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**DEVELOPMENT OF NUMERICAL MANIFOLD
METHOD FOR CRACKING PROCESS IN ROCK**

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**SCHOOL OF CIVIL AND ENVIRONMENTAL
ENGINEERING**

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DEVELOPMENT OF NUMERICAL MANIFOLD METHOD FOR CRACKING PROCESS IN ROCK

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SUMMARY

In this thesis, the numerical manifold method (NMM) has been extended for cracking problems, such as crack initiation, crack propagation, crack coalescence and structure safety problems associated with cracks.

Different failure criteria have been employed to predict the crack initiation and propagation. A crack representative strategy has been adopted to capture the discontinuity across the crack. Crack evolution techniques have been implemented to treat the manifold elements, the physical covers and the loops during the fracturing process.

Based on the partition of unity method, The NMM has been coupled with the fracture mechanics to simulate the linear-elastic fracture problems. Asymptotic crack tip functions extracted from the analytical solution have been incorporated into the local approximation spaces for the singular physical covers, which can overcome the limitations the conventional NMM suffers. Combining with the contact techniques inherited from discontinuous deformation analysis (DDA), the frictional crack propagation problems have been investigated.

By incorporating a new way to treat the material interface, the influences of the inclusion stiffness on the cracking behavior of specimen containing inclusions have been discussed. By incorporating an elastic-plastic constitutive model, the effects of the plasticity on the cracking behavior have been investigated. By incorporating a visco-elastic constitutive model, the effects of the loading rate on the cracking behavior have also been discussed.

The developed program has been applied to simulate the progressive failure of rock slopes. To overcome the limitation of the conventional NMM, which sometimes improperly removes the interface cohesion, the displacement-dependent cohesion removal method has been adopted. Numerical results indicate that the developed NMM is able to capture the entire processes of the progressive slide surface development related to crack initiation, propagation, coalescence and degradation to eventual catastrophic failure.

The present study showed that the NMM is promising for failure analysis in rock engineering and deserves to be further improved for more complex applications in the future.

NOTE

Chapter 5 includes part of content that has been published in the papers “Wu, Z.J., Wong, L.N.Y., 2012. Frictional crack initiation and propagation analysis using the numerical manifold method. **Computer and Geotechnics**. 39, 38-53.” and “Wu, Z.J., Wong, L.N.Y. available online. Modeling cracking behavior of rock mass containing inclusions using the enriched numerical manifold method. **Engineering Geology**. 162, 1-13”.

Chapter 6 includes part of content that has been published in the paper “Wu, Z.J., Wong, L.N.Y. 2013. Elastic-plastic cracking analysis for brittle-ductile rocks using manifold method. **International Journal of Fracture**. 180(1), 71-91.”

Chapter 7 includes part of content that has been published in the paper “Wu, Z.J., Wong, L.N.Y., Fan, L.F. available online. Dynamic study on fracture problems in visco-elastic sedimentary rocks using the numerical manifold method. **Rock Mechanics and Rock Engineering**. 46 (6), 1415-1427.”

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List of notations and symbols (For chapters 5, 6, 7 and 8)

$u(x,y)$ $v(x,y)$	Displacements of x , y directions	K	Total stiffness
E	Young's modulus	ν	Poisson's ratio
G	Shear modulus	G_f	Strain energy release rate
J	J-integral	c_i ϕ_i	Cohesion and friction angle of the intact material
c_j ϕ_j	Cohesion and friction angle of the joint material	μ	Friction coefficient
t f	Normal and shear forces between fracture surfaces	k_n k_s	Normal and tangential contact spring stiffness
c_y ϕ_y	Cohesion and friction angle for determining yield surface	c' ϕ'	Cohesion and friction angle for determining elasto-plastic cracking
c^* σ^*	Critical cohesion and tensile strength	σ_t	Tensile strength of material
ε^{ps}	Plastic strain	$D(\varepsilon^{ps})$	Damage function
γ_0	Plastic shear strain	ε^*	Limiting value of plastic shear strain
λ	Empirical parameter or wave length	h	Hardening parameter
F^s F^t	Drucker-Prager yield criterion and tensile failure criterion	I_1 J_2	First invariant of the stress state and the second invariant of the deviatoric stress tensor
ψ	Dilation angle of the material	g^s	Shear potential function

λ_s λ_t	Proportional constants	μ	Friction coefficient
ω	Basic frequency of the applied load	$E_v(\omega)$	Frequency dependent modulus
$\eta_v(\omega)$	Frequency dependent damping ratio	$E(t)$	Relaxation modulus
σ, ε	Stress and strain vectors	τ_l	Retardation time
t	Time	ζ	Integration variable
E_l	Dynamic modulus of Maxwell element	E_∞	Modulus under unrelaxed condition
D	Material elasticity matrix	$\tilde{E}$ γ_n	Variables for the incremental form of the constitutive relation
Δd ΔF	Incremental displacement and nodal force vector	M	Mass matrix
α	Newmark's integration parameter	$\dot{d}$ $\ddot{d}$	Velocity and acceleration vector
$\Delta \tilde{F}$	Equivalent incremental nodal force matrix	$\tilde{K}$	Equivalent stiffness matrix
K_I K_{II}	Stress intensity factor for mode I and mode II	θ	Crack initiation angle
$\varphi(x)$	Weighting function	K_{Ic}	Mode I fracture roughness
q_i^j	Local property around the crack tip	k_i^j	Enriched DOF
$J(t)$	Creep compliance	κ	Kolosov constant

δ	Fracture distance	C_0	Wave propagation velocity
σ_m	Peak pressure of the applied load	C	Continuity of the flaws
σ_θ	Tangential stress	σ_r	Radial stress
$\sum w_2$	Accumulated relative sliding displacement	E	Young's modulus
E_e	Elastic matrix	C_c	Hooke's tensor
β	Newmark's integration parameter or flaw inclination angle		

Note: Symbols used in chapters 2, 3 and 4 are not included in this list. Refer to the symbols definition in the relevant chapters.

CHAPTER 1 INTRODUCTION

1.1 Background and problem statement

Because of the presence of discontinuities which originate from fissures, cleavages, beddings, joints, and faults, natural rock masses are always inherently complex and inhomogeneous. Breakage usually takes place along discontinuities. Therefore, how to model the discontinuities and their growth is highly important for researchers and engineers to quantitatively evaluate the effect of cracks on the stability and life span of the relevant structure with cracks under service condition. For this purpose, a lot of numerical and mechanical pioneers have done plenty of work.

Since the studies carried by Inglis (1913), Griffith (1920) and Irwin (1957), the fracture problems have been extensively studied and led to a vast number of theories and applications. However, because of the complexities of the rock mass and arbitrary feature of the crack types, to obtain an analytical or a closed-form solution for a crack problem is always a very challenging work and to some extent impossible without a large number of assumptions. But with these assumptions, the solutions so obtained often deviate from reality.

The advantages of the numerical methods have alleviated the situation to a certain extent. For cracking problems, the numerical methods employed in the past can be roughly classified into either continuum or discontinuum methods. The most widely used continuum-based numerical methods are the finite element method (FEM), the finite difference method (FDM), the boundary element method (BEM) and various meshless or meshfree methods. Among the above mentioned methods, the FEM is the most widely used one for its flexibility in handling material heterogeneity and anisotropy, complex constitutive models, complex boundary conditions, and dynamic problems.

In contrast, the most widely used discontinuum-based methods are the discrete element method (DEM). The key feature of the DEM is that the problem domain is treated as an assemblage of rigid or deformable blocks, which makes the fracturing process modeling in DEM as easy as breaking the bond between two

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distinct elements (Jing 2003). Typical branches of the DEM family are the distinct element method proposed by Cundall (1971), the generalized discrete element method proposed by Williams et al. (1985), the discontinuous deformation analysis (DDA) proposed by Shi (1988) and the finite-discrete element method concurrently developed by several groups (e.g., Munjiza et al. 1995).

However, cracking problems are problems involving crack initiation either at new places or at the tip of pre-existing flaw, propagation direction and length, crack coalescence etc. These problems are actually a combination of continuum and discontinuum problems, hence termed as combined continuous-discontinuous problems. Combined continuous-discontinuous problems can be better simulated by a combined continuous and discontinuous method.

The numerical manifold method (NMM), originally proposed by Shi (1991, 1992) is such a hybrid method. The NMM combines the widely used FEM and DDA in a uniform framework. It provides a unified bridge for both continuous and discontinuous problems by adopting an additional cover called mathematical cover (MC), which is independent of the physical domain of the problem. The following features make NMM extremely suitable for crack problems.

- (1) In NMM, discontinuity can be introduced in a discrete manner, without remeshing.
- (2) In NMM, the same mesh can be used for various orders of cover displacement functions, while the order of displacement functions is limited by the node number in the finite element method.
- (3) The physical covers are separated by the discontinuities in geometry intersected with the mathematical covers, in such a simple way, the discontinuities across the cracks can be captured without further requirement of defining enriched functions to incorporate unknowns, which makes NMM very suitable for complicated crack problems.
- (4) The discontinuity representation algorithm employed is suitable for any number and any geometrical complexities like intersections, junctions and branching of discontinuities in the element;

- (5) It is possible to use different orders of cover displacement functions together in an element for processing stress concentration areas in the numerical manifold method.
- (6) Using a simplex integration method (Shi 1995), element subdivision for the numerical integration of weak form can be avoided in most of the cracked elements.

However, the NMM, which is still at its infant stage, also suffers from some limitations and needs to be further developed. The conventional NMM adopts the simplex integration method for polynomial approximations. However, polynomials cannot represent the stress singularities around the crack tips well. In addition, although a continuous block and a block with pre-existing discontinuities can be well represented, a crack evolution technique is needed to transform a continuous media to a discontinuous media. Another problem is that the contact algorithm the conventional NMM inherits from the DDA sometime improperly removes the interface cohesion. Furthermore, the elastic-plastic and viscoelastic constitutive models are not considered in the conventional NMM. This study aims to extend the conventional NMM to solve the aforementioned limitations.

1.2 Objectives of the work

The main purpose of the study is to develop the NMM as a tool for cracking problems, such as crack initiation, crack propagation, crack coalescence and crack related structure safety problems associated with cracks. More specifically, the studies are:

1. To develop a more advanced method for analyzing cracking problems. Comparing with other numerical methods, the advantages of NMM in dealing with crack problems will be illustrated.
2. To model mixed mode cracking problems in rocks. By incorporating different crack initiation criteria, such as stress-based criterion and energy-based criterion, the efficiency and accuracy of NMM used for predicting crack initiation problems will be investigated.
3. By incorporating the cohesive law for the cracked elements, the effect of the fracture process zone will be investigated.

4. To incorporate the conventional NMM with asymptotic crack-tip functions to overcome the limitations that the conventional NMM suffers, such as the singularities cannot be well captured and crack tip only be allowed to stop at the side of element which is caused by adopting polynomial local approximation spaces.
5. Based on the contact techniques inherited from DDA, the frictional crack propagation problems will be investigated.
6. By incorporating the Mohr-Coulomb crack initiation criterion into NMM, the cracking mechanisms observed in experimental tests will be investigated, especially the secondary developed cracks.
7. By a parametric analysis, the dependence of the crack growth will be investigated.
8. By incorporating a new way to treat the material interface, the influences of the inclusion stiffness on the cracking behavior are investigated.
9. By incorporating an elastic-plastic constitutive model, the effects of the plasticity on the cracking behavior are investigated.
10. By incorporating a viscoelastic constitutive model, the effects of the loading rate on the cracking behavior are investigated.
11. By incorporating the displacement-dependent cohesion removal method, the limitation of the conventional NMM which sometimes improperly removes the interface cohesion, is overcome.

1.3 Outline of the research work

The thesis contains seven chapters. The current chapter presents the research background and the objectives of this research work.

Chapter 2 presents an extensive review of the literature on numerical methods used for failure analysis of the rock mass. The review consists of two parts: the first part focuses on the popular developed methods used for continuous-discontinuous problems and their limitations; the second part reviews the recent developments of the NMM. The advantageous features of the NMM compared with other numerical methods are emphasized. The limitations of the conventional NMM are also briefly discussed.

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Chapter 3 introduces the basic concepts of the NMM, including construction of mathematical covers and their associated partition of unity functions, generation of physical covers and manifold elements, formulations of discrete system of equations, integration scheme, contact detection and modeling. Two examples are simulated to demonstrate the capability of the NMM in accurately describing the stress/displacement field within a continuous body and realistically modeling the interactions among discrete blocks.

The extension of the NMM for modelling cracking problems is presented in chapter 4. The calculation of the stress intensity factors, the crack identification, the crack evolution techniques, and the crack initiation and propagation criteria are discussed. The validity of the presented method is calibrated through a typical example under uniaxial loading.

Chapter 5 applies the developed NMM to linear-elastic fracture problems. Through the partition of the unity method, the analytical solutions around the crack tip are directly added to the displacement approximation functions. The effectiveness of such enriched method is illustrated by several examples. Based on the contact technique, the cracking behaviour of closed frictional pre-existing flaws is investigated. In addition, through a new way of treating the material interface, the cracking behaviour of specimens containing inclusions is studied.

Chapter 6 extends the NMM to elastic-plastic cracking analysis in brittle-ductile rocks. By incorporating the modified Mohr-Coulomb crack initiation criterion, the formation of localized deformation bands and failure processes of brittle-ductile materials (coarse and medium marbles) containing pre-existing flaws under various loading conditions are simulated numerically.

Chapter 7 studies the loading rate effect on fracture problems in viscoelastic sedimentary rocks. By incorporating a modified 3-element viscoelastic constitutive model in the NMM, the effects of the loading rates on the cracking behavior of a sedimentary rock, such as COD (crack open displacement), CSD (crack sliding displacement), crack initiation, crack propagation and final failure mode, are successfully modeled.

Chapter 8 applies the NMM to investigate the progressive failure in rock slopes. Based on the capability of the NMM in dealing with continuous-

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discontinuous problems, the entire processes of the progressive slide surface development related to crack initiation, propagation, coalescence and degradation to eventual catastrophic failure are successfully captured. To overcome the limitation of the original NMM associated with an improper removal of the interface cohesion of the discontinuities, the displacement-dependent cohesion removal method is adopted.

Chapter 9 summarizes the main findings from this research work, with recommendations for future studies.

CHAPTER 2 LITERATURE REVIEW

2.1 INTRODUCTION

Rock mass is always dissected by joints, faults, cracks or other discontinuities which control the mechanical behavior such as deformation and failure of the rock mass. Closed-form solutions do not exist for such complicated geometries and numerical methods must be used for solving practical problems. Due to the difference in underlying material assumptions, different numerical methods have been developed over last four decades for continuous and discrete problems.

For the continuum-based assumption, the most widely used numerical methods are:

- Finite Difference Method (FDM)
- Finite Element Method (FEM)
- Boundary Element Method (BEM)

For the discontinuum-based assumption, the most widely used numerical method is:

- Discrete Element Method (DEM)

For crack initiation and propagation problems, all the above mentioned continuum-based or discontinuum-based only methods more or less have some limitations. In order to overcome these limitations, many derivative methods (XFEM, GFEM etc) which are based on either continuum or discontinuum methods have been developed. Besides, some coupling methods, such as DEM/FEM, DEM/BEM and FEM/BEM, are also developed. NMM is such a method which inherits both the feature of DEM and FEM. In the following, based on the continuum and discontinuum categories, the most popular numerical methods for reproducing rock mass behavior especially the crack process will be reviewed. The review mainly focuses on their advantages and disadvantages in simulating crack behavior, such as initiation, propagation, interaction and coalescence.

2.2 The Finite Difference method (FDM)

2.2.1 Basic concepts

The finite difference method (FDM) was first proposed by Thom and Apelt (1961) in the 1920s under the title “the method of square” to solve nonlinear hydrodynamic equations. Its techniques are based upon the approximations that permit replacing differential equations by finite difference equations. These finite difference approximations are algebraic in form, and the solutions are related to grid points. Thus, a finite difference solution basically involves three steps:

1. Dividing the medium into grids of nodes.
2. Approximating the given differential equation by finite difference equivalence that relates the solutions to grid points.
3. Solving the difference equations subject to the prescribed boundary conditions and/or initial conditions.

The conventional FDM utilizes a regular grid of nodes, such as the grid patterns shown in Fig 2.2.1.

Fig 2.2.1 Common two-dimension grid patterns

2.2.2 Finite volume method

In FDM, the governing PDEs are directly discretized by replacing the partial derivatives by the differences at neighboring grid points. Unlike the finite element method and boundary element method, there are no trial functions or approximation functions employed to approximate the displacement field at the neighborhoods of the grid points. It is therefore the most direct technique to solve the PDEs. However, since the conventional FDM relies on regular grid systems, it will have difficulties in dealing with complex boundary, discontinuity and material inhomogeneity

problems. These make conventional FDM difficult to be applied to practical problems. In order to overcome these shortcomings, significant progress has been made in the FDM so that irregular grid can also be used (Brighi et al. 1998; Perrone and Kao 1975). Among them, the finite volume method (FVM) may be the most significant progress has ever made to extend conventional FDM to more complicated problems. Unlike the regular FDM which uses the pointwise approximations on a grid, the FVM approximates the integral value on a reference volume. With this modification, the FVM has the following advantages over the regular FDM.

1. Applicable to the integral form of the conservative law.
2. Able to handle discontinuities and complex boundary in solutions
3. Natural choice for heterogeneous material as each grid cell can be assigned different material parameters.

Compared with the FEM and BEM, no-matrix-equation-solving is required in FDM (Jing 2003). The integration of non-linear constitutive equations is an easy computer implementation step, rather than the iterative procedure required in FEM and BEM. This feature maybe the most attractive one of FVM.

2.2.3 Crack analyses with FDM/FVM

It is not easy for FDM/FVM to deal with discontinuous problems such as crack initiation and crack propagation since both of them require the functions to be continuous between the grid points. However, with the development of special technique (Kourdey et al. 2001; Marmo and Wilson 2001), it is now possible to use an alternative method which uses material failure or damage propagation at the grid points to simulate the process of crack initiation and crack propagation. One associated limitation of this method is that no crack surfaces are created in the models, so it can not truly simulate crack development phenomenon.

2.3 The finite element method

The concept of the finite element method, as introduced originally by Turner et al. (1956), assumes the real continuum to be divided into elements interconnected only at a finite number of nodal points at which some fictitious forces,

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representative of the distributed stresses actually acting on the element boundaries, were supposed to be introduced. The method was rapidly developed and adopted in many fields as illustrated in the books of Zienkiewicz et al. (1977) and Owen and Hinton (1980).

The finite element method nowadays becomes the most powerful and the best-developed numerical method in engineering science for the following reasons:

1: any shape of continuum including bodies with holes and openings can be simulated without difficulty.

2: a finite element analysis can produce a realistic stress field throughout the continuum.

3: type of material and geometrical representation can be made according to the real behavior of the structure, such as different properties or supporters, and loading conditions.

4: boundary conditions can be dealt with easily.

5: the finite element analysis is now well tested and proved to be a reliable tool for analysis.

The crack analysis using the FEM and its modified methods are described below

2.3.1 The traditional FEM

In order to tackle discontinuous problems (such as cracks and joints) two approaches are usually adopted in the traditional FEA continuum assumptions. The first one is the equivalent continuum model. The other is the joint element or interface element model.

When the joints or cracks are numerous in a particular model, the equivalent continuum model is used to represent the combined behavior of the rock mass and the rock joints. During the last decades, different approaches have been developed for the equivalent model. The analytical solution considering compliance and anisotropy induced by joints was proposed by Gerrard (1982), and Bernard and Amadei (1983). The multi-laminated model proposed by Zienkiewicz and Pande (1977) maybe the most comprehensive model in this category.

On the other hand, the joint or interface element approach was proposed to model problems with fewer joints or cracks. The first use of the joint element for

modeling rock joints can be attributed to the work by Goodman and John (1968). In their work, a linear line element without thickness was employed and is suitable for two-dimensional analysis with four nodes. In 1984, the concept of the thin layer element was developed by Desai and Zaman (1984). The thin layer element is a solid element with a very small thickness and a constitutive law for describing its contact, sliding, separation and rebounding. Katona (1983) developed an FEM interface element which is based on contact mechanics, while adopting the Coulomb friction law to model the states of sticking, slipping, and opening. A similar approach was further discussed by Wang and Yuan (1997). Though the interface element presents a significant improvement over the early joint element models, it still did not take kinematics (large deformation and large movement) into consideration.

Despite these efforts, the treatment of cracks and crack growth still remains a challenging task in the application of conventional FEM. When the number of cracks is large, the global stiffness matrix tends to be singular. When simulating the crack propagation problems, the FEM suffers from the fact that the mesh should conform with the cracks. The mesh size near the cracks should be small. Continuous remeshing is required during crack growth. These shortcomings make FEM very time consuming and less efficient in dealing with crack problems compared with its counterpart BEM.

Different modifications have been developed based on the Partition of Unity Method (PUM) (Babuska 1997) to the conventional FEM in order to overcome these shortcomings. Some typical examples are discussed in following parts.

2.3.2 The embedded FEM (EFEM) and the extended FEM (XFEM)

Belytschko et al (1988) were the first who adopted the local enrichment of the approximation field at an element level for localization problems. By this method, the finite element mesh can be constructed regardless of cracks. Embedded finite element method (EFEM) uses an element enrichment scheme, where the displacement field is approximated by the enriched functions at the element level. For example: $u \approx Nd + N_c d_c$. Then the strain field ε and stress field σ can be obtained by: $\varepsilon \approx Bd + Ge$, $\sigma \approx E\varepsilon$ where N , B are the standard FEM displacement

interpolation and strain interpolation matrices and d is the FEM standard degrees of freedom. N_c and G are the matrices containing enrichment terms for the displacement and strain fields. d_c and e are the enriched degrees of freedoms and unknowns. These unknowns are obtained by imposing traction continuity and compatibility within the element. However, since the EFEM is enriched at element level, the crack path must be continuous. On the other hand, extended finite element method (XFEM) developed by Belytschko and Black (1999), is also a local enrichment scheme but adopts the partition of unity to incorporate an enrichment to the approximation field. In XFEM, in contrast to the element enrichment scheme a node enrichment scheme is adopted. A prominent feature of using the notion of partition of unity in XFEM in particular, or in any partition of unity method in general, is that it automatically enforces the conformity of the global approximation space.

With jump functions and crack tip functions, the XFEM can simulate arbitrary single crack problems. Fig 2.3.1 shows the modeling of a single crack with XFEM, each square node whose element is completely cut by the crack is enriched with the jump function $H(x)$, while each circled node whose element is partially cut by the crack is enriched with the crack tip functions. Then the corresponding displacement approximation can be expressed as:

$$u_{XFEM} = \sum_{i=1}^N N_i u_i + \sum_{j=1}^{n^{FE}} N_j H(x) b_j + \sum_{k=1}^{n^{PE}} N_k \varphi(x) a_k \quad (2-1)$$

Where $H(x)$ is a step function, defined as:

$$H(x) = \begin{cases} 1, & \text{where the node is in the part} \\ -1, & \text{where the node is outside of the part} \end{cases}$$

$\varphi(x)$ is the crack tip enrichment function, usually including $\cos \theta/2$ that can differentiate the crack direction from 0 to 2π ($\cos \pi = -\cos 0$). By this way the XFEM can include the effect of the crack tip which may partially cut the front element. n^{FE} is the node set whose element is fully cut by the crack and n^{PE} is the node set whose element is partially cut by the crack. Then the discontinuity between

the crack interfaces for an element that is fully cut by a crack can be expressed as follows:

$$\begin{aligned}
 \llbracket \mathbf{u} \rrbracket &= u^+ - u^- \\
 &= \sum_{i=1}^N N_i^+ u_i + \sum_{j=1}^{n^{FE}} N_j^+ H(x^+) b_j - \sum_{i=1}^N N_i^- u_i + \sum_{j=1}^{n^{FE}} N_j^- H(x^-) b_j \\
 &= \sum_{j=1}^{n^{FE}} N_j^+ H(x^+) b_j - \sum_{j=1}^{n^{FE}} N_j^- H(x^-) b_j \\
 &= \sum_{j=1}^{n^{FE}} \bar{N}_j b_j
 \end{aligned} \tag{2-2}$$

Where N_j^+ and N_j^- are the shape functions evaluated just to the left and right side of the interface. $\bar{N}_j$ is the difference of the enriched shape functions on two sides of the discontinuity. When we adopt the step function for $H(x)$, $\bar{N}_j$ then can be expressed as:

$$\bar{N}_j = \sum_{j=1}^{n^{FE}} 2N_j(x_0) b_j \tag{2-3}$$

By the above procedure, XFEM can successfully model arbitrary single crack problems (Belytschko and Black 1999). However, since the step function XFEM adopted has two values 1 and -1, it can only describe a single crack discontinuity. In order to model multi-crack problem, additional enrichment functions must be adopted. So when the number of the crack becomes large, the efficiency of the XFEM will become very low. Though the XFEM has been successfully used for modeling numbers of arbitrary moving and intersecting cracks problems (Duax et al. 2000), to model complicated multi-crack problem is still very challenging for XFEM. A typical multi-crack problem with the XFEM is shown in Fig 2.3.2, from which we can easily see that with the increasing number of the cracks, the number of the enrichment functions will also increase. The enriched functions which are needed will become more and more complicated to be defined, which are sometimes even too complicated to be handled.

XFEM has also been extended to three-dimensional crack problems. In Sukumar et al. (2000) the same singular enrichment functions as used in two dimensions problems were used, but they were expressed in terms of polar coordinates of the plane normal to the crack front.

Fig 2.3.1 Modeling an arbitrary crack in a XFEM with jump functions (square nodes) or crack tip functions (circled nodes) enrichment [after (Moes et al. 1999)]

Fig 2.3.2 Modeling multi-crack problem with the XFEM

2.3.3 The generalized finite element method (GFEM)

The generalized finite element method (GFEM) (Babuska et al. 2000; Strouboulis et al. 2000a; Strouboulis et al. 2000b), which augments the finite

element space with analytically or numerically generated solution (or handbook functions) for a given problem, is also based on the PUM. The GFEM is in many ways similar to the “manifold method” except for the treatment of the cracks and discrete blocks. The GFEM also adopts two meshes. One is called approximation mesh which is used for the construction of the GFEM geometry approximation, while another one is called the integration mesh which is used by a sophisticated numerical quadrature algorithm, and can be easily constructed in each element of the approximation. With this feature, the mesh in GFEM can be independent of the geometry of the domain of interest and can be regular regardless of the geometry of the cracks. However, unlike the NMM, which directly cuts the mathematical cover* intersected by cracks into several physical covers and assigns each physical cover independent local functions, the GFEM also includes the nodes around the cracks with enriched functions to describe the cracks like the XFEM. A typical single crack problem by the GFEM is shown in Fig 2.3.3. For a single crack problem, the treatment of GFEM is similar to the XFEM. For adopting two meshes in the GFEM, the GFEM can capture the effect of the crack tip which may partially cut the element without the crack tip enrichment functions.¹

Fig 2.3.3 Representation of an arbitrary crack in a regular non-conformal mesh of GFEM with “enriched” nodes

*The Mathematical cover can be either a mesh of regular pattern or a combination of some arbitrary figures is used for building Physical covers. The physical cover, which includes the boundary of the material, joints, cracks, blocks and interfaces of material zones, is a unique portrait of the physical domain of a problem.

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However, when the number of the cracks becomes very large, especially for the complicated intersection problems, with the two meshes, the GFEM can relatively easily define the discontinuous enrichment functions and then assign them to the related nodes. A typical multi-crack problem example by the GFEM is shown in Fig 2.3.4. Compared with the XFEM shown in Fig 2.3.2, the GFEM first simply divides the domain of interest into four parts intersected by the cracks (Fig 2.3.4). It then assigns each part an independent local enrichment function, which is defined by the location of the part. As shown in Fig 2.3.4, the enrichment function is defined as:

$$H_i = \begin{cases} 1, & \Omega \in R_i \\ 0, & \Omega \notin R_i \end{cases} \quad (2-4)$$

In this way, the GFEM can easily solve the problem of defining the enrichment functions for the multi-intersected crack problems associated with the XFEM. Duarte et al. (2007) used such method successfully to model a branched crack problem.

Though the GFEM can alleviate the difficulty in defining the enrichment functions, compared with NMM, which directly introduces independent local functions to each physical cover, the efficiency of the GFEM is still a problem in dealing with complicated multi-crack problems.

Fig 2.3.4 Modeling multi-crack problem with the GFEM

2.4 The meshless methods

In simulations of failure processes, we need to model the propagation of cracks with arbitrary and complex paths. Such problems are not well easily handled by conventional computational methods because of their reliance on a mesh. The meshless method (Lucy 1977), on the other hand, is a comparably new class of numerical methods that approximates partial differential equations only based on a set of nodes without the need for an additional mesh. Therefore, it can alleviate difficulties related to mesh generation and remeshing associated with conventional methods.

Using the meshless method, for any given set of points, or nodes, there is no need to define elements in determining where the weighting function is zero or non-zero. Since no remeshing is required, the approach is very attractive for moving boundaries and strong discontinuous problems. Based on the moving least squares approximation, the Element-Free Galerkin (EFG) method (Belytschko et al. 1994a; Belytschko et al. 1994b) can predict the simple crack propagation with a high accuracy; similarly the Element-Free Galerkin Particle (EFG-P) method (Rabczuk and Belytschko 2004; Rabczuk and Zi 2007) is able to replicate the crack paths of

more complex crack problems although with a relative lower accuracy. However, the difficulties in enforcing essential boundary conditions and numerical integration, as well as the high computational expense, discourage the meshless method from being widely used.

2.5 The boundary element method

Like the FDM and FEM, the BEM is also a continuum based method which has had a certain success in recent years to solve various types of engineering problems (Beskos 1987, 1997). Compared with FDM and FEM, a remarkable advantage of the BEM is reducing the dimensionality of the problem by one. Thus, a three dimensional problem can be accurately solved by discretizing only two-dimensional surfaces surrounding the domain of interest. In the case where the problem is characterized by an axisymmetric geometry, the BEM further reduces the dimensionality of the problem, requiring just a discretization along a meridional line of the body. Using the same level of discretization, the BEM is often more accurate than FDM and FEM due to its semi-analytical nature and its direct integral formulation. These advantages in conjunction with the absence of the domain continuity requirements make the BEM ideal choice for linear-elastic fracture problems.

However, as the BEM needs to initially seek a weak solution for the problems, it is not as efficient as FEM to deal with material heterogeneity and non-linear material behaviors, for which the weak solutions are unavailable. The BEM is therefore more suitable for solving problems of cracks in linear-elastic and homogeneous bodies.

Due to its semi-analytical properties, the traditional BEM is very efficient to compute fracture parameters: the stress intensity factors (SIFs), T-stress (The first regular stress term in the Williams series expansion) and high order terms. Therefore, the BEM is very powerful for solving fracture mechanics problems and has also been applied in incremental analysis of crack-extension problems by Ingraffea et al. (1983). However the traditional BEM still suffers limitation when it is directly used for crack problems since the coincidences of the crack surfaces will give rise to a singular global stiffness matrix.

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Some special techniques have been developed to overcome this difficulty: the green function method (Snyder and Cruse 1975), the multi-domain approach MBEM (Blandford et al. 1981; Sollero and Aliabadi 1993; Wang et al. 1992), the dual boundary element method DBEM (Portela et al. 1992b, a, 1993), the displacement discontinuity method DDM (Crouch 1976).

The Green function method, which eliminates the need for discretization of the crack, is very accurate but limited to simple problems since analytical Green's functions are not available for a majority of problems. Jing (2002; 2003) mentioned that the multi-domain approach MBEM enables modeling arbitrary crack problems. However, it needs to artificially cut the domain into several sub-domains without cracks (Fig 2.5.1 a), which will not only give rise to additional degrees of freedom, but also make the crack path mesh dependent.

Jing (2003) has also reviewed the dual boundary element method DBEM, which incorporates two independent boundary integral equations for two crack surfaces which occupy nearly the same place in the domain (Fig 2.5.1 b); one is the displacement boundary integral equation and the other one is the traction boundary integral equation. Thus, general mixed-mode crack problems can be solved in a single-domain formulation naturally. At the crack tips, special crack tip elements will be adopted, such as Yamada et al. (1979) adopted for capturing the singularity at the crack tips.

The displacement discontinuity method DDM on the other hand does not need to separate the two opposite crack surfaces as done by MBEM and DBEM. By incorporating additional (Jing 2003) fictitious unknowns, the discontinuous displacement Δu_i , DDM can successfully model the crack problems with single crack elements which makes DDM very popular for crack problems (Crough 1976; Sato et al. 2001; Kayupov and Kuriyagawa 1996) (see Fig 2.5.1 c).

