Transceiver optimization for multi-user systems : rate improvement and degrees of freedom characterization

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**TRANSCEIVER OPTIMIZATION FOR MULTI-USER
SYSTEMS: RATE IMPROVEMENT AND DEGREES OF
FREEDOM CHARACTERIZATION**

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Abstract

This thesis is devoted to optimizing the transceiver strategies in multi-user wireless systems for maximizing the users' achievable rates and the degrees of freedom (DoF).

Multi-user wireless systems can be modeled as interference channels. Under the assumption that interference is treated as noise at each receiver, the prevalent transmission strategy in the literature is *proper* or circularly *symmetric* complex Gaussian (CSCG) signaling, i.e., the transmitted signal vector from each user is CSCG distributed. In the first part of this thesis, we study the achievable rates of Gaussian interference channels when the more general *improper* or circularly *asymmetric* complex Gaussian signaling is applied. For the Gaussian multiple-input multiple-output interference channel (MIMO-IC) with interference treated as noise, we show that the user's achievable rate with improper Gaussian signaling can be expressed as a summation of the rate achievable by the conventional proper Gaussian signaling in terms of the users' transmit *covariance* matrices, and an additional term, which is a function of both the users' transmit covariance and *pseudo-covariance* matrices. The additional degree of freedom in the pseudo-covariance matrix, which is conventionally set to be zero for the case of proper Gaussian signaling, provides an opportunity to further improve the achievable rates of Gaussian MIMO-ICs by employing improper Gaussian signaling. To this end, we propose *widely linear precoding*, which efficiently maps proper information-bearing signals to improper transmitted signals at each transmitter for any given pair of covariance and pseudo-covariance matrices. For the two-user single-input single-output interference channel (SISO-IC) and K -user multiple-input single-output interference channel (MISO-IC), efficient algorithms are proposed to optimize the transmit covariance and pseudo-covariance matrices. By utilizing the separable structure of the achievable rate expression, we demonstrate that improper complex Gaussian signaling can provide strict rate gains over the conventional proper Gaussian signaling.

In the second part of this thesis, we study the Gaussian interference channel from the DoF perspective. While most of such studies in the literature have assumed full-rank channel matrices; in certain practical scenarios, the channel matrices may be rank-deficient due to poor scattering and the presence of very few signal paths. We provide an exact DoF characterization for the 3-user $M_T \times M_R$

MIMO-IC with *rank-deficient* channel matrices, where each transmitter is equipped with M_T antennas and each receiver with M_R antennas, and the interfering channel matrices from each transmitter to the other two receivers are of ranks D_1 and D_2 , respectively. It is found that when the interfering links are weak in terms of the channel ranks, i.e., $D_1 + D_2 \leq \min(M_T, M_R)$, zero forcing is sufficient to achieve the optimal DoF. On the other hand, when $D_1 + D_2 > \min(M_T, M_R)$, a combination of zero forcing and interference alignment is required for DoF optimality. We show that the DoF characterization obtained in this thesis unifies several existing results in the literature.

Besides interference coordination under the interference channel model, a more aggressive interference mitigation scheme is known as network MIMO, or multi-cell cooperative processing (MCP) in cellular networks. In the last part of this thesis, we propose a modified block diagonalization (BD) scheme for MCP-enabled network, so that the achievable sum-rate in the low-to-medium signal-to-noise ratio (SNR) regime is improved as compared to the original BD scheme. Such a rate improvement is possible since the hard constraint of zero inter-user interference imposed by BD is relaxed; instead, interference leakage is allowed and carefully controlled via a rate-constrained power minimization problem. As such, the effective noise power can be suppressed, and the power enhancement problem associated with the BD scheme is mitigated.

Abbreviations

BC	Broadcast Channel.
BS	Base Station.
BD	Block Diagonalization.
CoMP	Coordinated MultiPoint.
CSCG	Circularly Symmetric Complex Gaussian.
CSI	Channel State Information.
DPC	Dirty Paper Coding.
EVD	Eigen Value Decomposition.
GSM	Global System for Mobile Communications.
IA	Interference Alignment.
i.i.d.	independent and identically distributed.
IC	Interference Channel.
KKT	Karush-Kuhn-Tucker.
LHS	Left Hand Side.
LTE	Long Term Evolution.
LP	Linear Programming.
MAC	Multiple Access Channel.
MCP	Multi-cell Cooperative Processing.
MIMO	Multiple-Input Multiple-Output.
MISO	Multiple-Input Single-Output.
MMSE	Minimum Mean Square Error.
MS	Mobile Station.
RV	Random Vector.
QCQP	Quadratically Constrained Quadratic Program.
RHS	Right Hand Side.
SDP	Semidefinite Programing.
SDR	Semidefinite Relaxation.
SIMO	Single-Input Multiple-Output.

SISO	Single-Input Single-Output.
SNR	Signal-to-Noise Ratio.
SINR	Signal-to-Interference-plus-Noise Ratio.
SOCP	Second-Order Cone Programming.
SUD	Single-User Detection.
SVD	Singular Value Decomposition.
TDMA	Time Division Multiple Access.
WSRMax	Weighted-Sum-Rate Maximization.
ZF	Zero Forcing.

Symbols

Throughout this thesis, scalars are denoted by italic letters. Boldface lower- and upper-case letters denote vectors and matrices, respectively.

$ x $	Magnitude of scalar x .
$(x)^+$	$\max(x, 0)$.
$\lceil x \rceil$	The smallest integer not less than x .
$\lfloor x \rfloor$	The largest integer not greater than x .
$\text{o}(x)$	Any function $f(x)$ such that $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = 0$.
$\log(\cdot)$	Logarithm function with base 2.
$\text{dom} f$	Domain of function f .
∇f	Gradient of function f .
$\mathbf{e}_k$	Unit vector with all the elements being zero except the k th element which is equal to one.
$[\mathbf{v}]_k$	The k th element of vector $\mathbf{v}$.
$\mathbf{x} \preceq \mathbf{y}$	The elements in vector $\mathbf{x}$ are no greater than those in $\mathbf{y}$.
$\mathbf{x} \succeq \mathbf{y}$	The elements in vector $\mathbf{x}$ are no smaller than those in $\mathbf{y}$.
$ \mathbf{d} $	Dimension of vector $\mathbf{d}$.
$\Re\{\cdot\}, \Im\{\cdot\}$	Real and imaginary parts of complex-valued numbers/matrices, respectively.
$\mathbf{I}_M$	M -dimensional identity matrix.
$\mathbf{0}$	All-zero matrix.
$\mathbf{A}^T, \mathbf{A}^*, \mathbf{A}^H$	Transpose, complex conjugate, and conjugate transpose of matrix $\mathbf{A}$, respectively.
$\text{rank}(\mathbf{A})$	Rank of matrix $\mathbf{A}$.
$\text{Tr}(\mathbf{S}), \mathbf{S} $	Trace and determinant of the square matrix $\mathbf{S}$, respectively.
$\mathbf{S}^{-1}, \mathbf{S}^\dagger$	The inverse and the Moore-Penrose pseudo-inverse of matrix $\mathbf{S}$, respectively.

$\mathbf{S}^{1/2}$	Hermitian square root of the Hermitian matrix $\mathbf{S}$ such that $\mathbf{S} = \mathbf{S}^{1/2}(\mathbf{S}^{1/2})^H$.
$\mathbf{S} \succeq \mathbf{0}$	$\mathbf{S}$ is positive semidefinite.
$\mathbf{S} \succ \mathbf{0}$	$\mathbf{S}$ is positive definite.
$\cap$	Intersection of two sets.
$[\cdot]_{\mathcal{C}}$	The projection onto the set $\mathcal{C}$.
$\sim$	Distributed according to.
$\mathbb{C}^{M \times N}, \mathbb{R}^{M \times N}$	Space of $M \times N$ complex-valued and real-valued matrices, respectively.
$\mathbb{S}^K, \mathbb{H}^K$	$K \times K$ real-valued symmetric and complex-valued Hermitian matrices, respectively.
$\text{vec}(\mathbf{A})$	A vector obtained by stacking all the columns of $\mathbf{A}$.
$\text{diag}\{\mathbf{x}\}$	A diagonal matrix with the elements in the main diagonal given by $\mathbf{x}$.
$\text{blkdiag}(\mathbf{A}, \mathbf{B})$	A block-diagonal matrix with the diagonal blocks given by $\mathbf{A}$ and $\mathbf{B}$.
$\mathcal{N}(\boldsymbol{\mu}, \mathbf{C})$	Real-valued Gaussian RV with mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{C}$.
$\mathcal{CN}(\mathbf{x}, \boldsymbol{\Sigma})$	CSCG RV with mean $\mathbf{x}$ and covariance matrix $\boldsymbol{\Sigma}$.
$\mathbb{E}(\cdot)$	Expectation of RVs.
$h(\mathbf{x})$	Entropy of RV $\mathbf{x}$.
$I(\mathbf{x}; \mathbf{y})$	Mutual information between two RVs $\mathbf{x}$ and $\mathbf{y}$.

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Chapter 1

Introduction

1.1 Motivation and Objectives

Interference is a key performance limiting factor in multi-user wireless communication systems. For conventional cellular networks such as GSM (Global System for Mobile Communications), while intra-cell interference can be completely avoided by ensuring orthogonality between signals using different time slots or frequency bands; the performance of cell-edge users is severely affected by inter-cell interference. Although the traditional approach of frequency reuse helps mitigating inter-cell interference, the overall spectrum efficiency is degraded drastically since each cell only utilizes a fraction of the available spectrum. Therefore, modern communication standards, such as 3GPP Long Term Evolution (LTE), advocate a universal frequency reuse, which makes inter-user interference, if not properly managed, a more severe problem. Therefore, effective interference management techniques are crucial to achieve high throughput in contemporary and future wireless communication systems.

1.1.1 Interference Coordination

In cellular networks where the base stations are connected via low-latency high-capacity backhaul links, base station cooperation is known to be an effective approach to combat against interference [1, 2]. There are several possible degrees of cooperation, which provide a tradeoff between performance gains and the overhead associated with the backhaul signaling and signal processing complexity. Probably

FIGURE 1.1: Illustration of interference coordination. The data symbol for each mobile user is transmitted from its corresponding base station only, and hence the system can be modeled as an interference channel.

the simplest form of base station cooperation is *interference coordination* [3] (see Figure 1.1), where the channel state information (CSI) is shared between the cooperating base stations via the backhaul links so that each base station has the CSI of both its serving users and the users in other cells. The availability of the global CSI enables interference-aware transmission/reception to be employed at the base station, so as to achieve a good balance between maximizing the received power to/from its serving users and minimizing the interference to/from other users [4]. This can be accomplished by strategies like coordinated scheduling [5], power allocation [6, 7], or beamforming [8–12] when multiple antennas are available. A useful mathematical model for multi-user systems with interference coordination is known as interfering broadcast channel [13], or *interference channel* [14–17] for the scenario when each base station serves one user only at each time/frequency resource block.

1.1.2 Multi-Cell Cooperative Processing

Apart from interference coordination, an even more aggressive form of base station cooperation is known as *multi-cell cooperative processing* (MCP) [1, 18–21], where the data symbols for each user are jointly transmitted/received by the cooperating base stations (see Figure 1.2). MCP requires the exchange of not only CSI as

FIGURE 1.2: Illustration of multi-cell cooperative processing. The data symbol for each mobile user is transmitted from all cooperating base stations, and hence the system can be modeled as a broadcast channel with per-group-of-antennas power constraint.

for interference coordination, but also the data symbols of the respective users, and hence it imposes a more stringent requirement on the backhaul links. In the ideal case with infinite-capacity and delay-free backhaul links, the downlink MCP-enabled network can be modeled as a broadcast channel [22–24], but with a per-group-of-antennas (or per-base station) power constraint due to the geographically distributed antenna elements. In contrast to interference coordination which essentially attempts to avoid the interfering links, MCP in fact tries to utilize the interfering links to carry useful symbols, and hence it is especially effective when the power gains of the interfering links are of the same order of magnitude as that of the direct links. MCP has also been studied under the more practical scenarios, such as with limited backhaul links [25–29] or with imperfect CSI sharing [30]. There are several alternative names for MCP, such as network MIMO [31, 32] and coordinated multi-point (CoMP) [33, 34].

In [35], a general base station cooperation model called *dynamic cooperation clusters* is introduced, which includes interference coordination and MCP as two special cases. To convey the most essential ideas, in this thesis, we will focus on interference coordination under the interference channel framework, as well as the MCP-enabled network with ideal backhaul links.

1.1.3 Rate Improvement and DoF Characterization

Although the study of the interference channel has a long history [14–17], it is far from being fully understood, both for the information-theoretic capacity limit and for the maximum rate achievable with the simple single-user detection (SUD) at the receivers [36]. For the former, even the optimal degrees of freedom (DoF) value [37, 38], a parameter that provides the first-order capacity approximation at the asymptotically high signal-to-noise ratio (SNR), is elusive for the general cases, i.e., K -user Gaussian MIMO interference channel (MIMO-IC) with arbitrary number of transmit and receive antennas, and with arbitrary ranks for the channel matrices.

In practical systems where the SNRs are always finite, a more appropriate performance measure than DoF is the users' achievable sum-rate or the rate region, the optimization of which has been pursued by many researchers (see Section 3.1 for related work). Unfortunately, due to the interference coupling between different users, the achievable rate region is a non-convex set in general, and the sum-rate maximization is a non-convex problem. It has been shown that finding the globally optimal transmission strategy for sum-rate maximization is NP-hard in general [39, 40]. While many researchers have targeted efficient sub-optimal solutions, global optimal algorithms do exist but at the expense of high complexity. For the achievable rate region, various characterizations have been obtained.

As for the MCP-enabled networks, the simple linear precoding schemes have received extensive research attentions in the past few years [41–46], due to their ability to achieve a good balance between complexity and performance. One linear precoding scheme of particular interest is block diagonalization (BD) [47–52], where inter-user interference is completely eliminated by restricting the precoding matrix for each mobile user to be orthogonal to the channels associated with all other users. The main advantages of BD are its simplicity and good performance at high SNR. However, its performance degrades in the low-to-medium SNR regime due to the transmit power boost problem, which is a direct consequence of the hard constraint for zero inter-user interference.

The objective of this thesis is to further improve the users' rate by optimizing the transceiver strategies for both interference channel and MCP-enabled network. For the former, a new ingredient called *improper* Gaussian signaling is utilized; and for the latter, the rate improvement is achieved by a modified BD scheme with

properly controlled interference leakage. Besides, the DoF characterization is also pursued for rank-deficient MIMO-IC.

1.2 Major Contribution of the Thesis

The main research contributions of this thesis are summarized as follows.

1.2.1 Improper Gaussian Signaling

Almost all existing research work on rate optimization for Gaussian interference channels has assumed, either consciously or not, *proper* or CSCG signaling, i.e., the transmitted signal vector for each user is CSCG distributed. As one of the main contributions of this thesis, we show that for Gaussian MIMO-IC with interference treated as noise, the users' achievable rates can be improved by applying the more general *improper* or circularly *asymmetric* complex Gaussian signaling. More specifically, the following results have been obtained:

- We derive a more general achievable rate expression for the K -user MIMO-IC, which is applicable when improper Gaussian signaling is employed at the transmitters. Our result shows that the user's achievable rate with improper Gaussian signaling can be expressed as a summation of the rate achievable by the conventional proper Gaussian signaling in terms of the transmit covariance matrices, and an additional term, which is a function of both the users' transmit covariance and *pseudo-covariance* matrices. This new result implies that the use of improper Gaussian signaling for MIMO-ICs is able to improve the achievable rate over the conventional proper Gaussian signaling, by further optimizing their pseudo-covariance matrices.
- For any given pair of signal covariance and pseudo-covariance matrices at each transmitter, we consider the problem of generating improper transmitted signals from proper information-bearing signals. An efficient method for this implementation is proposed, named as *widely linear precoding*.
- By adopting the rate-profile method, we formulate the optimization problem for the two-user SISO-IC to characterize the Pareto boundary of the achievable rate region with improper Gaussian signaling. By applying the

celebrated semidefinite relaxation (SDR) technique [53, 54], a joint covariance and pseudo-covariance optimization algorithm is proposed, which achieves near-optimal rate-pairs. Furthermore, by utilizing the separable structure of the achievable rate expression with improper Gaussian signaling, a separate covariance and pseudo-covariance optimization algorithm is also proposed, which guarantees the rate improvement over the conventional proper Gaussian signaling.

- For the more general K -user MISO-IC, by adopting a similar separate covariance and pseudo-covariance optimization approach as for the two-user SISO case, we develop an efficient sub-optimal solution for the transmit optimization problem with improper Gaussian signaling. Furthermore, we establish the optimality of rank-1 pseudo-covariance matrices, given rank-1 transmit covariance matrices have been applied. This thus greatly simplifies the pseudo-covariance optimization problem.

The main results related to improper Gaussian signaling have been presented in two journal and three conference papers listed in the following:

- Y. Zeng, C. M. Yetis, E. Gunawan, Y. L. Guan, and R. Zhang, “Transmit optimization with improper Gaussian signaling for interference channels,” *IEEE Trans. Signal Process.*, vol. 61, no. 11, pp. 2899-2913, Jun. 2013.
- Y. Zeng, R. Zhang, E. Gunawan, and Y. L. Guan, “Optimized transmission with improper Gaussian signaling in the K-user MISO interference channel,” *IEEE Trans. Wireless Commun.*, vol. 12, no. 12, pp. 6303-6313, Dec. 2013.
- Y. Zeng, R. Zhang, E. Gunawan, and Y. L. Guan, “MISO interference channel with improper Gaussian signaling,” in *IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP'13)*, Vancouver, Canada, May 26-31, 2013.
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- Y. Zeng, C. M. Yetis, E. Gunawan, Y. L. Guan, and R. Zhang, “Improving achievable rate for the two-user SISO interference channel with improper

Gaussian signaling,” in *Proc. IEEE Asilomar Conference on Signals, Systems, and Computers*, (**Invited**), Pacific Grove, CA, Nov. 4-7, 2012.

1.2.2 DoF of Rank-Deficient Interference Channel

Characterizing the optimal DoF value, or the maximum number of interference-free data symbols distinguishable by the receivers at the asymptotically high SNR, is one important step for understanding the ultimate capacity limit of a communication network. Most existing work in this area has assumed full-rank channel matrices, an assumption which is valid in rich scattering communication environment. However, in certain practical scenarios, the communication channel matrices may be rank-deficient due to poor scattering and the presence of very few signal paths. Recently, some researchers have started investigating the DoF for rank-deficient MIMO-IC (see Section 5.1.1 for related work). This thesis makes progress towards this direction by providing the complete DoF characterization for the 3-user $M_T \times M_R$ MIMO-IC with rank-deficient channel matrices, where each transmitter is equipped with M_T antennas and each receiver with M_R antennas, and the interfering channel matrices from each transmitter to the other two receivers are of ranks D_1 and D_2 , respectively.

One important intermediate step for both the converse and achievability arguments is to convert the fully-connected rank-deficient channel into an equivalent partially-connected full-rank MIMO-IC by invertible linear transformations. As such, existing techniques developed for the full-rank MIMO-IC can be incorporated to aid for the development for the rank-deficient MIMO-IC. Our result shows that when the interfering links are weak in terms of the channel ranks, i.e., $D_1 + D_2 \leq \min(M_T, M_R)$, zero forcing is sufficient to achieve the optimal DoF. On the other hand, when $D_1 + D_2 > \min(M_T, M_R)$, a combination of zero forcing and interference alignment is required for DoF optimality. The DoF characterization obtained in this thesis unifies and extends several existing results in the literature.

The main results for this part have been included in one journal paper and one conference paper given in the following:

- Y. Zeng, X. Xu, Y. L. Guan, E. Gunawan, and C. Wang, “Degrees of freedom of 3-user rank-deficient MIMO interference channel,” to appear in *IEEE Trans. Wireless Commun.*, available on-line at <http://arxiv.org/abs/1211.4198>.

- X. Xu, Y. Zeng, and Y. L. Guan, “On the degrees of freedom of 3-user rank-deficient MIMO interference channel,” in *Proc. IEEE 2013 International Conference on Communications (ICC’13)*, Budapest, Hungary, Jun. 9-13, 2013.

1.2.3 Modified Block Diagonalization

Lastly, to improve the achievable sum-rate of the BD scheme in the low-to-medium SNR regime while preserving its good performance at high SNR, we propose a modified BD scheme for MCP-enabled network with an effective-SNR enhancement technique. In contrast to the original BD scheme where zero-interference constraint is imposed, in the modified scheme, inter-user interference is allowed and properly controlled. This is achieved by firstly solving a rate-constrained power minimization problem, and then based on the optimization result, the transmit covariance matrices are scaled accordingly. As such, the effective noise power is suppressed and the effective SNR is enhanced, this thus mitigates the power enhancement problem associated with the original BD scheme and a rate improvement is achieved.

The main result has been published in the following journal paper:

- Y. Zeng, E. Gunawan, and Y. L. Guan, “Modified block diagonalization precoding in multi-cell cooperative networks,” *IEEE Trans. Veh. Technol.*, vol. 61, no. 8, pp. 3819-3824, Oct. 2012.

1.3 Organization of the Thesis

This thesis is organized as follows.

Chapter 1, the present chapter, gives the motivation, objectives, main research contributions, and the organization of this thesis.

Chapter 2 presents some mathematical preliminaries, i.e., the theory of complex random vectors and convex optimization, on which many results of this thesis are based.

Chapter 3 develops the general framework of improper complex Gaussian signaling for interference channels. A novel achievable rate expression is derived, which is a function of both the covariance and pseudo-covariance matrices of the

transmitted signals. Widely linear precoding is proposed to generate improper Gaussian signals with the optimized covariance and pseudo-covariance matrices. Furthermore, transmit optimization algorithms are proposed for the two-user SISO-IC.

Chapter 4 extends the results in Chapter 3, in which the transmit optimization problem for improper Gaussian signaling is considered for the more general K -user MISO-IC. By exploiting the separable structure of the achievable rate expression, a separate covariance and pseudo-covariance optimization algorithm is proposed.

Chapter 5 characterizes the DoF of the 3-user rank-deficient MIMO-IC. The outer bound is derived with the genie-aided signaling technique, and the achievability is based on a linear scheme that hybrids both interference alignment and zero-forcing strategies.

Chapter 6 proposes a modified BD scheme for MCP-enabled network so as to improve the low-to-medium SNR performance. The modified scheme is based on a rate-constrained power minimization problem, which is non-convex. An efficient sub-optimal solution with primal and dual updates is obtained for this problem.

Chapter 7 concludes the thesis and points out possible future working directions.

Chapter 2

Mathematical Preliminaries

In this chapter, some mathematical preliminaries are presented, i.e., the theory of complex random vectors (RVs) and convex optimization, on which many results of this thesis are based.

2.1 Complex Random Vectors

This section gives a brief introduction on complex RVs, with special emphasis on *improper* complex RVs, which have been probably overlooked by many researchers in communications and signal processing communities. For a comprehensive treatment on this topic, the readers may refer to the survey paper [55] and the textbook [56].

A complex-valued RV $\mathbf{z} \in \mathbb{C}^{n \times 1}$ is obtained by composing two n -dimensional real-valued RVs, $\mathbf{u}$ and $\mathbf{v}$, into its real and imaginary parts as

$$\mathbf{z} = \mathbf{u} + i\mathbf{v}. \quad (2.1)$$

The statistical properties of $\mathbf{z}$ are determined by the joint probability density function (pdf) of $\mathbf{u}$ and $\mathbf{v}$, which is denoted as $p_{\mathbf{u},\mathbf{v}}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$. The mean of $\mathbf{z}$ is given as $\mathbb{E}(\mathbf{z}) = \mathbb{E}(\mathbf{u}) + i\mathbb{E}(\mathbf{v})$. In this thesis, all RVs of interest are assumed to have zero mean, i.e. $\mathbb{E}(\mathbf{z}) = \mathbf{0}$. This is justified since from a communication perspective, a signaling RV with non-zero mean will always cost extra transmission power as compared to its zero-mean counterpart.

2.1.1 Real and Complex Representations

A straightforward approach for describing a complex-valued RV $\mathbf{z} = \mathbf{u} + i\mathbf{v} \in \mathbb{C}^{n \times 1}$ is to transform it into a $2n$ -dimensional real-valued RV $\bar{\mathbf{z}}$, obtained by stacking the real part $\mathbf{u}$ and the imaginary part $\mathbf{v}$ as

$$\bar{\mathbf{z}} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}. \quad (2.2)$$

Alternatively, it is sometimes useful to define an augmented RV $\underline{\mathbf{z}} \in \mathbb{C}^{2n \times 1}$ by stacking $\mathbf{z}$ on top of its complex conjugate $\mathbf{z}^*$ [56] as

$$\underline{\mathbf{z}} \triangleq \begin{bmatrix} \mathbf{z} \\ \mathbf{z}^* \end{bmatrix}. \quad (2.3)$$

It is easy to see that $\underline{\mathbf{z}}$ and $\bar{\mathbf{z}}$ are related as

$$\underline{\mathbf{z}} = \mathbf{M}_n \bar{\mathbf{z}}, \quad \bar{\mathbf{z}} = \frac{1}{2} \mathbf{M}_n^H \underline{\mathbf{z}}, \quad (2.4)$$

where

$$\mathbf{M}_n \triangleq \begin{bmatrix} \mathbf{I}_n & i\mathbf{I}_n \\ \mathbf{I}_n & -i\mathbf{I}_n \end{bmatrix} \in \mathbb{C}^{2n \times 2n} \quad (2.5)$$

is unitary up to a factor of 2, i.e.,

$$\mathbf{M}_n \mathbf{M}_n^H = \mathbf{M}_n^H \mathbf{M}_n = 2\mathbf{I}_{2n}. \quad (2.6)$$

It is obvious that $\underline{\mathbf{z}}$ is an equivalent redundant representation of $\bar{\mathbf{z}}$ in the complex domain. It turns out that such a redundant representation is very useful and convenient to deal with general complex RVs, for which some important insights and mathematical elegance would be lost if the real-valued representation $\bar{\mathbf{z}}$ had been adopted [56, 57].

2.1.2 Covariance and Pseudo-Covariance Matrices

For a zero-mean complex-valued RV $\mathbf{z}$, the covariance matrix $\mathbf{C}_{\mathbf{z}}$ and pseudo-covariance matrix¹ $\tilde{\mathbf{C}}_{\mathbf{z}}$ are defined as [61]

$$\mathbf{C}_{\mathbf{z}} \triangleq \mathbb{E}(\mathbf{z}\mathbf{z}^H), \quad \tilde{\mathbf{C}}_{\mathbf{z}} \triangleq \mathbb{E}(\mathbf{z}\mathbf{z}^T). \quad (2.7)$$

Note that the pseudo-covariance matrix $\tilde{\mathbf{C}}_{\mathbf{z}}$ is defined using a regular transpose rather than a Hermitian (conjugate) transpose. By definition, the covariance matrix $\mathbf{C}_{\mathbf{z}}$ is Hermitian and positive semidefinite, and the pseudo-covariance matrix $\tilde{\mathbf{C}}_{\mathbf{z}}$ is symmetric.

It is apparent that $\mathbf{C}_{\mathbf{z}}$ and $\tilde{\mathbf{C}}_{\mathbf{z}}$ are two different parameters for general complex RV $\mathbf{z}$. For a complete second-order characterization of $\mathbf{z}$, both the covariance matrix $\mathbf{C}_{\mathbf{z}}$ and the pseudo-covariance matrix $\tilde{\mathbf{C}}_{\mathbf{z}}$ are required [56, 60, 61, 63, 64]. By substituting $\mathbf{z} = \mathbf{u} + i\mathbf{v}$ into (2.7), we have

$$\mathbf{C}_{\mathbf{z}} = \mathbb{E}[(\mathbf{u} + i\mathbf{v})(\mathbf{u} - i\mathbf{v})^T] = (\mathbf{R}_{\mathbf{uu}} + \mathbf{R}_{\mathbf{vv}}) + i(\mathbf{R}_{\mathbf{vu}} - \mathbf{R}_{\mathbf{uv}}) \quad (2.8)$$

$$\tilde{\mathbf{C}}_{\mathbf{z}} = \mathbb{E}[(\mathbf{u} + i\mathbf{v})(\mathbf{u} + i\mathbf{v})^T] = (\mathbf{R}_{\mathbf{uu}} - \mathbf{R}_{\mathbf{vv}}) + i(\mathbf{R}_{\mathbf{vu}} + \mathbf{R}_{\mathbf{uv}}), \quad (2.9)$$

where

$$\begin{aligned} \mathbf{R}_{\mathbf{uu}} &\triangleq \mathbb{E}(\mathbf{u}\mathbf{u}^T), & \mathbf{R}_{\mathbf{vv}} &\triangleq \mathbb{E}(\mathbf{v}\mathbf{v}^T) \\ \mathbf{R}_{\mathbf{uv}} &\triangleq \mathbb{E}(\mathbf{u}\mathbf{v}^T), & \mathbf{R}_{\mathbf{vu}} &\triangleq \mathbb{E}(\mathbf{v}\mathbf{u}^T) \end{aligned} \quad (2.10)$$

are the auto- and cross-covariance matrices of the real and imaginary parts $\mathbf{u}$ and $\mathbf{v}$, respectively.

It can be obtained from (2.8) and (2.9) that

$$\mathbf{R}_{\mathbf{uu}} = \frac{1}{2}\Re\{\mathbf{C}_{\mathbf{z}} + \tilde{\mathbf{C}}_{\mathbf{z}}\}, \quad \mathbf{R}_{\mathbf{vv}} = \frac{1}{2}\Re\{\mathbf{C}_{\mathbf{z}} - \tilde{\mathbf{C}}_{\mathbf{z}}\} \quad (2.11)$$

$$\mathbf{R}_{\mathbf{vu}} = \frac{1}{2}\Im\{\mathbf{C}_{\mathbf{z}} + \tilde{\mathbf{C}}_{\mathbf{z}}\}, \quad \mathbf{R}_{\mathbf{uv}} = -\frac{1}{2}\Im\{\mathbf{C}_{\mathbf{z}} - \tilde{\mathbf{C}}_{\mathbf{z}}\}. \quad (2.12)$$

Therefore, (2.7) and (2.10) give two equivalent descriptions for the second-order statistics of the zero-mean complex-valued RV $\mathbf{z}$, in the complex and real domain, respectively.

¹Alternative terms used in the literature include relation matrix [58], conjugate covariance matrix [59], or complementary covariance matrix [55–57, 60]. Throughout this thesis, we will adhere to the term pseudo-covariance matrix as used in [61, 62].

Denote by $\bar{\mathbf{Q}}_{\mathbf{z}} \in \mathbb{R}^{2n \times 2n}$ and $\underline{\mathbf{C}}_{\mathbf{z}} \in \mathbb{C}^{2n \times 2n}$ the covariance matrix of $\bar{\mathbf{z}}$ and $\underline{\mathbf{z}}$, respectively. We then have

$$\bar{\mathbf{Q}}_{\mathbf{z}} \triangleq \mathbb{E}(\bar{\mathbf{z}}\bar{\mathbf{z}}^T) = \mathbb{E}\left(\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}^T\right) = \begin{bmatrix} \mathbf{R}_{\mathbf{uu}} & \mathbf{R}_{\mathbf{uv}} \\ \mathbf{R}_{\mathbf{vu}} & \mathbf{R}_{\mathbf{vv}} \end{bmatrix}, \quad (2.13)$$

$$\underline{\mathbf{C}}_{\mathbf{z}} \triangleq \mathbb{E}(\underline{\mathbf{z}}\underline{\mathbf{z}}^H) = \mathbb{E}\left(\begin{bmatrix} \mathbf{z} \\ \mathbf{z}^* \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}^* \end{bmatrix}^H\right) = \begin{bmatrix} \mathbf{C}_{\mathbf{z}} & \tilde{\mathbf{C}}_{\mathbf{z}} \\ \tilde{\mathbf{C}}_{\mathbf{z}}^* & \mathbf{C}_{\mathbf{z}}^* \end{bmatrix}. \quad (2.14)$$

Furthermore, it follows from (2.4) that $\bar{\mathbf{Q}}_{\mathbf{z}}$ and $\underline{\mathbf{C}}_{\mathbf{z}}$ are related as

$$\underline{\mathbf{C}}_{\mathbf{z}} = \mathbf{M}_n \bar{\mathbf{Q}}_{\mathbf{z}} \mathbf{M}_n^H, \quad \bar{\mathbf{Q}}_{\mathbf{z}} = \frac{1}{4} \mathbf{M}_n^H \underline{\mathbf{C}}_{\mathbf{z}} \mathbf{M}_n. \quad (2.15)$$

The augmented covariance matrix $\underline{\mathbf{C}}_{\mathbf{z}}$ given in (2.14) has some built-in redundancy for specifying the second-order statistics of $\mathbf{z}$, but it will turn out to be very convenient to deal with general complex RVs, as firstly illustrated by the following theorem.

Theorem 2.1. [56]: *Two given matrices $\mathbf{C}_{\mathbf{z}}$ and $\tilde{\mathbf{C}}_{\mathbf{z}}$ are a valid pair of covariance and pseudo-covariance matrices, i.e., there exists a RV with covariance and pseudo-covariance matrices given by $\mathbf{C}_{\mathbf{z}}$ and $\tilde{\mathbf{C}}_{\mathbf{z}}$, respectively, if and only if the augmented covariance matrix $\underline{\mathbf{C}}_{\mathbf{z}}$ formed in (2.14) is positive semidefinite, i.e., $\underline{\mathbf{C}}_{\mathbf{z}} \succeq \mathbf{0}$.*

It can be verified that the conditions for the covariance matrix $\mathbf{C}_{\mathbf{z}}$ being Hermitian and positive semidefinite, and the pseudo-covariance matrix $\tilde{\mathbf{C}}_{\mathbf{z}}$ being symmetric are implied by Theorem 2.1.

Two complex RVs $\mathbf{x}$ and $\mathbf{z}$ are uncorrelated if and only if both their real and imaginary parts are uncorrelated, i.e., $\mathbb{E}(\bar{\mathbf{x}}\bar{\mathbf{z}}^T) = \mathbf{0}$, where $\bar{\mathbf{x}}$ and $\bar{\mathbf{z}}$ are their corresponding real-valued representations as in (2.2). It is not difficult to verify that the *uncorrelatedness* of two complex RVs can be alternatively stated as follows.

Lemma 2.2. [61]: *Two zero-mean complex RVs $\mathbf{x}$ and $\mathbf{z}$ are uncorrelated if and only if $\mathbf{C}_{\mathbf{xz}} = \mathbf{0}$ and $\tilde{\mathbf{C}}_{\mathbf{xz}} = \mathbf{0}$, where $\mathbf{C}_{\mathbf{xz}} \triangleq \mathbb{E}(\mathbf{x}\mathbf{z}^H)$ and $\tilde{\mathbf{C}}_{\mathbf{xz}} \triangleq \mathbb{E}(\mathbf{x}\mathbf{z}^T)$.*

2.1.3 Proper and Improper Random Vectors

Definition 2.3. [61]: A complex RV $\mathbf{z}$ is called proper if its pseudo-covariance matrix vanishes, i.e., $\tilde{\mathbf{C}}_{\mathbf{z}} = \mathbf{0}$; otherwise, $\mathbf{z}$ is called improper.

It follows from (2.11) and (2.12) that the conditions for properness on the auto-covariance and cross-covariance of the real and imaginary parts $\mathbf{u}$ and $\mathbf{v}$ are

$$\mathbf{R}_{\mathbf{uu}} = \mathbf{R}_{\mathbf{vv}}, \quad (2.16)$$

$$\mathbf{R}_{\mathbf{uv}} = -\mathbf{R}_{\mathbf{vu}} = -\mathbf{R}_{\mathbf{uv}}^T. \quad (2.17)$$

Therefore, for a proper RV $\mathbf{z}$, its real and imaginary parts $\mathbf{u}$ and $\mathbf{v}$ have identical auto-covariance matrices and their cross-covariance matrix is skew-symmetric. When the dimension is 1, i.e., $z = u + iv$ is a proper random variable, then we have $R_{uu} = R_{vv}$ and $R_{uv} = 0$, i.e., the real and imaginary parts of a proper random variable have identical variance and they are uncorrelated.

A more restrictive definition than properness is known as *circularly* symmetric, which is defined as follows.

Definition 2.4. [56]: A complex RV $\mathbf{z}$ is circularly symmetric if its distribution is rotationally invariant, i.e., if $\mathbf{z}$ and $\hat{\mathbf{z}} = e^{i\alpha}\mathbf{z}$ have the same distribution for any real value α .

For a circularly symmetric RV $\mathbf{z}$, we have

$$\mathbb{E}(\mathbf{z}) = \mathbb{E}(\hat{\mathbf{z}}) = e^{i\alpha}\mathbb{E}(\mathbf{z}), \quad \forall \alpha, \implies \mathbb{E}(\mathbf{z}) = \mathbf{0}, \quad (2.18)$$

and

$$\tilde{\mathbf{C}}_{\mathbf{z}} = \tilde{\mathbf{C}}_{\hat{\mathbf{z}}} = \mathbb{E}(\hat{\mathbf{z}}\hat{\mathbf{z}}^T) = e^{i2\alpha}\tilde{\mathbf{C}}_{\mathbf{z}}, \quad \forall \alpha, \implies \tilde{\mathbf{C}}_{\mathbf{z}} = \mathbf{0}. \quad (2.19)$$

Thus, circularity implies zero-mean and properness, but the converse is not true in general. However, for the special case of zero-mean complex *Gaussian* RV as will be discussed in the next subsection, circularity and properness are equivalent.

2.1.4 Complex Gaussian Random Vectors and Entropy

A special class of complex RVs of particular interest is known as Gaussian RV (*not necessarily proper Gaussian*). A complex Gaussian RV $\mathbf{z} \in \mathbb{C}^{n \times 1}$ is defined as a RV with jointly Gaussian real and imaginary parts. Similar to real Gaussian RVs, the pdf of a zero-mean complex Gaussian RV is completely specified by its second-order statistics, which can be conveniently written in the following form by making use

of the augmented vector $\underline{\mathbf{z}}$ defined in (2.3) and the augmented covariance matrix $\underline{\mathbf{C}}_{\mathbf{z}}$ given in (2.14) [56]:

$$p(\mathbf{z}) = \frac{1}{\pi^n |\underline{\mathbf{C}}_{\mathbf{z}}|^{1/2}} \exp \left\{ -\frac{1}{2} \underline{\mathbf{z}}^H \underline{\mathbf{C}}_{\mathbf{z}}^{-1} \underline{\mathbf{z}} \right\}. \quad (2.20)$$

When $\mathbf{z}$ is a proper complex Gaussian RV, where $\tilde{\mathbf{C}}_{\mathbf{z}} = \mathbf{0}$ and $\underline{\mathbf{C}}_{\mathbf{z}}$ is block-diagonal, the pdf in (2.20) reduces to the following familiar form:

$$p(\mathbf{z}) = \frac{1}{\pi^n |\mathbf{C}_{\mathbf{z}}|} \exp \left\{ -\mathbf{z}^H \mathbf{C}_{\mathbf{z}}^{-1} \mathbf{z} \right\}. \quad (2.21)$$

Since a complex Gaussian RV is completely specified by its first- and second-order statistics, we have the following result.

Lemma 2.5. [56]: *A zero-mean complex Gaussian RV is circularly symmetric if and only if it is proper.*

For example, in wireless communications, it is commonly assumed that additive noise vectors $\mathbf{n}$ is zero-mean CSCG distributed. This is equivalent to that $\mathbf{n}$ is a proper Gaussian RV with zero pseudo-covariance matrix $\tilde{\mathbf{C}}_{\mathbf{n}} = \mathbf{0}$.

For general complex *Gaussian* RVs, the differential entropy is a function of both the covariance and pseudo-covariance matrices, which can be conveniently expressed in terms of the augmented covariance matrix $\underline{\mathbf{C}}_{\mathbf{z}}$ as shown by the following theorem.

Theorem 2.6. [56]: *The differential entropy of a complex Gaussian RV $\mathbf{z} \in \mathbb{C}^{n \times 1}$ with augmented covariance matrix $\underline{\mathbf{C}}_{\mathbf{z}}$ is given by*

$$h(\mathbf{z}) = \frac{1}{2} \log [(\pi e)^{2n} |\underline{\mathbf{C}}_{\mathbf{z}}|]. \quad (2.22)$$

Theorem 2.6 generalizes the entropy formula for the widely used proper Gaussian RVs. If $\tilde{\mathbf{C}}_{\mathbf{z}} = \mathbf{0}$, (2.22) reduces to the well-known formula $h(\mathbf{z}) = \log [(\pi e)^n |\mathbf{C}_{\mathbf{z}}|]$ for proper Gaussian RVs [61].

Proper Gaussian RVs are known to be entropy maximizers, as stated by the following theorem.

Theorem 2.7. [61]: Suppose that $\mathbf{z} \in \mathbb{C}^{n \times 1}$ is a zero-mean complex RV with nonsingular covariance matrix $\mathbf{C}_{\mathbf{z}}$. Then, the differential entropy of $\mathbf{z}$ satisfies

$$h(\mathbf{z}) \leq \log [(\pi e)^n |\mathbf{C}_{\mathbf{z}}|], \quad (2.23)$$

with equality if and only if $\mathbf{z}$ is proper and Gaussian distributed.

2.2 Convex Optimization

Many problems in communications and signal processing can be formulated as mathematical optimization problems. One paradigmatic example is the resource allocation problem (e.g., power, subcarrier, beamforming vectors, transmit covariance matrices) in multi-user communication systems. This section is devoted to giving a brief introduction on optimization theory. The focus is on convex optimization theory, which has found numerous engineering applications in the last few decades. The technique of semidefinite relaxation (SDR), which can be applied to efficiently find approximate solutions to non-convex quadratically constrained quadratic programs (QCQPs), is also reviewed.

