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On Micropolar Fluid Theory

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New fundamental solutions for an incompressible micropolar fluid have been obtained in an explicit form by considering a three-dimensional Oseen flow due to a point force and a point couple. Fundamental solutions that do not exist in classical flows have also emerged in this study owing to the existence of the microrotation velocity field in a micropolar fluid. The newly-found fundamental solutions, together with the representations that have been derived based on the Stokes approximation, are believed to be useful in the application of the boundary integral method and the singularity method in seeking for solutions to flow problems. These flow problems are not trivial, as they deal with rheologically complex fluids and the mechanics of fluids at microscale and nanoscale, where classical theories have been found to be invalid there.

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Chapter 1 Introduction

1.1 Background

Recent developments in the areas of biological “molecular machinery”, atherogenesis, microcirculation, microfluidic devices and systems and etcetera have prompted for research to be done on fluid behaviour at the microscale and nanoscale. Traditionally, theories in the well-established classical fluid mechanics, such as the Navier-Stokes equations, have been applied to model the behaviour of fluids in the above-mentioned situations.

However, there is now an understanding that the physical mechanisms of heat, mass and momentum transport in microscale and nanoscale fluid flows differ significantly from that of the macroscale. Many experimental studies done on microchannel flows have indicated that fluid flows on the microscale are different from that on the macroscale. For instance, studies have reported significant increases or decreases in the friction factor for microchannel flows (Jiang et al., 1995; Wilding et al., 1994; Wu & Little, 1983).

Further, studies done on the behaviour of other Non-Newtonian fluids, like polymeric fluids and human blood, are found to deviate from the prediction based on classical theories. Experiments on fluids that contain extremely small amount of polymeric additives indicated that the skin friction near a rigid surface are about

30% to 50% lower than those without additives (Hoyt & Fabula, 1964; Vogel & Patterson).

Such discrepancies in experimental results and theoretical predictions have been attributed to several factors, including surface effects and microrotational effects of the molecules (Figure 1.1). On a deeper level, these experimental findings seem to imply that the Navier-Stokes equation can no longer predict fluid behaviour accurately in the above-mentioned situations.

Figure 1.1 Schematic representation of various effects in microscale fully developed fluid flow (Papautsky, Brazzle, Ameel, & Frazier, 1999)

The Navier-Stokes equations are based on classical continuum mechanics. Within the continuum theory, one regards matter as a continuous entity, made up of 'particles'. A 'particle' in a continuum is an infinitesimal volume of material, which has properties such as position, mass and density defined but does not deform. Therefore, all material bodies possess continuous mass densities, and that constitutive equations are valid throughout an entire body.

However, as the size of a material volume element approaches zero, it is found that, past a certain volume, the mass density begins to show a dependence on its volume and the continuity assumption for mass density is no longer valid. Since the macroscopic limitation of the material volume element exists, the stress tensor is no longer symmetric, and hence couple stresses arise. The inadequacy of the classical continuum theory has led to the development of theories of microcontinua in which continuous media possess not only mass and velocity but also a substructure.

At scales larger than a few molecules ($\sim 1\text{nm}$), a fluid can be treated as a continuum, and its flow is governed by partial differential equations, expressing conservation of mass, momentum and energy. Within this framework of continuum mechanics, several theories, such as the theory of polar fluids (Cowin, 1974), the theory of anisotropic fluids (Ericksen, 1960) and the theory of simple microfluids (Eringen, 1964), have been developed to take into account geometry, deformation, and intrinsic motion of individual material particles.

Of interest here is the theory of simple microfluids that stems from the concept of a microcontinuum, which is a continuous collection of deformable point particles. In a microcontinuum, material particles (or microelements) are said to be contained within a particle (or macroelement). As a result, a macroelement is deformable. In a simple microfluid, its properties and behaviour are also affected by local motions of the material particles (or microelements) contained in each of its particle (or macroelement).

The theory of simple microfluids is a complicated one. Even in the simplest case of constitutively linear theory, a microfluid has twenty-two viscosity coefficients. The non-linear Stokesian fluids turn out to be a special class of the microfluids. The classification of microfluids is best summarized by Figure 1.2.

Figure 1.2 Classification of microfluids (Eringen, 1999)

Within the family of microfluids, there is a sub-class of fluids called the micropolar fluids (Eringen, 1967). The theory of micropolar fluids asserts that micropolar fluids can support couple stresses and body couples and exhibit microrotational effects. In such fluids, rigid particles contained in a small volume element can rotate about the centroid of the volume element, in an average sense, described by an independent micro-rotation vector.

The theory of micropolar fluids has shown promise in predicting fluid behaviour at the microscale and nanoscale. For example, a study done by Papautsky et al. (1999) revealed that a numerical model for water flows in microchannels, based on the theory of micropolar fluids, gives better predictions of experimental results than those given by the Navier-Stokes equations (Figure 1.3 and Figure 1.4).

Figure 1.3 Comparison of experimental data, Navier-Stokes theory, and the numerical model for water flows in microchannel. Microchannels used to obtain experimental data were 3000x600x30 μm^3 (LxWxH) (Papautsky et al., 1999)

Figure 1.4 Comparison of experimental data with the model predictions for water flows in microchannels. Channels used by Jiang et al. (1995) were 10000~60~25.4 μm^3 (LxWxH), while channels used by Wilding et al. (1994) were 11700x80x20 μm^3 (Wilding 1) and 11700~150~40 μm^3 (Wilding 2). Microchannels used to obtain experimental data were 3000x600x30 μm^3 (Papautsky et al., 1999)

In addition, Power (1995) modelled the low Reynolds number flow of the cerebrospinal fluid through the brain based on the micropolar fluid theory. The resulting system of integral equation possesses a unique continuous solution. The model also has the possibility of showing the different mechanisms that the brain has to control the flow of the cerebrospinal fluid.

Furthermore, micropolar fluids can also model anisotropic fluids, liquid crystals with rigid molecules, magnetic fluids, cloud with dust, muddy fluids and other biological fluids (Eringen, 1999). Therefore, in view of the potential applications of the micropolar fluid theory in microscale and nanoscale fluid mechanics and in non-Newtonian fluid mechanics, it seems justifiable to explore this theory in greater detail.

1.2 Objectives

This project aims to meet the following objectives:

- to explore the theory of micropolar fluids,
- to obtain the fundamental solutions of the micropolar fluid flows.

The relevant technical aspects of the micropolar fluids will be discussed in the subsequent chapters.

Chapter 2 Continuum theory of fluids

It is mentioned in Chapter 1 that the concept of a particle in a continuum and in a microcontinuum differs. However, many ideas of the microcontinuum theory are based largely on the classical continuum theory, which has gained familiarity with a larger group of readers owing to its history. This chapter highlights the ideas in continuum mechanics that are relevant to the microcontinuum theory. Important theoretical aspects of micropolar fluids, namely the constitutive equations and the governing equations, are also given.

2.1 Kinematics

This section deals with the kinematics of continuum mechanics, which has been well-established. In particular, some fundamental equations in classical fluid mechanics are derived. These equations serve as a foundation for the derivation of the governing equations of the micropolar fluids.

2.1.1 Material and spatial coordinates

In the study of continuum mechanics, matter is treated as a continuous entity. Such an entity can be represented by a *material body*, which is made up of a set of *particles* or *material points*. A *particle* in the Euclidean n -space $\mathcal{R}^n$ can be thought of as an infinitesimal volume of material with density defined.

Suppose at a certain time $t = t_0$, a material body occupies a certain region in $\mathcal{R}^3$ and a particle on it has position vector $\mathbf{X}$ measured from the origin (Figure 2.1). At this time, only one particle can occupy the position given by (X_1, X_2, X_3) . Thus, the set (X_1, X_2, X_3) identifies that particle and is known as *material coordinates*. Often, the particle is called $\mathbf{X}$ out of sheer convenience.

At a later time t , we let the position vector of the particle be $\mathbf{x}$. The set (x_1, x_2, x_3) is called *spatial coordinates* of particle $\mathbf{X}$. Then, with respect to an orthogonal basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, $\mathbf{X}$ and $\mathbf{x}$ can be written as

$$\mathbf{X} = X_1 \hat{\mathbf{e}}_1 + X_2 \hat{\mathbf{e}}_2 + X_3 \hat{\mathbf{e}}_3 = X_i \hat{\mathbf{e}}_i,$$

$$\mathbf{x} = x_1 \hat{\mathbf{e}}_1 + x_2 \hat{\mathbf{e}}_2 + x_3 \hat{\mathbf{e}}_3 = x_i \hat{\mathbf{e}}_i$$

respectively, adopting the Einstein's summation convention (Appendix B).

Figure 2.1 Material and Spatial coordinates

Hence, the path of the particle can be described by the mapping

$$(2.1.1) \quad \mathbf{x} = \mathbf{x}(\mathbf{X}, t) \quad \text{or} \quad x_i = x_i(X_1, X_2, X_3, t)$$

$$(2.1.2) \quad \text{with } \mathbf{x}(\mathbf{X}, 0) = \mathbf{X},$$

where t can be taken to be a parameter and t_0 is often taken to be zero, without any loss of generality.

It is commonly assumed that the motion of the body is continuous, and that (2.1.1) is a one-one function. Physically, the first assumption means that the particles in the neighbourhood of a particle continue to remain in that same neighbourhood during the motion. The second assumption restricts a particle to occupy only one position and no two different particles can occupy the same position $\mathbf{x}$ at any time t .

It is desirable for (2.1.1) to be invertible; that is

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t) \quad \text{or} \quad X_i = X_i(x_1, x_2, x_3, t).$$

The validity of (2.1.3) is governed by the following theorem, stated without proof:

Theorem 2.1 If $x_i = g_i(y_1, \dots, y_n)$, $i = 1, \dots, n$, are continuous functions of the variables $y_1, \dots, y_n$ with continuous first partial derivatives, and if the Jacobian

$$J = \frac{\partial(x_1, \dots, x_n)}{\partial(y_1, \dots, y_n)}$$

does not vanish, then the transformation from $\mathbf{y}$ to $\mathbf{x}$ can be

uniquely inverted to give $y_i = f_i(x_1, \dots, x_n)$ (Aris, 1989).

Based on the above theorem, for (2.1.3) to exist, it is assumed that J is non-zero ($0 < J < \infty$) and (2.1.3) is differentiable as many times as required, though the theorem does not require such high orders of differentiability.

2.1.2 Material derivative

In general, any scalar, vector or tensor representation of any physical property of a material body can be expressed as a function of $\mathbf{x}$ and t or of $\mathbf{X}$ and t . Let f be such a scalar function given by

If $\mathbf{X}$ is prescribed, then the value of f is what an observer attached to the particle will measure. The *material derivative* of the property f is defined as the time rate of change of f measured by the observer. In other words,

$$(2.1.5) \quad \frac{Df}{Dt} = \left. \frac{\partial f}{\partial t} \right|_{\mathbf{X}=\text{constant}},$$

where $\frac{D}{Dt}$ denotes the material derivative. Should f have been prescribed instead, then (2.1.5) becomes

$$(2.1.6) \quad \frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial t} + \frac{\partial f}{\partial x_3} \frac{\partial x_3}{\partial t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial t},$$

by virtue of the chain rule. Since $\frac{\partial x_i}{\partial t}$ gives the time rate of change of displacement

x_i , or the velocity in the x_i -direction, denoted by u_i , therefore

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial t} + (\mathbf{u} \cdot \nabla) f,$$

where ∇ is the del operator. The material derivative can then be generalized as

$$(2.1.7) \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla).$$

The first term, $\frac{\partial}{\partial t}$, is known as the local rate of change. It measures the rate of change at x . The second term, $u_i \frac{\partial}{\partial x_i}$, is known as the convective rate of change as it measures the rate of change as the particle changes its location.

2.1.3 Reynolds transport theorem

Figure 2.2 Volume V moving in region D

Consider a region D in $\mathcal{R}^3$ being filled with moving fluid. Let $V(t)$ be an arbitrary closed volume moving with the fluid in D (Figure 2.2). Introducing f as any function of x and t , the *Reynolds transport theorem* states that

$$\frac{d}{dt} \int_{V(t)} f(x, t) dx = \int_{V(t)} \left\{ \frac{\partial f(x, t)}{\partial t} + u(x, t) \cdot \nabla f(x, t) + f(x, t) \nabla \cdot u(x, t) \right\} dx.$$

To prove this theorem, the following lemmas are required (Aris, 1989):

Lemma 2.2

$$dx = J(X, t) dx_0, \text{ where } dx_0 = dX_1 dX_2 dX_3.$$

Lemma 2.3

$$\frac{d}{dt} J(X, t) = J(X, t) \nabla \cdot u(x(X, t), t).$$

Proof for transport theorem:

Let $G(t) = \int_{V(t)} f(\mathbf{x}, t) d\mathbf{x}$ and take derivative with respect to time. Thus, we have

$$\frac{d}{dt} G(t) = \frac{d}{dt} \int_{V(t)} f(\mathbf{x}, t) d\mathbf{x}.$$

The derivative on the right-hand side of the above equation cannot be brought through the integral sign as the integration is done throughout a varying volume $V(t)$. Hence, it is necessary to transform the domain of integration from $\mathbf{x}$ -space to $\mathbf{X}$ -space by using Lemma 2.2. Therefore,

$$\begin{aligned} \frac{d}{dt} G(t) &= \frac{d}{dt} \int_{V(0)} F(\mathbf{X}, t) J d\mathbf{x}_0 \\ &= \int_{V(0)} \frac{d}{dt} [F(\mathbf{X}, t) J(\mathbf{X}, t)] d\mathbf{x}_0 \\ &= \int_{V(0)} \left\{ J(\mathbf{X}, t) \frac{d}{dt} F(\mathbf{X}, t) + F(\mathbf{X}, t) \frac{d}{dt} J(\mathbf{X}, t) \right\} d\mathbf{x}_0. \end{aligned}$$

Finally, we apply Lemma 2.3 to the last equality to yield (2.1.8):

$$\begin{aligned} \frac{d}{dt} G(t) &= \int_{V(0)} \left\{ J(\mathbf{X}, t) \frac{d}{dt} F(\mathbf{X}, t) + F(\mathbf{X}, t) J(\mathbf{X}, t) \nabla \cdot \mathbf{u}(\mathbf{x}(\mathbf{X}, t), t) \right\} d\mathbf{x}_0 \\ &= \int_{V(0)} \left\{ \frac{d}{dt} F(\mathbf{X}, t) + F(\mathbf{X}, t) \nabla \cdot \mathbf{u}(\mathbf{x}(\mathbf{X}, t), t) \right\} J(\mathbf{X}, t) d\mathbf{x}_0 \\ &= \int_{V(t)} \left\{ \frac{D}{Dt} f(\mathbf{x}, t) + f(\mathbf{x}, t) \nabla \cdot \mathbf{u}(\mathbf{x}, t) \right\} d\mathbf{x} \\ &= \int_{V(t)} \left\{ \frac{\partial}{\partial t} f(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t) \cdot \nabla f(\mathbf{x}, t) + f(\mathbf{x}, t) \nabla \cdot \mathbf{u}(\mathbf{x}, t) \right\} d\mathbf{x} \end{aligned}$$

Q.E.D.

2.1.4 Equation of continuity

One of the axioms of continuum mechanics is the *principle of conservation of mass*, which states that the mass of fluid in a material volume does not change. Suppose

that $\rho(\mathbf{x}, t)$ is the mass per unit volume of the moving fluid at some point $\mathbf{x}$ and time t . Then, the mass of fluid, m , is given by

$$m = \int_{V(t)} \rho(\mathbf{x}, t) d\mathbf{x}$$

and that

$$\frac{dm}{dt} = 0.$$

If we replace f in (2.1.8) by ρ , then we have

$$\frac{dm}{dt} = \int_{V(t)} \left\{ \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} \right\} d\mathbf{x} = 0.$$

Since V is arbitrary, the above integrand must be zero. Therefore, we write

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0,$$

or simply,

$$(2.1.9) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \text{or} \quad \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0.$$

Equation (2.1.9) is better known as the *equation of continuity*.

If $f(\mathbf{x}, t) = \rho(\mathbf{x}, t)g(\mathbf{x}, t)$, then (2.1.8) becomes

$$\frac{d}{dt} \int_{V(t)} \rho g d\mathbf{x} = \int_{V(t)} \left\{ \rho \left[\frac{\partial g}{\partial t} + \mathbf{u} \cdot \nabla g \right] + g \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] \right\} d\mathbf{x}.$$

By virtue of (2.1.9), the above equation simplifies to

$$(2.1.10) \quad \frac{d}{dt} \int_{V(t)} \rho g d\mathbf{x} = \int_{V(t)} \rho \frac{Dg}{Dt} d\mathbf{x}.$$

2.2 Dynamics

In the previous section, we consider only the kinematic description of the motion of a continuum without considering the forces that cause motion and deformation. It is generally accepted that matter is formed by molecule, which in turn consist of atoms and subatomic particles. Hence, the internal forces in real matter are those between the mentioned particles. In this section, we shall derive equations that describe the interaction between the internal forces and the kinematic quantities of the body.

2.2.1 Forces in a continuum

In general, it is assumed that forces that can act on an element of a continuous media fall into two categories: *body forces* and *surface forces*. Body forces or external forces, like gravitational forces and electrostatic forces, for example, act throughout a body by a non-contact interaction with matter. On the other hand, forces that arise as a result of contact and act over a surface element of the body are called surface forces. The element can be part of the bounding surface or an arbitrary element on a surface within the body. Contact forces between the faces of two bodies, pressed against each other, is an example of surface forces.

2.2.2 Cauchy's principle

Consider a body acted upon by forces F_1 , F_2 and F_3 (Figure 2.3). Let the body be sliced into two portions, labelled I and II. An arbitrary point P lies on a small elemental area ΔA , shaded in Figure 2.4, on the surface Π of portion I. The elemental area has an outward normal unit vector $\hat{n}$. The resultant internal surface over ΔA , by portion II on portion I, is ΔF , is non-parallel to $\hat{n}$ in general.

Figure 2.3 Body acted upon by body forces & surface forces

Figure 2.4 Stress vector at interior point P

The stress vector at P , denoted by $\mathbf{t}_{(\hat{n})}$, is defined as

$$\mathbf{t}_{(\hat{n})} = \lim_{\Delta A \rightarrow 0} \frac{\Delta \mathbf{F}}{\Delta A}.$$

The *Cauchy's stress principle* states that $\mathbf{t}_{(\hat{n})}$ is a function of $\mathbf{x}$, t and $\hat{\mathbf{n}}$. More specifically, $\mathbf{t}_{(\hat{n})}$ can be expressed as

$$(2.2.1) \quad \mathbf{t}_{(\hat{n})} = \hat{\mathbf{n}} \cdot \mathbf{T} \quad \text{or} \quad t_{(\hat{n})i} = T_{ji} n_j,$$

where $\mathbf{T}$ is known as the stress tensor. The derivation of (2.2.1) is shown in the next subsection. Note that in (2.2.1), $\mathbf{t}_{(\hat{n})}$ and $\hat{\mathbf{n}}$ are represented as row matrices. They can also be written as column matrices by taking the transpose of (2.2.1), which gives

$$t_{(\hat{n})i} = T_{ij} n_j.$$

By Newton's third law, portion II experiences a stress vector pointing in the opposite direction of $\mathbf{t}_{(\hat{n})}$ at P' , a point that coincides with P when portions I and II are put back together. In other words,

(2.2.2)

$$\mathbf{t}_{(\hat{n})} = -\mathbf{t}_{(-\hat{n})}.$$

2.2.3 Principle of linear momentum

To derive (2.2.1), it is useful to employ the *principle of linear momentum*, which states that the time rate of change of linear momentum of a body is equal to the resultant force acting on it. Formally,

$$(2.2.3) \quad \frac{d}{dt} \int_V \rho \mathbf{u} d\mathbf{x} = \int_V \rho \mathbf{f} d\mathbf{x} + \int_S \mathbf{t}_{(\hat{n})} dS,$$

where $\mathbf{S}$, $\mathbf{V}$, $\mathbf{f}$ and $\mathbf{u}$ represent the bounding surface, volume, body force per unit mass and linear velocity of the body respectively.

Proof of (2.2.1):

Suppose a body in $\mathcal{R}^3$ has a certain shape with a characteristic length d such that $V \propto d^3$ and $S \propto d^2$. We divide (2.2.3) throughout by d^2 and then let the body shrink to a point, while preserving its shape. Since $\rho \mathbf{f}$ is approximately constant throughout the infinitesimal body, the volume integral on the right-hand side of (2.2.3) can be rewritten as $\lim_{d \rightarrow 0} \frac{1}{d^2} \rho \mathbf{f} (a' d^3)$, where a' is the proportionality constant. It is evident that this limit simplifies to zero. Similar reasoning can be applied to the term on the left-hand side of (2.2.3). Eventually, (2.2.3) reduces to

$$(2.2.4) \quad \lim_{d \rightarrow 0} \frac{1}{d^2} \int_S \mathbf{t}_{(\hat{n})} dS = \mathbf{0},$$

which says that the local resultant stress is always zero.

Next, consider a small tetrahedron with vertices $\mathbf{O}$, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as shown in Figure 2.5. Plane ABC has an outward normal $\hat{\mathbf{n}} = n_i \hat{\mathbf{e}}_i$ and an area of ΔA . By vector analysis, it can be shown that the area of each remaining plane is given by

$(\hat{n} \cdot \hat{e}_i) \Delta A$ or $n_i \Delta A$, where $\hat{n}$ is the unit vector normal to the plane. For instance, the area of plane OAB is $n_3 \Delta A$.

Figure 2.5 Stresses on an elementary tetrahedron

The tetrahedron is subjected to stresses $\mathbf{t}_{(-\hat{e}_1)}$, $\mathbf{t}_{(-\hat{e}_2)}$, $\mathbf{t}_{(-\hat{e}_3)}$ and $\mathbf{t}_{(\hat{n})}$. Applying (2.2.4) to the tetrahedron yields

$$(\mathbf{t}_{(\hat{n})} + \mathbf{t}_{(-\hat{e}_1)}n_1 + \mathbf{t}_{(-\hat{e}_2)}n_2 + \mathbf{t}_{(-\hat{e}_3)}n_3)\Delta A = \mathbf{0}.$$

Further, by employing (2.2.2), the above equation becomes

$$(2.2.5) \quad \mathbf{t}_{(\hat{n})} = \mathbf{t}_{(\hat{e}_1)}n_1 + \mathbf{t}_{(\hat{e}_2)}n_2 + \mathbf{t}_{(\hat{e}_3)}n_3 = n_j \mathbf{t}_{(\hat{e}_j)}.$$

Since $\mathbf{t}_{(\hat{e}_j)}$ is a stress vector, it can be expressed as $t_{(\hat{e}_j)1}\hat{e}_1 + t_{(\hat{e}_j)2}\hat{e}_2 + t_{(\hat{e}_j)3}\hat{e}_3$; its i^{th} -component is $t_{(\hat{e}_j)i}$. By defining a stress tensor $\mathbf{T}$ such that $T_{ji} = t_{(\hat{e}_j)i}$, (2.2.5) becomes

$$\mathbf{t}_{(\hat{n})} = \hat{n} \cdot \mathbf{T} \quad \text{or} \quad t_{(\hat{n})i} = T_{ji}n_j$$

Q.E.D.

2.2.4 Cauchy's equation of motion

Using (2.2.1) and the Divergence theorem, (2.2.3) can be rewritten as

$$(2.2.6) \quad \rho \frac{Du}{Dt} = \rho \mathbf{f} + \nabla \cdot \mathbf{T} \quad \text{or} \quad \rho \frac{Du_i}{Dt} = \rho f_i + T_{ji,j},$$

which is known as the *Cauchy's equation of motion*. It is clear that this equation relates the linear momentum of the body to the applied forces.

Proof:

The Divergence theorem states that

$$(2.2.7) \quad \int_V \nabla \cdot \mathbf{A} \, d\mathbf{x} = \int_S \hat{\mathbf{n}} \cdot \mathbf{A} \, dS \quad \text{or} \quad \int_V A_{j,j} \, d\mathbf{x} = \int_S A_j n_j \, dS ,$$

where A_j is the component of any vector $\mathbf{A}$ and all other notations have their usual meanings. Substituting (2.2.1) into (2.2.3) gives

$$\frac{d}{dt} \int_V \rho \mathbf{u} \, d\mathbf{x} = \int_V \rho \mathbf{f} \, d\mathbf{x} + \int_S \hat{\mathbf{n}} \cdot \mathbf{T} \, dS .$$

Referring to (2.2.7), the Divergence theorem allows the surface integral to be written as $\int_V \nabla \cdot \mathbf{T} \, d\mathbf{x}$. Further, the time derivative of the volume integral can be

replaced by $\int_V \rho \frac{D\mathbf{u}}{Dt} \, d\mathbf{x}$, by virtue of (2.1.10). Hence, we have

$$\int_V \rho \frac{D\mathbf{u}}{Dt} \, d\mathbf{x} = \int_V [\rho \mathbf{f} + \nabla \cdot \mathbf{T}] \, d\mathbf{x} .$$

For an arbitrary volume V , the integrands on both sides of the equation must be equal, thus giving (2.2.6).

Q.E.D.

2.2.5 Equation of angular momentum

While the Cauchy's equation governs the linear motion of a continuum, there also exists a similar equation governing the angular motion, which can be expressed as

$$(2.2.8) \quad \rho \frac{D\mathbf{W}}{Dt} = \rho \mathbf{l} + \nabla \cdot \mathbf{C} + \mathbf{T}_x ,$$

Chapter 7 Continuum theory of fluids

where $\rho \mathbf{W}$ is an intrinsic or internal angular momentum per unit volume, $\rho \mathbf{l}$ is a body torque per unit volume, $\mathbf{C}$ is a couple stress tensor and $\mathbf{T}_x$ is a vector with i^{th} -component as $\varepsilon_{ijk} \mathbf{T}_{jk}$.