Though significant progress has been achieved in the analysis of crack problems with BEM, due to the inherent deficiency in treatment of material non-homogeneity and non-linearity, and the complexities of crack problems, such as microscopic heterogeneity and non-linearity at the crack tip vicinity, to simulate the complex crack problems by BEM is still a challenging task.

Fig 2.5.1 Meshes for fracture analysis with BEM: (a) multi-domain approach (b) DBEM (c) DDM quoted from (Yamada et al. 1979)

2.6 The discrete element methods

Though many geomaterials, like intact rocks, do not behave like granular materials, the Discrete Element Method (DEM) is still widely applied to investigate their overall mechanical behavior, by assuming the whole mass can be approximated as assemblies of discrete elements which are bonded together by different models of cohesive frictional forces.

Although there are different versions of the Discrete Element Methods used in geotechnics, this thesis will consider only two of them (Donze et al. 2009).

The first one is the classical Discrete (or Distinct) Element Method which is first presented by Cundall and Strack (1979). For classical DEM, the interaction forces will be computed by the slight interpenetration between each contact pair. This “Force-Displacement” method is often referred as “smooth-contact” method, which is not appropriate since two discrete elements can penetrate each other in a mathematical sense.

There are other methods which exclude possible interpenetration between discrete elements. These methods are referred as “non-smooth contact” methods, for which there are two main classes of integrators: one is the event-driven integrators also referred as the Event-Driven Method (EDM) (Luding et al. 1996), the other one is time-stepping integrators, also referred as the Contact Dynamic Method (CDM)

(Moreau 1994; Jean 1999). The EDM is very accurate but it only treats one force at a time. Therefore, it is not suitable for problems with many contacts. The CDM, instead of working with the forces itself, works with the integral of the contact forces. Therefore, the CDM can deal with problems with many contacts.

For rigid block systems, the above mentioned methods often consider the discrete elements as rigid blocks in explicit time-marching schemes. However, they are not so efficient for problems when deformations and stresses must be considered. For deformable block systems, based on the difference of the solution strategies for the treatment of block deformability, the DEMs can be categorized as below.

One type is the explicit DEM, whose typical representative is the method created by (Cundall 1980, 1988) with the computer codes UDEC and 3DEC for 2D and 3D problems, respectively. Another type is the implicit DEM whose typical representative is the Discontinuous Deformation Analysis (DDA) approach (Shi and Goodman 1985, 1989) which uses a finite element method to solve for stress and deformation field inside the discrete element. Compared with explicit DEM, DDA permits relatively larger time steps and adopts closed-form integrations for the stiffness matrices of elements.

Crack analysis in DEM

In the DEM, the rock mass is represented as an assemblage of discrete elements that can be cemented together by cohesive frictional forces with a tensile threshold. As mentioned in Donze et al. (2009), the basic idea for DEM to deal with a crack problem is to reproduce the mechanical behavior of a rock mass by simulating the initiation, propagation and interaction of local cracks. These cracks are initiated when the cement strength (either the tensile strength or the shear strength) between discrete elements is exceeded, depending on the local stress conditions. As the local cracks grow, the macroscopic cracks will form when the local cracks coalesce, which can propagate further when new micro-cracks occur at the macro-crack tips (Fig 2.6.1).

Compared with continuous based methods, the DEMs have the following advantages (summarized by Donze et al., 2009):

1. During the crack propagation, no additional effort such as remeshing or application of special elements is required.

2. No complicated mathematical constitutive model with many material constants is needed. The user however needs to predefine the microproperties (particle and cement properties) and determines the macroproperties (elastic and strength parameters) from numerical lab experiments.
3. No assumption of where and how the cracks will appear is needed.

For the above reasons, the DEM has been widely used for crack problems. Yang and Lee (1999) and Jiang et al. (2009b) used UDEC to successfully investigate the failure mechanisms in a rock mass. Potyondy and Cundall (2004) used PFC2D to simulate biaxial and Brazilian tests of Lac du Bonnet granite and showed that the code was very efficient of reproducing rock crack behavior. The DDA approach was also successfully extended by Ning et al. (2011) to predict the failure processes of a rock mass under a blasting load.

Fig 2.6.1 Developed crack in the synthetic material in the Brazilian test by DEM: (a) before peak load (b) immediately after the peak load [from Fakhimi (2004)]

Despite the outstanding advantages in dealing with discontinuous and highly heterogeneous media, there are still some important issues for using the DEM for practical crack problems. Some of them are listed below:

1. As found in modeling by DEM, the fracture toughness depends on the radius of the discrete elements. It implies that when modeling crack problems, the resultant macroscopic fractures obtained are element size dependent. Therefore, the choice of the particle size should correlate the material fracture toughness which is hard to achieve.
2. As discussed by Donze et al (2009), when spherical discrete elements are adopted for representing rock material, an artifact numerical porosity will be introduced, which differs from the real porosity. In addition, the elements are non deformable and have a size scale different from that of real grains. It leads to cross effects with the local constitutive law on the macroscopic behavior.
3. Though there is no complicated constitutive law needed for DEM, there is a need for relationships between local and macroscopic constitutive laws. However, building these relationships seem hopeless by using only data obtained from classical geomechanical tests (like triaxial tests).

2.7 The coupling method

In order to overcome the drawback in each method, many approaches of coupling of two or more than two totally different methods have been developed. Among these coupling methods, the most widely used are the coupling FEM/BEM, DEM/BEM, DEM/FEM.

The coupling FEM/BEM method was first proposed by Zienkiewicz et al. (1977) which inherits the advantages of FEM in dealing with non-linearity and the advantages of BEM in large regions with relatively small boundaries (Nishioka 1999). By removing their respective disadvantages and keeping their advantages, the coupling FEM/BEM is capable of simulating problems involving discrete cracks and inhomogeneities or plastic deformations simultaneously (Kabele et al. 1999).

Lorig et al. (1986) used a hybrid method which incorporated both discrete element and boundary element methods to analyze the behavior of jointed rock. The coupling DEM/BEM uses the DEM to simulate the near-tip behavior in detail, while using the BEM to provide boundary condition for simulating the far-field

rock mass. Later, such technique was implemented into UDEC by Lemos (1987), which is now commercially used in many problems governed by discontinuity.

One of the limitations of DEM for fracture analysis is related to the large number of particles necessary in the discretization, which limits the use of particle systems in large structures. In order to overcome this limitation, a coupling DEM/FEM method which uses the DEM in the discretization of the fracture zone, and FEM in the surrounding areas is proposed by Bazant et al. (1990). Later, such method was further developed (Ariffin et al. 2006; Azevedo and Lemos 2006; Fakhimi 2009) for fracture analysis.

2.8 The numerical manifold method (NMM)

Although the coupling FEM/BEM and DEM/BEM methods enable a continuous-discontinuous analysis, the numerical manifold method (NMM), which is first presented by Shi (1991), represents a major leap in numerical analysis because it combines the widely used FEM and joint or block oriented DDA in a unified form. The core and most innovative feature of the NMM is the adoption of a two cover (mesh) system, on which the nodes and elements are generated. Similar to other PUMs, the finite covering of a problem domain is also the basic construction of the NMM. The MC, which is used for building PCs, can be either a mesh of regular pattern or a combination of some arbitrary figures. However, the whole mesh has to be large enough to cover the whole physical domain. The physical mesh, which includes the boundary of the material, joints, cracks, blocks and interfaces of material zones, is a unique portrait of the physical domain of a problem, and defines the integration fields. The intersection of the MC and the physical mesh, or the common area of the two systems, defines the region of the PCs. A common area of these overlapped PCs or an independent PC corresponds to an element in the NMM. With two covers system and the simplex integration method, the NMM can have meshes independent of the domain geometry, and therefore makes the meshing task very simple and the simulation of the fracturing process without remeshing. After the first appearance of NMM in 1991, many developments and extensions have been accomplished. Most of the relevant publications are included in the series of the proceedings of the international

conference on analysis of discontinuous deformation (ICADD) symposia (Amadei 1999; Kourepinis et al. 2003; Kourepinis et al. 2010a; Kourepinis et al. 2010b; Li et al. 1995; Miki et al. 2010; Ohnishi 1997; Salami and Banks 1996; Su and Xie 2005; Su et al. 2003b, a; Turner et al. 1956). The main developments and extensions will be reviewed in the following sub-sections.

2.8.1 Development of the cover elements

One of the difficulties in the application of NMM is the choice of suitable cover functions and weight functions. A simple and direct way is the use of finite element covers. Then in these covers, the weight functions of the covers are coincident with the shape functions of FEM. Shi (1996) developed the triangular finite element covers while Shyu and Salami (1995) implemented quadrilateral isoparametric elements into the NMM. Generally speaking, the quadrilateral elements are more accurate than the triangular elements. However, for some special problems such as the bending problems, the quadrilateral isoparametric elements have a higher precision and efficiency. Cheng et al. (2002) used Wilson non-conforming elements for a cantilever slab bending problem and showed that it was very efficient and precise. Ohtsubo et al (1997) incorporated different shape finite covers for NMM to control the accuracy. In order to avoid the distortion of mesh when extreme large deformation is encountered (Cheng et al. 2002), the mesh-free manifold method which is based on the partition of unity is developed by Li and Chen (2004). Since practical problems are actually three dimensional, the 2D NMM can only give approximation results. Therefore, based on the finite hexahedron finite covers, Luo et al. (2005) discussed the theory and numerical integration scheme for 3D NMM. Later, Jiang et al. (2009a) developed a 3D NMM based on the tetrahedral finite covers and He and Ma (2010) proposed a 3D block generation algorithm which allows for any arbitrary discrete structure or block system to be analyzed by NMM.

2.8.2 Development of high order cover functions

The original NMM adopts linear cover functions for two-dimensional problems. In order to improve its level of approximation, Chen et al. (1998) derived high-order displacement functions based on triangular meshes and programmed the

second-order NMM codes for verifying its efficiency through a beam bending problem under a central point loading. In order to avoid the cumbersome procedures in deriving the explicit expressions of element matrix for high-order NMM, Su et al. (2003b) proposed an automatic way for programming the high-order numerical manifold method using Mathematica. However, though the foundations for high-order NMM have been laid, there were few attempts to implement this in practice for any arbitrary level of approximation until Kourepinis (2008) developed a novel technique whereby the order of the displacement functions can be increased for a selected number of nodes. Based on such technique, Kourepinis et al. (2010a) successfully modeled the fracturing in concrete with high-order approximation improved locally.

2.8.3 Extension of enforcing essential boundary conditions

Due to the non-interpolating nature of the weight function with respect to nodal displacements, imposing the essential boundary conditions in NMM can be more difficult than it is in other numerical methods. For traditional NMM, the boundary conditions were usually imposed by penalty method (Terada et al. 2003), which can alleviate the difficulties discussed before. However, the penalty method suffers from the fact that the matrix system tends to be ill-conditioned. An alternative way to overcome such limitation is to use the Lagrange multiplier method as done by Terada et al. (2003) and Terada and Kurumatani (2005). In the analysis of problem involving multiple materials, debonding of material interfaces requires a high accuracy of the Lagrange multipliers. The augmented Lagrange multiplier method with the Uzawa update was discussed by Ma et al. (2010). This method is regarded as a combination of penalty method and Lagrange multiplier method.

2.8.4 Extension for modeling evolving discontinuities

In NMM, the partition of unity property of weight functions can be used to introduce arbitrary strong discontinuities, whereby jumps in the displacement field are modeled explicitly, without the requirement for a priori assumptions, without remeshing, and without the use of interface elements. This can easily be achieved by introducing independent discontinuous local functions to each physical cover,

without the need to introduce independent unknowns to the relative nodes. Therefore, the NMM is very suitable for modeling evolving discontinuities.

Tsay et al. (1999) have applied the numerical manifold method and the crack opening displacement method to predict crack propagation successfully. However, though the crack opening displacement method is an easier way to determine the stress intensity factors, it suffers from difficulty in determining the measured points for mixed mode cracks. Chiou et al. (2002) combined the virtual crack extension technique with the NMM to study the mixed mode crack propagation and showed that the virtual crack extension technique can accurately calculate the Mode-I stress intensity factor by the potential energy release rate even with coarse meshes. An (2010) incorporated the domain form of the interaction integral method which had been successfully adopted by Belytschko and Black (1999) in XFEM into the NMM to investigate the mixed mode crack propagation problem and found that the interaction integral method can accurately predict the stress intensity factors at the crack tips.

When the traditional NMM was adopted for discontinuous problems involving crack initiation and crack propagation problems, the crack tips were constrained to stop at the edges of the element, which of course reduces the accuracy of the results, especially when a coarse mesh was adopted. In order to overcome such limitations, Li and Cheng (2005), Gao and Cheng (2010), and Zhu et al. (2011) developed the enriched meshless manifold method by expanding the base functions with singular functions and applied it to two-dimensional crack modeling. The result showed that the enriched manifold method has a higher accuracy. As discussed earlier, the BEM is very efficient in dealing with linear homogeneous problems since it can reduce the dimensionality of the problem by one. The coupling manifold method with BEM is thus very attractive for combining their strengths. Zhang et al. (1999b); Zhang et al. (1999a); Zhang et al. (2003); Zhang et al. (2007); Zhang et al. (2010a) coupled NMM with boundary element method (BEM) and applied it to crack propagation problems.

2.8.5 Extension to large deformation problems

In spite of the suitability of the Lagrangian formulation for large deformation analysis, it often suffers from the highly distorted mesh and sometimes deviates

from the desired numerical accuracy. So for large deformation problems, Eulerian formulation is superior to the Lagrangian method, since the Eulerian mesh is fixed in space and can never be distorted. Su and Xie (2005) incorporated the numerical manifold method with fixed mathematical meshes to solve large displacement problems. In order to track moving boundaries, after each step, the physical covers and manifold elements will be updated by the new boundaries intersecting the fixed mathematical covers. Terada et al. (2007) used both the Lagrangian and the Eulerian mathematical meshes simultaneously for hyperelastic bodies undergoing large deformation. The current configuration is determined with the Lagrangian mesh, while the Eulerian mesh provides the reference configuration in the corresponding time interval. However, due to the nature of the Eulerian mesh for solid dynamics, it is hard to impose the external boundaries and material interfaces in the spatially fixed mesh. To overcome such limitation, Okazawa et al. (2010) developed an Eulerian finite cover method (an alias of NMM) for large deformation solid dynamics by incorporating the approximation strategy of the FCM into the existing Eulerian explicit FEM.

2.8.6 Development of three-dimensional NMM

Practical geotechnical problems are generally three-dimensional (3D) in nature, so there is a need to develop a 3D NMM. However, though there are lots of researchers working on 3D NMM, only preliminary achievements have been reached due to the difficulty in describing the complex geometry and developing a reliable 3D contact model. Terada and Kurumatani (2005) introduced an integrated procedure for a 3D structure analyses with the FCM, which used the 3D-CAD to model the geometry of the structure and formulated the FEM with interface elements for the static equilibrium state of a structure. Luo et al. (2005) explored the theory and numerical integration scheme of 3D NMM for structural analysis. Cheng and Zhang (2008) developed a 3D NMM based on tetrahedron and hexahedron elements and they also developed the corresponding 3D cover contact detection algorithms, which have some fundamental differences from the corresponding 2D analysis. Jiang et al. (2009a) developed a 3D NMM based on the tetrahedral finite element covers and the global equilibrium equations were established by minimizing the total potential energy. He and Ma (2010) and He et al. (2012) also

developed a 3D NMM and proposed a 3D block generation algorithm for any arbitrary discrete structure or block system.

2.9 Significant features of NMM

The NMM is a powerful numerical analysis technique based on the partition of unity concept and has similar ideas with those used in meshless methods (Askes 2000). Furthermore, NMM combines the widely used FEM and joint or block oriented DDA in a unified form. Fig 2.9.1 shows the relationship between NMM and other numerical techniques. The relationship between NNM and DDA can be described as: NMM is a generalized formulation of DDA, whereby blocks are substituted by elements formed by overlapping covers. However, regarding the way the approximation is constructed, NMM elements utilize multiple covers while DDA only utilizes a single cover. Furthermore, NMM does not normally include rotation terms. As illustrated by Fig 2.9.1, NMM not only inherits most of the advanced features from these widely used numerical methods, but also incorporates some special features that other numerical methods lack. The most attractive features of NMM can be summarized as:

1. The NMM employs two sets of meshes for formulating the physical problem: the mathematical cover and the physical cover. The mathematical meshes can be independent of the physical domain, thus a regular mesh can be adopted regardless of the complex geometry, which makes the meshing task very convenient. This feature makes NMM very suitable for crack problems since no remeshing is needed.
2. The physical covers are separated by the discontinuities intersected with the mathematical covers, so that discontinuities can be easily captured in a direct manner, even for very complex multi-crack problems.
3. The Partition of Unity (PU)-based approximation strategy of the NMM is very similar to that of the PU-FEM. The calculation accuracy can be improved by introducing special or high-order local approximation without remeshing.

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4. Higher-order displacement approximation can be implemented by simply choosing higher-order cover functions, instead of introducing more nodes within the element.
5. The level of approximation can be improved globally, locally or even in a single element. The level of approximation can be different which is hard to be implemented for FEM based partition unity method.
6. For any arbitrary level of the approximation, the simplex integration method that NMM adopted makes the integration associated with the discretisation procedure an analytical task rather than a numerical task.
7. Using the simplex integration method, element subdivision for the numerical integration of weak form can be avoided in most of the cracked elements.
8. With the contact technique inherited from DDA, large displacement sliding, shearing and even discrete blocky movement are allowed in the NMM.

Fig 2.9.1 Schematic relationship between NMM and other techniques (Kourepinis 2008)

2.10 Limitations of NMM

Compared with FEM, the NMM is still at its infant stage and needs to be further developed. As discussed by Kourepinis (2008) though the analytical integration scheme NMM is very powerful and gives NMM many advantages over other numerical methods which adopt numerical integration, NMM also suffers limitations associated with using the analytical integration:

1. The analytical integration scheme cannot deal with some special functions that can not be integrated analytically. As a result, some special functions such as the singular functions will be excluded by NMM, which will prevent NMM from modeling discontinuous problems that often results in singular characteristics.

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2. The analytical integration scheme makes NMM hard to deal with nonlinear problems, since the geometry or material nonlinearity often give rise to stress and strain patterns that cannot be integrated analytically.
3. When the number of terms in the approximation function over a cover increases, the cost of analytical integration increases sharply which makes the high-order approximation very costly.

CHAPTER 3 BASIC CONCEPTS OF THE NMM

3.1 Introduction

The numerical manifold method (NMM) is a newly developed numerical method (Shi 1991) that contains and combines FEM, DDA and analytical simplex integration method in one framework. Therefore, it can provide a unified framework for solving problems dealing with continuous media, discontinuous media or the combination of both. The contents in this chapter are mainly from the paper of Manifold method presented by Shi (1995, 1996).

Based upon finite cover systems and their associated partition of unity functions, the basic structure of this general numerical manifold method can be described in Fig 3.1.1.

Fig 3.1.1 Basic structure of the manifold method (modified from Cheng, 2001)

The NMM inherits all the contact-detection techniques and block kinematics from DDA. The contact detection has two constraint conditions: no tension, and no penetration. These two constrains are controlled by two inequities. The block kinematics inherited from DDA enables the NMM to deal with the mechanical response of a block system under general loading and boundary conditions with body movement and large deformation occurring simultaneously.

The most innovative feature of NMM is its incorporation of two layer covers to describe a problem. The first layer used for description is the physical cover, which is used to describe the physical boundaries including all discontinuities. The second layer of the description is the mathematical cover, which is used for calculation. The shape of the mathematical cover can be arbitrary and the size is

selected according to the computational accuracy requirements. Here we adopt the triangular element for the mathematical mesh due to its convenience to implement. Then by these two covers we can obtain manifold elements which later are used to form the global stiffness matrix and force vector.

The basic procedure of forming manifold elements by the mathematical cover and physical cover is shown in Fig 3.1.2. As illustrated in the figure, the MCs first form from the mathematical meshes, such as the rectangular MC M_1 and the circle MC M_2 . From the formed MCs and the physical meshes, the PCs are defined. For example, MC M_1 intersecting with the physical boundary Γ_u and the discontinuity boundary Γ_D forms the PCs P_1^1 , P_1^2 and P_1^3 , while the MC M_2 intersecting with the physical boundary Γ_u forms PCs P_2^1 and P_2^2 . Finally, the NMM elements are created by overlapping these PCs, such as the elements E_3 , E_4 form from the overlapping of PCs P_1^2 and P_2^2 , P_1^3 and P_2^2 , respectively. The left independent areas of PCs then form the other manifold elements, such as element E_1 from PC P_1^2 , E_2 from PC P_1^3 and E_5 from PC P_2^2 .

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Fig 3.1.2 An example to illustrate the basic concepts of the NMM

Unlike the finite element method, the NMM does not employ numerical integration such as Gauss integration method. It adopts the simplex integration method, so no mapping is required to convert an arbitrary shape into a regular pattern. Therefore, the physical boundary of a problem can be of any shape.

In the following part, only the basic knowledge of the NMM used for solving engineering problems is presented. The details can be found at Shi (1995). The open code of the original NMM can be downloaded at <http://www.ddamm.org/tiki-index.php?page=Software>.

3.2 Cover functions and weight functions based on finite element mesh

The NMM does not require a mathematical mesh to coincide with the physical boundary of a problem and the displacement functions are independent of the material boundary. If the material occupies only part of the element, the displacement functions are still the same. Manifold elements are the common part of physical covers, the global displacement functions are taken as a percentage from the displacement functions of these covers. The meaning of weight functions $w_{e(i)}(x, y)$ is weighted average, which is taken from a percentage from each corresponding cover displacement $u_{e(i)}, v_{e(i)}$. The way used to define a weight function is very important for the analysis. Assuming the element E is the common area of covers ($C_i, i=1, 2, \dots, m$), then the displacement function can be expressed as:

$$u(x, y) = \sum_{i=1}^m w_{e(i)}(x, y) u_e(i) \quad (3-1)$$

$$v(x, y) = \sum_{i=1}^m w_i(x, y) v_e(i) \quad (3-2)$$

Where $u_e(i), v_e(i)$ are the displacements of cover i related to element e . $w_i(x, y)$ is the weight function of cover i satisfying the following equations:

$$0 \leq w_i(x, y) \leq 1, \quad (x, y) \in C_i \quad (3-3)$$

$$w_i(x, y) = 0, \quad (x, y) \notin C_i \quad (3-4)$$

$$\sum_{i=1}^m w_i(x, y) v_i(i) \equiv 1; \quad (x, y) \in E \quad (3-5)$$

Currently, the triangular mesh is adopted as the mathematical mesh with a linear displacement field within each triangle. The first-order linear displacement function can be expressed as follows:

$$\begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{pmatrix} 1 & 0 & x & 0 & y & 0 \\ 0 & 1 & 0 & x & 0 & y \end{pmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{Bmatrix} \quad (3-6)$$

This can be easily extended to any order polynomial; for instance the use of second order displacement function will give the following:

$$\begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{pmatrix} 1 & 0 & x & 0 & y & 0 & x^2 & 0 & y^2 & 0 & xy & 0 \\ 0 & 1 & 0 & x & 0 & y & 0 & x^2 & 0 & y^2 & 0 & xy \end{pmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \\ d_1 \\ d_2 \\ e_1 \\ e_2 \\ f_1 \\ f_2 \end{Bmatrix} \quad (3-7)$$

Where $u(x, y)$ and $v(x, y)$ are the x and y displacements, respectively, at a point (x, y) of an element i. the $a_1, b_1, \dots, f_1, f_2$ are coefficients of the displacement function.

Evaluation of $u(x, y)$ at three nodes of a triangular element gives the following equations:

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad (3-8)$$

$$\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad (3-9)$$

Where $x_i, y_i, i=1,2,3$ are nodal coordinates and $u_i, v_i, i=1,2,3$ are nodal displacements. Then the displacement field $u(x, y)$ can be rewritten as follows:

$$u(x, y) = (1 \ x \ y) \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = (N_1 \ N_2 \ N_3) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad (3-10)$$

A similar relationship for the displacement field $v(x, y)$ can also be rewritten as follows:

$$v(x, y) = (1 \ x \ y) \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (N_1 \ N_2 \ N_3) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad (3-11)$$

By introducing the weight functions, the displacement field $u(x, y)$ and $v(x, y)$ for an element can be written as:

$$u(x, y) = \sum_{i=1}^n w_{e(i)}(x, y) u_{e(i)} \quad (3-12)$$

$$v(x, y) = \sum_{i=1}^n w_{e(i)}(x, y) v_{e(i)} \quad (3-13)$$

$$\begin{cases} w_{e(i)}(x, y) \geq 0 & (x, y) \in U_i \\ w_{e(i)}(x, y) = 0 & (x, y) \notin U_i \\ \sum_{(x, y) \in U_i} w_{e(i)}(x, y) = 1 \end{cases} \quad (3-14)$$

Where $w_{e(i)}(x, y)$ is the weighting function, n is the number of related physical covers of the element. Since the manifold method uses physical and mathematical meshes, by combining the weighting function to connect these two meshes, it is easy to deal with discontinuities within its domain. Satisfying Eq. (3-14), there are many ways to construct the weighting function. The simplest way is to adopt the weighting function as the interpolation function used by FEM. The weighting function has only the values of 0 and 1 in which 1 is defined within its corresponding component area, and 0 elsewhere.

From Eq. (3-10) to Eq. (3-13), for a triangular element, the weight function can be written as:

$$(w_{e(1)}(x, y) \quad w_{e(2)}(x, y) \quad w_{e(3)}(x, y)) = (1 \quad x \quad y) \begin{pmatrix} 1 & x_{e(1)} & y_{e(1)} \\ 1 & x_{e(2)} & y_{e(2)} \\ 1 & x_{e(3)} & y_{e(3)} \end{pmatrix}^{-1} \quad (3-15)$$

Assuming:

$$\begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}^T = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}^{-1} \quad (3-16)$$

Then Eq. (3-15) can be rewritten as:

$$\begin{pmatrix} w_{e(1)}(x, y) \\ w_{e(2)}(x, y) \\ w_{e(3)}(x, y) \end{pmatrix} = \begin{pmatrix} f_{11} + f_{12}x + f_{13}y \\ f_{21} + f_{22}x + f_{23}y \\ f_{31} + f_{32}x + f_{33}y \end{pmatrix} \quad (3-17)$$

3.3 Formulation on elements

Similar to the FEM, the stiffness matrix of an element due to the strain energy is derived here to illustrate the basic formulation. The strain energy over a triangular element is derived first, which is then modified by a weighting function to give the strain energy over its physical domain.

The strain–displacement relationship can be expressed as follows:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u(x, y)}{\partial x} \\ \frac{\partial v(x, y)}{\partial y} \\ \frac{\partial u(x, y)}{\partial y} + \frac{\partial v(x, y)}{\partial x} \end{Bmatrix} \quad (3-18)$$

From the equation we have obtained above, the displacement within an element has the following form:

$$\begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{Bmatrix} w_{e(1)} & 0 & w_{e(2)} & 0 & w_{e(3)} & 0 \\ 0 & w_{e(1)} & 0 & w_{e(2)} & 0 & w_{e(3)} \end{Bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad (3-19)$$

The element strain can be related to the nodal displacement as follows:

$$\{\varepsilon\} = [B_{e(i)}(x, y)] \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} \quad (3-20)$$

While the $B(x, y)$ is a linear derivative of $w(x, y)$:

$$[B_{e(i)}(x, y)] = \begin{bmatrix} f_{i2} & 0 \\ 0 & f_{i3} \\ f_{i3} & f_{i2} \end{bmatrix} \quad (3-21)$$

Therefore, the constitutive relationship can be illustrated as follows:

$$\{\sigma\} = [E]\{\varepsilon\} = [E][B] \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} \quad (3-22)$$

Where $[E]$ is the elasticity matrix.

Then the strain energy density Ω_e can be written as follows:

$$\Omega_e = \frac{1}{2} \{\varepsilon\}^T \{\sigma\} = \frac{1}{2} \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix}^T [B(x, y)]^T [E][B(x, y)] \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} \quad (3-23)$$

The total strain energy of an element can be obtained through integration. To account for the physical area of a generalized element, the weighting function is introduced. The final total strain energy of an element can be written as follows:

$$\Pi_e = \iint W(x, y) \Omega_e dA \quad (3-24)$$

Employing the minimum potential energy principle, the stiffness matrix associated with the strain of the element can be found as follows:

$$[K] = \iint_A W(x, y) [B(x, y)]^T [E] [B(x, y)] dA \quad (3-25)$$

In a similar way, other components of the stiffness matrix and the force vectors can be obtained.

3.4 Equilibrium equations

The total potential energy is the summation of all the potential energy sources: individual stresses and forces. In the following, the potential energy of each force or stress and its differentiations will be computed separately by the similar method used in section 3.3.

- (1) The strain potential energy Π_e produces the stiffness matrix.
- (2) The potential energy Π_σ of initial stresses produces the initial stress matrix.
- (3) The potential energy Π_i of inertia produces the mass matrix.
- (4) The potential energy Π_w of body force produces the body load matrix.
- (5) The potential energy Π_p of point loads produces the point load matrix.
- (6) The potential energy Π_U of displacement constraints produces the displacement load matrix.
- (7) The potential energy Π_s of contact springs produces contact matrix.
- (8) The potential energy Π_f of friction forces produces the friction matrix.

For a static problem, then the total potential energy can be described as:

$$\Pi = \frac{1}{2} (D_1^T \ D_2^T \ D_3^T \ \bullet \bullet \bullet \ D_n^T) \begin{pmatrix} K_{11} & K_{12} & K_{13} & \bullet \bullet \bullet & K_{1n} \\ K_{21} & K_{22} & K_{23} & \bullet \bullet \bullet & K_{2n} \\ K_{31} & K_{32} & K_{33} & \bullet \bullet \bullet & K_{3n} \\ \bullet \bullet \bullet & \bullet \bullet \bullet & \bullet \bullet \bullet & \bullet \bullet \bullet & \bullet \bullet \bullet \\ K_{n1} & K_{n2} & K_{n3} & \bullet \bullet \bullet & K_{nn} \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ \bullet \bullet \bullet \\ D_n \end{pmatrix} + \frac{1}{2} (D_1^T \ D_2^T \ D_3^T \ \bullet \bullet \bullet \ D_n^T) \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ \bullet \bullet \bullet \\ F_n \end{pmatrix} + C \quad (3-26)$$

Where D is the displacement matrix; K is the global stiffness matrix; F is the node (cover) force matrix; C is other energy such as water energy which is not considered in this thesis.

Based on the minimum potential energy principle, each matrix obtained previously will be assembled into systems of the following equilibrium equations.

$$\begin{pmatrix} K_{11} & K_{12} & K_{13} & \bullet \bullet \bullet & K_{1n} \\ K_{21} & K_{22} & K_{23} & \bullet \bullet \bullet & K_{2n} \\ K_{31} & K_{32} & K_{33} & \bullet \bullet \bullet & K_{3n} \\ \bullet \bullet \bullet & \bullet \bullet \bullet & \bullet \bullet \bullet & \bullet \bullet \bullet & \bullet \bullet \bullet \\ K_{n1} & K_{n2} & K_{n3} & \bullet \bullet \bullet & K_{nn} \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ \bullet \bullet \bullet \\ D_n \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ \bullet \bullet \bullet \\ F_n \end{pmatrix} \quad (3-27)$$

Similarly, for a dynamic problem, we finally can obtain the discrete equations for a dynamic analysis

$$K\Delta d + M\Delta \ddot{d} = \Delta F \quad (3-28)$$

In the dynamic analysis, the acceleration and the velocity can also be expressed by using the Newmark scheme, expressed as:

$$d_{n+1} = d_n + \Delta t \dot{d}_n + \frac{(\Delta t^2)}{2} ((1-2\beta) \ddot{d}_n + 2\beta \ddot{d}_{n+1}) \quad (3-29)$$

$$\dot{d}_{n+1} = \dot{d}_n + \Delta t ((1-\gamma) \ddot{d}_n + \gamma \ddot{d}_{n+1}) \quad (3-30)$$

Where γ and β are two constants.