For a comprehensive treatment on convex optimization theory, the readers may refer to the textbook [65]. For the applications of convex optimization in communications and signal processing, the reference [66] and the tutorial paper [67] provide good starting points. The main references for SDR technique are [53] and [54].

2.2.1 Convex Optimization Problem

The following basic concepts are necessary before introducing convex optimization problems.

- *Convex set:* A set is convex if it contains line segment between any two points in the set. Mathematically, set $S \in \mathbb{R}^n$ is convex if $\forall \mathbf{x}, \mathbf{y} \in S$ and $0 \leq \theta \leq 1$, $\theta \mathbf{x} + (1 - \theta) \mathbf{y} \in S$. Some examples of convex sets are the unit ball $\mathcal{B} = \{\mathbf{x} | \|\mathbf{x}\| \leq 1\}$, the second-order cone $\text{SOC}(n) = \{(\mathbf{x}, t) | \|\mathbf{x}\| \leq t\}$, and the positive semidefinite cone $\mathbb{S}_+^n = \{\mathbf{X} \in \mathbb{S}^n | \mathbf{X} \succeq \mathbf{0}\}$.

- *Convex function*: A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if $\text{dom} f$ is a convex set, and $\forall \mathbf{x}, \mathbf{y} \in \text{dom} f$ and $0 \leq \theta \leq 1$, $f(\theta \mathbf{x} + (1 - \theta) \mathbf{y}) \leq \theta f(\mathbf{x}) + (1 - \theta) f(\mathbf{y})$. Two examples of convex functions are the exponential function e^{ax} , and the quadratic function $\mathbf{x}^T \mathbf{A} \mathbf{x}$, with $\mathbf{A} \succeq \mathbf{0}$.
- *Quasi-convex function*: A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is quasi-convex if $\text{dom} f$ is convex and the sublevel sets $S_\alpha = \{\mathbf{x} \in \text{dom} f | f(\mathbf{x}) \leq \alpha\}$ are convex for all α . One such example is the linear-fractional function $f(\mathbf{x}) = \frac{\mathbf{a}^T \mathbf{x} + b}{\mathbf{c}^T \mathbf{x} + d}$, with $\text{dom} f = \{\mathbf{x} | \mathbf{c}^T \mathbf{x} + d > 0\}$.

A mathematical optimization problem can be represented in the following form

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & f_l(\mathbf{x}) \leq 0, \quad l = 1, \dots, m \\ & h_l(\mathbf{x}) = 0, \quad l = 1, \dots, p, \end{aligned} \tag{2.24}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the optimization variable, $f_0(\mathbf{x})$ is the cost function, $f_l(\mathbf{x}) \leq 0$ and $h_l(\mathbf{x}) = 0$ are the inequality and equality constraints, respectively. The domain $\mathcal{D}$ of the optimization problem (2.24) is the intersection of the domains of the cost and constraint functions, i.e.,

$$\mathcal{D} = \bigcap_{l=0}^m \text{dom} f_l \cap \bigcap_{l=1}^p \text{dom} h_l. \tag{2.25}$$

A point $\mathbf{x} \in \mathcal{D}$ is feasible if it satisfies all the equality and inequality constraints in (2.24). The optimization problem is feasible if there exists at least one feasible point; otherwise, it is infeasible. We denote the optimal value of (2.24) as p^* , which is achieved by an optimal solution $\mathbf{x}^*$. If the problem is infeasible, we have $p^* = \infty$.

The problem (2.24) is known as a convex optimization problem if the functions $f_0, f_1, \dots, f_m$ are convex and h_l are affine.² It is a quasi-convex problem if the cost function $f_0(\mathbf{x})$ is quasi-convex, and $f_1, \dots, f_m$ are convex and h_l are affine functions. Two examples of convex optimization problems that will be frequently used in this thesis are:

²An affine function can be written in the form $\mathbf{a}^T \mathbf{x} + b$.

- Second-order cone programming (SOCP):

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{q}^T \mathbf{x} \\ \text{s.t.} \quad & \|\mathbf{A}_l \mathbf{x} + \mathbf{b}_l\| \leq \mathbf{c}_l^T \mathbf{x} + d_l, \quad l = 1, \dots, m, \end{aligned} \quad (2.26)$$

where $\mathbf{A}_l \in \mathbb{R}^{n_l \times n}$.

- Semidefinite programming problem (SDP):

$$\begin{aligned} \min_{\mathbf{X}} \quad & \text{Tr}(\mathbf{A}_0 \mathbf{X}) \\ \text{s.t.} \quad & \text{Tr}(\mathbf{A}_l \mathbf{X}) \leq 1, \quad l = 1, \dots, m \\ & \mathbf{X} \succeq \mathbf{0}, \end{aligned} \quad (2.27)$$

where $\mathbf{A}_l \in \mathbb{R}^{n \times n}$.

One example of quasi-convex problem is the linear-fractional problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{G}\mathbf{x} \preceq \mathbf{h} \\ & \mathbf{A}\mathbf{x} = \mathbf{b}, \end{aligned} \quad (2.28)$$

where $f_0(\mathbf{x}) = \frac{\mathbf{a}^T \mathbf{x} + b}{\mathbf{c}^T \mathbf{x} + d}$, with $\text{dom} f_0 = \{\mathbf{x} | \mathbf{c}^T \mathbf{x} + d > 0\}$.

In some cases, convex optimization problems can be analytically solved and closed-form expressions can be obtained. In general, however, one must resort to iterative methods, such as the interior-point methods [68]. Some existing software tools such as CVX [69] are also available to find numerical solutions once a convex optimization problem has been formulated following a certain syntax. A quasi-convex optimization problem can be efficiently solved via convex feasibility problems with bisection method [65].

2.2.2 Lagrangian Duality and KKT Conditions

For the optimization problem (2.24) (not necessarily convex), by introducing dual variables $\boldsymbol{\lambda} \in \mathbb{R}^m$ and $\boldsymbol{\nu} \in \mathbb{R}^p$, we can form the Lagrangian function

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = f_0(\mathbf{x}) + \sum_{l=1}^m \lambda_l f_l(\mathbf{x}) + \sum_{l=1}^p \nu_l h_l(\mathbf{x}), \quad (2.29)$$

where λ_l is the Lagrange multiplier associated with the l th inequality constraint $f_l(\mathbf{x}) \leq 0$ and ν_l is the Lagrange multiplier for the l th equality constraint $h_l(\mathbf{x}) = 0$. The Lagrange dual function $g(\boldsymbol{\lambda}, \boldsymbol{\nu})$ associated with (2.24) is defined as

$$g(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \min_{\mathbf{x} \in \mathcal{D}} L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}). \quad (2.30)$$

As $g(\boldsymbol{\lambda}, \boldsymbol{\nu})$ is a pointwise minimum of a family of linear functions in $(\boldsymbol{\lambda}, \boldsymbol{\nu})$, the dual function $g(\boldsymbol{\lambda}, \boldsymbol{\nu})$ is always concave (even the original optimization problem is non-convex). The weak duality result says that for any primal feasible vector $\mathbf{x}$ and any dual feasible vectors $(\boldsymbol{\lambda}, \boldsymbol{\nu})$ such that $\boldsymbol{\lambda} \succeq \mathbf{0}$, we have

$$f_0(\mathbf{x}) \geq g(\boldsymbol{\lambda}, \boldsymbol{\nu}). \quad (2.31)$$

In particular, the optimal value p^* of (2.24) also satisfies (2.31), i.e., $p^* = f_0(\mathbf{x}^*) \geq g(\boldsymbol{\lambda}, \boldsymbol{\nu})$. In other words, the dual function $g(\boldsymbol{\lambda}, \boldsymbol{\nu})$ serves as a lower bound on the optimal value p^* of the original optimization problem. The best lower bound can be found by solving the following dual optimization problem:

$$\begin{aligned} \max_{\boldsymbol{\lambda}, \boldsymbol{\nu}} \quad & g(\boldsymbol{\lambda}, \boldsymbol{\nu}) \\ \text{s.t.} \quad & \boldsymbol{\lambda} \succeq \mathbf{0}. \end{aligned} \quad (2.32)$$

Problem (2.24) is usually referred as the primal problem and (2.32) the dual problem. Denote the optimal value of (2.32) by d^* . Then we have $p^* \geq d^*$, with the value $p^* - d^*$ known as the duality gap. For convex optimization problems satisfying some mild constraint qualification conditions such as the Slater's condition,³ we have zero duality gap, i.e., $p^* = d^*$.

One very useful result in optimization theory is known as the Karush-Kuhn-Tucker (KKT) conditions, which consist of the following four conditions:

1. Primal feasibility: $f_l(\mathbf{x}) \leq 0$, $l = 1, \dots, m$, $h_l(\mathbf{x}) = 0$, $l = 1, \dots, p$.
2. Dual feasibility: $\boldsymbol{\lambda} \succeq \mathbf{0}$.
3. Complementary slackness: $\lambda_l f_l(\mathbf{x}) = 0$, $l = 1, \dots, m$.
4. Zero gradient: $\nabla f_0(\mathbf{x}) + \sum_{l=1}^m \lambda_l \nabla f_l(\mathbf{x}) + \sum_{l=1}^p \nu_l \nabla h_l(\mathbf{x}) = \mathbf{0}$.

³Slater's condition requires the existence of a strict interior point for the set defined by the equality and inequality constraints in (2.24).

For general optimization problems (not necessarily convex), the KKT conditions are necessary for local optimality, i.e., $\mathbf{x}^*$ is a local optimal solution of (2.24) only if there exists some $(\boldsymbol{\lambda}^*, \boldsymbol{\nu}^*)$ such that the above KKT conditions are satisfied. For convex optimization problems (under mild constraint qualification conditions such as the Slater's condition), the KKT conditions are both necessary and sufficient for global optimality, based on which the optimal solutions can be analytically obtained for certain simple problems.

2.2.3 Subgradient Method and Dual Decomposition

It is sometimes preferable to solve an optimization problem (2.24) via solving its dual (2.32). One reason is that the dual problem is always concave, even for non-convex primal problems. Furthermore, the dual problem may have less constraints than the primal problem, and via dual decomposition, parallel computation becomes possible, which makes the solution much more efficient [70–73]. However, the dual function is in general non-differentiable, in which case the subgradient method can be quite useful.

For a concave function $g(\mathbf{z})$, a vector $\mathbf{s}$ is a subgradient of g at point $\mathbf{z}$ if

$$g(\bar{\mathbf{z}}) \leq g(\mathbf{z}) + \mathbf{s}^T(\bar{\mathbf{z}} - \mathbf{z}), \quad \forall \bar{\mathbf{z}}. \quad (2.33)$$

Subgradient generalizes the gradient concept to non-differentiable functions. If g is differentiable, the gradient $\nabla g(\mathbf{z})$ is a subgradient of g at $\mathbf{z}$.

Consider the following general convex optimization problem:

$$\begin{aligned} \max_{\mathbf{z}} \quad & g(\mathbf{z}) \\ \text{s.t.} \quad & \mathbf{z} \in \mathcal{C}, \end{aligned} \quad (2.34)$$

where $g(\mathbf{z})$ is a concave function and $\mathcal{C}$ is a closed convex set. With the subgradient method, a sequence of feasible points $\{\mathbf{z}(t)\}$ are generated as

$$\mathbf{z}(t+1) = [\mathbf{z}(t) + \alpha(t)\mathbf{s}(t)]_{\mathcal{C}}, \quad (2.35)$$

where $\mathbf{s}(t)$ is a subgradient of g evaluated at the point $\mathbf{z}(t)$, $[\cdot]_{\mathcal{C}}$ denotes the projection onto the feasible set $\mathcal{C}$, and $\alpha(t)$ is a positive step-size. Several step-size rules are available for the subgradient method, e.g., the constant step-size rule with $\alpha(t) = \alpha$

and the diminishing step-size rule with $\alpha(t) = (1 + b)/(t + b)$, where b is a fixed nonnegative number. The choice of step sizes in general affects the convergence of the subgradient method. An alternative subgradient-based approach for solving (2.34), for which the annoying requirement for choosing the step-size can be circumvented, is the ellipsoid method [74]. Ellipsoid method has been applied in solving several problems in communications and signal processing [52, 70].

For the dual problem (2.32), the subgradient can be readily obtained. From (2.30), we have

$$g(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \min_{\mathbf{x} \in \mathcal{D}} \left\{ f_0(\mathbf{x}) + \sum_{l=1}^m \lambda_l f_l(\mathbf{x}) + \sum_{l=1}^p \nu_l h_l(\mathbf{x}) \right\}. \quad (2.36)$$

Let $\mathbf{x}_{\boldsymbol{\lambda}, \boldsymbol{\nu}}$ minimize the above Lagrangian function for any given $(\boldsymbol{\lambda}, \boldsymbol{\nu})$. Then $\forall (\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\nu}})$ we have

$$\begin{aligned} g(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\nu}}) &= \min_{\mathbf{x} \in \mathcal{D}} \left\{ f_0(\mathbf{x}) + \sum_{l=1}^m \bar{\lambda}_l f_l(\mathbf{x}) + \sum_{l=1}^p \bar{\nu}_l h_l(\mathbf{x}) \right\} \\ &\leq f_0(\mathbf{x}_{\boldsymbol{\lambda}, \boldsymbol{\nu}}) + \sum_{l=1}^m \bar{\lambda}_l f_l(\mathbf{x}_{\boldsymbol{\lambda}, \boldsymbol{\nu}}) + \sum_{l=1}^p \bar{\nu}_l h_l(\mathbf{x}_{\boldsymbol{\lambda}, \boldsymbol{\nu}}) \\ &= f_0(\mathbf{x}_{\boldsymbol{\lambda}, \boldsymbol{\nu}}) + \sum_{l=1}^m \lambda_l f_l(\mathbf{x}_{\boldsymbol{\lambda}, \boldsymbol{\nu}}) + \sum_{l=1}^p \nu_l h_l(\mathbf{x}_{\boldsymbol{\lambda}, \boldsymbol{\nu}}) \\ &\quad + \sum_{l=1}^m (\bar{\lambda}_l - \lambda_l) f_l(\mathbf{x}_{\boldsymbol{\lambda}, \boldsymbol{\nu}}) + \sum_{l=1}^p (\bar{\nu}_l - \nu_l) h_l(\mathbf{x}_{\boldsymbol{\lambda}, \boldsymbol{\nu}}) \\ &= g(\boldsymbol{\lambda}, \boldsymbol{\nu}) + \sum_{l=1}^m (\bar{\lambda}_l - \lambda_l) f_l(\mathbf{x}_{\boldsymbol{\lambda}, \boldsymbol{\nu}}) + \sum_{l=1}^p (\bar{\nu}_l - \nu_l) h_l(\mathbf{x}_{\boldsymbol{\lambda}, \boldsymbol{\nu}}). \end{aligned} \quad (2.37)$$

From (2.37) and (2.33), the vector formed by $(\{f_l(\mathbf{x}_{\boldsymbol{\lambda}, \boldsymbol{\nu}})\}_{l=1}^m, \{h_l(\mathbf{x}_{\boldsymbol{\lambda}, \boldsymbol{\nu}})\}_{l=1}^p)$ is a subgradient of $g(\boldsymbol{\lambda}, \boldsymbol{\nu})$. This subgradient is obtained for free once the Lagrangian minimizer in (2.36) is obtained.

The dual-decomposition method for solving the convex problem (2.24) is then summarized as follows:

Algorithm 1 Dual-Decomposition Method for Solving (2.24)

-
- 1: Initialize $(\boldsymbol{\lambda}, \boldsymbol{\nu})$ with $\boldsymbol{\lambda} \succeq \mathbf{0}$.
 - 2: **repeat**
 - 3: For the given $(\boldsymbol{\lambda}, \boldsymbol{\nu})$, obtain $\mathbf{x}_{\boldsymbol{\lambda}, \boldsymbol{\nu}}$ by solving (2.36).
 - 4: Compute the subgradient of $g(\boldsymbol{\lambda}, \boldsymbol{\nu})$, and update $(\boldsymbol{\lambda}, \boldsymbol{\nu})$ based on (2.35) or the ellipsoid method.
 - 5: **until** $(\boldsymbol{\lambda}, \boldsymbol{\nu})$ converge to a prescribed accuracy.
-

Remark 2.8. For convex problems satisfying some mild constraint qualification conditions, the convergence to global optimality of Algorithm 1 is guaranteed.

2.2.4 Semidefinite Relaxation

Non-convex optimization problem is in general difficult to be optimally solved. In this subsection, we give a brief introduction on the SDR technique, which has been successfully applied to find high-quality approximate solutions for various non-convex problems in communications and signal processing [75–79].

Consider a complex-valued QCQP problem:

$$\begin{aligned}
 \min_{\mathbf{x} \in \mathbb{C}^n} \quad & \mathbf{x}^H \mathbf{C} \mathbf{x} \\
 \text{s.t.} \quad & \mathbf{x}^H \mathbf{A}_l \mathbf{x} \leq 1, \quad l = 1, \dots, m,
 \end{aligned} \tag{2.38}$$

where $\mathbf{C}, \mathbf{A}_l \in \mathbb{H}^n$, but *not* necessarily positive semidefinite. As a result, the problem (2.38) is non-convex in general. With the identities $\mathbf{x}^H \mathbf{C} \mathbf{x} = \text{Tr}(\mathbf{C} \mathbf{x} \mathbf{x}^H)$ and $\mathbf{x}^H \mathbf{A}_l \mathbf{x} = \text{Tr}(\mathbf{A}_l \mathbf{x} \mathbf{x}^H)$, the problem (2.38) can be equivalently written as

$$\begin{aligned}
 \min_{\mathbf{X} \in \mathbb{H}^n} \quad & \text{Tr}(\mathbf{C} \mathbf{X}) \\
 \text{s.t.} \quad & \text{Tr}(\mathbf{A}_l \mathbf{X}) \leq 1, \quad l = 1, \dots, m \\
 & \mathbf{X} \succeq \mathbf{0} \\
 & \text{rank}(\mathbf{X}) = 1.
 \end{aligned} \tag{2.39}$$

The constraints $\mathbf{X} \succeq \mathbf{0}$ and $\text{rank}(\mathbf{X}) = 1$ are imposed since we have used $\mathbf{X} = \mathbf{x} \mathbf{x}^H$. The only difficulty in solving (2.39) is the rank-1 constraint, which is non-convex. One important step of the SDR technique is to drop the non-convex rank-1

constraint to obtain the following relaxed version of (2.38):

$$\begin{aligned}
& \min_{\mathbf{X} \in \mathbb{H}^n} \quad \text{Tr}(\mathbf{C}\mathbf{X}) \\
& \text{s.t.} \quad \text{Tr}(\mathbf{A}_l \mathbf{X}) \leq 1, \quad l = 1, \dots, m \\
& \quad \mathbf{X} \succeq \mathbf{0}.
\end{aligned} \tag{2.40}$$

(2.40) is an SDP problem, which can be efficiently solved. Denote the optimal solution to (2.40) as $\mathbf{X}^*$. If $\text{rank}(\mathbf{X}^*) = 1$, in which case $\mathbf{X}^*$ can be written as $\mathbf{X}^* = \mathbf{x}^* \mathbf{x}^{*H}$, then $\mathbf{x}^*$ is the solution to the original problem (2.38). Otherwise, an approximate solution $\hat{\mathbf{x}}$ to (2.38) can be generated from $\mathbf{X}^*$ with the following Gaussian randomization procedure:

Algorithm 2 Gaussian Randomization Procedure for (2.38)

- 1: Given an SDR solution $\mathbf{X}^*$, and the number of randomizations T .
- 2: **for** $t = 1, \dots, T$ **do**
- 3: generate $\boldsymbol{\xi}_t \sim \mathcal{CN}(\mathbf{0}, \mathbf{X}^*)$, and construct a feasible point

$$\tilde{\mathbf{x}}_t = \frac{\boldsymbol{\xi}_t}{\sqrt{\max_{l=1, \dots, m} \boldsymbol{\xi}_t^H \mathbf{A}_l \boldsymbol{\xi}_t}} \tag{2.41}$$

- 4: **end for**
 - 5: determine $t^* = \arg \min_{t=1, \dots, T} \tilde{\mathbf{x}}_t^H \mathbf{C} \tilde{\mathbf{x}}_t$.
 - 6: Output $\hat{\mathbf{x}} = \tilde{\mathbf{x}}_{t^*}$ as an approximate solution to (2.38).
-

Denote the optimal values of the problem (2.38) and its SDR problem (2.40) as v^* and v_{sdr} , respectively. Further denote by $\hat{v}$ the objective value of (2.38) corresponding to the obtained approximate solution $\hat{\mathbf{x}}$. Then we have

$$v_{\text{sdr}} \leq v^* \leq \hat{v}. \tag{2.42}$$

The ratio $\hat{v}/v^*$ is called the approximation ratio. Since the optimal value v^* is in general difficult to be found, the ratio $\hat{v}/v_{\text{sdr}}$ is generally used to measure the approximation accuracy. The approximation accuracy bounds have been analytically obtained for various problems, see e.g. [54]. For a complex-valued homogeneous QCQP with 3 constraints, SDR is tight [77, 80], i.e., the optimal solution can be obtained for the original non-convex QCQP by solving its SDR problem.

Chapter 3

Improper Gaussian Signaling for Interference Channel

3.1 Introduction and Literature Review

An interference channel models a multi-user communication scenario where each transmitter is intended to send independent information to its corresponding receiver while causing interference to all other receivers. It provides a unified mathematical framework for many practical communication systems, e.g., the multi-user digital subscriber lines (DSLs) impaired by crosstalk [81], the coordinated multi-cell downlink transmission [9], and the single-cell downlink communication restricted to linear precoding schemes [43].

3.1.1 Information-Theoretic Study

The history of the information-theoretic study on interference channels can be traced back to 1960s due to Shannon [14]. In [82], Carleial showed that for the two-user Gaussian SISO-IC, the interference does not reduce the capacity if it is *very strong*. In this case, the capacity achieving strategy at the receivers is to firstly decode and subtract the interference prior to the decoding of the desired message. Similar results have been reported for the discrete memoryless interference channel [83, 84] and Gaussian MISO-IC [85]. Later, it was found that the very strong interference requirement can be relaxed while still being able to characterize the capacity region, which leads to the *strong* interference conditions [16, 85–87]. More

recently, another interference condition, termed *generally strong* interference, is introduced in [88], for which the capacity can still be achieved by jointly decoding the signal and the interference.

For the general case, the capacity region of the Gaussian interference channel remains unknown. Several outer bounds and achievable rate regions are available [16, 17, 89–94]. For the SISO-IC, the best achievable rate region to date is obtained by the celebrated Han-Kobayashi scheme [16]. Recently, it has been shown in [17] that a particular form of this scheme achieves within one bit to the capacity region of the two-user Gaussian SISO-IC. A key technique in the Han-Kobayashi scheme is to split each user's transmit signal into a common message, which is decodable at both receivers, and a private message, which is decodable at the intended receiver only. Since such capacity-approaching techniques require multi-user encoding/decoding, which are difficult to implement in practical systems, a more pragmatic approach is to employ single-user encoding and decoding by treating interference as noise at all receivers. In fact, this simplified approach has been shown to be sum-capacity optimal for Gaussian interference channels when the interference level is below a certain threshold (termed *noisy-interference*). Such results have been established for the two-user Gaussian SISO-IC [91, 95, 96], MISO-IC [97] and MIMO-IC [87, 98], as well as the parallel Gaussian interference channel [99] and the Gaussian interference channel with more than two users [96, 100].

3.1.2 Coordinated Resource Allocation

Under the assumption of single-user detection (SUD) with interference treated as noise, the transmit optimization problem for Gaussian interference channels reduces to coordinated resource allocation among the transmitters, which has received significant research attentions in the last few decades [35, 101, 102]. Early studies on resource allocation mostly focused on power control since SISO-IC with single-antenna terminals was considered [6, 7, 39, 103–107]. When transmitters/receivers are equipped with multiple antennas, the network performance can be further improved via transmit/receive beamforming. One useful technique for optimizing transmit beamformers is to transform the design problem into an equivalent receiver beamformer optimization problem via the so-called uplink-downlink or network duality principle [9, 24, 108–117].

Resource allocation problems in the literature typically belong to the following categories: i) quality-of-service (QoS) constrained transmit power minimization [101], where one commonly used QoS measure is the received signal-to-interference-plus-noise ratio (SINR) [9, 41, 103, 104, 109, 110, 114, 116, 118]; ii) worst-case QoS maximization subject to a power constraint [39–41]; iii) (weighted) sum-rate maximization (WSRMax) [7, 39, 107, 119–124]; iv) mean-square error (MSE) minimization [125, 126]; v) the achievable rate region characterization [36, 127–133]. For MISO-IC, both the SINR-constrained power minimization and the worst-case SINR maximization problems can be optimally solved, either by the aforementioned network duality principle or by convex optimization techniques, such as SOCP [41] or SDP [134]. In contrast, the WSRMax problems are known to be NP-hard in general [39, 40].

3.1.3 Sum-Rate Maximization

For the special case of two-user SISO-IC, it has been obtained that the sum-rate optimal power allocation strategy is binary, i.e., either both transmitters using maximum power or one of them transmitting with full power and the other being completely shut down [7]. For the more general scenario such as K -user SISO-IC or MISO-IC, it has been shown that finding the globally optimal transmit strategy for sum-rate maximization is an NP-hard problem [39, 40]. Many sub-optimal or heuristic algorithms have thus been proposed. In [135–137], algorithms based on *interference pricing* have been developed for achieving local optimums. The problem has also been investigated from egotistic versus altruistic points of view [4, 8]. Beamforming techniques based on virtual SINR framework have been applied in [138, 139]. In [140], the beamforming vectors are obtained by solving the KKT conditions of the non-convex sum-rate maximization problems. Gradient descent algorithms have also been proposed for Gaussian MIMO-IC over the transmit covariance matrices [119] or the precoding matrices [141]. More recently, for Gaussian SISO-IC, single-input multiple-output interference channel (SIMO-IC) and MISO-IC, globally optimal solutions to the WSRMax problem have been obtained under the *monotonic optimization* framework [120–124]. However, the complexity of such algorithms increases exponentially with the number of users, and their generalization to the more general MIMO-IC remains unknown. A useful technique to solve the WSRMax problems for MIMO-ICs is via iteratively minimizing the weighted MSE, which utilizes the inherent relationship between the

mutual information and MSE [13, 142, 143]. The study of interference channels with game-theoretic models has also been given in [144–147]. Besides, several sum-rate maximization algorithms inspired by interference alignment have been proposed [125, 141, 148–152], e.g., the maximum SINR algorithm [152] and the interference-plus-noise leakage minimization algorithm [151].

3.1.4 Achievable Rate Region

As for the achievable rate region with interference treated as noise, the complete characterization has been obtained for the Gaussian SISO-IC [127], SIMO-IC [122], and MISO-IC [36, 128–130]. For the more general Gaussian MIMO-IC, some recent results have also been reported [131–133]. In general, the achievable rate region of multi-user systems can be characterized by its Pareto boundary, which consists of all the achievable rate-tuples at each of which it is impossible to improve one user’s rate, without simultaneously decreasing the rate of at least one of the other users. A traditional approach for characterizing Pareto boundaries of Gaussian interference channels is via solving WSRMax problems. An alternative method based on the concept of “rate profile” was applied in [129, 153, 154], which eventually results in solving a sequence of SINR feasibility problems that are easier to handle than WSRMax problems. It is worth mentioning that for the MISO-IC setup, it has been shown by various methods in [36, 129, 130] that all the rate-tuples on the Pareto boundary can be achieved with transmit beamforming, i.e., with rank-1 transmit covariance matrices.

3.1.5 Improper Gaussian Signaling

It is necessary to point out that in all the aforementioned resource optimization work for Gaussian interference channels, the transmitted signals are assumed to be *proper* or CSCG distributed. A key property of proper Gaussian RVs is that their second-order statistics are completely specified by the conventional covariance matrix under a zero-mean assumption. In contrast, for the more general *improper* or circularly *asymmetric* complex Gaussian RVs, an extra parameter called *pseudo-covariance* matrix is required for the complete second-order characterization [56, 60, 61]. While the impropriety of the signals has been exploited in various areas in signal processing, such as widely linear estimation [155], most of the prior studies on Gaussian interference channels have adopted the proper Gaussian assumption

without any justification. This may be due to the common practice of modeling the additive receiver noise as proper Gaussian, and the well-known maximum-entropy theorem [61], i.e., proper Gaussian RVs maximize the differential entropy for any given covariance matrix. As a result, proper Gaussian signaling has been shown to be capacity optimal for the Gaussian point-to-point channel [156], multiple-access channel (MAC) [157] and broadcast channel [22–24]. Only recently it was revealed that for Gaussian interference channels with interference treated as noise, improper Gaussian signaling provides a new opportunity to further improve the achievable rates over the conventional proper Gaussian signaling [158]. For instance, it was shown in [159] that improper Gaussian signaling, together with symbol extensions and interference alignment, is able to improve the DoF for the three-user SISO-IC with time-invariant channel coefficients at the asymptotically high SNR. In [160, 161], it was shown that the achievable rate region can be enlarged with improper Gaussian signaling even for the two-user SISO-IC at finite SNR.

The prior studies [159–161] on improper Gaussian signaling are all based on the equivalent double-sized real-valued MIMO-IC by separately dealing with the real and imaginary parts. Although any complex-valued system can be transformed into an equivalent real-valued system, as pointed out in [56, 57], much of the elegance of the system description is lost. Therefore, in this thesis, we will mainly focus on the complex-valued channel model for studying improper Gaussian signaling in Gaussian interference channels to gain new insights.

3.2 System Model

Consider a K -user MIMO-IC as shown in Figure 3.1, where transmitter k is intended to send message W_k to receiver k , while possibly interfering with all other $K - 1$ receivers. Denote the number of transmit and receive antennas for each user by M and N , respectively.¹ Assuming narrow-band transmission, the equivalent baseband received signal at each receiver can be expressed as

$$\mathbf{y}_k(n) = \mathbf{H}_{kk}\mathbf{x}_k(n) + \sum_{j \neq k} \mathbf{H}_{kj}\mathbf{x}_j(n) + \mathbf{n}_k(n), \quad k = 1, \dots, K, \quad (3.1)$$

¹The techniques developed in this thesis are also applicable to the scenario when each terminal is equipped with different number of antennas.

FIGURE 3.1: A K -user MIMO-IC with M transmit and N receive antennas.

where n is the symbol index, $\mathbf{H}_{kk} \in \mathbb{C}^{N \times M}$ denotes the direct channel matrix from transmitter k to receiver k , while $\mathbf{H}_{kj}, j \neq k$, denotes the interference channel matrix from transmitter j to receiver k ; we assume quasi-static fading and thus all channels are constant over n 's in (3.1); $\mathbf{n}_k(n)$ represents the independent and identically distributed (i.i.d.) CSCG noise vector at receiver k with $\mathbf{n}_k(n) \sim \mathcal{CN}(\mathbf{0}, \sigma^2 \mathbf{I})$; and $\mathbf{x}_k(n) \in \mathbb{C}^{M \times 1}$ is the transmitted signal vector from transmitter k , which is independent of $\mathbf{x}_j(n), \forall j \neq k$. In this thesis, for the purpose of exposition, we assume that symbol extensions over time as in [159] are not used. Hence, $\mathbf{x}_k(n)$ is independent over n . For brevity, the symbol index n is omitted in the rest of this thesis.

3.3 Gaussian Signaling: The Three Schemes

In this section, we discuss the three Gaussian signaling schemes, namely, the conventional proper complex Gaussian signaling, Gaussian signaling in the real domain, and the covariance/pseudo-covariance based improper complex Gaussian signaling. The rate regions achieved by these three schemes are analytically compared.

3.3.1 Proper Gaussian Signaling

It has been widely assumed in the literature, either implicitly or explicitly, that proper complex Gaussian (or CSCG) signaling is employed for the MIMO-IC given

in (3.1), i.e, the transmitted signal vector $\mathbf{x}_k$ is assumed to be distributed as $\mathbf{x}_k \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_{\mathbf{x}_k})$, $\forall k$, where $\mathbf{C}_{\mathbf{x}_k}$ denotes the transmit covariance matrix of user k . Under the assumption that interference is treated as noise and perfect channel state information is available, the achievable rate for user k is given by the following well-known formula [162]:

$$R_{k,\text{proper}}(\{\mathbf{C}_{\mathbf{x}_j}\}) = \log \frac{|\sigma^2 \mathbf{I} + \sum_{j=1}^K \mathbf{H}_{kj} \mathbf{C}_{\mathbf{x}_j} \mathbf{H}_{kj}^H|}{|\sigma^2 \mathbf{I} + \sum_{j \neq k} \mathbf{H}_{kj} \mathbf{C}_{\mathbf{x}_j} \mathbf{H}_{kj}^H|}, \quad \forall k. \quad (3.2)$$

The achievable rate region for the K -user MIMO-IC consists of all the rate-tuples for all users that can be simultaneously achieved under a given set of transmit power constraints for each transmitter, denoted by P_j , $j = 1, \dots, K$. With (3.2), the achievable rate region for the MIMO-IC (3.1) with proper Gaussian signaling can be expressed as

$$\mathcal{R}_{\text{proper}} \triangleq \bigcup_{\{\mathbf{C}_{\mathbf{x}_j} \succeq \mathbf{0}: \text{Tr}(\mathbf{C}_{\mathbf{x}_j}) \leq P_j, \forall j\}} \left\{ R_{1,\text{proper}}(\{\mathbf{C}_{\mathbf{x}_j}\}), \dots, R_{K,\text{proper}}(\{\mathbf{C}_{\mathbf{x}_j}\}) \right\}. \quad (3.3)$$

For any given transmit covariance matrix $\mathbf{C}_{\mathbf{x}_k}$ with $\text{rank}(\mathbf{C}_{\mathbf{x}_k}) = r_k$, the transmitted signal $\mathbf{x}_k$ can be obtained by the following linear precoding scheme

$$\mathbf{x}_k = \mathbf{C}_{\mathbf{x}_k}^{\frac{1}{2}} \mathbf{d}_k, \quad (3.4)$$

where $\mathbf{d}_k \in \mathbb{C}^{r_k \times 1}$ is the information-bearing signal distributed according to $\mathbf{d}_k \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{r_k})$, and $\mathbf{C}_{\mathbf{x}_k}^{\frac{1}{2}} \in \mathbb{C}^{M \times r_k}$ is the precoding matrix satisfying $\mathbf{C}_{\mathbf{x}_k} = \mathbf{C}_{\mathbf{x}_k}^{\frac{1}{2}} \mathbf{C}_{\mathbf{x}_k}^{\frac{1}{2}H}$. In terms of the real-valued representation, (3.4) can be equivalently written as

$$\underbrace{\begin{bmatrix} \Re\{\mathbf{x}_k\} \\ \Im\{\mathbf{x}_k\} \end{bmatrix}}_{\bar{\mathbf{x}}_k} = \underbrace{\begin{bmatrix} \Re\{\mathbf{C}_{\mathbf{x}_k}^{\frac{1}{2}}\} & -\Im\{\mathbf{C}_{\mathbf{x}_k}^{\frac{1}{2}}\} \\ \Im\{\mathbf{C}_{\mathbf{x}_k}^{\frac{1}{2}}\} & \Re\{\mathbf{C}_{\mathbf{x}_k}^{\frac{1}{2}}\} \end{bmatrix}}_{\bar{\mathbf{W}}_k} \underbrace{\begin{bmatrix} \Re\{\mathbf{d}_k\} \\ \Im\{\mathbf{d}_k\} \end{bmatrix}}_{\bar{\mathbf{d}}_k}. \quad (3.5)$$

3.3.2 Gaussian Signaling in the Real Domain

By separating the real and imaginary parts of (3.1), we obtain the following equivalent $2N \times 2M$ real-valued MIMO-IC:

$$\underbrace{\begin{bmatrix} \Re\{\mathbf{y}_k\} \\ \Im\{\mathbf{y}_k\} \end{bmatrix}}_{\bar{\mathbf{y}}_k} = \sum_{j=1}^K \underbrace{\begin{bmatrix} \Re\{\mathbf{H}_{kj}\} & -\Im\{\mathbf{H}_{kj}\} \\ \Im\{\mathbf{H}_{kj}\} & \Re\{\mathbf{H}_{kj}\} \end{bmatrix}}_{\bar{\mathbf{H}}_{kj}} \underbrace{\begin{bmatrix} \Re\{\mathbf{x}_j\} \\ \Im\{\mathbf{x}_j\} \end{bmatrix}}_{\bar{\mathbf{x}}_j} + \underbrace{\begin{bmatrix} \Re\{\mathbf{n}_k\} \\ \Im\{\mathbf{n}_k\} \end{bmatrix}}_{\bar{\mathbf{n}}_k}, \quad (3.6)$$

$$\bar{\mathbf{y}}_k = \bar{\mathbf{H}}_{kk} \bar{\mathbf{x}}_k + \underbrace{\sum_{j \neq k} \bar{\mathbf{H}}_{kj} \bar{\mathbf{x}}_j}_{\bar{\mathbf{s}}_k} + \bar{\mathbf{n}}_k, \quad (3.7)$$

where $\bar{\mathbf{y}}_k \in \mathbb{R}^{2N \times 1}$ and $\bar{\mathbf{x}}_k \in \mathbb{R}^{2M \times 1}$ represent the received and transmitted signals of user k , respectively; $\bar{\mathbf{n}}_k \sim \mathcal{N}(\mathbf{0}, \frac{\sigma^2}{2} \mathbf{I}_{2N})$ is the real-valued Gaussian noise vector, and $\bar{\mathbf{H}}_{kj} \in \mathbb{R}^{2N \times 2M}$ is the real-valued channel matrix from transmitter j to receiver k , which has the specific structure as in (3.6). Note that in (3.7), we have denoted by $\bar{\mathbf{s}}_k$ the received interference plus noise at receiver k .

With Gaussian signaling applied to (3.7), $\bar{\mathbf{x}}_k$ is assumed to be a zero-mean real-valued Gaussian RV with covariance matrix $\bar{\mathbf{Q}}_{\mathbf{x}_k}$, i.e., $\bar{\mathbf{x}}_k \sim \mathcal{N}(\mathbf{0}, \bar{\mathbf{Q}}_{\mathbf{x}_k})$, $\forall k$. It then follows that both $\bar{\mathbf{y}}_k$ and $\bar{\mathbf{s}}_k$ are Gaussian distributed with

$$\bar{\mathbf{y}}_k \sim \mathcal{N}(\mathbf{0}, \bar{\mathbf{Q}}_{\mathbf{y}_k}), \quad \bar{\mathbf{s}}_k \sim \mathcal{N}(\mathbf{0}, \bar{\mathbf{Q}}_{\mathbf{s}_k}), \quad (3.8)$$

where

$$\bar{\mathbf{Q}}_{\mathbf{y}_k} = \sum_{j=1}^K \bar{\mathbf{H}}_{kj} \bar{\mathbf{Q}}_{\mathbf{x}_j} \bar{\mathbf{H}}_{kj}^T + \frac{\sigma^2}{2} \mathbf{I}_{2N}, \quad (3.9)$$

$$\bar{\mathbf{Q}}_{\mathbf{s}_k} = \sum_{j \neq k} \bar{\mathbf{H}}_{kj} \bar{\mathbf{Q}}_{\mathbf{x}_j} \bar{\mathbf{H}}_{kj}^T + \frac{\sigma^2}{2} \mathbf{I}_{2N}. \quad (3.10)$$

The achievable rate of user k for the MIMO-IC (3.7) can be written as [162]

$$R_{k,\text{real}}(\{\bar{\mathbf{Q}}_{\mathbf{x}_j}\}) = I(\bar{\mathbf{x}}_k; \bar{\mathbf{y}}_k) = h(\bar{\mathbf{y}}_k) - h(\bar{\mathbf{y}}_k | \bar{\mathbf{x}}_k) = h(\bar{\mathbf{y}}_k) - h(\bar{\mathbf{s}}_k) \quad (3.11)$$

$$= \frac{1}{2} \log((2\pi e)^{2N} |\bar{\mathbf{Q}}_{\mathbf{y}_k}|) - \frac{1}{2} \log((2\pi e)^{2N} |\bar{\mathbf{Q}}_{\mathbf{s}_k}|) \quad (3.12)$$

$$= \frac{1}{2} \log \frac{\left| \sum_{j=1}^K \bar{\mathbf{H}}_{kj} \bar{\mathbf{Q}}_{\mathbf{x}_j} \bar{\mathbf{H}}_{kj}^T + \frac{\sigma^2}{2} \mathbf{I}_{2N} \right|}{\left| \sum_{j \neq k} \bar{\mathbf{H}}_{kj} \bar{\mathbf{Q}}_{\mathbf{x}_j} \bar{\mathbf{H}}_{kj}^T + \frac{\sigma^2}{2} \mathbf{I}_{2N} \right|}. \quad (3.13)$$

The achievable rate region for the MIMO-IC (3.1) with Gaussian signaling applied in the real domain can then be expressed as

$$\mathcal{R}_{\text{real}} \triangleq \bigcup_{\{\bar{\mathbf{Q}}_{\mathbf{x}_j} \succeq \mathbf{0}: \text{Tr}(\bar{\mathbf{Q}}_{\mathbf{x}_j}) \leq P_j, \forall j\}} \left\{ R_{1,\text{real}}(\{\bar{\mathbf{Q}}_{\mathbf{x}_j}\}), \dots, R_{K,\text{real}}(\{\bar{\mathbf{Q}}_{\mathbf{x}_j}\}) \right\}. \quad (3.14)$$

For any given transmit covariance matrix $\bar{\mathbf{Q}}_{\mathbf{x}_k}$ for (3.7) with $\text{rank}(\bar{\mathbf{Q}}_{\mathbf{x}_k}) = \bar{r}_k$, the transmitted signal $\bar{\mathbf{x}}_k$ can be obtained by linear precoding in real domain as

$$\bar{\mathbf{x}}_k = \bar{\mathbf{Q}}_{\mathbf{x}_k}^{\frac{1}{2}} \bar{\mathbf{d}}_k, \quad (3.15)$$

where $\bar{\mathbf{d}}_k \in \mathbb{R}^{\bar{r}_k \times 1}$ is the information-bearing signal distributed according to $\bar{\mathbf{d}}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_{\bar{r}_k})$, and $\bar{\mathbf{Q}}_{\mathbf{x}_k}^{\frac{1}{2}} \in \mathbb{R}^{2M \times \bar{r}_k}$ is the real-valued precoding matrix satisfying $\bar{\mathbf{Q}}_{\mathbf{x}_k} = \bar{\mathbf{Q}}_{\mathbf{x}_k}^{\frac{1}{2}} \bar{\mathbf{Q}}_{\mathbf{x}_k}^{\frac{1}{2}T}$.