Proof:

The proof begins with the *principle of angular momentum*, which is given by

$$\frac{d}{dt} \int_V \rho (\mathbf{W} + \mathbf{x} \times \mathbf{u}) d\mathbf{x} = \int_V (\rho \mathbf{l} + \mathbf{x} \times \rho \mathbf{f}) d\mathbf{x} + \int_S (\mathbf{c}_{(\hat{n})} + \mathbf{x} \times \mathbf{t}_{(\hat{n})}) dS.$$

The term on the left-hand side of (2.2.9) represents the total angular momentum. The first term, $\rho \mathbf{W}$, has been explained earlier. The second term, $\rho (\mathbf{x} \times \mathbf{u})$, is recognisable as the moment of linear momentum. The term $\mathbf{c}_{(\hat{n})}$, in the surface integral, is a couple stress tensor that is analogous to the stress vector $\mathbf{t}_{(\hat{n})}$. Hence, it follows that $\mathbf{c}_{(\hat{n})} = \hat{\mathbf{n}} \cdot \mathbf{C}$, where $\mathbf{C}$ is a couple stress tensor.

By the Divergence theorem given by (2.2.7), we rewrite the surface integral of $\mathbf{c}_{(\hat{n})}$ as follows:

$$\int_S \mathbf{c}_{(\hat{n})} dS = \int_S \hat{\mathbf{n}} \cdot \mathbf{C} dS = \int_V \nabla \cdot \mathbf{C} d\mathbf{x}.$$

The other surface integral has i^{th} -component as $\int_S \varepsilon_{ijk} x_j t_{(\hat{n})k} dS$. Using (2.2.1) and the

Divergence theorem, we can write this surface integral as

$$\int_S \varepsilon_{ijk} x_j t_{(\hat{n})k} dS = \int_V [\varepsilon_{ijk} x_j T_{pk,p} + \varepsilon_{ijk} T_{jk}] d\mathbf{x}$$

or

$$\int_S \mathbf{x} \times \mathbf{t}_{(\hat{n})} dS = \int_V [\mathbf{x} \times (\nabla \cdot \mathbf{T}) + \mathbf{T}_x] d\mathbf{x},$$

where $\mathbf{T}_x$ is a vector with $\varepsilon_{ijk} \mathbf{T}_{jk}$ as its i^{th} -component.

Next, by virtue of (2.1.10), the term on the left-hand side of (2.2.9) can be replaced

by $\int_V \rho \frac{D}{Dt} (\mathbf{W} + \mathbf{x} \times \mathbf{u}) d\mathbf{x}$. Hence, (2.2.9) can be rewritten as

$$\int_V \rho \frac{D}{Dt} (\mathbf{W} + \mathbf{x} \times \mathbf{u}) d\mathbf{x} = \int_V [\rho \mathbf{l} + \mathbf{x} \times \rho \mathbf{f} + \nabla \cdot \mathbf{C} + \mathbf{x} \times (\nabla \cdot \mathbf{T}) + \mathbf{T}_x] d\mathbf{x}.$$

As V is arbitrary, the integrands on both sides of the above equation must be equal.

Therefore, we can write

$$\rho \frac{D}{Dt} (\mathbf{W} + \mathbf{x} \times \mathbf{u}) = \rho \mathbf{l} + \mathbf{x} \times \rho \mathbf{f} + \nabla \cdot \mathbf{C} + \mathbf{x} \times (\nabla \cdot \mathbf{T}) + \mathbf{T}_x.$$

Furthermore, since $\rho \frac{D}{Dt} (\mathbf{x} \times \mathbf{u}) = \rho \left(\mathbf{x} \times \frac{D\mathbf{u}}{Dt} \right)$, the above equation simplifies to

$$\mathbf{x} \times \left(\rho \frac{D\mathbf{u}}{Dt} - \rho \mathbf{f} - \nabla \cdot \mathbf{T} \right) = \rho \mathbf{l} + \nabla \cdot \mathbf{C} + \mathbf{T}_x - \rho \frac{D\mathbf{W}}{Dt}.$$

In view of the Cauchy's equation of motion, the term in the parenthesis goes to zero, thus, giving (2.2.8).

Q.E.D.

As illustrated in Figure 1.2, the classical fluid is a subset of the class of microfluids.

A classical fluid does not transmit couple stress $\mathbf{c}$, does not support body torque $\mathbf{l}$ and internal angular momentum $\rho \mathbf{W}$. Thus, for a classical fluid, (2.2.8) simplifies to

$$(2.2.10) \quad \mathbf{T}_x = \mathbf{0}.$$

This is an important result in the classical continuum theory. It implies that the stress tensor is always symmetric.

2.3 Constitutive equations for micropolar fluids

The equations derived in Sections 2.1 and 2.2 represent some of the basic axioms of classical mechanics. They are applicable to all materials so long as the materials can be modelled as a continuum. As such, they are too general to be used in describing the mechanical behaviour of a particular material. To address this inadequacy, *constitutive equations* are given in this section. They connect the dependency of stress acting on a material particle, with the history of both deformation and temperature. In general, a constitutive equation takes the following form:

$$(2.3.1) \quad \sigma_{ij} = f_{ij}(\text{history of deformation, history of temperature}),$$

where f_{ij} are the components of a second-order tensor.

The general form given by (2.3.1) can be tailored by assuming the independent variables in f_{ij} . The choices of such variables are motivated through experiments.

The forms of f_{ij} are restricted, as they have to conform to the following basic axioms:

- 1) Axiom of Causality
- 2) Axiom of Determinism
- 3) Axiom of Equipresence
- 4) Axiom of Objectivity
- 5) Axiom of Material Invariance

6) Axiom of Neighbourhood

7) Axiom of Memory

8) Axiom of Admissibility

Details of these axioms can be found in textbooks on classical continuum theory (Eringen, 1980). Based on these axioms and the constitutive equations, the governing equations of micropolar fluids are derived in this section.

2.3.1 Definition of a fluid

A *fluid* is any material that is unable to withstand shear forces when it is at rest. In other words, it can sustain shear stresses only when it is in motion and will continue to be in motion as long as the shearing stresses remain. This property can be demonstrated by a sudden tilting of a container filled with fluid that was at rest. The gravitational forces induce shear stresses on the fluid, causing it to flow till the free surface is again horizontal.

Though the presence of shear stresses is no longer felt by the fluid in the container, the fluid does not return to its initial configuration; it has no memory of its past states. Hence, a fluid is also a body with every configuration of it, leaving the density unchanged, taken to be the reference configuration.

Based on the definition of a fluid, the stress vector $\mathbf{t}_{(\hat{\mathbf{n}})}$ on an arbitrary surface element of any point in a fluid at rest has to be proportional to the normal $\hat{\mathbf{n}}$ of that element. Thus,

$$(2.3.2) \quad t_{(\hat{\mathbf{n}})i} = T_{ji} n_j = -p(\rho, \theta) n_i,$$

where p is known as the thermodynamic pressure or the hydrostatic pressure and is a function of density ρ and temperature θ . From (2.3.2),

$$(\mathbf{T}_{ji} + p\delta_{ij})n_j = 0 \Rightarrow \mathbf{T}_{ji} + p\delta_{ij} = 0$$

for all n_j . Thus, we have

$$(2.3.3) \quad \mathbf{T}_{ij} = -p\delta_{ji} = -p\delta_{ij} \quad \text{or} \quad \mathbf{T} = -p\mathbf{I}.$$

When a fluid is in motion, the shear stresses are usually non-zero. To account for that, (2.3.3) is modified as

$$(2.3.4) \quad \mathbf{T}_{ij} = -p\delta_{ij} + \mathbf{P}_{ij} \quad \text{or} \quad \mathbf{T} = -p\mathbf{I} + \mathbf{P},$$

where $\mathbf{P}_{ij}$ is called the viscous stress tensor.

2.3.2 Micropolar fluids

Micropolar fluids are fluids that exhibit microrotational effects and microinertial. Each material volume element contains rigid microelements that can rotate about the centroid of the material volume element. The material volume element does its usual rigid body motion. In addition, a microrotational vector describes the rotational motion of the microelements. Thus, against the three translational degrees of freedom of the classical theory, micropolar fluids possess six degrees of freedom: three translational degrees and three rotational degrees.

A micropolar fluid belongs to a class of *isotropic, polar* fluids. A fluid that is capable of transmitting stress couples and being subjected to body torques is said to be polar. An isotropic material is one that possesses no preferred direction and hence its mechanical properties are the same in all directions. Thus, if such a property is represented by a tensor, then its components remain unchanged under all

orthogonal transformations of coordinates. An isotropic tensor is deemed suitable for such a representation. Such a tensor is best expressed mathematically by the unit matrix $\mathbf{I}$ since its components δ_{ij} are the same for any Cartesian basis.

We assume that the intrinsic angular momentum per unit mass, denoted $\mathbf{W}$ in (2.2.9), is given by

$$(2.3.5) \quad W_i = I_{ik} v_k,$$

where v_k is the component of the vector field of microrotation, which represents the angular velocity of rotation of microelements and I_{ik} is known as the microinertia tensor. If the polar fluid is further assumed to be isotropic, as in the case of a micropolar fluid, then

where j is a scalar called the microinertial coefficient. This implies that

$$(2.3.6) \quad I_{ik} = j \delta_{ik},$$

The microinertial coefficient j is analogous to mass in linear momentum; it can be taken as the inertial of the microelements in a fluid particle to microrotation.

In addition, a micropolar fluid has stress tensor $\mathbf{T}$ and couple stress tensor $\mathbf{C}$ given as (Eringen, 1967):

$$(2.3.7a) \quad T_{ij} = (-p + \lambda_v u_{k,k}) \delta_{ij} + \mu_v (u_{i,j} + u_{j,i}) + \kappa_v (u_{j,i} - \varepsilon_{ijm} v_m),$$

$$(2.3.7b) \quad C_{ij} = \alpha_v v_{k,k} \delta_{ij} + \beta_v v_{i,j} + \gamma_v v_{j,i},$$

where $(\lambda_v, \mu_v, \kappa_v)$ represent the viscosity coefficients for the stress tensor and $(\alpha_v, \beta_v, \gamma_v)$ are the viscosity coefficients responsible for the gyrational dissipation. It

is evident that it is the presence of microrotation velocity $\mathbf{v}$ that gives rise to the non-symmetric stress tensor $\mathbf{T}$ and couple stress tensor $\mathbf{C}$.

An alternative form for $\mathbf{T}$ and $\mathbf{C}$ are given by Eukaszewicz (1999) as follows:

$$(2.3.8a) \quad T_{ij} = (-p + \lambda u_{k,k}) \delta_{ij} + \mu (u_{i,j} + u_{j,i}) + \mu_r (u_{j,i} - u_{i,j}) - 2\mu_r \varepsilon_{ijm} v_m,$$

$$(2.3.8b) \quad C_{ij} = c_0 v_{k,k} \delta_{ij} + c_d (v_{i,j} + v_{j,i}) + c_a (v_{j,i} - v_{i,j}),$$

where (λ, μ, μ_r) represents the second viscosity coefficient, the dynamic Newtonian viscosity and the dynamic microrotation viscosity respectively, and (c_0, c_a, c_d) represent the coefficients of angular viscosities. Both (2.3.7) and (2.3.8) can be made equivalent through the following relations:

$$(2.3.9) \quad \lambda_v = \lambda, \quad \kappa_v = 2\mu_r, \quad \mu_v = \mu - \mu_r,$$

$$\alpha_v = c_0, \quad \beta_v = c_d - c_a, \quad \gamma_v = c_d + c_a.$$

The existence of the second viscosity coefficient, λ , and the dynamic Newtonian viscosity, μ , brings about resistance in linear motion when there is dilation of the fluid particle and linear motion respectively. The dynamic microrotation viscosity, μ_r , can be thought of as the amount of coupling between $\mathbf{u}$ and $\mathbf{v}$ or the extent by which T_{ij} and C_{ij} are affected by the microrotation and the velocity field respectively (V. K. Stokes, 1984). Coupling between $\mathbf{u}$ and $\mathbf{v}$ brings about resistance in linear motion when there is linear motion. The coefficients of angular viscosities (c_0, c_a, c_d) in the microrotational motion can be taken to be analogous to (λ, μ, μ_r) in the linear motion.

For the local Clausius-Duhem inequality to be satisfied, the viscosity coefficients must be restricted as follows:

$$3\lambda + 2\mu \geq 0, \mu \geq 0, \mu_r \geq 0, c_d \geq 0,$$

$$c_a + c_d \geq 0, 3c_0 + 2c_d \geq 0, -(c_a + c_d) \leq c_d - c_a \leq (c_a + c_d).$$

Proof of (2.3.8a) and (2.3.8b):

We assume that

$$\mathbf{T}_{ij} = -p\delta_{ij} + F(u_{i,j} - \varepsilon_{kji}v_k),$$

$$\mathbf{C}_{ij} = G(v_{i,j}),$$

where F and G are linear functions of their arguments, that is,

$$F(u_{i,j} - \varepsilon_{kji}v_k) = A_{ijkl}(u_{i,j} - \varepsilon_{kji}v_k),$$

$$G(v_{i,j}) = B_{ijkl}v_{i,j},$$

with $\mathbf{A}$ and $\mathbf{B}$ as being fourth-order tensors. Assuming that the fluid is isotropic, $\mathbf{A}$ and $\mathbf{B}$ are then isotropic tensors. For such a tensor, it can be shown that it is of the form

$$\lambda_1(\delta_{ij}\delta_{kl}) + \mu_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \nu_1(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}),$$

where λ_1 , μ_1 and ν_1 are constants (Fung, 1994). Therefore, G may be written as

$$G = \lambda_1 v_{j,j} \delta_{kl} + \mu_1 (v_{k,l} + v_{l,k}) + \nu_1 (v_{l,k} - v_{k,l}).$$

Finally, letting $\lambda_1 = c_0$, $\mu_1 = c_d$ and $\nu_1 = c_a$, the above equation becomes (2.3.8b).

Equation (2.3.8a) can be derived in a similar manner.

Q.E.D.

With $\mathbf{W}$, $\mathbf{T}$ and $\mathbf{C}$ properly defined, the Cauchy's equation of motion, given by (2.2.6), and the equation of angular momentum, given by (2.2.8), can be expressed in terms of $\mathbf{u}$ and $\mathbf{v}$. Then, together with the equation of continuity, given by (2.1.9), they form a system of equations governing the hydrodynamics of micropolar fluids:

$$(2.3.10a) \quad \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0,$$

$$(2.3.10b) \quad -\nabla p + (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - (\mu + \mu_r) \nabla \times (\nabla \times \mathbf{u}) + 2\mu_r \nabla \times \mathbf{v} + \rho \mathbf{f} = \rho \frac{D\mathbf{u}}{Dt},$$

$$(2.3.10c) \quad (c_0 + 2c_d) \nabla (\nabla \cdot \mathbf{v}) - (c_a + c_d) \nabla \times (\nabla \times \mathbf{v}) + 2\mu_r (\nabla \times \mathbf{u} - 2\mathbf{v}) + \rho \mathbf{l} = \rho j \frac{D\mathbf{v}}{Dt}.$$

The equation of continuity has been previously derived in sub-section 2.1.4. The derivation of (2.3.10b) and (2.3.10c) from the laws of conservation of momentum and angular momentum are given as follows:

Proof of (2.3.10b):

First, we evaluate the divergence of $\mathbf{T}$ as follows:

$$\begin{aligned} \nabla \cdot \mathbf{T} &= \frac{\partial}{\partial x_k} \hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_i T_{ij} \hat{\mathbf{e}}_j = \delta_{ki} T_{ij,k} \hat{\mathbf{e}}_j = T_{ij,i} \hat{\mathbf{e}}_j, \\ \Rightarrow (\nabla \cdot \mathbf{T})_i &= \hat{\mathbf{e}}_i \cdot T_{kj,k} \hat{\mathbf{e}}_j = \delta_{ij} T_{kj,k} = T_{ki,k}. \end{aligned}$$

Then, based on the definition of $\mathbf{T}$ in (2.3.8a), $T_{ki,k}$ is found to be

$$T_{ki,k} = -p_{,i} + \lambda u_{n,ni} + \mu (u_{k,ki} + u_{i,kk}) + \mu_r (u_{i,kk} - u_{k,ki}) + 2\mu_r \varepsilon_{ikm} v_{m,k},$$

or equivalently,

$$\nabla \cdot \mathbf{T} = -\nabla p + (\lambda + \mu - \mu_r) \nabla (\nabla \cdot \mathbf{u}) + (\mu + \mu_r) \nabla^2 \mathbf{u} + 2\mu_r \nabla \times \mathbf{v}.$$

Using vector identities (Appendix C.7), the above equation can be rewritten as

$$\nabla \cdot \mathbf{T} = -\nabla p + (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - (\mu + \mu_r) \nabla \times (\nabla \times \mathbf{u}) + 2\mu_r \nabla \times \mathbf{v}.$$

By substituting the above expression for the divergence of $\mathbf{T}$ in the Cauchy's equation of motion, (2.3.10b) is yielded.

Q.E.D.

Proof of (2.3.10c)

Similarly, based on (2.3.8b), we evaluate the divergence of $\mathbf{C}$, which turns out to be

$$(\nabla \cdot \mathbf{C})_i = C_{ki,k} = c_0 v_{n,nk} + c_d (v_{k,ki} + v_{i,kk}) + c_a (v_{i,kk} - v_{k,ki})$$

or equivalently,

$$\nabla \cdot \mathbf{C} = (c_0 + c_d - c_a) \nabla (\nabla \cdot \mathbf{v}) + (c_d + c_a) \nabla^2 \mathbf{v}.$$

Using vector identities (Appendix C.7), the above equation can be rewritten as

$$\nabla \cdot \mathbf{C} = (c_0 + 2c_d) \nabla (\nabla \cdot \mathbf{v}) - (c_d + c_a) \nabla \times (\nabla \times \mathbf{v}).$$

As mentioned earlier, the vector, T_x , has $\varepsilon_{ijk} T_{jk}$ as its i^{th} -component, which can be evaluated using (2.3.8a) as follows:

$$\begin{aligned} (T_x)_i &= \varepsilon_{ijk} \{ (-p + \lambda u_{m,m}) \delta_{ij} + \mu (u_{j,k} + u_{k,j}) + \mu_r (u_{k,j} - u_{j,k}) - 2\mu_r \varepsilon_{jkm} v_m \} \\ &= 0 + \mu (\varepsilon_{ijk} u_{j,k} + \varepsilon_{ikj} u_{j,k}) + \mu_r (\varepsilon_{ijk} u_{k,j} - \varepsilon_{ikj} u_{k,j}) - 2\mu_r \varepsilon_{ijk} \varepsilon_{mjk} v_m \\ &= 0 + 2\mu_r \varepsilon_{ijk} u_{k,j} - 2\mu_r (2\delta_{im}) v_m \\ &= 2\mu_r \varepsilon_{ijk} u_{k,j} - 4\mu_r v_i, \end{aligned}$$

where the identity $\varepsilon_{jkm} \varepsilon_{mjk} = 2\delta_{ij}$ has been used in the third equality. Finally, by substituting the expressions for T_x and the divergence of $\mathbf{C}$, and (2.3.6) into (2.2.8), we obtain (2.3.10c).

Q.E.D.

2.3.3 Passage to classical fluid dynamics

Consider the special case where $\mu_r = 0$. Then, (2.3.10b) and (2.3.10c) reduce to:

$$(2.3.11b) \quad -\nabla p + (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times (\nabla \times \mathbf{u}) + \rho \mathbf{f} = \rho \frac{D\mathbf{u}}{Dt},$$

$$(2.3.11c) \quad (c_0 + 2c_d)\nabla(\nabla \cdot \mathbf{v}) - (c_a + c_d)\nabla \times (\nabla \times \mathbf{v}) + \rho \mathbf{l} = \rho \mathbf{j} \frac{D\mathbf{v}}{Dt},$$

respectively. It can be seen that the velocity field $\mathbf{u}$ and the microrotation velocity field $\mathbf{v}$ are decoupled and hence, the global motion is unaffected by microrotation. It is noted that (2.3.11b) is the well-known Navier-Stokes equation in classical fluid mechanics. Further, if $\mathbf{l}$, $\mathbf{v}$ and the viscosity coefficients c_0 , c_a , c_d are zero, then the system of field equations (2.3.10) reduces to that of classical hydrodynamics.

Chapter 3 Stokeslet and Couplet

It is well-known that all flow problems in classical hydrodynamics are governed by the continuity equation and the Navier-Stokes equation. However, the number of analytical solutions of the complete Navier-Stokes equations is few owing to its non-linearity. On top of that, at sufficiently high velocity flows, turbulence, which is associated with inherent instability of steady, laminar flow patterns, occurs. Flow quantities, such as velocity and pressure are no longer unique functions of space and time coordinates, but must be described by stochastic laws.

To obviate the difficulty of obtaining complete analytical solutions, various approximation schemes must be used. One common way of simplifying the Navier-Stokes equations at high Reynolds numbers $\mathbf{Re}$ is to neglect the viscous term, $\mu \nabla^2 \mathbf{u}$, in comparison to the inertial term, $\rho \mathbf{u} \cdot \nabla \mathbf{u}$. If the flow is further assumed to be irrotational, then one obtains the potential flow equations, which govern the ideal flows in classical hydrodynamics theory. Unfortunately, this theory does not provide any information on the drag experienced by submerged bodies. Furthermore, the omission of the viscous term will cause the solutions to be unable to satisfy the no-slip boundary condition at the boundaries. As such, solutions of this type carry no physical meaning, at least in the proximity of the boundary, even for large $\mathbf{Re}$.

Another method of simplification, known as the Stokes approximation (G. G. Stokes, 1851), which is discussed and demonstrated in this chapter, involves neglecting the inertial term, $\rho \mathbf{u} \cdot \nabla \mathbf{u}$, compared with the viscous term, $\mu \nabla^2 \mathbf{u}$, under some special conditions. Exact solutions, which carry physical meanings, are then feasible as the principle of superposition is applicable.

However, determination of the solutions for Stokes flows is still recognized to be difficult in general for arbitrary body types. As a result, not many exact solutions are known. One of the few analytical methods available for solving flow problems involving relatively complex geometries is the boundary integral method. Another method is the method of fundamental solutions or the singularity method. Both methods rely on the use of fundamental solutions, which come about from singularities in the flow field solutions that exist at the points where a point force or point couple exerts. A point force or point couple may be thought to be spherical in shape, size of which is infinitesimal. The fundamental solution due to a point force in a Stokes flow is known as *Stokeslet* (Hancock, 1953) and that of a point couple is known as rotlet (Chwang & Wu, 1974) or couplet by Batchelor (Batchelor, 1970). Other fundamental solutions include dipoles, stresslets, quadruples, etc..

The boundary integral methods form a class of numerical techniques that aims to extract quantitative information for simple and complex geometries. In the former, an appropriate coordinate system is chosen to facilitate the separation of the variables for the body of interest, yielding closed-form analytical solutions. The same is done for the latter. In addition, for complex body shapes, fundamental

solutions are used to describe the geometry of the body fully. These results in solving a 2-D (boundary) integral equation, in which the unknowns are densities of the fundamental solutions distributed over the boundary of the fluid domain, instead of solving a 3-D partial differential equation.

The boundary integral method was employed by Lawrence & Weinbaum (1986; 1988) and Lovalenti & Brady (1993) to study the unsteady hydrodynamic force on a spherical particle. Kim & Karrila (1991) adopted the same method to consider the micro-hydrodynamics of small particles in their book. Further, Power (1995) applied the method to model the flow of the cerebrospinal fluid in the brain.

On the other hand, the singularity method is based on choosing different fundamental solutions and then distributing spatially to model the body of interest, and hence obtain analytical solutions. Consequently, the success of this method relies on the choice of fundamental solutions chosen and their spatial distributions. Pozrikidis (1992) gave 3 advantages in the usage of the singularity method. Firstly, in problems involving suspended particles, one is interested in evaluating certain global variables such as the hydrodynamic force or torque acting on a suspended particle. By employing the singularity method, the variables of interest can be computed easily, without direct reference to the detailed structure of the flow. Secondly, generalized Faxen relations can be derived from singularity representations in a more direct manner. Thirdly, the singularity method may be combined effectively with the boundary integral method, giving a compound method that contains many of the advantages of its constituents.

The singularity method was adopted by Hancock (1953) to study the self-propulsion of microscopic organisms. Chwang and Wu (1974) used a spatial distribution of couplets to consider pure rotation of an axisymmetric body having an arbitrary prolate form. A year later in 1975, they derived other fundamental solutions, including stresslets, roton and stresson, by considering the Stokeslet, and employed them to construct exact solutions to some Stokes-flow problems. Blake and Chwang (1974) used fundamental solutions to obtain image systems in the vicinity of a stationary no-slip plane boundary. Dabroš (1985) used the Stokeslet as the base function to find the hydrodynamic forces and velocities of arbitrarily shaped particles in the vicinity of a wall. Kim (1986) also employed the singularity method to obtain the disturbance velocity fields due to a moving ellipsoid.

It is clear that both the boundary integral method and the singularity method have been well-received in the solution of classical flow problems, and that fundamental solutions representations form an important component. It is then necessary to derive fundamental solutions representations, corresponding to those in the classical flows, for micropolar fluid flows. The two most fundamental solutions representations are the Stokeslet and the couplet. Therefore, in this chapter, the Stokes approximation is introduced to derive them by considering flows due to a point force and a point couple.

