A characteristic time is defined as,

$$T \equiv l_0/c_p \quad (5C.4)$$

Consequently, the non-dimensional time is expressed by,

$$\tau \equiv tc_p/l_0 \quad (5C.5)$$

The non-dimensionalization for the initial density over the average density ρ_0 is,

$$\frac{\rho_0(X)}{\rho_0} = 1 - \Theta_\rho \left(\frac{1}{2} - \xi \right) \quad (5C.6)$$

$$\frac{\rho_0(Y)}{\rho_0} = 1 - \Theta_\rho \left(\eta - \frac{1}{2} \right) \quad (5C.7)$$

Correspondingly, the non-dimensionalization for the locking strain is,

$$\begin{aligned} \varepsilon_d(X) &= 1 - \frac{\rho_0(X)}{\rho_s} = 1 - \frac{\rho_0(X)}{\rho_0} \frac{\rho_0}{\rho_s} \\ &= 1 - \left[1 - \Theta_\rho \left(\frac{1}{2} - \xi \right) \right] (1 - \varepsilon_d) \end{aligned} \quad (5C.8)$$

$$\begin{aligned} \varepsilon_d(Y) &= 1 - \frac{\rho_0(Y)}{\rho_s} = 1 - \frac{\rho_0(Y)}{\rho_0} \frac{\rho_0}{\rho_s} \\ &= 1 - \left[1 - \Theta_\rho \left(\eta - \frac{1}{2} \right) \right] (1 - \varepsilon_d) \end{aligned} \quad (5C.9)$$

where $\varepsilon_d = 1 - \rho_0/\rho_s$ is the locking strain when the initial density is the average value

ρ_0 .

The non-dimensionalization of the quasi-static stress is,

$$\begin{aligned} \frac{\sigma_p(X)}{\sigma_0} &= C_4 \left(\frac{\rho_0(X)}{\rho_s} \right)^{3/2} \sigma_{ys} / C_4 \left(\frac{\rho_0}{\rho_s} \right)^{3/2} \sigma_{ys} \\ &= \left(\frac{\rho_0(X)}{\rho_0} \right)^{3/2} = \left[1 - \Theta_\rho \left(\frac{1}{2} - \xi \right) \right]^{3/2} \end{aligned} \quad (5C.10)$$

$$\begin{aligned} \frac{\sigma_p(Y)}{\sigma_0} &= C_4 \left(\frac{\rho_0(Y)}{\rho_s} \right)^{3/2} \sigma_{ys} / C_4 \left(\frac{\rho_0}{\rho_s} \right)^{3/2} \sigma_{ys} \\ &= \left(\frac{\rho_0(Y)}{\rho_0} \right)^{3/2} \left[1 - \Theta_\rho \left(\eta - \frac{1}{2} \right) \right]^{3/2} \end{aligned} \quad (5C.11)$$

Hence, the non-dimensional kinetic relationships for the Double Shock (DS) mode are non-dimensionalized as follows,

$$\frac{d\mu}{d\tau} = \nu_1 \quad (5C.12)$$

$$\frac{d\xi}{d\tau} = \frac{\nu_1 - \nu_2}{1 - \left[1 - \Theta_\rho \left(\frac{1}{2} - \xi\right)\right] (1 - \varepsilon_d)} \quad (5C.13)$$

$$\frac{d\eta}{d\tau} = \frac{\nu_2}{1 - \left[1 - \Theta_\rho \left(\eta - \frac{1}{2}\right)\right] (1 - \varepsilon_d)} \quad (5C.14)$$

$$\frac{d\lambda}{d\tau} = \frac{d\xi}{d\tau} - \frac{d\eta}{d\tau} = \frac{\nu_1 - \nu_2}{1 - \left[1 - \Theta_\rho \left(\frac{1}{2} - \xi\right)\right] (1 - \varepsilon_d)} - \frac{\nu_2}{1 - \left[1 - \Theta_\rho \left(\eta - \frac{1}{2}\right)\right] (1 - \varepsilon_d)} \quad (5C.15)$$

$$\frac{d\gamma}{d\tau} = \frac{\left[1 - \Theta_\rho \left(\frac{1}{2} - \xi\right)\right] (1 - \varepsilon_d)}{1 - \left[1 - \Theta_\rho \left(\frac{1}{2} - \xi\right)\right] (1 - \varepsilon_d)} (\nu_1 - \nu_2) \quad (5C.16)$$

$$\frac{d\zeta}{d\tau} = \frac{\left[1 - \Theta_\rho \left(\eta - \frac{1}{2}\right)\right] (1 - \varepsilon_d)}{1 - \left[1 - \Theta_\rho \left(\eta - \frac{1}{2}\right)\right] (1 - \varepsilon_d)} \nu_2 \quad (5C.17)$$

Non-dimensionalization of Eqs. (5.12) and (5.13) gives,

$$\frac{d\nu_2}{d\tau} = -2\Theta_\rho \frac{\left[1 - \Theta_\rho \left(\frac{1}{2} - \xi\right)\right]^{1.5} - \left[1 - \Theta_\rho \left(\eta - \frac{1}{2}\right)\right]^{1.5}}{\left[1 - \Theta_\rho \left(\frac{1}{2} - \xi\right)\right]^2 - \left[1 - \Theta_\rho \left(\eta - \frac{1}{2}\right)\right]^2} \quad (5C.18)$$

$$\frac{dv_1}{d\tau} = - \frac{\left[1 - \Theta_\rho \left(\frac{1}{2} - \xi\right)\right]^{1.5} + \frac{(\nu_1 - \nu_2)^2 \left[1 - \Theta_\rho \left(\frac{1}{2} - \xi\right)\right]}{1 - \left[1 - \Theta_\rho \left(\frac{1}{2} - \xi\right)\right](1 - \varepsilon_d)}}{\Gamma + \xi - \frac{1}{2}\xi\Theta_\rho + \frac{1}{2}\Theta_\rho\xi^2} \quad (5C.19)$$

Similarly, non-dimensional equations for the other cases can be obtained by applying the relevant initial conditions.