Based on the rate expressions (3.2) and (3.13), for the same set of transmit power constraints $P_1, \dots, P_K$, the relationship between the two achievable rate regions $\mathcal{R}_{\text{proper}}$ defined in (3.3) and $\mathcal{R}_{\text{real}}$ given in (3.14) is not immediately clear, e.g., whether or not one of them is a subset of the other. With the pseudo-covariance based improper complex Gaussian signaling introduced in the next subsection, we will show that the rate region achieved by the conventional proper Gaussian signaling can be a strictly subset of that achieved by the Gaussian signaling applied in the real domain, i.e., $\mathcal{R}_{\text{proper}} \subset \mathcal{R}_{\text{real}}$ holds.

3.3.3 Improper Gaussian Signaling: Pseudo-Covariance Based Approach

In this subsection, we derive the achievable rate with improper complex Gaussian signaling applied to the MIMO-IC (3.1) in terms of the transmit covariance and pseudo-covariance matrices. Denote the covariance and pseudo-covariance matrices of the zero-mean transmitted Gaussian RV $\mathbf{x}_k$ by $\mathbf{C}_{\mathbf{x}_k}$ and $\tilde{\mathbf{C}}_{\mathbf{x}_k}$, respectively, i.e.,

$$\mathbf{C}_{\mathbf{x}_k} = \mathbb{E}(\mathbf{x}_k \mathbf{x}_k^H), \quad \tilde{\mathbf{C}}_{\mathbf{x}_k} = \mathbb{E}(\mathbf{x}_k \mathbf{x}_k^T), \quad k = 1, \dots, K. \quad (3.16)$$

Since $\mathbf{x}_k$ and $\mathbf{x}_j$ are independent for $j \neq k$, then by using Lemma 2.2 and the fact that independence and uncorrelatedness are equivalent for Gaussian RVs, the covariance and pseudo-covariance matrices of the received signal vector $\mathbf{y}_k$, $k =$

$1, \dots, K$, can be obtained as

$$\mathbf{C}_{\mathbf{y}_k} = \mathbb{E}(\mathbf{y}_k \mathbf{y}_k^H) = \sum_{j=1}^K \mathbf{H}_{kj} \mathbf{C}_{\mathbf{x}_j} \mathbf{H}_{kj}^H + \sigma^2 \mathbf{I}, \quad (3.17)$$

$$\tilde{\mathbf{C}}_{\mathbf{y}_k} = \mathbb{E}(\mathbf{y}_k \mathbf{y}_k^T) = \sum_{j=1}^K \mathbf{H}_{kj} \tilde{\mathbf{C}}_{\mathbf{x}_j} \mathbf{H}_{kj}^T, \quad (3.18)$$

where in (3.18), we have used the fact that the pseudo-covariance of the CSCG noise vector $\mathbf{n}_k$ is a zero matrix. It can be observed from (3.17) that $\mathbf{C}_{\mathbf{y}_k}$ is a nonsingular square matrix. Then with the augmented covariance matrix $\underline{\mathbf{C}}_{\mathbf{y}_k}$ defined in (2.14) and using the Schur complement result [163], we obtain

$$\begin{aligned} |\underline{\mathbf{C}}_{\mathbf{y}_k}| &= \begin{vmatrix} \mathbf{C}_{\mathbf{y}_k} & \tilde{\mathbf{C}}_{\mathbf{y}_k} \\ \tilde{\mathbf{C}}_{\mathbf{y}_k}^* & \mathbf{C}_{\mathbf{y}_k}^* \end{vmatrix} = |\mathbf{C}_{\mathbf{y}_k}| |\mathbf{C}_{\mathbf{y}_k}^* - \tilde{\mathbf{C}}_{\mathbf{y}_k}^* \mathbf{C}_{\mathbf{y}_k}^{-1} \tilde{\mathbf{C}}_{\mathbf{y}_k}| \\ &= |\mathbf{C}_{\mathbf{y}_k}|^2 |\mathbf{I} - \mathbf{C}_{\mathbf{y}_k}^{-1} \tilde{\mathbf{C}}_{\mathbf{y}_k} \mathbf{C}_{\mathbf{y}_k}^{-T} \tilde{\mathbf{C}}_{\mathbf{y}_k}^H|, \end{aligned} \quad (3.19)$$

where in the last equality, we have used the identities $|\mathbf{A}| = |\mathbf{A}^T|$, $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$, and the facts that $\mathbf{C}_{\mathbf{y}_k}$ is Hermitian and $\tilde{\mathbf{C}}_{\mathbf{y}_k}$ is symmetric. With the transmitted signals being Gaussian, the received signal $\mathbf{y}_k$ is also Gaussian. Then based on Theorem 2.6, the differential entropy of $\mathbf{y}_k \in \mathbb{C}^{N \times 1}$ is given by

$$\begin{aligned} h(\mathbf{y}_k) &= \frac{1}{2} \log((\pi e)^{2N} |\underline{\mathbf{C}}_{\mathbf{y}_k}|) \\ &= \log((\pi e)^N |\mathbf{C}_{\mathbf{y}_k}|) + \frac{1}{2} \log |\mathbf{I} - \mathbf{C}_{\mathbf{y}_k}^{-1} \tilde{\mathbf{C}}_{\mathbf{y}_k} \mathbf{C}_{\mathbf{y}_k}^{-T} \tilde{\mathbf{C}}_{\mathbf{y}_k}^H|. \end{aligned} \quad (3.20)$$

For the input-output relationship in (3.1), denote $\mathbf{s}_k$ as the interference-plus-noise term at receiver k , i.e., $\mathbf{s}_k = \sum_{j \neq k} \mathbf{H}_{kj} \mathbf{x}_j + \mathbf{n}_k$. Then the covariance and pseudo-covariance matrices of $\mathbf{s}_k$ are given by

$$\mathbf{C}_{\mathbf{s}_k} = \mathbb{E}(\mathbf{s}_k \mathbf{s}_k^H) = \sum_{j \neq k} \mathbf{H}_{kj} \mathbf{C}_{\mathbf{x}_j} \mathbf{H}_{kj}^H + \sigma^2 \mathbf{I}, \quad (3.21)$$

$$\tilde{\mathbf{C}}_{\mathbf{s}_k} = \mathbb{E}(\mathbf{s}_k \mathbf{s}_k^T) = \sum_{j \neq k} \mathbf{H}_{kj} \tilde{\mathbf{C}}_{\mathbf{x}_j} \mathbf{H}_{kj}^T. \quad (3.22)$$

Similarly, the differential entropy of $\mathbf{s}_k$ can be expressed as

$$h(\mathbf{s}_k) = \log((\pi e)^N |\mathbf{C}_{\mathbf{s}_k}|) + \frac{1}{2} \log |\mathbf{I} - \mathbf{C}_{\mathbf{s}_k}^{-1} \tilde{\mathbf{C}}_{\mathbf{s}_k} \mathbf{C}_{\mathbf{s}_k}^{-T} \tilde{\mathbf{C}}_{\mathbf{s}_k}^H|. \quad (3.23)$$

Under the assumption that interference is treated as noise, the achievable rate at receiver k with improper Gaussian signaling can be obtained as

$$R_{k,\text{improper}}(\{\underline{\mathbf{C}}_{\mathbf{x}_j}\}) = I(\mathbf{x}_k; \mathbf{y}_k) = h(\mathbf{y}_k) - h(\mathbf{y}_k | \mathbf{x}_k) = h(\mathbf{y}_k) - h(\mathbf{s}_k) \quad (3.24)$$

$$= \frac{1}{2} \log((\pi e)^{2N} |\underline{\mathbf{C}}_{\mathbf{y}_k}|) - \frac{1}{2} \log((\pi e)^{2N} |\underline{\mathbf{C}}_{\mathbf{s}_k}|) \quad (3.25)$$

$$= \underbrace{\log \frac{|\sigma^2 \mathbf{I} + \sum_{j=1}^K \mathbf{H}_{kj} \mathbf{C}_{\mathbf{x}_j} \mathbf{H}_{kj}^H|}{|\sigma^2 \mathbf{I} + \sum_{j \neq k} \mathbf{H}_{kj} \mathbf{C}_{\mathbf{x}_j} \mathbf{H}_{kj}^H|}}_{\triangleq R_{k,\text{proper}}(\{\mathbf{C}_{\mathbf{x}_j}\})} + \frac{1}{2} \log \frac{|\mathbf{I} - \mathbf{C}_{\mathbf{y}_k}^{-1} \tilde{\mathbf{C}}_{\mathbf{y}_k} \mathbf{C}_{\mathbf{y}_k}^{-T} \tilde{\mathbf{C}}_{\mathbf{y}_k}^H|}{|\mathbf{I} - \mathbf{C}_{\mathbf{s}_k}^{-1} \tilde{\mathbf{C}}_{\mathbf{s}_k} \mathbf{C}_{\mathbf{s}_k}^{-T} \tilde{\mathbf{C}}_{\mathbf{s}_k}^H|}. \quad (3.26)$$

Note that $\underline{\mathbf{C}}_{\mathbf{x}_j}$ is the augmented covariance matrix of $\mathbf{x}_j$ as defined in (2.14), which includes as submatrices the covariance matrix $\mathbf{C}_{\mathbf{x}_j}$ and the pseudo-covariance matrix $\tilde{\mathbf{C}}_{\mathbf{x}_j}$.

With (3.26), the achievable rate region of (3.1) with improper complex Gaussian signaling can then be expressed as

$$\mathcal{R}_{\text{improper}} \triangleq \bigcup_{\{\underline{\mathbf{C}}_{\mathbf{x}_j} \succeq \mathbf{0}: \text{Tr}(\underline{\mathbf{C}}_{\mathbf{x}_j}) \leq 2P_j, \forall j\}} \left\{ R_{1,\text{improper}}(\{\underline{\mathbf{C}}_{\mathbf{x}_j}\}), \dots, R_{K,\text{improper}}(\{\underline{\mathbf{C}}_{\mathbf{x}_j}\}) \right\}, \quad (3.27)$$

where the constraint $\underline{\mathbf{C}}_{\mathbf{x}_j} \succeq \mathbf{0}$ follows from Theorem 2.1, and $\text{Tr}(\underline{\mathbf{C}}_{\mathbf{x}_j}) \leq 2P_j$ is equivalent to the power constraint $\text{Tr}(\mathbf{C}_{\mathbf{x}_j}) \leq P_j$ due to the identity $\text{Tr}(\underline{\mathbf{C}}_{\mathbf{x}_j}) = 2\text{Tr}(\mathbf{C}_{\mathbf{x}_j})$, as evident from (2.14).

Equation (3.26) shows that with improper complex Gaussian signaling, the achievable rate of the K -user MIMO-IC can be expressed as a summation of two terms. The first term, denoted by $R_{k,\text{proper}}(\{\mathbf{C}_{\mathbf{x}_j}\})$, is the rate achievable by the conventional proper Gaussian signaling as shown in (3.2), which is a function of the users' transmit covariance matrices only. The second term is a function of both the covariance and pseudo-covariance matrices of the transmitted signals. For the special case of $\tilde{\mathbf{C}}_{\mathbf{x}_j} = \mathbf{0}, \forall j$, the second term in (3.26) vanishes and it reduces to the rate expression for the conventional proper Gaussian signaling. Therefore, the rate formula (3.26) generalizes the well-known rate expression (3.2) in the sense that it is applicable to the more general improper Gaussian signaling with non-zero pseudo-covariance matrices. Furthermore, the separability of the achievable rate expression by improper Gaussian signaling provides a general method to improve

the achievable rate over the conventional proper Gaussian signaling, i.e., for any given covariance matrices obtained by existing proper Gaussian signaling schemes, the rate can be improved with improper Gaussian signaling by choosing the pseudo-covariance matrices that make the second term in (3.26) strictly positive.

Remark 3.1. From user k 's perspective, if improper complex Gaussian signaling has been employed by other transmitters and under the assumption that the resulting interference is treated as additional noise over existing background noise at the receiver, user k essentially communicates over a point-to-point MIMO channel corrupted by **improper** or circularly **asymmetric** complex Gaussian noise. Therefore, the well-known result that proper Gaussian signaling is optimal [156], which is applicable for point-to-point channels with circularly **symmetric** complex Gaussian noise only, does not apply here. This thus motivates our current work on investigating the more general improper Gaussian signaling for interference channels.

3.3.4 Rate Region Comparison

In this subsection, the rate regions achievable by the three Gaussian signaling schemes discussed previously are compared.

Lemma 3.2. *For the Gaussian MIMO-IC given in (3.1), the achievable rate regions $\mathcal{R}_{\text{proper}}$ and $\mathcal{R}_{\text{improper}}$ defined in (3.3) and (3.27) respectively have the following relationship:*

$$\mathcal{R}_{\text{proper}} \subset \mathcal{R}_{\text{improper}}. \quad (3.28)$$

Proof. To prove $\mathcal{R}_{\text{proper}} \subset \mathcal{R}_{\text{improper}}$, we will show that $\mathcal{R}_{\text{proper}} \subseteq \mathcal{R}_{\text{improper}}$ and $\mathcal{R}_{\text{improper}} \not\subseteq \mathcal{R}_{\text{proper}}$ in general. In fact, the statement $\mathcal{R}_{\text{proper}} \subseteq \mathcal{R}_{\text{improper}}$ readily follows from the rate expression (3.26). Specifically, for any rate-tuple $(R_1, \dots, R_K) \in \mathcal{R}_{\text{proper}}$ that is achievable by the conventional proper Gaussian signaling scheme with the set of transmit covariance matrices $\{\hat{\mathbf{C}}_{\mathbf{x}_j}\}$, it is also achievable by the more general improper Gaussian signaling with the same set of covariance matrices and zero pseudo-covariance matrices, i.e., with $\{\mathbf{C}_{\mathbf{x}_j} = \hat{\mathbf{C}}_{\mathbf{x}_j}, \tilde{\mathbf{C}}_{\mathbf{x}_j} = \mathbf{0}\}$, or $\{\underline{\mathbf{C}}_{\mathbf{x}_j} = \text{blkdiag}(\hat{\mathbf{C}}_{\mathbf{x}_j}, \hat{\mathbf{C}}_{\mathbf{x}_j}^*)\}$.

To show that $\mathcal{R}_{\text{improper}} \not\subseteq \mathcal{R}_{\text{proper}}$, it is sufficient to show that for one particular channel realization of (3.1), $\mathcal{R}_{\text{proper}}$ is a strictly subset of $\mathcal{R}_{\text{improper}}$. To this end, we

consider the special case of two-user SISO-IC ($K = 2$ and $M = 1$ in (3.1)), with

$$h_{kj} = 1, \quad \forall k, j = 1, 2, \quad \sigma^2 = 1, \quad P_1 = P_2 = 1. \quad (3.29)$$

Define the rate-pair $(R_1, R_2) = (\log \sqrt{3}, \log \sqrt{3})$. We will show that for the two-user SISO-IC given in (3.29), $(R_1, R_2) \in \mathcal{R}_{\text{improper}}$, but $(R_1, R_2) \notin \mathcal{R}_{\text{proper}}$. First, it is easy to verify based on (3.26) that with improper Gaussian signaling, the rate-pair $(\log \sqrt{3}, \log \sqrt{3})$ can be achieved by letting the transmit covariances and pseudo-covariances² respectively given by $C_{x_1} = C_{x_2} = 1$, $\tilde{C}_{x_1} = 1$, and $\tilde{C}_{x_2} = -1$. Next, we show by contradiction that $(R_1, R_2) \notin \mathcal{R}_{\text{proper}}$. Suppose, on the contrary, that the rate-pair $(\log \sqrt{3}, \log \sqrt{3})$ is also achievable by the proper complex Gaussian signaling scheme. It then follows from (3.2) that there exists a power assignment policy³ (C_{x_1}, C_{x_2}) such that

$$\log \left(1 + \frac{C_{x_1}}{1 + C_{x_2}} \right) \geq \log \sqrt{3}, \quad (3.30)$$

$$\log \left(1 + \frac{C_{x_2}}{1 + C_{x_1}} \right) \geq \log \sqrt{3}, \quad (3.31)$$

which leads to the following requirements on the power levels

$$C_{x_1} \geq \frac{\sqrt{3} - 1}{2 - \sqrt{3}} = 2.73 > P_1, \quad (3.32)$$

$$C_{x_2} \geq \frac{\sqrt{3} - 1}{2 - \sqrt{3}} = 2.73 > P_2. \quad (3.33)$$

(3.32) and (3.33) obviously violate the maximum power constraints $C_{x_1} \leq P_1$ and $C_{x_2} \leq P_2$. Therefore, the rate pair $(\log \sqrt{3}, \log \sqrt{3})$ is not achievable by the conventional proper Gaussian signaling scheme, i.e., $(R_1, R_2) \notin \mathcal{R}_{\text{proper}}$.

This thus completes the proof of Lemma 3.2. $\square$

Lemma 3.2 shows that the conventional proper complex Gaussian signaling scheme, which has been widely assumed in the literature, is in fact strictly sub-optimal in terms of the achievable rate region for general Gaussian MIMO-IC. To gain further insights, in Section 3.6.1, we analytically show that the maximum sum-rate achieved by the optimal proper Gaussian signaling can be strictly improved

²For the special case of single antenna at the transmitters, i.e., $M = 1$, we use the notation C_{x_j} and $\tilde{C}_{x_j}$ to denote the covariance and pseudo-covariance of the transmitted signal by user j .

³Note that when the transmitter has single antenna only ($M = 1$), the covariance C_{x_j} is equal to the power transmitted by user j .

if non-zero pseudo-covariances are allowed, which further affirms the superiority of the more general improper Gaussian signaling introduced in Section 3.3.3 over the conventional proper Gaussian signaling scheme for general interference channels.

Lemma 3.3. *For the Gaussian MIMO-IC given in (3.1), the achievable rate regions $\mathcal{R}_{\text{real}}$ and $\mathcal{R}_{\text{improper}}$ defined in (3.14) and (3.27) respectively satisfy*

$$\mathcal{R}_{\text{real}} = \mathcal{R}_{\text{improper}}. \quad (3.34)$$

Proof. To show $\mathcal{R}_{\text{real}} = \mathcal{R}_{\text{improper}}$, we will show that both $\mathcal{R}_{\text{real}} \subseteq \mathcal{R}_{\text{improper}}$ and $\mathcal{R}_{\text{improper}} \subseteq \mathcal{R}_{\text{real}}$ hold.

First, for any given rate-tuple $(R_1, \dots, R_K) \in \mathcal{R}_{\text{real}}$ achievable by the Gaussian signaling scheme applied in the real domain described in Section 3.3.2, we denote the set of transmit covariance matrices as $\{\bar{\mathbf{Q}}_{\mathbf{x}_j}\}$. We thus have

$$\bar{\mathbf{Q}}_{\mathbf{x}_j} \succeq \mathbf{0}, \quad \text{Tr}(\bar{\mathbf{Q}}_{\mathbf{x}_j}) \leq P_j, \quad j = 1, \dots, K, \quad (3.35)$$

$$R_{k,\text{real}}(\{\bar{\mathbf{Q}}_{\mathbf{x}_j}\}) = \frac{1}{2} \log \frac{|\bar{\mathbf{Q}}_{\mathbf{y}_k}|}{|\bar{\mathbf{Q}}_{\mathbf{s}_k}|} = R_k, \quad k = 1, \dots, K, \quad (3.36)$$

where (3.36) follows from (3.12) with $\bar{\mathbf{Q}}_{\mathbf{y}_k}$ and $\bar{\mathbf{Q}}_{\mathbf{s}_k}$ given by (3.9) and (3.10), respectively. We will show that the same rate-tuple is also achievable by the pseudo-covariance based improper complex Gaussian signaling discussed in Section 3.3.3, i.e., $(R_1, \dots, R_K) \in \mathcal{R}_{\text{improper}}$. To this end, the augmented transmit covariance matrix $\underline{\mathbf{C}}_{\mathbf{x}_j}$ should be set as

$$\underline{\mathbf{C}}_{\mathbf{x}_j} = \mathbf{M}_M \bar{\mathbf{Q}}_{\mathbf{x}_j} \mathbf{M}_M^H, \quad j = 1, \dots, K, \quad (3.37)$$

where the matrix $\mathbf{M}_M$ is defined in (2.5), which is unitary up to a factor of 2, i.e., $\mathbf{M}_M \mathbf{M}_M^H = \mathbf{M}_M^H \mathbf{M}_M = 2\mathbf{I}_{2M}$. It then follows from (3.35) that $\underline{\mathbf{C}}_{\mathbf{x}_j}$ satisfies the constraints specified in (3.27), i.e.,

$$\underline{\mathbf{C}}_{\mathbf{x}_j} \succeq \mathbf{0}, \quad (3.38)$$

$$\text{Tr}(\underline{\mathbf{C}}_{\mathbf{x}_j}) = \text{Tr}(\mathbf{M}_M \bar{\mathbf{Q}}_{\mathbf{x}_j} \mathbf{M}_M^H) = 2\text{Tr}(\bar{\mathbf{Q}}_{\mathbf{x}_j}) \leq 2P_j, \quad j = 1, \dots, K. \quad (3.39)$$

Furthermore, based on (3.25), the achievable rates with (3.37) are given by

$$R_{k,\text{improper}}(\{\underline{\mathbf{C}}_{\mathbf{x}_j}\}) = \frac{1}{2} \log \frac{|\underline{\mathbf{C}}_{\mathbf{y}_k}|}{|\underline{\mathbf{C}}_{\mathbf{s}_k}|}. \quad (3.40)$$

To relate (3.40) to (3.36), it is noted that $\underline{\mathbf{C}}_{\mathbf{y}_k} = \mathbb{E}(\underline{\mathbf{y}}_k \underline{\mathbf{y}}_k^H)$, where $\underline{\mathbf{y}}_k$ is the augmented vector of $\mathbf{y}_k$ as defined in (2.3). It then follows from (3.1) that

$$\underbrace{\begin{bmatrix} \mathbf{y}_k \\ \mathbf{y}_k^* \end{bmatrix}}_{\underline{\mathbf{y}}_k} = \sum_{j=1}^K \underbrace{\begin{bmatrix} \mathbf{H}_{kj} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{kj}^* \end{bmatrix}}_{\underline{\mathbf{H}}_{kj}} \underbrace{\begin{bmatrix} \mathbf{x}_j \\ \mathbf{x}_j^* \end{bmatrix}}_{\underline{\mathbf{x}}_j} + \underbrace{\begin{bmatrix} \mathbf{n}_k \\ \mathbf{n}_k^* \end{bmatrix}}_{\underline{\mathbf{n}}_k}, \quad (3.41)$$

or

$$\underline{\mathbf{y}}_k = \sum_{j=1}^K \underline{\mathbf{H}}_{kj} \underline{\mathbf{x}}_j + \underline{\mathbf{n}}_k. \quad (3.42)$$

It can be verified that $\underline{\mathbf{H}}_{kj}$ defined in (3.41) and $\bar{\mathbf{H}}_{kj}$ given in (3.6) are related as

$$\underline{\mathbf{H}}_{kj} = \frac{1}{2} \mathbf{M}_N \bar{\mathbf{H}}_{kj} \mathbf{M}_M^H, \quad \bar{\mathbf{H}}_{kj} = \frac{1}{2} \mathbf{M}_N^H \underline{\mathbf{H}}_{kj} \mathbf{M}_M. \quad (3.43)$$

With (3.37), (3.42), (3.43), and the fact that $\underline{\mathbf{C}}_{\mathbf{x}_j} = \mathbb{E}(\underline{\mathbf{x}}_j \underline{\mathbf{x}}_j^H)$, we have

$$\underline{\mathbf{C}}_{\mathbf{y}_k} = \sum_{j=1}^K \underline{\mathbf{H}}_{kj} \underline{\mathbf{C}}_{\mathbf{x}_j} \underline{\mathbf{H}}_{kj}^H + \sigma^2 \mathbf{I}_{2N} \quad (3.44)$$

$$= \sum_{j=1}^K \left(\frac{1}{2} \mathbf{M}_N \bar{\mathbf{H}}_{kj} \mathbf{M}_M^H \right) (\mathbf{M}_M \bar{\mathbf{Q}}_{\mathbf{x}_j} \mathbf{M}_M^H) \left(\frac{1}{2} \mathbf{M}_N \bar{\mathbf{H}}_{kj} \mathbf{M}_M^H \right)^H + \sigma^2 \mathbf{I}_{2N} \quad (3.45)$$

$$= \mathbf{M}_N \left(\sum_{j=1}^K \bar{\mathbf{H}}_{kj} \bar{\mathbf{Q}}_{\mathbf{x}_j} \bar{\mathbf{H}}_{kj}^T + \frac{\sigma^2}{2} \mathbf{I}_{2N} \right) \mathbf{M}_N^H \quad (3.46)$$

$$= \mathbf{M}_N \bar{\mathbf{Q}}_{\mathbf{y}_k} \mathbf{M}_N^H. \quad (3.47)$$

Similarly, we can obtain that

$$\underline{\mathbf{C}}_{\mathbf{s}_k} = \mathbf{M}_N \bar{\mathbf{Q}}_{\mathbf{s}_k} \mathbf{M}_N^H. \quad (3.48)$$

By substituting (3.47) and (3.48) into (3.40), we have

$$R_{k,\text{improper}}(\{\underline{\mathbf{C}}_{\mathbf{x}_j}\}) = \frac{1}{2} \log \frac{|\mathbf{M}_N \bar{\mathbf{Q}}_{\mathbf{y}_k} \mathbf{M}_N^H|}{|\mathbf{M}_N \bar{\mathbf{Q}}_{\mathbf{s}_k} \mathbf{M}_N^H|} = \frac{1}{2} \log \frac{|\bar{\mathbf{Q}}_{\mathbf{y}_k}|}{|\bar{\mathbf{Q}}_{\mathbf{s}_k}|} = R_k, \quad \forall k. \quad (3.49)$$

Therefore, the rate-tuple $(R_1, \dots, R_K)$ is also achievable with the pseudo-covariance based improper complex Gaussian signaling with the augmented covariance matrix $\underline{\mathbf{C}}_{\mathbf{x}_j}$ determined by (3.37). We thus have proved that $\mathcal{R}_{\text{real}} \subseteq \mathcal{R}_{\text{improper}}$.

Since the transformations (3.37), (3.47), and (3.48) are all invertible, we can similarly prove that $\mathcal{R}_{\text{improper}} \subseteq \mathcal{R}_{\text{real}}$. The details are omitted for brevity.

This thus completes the proof of Lemma 3.3. $\square$

By combining Lemma 3.2 and Lemma 3.3, we obtain the following result.

Theorem 3.4. *For the Gaussian MIMO-IC given in (3.1), the achievable rate regions $\mathcal{R}_{\text{proper}}$, $\mathcal{R}_{\text{real}}$, and $\mathcal{R}_{\text{improper}}$ defined in (3.3), (3.14), and (3.27), respectively, satisfy the following relationships:*

$$\mathcal{R}_{\text{proper}} \subset \mathcal{R}_{\text{improper}} = \mathcal{R}_{\text{real}}. \quad (3.50)$$

Theorem 3.4 shows that Gaussian signaling over real domain is essentially identical to the pseudo-covariance based improper complex Gaussian signaling, and they can both achieve strictly larger rate region than that by the conventional proper Gaussian signaling. Intuitively, it is easy to understand that $\mathcal{R}_{\text{proper}} \subset \mathcal{R}_{\text{improper}}$ since, based on the rate formulas (3.2) and (3.26), proper Gaussian signaling is just a special case of improper Gaussian signaling by restricting all the pseudo-covariance matrices of the transmitted signals to be zero. For the fundamental reasons for $\mathcal{R}_{\text{proper}} \subset \mathcal{R}_{\text{real}}$, one could resort to their corresponding linear precoding schemes expressed in the real domain as given by (3.5) and (3.15), respectively. It is observed that the precoding matrix $\bar{\mathbf{W}}_k$ in (3.5) resulting from the conventional proper Gaussian signaling must satisfy a very specific structure, with free variables only given by $\Re\{\mathbf{C}_{\mathbf{x}_k}^{\frac{1}{2}}\}$ and $\Im\{\mathbf{C}_{\mathbf{x}_k}^{\frac{1}{2}}\}$. In contrast, such a restriction does not exist for the precoding matrix in (3.15), which thus provides the opportunity for $\mathcal{R}_{\text{real}}$ to be strictly larger than $\mathcal{R}_{\text{proper}}$.

3.4 Problem Formulation

In the previous subsection, we have shown that improper complex Gaussian signaling is able to enlarge the achievable rate region over the conventional proper Gaussian signaling scheme. We are then interested in characterizing the maximum achievable rate region for the MIMO-IC (3.1) employed with improper complex Gaussian signaling. According to Theorem 3.4, for any given channel realization, there are in general two approaches for such characterizations. One is based on the Gaussian signaling scheme in the real domain as discussed in Section 3.1.4, and

the other is the pseudo-covariance based approach as presented in Section 3.3.3. For both methods, the rate regions can be fully characterized by determining the corresponding Pareto boundary, which consists of all the Pareto-optimal rate-tuples as defined in the following.

Definition 3.5. [128]: A rate-tuple $(R_1, \dots, R_K)$ is Pareto optimal if there is no other rate-tuple $(R'_1, \dots, R'_K)$ with $(R'_1, \dots, R'_K) \geq (R_1, \dots, R_K)$ and $(R'_1, \dots, R'_K) \neq (R_1, \dots, R_K)$, where the inequality is component-wise.

To characterize the Pareto boundary of $\mathcal{R}_{\text{real}}$ or $\mathcal{R}_{\text{improper}}$, we adopt the rate-profile approach as used in [129, 153, 154]. Specifically, for a given rate profile $\boldsymbol{\alpha} = (\alpha_1 \cdots \alpha_K)$ with $\alpha_k > 0$, $\forall k$, and $\sum_{k=1}^K \alpha_k = 1$, the corresponding Pareto-optimal rate-tuple in $\mathcal{R}_{\text{real}}$ or $\mathcal{R}_{\text{improper}}$ can be respectively found by solving the following optimization problems:

$$\begin{aligned} \max_{\{\bar{\mathbf{Q}}_{\mathbf{x}_k}\}, R} \quad & R \\ \text{s.t.} \quad & R_{k,\text{real}}(\{\bar{\mathbf{Q}}_{\mathbf{x}_j}\}) \geq \alpha_k R, \quad \forall k, \\ & \text{Tr}(\bar{\mathbf{Q}}_{\mathbf{x}_k}) \leq P_k, \quad \forall k, \\ & \bar{\mathbf{Q}}_{\mathbf{x}_k} \succeq \mathbf{0}, \quad \forall k, \end{aligned} \tag{3.51}$$

or

$$\begin{aligned} \max_{\{\underline{\mathbf{C}}_{\mathbf{x}_k}\}, R} \quad & R \\ \text{s.t.} \quad & R_{k,\text{improper}}(\{\underline{\mathbf{C}}_{\mathbf{x}_j}\}) \geq \alpha_k R, \quad \forall k, \\ & \text{Tr}(\underline{\mathbf{C}}_{\mathbf{x}_k}) \leq 2P_k, \quad \forall k, \\ & \underline{\mathbf{C}}_{\mathbf{x}_k} \succeq \mathbf{0}, \quad \forall k. \end{aligned} \tag{3.52}$$

It can be inferred from Lemma 3.3 that problems (3.51) and (3.52) are equivalent, in the sense that they have the same optimal value and their optimal solutions are related by invertible linear transformations as given by (3.37). Therefore, solving one of them is tantamount to solving the other. Denote the optimal value as R^* . Then the rate-tuple $R^* \cdot \boldsymbol{\alpha}$ must be on the Pareto boundary of the rate region $\mathcal{R}_{\text{real}}$, or equivalently $\mathcal{R}_{\text{improper}}$. Thereby, by solving (3.51) or (3.52) with different rate-profile vectors $\boldsymbol{\alpha}$, the complete Pareto boundary of the rate region achievable by improper complex Gaussian signaling scheme can be found.

Unfortunately, both problems (3.51) and (3.52) are non-convex due to the non-convex rate constraints. In fact, when the number of transmit and receive antennas are at least two ($M \geq 2$ and $N \geq 2$), (3.51) and (3.52) are NP-hard problems.

Such a conclusion is drawn based on the result obtained in [164], where it has been shown that the minimum rate maximization (max-min) problem is NP-hard for the MIMO-IC with the conventional proper complex Gaussian signaling when $M \geq 2$ and $N \geq 2$. Since the optimization problems (3.51) and (3.52) for improper complex Gaussian signaling are more general than that studied in [164], the NP-hardness result directly follows. Therefore, obtaining the globally optimal improper complex Gaussian signaling scheme will be quite difficult; instead, efficient sub-optimal solutions should be pursued, either by solving (3.51) or (3.52). In the remaining part of this thesis, we will focus on finding efficient approximate solutions to problem (3.52), instead of (3.51), mainly for two reasons. Firstly, the achievable rate with improper Gaussian signaling expressed in terms of the transmit covariance and pseudo-covariance matrices given in (3.26) provides more insights than its alternative form (3.13). In particular, the separable structure of (3.26) ensures that the proposed improper Gaussian signaling scheme based on solving (3.52) performs at least as good as the optimal proper Gaussian signaling scheme (see Sections 3.6.3 and 4.2); whereas such a desired result cannot be guaranteed for improper Gaussian signaling optimized in the real domain (see Figure 3.6). Secondly, due to the doubling of the system dimension when converting into real domain, for all setups with any M and N values, the rate expression (3.13) is always given as the difference of log-determinant of covariance matrices, which is in general difficult to handle. In contrast, for the SISO and MISO setups that are the main focus of this thesis, the achievable rate formula (3.26) reduces to a much simpler form given by the logarithm of scalars (see (3.75) and (4.6)), for which some celebrated optimization techniques, such as semidefinite relaxation, can be applied to find high quality approximate solutions, as will be shown in Section 3.6 and Chapter 4.

3.5 Widely Linear Precoding

Before solving the optimization problem (3.52), we consider in this section the problem how to generate the transmitted signal $\mathbf{x}_k$ at each transmitter given any valid pair of covariance matrix $\mathbf{C}_{\mathbf{x}_k}$ and pseudo-covariance matrix $\tilde{\mathbf{C}}_{\mathbf{x}_k}$, from an information-bearing signal $\mathbf{d}_k$ that is selected from conventional CSCG (proper Gaussian) codebooks. Without loss of generality, we assume $\mathbf{d}_k \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$; thus,

we have

$$\mathbf{C}_{\mathbf{d}_k} = \mathbf{I}, \quad \tilde{\mathbf{C}}_{\mathbf{d}_k} = \mathbf{0}, \quad k = 1, \dots, K. \quad (3.53)$$

First, consider the conventional linear precoding given by

$$\mathbf{x}_k = \mathbf{U}_k \mathbf{d}_k, \quad (3.54)$$

where $\mathbf{U}_k$ is the precoding matrix. Then the pseudo-covariance matrix of $\mathbf{x}_k$ is given by $\tilde{\mathbf{C}}_{\mathbf{x}_k} = \mathbf{U}_k \tilde{\mathbf{C}}_{\mathbf{d}_k} \mathbf{U}_k^T = \mathbf{0}$ for any precoding matrix $\mathbf{U}_k$. This implies that the conventional linear precoding is not able to map the proper Gaussian signal $\mathbf{d}_k$ to the improper transmitted Gaussian signal $\mathbf{x}_k$ with non-zero $\tilde{\mathbf{C}}_{\mathbf{x}_k}$.

Since the augmented covariance matrix defined in (2.14) contains both the covariance and pseudo-covariance matrices, a necessary condition for a RV $\mathbf{z}_k$ to have covariance matrix $\mathbf{C}_{\mathbf{x}_k}$ and pseudo-covariance matrix $\tilde{\mathbf{C}}_{\mathbf{x}_k}$ is that its augmented covariance matrix satisfies $\underline{\mathbf{C}}_{\mathbf{z}_k} = \underline{\mathbf{C}}_{\mathbf{x}_k}$. This is ensured by the transformation

$$\begin{bmatrix} \mathbf{z}_k \\ \mathbf{z}_k^* \end{bmatrix} = \underline{\mathbf{C}}_{\mathbf{x}_k}^{\frac{1}{2}} \begin{bmatrix} \mathbf{d}_k \\ \mathbf{d}_k^* \end{bmatrix}, \quad (3.55)$$

where $\underline{\mathbf{C}}_{\mathbf{x}_k}^{\frac{1}{2}}$ denotes the Hermitian square root of the positive semidefinite matrix $\mathbf{C}_{\mathbf{x}_k}$ so that $\underline{\mathbf{C}}_{\mathbf{x}_k} = \underline{\mathbf{C}}_{\mathbf{x}_k}^{\frac{1}{2}} (\underline{\mathbf{C}}_{\mathbf{x}_k}^{\frac{1}{2}})^H$. A common method for finding $\underline{\mathbf{C}}_{\mathbf{x}_k}^{\frac{1}{2}}$ is via eigenvalue decomposition (EVD). Specifically, let the EVD of $\mathbf{C}_{\mathbf{x}_k}$ be expressed as $\mathbf{C}_{\mathbf{x}_k} = \mathbf{U} \mathbf{D} \mathbf{U}^H$; then $\underline{\mathbf{C}}_{\mathbf{x}_k}^{\frac{1}{2}} = \mathbf{U} \mathbf{D}^{\frac{1}{2}}$ is obtained. However, it is worth pointing out that the above obtained $\underline{\mathbf{C}}_{\mathbf{x}_k}^{\frac{1}{2}}$ cannot satisfy (3.55) in general. This is because the two vectors $\mathbf{z}_k$ and $\mathbf{z}_k^*$ in (3.55) are complex conjugate of each other; therefore, the transformation matrix $\underline{\mathbf{C}}_{\mathbf{x}_k}^{\frac{1}{2}}$ should be designed with more care than the conventional EVD. Specifically, if we can find one $\underline{\mathbf{C}}_{\mathbf{x}_k}^{\frac{1}{2}}$ such that it has the following structure:

$$\underline{\mathbf{C}}_{\mathbf{x}_k}^{\frac{1}{2}} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_2^* & \mathbf{B}_1^* \end{bmatrix}, \quad (3.56)$$

i.e., the upper-left (upper-right) block is the complex conjugate of the lower-right (lower-left) block, then (3.55) is equivalent to the following two sets of equations:

$$\mathbf{z}_k = \mathbf{B}_1 \mathbf{d}_k + \mathbf{B}_2 \mathbf{d}_k^*, \quad (3.57)$$

$$\mathbf{z}_k^* = \mathbf{B}_2^* \mathbf{d}_k + \mathbf{B}_1^* \mathbf{d}_k^*. \quad (3.58)$$

It is obvious that the two equations (3.57) and (3.58) are equivalent, i.e., (3.58) is simply obtained by taking the complex conjugate on both sides of (3.57) and vice versa. Therefore, the remaining task is to find one $\underline{\mathbf{C}}_{\mathbf{x}_k}^{\frac{1}{2}}$ with the structure given by (3.56). To this end, the following $2M \times 2M$ unitary matrix is defined [57]:

$$\mathbf{T} \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_M & i\mathbf{I}_M \\ \mathbf{I}_M & -i\mathbf{I}_M \end{bmatrix}, \quad \mathbf{T}\mathbf{T}^H = \mathbf{T}^H\mathbf{T} = \mathbf{I}_{2M}. \quad (3.59)$$

For any *real-valued* matrix $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \in \mathbb{R}^{2M \times 2M}$, it can be verified that the matrix $\mathbf{T}\mathbf{A}\mathbf{T}^H$ has the structure given in (3.56), i.e.,

$$\mathbf{T}\mathbf{A}\mathbf{T}^H = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2^* & \mathbf{A}_1^* \end{bmatrix}, \quad (3.60)$$

with $\mathbf{A}_1 = \frac{1}{2}[(\mathbf{A}_{11} + \mathbf{A}_{22}) + i(\mathbf{A}_{21} - \mathbf{A}_{12})]$ and $\mathbf{A}_2 = \frac{1}{2}[(\mathbf{A}_{11} - \mathbf{A}_{22}) + i(\mathbf{A}_{21} + \mathbf{A}_{12})]$.