3.1 The classical Stokes approximation

The governing equations for a classical fluid are:

$$(3.1.1) \quad \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0,$$

$$(3.1.2) \quad -\nabla p + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} + \rho \mathbf{f} = \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right),$$

where all symbols are as defined in sub-section 2.3.2. It can be seen that this system of field equations are non-linear. To be exact, the non-linearity is contributed by the convection operator $(\mathbf{u} \cdot \nabla)$. In addition, this set of equations also contains unsteady terms, namely $\frac{\partial \rho}{\partial t}$ and $\frac{d\mathbf{u}}{dt}$. The nature of such non-linearity and unsteadiness has always posed problems when exact solutions are required. To obviate this difficulty, it is customary to make certain assumptions that aim to simplify the set of governing equations involved in the problem.

One such assumption is that the fluid is an incompressible homogenous fluid; the fluid density ρ is constant. Hence, the first and second terms in (3.1.1) are reduced to zero. We then have

$$(3.1.3) \quad \nabla \cdot \mathbf{u} = 0.$$

It should be mentioned here that the assumption of incompressibility is made for all analysis in this project for simplicity. We will also neglect body forces, again for simplicity, such that the Navier-Stokes equation becomes

$$(3.1.4) \quad -\nabla p + \mu \nabla^2 \mathbf{u} = \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right).$$

3.1.1 The Stokes equation

Equation (3.1.4) can be simplified further if we introduce the *Stokes approximation*. Referring to a frame of reference that is either fixed or translated in space, consider

a flow with a characteristic length L related to the size of the boundaries (such as a body dimension), and a characteristic velocity U determined by the particular mechanism driving the flow (such as the free-stream velocity). Let

$$(3.1.5) \quad \mathbf{u} = U\mathbf{u}', \quad p = \frac{\rho\nu U}{L} p', \quad x_i = Lx_i', \quad t = \frac{L^2}{\nu} t'.$$

The primed quantities are the dimensionless variables where the kinematic viscosity

$\nu = \frac{\mu}{\rho}$ has been used to nondimensionalize the pressure and the time. The time $\frac{L^2}{\nu}$

corresponds to the time required for viscous diffusion to transverse the distance L .

Substituting these dimensionless variables into the Navier-Stokes equation (or (3.1.4)) yields the following dimensionless equation:

$$-\frac{\nu U}{L^2} \nabla' p' + \frac{\nu U}{L^2} \nabla'^2 \mathbf{u}' = \frac{\nu U}{L^2} \frac{\partial \mathbf{u}'}{\partial t'} + \frac{U^2}{L} \mathbf{u}' \cdot \nabla' \mathbf{u}',$$

where the del and Laplacian operators are expressed in terms of the dimensionless

space variables x_i' . Multiplying the equation above by $\frac{L^2}{\nu U}$ and introducing the

Reynolds number $\mathbf{Re} = \frac{UL}{\nu}$, we have

$$-\nabla' p' + \nabla'^2 \mathbf{u}' = \frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{Re}(\mathbf{u}' \cdot \nabla') \mathbf{u}'.$$

For a low Reynolds number flow, $\mathbf{Re} \ll 1$, the inertial convective term, $\mathbf{Re}(\mathbf{u}' \cdot \nabla') \mathbf{u}'$, is small compared with the rest of the terms and thus can be neglected.

Reverting back to dimensional variables using (3.1.5), the *unsteady Stokes equation* or *linearized Navier-Stokes equation* results:

$$(3.1.6) \quad -\nabla p + \mu \nabla^2 \mathbf{u} = \frac{\partial \mathbf{u}}{\partial t}.$$

Further, if the flow is steady or quasi-steady, then equation (3.1.6) reduces to the *Stokes equation*:

$$(3.1.7a) \quad \nabla p = \mu \nabla^2 \mathbf{u},$$

which states that pressure and viscous forces balance at any instant in time. Using the vector identity

$$\nabla^2 \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \nabla \times \nabla \times \mathbf{a},$$

and in view of (3.1.3), equation (3.1.7a) can be rewritten as

$$(3.1.7b) \quad \nabla p = -\mu \nabla \times \nabla \times \mathbf{u}.$$

Some remarks about quasi-steady flows are deemed appropriate here. Equation (3.1.7) merely states that the forces exerted on fluid parcels are in a state of dynamic equilibrium and the flow may not, in actual fact, be steady. As a consequence, the instantaneous structure of the flow depends solely upon the boundary configuration and boundary conditions, and is independent of the history of the motion, hence the name "quasi-steady". Real life physical examples of steady and quasi-steady Stokes flows are given in the next sub-section.

3.1.2 Real fluid Stokes flows

A flow satisfying (3.1.7) is known as a Stokes or creeping flow. Although the Stokes equation is a consequence of small $\mathbf{Re}$, a Stokes flow need not necessarily be slow. Recall that since $\mathbf{Re} = \frac{\rho U L}{\mu}$, $\mathbf{Re}$ may be small because U , L , and/or ρ is small and/or μ is large. Examples of real fluid flows with small $\mathbf{Re}$ are as follows (O'Neill & Chorlton, 1989; Pozrikidis, 1997):

- (i) Sedimentation of solid particles in a liquid – here U is small and/or μ is large.
- (ii) Motion of dust in air, settling of water droplets in a cloud, motion of aerosols or suspensions of particles – here L is small and/or U is small.
- (iii) Flow of underground water, oil or natural gas through porous rock formations,
- (iv) Blood flow in capillaries.
- (v) Flow due to motion of ciliated micro-organisms.

3.2 The Dirac delta function

The point force or point couple can be represented mathematically by the *Dirac delta function* or simply, the *delta function* $\delta(x)$. For a delta function positioned at $x = y$, it is defined by the following properties:

$$\delta(x - y) = \begin{cases} 0, & \text{for } x \neq y, \\ \infty, & \text{for } x = y, \end{cases}$$

$$\int_a^b \delta(x - y) dx = \begin{cases} 1, & \text{for } y \in (a, b), \\ 0, & \text{for } y \notin [a, b]. \end{cases}$$

The delta function can be used as a limit to any sequence of functions as long as the defining properties are satisfied. For example, $\delta(x) = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\pi(x^2 + \epsilon^2)}$ and

$$\delta(x) = \lim_{N \rightarrow 0} \frac{\sin(Nx)}{\pi x}.$$

Mathematically, the delta function is not a function as it has no

finite value at $x = y$. Rather, it is called a distribution or a generalized function, which has been developed under the Theory of Distributions. According to this

theory, the delta function can be used only inside integrals. It also has the following property:

$$\int_{-\infty}^{\infty} f(y)\delta(x-y)dy = f(x).$$

In n -space, for a delta function positioned at $\mathbf{x} = \mathbf{y}$, the above property can be generalized as

$$(3.2.1) \quad \int_{\mathbb{R}^n} f(\mathbf{y})\delta(\mathbf{x} - \mathbf{y})d\mathbf{y} = f(\mathbf{x}),$$

which further suggests that

$$\delta(\mathbf{x} - \mathbf{y}) = \delta(x_1 - y_1) \times \delta(x_2 - y_2) \times \cdots \times \delta(x_n - y_n).$$

For $f(\mathbf{y}) = 1$, (3.2.1) becomes

$$(3.2.2) \quad \int_{\mathbb{R}^n} \delta(\mathbf{x} - \mathbf{y})d\mathbf{y} = 1.$$

For more mathematical exposition on the delta function, one may refer to Lighthill (1959).

3.3 The fundamental solution

It is stated previously that a point force or a point couple in a flow is represented by $\delta(\mathbf{x} - \mathbf{x}_0)$. The resulting solutions are known as *fundamental solutions* or *Green's functions*. Consider the following generic equation

$$(3.3.1) \quad L(u) = f,$$

where L is a differential operator acting on an unknown function u , with f as the forcing function. If f is a delta function, then the solution to (3.3.1) is called the fundamental solution or the Green's function, $\mathbf{G}(\mathbf{x}, \mathbf{x}_0)$. Formally,

$$L[G(\mathbf{x}, \mathbf{x}_0)] = \delta(\bar{\mathbf{x}}),$$

where $\bar{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$.

To briefly illustrate the application of the fundamental solution, consider the case where L is the Laplacian operator ∇^2 , for instance. Then (3.3.1) becomes the well-known Poisson's equation:

$$\nabla^2 u = f.$$

It can be shown that the solution to the Poisson's equation in infinite space is given by (Haberman, 1987)

$$(3.3.2) \quad u(\mathbf{x}) = \int_{\mathfrak{R}^3} f(\mathbf{x}_0) G(\mathbf{x}, \mathbf{x}_0) d\mathbf{x}_0,$$

where $\nabla^2 G(\mathbf{x}, \mathbf{x}_0) = \delta(\bar{\mathbf{x}})$. In other words, if the Green's function G is known, the solution u can be evaluated by the integral given by (3.3.2). In some situations, this method of solution might be easier than solving the Poisson's equation directly. It is to be noted that the fundamental solution takes different forms in different Euclidean space. In this project, we consider only fundamental solutions in $\mathfrak{R}^3$

3.4 Some Fourier transforms and their inverses

It will be convenient to apply the method of Fourier transform to seek fundamental solutions due to a point force and a point couple positioned at the pole $\mathbf{x} = \mathbf{x}_0$ in Euclidean space $\mathfrak{R}^N$, for $N = 2, 3$. Without any loss of generality, we take the pole to be the origin.

Suppose that f is an absolutely integrable function in $\mathfrak{R}^N$. Then, the N -dimensional complex Fourier transform of the function f , $\mathcal{F}\{f(\mathbf{x})\}$ and its inverse, $\mathcal{F}^{-1}\{\hat{f}(\boldsymbol{\xi})\}$, are defined by

$$(3.4.1) \quad \mathcal{F}\{f(\mathbf{x})\} = \hat{f}(\boldsymbol{\xi}) = (2\pi)^{-\frac{N}{2}} \int_{\mathfrak{R}^N} f(\mathbf{x}) e^{-i\boldsymbol{\xi} \cdot \mathbf{x}} d\mathbf{x},$$

$$(3.4.2) \quad \mathcal{F}^{-1}\{\hat{f}(\boldsymbol{\xi})\} = f(\mathbf{x}) = (2\pi)^{-\frac{N}{2}} \int_{\mathfrak{R}^N} \hat{f}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi} \cdot \mathbf{x}} d\boldsymbol{\xi},$$

where $\boldsymbol{\xi}$ is the transformed variable of $\mathbf{x}$, $\boldsymbol{\xi} \cdot \mathbf{x} = \xi_1 x_1 + \cdots + \xi_N x_N$, and i is complex number such that $i = \sqrt{-1}$. For $f(\mathbf{x})$ and its derivatives to be Fourier transformable, it is required that they decay at infinity, and are all absolutely integrable. Based on (3.4.1), the Fourier transform of grad f and of Laplacian of f are

$$(3.4.3) \quad \mathcal{F}\{\nabla f\} = i\boldsymbol{\xi} \hat{f}, \quad \mathcal{F}\{\nabla^2 f\} = -\boldsymbol{\xi}^2 \hat{f},$$

where $\boldsymbol{\xi}^2 = \xi_1^2 + \cdots + \xi_N^2$. We will derive (3.4.3) for $\mathfrak{R}^3$ here:

By definition, we have

$$\mathcal{F}\{\nabla f\} = \hat{\mathbf{e}}_j (2\pi)^{-\frac{3}{2}} \int_{\mathfrak{R}^3} \frac{\partial f}{\partial x_j} e^{-i\boldsymbol{\xi} \cdot \mathbf{x}} d\mathbf{x},$$

where the Einstein's summation convention is adopted. Using integration by parts, the component of $\hat{\mathbf{e}}_1$ can be evaluated as follows:

$$\begin{aligned} (2\pi)^{-\frac{3}{2}} \int_{\mathfrak{R}^3} \frac{\partial f}{\partial x_1} e^{-i\boldsymbol{\xi} \cdot \mathbf{x}} d\mathbf{x} &= (2\pi)^{-\frac{3}{2}} \int_{-\infty}^{\infty} e^{-i\xi_1 x_1} df \int_{-\infty}^{\infty} e^{-i\xi_2 x_2} dx_2 \int_{-\infty}^{\infty} e^{-i\xi_3 x_3} dx_3 \\ &= (2\pi)^{-\frac{3}{2}} \left[\left(f e^{-i\xi_1 x_1} \right) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -i\xi_1 f e^{-i\xi_1 x_1} dx_1 \right] \int_{-\infty}^{\infty} e^{-i\xi_2 x_2} dx_2 \int_{-\infty}^{\infty} e^{-i\xi_3 x_3} dx_3 \\ &= i\xi_1 (2\pi)^{-\frac{3}{2}} \int_{\mathfrak{R}^3} f e^{-i\boldsymbol{\xi} \cdot \mathbf{x}} d\mathbf{x} \\ &= i\xi_1 \hat{f}. \end{aligned}$$

By replacing x_1 and ξ_1 with other variables in the result above, we have

$$\mathcal{F}\{\nabla f\} = \hat{\mathbf{e}}_1 i \xi_1 \hat{f} + \hat{\mathbf{e}}_2 i \xi_2 \hat{f} + \hat{\mathbf{e}}_3 i \xi_3 \hat{f} = i \boldsymbol{\xi} \hat{f}.$$

Next, we evaluate the Fourier transform of the Laplacian of f . By definition,

$$\mathcal{F}\{\nabla^2 f\} = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \frac{\partial^2 f}{\partial x_j \partial x_j} e^{-i\boldsymbol{\xi} \cdot \mathbf{x}} d\mathbf{x}.$$

For $j = 1$, we have

$$\begin{aligned} (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \frac{\partial^2 f}{\partial x_1^2} e^{-i\boldsymbol{\xi} \cdot \mathbf{x}} d\mathbf{x} &= (2\pi)^{-\frac{3}{2}} \int_{-\infty}^{\infty} e^{-i\xi_1 x_1} d\left(\frac{\partial f}{\partial x_1}\right) \int_{-\infty}^{\infty} e^{-i\xi_2 x_2} dx_2 \int_{-\infty}^{\infty} e^{-i\xi_3 x_3} dx_3 \\ &= (2\pi)^{-\frac{3}{2}} \left[\frac{\partial f}{\partial x_1} e^{-i\xi_1 x_1} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -i\xi_1 e^{-i\xi_1 x_1} df \right] \int_{-\infty}^{\infty} e^{-i\xi_2 x_2} dx_2 \int_{-\infty}^{\infty} e^{-i\xi_3 x_3} dx_3 \\ &= (2\pi)^{-\frac{3}{2}} i \xi_1 \left[f e^{-i\xi_1 x_1} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -i\xi_1 f e^{-i\xi_1 x_1} dx_1 \right] \int_{-\infty}^{\infty} e^{-i\xi_2 x_2} dx_2 \int_{-\infty}^{\infty} e^{-i\xi_3 x_3} dx_3 \\ &= -(2\pi)^{-\frac{3}{2}} \xi_1^2 \int_{\mathbb{R}^3} f e^{-i\boldsymbol{\xi} \cdot \mathbf{x}} d\mathbf{x} \\ &= -\xi_1^2 \hat{f}. \end{aligned}$$

Similarly, by replacing x_1 and ξ_1 with other variables in the result above, we have

$$\mathcal{F}\{\nabla^2 f\} = -\xi_1^2 \hat{f} - \xi_2^2 \hat{f} - \xi_3^2 \hat{f} = -\boldsymbol{\xi}^2 \hat{f}.$$

Furthermore, based on (3.4.1) and (3.4.2), the following properties are assumed without proof (Kohr & Pop, 2004; Ladyzhenskaia, 1969):

$$(3.4.4) \quad \mathcal{F}\{\delta(\mathbf{x})\} = (2\pi)^{-\frac{N}{2}},$$

$$(3.4.5) \quad \left\{ \mathcal{F}^{-1} \left[\frac{(2\pi)^{-\frac{3}{2}}}{\xi^2} \right] \right\}(\mathbf{x}) = \frac{1}{4\pi r} \text{ for } N = 3, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

$$(3.4.6) \quad \left\{ \mathcal{F}^{-1} \left[\frac{(2\pi)^{-\frac{3}{2}}}{\xi^4} \right] \right\}(\mathbf{x}) = -\frac{r}{8\pi} \text{ for } N = 3, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$(3.4.7) \quad \left\{ \mathcal{F}^{-1} \left[\frac{(2\pi)^{-\frac{N}{2}}}{\xi^2 + \chi^2} \right] \right\}(\mathbf{x}) = \begin{cases} \frac{1}{2\pi} K_0(\chi r) & \text{for } N = 2, r = \sqrt{x_1^2 + x_2^2}, \\ \frac{e^{-\chi r}}{4\pi r} & \text{for } N = 3, r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \end{cases}$$

$$(3.4.8) \quad \left\{ \mathcal{F}^{-1} \left[\frac{(2\pi)^{-\frac{N}{2}}}{\xi^2 (\xi^2 + \chi^2)} \right] \right\}(\mathbf{x}) = \frac{\mathbf{Re}(\chi) > 0}{4\pi \chi^2 r}, \text{ for } N = 3, r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Note that χ is the particular square root of $\chi^2 \in C \setminus \{z \in C : \mathbf{Re}(z) \leq 0, \mathbf{Im}(z) = 0\}$, which has a positive real part, i.e. $\mathbf{Re}(\chi) > 0$, and K_0 denotes the modified Bessel function (Abramowitz, Stegun, & United States. National Bureau of Standards., 1972).

3.5 Classical Stokes flow due to a point force in $\mathbb{R}^3$

In this section, we shall consider the case of a point force applied at a certain point within a particular domain of a classical fluid flow to obtain the resulting pressure and velocity fields. Physically, it may be identified with the flow generated by the slow motion of a small particle in a quiescent fluid without boundaries.

The Stokes approximation is applied here to simplify the governing equations. Hence, the resulting velocity and pressure fields are found by solving the steady Stokes equations obtained in Section 3.1:

$$(3.5.1) \quad \nabla \cdot \mathbf{u} = 0,$$

$$(3.5.2) \quad -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{F} \delta(\mathbf{x}) = \mathbf{0},$$

where the constant point force $\rho \mathbf{F} \delta(\mathbf{x})$ is assumed to act at the origin $\mathbf{x} = \mathbf{0}$, without any loss of generality. For such an unbounded domain of flow, it is required that

$$p \rightarrow p_\infty, |\mathbf{u}| \rightarrow 0 \text{ as } |\mathbf{x}| \rightarrow \infty,$$

where p_∞ is the uniform pressure at far field.

To proceed, we take the divergence of (3.5.2), which simplifies under (3.5.1) as

$$(3.5.3) \quad \nabla^2 p = \nabla \cdot [\rho \mathbf{F} \delta(\mathbf{x})] = \rho \mathbf{F} \cdot \nabla \delta(\mathbf{x}).$$

To solve for p , it is convenient to take Fourier transform of (3.5.3). Using the properties in Section 3.4, the Fourier transform of p is found to be

$$(3.5.4) \quad \hat{p} = -\rho \mathbf{F} \cdot (i\boldsymbol{\xi}) \left[\frac{(2\pi)^{-\frac{3}{2}}}{\xi^2} \right].$$

Under the inverse Fourier transform, the pressure field p is then found to be

$$(3.5.5) \quad p = -\frac{\rho}{4\pi} \mathbf{F} \cdot \nabla \left(\frac{1}{r} \right) = \frac{\rho}{4\pi} \frac{\mathbf{F} \cdot \mathbf{x}}{r^3}.$$

The Fourier transform of (3.5.2) gives

$$-(i\boldsymbol{\xi})\hat{\mathbf{p}} - \mu \xi^2 \hat{\mathbf{u}} + \rho \mathbf{F} (2\pi)^{-\frac{3}{2}} = \mathbf{0}.$$

To solve for $\mathbf{u}$, we eliminate $\hat{\mathbf{p}}$ by substituting (3.5.4) into the above expression.

This gives the Fourier transform of $\mathbf{u}$ as

$$(3.5.6) \quad \hat{\mathbf{u}} = \frac{\rho}{\mu} \left\{ \mathbf{F} \left[\frac{(2\pi)^{-\frac{3}{2}}}{\xi^2} \right] + \mathbf{F} \cdot (i\boldsymbol{\xi}) (i\boldsymbol{\xi}) \left[\frac{(2\pi)^{-\frac{3}{2}}}{\xi^4} \right] \right\}.$$

Referring to (3.4.5) and (3.4.6), the velocity field $\mathbf{u}$ is then given by

$$(3.5.7) \quad \mathbf{u} = \frac{\rho}{\mu} \left[\frac{\mathbf{F}}{4\pi r} + (\mathbf{F} \cdot \nabla) \nabla \left(-\frac{r}{8\pi} \right) \right].$$

Using vector identities (Appendix C.7), the expression $(\mathbf{F} \cdot \nabla) \nabla \left(-\frac{r}{8\pi} \right)$ is evaluated as follows:

$$\begin{aligned}
 (\mathbf{F} \cdot \nabla) \nabla \left(-\frac{r}{8\pi} \right) &= (\mathbf{F} \cdot \nabla) \left(-\frac{\mathbf{x}}{8\pi r} \right) \\
 &= -\frac{\mathbf{F} \cdot \nabla \mathbf{x}}{8\pi r} - \mathbf{x} \mathbf{F} \cdot \nabla \left(\frac{1}{8\pi r} \right) \\
 &= -\frac{\mathbf{F} \cdot (\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3)}{8\pi r} + \frac{\mathbf{x}}{8\pi} \left(\frac{\mathbf{F} \cdot \mathbf{x}}{r^3} \right) \\
 &= -\frac{\mathbf{F}}{8\pi r} + \left(\frac{\mathbf{F} \cdot \mathbf{x}}{8\pi r^3} \right) \mathbf{x}.
 \end{aligned}$$

Finally, (3.5.7) gives the velocity field $\mathbf{u}$ due to point force to be

$$(3.5.8) \quad \mathbf{u} = \frac{\rho}{8\pi\mu} \left[\frac{\mathbf{F}}{r} + \frac{\mathbf{F} \cdot \mathbf{x}}{r^3} \mathbf{x} \right].$$

In most fluid dynamics literature, it is customary to define $\mathbf{u}$ such that

$$(3.5.9) \quad u_i = \frac{\rho}{8\pi\mu} S_{ij}(\mathbf{x}) F_j,$$

where S_{ij} is a second-order tensor. In fact, the pressure field p is also customary defined in a similar manner for the convenience of solving by, for instance, the method of Fourier transform. Based on (3.5.9), the tensor S_{ij} is given by

$$(3.5.10) \quad S_{ij}(\mathbf{x}) = \frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3},$$

which is well-known as the Oseen-Burgers tensor or the classical Stokeslet. However, we see no harm in referring (3.5.9) as the Stokeslet as S_{ij} is a result of it.

If the source point is located at $\mathbf{x} = \mathbf{x}_0$, then r will be defined by $r = |\bar{\mathbf{x}}|$ and $\mathbf{x}$ is replaced by $\bar{\mathbf{x}}$ in all the solutions.

3.6 Stokes flow of a micropolar fluid due to a point force in $\mathcal{R}^3$

The Stokes approximation is previously used in Section 3.1 to linearize the governing equations of classical fluids given by (3.1.1) and (3.1.2), for which solutions would otherwise be impossible to obtain. In this section, the Stokes approximation is similarly applied so that the flow quantities due to a point force in an unbounded micropolar fluid field can be yielded.

This problem was first treated by Ramkisson & Majumdar (1976), who first linearized the governing equations of micropolar fluids and then applied the method of Fourier transform to obtain the solutions for the flow field. Eringen (1999) addressed the same problem by obtaining the Laplace-transformed generalized solutions of the governing equations by the method of Laplace transforms and Fourier transforms, in which he acknowledged that inversion of the Laplace transforms is difficult. Nevertheless, he managed to obtain the same solution as Ramkisson & Majumdar for the problem of a point force.

We shall begin the solution of this problem by neglecting body forces and body torques for simplicity. Further, considering steady and quasi-steady flows, the governing equations then reduce to

$$(3.6.1) \quad \nabla \cdot \mathbf{u} = 0,$$

$$(3.6.2) \quad -\nabla p - (\mu + \mu_r)\nabla \times (\nabla \times \mathbf{u}) + 2\mu_r \nabla \times \mathbf{v} + \rho \mathbf{F} \delta(\mathbf{x}) = \mathbf{0},$$

$$(3.6.3) \quad -(c_a + c_d)\nabla \times (\nabla \times \mathbf{v}) + 2\mu_r(\nabla \times \mathbf{u} - 2\mathbf{v}) = \mathbf{0},$$

where the divergence of $\mathbf{v}$ is first assumed to be identically zero for simplicity.