Then,

$$\Delta \ddot{d} = \frac{1}{\beta(\Delta t)^2} [\Delta d - \Delta t \dot{d}_n - \frac{(\Delta t)^2}{2} \ddot{d}_n] \quad (3-31)$$

If we use the scheme of the average acceleration, i.e. $\beta = 1/4$ and $\gamma = 1/2$, then Eq. (3-31) becomes

$$\Delta \ddot{d} = \frac{4}{(\Delta t)^2} \Delta d - \frac{4}{\Delta t} \dot{d}_n - 2 \ddot{d}_n \quad (3-32)$$

Substituting Eq. (3-32) into Eq. (3-28), we have

$$\overline{K} \Delta d = \Delta \overline{F} \quad (3-33)$$

Where $\overline{K}$ and $\Delta \overline{F}$ are the equivalent stiffness matrix and the equivalent loading vector. They can be expressed as

$$\overline{K} = K + \frac{4}{(\Delta t)^2} M \quad (3-34)$$

And

$$\Delta \overline{F} = \Delta F + \frac{4}{\Delta t} M \dot{d}_n + 2M \ddot{d}_n \quad (3-35)$$

3.5 Integration method

The NMM adopts the simplex integration which can give accurate solutions over arbitrary n-dimensional generally shaped domains. The integrand can be any n-dimensional polynomials.

Fig 3.5.1 Triangulation of an arbitrary manifold element for simplex integration

The integration over an arbitrary manifold element with ordered vertices $V_1, V_2, \dots, V_n$ (Fig 3.5.1) can be expressed as:

$$\iint_A f(x, y) dx dy \quad (3-36)$$

Where $f(x, y)$ is an arbitrary polynomial function, A represents the area occupied by the vertices $V_1 V_2 V_3 \dots V_n V_{n+1}$.

For the two or higher dimensional ordinary integration, the integration domain has no orientation, therefore the integration has no algebraic addition of oriented domain. It is then necessary to introduce simplex integration on a simplex domain, which is a simple oriented domain.

Two dimensional simplex $V_0 V_1 V_2$ is an oriented triangle, Its volume is

$$V = \frac{1}{2} J = \frac{1}{2} \begin{vmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} \quad (3-37)$$

The volume of simplex $V_1 V_0 V_2$ is the negative volume of simplex $V_0 V_1 V_2$. Then the two dimensional simplex integration is defined as:

$$\int_{V_0 V_1 V_2} f(x, y) D(x, y) = \text{sgn}(J) \iint_{V_0 V_1 V_2} f(x, y) dx dy \quad (3-38)$$

CHAPTER 3 BASIC CONCEPTS OF THE NMM

Integration over a two-dimensional simplex can be obtained by the following two steps. First, by the following coordinate transformation, the two dimensional normal simplex with vertices of $V_0(x_0, y_0)$, $V_1(x_1, y_1)$, $V_2(x_2, y_2)$ can then be converted into a 2D coordinate simplex one

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = J^T \bullet \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_0 + u_1 + u_2 \\ x_0 u_0 + x_1 u_1 + x_2 u_2 \\ y_0 u_0 + y_1 u_1 + y_2 u_2 \end{bmatrix} \quad (3-39)$$

Where (u_1, u_2) are vertices of standard 2D coordinates simplex $\{U_0(0,0), U_1(1,0), U_2(0,1)\}$.

Then

$$\begin{aligned} \int_{V_0 V_1 V_2} x^{m_1} y^{m_2} D(x, y) &= \text{sign}(J) \int \int_{V_0 V_1 V_2} x^{m_1} y^{m_2} dx dy \\ &= \text{sign}(J) \int \int_{U_0 U_1 U_2} x^{m_1} y^{m_2} |J| du_1 du_2 \\ &= J \int \int_{U_0 U_1 U_2} x^{m_1} y^{m_2} du_1 du_2 \end{aligned} \quad (3-40)$$

Second, by the following basic forms of simplex integration:

$$\begin{aligned} S_1 &= \int_0^1 u_2^{i_2} du_2 \int_0^{1-u_2} u_1^{i_1} (1-u_2-u_1)^{i_0} du_1 \\ &= \int_0^1 u_2^{i_2} (1-u_2)^{i_0+i_1+1} du_2 \frac{i_0! i_1!}{(i_0+i_1+1)!} \\ &= \frac{i_0! i_1! i_2!}{(i_0+i_1+2)!} \end{aligned} \quad (3-41)$$

The two dimensional integration of polynomials of degree 0, 1 can be computed as:

$$\int_{V_0 V_1 V_2} 1 D(x, y) = \sum_{k=1}^n J \int \int_{U_0 U_1 U_2} du_1 du_2 = J \frac{0!}{(0+2)!} = \frac{J}{2} \quad (3-42)$$

$$\begin{aligned} \int_{V_0 V_1 V_2} x D(x, y) &= \sum_{k=1}^n J \int \int_{U_0 U_1 U_2} (x_0 u_0 + x_1 u_1 + x_2 u_2) du_1 du_2 \\ &= J \frac{1!}{(1+2)!} (x_0 + x_1 + x_2) = \frac{J}{6} (x_0 + x_1 + x_2) \end{aligned} \quad (3-43)$$

$$\begin{aligned} \int_{V_0 V_1 V_2} y D(x, y) &= \sum_{k=1}^n J \iint_{u_0 u_1 u_2} (y_0 u_0 + y_1 u_1 + y_2 u_2) du_1 du_2 \\ &= J \frac{1!}{(1+2)!} (y_0 + y_1 + y_2) = \frac{J}{6} (y_0 + y_1 + y_2) \end{aligned} \quad (3-44)$$

The simplex integration for higher-order polynomials can be derived in a similar way.

3.6 Contact detection technique

Discontinuous numerical analysis depends heavily on the contacts between blocks. Wrong detection or description may lead to wrong analysis, or breakdown of the computation. The NMM adopts an iterative scheme to account for the large displacement and deformation. The coordinates of the blocks and contacts between blocks can change at each iterative step.

In order to obtain a reliable result, the time steps should be small enough so that the displacements in the whole material body are less than a pre-defined limit value ρ . Therefore, the displacements $(u(x, y), v(x, y))$, the rotations $r(x, y)$, the deformations $(\varepsilon_x, \varepsilon_y, \varepsilon_{xy})$ can be accurately represented by linear functions of the cover unknowns.

Based on a small step time, every contact forms at the beginning of each time step. Each contact is formed with two sides. The two sides of the pair, which are possible to contact, penetrate or enter (very small penetration) from one to another at the end of the time step, are defined as contact pairs. Since practically no penetrations on the two sides are allowed on the contacts, there is only very small entrance for contacts.

There are three types of contacts: angle-to-angle, angle-to-edge and edge-to-edge, are shown in Fig 3.6.1. The edge-to-edge contact can be transferred to the angle-to-edge contact. The criteria for the contacts are:

- (1) For angle-to-angle contact, the minimum distance of two angle vertices of the contacts is less than 2ρ .
- (2) For angle-to-edge contact, the minimum distance of angle vertex to the edge of the contacts is less than 2ρ .

- (3) For angle-to-angle contact, when the angle vertex translate to another angle vertex without rotation, the maximum of the overlapping angle of the two angle is less than 2δ
- (4) For angle-to-edge contact, when the angle vertex translate to the edge without rotation, the maximum of the overlapping angle of the angle and the edge is less than 2δ .

Where ρ and δ are the two constants that should be given by analysts. In this thesis, ρ and δ are set to be $1e-5 \times (\text{length of the model})$ and 1.5° , respectively.

The common sense requires no penetration during contacts. In the computation, an angle-to-angle contact will be transferred to one or two angle-to-edge contacts. As shown in Fig 3.6.1 (c), the edge-to edge contact can be transferred into two angle-to-edge contacts (P_2 -to- P_1P_3 and P_3 -to- P_2P_4). Based on the orientation, the following is used to judge the angle-to-edge contact penetration.

As shown in Fig 3.6.2 P_1 is a point moving to point P_1' after deformation; P_2P_3 is the entrance line and the (x_i, y_i) and (u_i, v_i) are the coordinates and the displacement increments of P_i ($i=1,2,3$). If P_1 , P_2 and P_3 rotate in the same sense as the rotation of ox to oy (Fig 3.6.2), whether P_1' has passed the entrance line P_2P_3 or not can be judged by the inequality:

$$\Delta = \begin{vmatrix} 1 & x_1 + u_1 & y_1 + v_1 \\ 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{vmatrix} \leq 0 \quad (3-45)$$

Fig 3.6.1 Three types of contacts: (a) angle-to -angle (b) angle-to-edge (c) edge-to-edge

Fig 3.6.2 Judging penetration by rotation

For the contacts, the “entrance” lines can be defined. A contact between two convex angles is shown by Fig 3.6.3, where the two bold lines are the entrance lines (Table 3.6.1). Penetration will occur if the two entrance lines are passed by the vertices of the other angle simultaneously.

- (1) For the angle-to-angle contact, if both angles are less than 180° , the two entrance lines are defined by the bold line in Fig 3.6.3.
- (2) For the angle-to-angle contact, if an angle is larger than 180° , the two entrance lines are the two edges of the angle greater than 180° .
- (3) For the angle-to-edge contact, the only entrance line is the edge.
- (4) For the edge-to-edge contact, the two edges are parallel, there is only one entrance line in this case. The entrance line can be any of the two edges.

Table 3.6.1 Definition of two entrance lines when both angles are less than 180°

Associated angles		Two entrance lines	
$\alpha \leq 180^\circ$	$\beta \leq 180^\circ$	OE_3	OE_2
$\alpha \leq 180^\circ$	$\beta > 180^\circ$	OE_3	OE_4
$\alpha > 180^\circ$	$\beta \leq 180^\circ$	OE_1	OE_2
$\alpha > 180^\circ$	$\beta > 180^\circ$	OE_1	OE_4

Fig 3.6.3 Entrance line finding

The computation of manifold method follows time steps. The closed contact points will be inherited to next time step and judged again in the new step. The closed contacts of the previous time step will be transferred to the next time step, if the contacts are found in the same contact position. The entrance lines will be transferred if possible. The angle-to-angle contacts and angle-to-edge contacts have

different contact parameters and different springs (normal and shear springs). The same contact may also have different contact parameters in different contact states.

1. For the springs of closed angle-to-angle contacts:

Normal stiff spring.

2. The springs of closed angle-to-edge contacts in the sliding mode:

Normal stiff spring and a pair of shear sliding forces.

3. The springs of the closed angle-to-edge contacts locked in both normal and shear directions.

Normal stiff spring and shear stiff spring.

4. The closed angle-to-angle contacts have the following contact parameters:

Normal forces and normal displacements.

5. The closed angle-to-edge contacts in the sliding mode have the following contact parameters:

Normal forces; normal displacements; one of the two possible sliding directions; and contact positions on the edges.

6. The closed angle-to-edge contacts locked in both normal and shear directions have the following contact parameters:

Normal forces; normal displacements; shear forces; shear displacements; and contact positions on the edges.

Transferring the contacts to next step, an angle-to-angle contact may be transferred to an angle-to-edge contact. Also an angle-to-edge contact may be transferred to angle-to-angle contact. Upon transferring to different kinds of contacts, some contact parameters may not be needed.

3.7 Calibration and comparison with other methods

Based on the theory as discussed above, the NMM has been coded by using the program C which is based on the previous work done by Shi (1991) and has been applied to several examples below. The first example is designed to verify the capability of the NMM for describing the displacement field and stress field. A comparison against the results obtained from UDEC (which is the most representative explicit DEM method) will be performed. The second example is

designed to check the ability of the NMM in dealing with the frictional contact between two blocks.

3.7.1 Example of a cantilever beam

The beam used for this study is shown in Fig 3.7.1. The dimensions of the beam are $10\text{m} \times 2\text{m}$, The Young's modulus and Poisson's's ratio of the beam are $E=1.5 \times 10^9 \text{ N/m}^2$ and $\nu=0.24$, respectively.

As shown in Fig 3.7.2, the left side is fixed and a vertical loading is applied at the top right corner A. Based on the theory of elasticity, a theoretical result based on Timoshenko beam theory that takes the nonlinear stress distribution along the beam height direction and shear forces effect into consideration can be obtained. However, for a beam with a span-height ratio ≥ 5 , smaller than 1% differences will occur when using the simplified Euler-Bernoulli beam. Therefore, for easy understanding, the theoretical solution of the vertical displacement (neglecting the shear deformation) of point A is given based on Euler-Bernoulli beam theory:

$$V(A) = \frac{fL^3}{3EI} \quad (3-46)$$

Where $f=300 \text{ kN}$, $L=10\text{m}$, $I=2/3\text{m}^4$, $E=1.5 \times 10^9 \text{ N/m}^2$.

Two thousand time steps, with a time step interval $\Delta t = 0.001\text{s}$, are used in the calculations. The displacements of point A obtained from different meshes and the theoretical solution are listed in Table 3.7.1. As illustrated in the table, the displacement predicted by NMM becomes more accurate when the mesh size decreases.

Fig 3.7.1 The cantilever beam

Fig 3.7.2 Models of different meshes

Table 3.7.1 The displacement of point A by different meshes

The type used for calculation	Displacement of point A(mm)
Theoretical result	100
NMM based on Fig 3.7.2 (a)	82.3
NMM based on Fig 3.7.2 (b)	95.6
NMM based on Fig 3.7.2 (c)	101.1

3.7.2 Direct shear test

In order to verify the NMM in dealing with discontinuous problems, a comparison with the most representative explicit Distinct Element Method UDEC, which is famous for its power and efficiency in dealing with discontinuities, will be done. A simple direct shear test model is created and studied by UDEC and NMM

respectively. The model is then used to evaluate the adequacy of the Coulomb slip model to describe the response of a joint subjected to shear loading. The test that consists of a single horizontal joint is first subjected to a vertical normal confining stress, and then to a horizontal unidirectional shear displacement. The results can be applied to the analysis of joint slip around an underground excavation. From the result, we can see that NMM is very capable and convenient in dealing with discontinuous problems. Fig 3.7.3 shows the model for direct shear test.

For both the NMM and UDEC simulations, first, a normal stress of 10 MPa which represents the confining stress acting on the joint is applied. A horizontal velocity is then applied to the top block to produce a relative shear displacement between the two blocks. For demonstration purposes, we only apply a small shear displacement of less than 1 mm to this model. For both simulations, the average normal and shear stresses and normal and shear displacements along the joint are measured.

The rock material is assumed to be elastic without damage with the following parameters: mass density = 2600 kg/m³; Young's Modulus = 7.2 GPa; Poisson's Ratio $\nu = 0.24$. The joint obeys the Coulomb slip criterion with the following parameters: friction angle = 30°; tensile strength = 0 MPa; cohesion = 0 MPa. The normal stiffness between two blocks is $k_n = 72 \text{ GPa}$ and the normal to shear stiffness ratio is $k_n/k_s = 0.25$.

Fig 3.7.3 Direct shear test model (a) geometry model (b) UDEC model (c) NMM model

Fig 3.7.4 Average shear stress vs shear displacement: (a) result from NMM (b) result from UDEC

From Fig 3.7.4, there is slight difference in the slip distance when the maximum shear stress is reached, the maximum shear strength from two different models are nearly the same. The slight difference of the slip distance predicted may be due to the difference of the contact technique and inherent theory adopted by these two methods.

In addition, the influence of normal stresses is also investigated by NMM. The theoretical shear strength τ for direct shear test without cohesion can be expressed as:

$$\tau = \sigma \tan(\phi) \quad (3-47)$$

Where σ is the normal stress, ϕ is the friction angle.

Eight cases with various normal stresses of 2 MPa, 4 MPa, 6 MPa, 8 MPa, 10 MPa, 12 MPa, 14 MPa and 16 MPa are simulated to investigate the influences of normal stress on the joint shear strength, during which the friction angle is fixed to be 30° . The numerical result matches well with the theoretical solutions (as shown in Fig 3.7.5), i.e. shear strength increases linearly with the normal stresses.

Fig 3.7.5 Comparison of the NMM simulation result with the theoretical result of shear strength for direct shear test

CHAPTER 4 EXTENSION OF NMM FOR FRACTURE PROBLEMS

4.1 Introduction

The crack initiation and propagation problems have been extensively studied for the past several decades. There are two major research directions. One direction is to develop fracture propagation theories (Erdogan and Sih 1963; Hussian 1974; Sih 1974), the other one is to develop techniques to calculate the stress intensity factors (Chan 1970; Ishikawa 1980; Sha 1984; Sha 1990). However, there are few methods available to model arbitrary fracture propagation efficiently.

One of the main challenges to describe such discontinuous behavior numerically comes from the difficulties in describing evolving discontinuities topologically and adopting the approximation locally in order to accurately capture the evolving stress field around discontinuities. In finite element methods (FEM), this was accomplished by remeshing, though it is very time consuming. While in discrete methods, it was done by using artificial connection of discrete bodies which are identified a priori to act as continua (continuous bodies). In boundary element method, the discontinuous behavior can be simulated successfully. However, since the boundary element method requires a fundamental solution to formulate the governing equations, the scope of problems that it can solve is quite limited.

Belytschko et al. (1994a) applied the moving least square interpolation concept to develop the element-free Galerkin method, which is considered to be a generalized finite element method, to investigate the crack initiation and propagation problem. This method can avoid the burden of remeshing compared with traditional finite element method. By the element-free Galerkin method, Belytschko et al. (1995) successfully simulated the crack initiation and growth behavior and calculated the stress intensity factor at the crack tip for single arbitrary crack problem. However, this method needs to incorporate unknowns through enriched functions. For a problem with many cracks or complicated branch cracks whose enriched functions are hard to define, the enriching process becomes very complicated. Alternatively, Shi (1995) conceptualized a framework, which he called

manifold method, made it possible to deal with such discontinuity with a higher flexibility. The NMM represents a major leap in numerical analysis because it incorporates an innovative discontinuity modeling technique which is based on two meshes.

In this chapter, the basic concepts of linear elastic fracture mechanics (LEFM) are first introduced. Then the detailed procedures of extending the NMM to fracture problems are presented.

4.2 Basic modes of fracture

There are three basic types of fracture modes, as shown in Fig 4.2.1. For a crack in an ideal elastic brittle material, these modes are mode I, tensile (or opening); mode II, in plane shear (or sliding); and mode III, out-of-plane (or tearing mode). A combination of any of two of the above three basic fracture modes constitutes a mixed mode. Such problems are called mixed-mode problems. In this research, only planar components are considered and hence further discussion would be limited to the first two modes.

Fig 4.2.1 Basic modes of fracture

4.3 Mixed-mode crack tip fields

The strength of stress singularity near the crack-tip region in each mode is characterized by the stress intensity factor (SIF) for that mode. The SIFs for the three modes are denoted by K_I , K_{II} and K_{III} . According to LEFM theory, the crack-tip fields for all three basic fracture modes can be expressed as follows (according to Fig 4.3.1).

Stress fields near a crack tip under pure mode I loading are given by:

$$\begin{aligned}
 \sigma_x &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) \\
 \sigma_y &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) \\
 \sigma_{xy} &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(\sin \frac{\theta}{2} \cos \frac{3\theta}{2}\right)
 \end{aligned} \tag{4-1}$$

Fig 4.3.1 Polar coordinate with respect to the crack tip

And the associated compatible displacement fields are as follows:

$$\begin{aligned}
 u &= \frac{K_I}{4G} \sqrt{\frac{r}{2\pi}} \left[(2k-1) \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \right] \\
 v &= \frac{K_I}{4G} \sqrt{\frac{r}{2\pi}} \left[(2k-1) \sin \frac{\theta}{2} - \sin \frac{3\theta}{2} \right]
 \end{aligned} \tag{4-2}$$

where:

$$k = 3 - 4\nu \text{ for plane strain}$$

$$k = \frac{3-\nu}{1+\nu} \text{ for plane stress}$$

is the Kolosov coefficient· ν is the Poisson's's ratio. G is the shear modulus. θ , r are polar coordinates with respect to the crack tip as shown in Fig 4.3.1.

Stress fields near a crack tip for a crack under pure mode II loading are given by:

$$\begin{aligned}\sigma_x &= -\frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2}\right) \\ \sigma_y &= \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left(\cos \frac{\theta}{2} \cos \frac{3\theta}{2}\right) \\ \sigma_{xy} &= \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \cos \frac{3\theta}{2}\right)\end{aligned}\tag{4-3}$$

And the associated compatible displacement fields are as follows:

$$\begin{aligned}u &= \frac{K_{II}}{4G} \sqrt{\frac{r}{2\pi}} \left[(2k+3) \sin \frac{\theta}{2} + \sin \frac{3\theta}{2}\right] \\ v &= \frac{K_{II}}{4G} \sqrt{\frac{r}{2\pi}} \left[(2k-3) \cos \frac{\theta}{2} + \cos \frac{3\theta}{2}\right]\end{aligned}\tag{4-4}$$

It can be observed from these equations that the near-tip stresses and displacements are completely determined by the stress intensity factors. The numerical computation of SIFs is discussed in the following section.

4.4 Method for determination of stress intensity factor

The stress intensity factors depend on geometry, loading, and initial crack geometry. For simple geometries (plates; shafts; pipes; etc.), SIFs can be determined using the vast pool of empirical data. However, accuracy of these empirical relations fades away as geometric complexity increases. As a result, different techniques are needed. Because of the advancements in numerical techniques, such as FEM, the meshless element method and the NMM, several techniques have been proposed to evaluate the SIFs in LEFM. Lin and Johnston (1992) and Wilson and Meguid (1995) have suggested several most commonly used methods for the determination of SIFs. According to their research, the following four methods can be used to determine the mixed-mode stress intensity factors. They are:

- (1) singular element method
- (2) direct stress and displacement extrapolation methods
- (3) virtual crack extension(VCE)method
- (4) J-integral method

Because of the relationship between the energy release rate and the J-integral in LEFM and the convenience of the achievement of J-integral, J-integral will be adopted for this study. In the following sections, the possibility of combining virtual crack extension method and J-integral with NMM will be investigated.

4.4.1 Virtual crack extension method

The virtual crack extension method (VCE) based on the energy concept is an efficient way to calculate the SIFs. The main advantage is that relatively coarse meshes can give an accurate estimation of SIFs. The virtual crack extension technique first proposed by Park (1974) provides a convenient finite element approach without using the singular element or extreme mesh refinement in the crack tip region for the determination of the strain energy release rate G_f in cracked bodies.

The relationship between G_f and mode I and mode II SIFs can be expressed as:

$$G_f = \frac{1-\nu^2}{E} (K_I^2 + K_{II}^2) \quad \text{for plane strain} \quad (4-5)$$

$$G_f = \frac{1}{E} (K_I^2 + K_{II}^2) \quad \text{for plane stress} \quad (4-6)$$

The energy release rate of the system for the unit crack extension can be calculated by differentiating the total potential energy with respect to the crack length.

$$\begin{aligned} G_f &= -\frac{\partial \Pi}{\partial a} = -\frac{\partial \left\{ \frac{1}{2} U^T K U + U^T f \right\}}{\partial a} \\ &= -\frac{\partial U^T}{\partial a} (K U - f) - \frac{1}{2} U^T \frac{\partial K}{\partial a} U - U^T \frac{\partial f}{\partial a} \end{aligned} \quad (4-7)$$

Where U = nodal displacement vector; K = global stiffness matrix; f = equivalent nodal force vector;

Since $K U - f = 0$, so the equation above can be written as:

$$G_f = -\frac{1}{2} U^T \frac{\partial K}{\partial a} U - U^T \frac{\partial f}{\partial a} \quad (4-8)$$

The above procedures give us a general framework for obtaining the potential energy release rate. Park (1974) and Hellen (1975) prove that the VCE method is equivalent to the J-integral when expressed in terms of stress and displacement using constant stress triangular finite element method.

4.4.2 J-integral method

J-integral, which was first proposed by Rice (1968), is basically a path independent contour integral around the crack tip. Similar to the VCE method, the J-integral method is also an energy-based method for the determination for the potential energy release rate G_f . According to Rice, assuming small displacement gradients and neglecting body forces and crack-face tractions, the path-independent J integral under a 2D simple conditions is defined as:

$$J = \int_{\Gamma} \left(W n_x - T \frac{\partial U}{\partial x} \right) ds \quad (4-9)$$

Where $U = \{u \ v\}$ is the displacement vector, $\{n\} = \{n_x \ n_y\}$ is the outward normal of the contour Γ (Fig 4.4.1). And T is the traction vector which can be expressed as:

$$\{T\} = \begin{Bmatrix} \sigma_{xx} n_x + \tau_{xy} n_y \\ \sigma_{yy} n_x + \tau_{xy} n_y \end{Bmatrix} \quad (4-10)$$

In the finite element method, there are usually two ways to calculate the J-integral along the contour Γ . First, the contour Γ , which passes through the Gauss integration point where the stresses and strains have already been obtained. Second, along the contour Γ , which follows the edges of the element. The extrapolation of the strains and stresses to the points is thus required. However, in the NMM, the strain and stress at any points within the physical domain can be obtained analytically. Therefore, the integration path can be chosen arbitrarily without losing the accuracy.

Fig 4.4.1 Crack tip coordinates and contour for the evaluation of the J integral

To implement the computation of the J-integral using the NMM, first of all, the element form of the J-integral has to be formulated. The displacement function of the constant strain triangular element in the NMM can be expressed as follows:

$$\begin{aligned} u &= a_1 + b_1 x + c_1 y \\ v &= a_2 + b_2 x + c_2 y \end{aligned} \quad (4-11)$$

Therefore,

$$\frac{\partial u}{\partial x} = b_1, \quad \frac{\partial v}{\partial x} = b_2 \quad (4-12)$$

The strain energy of the element for linear elasticity is:

$$W = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} = \frac{1}{2} (\sigma_{xx} \varepsilon_{xx} + \sigma_{xy} \varepsilon_{xy} + \sigma_{yy} \varepsilon_{yy}) \quad (4-13)$$

Then the first term of J-integral can be rewritten as:

$$W n_x = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \frac{(y_{n+1} - y_n)}{\sqrt{(y_{n+1} - y_n)^2 + (x_{n+1} - x_n)^2}} \quad (4-14)$$

Where

$$\begin{bmatrix} n_x & n_y \end{bmatrix} = \frac{1}{\sqrt{[(y_{n+1} - y_n)^2 + (x_{n+1} - x_n)^2]}} [(y_{n+1} - y_n) \quad -(x_{n+1} - x_n)] \quad (4-15)$$

$$\Delta S = \sqrt{[(y_{n+1} - y_n)^2 + (x_{n+1} - x_n)^2]} \quad (4-16)$$

By combining Eq. (4-10) and Eq. (4-14), the second term of J integral can be expressed as:

$$\left[\sigma_{xx} b_1 + \tau_{xy} b_2 \right] (y_n - y_{n+1}) + \left[\sigma_{yy} b_2 + \tau_{xy} b_1 \right] (x_{n+1} - x_n) \quad (4-17)$$

Therefore, the J integral for the constant stress triangular element is:

$$J = W(y_{n+1} - y_n) + [\sigma_{xx} b_1 + \tau_{xy} b_2] (y_n - y_{n+1}) + [\sigma_{yy} b_2 + \tau_{xy} b_1] (x_{n+1} - x_n) \quad (4-18)$$

For an elastic material, the J integral is identical to the energy release rate, G. The relation between J integral and mode I and II SIFs can be written as:

$$J = G_f = \frac{1-\nu^2}{E} (K_I^2 + K_{II}^2) \quad \text{for plane strain} \quad (4-19)$$

$$J = G_f = \frac{1}{E} (K_I^2 + K_{II}^2) \quad \text{for plane stress} \quad (4-20)$$

It should be noted, however, that for a non-elastic material, the J integral is not equal to G.

In order to solve the mixed-mode problem, the decomposition of the displacement field into the mode-I and mode-II fields is carried out by decomposing the displacement components of an arbitrary near-tip nodal point and its mirror image into symmetric and antisymmetric conditions (Xie et al. 1995). Fig 4.4.2 shows the concept of displacement field decomposition, the decomposed nodal displacement at point A can be written in local crack-tip coordinates system as:

$$\{u\} = \{u_I\} + \{u_{II}\} = \begin{Bmatrix} u_I \\ v_I \end{Bmatrix} + \begin{Bmatrix} u_{II} \\ v_{II} \end{Bmatrix} \quad (4-21)$$

Where

$$\{u_{IA}\} = \begin{Bmatrix} u_{IA} \\ v_{IA} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} u_A + u_B \\ v_A - v_B \end{Bmatrix} \quad (4-22)$$

$$\{u_{IIA}\} = \begin{Bmatrix} u_{IIA} \\ v_{IIA} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} u_A - u_B \\ v_A + v_B \end{Bmatrix} \quad (4-23)$$

Fig 4.4.2 Decomposition of displacement field into Mode-I and Mode-II fields with respect to local crack-tip x-y coordinates system (Xie et al. 1995)

where u_A, v_A, u_B and v_B can be expressed in global coordinates as:

$$\begin{Bmatrix} u_A \\ v_A \end{Bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} U_A \\ V_A \end{Bmatrix} \quad (4-24)$$

$$\begin{Bmatrix} u_B \\ v_B \end{Bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} U_B \\ V_B \end{Bmatrix} \quad (4-25)$$

The displacement components in Eq. (4-22) and Eq. (4-23) are then transformed into the global coordinate system.

$$\begin{Bmatrix} U_{IA} \\ V_{IA} \end{Bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} u_{IA} \\ v_{IA} \end{Bmatrix} \quad (4-26)$$

$$\begin{Bmatrix} U_{IIA} \\ V_{IIA} \end{Bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} u_{IIA} \\ v_{IIA} \end{Bmatrix}$$

Then, we can get the decomposed nodal displacement vector at point A in the global coordinate system:

$$U_A = U_{IA} + U_{IIA} = \begin{Bmatrix} U_{IA} \\ V_{IA} \end{Bmatrix} + \begin{Bmatrix} U_{IIA} \\ V_{IIA} \end{Bmatrix} \quad (4-27)$$

where:

$$\begin{aligned} U_{IA} &= \frac{1}{2} [U_A + (\cos^2 \theta - \sin^2 \theta)U_B + 2 \cos \theta \sin \theta V_B] \\ V_{IA} &= \frac{1}{2} [V_A + (\sin^2 \theta - \cos^2 \theta)V_B + 2 \cos \theta \sin \theta U_B] \\ U_{IIA} &= \frac{1}{2} [U_A + (\sin^2 \theta - \cos^2 \theta)U_B - 2 \cos \theta \sin \theta V_B] \\ V_{IIA} &= \frac{1}{2} [V_A + (\cos^2 \theta - \sin^2 \theta)V_B - 2 \cos \theta \sin \theta U_B] \end{aligned} \quad (4-28)$$

By using the separated $\{U_I\}$ and $\{U_{II}\}$, the stress and strains of mode I and II for all elements along the J integral path can be found. Hence, using the same procedure, the decoupled J_1 and J_2 integral can be obtained as follows:

$$\begin{aligned} J_I &= W_I(y_{n+1} - y_n) + [\sigma_{I,xx} b_{I,1} + \tau_{I,xy} b_{I,2}](y_n - y_{n+1}) \\ &\quad + [\sigma_{I,yy} b_{I,1} + \tau_{I,xy} b_{I,2}](x_{n+1} - x_n) \end{aligned} \quad (4-29)$$

$$\begin{aligned} J_{II} &= W_{II}(y_{n+1} - y_n) + [\sigma_{II,xx} b_{II,1} + \tau_{II,xy} b_{II,2}](y_n - y_{n+1}) \\ &\quad + [\sigma_{II,yy} b_{II,1} + \tau_{II,xy} b_{II,2}](x_{n+1} - x_n) \end{aligned} \quad (4-30)$$

Where:

$$b_{I,1} = \frac{\partial U_I}{\partial x}, \quad b_{I,2} = \frac{\partial V_I}{\partial x}, \quad b_{II,1} = \frac{\partial U_{II}}{\partial x}, \quad b_{II,2} = \frac{\partial V_{II}}{\partial x} \quad (4-31)$$

Since the integral path is always defined under the local crack-tip coordinate system, we need to change the local coordinate to the global one. The relationship between the global coordinates of x - y and the local crack-tip coordinates x_1 - x_2 can be expressed as follows:

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \quad (4-32)$$

4.5 NMM for crack initiation and propagation under mixed mode

4.5.1 Crack modeling in NMM

The most advanced feature of the NMM is the adoption of two covers to describe a problem. Unlike the enriched functions used in XFEM and GFEM, in which a prior definition is required to capture the discontinuities, the two-layered approach and the loop concept in the NMM allow for easy crack identification and direct crack propagation simulation without further incorporating unknowns to the related nodes through enrichment functions. A detailed comparison between the NMM and other partition of unity methods can be found in previous work (An et al. 2012).