Theorem 3.6. [57] *There exists one form of EVD for the augmented covariance matrix $\underline{\mathbf{C}}_{\mathbf{x}_k} \in \mathbb{C}^{2M \times 2M}$ defined in (2.14), which is given by*

$$\underline{\mathbf{C}}_{\mathbf{x}_k} = (\mathbf{T}\mathbf{V})\mathbf{\Lambda}(\mathbf{T}\mathbf{V})^H, \quad (3.61)$$

where $\mathbf{V} \in \mathbb{R}^{2M \times 2M}$ is a *real-valued orthogonal matrix* and $\mathbf{\Lambda} = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_{2M}\}$ contains the *eigenvalues* of $\underline{\mathbf{C}}_{\mathbf{x}_k}$.

In fact, (3.61) can be obtained by considering $\mathbf{T}^H \underline{\mathbf{C}}_{\mathbf{x}_k} \mathbf{T}$, which is a *real-valued* matrix given by

$$\mathbf{T}^H \underline{\mathbf{C}}_{\mathbf{x}_k} \mathbf{T} = \begin{bmatrix} \Re\{\mathbf{C}_{\mathbf{x}_k} + \tilde{\mathbf{C}}_{\mathbf{x}_k}\} & \Im\{-\mathbf{C}_{\mathbf{x}_k} + \tilde{\mathbf{C}}_{\mathbf{x}_k}\} \\ \Im\{\mathbf{C}_{\mathbf{x}_k} + \tilde{\mathbf{C}}_{\mathbf{x}_k}\} & \Re\{\mathbf{C}_{\mathbf{x}_k} - \tilde{\mathbf{C}}_{\mathbf{x}_k}\} \end{bmatrix}. \quad (3.62)$$

Furthermore, since $\mathbf{C}_{\mathbf{x}_k}$ is Hermitian and $\tilde{\mathbf{C}}_{\mathbf{x}_k}$ is symmetric, it can be verified that the matrix $\mathbf{T}^H \underline{\mathbf{C}}_{\mathbf{x}_k} \mathbf{T}$ given above is *symmetric* as well. Therefore, its *real-valued*

EVD can be written as

$$\mathbf{T}^H \underline{\mathbf{C}}_{\mathbf{x}_k} \mathbf{T} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^T. \quad (3.63)$$

The EVD in (3.61) is then obtained by applying a unitary transformation $\mathbf{T}$ to (3.63).

For any given $\underline{\mathbf{C}}_{\mathbf{x}_k} \succeq \mathbf{0}$, all the eigenvalues are nonnegative [57], i.e., $\lambda_l \geq 0, l = 1, \dots, 2M$. Thus (3.61) can be written as

$$\underline{\mathbf{C}}_{\mathbf{x}_k} = \mathbf{T} \mathbf{V} \mathbf{\Lambda} \mathbf{V}^H \mathbf{T}^H = (\mathbf{T} \mathbf{V} \mathbf{\Lambda}^{1/2} \mathbf{T}^H) (\mathbf{T} \mathbf{V} \mathbf{\Lambda}^{1/2} \mathbf{T}^H)^H. \quad (3.64)$$

Then we have

$$\underline{\mathbf{C}}_{\mathbf{x}_k}^{1/2} = \mathbf{T} (\mathbf{V} \mathbf{\Lambda}^{1/2}) \mathbf{T}^H = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_2^* & \mathbf{B}_1^* \end{bmatrix}, \quad (3.65)$$

where the last equality follows from (3.60) and the fact that $\mathbf{V} \mathbf{\Lambda}^{1/2}$ is a real-valued matrix.

From (3.55) and (3.65), it follows that to obtain the transmitted signal vector $\mathbf{x}_k$, which is generally improper with the covariance and pseudo-covariance matrices specified by $\underline{\mathbf{C}}_{\mathbf{x}_k}$, the following precoding needs to be applied to the proper information-bearing signal $\mathbf{d}_k$:

$$\mathbf{x}_k = \mathbf{B}_1 \mathbf{d}_k + \mathbf{B}_2 \mathbf{d}_k^*, \quad (3.66)$$

where $\mathbf{B}_1$ and $\mathbf{B}_2$ are the corresponding blocks in $\mathbf{T} \mathbf{V} \mathbf{\Lambda}^{1/2} \mathbf{T}^H$ as shown in (3.65), with $\mathbf{V}$ and $\mathbf{\Lambda}$ obtained by the particular form of EVD in (3.61). Following similar terminologies used in existing literatures on improper signal processing such as [155], we refer to the precoding given in (3.66) as *widely linear precoding*. Note that if $\mathbf{B}_2 = \mathbf{0}$, which is the case when $\underline{\mathbf{C}}_{\mathbf{x}_k}$ is block-diagonal (i.e., $\tilde{\mathbf{C}}_{\mathbf{x}_k} = \mathbf{0}$), (3.66) reduces to the conventional linear precoding for proper Gaussian signaling given by (3.54). Last, in terms of the real-valued representation, (3.66) can be re-expressed as

$$\begin{bmatrix} \Re\{\mathbf{x}_k\} \\ \Im\{\mathbf{x}_k\} \end{bmatrix} = \begin{bmatrix} \Re\{\mathbf{B}_1 + \mathbf{B}_2\} & \Im\{\mathbf{B}_2 - \mathbf{B}_1\} \\ \Im\{\mathbf{B}_2 + \mathbf{B}_1\} & \Re\{\mathbf{B}_1 - \mathbf{B}_2\} \end{bmatrix} \begin{bmatrix} \Re\{\mathbf{d}_k\} \\ \Im\{\mathbf{d}_k\} \end{bmatrix}. \quad (3.67)$$

3.6 Transmit Optimization for Two-User SISO-IC

In this section, we focus on the special case of two-user SISO-IC, for which the input-output relationship in (3.1) can be simplified as

$$\begin{aligned} y_1 &= h_{11}x_1 + h_{12}x_2 + n_1, \\ y_2 &= h_{21}x_1 + h_{22}x_2 + n_2, \end{aligned} \quad (3.68)$$

where $h_{kj} = |h_{kj}|e^{i\phi_{kj}}$, $k, j \in \{1, 2\}$, is the complex scalar channel from transmitter j to receiver k .⁴ Denote the covariance and pseudo-covariance of the transmitted signal x_k as C_{x_k} and $\tilde{C}_{x_k}$, respectively⁵, i.e.,

$$C_{x_k} = \mathbb{E}(x_k x_k^*), \quad \tilde{C}_{x_k} = \mathbb{E}(x_k x_k), \quad k = 1, 2. \quad (3.69)$$

Note that C_{x_k} is a nonnegative real number equals to the transmit power of user k , while $\tilde{C}_{x_k}$ is a complex number in general. From Theorem 2.1 and by applying the Shur complement result, it can be obtained that the following conditions are necessary and sufficient for C_{x_k} and $\tilde{C}_{x_k}$ to be a valid pair of covariance and pseudo-covariance:

$$C_{x_k} \geq 0, \quad |\tilde{C}_{x_k}|^2 \leq C_{x_k}^2, \quad k = 1, 2. \quad (3.70)$$

The covariance and pseudo-covariance of the received signal y_k can be written as

$$C_{y_k} = \mathbb{E}(y_k y_k^*) = |h_{k1}|^2 C_{x_1} + |h_{k2}|^2 C_{x_2} + \sigma^2, \quad (3.71)$$

$$\tilde{C}_{y_k} = \mathbb{E}(y_k y_k) = h_{k1}^2 \tilde{C}_{x_1} + h_{k2}^2 \tilde{C}_{x_2}. \quad (3.72)$$

For the interference-plus-noise term $s_k = h_{k\bar{k}}x_{\bar{k}} + n_k$, $\bar{k} \neq k$, we have

$$C_{s_k} = |h_{k\bar{k}}|^2 C_{x_{\bar{k}}} + \sigma^2, \quad \tilde{C}_{s_k} = h_{k\bar{k}}^2 \tilde{C}_{x_{\bar{k}}}. \quad (3.73)$$

⁴Since a phase rotation can be applied at each of the receivers with coherent demodulation, without loss of generality, the direct channel gains h_{11} and h_{22} can be assumed to be real values. However, this assumption will not change the remaining results in this chapter.

⁵Note that we still use the terms covariance and pseudo-covariance for the case of scalar random variables, although probably the more appropriate terms in this case should be variance and pseudo-variance, respectively.

Then the rate expression in (3.26) for the two-user SISO-IC reduces to

$$R_k^{\text{SISO}} = \frac{1}{2} \log \frac{C_{y_k}^2 - |\tilde{C}_{y_k}|^2}{C_{s_k}^2 - |\tilde{C}_{s_k}|^2} \quad (3.74)$$

$$= \underbrace{\log \left(1 + \frac{|h_{kk}|^2 C_{x_k}}{\sigma^2 + |h_{k\bar{k}}|^2 C_{x_{\bar{k}}}} \right)}_{\triangleq R_{k,\text{proper}}^{\text{SISO}}(C_{x_1}, C_{x_2})} + \frac{1}{2} \log \frac{1 - C_{y_k}^{-2} |\tilde{C}_{y_k}|^2}{1 - C_{s_k}^{-2} |\tilde{C}_{s_k}|^2}. \quad (3.75)$$

Problem (3.52) for characterizing the achievable rate region for the two-user SISO-IC can be re-expressed as

$$\begin{aligned} \max. \quad & R \\ \text{s.t.} \quad & R_k^{\text{SISO}} \geq \alpha_k R, \quad k = 1, 2 \\ & 0 \leq C_{x_k} \leq P_k, \quad k = 1, 2 \\ & |\tilde{C}_{x_k}|^2 \leq C_{x_k}^2, \quad k = 1, 2. \end{aligned} \quad (3.76)$$

3.6.1 Strict Sum-Rate Improvement by Improper Gaussian

To further illustrate the strict performance improvement by improper Gaussian signaling over the conventional proper Gaussian signaling, in this subsection, we show that the maximum sum-rate achieved by the optimal proper Gaussian signaling can be further increased if non-zero pseudo-covariances are allowed. For simplicity, we consider a symmetric two-user SISO-IC with real-valued channel coefficients, i.e.,

$$h_{11} = h_{22} = h_d, \quad h_{12} = h_{21} = h_c, \quad P_1 = P_2 = P, \quad (3.77)$$

where h_d and h_c are real values denoting the direct and cross channel gains, respectively.

With proper Gaussian signaling, it is known that the sum-rate *optimal* power allocation for the two-user SISO-IC is binary [7], i.e., either both transmitters use the maximum power or one of them is completely off. In this case, the optimal power allocation $(C_{x_1}^*, C_{x_2}^*)$ satisfies

$$(C_{x_1}^*, C_{x_2}^*) \in \{(P, 0), (0, P), (P, P)\}. \quad (3.78)$$

If one of the users keeps silence, i.e., $(C_{x_1}^*, C_{x_2}^*) = (P, 0)$ or $(C_{x_1}^*, C_{x_2}^*) = (0, P)$, the two-user SISO-IC reduces to a single-user point-to-point channel, where proper Gaussian signaling is known to be optimal [156]. In this case, further optimizing the pseudo-covariances $\tilde{C}_{x_1}$ and $\tilde{C}_{x_2}$ will not provide any rate gain. Therefore, the scenario of interest is when $(C_{x_1}^*, C_{x_2}^*) = (P, P)$, in which case the channel coefficients satisfy

$$1 + \frac{h_d^2 P}{\sigma^2} \leq \left(1 + \frac{h_d^2 P}{\sigma^2 + h_c^2 P}\right)^2. \quad (3.79)$$

Define the signal-to-noise ratio (SNR) and interference-to-noise ratio (INR) respectively as

$$\text{SNR} \triangleq \frac{h_d^2 P}{\sigma^2}, \quad \text{INR} \triangleq \frac{h_c^2 P}{\sigma^2}. \quad (3.80)$$

Then (3.79) can be equivalently written as

$$1 + \text{SNR} \leq \left(1 + \frac{\text{SNR}}{1 + \text{INR}}\right)^2 \quad (3.81)$$

or

$$\text{INR}^2 - 1 \leq \text{SNR}. \quad (3.82)$$

The condition given in (3.82) quantifies the intuition that in the sum-rate optimal strategy of the two-user SISO-IC, both users will use maximum transmission power only when the interfering links are sufficiently weak.

Next, improper Gaussian signaling is considered. With the covariances fixed to the sum-rate optimal proper Gaussian solution $(C_{x_1}^*, C_{x_2}^*) = (P, P)$, a strict sum-rate improvement is achieved by improper Gaussian signaling if the pseudo-covariances $\tilde{C}_{x_1}$ and $\tilde{C}_{x_2}$ can be found to make the second term in (3.75) strictly positive. For the channel given in (3.77), this is equivalent to solve the following feasibility problem

$$\begin{aligned} \text{find } & \tilde{C}_{x_1} \in \mathbb{C}, \tilde{C}_{x_2} \in \mathbb{C} \\ \text{s.t. } & C_{y_1}^{-2} |h_d^2 \tilde{C}_{x_1} + h_c^2 \tilde{C}_{x_2}|^2 < C_{s_1}^{-2} h_c^4 |\tilde{C}_{x_2}|^2, \\ & C_{y_2}^{-2} |h_c^2 \tilde{C}_{x_1} + h_d^2 \tilde{C}_{x_2}|^2 < C_{s_2}^{-2} h_c^4 |\tilde{C}_{x_1}|^2, \\ & |\tilde{C}_{x_1}| \leq P, \quad |\tilde{C}_{x_2}| \leq P, \end{aligned} \quad (3.83)$$

where $C_{y_1} = C_{y_2} = (h_d^2 + h_c^2)P + \sigma^2$ is the total received power, and $C_{s_1} = C_{s_2} = h_c^2 P + \sigma^2$ is the interference-plus-noise power. Instead of solving (3.83) to find a feasible pair of $\tilde{C}_{x_1}$ and $\tilde{C}_{x_2}$, to gain some more insights, we derive a sufficient condition over the channel coefficients such that for a particular choice of $(\tilde{C}_{x_1}, \tilde{C}_{x_2})$, (3.83) is feasible. A heuristic choice of $(\tilde{C}_{x_1}, \tilde{C}_{x_2})$ is obtained as follows. It is firstly noted that a common phase rotation to $\tilde{C}_{x_1}$ and $\tilde{C}_{x_2}$ does not alter the feasibility of (3.83). Therefore, without loss of generality, we can choose $\tilde{C}_{x_1}$ such that it is a nonnegative real number. Furthermore, in order to maximize the feasible region, the phase of $\tilde{C}_{x_2}$ should be chosen so as to minimize the left hand side (LHS) of the first two constraints in (3.83), and the magnitudes of $\tilde{C}_{x_1}$ and $\tilde{C}_{x_2}$ should be chosen so as to maximize the right hand side (RHS) of the first two constraints in (3.83). We therefore choose the pseudo-covariance pair $(\tilde{C}_{x_1}, \tilde{C}_{x_2}) = (P, -P)$, in which case, (3.83) is feasible if the following conditions are satisfied:

$$\frac{(h_d^2 - h_c^2)^2}{(h_d^2 P + h_c^2 P + \sigma^2)^2} < \frac{h_c^4}{(h_c^2 P + \sigma^2)^2}. \quad (3.84)$$

With (3.80), (3.84) can be simplified as

$$\left(1 - \frac{\text{SNR}}{\text{INR}}\right)^2 < \left(1 + \frac{\text{SNR}}{1 + \text{INR}}\right)^2. \quad (3.85)$$

or

$$\text{SNR} < 2\text{INR}(1 + \text{INR}). \quad (3.86)$$

The condition given in (3.86) shows that improper Gaussian signaling is superior when the interfering links are sufficiently strong, which is expected since it is the non-circularity of the interfering signals that is exploited for each user to outperform the conventional proper Gaussian signaling. By combining (3.82) and (3.86), we get the following result:

Theorem 3.7. *For the symmetric real-valued SISO-IC with the channels satisfying $\text{INR}^2 - 1 \leq \text{SNR} < 2\text{INR}(1 + \text{INR})$, by applying improper Gaussian signaling with $(C_{x_1}, C_{x_2}, \tilde{C}_{x_1}, \tilde{C}_{x_2}) = (P, P, P, -P)$, a **strict** sum-rate improvement is achieved over the **optimal** conventional proper Gaussian signaling, in which case the sum-rate optimal power allocation is given by $(C_{x_1}, C_{x_2}) = (P, P)$.*

It can be easily verified that the region specified by $\text{INR}^2 - 1 \leq \text{SNR} < 2\text{INR}(1 + \text{INR})$ is non-empty. A graphical illustration is given in Figure 3.2.

FIGURE 3.2: An SNR – INR region where improper Gaussian signaling provides strict sum-rate improvement over the *optimal* proper Gaussian signaling for the symmetric real-valued two-user SISO-IC.

Remark 3.8. Theorem 3.7 only gives a sufficient but not necessary condition for a strict sum-rate improvement by employing improper Gaussian signaling over the conventional proper Gaussian signaling. The actual region can be larger if the covariances and pseudo-covariances are optimized.

Remark 3.9. In practice, the channels are complex-valued in general. The optimization for improper complex Gaussian signaling for general complex-valued channels is the task to be pursued next for the two-user SISO-IC, as well as in Chapter 4 for K -user MISO-IC.

3.6.2 Joint Covariance and Pseudo-Covariance Optimization

In this subsection, by utilizing the rate formula (3.74), we propose an approximate solution to the non-convex optimization problem (3.76). The covariance and pseudo-covariance of the transmitted signals are jointly optimized by applying the SDR technique. By treating R as a slack variable and substituting R_k^{SISO} with

(3.74), problem (3.76) can be equivalently written as

$$\begin{aligned} \max_{\{C_{x_k}\}, \{\tilde{C}_{x_k}\}} \min_{k=1,2} & \frac{1}{2\alpha_k} \log \frac{C_{y_k}^2 - |\tilde{C}_{y_k}|^2}{C_{s_k}^2 - |\tilde{C}_{s_k}|^2} \\ \text{s.t.} \quad & 0 \leq C_{x_k} \leq P_k, \quad k = 1, 2 \\ & |\tilde{C}_{x_k}|^2 \leq C_{x_k}^2, \quad k = 1, 2. \end{aligned} \quad (3.87)$$

Problem (3.87) is a minimum-weighted-rate maximization (MinWR-Max) problem, where the weights are related to the rate-profile $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$. The following result will be used for solving (3.87).

Lemma 3.10. *For any set of $\{C_{x_k}\}$ and $\{\tilde{C}_{x_k}\}$ that is feasible for (3.87), the following inequalities hold:*

$$C_{y_k}^2 - |\tilde{C}_{y_k}|^2 \geq \sigma^4 > 0, \quad k = 1, 2, \quad (3.88)$$

$$C_{s_k}^2 - |\tilde{C}_{s_k}|^2 \geq \sigma^4 > 0, \quad k = 1, 2. \quad (3.89)$$

Proof. Please refer to Appendix A. □

Define the following 2-dimensional real-valued vectors:

$$\begin{aligned} \mathbf{c} &= \begin{bmatrix} C_{x_1} & C_{x_2} \end{bmatrix}^T, \\ \mathbf{a}_1 &= \begin{bmatrix} |h_{11}|^2 & |h_{12}|^2 \end{bmatrix}^T, \quad \mathbf{b}_1 = \begin{bmatrix} 0 & |h_{12}|^2 \end{bmatrix}^T, \\ \mathbf{a}_2 &= \begin{bmatrix} |h_{21}|^2 & |h_{22}|^2 \end{bmatrix}^T, \quad \mathbf{b}_2 = \begin{bmatrix} |h_{21}|^2 & 0 \end{bmatrix}^T. \end{aligned}$$

Then from (3.71) and (3.73), we have

$$C_{y_k}^2 = (\sigma^2 + \mathbf{a}_k^T \mathbf{c})^2, \quad C_{s_k}^2 = (\sigma^2 + \mathbf{b}_k^T \mathbf{c})^2, \quad k = 1, 2. \quad (3.90)$$

Further define the following 2-dimensional complex-valued vectors:

$$\begin{aligned} \mathbf{q} &= \begin{bmatrix} \tilde{C}_{x_1} & \tilde{C}_{x_2} \end{bmatrix}^T, \\ \mathbf{f}_1 &= \begin{bmatrix} h_{11}^2 & h_{12}^2 \end{bmatrix}^H, \quad \mathbf{g}_1 = \begin{bmatrix} 0 & h_{12}^2 \end{bmatrix}^H, \\ \mathbf{f}_2 &= \begin{bmatrix} h_{21}^2 & h_{22}^2 \end{bmatrix}^H, \quad \mathbf{g}_2 = \begin{bmatrix} h_{21}^2 & 0 \end{bmatrix}^H. \end{aligned}$$

Then from (3.72) and (3.73), we have

$$|\tilde{C}_{y_k}|^2 = |\mathbf{f}_k^H \mathbf{q}|^2 = \mathbf{q}^H \mathbf{F}_k \mathbf{q}, \quad (3.91)$$

$$|\tilde{C}_{s_k}|^2 = |\mathbf{g}_k^H \mathbf{q}|^2 = \mathbf{q}^H \mathbf{G}_k \mathbf{q}, \quad (3.92)$$

where $\mathbf{F}_k = \mathbf{f}_k \mathbf{f}_k^H$ and $\mathbf{G}_k = \mathbf{g}_k \mathbf{g}_k^H$, $k = 1, 2$. By substituting (3.90)-(3.92) into (3.87), we obtain the following equivalent problem

$$\begin{aligned} \max_{\mathbf{c} \in \mathbb{R}^2, \mathbf{q} \in \mathbb{C}^2} \min_k & \frac{1}{2\alpha_k} \log \frac{(\sigma^2 + \mathbf{a}_k^T \mathbf{c})^2 - \mathbf{q}^H \mathbf{F}_k \mathbf{q}}{(\sigma^2 + \mathbf{b}_k^T \mathbf{c})^2 - \mathbf{q}^H \mathbf{G}_k \mathbf{q}} \\ \text{s.t.} \quad & \mathbf{c}^T \mathbf{E}_k \mathbf{c} \leq P_k^2, \quad k = 1, 2 \\ & \mathbf{e}_k^T \mathbf{c} \geq 0, \quad k = 1, 2 \\ & \mathbf{q}^H \mathbf{E}_k \mathbf{q} \leq \mathbf{c}^T \mathbf{E}_k \mathbf{c}, \quad k = 1, 2, \end{aligned} \quad (3.93)$$

where $\mathbf{e}_k$ is the k th column of the identity matrix $\mathbf{I}_2$, and $\mathbf{E}_k = \mathbf{e}_k \mathbf{e}_k^T$. The objective function of (3.93) is given by the minimum of weighted log-fraction of quadratic functions of $\mathbf{c}$ and $\mathbf{q}$. Due to the noise power σ^2 , the quadratics are non-homogeneous.⁶ By introducing a new variable t , we obtain the homogenized quadratics, which yields

$$\begin{aligned} \max_{\mathbf{c}, \mathbf{q}, t} \min_k & \frac{1}{2\alpha_k} \log \frac{(\sigma^2 t + \mathbf{a}_k^T \mathbf{c})^2 - \mathbf{q}^H \mathbf{F}_k \mathbf{q}}{(\sigma^2 t + \mathbf{b}_k^T \mathbf{c})^2 - \mathbf{q}^H \mathbf{G}_k \mathbf{q}} \\ \text{s.t.} \quad & \mathbf{c}^T \mathbf{E}_k \mathbf{c} \leq P_k^2, \quad k = 1, 2 \\ & \mathbf{e}_k^T \mathbf{c} t \geq 0, \quad k = 1, 2 \\ & \mathbf{q}^H \mathbf{E}_k \mathbf{q} \leq \mathbf{c}^T \mathbf{E}_k \mathbf{c}, \quad k = 1, 2 \\ & t^2 = 1. \end{aligned} \quad (3.94)$$

Problem (3.94) is equivalent to (3.93) in the sense that if it has an optimal solution $(\mathbf{c}^*, \mathbf{q}^*, t^*)$, then $(\mathbf{c}^*/t^*, \mathbf{q}^*/t^*)$ is an optimal solution to (3.93) with the same optimal value. Therefore, (3.93) can be solved by solving (3.94). Next, we show that the celebrated SDR technique can be applied to find an approximate

⁶A homogeneous quadratic function has no linear terms.

solution to (3.94). Define

$$\mathbf{C} = \begin{bmatrix} t \\ \mathbf{c} \end{bmatrix} \begin{bmatrix} t \\ \mathbf{c} \end{bmatrix}^T, \quad \mathbf{Q} = \mathbf{q}\mathbf{q}^H, \quad (3.95)$$

$$\mathbf{A}_k = \begin{bmatrix} \sigma^2 \\ \mathbf{a}_k \end{bmatrix} \begin{bmatrix} \sigma^2 \\ \mathbf{a}_k \end{bmatrix}^T, \quad \mathbf{B}_k = \begin{bmatrix} \sigma^2 \\ \mathbf{b}_k \end{bmatrix} \begin{bmatrix} \sigma^2 \\ \mathbf{b}_k \end{bmatrix}^T \quad (3.96)$$

$$\mathbf{K}_k = \begin{bmatrix} 0 & \frac{1}{2}\mathbf{e}_k^T \\ \frac{1}{2}\mathbf{e}_k & \mathbf{0} \end{bmatrix}, \quad \hat{\mathbf{E}}_k = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_k \end{bmatrix}. \quad (3.97)$$

With the identity $\mathbf{x}^H \mathbf{A} \mathbf{x} = \text{Tr}(\mathbf{A} \mathbf{x} \mathbf{x}^H)$, (3.94) is recast as

$$\begin{aligned} & \max_{\mathbf{C} \in \mathbb{S}^3, \mathbf{Q} \in \mathbb{H}^2} \min_k \frac{1}{2\alpha_k} \log \frac{\text{Tr}(\mathbf{A}_k \mathbf{C}) - \text{Tr}(\mathbf{F}_k \mathbf{Q})}{\text{Tr}(\mathbf{B}_k \mathbf{C}) - \text{Tr}(\mathbf{G}_k \mathbf{Q})} \\ & \text{s.t.} \quad \text{Tr}(\hat{\mathbf{E}}_k \mathbf{C}) \leq P_k^2, \quad k = 1, 2 \\ & \quad \text{Tr}(\mathbf{K}_k \mathbf{C}) \geq 0, \quad k = 1, 2 \\ & \quad \text{Tr}(\mathbf{E}_k \mathbf{Q}) \leq \text{Tr}(\hat{\mathbf{E}}_k \mathbf{C}), \quad k = 1, 2 \\ & \quad \mathbf{C}_{11} = 1 \\ & \quad \mathbf{C} \succeq \mathbf{0}, \quad \mathbf{Q} \succeq \mathbf{0} \\ & \quad \text{rank}(\mathbf{C}) = 1, \quad \text{rank}(\mathbf{Q}) = 1, \end{aligned} \quad (3.98)$$

where $\mathbf{C}_{11}$ denotes the $(1, 1)$ -th entry of $\mathbf{C}$; the positive semidefinite constraints and the rank-1 constraints are due to (3.95). With such a reformulation, the objective function of (3.98) is now a log-fraction of linear functions of $\mathbf{C}$ and $\mathbf{Q}$. With Lemma 3.10 and the equivalence between (3.87) and (3.98), for any pair of $\mathbf{C}$ and $\mathbf{Q}$ that is feasible for (3.98), we have

$$\text{Tr}(\mathbf{A}_k \mathbf{C}) - \text{Tr}(\mathbf{F}_k \mathbf{Q}) \geq \sigma^4 > 0, \quad k = 1, 2, \quad (3.99)$$

$$\text{Tr}(\mathbf{B}_k \mathbf{C}) - \text{Tr}(\mathbf{G}_k \mathbf{Q}) \geq \sigma^4 > 0, \quad k = 1, 2. \quad (3.100)$$

The SDR problem of (3.98) is obtained by discarding the non-convex rank-1 constraints, and by including the extra constraints (3.99) and (3.100), which gives

$$\begin{aligned}
& \max_{\mathbf{C} \in \mathbb{S}^3, \mathbf{Q} \in \mathbb{H}^2} \min_k \frac{1}{2\alpha_k} \log \frac{\text{Tr}(\mathbf{A}_k \mathbf{C}) - \text{Tr}(\mathbf{F}_k \mathbf{Q})}{\text{Tr}(\mathbf{B}_k \mathbf{C}) - \text{Tr}(\mathbf{G}_k \mathbf{Q})} \\
& \text{s.t.} \quad \text{Tr}(\hat{\mathbf{E}}_k \mathbf{C}) \leq P_k^2, \quad k = 1, 2 \\
& \quad \text{Tr}(\mathbf{K}_k \mathbf{C}) \geq 0, \quad k = 1, 2 \\
& \quad \text{Tr}(\mathbf{E}_k \mathbf{Q}) \leq \text{Tr}(\hat{\mathbf{E}}_k \mathbf{C}), \quad k = 1, 2 \\
& \quad \mathbf{C}_{11} = 1, \quad \mathbf{C} \succeq \mathbf{0}, \quad \mathbf{Q} \succeq \mathbf{0} \\
& \quad \text{Tr}(\mathbf{A}_k \mathbf{C}) - \text{Tr}(\mathbf{F}_k \mathbf{Q}) \geq \sigma^4, \quad k = 1, 2 \\
& \quad \text{Tr}(\mathbf{B}_k \mathbf{C}) - \text{Tr}(\mathbf{G}_k \mathbf{Q}) \geq \sigma^4, \quad k = 1, 2.
\end{aligned} \tag{3.101}$$

Note that although the constraints (3.99) and (3.100) are redundant in the rank-constrained problem (3.98), the advantages of including them in the rank-relaxed problem (3.101) are twofold. Firstly, it makes (3.101) a problem with less relaxation of (3.98) as compared to that without the two constraints, since without the rank-1 constraints, (3.99) and (3.100) cannot be automatically guaranteed. Secondly, the strict positivity of (3.100) makes problem (3.101) quasi-convex and hence can be solved with the standard bisection method [65]. Since any point feasible for (3.98) is also feasible for (3.101), the optimal objective value of (3.101) provides an upper bound on that of (3.98). To solve the quasi-convex problem (3.101), consider the following feasibility problem for a fixed R :

$$\begin{aligned}
& \text{find } \mathbf{C} \in \mathbb{S}^3, \mathbf{Q} \in \mathbb{H}^2 \\
& \text{s.t.} \quad \text{Tr}(\hat{\mathbf{E}}_k \mathbf{C}) \leq P_k^2, \quad k = 1, 2 \\
& \quad \text{Tr}(\mathbf{K}_k \mathbf{C}) \geq 0, \quad k = 1, 2 \\
& \quad \text{Tr}(\mathbf{E}_k \mathbf{Q}) \leq \text{Tr}(\hat{\mathbf{E}}_k \mathbf{C}), \quad k = 1, 2 \\
& \quad \mathbf{C}_{11} = 1, \quad \mathbf{C} \succeq \mathbf{0}, \quad \mathbf{Q} \succeq \mathbf{0} \\
& \quad \text{Tr}(\mathbf{B}_k \mathbf{C}) - \text{Tr}(\mathbf{G}_k \mathbf{Q}) \geq \sigma^4, \quad k = 1, 2 \\
& \quad \text{Tr}(\mathbf{A}_k \mathbf{C}) - \text{Tr}(\mathbf{F}_k \mathbf{Q}) \geq 2^{2\alpha_k R} [\text{Tr}(\mathbf{B}_k \mathbf{C}) - \text{Tr}(\mathbf{G}_k \mathbf{Q})], \quad k = 1, 2.
\end{aligned} \tag{3.102}$$

(3.102) is an SDP problem, which can be efficiently solved [65]. If (3.102) is feasible, then the optimal objective value R_{sdr} of (3.101) satisfies $R_{\text{sdr}} \geq R$; otherwise $R_{\text{sdr}} < R$. Therefore, (3.101) can be solved by solving the SDP problem (3.102) together with a bisection search over R . The algorithm for solving (3.101) is summarized as follows:

Algorithm 3 Bisection Method for Solving (3.101)

-
- 1: Initialize $R_L \leftarrow 0$ and R_U large enough such that $R_{\text{sdr}} \in [R_L, R_U]$.
 - 2: **repeat**
 - 3: Set $R \leftarrow (R_L + R_U)/2$, and solve the SDP feasibility problem (3.102).
 - 4: If (3.102) is feasible, then set $R_L \leftarrow R$; otherwise, set $R_U \leftarrow R$.
 - 5: **until** $R_U - R_L \leq \epsilon$, where ϵ is the tolerance.
-

Note that since problem (3.101) is quasi-convex, Algorithm 3 is guaranteed to converge to the optimal solution of (3.101) [65]. Denote the solution to (3.101) by $(\mathbf{C}^*, \mathbf{Q}^*)$. If $\text{rank}(\mathbf{C}^*) = 1$ and $\text{rank}(\mathbf{Q}^*) = 1$, then $(\mathbf{C}^*, \mathbf{Q}^*)$ is also the optimal solution to the rank-constrained problem (3.98). In this case, SDR is tight and the optimal solution to (3.94) is given by the principal components of $\mathbf{C}^*$ and $\mathbf{Q}^*$, from which the solution to (3.93) can be obtained; otherwise, we apply the following Gaussian randomization procedure customized to our problem to obtain an approximate solution to (3.93) [53].

Algorithm 4 Gaussian Randomization Procedure for (3.93)

Input: The solution $(\mathbf{C}^*, \mathbf{Q}^*)$ to (3.101) and the number of randomization trials L .

- 1: **for** $l = 1, \dots, L$ **do**
- 2: Generate $\begin{bmatrix} t_l \\ \boldsymbol{\xi}_l \end{bmatrix} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}^*), \boldsymbol{\beta}_l \sim \mathcal{CN}(\mathbf{0}, \mathbf{Q}^*)$.
- 3: Let $\mathbf{c}_l = \boldsymbol{\xi}_l/t_l$, and $\mathbf{q}_l = \boldsymbol{\beta}_l/t_l$.
- 4: Construct a feasible point $(\check{\mathbf{c}}_l, \check{\mathbf{q}}_l)$ for (3.93) as follows:

$$[\check{\mathbf{c}}_l]_k = \max(0, \min([\mathbf{c}_l]_k, P_k)), \quad (3.103)$$

$$[\check{\mathbf{q}}_l]_k = \eta_k[\mathbf{q}_l]_k, \quad (3.104)$$

$$\text{where } \eta_k = \min \left\{ 1, \frac{[\check{\mathbf{c}}_l]_k}{|[\mathbf{q}_l]_k|} \right\}, \quad k = 1, 2.$$

- 5: **end for**
- 6: Let $(\hat{\mathbf{c}}, \hat{\mathbf{q}})$ be the solution to

$$\max_{\check{\mathbf{c}}_l, \check{\mathbf{q}}_l, l=1, \dots, L} \min_k \frac{1}{2\alpha_k} \log \frac{(\sigma^2 + \mathbf{a}_k^T \check{\mathbf{c}}_l)^2 - \check{\mathbf{q}}_l^H \mathbf{F}_k \check{\mathbf{q}}_l}{(\sigma^2 + \mathbf{b}_k^T \check{\mathbf{c}}_l)^2 - \check{\mathbf{q}}_l^H \mathbf{G}_k \check{\mathbf{q}}_l}$$

Output: $(\hat{\mathbf{c}}, \hat{\mathbf{q}})$ as an approximate solution for (3.93).

Note that (3.103) makes sure that the covariance vector $\check{\mathbf{c}}_l$ satisfies the first two constraints of (3.93), and (3.104) ensures that the last constraint in (3.93) is satisfied by $(\check{\mathbf{c}}_l, \check{\mathbf{q}}_l)$.

The proposed algorithm for (3.76) by jointly optimizing the covariance and pseudo-covariance is summarized as follows:

Algorithm 5 Joint Covariance and Pseudo-Covariance Optimization for (3.76)

- 1: Solve the SDR problem (3.101) with Algorithm 3, and denote the solution as $(\mathbf{C}^*, \mathbf{Q}^*)$.
 - 2: If $\text{rank}(\mathbf{C}^*) = 1$ and $\text{rank}(\mathbf{Q}^*) = 1$, then the optimal solution to (3.93) is obtained based on the principal components of $\mathbf{C}^*$ and $\mathbf{Q}^*$; otherwise, obtain an approximate solution $(\hat{\mathbf{c}}, \hat{\mathbf{q}})$ to (3.93) with Algorithm 4.
 - 3: Then the covariances $\{C_{x_k}\}$ and pseudo-covariances $\{\tilde{C}_{x_k}\}$ are obtained from $\hat{\mathbf{c}}$ and $\hat{\mathbf{q}}$ accordingly.
-

3.6.3 Separate Covariance and Pseudo-Covariance Optimization

For the algorithm proposed in the preceding subsection, although the transmit covariances and pseudo-covariances are jointly optimized, it is not clear whether a rate gain over the conventional proper Gaussian signaling is attainable since the obtained solutions are not necessarily globally optimal. In this section, by utilizing the result that the user's achievable rate is separable as shown in (3.75), we propose a separate covariance and pseudo-covariance optimization algorithm for solving (3.76). Specifically, the covariances of the transmitted signals are first optimized by setting the pseudo-covariances to be zero, i.e., proper Gaussian signaling is applied. Then, the pseudo-covariances are optimized with the covariances fixed as the previously optimized values. With such a separation approach, the obtained improper signaling scheme is guaranteed to be no smaller than the rate over proper Gaussian signaling scheme.

3.6.3.1 Covariance Optimization

When restricted to proper Gaussian signaling with $\tilde{C}_{x_1} = 0$ and $\tilde{C}_{x_2} = 0$, the second term in (3.75) vanishes to zero. As a result, problem (3.76) reduces to

$$\begin{aligned}
 & \max_{r, C_{x_1}, C_{x_2}} \quad r \\
 & \text{s.t.} \quad \log\left(1 + \frac{|h_{11}|^2 C_{x_1}}{\sigma^2 + |h_{12}|^2 C_{x_2}}\right) \geq \alpha_1 r, \\
 & \quad \log\left(1 + \frac{|h_{22}|^2 C_{x_2}}{\sigma^2 + |h_{21}|^2 C_{x_1}}\right) \geq \alpha_2 r, \\
 & \quad 0 \leq C_{x_1} \leq P_1, \quad 0 \leq C_{x_2} \leq P_2.
 \end{aligned} \tag{3.105}$$

For any fixed target rate r , problem (3.105) can be transformed to the following feasibility problem:

$$\begin{aligned}
 & \text{Find} \quad C_{x_1} \in \mathbb{R}, \quad C_{x_2} \in \mathbb{R} \\
 & \text{s.t.} \quad |h_{11}|^2 C_{x_1} \geq (\sigma^2 + |h_{12}|^2 C_{x_2})(2^{\alpha_1 r} - 1), \\
 & \quad |h_{22}|^2 C_{x_2} \geq (\sigma^2 + |h_{21}|^2 C_{x_1})(2^{\alpha_2 r} - 1), \\
 & \quad 0 \leq C_{x_1} \leq P_1, \quad 0 \leq C_{x_2} \leq P_2.
 \end{aligned} \tag{3.106}$$

(3.106) is a linear programming (LP) problem, which can be efficiently solved with existing algorithms such as the simplex method [165]. An alternative way for solving (3.106) is by using the existing power control results [106]. Specifically, given the SINR targets $\Gamma_k = 2^{\alpha_k r} - 1$, $k = 1, 2$, define the normalized channel gain matrix $\mathbf{G}$ and the normalized noise vector $\mathbf{u}$ respectively as

$$\mathbf{G} = \begin{bmatrix} 0 & \frac{\Gamma_1 |h_{12}|^2}{|h_{11}|^2} \\ \frac{\Gamma_2 |h_{21}|^2}{|h_{22}|^2} & 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \frac{\Gamma_1 \sigma^2}{|h_{11}|^2} \\ \frac{\Gamma_2 \sigma^2}{|h_{22}|^2} \end{bmatrix}.$$

To achieve the rate constraints in (3.106) with equality, the covariances (or the transmit powers) are given by

$$\mathbf{c} = (\mathbf{I} - \mathbf{G})^{-1} \mathbf{u}, \tag{3.107}$$

where $\mathbf{c} \triangleq [C_{x_1} \ C_{x_2}]^T$. Following the results in [106], the problem (3.106) is feasible if and only if the following two conditions are satisfied: (1) $\rho(\mathbf{G}) < 1$, where $\rho(\mathbf{G})$ denotes the spectral radius, or the maximum modulus eigenvalue, of $\mathbf{G}$; (2) the solution obtained by (3.107) satisfies $C_{x_1} \leq P_1$ and $C_{x_2} \leq P_2$.

If r is feasible for (3.106), then the optimal value of (3.105) satisfies $r^* \geq r$; otherwise, $r^* < r$. Thus, the covariance optimization problem (3.105) can be efficiently solved by solving (3.106) together with the bisection method for updating r .