Similar to the classical case, the pressure, velocity and microrotation velocity fields are required to decay at far field. Formally,

$$p \rightarrow p_\infty, |\mathbf{u}| \rightarrow 0, |\mathbf{v}| \rightarrow 0 \text{ as } |\mathbf{x}| \rightarrow \infty.$$

To begin, the divergence of (3.6.2) is taken, yielding

$$\nabla^2 p = \rho \nabla \cdot [\mathbf{F} \delta(\mathbf{x})],$$

which is exactly (3.5.3). The solution of p is then given by (3.5.5) as

$$(3.6.4) \quad p = -\frac{\rho}{4\pi} \mathbf{F} \cdot \nabla \left(\frac{1}{r} \right) = \frac{\rho \mathbf{F} \cdot \mathbf{x}}{4\pi r^3}.$$

Next, we proceed to solve for $\mathbf{v}$ by taking the curl of (3.6.3), which is

$$(3.6.5) \quad \nabla \times \nabla \times \mathbf{u} = 2\nabla \times \mathbf{v} + \frac{c_a + c_d}{2\mu_r} \nabla \times \nabla \times \nabla \times \mathbf{v}.$$

Substituting (3.6.5) into (3.6.2), and using vector identities (Appendix C.7), the following equation is obtained:

$$(3.6.6) \quad (\nabla^2 - \lambda^2) \mathbf{a} = a_4 \left[\frac{\nabla p}{\rho} - \mathbf{F} \delta(\mathbf{x}) \right],$$

where $\mathbf{a} = \nabla \times \mathbf{v}$, $\lambda^2 = \frac{4\mu_r \mu}{(c_a + c_d)(\mu + \mu_r)}$ and $a_4 = \frac{2\rho\mu_r}{(c_a + c_d)(\mu + \mu_r)}$. As p has been solved, the above equation becomes a partial differential equation in terms of only one unknown: $\mathbf{a}$.

Under Fourier transform, we have

$$\hat{\mathbf{a}} = a_4 \left\{ (i\boldsymbol{\xi}) [\mathbf{F} \cdot (i\boldsymbol{\xi})] \frac{(2\pi)^{-\frac{3}{2}}}{\xi^2 (\xi^2 + \lambda^2)} + \mathbf{F} \frac{(2\pi)^{-\frac{3}{2}}}{\xi^2 + \lambda^2} \right\}.$$

Recall that the vector $\mathbf{a}$ is defined to be the curl of $\mathbf{v}$. From Appendix C.7, we have

$$\nabla \times \mathbf{a} = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v} = -\nabla^2 \mathbf{v}$$

since the divergence of $\mathbf{v}$ has been assumed to be zero. Therefore, under Fourier transform,

$$\hat{\mathbf{v}} = \frac{1}{\xi^2} [(i\boldsymbol{\xi}) \times \hat{\mathbf{a}}],$$

which in view of (3.6.7) becomes

$$(3.6.8) \quad \hat{\mathbf{v}} = a_4 (i\boldsymbol{\xi}) \times \mathbf{F} \frac{(2\pi)^{-\frac{3}{2}}}{\xi^2 (\xi^2 + \lambda^2)}.$$

Using the properties in Section 3.5, the inverse Fourier transform of the (3.6.8) gives

$$(3.6.9) \quad \mathbf{v} = \frac{\rho}{8\pi\mu} \nabla \times \left(\frac{1 - e^{-\lambda r}}{r} \mathbf{F} \right).$$

It is noted that the divergence of $\mathbf{v}$ is identically zero. Hence, the assumption of the divergence of $\mathbf{v}$ being zero is reasonable as there is no contradiction. Equation (3.6.7) represents the solution for microrotation velocity in the presence of a point force, which has not been named by either Ramkissoon & Majumdar (1976) or Eringen (1999). We shall call it the “micropolar micro-Stokeslet”.

By virtue of (3.6.1) and using vector identities (Appendix C.7), (3.6.5) becomes

$$-\nabla^2 \mathbf{u} = 2\nabla \times \mathbf{v} - \frac{c_a + c_d}{2\mu_r} \nabla^2 (\nabla \times \mathbf{v}).$$

Under Fourier transform,

$$\hat{\mathbf{u}} = 2(i\xi) \times \frac{\hat{\mathbf{v}}}{\xi^2} + \frac{c_a + c_d}{2\mu_r} (i\xi) \times \hat{\mathbf{v}},$$

Substituting (3.6.8) into the above equation gives

$$(3.6.10) \quad \hat{\mathbf{u}} = \frac{2a_d}{\lambda^2} (i\xi) \times (i\xi) \times \left\{ \mathbf{F} \left[\frac{(2\pi)^{-\frac{3}{2}}}{\xi^4} - \frac{(2\pi)^{-\frac{3}{2}}}{\xi^2(\xi^2 + \lambda^2)} \right] \right\} + \frac{c_a + c_d}{2\mu_r} (i\xi) \times \hat{\mathbf{v}},$$

which upon applying inverse Fourier transform, and simplification, gives $\mathbf{u}$ to be

$$(3.6.11) \quad \mathbf{u} = -\frac{\rho(c_a + c_d)}{16\pi\mu^2} \nabla \times \nabla \times \left(\frac{1 - e^{-\lambda r}}{r} \mathbf{F} \right) + \frac{\rho}{8\pi\mu} \left[\frac{\mathbf{F}}{r} + \frac{(\mathbf{F} \cdot \mathbf{x})\mathbf{x}}{r^3} \right].$$

It is clear that the solution of $\mathbf{u}$ for a micropolar fluid is much more complicated than that of the classical. Hence, it is not feasible to express $\mathbf{u}$ for a micropolar fluid in an expression similar to (3.5.10), and there is then no harm in calling (3.6.11) the “micropolar Stokeslet”. Note that if the source point is located at $\mathbf{x} = \mathbf{x}_0$, then r will be defined by $r = |\bar{\mathbf{x}}|$ and $\mathbf{x}$ is replaced by $\bar{\mathbf{x}}$ in all the solutions.

It is noted that when $\mu_r \rightarrow 0$, equation (3.6.2) reduces to the classical Navier-Stokes equation given previously by (3.5.2). Not surprisingly, by inspection, (3.6.11) tends to

$$\mathbf{u} = \frac{\rho}{8\pi\mu} \left[\frac{\mathbf{F}}{r} + \frac{(\mathbf{F} \cdot \mathbf{x})\mathbf{x}}{r^3} \right],$$

the classical Stokeslet given previously by (3.5.8).

3.6.1 Drag on axisymmetric bodies based on Stokeslets

Having obtained expressions for the Stokeslets, we will examine an application of it, that is, to derive a general expression for the calculation of drag force on axisymmetric bodies. After that, the expression will be used to derive the drag force

experienced by a translating sphere in a quiescent classical fluid and a quiescent micropolar fluid.

For axisymmetric flows, it is convenient to introduce the spherical coordinates (r, θ, ϕ) (Figure 3.1).

Figure 3.1 Frame of spherical coordinates (Kohr & Pop, 2004)

In an axisymmetric flow, all flow quantities are independent of the azimuthal angle ϕ . Therefore, the pressure, velocity and microrotation velocity fields have the following components:

$$\begin{aligned} p &= p(r, \theta), \\ \mathbf{u} &= \mathbf{u}(r, \theta) = (u_r, u_\theta, 0), \\ \mathbf{v} &= v(r, \theta) \hat{\mathbf{e}}_\phi = (0, 0, v). \end{aligned}$$

In addition, the radial and meridional components of the velocity field can be expressed in terms of a scalar function $\Psi(r, \theta)$, better known as the stream function, as follows:

$$(3.6.12) \quad u_r = \frac{-1}{r^2 \sin \theta} \frac{\partial \Psi}{\partial \theta}, \quad u_\theta = \frac{1}{r \sin \theta} \frac{\partial \Psi}{\partial r}.$$

Based on the definition of $\Psi(r, \theta)$ in (3.6.12), and after performing integration, the stream function of a micropolar Stokeslet Ψ_{PS}^* is

$$(3.6.13) \quad \Psi_{PS}^* = \frac{F}{8\pi\mu} \left[\frac{c_a + c_d}{2\mu} \frac{1}{r} - r - \frac{c_a + c_d}{2\mu} \left(\lambda + \frac{1}{r} \right) e^{-\lambda r} \right] \sin^2 \theta.$$

Let Ψ be the stream function of the axisymmetric flow generated by an axisymmetric body. At distances far away from the body in an unbounded medium, the stream function Ψ must become identical to that of a Stokeslet (Ramkissoon & Majumdar, 1976). This is because at far field, the dimensions of the body become insignificant and the effect of the body is as though it were a point force (O'Neill & Chorlton, 1989). Therefore,

$$\lim_{r \rightarrow \infty} \Psi_{PS}^* = \lim_{r \rightarrow \infty} \Psi.$$

This implies that (Ramkissoon & Majumdar, 1976)

$$(3.6.14) \quad F = -8\pi\mu \lim_{r \rightarrow \infty} \frac{\Psi}{r \sin^2 \theta}.$$

To illustrate the effect of (3.6.14), consider the problem of the translation of a sphere, which is axisymmetric, with a constant velocity $U_\infty \hat{e}_1$ through a quiescent and incompressible micropolar fluid of infinite expanse past a stationary sphere of radius a . Assume that the resulting flow caused by the presence of this sphere is a Stokes flow. Then, the associated stream function Ψ_p^* can be inferred from Ramkissoon & Majumdar (1976) as

$$\Psi_p^* = \frac{1}{2} U_\infty \left[\frac{A_1}{r} + B_1 r + B_2 \left(\lambda + \frac{1}{r} \right) e^{-\lambda r} \right] \sin^2 \theta$$

$$\text{where } A_1 = \frac{3}{4}a^3\left(\lambda_2 + \frac{2}{3}\right) + \frac{3a^2\lambda_2}{2\lambda^2}\left(\lambda + \frac{1}{a}\right), \quad B_1 = -\frac{3}{4}a(2 + \lambda_2), \quad B_2 = -\frac{3a\lambda_2 e^{\lambda a}}{2\lambda^2},$$

$$\lambda_2 = \frac{2\mu_r}{\mu + a\lambda(\mu + \mu_r)} \quad \text{and} \quad \lambda^2 = \frac{4\mu_r\mu}{(c_a + c_d)(\mu + \mu_r)}.$$

Hence, based on (3.6.14),

$$F = 4\pi\mu U_\infty B_1 = 6\pi a U_\infty \left[\frac{\mu(\mu + \mu_r)(1 + a\lambda)}{\mu + a\lambda(\mu + \mu_r)} \right]$$

Using Newton's third law, the drag D acting on the sphere is given as

$$(3.6.15) \quad D = -F = -6\pi a U_\infty \left[\frac{\mu(\mu + \mu_r)(1 + a\lambda)}{\mu + a\lambda(\mu + \mu_r)} \right].$$

The negative sign in the drag force means that the force exerted on the sphere opposes its motion, as expected.

Taking the limit as $\mu_r \rightarrow 0$, (3.6.15) approaches

$$(3.6.16) \quad D = -6\pi a \mu U_\infty,$$

which is the well-known Stokes *law* for the drag force acting on a translating sphere in a classical fluid (Figure 3.2).

Figure 3.2 Streamlines for a translating sphere (Happel & Brenner, 1973)

3.7 Fluid flows due to a point couple in $\mathbb{R}^3$

In this section, we consider the problem of a point couple acting in a flow field. For the classical case, we follow the treatment given by Chwang & Wu (1974). As for the micropolar fluid case, the problem has been treated by Eringen (1999). As mentioned at the beginning of Section 3.6, Eringen (1999) obtained the Laplace-transformed generalized solutions of the governing equations by the method of Laplace transforms and Fourier transforms, in which he also managed to obtain the inverse Laplace transform solution for the problem of a point couple.

We will present the problem and apply the method of Fourier transform to derive the classical couplet. The unbounded classical Stokes flow is being considered first, followed by the unbounded micropolar fluid Stokes flow. Without any loss of generality, the source point is assumed to be at the origin $\mathbf{x} = \mathbf{0}$.

3.7.1 Classical Stokes flow due to a point couple

Consider steady or quasi-steady Stokes flows, the governing equations are given by

$$(3.7.1) \quad \nabla \cdot \mathbf{u} = 0 ,$$

$$(3.7.2) \quad -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{F} = \mathbf{0} ,$$

where $\mathbf{F}$ is the external forcing function to the unbounded flow and is expressed as

$$\mathbf{F} = \nabla \times \left[\frac{1}{2} \mathbf{q} \delta(\mathbf{x}) \right] .$$

Evidently, the forcing function $\mathbf{F}$ is solenoidal. The vector $\mathbf{q}$ is a constant and the product $\mathbf{q} \delta(\mathbf{x})$ is a vector potential. For an unbounded flow, it is required that

$$p \rightarrow p_\infty , |\mathbf{u}| \rightarrow 0 \text{ as } |\mathbf{x}| \rightarrow \infty .$$

The divergence of (3.7.2) gives

$$\nabla^2 p = 0,$$

which states that p is harmonic in the entire space. By the well-known theorem for harmonic functions (Kellogg, 1929),

$$(3.7.3) \quad p = p_\infty = \text{constant}.$$

It is to be noted that (3.7.3) is no longer valid when the flow is bounded by additional material surfaces (Chwang & Wu, 1974).

With a constant pressure field, equation (3.7.2) now simplifies as

$$\nabla^2 \mathbf{u} = -\frac{\rho}{\mu} \nabla \times \left[\frac{1}{2} \mathbf{q} \delta(\mathbf{x}) \right].$$

Under Fourier transform,

$$\hat{\mathbf{u}} = \frac{\rho}{2\mu} (i\boldsymbol{\xi}) \times \left[\mathbf{q} \frac{(2\pi)^{-\frac{3}{2}}}{\xi^2} \right].$$

Using the properties in Section 3.4, the inverse Fourier transform of $\hat{\mathbf{u}}$ is given by

$$(3.7.4) \quad \mathbf{u} = \nabla \times \left(\frac{\rho}{\mu} \frac{\mathbf{q}}{8\pi r} \right) = \frac{\rho}{8\pi\mu} \frac{\mathbf{q} \times \mathbf{x}}{r^3}.$$

It is also customary in classical fluid dynamics to express $\mathbf{u}$ as

$$u_i = \frac{\rho}{8\pi\mu} C_{im} q_m,$$

where based on (3.7.4), the tensor C_{im} is given by

$$C_{im} = \epsilon_{iml} \frac{x_l}{r^3},$$

better known as the rotlet; it is also named couplet by Batchelor (1970). Again, we see no harm in referring (3.7.4) as the couplet as C_{im} is a result of it.

If the source point is located at $\mathbf{x} = \mathbf{x}_0$, then $\mathbf{r}$ will be defined by $\mathbf{r} = |\bar{\mathbf{x}}|$ and $\mathbf{x}$ is replaced by $\bar{\mathbf{x}}$ in all the solutions.

3.7.2 Stokes flow of a micropolar fluid due to a point couple

As with in Section 3.6, the Stokes approximation is applied here so that the solution due to a point couple in a micropolar fluid can be yielded. Similarly, by considering steady and quasi-steady flows, and neglecting body forces and body torques, the governing equations (2.3.10) then reduce to

$$(3.7.5) \quad \nabla \cdot \mathbf{u} = 0,$$

$$(3.7.6) \quad -\nabla p - (\mu + \mu_r)\nabla \times (\nabla \times \mathbf{u}) + 2\mu_r \nabla \times \mathbf{v} = \mathbf{0},$$

$$(3.7.7) \quad (c_0 + 2c_d)\nabla(\nabla \cdot \mathbf{v}) - (c_a + c_d)\nabla \times (\nabla \times \mathbf{v}) + 2\mu_r(\nabla \times \mathbf{u} - 2\mathbf{v}) + \rho \mathbf{Q} \delta(\mathbf{x}) = \mathbf{0},$$

where $\rho \mathbf{Q} \delta(\mathbf{x})$ is a point couple, with $\mathbf{Q}$ as a constant vector. The pressure, velocity and microrotation velocity fields are required to decay at far field. Formally,

$$p \rightarrow p_\infty, |\mathbf{u}| \rightarrow 0, |\mathbf{v}| \rightarrow 0 \text{ as } |\mathbf{x}| \rightarrow \infty.$$

The pressure field can be obtained by taking the divergence of (3.7.6), which gives

$$\nabla^2 p = 0.$$

The pressure field p turns out to be a harmonic function. Following a similar argument in sub-section 3.7.1, p is thus known to be

$$(3.7.8) \quad p = p_\infty = \text{constant}.$$

Next, we take the divergence of (3.7.7), which simplifies to the following equation, with the divergence of $\mathbf{v}$ as the only unknown:

$$(\nabla^2 - \lambda_0^2)f = -c_3 \nabla \cdot [\mathbf{Q}\delta(\mathbf{x})],$$

where $f = \nabla \cdot \mathbf{v}$, $\lambda_0^2 = \frac{4\mu_r}{c_0 + 2c_d}$ and $c_3 = \frac{\rho}{c_0 + 2c_d}$. Under Fourier transform, the

above equation becomes

$$(3.7.9) \quad \hat{f} = c_3(i\xi) \cdot \left[\mathbf{Q} \frac{(2\pi)^{-\frac{3}{2}}}{\xi^2 + \lambda_0^2} \right].$$

We proceed to determine the curl of $\mathbf{v}$. To do so, we take the curl of (3.7.7) is taken.

We have

$$(3.7.10) \quad (c_a + c_d)\nabla^2(\nabla \times \mathbf{v}) + 2\mu_r(\nabla \times \nabla \times \mathbf{u} - 2\nabla \times \mathbf{v}) + \rho \nabla \times [\mathbf{Q}\delta(\mathbf{x})] = \mathbf{0}.$$

We shall eliminate the curl of curl of $\mathbf{u}$ in the above equation so as to arrive at an equation in terms of only one unknown. From (3.7.6), the curl of curl of $\mathbf{u}$ can be expressed as

$$(3.7.11) \quad \nabla \times \nabla \times \mathbf{u} = \frac{2\mu_r}{\mu + \mu_r} \nabla \times \mathbf{v}$$

since the divergence of p is zero. With the substitution of (3.7.11) into (3.7.10), a partial differential equation of only one unknown is yielded:

$$(\nabla^2 - \lambda^2)\mathbf{a} = -\frac{\rho}{c_a + c_d} \nabla \times [\mathbf{Q}\delta(\mathbf{x})],$$

where $\mathbf{a} = \nabla \times \mathbf{v}$ and $\lambda^2 = \frac{4\mu\mu_r}{(c_a + c_d)(\mu + \mu_r)}$, as usual. Under Fourier transform, the

above equation becomes

$$(3.7.12) \quad \hat{\mathbf{a}} = \frac{\rho}{c_a + c_d} (i\xi) \times \left[\mathcal{Q} \frac{(2\pi)^{-\frac{3}{2}}}{\xi^2 + \lambda^2} \right].$$

To evaluate the expression for $\mathbf{v}$, recall that $\mathbf{a}$ is defined as the curl of $\mathbf{v}$. Hence,

$$\nabla^2 \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla \times \nabla \times \mathbf{v} = \nabla f - \nabla \times \mathbf{a},$$

which under Fourier transform becomes

$$-\xi^2 \hat{\mathbf{v}} = (i\xi) \hat{f} - (i\xi) \times \hat{\mathbf{a}}.$$

By using (3.7.9) and (3.7.12), $\hat{f}$ and $\hat{\mathbf{a}}$ are eliminated from the above equation to give $\hat{\mathbf{v}}$ as

$$\begin{aligned} \hat{\mathbf{v}} = & -c_3 (i\xi)(i\xi) \cdot \left[\mathcal{Q} \frac{(2\pi)^{-\frac{3}{2}}}{\xi^2 (\xi^2 + \lambda_0^2)} \right] + \frac{\rho}{c_a + c_d} (i\xi) \times (i\xi) \\ & \times \left[\mathcal{Q} \frac{(2\pi)^{-\frac{3}{2}}}{\xi^2 (\xi^2 + \lambda^2)} \right]. \end{aligned}$$

Using the properties in Section 3.4, the inverse Fourier transform of $\hat{\mathbf{v}}$ is given by

$$(3.7.13) \quad \mathbf{v} = \frac{\rho(\mu + \mu_r)}{16\pi\mu_r\mu} \nabla \times \nabla \times \left(\frac{1 - e^{-\lambda r}}{r} \mathcal{Q} \right) - \frac{\rho}{16\pi\mu_r} \nabla \left[\nabla \cdot \left(\frac{1 - e^{-\lambda_0 r}}{r} \mathcal{Q} \right) \right].$$

Since (3.7.13) corresponds to the solution due to a point couple in a micropolar fluid, we shall then call it the “micropolar micro-couplet”.

The velocity field $\mathbf{u}$ can now be easily worked out by considering (3.7.11). Under the continuity equation, given by (3.7.5), equation (3.7.11) can be rewritten as

$$-\nabla^2 \mathbf{u} = \frac{2\mu_r}{\mu + \mu_r} \mathbf{a}$$

Taking Fourier transform of the above equation, and substituting (3.7.12) into it gives

$$\hat{\mathbf{u}} = \frac{2\rho\mu_r}{(\mu + \mu_r)(c_a + c_d)} (i\xi) \times \left[\mathbf{Q} \frac{(2\pi)^{-\frac{3}{2}}}{\xi^2 (\xi^2 + \lambda^2)} \right]$$

Using the properties in Section 3.4, the inverse Fourier transform of $\hat{\mathbf{u}}$ is given by

$$(3.7.14) \quad \mathbf{u} = \frac{\rho}{8\pi\mu} \nabla \times \left(\frac{1 - e^{-\lambda r}}{r} \mathbf{Q} \right).$$

We shall call (3.7.14) the “micropolar couplet” since it is analogous to classical couplet.

When c_0 , c_a and c_d tend to zero, equations (3.7.6) reduces to the classical Navier-Stokes equation given previously by (3.5.2) under (3.7.7). Then, by inspection, (3.7.14) tends to

$$\mathbf{u} = \frac{\rho}{8\pi\mu} \nabla \times \left(\frac{\mathbf{Q}}{r} \right),$$

which is identical to the classical couplet given in (3.7.4) is recovered. The micropolar couplet agrees with that of Eringen (1999), but not the micro-couplet as it is noted that there's a minor mistake in his solution for $\mathbf{v}$.

As before, if the source point is located at $\mathbf{x} = \mathbf{x}_0$, then r will be defined by $r = |\bar{\mathbf{x}}|$ and $\mathbf{x}$ is replaced by $\bar{\mathbf{x}}$ in all the solutions.

3.8 Other fundamental solutions

If the source is located at $\mathbf{x} = \mathbf{x}_0$, the classical Stokeslet, given by (3.5.9), becomes

$$u_i = \frac{\rho}{8\pi\mu} S_{ij}(\bar{\mathbf{x}}) F_j,$$

where the Stokeslet $S_{ij}(\bar{\mathbf{x}})$ turns out to be

$$S_{ij}(\bar{\mathbf{x}}) = \frac{\delta_{ij}}{|\bar{\mathbf{x}}|} + \frac{\bar{x}_i \bar{x}_j}{|\bar{\mathbf{x}}|^3}.$$

By successive differentiation of $S_{ij}(\bar{\mathbf{x}})$ with respect to the source point $\mathbf{x}_0$, and after some tensor manipulation, a family of fundamental solutions consisting of the Stokeslet doublet, the couplet, the Stresslet, etcetera for the Stokes flow can be built up (Chwang & Wu, 1975; Pozrikidis, 1997). It is noted that the introduction of $S_{ij}(\bar{\mathbf{x}})$ allows differentiation with respect to $x_{0,j}$ to be done with more ease, as compared to differentiating $\mathbf{u}$ directly with respect to $\mathbf{x}_0$.

It is expected that a corresponding set of fundamental solutions for the micropolar fluid can also be formed using the micropolar Stokeslet through similar methods. In addition, a new family of fundamental solutions can be obtained from the micropolar micro-Stokeslet and micropolar micro-couplet, though it is expected to be tedious, given that the expressions of solutions of the Stokes micropolar fluid flow are complicated. Hence, deriving the micropolar couplet separately, as demonstrated in Section 3.7, might be an easier alternative.

Chapter 4 Oseenlet and Oseen's couplet

The Stokes approximation has been used to linearize the governing equations of both classical fluids and micropolar fluids to obviate the difficulties of obtaining analytical solutions. In Chapter 3, the fundamental solutions of classical Stokes flow and micropolar fluid Stokes flow have been successfully obtained.

Though the applicability of the Stokes approximation is wide, Stokes (1851) acknowledged that under his linearization, it was impossible to obtain solutions for 2D viscous flow over a finite body. This is well-known as Stokes paradox. Then, Oseen (1910) discovered the non-uniform character of Stokes solutions and provided a modified approximation that is well-known as Oseen approximation. The singularity representation based on the Oseen approximation is known as the Oseenlet in classical fluid dynamics.

Much has been done on the classical Oseenlet. Oseen (1927) obtained the steady Oseenlet while Chwang & Chan (2000) obtained the unsteady Stokeslet and Oseenlet for classical flows. Then, Shu & Chwang (2001) went a step further by deriving the generalized fundamental solutions for unsteady viscous flows, which include both translational and rotational motions.

Olmstead & Majumdar (1983) pioneered the derivation of the Oseenlet for micropolar fluid flows by considering a steady, incompressible micropolar fluid flow in $\mathcal{R}^2$. They considered a total of 3 problems: the problem of a point force acting in the x_1 -direction, then in x_2 -direction, and a point couple pointing in the x_3 -direction. However, there has been no further work done after that. Hence, we aim to derive the fundamental solutions of a micropolar fluid flow in $\mathcal{R}^3$, so that the point force and point couple can be prescribed in any direction in $\mathcal{R}^3$.

In this chapter, the paradoxes related to 2D viscous flows based on Stokes approximation are briefly discussed. Then, the governing equations of both classical fluids and micropolar fluids are linearized using the Oseen approximation to derive another singularity representation: the *Oseenlet*. The analysis is then extended to the case of a point couple in an unbounded flow to derive the couplet.