To identify cracks in the NMM, the loop concept is incorporated in the NMM. The loop is also the basis of the NMM contact technique. A loop is an independent closed domain, which can be either a block (an independent closed physical area) or a crack. As shown in Fig 4.5.1, Loop 1 is defined as the closed domain encompassed by five vertices (1, 2, 3, 4 and 1), while loop 2 is defined as the closed domain encompassed by three vertices (5, 6 and 5). The crack in the NMM has two faces: one is from vertex 5 to vertex 6, while the other one is from vertex 6 to vertex 5. This is how a closed pre-existing flaw is defined in the NMM. As shown in Fig 4.5.2, the closed flaw has two faces, while each face has its own vertices that may occupy the same place. When the flaw intersects the mathematical covers (regular hexagonal units as shown in Fig 4.5.3), each vertex that is on the flaw, will be assigned to form different elements. The other vertices that are related to the elements, which are not cut by the crack, will remain unchanged. Because the contact technique in the NMM is based on the loop vertices, the contact between

two flaw faces can be captured. For example, as shown in Fig 4.5.2, the distance between loop vertices 11 and 20 is so small that they are considered as a contact pair. Similarly, vertices 12 and 19, 13 and 18, 14 and 17, and 15 and 16 are all possible contact pairs. Their contact status will be judged and updated after every step. In such a simple way, the arbitrary crack trajectory and the interaction between the crack surfaces can be captured, which typically is not easily handled by conventional finite element method.

The identification of a closed crack in the NMM has just been discussed. The following section focuses on how the jump or discontinuity of displacements across the crack surfaces can be captured. As shown in Fig 4.5.4 a, when no crack is present in a mathematical cover, a hexagonal mathematical cover has six triangular elements sharing the same vertex (P_1 , cover center) and each element is constructed by three different mathematical covers. Crack initiation or propagation can be associated with one of two scenarios: either the mathematical cover is fully cut by the crack (Fig 4.5.4 b) or the mathematical cover is partially cut by the crack (Fig 4.5.4 c).

For the condition shown in Fig 4.5.4 b, the mathematical cover P_1 is split into two parts. As a result, two physical covers, P_1^1 (in shade) and P_1^2 (the left part) form. By the knowledge that the elements in NMM are the common part of physical covers, the original elements 1, 5 and 6 are then substituted by elements 7, 8, 9, 10, 11 and 12. After the elements have been updated, the covers related to the elements also need to be updated. The original related physical cover P_1 will be substituted by P_1^2 for the elements in the shaded area, while for elements 10, 11 and 12, the original related physical cover P_1 will be substituted by P_1^1 . Using the above procedures, the jump or discontinuity across the crack surface can be captured. For example, the jump across the interface between element 7 and element 10 can be expressed as:

$$\left[\mathbf{u}^h \right] = \sum_i^3 \varphi_{i(ele\ 7)} \mathbf{u}_{i(ele\ 7)} - \sum_j^3 \varphi_{j(ele\ 10)} \mathbf{u}_{j(ele\ 10)} \quad (4-33)$$

where $[[\cdot]]$ represents the jump of a function; φ_i ($i=1, 2, 3$) is the weight functions for the corresponding physical cover; u_i ($i=1, 2, 3$) is the local displacement functions for the related physical covers.

For elements 7 and 10, the related covers are nearly the same except for the original related cover P_1 . After updating the physical covers, element 7 is related to P_1^2 , while element 10 is related to P_1^1 . Therefore, Eq. (4-33) can be simplified to

$$[[u]] = \varphi_3 [u_3(P_1^2) - u_3(P_1^1)] \quad (4-34)$$

With the knowledge that $u_3(P_1^2)$ and $u_3(P_1^1)$ are individually defined as local displacement functions, which contain independent degrees of freedom (DOFs), the displacement discontinuity across the crack surfaces between element 7 and element 10 can be described. In such a direct way, the NMM can not only solve simple crack problems, but the NMM can also solve complicated crack problems without difficulty in defining the enrichment functions.

For the case shown in Fig 4.5.4 c, when the mathematical cover is partially cut by a crack, the discontinuity cannot be captured without further incorporating unknowns to the related covers. Based on the partition of unity method (Babuska and Melenk 1997), instead of using the conventional polynomial cover functions in the local displacement approximations, more accurate functions are incorporated for the cover functions to enhance the approximations, such as functions with terms of $\sin(n\theta/2)$ ($n=2k+1$, where θ is the angle from the tangent of the flaw to the crack path at the crack tip, as shown in Fig 4.5.5.), to capture the discontinuity between $\theta=+\pi$ and $\theta=-\pi$. Detailed enhancement methods have been discussed by Li and Cheng (2005). In this study, the extrinsic enrichment method is adopted by directly adding the extrinsic basis to the approximations through the partition of unity. Then the enriched approximations can be expressed as:

$$u^h = \sum_i^3 \varphi_i u_i + \sum_j^N k_j q_j \quad (4-35)$$

where k_j is the array of additional unknowns; N is the number of additional unknowns; q_j is the enriched functions including the terms of $\sin(n\theta/2)$ and $\cos(n\theta/2)$.

In this chapter, for simplicity, the assumptions that $N = 2$, $q_1 = \sqrt{r} \cos(\theta/2)$ and $q_2 = \sqrt{r} \sin(\theta/2)$ are adopted for the enrichment functions to capture the discontinuity across the crack surfaces that partially cuts the mathematical cover. The displacement for element 3 in Fig 4.5.4 c can then be expressed as:

$$u^h = \sum_i \varphi_i(x) u_i + \sum_i \sum_j^N \varphi_i(x) k_i^j q_j(x) \quad (4-36)$$

where k_i^j is the enriched DOF; N is the number of additional terms; $q_j(x)$ is the enrichment function including the terms of $\sin(\theta/2)$ and $\cos(\theta/2)$; θ is the polar coordinate in the local coordinates system with the origin at the crack tip, as shown in Fig 4.5.5.

Fig 4.5.1 Representation of a physical domain with a crack in NMM

Fig 4.5.2 Loop vertexes in NMM

Fig 4.5.3 Mathematical cover system in NMM

Fig 4.5.4 Sketches of crack modeling by NMM (a) The physical covers and elements before cracking (b) Physical covers and elements after cracking when the mathematical cover is fully cut by the crack (c) The physical covers and elements after cracking when the mathematical cover is partially cut by the crack

Fig 4.5.5 Local co-ordinate system at the crack tip

In this study, for simplicity, the assumptions of $N = 2$, $q_i^1 = \cos(\theta/2)$ and $q_i^2 = \sin(\theta/2)$, $k_i^1 = k_1$ ($i=1, 2, 3$) and $k_i^2 = k_2$ ($i=1, 2, 3$) are adopted for the enrichment functions to capture the discontinuity across the crack surfaces that partially cuts the mathematical cover. Since $\sum_i^3 \varphi_i(x) = 1$, the displacements for element 4 in Fig 4.5.4 c can then be expressed as:

$$u^h = \sum_i^3 \varphi_i(x) u_i + k_1 \cos(\theta/2) + k_2 \sin(\theta/2) \quad (4-37)$$

Since $\varphi_i(x)$ and u_i are the same for the upper and bottom crack surfaces, while the upper crack surface has an angle $\theta = \pi$ and the bottom crack surface has an angle $\theta = -\pi$, the discontinuity jump across the crack surfaces can then be calculated as:

$$[[u^h]] = 2\sqrt{r} \times k_2 \quad (4-38)$$

4.5.2 Criteria for crack initiation

For a given cracking problem, crack initiation and propagation criteria are needed to judge whether a new crack will develop or not at the beginning of each time step. When a crack initiation or propagation is predicted to occur, the crack growth direction has to be determined also.

The criteria used for crack initiation and propagation so far can be grouped into three headings: stress-based criterion, energy-based criterion, and strain-based criterion. The critical condition refers to the extremity of one of the stated parameters, i.e., stress, energy, or strain. The most commonly used criteria are the ones based on stress and energy, such as the maximum tangential stress (MTS) criterion and the minimum strain energy density (S) criterion. However, these two criteria can only handle crack initiation and propagation emanating from pre-existing cracks. To model a new crack development in the intact flawless rock material, an additional criterion is needed. In this study, the Mohr-Coulomb with a

tensile cut-off criterion the MTS-criterion and the S-criterion are incorporated into the NMM.

4.5.2.1 Maximum Tangential Stress (MTS)-criterion

Using stress as a parameter, MTS ($\sigma_{\theta, \max}$) criterion was the first one presented by Erdogan and Sih (1963). It states that the direction of crack initiation coincides with the direction of the maximum tangential stress along a constant radius around the crack tip. It can be stated mathematically as:

$$\frac{\partial \sigma_{\theta}}{\partial \theta} = 0, \quad \frac{\partial^2 \sigma_{\theta}}{\partial \theta^2} < 0 \quad (4-39)$$

By expressing the stress field in Eqs. (4-1) and (4-3) in polar coordinates and applying the MTS-criterion, the following equation can be obtained:

$$\begin{aligned} \tan^2 \frac{\theta}{2} - \frac{\mu}{2} \tan \frac{\theta}{2} - \frac{1}{2} &= 0 \\ -\frac{3}{2} \left[\left(\frac{1}{2} \cos^3 \frac{\theta}{2} - \cos \frac{\theta}{2} \sin^2 \frac{\theta}{2} \right) + \frac{1}{\mu} \left(\sin^3 \frac{\theta}{2} - \frac{7}{2} \sin \frac{\theta}{2} \cos^3 \frac{\theta}{2} \right) \right] &< 0 \end{aligned} \quad (4-40)$$

where μ is defined as:

$$\mu = \frac{K_I}{K_{II}} \quad (4-41)$$

Solving Eq. (4-40), the critical angle θ_c defining the radial direction of propagation can be obtained:

$$\theta_c = \begin{cases} 2 \arctan \left(\frac{1}{4} \left(\frac{K_I}{K_{II}} + \sqrt{\left(\frac{K_I}{K_{II}} \right)^2 + 8} \right) \right) & K_{II} < 0 \\ 0 & K_{II} = 0 \\ 2 \arctan \left(\frac{1}{4} \left(\frac{K_I}{K_{II}} - \sqrt{\left(\frac{K_I}{K_{II}} \right)^2 + 8} \right) \right) & K_{II} > 0 \end{cases} \quad (4-42)$$

4.5.2.2 Minimum Strain Energy Density (S)-criterion

S-criterion which was first presented by Sih (1974) states that the direction of crack initiation coincides with the direction of minimum strain energy density

along a constant radius around the crack tip. In mathematical form, S- criterion can be stated as:

$$\frac{\partial S}{\partial \theta} = 0, \quad \frac{\partial^2 S}{\partial \theta^2} > 0 \quad (4-43)$$

where S is the strain energy density factor, defined as:

$$S = r_0 \frac{dW}{dV} \quad (4-44)$$

where dW/dV is the strain energy density function per unit volume and r_0 is a finite distance from the point of failure initiation. For slit cracks, the crack tip is assumed to be the point of failure initiation. Using the stress field Eq. (4-1) in Cartesian co-ordinates, we can obtain the strain energy density function per unit volume, and using Eq. (4-44), we can write the strain energy density function as:

$$S = a_{11}K_I^2 + 2a_{12}K_I K_{II} + a_{22}K_{II}^2 \quad (4-45)$$

where the factors a_{ij} are functions of the angle θ and are defined as

$$\begin{aligned} a_{11} &= \frac{1}{16E\pi} [(1 + \cos \theta)(\kappa - \cos \theta)], \\ a_{12} &= \frac{1}{16E\pi} \sin \theta [2 \cos \theta - (\kappa - 1)], \\ a_{22} &= \frac{1}{16E\pi} [(\kappa + 1)(1 - \cos \theta) + (1 + \cos \theta)(3 \cos \theta - 1)], \end{aligned} \quad (4-46)$$

where E is the modulus of the material, and κ is a constant depending on stress state, and defined as:

$$\begin{aligned} \kappa &= \frac{(3 - \nu)}{(1 + \nu)} \quad \text{for plane stress} \\ \kappa &= (3 - 4\nu) \quad \text{for plane strain} \end{aligned} \quad (4-47)$$

From Eq. (4-43) and Eq. (4-45), we can get:

$$\begin{aligned}
 & [2(1+\kappa)\mu]\tan^4\frac{\theta}{2} + [2\kappa(1-\mu^2) - 2\mu^2 + 10]\tan^3\frac{\theta}{2} - 24\mu\tan^2\frac{\theta}{2} \\
 & + [2\kappa(1-\mu^2) + 6\mu^2 - 14]\tan\frac{\theta}{2} + 2(3-\kappa)\mu = 0 \\
 & [2(\kappa-1)\mu]\sin\theta - 8\mu\sin 2\theta + [(\kappa-1)(1-\mu^2)]\cos\theta + [2(\mu^2-3)]\cos 2\theta > 0
 \end{aligned} \tag{4-48}$$

Where μ is defined as in Eq. (4-41).

4.5.2.3 Mohr-Coulomb criterion

The Mohr-Coulomb criterion is able to recover the salient features of the quasi-brittle response within engineering accuracy. In addition, its parameters can be easily determined by conventional uniaxial compression (with axial and lateral strain measurement) and indirect tensile testing. The Mohr-Coulomb criterion with a tensile cut-off which was first presented by Brady and Brown (1993) is adopted for the crack initiation study in this section.

As shown in Fig 4.5.6, a crack will initiate when the following conditions are satisfied.

$$\begin{cases} R \leq r \\ \sigma_3 > -\sigma_t \end{cases} \tag{4-49}$$

Or

$$\sigma_3 \leq -\sigma_t \tag{4-50}$$

where σ_t is the tensile strength of the material, σ_1, σ_3 are the principal stresses and can be expressed as:

$$\begin{aligned}
 \sigma_1 &= \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\
 \sigma_3 &= \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}
 \end{aligned} \tag{4-51}$$

R and r can be expressed as:

$$\begin{aligned}
 R &= c_i \cos \phi + \frac{\sigma_x + \sigma_y}{2} \sin \phi_i \\
 r &= \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}
 \end{aligned} \tag{4-52}$$

where c_i and ϕ_i are the cohesion and friction angle of the material.

Fig 4.5.6 The Mohr-Coulomb criterion for the elements

From Fig 4.5.6, when the condition expressed in Eq. (4-49) is satisfied, the crack initiation angle θ will be:

$$\theta = \pi/4 + \phi_i/2 \quad (4-53)$$

where θ is defined as the angle measured counter-clockwise from the first principal stress to the crack initiation direction.

When the condition expressed in Eq. (4-50) is satisfied, the crack initiation direction will be perpendicular to the direction of the third principal stress.

4.5.3 Treatment of manifold elements during cracking process

For the crack initiation case, when the elements without a crack tip satisfy the failure criterion Eq. (4-49) or Eq. (4-50), a crack will initiate. However, using only the crack initiation direction discussed in section 4.5.2.3, the location of the crack is still indeterminate. We hereby further assume that the crack passes through the center of the element. Then if the failure criterion Eq. (4-50) is met, a tensile crack perpendicular to σ_3 as shown in Fig 4.5.7 a will develop. If the principal

stresses of the elements satisfy the condition expressed in Eq. (4-49), a shear crack at an angle of $(\pi/4 + \phi_i/2)$ with the direction of σ_1 as shown in Fig 4.5.7 b will develop.

Fig 4.5.7 Illustration of crack initiation for elements (Wu and Wong 2013)(a) tensile crack; (b) shear crack (O represents the center of the elements)

Complications arise when an element contains more than one crack tip satisfying the failure criterion in section 4.5.2. Three different scenarios are described below. If an element contains only one crack tip that reaches the failure criterion, the crack will propagate in a direction predicted by failure criteria discussed in section 4.5.2, as shown in Fig 4.5.8 a. If an element, that contains two crack tips, reaches the failure criterion, the newly formed crack is assumed to link the original two cracks together, as illustrated in Fig 4.5.8 b. If an element, that contains more than two crack tips, reaches the failure criterion, the newly formed crack will link up the crack tip that has a minimum included angle θ_1 with the crack propagation direction predicted according to the failure criterion (Fig 4.5.8 c).

After crack initiation and propagation, the newly formed cracks then cut the MCs into certain sub-domains. The PCs and the manifold elements will be subsequently updated automatically according to the procedure discussed in section 4.5.1.

4.5.4 Treatment of loops after crack initiation and propagation

In the NMM, the contact detection and crack representation are both based on loops (An 2010; Ning et al. 2011; Wu and Wong 2012, 2013). Therefore, once crack initiation or propagation occurs, the loops need to be updated. Based on the

relative position between the newly developed crack and the original loop, the modification to the loops can be categorized into the following four types: Type I: A new crack forms at an independent location without any intersection with the original loops, thus a new loop is added (Fig 4.5.9 a). Type II: A new crack intersects the original loop, but it does not cut the loop into two parts. The new crack is added to the original loop (Fig 4.5.9 b). Type III: A new crack cuts the original loop into two loops (Fig 4.5.9 c). Type IV: A new crack combines the two original loops into a single loop (Fig 4.5.9 d). Other types of loop updating can be deduced from the above four basic loop updating scenarios.

Fig 4.5.8 Illustration of crack propagation in elements containing one or more crack tips (Wu and Wong 2013) (a) crack propagation in elements containing single crack tips; (b) new formed crack in element containing two crack tips; (c) linking the crack tip which has a minimum included angle θ_1 with the originally predicted crack trajectory in elements containing more than two crack tips.

Fig 4.5.9 Updating of loops after crack initiation and propagation (An 2010; Ning et al. 2011; Wu and Wong 2012, 2013): (a) a new loop forms, (b) a crack is added to the pre-existing loop, (c) the pre-existing loop is cut into two new loops, (d) the pre-existing two loops combine into one.

4.5.5 Calibration of the NMM with different criteria for uniaxial loading

In order to solve the equations for the crack initiation angles, the expressions for the stress intensity factors for the angled crack problem for different loading conditions are needed. The general forms of stress intensity factors are given by (Rooke and Cartwright, 1976):

$$K_I = \sigma_n \sqrt{\pi a}, \quad K_{II} = \tau_n \sqrt{\pi a} \quad (4-54)$$

Where σ_n and τ_n are the normal and the tangential stresses to the crack plane respectively, as illustrated in Fig 4.5.10. To obtain expressions for σ_n and τ_n for the slant crack problem, the most general loading case is considered in Fig 4.5.10.

$$\sigma_n = \sigma_x \cos^2 \beta + \sigma_y \sin^2 \beta - \tau_{xy} \sin 2\beta \quad (4-55)$$

$$\tau_n = \frac{(\sigma_y - \sigma_x)}{2} \sin 2\beta - \tau_{xy} \cos 2\beta \quad (4-56)$$

Fig 4.5.10 General loading condition for angled crack

For uniaxial loading:

$$\sigma_x = 0, \quad \sigma_y = \sigma, \quad \tau_{xy} = 0 \quad (4-57)$$

Substituting Eq. (4-57) into Eq. (4-53), (4-56), we can obtain:

$$K_I = \sigma_y \sin^2 \beta \sqrt{\pi a}, \quad K_{II} = \sigma_y \sin \beta \cos \beta \sqrt{\pi a} \quad (4-58)$$

Using these stress intensity factors for uniaxial loading, software Mathematica version 7.0 is used to find the roots of the equation for each criterion. The real roots are separated from the complex roots, which are then tested for extreme condition (Khan and Khraisheh 2000).

Since the equation for the MTS-criterion is of second degree, two roots are obtained. The roots that yield the maximum tangential stress are for uniaxial tension case, while the roots that yield the minimum tangential stress are for uniaxial compression case. Since for uniaxial compression, σ is to be replaced by $-\sigma$. Therefore, the roots for uniaxial tension are negative, and those for uniaxial compression are positive.

For the S-criterion, four roots from the equation can be obtained, two are complex conjugates and two are real. One set of the real roots (negative ones) is for uniaxial tension, and the other set of the real roots (positive ones) is for uniaxial compression. Both roots for uniaxial tension and compression yield minimum strain energy density. This is in contrast to the MTS-criterion, where one set of the roots yields maximum value and the other set yields minimum value. Since the S-criterion depends on σ^2 , the solution is for both uniaxial tension ($+\sigma$) and uniaxial compression ($-\sigma$), as indicated by Sih (1974). From section 4.5, the S-criterion depends on the Poisson's ratio (ν) of the material. Here we take $\nu=0.3$ for analysis. Since the Mohr-coulomb criterion largely depends on the friction angle of the material, for comparison reasons, in the following section, only the MTS and S criteria are conducted.

For verifying the efficiency of the NMM used for estimating the crack initiation angle, the MTS-criterion and the S-criterion were incorporated into NMM and used for simulating the crack initiation problem. For comparison, 9 models with flaw inclination angle from 5 degrees to 85 degrees with 10 degree interval were run for each criterion and each loading condition. The dimensions of the pre-cracked model for analysis are 12 cm in width and 20 cm in height. The half-length of the crack is chosen to be $a=1$ cm. The load applied on the top and bottom of the models for analysis is 100 kPa in magnitude (100 kPa for tension, -100 kPa for compression). The Young's modulus is taken as $E=10^9$ Pa. Some models of element meshes adopted in present study are shown in Fig 4.5.13.

As illustrated by part of the simulation results shown from Fig 4.5.12 to Fig 4.5.14, we can find that with the incorporation of crack initiation criteria, the NMM can solve the crack initiation problem and provide a similar trend with the analytical solution for predicting the crack initiation angle with the pre-existing flaw angle. For compression, the crack initiation angle is larger under the S-criterion than the MTS criterion. However, for tension, such phenomenon does not exist. And under the same circumstances, the crack initiation angle will be larger for compression than for tension.

To solve a problem numerically, errors may be introduced during the discretization of the space and the approximation of the displacement function as

well as the computation of the functions. As the degree of the accuracy of the results predicted by NMM mainly depends on the size of the cover and the displacement approximation functions adopted. A crack tip enrichment technique is adopted for the covers around the crack tip to improve the accuracy. However, as illustrated in Fig 4.5.13 to Fig 4.5.14, under a small angle for tension and a large angle for compression, the NMM fails to predict a good match compared with the theoretical results. It is because under such circumstances, the stress intensity factor approaches zero according to equation (4-58). Therefore, the proportion of the numerical error may be exaggerated due to the error related to the stress intensity factor in the denominator.

Model with 5 degree inclined crack

Model with 45 degree inclined crack

Fig 4.5.11 Sketches of some models used for analysis

Result of crack initiation under uniaxial vertical compression for 5 degree inclined crack using MTS-criterion

Result of crack initiation under uniaxial vertical tension for 45 degree inclined crack using S-criterion

Fig 4.5.12 Part of simulation results by NMM

Fig 4.5.13 Uniaxial loading , positive roots for compression

Fig 4.5.14 Uniaxial loading, negative roots for tension

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5.1 Introduction

Linear Elastic Fracture Mechanics (LEFM) is a branch of fracture mechanics that deals with problems in which the size of the plastic zone around the crack tip is very small in comparison with the domain size. For brittle material, with little plastic deformation, the fracture initiates. Therefore, LEFM can well represent the brittle materials for analyzing their cracking performance under different loads. In this chapter, we only focus on the LEFM. A detailed and authoritative discussion on LEFM theory can be found in Anderson (1985).

5.2 Enriched NNM for linear-elastic fracture

For the traditional NMM used for a crack problem, the polynomial displacement approximation function is used and a cover is divided into two irregular sub-covers at the discontinuity. As a result the traditional method sometimes gives an unsatisfactory result at the tip of a crack since the real displacement field near the crack tip may be singular. To improve the accuracy of the traditional manifold method, the enriched manifold method is developed and presented in this section.

The enriched manifold method is based on the partition of unity method (Melenk and Babuska 1996). The enrichment for the manifold method approximations near the tip of a linear elastic crack is achieved by the expansion of the base functions with special functions. Both extrinsic enrichment and intrinsic enrichment can be made to the base functions. In this thesis, the extrinsic enrichment approach is adopted by directly adding the analytical solution around the crack tip to extend the trial function for the displacement approximation. Details of the method are introduced below.

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5.2.1 Enriched numerical manifold method

In the enriched NMM, the additional displacements $\bar{T}\bar{D}$ are added to the traditional displacement functions TD . The displacement functions can then be expressed as:

$$u^h = TD + \bar{T}\bar{D} \quad (5-1)$$

where TD is the traditional displacement function and $\bar{T}\bar{D}$ is the enriched displacement function, $\bar{D}$ is the additional degree of freedom.

Substituting Eq. (5-1) into the strain-displacement Eq. (3-18), the strains for an enriched element are:

$$\varepsilon^h = \varepsilon + \bar{\varepsilon} = BD + \bar{B}\bar{D} \quad (5-2)$$

Where ε and $\bar{\varepsilon}$ are strains for traditional part and enriched part, respectively, and B and $\bar{B}$ are strain matrix for traditional part and enriched part, respectively.

Substituting Eq. (5-1) into Eq. (5-2) into the minimum potential energy principle, we can obtain the following discrete form:

$$\begin{bmatrix} K_{DD} & K_{D\bar{D}} \\ K_{\bar{D}D} & K_{\bar{D}\bar{D}} \end{bmatrix} \begin{pmatrix} D \\ \bar{D} \end{pmatrix} = \begin{pmatrix} P_D \\ P_{\bar{D}} \end{pmatrix} \quad (5-3)$$

$$K_{DD} = \int_V B^T E_e B dV \quad (5-4)$$

$$K_{D\bar{D}} = \int_V B^T E_e \bar{B} dV \quad (5-5)$$

$$K_{\bar{D}D} = \int_V \bar{B}^T E_e B dV \quad (5-6)$$

$$P_D = \int_V N^T f dV + \int_S N^T T dS \quad (5-7)$$

$$P_{\bar{D}} = \int_V \bar{N}^T f dV + \int_S \bar{N}^T T dS \quad (5-8)$$

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where E_e is the elastic matrix, f and T are the body force and the prescribed tractions, respectively. From Eqs. (5-3) to (5-8), we can get:

$$\bar{D} = (K_{\bar{D}\bar{D}})^{-1} (P_{\bar{D}} - K_{D\bar{D}} D) \quad (5-9)$$

Substituting Eq. (5-9) into Eq. (5-3) we can eliminate an additional degree of freedom $\bar{D}$, and obtain the system of equations for the enriched NMM as:

$$K^e D = P^e \quad (5-10)$$

where:

$$K^e = K_{DD} - K_{D\bar{D}} (K_{\bar{D}\bar{D}})^{-1} K_{\bar{D}D} \quad (5-11)$$

$$P^e = P_D - K_{D\bar{D}} (K_{\bar{D}\bar{D}})^{-1} P_{\bar{D}} \quad (5-12)$$

The above expressions are the explicit forms containing additional displacement terms.

For two-dimensional fracture problems, the displacement field near the tip of a mixed mode crack is:

$$u(x, y) = \frac{K_I}{2G} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} (k - 1 + 2 \sin^2 \frac{\theta}{2}) + \frac{K_{II}}{2G} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} (k + 1 + 2 \cos^2 \frac{\theta}{2}) \quad (5-13)$$

$$v(x, y) = \frac{K_I}{2G} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} (k + 1 - 2 \cos^2 \frac{\theta}{2}) - \frac{K_{II}}{2G} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} (k - 1 - 2 \sin^2 \frac{\theta}{2}) \quad (5-14)$$

In which k is the Kolosov constant as defined previously. So we can assume the enriched displacement matrix to be:

$$\bar{N} = \begin{bmatrix} q_{11}(x, y) & q_{12}(x, y) \\ q_{21}(x, y) & q_{22}(x, y) \end{bmatrix} \quad (5-15)$$

Where

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$$\begin{aligned}
 q_{11}(x, y) &= \frac{1}{2G} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} (k-1+2\sin^2 \frac{\theta}{2}) \\
 q_{12}(x, y) &= \frac{1}{2G} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} (k+1+2\cos^2 \frac{\theta}{2}) \\
 q_{21}(x, y) &= \frac{1}{2G} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} (k+1-2\cos^2 \frac{\theta}{2}) \\
 q_{22}(x, y) &= -\frac{1}{2G} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} (k-1-2\sin^2 \frac{\theta}{2})
 \end{aligned} \tag{5-16}$$

Therefore, substituting Eq. (5-16) into the strain equation we can obtain:

$$\bar{B} = \begin{bmatrix} q_{11,x} & q_{12,x} \\ q_{21,y} & q_{22,y} \\ q_{11,y} + q_{21,x} & q_{12,y} + q_{22,x} \end{bmatrix} \tag{5-17}$$

From Eq. (5-16), (5-17), we found that the strain matrix $\bar{B}$ is no longer polynomial. As a result, the simplex integration method that the traditional NMM adopts will no longer work. In order to maintain the advantage of the simplex integration method but also make the non-polynomial displacement functions work well, the gauss- simplex integration method is adopted. This method uses the simplex integration method for most of the elements except for the elements with the enriched displacement functions, while for enriched elements, the gauss integration method is used. Since the enriched functions are singular, we need to distribute plenty of gauss points for capturing the singular characteristics near the crack tips. In this thesis, n=12 gauss points are used for integration. For implementation, if the enriched elements are not triangles, a division to cut the arbitrary shape elements into several sub-triangles is needed. This procedure is completed by connecting each vertex to the center of the elements as shown in Fig 5.2.1.

Through the edge crack model shown in Fig 5.2.2, the effectiveness of the enriched manifold method for a singular crack problem is illustrated.

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Fig 5.2.1 Partition of non-triangle elements into sub-triangles by connecting vertex to the center of the original elements. (a) an element is fully cut by a crack (b) an element is partially cut by a crack.

Fig 5.2.2 Edge crack model

The model is loaded in tension at the top with $\sigma = 0.2$ GPa, the essential boundary conditions are applied at the bottom of the plate with y displacement fixed to be zero. The parameters used in the numerical simulations are $L=52\text{mm}$, $D=20\text{mm}$, $a=12\text{mm}$; $E=76\text{ GPa}$ and $\nu=0.286$, and a plane strain condition is assumed. The closed-form solution for the problem is given in Eq. 4-1 for mode I. The computed mode I stress intensity factors are compared with a finite geometry corrected value $K_I = C_1 \sigma \sqrt{a\pi}$, where the correction parameter C is given by Sih (1991) as:

$$C_1 = 1.12 - 0.231(a/L) + 10.55(a/L)^2 - 21.72(a/L)^3 + 30.39(a/L)^4 \quad (5-18)$$

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where a is the length of the crack, L is the length of the model.

The stress intensity factor K_I is normalized by $\sigma\sqrt{a\pi}$, and the normalized K_I value obtained by the linear basis (linear displacement approximations without enriched functions) and the enriched functions are 1.38 and 1.43, respectively. The analytical solution is $C_1=1.44$. It is clearly demonstrated that the enrichment of basis function near the crack tip results in a higher accuracy. Fig 5.2.3 shows the principal stress direction around the crack tip, as well as the stress concentration at the crack tip. Fig 5.2.4 and Fig 5.2.5 compare the σ_x and σ_y values ahead of crack tip obtained by different methods, which demonstrate that the incorporation of the enriched functions for the crack tip elements can better capture the singularity at the crack tip than linear base functions. Furthermore, as discussed in section 4.5.1, with the enriched functions, the jump or discontinuity across the crack surface for element which is partially cut by a crack can be captured.

Fig 5.2.3 The principal stress direction near the crack tip

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Fig 5.2.4 σ_x versus the distance from the crack tip

Fig 5.2.5 σ_y versus the distance from the crack tip

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5.2.2 Verification of the enriched NMM for cracking problems

In the previous section, the extrinsic enrichment function is added to the approximation through the partition of unity method. The effectiveness and accuracy of the proposed method in capturing the stresses around the flaw tip are demonstrated. In this section, two Brazilian tensile disc tests with or without pre-existing flaws and a four point shear specimen with pre-existing flaws are presented to demonstrate the capability of the enriched NMM in capturing the cracking processes. The Mohr-Coulomb criterion (section 4.5.2.3) is used for crack initiation case (elements without flaw tips) and the MTS-criterion (section 4.5.2.1) is used for propagation case (elements with flaw tips).

5.2.2.1 Brazilian tensile disc tests

The diameter of the disc is 5cm, while the length and width of the rigid platens are 5cm and 0.2cm, respectively for both intact and pre-fractured tests (Fig 5.2.6 a, Fig 5.2.7 a).

Fig 5.2.6 shows the simulated failure process of the intact disc model with 2358 manifold elements, 1300 MCs 1300 PCs and three loops. As illustrated in the figure, a tensile crack first initiates at the disc centre parallel to the loading due to the tensile stress concentration there (Fig 5.2.6 b). As the load increases, the crack keeps propagating along the loading direction until it reaches the top and bottom boundaries of the disc. Several scattered cracks with limited lengths also develop around the loading ends due to the stress concentrations there (Fig 5.2.6 c). The final failure pattern predicted by the developed NMM closely resembles that observed experimentally by Liu (2004) (Fig 5.2.6 d).