3.6.3.2 Pseudo-Covariance Optimization

Denote the optimal solution to the covariance optimization problem (3.105) as $\{r^*, C_{x_1}^*, C_{x_2}^*\}$. By fixing the covariances as $C_{x_1}^*$ and $C_{x_2}^*$, (3.76) is then optimized over the pseudo-covariances $\tilde{C}_{x_1}$ and $\tilde{C}_{x_2}$. By substituting the first term in the rate expression (3.75) with $\alpha_k r^*$, the problem for pseudo-covariance optimization is formulated as

$$\begin{aligned}
& \max_{\tilde{C}_{x_1}, \tilde{C}_{x_2}, R} R \\
& \text{s.t.} \quad \alpha_1 r^* + \frac{1}{2} \log \frac{1 - C_{y_1}^{-2} |\tilde{C}_{y_1}|^2}{1 - C_{s_1}^{-2} |\tilde{C}_{s_1}|^2} \geq \alpha_1 R, \\
& \quad \alpha_2 r^* + \frac{1}{2} \log \frac{1 - C_{y_2}^{-2} |\tilde{C}_{y_2}|^2}{1 - C_{s_2}^{-2} |\tilde{C}_{s_2}|^2} \geq \alpha_2 R, \\
& \quad |\tilde{C}_{x_1}|^2 \leq C_{x_1}^{*2}, \\
& \quad |\tilde{C}_{x_2}|^2 \leq C_{x_2}^{*2},
\end{aligned} \tag{3.108}$$

where $C_{y_1}, C_{s_1}, C_{y_2}$ and C_{s_2} are fixed covariance terms corresponding to the transmit covariances $C_{x_1}^*$ and $C_{x_2}^*$, and $\tilde{C}_{y_1}, \tilde{C}_{s_1}, \tilde{C}_{y_2}$ and $\tilde{C}_{s_2}$ are the received pseudo-covariances that are related to $\tilde{C}_{x_1}$ and $\tilde{C}_{x_2}$ via (3.72) and (3.73), respectively. Again, if a given R is achievable for certain pair of $\tilde{C}_{x_1}$ and $\tilde{C}_{x_2}$, then the optimal value of (3.108) satisfies $R^* \geq R$; otherwise, $R^* < R$. Therefore, (3.108) can be solved via solving a set of feasibility problems together with the bisection method. It can be easily observed that $\{\tilde{C}_{x_1} = 0, \tilde{C}_{x_2} = 0, R = r^*\}$ is feasible for (3.108). Therefore, $R^* \geq r^*$ is satisfied, i.e., with the proposed separate covariance and pseudo-covariance optimization approach, the users' sum-rate corresponding to the rate-profile given by (α_1, α_2) with improper Gaussian signaling is guaranteed to be no smaller than that obtained with the optimal proper Gaussian signaling obtained by solving (3.105). The remaining problem to be solved is the feasibility problem resulting from (3.108) for a given R . By substituting $\tilde{C}_{y_k}$ and $\tilde{C}_{s_k}$ into (3.108) and after some manipulations, the feasibility problem for a given R can be

formulated as

$$\begin{aligned}
& \text{Find } \tilde{C}_{x_1} \in \mathbb{C}, \tilde{C}_{x_2} \in \mathbb{C} \\
& \text{s.t. } a_1 |h_{11}^2 \tilde{C}_{x_1} + h_{12}^2 \tilde{C}_{x_2}|^2 + b_1 \leq |\tilde{C}_{x_2}|^2, \\
& \quad a_2 |h_{21}^2 \tilde{C}_{x_1} + h_{22}^2 \tilde{C}_{x_2}|^2 + b_2 \leq |\tilde{C}_{x_1}|^2, \\
& \quad |\tilde{C}_{x_1}|^2 \leq C_{x_1}^{\star 2}, \\
& \quad |\tilde{C}_{x_2}|^2 \leq C_{x_2}^{\star 2},
\end{aligned} \tag{3.109}$$

where $a_k = \frac{C_{s_k}^2}{\beta_k C_{y_k}^2 |h_{k\bar{k}}|^4}$, $b_k = \frac{(1-1/\beta_k)C_{s_k}^2}{|h_{k\bar{k}}|^4}$, $\beta_k = 2^{2\alpha_k(R-r^*)}$, $k = 1, 2, \bar{k} \neq k$. Since the optimal value of (3.108) satisfies $R^* \geq r^*$, we can assume that $R \geq r^*$ in (3.109) without loss of optimality. We then have $\beta_k \geq 1, b_k \geq 0, \forall k$. Next, we show that (3.109) can be efficiently solved via solving a few number of SOCP problems.

First, it can be verified that if $\{\tilde{C}_{x_1}, \tilde{C}_{x_2}\}$ is feasible for (3.109), then so is $\{\tilde{C}_{x_1}e^{i\omega}, \tilde{C}_{x_2}e^{i\omega}\}$ for any ω . Therefore, without loss of generality, we may choose the common phase rotation ω so that $\tilde{C}_{x_1}$ is real and nonnegative. Denote the magnitude and phase of $\tilde{C}_{x_2}$ by t and θ , respectively, i.e., $\tilde{C}_{x_2} = te^{i\theta}$. Then for any fixed value of θ , (3.109) can be transformed into an SOCP feasibility problem given by

$$\begin{aligned}
& \text{Find } \tilde{C}_{x_1} \in \mathbb{R}, t \in \mathbb{R} \\
& \text{s.t. } \left\| \frac{\sqrt{a_1}(h_{11}^2 \tilde{C}_{x_1} + h_{12}^2 te^{i\theta})}{\sqrt{b_1}} \right\| \leq t, \\
& \quad \left\| \frac{\sqrt{a_2}(h_{21}^2 \tilde{C}_{x_1} + h_{22}^2 te^{i\theta})}{\sqrt{b_2}} \right\| \leq \tilde{C}_{x_1}, \\
& \quad \tilde{C}_{x_1} \leq C_{x_1}^*, t \leq C_{x_2}^*.
\end{aligned} \tag{3.110}$$

Theorem 3.11. *The feasibility problem (3.109) can be optimally solved by solving a few number of SOCP problems (3.110), each for a fixed value θ , where without loss of optimality, θ can be restricted to the following discrete set:*

$$\theta \in \{\pi + 2(\phi_{11} - \phi_{12}), \pi + 2(\phi_{21} - \phi_{22})\} \cup \Theta_A \cup \Theta_B,$$

where Θ_A and Θ_B are the solution sets for θ to the following two sets of equations with variables (θ, t) and $(\theta, \tilde{C}_{x_1})$, respectively:

$$\Theta_A : \begin{cases} a_1 |h_{11}^2 C_{x_1}^* + h_{12}^2 te^{i\theta}|^2 + b_1 = t^2 \\ a_2 |h_{21}^2 C_{x_1}^* + h_{22}^2 te^{i\theta}|^2 + b_2 = C_{x_1}^{\star 2} \end{cases} \tag{3.111}$$

$$\Theta_{\mathcal{B}} : \begin{cases} a_1 |h_{11}^2 \tilde{C}_{x_1} + h_{12}^2 C_{x_2}^* e^{i\theta}|^2 + b_1 = C_{x_2}^{*2} \\ a_2 |h_{21}^2 \tilde{C}_{x_1} + h_{22}^2 C_{x_2}^* e^{i\theta}|^2 + b_2 = \tilde{C}_{x_1}^2 \end{cases} \quad (3.112)$$

Proof. Please refer to Appendix B. $\square$

Theorem 3.11 can be intuitively understood as follows. For the feasibility problem (3.109), if the first constraint is more “restrictive” than the second one, then θ should have a value such that the LHS of the first constraint is minimized. This corresponds to $\theta = \pi + 2(\phi_{11} - \phi_{12})$ so that $h_{11}^2 \tilde{C}_{x_1}$ and $h_{12}^2 \tilde{C}_{x_2}$ are antiphase. Similar argument for $\theta = \pi + 2(\phi_{21} - \phi_{22})$ can be made. On the other hand, if the first two constraints of (3.109) are equally “restrictive”, then a feasible solution tends to make both constraints satisfied with equality, as given by (3.111) and (3.112). $\Theta_{\mathcal{A}}$ and $\Theta_{\mathcal{B}}$ correspond to the cases where either the third or the fourth constraint of (3.109) is active, which can be assumed without loss of generality as shown by Proposition B.1 in Appendix B. The elements in $\Theta_{\mathcal{A}}$ and $\Theta_{\mathcal{B}}$ can be efficiently obtained by following the procedures given in Appendix C.

Remark 3.12. Based on Appendix C, it follows that the cardinalities of the two sets $\Theta_{\mathcal{A}}$ and $\Theta_{\mathcal{B}}$ satisfy $|\Theta_{\mathcal{A}}| \leq 4$ and $|\Theta_{\mathcal{B}}| \leq 4$. Therefore, at most 10 SOCP problems as given by (3.110) need to be solved in order to optimally solve the non-convex problem (3.109). This number can be lesser, since any feasible solution to (3.110) for a particular θ value is sufficient to declare that (3.109) is feasible, and hence solves the problem.

The proposed algorithm for (3.76) by separately optimizing the transmit covariance and pseudo-covariance is summarized as follows:

Algorithm 6 Separate Covariance and Pseudo-covariance Optimization for (3.76)

- 1: Solve the covariance optimization problem (3.105) via bisection method and the LP problem (3.106), and denote the optimal solution as $\{r^*, C_{x_1}^*, C_{x_2}^*\}$.
 - 2: Initialize $R_L = r^*$, and R_U to be the value given by (D.4) in Appendix D.
 - 3: **repeat**
 - 4: Set $R \leftarrow (R_L + R_U)/2$ and solve the feasibility problem (3.109) based on Theorem 3.11.
 - 5: If (3.109) is feasible, then set $R_L \leftarrow R$; otherwise, $R_U \leftarrow R$.
 - 6: **until** $R_U - R_L \leq \epsilon$, where ϵ is the error tolerance.
-

FIGURE 3.3: Empirical ratio $R^*/\hat{R}$ for the two-user SISO-IC over 500 random channel realizations.

3.7 Numerical Results

In this section, we evaluate the performance of the proposed algorithms for the two-user SISO-IC with numerical examples. Both transmitters are assumed to have the same power constraint P , i.e., $P_1 = P_2 = P$. SNR is defined as P/σ^2 . For the SDR-based joint covariance and pseudo-covariance optimization algorithm, $L = 1000$ is used for the Gaussian randomization procedure in Algorithm 4.

3.7.1 Approximation Ratio for SDR

In this subsection, we evaluate the quality of the approximate solution obtained by the SDR-based joint covariance and pseudo-covariance optimization algorithm proposed in Section 3.6.2. Denote R^* and R_{sdr} as the optimal objective values of (3.93) and its relaxation (3.101), respectively. Further denote $\hat{R}$ as the objective value of (3.93) corresponding to the approximate solution obtained by Algorithm 4. Then we have

$$\hat{R} \leq R^* \leq R_{\text{sdr}}, \text{ or } 1 \leq R^*/\hat{R} \leq R_{\text{sdr}}/\hat{R}, \quad (3.113)$$

FIGURE 3.4: Achievable rate region for the two-user SISO-IC with channel realization $\mathbf{H}^{(1)}$, and SNR = 0 dB.

where $R^*/\hat{R}$ is the approximation ratio. Since in general the optimal value R^* is difficult to be found, the upper bound $R_{\text{sdr}}/\hat{R}$ of the approximation ratio is usually used to evaluate the quality of the obtained approximate solutions [53]. Fortunately, for the two-user SISO-IC considered herein, the optimal value R^* of (3.87) and hence that of its equivalent problem (3.93) can be obtained by the exhaustive search method proposed in [161]. With the rate-profile α setting to $(1/2, 1/2)$, the empirical ratios of $R^*/\hat{R}$ over 500 random channel realizations at SNR= 0 dB are plotted in Figure 3.3, where the channel coefficients are generated from i.i.d. CSCG random variables with zero-mean and unit-variance. It is found that the mean of $R^*/\hat{R}$ is close to 1, which demonstrates the high quality of the approximate solution obtained by the SDR-based solution.

3.7.2 Rate Region Comparison

In Figure 3.4 and Figure 3.5, the achievable rate regions for an example two-user SISO-IC are plotted for SNR=0 dB and 10 dB, respectively. The channel matrix for both plots is given by $\mathbf{H}^{(1)} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 2.0310e^{-i0.6858} & 1.4766e^{i2.6452} \\ 0.7280e^{i1.9726} & 0.9935e^{-i0.6676} \end{bmatrix}$. The proposed improper Gaussian signaling schemes with joint and separate covariance

FIGURE 3.5: Achievable rate region for the two-user SISO-IC with channel realization $\mathbf{H}^{(1)}$, and SNR = 10 dB.

and pseudo-covariance optimizations are compared with other existing schemes, including the optimal proper Gaussian signaling scheme by solving problem (3.105), the optimal improper Gaussian signaling obtained by the exhaustive search method [161], and the rank-1 improper Gaussian signaling scheme based on the equivalent real-valued MIMO-IC [160].⁷ Both figures reveal that for the given channel $\mathbf{H}^{(1)}$, the achievable rate regions are significantly enlarged with improper Gaussian signaling over the conventional proper Gaussian signaling. The plots also demonstrate that the SDR-based joint covariance and pseudo-covariance optimization algorithm yields almost the optimal rates given by the exhaustive search, which is consistent with the observation in Figure 3.3. Moreover, it is observed that the separate covariance and pseudo-covariance optimization algorithm performs close to the optimal solution, and also always outperforms the optimal proper Gaussian signaling. It is worth remarking that, even with time-sharing (TS),⁸ improper Gaussian

⁷The curves denoted by “improper, rank-1” are given after taking the normal-hull of all the achievable rate-pairs obtained by the rank-1 scheme proposed in [160]. The normal-hull of a set $\mathcal{S} \in \mathbb{R}_+^n$ is given by $\text{NHull}(\mathcal{S}) = \{\mathbf{y} \in \mathbb{R}_+^n | \mathbf{y} \preceq \mathbf{x}, \forall \mathbf{x} \in \mathcal{S}\}$.

⁸The achievable rate region with TS is obtained by taking the convex-hull operation over all the achievable rate-pairs. A convex hull of a set $\mathcal{A}$ is the set of all convex combinations of points in $\mathcal{A}$, where a convex combination of $\mathbf{x}_1, \dots, \mathbf{x}_k$ is any point $\mathbf{x}$ of the form $\mathbf{x} = \sum_{l=1}^k \theta_l \mathbf{x}_l$, with $\theta_l \geq 0$ and $\sum_{l=1}^k \theta_l = 1$.

FIGURE 3.6: Achievable rate region for the two-user SISO-IC with channel realization $\mathbf{H}^{(2)}$, and $\text{SNR} = 0$ dB.

signaling still outperforms proper Gaussian signaling, as shown by the dashed lines in the two figures. For this particular channel realization, the Pareto boundary points of the achievable rate region with TS using improper Gaussian signaling can be obtained by the TS between the two single-user maximum rate points, and the largest rate corner point by the existing rank-1 scheme [160]. However, this is not always the case, as illustrated by the next example.

Consider the channel given by $\mathbf{H}^{(2)} = \begin{bmatrix} 4.0e^{i1.7730} & 0.90e^{i1.6744} \\ 0.80e^{i0.6249} & 1.50e^{i2.1057} \end{bmatrix}$. It is observed from Figure 3.6 that for this particular channel realization at $\text{SNR}=0$ dB, there is no notable performance gain by using improper Gaussian signaling over proper signaling, which is in contrast to that observed in Figure 3.4 with channel $\mathbf{H}^{(1)}$. This is mainly due to the relatively weaker interfering link in this channel setup. For example, the interference-to-signal power gain ratio at user 1's receiver is given by $|h_{12}|^2/|h_{11}|^2 = 0.051$, which is much smaller than 0.53 in channel $\mathbf{H}^{(1)}$. Note that intuitively, it is the non-negligible mutual interference among the users that is exploited by improper Gaussian signaling to outperform the conventional proper

FIGURE 3.7: Achievable rate region for the two-user SISO-IC with channel realization $\mathbf{H}^{(2)}$ and SNR = 10 dB.

Gaussian signaling.⁹ Therefore, for channel realization $\mathbf{H}^{(2)}$ with almost negligible interfering link for user 1, no observable performance gain can be achieved by improper Gaussian signaling at SNR=0 dB. As SNR increased to 10 dB, the interference power increases and a notable gain by improper Gaussian signaling is again achieved, as shown in Figure 3.7. Another observation from Figure 3.6 and Figure 3.7 is that the rank-1 improper signaling scheme [160], which is based on the equivalent real-valued 2×2 MIMO-IC, gives smaller rate region than that by proper Gaussian signaling. In contrast, our proposed improper signaling schemes with either joint or separate covariance and pseudo-covariance optimizations, are observed to perform no worse than the optimal proper Gaussian signaling, in accordance with our previous discussion.

3.7.3 Max-Min Rate Comparison

The rate-profile technique used in characterizing the Pareto boundary of the achievable rate region can be directly applied for maximizing the minimum (max-min)

⁹Consider the extreme case where all the interfering link gains vanish to zero and the SISO-IC reduces to K decoupled Gaussian point-to-point channels. In this case, proper Gaussian signaling is known to be optimal.

FIGURE 3.8: Average max-min rate for the two-user SISO-IC.

rate of the two users without TS. Specifically, the max-min problem for the two-user SISO-IC is equivalent to solving (3.76) by using the rate-profile $\alpha = (1/2, 1/2)$. An alternative max-min solution with improper Gaussian signaling was proposed in [161], where based on the equivalent real-valued 2×2 MIMO-IC, the transmit covariance matrix of the equivalent real-valued signal vector for each user is assumed to be of rank-1. Note that the use of rank-1 transmission in both [160] and [161] can be justified by the fact that the total DoF of two-user 2×2 MIMO-ICs exactly equals to 2 [38]. For the ease of precoder design, zero-forcing (ZF) receivers were applied in [161]. As a benchmark comparison, we also plot the max-min rate achievable by the simple time division multiple access (TDMA) scheme, where for simplicity, each user is assumed to access the channel for half of the time.

To evaluate the average max-min rates, 500 random channel realizations are simulated, where the channel coefficients are drawn from independent zero-mean CSCG random variables. For this example, asymmetric channels are considered, where the average power values of the direct and interfering channels are 1 and 0.2, respectively, i.e., $h_{kk} \sim \mathcal{CN}(0, 1)$, $h_{k\bar{k}} \sim \mathcal{CN}(0, 0.2)$, $k = 1, 2$, $\bar{k} \neq k$. The obtained results are shown in Figure 3.8. The optimal max-min rate achievable by proper Gaussian signaling and that by improper Gaussian signaling obtained by the exhaustive search method [161] are also included in the figure. It is observed that in

the low-SNR regime, there is no notable gain by improper Gaussian signaling over conventional proper Gaussian signaling, which is due to the negligible interference levels at low SNRs. As SNR increases, the max-min rate by proper Gaussian signaling saturates since the total number of data streams transmitted, which is 2, exceeds the total number of DoF of the two-user SISO-IC, which is 1. In contrast, the linear increase of the max-min rates with respect to the logarithm of SNR can be achieved either by TDMA, or by improper Gaussian signaling. It is worth remarking that over the entire SNR range, the proposed algorithms based on covariance and pseudo-covariance optimizations yield close-to-optimal performance obtained by exhaustive search method. On the other hand, the rank-1 transmission with ZF receivers based on the equivalent real-valued MIMO-IC gives a near-optimal performance in the high-SNR regime, which is expected due to the optimality of ZF receivers at high SNR as well as the DoF optimality of rank-1 transmission as pointed out in [160, 161]; however, in the low-to-medium SNR regime, the rank-1 transmission scheme results in strictly sub-optimal performance, which may be due to the noise enhancement issue associated with the ZF receivers applied in [161].

3.7.4 Sum-Rate Comparison

In this subsection, the sum-rate of the two-user SISO-IC with improper Gaussian signaling is considered. By using the equivalent real-valued MIMO-IC as given in (3.7), existing sum-rate maximization algorithms in the literature, such as the one via the iterative weighted MSE minimization (WMMSE) [13], can be applied for maximizing the sum-rate of the two-user SISO-IC when improper Gaussian signaling is employed. However, although the WMMSE algorithm is guaranteed to converge to a local maximum of the sum-rate, it is not guaranteed to achieve the global sum-rate maximum. We will illustrate using the following example that with the proposed covariance and pseudo-covariance based algorithms, it is possible to strictly improve the sum-rate obtained by the WMMSE approach applied in the real domain.

Without loss of generality, denote the rate-pair obtained by the iterative WMMSE scheme by $r \cdot (\delta_1, \delta_2)$, where $\delta_1 \geq 0, \delta_2 \geq 0$, and $\delta_1 + \delta_2 = 1$. With the rate-profile $\boldsymbol{\alpha} = (\delta_1, \delta_2)$, problem (3.76) is solved to obtain a new sum-rate R . If $R > r$, then the rate-pair obtained by the WMMSE algorithm cannot be sum-rate optimal, and

TABLE 3.1: Equivalence between complex-valued SISO-IC and real-valued MIMO-IC

	Complex-valued SISO-IC	Real-valued MIMO-IC
Channel	$h_{11} = 2.7388 - 0.2498i, h_{12} = 0.9956 + 1.8047i$ $h_{21} = 0.6680 - 1.6470i, h_{22} = 0.4760 + 1.2706i$	$\mathbf{H}_{11} = \begin{bmatrix} 2.7388 & 0.2498 \\ -0.2498 & 2.7388 \end{bmatrix}, \mathbf{H}_{12} = \begin{bmatrix} 0.9956 & -1.8047 \\ 1.8047 & 0.9956 \end{bmatrix}$ $\mathbf{H}_{21} = \begin{bmatrix} 0.6680 & 1.6470 \\ -1.6470 & 0.6680 \end{bmatrix}, \mathbf{H}_{22} = \begin{bmatrix} 0.4760 & -1.2706 \\ 1.2706 & 0.4760 \end{bmatrix}$
WMMSE Init.	$C_{x_1} = 10.000, \tilde{C}_{x_1} = 9.546e^{i0.5512}$ $C_{x_2} = 10.000, \tilde{C}_{x_2} = 7.6118e^{i2.8995}$	$\mathbf{Q}_1 = \begin{bmatrix} 9.0660 & 2.4998 \\ 2.4998 & 0.9340 \end{bmatrix}, \mathbf{Q}_2 = \begin{bmatrix} 1.3051 & 0.9125 \\ 0.9125 & 8.6949 \end{bmatrix}$
WMMSE Sol.	$C_{x_1} = 9.9981, \tilde{C}_{x_1} = 9.9981e^{i0.4575}$ $C_{x_2} = 9.9800, \tilde{C}_{x_2} = 9.9800e^{-i3.0980}$	$\mathbf{Q}_1 = \begin{bmatrix} 9.4840 & 2.2081 \\ 2.2081 & 0.5141 \end{bmatrix}, \mathbf{Q}_2 = \begin{bmatrix} 0.0047 & -0.2174 \\ -0.2174 & 9.9752 \end{bmatrix}$
Separate opt.	$C_{x_1} = 8.7366, \tilde{C}_{x_1} = 8.7366$ $C_{x_2} = 9.9887, \tilde{C}_{x_2} = 9.9885e^{i0.0142}$	$\mathbf{Q}_1 = \begin{bmatrix} 8.7366 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{Q}_2 = \begin{bmatrix} 9.9881 & 0.0708 \\ 0.0708 & 0.0006 \end{bmatrix}$
Joint opt.	$C_{x_1} = 10.00, \tilde{C}_{x_1} = 10.00e^{i1.1204}$ $C_{x_2} = 10.00, \tilde{C}_{x_2} = 10.00e^{-i1.0440}$	$\mathbf{Q}_1 = \begin{bmatrix} 7.1768 & 4.5013 \\ 4.5013 & 2.8232 \end{bmatrix}, \mathbf{Q}_2 = \begin{bmatrix} 7.5137 & 4.3221 \\ 4.3221 & 2.4863 \end{bmatrix}$

TABLE 3.2: Sum-Rate comparison

	WMMSE	Separate opt.	Joint opt.
Rate-pair	(2.8673, 1.8102)	(3.4079, 2.1515)	(3.4761, 2.2078)
Sum-rate	4.6775	5.5594	5.6839
Improvement	—	18.85%	21.52%

a strictly improved sum-rate has been obtained. Through our extensive numerical simulations, it is found that although the iterative WMMSE approach applied in the real domain is able to find high quality local maximum of the sum-rate for most instances, there do exist some scenarios that a strict sum-rate improvement can be achieved by optimizing the transmit covariance and pseudo-covariance. One such example is shown in Table 3.1, where the sum-rate results obtained by the various algorithms are given in Table 3.2 for SNR= 10 dB.

3.8 Chapter Summary

In this chapter, we have investigated the achievable rate of Gaussian interference channel with improper or circularly asymmetric complex Gaussian signaling applied at each transmitter. Under the assumption that interference is treated as additive noise at each receiver, we have derived a new rate expression, which is more general than the widely used rate formula for the conventional proper Gaussian signaling. Our result shows that the achievable rate with improper Gaussian signaling can be expressed as a summation of the rate achievable by the conventional proper Gaussian signaling in terms of the transmit covariance matrices, and an additional term that is a function of both the users' transmit covariance and pseudo-covariance matrices. Such a separable structure provides a

useful method to improve the rate over the conventional proper Gaussian signaling by further optimizing the pseudo-covariance matrices, as has been illustrated in Section 3.6.3 for the two-user SISO-IC. An extension of such a technique to the K -user MISO-IC will be given in Chapter 4. We have also proposed the technique of widely linear precoding, which efficiently maps the proper Gaussian information-bearing signals to the improper Gaussian transmitted signals with any given valid pair of covariance and pseudo-covariance matrices. Numerical results have been provided to verify the performance of the proposed algorithms.

Chapter 4

Optimized Improper Gaussian Signaling for MISO-IC

In Chapter 3, we have developed a covariance and pseudo-covariance based optimization framework for interference channels employing improper Gaussian signaling. The separable structure of the novel rate expression derived therein has been exploited to improve the users' achievable rates for two-user SISO-IC. However, the algorithms proposed in Chapter 3 cannot be applied when there are more than two users and/or multiple transmit antennas. This thus motivates our work in the current chapter that extends the results in Chapter 3 to optimize improper Gaussian signaling to the more general K -user MISO-IC. We also show that the improper Gaussian signaling scheme developed for the MISO-IC can be also applied to the K -user MISO broadcast channel (MISO-BC), under the assumption of employing (widely) linear precoding at the multi-antenna transmitter.¹

4.1 MISO-IC with Improper Gaussian Signaling

For a K -user MISO-IC where each transmitter is equipped with M antennas and each receiver with one single antenna, the input-output relationship shown in (3.1)

¹It is worth noting that if the optimal non-linear “dirty paper coding (DPC)” based precoding is applied for the MISO-BC, the achievable rate region can be equivalently characterized in the dual uplink SIMO multiple-access channel (MAC) via the celebrated BC-MAC duality result [22–24], for which it can be shown that proper Gaussian signaling is optimal.

reduces to

$$y_k = \mathbf{h}_{kk}\mathbf{x}_k + \sum_{j \neq k} \mathbf{h}_{kj}\mathbf{x}_j + v_k, \quad k = 1, \dots, K, \quad (4.1)$$

where $\mathbf{h}_{kk} \in \mathbb{C}^{1 \times M}$ denotes the direct channel from transmitter k to receiver k , while $\mathbf{h}_{kj} \in \mathbb{C}^{1 \times M}, j \neq k$, denotes the interference channel from transmitter j to receiver k ; v_k represents the i.i.d. zero-mean CSCG noise with variance σ^2 , i.e., $v_k \sim \mathcal{CN}(0, \sigma^2)$; and $\mathbf{x}_k \in \mathbb{C}^{M \times 1}$ is the transmitted signal vector from transmitter k , which is independent of $\mathbf{x}_j$ for $j \neq k$.

With improper Gaussian signaling applied at each transmitter, the covariance and pseudo-covariance of the received signal y_k for the MISO-IC can be written as

$$C_{y_k} = \mathbb{E}(y_k y_k^*) = \sum_{j=1}^K \mathbf{h}_{kj} \mathbf{C}_{\mathbf{x}_j} \mathbf{h}_{kj}^H + \sigma^2, \quad (4.2)$$

$$\tilde{C}_{y_k} = \mathbb{E}(y_k y_k) = \sum_{j=1}^K \mathbf{h}_{kj} \tilde{\mathbf{C}}_{\mathbf{x}_j} \mathbf{h}_{kj}^T, \quad (4.3)$$

where $\mathbf{C}_{\mathbf{x}_j} = \mathbb{E}(\mathbf{x}_j \mathbf{x}_j^H)$ and $\tilde{\mathbf{C}}_{\mathbf{x}_j} = \mathbb{E}(\mathbf{x}_j \mathbf{x}_j^T)$ are the covariance and pseudo-covariance matrices of the transmitted signal $\mathbf{x}_j$, respectively. It can be seen from (4.2) that with single-antenna receivers, the covariance C_{y_k} of the received signal is a positive real number whose value equals to the total received power at receiver k . Denote the interference-plus-noise term at receiver k as s_k , i.e., $s_k = \sum_{j \neq k} \mathbf{h}_{kj}\mathbf{x}_j + v_k$. Then the covariance and pseudo-covariance of s_k are respectively given by

$$C_{s_k} = \mathbb{E}(s_k s_k^*) = \sum_{j \neq k} \mathbf{h}_{kj} \mathbf{C}_{\mathbf{x}_j} \mathbf{h}_{kj}^H + \sigma^2, \quad (4.4)$$

$$\tilde{C}_{s_k} = \mathbb{E}(s_k s_k) = \sum_{j \neq k} \mathbf{h}_{kj} \tilde{\mathbf{C}}_{\mathbf{x}_j} \mathbf{h}_{kj}^T. \quad (4.5)$$

With improper Gaussian signaling at all transmitters, the resulting interference at each receiver is improper Gaussian as well. Under the assumption that the improper Gaussian interference is treated as additional noise over existing proper Gaussian background noise at all receivers, the achievable rate is given by (3.26) for the general K -user MIMO-IC. For the K -user MISO-IC setup considered in

this chapter, the achievable rate expression (3.26) reduces as

$$R_k^{\text{MISO}} = \underbrace{\log \left(1 + \frac{\mathbf{h}_{kk} \mathbf{C}_{\mathbf{x}_k} \mathbf{h}_{kk}^H}{\sigma^2 + \sum_{j \neq k} \mathbf{h}_{kj} \mathbf{C}_{\mathbf{x}_j} \mathbf{h}_{kj}^H} \right)}_{\triangleq R_k^{\text{proper}}(\{\mathbf{C}_{\mathbf{x}_j}\})} + \frac{1}{2} \log \frac{1 - C_{y_k}^{-2} |\tilde{C}_{y_k}|^2}{1 - C_{s_k}^{-2} |\tilde{C}_{s_k}|^2}. \quad (4.6)$$

The optimization problem (3.52) to find Pareto-optimal rate-tuples for the K -user MISO-IC can be rewritten as

$$\begin{aligned} \max_{\{\mathbf{C}_{\mathbf{x}_k}\}, \{\tilde{\mathbf{C}}_{\mathbf{x}_k}\}, R} \quad & R \\ \text{s.t.} \quad & R_k^{\text{MISO}} \geq \alpha_k R, \quad \forall k, \\ & \text{Tr}(\mathbf{C}_{\mathbf{x}_k}) \leq P_k, \quad \forall k, \\ & \begin{bmatrix} \mathbf{C}_{\mathbf{x}_k} & \tilde{\mathbf{C}}_{\mathbf{x}_k} \\ \tilde{\mathbf{C}}_{\mathbf{x}_k}^* & \mathbf{C}_{\mathbf{x}_k}^* \end{bmatrix} \succeq \mathbf{0}, \quad \forall k. \end{aligned} \quad (4.7)$$

Problem (4.7) is non-convex due to the non-convex rate constraint; thus it is difficult to find the globally optimal solution. In fact, the complexity status of (4.7) remains an open problem, even for the SISO case ($M = 1$). This is in contrast to the MIMO-IC setup that is known to be NP-hard (see Section 3.4), as well as to the case with the conventional proper Gaussian signaling in the SISO/MISO setup, which is known to be polynomial-time solvable (see [41] and Section 4.2.1).

Remark 4.1. Determining the complexity status of problem (4.7) is a non-trivial task that worth further studies. Note that since (4.7) is equivalent to problem (3.51) corresponding to $N = 1$, determining the complexity status of (3.51) for the *real-valued* $2 \times 2M$ MIMO-IC is tantamount to determining the NP-hardness of (4.7). In [166], the authors have shown that the max-min problem is NP-hard for the *real-valued* MIMO-IC with at least two transmitter (receiver) antennas and at least three receiver (transmitter) antennas. However, the techniques used in [166] cannot be directly applied to (3.51) since the channel matrices $\bar{\mathbf{H}}_{kj}$ of (3.51) have the very specific structure as shown in (3.6).

4.2 Separate Covariance and Pseudo-Covariance Optimization

In this section, we develop an efficient sub-optimal solution for problem (4.7) by adopting a similar separate covariance and pseudo-covariance optimization approach as in Section 3.6.3. Specifically, the optimal transmit covariance matrices are firstly obtained by setting the pseudo-covariance matrices to be zero, i.e., proper Gaussian signaling is applied. With the covariance matrices fixed to the previously optimized values, the pseudo-covariance matrices are then optimized for further rate improvement. For the pseudo-covariance optimization sub-problem, we first establish the optimality of rank-1 pseudo-covariance matrices, given that rank-1 transmit covariance matrices have been applied. Moreover, each of such rank-1 pseudo-covariance matrices is parameterized by one single complex scalar. As a result, the original matrix optimization problem can be reformulated as an equivalent vector optimization problem with a much lower dimension. The SDR technique is then applied to find an efficient approximate solution to the reformulated problem.

4.2.1 Covariance Optimization

When restricted to proper Gaussian signaling by setting $\tilde{\mathbf{C}}_{\mathbf{x}_k} = \mathbf{0}, \forall k$, the second term in (4.6) vanishes to zero and the problem (4.7) reduces to

$$\begin{aligned}
 & \max_{r, \{\mathbf{C}_{\mathbf{x}_k}\}} r \\
 & \text{s.t. } \log \left(1 + \frac{\mathbf{h}_{kk} \mathbf{C}_{\mathbf{x}_k} \mathbf{h}_{kk}^H}{\sigma^2 + \sum_{j \neq k} \mathbf{h}_{kj} \mathbf{C}_{\mathbf{x}_j} \mathbf{h}_{kj}^H} \right) \geq \alpha_k r, \quad \forall k, \\
 & \quad \text{Tr}(\mathbf{C}_{\mathbf{x}_k}) \leq P_k, \quad \forall k, \\
 & \quad \mathbf{C}_{\mathbf{x}_k} \succeq \mathbf{0}, \quad \forall k.
 \end{aligned} \tag{4.8}$$

Denote the optimal value of problem (4.8) as r^* . Then the rate-tuple $r^* \cdot \boldsymbol{\alpha}$ is on the Pareto boundary of the achievable rate region with proper Gaussian signaling. It has been shown in [36, 129, 130] that all the Pareto-optimal rate-tuples with proper Gaussian signaling can be achieved by transmit beamforming, i.e., with rank-1 transmit covariance matrices. Therefore, without loss of optimality for

problem (4.8), we let

$$\mathbf{C}_{\mathbf{x}_k} = \mathbf{t}_k \mathbf{t}_k^H, \quad \forall k, \quad (4.9)$$

where $\mathbf{t}_k$ is the transmit beamforming vector for user k . As a result, problem (4.8) can be equivalently written as

$$\begin{aligned} \max_{r, \{\mathbf{t}_k\}} \quad & r \\ \text{s.t.} \quad & \log \left(1 + \frac{|\mathbf{h}_{kk} \mathbf{t}_k|^2}{\sigma^2 + \sum_{j \neq k} |\mathbf{h}_{kj} \mathbf{t}_j|^2} \right) \geq \alpha_k r, \quad \forall k, \\ & \|\mathbf{t}_k\|^2 \leq P_k, \quad \forall k. \end{aligned} \quad (4.10)$$

For any fixed target rate r , denote the target SINR as γ_k , i.e., $\gamma_k = 2^{\alpha_k r} - 1$. Then the rate constraint in (4.10) can be expressed as

$$\sigma^2 + \sum_{j=1}^K |\mathbf{h}_{kj} \mathbf{t}_j|^2 \leq (1 + 1/\gamma_k) |\mathbf{h}_{kk} \mathbf{t}_k|^2. \quad (4.11)$$

Furthermore, it is observed that any phase rotation to the beamforming vectors $\{\mathbf{t}_k\}$ does not change the achievable rate and the transmit power. Therefore, without loss of generality, we may assume that $\mathbf{h}_{kk} \mathbf{t}_k$ is real and nonnegative for all k [41, 114]. As a result, the feasibility problem for (4.10) to achieve a given rate r can be formulated as

$$\begin{aligned} & \text{Find } \{\mathbf{t}_k\} \\ & \text{s.t.} \quad \left\| \begin{array}{c} \mathbf{h}_{k1} \mathbf{t}_1 \\ \vdots \\ \mathbf{h}_{kK} \mathbf{t}_K \\ \sigma \end{array} \right\| \leq \sqrt{1 + 1/\gamma_k} (\mathbf{h}_{kk} \mathbf{t}_k), \quad \forall k \\ & \quad \Im\{\mathbf{h}_{kk} \mathbf{t}_k\} = 0, \quad \|\mathbf{t}_k\| \leq \sqrt{P_k}, \quad \forall k. \end{aligned} \quad (4.12)$$

(4.12) is an SOCP problem, which can be efficiently solved [65]. If problem (4.12) for the given value r is feasible, then the optimal value of (4.8) satisfies $r^* \geq r$; otherwise, $r^* < r$. Therefore, problem (4.8) can be optimally solved by solving (4.12) with different values of r , and applying a bisection update over r .

4.2.2 Pseudo-Covariance Optimization

Denote the optimal solution to the covariance optimization problem (4.8) as $\{r^*, \mathbf{C}_{\mathbf{x}_k}^* = \mathbf{t}_k \mathbf{t}_k^H\}$. By fixing the transmit covariance matrices as $\{\mathbf{C}_{\mathbf{x}_k}^* = \mathbf{t}_k \mathbf{t}_k^H\}$, problem (4.7) is further optimized over the pseudo-covariance matrices $\{\tilde{\mathbf{C}}_{\mathbf{x}_k}\}$ in this subsection. By replacing the first term in the rate expression (4.6) by $\alpha_k r^*$, the problem to optimize the pseudo-covariance matrices is formulated as

$$\begin{aligned} & \max_{R, \{\tilde{\mathbf{C}}_{\mathbf{x}_k}\}} R \\ \text{s.t. } & \alpha_k r^* + \frac{1}{2} \log \frac{1 - C_{y_k}^{-2} |\tilde{C}_{y_k}|^2}{1 - C_{s_k}^{-2} |\tilde{C}_{s_k}|^2} \geq \alpha_k R, \quad \forall k, \\ & \begin{bmatrix} \mathbf{t}_k \mathbf{t}_k^H & \tilde{\mathbf{C}}_{\mathbf{x}_k} \\ \tilde{\mathbf{C}}_{\mathbf{x}_k}^* & (\mathbf{t}_k \mathbf{t}_k^H)^* \end{bmatrix} \succeq \mathbf{0}, \quad \forall k, \end{aligned} \quad (4.13)$$

where C_{y_k} and C_{s_k} are the fixed covariances with the previously optimized transmit covariance matrices $\{\mathbf{C}_{\mathbf{x}_k}^* = \mathbf{t}_k \mathbf{t}_k^H\}$; $\tilde{C}_{y_k}$ and $\tilde{C}_{s_k}$ are the pseudo-covariances, which are related to transmit pseudo-covariance matrices $\{\tilde{\mathbf{C}}_{\mathbf{x}_k}\}$ via (4.3) and (4.5), respectively. By treating R as a slack variable and discarding irrelevant terms, problem (4.13) can be reformulated as

$$\begin{aligned} & \max_{\{\tilde{\mathbf{C}}_{\mathbf{x}_k}\}} \min_{k=1, \dots, K} \frac{1}{2\alpha_k} \log \frac{1 - C_{y_k}^{-2} |\tilde{C}_{y_k}|^2}{1 - C_{s_k}^{-2} |\tilde{C}_{s_k}|^2} \\ \text{s.t. } & \begin{bmatrix} \mathbf{t}_k \mathbf{t}_k^H & \tilde{\mathbf{C}}_{\mathbf{x}_k} \\ \tilde{\mathbf{C}}_{\mathbf{x}_k}^* & (\mathbf{t}_k \mathbf{t}_k^H)^* \end{bmatrix} \succeq \mathbf{0}, \quad \forall k. \end{aligned} \quad (4.14)$$

(4.14) is a matrix optimization problem that deals with the additional rate term in (4.6) due to the use of improper Gaussian signaling. Denote the optimal value of (4.14) as τ^* . If $\tau^* > 0$, then a strict rate improvement corresponding to rate-profile $\boldsymbol{\alpha}$ over the optimal proper Gaussian signaling is achieved. The following result will be used for solving (4.14).

Lemma 4.2. *The positive semidefinite constraint in (4.14) is satisfied if and only if*

$$\tilde{\mathbf{C}}_{\mathbf{x}_k} = Z_k \tilde{\mathbf{t}}_k \tilde{\mathbf{t}}_k^T, \quad \forall k, \quad (4.15)$$

where Z_k is a complex scalar variable satisfying $|Z_k| \leq \|\mathbf{t}_k\|^2$, and $\tilde{\mathbf{t}}_k = \mathbf{t}_k / \|\mathbf{t}_k\|$ denotes the normalized transmit beamforming vector for user k with proper Gaussian

signaling.

Proof. Please refer to Appendix E. □

It is noted that $\tilde{\mathbf{C}}_{\mathbf{x}_k}$ in (4.15) is a rank-1 symmetric matrix; thus Lemma 4.2 has established the optimality of rank-1 pseudo-covariance matrices for the K -user MISO-IC, if rank-1 transmit covariance matrices have been applied. Furthermore, it follows from (4.15) that each of such rank-1 pseudo-covariance matrices is parameterized by one single complex scalar Z_k . As a result, the number of variables for the pseudo-covariance optimization problem can be significantly reduced from KM^2 in the original matrix optimization problem (4.14) to K by applying Lemma 4.2, as will be shown next.