4.1 The paradoxes of Stokes and Whitehead

Recall that in sub-section 3.1.1, the dimensionless Navier-Stokes equation is shown to be

$$(4.1.1) \quad -\nabla' p' + \nabla'^2 \mathbf{u}' = \frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{Re}(\mathbf{u}' \cdot \nabla') \mathbf{u}',$$

so that the Stokes equation corresponds to the approximation of $\mathbf{Re} \ll 1$. Thus, a more accurate solution for the stream function Ψ for low-Re flows could be sought in the form

$$\Psi = \Psi_0 + \mathbf{Re} \Psi_1 + O(\mathbf{Re}^2),$$

which represents an asymptotic expansion of Ψ . By employing a limiting procedure, the problem for Ψ_0 may be solved and so may the problem for Ψ_1 and so on (Van Dyke, 1964). The expression for Ψ_0 for the problem of Stokes flow over a sphere is given in Chapter 5. However, it is found that a solution for Ψ_0 does not exist for the same problem over the 2D cylinder. This is documented in many fluid dynamics literature like Pnueli and Gutfinger (1997), for example. The lack of such a solution is well-known as Stokes paradox. Interestingly, in the problem for Ψ_1 , a solution exists for a cylinder but not for a sphere. This is well-known as Whitehead's paradox.

4.2 The Oseen approximation

The difficulty mentioned in the previous section is referred to as a singular perturbation (Van Dyke, 1964). The Stokes approximation is really the first-order problem arising out of a perturbation type of solution to the Navier-Stokes equations. The inability of this type of solution to match the required boundary conditions renders the perturbation singular. In 2D, the difficulty associated with this singular perturbation appears immediately, whereas in 3D, the difficulty is postponed to the second-order term in the expansion.

The physics behind the Stokes paradox is due to the negligence of the convection of momentum of the fluid, an assumption that is invalid far from the body. Owing to the nature of the viscous boundary conditions near the body, viscous diffusion, represented by the term $\nabla'^2 \mathbf{u}'$ in (4.1.1), dominates while convective effects,

represented by the term $(\mathbf{u}' \cdot \nabla') \mathbf{u}'$ are small because the fluid decelerates in that region. Thus, neglecting the convective term is reasonable.

However, far from the body, convective effects are dominant as the fluid velocity approaches the free-stream velocity and the decay of velocity gradients reduces the effects of viscous diffusion. This constitutes a contradiction to the underlying principle of Stokes approximation, causing the perturbation to be singular.

Oseen, recognizing the discrepancy that exists in the Stokes approximation, proposed linearizing the Navier-Stokes equations by introducing a linear operator, $\mathbf{U}_\infty \cdot \nabla$, to replace the original convection operator $\mathbf{u} \cdot \nabla$. Thus, the governing equations for an incompressible classical uniform flow with velocity $\mathbf{U}_\infty$ are

$$(4.2.1) \quad \nabla \cdot \mathbf{u} = 0 ,$$

$$(4.2.2) \quad -\nabla p + \mu \nabla^2 \mathbf{u} = \rho \mathbf{U}_\infty \cdot \nabla \mathbf{u} .$$

Near the body, the linearized convective term makes very little contribution to the flow for small $\mathbf{Re}$, and for reasons previously stated. Hence, both the Stokes approximation and Oseen approximation are acceptable in this region. At far field, the fluid velocity only differs slightly from the free-stream velocity $\mathbf{U}_\infty$, so Oseen linearized convective operator is a valid approximation. Therefore, it can be said that Oseen achieved a remarkable improvement over the Stokes equations.

Equations (4.2.1) and (4.2.2) are known as the Oseen equations. For a body translating with a constant velocity $\mathbf{U}_\infty$ in an unbounded flow, the flow is governed by the unsteady linearized Navier-Stokes equation

$$(4.2.3) \quad -\nabla p + \mu \nabla^2 \mathbf{u} = \rho \frac{\partial \mathbf{u}}{\partial t},$$

which is to be solved subject to the conditions that $\mathbf{u}$ is equal to $\mathbf{U}_\infty$ at the instantaneous location of the body surface, and $\mathbf{u}$ decays at far field. Since in a frame of reference moving with the body the velocity field is steady, we can write

$$\frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{U}_\infty \cdot \nabla \mathbf{u} = \mathbf{0},$$

which demonstrates that (4.2.3) is identical to (4.2.2) with the direction of $\mathbf{U}_\infty$ being the opposite.

4.3 Fundamental solutions of some partial differential operators

In this section, we make use of some of the properties of the Fourier transform given in Section 3.4 to derive the fundamental solutions of some partial differential operators. These fundamental solutions are important as they provide the inverse Fourier transforms required in the following sections.

The following Fourier transform properties are assumed without proof (Kohr & Pop, 2004; Ladyzhenskaia, 1969) in $\mathfrak{R}^N$:

$$(4.3.1) \quad \mathcal{F}\{\delta(\mathbf{x})\} = (2\pi)^{-\frac{N}{2}},$$

$$(4.3.2) \quad \left\{ \mathcal{F}^{-1} \left[\frac{(2\pi)^{-\frac{3}{2}}}{\xi^2} \right] \right\}(\mathbf{x}) = \frac{1}{4\pi r} \text{ for } N=3, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

$$(4.3.3) \quad \left\{ \mathcal{F}^{-1} \left[\frac{(2\pi)^{-\frac{N}{2}}}{\xi^2 + i2\chi\xi_1} \right] \right\}(\mathbf{x}) = \begin{cases} \frac{e^{\chi x_1}}{2\pi} K_0(\chi r) \text{ for } N=2, r = \sqrt{x_1^2 + x_2^2}, \\ \frac{e^{\chi(x_1-r)}}{4\pi r} \text{ for } N=3, r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \end{cases}$$

$$(4.3.4) \quad \left\{ \mathcal{F}^{-1} \left[\frac{(2\pi)^{-\frac{N}{2}}}{\xi^2 + \chi^2} \right] \right\}(\mathbf{x}) = \begin{cases} \frac{1}{2\pi} K_0(\chi r) & \text{for } N=2, r = \sqrt{x_1^2 + x_2^2}, \\ \frac{e^{-\chi r}}{4\pi r} & \text{for } N=3, r = \sqrt{x_1^2 + x_2^2 + x_3^2}. \end{cases}$$

Note that χ is the particular square root of $\chi^2 \in C \setminus \{z \in C : \mathbf{Re}(z) \leq 0, \mathbf{Im}(z) = 0\}$, which has a positive real part, i.e. $\mathbf{Re}(\chi) > 0$, and K_0 denotes the modified Bessel function (Abramowitz et al., 1972).

I) Consider the following partial differential equation:

$$(4.3.5) \quad \left(\nabla^2 - 2\chi \frac{\partial}{\partial x_1} \right) \Phi_0 = -\delta(\mathbf{x}),$$

where χ is the particular square root of $\chi^2 \in C \setminus \{z \in C : \mathbf{Re}(z) \leq 0, \mathbf{Im}(z) = 0\}$, which has a positive real part, i.e. $\mathbf{Re}(\chi) > 0$. To obtain the fundamental solution Φ_0 , it is convenient to take Fourier transform of (4.3.5) which turns out to be

$$\hat{\Phi}_0 = \frac{(2\pi)^{-\frac{N}{2}}}{\xi^2 + i2\chi\xi_1}.$$

According to (4.3.3), the inverse Fourier transform of $\hat{\Phi}_0$ or the fundamental solution Φ_0 is given by

$$(4.3.6) \quad \Phi_0 = \left\{ \mathcal{F}^{-1} \left[\frac{(2\pi)^{-\frac{N}{2}}}{\xi^2 + i2\chi\xi_1} \right] \right\}(\mathbf{x}) = \begin{cases} \frac{e^{\chi x_1}}{2\pi} K_0(\chi r) & \text{for } N=2, \\ \frac{e^{\chi(x_1-r)}}{4\pi r} & \text{for } N=3. \end{cases}$$

II) Consider the following partial differential equation:

$$(4.3.7) \quad \left(\nabla^2 - 2m \frac{\partial}{\partial x_1} - a \right) \Phi_1 = -\delta(\mathbf{x}),$$

where $m, a > 0$. Applying the method of Fourier transform, we obtain

$$\hat{\Phi}_1 = \frac{(2\pi)^{-\frac{N}{2}}}{\xi^2 + i2m\xi_1 + a}.$$

To obtain the fundamental solution Φ_1 , first let $\Phi_1 = ge^{mx_1}$, where g is a scalar function. Then,

$$\left(\nabla^2 - 2m \frac{\partial}{\partial x_1} - a \right) \Phi_1 = e^{mx_1} [\nabla^2 - (m^2 + a)]g.$$

So, (4.3.7) becomes

$$(\nabla^2 - w^2)g = -e^{-mx_1}\delta(\mathbf{x}),$$

for $w = \sqrt{m^2 + a}$. To obtain g , it is convenient to take the Fourier transform of the above equation. In view of (3.2.1) and the properties stated in Section 3.4, we have

$$\hat{g} = \frac{(2\pi)^{-\frac{N}{2}}}{\xi^2 + w^2}.$$

Based on (4.3.4), the inverse Fourier transform of $\hat{g}$ is

$$g = \begin{cases} \frac{1}{2\pi} K_0(wr) & \text{for } N = 2, \\ \frac{e^{-wr}}{4\pi r} & \text{for } N = 3. \end{cases}$$

Therefore, the fundamental solution Φ_1 is

$$\Phi_1 = \begin{cases} \frac{e^{mx_1}}{2\pi} K_0(wr) & \text{for } N = 2, \\ \frac{e^{(mx_1 - wr)}}{4\pi r} & \text{for } N = 3. \end{cases}$$

The solution Φ_1 in $\mathcal{R}^2$ agrees with the one given by Olmstead and Majumdar (1983). Thus, the method of derivation is reasonable and so is the solution Φ_1 in $\mathcal{R}^3$. In other words,

$$(4.3.8) \quad \Phi_1 = \mathcal{F}^{-1} \left\{ \frac{(2\pi)^{-\frac{N}{2}}}{\xi^2 + i2m\xi_1 + a} \right\} = \begin{cases} \frac{e^{mx_1}}{2\pi} K_0(wr) & \text{for } N = 2, \\ \frac{e^{(mx_1 - wr)}}{4\pi r} & \text{for } N = 3. \end{cases}$$

where $w = \sqrt{m^2 + a}$.

III) Next, consider the following partial differential equation:

$$(4.3.9) \quad \left(\nabla^2 - 2n \frac{\partial}{\partial x_1} \right) \left(\nabla^2 - 2m \frac{\partial}{\partial x_1} - a \right) \Phi_2 = -\delta(x),$$

where $n, m, a > 0$. Taking the Fourier transform of (4.3.9) gives

$$(4.3.10) \quad \begin{aligned} \hat{\Phi}_2 &= \frac{-(2\pi)^{-\frac{N}{2}}}{(\xi^2 + i2n\xi_1)(\xi^2 + i2m\xi_1 + a)} \\ &= \frac{-(2\pi)^{-\frac{N}{2}}}{2(n-m)(-\alpha + i\xi_1)} \left(\frac{1}{\xi^2 + i2m\xi_1 + a} - \frac{1}{\xi^2 + i2n\xi_1} \right), \end{aligned}$$

where $\alpha = \frac{a}{2(n-m)}$ and $n-m > 0$. Olmstead and Majumdar (1983) gave the

solution Φ_2 in $\mathcal{R}^2$ as

$$\Phi_2(x_1, x_2) = \frac{e^{\alpha x_1}}{4\pi(n-m)} \int_{x_1}^{\infty} e^{-\alpha t} \left\{ e^{mt} K_0 \left[w(t^2 + x_2^2)^{\frac{1}{2}} \right] - e^{nt} K_0 \left[n(t^2 + x_2^2)^{\frac{1}{2}} \right] \right\} dt,$$

where all symbols are as previously defined. By observing (4.3.6), (4.3.8), (4.3.10)

and $\Phi_2(x_1, x_2)$, an educated guess of Φ_2 in $\mathcal{R}^3$ is given by

$$(4.3.11) \quad \Phi_2 = \frac{e^{\alpha x_1}}{8\pi(n-m)} \int_{x_1}^{\infty} e^{-\alpha t} \left[\frac{e^{(mt - ws)} - e^{n(t-s)}}{s} \right] dt,$$

where $s = \sqrt{t^2 + x_2^2 + x_3^2}$.

To verify that (4.3.11) is indeed the solution, we first note that

$$(4.3.12) \quad \left(\nabla^2 - 2n \frac{\partial}{\partial x_1} \right) \left(\nabla^2 - 2m \frac{\partial}{\partial x_1} - a \right) \Phi_2 = \left(\nabla^2 - 2m \frac{\partial}{\partial x_1} - a \right) \\ \times \left\{ e^{\alpha x_1} \left[\nabla^2 + (2\alpha - 2n) \frac{\partial}{\partial x_1} + (\alpha^2 - 2\alpha n) \right] (e^{-\alpha x_1} \Phi_2) \right\},$$

since

$$\left[\nabla^2 + (2\alpha - 2n) \frac{\partial}{\partial x_1} + (\alpha^2 - 2\alpha n) \right] (e^{-\alpha x_1} \Phi_2) = e^{-\alpha x_1} \left(\nabla^2 - 2n \frac{\partial}{\partial x_1} \right) \Phi_2.$$

Next, we substitute (4.3.11) into the right-hand side of the above equation in replacement of Φ_2 . We find that

$$\left[\nabla^2 + (2\alpha - 2n) \frac{\partial}{\partial x_1} + (\alpha^2 - 2\alpha n) \right] (e^{-\alpha x_1} \Phi_2) \\ = \frac{1}{8\pi(n-m)} \left[\nabla^2 + (2\alpha - 2n) \frac{\partial}{\partial x_1} + (\alpha^2 - 2\alpha n) \right] \int_{x_1}^{\infty} e^{-\alpha t} \left[\frac{e^{(mt-ws)} - e^{n(t-s)}}{s} \right] dt.$$

Consequently, we have

$$\left[\nabla^2 + (2\alpha - 2n) \frac{\partial}{\partial x_1} + (\alpha^2 - 2\alpha n) \right] (e^{-\alpha x_1} \Phi_2) \\ = \frac{1}{8\pi(n-m)} \int_{x_1}^{\infty} \left[\bar{\nabla}^2 + (2\alpha - 2n) \frac{\partial}{\partial t} + (\alpha^2 - 2\alpha n) \right] e^{-\alpha t} \left[\frac{e^{(mt-ws)} - e^{n(t-s)}}{s} \right] dt,$$

where $\bar{\nabla}^2$ is defined in terms of variables (t, x_2, x_3) . After much simplification, we have

$$\left[\nabla^2 + (2\alpha - 2n) \frac{\partial}{\partial x_1} + (\alpha^2 - 2\alpha n) \right] (e^{-\alpha x_1} \Phi_2) = \frac{1}{8\pi(n-m)} \\ \times \int_{x_1}^{\infty} e^{-\alpha t} \left(\bar{\nabla}^2 - 2n \frac{\partial}{\partial t} \right) \left[\frac{e^{(mt-ws)} - e^{n(t-s)}}{s} \right] dt.$$

For ease of manipulation, the previous equation is rewritten as

$$\left[\nabla^2 + (2\alpha - 2n) \frac{\partial}{\partial x_1} + (\alpha^2 - 2\alpha n) \right] \left(e^{-\alpha x_1} \Phi_2 \right) = \frac{1}{2(n-m)} \int_{x_1}^{\infty} e^{-\alpha t} \left\{ \left(\nabla^2 - 2m \frac{\partial}{\partial t} - a \right) \right. \\ \left. \times \frac{e^{(mt-ws)}}{4\pi s} + \left[2(m-n) \frac{\partial}{\partial t} + 2\alpha(n-m) \right] \frac{e^{(mt-ws)}}{4\pi s} - \left(\nabla^2 - 2n \frac{\partial}{\partial t} \right) \frac{e^{n(t-s)}}{4\pi s} \right\} dt,$$

which in view of (4.3.5), (4.3.6), (4.3.7) and (4.3.8), simplifies to

$$\left[\nabla^2 + (2\alpha - 2n) \frac{\partial}{\partial x_1} + (\alpha^2 - 2\alpha n) \right] \left(e^{-\alpha x_1} \Phi_2 \right) = \int_{x_1}^{\infty} e^{-\alpha t} \left(\alpha - \frac{\partial}{\partial t} \right) \frac{e^{(mt-ws)}}{4\pi s} dt.$$

Then, applying integration by parts, the integral above can be evaluated to be

$$(4.3.13) \quad \left[\nabla^2 + (2\alpha - 2n) \frac{\partial}{\partial x_1} + (\alpha^2 - 2\alpha n) \right] \left(e^{-\alpha x_1} \Phi_2 \right) \\ = - \left[\frac{e^{-\alpha t} e^{(mt-ws)}}{4\pi s} \right]_{x_1}^{\infty} = \frac{e^{-\alpha x_1} e^{(mx_1-wr)}}{4\pi r}.$$

Finally, substituting (4.3.13) back into (4.3.12), and in view of (4.3.7) and (4.3.8), (4.3.9) is obtained.

Returning to (4.3.10), it can be concluded that

$$(4.3.14) \quad \Phi_2 = \mathcal{F}^{-1} \left\{ \frac{-(2\pi)^{-\frac{3}{2}}}{(\xi^2 + i2n\xi_1)(\xi^2 + i2m\xi_1 + a)} \right\} \\ = \frac{e^{\alpha x_1}}{8\pi(n-m)} \int_{x_1}^{\infty} e^{-\alpha t} \left[\frac{e^{(mt-ws)} - e^{n(t-s)}}{s} \right] dt$$

in $\mathfrak{R}^3$, where $\alpha = \frac{a}{2(n-m)}$, $w = \sqrt{m^2 + a}$ and $s = \sqrt{t^2 + x_2^2 + x_3^2}$.

IV) Suppose that $n=0$ in (4.3.9). Then, the partial differential equation becomes

$$(4.3.15) \quad \nabla^2 \left(\nabla^2 - 2m \frac{\partial}{\partial x_1} - a \right) \Phi_3 = -\delta(\mathbf{x}).$$

The Fourier transform of Φ_3 in $\mathfrak{R}^3$ is then given by

$$\hat{\Phi}_3 = \frac{-(2\pi)^{-\frac{3}{2}}}{\xi^2(\xi^2 + i2m\xi_1 + a)} = \frac{-(2\pi)^{-\frac{3}{2}}}{2m(\beta + i\xi_1)} \left(\frac{1}{\xi^2} - \frac{1}{\xi^2 + i2m\xi_1 + a} \right),$$

where $\beta = \frac{a}{2m}$. Based on guesswork, the inverse Fourier transform of the above is

$$(4.3.16) \quad \Phi_3 = \frac{e^{-\beta x_1}}{8\pi m} \int_{-\infty}^{x_1} e^{\beta t} \left[\frac{e^{(mt-ws)} - 1}{s} \right] dt,$$

where $\beta = \frac{a}{2m}$, $w = \sqrt{m^2 + a}$ and $s = \sqrt{t^2 + x_2^2 + x_3^2}$. We verify our guess as

follows:

We take the operator $\left(\nabla^2 - 2m \frac{\partial}{\partial x_1} - a \right)$ and operate it on Φ_3 . We arrive at

$$\begin{aligned} \left(\nabla^2 - 2m \frac{\partial}{\partial x_1} - a \right) \Phi_3 &= \frac{e^{-\beta x_1}}{8\pi m} \left[\nabla^2 - (2\beta + 2m) \frac{\partial}{\partial x_1} + \beta^2 \right] \\ &\quad \times \int_{-\infty}^{x_1} e^{\beta t} \left[\frac{e^{(mt-ws)} - 1}{s} \right] dt. \end{aligned}$$

Further, we find that

$$\begin{aligned} \left(\nabla^2 - 2m \frac{\partial}{\partial x_1} - a \right) \Phi_3 &= \frac{e^{-\beta x_1}}{2m} \int_{-\infty}^{x_1} \left[\bar{\nabla}^2 - (2\beta + 2m) \frac{\partial}{\partial t} + \beta^2 \right] e^{\beta t} \left[\frac{e^{(mt-ws)} - 1}{4\pi s} \right] dt \\ &= \frac{e^{-\beta x_1}}{2m} \int_{-\infty}^{x_1} e^{\beta t} \left(\bar{\nabla}^2 - 2m \frac{\partial}{\partial t} - a \right) \left[\frac{e^{(mt-ws)} - 1}{4\pi s} \right] dt, \end{aligned}$$

where $\bar{\nabla}^2$ is defined in terms of variables (t, x_2, x_3) . In view of the fundamental

solution Φ_1 in (4.3.8), and this well-known relation $\nabla^2 \left(\frac{1}{4\pi r} \right) = -\delta(\mathbf{x})$ in $\mathfrak{R}^3$, the

above equation simplifies to

$$\left(\nabla^2 - 2m \frac{\partial}{\partial x_1} - a \right) \Phi_3 = \frac{e^{-\beta x_1}}{2m} \int_{-\infty}^{x_1} e^{\beta t} \left(2m \frac{\partial}{\partial t} + a \right) \left(\frac{1}{4\pi s} \right) dt.$$

Using integration by parts, the above integral can be evaluated as

$$\left(\nabla^2 - 2m \frac{\partial}{\partial x_1} - a \right) \Phi_3 = e^{-\beta x_1} \frac{e^{\beta t}}{4\pi s} \Big|_{-\infty}^{x_1} = \frac{1}{4\pi r}.$$

Finally, we yield

$$\nabla^2 \left(\nabla^2 - 2m \frac{\partial}{\partial x_3} - a \right) \Phi_3 = \nabla^2 \left(\frac{1}{4\pi r} \right) = -\delta(\mathbf{x}).$$

Our guess of Φ_3 given by (4.3.16) is verified as correct. Hence, we can write

$$(4.3.17) \quad \Phi_3 = \mathcal{F}^{-1} \left\{ \frac{-(2\pi)^{-\frac{3}{2}}}{\xi^2 (\xi^2 + i2m\xi_1 + a)} \right\} = \frac{e^{-\beta x_1}}{8\pi m} \int_{-\infty}^{x_1} e^{\beta t} \left[\frac{e^{(mt - ws)} - 1}{s} \right] dt$$

in $\mathfrak{R}^3$, where $\beta = \frac{a}{2m}$, $w = \sqrt{m^2 + a}$ and $s = \sqrt{t^2 + x_2^2 + x_3^2}$.

V) Now, suppose that $m = a = 0$ in (4.3.9). Then, the partial differential equation becomes

$$(4.3.18) \quad \nabla^2 \left(\nabla^2 - 2n \frac{\partial}{\partial x_1} \right) \Phi_4 = -\delta(\mathbf{x}).$$

Applying the method of Fourier transform in $\mathfrak{R}^3$, we find that

$$\hat{\Phi}_4 = \frac{-(2\pi)^{-\frac{3}{2}}}{\xi^2 (\xi^2 + i2n\xi_1)} = \frac{-(2\pi)^{-\frac{3}{2}}}{i2n\xi_1} \left(\frac{1}{\xi^2} - \frac{1}{\xi^2 + i2n\xi_1} \right).$$

Referring to (4.3.2) and (4.3.3), an educated guess of the inverse Fourier transform of $\hat{\Phi}_4$ is

$$(4.3.19) \quad \Phi_4 = \frac{1}{8\pi n} \int_{x_1}^{\infty} \frac{1 - e^{n(t-s)}}{s} dt.$$

Again, we have to verify our guess. We start by operating $\left(\nabla^2 - 2n \frac{\partial}{\partial x_1}\right)$ on

(4.3.19). Then, we have

$$\left(\nabla^2 - 2n \frac{\partial}{\partial x_1}\right)\Phi_4 = \frac{1}{2n} \int_{x_1}^{\infty} \bar{\nabla}^2 \left(\frac{1}{4\pi s}\right) - 2n \frac{\partial}{\partial t} \left(\frac{1}{4\pi s}\right) - \left(\bar{\nabla}^2 - 2n \frac{\partial}{\partial t}\right) \frac{e^{n(t-s)}}{4\pi s} dt.$$

In view of this well-known relation $\nabla^2 \left(\frac{1}{4\pi r}\right) = -\delta(\mathbf{x})$ in $\mathfrak{R}^3$, (4.3.5) and (4.3.6),

the equation above simplifies to

$$\left(\nabla^2 - 2n \frac{\partial}{\partial x_1}\right)\Phi_4 = -\int_{x_1}^{\infty} d\left(\frac{1}{4\pi s}\right) = -\frac{1}{4\pi s}\Big|_{x_1}^{\infty} = \frac{1}{4\pi r}.$$

Finally, by taking Laplacian, we have

$$\nabla^2 \left(\nabla^2 - 2n \frac{\partial}{\partial x_1}\right)\Phi_4 = -\delta(\mathbf{x}).$$

Hence, (4.3.19) is indeed the fundamental solution of (4.3.18) and thus, we can write

$$(4.3.20) \quad \Phi_4 = \mathcal{F}^{-1} \left\{ \frac{-(2\pi)^{-\frac{3}{2}}}{\xi^2 (\xi^2 + i2n\xi_1)} \right\} = \frac{1}{8\pi n} \int_{x_1}^{\infty} \frac{1 - e^{n(t-s)}}{s} dt$$

in $\mathfrak{R}^3$ where $s = \sqrt{t^2 + x_2^2 + x_3^2}$.