Equation Chapter (Next) Section 1

Chapter 6 The cellular material with varying cross-section under impact

6.1 Introduction

Few at all experimental studies on graded material appeared in literature. Previous chapters only presented numerical and analytical analysis. Hence, it is necessary to conduct an experimental study to somehow validate the analytical result, described in the previous studies.

In this chapter, a cellular rod with graded cross-section is proposed, shown in Figure 6.2. Similar to the cases in Chapters 4 and 5, we consider the impact scenario, in which the cellular rod is impinged upon by a rigid mass with an initial velocity. Gas gun is employed to assign a high impact velocity to a steel block and the crushing process is captured by a high speed camera. The experimental result states that the DS mode appears in the rod with negative gradient while the SS mode takes place in the case with positive gradient.

Due to the limitation of the experiment equipment, finite element simulations are employed to extend the experimental study to the conditions with higher impact velocities. Comparison between the experimental and FE result shows good consistence, validating the FE model.

Afterwards, an analytical derivation is presented by using the simple shock theory. Comparison is made between the analytical and FE result. Then, the energy absorption capacity and the force transmitted to the distal end are analyzed.

In this chapter, it should be noted that the rod is made of homogenous foam material, but has a variable cross-section; while the properties of the graded foam, such as the density and the yielding stress, vary along the loading direction in the previous chapters.

6.2 Experiment

6.2.1 Experiment set up

In this experiment, the closed-cell aluminum foam, Alporas, was used, which is basically made from aluminum with 1.5% of Ca and 1.5% of Ti. Quasi-static compression was first carried out to obtain the stress-strain relationship of the foam, shown in Figure 6.1. The relative density of the tested foam is 8.4%. Efficiency method is used to determine the locking strain and the quasi-static plateau stress of the foam. For the foam with 8.4% relative density, the locking strain ε_d is 0.5 and the quasi-static plateau stress $\sigma_0 = 0.62$ MPa. Due to the inhomogeneity of the foam, specimens in the dynamic test may have different relative densities. In the dynamic test, each specimen is individually measured and density calculated. On the other hand, the stress-strain relationship in Figure 6.1 is only tested for the foam with the relative density 8.4%. Hence, we employed the same scaling method described in Chapter 5 to obtain the stress-strain relationship for the foam material with other relative density (e.g. 12% shown in Figure 6.1). Figure 6.2 shows the foam block with a cross-section gradient. The height “a” is a constant along the loading direction while the width “b” linearly increases or decreases.

Figure 6.1 Stress-strain relationship of Alporas foam

Figure 6.2 (a) The dimension of the graded foam block; (b) graded aluminum foam specimen

The gradient in the cross-section, Θ_A is defined by,

$$\Theta_A = \Delta A / A_0 \quad (6.1)$$

where ΔA is the difference of the cross-section area at the two ends of the foam and A_0 is the average cross-section of the specimen. Two different gradients of foam block were used in the experiment, shown in Table 6.1. It is defined that $\Theta_A > 0$ when the proximal end has the largest cross-section; i.e. the cross-section increases from the proximal end along the rod; and $\Theta_A < 0$ when the end with smallest cross-section is at the proximal end.

Table 6.1 Dimension of the graded aluminum block

Gradient	Dimensions (mm)			
	a	b	b'	l_0
0.5	35.56	71.12	10.13	127.00
1.5	35.56	50.80	30.37	127.00

Dynamic crushing processes of these graded cellular blocks were measured from the apparatus shown in Figure 6.3. It consists of a horizontal base plate with two guide plates attached to it. The initial velocity of the rigid mass was obtained by the impact of the bullet from the gas gun. After the impact, the rigid mass moved along the guide plate and impinged on the specimen. High speed camera was used capture the crushing process of the cellular specimen. The loading frame of the high speed camera is chosen as 36000 frames per second. By analyzing the figures of the two instants during the impact, the velocity of the impinging mass can be obtained.

Figure 6.3 Set up of the high speed impact of the graded foam block. A high speed camera is used to record the crushing process.

In this experiment, three different impinging mass were used, which are shown in Figure 6.4. The weights of these masses are 350g, 300g and 260g, respectively. A summary of the graded foam block tested are listed in Table 6.2, including the specimen dimensions, impact velocities, relative density, imping mass, gradient, locking strain and plateau stress. The initial velocities are varying from 21m/s to 40m/s.