By substituting (4.15) into (4.3) and (4.5), we get

$$\tilde{C}_{y_k} = \sum_{j=1}^K (\mathbf{h}_{kj} \tilde{\mathbf{t}}_j)^2 Z_j, \quad \tilde{C}_{s_k} = \sum_{j \neq k} (\mathbf{h}_{kj} \tilde{\mathbf{t}}_j)^2 Z_j, \quad \forall k. \quad (4.16)$$

Define the following K -dimensional complex-valued vectors:

$$\mathbf{z} = [Z_1 \quad \cdots \quad Z_K]^T, \quad (4.17)$$

$$\mathbf{m}_k = C_{y_k}^{-1} [(\mathbf{h}_{k1} \tilde{\mathbf{t}}_1)^2 \quad \cdots \quad (\mathbf{h}_{kK} \tilde{\mathbf{t}}_K)^2]^H, \quad (4.18)$$

$$\mathbf{w}_k = C_{s_k}^{-1} [\cdots (\mathbf{h}_{k(k-1)} \tilde{\mathbf{t}}_{k-1})^2 \quad 0 \quad (\mathbf{h}_{k(k+1)} \tilde{\mathbf{t}}_{k+1})^2 \quad \cdots]^H. \quad (4.19)$$

Then we have

$$C_{y_k}^{-2} |\tilde{C}_{y_k}|^2 = |\mathbf{m}_k^H \mathbf{z}|^2, \quad (4.20)$$

$$C_{s_k}^{-2} |\tilde{C}_{s_k}|^2 = |\mathbf{w}_k^H \mathbf{z}|^2. \quad (4.21)$$

Therefore, problem (4.14) can be reformulated as

$$\begin{aligned} \max_{\mathbf{z} \in \mathbb{C}^K} \min_{k=1, \dots, K} & \frac{1}{2\alpha_k} \log \frac{1 - |\mathbf{m}_k^H \mathbf{z}|^2}{1 - |\mathbf{w}_k^H \mathbf{z}|^2} \\ \text{s.t.} & |\mathbf{e}_k^T \mathbf{z}|^2 \leq \|\mathbf{t}_k\|^4, \quad \forall k, \end{aligned} \quad (4.22)$$

where $\mathbf{e}_k$ is the k th column of the K -dimensional identity matrix. Note that the constraint in (4.22) corresponds to the condition $|Z_k| \leq \|\mathbf{t}_k\|^2$ given in Lemma 4.2. Before solving the optimization problem (4.22), we first give an intuitive discussion

on when it is possible for (4.22) to have a strictly positive objective value, or in other words, to achieve a strict rate improvement over the optimal proper Gaussian signaling by further optimizing the pseudo-covariance matrices.

Denote the second term of the rate expression in (4.6) as ΔR_k , which is the additional rate term due to the use of improper Gaussian signaling. With the rank-1 transmit covariance and pseudo-covariance matrices given above, a simple upper bound on ΔR_k for (4.22) can be obtained as

$$\Delta R_k = \frac{1}{2} \log \frac{1 - |\mathbf{m}_k^H \mathbf{z}|^2}{1 - |\mathbf{w}_k^H \mathbf{z}|^2} \quad (4.23)$$

$$\leq \frac{1}{2} \log \frac{1}{1 - |\mathbf{w}_k^H \mathbf{z}|^2} \quad (4.24)$$

$$\leq -\frac{1}{2} \log(1 - \|\mathbf{w}_k\|^2 \|\mathbf{z}\|^2), \quad (4.25)$$

where in order for the upper bound in (4.25) to be meaningful, $\|\mathbf{w}_k\|^2$ is assumed to be small enough so that $\|\mathbf{w}_k\|^2 \|\mathbf{z}\|^2 < 1$. Note that $\|\mathbf{w}_k\|^2 = C_{s_k}^{-2} \sum_{j \neq k} |\mathbf{h}_{kj} \tilde{\mathbf{t}}_j|^4$, which reflects the interference level experienced at receiver k given the beamforming vectors $\{\mathbf{t}_j\}$. As $\|\mathbf{w}_k\|^2 \rightarrow 0$, the upper bound in (4.25) approaches to 0, which leads to the following remark.

Remark 4.3. Improper Gaussian signaling is more advantageous than proper Gaussian signaling in the MISO-IC only when there is non-negligible inter-user interference. For example, with zero-forcing transmit beamforming vectors $\{\mathbf{t}_j^{\text{ZF}}\}$, i.e., $\mathbf{h}_{kj} \mathbf{t}_j^{\text{ZF}} = 0$, $\forall j \neq k$, we have $\|\mathbf{w}_k\|^2 = 0$ and thus $\Delta R_k = 0$, $\forall k$. As a result, no rate improvement can be achieved by further optimizing the pseudo-covariance matrices. This is as expected since ZF transmit beamforming essentially results in K independent point-to-point MISO channels, where proper Gaussian signaling is known to be optimal [156].

For the general K -user MISO-IC, ZF transmit beamforming is feasible only when the number of transmit antennas at each transmitter is no smaller than the total number of users, i.e., $M \geq K$. For the general case when the users are interfered by each other, i.e., $\mathbf{w}_k \neq \mathbf{0}$, there is a potential to improve the achievable rates over the optimal proper Gaussian signaling by solving the pseudo-covariance optimization problem in (4.22), as will be discussed next.

Problem (4.22) maximizes the minimum of K weighted log-fractions of quadratic functions over $\mathbf{z}$, for which the well-known SDR technique can be applied to find an efficient approximate solution [53, 54]. Let $\mathbf{M}_k = \mathbf{m}_k \mathbf{m}_k^H$ and $\mathbf{W}_k = \mathbf{w}_k \mathbf{w}_k^H$, $\forall k$,

and with the identity $\mathbf{x}^H \mathbf{A} \mathbf{x} = \text{Tr}(\mathbf{A} \mathbf{x} \mathbf{x}^H)$, the SDR problem of (4.22) is given by

$$\begin{aligned} \max_{\mathbf{Z} \succeq \mathbf{0}} \min_{k=1, \dots, K} & \frac{1}{2\alpha_k} \log \frac{1 - \text{Tr}(\mathbf{M}_k \mathbf{Z})}{1 - \text{Tr}(\mathbf{W}_k \mathbf{Z})} \\ \text{s.t.} & \text{Tr}(\mathbf{E}_k \mathbf{Z}) \leq \|\mathbf{t}_k\|^4, \quad \forall k, \end{aligned} \quad (4.26)$$

where $\mathbf{E}_k = \mathbf{e}_k \mathbf{e}_k^T$. It is easy to see that problem (4.22) is equivalent to (4.26) with the additional constraint $\text{rank}(\mathbf{Z}) = 1$, under which $\mathbf{Z}$ can be expressed as $\mathbf{Z} = \mathbf{z} \mathbf{z}^H$. Therefore, the optimal value of (4.26), denoted as τ_{sdr} , provides an upper bound on that of (4.22). Since $\mathbf{Z} = \mathbf{0}$ is always feasible for (4.26), which corresponds to the objective value equals to 0, we have $\tau_{\text{sdr}} \geq 0$. In other words, the proposed pseudo-covariance optimization is guaranteed to perform no worse than the optimal proper Gaussian signaling obtained in Section 4.2.1.

By introducing a slack variable τ , problem (4.26) can be recast as

$$\begin{aligned} \max_{\mathbf{Z} \succeq \mathbf{0}, \tau} & \tau \\ \text{s.t.} & \frac{1}{2\alpha_k} \log \frac{1 - \text{Tr}(\mathbf{M}_k \mathbf{Z})}{1 - \text{Tr}(\mathbf{W}_k \mathbf{Z})} \geq \tau, \quad \forall k, \\ & \text{Tr}(\mathbf{E}_k \mathbf{Z}) \leq \|\mathbf{t}_k\|^4, \quad \forall k. \end{aligned} \quad (4.27)$$

Theorem 4.4. *For any matrix $\mathbf{Z}$ that is feasible for problem (4.26), the following inequalities hold:*

$$1 - \text{Tr}(\mathbf{W}_k \mathbf{Z}) \geq C_{s_k}^{-2} \sigma^4 > 0, \quad \forall k, \quad (4.28)$$

$$1 - \text{Tr}(\mathbf{M}_k \mathbf{Z}) \geq C_{y_k}^{-2} \sigma^4 > 0, \quad \forall k. \quad (4.29)$$

Proof. Please refer to Appendix F. □

With the inequality (4.28), problem (4.26) is quasi-convex. For any given $\tau \geq 0$, consider the following optimization problem:

$$\begin{aligned} \min_{\mathbf{Z} \succeq \mathbf{0}} & \text{Tr}(\mathbf{E}_1 \mathbf{Z}) \\ \text{s.t.} & 1 - \text{Tr}(\mathbf{M}_k \mathbf{Z}) \geq 2^{2\alpha_k \tau} (1 - \text{Tr}(\mathbf{W}_k \mathbf{Z})), \quad \forall k, \\ & \text{Tr}(\mathbf{E}_k \mathbf{Z}) \leq \|\mathbf{t}_k\|^4, \quad k = 2, \dots, K. \end{aligned} \quad (4.30)$$

(4.30) is an SDP problem, which minimizes the LHS of the second constraint in (4.27) corresponding to $k = 1$, subject to a target objective value τ for problem

(4.27). Denote the optimal value of (4.30) as $f(\tau)$. If $f(\tau) \leq \|\mathbf{t}_1\|^4$, then the optimal value of (4.27) satisfies $\tau_{\text{sdr}} \geq \tau$; otherwise, $\tau_{\text{sdr}} < \tau$. Therefore, (4.27) can be optimally solved by solving the SDP problem (4.30) with different values of τ , and applying a bisection update for τ .

Denote the solution to (4.27) by $\mathbf{Z}^*$. If $\text{rank}(\mathbf{Z}^*) = 1$, i.e., $\mathbf{Z}^* = \mathbf{z}\mathbf{z}^H$, then $\mathbf{z}$ is the optimal solution to (4.22). In this case, the proposed SDR is tight. On the other hand, if $\text{rank}(\mathbf{Z}^*) > 1$, we then apply the following Gaussian randomization procedure customized to our problem to find an approximate solution to problem (4.22).

Algorithm 7 Gaussian Randomization Procedure for (4.22)

Input: The solution $\mathbf{Z}^*$ to the SDR problem (4.27), and the number of randomization trials L .

- 1: **for** $l = 1, \dots, L$ **do**
- 2: Generate $\boldsymbol{\xi}_l \sim \mathcal{CN}(\mathbf{0}, \mathbf{Z}^*)$, and construct a feasible point $\mathbf{z}_l$ to (4.22) as follows:

$$[\mathbf{z}_l]_k = \kappa_k [\boldsymbol{\xi}_l]_k, \text{ with } \kappa_k = \min \left\{ 1, \frac{\|\mathbf{t}_k\|^2}{|[\boldsymbol{\xi}_l]_k|} \right\}, \forall k. \quad (4.31)$$

- 3: **end for**
- 4: determine $l^* = \arg \max_{l=1, \dots, L} \min_{k=1, \dots, K} \frac{1 - \mathbf{z}_l^H \mathbf{M}_k \mathbf{z}_l}{1 - \mathbf{z}_l^H \mathbf{W}_k \mathbf{z}_l}$

Output: $\hat{\mathbf{z}} = \mathbf{z}_{l^*}$ as an approximate solution for (4.22).

To summarize, the proposed algorithm for problem (4.7) by separately optimizing the transmit covariance and pseudo-covariance matrices is given in Algorithm 8.

Algorithm 8 Separate Covariance and Pseudo-Covariance Optimization for (4.7)

- 1: Solve the covariance optimization problem (4.8), and denote the solution as $\{r^*, \mathbf{C}_{\mathbf{x}_k}^* = \mathbf{t}_k \mathbf{t}_k^H\}$.
 - 2: Solve problem (4.27) using bisection search method over (4.30), and denote the solution as $\mathbf{Z}^*$.
 - 3: If $\text{rank}(\mathbf{Z}^*) = 1$, then its principal component $\mathbf{z}$ with $\mathbf{Z}^* = \mathbf{z}\mathbf{z}^H$ is the optimal solution to (4.22); otherwise, find an approximate solution $\hat{\mathbf{z}}$ using Algorithm 7.
 - 4: Obtain the pseudo-covariance matrix solution $\{\tilde{\mathbf{C}}_{\mathbf{x}_k}^*\}$ using (4.15), and the maximum sum-rate R^* using (4.6).
-

Remark 4.5. As pointed out in [76], the complexity of the Gaussian randomization process is almost negligible as compared with that of solving the SDP problem. The complexity of the latter is known to be polynomial with respect to the number of users K and the number of antennas M . Therefore, the proposed separate covariance and pseudo-covariance optimization algorithm has polynomial complexity in K and M .

Remark 4.6. In the special case of $K = 2$, (4.30) is a complex-valued SDP problem with three affine constraints. It is known that if such a problem is feasible, there is always a rank-1 optimal solution [77, 80]. Therefore, for the two-user MISO-IC with rank-1 transmit covariance matrices, the SDR-based solution will give the optimal pseudo-covariance matrices. In Chapter 3, an SOCP-based algorithm has been proposed, which is able to find the optimal pseudo-covariances for the two-user SISO-IC. However, the SOCP-based algorithm is difficult to be extended to the case of $K \geq 2$ and/or MISO-IC with $M > 1$.

4.3 Improper Gaussian Signaling for MISO-BC

The improper Gaussian signaling scheme discussed in the preceding section can also be applied to the Gaussian MISO-BC under the assumption of (widely) linear precoding employed at the base station transmitter.² Consider a Gaussian MISO-BC where a transmitter with $M > 1$ transmit antennas sends independent information to K single-antenna receivers. The signal received by the k th user can be written as

$$\begin{aligned} y_k &= \mathbf{h}_k \mathbf{x} + v_k \\ &= \mathbf{h}_k \mathbf{x}_k + \sum_{j \neq k} \mathbf{h}_k \mathbf{x}_j + v_k, \quad k = 1, \dots, K, \end{aligned}$$

where $\mathbf{h}_k \in \mathbb{C}^{1 \times M}$ denotes the channel vector from the transmitter to user k ; $v_k \sim \mathcal{CN}(0, \sigma^2)$ represents the CSCG noise; and $\mathbf{x} = \sum_{k=1}^K \mathbf{x}_k$ is the aggregated transmitted signal vector from the transmitter, with $\mathbf{x}_k$ denoting the transmitted signal intended for user k .

It is known that the capacity of the MISO-BC can be achieved by employing the DPC technique at the transmitter [22–24]. DPC is a nonlinear precoding technique by which the information for different users is encoded in a sequential manner, so

²Note that if proper Gaussian signaling is to be employed at the transmitter, the conventional (strictly) linear precoding is sufficient. On the other hand, if improper Gaussian signaling is to be applied, widely linear precoding is required, as has been discussed in Section 3.5.

that the interference caused by earlier encoded users can be completely removed with the non-causal information of the earlier encoded users' data streams. Since the nonlinear capacity-achieving DPC scheme is difficult to implement in practical systems, a great deal of research has focused on simpler linear precoding schemes at the transmitter. In this case, all the inter-user interferences are treated as additive Gaussian noise, thus resembling a MISO-IC (but with the transmitter's sum-power constraint as opposed to transmitters' individual power constraints in the general MISO-IC defined in Section 4.1). Denote by $\mathbf{t}_k$ the transmit beamforming vector for user k in a K -user MISO-BC with linear precoding, we then have

$$\mathbf{x}_k = \mathbf{t}_k d_k, \quad \forall k, \quad (4.32)$$

where d_k is a CSCG random variable representing the information signal of user k , i.e., $d_k \sim \mathcal{CN}(0, 1)$. The signal vector $\mathbf{x}_k$ in (4.32) is a proper Gaussian RV with covariance and pseudo-covariance matrices respectively given by

$$\mathbf{C}_{\mathbf{x}_k} = \mathbf{t}_k \mathbf{t}_k^H, \quad \tilde{\mathbf{C}}_{\mathbf{x}_k} = \mathbf{0}, \quad \forall k. \quad (4.33)$$

However, it is not immediately clear from (4.32) whether the restriction of proper Gaussian signaling, or zero pseudo-covariance matrices of the transmitted signals, will incur any rate loss under the assumption of (widely) linear precoding in the MISO-BC. Therefore, we consider the more general improper Gaussian signaling similar to the MISO-IC, where the pseudo-covariance matrices $\{\tilde{\mathbf{C}}_{\mathbf{x}_k}\}$ are further optimized. Following a similar derivation as for the MISO-IC, the achievable rate for user k in a MISO-BC with improper Gaussian signaling can be expressed as

$$R_{k,\text{BC}} = \underbrace{\log \left(1 + \frac{|\mathbf{h}_k \mathbf{t}_k|^2}{\sigma^2 + \sum_{j \neq k} |\mathbf{h}_k \mathbf{t}_j|^2} \right)}_{\triangleq R_{k,\text{BC}}^{\text{proper}}(\{\mathbf{t}_j\})} + \frac{1}{2} \log \frac{1 - C_{y_k}^{-2} |\tilde{C}_{y_k}|^2}{1 - C_{s_k}^{-2} |\tilde{C}_{s_k}|^2}, \quad (4.34)$$

where C_{y_k} , $\tilde{C}_{y_k}$, C_{s_k} , and $\tilde{C}_{s_k}$ are defined similarly as in (4.2)-(4.5). Note that the rate expression (4.34) bears the same structure as formula (4.6) for the MISO-IC. For a given rate-profile $\boldsymbol{\alpha} = (\alpha_1 \cdots \alpha_K)$, the problem to find a Pareto-optimal

rate-tuple for the MISO-BC can thus be formulated as

$$\begin{aligned}
& \max_{\{\mathbf{t}_k\}, \{\tilde{\mathbf{C}}_{\mathbf{x}_k}\}, R} R \\
& \text{s.t.} \quad R_{k,\text{BC}} \geq \alpha_k R, \quad \forall k, \\
& \quad \begin{bmatrix} \mathbf{t}_k \mathbf{t}_k^H & \tilde{\mathbf{C}}_{\mathbf{x}_k} \\ \tilde{\mathbf{C}}_{\mathbf{x}_k}^* & (\mathbf{t}_k \mathbf{t}_k^H)^* \end{bmatrix} \succeq \mathbf{0}, \quad \forall k, \\
& \quad \sum_{k=1}^K \|\mathbf{t}_k\|^2 \leq P.
\end{aligned} \tag{4.35}$$

Note that from the optimization perspective, problem (4.35) is identical to (4.7). Therefore, the separate covariance and pseudo-covariance optimization algorithm presented in the preceding section for the MISO-IC can be directly applied to solve (4.35).

4.4 Numerical Results

In this section, we evaluate the performance of the proposed algorithm by numerical simulations. All the transmitters are assumed to have the same power constraint P , i.e., $P_k = P$, $\forall k$. The average SNR is defined as P/σ^2 . For the Gaussian randomization process described in Algorithm 7, $L = 1000$ is used.

4.4.1 Approximation Ratio for SDR

First, we evaluate the quality of the SDR-based approximate solution for the pseudo-covariance optimization proposed in Section 4.2.2. Denote τ^* and τ_{sdr} as the optimal objective values of problem (4.22) and its relaxation (4.26), respectively. Also denote $\hat{\tau}$ as the objective value of (4.22) with the approximate solution obtained by Algorithm 7. Then we have

$$\hat{\tau} \leq \tau^* \leq \tau_{\text{sdr}}, \quad \text{or} \quad 1 \leq \tau^*/\hat{\tau} \leq \tau_{\text{sdr}}/\hat{\tau},$$

where $\tau^*/\hat{\tau}$ is the approximation ratio. Since in general the optimal value τ^* is difficult to be found, the upper bound $\tau_{\text{sdr}}/\hat{\tau}$ of the approximation ratio can be used to evaluate the quality of the SDR-based approximate solution [54]. In particular, if $\tau_{\text{sdr}}/\hat{\tau} = 1$, then the obtained SDR-based solution is optimal.

FIGURE 4.1: Empirical ratio of $\tau_{\text{sdr}}/\hat{\tau}$ for the four-user MISO-IC with $M = 2$ over 500 channel realizations.

TABLE 4.1: Mean and standard deviation (std) of the approximation ratio upper bound $\tau_{\text{sdr}}/\hat{\tau}$.

	K	2	3	4	5	6
$M = 1$	mean	1.0	1.032	1.138	1.267	1.391
	std	0	0.092	0.245	0.350	0.441
$M = 2$	mean	1.0	1.012	1.162	1.401	1.640
	std	0	0.068	0.388	0.621	0.691

We first consider the four-user MISO-IC ($K = 4$) with each transmitter equipped with two antennas ($M = 2$). With the rate-profile in (4.7) set as $\boldsymbol{\alpha} = (1/K)\mathbf{1}$, where $\mathbf{1}$ is an all-one vector, the empirical ratios of $\tau_{\text{sdr}}/\hat{\tau}$ over 500 random channel realizations at SNR= 10 dB are plotted in Figure 4.1, where the channel coefficients are generated from i.i.d. CSCG random variables with zero-mean and unit-variance. It is observed that for a majority of channel realizations, the approximation ratio is close to 1, which demonstrates the high quality of the approximate solution obtained by the SDR-based pseudo-covariance optimization.

With different pairs of values for M and K , Table 4.1 summarizes the mean and standard deviation of $\tau_{\text{sdr}}/\hat{\tau}$ at SNR = 10 dB over 500 random channel realizations. It is observed that for all the setups considered, the mean values of the approximation ratio upper bounds are between 1 and 1.64. In particular,

TABLE 4.2: Channel realizations for Figure 4.2 and Figure 4.3.

	channel realization $\mathbf{H}^{(1)}$	channel realization $\mathbf{H}^{(2)}$
$\mathbf{h}_{11}$	$[0.3676e^{-1.7037i} \ 0.4993e^{1.6076i}]$	$[0.3676e^{-1.7037i} \ 0.4993e^{1.6076i}]$
$\mathbf{h}_{21}$	$[0.2526e^{-1.8997i} \ \mathbf{0.3270e^{1.5884i}}]$	$[0.2526e^{-1.8997i} \ \mathbf{0.3270e^{-0.3810i}}]$
$\mathbf{h}_{22}$	$[0.4694e^{-0.1915i} \ 0.5682e^{0.5302i}]$	$[0.4694e^{-0.1915i} \ 0.5682e^{0.5302i}]$
$\mathbf{h}_{12}$	$[0.2885e^{-0.2454i} \ \mathbf{0.3656e^{0.4710i}}]$	$[0.2885e^{-0.2454i} \ \mathbf{0.3656e^{1.8673i}}]$
$\cos \theta_1$	0.9961	0.6601
$\cos \theta_2$	0.9997	0.7793

FIGURE 4.2: Achievable rate region for the two-user MISO-IC with $M = 2$, SNR = 10 dB, and channel realization $\mathbf{H}^{(1)}$.

for $K = 2$, $\tau_{\text{sdr}}/\hat{\tau} = 1$ is guaranteed, which verifies the optimality of the SDR-based pseudo-covariance optimization for the two-user MISO-IC, as discussed in Remark 4.6.

4.4.2 Rate Region Comparison

In Figure 4.2, the achievable rate region for an example two-user MISO-IC with $M = 2$ is plotted for SNR=10 dB, with the channel realization (denoted as $\mathbf{H}^{(1)}$) given in the left column of Table 4.2. The proposed improper Gaussian signaling with the SDR-based separate covariance and pseudo-covariance optimization is compared against the optimal proper Gaussian signaling obtained by solving (4.8).

FIGURE 4.3: Achievable rate region for the two-user MISO-IC with $M = 2$, $\text{SNR} = 10$ dB, and channel realization $\mathbf{H}^{(2)}$.

It is observed that for this channel setup, the achievable rate region has been significantly enlarged by the proposed pseudo-covariance optimization for improper Gaussian signaling.

Next, we consider a different channel realization $\mathbf{H}^{(2)}$, which differs from $\mathbf{H}^{(1)}$ only in the phases of the second elements in $\mathbf{h}_{21}$ and $\mathbf{h}_{12}$, as shown in the right column of Table 4.2. It is observed from Figure 4.3 that for this new channel setup, there is no notable rate gain by the proposed improper Gaussian signaling over the optimal proper signaling, which is in sharp contrast to the result in Figure 4.2. This can be explained by comparing the residue interference levels after applying the optimal *proper* Gaussian signaling in the two cases. It is worth noting that for the two-user MISO-IC, the optimal proper Gaussian signaling at each transmitter is known to be a linear combination of the ZF beamforming and the maximum ratio transmission (MRT) beamforming [128]. Applying this result to the two-user MISO-IC in our context, it then follows that one user will generate lower interference to the other user if its direct channel and interfering channel vectors are more close to be orthogonal.³ Let θ_1 denote the angle between the direct and interfering channel vectors for transmitter 1, i.e., $\cos \theta_1 \triangleq |\mathbf{h}_{21} \mathbf{h}_{11}^H| / (\|\mathbf{h}_{21}\| \|\mathbf{h}_{11}\|)$,

³Consider one extreme case when the direct and interfering channel vectors are orthogonal, then the MRT and ZF beamforming vectors coincide, and thus no interference is generated to the

FIGURE 4.4: Average max-min rate for the two-user SISO-IC.

and define θ_2 for transmitter 2 similarly. As shown in Table 4.2, since θ_1 and θ_2 are smaller in the case of $\mathbf{H}^{(1)}$ than that in the case of $\mathbf{H}^{(2)}$, more substantial interference is resulted for channel realization $\mathbf{H}^{(1)}$ than that of $\mathbf{H}^{(2)}$ after applying the optimal proper transmit beamforming. As a result, further optimization over the pseudo-covariance matrices provides more notable rate gains in the case of $\mathbf{H}^{(1)}$ than $\mathbf{H}^{(2)}$, as discussed in Remark 4.3.

4.4.3 Max-Min Rate Comparison

In this subsection, the average max-min rate achievable over 500 random channel realizations by the proposed improper Gaussian signaling is compared with that by the optimal proper Gaussian signaling obtained by solving (4.8).

We first consider the special case of the two-user SISO-IC, where two other improper Gaussian signaling designs have also been proposed in Chapter 3, i.e., the SDR-based joint covariance and pseudo-covariance optimization in Section 3.6.2 and the SOCP-based separate covariance and pseudo-covariance optimization in Section 3.6.3. It is observed in Figure 4.4 that improper Gaussian signaling

other user; however, for the other extreme case when the direct and interfering channel vectors are linearly dependent, a nonzero interference is always resulted.

FIGURE 4.5: Average max-min rate for the three-user MISO-IC with $M = 2$.

provides significant gains over the optimal proper Gaussian signaling. At high SNR, the max-min rate by proper Gaussian signaling saturates, since the total number of data streams transmitted, which is 2, exceeds the total number of DoF of the two-user SISO-IC, which is known to be 1. In contrast, a linear increase of the max-min rate with respect to the logarithm of SNR is observed by improper Gaussian signaling. It is also observed that the two separate covariance and pseudo-covariance optimization algorithms, SDR-based or SOCP-based, provide the same max-min rate in the case of $K = 2$. This is as expected since both algorithms achieve the global optimality for the covariance and pseudo-covariance optimization sub-problems in the two-user SISO-IC case. Moreover, it is observed in Figure 4.4 that the SDR-based joint covariance and pseudo-covariance optimization algorithm proposed in Chapter 3 achieves additional rate gains over the SDR/SOCP-based separate optimization. However, the extension of such a joint optimization to the general K -user MISO-IC remains unknown.

To show the max-min rate performance with improper Gaussian signaling when there are multiple transmitting antennas, we consider a three-user MISO-IC with $M = 2$. As shown in Figure 4.5, a significant rate improvement is again observed by the proposed improper Gaussian signaling over the optimal proper Gaussian signaling. In Figure 4.6, the MISO-IC with two transmit antennas and a relatively

FIGURE 4.6: Average max-min rate for the MISO-IC with $K = 20$ and $M = 2$.

large number of users $K = 20$ is considered. It is observed that the max-min rate drops significantly as compared to Figure 4.5 due to the excessive interference incurred by each user. This implies that for practical systems with limited number of antennas at each terminal, appropriate user scheduling should be performed to avoid user overloading in each spectrum band, an interesting topic which is out of the scope of this thesis.

4.4.4 Rate Region of MISO-BC

At last, MISO-BC with (widely) linear precoding as discussed in Section 4.3 is considered. In Figure 4.7 and Figure 4.8, the achievable rate regions for an example two-user MISO-BC with $M = 2$ are plotted for SNR=0 dB and SNR = 10 dB, respectively. The channel vectors of the two users are respectively given by $\mathbf{h}_1 = [1.1741e^{1.0030i} \ 0.8064e^{2.8642i}]$, $\mathbf{h}_2 = [1.8116e^{2.0647i} \ 0.9209e^{-2.4167i}]$. Under the assumption of (widely) linear precoding, the proposed improper Gaussian signaling is compared with the optimal proper Gaussian signaling with linear beamforming. As a benchmark, the capacity region achieved by DPC is also included. It is observed that without time sharing (TS) of the achievable users' rates by a convex-hull operation, the achievable rate region for the MISO-BC is

FIGURE 4.7: Achievable rate region for the two-user MISO-BC employing (widely) linear precoding with $M = 2$, and $\text{SNR} = 0$ dB.

significantly enlarged by employing improper Gaussian signaling over the optimal proper Gaussian signaling. However, such a performance gain no longer exists if TS is performed.⁴

4.5 Chapter Summary

In this chapter, we have studied the transmit optimization problem for the K -user MISO-IC when improper or circularly asymmetric complex Gaussian signaling is applied. By exploiting the separable achievable rate structure, a separate transmit covariance and pseudo-covariance optimization algorithm has been proposed. For the pseudo-covariance optimization, the optimality of rank-1 pseudo-covariance matrices has been established, given that rank-1 transmit covariance matrices have been applied. Moreover, each of such rank-1 pseudo-covariance matrices can be parameterized by one single complex scalar; therefore, the original matrix

⁴Based on our simulations over random channels with different setups, it is found that the rate gain by the improper signaling over the optimal proper signaling diminishes after the TS is applied. In the recent work [167], by constructing a specific channel realization, the authors have shown that improper Gaussian signaling is able to provide better achievable rates than proper Gaussian signaling even with TS.

FIGURE 4.8: Achievable rate region for the two-user MISO-BC employing (widely) linear precoding with $M = 2$, and $\text{SNR} = 10$ dB.

optimization problem for pseudo-covariances can be reformulated to vector optimization problem with a much lower dimension. An efficient approximate solution is then obtained by applying the SDR technique for the reformulated pseudo-covariance optimization problem. The proposed improper Gaussian signaling has also been extended to the case of MISO-BC with (widely) linear precoding. Finally, numerical results have been provided to demonstrate the achievable rate gains over the conventional proper Gaussian signaling.

Chapter 5

Degrees of Freedom of 3-User Rank-Deficient MIMO-IC

5.1 Introduction and Literature Review

The degrees of freedom (DoF) value provides one useful alternative metric in understanding the ultimate rate limit of a communication network where the exact capacity is unknown. Characterizing the optimal DoF for various interference-limited networks has been an extensive research topic in the last few years. While most of such studies have assumed full-rank channel matrices, some researchers have started investigating the DoF for *rank-deficient* interference channel, which is a more suitable model for certain practical communication scenario with poor scattering environment. This chapter makes progress towards this direction by providing the complete DoF characterization for the 3-user $M_T \times M_R$ rank-deficient MIMO-IC.

5.1.1 Related Work

Under the assumption of full-rank channel matrices, the DoF characterizations have been obtained for various interference channels under different settings. For the two-user MIMO-IC, where user 1 has M_1 transmit and N_1 receive antennas, and user 2 has M_2 transmit and N_2 receive antennas, the DoF value is given by $\min(M_1 + M_2, N_1 + N_2, \max(M_1, N_2), \max(M_2, N_1))$ [38]. Furthermore, it has been shown that zero forcing is sufficient to achieve this optimal DoF value. When the

number of users is greater than 2, and hence each receiver is possibly interfered by more than one interfering sources, a novel technique known as interference alignment has been shown to achieve higher DoF than previously believed [158, 159, 168–171]. The key for interference alignment is to carefully design the transmitters so that the signal spaces are aligned at the receivers where they constitute interference, yet remain distinguishable at the receivers where they are desired. In [169], Cadambe and Jafar have shown that for the K -user MIMO-IC where all nodes are equipped with the same number of antennas M , each user is able to achieve $M/2$ DoF. In [170], Gou and Jafar studied the K -user $M_T \times M_R$ MIMO-IC with M_T antennas at each transmitter and M_R antennas at each receiver, where the optimal DoF value was obtained when the ratio $\frac{\max(M_T, M_R)}{\min(M_T, M_R)}$ is an integer. The results of [170] have been extended in [172] by employing the rational dimensions framework. The feasibility of interference alignment for MIMO-ICs has been investigated in [173–175]. In [176], interference alignment has been formulated as a rank-constrained rank minimization problem, where heuristic solutions based on convex optimization techniques have been proposed. The DoF of systems with CoMP transmission and reception has also been investigated in [177].

More recently, for the 3-user $M_T \times M_R$ interference channel with M_T antennas at each transmitter and M_R antennas at each receiver, a novel concept of “subspace alignment chains” was introduced in [171], where ideally each transmitted signal subspace is aligned with another interference subspace at both undesired receivers and an alignment chain is formed. The finite length of such an alignment chain ultimately limits the DoF value. For the DoF outer bound in [171], a technique termed “onion peeling” has been introduced, where an equivalent channel with layered structure was obtained by applying change of basis operations at the transmitters and receivers. As such, it helps in determining the appropriate genie signals to be provided to each receiver for deriving the DoF outer bounds. Another useful technique used in [171] is the concept of *spatially-normalized DoF*, which is the DoF value obtained by spatial extension, i.e., proportional scaling of the number of antennas at each node and a corresponding normalization of DoF by the same factor. Spatial extension makes it easier to deal with non-integer DoF values without having to deal with the added complexity of diagonal or block diagonal channel structures that would result from time or frequency extensions. The idea of spatially-normalized DoF has also been previously used in [178] for the study of the MIMO X channel.

As for the DoF characterizations for rank-deficient MIMO-ICs, there have been a few results reported. In [179], under the assumption that all channel matrices have the same rank, an achievable DoF value was obtained for the K -user time-varying interference channel. The achievable scheme proposed therein is based on either zero forcing or interference alignment, whichever gives a higher DoF value. In [180], the authors investigated the DoF of the 2-user and 3-user rank-deficient MIMO-IC. While the optimal DoF value has been obtained for the 2-user case, only symmetric setup with equal antenna terminals was considered for the 3-user case. For the 2-user rank-deficient MIMO-IC, if in addition the channel matrices are also correlated, which is indeed the case for the equivalent interference channel model resulting from the double unicast network with linear network coding, the DoF region has been independently obtained in [181] and [182]. The rank-deficiency of the channel matrices has also been addressed in the cooperative relaying setup [183].

5.1.2 DoF of 3-User $M_T \times M_R$ Rank-Deficient Channel

In this chapter, we investigate the DoF of the 3-user $M_T \times M_R$ rank-deficient MIMO-IC where each transmitter is equipped with M_T antennas and each receiver with M_R antennas. All the direct channel matrices are of rank D_0 , and the interfering channel matrices from each transmitter to the other two receivers are of ranks D_1 and D_2 , respectively. As compared with the full-rank case, the DoF characterization for the rank-deficient MIMO-IC is complicated by the additional rank parameters D_0 , D_1 and D_2 , as well as their relations with the number of antennas M_T and M_R . By adopting the technique of spatial extension to deal with non-integer DoF values for inner bound [171], we provide a complete DoF characterization for all possible settings of the 3-user rank-deficient channel parameterized by $(M_T, M_R, D_0, D_1, D_2)$. The result obtained in this chapter unifies several existing DoF characterizations in the literature.

To achieve the maximum possible DoF value, intuitively, one should zero force as many interfering links as possible. One distinguishing feature of the rank-deficient MIMO-IC, as compared with the full-rank counterparts, is that *both* the left and right null space of the interfering channel matrices are non-empty, and hence should be utilized for zero forcing purposes. Therefore, a critical first step for the DoF characterization is to appropriately exploit the rank deficiency of

the interfering channel matrices. To this end, we first transform the original fully-connected rank-deficient MIMO-IC into an equivalent partially-connected *full-rank* MIMO-IC. This in effect extracts the rank-deficiency of the interfering channel matrices embedded in all antennas to a subset of antennas so that the corresponding interfering links are nullified. As such, existing techniques developed for the full-rank MIMO-IC, such as “onion peeling” and “subspace alignment chains”, can be incorporated to aid for the derivations of the DoF outer and inner bounds.

The outer bound is derived with the genie-aided signaling technique. For the achievability, a two-layered linear processing scheme is proposed, with zero forcing in the inner layer and a *block-diagonal* precoding and interference cancelation, if necessary, in the outer layer. The using of block-diagonal precoding ensures that each information-bearing symbol only interferes a particular group of receive antennas, which in turn makes interference cancelation possible. When the interfering links are weak in terms of the channel ranks, i.e., $D_1 + D_2 \leq \min(M_T, M_R)$, zero forcing together with a random block-diagonal precoding is sufficient to achieve the optimal DoF; whereas when $D_1 + D_2 > \min(M_T, M_R)$, the block-diagonal precoding in the outer layer must be carefully designed so that interference alignment is achieved, i.e., in this case, a combination of zero forcing and interference alignment is required to achieve the optimal DoF value.

5.2 System Model

Consider a 3-user MIMO-IC where transmitter k is intended to send message W_k to receiver k , $k = 1, 2, 3$. Each transmitter is equipped with M_T antennas and each receiver has M_R antennas. The received signal at user k is given by

$$\mathbf{y}_k = \mathbf{H}_{kk}\mathbf{x}_k + \sum_{j=1, j \neq k}^3 \mathbf{H}_{kj}\mathbf{x}_j + \mathbf{n}_k, \quad k = 1, 2, 3, \quad (5.1)$$

where $\mathbf{H}_{kk} \in \mathbb{C}^{M_R \times M_T}$ denotes the direct channel matrix from transmitter k to receiver k , while $\mathbf{H}_{kj} \in \mathbb{C}^{M_R \times M_T}$, $j \neq k$, denotes the interfering channel matrix from transmitter j to receiver k ; $\mathbf{y}_k \in \mathbb{C}^{M_R \times 1}$ represents the received signal vector at receiver k ; $\mathbf{x}_k \in \mathbb{C}^{M_T \times 1}$ is the transmitted signal vector from transmitter k ; and $\mathbf{n}_k \in \mathbb{C}^{M_R \times 1}$ denotes the circularly symmetric complex additive white Gaussian noise (AWGN) vector with $\mathbf{n}_k \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{M_R})$.

FIGURE 5.1: 3-user rank-deficient MIMO-IC parameterized by $(M_T, M_R, D_0, D_1, D_2)$.

Different from most of the existing work where full-rank channel matrices have been assumed, here the more general scenario with possibly *rank-deficient* channel matrices is considered. Specifically, all the direct channel matrices $\mathbf{H}_{kk}$ are assumed to be of rank D_0 , and the interfering channel matrices $\mathbf{H}_{(k-1)k}$ and $\mathbf{H}_{(k+1)k}$ are of ranks D_1 and D_2 , respectively, as shown in Figure 5.1. We then have $D_i \leq \min(M_R, M_T)$, $i = 0, 1, 2$. Note that the user index k is interpreted modulo 3, e.g., user 0 is the same as user 3. Furthermore, the channel matrices are assumed to be independent and randomly generated, where a random rank-deficient channel matrix of size $M_R \times M_T$ with rank D can be seen as a product of two full-rank generic matrices of size $M_R \times D$ and $D \times M_T$. In the following, this channel model is expressed compactly as $(M_T, M_R, D_0, D_1, D_2)$.

Denote by ρ the average power constraint at each transmitter. The sum rate $R_\Sigma(\rho)$ and the DoF $d_\Sigma(M_T, M_R, D_0, D_1, D_2)$ follow from the standard definition in information theory. Further denote by $d = d_\Sigma/3$ the normalized DoF per user. Similar to [171], the *spatially-normalized* sum DoF value is defined as

$$\bar{d}_\Sigma = \max_{q \in \mathbb{Z}^+} \frac{d_\Sigma(qM_T, qM_R, qD_0, qD_1, qD_2)}{q}. \quad (5.2)$$

Note that in (5.2), both the number of antennas and the ranks of the channel matrices are scaled by a factor q , and the DoF value is normalized by the same factor. While such a spatial extension is not necessary for the DoF outer bound, it is quite useful for the achievability result to deal with non-integer DoF values. As argued in [171] and [178], the definition in (5.2) is justified by the fact that

for every wireless network where the DoF characterizations are available, the DoF result is unaffected by spatial normalization. We further denote by $\bar{d}$ the spatially-normalized DoF per user, i.e., $\bar{d} = \bar{d}_\Sigma/3$.

For notational convenience, in the remaining part of this chapter, we let $D_t \triangleq D_1 + D_2$, $N \triangleq \max(M_T, M_R)$, and $M \triangleq \min(M_T, M_R)$.

5.3 Main Result and Discussions

5.3.1 Main Result

Theorem 5.1. (*Spatially-Normalized DoF*) For the 3-user rank-deficient MIMO-IC parameterized by $(M_T, M_R, D_0, D_1, D_2)$, the spatially-normalized DoF value per user is given by

$$\bar{d} = \begin{cases} \min\left(D_0, \frac{N+M-D_t}{2}\right), & D_t \leq M \\ \min\left(D_0, \frac{N}{2}, \frac{pM}{2p-1}, \frac{pN+2M-D_t}{2p+1}\right), & D_t > M \end{cases}, \quad (5.3)$$

where $p = \lceil \frac{D_t-M}{N-M} \rceil$.