VI) Lastly, consider the following partial differential equation:

$$(4.3.21) \quad \nabla^2 \left(\nabla^2 - 2n \frac{\partial}{\partial x_1}\right) \left(\nabla^2 - 2m \frac{\partial}{\partial x_1} - a\right) \Phi_5 = -\delta(\mathbf{x}),$$

Taking the Fourier transform of (4.3.21) in $\mathfrak{R}^3$ gives

$$\hat{\Phi}_5 = \frac{(2\pi)^{-\frac{3}{2}}}{\xi^2(\xi^2 + i2n\xi_1)(\xi^2 + i2m\xi_1 + a)}.$$

To obtain the inverse Fourier transform of $\hat{\Phi}_5$, we first rewrite $\hat{\Phi}_5$ in terms of the following partial fractions:

$$\hat{\Phi}_5 = (2\pi)^{-\frac{3}{2}} \left(\frac{A}{\xi^2} + \frac{B}{\xi^2 + i2n\xi_1} + \frac{C}{\xi^2 + i2m\xi_1 + a} \right),$$

where A , B and C are then found to be given by

$$A = \frac{1}{2na} \left(\frac{1}{i\xi_1} - \frac{1}{\beta + i\xi_1} \right), \quad B = -\frac{1}{2na} \left(\frac{1}{i\xi_1} - \frac{1}{-\alpha + i\xi_1} \right),$$

$$C = \frac{1}{2na} \left(\frac{1}{\beta + i\xi_1} - \frac{1}{-\alpha + i\xi_1} \right),$$

where $\alpha = \frac{a}{2(n-m)}$, $n-m > 0$ and $\beta = \frac{a}{2m}$, as defined previously. Hence, we can

rewrite $\hat{\Phi}_5$ in a more convenient form as follows:

$$\begin{aligned} \hat{\Phi}_5 = \frac{(2\pi)^{-\frac{3}{2}}}{2na} & \left[\frac{1}{i\xi_1} \left(\frac{1}{\xi^2} - \frac{1}{\xi^2 + i2n\xi_1} \right) + \frac{1}{\beta + i\xi_1} \left(\frac{1}{\xi^2 + i2m\xi_1 + a} - \frac{1}{\xi^2} \right) \right. \\ & \left. - \frac{1}{-\alpha + i\xi_1} \left(\frac{1}{\xi^2 + i2m\xi_1 + a} - \frac{1}{\xi^2 + i2n\xi_1} \right) \right], \end{aligned}$$

then more compactly as

$$\begin{aligned} \hat{\Phi}_5 = \frac{1}{a} & \left[\frac{(2\pi)^{-\frac{3}{2}}}{\xi^2(\xi^2 + i2n\xi_1)} \right] - \frac{m}{an} \left[\frac{(2\pi)^{-\frac{3}{2}}}{\xi^2(\xi^2 + i2m\xi_1 + a)} \right] \\ & + \frac{m-n}{an} \left[\frac{(2\pi)^{-\frac{3}{2}}}{(\xi^2 + i2n\xi_1)(\xi^2 + i2m\xi_1 + a)} \right]. \end{aligned}$$

The inverse Fourier transform of $\hat{\Phi}_5$ can now be easily obtained by using (4.3.14), (4.3.17) and (4.3.20). Thus,

(4.3.22)

$$\begin{aligned}\Phi_5 &= \mathcal{F}^{-1} \left\{ \frac{(2\pi)^{-\frac{3}{2}}}{\xi^2 (\xi^2 + i2n\xi_1)(\xi^2 + i2m\xi_1 + a)} \right\} \\ &= \frac{1}{2na} \int_{x_1}^{\infty} \frac{e^{n(t-s)} - 1}{4\pi s} dt + \frac{e^{-\beta x_1}}{2na} \int_{-\infty}^{x_1} e^{\beta t} \left[\frac{e^{(mt-ws)} - 1}{4\pi s} \right] dt + \frac{e^{\alpha x_1}}{2na} \int_{x_1}^{\infty} e^{-\alpha t} \left[\frac{e^{(mt-ws)} - e^{n(t-s)}}{4\pi s} \right] dt\end{aligned}$$

in $\mathfrak{R}^3$, where $\alpha = \frac{a}{2(n-m)}$, $n-m > 0$, $\beta = \frac{a}{2m}$, $w = \sqrt{m^2 + a}$ and $s = \sqrt{t^2 + x_2^2 + x_3^2}$. With the derivation of Φ_0 to Φ_5 , we are now in a position to obtain fundamental solutions representations of Oseen flows.

4.4 Classical Oseen flow due to a point force in $\mathfrak{R}^3$

As suggested by the above heading, this section aims to derive the free-space fundamental solution of classical Oseen flow. Suppose that in a quiescent incompressible classical fluid of infinite expanse, a constant force $\mathbf{F}$ is exerted at the origin $\mathbf{x} = \mathbf{0}$. Based on Oseen approximation, the governing equations (3.1.3) and (3.1.4) become

$$(4.4.1) \quad \nabla \cdot \mathbf{u} = 0,$$

$$(4.4.2) \quad -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{F} \delta(\mathbf{x}) = \rho \mathbf{U}_{\infty} \cdot \nabla \mathbf{u},$$

where all notations take their usual meanings. As the domain of flow is unbounded, the pressure and velocity fields are required to vanish as $|\mathbf{x}| \rightarrow \infty$.

Without any loss of generality, it is assumed that $\mathbf{U}_{\infty}$ is given by $(U_{\infty}, 0, 0)$, where $U_{\infty} \in \mathfrak{R}$. Hence, (4.4.2) is rewritten as

$$(4.4.3) \quad -\nabla p + \mu \nabla^2 \mathbf{u} - \rho U_\infty \frac{\partial \mathbf{u}}{\partial x_1} + \rho \mathbf{F} \delta(\mathbf{x}) = \mathbf{0}.$$

In view of (4.4.1), the divergence of (4.4.3) yields

$$(4.4.4) \quad \nabla^2 p = \nabla \cdot [\rho \mathbf{F} \delta(\mathbf{x})] = \rho \mathbf{F} \cdot \nabla \delta(\mathbf{x}),$$

which states that p is harmonic everywhere except at the pole. Since (4.4.4) is exactly the same as (3.5.3), the pressure field p is then given by

$$(4.4.5) \quad p = -\frac{\rho}{4\pi} \mathbf{F} \cdot \nabla \left(\frac{1}{r} \right) = \frac{\rho}{4\pi} \frac{\mathbf{F} \cdot \mathbf{x}}{r^3}.$$

To solve for $\mathbf{u}$, it is convenient to take Fourier transform of (4.4.3). Using the properties in Section 3.4, the Fourier transform of (4.4.3) reads

$$(4.4.6) \quad -(i\xi) \hat{p} - \mu \xi^2 \hat{\mathbf{u}} - \rho U_\infty (i\xi_1) \hat{\mathbf{u}} + \rho \mathbf{F} (2\pi)^{-\frac{3}{2}} = \mathbf{0}.$$

Further, from (4.4.5), the Fourier transform of p is found to be

$$(4.4.7) \quad \hat{p} = -\rho \mathbf{F} \cdot (i\xi) \left[\frac{(2\pi)^{-\frac{3}{2}}}{\xi^2} \right].$$

Substituting $\hat{p}$ into (4.4.6) and rearranging, the Fourier transform of $\mathbf{u}$ is found to be given by

$$\hat{\mathbf{u}} = \frac{\rho}{\mu} \left\{ \frac{(2\pi)^{-\frac{3}{2}}}{\xi^2 + i2n_0\xi_1} \mathbf{F} + \mathbf{F} \cdot (i\xi) (i\xi) \frac{(2\pi)^{-\frac{3}{2}}}{\xi^2 (\xi^2 + i2n_0\xi_1)} \right\},$$

where $2n_0 = \frac{\rho U_\infty}{\mu}$. Referring to (4.3.6) and (4.3.20), the inverse Fourier transform

of the above equation is

$$(4.4.8) \quad \mathbf{u} = \frac{\rho}{\mu} \frac{e^{n_0(x_1-r)}}{4\pi r} \mathbf{F} + (\mathbf{F} \cdot \nabla) \nabla [\mathbf{K}(\mathbf{x}; n_0, n_0) - \mathbf{K}(\mathbf{x}; 0, 0)],$$

where $2n_0 = \frac{\rho U_\infty}{\mu}$, $s = \sqrt{t^2 + x_2^2 + x_3^2}$ and we define a function K such that

$$(4.4.9) \quad K(\mathbf{x}; a_0, b_0) = \int_{x_1}^{\infty} \frac{e^{a_0 t - b_0 s}}{4U_\infty \pi s} dt.$$

It shall be reiterated that (4.4.8) is called the Oseenlet by Oseen (1927). If the source point is located at $\mathbf{x} = \mathbf{x}_0$, then $\mathbf{r}$ will be defined by $r = |\bar{\mathbf{x}}|$ and $\mathbf{x}$ is replaced by $\bar{\mathbf{x}}$ in all the solutions.

It is clear that in the limit where U_∞ tends to 0, (4.4.2) becomes the Stokes equation, given in (3.5.2). It is obvious that

$$(4.4.10) \quad \lim_{U_\infty \rightarrow 0} \frac{e^{n_0(x_1 - r)}}{4\pi r} = \frac{1}{4\pi r} \lim_{U_\infty \rightarrow 0} [1 + O(n_0)] = \frac{1}{4\pi r}.$$

Expanding the difference between the two K functions in (4.4.8), we have

$$K(\mathbf{x}; n_0, n_0) - K(\mathbf{x}; 0, 0) = \int_{x_1}^{\infty} \frac{1}{4U_\infty \pi s} [1 + n_0(t - s) + O(n_0^2) - 1] dt,$$

which in the limit $U_\infty \rightarrow 0$ becomes

$$(4.4.11) \quad \lim_{U_\infty \rightarrow 0} [K(\mathbf{x}; n_0, n_0) - K(\mathbf{x}; 0, 0)] = \frac{\rho}{8\pi\mu} \int_{x_1}^{\infty} \left(\frac{t}{\sqrt{t^2 + x_2^2 + x_3^2}} - 1 \right) dt.$$

From integration tables (Abramowitz et al., 1972), we have

$$\int_{x_1}^{\infty} \frac{t}{\sqrt{t^2 + x_2^2 + x_3^2}} dt = \sqrt{t^2 + x_2^2 + x_3^2} \Big|_{x_1}^{\infty}.$$

Hence, the integral in (4.4.11) becomes

$$\begin{aligned}
 \int_{x_1}^{\infty} \left[\frac{t}{\sqrt{t^2 + x_2^2 + x_3^2}} - 1 \right] dt &= \lim_{t \rightarrow \infty} \left(\sqrt{t^2 + x_2^2 + x_3^2} - t \right) - \left(\sqrt{x_1^2 + x_2^2 + x_3^2} - x_1 \right) \\
 &= \lim_{t \rightarrow \infty} \left[\left(\sqrt{t^2 + x_2^2 + x_3^2} - t \right) \times \frac{\left(\sqrt{t^2 + x_2^2 + x_3^2} + t \right)}{\left(\sqrt{t^2 + x_2^2 + x_3^2} + t \right)} \right] - (r - x_1) \\
 &= \lim_{t \rightarrow \infty} \frac{x_2^2 + x_3^2}{\sqrt{t^2 + x_2^2 + x_3^2} + t} - (r - x_1) \\
 &= x_1 - r.
 \end{aligned}$$

Finally, the expression in (4.4.11) tends to

$$(4.4.12) \quad \lim_{U_{\infty} \rightarrow 0} [K(\mathbf{x}; n_0, n_0) - K(\mathbf{x}; 0, 0)] = \frac{\rho}{8\pi\mu} (x_1 - r).$$

Finally, in view of (4.4.10) and (4.4.11), we see that (4.4.9) simplifies to

$$\begin{aligned}
 \mathbf{u} &= \frac{\rho}{4\pi\mu r} \mathbf{F} + \frac{\rho}{8\pi\mu} (\mathbf{F} \cdot \nabla) \nabla (x_1 - t) \\
 &= \frac{\rho}{8\pi\mu} \left(\frac{\mathbf{F}}{r} + \frac{\mathbf{F} \cdot \mathbf{x}}{r^3} \mathbf{x} \right),
 \end{aligned}$$

which is the classical Stokeslet, given by (3.5.8).

4.5 Oseen flow of a micropolar fluid due to a point force in $\mathbb{R}^3$

Consider a similar point force mentioned in the previous section acting in an unbounded quiescent, incompressible micropolar fluid. The resulting fluid flow is assumed steady or quasi-steady, with negligible body forces and body torques. Then, based on Oseen approximation, the governing equations (2.3.10) reduce to

$$(4.5.1) \quad \nabla \cdot \mathbf{u} = 0,$$

$$(4.5.2) \quad -\nabla p - (\mu + \mu_r) \nabla \times (\nabla \times \mathbf{u}) + 2\mu_r \nabla \times \mathbf{v} + \rho \mathbf{F} \delta(\mathbf{x}) = \rho (\mathbf{U}_{\infty} \cdot \nabla) \mathbf{u},$$

$$(4.5.3) \quad -(c_a + c_d) \nabla \times (\nabla \times \mathbf{v}) + 2\mu_r (\nabla \times \mathbf{u} - 2\mathbf{v}) = \rho j (\mathbf{U}_{\infty} \cdot \nabla) \mathbf{v},$$

where the divergence of $\mathbf{v}$ is assumed zero for simplicity. The pressure, velocity and microrotation velocity fields are required to decay as $|\mathbf{x}| \rightarrow \infty$ in such an unbounded flow. In addition, without any loss of generality, the free-stream velocity is taken to be $(U_\infty, 0, 0)$, where $U_\infty \in \Re$.

To begin, the divergence of (4.5.2) is taken, yielding

$$\nabla^2 p = \nabla \cdot [\rho \mathbf{F} \delta(\mathbf{x})] = \rho \mathbf{F} \cdot \nabla \delta(\mathbf{x}),$$

which is identical to (4.4.4). Thus, referring to (4.4.5), the pressure field p is given by

$$(4.5.4) \quad p = -\frac{\rho}{4\pi} \mathbf{F} \cdot \nabla \left(\frac{1}{r} \right) = \frac{\rho}{4\pi} \frac{\mathbf{F} \cdot \mathbf{x}}{r^3}.$$

It is noted that both the Stokes and Oseen flows due to a point force have the same pressure field, irregardless of whether the fluid is classical or micropolar in nature.

Making the curl of $\mathbf{u}$ the subject of (4.5.3), we have

$$(4.5.5) \quad \nabla \times \mathbf{u} = 2\mathbf{v} + \frac{c_a + c_d}{2\mu_r} \nabla \times (\nabla \times \mathbf{v}) + \frac{\rho j U_\infty}{2\mu_r} \frac{\partial \mathbf{v}}{\partial x_1}.$$

Consequently, the curl of the above gives the curl of curl of $\mathbf{u}$ as

$$(4.5.6) \quad \nabla \times \nabla \times \mathbf{u} = 2\nabla \times \mathbf{v} + \frac{c_a + c_d}{2\mu_r} \nabla \times \nabla \times (\nabla \times \mathbf{v}) + \frac{\rho j U_\infty}{2\mu_r} \frac{\partial}{\partial x_1} (\nabla \times \mathbf{v}).$$

Taking the curl of (4.5.2) gives

$$(4.5.7) \quad \begin{aligned} & -(\mu + \mu_r) \nabla \times \nabla \times \nabla \times \mathbf{u} + 2\mu_r \nabla \times \nabla \times \mathbf{v} + \nabla \times [\rho \mathbf{F} \delta(\mathbf{x})] \\ & = \rho U_\infty \frac{\partial (\nabla \times \mathbf{u})}{\partial x_1}. \end{aligned}$$

By substituting (4.5.5) and (4.5.6) into (4.5.7), a partial differential vector equation containing only one unknown, $\mathbf{v}$, results:

$$(4.5.8) \quad \left(\nabla^4 - \lambda^2 \nabla^2 - a_1 \frac{\partial}{\partial x_1} \nabla^2 + a_2 \frac{\partial}{\partial x_1} + a_3 \frac{\partial^2}{\partial x_1^2} \right) \mathbf{v} = a_4 \nabla \times [F \delta(\mathbf{x})],$$

$$\text{where } \lambda^2 = \frac{4\mu_r \mu}{(c_a + c_d)(\mu + \mu_r)}, \quad a_1 = \rho U_\infty \left(\frac{j}{c_a + c_d} + \frac{1}{\mu + \mu_r} \right), \quad a_2 = \frac{4\rho U_\infty \mu_r}{(c_a + c_d)(\mu + \mu_r)},$$

$$a_3 = \frac{j\rho^2 U_\infty^2}{(c_a + c_d)(\mu + \mu_r)} \quad \text{and} \quad a_4 = \frac{2\rho\mu_r}{(c_a + c_d)(\mu + \mu_r)}.$$

Clearly, the process of solving for $\mathbf{v}$ is a formidable one.

It is mentioned in the book by Zwillinger (1989) that one of the methods of solving partial differential equations of high orders, such as (4.5.8), is to factorize the high order partial differential operator into products of lower order operators. This method was also used by Olmstead & Majumdar (1983). Formally, they proposed that

$$(4.5.9) \quad L = \left(\nabla^2 + A_1 \frac{\partial}{\partial x_1} + B_1 \right) \left(\nabla^2 + A_2 \frac{\partial}{\partial x_1} + B_2 \right),$$

where L is a fourth order partial differential operator and A_1 , A_2 , B_1 , and B_2 are constants. While the method of factorization is attractive, it was noted that a certain relationship between the parameters must exist for L to admit the desired factorization.

To proceed from (4.5.8), one has little choice but to attempt to factorize the differential operator in that equation. For (4.5.9) to exist for the differential operator in (4.5.8), the following must be true:

$$\begin{aligned}
 (4.5.10) \quad L &= \nabla^4 - \lambda^2 \nabla^2 - a_1 \frac{\partial}{\partial x_1} \nabla^2 + a_2 \frac{\partial}{\partial x_1} + a_3 \frac{\partial^2}{\partial x_1^2} \\
 &= \nabla^4 + (B_1 + B_2) \nabla^2 + (A_1 + A_2) \frac{\partial}{\partial x_1} \nabla^2 + (A_1 B_2 + B_1 A_2) \frac{\partial}{\partial x_1} + A_1 A_2 \frac{\partial^2}{\partial x_1^2} + B_1 B_2.
 \end{aligned}$$

Consequently, it is required that

$$\begin{aligned}
 (4.5.11) \quad B_1 + B_2 &= -\lambda^2, & A_1 + A_2 &= -a_1, & A_1 B_2 + B_1 A_2 &= a_2, \\
 A_1 A_2 &= a_3, & B_1 B_2 &= 0.
 \end{aligned}$$

There are five equations, with only four unknowns. To expedite the solution process, the value of B_1 is taken to be zero since $B_1 B_2 = 0$. Therefore,

$$B_1 = 0 \Rightarrow B_2 = -\lambda^2.$$

Then, from $A_1 B_2 + B_1 A_2 = a_2$, A_1 is found to be

$$A_1 = -\frac{a_2}{\lambda^2} = -\frac{\rho U_\infty}{\mu}.$$

Consequently,

$$A_2 = \frac{a_3}{A_1} = -\frac{j\rho\mu U_\infty}{(c_a + c_d)(\mu + \mu_r)}.$$

However, the sum of A_1 and A_2 must equal to $-a_1$, or

$$\frac{\rho U_\infty}{\mu} + \frac{j\rho\mu U_\infty}{(c_a + c_d)(\mu + \mu_r)} = \rho U_\infty \left(\frac{j}{c_a + c_d} + \frac{1}{\mu + \mu_r} \right),$$

which implies that

$$(4.5.12) \quad j = \frac{c_a + c_d}{\mu}.$$

Hence, in view of (4.5.10), the partial differential operator in (4.5.9) can be factorized into

$$\begin{aligned}
 (4.5.13) \quad & \nabla^4 - \lambda^2 \nabla^2 - a_1 \frac{\partial}{\partial x_1} \nabla^2 + a_2 \frac{\partial}{\partial x_1} + a_3 \frac{\partial^2}{\partial x_1^2} \\
 & = \left(\nabla^2 - \frac{a_2}{\lambda^2} \frac{\partial}{\partial x_1} \right) \left(\nabla^2 - \frac{a_3 \lambda^2}{a_2} \frac{\partial}{\partial x_1} - \lambda^2 \right).
 \end{aligned}$$

This allows (4.5.8) to be rewritten as

$$(4.5.14) \quad \left(\nabla^2 - 2n_0 \frac{\partial}{\partial x_1} \right) \left(\nabla^2 - 2m_0 \frac{\partial}{\partial x_1} - \lambda^2 \right) \mathbf{v} = a_4 \nabla \times [\mathbf{F} \delta(\mathbf{x})],$$

where $2n_0 = \frac{\rho U_\infty}{\mu}$ and $2m_0 = \frac{\rho U_\infty}{\mu + \mu_r}$. It must be stressed that the above

factorization is valid under the physical constraint of the parameters given by (4.5.12).

To solve for $\mathbf{v}$, it seems convenient to take Fourier transform of (4.5.14). Using the properties defined in Section 3.4, the Fourier transform of (4.5.14) is

$$(\xi^2 + i2n_0 \xi_1) (\xi^2 + i2m_0 \xi_1 + \lambda^2) \hat{\mathbf{v}} = (2\pi)^{-\frac{3}{2}} a_4 (i\xi) \times \mathbf{F},$$

which implies that

$$(4.5.15) \quad \hat{\mathbf{v}} = \frac{(2\pi)^{-\frac{3}{2}} a_4 (i\xi) \times \mathbf{F}}{(\xi^2 + i2n_0 \xi_1) (\xi^2 + i2m_0 \xi_1 + \lambda^2)}.$$

The inverse Fourier transform of the above gives $\mathbf{v}$ to be

$$\mathbf{v} = a_4 \nabla \times \left\{ \mathcal{F}^{-1} \left[\frac{(2\pi)^{-\frac{3}{2}}}{(\xi^2 + i2n_0 \xi_1) (\xi^2 + i2m_0 \xi_1 + \lambda^2)} \right] \mathbf{F} \right\},$$

which by reference to (4.3.14) is

$$(4.5.16) \quad \mathbf{v} = \frac{2\mu}{c_a + c_d} \nabla \times \left\{ \mathbf{F} e^{\alpha_0 x_1} [\mathbf{K}(\mathbf{x}; n_0 - \alpha_0, n_0) - \mathbf{K}(\mathbf{x}; m_0 - \alpha_0, w_0)] \right\}$$

where $2n_0 = \frac{\rho U_\infty}{\mu}$, $2m_0 = \frac{\rho U_\infty}{\mu + \mu_r}$, $\alpha_0 = \frac{\lambda^2}{n_0 - m_0} = \frac{4\mu^2}{\rho U_\infty (c_a + c_d)}$, $w_0 = \sqrt{m_0^2 + \lambda^2}$,

$s = \sqrt{t^2 + x_2^2 + x_3^2}$ and $K(\mathbf{x}; a_0, b_0) = \int_{x_1}^{\infty} \frac{e^{a_0 t - b_0 s}}{4U_\infty \pi s} dt$. As $\mathbf{v}$ can be expressed as the curl

of the product of a scalar function and the constant vector $\mathbf{F}$, it is clear that the divergence of $\mathbf{v}$ is zero. Hence, the assumption made earlier about the divergence of $\mathbf{v}$ being zero is reasonable. We shall call (4.5.16) the micropolar micro-Oseenlet.

We shall now show that as $U_\infty \rightarrow 0$, (4.5.16) tends to the micropolar micro-Stokeslet in (3.6.9). Firstly, we rewrite K in (4.5.16) so that

$$\begin{aligned} & K(\mathbf{x}; n_0 - \alpha_0, n_0) - K(\mathbf{x}; m_0 - \alpha_0, w_0) \\ &= \int_{x_1}^{\infty} \left[\frac{e^{n_0(t-s)} - e^{(m_0 t - w_0 s)}}{4U_\infty \pi s} \right] d \left(\frac{e^{-\alpha_0 t}}{-\alpha_0} \right) \\ &= -\frac{1}{\alpha_0} \left[e^{-\alpha_0 t} \frac{e^{n_0(t-s)} - e^{(m_0 t - w_0 s)}}{4U_\infty \pi s} \right]_{x_1}^{\infty} + \frac{1}{\alpha_0 U_\infty} \int_{x_1}^{\infty} e^{-\alpha_0 t} d \left[\frac{e^{n_0(t-s)} - e^{(m_0 t - w_0 s)}}{4\pi s} \right]. \end{aligned}$$

Hence,

$$\begin{aligned} & e^{\alpha_0 x_1} [K(\mathbf{x}; n_0 - \alpha_0, n_0) - K(\mathbf{x}; m_0 - \alpha_0, w_0)] \\ (4.5.17) \quad &= \frac{\rho(c_a + c_d)}{16\mu^2 \pi} \left\{ \frac{e^{n_0(x_1-r)} - e^{(m_0 x_1 - w_0 r)}}{r} + \int_{x_1}^{\infty} e^{\alpha_0(x_1-t)} d \left[\frac{e^{n_0(t-s)} - e^{(m_0 t - w_0 s)}}{s} \right] \right\}. \end{aligned}$$

Let's examine the behaviour of (4.5.17) as $U_\infty \rightarrow 0$. The first term in the braces tends to

$$\begin{aligned} (4.5.18) \quad \lim_{U_\infty \rightarrow 0} \frac{e^{n_0(x_1-r)} - e^{(m_0 x_1 - w_0 r)}}{r} &= \frac{1}{r} \lim_{U_\infty \rightarrow 0} \left\{ [1 + O(n_0)] - e^{-r\sqrt{O(m_0^2) + \lambda^2}} [1 + O(m_0)] \right\} \\ &= \frac{1 - e^{-\lambda r}}{r}. \end{aligned}$$

The second term in the braces in (4.5.17) tends to

$$\begin{aligned}
 (4.5.19) \quad & \lim_{U_\infty \rightarrow 0} \int_{x_1}^{\infty} e^{\alpha_0(x_1-t)} d \left[\frac{e^{n_0(t-s)} - e^{(m_0 t - w_0 s)}}{s} \right] \\
 &= \lim_{U_\infty \rightarrow 0} \int_{x_1}^{\infty} e^{\alpha_0(x_1-t)} d \left[\frac{1 + O(n_0)}{s} \right] - \lim_{U_\infty \rightarrow 0} \int_{x_1}^{\infty} e^{\alpha_0(x_1-t)} d \left[\frac{e^{-s\sqrt{O(m_0^2) + \lambda^2}} (1 + O(m_0))}{s} \right] \\
 &= 0
 \end{aligned}$$

since $x_1 < t$ in the domain of integration. Therefore, the behaviour of (4.5.17) as

$U_\infty \rightarrow 0$ is

(4.5.20)

$$\lim_{U_\infty \rightarrow 0} e^{\alpha_0 x_1} [K(\mathbf{x}; n_0 - \alpha_0, n_0) - K(\mathbf{x}; m_0 - \alpha_0, w_0)] = \frac{\rho(c_a + c_d)}{16\mu^2\pi} \left(\frac{1 - e^{-\lambda r}}{r} \right).$$

Finally, as $U_\infty \rightarrow 0$, (4.5.16) tends to

$$\mathbf{v} = \frac{2\mu}{c_a + c_d} \frac{\rho(c_a + c_d)}{16\mu^2\pi} \nabla \times \left[\mathbf{F} \left(\frac{1 - e^{-\lambda r}}{r} \right) \right] = \frac{\rho}{8\pi\mu} \nabla \times \left(\frac{1 - e^{-\lambda r}}{r} \mathbf{F} \right)$$

which is the micropolar micro-Stokeslet in (3.6.9).