Figure 6.4 Three different impinging mass

Table 6.2 Summary of the impact tests

Spec. NO.	Dimension (mm)				Spec. Mass (g)	Rig. mass G (kg)	V_0 (m/s)	ρ_0 (g/cm ³)	Θ_A	Locking strain ε_d	Plateau stress(MPa) σ_0
	a	b	b'	l_0							
1	35.75	70.25	10.15	127.20	41.68	0.35	21.60	0.23(8.4%)	-1.5	0.50	0.62
2	35.65	50.75	30.85	127.20	63.38	0.35	25.20	0.34(12.7%)	-0.49	0.24	1.15
3	35.60	70.5	10.65	127.20	40.62	0.35	25.20	0.22(8.1%)	-1.48	0.52	0.59
4	35.70	70.75	11.30	127.20	39.17	0.35	25.20	0.21(7.8%)	-1.45	0.54	0.55
5	35.65	70.75	10.75	127.20	40.29	0.35	28.80	0.22(8.1%)	-1.47	0.52	0.59
6	35.60	50.85	30.50	127.25	52.36	0.30	32.40	0.28(10.5%)	-0.50	0.38	0.87
7	35.60	50.70	30.70	127.20	61.12	0.30	32.40	0.33(12.3%)	0.49	0.27	1.10
8	35.30	50.70	30.55	127.20	63.69	0.26	39.60	0.35(13.0%)	-0.50	0.23	1.19
9	35.20	70.00	11.50	127.20	45.54	0.26	39.60	0.25(9.2%)	-1.44	0.45	0.71
10	35.75	70.25	11.25	127.10	40.24	0.30	28.80	0.22(8.1%)	-1.45	0.52	0.59
11	35.45	50.75	30.55	127.10	54.00	0.26	36.00	0.29(10.9%)	0.50	0.35	0.92
12	35.50	70.25	11.20	127.20	48.86	0.26	36.00	0.27(9.8%)	1.45	0.42	0.78
13	35.50	70.50	11.20	127.20	45.87	0.26	32.40	0.25(9.2%)	-1.45	0.45	0.71
14	35.50	50.75	30.75	127.10	53.37	0.26	32.40	0.29(10.8%)	-0.49	0.36	0.9

6.2.2 Experimental results

The crushing process of the graded foam block was recorded by the high speed camera. The response of the specimen with positive gradient is similar while the response of the specimen with negative gradient is more complex. Hence, we just present a typical deformation process of the case with positive gradient, as shown in Figure 6.5. The crushing process shows that the deformation initiates from the impinging end and then propagates to the distal end, which is the same with the homogenous condition. Only one compaction zone is found adjacent to the proximal end and no obvious deformation is found at the distal end.

Figure 6.5 Crushing process of the graded foam block with $\Theta_A = -0.5$, $V_0 = 36$ m/s and $G = 0.26$ kg

As for the response of the specimen with negative gradient, the deformation process varies due to the different impact velocity. Figure 6.6 shows the crushing process of the graded foam block with $\Theta_A = -0.5$, $V_0 = 25.2$ m/s and $G = 0.35$ kg. The response seems

like the quasi-static behavior. Due to the variation of the cross-section, most deformation area is concentrated at the distal end due to the variation of cross-section. However, focusing on the area at the proximal end, we may find that initial deformation takes place from the proximal end. However, the deformation ceases quickly after the impact and further deformation concentrates at the distal end.

Figure 6.6 Crushing process of the graded foam block with $\Theta_A = -0.5$, $V_0 = 25.2 \text{ m/s}$ and $G = 0.35 \text{ kg}$

Figure 6.7 shows an impact case with a higher velocity but lower impinging mass. Two compaction zones appear in the specimen with negative gradient under impact. Comparing with the one described in Figure 6.6, the deformation zone at the proximal end is much larger. Due to the limitation of the experiment equipment (gas gun), the maximum velocity from the test is 40m/s. As a result, the densification zone at the proximal end is still not very significant. Hence, FE tools are employed to carry out the

numerical simulation to investigate the behavior of the graded specimen under higher impact velocities.

Figure 6.7 Crushing process of the graded foam block with $\Theta_A = -0.5$, $V_0 = 32.4$ m/s and $G = 0.26$ kg

6.3 Numerical simulation

In this section, FE simulations are performed using the explicit finite element code ABAQUS/EXPLICIT. Validation is first implemented with the experiment result for the condition with low velocity impact. Afterwards, simulations with higher velocities are carried out.

6.3.1 Finite element model

Figure 6.8 depicts the FE modeling in ABAQUS/EXPLICIT, including three parts, namely, impinging rigid mass, graded cellular block and the fixed support. The impinging mass is assigned an initial velocity and it can only move along the loading direction; the graded cellular block is stationary initially; and the support plate is fixed by constraining all the degrees.

Figure 6.8 FE modeling in ABAQUS/EXPLICIT

The cellular material is prescribed by the material model, so called “crushable foam”. Material properties (i.e. density, elastic property and plastic hardening property) are obtained from the quasi-static stress-strain curve, shown in Figure 6.1. The dimension of the cellular block, the mass of the impinging mass and the initial velocity are the same with the experiment. Similar to Chapter 5, isotropic hardening property is employed; the compression yield stress ratio is defined as 1.1; and both the elastic and plastic Poisson ratios are set to be zero so that the deformation in the lateral direction is negligible. As for

the mesh sensitivity analysis, three global mesh sizes are chosen, 2mm, 3.6mm and 5mm. Figure 6.9 shows the mesh sensitivity investigation by comparing the velocity time history of the imping plate with three mesh sizes under the condition of $V_0 = 104.6\text{m/s}$, $\Theta_A = -1.5$ and mass ratio $\Gamma=1$. From the sensitivity analysis, it is found that the global size 3.6mm is sufficient enough for the FE simulation.

Figure 6.9 Mesh size sensitivity investigation: comparison of the time history of the velocity of the impinging plate under $V_0 = 104.6\text{m/s}$, $\Theta_A = -1.5$ and $\Gamma=1$.

6.3.2 Validation of the FE simulation

Figure 6.10 and Figure 6.11 show the comparison of the effective plastic strain between the FE simulation and the experiment for two cases, respectively. The blue color represents no deformation while red color means that the material reaches the plastic strain limit (locking strain). However, we should note that the compression does not

proceed in the same way as R-PP-L assumes, that is the deformation does not cease at the lock strain. The grey color represents that the deformation exceeds the locking strain. The deformation processes are similar in the FE simulation and the experiment. The deformation process is a typical DS mode, investigated in the previous chapters. Two plastic deformation zones are found at the proximal and distal ends of the foam. After impact, plastic deformation takes place immediately at the proximal end while no deformation is observed at the distal end. With the increase of the displacement of the impinging mass, the deformation at the proximal end gradually ceases while the plastic deformation begins to concentrate at the distal end until the end of the crushing.