The converse proof of Theorem 5.1 is based on the genie-aided signaling technique. The details are given in Section 5.5 with a preliminary step presented in Section 5.4. Note that the DoF converse is established for arbitrary values of M_T , M_R , D_0 , D_1 and D_2 , without the need for spatial extension. However, it can be verified that if the number of antennas and the ranks of the channel matrices are scaled by a factor q , the outer bound for the DoF value would be scaled by the same factor as well. Therefore, the outer bound for the rank-deficient MIMO-IC scales with spatial dimension, and hence holds both with and without spatial normalization.

The achievability of Theorem 5.1 is based on a two-layered linear processing scheme as will be presented in Section 5.6.

Remark 5.2. Due to symmetry, the DoF value depends only on the *sum* of the ranks of the interfering channel matrices $D_t = D_1 + D_2$, instead of the individual ranks D_1 and D_2 . Therefore, a network parameterized by $(M_T, M_R, D_0, D_1, D_2)$ would have identical spatially-normalized DoF value as that parameterized by $(M_T, M_R, D_0, D_1 - \Delta_D, D_2 + \Delta_D)$, for any valid Δ_D .

Remark 5.3. For the rank-deficient MIMO-IC with the number of antennas given by M and N , the DoF value is determined by either the rank of the direct link or that of the interfering links, depending on which one is the bottleneck. While the direct link affects the DoF via the simple quantity D_0 , the effect of the interfering links is complicated by the relations between D_t , M and N . With $D_0 = M$, Fig. 5.2 plots $\bar{d}/N$ versus M/N for different D_t values. It is observed that in the sufficiently low M/N regime where either the transmitters (when $M_T > M_R$) or the receivers (when $M_R > M_T$) have enough number of antennas to zero force all interfering links, $\bar{d}$ is solely determined by the rank of the direct links, i.e., $\bar{d} = D_0 = M$; otherwise, the interfering links will become the bottleneck and limit the maximum achievable $\bar{d}$. It is also observed for all $D_t \geq M$, the DoF converges to the “half-cake” result [169] as M approaches N . On the other hand, with $D_t < M$, higher DoF can be achieved, which is expected.

FIGURE 5.2: $\bar{d}/N$ as a function of M/N for different D_t values, with $D_0 = M$.

5.3.2 Comparison with Existing Results

The DoF characterization given in Theorem 5.1 unifies several existing results in the literature.

5.3.2.1 $D_t = 0$

In the extreme case when the interfering links vanish, i.e., $D_1 = D_2 = 0$, the model in (5.1) reduces to the 3-user independent point-to-point MIMO channels with the direct channel matrices having rank D_0 . In this case, it is well known that the DoF value per user is D_0 [156], which is equal to the value obtained by evaluating (5.3) with $D_t = 0$.

5.3.2.2 $M_T = M_R$

When restricting to the case where all terminals are equipped with the same number of antennas, i.e., $M_T = M_R$ or $M = N$, the spatially-normalized DoF value given in Theorem 5.1 reduces to that obtained in [180], which is

$$\bar{d} = \min \left\{ D_0, \max \left(\frac{M}{2}, M - \frac{D_t}{2} \right) \right\}. \quad (5.4)$$

5.3.2.3 $D_0 = D_1 = D_2 = M$

For the setup when all channel matrices have full rank M , the spatially-normalized DoF have been obtained in [171]. By substituting D_t with $2M$, the result given in Theorem 5.1 can be simplified to

$$\bar{d} = \min \left\{ \frac{pM}{2p-1}, \frac{pN}{2p+1} \right\}, \text{ where } p = \left\lceil \frac{M}{N-M} \right\rceil, \quad (5.5)$$

which matches with that obtained in [171].

5.3.2.4 $D_0 = D_1 = D_2 = D$

For the scenario where all channel matrices are of the same rank D , the authors in [179] derived an achievable DoF value for the K -user $M_T \times M_R$ rank-deficient MIMO-IC. For $K = 3$, the achievable DoF per user obtained in [179] can be written as

$$\bar{d} = \min \left\{ D, \max \left(\frac{M+N-D}{3}, \frac{DQ}{Q+1} \right) \right\}, \quad (5.6)$$

FIGURE 5.3: DoF Comparison for the special case of 3-user rank-deficient MIMO-IC with $D_0 = D_1 = D_2 = D$.

where $Q \triangleq \lfloor \frac{N}{D} \rfloor$. By evaluating the result in Theorem 5.1 with $D_0 = D_1 = D_2 = D$, the exact DoF for this setup can be obtained as

$$\bar{d} = \begin{cases} D, & D \leq \frac{M}{2} \\ \min \left(D, \frac{N}{2}, \frac{pM}{2p-1}, \frac{pN+2M-2D}{2p+1} \right), & D > \frac{M}{2} \end{cases}, \quad (5.7)$$

where $p = \lceil \frac{2D-M}{N-M} \rceil$. By comparing (5.6) and (5.7), it can be verified that the achievable scheme in [179], which is based on either zero forcing or interference alignment, is sub-optimal, as illustrated in Fig 5.3 with four different values of D . This implies that for rank-deficient MIMO-ICs, zero forcing or interference alignment alone is not sufficient to achieve the optimal DoF in general; instead, a combination of them is required, as will be shown in Section 5.6.

5.4 Equivalent Partially-Connected Full-Rank MIMO-IC

In this section, we present a preliminary step for the proof of Theorem 5.1, i.e., the conversion of the original fully-connected rank-deficient MIMO-IC into an equivalent partially-connected *full-rank* MIMO-IC by exploiting the rank deficiency of the interfering channel matrices and applying *invertible* linear transformations at the transmitters and receivers. While *invertible* linear transformations, or change of basis operations, do not affect the DoF, they create an equivalent channel such that the existing techniques developed for the full-rank MIMO-IC can be incorporated to aid for the development of the rank-deficient channel.

Due to the reciprocity of linear transformations [184], without loss of generality, we assume in the remaining part of this section that $M_T \leq M_R$, so that $M = M_T$ and $N = M_R$. Furthermore, since different linear transformations are required for different interference levels, we distinguish the low- and high-interference cases, where the interference level is measured by the ranks of the interfering channel matrices as far as the DoF are concerned. Specifically, we have

1. Low-interference case: $0 \leq D_t \leq M$.
2. High-interference case: $M < D_t \leq 2M$.

5.4.1 Low-Interference Case: $D_t \leq M$

For the low-interference case with $D_t \leq M$, we start with the design of the invertible linear transformation $\mathbf{R}_k \in \mathbb{C}^{N \times N}$ at receiver k , which is based on the interfering channel matrices $\mathbf{H}_{k(k+1)}$ from transmitter $k+1$ and $\mathbf{H}_{k(k-1)}$ from transmitter $k-1$. $\mathbf{R}_k$ is partitioned as

$$\mathbf{R}_k = \begin{bmatrix} \mathbf{U}_{k(k+1)} \\ \mathbf{U}_k^c \\ \mathbf{U}_{k(k-1)} \end{bmatrix}, \quad (5.8)$$

where the central block $\mathbf{U}_k^c$ is designed so that it zero forces the signals from *both* interfering transmitters, i.e.,

$$\mathbf{U}_k^c [\mathbf{H}_{k(k+1)} \quad \mathbf{H}_{k(k-1)}] = \mathbf{0}. \quad (5.9)$$

Since $\text{rank}(\mathbf{H}_{k(k+1)}) = D_1$, $\text{rank}(\mathbf{H}_{k(k-1)}) = D_2$, and the channels are independent and randomly generated, then with probability 1, we have

$$\text{rank}[\mathbf{H}_{k(k+1)} \quad \mathbf{H}_{k(k-1)}] = \min(N, 2M, D_t) = D_t. \quad (5.10)$$

Therefore, the left null space of the concatenated $N \times 2M$ matrix $[\mathbf{H}_{k(k+1)} \quad \mathbf{H}_{k(k-1)}]$ has dimension $N - D_t$. The rows of $\mathbf{U}_k^c$ are then selected to be the basis spanning this null space; and hence the size of $\mathbf{U}_k^c$ can be chosen as $(N - D_t) \times N$.

The two remaining blocks in $\mathbf{R}_k$ are designed such that $\mathbf{U}_{k(k+1)}$ and $\mathbf{U}_{k(k-1)}$ zero force the interfering signals from transmitter $k + 1$ and transmitter $k - 1$, respectively, i.e.,

$$\mathbf{U}_{k(k+1)}\mathbf{H}_{k(k+1)} = \mathbf{0}, \quad \mathbf{U}_{k(k-1)}\mathbf{H}_{k(k-1)} = \mathbf{0}. \quad (5.11)$$

Furthermore, since $\mathbf{U}_k^c$ satisfying (5.9) also satisfies the zero-forcing conditions in (5.11), to ensure the invertibility of $\mathbf{R}_k$, the rows of $\mathbf{U}_{k(k+1)}$ are then selected from the left null space of $\mathbf{H}_{k(k+1)}$, but *not* in the subspace spanned by the rows of $\mathbf{U}_k^c$. With $\text{rank}(\mathbf{H}_{k(k+1)}) = D_1$, the dimension of the subspace where $\mathbf{U}_{k(k+1)}$ can be selected from is then given by $(N - D_1) - (N - D_t) = D_2$. Therefore, the size of $\mathbf{U}_{k(k+1)}$ is given by $D_2 \times N$. $\mathbf{U}_{k(k-1)}$ can be designed similarly and its size is $D_1 \times N$. Under the assumption that the channel matrices are independent and randomly generated, it follows that with probability 1, the resulting $N \times N$ matrix $\mathbf{R}_k$ is of full rank, and hence invertible.

After $\mathbf{R}_k$ at receiver k is determined, we switch to the design of the invertible linear transformation $\mathbf{T}_k$ at transmitter k . We have $\mathbf{T}_k \in \mathbb{C}^{M \times M}$ and it is partitioned as

$$\mathbf{T}_k = \begin{bmatrix} \mathbf{V}_{(k-1)k} & \mathbf{G}_k & \mathbf{V}_{(k+1)k} \end{bmatrix}, \quad (5.12)$$

where the central block $\mathbf{G}_k$ is designed so that it zero forces the interfering signals to *both* receiver $k - 1$ and $k + 1$, i.e.,

$$\begin{bmatrix} \mathbf{H}_{(k-1)k} \\ \mathbf{H}_{(k+1)k} \end{bmatrix} \mathbf{G}_k = \mathbf{0}. \quad (5.13)$$

The columns of $\mathbf{G}_k$ are then selected to be the basis spanning the null space of the concatenated matrix in (5.13). Since $\text{rank}(\mathbf{H}_{(k-1)k}) = D_1$ and $\text{rank}(\mathbf{H}_{(k+1)k}) = D_2$,

we have

$$\text{rank} \begin{bmatrix} \mathbf{H}_{(k-1)k} \\ \mathbf{H}_{(k+1)k} \end{bmatrix} = \min(2N, M, D_t) = D_t.$$

Therefore, the null space of the concatenated $2N \times M$ matrix in (5.13) would have dimension $M - D_t$ and the size of $\mathbf{G}_k$ is $M \times (M - D_t)$.

Next, the two blocks $\mathbf{V}_{(k-1)k}$ and $\mathbf{V}_{(k+1)k}$ in $\mathbf{T}_k$ are designed so that they zero force the interfering signals to receiver $k - 1$ and receiver $k + 1$, respectively, i.e.,

$$\mathbf{H}_{(k-1)k} \mathbf{V}_{(k-1)k} = \mathbf{0}, \quad \mathbf{H}_{(k+1)k} \mathbf{V}_{(k+1)k} = \mathbf{0}. \quad (5.14)$$

Furthermore, since $\mathbf{G}_k$ satisfying (5.13) also satisfies the zero-forcing conditions in (5.14), to ensure the invertibility of $\mathbf{T}_k$, the columns of $\mathbf{V}_{(k-1)k}$ are chosen from the null space of $\mathbf{H}_{(k-1)k}$, but *not* in the subspace spanned by the columns of $\mathbf{G}_k$. With $\text{rank}(\mathbf{H}_{(k-1)k}) = D_1$, the dimension of the subspace where the columns of $\mathbf{V}_{(k-1)k}$ can be selected from is then given by $(M - D_1) - (M - D_t) = D_2$; and hence $\mathbf{V}_{(k-1)k}$ is of size $M \times D_2$. A similar design for $\mathbf{V}_{(k+1)k}$ can be obtained and its size is $M \times D_1$. Under the assumption that the channel matrices are independent and randomly generated, the resulting $M \times M$ matrix $\mathbf{T}_k$ is of full rank, and hence is invertible.

Next, we derive the resulting channel model with the linear transformations $\mathbf{T}_k$ and $\mathbf{R}_k$ applied at the transmitters and receivers, respectively. Denote by $\tilde{\mathbf{x}}_k \in \mathbb{C}^{M \times 1}$ the signal vector at transmitter k before applying $\mathbf{T}_k$. Then the transmitted signal $\mathbf{x}_k$ is given by $\mathbf{x}_k = \mathbf{T}_k \tilde{\mathbf{x}}_k$. Based on the partition of $\mathbf{T}_k$ in (5.12), $\tilde{\mathbf{x}}_k$ is decomposed into head, middle and tail parts as $\tilde{\mathbf{x}}_k = \begin{bmatrix} \tilde{\mathbf{x}}_{kh}^T & \tilde{\mathbf{x}}_{km}^T & \tilde{\mathbf{x}}_{kt}^T \end{bmatrix}^T$, where the dimensions are given by $|\tilde{\mathbf{x}}_{kh}| = D_2$, $|\tilde{\mathbf{x}}_{km}| = M - D_t$, and $|\tilde{\mathbf{x}}_{kt}| = D_1$. Further denote by $\tilde{\mathbf{y}}_k \in \mathbb{C}^{N \times 1}$ the signal vector at receiver k after applying linear transformation $\mathbf{R}_k$, i.e., $\tilde{\mathbf{y}}_k = \mathbf{R}_k \mathbf{y}_k$, where $\mathbf{y}_k$ is the received vector at receiver k . By substituting $\mathbf{y}_k$ with (5.1) and using $\mathbf{x}_k = \mathbf{T}_k \tilde{\mathbf{x}}_k$, we get the input-output relationship after the linear transformations $\mathbf{T}_k$ and $\mathbf{R}_k$ as

$$\tilde{\mathbf{y}}_k = \mathbf{R}_k \mathbf{H}_{kk} \mathbf{T}_k \tilde{\mathbf{x}}_k + \mathbf{R}_k \mathbf{n}_k + \underbrace{\mathbf{R}_k (\mathbf{H}_{k(k-1)} \mathbf{T}_{k-1} \tilde{\mathbf{x}}_{k-1} + \mathbf{H}_{k(k+1)} \mathbf{T}_{k+1} \tilde{\mathbf{x}}_{k+1})}_{\tilde{\mathbf{s}}_k}, \quad (5.15)$$

FIGURE 5.4: The equivalent partially-connected full-rank MIMO-IC for the low-interference case.

where we have denoted by $\tilde{\mathbf{s}}_k$ the interference vector at receiver k . By substituting $\mathbf{R}_k$ with (5.8) and $\mathbf{T}_k$ with (5.12), $\tilde{\mathbf{s}}_k$ can be written as $\tilde{\mathbf{s}}_k = \begin{bmatrix} \tilde{\mathbf{s}}_{kh}^T & \mathbf{0}^T & \tilde{\mathbf{s}}_{kt}^T \end{bmatrix}^T$, where

$$\tilde{\mathbf{s}}_{kh} = \underbrace{\mathbf{U}_{k(k+1)} \mathbf{H}_{k(k-1)} \mathbf{V}_{(k+1)(k-1)}}_{\tilde{\mathbf{H}}_{kh}^{(L)}: D_2 \times D_2} \tilde{\mathbf{x}}_{(k-1)h} \quad (5.16)$$

$$\tilde{\mathbf{s}}_{kt} = \underbrace{\mathbf{U}_{k(k-1)} \mathbf{H}_{k(k+1)} \mathbf{V}_{(k-1)(k+1)}}_{\tilde{\mathbf{H}}_{kt}^{(L)}: D_1 \times D_1} \tilde{\mathbf{x}}_{(k+1)t} \quad (5.17)$$

The above result shows that by applying the carefully designed liner transformations $\mathbf{T}_k$ and $\mathbf{R}_k$, the middle $N - D_t$ antennas of receiver k are interference-free, the head D_2 antennas are interfered only by the head D_2 antennas from transmitter $k - 1$ via matrix $\tilde{\mathbf{H}}_{kh}^{(L)}$, and the tail D_1 antennas are interfered only by the tail D_1 antennas from transmitter $k + 1$ via matrix $\tilde{\mathbf{H}}_{kt}^{(L)}$. Furthermore, since the channels are independent and randomly generated, with probability 1, the $D_2 \times D_2$ matrix $\tilde{\mathbf{H}}_{kh}^{(L)}$ and the $D_1 \times D_1$ matrix $\tilde{\mathbf{H}}_{kt}^{(L)}$ are both of full rank and hence invertible. Such an equivalent partially-connected full-rank MIMO-IC for the low-interference case is illustrated in Figure 5.4.¹

¹Note that only the residue interfering links are shown in Figure 5.4 and in all similar plots hereafter. The direct links are omitted for conciseness.

5.4.2 High-Interference Case: $M < D_t \leq 2M$

When $D_t > M$, the designs of $\mathbf{R}_k$ and $\mathbf{T}_k$ are slightly different from that for the low-interference case. The linear transformation $\mathbf{R}_k$ at receiver k is partitioned as

$$\mathbf{R}_k = \begin{bmatrix} \mathbf{U}_{k(k+1)} \\ \mathbf{J}_k \\ \mathbf{U}_{k(k-1)} \end{bmatrix}, \quad (5.18)$$

where $\mathbf{U}_{k(k+1)} \in \mathbb{C}^{(M-D_1) \times N}$, $\mathbf{U}_{k(k-1)} \in \mathbb{C}^{(M-D_2) \times N}$, and $\mathbf{J}_k \in \mathbb{C}^{(N+D_t-2M) \times N}$. The rows of $\mathbf{U}_{k(k+1)}$ and $\mathbf{U}_{k(k-1)}$ are selected from the left null spaces of $\mathbf{H}_{k(k+1)}$ and $\mathbf{H}_{k(k-1)}$, respectively, but *not* in the intersection of these two null spaces. With $D_t > M$, it can be verified that the specified number of rows for $\mathbf{U}_{k(k+1)}$ and $\mathbf{U}_{k(k-1)}$ can always be found. Lastly, the central block $\mathbf{J}_k$ is generated randomly. It then follows that the resulting $N \times N$ matrix $\mathbf{R}_k$ is full rank, and hence invertible.

With $\mathbf{R}_k$ specified, we switch to the linear transformation $\mathbf{T}_k \in \mathbb{C}^{M \times M}$ at transmitter k , which is partitioned as

$$\mathbf{T}_k = \begin{bmatrix} \mathbf{V}_{(k-1)k} & \mathbf{Q}_k & \mathbf{V}_{(k+1)k} \end{bmatrix}, \quad (5.19)$$

where $\mathbf{V}_{(k-1)k} \in \mathbb{C}^{M \times (M-D_1)}$ and $\mathbf{V}_{(k+1)k} \in \mathbb{C}^{M \times (M-D_2)}$ are selected to be the basis of the null spaces of $\mathbf{H}_{(k-1)k}$ and $\mathbf{H}_{(k+1)k}$, respectively. Note that since $D_t > M$, we have $(M - D_1) + (M - D_2) < M$. As a result, these two null spaces have empty intersection, and hence the columns of $\mathbf{V}_{(k-1)k}$ and $\mathbf{V}_{(k+1)k}$ are linearly independent. Next, the central block $\mathbf{Q}_k$ of $\mathbf{T}_k$ is designed so that, together with the receiver matrix $\mathbf{R}_{k-1}$ or $\mathbf{R}_{k+1}$, the interferences to a subset of antennas of receiver $k-1$ and receiver $k+1$ are zero forced. Specifically, we have

$$\begin{bmatrix} \mathbf{U}_{(k-1)(k+1)} \mathbf{H}_{(k-1)k} \\ \mathbf{U}_{(k+1)(k-1)} \mathbf{H}_{(k+1)k} \end{bmatrix} \mathbf{Q}_k = \mathbf{0}. \quad (5.20)$$

Since $\mathbf{U}_{(k-1)(k+1)}$ is designed solely based on $\mathbf{H}_{(k-1)(k+1)}$, it is independent of $\mathbf{H}_{(k-1)k}$. Thus, we have

$$\begin{aligned} \text{rank}(\mathbf{U}_{(k-1)(k+1)} \mathbf{H}_{(k-1)k}) &= \min(\text{rank}(\mathbf{U}_{(k-1)(k+1)}), \text{rank}(\mathbf{H}_{(k-1)k})) \\ &= \min(M - D_2, D_1) = M - D_2. \end{aligned}$$

Similarly, we have $\text{rank}(\mathbf{U}_{(k+1)(k-1)}\mathbf{H}_{(k+1)k}) = M - D_1$. Since $\mathbf{U}_{(k-1)(k+1)}\mathbf{H}_{(k-1)k}$ and $\mathbf{U}_{(k+1)(k-1)}\mathbf{H}_{(k+1)k}$ are independent, we have

$$\begin{aligned} & \dim \left(\mathcal{N} \left(\begin{bmatrix} \mathbf{U}_{(k-1)(k+1)}\mathbf{H}_{(k-1)k} \\ \mathbf{U}_{(k+1)(k-1)}\mathbf{H}_{(k+1)k} \end{bmatrix} \right) \right) \\ &= \dim \left(\mathcal{N}(\mathbf{U}_{(k-1)(k+1)}\mathbf{H}_{(k-1)k}) \right) + \dim \left(\mathcal{N}(\mathbf{U}_{(k+1)(k-1)}\mathbf{H}_{(k+1)k}) \right) - M \\ &= D_t - M. \end{aligned}$$

Therefore, the columns of $\mathbf{Q}_k$ can be chosen to be the basis spanning the null space of the concatenated matrix in (5.20) and its size is given by $M \times (D_t - M)$. It follows that the resulting $M \times M$ matrix $\mathbf{T}_k$ is full-rank and hence invertible.

Following similar analysis as for the low-interference case, the interference vector $\tilde{\mathbf{s}}_k = [\tilde{\mathbf{s}}_{kh}^T \quad \tilde{\mathbf{s}}_{km}^T \quad \tilde{\mathbf{s}}_{kt}^T]^T$ at receiver k with transformations $\mathbf{R}_k$ and $\mathbf{T}_k$ can be obtained, where

$$\tilde{\mathbf{s}}_{kh} = \underbrace{\mathbf{U}_{k(k+1)}\mathbf{H}_{k(k-1)}\mathbf{V}_{(k+1)(k-1)}}_{\tilde{\mathbf{H}}_{kh}^{(H)}: (M-D_1) \times (M-D_1)} \tilde{\mathbf{x}}_{(k-1)h} \quad (5.21)$$

$$\begin{aligned} \tilde{\mathbf{s}}_{km} &= \underbrace{\mathbf{J}_k\mathbf{H}_{k(k+1)}\mathbf{Q}_{k+1}}_{\mathbf{P}_{k(k+1)}} \tilde{\mathbf{x}}_{(k+1)m} + \underbrace{\mathbf{J}_k\mathbf{H}_{k(k-1)}\mathbf{Q}_{k-1}}_{\mathbf{P}_{k(k-1)}} \tilde{\mathbf{x}}_{(k-1)m} \\ &\quad + \mathbf{J}_k\mathbf{H}_{k(k+1)}\mathbf{V}_{(k-1)(k+1)}\tilde{\mathbf{x}}_{(k+1)t} + \mathbf{J}_k\mathbf{H}_{k(k-1)}\mathbf{V}_{(k+1)(k-1)}\tilde{\mathbf{x}}_{(k-1)h} \end{aligned} \quad (5.22)$$

$$\tilde{\mathbf{s}}_{kt} = \underbrace{\mathbf{U}_{k(k-1)}\mathbf{H}_{k(k+1)}\mathbf{V}_{(k-1)(k+1)}}_{\tilde{\mathbf{H}}_{kt}^{(H)}: (M-D_2) \times (M-D_2)} \tilde{\mathbf{x}}_{(k+1)t} \quad (5.23)$$

It can be obtained that the square matrices $\tilde{\mathbf{H}}_{kh}^{(H)}$ and $\tilde{\mathbf{H}}_{kt}^{(H)}$ defined above are both of full rank. Furthermore, the $(N + D_t - 2M) \times (D_t - M)$ matrices $\mathbf{P}_{k(k+1)}$ and $\mathbf{P}_{k(k-1)}$ are both of full-column rank.

The residue interfering links after linear transformations $\mathbf{R}_k$ and $\mathbf{T}_k$ for the high-interference case are plotted in Figure 5.5. The channel shown in Figure 5.5 can be viewed as consisting of two layers. The outer layer includes the head and tail antennas (antennas kh and kt for user k), and each receive antenna block is interfered by a single source only via a full-rank square interfering matrix $\tilde{\mathbf{H}}_{kh}^{(H)}$ and $\tilde{\mathbf{H}}_{kt}^{(H)}$, respectively. The middle antennas (antennas km for user k) form the inner layer, where the three users are fully connected via full-rank matrices $\mathbf{P}_{kj}$ defined above. Note that while the antennas in the inner layer are interfered by the signals from the outer layer, there is no interfering link from the inner layer to the

FIGURE 5.5: The equivalent partially-connected full-rank MIMO-IC for the high-interference case with two-layered interpretation.

outer layer. With such a layered structure in the equivalent channel, the existing techniques developed for the full-rank MIMO-IC can be incorporated for the outer and inner bounds derivations, as will be shown in the following two sections.

5.5 Information Theoretic DoF Outer Bound

In this section, we present the detailed proof for the converse part of Theorem 5.1. First, with single-user bounding, we have

$$d \leq D_0. \quad (5.24)$$

The remaining bounds in Theorem 5.1 are obtained by using the genie-aided signaling technique. To identify the appropriate side information to be provided by the genie, we use the equivalent channel developed in the preceding section. As pointed out in [171], since no claim to duality can be made a-priori for information theoretic DoF results, in principle, the two cases $M_T \leq M_R$ and $M_T > M_R$ need to be discussed separately. However, in this thesis, we only present the detailed derivations for $M_T \leq M_R$, or $M = M_T$ and $N = M_R$, as the reciprocal channel can be handled similarly. Similar as in Section 5.4, we need to distinguish the low- and high-interference cases.

5.5.1 Low-Interference Case: $D_t \leq M$

For the low-interference case, the interfering links of the equivalent partially-connected full-rank MIMO-IC are plotted in Figure 5.4. Since we are dealing with a converse argument, it follows by assumption that each receiver is able to decode and subtract out its desired signal. Consider the interference vector $\tilde{\mathbf{s}}_{1h}$ received by the head antennas of receiver 1. As given in (5.16), $\tilde{\mathbf{s}}_{1h}$ is a function of $\tilde{\mathbf{x}}_{3h}$ via the $D_2 \times D_2$ matrix $\tilde{\mathbf{H}}_{1h}^{(L)}$. Since $\tilde{\mathbf{H}}_{1h}^{(L)}$ is full rank with probability 1, receiver 1 is able to distinguish $\tilde{\mathbf{x}}_{3h}$ based on $\tilde{\mathbf{s}}_{1h}$ subject to the noise distortion. With a similar argument, receiver 1 is also able to demodulate $\tilde{\mathbf{x}}_{2t}$ subject to the noise distortion after subtracting its desired signal. Hence, if a genie provides the signal $G_{1L} = \{\tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{2m}^n, \tilde{\mathbf{x}}_{3m}^n, \tilde{\mathbf{x}}_{3t}^n\}$ to receiver 1, receiver 1 should be able to decode all the three messages W_1 , W_2 and W_3 subject to the noise distortion. Let R_k denote the rate for user k and R_Σ the sum rate of the three users. By Fano's inequality, we have

$$\begin{aligned}
nR_\Sigma &\leq I(W_1, W_2, W_3; \tilde{\mathbf{y}}_1^n, G_{1L}) + no(\log \rho) + o(n) \\
&= h(\tilde{\mathbf{y}}_1^n, G_{1L}) - h(\tilde{\mathbf{y}}_1^n, G_{1L} | W_1, W_2, W_3) + no(\log \rho) + o(n) \\
&\leq h(\tilde{\mathbf{y}}_1^n) + h(G_{1L} | \tilde{\mathbf{y}}_1^n) + no(\log \rho) + o(n) \\
&\stackrel{(a)}{\leq} Nn \log \rho + h(\tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{2m}^n, \tilde{\mathbf{x}}_{3m}^n, \tilde{\mathbf{x}}_{3t}^n | \tilde{\mathbf{y}}_1^n) + no(\log \rho) + o(n) \\
&\leq Nn \log \rho + h(\tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{2m}^n | \tilde{\mathbf{y}}_1^n) + h(\tilde{\mathbf{x}}_{3m}^n, \tilde{\mathbf{x}}_{3t}^n | \tilde{\mathbf{y}}_1^n) + no(\log \rho) + o(n) \\
&\stackrel{(b)}{\leq} Nn \log \rho + h(\tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{2m}^n | \tilde{\mathbf{x}}_{2t}^n) + h(\tilde{\mathbf{x}}_{3m}^n, \tilde{\mathbf{x}}_{3t}^n) + no(\log \rho) + o(n) \\
&\stackrel{(c)}{\leq} Nn \log \rho + nR_2 - h(\tilde{\mathbf{x}}_{2t}^n) + h(\tilde{\mathbf{x}}_{3m}^n, \tilde{\mathbf{x}}_{3t}^n) + no(\log \rho) + o(n), \tag{5.25}
\end{aligned}$$

where (a) is true since receiver 1 has only N antennas, (b) follows since conditioning does not increase the entropy, and (c) is true as the message W_2 can be decoded from the observation of $\{\tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{2m}^n, \tilde{\mathbf{x}}_{2t}^n\}$.

By symmetry, all three messages can be decoded subject to the noise distortion if a genie provides the signals $G_{2L} = \{\tilde{\mathbf{x}}_{3h}^n, \tilde{\mathbf{x}}_{3m}^n, \tilde{\mathbf{x}}_{1m}^n, \tilde{\mathbf{x}}_{1t}^n\}$ and $G_{3L} = \{\tilde{\mathbf{x}}_{1h}^n, \tilde{\mathbf{x}}_{1m}^n, \tilde{\mathbf{x}}_{2m}^n, \tilde{\mathbf{x}}_{2t}^n\}$ to receiver 2 and receiver 3, respectively. We thus have

$$nR_\Sigma \leq Nn \log \rho + nR_3 - h(\tilde{\mathbf{x}}_{3t}^n) + h(\tilde{\mathbf{x}}_{1m}^n, \tilde{\mathbf{x}}_{1t}^n) + no(\log \rho) + o(n) \tag{5.26}$$

$$nR_\Sigma \leq Nn \log \rho + nR_1 - h(\tilde{\mathbf{x}}_{1t}^n) + h(\tilde{\mathbf{x}}_{2m}^n, \tilde{\mathbf{x}}_{2t}^n) + no(\log \rho) + o(n) \tag{5.27}$$

By summing up (5.25)-(5.27), we get

$$\begin{aligned}
2nR_\Sigma &\leq 3Nn \log \rho + \sum_{k=1}^3 h(\tilde{\mathbf{x}}_{km}^n | \tilde{\mathbf{x}}_{kt}^n) + no(\log \rho) + o(n) \\
&\leq 3Nn \log \rho + \sum_{k=1}^3 h(\tilde{\mathbf{x}}_{km}^n) + no(\log \rho) + o(n) \\
&\stackrel{(d)}{\leq} 3Nn \log \rho + 3(M - D_t)n \log \rho + no(\log \rho) + o(n), \tag{5.28}
\end{aligned}$$

where (d) follows since the dimension of $\tilde{\mathbf{x}}_{km}$ is $M - D_t$ as given in Section 5.4.1.

Let R denote the rate normalized per user, i.e., $R = R_\Sigma/3$. Then (5.28) can be equivalently written as

$$6nR \leq 3Nn \log \rho + 3(M - D_t)n \log \rho + no(\log \rho) + o(n). \tag{5.29}$$

By dividing $\log \rho$ and n on both sides of (5.29), and letting $\rho \rightarrow \infty$ and $n \rightarrow \infty$, we obtain the following DoF outer bound for the low-interference case:

$$d \leq \frac{N + M - D_t}{2}. \tag{5.30}$$

5.5.2 High-Interference Case: $M < D_t \leq 2M$

For the high-interference case, the interfering links of the equivalent channel after invertible linear transformations are plotted in Figure 5.5. Consider the interference vector $\tilde{\mathbf{s}}_{1h}$ received by the head antennas of receiver 1, which is a function of $\tilde{\mathbf{x}}_{3h}$ via the square matrix $\tilde{\mathbf{H}}_{1h}^{(H)}$ as shown in (5.21). Since $\tilde{\mathbf{H}}_{1h}^{(H)}$ is full rank, after decoding and subtracting its desired signal, receiver 1 is able to distinguish $\tilde{\mathbf{x}}_{3h}$ subject to the noise distortion based on the observation of $\tilde{\mathbf{s}}_{1h}$. Similarly, receiver 1 can also demodulate $\tilde{\mathbf{x}}_{2t}$ subject to the noise distortion. With $\tilde{\mathbf{x}}_{3h}$ and $\tilde{\mathbf{x}}_{2t}$ obtained, the interference terms at the middle antennas of receiver 1 due to these two signals can be subtracted out from $\tilde{\mathbf{s}}_{1m}$, leaving it a linear function of $\tilde{\mathbf{x}}_{2m}$ and $\tilde{\mathbf{x}}_{3m}$ via the full column rank matrices $\mathbf{P}_{12}$ and $\mathbf{P}_{13}$, respectively. Therefore, if receiver 1 is provided with the side information $\tilde{\mathbf{x}}_{2m}$, it will be able to get $\tilde{\mathbf{x}}_{3m}$ subject to the noise distortion; and vice versa.

Based on the above analysis, if a genie provides the signals $G_{1H} = \{\tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{2m}^n, \tilde{\mathbf{x}}_{3t}^n\}$ to receiver 1, receiver 1 would be able to decode all the three messages subject to

the noise distortion. With Fano's inequality, we have

$$\begin{aligned}
nR_\Sigma &\leq I(W_1, W_2, W_3; \tilde{\mathbf{y}}_1^n, G_{1H}) + no(\log \rho) + o(n) \\
&\leq h(\tilde{\mathbf{y}}_1^n) + h(G_{1H}|\tilde{\mathbf{y}}_1^n) + no(\log \rho) + o(n) \\
&\leq Nn \log \rho + h(\tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{2m}^n, \tilde{\mathbf{x}}_{3t}^n|\tilde{\mathbf{y}}_1^n) + no(\log \rho) + o(n) \\
&= Nn \log \rho + h(\tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{2m}^n|\tilde{\mathbf{x}}_{3t}^n, \tilde{\mathbf{y}}_1^n) + h(\tilde{\mathbf{x}}_{3t}^n|\tilde{\mathbf{y}}_1^n) + no(\log \rho) + o(n) \\
&\leq Nn \log \rho + h(\tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{2m}^n|\tilde{\mathbf{x}}_{2t}^n) + h(\tilde{\mathbf{x}}_{3t}^n) + no(\log \rho) + o(n) \\
&\leq Nn \log \rho + nR_2 - h(\tilde{\mathbf{x}}_{2t}^n) + h(\tilde{\mathbf{x}}_{3t}^n) + no(\log \rho) + o(n) \tag{5.31}
\end{aligned}$$

Due to symmetry, by advancing the user indices to receiver 2 and receiver 3, the following inequalities can be obtained:

$$nR_\Sigma \leq Nn \log \rho + nR_3 - h(\tilde{\mathbf{x}}_{3t}^n) + h(\tilde{\mathbf{x}}_{1t}^n) + no(\log \rho) + o(n) \tag{5.32}$$

$$nR_\Sigma \leq Nn \log \rho + nR_1 - h(\tilde{\mathbf{x}}_{1t}^n) + h(\tilde{\mathbf{x}}_{2t}^n) + no(\log \rho) + o(n) \tag{5.33}$$

By summing up (5.31)-(5.33) and with $R_\Sigma = 3R$, we have

$$6nR \leq 3Nn \log \rho + no(\log \rho) + o(n) \tag{5.34}$$

By dividing $\log \rho$ and n on both sides of (5.34), and letting $\rho \rightarrow \infty$ and $n \rightarrow \infty$, we obtain the following DoF outer bound for the high-interference case:

$$d \leq \frac{N}{2}. \tag{5.35}$$

To prove the remaining two bounds in Theorem 5.1, i.e.,

$$d \leq \min \left(\frac{pM}{2p-1}, \frac{pN+2M-D_t}{2p+1} \right), \tag{5.36}$$

where $p = \lceil \frac{D_t-M}{N-M} \rceil$, we will first give the detailed derivations for $p = 1, 2, 3$, followed by a brief description for general p values as the main technique follows from [171].

5.5.2.1 $p = 1$

For $p = \lceil \frac{D_t-M}{N-M} \rceil = 1$, we have $0 < \frac{D_t-M}{N-M} \leq 1$, or equivalently

$$M < D_t \leq N. \tag{5.37}$$

In this case, we need to show that $d \leq \min\left(M, \frac{N+2M-D_t}{3}\right)$ according to (5.36). Note that the bound $d \leq M$ follows trivially from the single-user bound shown in (5.24) since $D_0 \leq M$. To show $d \leq \frac{N+2M-D_t}{3}$ under the condition given in (5.37), we again use the equivalent channel given in Figure 5.5.

As discussed previously, after subtracting its desired signal, receiver 1 is able to obtain $\tilde{\mathbf{x}}_{3h}$ and $\tilde{\mathbf{x}}_{2t}$ subject to the noise distortion. Therefore, the interference caused by these two signals can be subtracted out from $\tilde{\mathbf{s}}_{1m}$, leaving the middle antennas interfered by $\tilde{\mathbf{x}}_{2m}$ and $\tilde{\mathbf{x}}_{3m}$ only via full column rank matrices. Furthermore, we have $|\tilde{\mathbf{x}}_{2m}| + |\tilde{\mathbf{x}}_{3m}| = 2(D_t - M)$, which is no greater than $|\tilde{\mathbf{s}}_{1m}| = N + D_t - 2M$ under the condition given in (5.37). Therefore, $\tilde{\mathbf{x}}_{2m}$ and $\tilde{\mathbf{x}}_{3m}$ are also distinguishable by receiver 1 subject to the noise distortion. As a result, if a genie provides $\{\tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{3t}^n\}$ to receiver 1, receiver 1 will be able to decode all the three messages subject to the noise distortion. Therefore, we have

$$\begin{aligned}
3nR &\leq I(W_1, W_2, W_3; \tilde{\mathbf{y}}_1^n, \tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{3t}^n) + no(\log \rho) + o(n) \\
&\leq h(\tilde{\mathbf{y}}_1^n) + h(\tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{3t}^n | \tilde{\mathbf{y}}_1^n) + no(\log \rho) + o(n) \\
&\leq Nn \log \rho + h(\tilde{\mathbf{x}}_{2h}^n) + h(\tilde{\mathbf{x}}_{3t}^n) + no(\log \rho) + o(n) \\
&\leq n(N + 2M - D_t) \log \rho + no(\log \rho) + o(n), \tag{5.38}
\end{aligned}$$

where the last inequality follows since $|\tilde{\mathbf{x}}_{2h}| = M - D_1$ and $|\tilde{\mathbf{x}}_{3t}| = M - D_2$. (5.38) leads to the desired DoF outer bound

$$d \leq \frac{N + 2M - D_t}{3}.$$

To prove the upper bound (5.36) for $p \geq 2$, we need to further decompose the middle antennas in Figure 5.5 (i.e., $\tilde{\mathbf{x}}_{km}$ and $\tilde{\mathbf{s}}_{km}$) by making use of the equivalent channel structure developed in [171] for the full-rank MIMO-IC. Specifically, with the two-layered interpretation on the channel given in Figure 5.5, the inner layer consisting of $\tilde{\mathbf{x}}_{km}$ and $\tilde{\mathbf{s}}_{km}$ can be firstly treated separately, which gives a 3-user fully-connected *full-rank* MIMO-IC with $\tilde{M}$ transmit antennas and $\tilde{N}$ receive antennas, where $\tilde{M} = D_t - M$ and $\tilde{N} = N + D_t - 2M$. As such, the equivalent channel model given in [171] can be applied to the middle antennas, after which the outer layer formed by the head and tail antennas (kh and kt for user k) can be integrated seamlessly. We start with the detailed discussion for $p = 2$ in the following.