To solve for $\mathcal{U}$, we return to (4.5.6), which can be written as

$$-\nabla^2 \mathbf{u} = 2\nabla \times \mathbf{v} - \frac{c_a + c_d}{2\mu_r} \nabla^2 (\nabla \times \mathbf{v}) + \frac{\rho j U_\infty}{2\mu_r} \frac{\partial}{\partial x_1} (\nabla \times \mathbf{v})$$

using vector identities (Appendix C.7), keeping in view of (4.5.1), and the assumption that the divergence of $\mathbf{v}$ is zero. Under Fourier transform, the above equation transforms to

$$\xi^2 \hat{\mathbf{u}} = \left(2 + \frac{c_a + c_d}{2\mu_r} \xi^2 + i \frac{\rho j U_\infty}{2\mu_r} \xi_1 \right) (i\xi) \times \hat{\mathbf{v}}.$$

Substituting (4.5.12) and (4.5.15) into the above equation leads to

$$\hat{\mathbf{u}} = (2\pi)^{-\frac{3}{2}} a_4(i\xi) \times (i\xi) \times \left\{ \left[\frac{2}{\xi^2(\xi^2 + i2n_0\xi_1)(\xi^2 + i2m_0\xi_1 + \lambda^2)} + \frac{c_a + c_d}{2\mu_r} \frac{1}{\xi^2(\xi^2 + i2m_0\xi_1 + \lambda^2)} \right] \mathbf{F} \right\},$$

which upon applying inverse Fourier transform gives $\mathbf{u}$ to be

$$\mathbf{u} = a_4 \nabla \times \nabla \times \left\{ 2\mathcal{F}^{-1} \left[\frac{(2\pi)^{-\frac{3}{2}}}{\xi^2(\xi^2 + i2n_0\xi_1)(\xi^2 + i2m_0\xi_1 + \lambda^2)} \right] \mathbf{F} + \frac{c_a + c_d}{2\mu_r} \mathcal{F}^{-1} \left[\frac{(2\pi)^{-\frac{3}{2}}}{\xi^2(\xi^2 + i2m_0\xi_1 + \lambda^2)} \right] \mathbf{F} \right\}.$$

Finally, by referring to (4.3.17) and (4.3.22), we find that the linear velocity $\mathbf{u}$ is given by

$$(4.5.21) \quad \mathbf{u} = \nabla \times \nabla \times \{ \mathbf{F} [\mathbf{K}(\mathbf{x}; n_0, n_0) - \mathbf{K}(\mathbf{x}; 0, 0)] + \mathbf{F} e^{\alpha_0 x_1} \times [\mathbf{K}(\mathbf{x}; m_0 - \alpha_0, w_0) - \mathbf{K}(\mathbf{x}; n_0 - \alpha_0, n_0)] \}.$$

We shall call (4.5.21) the micropolar Oseenlet, a name that is analogous to the classical oseenlet. If the source point is located at $\mathbf{x} = \mathbf{x}_0$, then $\mathbf{r}$ will be defined by $r = |\bar{\mathbf{x}}|$ and $\mathbf{x}$ is replaced by $\bar{\mathbf{x}}$ in all the solutions.

Consider the case where $\mu_r \rightarrow 0$, which represents decoupling of the velocity and microrotation velocity fields. Then, $m_0, w_0 \rightarrow n_0$, $\alpha_0 \rightarrow 0$ and (4.5.21) reduces to

$$\mathbf{u} = \nabla \times \nabla \times \{ \mathbf{F} [\mathbf{K}(\mathbf{x}; n_0, n_0) - \mathbf{K}(\mathbf{x}; 0, 0)] \}.$$

Using vector identities in Appendix C.7, we can rewrite the equation above in terms of its integral form as

$$\mathbf{u} = \frac{1}{U_\infty} (\mathbf{F} \cdot \nabla) \nabla \int_{x_1}^{\infty} \frac{e^{n_0(t-s)} - 1}{4\pi s} dt + \frac{1}{U_\infty} \nabla^2 \left[\mathbf{F} \int_{x_1}^{\infty} \frac{1 - e^{n_0(t-s)}}{4\pi s} dt \right].$$

After the Laplacian has been brought through the integral sign, we have

$$\nabla^2 \left[\mathbf{F} \int_{x_1}^{\infty} \frac{1 - e^{n_0(t-s)}}{4\pi s} dt \right] = \mathbf{F} \int_{x_1}^{\infty} \bar{\nabla}^2 \left(\frac{1}{4\pi s} \right) - \left(\bar{\nabla}^2 - 2n_0 \frac{\partial}{\partial t} \right) \frac{e^{n_0(t-s)}}{4\pi s} - 2n_0 \frac{\partial}{\partial t} \left[\frac{e^{n_0(t-s)}}{4\pi s} \right] dt,$$

which in view of this well-known relation $\nabla^2 \left(\frac{1}{4\pi r} \right) = -\delta(\mathbf{x})$ in $\Re^3$, (4.3.5) and

(4.3.6), simplifies to

$$\nabla^2 \left[\mathbf{F} \int_{x_1}^{\infty} \frac{1 - e^{n_0(t-s)}}{4\pi s} dt \right] = \mathbf{F} \int_{x_1}^{\infty} -2n_0 d \left[\frac{e^{n_0(t-s)}}{4\pi s} \right] = \mathbf{F} n_0 \frac{e^{n_0(x_1-r)}}{2\pi r}$$

Hence, the expression for $\mathbf{u}$ simplifies to

$$\mathbf{u} = \frac{\rho}{\mu} \frac{e^{n_0(x_1-r)}}{4\pi r} \mathbf{F} + (\mathbf{F} \cdot \nabla) \nabla [\mathbf{K}(\mathbf{x}; n_0, n_0) - \mathbf{K}(\mathbf{x}; 0, 0)].$$

We see that the classical Oseenlet given by (4.4.8) is being recovered.

It can also be shown that in the limit when $U_{\infty} \rightarrow 0$, the micropolar Oseenlet (4.5.21) becomes the micropolar Stokeslet (3.6.11). In view of (4.4.12) and (4.5.20), (4.5.21) tends to

$$\begin{aligned} \mathbf{u} &= \nabla \times \nabla \times \left[\frac{\rho}{8\pi\mu} (x_1 - r) \mathbf{F} - \frac{\rho(c_a + c_d)}{16\mu^2\pi} \left(\frac{1 - e^{-\lambda r}}{r} \mathbf{F} \right) \right] \\ &= -\frac{\rho(c_a + c_d)}{16\mu^2\pi} \nabla \times \nabla \times \left(\frac{1 - e^{-\lambda r}}{r} \mathbf{F} \right) + \frac{\rho}{8\pi\mu} \left[\frac{\mathbf{F}}{r} + \frac{\mathbf{F} \cdot \mathbf{x}}{r^3} \mathbf{x} \right] \end{aligned}$$

which is the micropolar Stokeslet (3.6.11).

4.6 Oseen flows due to a point couple in $\Re^3$

We extend the applicability of Oseen approximation in deriving the flow properties due to a point force to deriving the flow properties due to a point couple in this

section. We shall consider a point couple in a classical fluid first, followed by a point couple in a micropolar fluid.

4.6.1 Classical Oseen flow due to a point couple

Consider a quiescent incompressible classical fluid of infinite expanse, with a constant forcing function $\mathbf{F}$. Based on Oseen approximation, the governing equations are

$$(4.6.1) \quad \nabla \cdot \mathbf{u} = 0 ,$$

$$(4.6.2) \quad -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{F} = \rho \mathbf{U}_\infty \cdot \nabla \mathbf{u} .$$

As with in sub-section 3.7.1, the external forcing function $\mathbf{F}$ is expressed as

$$\mathbf{F} = \nabla \times \left[\frac{1}{2} \mathbf{q} \delta(\mathbf{x}) \right] .$$

We reiterate here $\mathbf{q}$ is a constant vector and the product $\mathbf{q} \delta(\mathbf{x})$ is a vector potential.

The velocity and pressure fields are required to decay at far field. Formally,

$$|\mathbf{u}| \rightarrow 0, \quad p \rightarrow p_\infty \quad \text{as } |\mathbf{x}| \rightarrow \infty .$$

Similar to Stokes flow, the divergence of (4.6.2) gives

$$\nabla^2 p = 0 .$$

The pressure field is also harmonic in nature. Referring to sub-section 3.7.1, p is known to be

$$(4.6.3) \quad p = p_\infty = \text{constant} .$$

Again, we state that (4.6.3) is only valid for unbounded flows.

Then, (4.6.2) can be rewritten as

$$(4.6.4) \quad \left(\nabla^2 - 2n_0 \frac{\partial}{\partial x_1} \right) \mathbf{u} = -\frac{\rho}{2\mu} \nabla \times [\mathbf{q} \delta(\mathbf{x})],$$

where $2n_0 = \frac{\rho U_\infty}{\mu}$ and the free-stream velocity is taken to be $(U_\infty, 0, 0)$, where

$U_\infty \in \Re$, as defined in the previous section. To expedite the solution process, we take Fourier transform of (4.6.4). This turns out to be

$$(\xi^2 + i2n_0\xi_1)\hat{\mathbf{u}} = \frac{\rho}{2\mu} (2\pi)^{-\frac{3}{2}} (i\xi) \times \mathbf{q}$$

This implies that the Fourier transform of $\mathbf{u}$ is

$$\hat{\mathbf{u}} = \frac{\rho}{2\mu} (i\xi) \times \left[\frac{(2\pi)^{-\frac{3}{2}} \mathbf{q}}{\xi^2 + i2n_0\xi_1} \right].$$

Referring to (4.3.3), the inverse Fourier transform of $\hat{\mathbf{u}}$ is given by

$$(4.6.5) \quad \mathbf{u} = \frac{\rho}{8\pi\mu} \nabla \times \left[\mathbf{q} \frac{e^{n_0(x_3-r)}}{r} \right].$$

Since (4.6.5) is derived on Oseen approximation, we shall call it the Oseen's couplet. By inspection, for $U_\infty \rightarrow 0$ or $n_0 \rightarrow 0$, the classical couplet, given by (3.7.4), is recovered.

4.6.2 Oseen flow of a micropolar fluid due to a point couple

Consider a point couple acting on an unbounded quiescent, incompressible micropolar fluid. The fluid has negligible body forces and body torques. Then, based on Oseen approximation, the governing equations (2.3.10) can be linearized as

$$(4.6.6) \quad \nabla \cdot \mathbf{u} = 0,$$

$$(4.6.7) \quad -\nabla p - (\mu + \mu_r) \nabla \times (\nabla \times \mathbf{u}) + 2\mu_r \nabla \times \mathbf{v} = \rho (\mathbf{U}_\infty \cdot \nabla) \mathbf{u},$$

$$(4.6.8) \quad (c_0 + 2c_d)\nabla(\nabla \cdot \mathbf{v}) - (c_a + c_d)\nabla \times (\nabla \times \mathbf{v}) + 2\mu_r(\nabla \times \mathbf{u} - 2\mathbf{v}) + \rho \mathbf{Q} \delta(\mathbf{x}) = \rho j(\mathbf{U}_\infty \cdot \nabla) \mathbf{v},$$

where $\rho \mathbf{Q} \delta(\mathbf{x})$ is a point couple, with $\mathbf{Q}$ as a constant vector. Without loss of generality, the couple is assumed to be positioned at the origin. Further, $\mathbf{U}_\infty$ is a constant vector assumed to be $(U_\infty, 0, 0)$, where $U_\infty \in \mathfrak{R}$. The pressure, velocity and microrotation velocity fields have to decay as $|\mathbf{x}| \rightarrow \infty$. Formally,

$$p \rightarrow p_\infty, |\mathbf{u}| \rightarrow 0, |\mathbf{v}| \rightarrow 0 \text{ as } |\mathbf{x}| \rightarrow \infty.$$

We begin by taking the divergence of (4.6.7), which states

$$\nabla^2 p = 0.$$

As before, the pressure field p is such that

$$(4.6.9) \quad p = p_\infty = \text{constant}.$$

This reduces the gradient of p in (4.6.7) to be zero.

As a point couple now acts on the fluid, obtaining the linear velocity field $\mathbf{u}$ should be easier. This can be done by first taking the curl of (4.6.8), which gives

$$(4.6.10) \quad (c_a + c_d)\nabla^2 \mathbf{a} - 2\mu_r \nabla^2 \mathbf{u} - 4\mu_r \mathbf{a} + \rho \nabla \times [\mathbf{Q} \delta(\mathbf{x})] = \rho j U_\infty \frac{\partial \mathbf{a}}{\partial x_1},$$

where $\mathbf{a} = \nabla \times \mathbf{v}$. Note that vector identities (Appendix C.7) and (4.6.6) are used to express the curl of (4.6.8) in the above form. To express (4.6.10) solely in terms of $\mathbf{u}$, we make use of (4.6.7), which can be rewritten as

$$(4.6.11) \quad \mathbf{a} = \frac{1}{2\mu_r} \left[\rho U_\infty \frac{\partial}{\partial x_1} - (\mu + \mu_r) \nabla^2 \right] \mathbf{u}.$$

Substituting (4.6.11) into (4.6.10) leads to

$$(4.6.12) \quad \left(\nabla^4 - \lambda^2 \nabla^2 - a_1 \frac{\partial}{\partial x_1} \nabla^2 + a_2 \frac{\partial}{\partial x_1} + a_3 \frac{\partial^2}{\partial x_1^2} \right) \mathbf{u} = a_4 \nabla \times [\mathbf{Q} \delta(\mathbf{x})],$$

where $\lambda^2 = \frac{4\mu_r \mu}{(c_a + c_d)(\mu + \mu_r)}$, $a_1 = \rho U_\infty \left(\frac{j}{c_a + c_d} + \frac{1}{\mu + \mu_r} \right)$, $a_2 = \frac{4\rho U_\infty \mu_r}{(c_a + c_d)(\mu + \mu_r)}$,
 $a_3 = \frac{j\rho^2 U_\infty^2}{(c_a + c_d)(\mu + \mu_r)}$ and $a_4 = \frac{2\rho \mu_r}{(c_a + c_d)(\mu + \mu_r)}$, as defined previously in Section

4.5. As (4.6.12) is identical to (4.5.8), we can then factorize the partial differential operator, just like in (4.5.14), under the physical constraint given by (4.5.12). Then, we can write

$$\left(\nabla^2 - 2n_0 \frac{\partial}{\partial x_1} \right) \left(\nabla^2 - 2m_0 \frac{\partial}{\partial x_1} - \lambda^2 \right) \mathbf{u} = a_4 \nabla \times [\mathbf{Q} \delta(\mathbf{x})],$$

whose Fourier transform is

$$(4.6.13) \quad \hat{\mathbf{u}} = a_4 (i\xi) \times \left\{ \frac{(2\pi)^{-\frac{3}{2}} \mathbf{Q}}{(\xi^2 + i2n_0 \xi_1)(\xi^2 + i2m_0 \xi_1 + \lambda^2)} \right\},$$

where $2n_0 = \frac{\rho U_\infty}{\mu}$, $2m_0 = \frac{\rho U_\infty}{\mu + \mu_r}$ and $\alpha_0 = \frac{\lambda^2}{2(n_0 - m_0)}$. In view of (4.3.14), the

inverse Fourier transform of $\hat{\mathbf{u}}$ is found to be

$$(4.6.14) \quad \mathbf{u} = \frac{2\mu}{c_a + c_d} \nabla \times \left\{ \mathbf{Q} e^{\alpha_0 x_1} [\mathbf{K}(\mathbf{x}; n_0 - \alpha_0, n_0) - \mathbf{K}(\mathbf{x}; m_0 - \alpha_0, w_0)] \right\},$$

where $w_0 = \sqrt{m_0^2 + \lambda^2}$, $s = \sqrt{t^2 + x_2^2 + x_3^2}$ and $\mathbf{K}(\mathbf{x}; a_0, b_0) = \int_{x_1}^{\infty} \frac{e^{a_0 t - b_0 s}}{4U_\infty \pi s} dt$. We shall

call (4.6.14) the micropolar Oseen's couplet as it is derived from the Oseen approximation.

Recall that the micropolar micro-Stokeslet is given as

$$\mathbf{v} = \frac{\rho}{8\pi\mu} \nabla \times \left(\frac{1 - e^{-\lambda r}}{r} \mathbf{F} \right)$$

in (3.6.9), and the micropolar couplet is given as

$$\mathbf{u} = \frac{\rho}{8\pi\mu} \nabla \times \left(\frac{1 - e^{-\lambda r}}{r} \mathbf{Q} \right)$$

in (3.7.14). Clearly, both solutions are identical, except that the former is caused by a point force while the latter is due to a point couple. Therefore, it is no surprise that the micropolar Oseen's couplet, given by (4.6.14), has an identical expression as the micropolar micro-Oseenlet in (4.5.16), which is due to a point force.

It is clear that in the limit $U_\infty \rightarrow 0$, the micropolar Oseen's couplet (4.6.14) becomes the micropolar couplet (3.7.14) based on (4.5.20):

$$\begin{aligned} \mathbf{u} &= \frac{2\mu}{c_a + c_d} \nabla \times \left\{ \lim_{U_\infty \rightarrow 0} Q e^{\alpha_0 x_1} [K(\mathbf{x}; n_0 - \alpha_0, n_0) - K(\mathbf{x}; m_0 - \alpha_0, w_0)] \right\} \\ &= \frac{\rho}{8\pi\mu} \nabla \times \left(\frac{1 - e^{-\lambda r}}{r} \mathbf{F} \right). \end{aligned}$$

Next, we take the divergence of (4.6.8) to evaluate the divergence of $\mathbf{v}$. We find that

$$\left(\nabla^2 - 2c_1 \frac{\partial}{\partial x_1} - \lambda_0^2 \right) f = -c_2 \nabla \cdot [\mathbf{Q} \delta(\mathbf{x})],$$

where $f = \nabla \cdot \mathbf{v}$, $2c_1 = \frac{j\rho U_\infty}{c_0 + 2c_d}$, $\lambda_0^2 = \frac{4\mu_r}{c_0 + 2c_d}$ and $c_2 = \frac{\rho}{c_0 + 2c_d}$. To expedite the

solution process, we take Fourier transform of the above equation. We end up with

$$(4.6.15) \quad \hat{f} = c_2(i\xi) \cdot \left[\frac{(2\pi)^{-\frac{3}{2}} \mathbf{Q}}{\xi^2 + i2c_1\xi_1 + \lambda_0^2} \right],$$

whose inverse Fourier transform is given by (4.3.8) as

$$(4.6.16) \quad f = c_2 \nabla \cdot \left[\mathbf{Q} \frac{e^{(c_1 x_1 - w_1 r)}}{4\pi r} \right],$$

$$\text{where} \quad w_1 = \sqrt{c_1^2 + \lambda_0^2}.$$

We next proceed to determine the curl of $\mathbf{v}$. This can be done by taking the Fourier transform of (4.6.11) and then substitute (4.6.13) into it. This result in

$$(4.6.17) \quad \hat{\mathbf{a}} = \frac{a_4(\mu + \mu_r)}{2\mu_r} (i\xi) \times \left[\frac{(2\pi)^{-\frac{3}{2}} (\xi^2 + i2m_0\xi_1) \mathbf{Q}}{(\xi^2 + i2n_0\xi_1)(\xi^2 + i2m_0\xi_1 + \lambda^2)} \right].$$

To relate f and $\mathbf{a}$, we make use of the following vector identity:

$$\nabla^2 \mathbf{v} = \nabla f - \nabla \times \mathbf{a},$$

whose Fourier transform is

$$-\xi^2 \hat{\mathbf{v}} = (i\xi) \hat{f} - (i\xi) \times \hat{\mathbf{a}}.$$

Next, we substitute (4.6.15) and (4.6.17) into the above equation to yield $\hat{\mathbf{v}}$ in terms of variables (ξ_1, ξ_2, ξ_3) . We find that

$$\begin{aligned} \hat{\mathbf{v}} = c_2 (i\xi) (i\xi) \cdot & \left[\frac{-(2\pi)^{-\frac{3}{2}} \mathbf{Q}}{\xi^2 (\xi^2 + i2c_1\xi_1 + \lambda_0^2)} \right] + a_4 \left(\frac{\mu + \mu_r}{2\mu_r} \right) (i\xi) \times (i\xi) \times \left[\frac{(2\pi)^{-\frac{3}{2}} \mathbf{Q}}{\xi^2 (\xi^2 + i2n_0\xi_1)} \right. \\ & \left. - \frac{\lambda^2 (2\pi)^{-\frac{3}{2}} \mathbf{Q}}{\xi^2 (\xi^2 + i2n_0\xi_1)(\xi^2 + i2m_0\xi_1 + \lambda^2)} \right]. \end{aligned}$$

Making use of the relations derived in Section 4.3, we find the inverse Fourier transform of $\hat{\mathbf{v}}$ to be

(4.6.18)

$$\begin{aligned} \mathbf{v} = \frac{\mu}{c_a + c_d} \nabla \left\{ \nabla \cdot \left[\mathbf{Q} e^{-\beta_0 x_1} (L(\mathbf{x}; \beta_0 + c_1, w_1) - L(\mathbf{x}; \beta_0, 0)) \right] \right\} - \frac{\mu}{c_a + c_d} \nabla \times \nabla \\ \times \left\{ \mathbf{Q} e^{-\beta_0 x_1} [L(\mathbf{x}; \beta_0 + m_0, w_0) - L(\mathbf{x}; \beta_0, 0)] \right. \\ \left. + \mathbf{Q} e^{\alpha_0 x_1} [K(\mathbf{x}; m_0 - \alpha_0, w_0) - K(\mathbf{x}; n_0 - \alpha_0, n_0)] \right\}, \end{aligned}$$

$$\text{where } \beta_0 = \frac{4\mu_r\mu}{\rho U_\infty(c_a + c_d)} \quad , \quad \lambda^2 = \frac{4\mu_r\mu}{(c_a + c_d)(\mu + \mu_r)} \quad , \quad w_0 = \sqrt{m_0^2 + \lambda^2} \quad ,$$

$$\alpha_0 = \frac{4\mu^2}{\rho U_\infty(c_a + c_d)} \quad , \quad s = \sqrt{t^2 + x_2^2 + x_3^2} \quad \text{and } L \text{ is defined as}$$

$$(4.6.19) \quad L(\mathbf{x}; a_0, b_0) = \int_{-\infty}^{x_1} \frac{e^{a_0 t - b_0 s}}{4U_\infty \pi s} dt .$$

We shall call (4.6.18) the micropolar Oseen's micro-couplet. If the source point $\mathbf{s}$ located is $\mathbf{x} = \mathbf{x}_0$, then r will be defined by $r = |\bar{\mathbf{x}}|$ and $\mathbf{x}$ is replaced by $\bar{\mathbf{x}}$ in all the solutions.