Figure 6.10 Comparison of the deformation process (effective plastic strain) for the foam with 12% relative density and $\Theta = -0.5$, $V_0 = 25.2$ m/s (the displacement of the rigid mass $u = 3.7, 16.6, 27.6, 31.2$ mm)

Figure 6.11 Comparison of the deformation process for the foam with 8.4% relative density and $\Theta = -1.5$, $V_0 = 21.6$ m/s. (the displacement of the rigid mass $u = 3.2, 12.2, 23.5, 31.8$ mm)

From the deformation profile, we may obtain the time history of the velocity. Comparison of the time history of the velocity between FE and experimental results shows good agreement (see Figure 6.12). Hence, we may conclude that the FE model is able to provide reliable results to the dynamic impact on the graded foam block.

Figure 6.12 Comparison of the velocity of the impinging mass

6.3.3 FE simulation with higher impact velocity

In this section, the FE result with higher impact velocities is presented to study the influence of the initial velocity. The employed foam material has a relative density of 8.4% and the impinged mass is the same with the experiment (350g). Two velocities are

employed, 50 and 100m/s. Figure 6.13(a-b) shows the contours of the effective plastic strain distribution of the specimen under higher impact velocities. The corresponding distribution of the compressive strain is shown in Figure 6.13(c-d). The deformation process is similar to the cases with low impact velocity, i.e. plastic deformation initiates from the proximal end; afterwards, the material at the distal end begins to collapse. As predicted, with the increase of the initial velocity, the compaction zone at the proximal end increases and the zone at distal end decreases. Moreover, it is found that the compressive strain is beyond its locking strain ($\varepsilon_d = 0.5$ for 8.4% foam) for case with the initial velocity of 100m/s, indicating that the locking strain may also be related to the impact velocity. Actually, a similar phenomenon has also been reported in the work of Zou et al. (2009) for the in-plane compression of the honeycomb structure.

(a)

(b)

Figure 6.13 (a), (b) Contours of the effective plastic strain in the deformation process of the graded cellular block under the initial velocity of 50m/s and 100m/s. (The displacement of the rigid mass $u = 14.6, 39.8, 58.5, 74.5 \text{ mm}$); (c), (d) The corresponding distribution of the compressive strain for 50 and 100m/s.

Note: δ is the non-dimensional displacement, which is the displacement, u over the original length of the rod, l_0 .

6.4 Analytical modeling

To understand the underlying mechanism of the deformation in the foam block with graded cross-section, analytical analysis is presented in this section. Similar to the previous study, we propose the impact scenario, in which a stationary cellular rod with initial length l_0 , initial density ρ_0 and the cross-section gradient Θ_A is impinged by a rigid mass G with initial impact velocity V_0 . The distal end of the rod is fixed on the support.

Figure 6.14 (a) Double Shock (DS) mode ($\Theta < 0$); (b) Single Shock(SS) mode ($\Theta > 0$)

The material of the cellular rod is assumed to be uniform. The mechanical behavior of the foam is described by R-PP-L material model. Hence, in the densification zone, the densified material has the density ρ_s and the dynamic stress behind the shock front is σ_d .

The cross-section areas are expressed as,

$$\begin{aligned}
\frac{A_{bs}}{A_0} &= 1 + \left(\frac{1}{2} - \frac{1}{1 - \varepsilon_d} \left(\frac{y}{l_0} \right) \right) \Theta_A \\
\frac{A_m}{A_0} &= \frac{1}{2} \left(2 + \left(\frac{x - y}{l_0} \right) \frac{\Theta_A}{1 - \varepsilon_d} \right) \\
\frac{A_f}{A_0} &= \frac{1}{2} \left(2 + \left(\frac{1}{1 - \varepsilon_d} \left(\frac{x}{l_0} \right) - 1 \right) \Theta_A \right) \\
\frac{A_{fs}}{A_0} &= 1 + \left(\frac{1}{1 - \varepsilon_d} \left(\frac{x}{l_0} \right) - \frac{1}{2} \right) \Theta_A
\end{aligned} \tag{6.2}$$

where A_{bs} and A_{fs} are the cross-section areas at the backward and forward shock front, respectively; A_m and A_f are the average cross-section areas in the undeformed zone and forward densified zone, respectively; and A_0 is the average cross-section of the rod. Regardless of the influence of the initial density on the quasi-static plateau stress and the locking strain, the effect of the cross-section gradient is equivalent to the case with the same gradient in the quasi-static plateau stress and the initial density. Due to the disequilibrium of the force at the two ends of the foam block, shock fronts at both the proximal end and the distal end may co-exist from the very beginning when the gradient is negative (see Figure 6.14a); while only a single shock (SS) mode appears in the rod with positive gradient (see Figure 6.14b).

Before the analytical derivation, some assumptions are made: the cross-section remains the same at each material point after densification; no buckling or wrapping occurs in the lateral direction; and the cellular material in the densified zone reaches the locking strain, ε_d .

Similar to the previous studies, non-dimensionalization of parameter is implemented for the analytical derivations. The length parameters are all non-dimensionalized with respect to the rod initial length, l_0 , as,

$$x/l_0 = \xi, \quad u/l_0 = \mu, \quad l/l_0 = \lambda, \quad y/l_0 = \eta \quad (6.3)$$

where x, y are the length of the forward and backward densification zones, respectively; u is the displacement of the impinging mass and l is the length of the undeformed length. Also, the characteristic velocity of the cellular rod c_p , the non-dimensional velocity v and the non-dimensional time τ are defined, as follow,

$$c_p \equiv \sqrt{\sigma_0/\rho_0}, \quad v \equiv V/c_p, \quad \tau \equiv tc_p/l_0 \quad (6.4)$$

The characteristic velocity $c_p = 52.3$ m/s for the 8.4% foam and 57.14m/s for 12% foam.