A 45° inclined pre-existing flaw with a length of 0.16 cm is cut at the disc centre for the fractured specimen (Fig 5.2.7 a). The vertical point loads are applied through platens at both ends of the discs. The material parameters of the intact samples are 10GPa for Young's modulus, 0.25 for Poisson's ratio, 35° for internal friction angle, 5.0MPa for cohesion, and 1.0MPa for tensile strength. The parameters of the joints are 35° for friction angle, 0.05 MPa for cohesion, and 1.0 MPa/m^{1/2} for the fracture toughness. The contact surfaces between the loading plates and the discs are assumed to be smooth, without friction and cohesion.

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Fig 5.2.7 shows the simulated failure process of the pre-fractured disc model with 2376 manifold elements, 1300 MCs, 1318 PCs and four loops. Due to the stress concentration around the pre-existing crack tips, two tensile wing cracks first initiate (Fig 5.2.7 b). With the increase of the applied load, the tensile wing cracks keep propagating and gradually curve towards the direction of the external loads. The final failure pattern obtained from the NMM is shown in Fig 5.2.7 c, which is very similar to the experimental result obtained by Al-Shayea (2005) (Fig 5.2.7 d).

5.2.2.2 A four-point shear specimen

A single-edge notched concrete specimen subjected to four-point shear (Fig 5.2.8), where a mixed-mode crack propagation is achieved, is numerically investigated by the NMM. The dimensions of the problem are:

$$b=0.2m, a=0.2b, c=0.2b, d=1.8b, e=2.2b, t=0.10m$$

where t is the specimen thickness and the material parameters are:

$$E=28,000 \text{ MPa}, \nu=0.1, G_f=145 \text{ N/m}$$

where E is the Young's modulus, ν is the Poisson's's ratio and G_f is the fracture energy.

Fig 5.2.9 shows the failure processes of four-point shear specimen predicted by the NMM with 1715 manifold elements, 927 MCs, 933 PCs and one loop. As illustrated in the figure, the crack starts propagating from the flaw tip in an inclined direction and when it leaves the region of high shear, it approaches the vertical direction and reaches the surface of the specimen to the left of the applied force F . The final crack trajectory predicted by the NMM model slightly deviates from that of the experimental result. That may be induced by the linear-elastic assumption associated with no traction forces at the fracture process zone in the model. Nevertheless, the final crack trajectory of a curvilinear shape predicted by the NMM generally fits the experimental result obtained by Carpinteri et al. (1993), as shown in Fig 5.2.10.

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Fig 5.2.6 Failure of Brazilian tensile disc test in intact specimen (a) NMM model (b) Crack initiation (c) Final failure (d) Experimental result by Liu (2004)

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Fig 5.2.7 Failure of Brazilian tensile disc test in pre-fractured specimen (a) NMM model (b) Crack initiation (c) Final failure (d) Experimental result by Al-Shayea (2005)

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Fig 5.2.8 A four-point shear specimen

Fig 5.2.9 Failure processes of a four-point shear specimen predicted by the NMM (a) before crack initiation (b) crack initiation (c) before failure (d) final failure

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Fig 5.2.10 Comparison of the crack path obtained from the NMM and the experimental test (Carpinteri et al. 1993)

5.3 Frictional crack initiation and propagation analysis using the enriched NMM

5.3.1 Introduction

Extensive and uncontrolled propagation of cracks in rocks can cause rock mass failure. Crack growth prediction has been a significant research topic and many studies have concentrated on either developing crack propagation theories (Erdogan and Sih 1963; Hussian 1974; Sih 1974) or developing techniques to calculate the stress intensity factors ahead of crack tips (Cartwright and Rooke 1974; Paris and Sih 1965; Smith and Raju 1998; Rice 1968; Rybicki and Kanninen 1977; Morais 2007).

Most of the previous studies have been focused on the problems associated with open flaws. According to Vallejo (1988), pre-existing flaws will close at a certain level of compressive stress and the flaw contact will become important when one considers considerable crack extension and propagation. Although great progress has been made in understanding the mechanisms of crack formation developed from flaws under compression (Horii and Nematnasser 1986; Bobet 2000;

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Bourne and Willemse 2001; Park and Bobet 2009), the dependence of the crack growth on factors such as the strength of the rock mass, the flaw inclination angle, the flaw friction angle and the loading condition, the evolution of secondary cracks and the effect of interactive forces between the flaw surfaces are poorly understood when dealing with frictional crack problems. Therefore, further investigation is needed. In this study, the pre-existing cracks are termed “flaws”. The Mohr-Coulomb criterion is adopted for both crack initiation and propagation.

5.3.2 Variational formulation of frictional cracks

Consider an elastic domain Ω with a pre-existing flaw as shown in Fig 5.3.1. The prescribed tractions F are imposed on the boundary Γ_F , whereas the prescribed displacements u (usually assumed to be zero) are imposed on the boundary Γ_u . Therefore, the stress field in domain Ω , σ , is related to the external loading F and the interaction forces (Fig 5.3.2) along the flaw Γ_c through the equilibrium equations:

$$\nabla \cdot \sigma + b = 0 \text{ on } \Omega \quad (5-19)$$

where b is the body force. With boundary conditions being:

$$u = 0 \text{ on } \Gamma_u, \quad \sigma \cdot n = F \text{ on } \Gamma_F \quad (5-20)$$

$$\sigma \cdot n^+ = -\sigma \cdot n^- = t^+ = -t^- = t, \quad \sigma \cdot \tau^+ = -\sigma \cdot \tau^- = f^+ = -f^- = f \text{ on } \Gamma_c \quad (5-21)$$

The orientation of the normal and the tangential stresses n^+ , n^- , τ^+ , τ^- are shown in Fig 5.3.2. t and f are the normal and tangential forces between the flaw surfaces and are dependent on the contact status. The NMM, which is based on the contact detection technique inherited from DDA, can accurately predict the contact status of the flaw faces. Based on the contact status, t and f can be obtained from the following criteria:

For open status ($w_1 > 0$), the normal and tangential forces t and f are:

$$t = f = 0 \quad (5-22)$$

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and for closed status, the normal force t is:

$$t = k_n (v^+ - v^-) = k_n w_1 \quad (5-23)$$

the tangential force f is dependent on the Coulomb friction law:

for the “sticking” state, i.e. $f \leq t \times \tan(\phi_j) + c_j$,

$$f = k_s (u^+ - u^-) = k_s w_2 \quad (5-24)$$

for the “slipping” state, i.e. $f > t \times \tan(\phi_j) + c_j$,

$$f = t \times \tan(\phi_j) + c_r \quad (5-25)$$

where k_n and k_s are the normal and tangential contact spring stiffness; w_1 and w_2 are the relative normal and tangential displacements of the contact pair; c_j and ϕ_j are the cohesion and the friction angle of the flaw, respectively; and c_r is the residual cohesion of the flaw after debonding, which is set to be zero in this study.

For an elastic material, the constitutive relationship between the stress and strain follows the generalized Hooke’s law:

$$\sigma = C_c \otimes \varepsilon \quad (5-26)$$

where C_c is Hooke’s tensor and ε is the infinitesimal strain tensor.

Assuming small deformations, the strain-displacement relation is:

$$\varepsilon = \nabla_s u \quad (5-27)$$

where ∇_s is the symmetric gradient tensor and u is the displacement vector.

For the numerical implementation, we now transform the strong form of the problem into the weak form. Using the principle of virtual work, the weak form of the problem is:

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$$\begin{aligned} \int_{\Omega} \delta \varepsilon^T \cdot \sigma d\Omega + \int_{\Omega} \delta u^T \cdot b d\Omega = \int_{\Gamma_F} \delta u^T \cdot F d\Gamma + \\ \int_{\Gamma_c^-} \delta u^T \cdot t^+ d\Gamma + \int_{\Gamma_c^-} \delta u^T \cdot t^- d\Gamma + \int_{\Gamma_c^+} \delta u^T \cdot f^+ d\Gamma + \int_{\Gamma_c^-} \delta u^T \cdot f^- d\Gamma \end{aligned} \quad (5-28)$$

Or, in a more compact form, given that $t = t^+ = -t^-$ and $f = f^+ = -f^-$:

$$\int_{\Omega} \delta \varepsilon^T \cdot \sigma d\Omega + \int_{\Omega} \delta u^T \cdot b d\Omega = \int_{\Gamma_F} \delta u^T \cdot F d\Gamma + \int_{\Gamma_c} \delta w_1^T \cdot t d\Gamma + \int_{\Gamma_c} \delta w_2^T \cdot f d\Gamma \quad (5-29)$$

Fig 5.3.1 Notations for a specimen containing a pre-existing flaw

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Fig 5.3.2 Close-up view of the flaw tip region

5.3.3 Numerical predictions for single closed flaw under compression

The model used for simulation represents a specimen geometry of 152.4×76.2 mm with a flaw length $2a=12.7$ mm (see Fig 5.3.3), which is of the same dimension as those studied by Bobet and Einstein (1998a, b) and Shen and Stephansson (1993). However, compared with the displacement discontinuity method they used, the NMM can give another insight and enable complete freedom of fracture initiation within the problem region. The material properties are listed in Table 5.3.1, which are based mostly on the work done by Bobet and Einstein (1997; 1998b). Some parameters will be controlled for parameter sensitivity analysis.

Table 5.3.1 Material properties of numerical simulation

	Modeling material
Young's modulus E (MPa)	5960
Poisson's's ratio ν	0.15
Tensile strength σ_t (MPa)	-18.1
Compressive strength	*
Cohesion of the flaw c_j (MPa)	*
Friction angle of the flaw ϕ_j	*

* indicates that those values will be adjusted during simulation.

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From σ_0 and σ_t , the Mohr-Coulomb criterion parameters c_i and ϕ_i , can be obtained using:

$$\phi_i = \sin^{-1}[(\sigma_0 + \sigma_t) / (\sigma_0 - \sigma_t)] \quad (5-30)$$

$$c_i = [-\sigma_0 \sigma_t / (\sigma_0 + \sigma_t)] \tan \phi_i \quad (5-31)$$

The mode II stress intensity K_{II} for a single closed flaw as shown in Fig 5.3.3 is (Maugis 1992):

$$K_{II} = -\sigma_v \sqrt{\pi a} / 2 \times \{ (1-k) \sin 2\beta - \mu [(1+k) + (1-k) \cos 2\beta] \} \quad (5-32)$$

where $k = \sigma_h / \sigma_v$ is the ratio of horizontal confining stress to vertical stress, a is half flaw length, β is the flaw inclination angle, and $\mu = \tan(\phi_i)$ is the friction coefficient for closed flaw (Fig 5.3.3).

Fig 5.3.3 Model geometry

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Fig 5.3.4 shows the variation of K_{II} with β , while keeping $\sigma_v = 20\text{MPa}$, $\phi_i = 30^\circ$ and $c_i = 0$. The results obtained from the NMM agree well with the theoretical results. A similarly good agreement is also shown in Fig 5.3.5, where the influence of the applied external stress on K_{II} is studied for two different flaw friction angles, $\phi_i = 0^\circ$ and 20° , while $\beta = 45^\circ$ and $c_i = 0$.

To examine the slippage and debonding processes of the flaw faces, another run consisting of $\beta = 45^\circ$, $\phi_i = 20^\circ$, with a different cohesion value ($c_i = 7.6\text{ MPa}$) is performed (Fig 5.3.6). Compared with the results shown in Figure Fig 5.3.5, due to the presence of cohesion and friction along the flaw faces, K_{II} will remain zero until the external stress reaches a critical magnitude when the Mohr-Coulomb law for the shear resistance in the crack plane is overcome, i.e, for run two (Fig 5.3.6), the normal stress between the flaw surfaces is $\sigma_n = 1/2\sigma$ and the shear stress along the crack surfaces is $\tau = 1/2\sigma$. The closed flaw will debond when:

$$\tau \geq \sigma_n \tan \phi_i + c_i \Rightarrow 0.182\sigma + 7.6 = 0.5\sigma \quad (5-33)$$

From this, the critical stress for debonding is determined to be $\sigma_v = 23.89\text{ MPa}$. Once the critical point is reached, the cohesion will be reduced and debonding will occur. Refer to the abrupt drop of K_{II} in Fig 5.3.6. After debonding, the variation of K_{II} follows Eq. (5-32).

The above comparison demonstrates that the NMM is capable of analyzing crack problems with a high degree of accuracy.

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Fig 5.3.4 Comparison of K_{II} for a single closed flaw in an infinite medium under uniaxial compression ($\sigma_v = 20$ MPa, $\phi_i = 30^\circ$, $c_i = 0$)

Fig 5.3.5 Comparison of K_{II} for a single closed crack in an infinite medium under uniaxial compression ($\beta = 45^\circ$, $\phi_i = 0$ or 20° , $c_i = 0$)

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Fig 5.3.6 Comparison of K_{II} for a single closed crack in an infinite medium ($\beta = 45^\circ$, $\phi_i = 20^\circ$, $c_i = 7.6$ MPa) under uniaxial compression

In Fig 5.3.7, the effect of the flaw inclination angles on K_{II} is demonstrated. For different flaw inclination angles, the respective maximum K_{II} occurs at different friction angles. The larger the friction angle, the larger the inclination angle is needed to achieve the maximum K_{II} (Fig 5.3.8). Fig 5.3.9 shows the variation of crack initiation stress with the flaw friction angle for three different flaw inclination angles. The numerical results indicate that the crack initiation stress increases with the flaw friction angle, which implies that the larger the flaw friction angle, the more difficult the growth of the flaw becomes.

To further investigate the crack growth phenomenon, the influences of the flaw friction angle, the ratio of the compressive strength σ_0 to the tensile strength σ_t , and the lateral confining stress σ_h are studied. The focus is placed on the effect of these factors on the crack initiation stress and the developed crack types. The crack initiation stress is defined as the applied vertical stress when the first crack initiates based on Mohr-Coulomb criterion. The flaw inclination angle is fixed to be $\beta = 45^\circ$ and the other parameters are chosen as follows: $\sigma_t = -10$ MPa, kept as constants for all the simulations; the ratio of the compressive strength to the tensile

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strength $|\sigma_0 / \sigma_t| = 1, 2, 3, 4, 5, 5.5, 6, 7, 8, 9, 10, 11$ and 12 , the friction angle $\phi_i = 0^\circ, 5^\circ, 10^\circ, 15^\circ, 20^\circ, 25^\circ, 30^\circ, 35^\circ, 40^\circ, 45^\circ$; the lateral confining stress $\sigma_h = 0, 1, 2.5, 4, 5, 7.5$ and 10 MPa.

The results obtained from uniaxial compression are plotted in Fig 5.3.10 and Fig 5.3.11. As shown in the figures, the flaw friction angle not only affects the crack initiation stress, but also sometimes changes the developed crack type, especially when the ratio of the compressive strength to the tensile strength $|\sigma_0 / \sigma_t|$ is between 5 and 7.

For uniaxial compression (Fig 5.3.10), when the ratio of $|\sigma_0 / \sigma_t|$ is smaller than 5, the initiation of shear cracks is preferred rather than tensile ones; when the ratio is larger than 7, tensile cracks will initiate first. Fig 5.3.10 also shows that the friction along the flaw surfaces tends to inhibit the crack from developing. With larger friction, a larger crack initiation stress is needed.

For the ratio of $|\sigma_0 / \sigma_t|$ between 5 and 7, either tensile cracks or shear cracks can initiate first depending on the flaw friction angle. This phenomenon is further illustrated in Fig 5.3.11 for $|\sigma_0 / \sigma_t| = 5.5$. As demonstrated in the figure, with an increase in the flaw friction angle, the crack initiation stress will also increase, which is also demonstrated in Fig 5.3.11. Such an effect is more prominent in the case of tensile cracks than shear cracks. When the friction angle is smaller than 10° , tensile cracks will initiate first. On the other hand, when the friction angle is larger than 10° , shear cracks will be the more favorable crack type. To conclude, the friction between the flaw surfaces not only inhibits the crack from developing, but also controls the type of crack initiated from the flaw.

The simulation results from biaxial loading are shown in Fig 5.3.12. It can be observed from the figure that the lateral confining stress can inhibit cracks from developing. A larger lateral confining stress leads to a larger crack initiation stress. Such a phenomenon is especially prominent for the case of tensile crack formation. When the lateral confining stress is increased to 5 MPa, the initiation of tensile cracks will be suppressed.

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Fig 5.3.7 Variation of K_{II} predicted by the NMM with the flaw inclination angle under uniaxial compression ($c_i=0$, $\phi_i=0^\circ, 15^\circ, 30^\circ, 45^\circ$)

Fig 5.3.8 Variation of the flaw inclination angle corresponding to K_{II} with the crack friction angle under uniaxial compression ($c_i=0$)

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Fig 5.3.9 Crack initiation stress of a single closed flaw under uniaxial compression ($c_i=0$)

Fig 5.3.10 Influence of the ratio of the compressive strength to the tensile strength $|\sigma_0 / \sigma_t|$ on the first crack initiation stress. The solid symbols indicate that the first cracks are shear, while the open symbols indicate that the first cracks are tensile ($c_i=0$)

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Fig 5.3.11 Tensile and shear crack initiation stress under uniaxial compression
($|\sigma_0 / \sigma_t| = 5.5$, $c_i = 0$)

Fig 5.3.12 Tensile and shear crack initiation stress under biaxial compression
($|\sigma_0 / \sigma_t| = 7.0$, $c_i = 0$, $\phi_i = 0$)

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5.3.3.1 Comparison with experimental results

To investigate the Mohr-Coulomb crack initiation criterion in predicting rock cracking behavior, the relevant predictions obtained from NMM model are compared with physical experimental results. As summarized by Wong and Einstein (2009b), the first observation of tensile crack initiation from pre-existing straight flaws under uniaxial compression was seen in Columbia Resin 39 (Bombolakis 1963). Crack initiation and propagation in pre-cracked samples under compression have been extensively studied experimentally on different materials, including both rock-like material, such as molded gypsum (Bobet and Einstein 1998b; Shen 1995; Reyes et al. 1991; Wong and Einstein 2007; sandstone-like molded barite (Wong and Chau 1998; Wong et al. 2001 etc. and natural rocks (such as sandstone (Petit and Barquins 1988); limestone (Ingraffea and Heuze 1980); granite (Martinez 1999); marble (Huang et al. 1990; Li et al. 2005) etc. In parallel with the above experimental studies on open flaws, physical tests have also been performed on closed frictional flaws (Shen 1995; Bobet and Einstein 1998b; Yang et al. 2008; Park and Bobet 2010).

With the use of high speed video technology (Wong and Einstein 2009a) during uniaxial compression loading tests on pre-cracked prismatic rock specimens, the details of the cracking mechanism (tensile/shear) and the development of the entire sequence of cracking, which often consists of a variety of different cracks, can be observed. Based on these observations, Wong and Einstein (2009b) classified the cracks emanating from pre-existing flaws into seven basic types (Fig 5.3.13). Unlike in other references which classify newly initiated cracks based on their temporal relationship, i.e. primary cracks and secondary cracks, the classification in Fig 5.3.13 is strictly in accordance with the crack trajectories and initiation mechanism (tensile/shear).

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Fig 5.3.13 Various crack types initiated from pre-existing flaws Wong and Einstein (2009b)

With reference to the above experimental observations, numerical simulations are performed using the NMM based on the Mohr-Coulomb criterion to model the cracking processes in rock specimens containing a single pre-existing flaw. The crack types obtained from this modeling are analyzed and discussed with regard to the initiation mechanism (tensile/shear).

1. Pure tensile wing cracks

The initiation of tensile wing cracks has been widely observed in both brittle and semi-brittle materials, such as Plaster of Paris (Lajtai 1974), glass (Ingraffea and Heuze 1980), and rocks: low and high porosity sandstone (Petit and Barquins 1988), marble (Li et al. 2005; Martinez 1999) molded gypsum and Carrara marble

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under uniaxial compression (Wong and Einstein 2009). According to the NMM predictions shown in section 5.3.3, when the ratio of the compressive strength to the tensile strength $|\sigma_0 / \sigma_t|$ is greater than 7, only tensile wing cracks will develop. The numerically modeled tensile wing cracks are initiated at the tip of the flaws and propagate in the direction of the loading in a stable manner.

Table 5.3.2 shows a comparison of the initiation stresses and crack initiation angles obtained from the physical experimental tests and our present numerical studies. It is observed that the experimental uncertainty of crack initiation stress is 1.6 MPa (Bobet 2000) and the uncertainty associated with the load step in the numerical simulation is 0.1 MPa. The predictions shown in Table 5.3.2 are thus comparable to those obtained from the physical experimental tests. Therefore, our study demonstrates that with the Mohr-Coulomb criterion, the NMM can accurately predict the crack initiation stress and propagation direction. The table also illustrates that, in agreement with the physical test results, the crack initiation angle decreases when the flaw inclination angle increases. The crack initiation angle θ is defined as shown in the schematic sketch in Fig 5.3.14. Fig 5.3.15 and Fig 5.3.16 show the crack trajectories predicted by the NMM for two models: one with a frictionless flaw and the other one with a frictional flaw. As shown in the figures, the model with a frictional flaw is associated with a larger crack initiation stress and a smaller crack initiation angle. Both the cracks shown in Fig 5.3.15 and Fig 5.3.16 can be categorized as a type 1 tensile crack according to Fig 5.3.13.

2. Pure shear crack

Pure shear cracks are not commonly observed in the physical tests. However, Wong and Einstein (2009b, c) observed such a crack type in pre-cracked Carrara marble. Previous work shows that when $|\sigma_0 / \sigma_t|$ is smaller than 5, shear cracks instead of tensile cracks will initiate first (Fig 5.3.10 and Fig 5.3.17). The pure shear crack initiation angles predicted by the Mohr-Coulomb criterion are material dependent ($|\sigma_0 / \sigma_t|$), as shown in Fig 5.3.18 for $\beta = 45^\circ$. The pure shear crack initiation angles predicted by the NMM for $\beta = 45^\circ$ ranged from 0° to 20° . Thus, when the shear cracks propagate, they tend to align with the pre-existing crack. As a

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result, the final pure shear cracks are usually quasi-parallel to the pre-existing crack as shown in Fig 5.3.17. This type of cracks can be categorized as a type 2 shear crack according to Fig 5.3.13.

Fig 5.3.14 Definition of crack initiation angle (θ)

Table 5.3.2 Comparison between physical tests and the NMM numerical models under uniaxial compression

Flaw inclination angle	Physical tests (from Bobet 2000)		NMM Model ($c_i=0, \phi_i=30^\circ, \sigma_o/\sigma_t =7$)	
	Initiation stress (MPa)	Initiation angle, ($^\circ$)	Initiation stress (MPa)	Initiation angle, ($^\circ$)
30°	25.9	85	25.6	79
45°	24.4	67	31.2	65
60°	28.7	54	38.8	52

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Fig 5.3.15 Numerical simulation of the development of tensile wing cracks under uniaxial compression ($c_i = 0, \phi_i = 0^\circ, \beta = 45^\circ, |\sigma_0 / \sigma_i| = 8$)

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Fig 5.3.16 Numerical simulation of the development of tensile wing cracks under uniaxial compression ($c_i = 0, \phi_i = 30^\circ, \beta = 45^\circ, |\sigma_0 / \sigma_t| = 8$)

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Fig 5.3.17 Numerical simulation of the development of pure shear cracks under uniaxial compression ($c_i=0, \phi_i=30^\circ, \beta=45^\circ, |\sigma_0 / \sigma_t|=3$)

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Fig 5.3.18 Variation of the pure shear crack initiation angle as a function of $|\sigma_0/\sigma_t|$ under uniaxial compression ($c_i=0$, $\phi_i=30^\circ$, $\beta=45^\circ$)

3. Mixed cracks

In addition to the pure tensile cracks, and shear cracks, mixed cracks consisting of both tensile and shear segments are also observed in the physical tests (Wong and Einstein 2009b). Two types of mixed crack types, namely coplanar shear-tensile cracks and tensile-oblique shear cracks, were obtained in our simulations.

The first type of mixed crack is a coplanar shear-tensile crack (Fig 5.3.19), which usually appears when the ratio of the compressive strength to the tensile strength $|\sigma_0/\sigma_t|$ is between 5 and 7, and the friction angle of the flaw is large and the inclination angle of the flaw is small. Under these conditions, the large frictional forces between the flaw surfaces will inhibit the tensile cracks from developing. Instead, shear cracks, which are generally co-planar with the pre-existing flaw, will initiate first (Fig 5.3.19 b). However, after the shear cracks have propagated for a short distance, the tensile cracks will take their preference (Fig 5.3.19 c, d), i.e. the further crack segments propagated from the tip of shear cracks are tensile in nature. This type of cracks can be categorized as a mixed tensile-shear crack according to Fig 5.3.13.

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Fig 5.3.19 Numerical simulation of mixed shear-tensile crack under uniaxial compression ($c_i=0$, $\phi_i=30^\circ$, $\beta=30^\circ$ $|\sigma_0 / \sigma_t| = 5$)

The second type of mixed crack is a tensile-oblique shear crack (Fig 5.3.20). This crack type usually develops when the ratio of the compressive strength to the tensile strength $|\sigma_0 / \sigma_t|$ is between 5 and 7, with a small flaw friction angle ϕ_i and a large flaw inclination angle. Under these conditions, the tensile wing cracks usually first initiate from the pre-existing flaw tips (Fig 5.3.20 b). After the tensile wing cracks propagate for a short distance (Fig 5.3.20 c), a pair of coplanar shear cracks will initiate from the tips of the pre-existing flaw (Fig 5.3.20 d). However, the shear cracks usually cannot propagate a long distance, because after the shear crack initiation, the previously developed tensile wing cracks will be pulled apart and the entire system will become unstable. In other words, both a type 1 tensile

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crack and a type 2 shear crack, as classified according to Fig 5.3.13, can develop from the same flaw tips.

From the above numerical analysis, the secondary cracks can be either tensile or shear in nature, depending mostly on the ratio of the compressive strength to the tensile strength of the material. The frictional properties and the inclination angle of the flaws may also affect the type and trajectory of the secondary cracks.

Fig 5.3.20 Numerical simulation of mixed tensile-oblique shear crack under uniaxial compression ($c_i = 0$, $\phi_i = 0^\circ$, $\beta = 60^\circ$, $|\sigma_0 / \sigma_t| = 6.5$)

4. More complicated crack types

In the present study, under certain conditions, both tensile cracks and shear cracks can initiate from the same flaw tip (see Fig 5.3.21). This complicated crack type usually appears when the ratio of the compressive strength to the tensile strength $|\sigma_0 / \sigma_t|$ is between 5 and 7, when the flaw friction and the flaw inclination angle are both small. Under these conditions, tensile wing cracks usually initiate

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and propagate first (Fig 5.3.21 b). Afterwards, a pair of oblique shear cracks will initiate from the tips of the pre-existing flaw and propagate away from the flaw (Fig 5.3.21 c). These oblique shear cracks can later either change into tensile cracks or keep on propagating as shear cracks, depending on the ratio of $|\sigma_0 / \sigma_t|$. For the ratio of $|\sigma_0 / \sigma_t|$ close to 7, the later developed oblique cracks will have a higher tendency to change into tensile cracks in response to further loading (Fig 5.3.21 d). The initial segment of this type of mixed crack belongs to type 3 shear crack according to Fig 5.3.13. In the literature, this type of crack is also commonly known as an anti-wing crack.

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Fig 5.3.21 Numerical simulation of the development of a tensile crack and a shear crack from the same flaw tip under uniaxial compression ($c_i = 0$, $\phi_i = 0^\circ$, $\beta = 30^\circ$, $|\sigma_0 / \sigma_t| = 5$)

5.3.4 Summary

In this section, based on the contact techniques inherited from DDA, special attention has been focused on the closed flaw problems to investigate the effects of the friction between the flaw surfaces on the crack growth such as the crack initiation stress, propagation direction and crack type. A parametric study has also been implemented to investigate the factors affecting the crack growth. Finally, the Mohr-Coulomb crack propagation criterion has been investigated by comparing the

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predictions obtained from the NMM with the experimental results. From the results, the following conclusions can be drawn:

1. Using the Mohr-Coulomb crack propagation criterion, the NMM can not only model the initiation of tensile wing cracks or shear cracks (initiation stresses and angles), but also predict mixed crack types.
2. Mode II stress intensity factor K_{II} decreases with an increase in the flaw friction.
3. Mode II stress intensity factor K_{II} increases with an increase in the flaw inclination angle first, which will decrease after reaching maximum.
4. An increase in the friction angle between the flaw surfaces will increase the inclination angle of the pre-existing flaw required to reach the maximum stress intensity factor.
5. The crack initiation stress will increase with an increase in the friction angle between the flaw surfaces. The effect is more prominent in tensile cracks.
6. The crack type observed is mostly dependent on the ratio of the compressive strength to the tensile strength of the material. However when the ratio of the compressive strength to the tensile strength is between 5 and 7, the lateral confining stress, the friction angle between the flaw surfaces and the inclination angle of the flaw may also affect the type of the crack developed.
7. When the ratio of the compressive strength to the tensile strength is between 5 and 7, the mixed type of cracks will typically develop. Oblique-type cracks, usually develop when the flaw inclination angle is large. However, coplanar-type cracks usually form when a small flaw inclination angle is encountered.
8. The secondary cracks are not always shear cracks but can also be tensile cracks.

5.4 Modeling cracking behavior of rock mass containing inclusions using the enriched NMM

5.4.1 Introduction

Inclusions are commonly found in natural materials such as rocks (hard or soft other minerals), as well as, man-made structure materials such as concrete (aggregate pieces). It is highly worthwhile to investigate the effects of these inclusions on the overall behavior (weakening or strengthening) of the materials, in particular the cracking process (crack initiation, propagation and coalescence) which usually finally leads to the failure of the structures such as slope sliding and rock fall. Despite that cracks and inclusions are both treated as discontinuities in the medium, the nature of these two discontinuities are different. The first one is obvious and denoted as “strong discontinuity”. The latter is not obvious (the displacement field is continuous but the stress and strain are sometimes discontinuous) and denoted as “weak discontinuity”. Hence, the treatment of these discontinuities in numerical modeling is of great interest.

Due to the capabilities of NMM in dealing with discontinuous problems, NMM has been successfully used for modeling both strong discontinuous problems (illustrated in last section) and weak discontinuous problems (Terada et al. 2007; Kurumatani and Terada 2009). However, in previous work, the weak discontinuities are treated the same as strong discontinuities except for the enforcement of the additional displacement compatibility condition along the interface. In the present work, the enriched method as adopted by An et al. (2013) and Nielsen (2012), which can conveniently treat the material discontinuities, is proposed to handle the material interfaces (weak discontinuities) within the NMM framework. Based on the crack initiation and propagation criteria, the crack identification method and the crack evolution technique developed in our previous studies, the cracking processes and the final failure patterns in the specimens containing single or double inclusions under uniaxial compression are investigated by the developed NMM. The feasibility of the proposed enriched method for simulating the material discontinuities is discussed by comparing the numerical results with the theoretical and laboratory test results.

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5.4.2 Treatment of weak discontinuity (material interface) in the NMM

In the traditional NMM, the material interface, which cuts the Mathematical covers into different parts, is treated equally as the strong discontinuity. To handle the displacement discontinuity across the material interface, additional techniques, such as the penalty method, the Lagrange multiplier method or the augmented Lagrange method are used (Terada et al. 2003; Terada et al. 2007; Kurumatani and Terada 2009). However, these techniques require a longer time for iteration and sometimes even lead to diverging results.

In this section, a more natural way to treat the material interfaces, which is adopted by XFEM (Sukumar et al. 2001; An et al. 2013) is employed. Instead of being split into two PCs, the PC which is fully cut by the material interface (weak discontinuity) keeps its integrity but its local displacement function is enriched. As shown in Fig 5.4.1, the local functions for the PCs which are intersected by the material interfaces (PCs marked by solid circles) are enriched. The associated enrichment function is based on the level set function (Sukumar et al. 2001) identifying the location of the material interfaces. The level set function for the circular inclusions adopted in the present work is:

$$f(x) = \|x - x_c\| - r_c \quad (5-34)$$

where x_c is the inclusion center and r_c is the inclusion radius. The interface is defined by $f(x) = 0$. Based on the level set function, several possible enrichment functions can be constructed, such as the absolute value of the level set function. However, a direct use of this function results in a convergence problem associated with blending elements as discussed by Fries (2008). Thus, a smoothing as suggested by Moes et al. (2003) is adopted, such that the enrichment function becomes:

$$F(x) = \sum_i \varphi_i(x) |f_i| - \left| \sum_i \varphi_i(x) f_i \right| \quad (5-35)$$

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The above not only solves the convergence problem associated with blending elements, but also achieves a satisfactory convergence rate as discussed by Moes et al. (2003). As a result, the local displacement function for the PCs containing a weak discontinuity (material interface) becomes:

$$u(x) = \sum_i \varphi_i(x) (d_i + F(x)a_i) \quad (5-36)$$

where d_i , are the classical degrees of freedom, and a_i are the additional degrees of freedom.