We shall now show that as $U_\infty \rightarrow 0$, (4.6.18) tends to the micropolar micro-couplet in (3.7.13). Firstly, we rewrite the first term in (4.6.18) so that

$$\begin{aligned} & L(\mathbf{x}; \beta_0 + c_1, w_1) - L(\mathbf{x}; \beta_0, 0) \\ &= \int_{-\infty}^{x_1} \left[\frac{e^{(c_1 t - w_1 s)} - 1}{4U_\infty \pi s} \right] d \left(\frac{e^{\beta_0 t}}{\beta_0} \right) \\ &= \frac{1}{\beta_0} \left[e^{\beta_0 t} \frac{e^{(c_1 t - w_1 s)} - 1}{4U_\infty \pi s} \right]_{-\infty}^{x_1} - \frac{1}{\beta_0 U_\infty} \int_{-\infty}^{x_1} e^{\beta_0 t} d \left[\frac{e^{(c_1 t - w_1 s)} - 1}{4\pi s} \right]. \end{aligned}$$

Hence,

$$(4.6.20) \quad \begin{aligned} & e^{-\beta_0 x_1} [L(\mathbf{x}; \beta_0 + c_1, w_1) - L(\mathbf{x}; \beta_0, 0)] \\ &= \frac{\rho(c_a + c_d)}{16\mu_r\mu\pi} \left\{ \frac{e^{(c_1 x_1 - w_1 r)} - 1}{r} - \int_{-\infty}^{x_1} e^{\beta_0(t - x_1)} d \left[\frac{e^{(c_1 t - w_1 s)} - 1}{s} \right] \right\}. \end{aligned}$$

Let's examine the behaviour of (4.6.20) as $U_\infty \rightarrow 0$. The first term in the braces tends to

$$\begin{aligned} \lim_{U_\infty \rightarrow 0} \frac{e^{(c_1 x_1 - w_1 r)} - 1}{r} &= \frac{1}{r} \lim_{U_\infty \rightarrow 0} \left\{ e^{-r\sqrt{O(c_1^2) + \lambda_0^2}} [1 + O(c_1)] - 1 \right\} \\ &= \frac{e^{-\lambda_0 r} - 1}{r}. \end{aligned}$$

The second term in the braces in (4.6.20) tends to

$$\begin{aligned}
 (4.6.21) \quad & \lim_{U_\infty \rightarrow 0} \int_{-\infty}^{x_1} e^{\beta_0(t-x_1)} d \left[\frac{e^{(c_1 t - w_1 s)} - 1}{s} \right] \\
 &= \lim_{U_\infty \rightarrow 0} \int_{-\infty}^{x_1} e^{\beta_0(t-x_1)} d \left[\frac{e^{-s \sqrt{O(c_1^2) + \lambda_0^2}} (1 + O(c_1))}{s} \right] - \lim_{U_\infty \rightarrow 0} \int_{-\infty}^{x_1} e^{\beta_0(t-x_1)} d \left[\frac{1}{s} \right] \\
 &= 0
 \end{aligned}$$

since $x_1 > t$ in the domain of integration. Therefore, the behaviour of (4.6.19) as

$U_\infty \rightarrow 0$ is

$$(4.6.22) \quad \lim_{U_\infty \rightarrow 0} e^{-\beta_0 x_1} [L(\mathbf{x}; \beta_0 + c_1, w_1) - L(\mathbf{x}; \beta_0, 0)] = \frac{\rho(c_a + c_d)}{16\mu_r \mu \pi} \left(\frac{e^{-\lambda_0 r} - 1}{r} \right).$$

By a similar reasoning, as $U_\infty \rightarrow 0$, the second term in (4.6.18) tends to

$$(4.6.23) \quad \lim_{U_\infty \rightarrow 0} e^{-\beta_0 x_1} [L(\mathbf{x}; \beta_0 + m_0, w_0) - L(\mathbf{x}; \beta_0, 0)] = -\frac{\rho(c_a + c_d)}{16\pi \mu_r \mu} \left(\frac{1 - e^{-\lambda r}}{r} \right).$$

Referring to (4.5.20), the third integral in (4.6.18) tends to

(4.6.24)

$$\lim_{U_\infty \rightarrow 0} e^{\alpha_0 x_1} [K(\mathbf{x}; n_0 - \alpha_0, n_0) - K(\mathbf{x}; m_0 - \alpha_0, w_0)] = -\frac{\rho(c_a + c_d)}{16\mu^2 \pi} \left(\frac{1 - e^{-\lambda r}}{r} \right)$$

Finally, with reference to (4.6.22) to (4.6.24), as (4.6.18) tends to

$$\mathbf{v} = \frac{\rho(\mu + \mu_r)}{16\pi \mu_r \mu} \nabla \times \nabla \times \left(\frac{1 - e^{-\lambda r}}{r} \mathbf{Q} \right) - \frac{\rho}{16\pi \mu_r} \nabla \left[\nabla \cdot \left(\frac{1 - e^{-\lambda_0 r}}{r} \mathbf{Q} \right) \right],$$

which is the micropolar micro-couplet in (3.7.13).

Chapter 5 Concluding remarks

New fundamental solutions for an incompressible micropolar fluid have been derived and expressed in an explicit form. We considered the problem of a three-dimensional, steady, unbounded Oseen flow of a micropolar fluid due to a point force and a point couple separately. The resulting new fundamental solutions for linear motions in Oseen flows are named the micropolar Oseenlet, given by (4.5.21), and the micropolar Oseen's couplet, given by (4.6.14). In addition, fundamental solutions representations that do not exist for classical flows have emerged from our study. They are due to the existence of microrotation velocity fields in a micropolar fluid. The microrotation fundamental solution due to a point force is named the micropolar micro-Oseenlet, given by (4.5.16), and the one due to a point couple is named the micropolar Oseen's micro-couplet, given by (4.6.18). Furthermore, the newly-found fundamental solutions can also generate other new fundamental solutions by successive differentiation with respect to the source point $\mathbf{x}_0$.

The new fundamental solutions, together with the already known basic fundamental solutions, namely the micropolar Stokeslet (given by (3.6.11)), the micropolar micro-Stokeslet (given by (3.6.9)), the micropolar couplet (given by (3.7.14)) and the micropolar micro-couplet (given by (3.7.13)), will act as tools that could be useful for researchers who employ the boundary integral method or the singularity method, for example, in seeking for solutions to flow problems. The areas of

applications of these methods are believed to have also widened as the micropolar fluid theory seems to show promise in characterizing fluid behaviour on microscale and nanoscale fluid flows and of other non-Newtonian fluids.

It is to be noted that the new fundamental solutions developed in this project are based on a certain physical constraint.

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Appendix A

Basic matrix algebra

This section summarizes the basic properties and terminology in matrix algebra that is required to understand Chapter 2 of this report. The reader may consult any other textbooks on linear algebra for further elaboration if necessary.

A.1 Definition of matrix

A *matrix* is an ordered rectangular array of elements. It is said to be an “ m by n ” matrix or $m \times n$ matrix or of *order* $m \times n$ if it consists of m rows and n columns.

Such a matrix can be expressed as

$$(A.1) \quad \mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{m1} & T_{m2} & \cdots & T_{mn} \end{bmatrix}.$$

The first subscript of each element indicates the row it is positioned at, and the second subscript indicates the column. In general, (A.1) can be written as

$$\mathbf{T} = \{T_{ij}\},$$

where $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$ and T_{ij} are the *components* or *elements* of matrix

$\mathbf{T}$. For clarity and convenience, the notation $\mathbf{T}^{m \times n}$ is sometimes used to indicate that it represents an $m \times n$ matrix.

A.2 Matrix addition, subtraction and scalar multiplication

Any two matrices of the same order are said to be *conformable for addition*. If

$\mathbf{T} = \{T_{ij}\}$ and $\mathbf{S} = \{S_{ij}\}$ are two $m \times n$ matrices, then denoting $\mathbf{C} = \mathbf{T} + \mathbf{S}$, the elements of $\mathbf{C}$ given by

$$C_{ij} = T_{ij} + S_{ij}$$

and $\mathbf{C}$ is also of the order $m \times n$.

For any scalar $\lambda \in \mathfrak{R}$ multiplied to a matrix, we have

$$\lambda \mathbf{S} = \mathbf{S} \lambda = \{\lambda S_{ij}\}.$$

Hence, for a matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{T} + \lambda \mathbf{S}$, the components of $\mathbf{D}$ are then given by

$$D_{ij} = T_{ij} + \lambda S_{ij}.$$

In particular, for $\lambda = -1$, $\mathbf{D}$ becomes the subtraction of matrices $\mathbf{T}$ and $\mathbf{S}$; that is

$$D_{ij} = T_{ij} - S_{ij}.$$

As with matrix addition, $\mathbf{D}$ is also of the same order as $\mathbf{T}$ and $\mathbf{S}$.

In general, matrix addition, subtraction and scalar multiplication are commutative; that is

$$\mathbf{T} + \mathbf{S} = \mathbf{S} + \mathbf{T},$$

$$\mathbf{T} - \mathbf{S} = -\mathbf{S} + \mathbf{T},$$

$$\lambda \mathbf{S} = \mathbf{S} \lambda.$$

A.3 Matrix multiplication

Consider an $m \times n$ matrix $\mathbf{T}$, with elements T_{ij} and an $n \times p$ matrix $\mathbf{S}$, with elements S_{ij} . Then, the product $\mathbf{TS}$ is defined as

$$\mathbf{TS} = \sum_{k=1}^n T_{ik} S_{kj} \quad (1 \leq i \leq m, 1 \leq j \leq p).$$

Let $\mathbf{C} = \mathbf{TS}$ such that $\mathbf{C} = \{C_{ij}\}$. Then, the components of $\mathbf{C}$ are given by

$$C_{ij} = \sum_{k=1}^n T_{ik} S_{kj} \quad (1 \leq i \leq m, 1 \leq j \leq p).$$

Both matrices $\mathbf{T}$ and $\mathbf{S}$ are said to be *conformable for multiplication*. Otherwise, the product $\mathbf{TS}$ is not defined. The order of $\mathbf{C}$ is $m \times p$, which can be easily seen if the orders of $\mathbf{T}$ and $\mathbf{S}$ are also included as follows:

$$\mathbf{C}^{m \times p} = \mathbf{T}^{m \times n} \mathbf{S}^{n \times p}.$$

In general, matrix multiplication is not commutative; that is, $\mathbf{TS} \neq \mathbf{ST}$, with the exception of multiplication with a unit matrix (section A.8). Matrix multiplication is, however, associative and distributive. Hence, for any three matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, which are mutually conformable for multiplication, we have

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C},$$

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC},$$

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}.$$

A.4 Transpose of a matrix

Suppose that the rows and columns of $\mathbf{T}$ in (A.1) are interchanged. What results is an $n \times m$ matrix $\mathbf{S}$ where

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{21} & \dots & T_{m1} \\ T_{12} & T_{22} & \dots & T_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ T_{1n} & T_{2n} & \dots & T_{mn} \end{bmatrix} \quad \text{or} \quad \mathbf{S} = \{S_{ij}\} = \{T_{ji}\}.$$

Thus, $\mathbf{S}$ is a matrix with elements T_{ji} . Such a matrix $\mathbf{S}$ is known as the *transpose* of $\mathbf{T}$ and is denoted by $\mathbf{T}^T$. In other words,

$$\mathbf{S} = \mathbf{T}^T \quad \text{or} \quad S_{ij} = T_{ji} = T_{ij}^T.$$

For instance, $\begin{pmatrix} 2 & 0 & -1 \\ 1 & 4 & 0 \end{pmatrix}$ is the transpose of $\begin{pmatrix} 2 & 1 \\ 0 & 4 \\ -1 & 0 \end{pmatrix}$. Further, the following

properties govern the operations involving the transpose of matrices:

$$(\mathbf{T} + \mathbf{S})^T = \mathbf{T}^T + \mathbf{S}^T,$$

$$(\mathbf{TS})^T = \mathbf{S}^T \mathbf{T}^T,$$

$$(\lambda \mathbf{S})^T = \lambda \mathbf{S}^T,$$

$$(\mathbf{S}^T)^T = \mathbf{S}.$$

A.5 Row matrix and column matrix

An $m \times 1$ matrix is called a *column matrix* or *column vector* as it only consists of one column of m elements. Similarly, an $1 \times m$ matrix is called a *row matrix* or *row vector* since it is made up of only one row of m elements. Consider the following row matrix

$$\mathbf{t} = (t_{11} \quad t_{12} \quad \dots \quad t_{1m}),$$

whereby the transpose of it gives the following column matrix:

$$\mathbf{t}^T = \begin{pmatrix} t_{11} \\ t_{12} \\ \vdots \\ t_{1m} \end{pmatrix}.$$

Conversely, the transpose of the column matrix is the row matrix $\mathbf{t}$.

A.6 Square matrix

A matrix with the same number of rows and columns is called a *square matrix*. Therefore, an $m \times m$ matrix such as

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{21} & \cdots & T_{1m} \\ T_{12} & T_{22} & \cdots & T_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ T_{m1} & T_{m2} & \cdots & T_{nm} \end{bmatrix}$$

is a typical square matrix. The elements located along the diagonal of the matrix are called the *diagonal* elements. The rest of the elements are referred to as *non-diagonal*.

A.7 Symmetrical matrix and anti-symmetrical matrix

Any square matrix $\mathbf{T}$ that has the property

$$\mathbf{T} = \mathbf{T}^T \quad \text{or} \quad T_{ij} = T_{ji}$$

is said to be *symmetric*. The matrix

$$\begin{bmatrix} T_{11} & 1 & 2 \\ 1 & T_{22} & -4 \\ 2 & -4 & T_{33} \end{bmatrix}$$

is one such example. On the other hand, an *anti-symmetric* or *skew-symmetric* matrix has the following property:

$$\mathbf{T} = -\mathbf{T}^T \quad \text{or} \quad T_{ij} = -T_{ji}.$$

For example, the matrix

$$\begin{bmatrix} T_{11} & 1 & 2 \\ -1 & T_{22} & -4 \\ 2 & 4 & T_{33} \end{bmatrix}$$

is anti-symmetric.

A.8 Diagonal matrix, unit matrix and zero matrix

A square matrix with all elements as zero except the diagonal elements is called a *diagonal matrix*. For instance, a 3x3 diagonal matrix always has the form

$$\begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix},$$

where T_{11} , T_{22} and T_{33} are non-zero. In particular, if $T_{11} = T_{22} = T_{33} = 1$, then the diagonal matrix is called the *unit matrix* or *identity matrix*, denoted universally by $\mathbf{I}$. Any matrix $\mathbf{T}$ multiplied to it remains unchanged, as long as $\mathbf{T}$ and $\mathbf{I}$ are conformable for multiplication. Formally,

$$\mathbf{IT} = \mathbf{TI} = \mathbf{T}.$$

Finally, a matrix with all elements as zero is called the *zero matrix*.

Appendix B

Einstein's summation convention

Consider the following sum

$$s = a_1b_1 + a_2b_2 + \cdots + a_nb_n = \sum_{i=1}^n a_ib_i .$$

Each term in the series s is represented by a_ib_i , and the index i appears twice. The summation sign can then be omitted if the *Einstein's summation convention* is adopted. Under this convention, the summation sign can be omitted as long as there is an index in the expression of the general term that occurs only twice. Thus, s can be written as

$$s = a_ib_i ,$$

whereas the summation sign for the sum $\sum_{i=1}^n a_ib_ic_i$ must be retained as the index i appears thrice.

As a consequence of the Einstein's summation convention, the sum s can be rewritten in the following forms:

$$s = a_ib_i = a_jb_j = a_mb_m .$$

The symbol that represents the repeating index is immaterial. Hence, the repeating index or the index that is being summed over is called the *dummy index*. Any other indices that are not summed over in the expression are called *free indices*. For example, for the expression

$$a_i = \sum_{j=1}^n U_{ij} b_j = U_{ij} b_j,$$

the free index is i while the dummy index is j . The free index can also be represented by some other symbol, say k . However, the free index appearing in every term of the equation must be the same; that is

$$a_k = U_{kj} b_j.$$

It should be noted that no index should appear more than twice under the summation convention.

By now, it should be clear that the summation convention allows expression involving the summation sign to be written more compactly. In this report, the summation convention is adopted and all indices are assumed to run from one to three, unless stated otherwise. For more exposition on the summation convention, one might want to refer to Bourne & Kendall (1992) or Matthews (1998).

Appendix C

Scalars, vectors and Cartesian tensors

"*Tensors*" is the generic name for mathematical entities that represent certain physical quantities in the physical world. In general, a tensor of order n , denoted by $\mathbf{T}$, is a quantity defined by 3^n components, which may be written as $T_{rst\dots}$ with n indices, provided that under rotation to a new coordinate system, they transform according to the law

where $T'_{ijk\dots}$ are the components of $\mathbf{T}$ as viewed from the new coordinate system.

Note that the tensor $\mathbf{T}$ does not undergo any transformation. Rather, it is its components that are changed as they are viewed from different coordinate systems. In this report, both matrices and tensors are denoted by upper case Latin letters in bold print. This is because tensors can be represented mathematically by matrices.

Based on definition of a tensor, a tensor can be of high order. In this report, Cartesian tensors or tensors associated with different Cartesian frames of reference, up to the order of four will be encountered. Tensors up to the order of three will be elaborated in this section as they are encountered more frequently.

C.1 Scalars – zeroth-order tensors

Scalars are known as the zeroth-order tensors. They represent quantities that can be characterized entirely by their magnitudes. Relevant examples are temperature and density. A zeroth-order tensor T consists of only one component such that

$$T = T'.$$

In general, scalars are not constants and can be functions of position and time. Though (C.2) asserts that the numerical value of T is independent of the choice of coordinate system, the functional dependence of $\mathbf{T}$ on position under different coordinate systems are different in general. In this report, scalars are mostly symbolized by Greek letters and lower case Latin letters such as ρ, μ, c_a , etc.. .

C.2 Vectors – first-order tensors

Figure C.1a Cartesian coordinate system

Vectors, or first-order tensors, are used to represent quantities that possess both magnitudes and directions. In this report, vectors are symbolized by lower-case Latin letters in bold-faced italic print. Consider a Cartesian coordinate system 123 in the Euclidean space $\mathfrak{R}^3$ with an orthonormal basis of unit vectors $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ (Figure C.1a).

Figure C.1b Position of P represented by vector $\mathbf{x}$

Consider an arbitrary point P in the $\Re^3$ space (Figure C.1b). Its position can be represented by the set of coordinates (x_1, x_2, x_3) . By virtue of the law of parallelogram addition, the position of P is given by

$$(C.3) \quad \mathbf{x} = x_1 \hat{\mathbf{e}}_1 + x_2 \hat{\mathbf{e}}_2 + x_3 \hat{\mathbf{e}}_3 = x_i \hat{\mathbf{e}}_i .$$

Suppose that there are two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} = a_i \hat{\mathbf{e}}_i$ and $\mathbf{b} = b_i \hat{\mathbf{e}}_i$. Then, the dot product of $\mathbf{a}$ and $\mathbf{b}$ is defined as

$$(C.4) \quad \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i ,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, both pointing outwards. Taking the dot product of (C.3) with $\hat{\mathbf{e}}_i$, the following relations are obtained:

$$x_1 = \hat{\mathbf{e}}_1 \cdot \mathbf{x} , \quad x_2 = \hat{\mathbf{e}}_2 \cdot \mathbf{x} , \quad x_3 = \hat{\mathbf{e}}_3 \cdot \mathbf{x} .$$

Thus, in general, the component of $\mathbf{x}$ in $\hat{\mathbf{e}}_i$ -direction is given by

$$x_i = \hat{\mathbf{e}}_i \cdot \mathbf{x} .$$

Further, based on (C.4), the magnitude of vector $\mathbf{x}$, denoted by $|\mathbf{x}|$, is given by

$$|\mathbf{x}| = \sqrt{x_i x_i} .$$

C.3 The Kronecker delta

When $\mathbf{a}$ and $\mathbf{b}$ in (C.4) are replaced with $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}_j$, the result of the dot product gives the *Kronecker delta*, symbolized by δ_{ij} , defined as

$$\delta_{ij} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j. \end{cases}$$

The Kronecker delta possesses a substitution property such that if it is multiplied to any term in an expression with a same index, that index is replaced by the dissimilar

index of the Kronecker delta. For instance, $\delta_{ij}x_j = x_i$ while $\delta_{ij}x_i = x_j$. For more exposition on the Kronecker delta, see Matthews (1998).

C.4 The cross product and the alternating symbol

The *cross* product of vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is a unit vector normal to both $\mathbf{a}$ and $\mathbf{b}$, pointing in a direction dictated by the right-hand rule. All other notations are as defined in Section C.2.

For any right-handed coordinate system, the vector $\mathbf{a} \times \mathbf{b}$ can be given by

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\hat{\mathbf{e}}_1 + (a_3b_1 - a_1b_3)\hat{\mathbf{e}}_2 + (a_1b_2 - a_2b_1)\hat{\mathbf{e}}_3.$$

The above expression can be written more compactly by introducing the permutation symbol ϵ_{ijk} , defined as

$$(C.5) \quad \begin{cases} \epsilon_{ijk} = 1 & \text{if } ijk \text{ is an even permutation; i.e. } \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \\ \epsilon_{ijk} = -1 & \text{if } ijk \text{ is an odd permutation; i.e. } \epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1, \\ \epsilon_{ijk} = 0 & \text{if any of } i, j, k \text{ are equal; i.e. } \epsilon_{112} = \epsilon_{223} = \dots = 0. \end{cases}$$

By (C.5) and the right-hand rule, the cross product of $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}_j$ can be written as

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \epsilon_{ijk}\hat{\mathbf{e}}_k.$$

Thus, the cross product of $\mathbf{a}$ and $\mathbf{b}$ is

$$(C.6) \quad \mathbf{a} \times \mathbf{b} = a_i b_j \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \epsilon_{ijk} a_i b_j \hat{\mathbf{e}}_k.$$

Using the even permutation jki instead of ijk , (C.6) can be rewritten as

$$\mathbf{a} \times \mathbf{b} = \epsilon_{jki} a_j b_k \hat{\mathbf{e}}_i = \epsilon_{ijk} a_j b_k \hat{\mathbf{e}}_i.$$

The following identity relates the Kronecker delta and permutation symbol:

$$\epsilon_{jkq} \epsilon_{mkq} = 2\delta_{mj}.$$

C.5 The orthogonal matrix

Suppose that a coordinate system 123 undergoes a rotation about its origin to another coordinate system 1'2'3' (Figure C.2).

Figure C.2 New coordinate system 1'2'3'

Under the new coordinate system 1'2'3', we let $\mathbf{x} = x'_i \hat{\mathbf{e}}'_i$. In other words,

$$(C.7) \quad \mathbf{x} = x'_i \hat{\mathbf{e}}'_i = x_i \hat{\mathbf{e}}_i.$$

Taking dot product of (C.7) with $\hat{\mathbf{e}}_j$ yields

$$(C.8) \quad x_j = a_{ji} x'_i,$$

where $a_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$. Let us define a matrix $\mathbf{A}$ such that $\mathbf{A} = \{a_{ij}\}$. Next, we take dot product of (C.7) with $\hat{\mathbf{e}}'_j$. This gives

$$x'_j = a_{ji} x_i.$$

Both (C.8) and (C.9) represent the transformation laws of a first-order tensor.

Substituting (C.8) into (C.9) leads to

$$x'_j = a_{ji} (a_{ki} x'_k).$$

Since the above equation is an identity, therefore,

$$(C.10) \quad a_{ji} a_{ik} = \delta_{jk} \quad \text{or} \quad \mathbf{A} \mathbf{A}^T = \mathbf{I}.$$

Any square matrix that satisfies (C.10) is known as an *orthogonal* matrix.

C.6 The second-order tensor

A second-order tensor satisfies the following transformation laws:

$$T_{ij} = a_{qi} a_{mj} T'_{qm} \quad \text{or} \quad \mathbf{T} = \mathbf{A}^T \mathbf{T}' \mathbf{A},$$

$$T'_{ij} = a_{iq} a_{jm} T_{qm} \quad \text{or} \quad \mathbf{T}' = \mathbf{A} \mathbf{T} \mathbf{A}^T.$$

As there are two indices, a second-order tensor has 3^2 or 9 components. Thus, it is represented by a 3x3 matrix. Sometimes, we find it useful to write it as the sum of a symmetric part and an anti-symmetric part:

$$T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}) \quad \text{or} \quad \mathbf{T} = \frac{1}{2}(\mathbf{T} + \mathbf{T}^T) + \frac{1}{2}(\mathbf{T} - \mathbf{T}^T).$$

C.7 Tensor fields, tensor calculus and vector identities

A *tensor field* assigns to every location $\mathbf{x}$, at every instant of time t , a tensor $T_{ij\dots k}(\mathbf{x}, t)$, for which $\mathbf{x}$ ranges over a finite region of space, like in Figure C.1b, and t varies over some interval of time. The field is continuous and hence differentiable if the components $T_{ij\dots k}(\mathbf{x}, t)$ are continuous functions of $\mathbf{x}$ and t .

Partial differentiation of a tensor field with respect to time, t , follows the usual rules of calculus and is symbolized by the operator $\frac{\partial}{\partial t}$. However, partial differentiation with respect to the coordinate x_i is usually abbreviated through the introduction of the *subscript comma*. Consider the del operator $\nabla = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}$ operating on a scalar function f and vectors $\mathbf{u}$ and $\mathbf{v}$. Then, it is customary to write the grad of f , ∇f , as $f_{,i}$, the vector gradient, $\nabla \mathbf{u}$, as $u_{j,i}$, the divergence of $\mathbf{u}$, $\nabla \cdot \mathbf{u}$, as $u_{i,i}$ and the curl of $\mathbf{u}$, $\nabla \times \mathbf{u}$, as $\varepsilon_{ijk} u_{k,j}$, where ε_{ijk} is the permutation symbol defined in (C.4).

Finally, we give the following vector identities for the del operator:

$$\nabla \cdot (\mathbf{u}f) = f\nabla \cdot \mathbf{u} + (\mathbf{u} \cdot \nabla)f,$$

$$\nabla \times (\mathbf{u}f) = f\nabla \times \mathbf{u} + \nabla f \times \mathbf{u},$$

$$\nabla \cdot (\nabla \times \mathbf{u}) = 0,$$

$$\nabla \times (\nabla f) = 0,$$

$$\nabla(fg) = f\nabla g + g\nabla f,$$

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}),$$

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v}),$$

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{u}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{u}),$$

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v},$$

$$(\nabla \times \mathbf{v}) \times \mathbf{v} = (\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v}).$$