6.4.1 Negative gradient – Double shock (DS) mode

Figure 6.14a depicts the DS mode with a negative gradient, in which shock fronts initiate from both the proximal and distal end of the foam rod.

From kinetic considerations, the following geometrical equations are easily obtained,

$$\frac{d\mu}{d\tau} = v_1, \quad \frac{d\xi}{d\tau} = \frac{1-\varepsilon_d}{\varepsilon_d}(v_1 - v_2), \quad \frac{d\lambda}{d\tau} = -\frac{1}{\varepsilon_d}v_1, \quad \frac{d\eta}{d\tau} = \frac{1-\varepsilon_d}{\varepsilon_d}v_2 \quad (6.5)$$

Applying the conservation of momentum for the whole system over a small time dt gives,

$$-A_{bs}\sigma_d dt = d\left[(G + \rho_s x A_f)V_1\right] + d(\rho_0 l V_2 A_m) \quad (6.6)$$

where A_{bs} is the cross-section at the backward shock front, A_m is the average cross-section of the undeformed part, A_f is the average cross-section of the forward densified zone and $\sigma_d = \sigma_0 + \rho_0 V^2 / \varepsilon_d$ is the dynamic stress at the compaction front (see ref. Reid and Peng (1997), Tan et al. (2005b)).

Also, at the forward shock front, the conservation of momentum gives,

$$-A_{fs}\sigma_0 dt = d\left[(G + \rho_s x A_f)V_1\right] - \frac{\rho_0}{1 - \varepsilon_d} V_2 A_{fs} dx \quad (6.7)$$

Combining Eqs. (6.6) and (6.7) gives the differential equations for the two velocities, namely velocity of the impinging mass and the undeformed zone. The non-dimensional expressions are shown as follows,

$$\frac{dv_1}{d\tau} = - \frac{\left((v_1 - v_2)^2 / \varepsilon_d + 1\right) \left(1 + \left(\frac{\xi}{1 - \varepsilon_d} - \frac{1}{2}\right) \Theta_A\right)}{\Gamma + \frac{1}{1 - \varepsilon_d} \xi \left(1 + \frac{1}{2} \left(\frac{\xi}{1 - \varepsilon_d} - 1\right) \Theta_A\right)} \quad (6.8)$$

$$\frac{dv_2}{d\tau} = - \frac{\Theta_A}{1 + \frac{1}{2(1 - \varepsilon_d)} (\xi - \eta) \Theta_A} \quad (6.9)$$

where Γ is the mass ratio of the impinging mass G over the mass of the cellular rod. It is seen that $dv_2/d\tau < 0$ when the gradient is positive. Initially, $v_2 = 0$ and the velocity is

impossible to decrease less than zero. When the gradient is negative, $dv_2/d\tau > 0$, indicating that v_2 increases with time, which is reasonable.

Two phases exist during the whole crushing process. In Phase I, both the forward and backward shock fronts appear until the velocity of the impinging mass v_1 decreases to the value of v_2 . Phase II starts immediately after Phase I ends. In Phase II, further deformation occurs at the backward shock front while the forward shock fronts ceases. In Phase II, $v_1 \equiv v_2 = v$. Hence, the governing equation is,

$$\frac{dv}{d\tau} = - \frac{1 + \left(\frac{1}{2} - \frac{\eta}{1 - \varepsilon_d} \right) \Theta_A}{\Gamma + \frac{\xi}{1 - \varepsilon_d} \left(1 + \frac{1}{2} \left(\frac{\xi}{1 - \varepsilon_d} - 1 \right) \Theta_A \right) + \lambda \left(1 + \frac{1}{2(1 - \varepsilon_d)} (\xi - \eta) \Theta_A \right)} \quad (6.10)$$

By using the governing Eqs. (6.8) - (6.10), one typical deformation process is sketched in Figure 6.15. The deformation process is similar to those cases with strength and density gradient, described in Chapter 4 and 5, respectively.

Figure 6.15 (a) History of the velocities versus with time; (b) history of the position versus with time (The physical parameters are $v_0 = 1$, $\Gamma = 1$ and $\varepsilon_d = 0.8$.)

6.4.2 Positive gradient – Single Shock (SS) mode

For the SS mode, only one shock front appears from the loading end. It can be regarded as a degenerated DS mode with the condition that $\nu_2 \equiv 0$. Hence, the governing equation for the SS mode can be obtained from Eq. (6.8) by substituting the condition, $\nu_2 \equiv 0$,

$$\frac{d\nu}{d\tau} = -\left(1 + \frac{\nu^2}{\varepsilon_d}\right) \left[1 + \left(\frac{\xi}{1 - \varepsilon_d} - \frac{1}{2} \right) \Theta_A \right] \bigg/ \left[\Gamma + \frac{1}{1 - \varepsilon_d} \xi \left(1 + \frac{1}{2} \left(\frac{\xi}{1 - \varepsilon_d} - 1 \right) \Theta_A \right) \right] \quad (6.11)$$

Solving the above governing equation gives a closed form equation for the instant velocity of the impinging mass, in terms of the displacement of the impinging mass,

$$\nu = \frac{\sqrt{4\nu_0^2 \Gamma^2 \varepsilon_d^4 - \mu \varepsilon_d (\Theta_A(u - \varepsilon_d) + 2\varepsilon_d)(\Theta_A \mu(u - \varepsilon_d) + 2\varepsilon_d(\mu + 2\Gamma \varepsilon_d))}}{\Theta_A \mu(\mu - \varepsilon_d) + 2\varepsilon_d(\mu + \Gamma \varepsilon_d)} \quad (6.12)$$

6.4.3 Comparison and discussion

6.4.3.1 Comparison with FE results

The distribution of the compressive strain from Figure 6.10 and Figure 6.11 is compared with the FE result first, shown in Figure 6.16. From Figure 6.16, two compaction zones are identified for both the two cases. Little deviation is found from the comparison of the compaction fronts. The main reason is that the material in the densification zone at the FE model is not exact “locked” at the locking strain (e.g. In Figure 6.16a, the densification strain at the distal end is beyond the locking strain $\varepsilon_d = 0.5$ while it is smaller than the locking strain at the proximal end). Overall, the comparison shows the analytical model is able to capture the characteristics of the crushing process.