5.4.3 Governing equations

5.4.3.1 Weak form

Let u be the displacement solution for the stated linear elastostatic problem. Let $u \in V$ be the displacement trial function, and $\delta u \in V_0$ be any set of admissible test functions. The space $V = H_1(\Omega)$ is the Sobolev space of functions with square-integrable first derivatives in Ω and $V_0 = H_1^0(\Omega)$ is the Sobolev space functions with square-integrable first derivatives in Ω and vanishing values on the essential boundary Γ_u . The weak form of the governing equation and associated boundary conditions can be expressed as:

$$\int_{\Omega} \sigma(u) : \varepsilon(\delta u) d\Omega + \lambda \int_{\Gamma_u} (u - \bar{u}) \cdot \delta u d\Gamma = \int_{\Gamma_t} \bar{t} \cdot \delta u d\Gamma + \int_{\Omega} b : \varepsilon(\delta u) d\Omega \quad \forall \delta u \in V_0 \quad (5-37)$$

where σ and ε are stress and strain tensor, respectively. b is the body force and $\bar{u}$, $\bar{t}$ are the essential boundary condition on Γ_u and traction boundary condition on Γ_t , respectively. λ is the real penalty number which is chosen to superimpose the essential boundary conditions.

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5.4.3.2 Discrete system

In the NMM, finite-dimensional subspaces $V^h \subset V$ and $V_0^h \subset V_0$ are used as the approximating trial and test spaces. Using the Galerkin method (Lin 2003), the weak form for the discrete problem can be stated as:

Find $u^h \in V^h \subset V$ such that

$$\int_{\Omega^h} \sigma(u^h) : \varepsilon(\delta u^h) d\Omega + \lambda \int_{\Gamma_u} (u^h - \bar{u}) \cdot \delta u^h d\Gamma = \int_{\Gamma_t} \bar{t} \cdot \delta u^h d\Gamma + \int_{\Omega^h} b \cdot \delta u^h d\Omega \quad \forall \delta u^h \in V_0^h \subset V_0 \quad (5-38)$$

The trial function u^h as well as the test functions δu^h are expressed as

$$u^h(x) = \sum_i \varphi_i(x)(d_i + F(x)a_i) \quad (5-39)$$

$$\delta u^h(x) = \sum_i \varphi_i(x)(\delta d_i + F(x)\delta a_i) \quad (5-40)$$

Substituting the trial and test functions of Eq. (5-39) and (5-40) into Eq. (10), and using the arbitrariness of the test functions, the following discrete system is obtained:

$$Kd=f \quad (5-41)$$

where

$$K_{ij} = \int_{\Omega^h} B_i^T D B_j d\Omega + \lambda \int_{\Gamma_u} (\varphi_i)^T \cdot (\varphi_j) d\Gamma \quad (5-42)$$

$$f_i = \int_{\Gamma_t^h} \hat{\phi}_i \cdot \bar{t} d\Gamma + \int_{\Omega^h} \hat{\phi}_i \cdot b d\Omega + \lambda \int_{\Gamma_t^h} \hat{\phi}_i \cdot \bar{u} d\Gamma \quad (5-43)$$

where $\hat{\phi}_i \equiv \varphi_i$ for a finite-element displacement degree of freedom, and $\hat{\phi}_i \equiv \varphi_i F$ for an enriched degree of freedom. D is the constitutive matrix for an isotropic linear elastic material, and the matrix B_i is:

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$$B_i = \begin{bmatrix} \hat{\phi}_{i,x} & 0 \\ 0 & \hat{\phi}_{i,y} \\ \hat{\phi}_{i,y} & \hat{\phi}_{i,x} \end{bmatrix} \quad (5-44)$$

5.4.4 Calibration of the proposed enriched method for inclusions

When a specimen containing an inclusion is loaded, cracking occurs at the location of largest tensile or shear stress concentration when the Mohr-Coulomb strength of the material at that point is exceeded. Since the inclusion and matrix typically have different mechanical properties, especially the modulus of elasticity, thermal coefficient, and hardening rate in addition to strength (Mitsui et al. 1994), it is, therefore, important to understand if any stress concentrations develop at the interface because of these mechanical differences. Under the linear elastic assumption, the property that has the greatest influence on the internal stress distribution in a composite material is the difference between the elastic constants of the inclusion and the surrounding matrix (Neville 1998).

Before using the developed NMM to analyze the cracking behavior associated with a plate containing one or two circular inclusions with different stiffness under compression, we first verify the capability of the developed NMM in capturing the weak discontinuity across the interface. Then the cracking events (crack initiation and propagation), crack mechanisms (tensile or shear) and the coalescence processes associated with an inclusion and its surrounding matrix are investigated in specimens containing either one or two inclusions.

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Fig 5.4.1 Illustration of treatment of an inclusion in the NMM. PCs marked by solid circles are enriched by the interface enrichment function.

5.4.4.1 Verification of the developed NMM in capturing the weak discontinuity

A plate containing a circular inclusion under uniaxial compression (see Fig 5.4.2) is studied for verification purpose. The dimensions of the plate are $W_1=W_2=2\text{m}$. The radius (r) of the inclusion is 0.1m . The Poisson's ratio (ν) of both the plate and the inclusion is 0.25 , and the Young's modulus of the plate E_1 is 1 MPa and that of the inclusion E_2 is varied during simulation. A mesh with 7538 MCs, 7490 common PCs, 48 enriched PCs and 14678 manifold elements is selected. Four different cases of different stiffness ratio between the plate and the inclusion E_1/E_2 are considered. Soft inclusion problems with $E_1/E_2=10$ and a hole $E_2=0$, and stiff inclusion problems with $E_1/E_2=0.1$ and a rigid inclusion $E_2=\infty$.

By sampling the measurement points along the direction of $\theta=0$ from the center of the inclusion to the edge of the plate at an interval of 0.05m , the circumferential stresses σ_θ at the measurement points due to different inclusion stiffness are obtained and compared with the theoretical results (Sukumar et al. 2001) as shown in Fig 5.4.3, Fig 5.4.4 and Fig 5.4.5. As illustrated in the figures, by incorporating the enriched method to treat the weak discontinuity, the NMM can

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accurately capture the stress discontinuity (weak across the interface for inclusions with varied stiffness. In particular, for soft inclusions, the maximum stress concentration appears at the interface along direction of $\theta=0$ (and $\theta=\pi$). For hard inclusions, instead of the stress concentration occurred at the interface along direction of $\theta=0$ (and $\theta=\pi$), a stress dissipation as stress drop at interface surface (Fig 5.4.3) is obtained.

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(b)

Fig 5.4.2 Circular inclusion in a plate under uniaxial compression (a) schematic illustration (b) NMM model

Fig 5.4.3 σ_θ variation along direction of $\theta=0$ for a rigid inclusion ($E_2 = \infty$)

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Fig 5.4.4 σ_θ variation along direction of $\theta=0$ for a hole ($E_2 = 0$)

Fig 5.4.5 σ_θ variation along direction of $\theta=0$ for different stiffness ratio

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5.4.5 Numerical results

Based on the enriched method to treat the material interface, the Mohr-Coulomb crack initiation criterion, and the crack treating technique, the cracking behavior of specimens containing single or double inclusions under uniaxial compression is numerically studied by the developed NMM. The crack initiation and propagation of specimens containing single inclusions with varied stiffness are studied first. Thereafter an investigation of the crack coalescence of double inclusions with weak stiffness is investigated. The relevant predictions obtained from the NMM are compared with the physical experimental results obtained by Janeiro (2009) and Janeiro and Einstein (2010).

The dimensions of the numerical model are the same as those of the physical models adopted by Janeiro (2009) which have a width W_1 of 76.2 mm and a length W_2 of 152.4 mm. The radius of the inclusion is 12.7 mm for single inclusion (as shown in Figure 8) and 6.35 mm for double inclusions (as shown in Fig 5.4.6). For the single inclusion problem, a mesh with 1668 MCs, 1632 common PCs, 36 enriched PCs and 3110 manifold elements is selected, while for the double inclusions problem, a mesh with 1674 MCs, 1634 common PCs, 40 enriched PCs and 3116 manifold elements is selected. The material properties are listed in Table 5.4.1. The ultracal[®] and plaster material represent the stiff and weak inclusions respectively.

5.4.5.1 Single inclusion

Fig 5.4.7 presents the typical cracking sequences and the respective applied stress σ_v at each stage predicted by the NMM in a specimen containing a single weak (plaster) inclusion under uniaxial vertical loading. As shown in the figure, the cracking behavior can be categorized into the following stages. First, vertical tensile cracks initiate at the top and bottom of the interface and propagate to the direction of the external load (Fig 5.4.7 a). As the load increases, shear cracks initiate at the two sides (beside $\theta=0$) of the interface and additional tensile cracks initiate within the inclusion (Fig 5.4.7 b). As the load continues to increase, the previously developed cracks keep on propagating. Particularly the central tensile crack initiated within the inclusion reaches the interface and coalesces with the earlier initiated

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tensile cracks (Fig 5.4.7 c). Finally, the specimen fails when the maximum stress is reached (Fig 5.4.7 d). These cracking processes predicted by NMM generally agree with the experimental results obtained by Janeiro (2009), as shown in Fig 5.4.9. As demonstrated in the figure, the crack initiation stress and maximum stress predicted by the NMM are 18.2 MPa and 32.0 MPa respectively, which are only 9% and 6% different from the physical results of 20.0 MPa and 34.0 MPa. Considering the inherent uncertainty of the modeled rock properties, the predictions by the NMM simulation are generally acceptable. However, rock failure problems with inclusions are always more complicated. The assumptions adopted in this study may still deviate from the material behavior in reality. Therefore, more thorough validations of the developed NMM are still needed.

Similarly, the cracking sequences in a specimen containing a stiff inclusion (Ultracal) obtained from the NMM and experiments are presented in Fig 5.4.8 and Fig 5.4.9 b respectively. As illustrated in the figures, for both models, tensile cracks first initiate in the interior of the matrix instead of from the inclusion interface. This can be explained by the theoretical results shown in Fig 5.4.10, from which the maximum tensile stress is found to occur in the interior of the matrix, but a short distance away from the interface. The different location of the first cracks in numerical and physical tests may be caused by the loading condition, particularly the edge effect in the physical tests. After these first tensile cracks have propagated for several steps, several new tensile cracks initiate at a distance away from the top and bottom of the interface in the numerical model (Fig 5.4.8 b), while in the physical test, debonding along the interface occurs prior to the initiation of new tensile cracks. When the load increases, in the numerical model, the previously developed cracks keep on propagating until those cracks above and below the inclusion reach the interface (Fig 5.4.8 c, d). In the physical test, after debonding, tensile cracks and shear cracks, which are similar to those observed in the specimen containing a weak inclusion as shown in Fig 5.4.7, develop. This discrepancy between the numerical and physical test results could be caused by the debonding which is generally observed during the physical tests. Although the enriched method used in this chapter can capture the weak discontinuity across the interface, there is still a discrepancy between the numerically modeled interface and the real

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grain interface. Except the weak discontinuity across the interface, the elements containing the interface are the same as the other intact elements. Therefore, the enriched method used in this study is hard to capture the debonding along the interface. After debonding, the stiff inclusion becomes more like a hole which makes the subsequent cracking behavior similar to that of a weak inclusion. Note also that the crack initiation stress and the maximum stress predicted by the NMM are 27.8 MPa and 34.0 MPa respectively, which are only 4.1% and 6.3% different from the physical results of 29.0 MPa and 32.0 MPa.

Table 5.4.1 Material properties of numerical simulation (Janeiro 2009)

	Hydrocal [®]	Ultracal [®]	Plaster
Young's modulus E (GPa)	15.0	29	10.5
Poisson's's ratio ν	0.25	0.25	0.25
Tensile strength σ_t (MPa)	-2.4	-3.9	-2.9
Compressive strength σ_0 (MPa)	37.2	91.1	28.6

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Fig 5.4.6 Physical model of specimen containing double inclusions

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Fig 5.4.7 Crack sequences for the weak (Plaster) single inclusion obtained by NMM. T=tensile crack, S=shear crack (a) initiation of first tensile cracks (b) initiation of shear cracks (c) status before coalescence (d) status at maximum stress

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Fig 5.4.8 Crack sequences for the stiff (Ultracal) single inclusion obtained by NMM. T=tensile crack, S=shear crack (a) initiation of first tensile cracks (b) initiation of secondary tensile cracks (c) status before coalescence (d) status at maximum stress

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Fig 5.4.9 Physical test results of hydrocal specimens containing a single inclusion (a) Plaster inclusion (b) Ultracal inclusion (Janeiro 2009) All the identifiable new cracks are identified by reference letters, e.g., B, C, etc. Each letter is then followed by a letter T or S in parentheses, which refers to either the tensile mode or the shear mode of crack initiation, respectively. The sequence of crack initiation is indicated by numbers shown as subscripts. The first crack to initiate is designated as 1, the second crack as 2, etc (Wong and Einstein 2009a).

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Fig 5.4.10 The tangential σ_θ and radial σ_r stress concentration factors for a stiff inclusion ($E_2/E_1=2$). (Compression is negative and tension is positive).

5.4.5.2 Double inclusions

The cracking sequences and respective applied stresses in a specimen containing double weak inclusions obtained from the NMM are presented in Fig 5.4.11. As shown in the figure, when the applied stress is smaller than 29.6MPa, the cracking behavior of the double inclusions behaves similar to that of the single inclusion (Fig 5.4.7). First, tensile cracks initiate at the top and bottom of the interfaces (Fig 5.4.11 a). Later, additional tensile cracks initiate at the center of the inclusions and shear cracks initiate at the two sides of the interfaces (Fig 5.4.11 b, c). However, as the applied load further increases ($\sigma_v=29.6\text{MPa}$), the previously developed neighboring shear cracks emanating from the two interfaces coalesce. After the coalescence, two new tensile cracks initiate at the junctions of the coalesced shear cracks and the cracks at the side of both interfaces open. The specimen then reaches the maximum applied stress (Fig 5.4.11 d).

Compared with the experimental result shown in Fig 5.4.12, the NNM combined with the enriched method can predict not only the simple cracking

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behavior (crack initiation and propagation) for double weak inclusions, but also the complex cracking behavior (crack coalescence). The final mode of specimen failure is predicted as well. Based on simplified assumptions, the crack initiation stress and the maximum stress predicted by the NMM are 28.0 MPa and 32.0 MPa respectively, which are 7.6% and 4.3% different from the physical results of 30.32 MPa and 30.64 MPa (Janeiro 2009, 2010). However, further validations are still needed to ensure the good results are not obtained by coincidence.

5.4.6 Summaries

In this section, the NMM has been extended to investigate the cracking behavior of specimens containing either one or double inclusions under uniaxial compression. We incorporated the enriched method as presented by XFEM into the NNM framework to treat the weak discontinuities across the material interfaces. For validating the efficiency of the proposed method, the numerical results are compared with both the theoretical and physical test results, based on which the following conclusions can be drawn:

1. The enriched method proposed to treat the material discontinuities within the NMM framework is shown to be able to capture the weak discontinuities across the interfaces for varied stiffness ratios. Since the enriched method treats the material discontinuities in a natural way, no additional PCs and manifold elements are formed, as well as no additional iterations are induced by enforcing the displacement compatibility condition as in other penalty weak discontinuity methods.

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Fig 5.4.11 Crack sequences for the weak (Plaster) double inclusions obtained by NMM. T=tensile crack, S=shear crack (a) initiation of first tensile cracks (b) initiation of shear cracks (c) status before shear coalescence (d) status at maximum stress

2. The comparisons between the numerical results and the physical tests (Janeiro 2009) reveal that by incorporating the proposed enriched method and the crack treating techniques, the NNM can predict the cracking behavior such as crack initiation stress, peak stress and crack type in specimens containing either single or double weak inclusions. However, the proposed enriched method still cannot realistically represent the material interface. Although it can successfully capture the stress discontinuities

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across the interface, no “real” interface exists in the NMM treatment. As a result, the debonding phenomenon which is frequently observed in physical tests is hard to be captured by the proposed method. Since after debonding, the inclusion behaves like a hole (similar to weak inclusion), the proposed method cannot satisfactorily predict the cracking behavior in specimens containing stiff inclusions.

Fig 5.4.12 Test results of Hydrocal specimen containing double weak inclusions (Janeiro 2009)

5.5 Conclusions

Due to the capabilities of NMM in dealing with discontinuous problems, NMM has been successfully extended for modeling both strong discontinuum and weak discontinuum problems. Based on the partition of unity method, using the extrinsic enrichment method, the analytical solution around the crack tip is added to the trial function for the displacement approximation. Furthermore, an enriched

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method which can conveniently treat the material discontinuities, is also incorporated to handle the material interfaces (weak discontinuities). These methods combining the crack initiation and propagation criteria, the crack identification method and the crack evolution technique, the cracking processes and the final failure patterns in the specimens containing pre-existing closed flaw or inclusions under uniaxial compression are investigated. The numerical results illustrated that:

1. By adopting a two-cover system, the NMM can easily capture the strong discontinuity between the crack surfaces in a direct way without further incorporating enrichment functions for most cases, which is typically necessary for other partition of unity methods.
2. With the crack identification method and crack evolution technique, the NMM is able to model not only simple cracking problems (single crack initiation and propagation) but also coalescence of cracks emanating from the inclusions.
3. The enriched method proposed to treat the material discontinuities within the NMM framework is shown to be able to capture the weak discontinuities across the interfaces with varied stiffness ratios. Since the enriched method treats the material discontinuities in a natural way, no additional PCs and manifold elements are formed, as well as no additional iterations are induced by enforcing the displacement compatibility condition as in other penalty weak discontinuity methods.
4. Rock failure problems with inclusions are always more complicated. The assumptions adopted in this study may still deviate from the material behavior in reality. Therefore, more thorough validations of the developed NMM are still needed.

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6.1 Introduction

Geomaterials are typically heterogeneous containing pores and discontinuities of different scales, which result in a complex stress-strain state and different damage behaviors. When describing the behavior of geological materials under different loading conditions, various forms of localized plastic deformation, including loosening and compaction, development of dilatancy zones, as well as microscopic fracturing should be taken into consideration. With the increase of plastic deformation, damage accumulates. As such, the geomaterial loses its strength which is accompanied by the initiation, propagation and coalescence of macro-fractures, leading to eventual failure.

The analysis of materials that exhibit inelastic and plastic behavior through strain localization, followed by the development and coalescence of micro-fractures into macro-fractures has challenged the researchers for several decades. Reproducing the gradual transition from a continuous to a discontinuous matter, and the interaction between natural fractures and induced fractures possessing micro and macro components is not an easy task.

In this chapter, the Drucker-Prager yield model (Drucker and Prager 1952) with a tensile cut-off is adopted to describe the geomaterial behavior beyond the elastic limit. A damage function is used to describe the process of damage accumulation and material deterioration associated with plastic deformation. As damage accumulates, a macro-crack will develop along the damage zone when the crack initiation criterion is satisfied. Based on the merits of NMM in dealing with discontinuities, fracture propagation is investigated. Towards the end of this chapter, the failure modes of pre-flawed marble specimens are studied using the developed NMM and compared with the laboratory test results.

6.2 Constitutive model

The material considered in the present study is assumed to be an elastic-plastic continuum, which yields according to the Drucker-Prager criterion with a tensile cut-off. For a more realistic simulation of the dilation property of the rock, the shear flow rule is non-associated and the tensile flow rule is associated (Melenk and Babuska 1996; Chen and Han 1988). Thus,

$$F^s = \alpha_\phi I_1 + \sqrt{J_2} - Y = 0, \quad F^t = \sigma_1 - \sigma_t = 0 \quad (6-1)$$

F^s and F^t are the Drucker-Prager yield criterion and the tensile failure criterion, respectively. I_1 and J_2 are the first invariant of the stress state and the second invariant of the deviatoric stress tensor, respectively. σ_1 is the maximum principal stress component (assuming the tensile stress to be positive). α_ϕ and Y are the material constants determined by the angle of internal friction ϕ_y and the cohesive strength c_y of the material for determining the yield surface. The Drucker-Prager yield criterion in the principal stress space is a right circular cone equally inclined to the principal stress axes as shown in Fig 6.2.1. The tensile cut-off truncates all parts of the cone for which the principal stress σ_1 is larger than the tensile yield strength σ_t . The yield state depends on the location of the stress state point. In the analysis, four stress states can be classified (Fig 6.2.1).

- (1) elastic state for which $F^s < 0$ and $F^t < 0$, corresponding to domain ①
- (2) elastic-plastic state for which $F^s = 0$ and $F^t < 0$, corresponding to domain ②, in which shear failure is declared and the stress point is brought back to the surface $F^s = 0$ using a shear flow rule.
- (3) elastic-plastic state for which $F^t = 0$ and $F^s < 0$, corresponding to domain ③, in which tensile failure takes place, and the stress point is brought back to the surface $F^t = 0$ using a tensile flow rule.
- (4) elastic-plastic state for which $F^s = 0$ and $F^t = 0$, corresponding to domain ④, in which tensile and shear failures take place simultaneously, and the mixed flow rule is used to bring back the stress point to domain ④.

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Fig 6.2.1 Drucker-Prager yield surface with tensile cut-off

As stress states (1), (2) and (3) can be considered as special cases of stress state (4), the constitutive relation for stress state (4) is discussed in detail below. After the initial yielding, the material behavior will become partially elastic and partially plastic. For any stress increment, any change of strain is assumed to be attributed to both elastic and plastic components, so that

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p \quad (6-2)$$

The elastic strain increment $d\varepsilon^e$ is simply related to the stress increment $d\sigma$ by the Hooke's law

$$d\varepsilon^e = D^{-1}d\sigma \quad (6-3)$$

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where D is the usual material elasticity matrix. The plastic strain can then further be decomposed into a shear component $d\varepsilon^{sp}$ ($d\varepsilon_1^p - d\varepsilon_3^p$) and a tensile component $d\varepsilon^{tp}$ ($d\varepsilon_3^p$). As mentioned earlier, the shear yield obeys the non-associated plastic flow and the tensile yield obeys the associated plastic flow, thus

$$d\varepsilon^{sp} = \lambda_s \frac{\partial g^s}{\partial \sigma}, \quad d\varepsilon^{tp} = \lambda_t \frac{\partial F^t}{\partial \sigma} \quad (6-4)$$

where λ_s and λ_t are proportionality constants. g^s is the shear potential function, corresponding, in general, to a non-associated flow rule with the form

$$g^s = \alpha_\psi I_1 + \sqrt{J_2} \quad (6-5)$$

where α_ψ is the material constant determined by the dilation angle of the material ψ .

Substituting Eq. (6-3) and Eq. (6-4) into Eq. (6-2) yields

$$d\sigma = Dd\varepsilon - \lambda_s D \frac{\partial g^s}{\partial \sigma} - \lambda_t D \frac{\partial F^t}{\partial \sigma} \quad (6-6)$$

Using Eq. (6-6) together with the yield conditions, we obtain

$$dF^s = \frac{\partial F^s}{\partial \sigma} d\sigma - A_s \lambda_s = 0, \quad dF^t = \frac{\partial F^t}{\partial \sigma} d\sigma - A_t \lambda_t = 0 \quad (6-7)$$

where A_s, A_t are the hardening/softening parameters for shear yielding and tensile yielding, respectively. Then

$$\begin{aligned} \left(\frac{\partial F^s}{\partial \sigma}\right)^T Dd\varepsilon - \lambda_s \left(\frac{\partial F^s}{\partial \sigma}\right)^T D \frac{\partial g^s}{\partial \sigma} - \lambda_t \left(\frac{\partial F^s}{\partial \sigma}\right)^T D \frac{\partial F^t}{\partial \sigma} - A_s \lambda_s &= 0 \\ \left(\frac{\partial F^t}{\partial \sigma}\right)^T Dd\varepsilon - \lambda_s \left(\frac{\partial F^t}{\partial \sigma}\right)^T D \frac{\partial g^s}{\partial \sigma} - \lambda_t \left(\frac{\partial F^t}{\partial \sigma}\right)^T D \frac{\partial F^t}{\partial \sigma} - A_t \lambda_t &= 0 \end{aligned} \quad (6-8)$$

In this study, assuming the shear and tensile hardening/softening parameters are dependent on the shear plastic strain and the tensile plastic strain, then

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$$A_s = (-1) \frac{\partial F^s}{\partial \varepsilon_s^p} \frac{\partial g^s}{\partial \tau}, \quad A_t = (-1) \frac{\partial F^t}{\partial \varepsilon_1^{pt}} \frac{\partial F^t}{\partial \sigma_1} \quad (6-9)$$

where ε_1^{pt} is the principal plastic tensile strain; ε_s^p and τ are the plastic shear strain and shear stress respectively which can be expressed as (Venneer and Borst 1984):

$$\begin{aligned} \varepsilon^{ps} &= \left\{ \frac{1}{2} (\varepsilon_1^{ps} - \varepsilon_m^{ps})^2 + \frac{1}{2} (\varepsilon_m^{ps})^2 + \frac{1}{2} (\varepsilon_3^{ps} - \varepsilon_m^{ps})^2 \right\}^{1/2} \\ \varepsilon_m^{ps} &= \frac{1}{3} (\varepsilon_1^{ps} + \varepsilon_3^{ps}) \\ \tau &= \sqrt{J_2} \end{aligned} \quad (6-10)$$

where ε_j^{ps} , $j=1, 3$ are the principal plastic shear strain. From Eq. (6-6) and Eq. (6-8), we can obtain

$$\begin{aligned} d\sigma &= \left[D - \frac{Dm'(m^T - \gamma_t)D}{(m^T - \gamma_t)Dm' + A_s} - \frac{Dn(n^T - \gamma_s)D}{(n^T - \gamma_s)Dn + A_t} \right] d\varepsilon \\ &= (D - D_{st})d\varepsilon \end{aligned} \quad (6-11)$$

where

$$\begin{aligned} m &= \frac{\partial F^s}{\partial \sigma}, \quad m' = \frac{\partial g^s}{\partial \sigma}, \quad n = \frac{\partial F^t}{\partial \sigma} \\ \gamma_s &= \frac{n^T Dm' m^T}{m^T Dm' + A_s}, \quad \gamma_t = \frac{m^T Dnn^T}{n^T Dn + A_s} \end{aligned} \quad (6-12)$$

Now for the case where the material is in an elastic-plastic state (2), $\lambda_t = 0$, and from the first equation of Eq. (6-8), the following is obtained

$$d\sigma = \left(D - \frac{Dm' m^T D}{A_s + m^T Dm'} \right) d\varepsilon = (D - D_s) d\varepsilon \quad (6-13)$$

For the case where the material is in an elastic-tensile yield state (3), $\lambda_s = 0$, and from the second equation of Eq. (6-8) the following is similarly obtained

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$$d\sigma = (D - \frac{Dnn^T D}{A_t + n^T Dn})d\varepsilon = (D - D_t)d\varepsilon \quad (6-14)$$

6.3 Crack initiation criterion

As mentioned in section 4.5.2, many crack initiation criteria have been developed. However, most of the existing criteria do not differentiate the development process of tensile crack from that of shear crack. As mentioned by Stefanov (2004), the tensile crack growth is usually caused by a rise of tensile stress near pores and cracks, while the shear crack growth is usually caused by the coalescences of micro-tensile cracks. Using the high speed camera and the scanning electron microscope techniques, Wong and Einstein (2009a) also observed similar phenomena in pre-cracked Carrara marble loaded under uniaxial compression.

In this chapter, the modified Mohr-Coulomb criterion with a tensile cut-off which is based on the premise that the crack initiation depends on the local stress relative to the strength of material rather than on the stress intensity factors, is hereby adopted. For modeling the course of shear crack evolution, a damage function is incorporated. Using such a crack initiation criterion, the significant differences in response to tensile and compressive loadings are captured. The following gives a detailed description of the crack initiation criterion.

As shown in Fig 6.3.1, a crack will initiate when the following conditions are satisfied.

For shear cracking (Stefanov 2004)

$$\begin{cases} R(1-D_\varepsilon) = r \\ \sigma_3 \geq -\sigma_t^* \end{cases} \quad (6-15)$$

For tensile cracking

$$\sigma_3 = -\sigma_t^* \quad (6-16)$$

where σ_t^* is the peak tensile strength of the material, $D_\varepsilon = \lambda \varepsilon^{ps} / \varepsilon^*$ is the damage function, ε^{ps} is the accumulated plastic shear strain, ε^* is the limiting value of

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plastic shear strain, λ is an empirical parameter, and R and r (Fig 6.3.1) can be expressed as:

$$R = c^* \cos \phi' + \frac{\sigma_x + \sigma_y}{2} \sin \phi' \quad (6-17)$$

$$r = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

where c^* and ϕ' are the critical cohesion and friction angle respectively, related to shear crack initiation.

From Fig 6.3.1, when the conditions expressed in Eq. (6-15) are satisfied, the crack initiation angle θ will be:

$$\theta = \pi/4 + \phi'/2 \quad (6-18)$$

where θ is defined as the angle measured counter-clockwise from the first principal stress to the crack initiation direction.

When the condition expressed in Eq. (6-16) is satisfied, the crack initiation direction will be perpendicular to the direction of the third principal stress.

Fig 6.3.1 Illustration of the modified Mohr-Coulomb crack initiation criterion

6.4 Numerical results

The numerical examples in this section are intended to highlight the advanced features of the NMM for both continuum and discontinuum problems and to investigate the cracking process under various loading conditions in elastic brittle-plastic materials. To demonstrate this, the cracking and failure processes in coarse and medium marbles containing different pre-existing discontinuities subjected to triaxial tests are modeled. Using the Drucker-Prager model, the post-limiting elastic deformation and the damage accumulation procedure are simulated. Using the modified Mohr-Coulomb crack initiation criterion, the tensile and shear crack development, as well as the mechanisms of the secondary crack development are investigated. After the crack extension, the coalescences of pre-existing flaws and the failure behaviors of the specimen are discussed.

6.4.1 Geometry and material parameters of numerical specimens

The axisymmetrical model investigated in the present study is based on the flawed marble specimens tested by Yang et al. (2008). In the test, the cylinder samples are used. Due to the complication of the 3D model, 2D axisymmetrical NMM model is used to approximate the cylinder sample. The geometry of the model is described in Fig 6.4.1 and Fig 6.4.2. The flaw length is $2a$, the flaw inclination angle is β_1 , the ligament angle is β_2 , and the ligament length is $2b$. To simplify the present analysis, the flaw length $2a$ and the ligament length are fixed to be 24 and 33 mm, respectively.

The effects of four different pre-existing flaw geometries on the cracking process and final failure behavior of the marble specimen under different confining pressures are studied (Table 6.4.1, Fig 6.4.2). Notice that type A flaw geometry represents an intact material specimen without pre-existing flaws.

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Table 6.4.1 Geometries of flawed marble specimen (same as Yang et al. (2008))

Marble Type	Flaw geometry	$\beta_1/^\circ$	$\beta_2/^\circ$	$2a/\text{mm}$	$2b/\text{mm}$	σ_3/MPa
Coarse marble	Type A	N/A	N/A	N/A	N/A	0-30
	Type B	30	38	24	33	0-30
	Type C	45	61	24	33	0-30
	Type D	60	75	24	33	0-30
Medium marble	Type A	N/A	N/A	N/A	N/A	0-40
	Type B	30	38	24	33	0-30
	Type C	45	61	24	33	0-30
	Type D	60	75	24	33	0-30

Fig 6.4.1 Geometry of flawed marble specimen

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Fig 6.4.2 Geometry of flaws in the flawed specimen

The development of a tensile crack in rock is generally caused by the increase of the local tensile stresses. When the tensile stress rises to a certain extent, the tensile crack develops and the rock usually exhibits a brittle behavior. While a shear crack usually forms by the linkage of micro-tensile cracks and exhibits a ductile behavior in the specimen (Stead et al. 2001; Stefanov 2004, 2008). Therefore, in this study, we assume that the tensile yield strength of the material σ_t equals the peak tensile strength σ^* , and the shear strength of the material depends on the accumulated plastic shear strain which can be written in the form (Stefanov 2004, 2008)

$$Y = Y_0[1 + h(A(\varepsilon^{ps}) - D(\varepsilon^{ps}))] \quad (6-19)$$

where $A(\varepsilon^{ps}) = \varepsilon^{ps} / \gamma_0$ is the linear hardening function; $D(\varepsilon^{ps}) = (\varepsilon^{ps} / \gamma_0)^2$ is the quadratic function that describes softening or damaging; γ_0 is the plastic shear strain at the start of softening process; h is the parameter for hardening or softening. In this thesis, only cohesion damage is considered. Therefore, $Y = c^*$ and $Y_0 = c'$ are the critical cohesion and cohesion at the start of damage process, respectively.