Figure 6.16 Comparison of the distribution of the compressive strain from Figure 6.10 and Figure 6.11

Also, a comparison of the velocity versus displacement between the FE and analytical results is shown in Figure 6.17. Four more cases (See Figure 6.17) are considered. Before the deformation reaches the locking strain ε_d , good agreement is achieved. Afterwards, the deformation continues growing while no prediction is given by the analytical model. Also, the reason is ideally locking densification in the analytical modeling. Similar aspects were reported in the former studies (Zou et al. 2009, Karagiozova et al. 2012). In spite of this defect, the analytical model gives good understanding of the deformation process of the graded cellular rod under impact.

Figure 6.17 Comparison of the velocity versus displacement of the impinging mass

6.4.3.2 Energy absorption capacity

By considering a critical case, in which the initial kinetic energy is exhausted just when the cellular rod is fully crushed, which is $v = 0, \mu = \varepsilon_d$, the densification velocity, v_d can be determined. For the SS mode, the densification velocity is obtained from Eq. (6.12),

$$v_d = \sqrt{(1 + 2\Gamma)\varepsilon_d} / \Gamma \quad (6.13)$$

Eq. (6.13) shows that the densification velocity is determined by the locking strain and the mass ratio, regardless of the influence of the gradient. Consequently, the energy ratio between the absorbed energy under dynamic and the quasi-static condition is given by,

$$\psi = E_d/E_q = \Gamma v_d^2/(2\varepsilon_d) = 1 + 1/(2\Gamma) \quad (6.14)$$

Eq. (6.14) reveals that the energy absorption capacity of the graded cellular rod in the SS mode is only related to the mass ratio. The above conclusions are the same with the homogenous cellular rod (Harrigan et al. 2010).

For the DS mode, the finite difference method is applied to Eqs. (6.8)-(6.10) to obtain the densification velocity. In this analytical analysis, the locking strain is defined as a constant, 0.5. Figure 6.18 shows the densification velocity and the ratio of the specific energy. As predicted, the densification velocity and the specific energy keep as constant when the gradient is positive, showing independence of the energy absorption capacity on the gradient; they decrease with the increase of the magnitude of the gradient when it is negative, indicating that the negative gradient weakens the energy absorption.

Similar to the SS mode, only little influence of the locking strain on the energy ratio is found in the DS mode (see Figure 6.19).

Figure 6.18 (a) Densification velocity versus with gradient; (b) energy ratio versus with gradient

Figure 6.19 The influence of the locking strain on the energy ratio in the DS mode

6.4.3.3 Reaction force at the distal end

To protect the structure from the impact or blast loading, the transmitted force of the cellular rod at the distal end is also considered.

For the SS mode, the force at the distal end is,

$$F_{dis} = \sigma_0 A_{fs} \quad (6.15)$$

where σ_0 is the quasi-static plateau stress and A_{fs} is the cross-section area at the shock front (see Eq. (6.2)). The non-dimensional force f_{dis} (i.e. $F_{dis}/\sigma_0 A_0$) is expressed as follows,

$$f_{dis} = 1 + \left(\mu - \frac{1}{2} \right) \Theta_A \quad (6.16)$$

From Eq. (6.16), it is noted that the distal end force in the SS mode only relates to the gradient, Θ_A , regardless of the initial velocity, impinging mass and locking strain. For a homogenous cellular rod, the force at the distal end is a constant, which only relates to its quasi-static plateau stress and its cross-section area.

For the DS mode, the force at the distal end is,

$$F_{dis} = \sigma_b^d A_{bs} \quad (6.17)$$

where σ_b^d and A_{bs} are the dynamic stress and A_{bs} is the cross-section area at the backward shock front, respectively. Hence, the non-dimensional force f_{dis} is expressed as follows,

$$f_{dis} = \left(1 + \frac{\nu_2^2}{\varepsilon_d} \right) \left(1 + \left(\frac{1}{2} - \frac{\eta}{1 - \varepsilon_d} \right) \Theta_A \right) \quad (6.18)$$

By using Eqs. (6.16) and (6.18), we may plot the non-dimensional curve of the force versus the displacement, μ . Figure 6.20a shows a typical relationship of the distal force and the displacement of the impinging mass under varies velocities. Figure 6.20b shows the corresponding velocity (ν_2) versus the displacement. The locking strain is defined as 0.5 and the mass ratio is 2. For the SS mode, the force curve is the same regardless of the variation of the impact velocity. The only difference is the densification length. Hence, we only plot the curve in a fully crushing case. In the beginning, the force at the distal end is the same regardless of the impact velocity and no matter if the gradient is positive

or negative. Afterwards, the distal force decreases with the increase of the impact velocity in the DS mode.

For the partially densified cases (i.e. $\nu_0 \leq \nu_d$), it is noted that the distal force in the DS mode is always lower than the one in the SS mode in the early stage, indicating that the protected structure may benefit from the less transmitted force. We may find the reason from Eq. (6.18). It states that the force at distal end in the DS mode is determined by two parameters, namely the velocity, ν_2 and the backward densification length, η . Figure 6.20b shows that the velocity increases more rapidly with a lower impact velocity. On the other hand, under a lower impact velocity, more deformation occurs at the distal end.

In the fully densified case (i.e. $\nu_0 > \nu_d$), the distal force is larger than the positive case at the final stage (e.g. $\nu_0 = 1.5$). For the case when the initial velocity $\nu_0 = 2$, the distal force is smaller than the other cases at the final stage but it rapidly increases to a higher value when the deformation reaches the locking strain. In such a case, the two shock fronts collide with each other not until ν_1 decreases to ν_2 . At the instant when the material is just densified ($\varepsilon = \varepsilon_d$), the force increases rapidly to the dynamic stress in the forward shock front. It should be noted in these fully crushing cases, the cellular material loses its ability to absorb the energy and protect the structure.

Figure 6.20 (a) Non-dimensional distal force versus displacement under varies impact velocity; (b) corresponding velocity v_2 versus displacement ($\Gamma = 2$, $\varepsilon_d = 0.5$, $|\Theta| = 1.5$)

6.5 Conclusion

In this study, the influence of the gradient in the cross-section area on the behavior of the graded cellular rod under impact has been experimentally investigated.

Experimental studies are first implemented by using a gas gun; and the impact velocity ranges from 21-40m/s. Observation of the deformation process identifies that only one single shock occurs when the gradient is positive while two densification zones appear in the case with negative gradient. It is noted from Figure 6.5-Figure 6.7 that the impact plate has been buckled before it impinges onto the foam due to the high pressure from the gas gun. This may lead uniform collapse of the foam at the proximal end. Although the comparison between the FE and experiment results shows good agreement, the influence of the buckle of impact plate may not be neglected. To prevent this buckling, new design of the experiment need to be present and this will be conducted in our future work.

Finite element simulations are employed to analysis the case with higher initial velocities. The FE model is validated by the good agreement with the experimental result. The result from the FE simulation with higher velocity shows that the higher velocity is, the larger densification zone appears at the proximal end.

Then, analytical modeling is implemented by employing R-PP-L material model. Purely using the mass and momentum conservation at the shock front gives the governing differential equation. The analytical modeling successfully predicts the existence of the two modes in the graded cellular rod. The investigation into the energy absorption capacity shows that the positive gradient has no influence and the negative gradient weakens the energy absorption capacity. By analyzing the force at the distal end, it is

found that transmitted force is smaller in the case with negative gradient, which may benefit the protected structure. Hence, which gradient is beneficial to a protective structure depends on the loading scenario as well as the design criteria/expectation.

Chapter 7 Conclusions and future work

7.1 Conclusions

This thesis focuses on the investigation into the dynamic behavior of the graded cellular material under impact from three aspects, namely the numerical simulation (graded honeycomb structure), analytical modeling (graded foam rod) and the experimental study (graded foam block). A summary of current research work is presented and future work is suggested in this chapter.

7.1.1 Finite element study - honeycomb structure

At the beginning, the investigation was first made into the dynamic behavior of honeycomb structures, because the honeycomb structure has regular cells and is easy to model in the finite element software (ABAQUS/EXPLICIT). An artificial case was proposed in which the yielding stress of the parent material of the cell varies layer by layer, introducing a gradient in the quasi-static plateau stress while other parameters (i.e. locking strain and initial density) are not affected.

Constant velocities in the range of 5-100m/s were applied to a rigid impinging plate. Basic deformation modes were identified by observation of the crushing process of the honeycomb structure. Critical velocities at which the deformation mode changes were summarized in terms of the ratio of the thickness, gradient and the quasi-static plateau stress. The analysis of the dynamic stress indicated that the conventional shock theory

may not be applicable to the case with a negative gradient, since two densification zones were found there.

Next, the influence of stress gradient on the energy absorption capacity of the structure was investigated by applying an initial velocity to the rigid impinged plate. Densification velocity, under which the structure is just fully crushed, was obtained. The FE simulation showed that the positive gradient enhanced the energy absorption capacity while the negative gradient weakened the capacity.

7.1.2 Analytical investigation - gradient in quasi-static plateau stress

Given that the honeycomb structure is two-dimensional, the Poisson ratio effect and the boundary effect are inevitable in the problem. One-dimensional chains with hexagonal and circular cells were employed to prevent the above effect. The result from the FE simulation showed that only one shock front appeared when the weakest cell was placed at the proximal end while two shock fronts initiated from both ends when the strongest cell was placed at the proximal end.

Based on these deformation modes, analytical modeling was presented. Two deformation modes were proposed, namely the Double Shock (DS) mode and the Single Shock (SS) mode. The behavior of the cellular material was described by using the R-PP-L material model. Two impact scenarios were studied. First, a stationary cellular rod with stress gradient was impinged by a rigid mass with an initial velocity; and in the second one, the cellular rod together with an attached mass impinged onto a rigid target with an initial velocity. By using the conservation of the mass and momentum at the shock fronts,

differential governing equations were obtained. Both deformation modes were identified in the two impact scenarios. Compared with the FE result, good agreement was achieved for both the hexagonal and circular chain, validating the analytical modeling. The energy absorption capacity was analyzed by considering the densification velocity. The analytical result showed that the same conclusion with the study of the honeycomb structure; i.e. the negative gradient weakens the capacity of the energy absorption while the positive gradient enhanced the capacity.