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The material parameters used in the simulation are listed in Table 6.4.2. The density, Young's modulus and the material yield strength parameters of cohesion (c_y) and friction angle (ϕ_y) are taken from the experiments conducted by Yang et al. (2008). The other parameters were chosen such that the predictions of the stress-strain curves of intact marble specimens (Type A flaw geometry) loaded under different conditions closely match those obtained from the experiments (Yang et al. 2008). c' and ϕ' are parameters related to shear crack initiation, which are not necessarily related to the macroscopic shear strength of the material; σ_t , plastic shear strain γ_0 at the start of softening process and limiting value of plastic shear strain ε^* are estimated from the experimental stress-strain results; λ and h are empirical parameters.

Fig 6.5.1 and Fig 6.5.2 show the stress-strain curves corresponding to different confining pressures for coarse and medium marble specimens respectively. With the increase of the confining stress, the deformation behavior of the specimen changes from elastic into plastic, and the strength of the specimen increases. A higher confining pressure leads to a later onset of plasticity for marble specimens under compression. In addition, with the increase of the confining pressure, the specimens undergo a larger pre-failure plastic deformation and the post-peak stress drop becomes less abrupt. All the above demonstrate that the deformation behavior generally complies with the experimental observations of a transition from the brittle to ductile behavior associated with an increasing confining stress (Figs 6.5.12 and 6.5.13).

Fig 6.5.3 shows the variation of element plastic shear strain in a coarse marble specimen vertically loaded under the confining stress $\sigma_3=30$ MPa before the formation of macro-cracks. The elements in red color experience the most severe plastic shear deformation, while the elements in cyan color have not developed any plastic shear strain. As illustrated in Fig 6.5.3, under a high confining pressure, the localized conjugate bands (shear bands) form, which later evolve to macro-shear cracks, leading to final failure.

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Table 6.4.2 Material properties for numerical simulations

Marble type	Density (kg/m ³)	E /GPa	μ	c_y /MPa	$\phi_y/^\circ$	ϕ^0	c'/MPa	$\phi'/^\circ$	λ	h	$\varepsilon^*/\%$	$\gamma_0/\%$	σ_t /MPa
Coarse marble	2704	45.54	0.2	22.14	28.0	25	27.0	28.0	0.5	0.6	2.0	0.3	9.8
Medium marble	2704	49.90	0.15	31.71	34.4	30	37.5	34.4	0.5	0.6	3.0	0.4	16.2

6.5 Failure mode of pre-cracked marble under compression

6.5.1 Introduction

As discussed in section 5.3.3, the crack initiation, propagation and failure mode from pre-existing straight flaws under uniaxial compression have been widely investigated. However, for triaxial compression on cylindrical and rectangular specimens, the triaxial steel cell used does not allow a direct observation of the cracking processes on the specimens' surfaces. To distinguish shear and tensile crack mechanisms based on direct observation during loading processes is thus very challenging.

By incorporating the modified Mohr-Coulomb crack initiation criterion, the NMM can not only distinguish the development mechanisms of the cracks (tensile/shear), but also capture the entire cracking process. Using the Drucker-Prager model with a non-associated flow rule, the process of shear band formation and shear crack generation in coarse and medium marbles are simulated numerically under various confining conditions. The different processes of crack initiation, propagation, coalescence and subsequent failure modes of flawed specimens (Types B, C and D flaw pair geometries) under different confining pressures are discussed in the next section. The mesh size adopted in this chapter is not very fine, that is due to the high consumption of the simulation when the plastic analysis is involved. For plastic fracturing analysis, three types of iterations related to contact detection, plastic convergence and cracking process are needed, which increases the simulation time significantly with the increase of the cover number.

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Fig 6.5.1 Typical triaxial stress-strain curves at different confining pressures predicted by NMM for intact coarse marble based on 2D axisymetrical model

Fig 6.5.2 Typical triaxial stress-strain curves at different confining pressures predicted by NMM for intact medium marble based on 2D axisymetrical model

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Fig 6.5.3 Distribution of element plastic shear strain in the deformed intact coarse marble specimen under 30 MPa confining stress before the development of macro-cracks.

In practice, to improve the accuracy the enriched method is adopted for covers around the crack tips. To avoid mesh effect especially the softening related simulation, a very coarse mesh is first used to investigate the failure processes of ductile rock. The mesh size is then decreased gradually for the numerical simulation until the differences of the finally macro-fracture growth trajectory and stress-strain curve predicted by the last two meshes are small enough. Therefore, though the mesh size has an effect on the accuracy of the numerical results, the mesh size chosen in the next work can provide an accurate result.

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6.5.2 Numerical predictions

Three failure modes are observed in the numerical simulations for coarse and medium marble subjected to triaxial compression: tensile mode (failure cracks consist of only tensile cracks), shear mode (failure cracks consist of only shear cracks) and mixed mode (failure cracks consist of both tensile and shear cracks). In the following section, the crack initiation mechanism (tensile/shear), the cracking process and the failure phenomenon of specimens consisting of different flaw geometries under different confining pressures are discussed. The major principal stress (σ_1) is in the vertical direction.

1. Type B flaw geometry

For coarse marble under a low confining pressure ($\sigma_3 \leq 10$ MPa) type 1 tensile cracks (as show in Fig 5.3.13) first initiate at the internal flaw tips, as shown in Fig 6.5.4(a). With the increase of the vertical applied load, the tensile cracks keep propagating and gradually curve towards the direction of the major external load without coalescence. Finally, these tensile cracks propagate to the top and bottom of the specimen, which lead to failure. Such a failure is classified as tensile failure mode.

For medium marble under a low confining pressure ($\sigma_3 \leq 15$ MPa), type 1 tensile cracks also first initiate at the internal flaw tips similar to those in the coarse marble. As the axial load increases and after tensile wing cracks have propagated for several steps, however, type 3 shear cracks also initiate at the internal flaw tips in a direction opposite to the propagation direction of the type 1 tensile cracks. After the shear cracks initiate, the pre-existing flaws and the other previously developed cracks will be pulled apart (aperture increases) and the entire system will become unstable. Therefore, such shear cracks cannot propagate for a long distance (Fig 6.5.4(b)). The specimen failure is thus a mixed failure mode, which is attributed to the development of tensile cracks and shear cracks.

Under a high confining pressure (for coarse marble, $\sigma_3 > 10$ MPa and medium marble, $\sigma_3 > 15$ MPa), the development of tensile cracks, which are observed under lower confining pressures is inhibited. Type 2 shear cracks will initiate instead, as

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shown in Fig 6.5.5(a), which eventually lead to a shear failure mode. Before such shear cracks appear, damage has usually already accumulated, leading to the formation of localized bands of plastic shear strain shown in Fig 6.5.5(b).

Fig 6.5.4 Typical cracking processes and failure modes of specimens with type B flaw geometry under lower confining pressure in (a) coarse marble ($\sigma_3=10\text{MPa}$) (b) medium marble ($\sigma_3=15\text{MPa}$)

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Fig 6.5.5 Typical failure modes and plastic shear strains of medium marble specimens with type B flaw geometry under confining pressure $\sigma_3=30\text{MPa}$ (a) shear failure mode (b) development of plastic shear strain elements prior to shear crack development (c) final failure mode.

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2. Type C flaw geometry

Fig 6.5.6 shows the failure modes predicted by the NMM for coarse and medium marble specimens with type C flaw geometry under a low confining pressure ($\sigma_3 < 15$ MPa), which generally agree with the experimental results (Fig 6.5.7) obtained by Yang et al (2008). As illustrated in Fig 6.5.6(a), for the coarse marble loaded under a low confining pressure, type 2 tensile cracks initiate at the two internal flaw tips first. With the increase of the applied load, the initiated tensile cracks keep on propagating towards each other until a shallowly-inclined shear crack coalesces them together. However, since this type of failure mode is attributed to the coalescence of two major tensile cracks by a short shear crack, the failure mode is a mixed failure mode. No significant localized band has developed.

For medium marble loaded under a low confining pressure, two type 2 tensile cracks initiate at the flaw tips and propagate towards each other, which are indicated as ① in Fig 6.5.6 (b). After the cracks have propagated for a certain distance, two secondary type 1 tensile cracks initiate around the original internal flaw tips, which are indicated as ② in Fig 6.5.6 (b). Therefore, the secondary cracks, which are typically hard to determine whether they are tensile or shear experimentally, are numerically found to be not necessarily shear cracks, but can also be tensile cracks. Finally, a shallowly-inclined shear crack, which is indicated as ③ in Fig 6.5.6 (b), links the two previously developed type 2 tensile cracks together. Hence, a mixed failure mode is obtained.

For cases when the confining pressure is high ($\sigma_3 > 15$ MPa), the failure modes in coarse marble and medium marble are both shear failure as shown in Fig 6.5.8 and Fig 6.5.9, respectively. However, the detailed coalescence processes in coarse marble and medium marble are different. For the coarse marble, after the initiated cracks propagate for a few steps, a significant localized deformation band develops between the two crack tips (Fig 6.5.8(b)). With the increase of the applied load, shear cracks will initiate in the elements in the localized deformation band, which have reached the critical state. The propagation of the shear cracks leads to a shear failure mode, as shown in Fig 6.5.8. For the medium marble, localized shear

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bands also develop around the cracks (Fig 6.5.9 (b)). However, due to the steeper orientation of the coalescence cracks as compared with those in the coarse marble, the coalescence shear crack can coalesce the flaw tips directly, resulting in a shear failure mode as shown in Fig 6.5.9 (a).

Fig 6.5.6 Typical failure modes of flawed specimens with type C flaw geometry under confining pressure $\sigma_3=10\text{MPa}$ (a) coarse marble (b) medium marble

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Fig 6.5.7 Typical failure modes of flawed specimens with type C flaw geometry under confining pressure $\sigma_3=10\text{MPa}$ obtained by Yang et al. (2008) (a) coarse marble (b) medium marble

Fig 6.5.8 Typical failure mode and plastic deformation of flawed coarse marble specimens with type C flaw geometry under confining pressure $\sigma_3=30\text{MPa}$ (a) shear failure mode (b) contour of element plastic shear strain before coalescence

Fig 6.5.9 Typical failure mode and plastic deformation of flawed medium marble specimens with type C flaw geometry under confining pressure $\sigma_3=30\text{MPa}$ (a) shear failure mode (b) contour of element plastic shear strain after coalescence

3. Type D flaw geometry

For coarse marble with type D flaw geometry under a low confining pressure ($\sigma_3 \leq 10\text{MPa}$), type 1 tensile cracks first initiate at the two internal flaw tips and keep on propagating towards each other as the axial load increases. Finally, the two tensile cracks are coalesced by a shallowly-inclined shear crack, which results in a mixed failure mode, as shown in Fig 6.5.10 (a). Since the mixed failure mode is mainly attributed to the two tensile cracks with only a short shear crack segment, no significant localized band develops. The deformation behavior tends to be brittle.

For medium marble with type D flaw geometry under a low confining pressure ($\sigma_3 \leq 10\text{MPa}$), similar to coarse marble specimen, type 1 tensile cracks first initiate at the internal flaw tips. After the tensile cracks propagate for a certain distance, two steeper secondary type 1 tensile cracks which are not observed in the coarse marble initiate around the original internal flaw tips (② in Fig 6.5.10 (b)). At

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last, the primary type 1 tensile cracks are connected by a shallowly-inclined shear crack (③ in Fig 6.5.10 (b)), which results in a mixed failure mode.

Fig 6.5.10 Typical failure modes of flawed specimens with type D flaw geometry under confining pressure $\sigma_3=10\text{MPa}$ (a) coarse marble (b) medium marble

Fig 6.5.11 Typical failure mode of flawed specimens with type D flaw geometry under confining pressure $\sigma_3=30\text{MPa}$ (a) contour of element plastic shear strain before coalescence (b) final failure mode after coalescence

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For specimens with type D flaw geometry under a high confining pressure ($\sigma_3 \geq 15\text{MPa}$), the cracking process and final failure mode of coarse marble and medium marble are similar, as shown in Fig 6.5.11. For both marbles under a high confining pressure, the development of tensile cracks is inhibited, while type 2 shear cracks first initiate at the internal flaw tips. The latter continue to propagate with the increasing external load. As shown in Fig 6.5.11 (a), before the two type 2 shear cracks coalesce, a localized shear deformation band has already formed. As the load further increases, the plastic shear strain in the deformation band also increases and the damage accumulates. Eventually, the material in the bridging region loses its strength and the sub-parallel shear cracks initiate to coalesce with the previously developed type 2 shear cracks. A shear failure mode as shown in Fig 6.5.11 (b) thus results.

From the above analysis, different fracture patterns and failure modes, which are dependent on the material parameters, loading conditions and flaw geometries, have been obtained. At a low confining pressure, usually tensile cracks initiate and propagate, which lead to a more brittle deformation behavior. When the confining pressure is high, usually more pronounced localization bands are formed before macro-shear cracks develop, which result in a more ductile deformation behavior. Compared with medium marble, the cracking mechanisms of coarse marble are more sensitive to the confining pressure. For both marble specimens with type B flaw geometry, under both high and low confining pressures, no direct coalescences occur. For type C and type D flaw geometries under different loading conditions, crack coalescences are observed in both types of marble. In addition, the later developed cracks, which are usually referred as the secondary cracks, are not always shear cracks, but can also be tensile cracks.

The stress–strain curves of both pre-cracked marble specimens with type C flaw geometry under different confining pressures are shown in Figs 6.5.12 and 6.5.13. As illustrated in the figures, when the confining pressure σ_3 increases, the strengths of both coarse and medium specimens increase. Compared with the coarse marble, the medium marble shows a higher strength and a higher strength increment with the increase of the confining pressure. It can also be observed from these two

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figures that when the confining pressure increases, the softening behavior of both specimens becomes weaker. Under a higher confining pressure, both specimens show a high deformation capability.

Fig 6.5.12 Stress–strain curves of pre-cracked marble specimen with type C flaw geometry under confining pressure $\sigma_3=10\text{MPa}$

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Fig 6.5.13 Stress–strain curves of pre-cracked marble specimen with type C flaw geometry under confining pressure $\sigma_3=30\text{MPa}$

6.6 Conclusions

In this chapter, the NMM has been extended and enhanced by incorporating the modified Mohr-Coulomb crack initiation criterion, the Drucker-Prager constitutive model with a tensile cut-off and the crack evolution techniques for rock failure analysis. Using the developed NMM, the deformation and failure processes of rock specimens, such as initial elastic deformation, secondary plastic deformation, large displacement (sliding or even discrete blocky movements) and multiple crack propagation and coalescences, which are typically observed in experimentally studies, have been reproduced.

The behavior of coarse and medium marbles with different flaw geometries under triaxial compression is numerically modeled, which generally agrees with the experimental findings obtained by Yang et al (2008). By incorporating the damage parameter D into the modified Mohr-Coulomb crack initiation criterion, not only

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the cracking mechanism (tensile/shear) can be identified, but also the different development processes of tensile and shear cracks can be modeled. When the failure is dominated by the development of tensile cracks, which is generally caused by the local increase of tensile stresses, a brittle deformation behavior results. The shear crack growth is usually preceded by a formation of localized deformation bands, which are due to the process of damage accumulation and material deterioration. The development of shear cracks thus generally leads to an exhibition of ductile deformation behavior of specimens. The results also show that the failure modes of pre-flawed marble specimens are dependent on the material parameters, loading conditions and flaw geometries. In addition, in agreement with the experimental results, the secondary cracks observed during the numerical simulations are not necessarily shear cracks, but they can also be tensile cracks.

However, the work done in this study is still at its preliminary stage. The failure of cylindrical shape specimens is actually a 3D problem. However, for simplicity, the 2D axial-symmetric models are used to approximate the real 3D models. The “planar” fractures obtained from 2D axial-symmetric models actually represent “conical” shapes in 3D, which of course causes some discrepancies with the real 3D tests. This is the limitation of the 2D model to approximate 3D problem. Actually, cracking problems in 3D are always more complicated and deviate from planar failure. Nonetheless, the 2D analysis presented in the thesis can be useful in identifying general cracking behavior in rocks. To accurately evaluate these damage related parameters c' , ϕ' , γ_0 , ε^* , λ and h are still very challenging. Currently most of them can only be estimated empirically. Therefore, for wider application, comprehensive determination of these parameters is highly necessary.

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7.1 Introduction

In blasting, earthquake, landslides, rock bursts and defense engineering involving nuclear blasting, the cracking processes and damage developments of rocks are under a wide range of loading rates. Therefore, the dynamic behavior of rock materials under different loading rates is of great concern.

Since the early studies of Rinehart (1965), the loading rate-dependent dynamic strength, which is one of the main features of dynamic behavior of rock materials, has been widely studied experimentally, as well as numerically. Dynamic laboratory measurements include compression (Green and Perkins 1968; Kumar 1968; Lindholm et al. 1974; Blanton 1981; Olsson 1991; Li et al. 2000; Li et al. 2004), torsion (Lipkin and Grady 1977) and tension (Khan and Irani 1982; Wang et al. 2006). Hopkinson bar techniques have identified that rock material strengths generally increase with increasing loading rate. However, according to the work done by Bantia et al. (1987), it is experimentally difficult to meet the energy balance for very high loading rates. In contrary to this, numerical simulation allows a relative simple transfer of the kinetic energy to other energy forms and to achieve an energy balance state. Therefore, it is highly worthwhile to perform numerical simulations to study the effect of loading rate on the crack mechanism (tensile or shear), crack evolution process, and failure modes of rocks.

The fracture process under dynamic load itself is a very complex, multi-scale physical phenomenon. One of the main challenges in fracture simulation is the need for a continuous updating of geometry and discretization of the cracked structure due to crack growth. To numerically model crack propagation, a number of numerical approaches have been proposed as discussed in chapter 2 and some of them have already been extended for dynamic cracking problems.

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Based on the conventional FEM framework, the inter-element crack method (cohesive zone method), in which the crack is modeled by separation along the element edges, has proved effective for localized dynamic fracture problems (Xu and Needleman 1994; Camacho and Ortiz 1996; Falk et al. 2001). However, Song et al. (2008) discovered for a different problem that the restriction of crack paths to specified angles in the inter-element crack method can result in a rather severe underestimation of the energy dissipation.

Through the partition of unity framework (Melenk and Babuska 1996), the XFEM has been successfully applied to dynamic crack propagation problems (Belytschko et al. 2003; Menouillard and Belytschko 2010; Menouillard et al. 2010; Liu et al. 2011). However, for complicated and extensive crack patterns, the level set description is still not sufficiently robust (Rabczuk et al. 2009). Furthermore, defining the enrichment functions is also tedious (Ma et al. 2009) in the XFEM. In addition, without the contact technique, it will be difficult to model the stick, slip and separation along the flaw surfaces accurately.

The molecular dynamics (MD) method which treats material as an assemblage of independent nano-elements has also become popular for studying the dynamic cracking behavior in rocks. By simply de-bonding the elements, the nucleation of cracks is simulated. Therefore, MD method is very convenient to deal with cracking problems and has been successfully used for dynamic fracture behavior (Abraham et al. 1994; Abraham and Broughton 1998; Resende et al. 2010). However, as mentioned by Thiagarajan (2007), the present computational technologies are still way behind for simulating elements at atomic level. Since the event of fracture is highly dynamic and its emergence and evolution depends on the localization of materials and specific loading conditions, these micromechanics models cannot be used to predict a fully correct global fracture phenomenon (Wang et al. 2008).

Owing to the capabilities of NMM in dealing with discontinuous problems, NMM has been successfully used for dynamic fracture mechanics analysis (Chen et al. 2006; Li and Cheng 2006). However, the parameters adopted by these simulations were obtained under quasi-static loadings and no strain-rate-dependent

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effect was considered. Therefore, these models could not give a realistic description of the behavior of rock mass under dynamic loadings, especially impact loading.

In this chapter, a modified three-element viscoelastic constitutive model with frequency-dependent parameters (Fan et al. 2012) is adopted to investigate the rate-dependent deformation behavior of a sedimentary rock under dynamic loading. Numerical examples of structural behavior of rock mass in a square plate with a pre-existing flaw and crack development process in a long thin bar are presented.

7.2 The constitutive relation and its incremental form

In this chapter, the rate-dependent behavior of a rock mass under dynamic loads is modeled by a modified three-element viscoelastic model (Fig 7.2.1). The model is constituted by a spring in parallel with a modified Maxwell element. The spring with constant elastic modulus E_∞ is used to describe the modulus of rock under an unrelaxed condition. The modified Maxwell element, which consists of a spring and a dashpot of loading frequency dependent modulus E_v and loading frequency dependent damping ratio η_v , is used to describe the rate-dependent property of rock. When the loading rate is low enough, the Maxwell term can be neglected. As given by Fan et al. (2012), the frequency-dependent modulus $E_v(\omega)$ and frequency dependent damping ratio $\eta_v(\omega)$ can be expressed as the following more practicable expression:

$$y(\omega) = A_0 + A_1 \exp(-\omega / B_1) + A_2 \exp(-\omega / B_2) \quad (7-1)$$

where $y(\omega)$ can be either $E_v(\omega)$ or $\eta_v(\omega)$ (Fig 7.2.1), which are determined by different coefficients. A_i and B_j ($i=1,2$ and $j=1,2$) are the coefficients and ω is the basic frequency of the applied load.

For a two dimensional case, the linear viscoelastic constitutive relation can be expressed in a weak form as (Christensen 1982; Ling and Xu 2004; Zhang and Li 2009; Zhang et al. 2010b)

$$\sigma(t) = \int_0^t E(t-\varsigma) D \frac{\partial \varepsilon(\varsigma)}{\partial \varsigma} d\varsigma \quad (7-2)$$

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where σ and ε are stress and strain tensor in vector form, respectively, t denotes time. D is the elastic matrix dependent only on the Poisson's ratio ν and $E(t)$ is the relaxation modulus. Mathematically, the relaxation modulus is often expanded in a Prony series as Wang (1985)

$$E(t) = E_{\infty} + E_1 \exp(-t / \tau_1) \quad (7-3)$$

where the dynamic elastic modulus of Maxwell element E_1 can be approximated as (Fan et al. 2012)

$$E_1 = \frac{\omega^2 \tau_1^2}{1 + \omega^2 \tau_1^2} E_v \quad (7-4)$$

where the retardation time τ_1 of the Maxwell element is η_v/E_v (Christensen 1982).

The incremental method is commonly adopted to solve time-dependent problems. Within this framework, the field variables σ_n and ε_n at $t=t_n$ are known. Assuming the strain rate in the time interval $\Delta t = t_{n+1} - t_n$ to be $\partial \varepsilon / \partial t = \Delta \varepsilon_n / \Delta t_n$, the discrete form of constitutive relation in Eq. (7-2) can be expressed as

$$\sigma_{n+1} = E_{\infty} D \varepsilon_{n+1} + E_1 D e^{-\Delta t_n / \tau_1} \int_0^{t_n} e^{-(t_n - \zeta) / \tau_1} \frac{\partial \varepsilon(\zeta)}{\partial \zeta} d\zeta + \frac{\Delta \varepsilon_n}{\Delta t_n} \int_0^{\Delta t_n} e^{-(\Delta t_n - \zeta) / \tau_1} d\zeta \quad (7-5)$$

From Eq. (7-5), the incremental form of constitutive relation can be obtained as

$$\Delta \sigma_{n+1} = \tilde{E} D \Delta \varepsilon_{n+1} + \sigma_{n+1}^0 \quad (7-6)$$

where $\Delta \sigma_{n+1}$ and $\Delta \varepsilon_{n+1}$ are the incremental stresses and strains, respectively; The explicit forms of $\tilde{E}$ and σ_{n+1}^0 are (Ling and Xu 2004; Zhang and Li 2009; Zhang et al. 2010b)

$$\tilde{E} = E_{\infty} + \frac{E_1 \tau_1}{\Delta t_{n+1}} (1 - \exp(-\Delta t_{n+1} / \tau_1)) \quad (7-7)$$

$$\sigma_{n+1}^0 = [\exp(-\Delta t_{n+1} / \tau_1) - 1] \gamma_n \quad (7-8)$$

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The parameter γ_n in Eq. (7-8) can be expressed in a recursive form

$$\gamma_n = \exp(-\Delta t_n / \tau_1) \gamma_{n-1} + \frac{E_1 \tau_1 [1 - \exp(-\Delta t_n / \tau_1)]}{\Delta t_n} D \Delta \varepsilon_n \quad (7-9)$$

Fig 7.2.1 Modified three-element viscoelastic model

7.3 Discrete system of equations

In the NMM, the discrete form of the governing equations without physical damping can be expressed as

$$K \Delta d + M \ddot{d} = \Delta F \quad (7-10)$$

where K is the stiffness matrix, M is the mass matrix, ΔF is the incremental load matrix, Δd is the incremental displacement, $\dot{d}$ is the velocity matrix (if any) and $\ddot{d}$ is the acceleration matrix.

For solving Eq. (7-10), Newmark's scheme relates displacement and velocity at time t_{n+1} to known values at time t_n and acceleration at time t_{n+1} by the expressions respectively of

$$\begin{aligned} \dot{d}_{n+1} &= \dot{d}_n + [(1 - \beta) \ddot{d}_n + \beta \ddot{d}_{n+1}] \Delta t \\ d_{n+1} &= d_n + \dot{d}_n \Delta t + [(1/2 - \alpha) \ddot{d}_n + \alpha \ddot{d}_{n+1}] \Delta t^2 \end{aligned} \quad (7-11)$$

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here α and β are two Newmark's integration parameters. In this study, the scheme of the average acceleration ($\alpha=1/4$; $\beta=1/2$) is adopted. Then substituting Eq. (7-11) into Eq. (7-10) gives

$$\tilde{K}\Delta d = \Delta\tilde{F} \quad (7-12)$$

where

$$\tilde{K} = \frac{1}{\alpha\Delta t^2}M + K, \quad \Delta\tilde{F} = \Delta F + \left[\left(\frac{1}{2\alpha} - 1 \right) \ddot{d}_n + \frac{1}{\alpha\Delta t} \dot{d}_n \right] M \quad (7-13)$$

7.4 Numerical examples

In this section, two numerical examples based on the 3-element viscoelastic model introduced earlier are presented and discussed. The first example is a pure mode crack problem of an edge crack in a square plate. The second example is related to segmental breakage of a long rod under a dynamic impact. The material parameters in both examples are based on the same type of sedimentary rock. The mechanical response of the rock under different loading rates is examined in these examples. The Mohr-Coulomb criterion is used for crack initiation and the MTS criterion is used for crack propagation judgment.

7.4.1 Pure mode crack problem

A square plate containing a pre-existing edge crack subjected to pure mode I and pure mode II loading under different loading rates is studied. The crack opening displacement (COD) and the crack sliding displacement (CSD), which represent the magnitude of normal and tangential deformations of the crack surface, respectively, are investigated. According to the viscoelastic principle, the displacement fields around the crack tip for the problem shown in Fig 8 when w is infinite are derived as (Zhang et al. 2010b)

For mode I crack

$$u(r, \theta, t) = (1 + \nu)J(t)K_I(t)\sqrt{\frac{r}{2\pi}} \left[\kappa - 1 + 2\sin^2\left(\frac{\theta}{2}\right) \right] \cos \frac{\theta}{2} \quad (7-14)$$

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$$v(r, \theta, t) = (1 + \nu)J(t)K_I(t)\sqrt{\frac{r}{2\pi}}\left[\kappa + 1 - 2\cos^2\left(\frac{\theta}{2}\right)\right]\sin\frac{\theta}{2} \quad (7-15)$$

For mode II crack

$$u(r, \theta, t) = (1 + \nu)J(t)K_{II}(t)\sqrt{\frac{r}{2\pi}}\left[\kappa + 1 + 2\cos^2\left(\frac{\theta}{2}\right)\right]\sin\frac{\theta}{2} \quad (7-16)$$

$$v(r, \theta, t) = (1 + \nu)J(t)K_{II}(t)\sqrt{\frac{r}{2\pi}}\left[\kappa - 1 - 2\sin^2\left(\frac{\theta}{2}\right)\right]\cos\frac{\theta}{2} \quad (7-17)$$

where u and v are the displacement components along the x and y direction in the Cartesian coordinate system at the crack tip. (r, θ) are the polar coordinates defined around the crack tip. κ is the Kolosov constant, and takes the value $\kappa=3-4\nu$ for the plane strain state. $J(t)$ is the creep compliance which equals (Zhang et al. 2010b)

$$J(t) = \frac{1}{E_\infty} - \frac{E_1}{E_\infty[E_\infty + E_1 \exp(-t/\tau_1)]} \exp(-t/\tau_1) \quad (7-18)$$

In our present study, a square plate of $w=5.0\text{m}$, containing a pre-existing edge crack ($a=1.0\text{m}$) is considered. According to equation (7-3), attributing to its viscoelastic property, the sedimentary rock takes a very long time to reach a fully relaxed state (reaching $E(t) = E_\infty$). Due to time limitation, only the loading rate effect on the state right after the loading is completed is investigated. The material parameters of a sedimentary rock as listed in Table 7.4.1 and Table 7.4.2 (Fan et al. 2012) are used. In this example, only loading rate effect on the edge crack COD and CSD is investigated without considering the crack initiation and propagation. A load equal to 20 MPa is applied to the plate at different loading rates. Mode I and mode II loading are separately investigated. The final COD ($v(r, \pi) - v(r, -\pi)$) and CSD ($u(r, \pi) - u(r, -\pi)$) are computed at a pair of reference points, i.e. A(0.5, π) and B(0.5, $-\pi$) as shown in Fig 7.4.1, where a triangular mesh with 5000 MCs, 4988 PCs and 9688 elements is adopted. The variations of COD and CSD with loading rate are plotted in Fig 7.4.2 and Fig 7.4.3, respectively, which reveal that the viscoelastic behavior of the sedimentary rock depends on the loading rate. As

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illustrated in the figures, the results predicted by the NMM closely match the theoretical results. When the loading rate is low ($<1000\text{MPa/s}$), the effect of the loading rate is negligible. The behavior of the sedimentary rock still exhibits an elastic property. However, with the increase of the loading rate ($>1000\text{MPa/s}$), the behavior of the rock mass will deviate from elasticity and exhibit a viscoelastic property. Therefore, for a more realistic representation of the behavior of sedimentary rock under dynamic loading, a viscoelastic model is needed.

Table 7.4.1 Material parameters of a sedimentary rock

Unit mass ρ (T/m^3)	Possion ratio	Spring elastic modulus E_{∞} (GPa)	Tensile strength σ_t (MPa)	Cohesion c_i (MPa)	Friction angle ϕ ($^{\circ}$)	Fracture toughness ($\text{MN/m}^{3/2}$)
2.68	0.3	55.77	18	20	30	2.6

Table 7.4.2 Coefficients for $E_v(\omega) \sim \omega, \eta_v(\omega) \sim \omega$

	A_0	A_1	A_2	B_1	B_2
$E_v(\omega)$	15.54e9	13.99e10	10.05e13	60.77	9.23
$\eta_v(\omega)$	24.70e5	20.45e6	27.89e7	1868.11	37.38

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Fig 7.4.1 Representation of an edge crack in the NMM mesh

Fig 7.4.2 Variation of the COD subjected to different loading rate

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Fig 7.4.3 Variation of the CSD subjected to different loading rate

7.4.2 Dynamic crack problem

To better understand the influences of the viscoelastic behavior of sedimentary rock in response to dynamic loading on the crack problems, such as crack initiation, propagation and final failure pattern, a long thin bar (as shown in Fig 7.4.4 with 524 MCs, 524 PCs and 836 elements) subjected to different loading rates is investigated. Both the elastic model and the viscoelastic model are considered and compared. The length of the bar is 1m and its width is 0.03m. A triangular wave load P with a peak value $P_m=28\text{MPa}$ is uniformly applied at the left end of the bar and the other end is free. As shown in Fig 7.4.5, because when the small tensile wave reflected at the free end superposes with the big input compressive wave, there is no net tensile wave created when the ascending part of the incident wave is reflected at the free end. Therefore, the descending loading time t can be taken as the load cycle time T for the Fourier Transform. For convenience, the ascending and descending loading time t of the incident wave is set to be the same, which are 0.03s, 3ms, 0.3ms and 0.03ms for modeling different loading conditions (loading rate is defined as P_m/T). The material parameters used are listed in Tables 7.4.1 and 7.4.2. Three

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measurement points A, B and C, are set at 0.3m, 0.6m and 0.9m away from the left end of the bar along the central line of the bar.

Fig 7.4.4 NMM model of a long bar

Fig 7.4.5 Illustration of an input wave and its reflection at the free end

Based on the stress wave at points A, B and C (as shown in Fig 7.4.6 for descending time $t=0.03\text{ms}$), the P wave propagation velocity C_0 can be calculated according to the arrival time of the peak stress σ_x at these points. The theoretical P wave propagation velocity C_0 for elastic and viscoelastic model can be calculated as (Wang 1985)

$$C_0 = \sqrt{E_\infty / \rho} \text{ for elastic model} \quad (7-19a)$$

$$C_0 = \sqrt{E(t) / \rho} \text{ for viscoelastic model} \quad (7-19b)$$

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Fig 7.4.7 plots the P wave propagation velocity C_0 versus the loading rate obtained by different methods. As shown in Fig 7.4.7, for both the elastic model and the viscoelastic model, the NMM can give results closely matching the theoretical results. The figure also illustrates that when the loading rate becomes very high ($>1000\text{MPa/s}$), the viscosity of the rock mass will also become strong and the effect of the viscosity on the wave propagation velocity C_0 will become significant. According to the theory of stress wave (Kolsky 1963; Wang 1985) under the triangular wave load, the distance of the first fracture away from the free end δ can be calculated as:

$$\delta = \frac{\lambda}{2} \cdot \frac{\sigma_t}{\sigma_m} \quad (7-20)$$

where σ_t is the dynamic tensile strength of the rock mass, σ_m is the peak pressure of the triangular load and λ is the wave length which is dependent on the wave velocity C_0 and the cycle time T . Since σ_t and σ_m are constant, and the cycle time T is also a constant for one time loading, under a certain load, the distance δ is determined by the P wave velocity C_0 . Because the wave velocity C_0 is in turn affected by the viscosity of the rock mass especially under high loading rate, the viscosity effect on the fracture behavior of the sedimentary rock bar under a high loading rate are simulated according to the two different models.

Comparisons of the simulation results between the viscoelastic model and elastic model for a descending loading time $t=0.03\text{ms}$ are shown in Fig 7.4.8. The fracture distance δ predicted by the NMM are 0.082m and 0.118m for the elastic model and the viscoelastic model, respectively, which are 6.8% and 3.5% away from the theoretical results of 0.088m and 0.114m. Since the theoretical results are based on the ideal one-dimension assumption, which is different from the numerical models, such a discrepancy is considered acceptable. As illustrated in the figure, generally, both models are able to predict the spalling phenomenon, i.e. the failure

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of the long bar into multiple segments. This is attributed to the continual propagation of the loading wave, and its reflection at the crack interfaces and bar surfaces. However, as demonstrated in the figure, the viscoelastic property of the rock mass, which is revealed in the viscoelastic model, affects not only the distance δ , but also the final failure pattern. Compared with the viscoelastic model, the crack initiation and penetration times in the elastic model are delayed. It is attributed to the different P-wave propagation velocity C_0 and fracture distance δ associated with different models, which lead to different stress dissipation after the first fracture development (crack penetration). As a result, the subsequent crack development process and final failure pattern are different. For the elastic model, only the first developed penetrative fracture is constituted by pure tensile cracks. The later developed penetrative fracture is constituted by two tensile cracks which are then linked together by a shear crack, as shown in Fig 7.4.8 a. However, for the viscoelastic model, since the wave velocity C_0 is higher and the fracture distance δ is longer, the reflecting tensile stress at the first fracture is higher than that in the elastic model. As a result, no mixed penetrative fracture (fracture consisting of tensile and shear cracks) develops. The two developed penetrative fractures are both constituted by pure tensile cracks, as shown in Fig 7.4.8 b.

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Fig 7.4.6 Variation of σ_x with time at measurement points A, B and C for descending load time $t=0.03$ ms

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Fig 7.4.7 Variation of wave velocity C_0 with loading ratio by different models

7.5 Conclusions

The present study applies the NMM to investigate the cracking behavior of a sedimentary rock under dynamic loading. A modified 3-element viscoelastic model was adopted to study the rate dependence of the viscoelastic property of the sedimentary rock under dynamic loading. From the simulation results, the following conclusions can be drawn:

1. The modified 3-element viscoelastic model can reflect the rate dependence viscoelastic property of the sedimentary rock. With the increase of the loading rate, especially when the loading rate is larger than 1000MPa/s, the stiffness of the rock mass will increase significantly. As a result, the structure behaviors of the rock mass are affected by the loading rate, as reflected by a decrease of COD and CSD with the increase of the loading rate.
2. The numerical results reveal that under a high loading rate ($>1000\text{MP/s}$), due to the viscoelastic property of the sedimentary rock, not only

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the structural behavior deviates from that of elastic model, but also different cracking processes and final modes are obtained.

3. However, the dynamic cracking behavior is a very important and complex problem. This study only involves the relevant preliminary work which provides a foundation for extending the NMM to investigate wave propagation through jointed rocks and stress wave induced failure problems. Further development is still needed to target at more complicated dynamic cracking problems in rock.

Fig 7.4.8 Simulation results of fracture processes of a long bar under high loading rate ($t=0.03\text{ms}$) by different models. (a) Elastic model (b) Viscoelastic model

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8.1 Introduction

In rock slope stability analysis, the failure mechanism is often assumed to be linked to the sliding of a rock along pre-existing discontinuities, where instability is structurally controlled. However, instability mechanism occurs not only along existing discontinuities, but also as a complex internal deformation associated with shear or tensile fracturing in the intact rock, particularly in massive natural rock slopes and deep, engineered slopes (e.g. open pit mines) (Eberhardt et al. 2004).

Terzaghi (1962) and Einstein et al. (1983) suggested that the persistence of key discontinuity sets is typically limited and the failure of the slope thus requires a complex interaction (such as coalescence) between pre-existing discontinuities, as well as crack initiation and propagation through the intact rock. In other words, massive rock slope instability inherently requires the evolution of natural discontinuities through a gradual transition from a discontinuous-continuous to a fully discontinuous medium. Therefore, a model to study rock slope stability should encompass the nucleation or activation of cracks within the rock matrix and their possible coalescence which would then lead to the creation of critical fractures connecting the pre-existing ones.

Due to the advances of computer power, there has been a dramatic increase of the numerical methods (Stead et al. 2001; Jing and Hudson 2002; Bobet et al. 2009) in the potential for understanding both the mechanisms/processes involved and the associated risk of complex rock slope failures. They range from simple infinite slope and planar failure limit equilibrium method to sophisticated coupled method. Though the initiation or trigger mechanisms that may involve sliding movements can be analysed as a limit equilibrium problem, most failures involve complex deformation and extensive internal disruption. This phenomenon is thus different from the rigid body assumptions required by most limit equilibrium back-analysis. Although both continuum-based and discontinuum-based methods provide useful means to analyse rock slope stability problems, modeling complex failures

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related to both interaction along pre-existing discontinuities and the creation of new ones through fracturing of intact rock are still challenging.

In this chapter, based on the contact technique inherited from DDA, a displacement-dependent criterion proposed by Wang et al. (2013) is incorporated into the NMM for the removal of cohesion between the contact interface. This technique to some extent avoids the improper removal of the cohesion, hence providing more accurate results for predicting the movement along a failure plane. The fracture mechanics model as discussed in chapter 5 is adopted to account for the stress concentration at the tip of a discontinuity and the MTS criterion (details discussed in chapter 4) is adopted for simulating a continuous stepped failure surface produced by the propagation of pre-existing discontinuities. The Mohr-Coulomb failure criterion (details discussed in chapter 4) with a tensile cut-off is also adopted to account for the failure of the intact rock. Combining these techniques with the crack evolution techniques developed in the previous chapters, the progressive failure of natural rock slopes involving the propagation of cracks within intact rock masses is investigated.

8.2 Contact mechanics

Discontinuous numerical analysis depends heavily on the accurate treatment of contacts between blocks. Wrong detection or description may lead to wrong analysis, or breakdown of the computation. The NMM adopts a rigorous contact searching method as used in DDA to account for the interactions between loops (details discussed in chapter 4).

The original two-dimensional NNM adopts the following three types of contacts: angle-to-angle, angle-to-edge, edge-to-edge (details discussed in chapter 3). When the contact status transforms from 'sticking' to 'sliding' or 'open', the cohesion between the contact pair is then removed. Since the spring contact stiffness is often very large, which means the original NMM assumes that once the contact pair experiences a relative movement, the cohesion will no longer exist at the interface. However, this assumption sometime leads to the improper removal of the interface cohesion.

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Consider an edge-to-edge contact as shown in Fig 8.2.1. Loop 1 represents a 45^0 inclined discontinuity, which is assumed to be fixed. Loop 2 represents a right-angle isosceles triangle, which is assumed to be free. During calculation, the edge-to-edge contact transforms into two angle-to-edge contacts (P_1 to P_3P_4 and P_2 to P_3P_4) and the force equilibrium analysis is performed for each contact pair. If the driving shear force exceeds the resisting shear force ($F_{si}=F_{ni} \tan \phi + c l_{p_1 p_2} / 2$, the cohesion resisting force is evenly distributed to each contact point, hence a factor of 2), the contact point in the NMM cannot maintain balance and the cohesion will be removed. In case of the following condition:

$$F_{s1} > F_{n1} \tan \phi_j + c_j l_{p_1 p_2} / 2 \quad (8-1a)$$

$$(F_{n1} + F_{n2}) \tan \phi_j + c_j l_{p_1 p_2} / 2 < F_{s1} + F_{s2} < (F_{n1} + F_{n2}) \tan \phi_j + c_j l_{p_1 p_2} \quad (8-1b)$$

According to the right part of Eq. (8-1b), Loop 2 as a whole should clearly be in balance. However, according to the status shown in Eq. (8-1a), the cohesion at point P_1 will be removed. As a result, the driving force at point P_2 will increase to $F_{s2} + (F_{s1} - F_{n1} \tan \phi)$, which is also bigger than the resistance force at point P_2 ($F_{n2} \tan \phi + c l_{p_1 p_2} / 2$) according to the left part of Eq. (8-1b). Therefore, the contact states at both points are considered as “sliding” and the cohesions are completely removed. In fact, materials such as rock (Barton 1976; Gehle and Kutter 2003) may have a displacement-dependent shear strength. In this study, a displacement-dependent cohesion and friction angle reduction (Wang et al. 2013) are incorporated into the NMM. In this thesis, only cohesion reduction is simulated. Displacement-dependent cohesion reduction means the cohesion between the discontinuity surfaces degrades with the increasing movement along the surfaces. Instead of removing the cohesion determined by the contact states, this technique enables relative movements between contact pairs and the removal of the cohesion based on the accumulated relative sliding parameters $\sum w_2$.

For validating the efficiency of the proposed displacement-dependent method, the critical stabilities of a block-on-inclined surface case (Fig 8.2.1 and Fig 8.2.2) with the consideration of cohesion c_j and friction angle ϕ_j are calculated. The

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unity weight, Young's modulus and Poisson's's ratio of the material are 20 kN/m^3 , 10 GPa and 0.25 respectively, which remain unchanged during calculation. The friction angle ϕ_j is kept constant throughout the simulation. The cohesion c_j is initially kept as a constant and later removed once the relative sliding displacement of the contact pairs exceeds the threshold of 0.002 m (Gehle and Kutter 2003). Since the block-on-inclined surface is $1 \text{ m} \times 1 \text{ m}$ in dimensions, the elastic modulus 10 GPa adopted in this paper is large enough to ensure that the block behaves as a rigid block (maximum strain is around $1.0 \text{ e-}6$). Assuming the blocks to be rigid, the analytical solution for the critical state can be obtained by assuming that the sliding force equals the resistant force as:

$$W \tan \phi_j \cos \alpha + c_j l_{p_1 p_2} = W \sin \alpha \quad (8-1c)$$

The critical states predicted by the original NNM and the NMM with displacement-dependent criterion are compared in Fig 8.2.3, which illustrates that using the relative sliding displacement along the interface gives a more accurate result between contacted loops than by using the contact states as the criterion for the removal of cohesion.

Fig 8.2.1 Schematic model of an inclined edge-edge contact

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Fig 8.2.2 NMM model of the block-on-inclined surface case

Fig 8.2.3 Comparisons of the critical stability analysis results of the block-on-inclined surface case obtained by different methods.

8.3 Progressive failure of discontinuous rock slopes

Terzaghi (1962) and Einstein et al. (1983) suggested that most rock slope failures involve a complex interaction between pre-existing discontinuities and brittle fracture propagation through intact rock bridges. Therefore, the failure surface of a slope is generated by the connection of pre-existing discontinuities in the rock mass. The surface depends on the persistence of the discontinuities, with a smaller persistence of discontinuities corresponding to a greater irregularity of the final failure surface. During uniaxial compression tests on pre-flawed prismatic rock specimens, multiple types of coalescence were observed by Park and Bobet (2009) and Wong and Einstein (2009c) which were mainly dependent on the position and separation distance of the pre-existing discontinuities.

In this section, using the developed NMM, details of various types of crack propagation involved in the formation of the failure surface are discussed. To

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compare the model with the results presented by Jennings (1970) and Jaeger (1971), it is necessary to determine the safety factor (SF). Because some parameters of the NMM model are not related to shear strength, the application of the strength reduction technique, often used in the FEM, is difficult to implement. Another method that could be used is based on increasing the acceleration of gravity, which is achieved by successively incrementing its value until failure of the slope. The safety factors of slopes are calculated with different configurations of discontinuities. The material properties are list in Table 8.3.1.

Table 8.3.1 Material properties (Camones et al. 2013)

Intact						Discontinuity		
Density kg/m ³	Young's modulus E(GPa)	Poisson's's ratio ν	Tensile strength σ_t (MPa)	Friction angle φ_i	Cohesion c_i (MPa)	Friction Angle φ_j	Cohesion c_j (MPa)	Fracture toughness K_{IC} (MPa/m ^{1/2})
3100	33.7	0.15	13.1	23.5	18.1	33	0-0.1	1.0

“Intact” means the parameters are for the intact rocks mass. “Discontinuity” means the parameters are for rock discontinuities. 0-0.1 for c_j means c_j is initially kept as a constant of 0.1MPa, which is removed once the relative sliding displacement of the contact pairs exceeds the threshold of 0.002m.

8.3.1 Modeling of the progressive failure mechanism

The geometry of the slope studied is shown in Fig 8.3.1. Simulations were performed for two different distributions of discontinuities. Failure was achieved in both cases due to the applied load P .

In the first case shown in Fig 8.3.2, only one discontinuity daylight on the slope face, which however does not outcrop at the top of the slope. The model contains 627MCs, 679 PCs and 1194 Manifold elements. With the increase of the external load P , a tensile wing crack (Wong and Einstein 2009b) first initiated at the tip of the pre-existing discontinuity (Fig 8.3.2 b). As the load P kept increasing, the previously initiated crack kept on propagating and exhibited a tendency to become vertical near the surface (Fig 8.3.2 c). Such numerical results can be commonly observed as the tension cracks on the rock slope in the field. The coalescence of the

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developed crack with the top slope surface led to a kinematic release of the slope. When the load P continued to increase, a sliding failure occurred (Fig 8.3.2 d).

For the case shown in Fig 8.3.3, four parallel discontinuities of the same dip angle were introduced in the slope. The model contains 1268MCs, 1342 PCs and 2475 Manifold elements. The objective of this simulation was to investigate and compare the coalescence types associated with different arrangements of the pre-existing discontinuities in rock slopes with those obtained by Park and Bobet (2009) experimentally (Fig 8.3.4). According to the flaw pair geometry defined by Park and Bobet (2009) (Fig 8.3.5), the upper two discontinuities have a spacing S of 0m and a continuity C of 3m (ligament angle $\alpha > 90^\circ$), the two central discontinuities have a spacing S of 3m and a continuity C of 5m (ligament angle $\alpha < 90^\circ$) and the lower two discontinuities have a spacing S of 3m and a continuity C of 0m (ligament angle $\alpha < 90^\circ$). When the load P increased, the upper two discontinuities were first observed to propagate and led to Type I coalescence (Fig 8.3.3 b). The coalescence crack consists of a shear crack linking up the two discontinuity tips. The coalescence between the two central discontinuities was observed to correspond to Type IV coalescence, attributing to the initiation and propagation of two tensile wing cracks. The coalescence of the lower two discontinuities belongs to Type VIIb coalescence (Fig 8.3.3 b), attributing to a tensile crack. These coalescences then led to a kinematic release of the slope. When the load kept on increasing, a shear zone was formed between the two central discontinuities. Two shear cracks later developed in this bridge zone (Fig 8.3.3 c). Finally, the above progressively developed cracking processes resulted in a multi-step-path slope failure (Fig 8.3.3 d).

As the above two cases illustrated, the failure of the rock slope depends to a great extent on the complexity of the rock mass geology especially the pre-existing discontinuities. When a kinematic release does not occur along fully persistent discontinuities, a failure of the intact rock bridges is required in order to fail the slope catastrophically. The coalescence type between discontinuities is mainly dependent on their relative positions (spacing S and continuity C). Using the developed NMM as a tool, not only the slope deformation but also the crack propagation, coalescence and final failure can be captured.

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Fig 8.3.1 Geometry of the slope

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Fig 8.3.2 Failure processes associated with a planar failure on the slope containing one pre-existing discontinuity (a) initial geometry of the model (b) crack initiation (c) crack coalescence with the slope surface (d) slope failure

Fig 8.3.3 Failure processes associated with a step failure on the slope containing four pre-existing discontinuities.

Fig 8.3.3 (continued)

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Fig 8.3.4 Different coalescence types observed in experimental studies (Park and Bobet 2009)

Fig 8.3.5 Geometry of a flaw pair (Park and Bobet 2009)

8.3.2 Evaluation of the model for the progressive failure mechanism

For evaluating the model in predicting the progressive failure of a rock slope, the safety factors (SF) predicted by the developed NMM are compared with the results presented by Jennings (1970) (Fig 8.3.6 a,b) and Jaeger (1971) (Fig 8.3.6 c). The SF calculation is based on successively increasing the acceleration of gravity until the slope fails. Assuming g_0 is the acceleration of gravity in the initial state and

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g_{trial} is the acceleration of gravity at failure, the SF can be defined as follows (Li et al. 2009):

$$SF = \frac{g_{\text{trial}}}{g_0} \quad (8-2)$$

Five examples of rock slopes containing different number of discontinuities separated at different spacing and continuity values are analyzed (Table 8.3.2). Example 1 represents a typical planar failure with one fully persistent discontinuity (Fig 8.3.7). In this particular case, the SF obtained by the NMM is very close to that obtained from the Jennings model (1970), especially for the model with the displacement-dependent cohesion removal method. The progressive failure type in slopes containing coplanar discontinuities is modeled in examples 2, 3 and 4 (Fig 8.3.8, Fig 8.3.9, Fig 8.3.10). As illustrated in these figures, the approximations of the examples to the result of limit equilibrium model are dependent on the continuity C of the discontinuities. According to Jennings (1970), the intact rock is assumed to fail in shear as a Mohr-Coulomb material and to contribute a strength component proportional to the fraction of the failure plane that is intact rock. However, Park and Bobet (2010) and Wong and Einstein (2009c, 2007) investigated this mechanism, and found that different coalescences types (Fig 8.3.4) can be generated, which are mainly dependent on the arrangement of the pre-existing discontinuities. For coplanar discontinuities (Spacing $S=0$), when the Continuity C is relatively small, a shear coalescence type as predicted by the Jennings model will develop. Therefore, examples 2 and 3 produce close approximations to the results of the Jennings model. However, since the Jennings model fails to account for the stress concentration at the tip of a discontinuity, the results from the NMM based on the fracture mechanics model are more conservative. As illustrated in examples 2 and 3, the stress concentration effect is more significant when the Continuity C is small. For coplanar discontinuities (Spacing $S=0$), which have a large Continuity C , the coalescence type will deviate from the shear type according to the experimental results obtained by Wong and Einstein (2009c, 2007). As illustrated in example 4 (Fig 8.3.10), tensile wing cracks instead of shear cracks initiate, which result in a different failure path as predicted

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by the Jennings model. As a result, the NMM results for such an example differ significantly from that of the Jennings model.

The second type of model to consider, as shown in example 5 (Fig 8.3.11), contains non-coplanar discontinuities, leading to a step path failure model. This example makes reference to the Jaeger's model (Fig 8.3.6c), which considers the development of tensile cracks as a necessary condition for the stepping failure to occur. The final failure path predicted by the NMM is slightly different from the equivalent path according to the Jaeger's model. However, since the Jaeger's model considers the tensile strength of the intact rock in the bridge region, the SF values obtained from these two methods are close. According to the experimental results obtained by Park and Bobet (2009) (Fig 8.3.4), the crack path predicted by the NMM is reasonable. The tensile cracks first initiate at both flaw tips, which later propagate and coalesce (type VIIb) to form the failure path.

To summarise, in the case with a persistent discontinuity, the Jennings method and the NMM provide very close SF values. In cases of non-persistent coplanar discontinuities, the difference of SF values increases with the Continuity C between the discontinuities. When the Continuity C is small, the failure mechanism across the rock bridge predicted by the NMM generally agrees with that by the Jennings model. When the Continuity C becomes larger, tensile cracks instead of shear cracks develop, which leads to a discrepancy of the results between the NMM and the Jennings model. In the case of non-coplanar discontinuities, the SF value predicted by the NNM is similar to that by the Jaeger's model despite the difference in failure path trajectories. In all examples, the SF values obtained by the NMM without the displacement-dependent cohesion removal method are more conservative than the NMM with the displacement-dependent cohesion removal method.

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Table 8.3.2 Arrangement of the discontinuities of the five examples (the ligament angle α are all not bigger than 90°)

Example	Nature of discontinuities
1	One persistent discontinuity
2	Three coplanar discontinuities separated at a spacing $S = 0\text{m}$, continuity $C = 2.0\text{m}$
3	Two coplanar discontinuities separated at a spacing $S = 0\text{m}$, continuity $C = 4.0\text{m}$
4	Three coplanar discontinuities separated at a spacing $S = 0\text{m}$, continuity $C = 7.0\text{m}$
5	Three stepped discontinuities separated at a spacing $S = 3.0\text{m}$, continuity $C = 7.0\text{m}$

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Fig 8.3.6 Force-balance limit equilibrium solutions for planar-type failures, where: FS the factor of safety, c_i and ϕ_i the cohesion and friction of intact rock, c_j and ϕ_j the discontinuity cohesion and discontinuity friction, A the surface area of failure, W the sliding rock mass weight, α the slope of failure plane, k the coefficient of continuity (after Jennings (1970)). T_0 the intact tensile strength; and β the angle of discontinuity in step path (after Jaeger (1971))

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Fig 8.3.7 Evaluation of the safety factor of example 1 (a) initial geometry (b) final failure

Fig 8.3.8 Evaluation of the safety factor of example 2 (a) initial geometry (b) final failure

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Example 3
 $SF(Jennings)=6.32$
 $SF(NMM \text{ with displacement-dependent method})=5.85$
 $SF(NMM \text{ without displacement-dependent method})=5.62$

Fig 8.3.9 Evaluation of the safety factor of example 3 (a) initial geometry (b) final failure

Example 4
 $SF(Jennings)=15.4$
 $SF(NMM \text{ with displacement-dependent method})=19.9$
 $SF(NMM \text{ without displacement-dependent method})=18.7$

Fig 8.3.10 Evaluation of the Safety factor of example 4 (a) initial geometry (b) final failure

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Fig 8.3.11 Evaluation of the safety factor of example 5 (a) initial geometry (b) final failure

8.4 Conclusions

As demonstrated in this study, the progressive failure of a rock slope requires the consideration of two factors: slide plane development and internal rock mass deformation/degradation. To realistically model such a complicated process, neither the continuous nor the discontinuous approaches, but a hybrid method which combines both continuous and discontinuous methodologies can provide a comprehensive analysis. The NNM is such a method which combines the widely used FEM and DDA in a uniform frame. By incorporating the crack evolution technique and crack initiation criterion, the NMM is able to represent the cracks or discontinuities, as well as to replicate the processes of crack propagation and coalescence.

The developed NMM appears to be a useful tool for modeling the progressive slope failure mechanism,. The types of crack propagation and coalescence mentioned in the previous studies (Park and Bobet 2009; Wong and Einstein 2007, 2009b) were observed in failure analysis in rock slopes containing non-persistent discontinuities. The entire processes of the crack propagation and slide plane development can be captured. Compared with the Jennings model which assumes that the shear failure occurs in portions of the intact rock bridge, the crack

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propagation and coalescence styles obtained by the developed NMM are much more commonly observed in nature and experimental tests.

For stability analysis, in slopes containing persistent discontinuities, the developed NMM produces results very close to that obtained based on the Jennings limit equilibrium model. In cases of coplanar non-persistent discontinuities, when the Continuity C between the discontinuities is small, close results between developed NMM and Jennings's model are obtained despite the differences in the consideration of the stress concentrations at the crack/discontinuity tips. That is because under such circumstances, the shear type coalescence between the discontinuities is predicted by both models. Without considering the stress concentrations at the crack/discontinuity tips, the Jennings model usually overestimates the resistance of the rock mass over failure. When the Continuity C becomes larger for coplanar discontinuities, the coalescence types predicted by the NMM are not shear but tensile ones. The SF values predicted by the NMM thus deviate from those predicted by the Jennings model. In cases of non-coplanar discontinuities, since the influence of the tensile strength is considered by both the Jaegers model and the NMM, the SF values are similar despite the difference in the failure path trajectories predicted by these two models. Compared with the physical tests performed by Park and Bobet (2009) the failure paths predicted by the developed NMM are more reasonable.

The results also reveal that the original NMM without the displacement-dependent cohesion removal method is always more conservative due to the improper removal of the interface cohesion. The displacement-dependent cohesion method can to some extent alleviate such a limitation. However, this method still needs to be further verified in the future. Most importantly, the relation between the cohesion with the shear displacements of rocks needs to be further investigated. Generally, the NMM developed in this chapter shows its preliminary capability to simulate the progressive failure of rock slopes. A future development is warranted to target at more complicated practical rock slope stability assessments.

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CHAPTER 9 CONCLUSIONS AND FUTURE WORK

9.1 Conclusions

This thesis mainly focuses on the extension of the NMM for cracking problems in rocks, such as crack initiation and propagation analysis. Based on the work done in this thesis, the following conclusions can be drawn:

1. Based on the advantages of the NMM in treating discontinuities, different crack initiation criteria have been incorporated into the NMM, such as the MTS-criterion, the S-criterion and Mohr-Coulomb criterion, to investigate the effect of different criteria on the crack propagation direction. The results showed that the crack initiation criteria do have a significant effect on the crack initiation direction. By comparing the results with the analytical solutions, the efficiency and accuracy of the NMM in predicting the crack initiation direction have been illustrated.
2. For the conventional NMM, the element which is partially cut by the crack usually cannot be represented well since the singularity at the crack tip cannot be captured by the polynomial approximation. However, in the NMM the approximation can be improved globally or locally, for arbitrary level. Therefore, to overcome such shortcomings, the enriched analytical solutions which include the singularity have been added to the cover displacement functions of the crack tip element. The results showed that the enriched manifold method not only improves the accuracy of the results but also solves the inconvenience that the conventional NMM requires the crack tip to be stopped at the side of elements.
3. By adopting a two-cover system, the NMM can easily capture the discontinuity between the crack surfaces in a direct way without further incorporating enrichment functions for most cases, which is necessary for other partition of unity methods. With the crack identification method and crack evolution technique, the NMM is able to model not only simple cracking problems (single crack initiation and propagation) but also crack

coalescence problems, which may not be easy for the partition of unity methods.

4. Based on the contact technique inherited from the DDA, it is very easy for NMM to consider the interactions between the flaw surfaces. The NMM thus has been used to investigate the cracking behavior of the specimens containing closed pre-existing flaws. Results illustrated that the friction between the flaw surfaces not only affects the crack initiation stress and propagation path, but sometime even changes the type of crack developed.
5. The Mohr-Coulomb with a cut-off crack criterion has been incorporated in the NMM to investigate the mechanism of the cracks observed in laboratory tests, especially the secondary cracks which sometimes are hard to determine whether they are tensile or shear by their appearances. The incorporation of the Mohr-Coulomb propagation criterion has also been justified by comparing the numerical predictions with the experimental results, which revealed that with Mohr-Coulomb with a cut-off crack criterion, the NMM can not only predict the tensile wing or shear cracks (initiation stresses and angles) accurately, but also the mixed type of cracks satisfactorily.
6. The NMM has also been extended and enhanced by incorporating the modified Mohr-Coulomb crack initiation criterion and the Drucker-Prager constitutive model with a tensile cut-off. Using such enhanced NMM, the deformation and failure processes of rock specimens, such as initial elastic deformation, secondary plastic deformation, large displacement (sliding or even discrete blocky movements) and multiple crack propagation and coalescences, which are typically observed in experimentally studies, have been simulated.
7. By incorporating a modified 3-element viscoelastic constitutive mode in the NMM, the viscoelastic deformation behavior of a sedimentary rock under different loading rates has been modeled and investigated. The numerical results reveal that under a high loading rate ($>1000\text{MPa/s}$), due to the viscoelastic property of the sedimentary rock, not only the structural

behavior deviates from that of elastic model, but also different cracking processes and final failure modes are obtained.

8. The displacement-dependent cohesion removal method has been adopted to overcome the limitation of the original NMM associated with an improper removal of the interface cohesion of the discontinuities. Using the modified NMM, the crack propagation and coalescence processes which represent the fundamental mechanisms of the progressive failure processes for rock slopes have been investigated. Simple examples of failure of rock slopes containing different arrangements of discontinuities have been modeled and their results have been compared with those based on the limit equilibrium method proposed by Jennings (1970) and Jaeger (1971). The results have illustrated that the types of crack propagation and coalescence obtained by the developed NMM are generally reasonable and worth to be further developed to represent the processes in nature and experimental studies.

9.2 Limitations and future work

The studies in this thesis have demonstrated the promising capability of the NMM to simulate the cracking behavior of rock material. However, compared with other numerical techniques such as FEM, XFEM and DEM, NMM is still at its infancy in terms of development, application and validation.

In this thesis, due to the complications of rock cracking related failure problems, many assumptions have been introduced in the developed model. The damping effect is not considered in this study. One of the reasons is due to the main focus of this thesis on the cracking problems under quasi-static load. While for the cracking problems under dynamic load, an equivalent viscoelastic model which has already partially considered the effect of the damping under micro-scale is adopted. Another reason is that according to the work done by Lin and Chang (2004), for damping ratio smaller than 20%, the effect on the analytic error is within 5%. Considering the rock material adopted in this study, the damping ratio is usually smaller than 10%. The effect of the damping on simulation results is therefore neglected. However, for cases related to soft rock or rock containing multiple joints, such assumption may make the results unreliable. For the case under cyclic loading,

such assumption may lead to unreliable results. Another big issue about the work done in this thesis is that only 2 D problems are investigated, which cannot realistically represent the real 3 D problems.

Considering only preliminary validation work which is mostly based on the back-calculation parameters has been done, the models developed in this thesis still have to be further validated and developed. For modeling more complicated practical rock failure process, the following future research works are recommended:

1. Development of a high-order approximation to improve the accuracy, which unifies the continuous-discontinuous and higher-order aspects of NMM together.
2. To incorporate probabilistic algorithms in which variations of the material properties (e.g. Young's modulus, cohesion and friction angle) and geometrical characteristics (e.g. joints dip angle, dip direction) can be considered for slope risk analysis.
3. To couple the current NMM with fluid mechanics to consider the seepage effects on the stability of rock slopes.
4. Based on the classical rock mechanics stability approaches such as key block analysis, extend the NMM for determining the stability and deformation of critical engineering decisions for civil structures such as underground caverns, tunnels, rock slopes, and dam foundations.
5. To incorporate the damping effect into the developed NMM to target at a wider range of application and to investigate seismic effect on cracking related failure problems such as earthquake triggered slope failure.
6. The material parameters used in this study need a thorough experimental or filed investigation support. Based on the investigation, the developed models in this thesis can then further be validated.

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APPENDICES

Flow chart of original NMM

Flow chart of NMM for linear-elastic cracking problems

Flow chart of NMM for elastic-plastic cracking problems

Flow chart of NMM for viscoelastic cracking problems

Flow chart of NMM for linear-elastic problems with displacement dependence cohesion removal method

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