7.1.3 Further analytical study - gradient in initial density

In this analysis, not only the initial density itself but also the quasi-static plateau stress and locking strain varied in the cellular rod, since these two parameters are related to the initial density. The deformation processes were similar to the case with stress gradient; i.e. the DS mode in the rod with negative gradient and the SS mode in the rod with positive gradient. Employing the concept of the deformation zone (i.e. original zone of the densified part) and the simple shock theory gave the general solution for the two modes.

The analytical study revealed the essential reason of the DS mode, which was the difference of the quasi-static plateaus stress between the proximal and distal end. Next, dynamic stresses at the proximal and distal ends were analyzed, showing that the transmitted stress at the distal end is smaller when the gradient is negative, which may benefit the structure to be protected. However, the investigation into the energy absorption capacity shows that the negative gradient weakens the capacity. Compared with the investigation into the stress gradient, more attention was paid on decoupling

effect of the density gradient. The analytical result showed that the DS mode was largely determined by the variation of the quasi-static plateau stress, while both the quasi-static plateau stress effect and dynamic enhancement effect were shown in the SS mode.

Numerical simulations were carried out by using ABAQUS/EXPLICIT. The material property was obtained from the stress-strain relationship of the practical aluminum foam (Alporas). Compared with the FE result, similar trends of the curves showed that current analytical modeling could capture the basic characteristics of the response of the graded cellular material under impact. However, due to the simplification of modeling of the cellular material (R-PP-L material model), deviation took place on the final length of the densified region. To give a better prediction, it is necessary to involve hardening property at the densification stage, instead of simple locking.

7.1.4 Experimental study - gradient in cross-sectional area

Experimental study was carried out with the impact of a foam block with graded cross-sectional area. Gas gun was employed to obtain initial velocity for a steel block and a high speed camera was used to record the crushing process. The experimental result confirmed the existence of the two basic deformation modes (i.e. the DS and SS modes) in the graded cellular material under impact.

Numerical simulations were used to extend the study to the impact condition with higher impact velocities. Comparison between the numerical and experimental result showed good agreement, validating the FE modeling.

Simple analytical modeling employing the R-PP-L material model was presented. Similar to the previous study, basic features of the response in the graded cellular material were captured and deviation appeared at the densification stage. Efforts were put on the influence of the gradient on the energy absorption capacity and the transmitted force at the distal end. Similar conclusion was obtained: negative gradient weakened the energy absorption capacity but minimized the transmitted force on the protect structure.

7.2 Summary of contributions

This study is concerned with the response of the graded cellular material under impact loadings. Previous study focused on the study of the homogenous cellular material and little research work was related to current research. Main contributions as a result of this research are summarized, as follows,

- In the study of the graded honeycomb structure, previous study mainly focused on the elastic property and the energy absorption capacity, neglecting the influence of the impact velocity and the deformation process. In the current study, basic deformation modes were identified; empirical equations were summarized for the critical velocities, in terms of the property of the graded honeycomb.
- No previous study was found in analytically investigating the behavior of the graded cellular material under impact. The author proposed an analytical model to predict the crushing process of the graded cellular rod. With this model, three gradients, namely stress gradient, density gradient and the cross-section gradients were investigated. Both the DS and SS modes were identified in the cellular rod

with these gradients. Energy absorption capacity and the transmitted force were analyzed. For the density gradient, the effect of the density gradient was decoupled into two parts, namely quasi-static plateau stress and the dynamic enhancement.

- Experimental study on the foam block with varying cross-section was first achieved in this research. The experimental result confirmed basic deformation modes in the graded cellular material.

7.3 Future work

Future study will focus on the following issues:

- The analytical model employs the R-PP-L material model, involving two material properties, which are the quasi-static plateau stress and the locking strain. The analytical model neglects the elastic property and employed a locking property at the densification stage. Furthermore, no plastic hardening property is taken into account. To give a better prediction, more efforts need to be put on involving these properties mentioned above into the analytical model.
- Due the limitation of the experiment equipment, the highest velocity of the mass block is 40m/s, which is not sufficient high to study the response of the graded cellular material. Hence, to achieve a higher impact velocity, the design of the experiment is necessary to be refined.
- More experimental studies are needed. In this research, the experiment is the study of the foam block with a gradient in the cross-section area. Experiments on

the investigation into gradients in the quasi-static plateau stress and initial density should be designed and carried out.

- Future work will investigate the application of the graded foam in engineering structures. For example, the graded cellular material can be filled as a foam core in the thin-walled structure like the traditional homogenous cellular material. Analysis and experimental work needs to conduct to capture the response and the energy absorption capacity of such structures.

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List of Publications

1. **Shen, C. J.**, G. Lu and T. X. Yu (2013). "Dynamic behavior of graded honeycombs - A finite element study." *Composite Structures* 98: 282-293.
2. **Shen, C. J.**, T. X. Yu and G. Lu (2013). "Double shock mode in graded cellular rod under impact." *International Journal of Solids and Structures* 50(1): 217-233.
3. **Shen, C. J.**, G. Lu and T. X. Yu. "Investigation into the Behavior of a Graded Cellular Rod under Impact." Accepted by *International Journal of Impact Engineering*.
4. **Shen, C. J.**, G. Lu and T. X. Yu. "Dynamic Response of a Cellular Block with Varying Cross-Section." Submitted to *International Journal of Impact Engineering*.
5. **Shen, C. J.**, G. Lu and T. X. Yu (2012). "Cellular Rod with Varying Cross-Section under Impact." *Key Engineering Materials* 535-536, 94
6. **Shen, C. J.**, G. Lu and T. X. Yu (2012). "Impact deformation of functionally graded cellular solids." Presented in ICTAM, Beijing, China.
7. **Shen, C. J.**, G. Lu and T. X. Yu (2013). "Response of a Cellular Rod with Varying Cross-Section under Impact." IUTAM Symposium, Xi'an, China.