5.5.2.2 $p = 2$

For $p = \lceil \frac{D_t - M}{N - M} \rceil = \lceil \frac{\tilde{M}}{\tilde{N} - \tilde{M}} \rceil = 2$, we have

$$N < D_t \leq 2N - M, \quad (5.39)$$

or $\tilde{M}/\tilde{N} \in (1/2, 2/3]$. For the 3-user $\tilde{M} \times \tilde{N}$ full-rank MIMO-IC with $\tilde{M}/\tilde{N} \in (1/2, 2/3]$, an equivalent channel obtained by appropriate linear transformations is given in Appendix A.2.2 of [171], which is reproduced in Figure 5.6.² It is observed that in the equivalent channel, the bottom $\tilde{N} - \tilde{M}$ antennas of receiver k (i.e., $\tilde{\mathbf{s}}_{kc_1}$) are interfered only by the last $\tilde{N} - \tilde{M}$ antennas of transmitter $k + 1$ (i.e. $\tilde{\mathbf{x}}_{(k+1)c_1}$); and the top $\tilde{N} - \tilde{M}$ antennas of receiver k (i.e., $\tilde{\mathbf{s}}_{ka_1}$) are interfered by transmitter $k - 1$ only. By applying this equivalent channel to the $\tilde{M} \times \tilde{N}$ full-rank MIMO-IC formed by the middle antennas of Figure 5.5, and after combining with the head and tail antennas, we obtain an equivalent channel for the rank-deficient MIMO-IC corresponding to $p = 2$, as shown in Figure 5.7.³

FIGURE 5.6: An equivalent channel for the 3-user $\tilde{M} \times \tilde{N}$ full-rank MIMO-IC with $\tilde{M}/\tilde{N} \in (1/2, 2/3]$.

The DoF outer bound in (5.36) corresponding to $p = 2$ is then derived based on Figure 5.7. Firstly, $\tilde{\mathbf{x}}_{3h}$ and $\tilde{\mathbf{x}}_{2t}$ can be demodulated by receiver 1 subject to the noise distortion, and hence can be subtracted out from $\tilde{\mathbf{s}}_{1c_1}$. This in turn makes $\tilde{\mathbf{x}}_{2c_1}$ distinguishable as well. Furthermore, if a genie provides $\tilde{\mathbf{x}}_{2b}$ to receiver 1, the

²Note that a slightly different form from that in [171] is given in Figure 5.6, where the transmitted signal by user k is decomposed into $\tilde{\mathbf{x}}_{kb}$ and $\tilde{\mathbf{x}}_{kc_1}$ for easier reference.

³Note that due to symmetry, we only show the signals for user 1 in Figure 5.7 and all similar figures hereafter.

$M - D_1$	$\tilde{\mathbf{x}}_{1h}$	$\mathbf{O}$	$\mathbf{O}$	$\tilde{\mathbf{s}}_{1h}(\tilde{\mathbf{x}}_{3h})$	$M - D_1$
$D_t - N$	$\tilde{\mathbf{x}}_{1b}$	$\mathbf{O}$	$\mathbf{O}$	$\tilde{\mathbf{s}}_{1a_1}(\tilde{\mathbf{x}}_{3h}, \tilde{\mathbf{x}}_{2t}, \tilde{\mathbf{x}}_{3b}, \tilde{\mathbf{x}}_{3c_1})$	$N - M$
$N - M$	$\tilde{\mathbf{x}}_{1c_1}$	$\mathbf{O}$	$\mathbf{O}$	$\tilde{\mathbf{s}}_{1b}(\tilde{\mathbf{x}}_{3h}, \tilde{\mathbf{x}}_{2t}, \tilde{\mathbf{x}}_{2b}, \tilde{\mathbf{x}}_{2c_1}, \tilde{\mathbf{x}}_{3b}, \tilde{\mathbf{x}}_{3c_1})$	$D_t - N$
$M - D_2$	$\tilde{\mathbf{x}}_{1t}$	$\mathbf{O}$	$\mathbf{O}$	$\tilde{\mathbf{s}}_{1c_1}(\tilde{\mathbf{x}}_{3h}, \tilde{\mathbf{x}}_{2t}, \tilde{\mathbf{x}}_{2c_1})$	$N - M$
			$\mathbf{O}$	$\tilde{\mathbf{s}}_{1t}(\tilde{\mathbf{x}}_{2t})$	$M - D_2$

FIGURE 5.7: An equivalent channel for the 3-user rank-deficient MIMO-IC when $p = 2$.

two remaining signals $\tilde{\mathbf{x}}_{3b}$ and $\tilde{\mathbf{x}}_{3c_1}$ can be demodulated based on $\tilde{\mathbf{s}}_{1a_1}$ and $\tilde{\mathbf{s}}_{1b}$. As a result, receiver 1 will be able to decode all the three messages subject to the noise distortion if it is provided with the side information $\{\tilde{\mathbf{x}}_{2b}^n, \tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{3t}^n\}$. With Fano's inequality, we have

$$\begin{aligned}
3nR &\leq I(W_1, W_2, W_3; \tilde{\mathbf{y}}_1^n, \tilde{\mathbf{x}}_{2b}^n, \tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{3t}^n) + no(\log \rho) + o(n) \\
&\leq h(\tilde{\mathbf{y}}_1^n) + h(\tilde{\mathbf{x}}_{2b}^n, \tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{3t}^n | \tilde{\mathbf{y}}_1^n) + no(\log \rho) + o(n) \\
&\leq Nn \log \rho + h(\tilde{\mathbf{x}}_{3t}^n | \tilde{\mathbf{y}}_1^n, \tilde{\mathbf{x}}_{2b}^n, \tilde{\mathbf{x}}_{2h}^n) + h(\tilde{\mathbf{x}}_{2b}^n, \tilde{\mathbf{x}}_{2h}^n | \tilde{\mathbf{y}}_1^n) + no(\log \rho) + o(n) \\
&\leq Nn \log \rho + h(\tilde{\mathbf{x}}_{3t}^n | \tilde{\mathbf{x}}_{3h}^n, \tilde{\mathbf{x}}_{3b}^n, \tilde{\mathbf{x}}_{3c_1}^n) + h(\tilde{\mathbf{x}}_{2b}^n, \tilde{\mathbf{x}}_{2h}^n | \tilde{\mathbf{x}}_{2c_1}^n) + no(\log \rho) + o(n) \\
&\leq Nn \log \rho + nR_3 - h(\tilde{\mathbf{x}}_{3h}^n, \tilde{\mathbf{x}}_{3b}^n, \tilde{\mathbf{x}}_{3c_1}^n) + h(\tilde{\mathbf{x}}_{2b}^n, \tilde{\mathbf{x}}_{2h}^n | \tilde{\mathbf{x}}_{2c_1}^n) + no(\log \rho) + o(n).
\end{aligned} \tag{5.40}$$

By advancing user indices, we can obtain two other inequalities similar to (5.40) at receiver 2 and 3, respectively. By summing them up, we get

$$6nR \leq 3Nn \log \rho - \sum_{k=1}^3 h(\tilde{\mathbf{x}}_{kc_1}^n) + no(\log \rho) + o(n). \tag{5.41}$$

Moreover, we have

$$\begin{aligned}
3nR &\leq \sum_{k=1}^3 h(\tilde{\mathbf{x}}_{kh}^n, \tilde{\mathbf{x}}_{kb}^n, \tilde{\mathbf{x}}_{kc_1}^n, \tilde{\mathbf{x}}_{kt}^n) + no(\log \rho) + o(n) \\
&\leq \sum_{k=1}^3 [h(\tilde{\mathbf{x}}_{kh}^n, \tilde{\mathbf{x}}_{kb}^n, \tilde{\mathbf{x}}_{kt}^n) + h(\tilde{\mathbf{x}}_{kc_1}^n)] + no(\log \rho) + o(n) \\
&\leq 3(2M - N)n \log \rho + \sum_{k=1}^3 h(\tilde{\mathbf{x}}_{kc_1}^n) + no(\log \rho) + o(n).
\end{aligned} \tag{5.42}$$

By summing up (5.41) and (5.42), we have

$$9nR \leq 6Mn \log \rho + no(\log \rho) + o(n), \quad (5.43)$$

which gives the following DoF outer bound:

$$d \leq \frac{2M}{3}. \quad (5.44)$$

Furthermore, it can be verified that under condition (5.39), receiver 1 will be able to decode all three messages subject to the noise distortion if a genie provides the signal $\{\tilde{\mathbf{x}}_{3c1}^n, \tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{3t}^n\}$. Then with Fano's inequality, we have

$$\begin{aligned} 3nR &\leq I(W_1, W_2, W_3; \tilde{\mathbf{y}}_1^n, \tilde{\mathbf{x}}_{3c1}^n, \tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{3t}^n) + no(\log \rho) + o(n) \\ &\leq h(\tilde{\mathbf{y}}_1^n) + h(\tilde{\mathbf{x}}_{3c1}^n, \tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{3t}^n | \tilde{\mathbf{y}}_1^n) + no(\log \rho) + o(n) \\ &\leq Nn \log \rho + h(\tilde{\mathbf{x}}_{2h}^n | \tilde{\mathbf{y}}_1^n, \tilde{\mathbf{x}}_{3c1}^n, \tilde{\mathbf{x}}_{3t}^n) + h(\tilde{\mathbf{x}}_{3c1}^n, \tilde{\mathbf{x}}_{3t}^n | \tilde{\mathbf{y}}_1^n) + no(\log \rho) + o(n) \\ &\leq Nn \log \rho + h(\tilde{\mathbf{x}}_{2h}^n | \tilde{\mathbf{x}}_{2c1}^n, \tilde{\mathbf{x}}_{2t}^n) + h(\tilde{\mathbf{x}}_{3c1}^n, \tilde{\mathbf{x}}_{3t}^n) + no(\log \rho) + o(n). \end{aligned} \quad (5.45)$$

By advancing user indices, we can obtain two other inequalities similar to (5.45). By summing them up, we get

$$9nR \leq 3Nn \log \rho + \sum_{k=1}^3 h(\tilde{\mathbf{x}}_{kh}^n, \tilde{\mathbf{x}}_{kc1}^n, \tilde{\mathbf{x}}_{kt}^n) + no(\log \rho) + o(n). \quad (5.46)$$

Summing up (5.41) and (5.46) yields

$$\begin{aligned} 15nR &\leq 6Nn \log \rho + \sum_{k=1}^3 h(\tilde{\mathbf{x}}_{kh}^n, \tilde{\mathbf{x}}_{kt}^n | \tilde{\mathbf{x}}_{kc1}^n) + no(\log \rho) + o(n) \\ &\leq 6Nn \log \rho + 3(2M - D_t) + no(\log \rho) + o(n), \end{aligned} \quad (5.47)$$

which gives the following DoF outer bound:

$$d \leq \frac{2N + 2M - D_t}{5}. \quad (5.48)$$

By combining (5.44) and (5.48), the upper bound (5.36) for $p = 2$ follows.

5.5.2.3 $p = 3$

For $p = \lceil \frac{D_t - M}{N - M} \rceil = \lceil \frac{\tilde{M}}{\tilde{N} - \tilde{M}} \rceil = 3$, we have

$$2N - M < D_t \leq 3N - 2M. \quad (5.49)$$

or $\tilde{M}/\tilde{N} \in (2/3, 3/4]$. In this case, an equivalent channel for the 3-user $\tilde{M} \times \tilde{N}$ full-rank MIMO-IC is given in Appendix A.2.1 of [171] and reproduced in Figure 5.8. Similar to the case with $p = 2$, it is observed that in the equivalent channel for $p = 3$, the bottom $\tilde{N} - \tilde{M}$ antennas of receiver k are interfered only by the last $\tilde{N} - \tilde{M}$ antennas of transmitter $k+1$; and the top $\tilde{N} - \tilde{M}$ antennas of receiver k are interfered only by the first $\tilde{N} - \tilde{M}$ antennas of transmitter $k-1$. By applying this equivalent channel model to the $\tilde{M} \times \tilde{N}$ full-rank MIMO-IC formed by the middle antennas of Figure 5.5, and after combining with the head and tail antennas, we obtain an equivalent channel for the rank-deficient MIMO-IC corresponding to $p = 3$, as shown in Figure 5.9, based on which the upper bound (5.36) for $p = 3$ can be obtained as shown next.

$\tilde{N} - \tilde{M}$	$\tilde{\mathbf{x}}_{1a_1}$	$\mathbf{O}$	$\mathbf{O}$	$\tilde{\mathbf{s}}_{1a_1}(\tilde{\mathbf{x}}_{3a_1})$	$\tilde{N} - \tilde{M}$
$3\tilde{M} - 2\tilde{N}$	$\tilde{\mathbf{x}}_{1b}$	$\mathbf{O}$	$\mathbf{O}$	$\tilde{\mathbf{s}}_{1b}(\tilde{\mathbf{x}}_{3a_1}, \tilde{\mathbf{x}}_{3b}, \tilde{\mathbf{x}}_{3c_1}, \tilde{\mathbf{x}}_{2a_1}, \tilde{\mathbf{x}}_{2b}, \tilde{\mathbf{x}}_{2c_1})$	$2\tilde{M} - \tilde{N}$
$\tilde{N} - \tilde{M}$	$\tilde{\mathbf{x}}_{1c_1}$	$\mathbf{O}$	$\mathbf{O}$	$\tilde{\mathbf{s}}_{1c_1}(\tilde{\mathbf{x}}_{2c_1})$	$\tilde{N} - \tilde{M}$

FIGURE 5.8: An equivalent channel for the 3-user $\tilde{M} \times \tilde{N}$ full-rank MIMO-IC with $\tilde{M}/\tilde{N} \in (2/3, 3/4]$.

$M - D_1$	$\tilde{\mathbf{x}}_{1h}$	$\mathbf{O}$	$\mathbf{O}$	$\tilde{\mathbf{s}}_{1h}(\tilde{\mathbf{x}}_{3h})$	$M - D_1$
$N - M$	$\tilde{\mathbf{x}}_{1a_1}$	$\mathbf{O}$	$\mathbf{O}$	$\tilde{\mathbf{s}}_{1a_1}(\tilde{\mathbf{x}}_{3h}, \tilde{\mathbf{x}}_{2t}, \tilde{\mathbf{x}}_{3a_1})$	$N - M$
$D_t - 2N + M$	$\tilde{\mathbf{x}}_{1b}$	$\mathbf{O}$	$\mathbf{O}$	$\tilde{\mathbf{s}}_{1b}(\tilde{\mathbf{x}}_{3h}, \dots, \tilde{\mathbf{x}}_{3c_1}, \tilde{\mathbf{x}}_{2a_1}, \dots, \tilde{\mathbf{x}}_{2t})$	$D_t - N$
$N - M$	$\tilde{\mathbf{x}}_{1c_1}$	$\mathbf{O}$	$\mathbf{O}$	$\tilde{\mathbf{s}}_{1c_1}(\tilde{\mathbf{x}}_{3h}, \tilde{\mathbf{x}}_{2t}, \tilde{\mathbf{x}}_{2c_1})$	$N - M$
$M - D_2$	$\tilde{\mathbf{x}}_{1t}$	$\mathbf{O}$	$\mathbf{O}$	$\tilde{\mathbf{s}}_{1t}(\tilde{\mathbf{x}}_{2t})$	$M - D_2$

FIGURE 5.9: An equivalent channel for the 3-user rank-deficient MIMO-IC when $p = 3$.

Following similar analysis as in the previous cases, we have

$$\begin{aligned}\tilde{\mathbf{y}}_1 &\rightarrow \{\tilde{\mathbf{x}}_{3h}, \tilde{\mathbf{x}}_{2t}, \tilde{\mathbf{x}}_{3a_1}, \tilde{\mathbf{x}}_{2c_1}\} \\ \{\tilde{\mathbf{y}}_1, \tilde{\mathbf{x}}_{3b}, \tilde{\mathbf{x}}_{3c_1}\} &\rightarrow \{\tilde{\mathbf{x}}_{3h}, \tilde{\mathbf{x}}_{2t}, \tilde{\mathbf{x}}_{3a_1}, \tilde{\mathbf{x}}_{3b}, \tilde{\mathbf{x}}_{3c_1}, \tilde{\mathbf{x}}_{2a_1}, \tilde{\mathbf{x}}_{2b}, \tilde{\mathbf{x}}_{2c_1}\} \\ \{\tilde{\mathbf{y}}_1^n, \tilde{\mathbf{x}}_{3b}^n, \tilde{\mathbf{x}}_{3c_1}^n, \tilde{\mathbf{x}}_{3t}^n, \tilde{\mathbf{x}}_{2h}^n\} &\rightarrow \{W_1, W_2, W_3\}\end{aligned}\quad (5.50)$$

$$\begin{aligned}\{\tilde{\mathbf{y}}_1, \tilde{\mathbf{x}}_{2a_1}, \tilde{\mathbf{x}}_{2b}\} &\rightarrow \{\tilde{\mathbf{x}}_{3h}, \tilde{\mathbf{x}}_{2t}, \tilde{\mathbf{x}}_{3a_1}, \tilde{\mathbf{x}}_{3b}, \tilde{\mathbf{x}}_{3c_1}, \tilde{\mathbf{x}}_{2a_1}, \tilde{\mathbf{x}}_{2b}, \tilde{\mathbf{x}}_{2c_1}\} \\ \{\tilde{\mathbf{y}}_1^n, \tilde{\mathbf{x}}_{2a_1}^n, \tilde{\mathbf{x}}_{2b_1}^n, \tilde{\mathbf{x}}_{3t}^n, \tilde{\mathbf{x}}_{2h}^n\} &\rightarrow \{W_1, W_2, W_3\},\end{aligned}\quad (5.51)$$

where $\mathcal{A} \rightarrow \mathcal{B}$ denotes that the signals in $\mathcal{B}$ can be decoded subject to the noise distortion based on the observation on $\mathcal{A}$.

Furthermore, since $|\tilde{\mathbf{x}}_{2b}| + |\tilde{\mathbf{x}}_{3b}| = 2(D_t - 2N + M)$, which is no greater than $|\tilde{\mathbf{s}}_{1b}| = D_t - N$ under condition (5.49), it can be verified that

$$\{\tilde{\mathbf{y}}_1^n, \tilde{\mathbf{x}}_{2a_1}^n, \tilde{\mathbf{x}}_{3c_1}^n, \tilde{\mathbf{x}}_{3t}^n, \tilde{\mathbf{x}}_{2h}^n\} \rightarrow \{W_1, W_2, W_3\}.\quad (5.52)$$

From (5.50), we have

$$\begin{aligned}3nR &\leq I(W_1, W_2, W_3; \tilde{\mathbf{y}}_1^n, \tilde{\mathbf{x}}_{3b}^n, \tilde{\mathbf{x}}_{3c_1}^n, \tilde{\mathbf{x}}_{3t}^n, \tilde{\mathbf{x}}_{2h}^n) + no(\log \rho) + o(n) \\ &\leq h(\tilde{\mathbf{y}}_1^n) + h(\tilde{\mathbf{x}}_{3b}^n, \tilde{\mathbf{x}}_{3c_1}^n, \tilde{\mathbf{x}}_{3t}^n, \tilde{\mathbf{x}}_{2h}^n | \tilde{\mathbf{y}}_1^n) + no(\log \rho) + o(n) \\ &\leq Nn \log \rho + h(\tilde{\mathbf{x}}_{2h}^n | \tilde{\mathbf{y}}_1^n, \tilde{\mathbf{x}}_{3b}^n, \tilde{\mathbf{x}}_{3c_1}^n, \tilde{\mathbf{x}}_{3t}^n) + h(\tilde{\mathbf{x}}_{3b}^n, \tilde{\mathbf{x}}_{3c_1}^n, \tilde{\mathbf{x}}_{3t}^n | \tilde{\mathbf{y}}_1^n) + no(\log \rho) + o(n) \\ &\leq Nn \log \rho + h(\tilde{\mathbf{x}}_{2h}^n | \tilde{\mathbf{x}}_{2a_1}^n, \tilde{\mathbf{x}}_{2b}^n, \tilde{\mathbf{x}}_{2c_1}^n, \tilde{\mathbf{x}}_{2t}^n) + h(\tilde{\mathbf{x}}_{3b}^n, \tilde{\mathbf{x}}_{3c_1}^n, \tilde{\mathbf{x}}_{3t}^n | \tilde{\mathbf{x}}_{3a_1}^n) + no(\log \rho) + o(n) \\ &\leq Nn \log \rho + nR_2 - h(\tilde{\mathbf{x}}_{2a_1}^n, \tilde{\mathbf{x}}_{2b}^n, \tilde{\mathbf{x}}_{2c_1}^n, \tilde{\mathbf{x}}_{2t}^n) + h(\tilde{\mathbf{x}}_{3b}^n, \tilde{\mathbf{x}}_{3c_1}^n, \tilde{\mathbf{x}}_{3t}^n | \tilde{\mathbf{x}}_{3a_1}^n) + no(\log \rho) + o(n)\end{aligned}\quad (5.53)$$

By advancing user indices, we can obtain another two similar inequalities at receiver 2 and 3, respectively. Summing up the three inequalities yields

$$6nR \leq 3Nn \log \rho - \sum_{k=1}^3 h(\tilde{\mathbf{x}}_{ka_1}^n) + no(\log \rho) + o(n)\quad (5.54)$$

Similarly, based on (5.51), we can obtain

$$6nR \leq 3Nn \log \rho - \sum_{k=1}^3 h(\tilde{\mathbf{x}}_{kc_1}^n) + no(\log \rho) + o(n)\quad (5.55)$$

Moreover, we have

$$\begin{aligned}
3nR &\leq \sum_{k=1}^3 h(\tilde{\mathbf{x}}_{kh}^n, \tilde{\mathbf{x}}_{ka_1}^n, \tilde{\mathbf{x}}_{kb}^n, \tilde{\mathbf{x}}_{kc_1}^n, \tilde{\mathbf{x}}_{kt}^n) + no(\log \rho) + o(n) \\
&\leq \sum_{k=1}^3 [h(\tilde{\mathbf{x}}_{kh}^n, \tilde{\mathbf{x}}_{kb}^n, \tilde{\mathbf{x}}_{kt}^n) + h(\tilde{\mathbf{x}}_{ka_1}^n) + h(\tilde{\mathbf{x}}_{kc_1}^n)] + no(\log \rho) + o(n) \\
&\leq 3(3M - 2N)n \log \rho + \sum_{k=1}^3 [h(\tilde{\mathbf{x}}_{ka_1}^n) + h(\tilde{\mathbf{x}}_{kc_1}^n)] + no(\log \rho) + o(n). \quad (5.56)
\end{aligned}$$

By summing up (5.54), (5.55) and (5.56), we have

$$15nR \leq 9Mn \log \rho + no(\log \rho) + o(n),$$

which gives the following DoF upper bound:

$$d \leq \frac{3M}{5}. \quad (5.57)$$

Similarly, from (5.52), we have

$$\begin{aligned}
3nR &\leq I(W_1, W_2, W_3; \tilde{\mathbf{y}}_1^n, \tilde{\mathbf{x}}_{2a_1}^n, \tilde{\mathbf{x}}_{3c_1}^n, \tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{3t}^n) + no(\log \rho) + o(n) \\
&\leq h(\tilde{\mathbf{y}}_1^n) + h(\tilde{\mathbf{x}}_{2a_1}^n, \tilde{\mathbf{x}}_{3c_1}^n, \tilde{\mathbf{x}}_{2h}^n, \tilde{\mathbf{x}}_{3t}^n | \tilde{\mathbf{y}}_1^n) + no(\log \rho) + o(n) \\
&\leq Nn \log \rho + h(\tilde{\mathbf{x}}_{3c_1}^n, \tilde{\mathbf{x}}_{3t}^n | \tilde{\mathbf{y}}_1^n) + h(\tilde{\mathbf{x}}_{2a_1}^n, \tilde{\mathbf{x}}_{2h}^n | \tilde{\mathbf{y}}_1^n) + no(\log \rho) + o(n) \\
&\leq Nn \log \rho + h(\tilde{\mathbf{x}}_{3c_1}^n, \tilde{\mathbf{x}}_{3t}^n | \tilde{\mathbf{x}}_{3a_1}^n, \tilde{\mathbf{x}}_{3h}^n) + h(\tilde{\mathbf{x}}_{2a_1}^n, \tilde{\mathbf{x}}_{2h}^n) + no(\log \rho) + o(n) \quad (5.58)
\end{aligned}$$

By advancing user indices, we can obtain two other similar inequalities. By summing them up, we have

$$9nR \leq 3Nn \log \rho + \sum_{k=1}^3 h(\tilde{\mathbf{x}}_{kh}^n, \tilde{\mathbf{x}}_{ka_1}^n, \tilde{\mathbf{x}}_{kc_1}^n, \tilde{\mathbf{x}}_{kt}^n) + no(\log \rho) + o(n) \quad (5.59)$$

By summing up (5.54), (5.55), and (5.59), we get

$$\begin{aligned}
21nR &\leq 9Nn \log \rho + \sum_{k=1}^3 \left[h(\tilde{\mathbf{x}}_{kh}^n, \tilde{\mathbf{x}}_{ka_1}^n, \tilde{\mathbf{x}}_{kc_1}^n, \tilde{\mathbf{x}}_{kt}^n) \right. \\
&\quad \left. - h(\tilde{\mathbf{x}}_{ka_1}^n) - h(\tilde{\mathbf{x}}_{kc}^n) \right] + no(\log \rho) + o(n) \\
&\leq 9Nn \log \rho + \sum_{k=1}^3 h(\tilde{\mathbf{x}}_{kh}^n, \tilde{\mathbf{x}}_{kt}^n) + no(\log \rho) + o(n) \\
&\leq (9N + 6M - 3D_t)n \log \rho + no(\log \rho) + o(n),
\end{aligned}$$

which gives the following DoF upper bound:

$$d \leq \frac{3N + 2M - D_t}{7}. \quad (5.60)$$

By combining (5.57) and (5.60), the upper bound (5.36) for $p = 3$ follows.

5.5.2.4 Intuition of “onion peeling” for general p values

We have shown the bound (5.36) for $p = 1, 2, 3$. The proof for general p values can be obtained similarly. Since the mathematical derivations overlap largely with the full-rank case [171], we only give a brief description of the techniques used. It is not difficult to see that for all the p values discussed, the proof of (5.36) follows the “onion peeling” intuition, as termed in [171]. Specifically, depending on the relationship between D_t , M and N , an equivalent channel with *layered* structure is developed, where the antennas in a particular layer are only interfered by the outer layers, but not by the inner layers. For instance, the signals shown in red in Figure 5.7 and Figure 5.9 can be viewed as the innermost layer and those in black as the outermost layer. With the equivalent layered channel structure, the genie information is provided such that the signals in the outer layer can be always demodulated prior to the processing of the received signals in the inner layer. As such, the interference caused by the outer-layer signals can be subtracted out from the inner-layer antennas. This in effect peels out the outer layer antennas; and hence is given the term “onion peeling”. As p increases, the antenna blocks involving D_t can be further decomposed (see, e.g., Figure 5.5 for $p = 1$ and Figure 5.9 for $p = 3$). Therefore, the number of layers increases and the onion peeling technique applies recursively. Note that as compared to the full-rank case, the equivalent layered channel structure for the rank-deficient MIMO-IC

has one additional layer, i.e., the outermost layer formed by the head and tail antennas, which is obtained by extracting the rank-deficiency of the interfering channel matrices as discussed previously. When reducing to the full-rank case, i.e., $D_1 = D_2 = M$, the outer layer vanishes as evidenced from Figure 5.5, in which case exactly the same equivalent layered channel given in [171] will be resulted.

5.6 Achievability Proof

In this section, we show the achievability of the spatially-normalized DoF given in Theorem 5.1. To convey the most interesting ideas, we only show the achievability through linear dimension counting approach. Figure 5.10 gives an overview of our proposed achievability scheme, which has two-layered linear processes. The zero forcing achieved by linear transformations $\mathbf{R}_k$ and $\mathbf{T}_k$ discussed in Section 5.4 constitutes the inner layer, with which an equivalent partially-connected full-rank MIMO-IC is obtained, as shown in Figure 5.4 and Figure 5.5 for the low- and high-interference cases, respectively. The outer layer of the proposed scheme consists of a *block-diagonal* precoding matrix $\mathbf{E}_k$ at the transmitter and interference cancelation, if necessary, at the receiver. For the high-interference case, $\mathbf{E}_k$ is carefully designed so that interference alignment is achieved; whereas for the low-interference case, a random $\mathbf{E}_k$ is sufficient since most interfering links are zero forced by the inner layer. In the remaining part of this section, we present the detailed design for the outer precoding matrix $\mathbf{E}_k$ for both the low- and high-interference cases. Due to the reciprocity of linear schemes [184], without loss of generality, we assume in the following that $M_R \geq M_T$, so that $M = M_T$ and $N = M_R$.

It is worth mentioning that we assume the DoF quantities in the following are all integer values, so that each signaling block is assigned with integer number of symbols. If this is not the case, a spatial extension by an appropriate factor is required, and all the arguments presented in the following still hold in the spatially-extended channel. After DoF normalization by the same factor, the achievable result for the spatially-normalized DoF in Theorem 5.1 follows.

FIGURE 5.10: The proposed two-layered achievability scheme.

5.6.1 Low-Interference Case: $D_t \leq M$

For the low-interference case, the spatially-normalized DoF value is given by

$$\bar{d} = \min \left(D_0, \frac{N + M - D_t}{2} \right). \quad (5.61)$$

In the following, we will consider when each term in (5.61) is active, respectively.

5.6.1.1 $\bar{d} = \frac{N+M-D_t}{2}$

First, we assume that the rank of the direct channel D_0 is sufficiently large so that $\bar{d} = \frac{N+M-D_t}{2}$. In this case, we have $\frac{N+M-D_t}{2} \leq D_0 \leq M$, which implies $D_t \geq N - M$. We show that each user is able to transmit $\frac{N+M-D_t}{2}$ data symbols, denoted as $\mathbf{d}_k \in \mathbb{C}^{\bar{d} \times 1}$, $k = 1, 2, 3$.

The information-bearing signal vector $\mathbf{d}_k$ is decomposed as $\mathbf{d}_k = [\mathbf{d}_{kh}^T \ \mathbf{d}_{km}^T \ \mathbf{d}_{kt}^T]^T$, and it is precoded with a block-diagonal matrix $\mathbf{E}_k \in \mathbb{C}^{M \times \bar{d}}$, which gives $\tilde{\mathbf{x}}_k$ as

$$\tilde{\mathbf{x}}_k = \begin{bmatrix} \tilde{\mathbf{x}}_{kh} \\ \tilde{\mathbf{x}}_{km} \\ \tilde{\mathbf{x}}_{kt} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{kh} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_{km} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{E}_{kt} \end{bmatrix} \begin{bmatrix} \mathbf{d}_{kh} \\ \mathbf{d}_{km} \\ \mathbf{d}_{kt} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{kh} \mathbf{d}_{kh} \\ \mathbf{E}_{km} \mathbf{d}_{km} \\ \mathbf{E}_{kt} \mathbf{d}_{kt} \end{bmatrix}. \quad (5.62)$$

Based on the partition of $\tilde{\mathbf{x}}_k$ as in Figure 5.4, the sizes of $\mathbf{E}_{kh}$, $\mathbf{E}_{km}$, $\mathbf{E}_{kt}$ are given by $D_2 \times |\mathbf{d}_{kh}|$, $(M - D_t) \times |\mathbf{d}_{km}|$, and $D_1 \times |\mathbf{d}_{kt}|$, respectively, where $|\mathbf{d}_{kh}|$, $|\mathbf{d}_{km}|$, and $|\mathbf{d}_{kt}|$ are chosen to satisfy

$$|\mathbf{d}_{kh}| + |\mathbf{d}_{km}| + |\mathbf{d}_{kt}| = \bar{d} = \frac{N + M - D_t}{2}, \quad (63a)$$

$$|\mathbf{d}_{kh}| \leq D_2, \quad (63b)$$

$$|\mathbf{d}_{km}| \leq M - D_t, \quad (63c)$$

$$|\mathbf{d}_{kt}| \leq D_1, \quad k = 1, 2, 3. \quad (63d)$$

With each diagonal block in $\mathbf{E}_k$ randomly generated, the conditions given in (5.63) ensure that $\mathbf{E}_k$ is of full column rank, which is necessary for the full decodability of the desired symbols. It can be verified that with $D_t \geq N - M$, one feasible choice to satisfy (5.63) is

$$\begin{aligned} |\mathbf{d}_{kh}| &= \min \left(D_2, \frac{N - M + D_t}{2} \right), \\ |\mathbf{d}_{km}| &= M - D_t, \\ |\mathbf{d}_{kt}| &= \frac{N - M + D_t}{2} - |\mathbf{d}_{kh}|, \quad k = 1, 2, 3. \end{aligned} \quad (5.64)$$

Next, we show that at each receiver, the $\bar{d}$ desired data symbols can be decoded. Due to symmetry, we only need to consider receiver 1. We denote by Z_1 the total dimension occupied by the interfering signals. Note that the proposed transmit and receive matrices $\mathbf{E}_k$, $\mathbf{T}_k$, and $\mathbf{R}_k$ are solely based on the interfering channel matrices, which are independent of the direct channel matrices $\mathbf{H}_{kk}$. Therefore, with probability 1, the subspaces occupied by the desired data symbols and that by the interference have no overlapping if

$$\bar{d} + Z_1 \leq N, \quad (5.65)$$

where N is the total dimension available at the receiver. If the above relationship holds, then receiver 1 can discard the Z_1 dimensions that contain interference and the remaining dimensions are enough to decode the $\bar{d}$ desired data symbols.

Due to zero forcing achieved by $\mathbf{T}_1$ and $\mathbf{R}_1$ in the inner layer, it is observed from Figure 5.4 that receiver 1 only sees the interfering signals $\tilde{\mathbf{x}}_{3h}$ and $\tilde{\mathbf{x}}_{2t}$. Therefore, we have

$$\begin{aligned} Z_1 &= |\tilde{\mathbf{x}}_{3h}| + |\tilde{\mathbf{x}}_{2t}| = |\mathbf{d}_{3h}| + |\mathbf{d}_{2t}| \\ &= \frac{N - M + D_t}{2}. \end{aligned} \quad (5.66)$$

As a result, condition (5.65) is satisfied with equality; and hence $\bar{d} = \frac{N+M-D_t}{2}$ is achievable.

5.6.1.2 $\bar{d} = D_0$

In this case, we have $D_0 < \frac{N+M-D_t}{2}$. Then the DoF value D_0 per user can be achieved by reducing the number of data streams of any information sub-blocks given in (5.64), till we have $|\mathbf{d}_{kh}| + |\mathbf{d}_{km}| + |\mathbf{d}_{kt}| = D_0$.

5.6.2 High-Interference Case: $M < D_t \leq 2M$

Similar to the low-interference case, we denote the information-bearing signal for the high-interference case as $\mathbf{d}_k$ and it is decomposed as $\mathbf{d}_k = [\mathbf{d}_{kh}^T, \mathbf{d}_{km}^T, \mathbf{d}_{kt}^T]^T$. Furthermore, $\mathbf{d}_k$ is precoded with a block-diagonal matrix $\mathbf{E}_k$ as shown in (5.62). The sizes of $\mathbf{E}_{kh}$, $\mathbf{E}_{km}$, and $\mathbf{E}_{kt}$ are now given by $(M - D_1) \times |\mathbf{d}_{kh}|$, $(D_t - M) \times |\mathbf{d}_{km}|$, and $(M - D_2) \times |\mathbf{d}_{kt}|$, respectively, which correspond to the partition of $\tilde{\mathbf{x}}_k$ as in Figure 5.5. To ensure the decodability of $\bar{d}$ desired data symbols at each receiver, the number of symbols $|\mathbf{d}_{kh}|$, $|\mathbf{d}_{km}|$, and $|\mathbf{d}_{kt}|$ of each sub-block in $\mathbf{d}_k$ are chosen to satisfy

$$|\mathbf{d}_{kh}| + |\mathbf{d}_{km}| + |\mathbf{d}_{kt}| = \bar{d}, \quad (67a)$$

$$|\mathbf{d}_{kh}| \leq M - D_1, \quad (67b)$$

$$|\mathbf{d}_{km}| \leq D_t - M, \quad (67c)$$

$$|\mathbf{d}_{kt}| \leq M - D_2, \quad k = 1, 2, 3. \quad (67d)$$

The blocks $\mathbf{E}_{kh}$ and $\mathbf{E}_{kt}$ for the head and tail antennas are randomly generated, so that under the conditions given in (5.67), they are of full column rank almost surely. In the remaining part of this subsection, for different $\bar{d}$ values activated in (5.3), we present a feasible selection of $|\mathbf{d}_{kh}|$, $|\mathbf{d}_{km}|$, and $|\mathbf{d}_{kt}|$ to satisfy (5.67), as well as the designs of the precoding matrices $\mathbf{E}_{km}$ for the middle antennas so that the full decodability condition given in (5.65) is satisfied.

Similar to the low-interference case, if the DoF is limited by the direct channel, i.e. $\bar{d} = D_0$, the achievability scheme can be simply obtained by reducing the number of data symbols transmitted for the interfering-link-limited cases. Therefore, in the following, we assume without loss of generality that the rank of the direct channel D_0 is large enough, i.e., $\frac{N}{2} < D_0 \leq M$, so that the achievable DoF $\bar{d}$ is limited by the interfering links. In this case, it can be verified that $\bar{d}$ in (5.3) for

the high-interference case can be further written as

$$\bar{d} = \begin{cases} \frac{N}{2}, & D_t \leq 2M - \frac{N}{2} \\ \min\left(\frac{pN+2M-D_t}{2p+1}, \frac{pM}{2p-1}\right), & D_t > 2M - \frac{N}{2} \end{cases} \quad (5.68)$$

where $p = \lceil \frac{D_t-M}{N-M} \rceil$. In the following, we will consider the cases separately when each term in (5.68) is active.

5.6.2.1 $\bar{d} = \frac{N}{2}$

With $D_t \leq 2M - \frac{N}{2}$, it can be verified that one feasible choice to (5.67) is

$$\begin{aligned} |\mathbf{d}_{kh}| &= \min\left(M - D_1, \frac{N}{2}\right) \\ |\mathbf{d}_{km}| &= 0 \\ |\mathbf{d}_{kt}| &= \frac{N}{2} - |\mathbf{d}_{kh}|, \quad k = 1, 2, 3. \end{aligned} \quad (5.69)$$

Since $|\mathbf{d}_{km}| = 0$, $\mathbf{E}_{km}$ is an empty matrix. Based on Fig 5.5, the total dimension Z_1 occupied by the interference is

$$Z_1 = |\tilde{\mathbf{x}}_{3h}| + |\tilde{\mathbf{x}}_{2t}| = |\mathbf{d}_{3h}| + |\mathbf{d}_{2t}| = \frac{N}{2}.$$

As a result, the decodability condition (5.65) is satisfied with equality; and hence the DoF value $\bar{d} = \frac{N}{2}$ is achievable.

5.6.2.2 $\bar{d} = \min\left(\frac{pN+2M-D_t}{2p+1}, \frac{pM}{2p-1}\right)$

In this case, $\bar{d}$ can be further written as

$$\bar{d} = \begin{cases} \frac{pN+2M-D_t}{2p+1}, & D_t \geq D^* \\ \frac{pM}{2p-1}, & D_t < D^*, \end{cases} \quad (5.70)$$

where $D^* \triangleq pN - \frac{2p^2-3p+2}{2p-1}M$. It can be verified that a feasible choice to (5.67) is

$$|\mathbf{d}_{kh}| = M - D_1, \quad (5.71)$$

$$|\mathbf{d}_{km}| = \bar{d} - (2M - D_t), \quad (5.72)$$

$$|\mathbf{d}_{kt}| = M - D_2. \quad (5.73)$$

With such a symbol assignment for each sub-block of $\mathbf{d}_k$, the total dimension Z_1 occupied by the interference at receiver 1 can be calculated based on Figure 5.5. Since $\tilde{\mathbf{H}}_{1h}^{(H)}$ and $\tilde{\mathbf{H}}_{1t}^{(H)}$ as defined in (5.21) and (5.23) are full-rank square matrices, interference cancelation can be applied, i.e., the interference caused by $\tilde{\mathbf{x}}_{3h}$ and $\tilde{\mathbf{x}}_{2t}$ can be subtracted out from $\tilde{\mathbf{s}}_{1m}$ by appropriate linear processing with $\tilde{\mathbf{s}}_{1h}$ and $\tilde{\mathbf{s}}_{1t}$, respectively. Therefore, we have

$$\begin{aligned} Z_1 &= |\tilde{\mathbf{x}}_{3h}| + |\tilde{\mathbf{x}}_{2t}| + |\mathbf{P}_{12}\tilde{\mathbf{x}}_{2m} + \mathbf{P}_{13}\tilde{\mathbf{x}}_{3m}| \\ &= |\mathbf{E}_{3h}\mathbf{d}_{3h}| + |\mathbf{E}_{2t}\mathbf{d}_{2t}| + \underbrace{|\mathbf{P}_{12}\mathbf{E}_{2m}\mathbf{d}_{2m} + \mathbf{P}_{13}\mathbf{E}_{3m}\mathbf{d}_{3m}|}_{Z_{1m}} \\ &= 2M - D_t + Z_{1m}, \end{aligned} \quad (5.74)$$

where $\mathbf{P}_{12}$ and $\mathbf{P}_{13}$ are full column rank matrices given in (5.22).

Next, the middle precoding matrices $\mathbf{E}_{km} \in \mathbb{C}^{(D_t-M) \times |\mathbf{d}_{km}|}$, with $k = 1, 2, 3$ and $|\mathbf{d}_{km}|$ given by (5.72), are carefully designed so that the interference dimension Z_{1m} is small enough to ensure the decodability condition (5.65) is satisfied. To this end, interference alignment is required in general. We will illustrate the main idea with the cases $p = 1$ and $p = 2$. The design of $\mathbf{E}_{km}$ for general p values is deferred to Appendix G.

Case I: $p = 1$. By substituting with $p = 1$, (5.70) reduces to

$$\bar{d} = \begin{cases} \frac{N+2M-D_t}{3}, & D_t \geq N - M \\ M, & D_t < N - M, \end{cases} \quad (5.75)$$

In this case, the matrices $\mathbf{E}_{km}$ are randomly generated from $\mathbb{C}^{(D_t-M) \times (\bar{d}-2M+D_t)}$. Then we have

$$Z_{1m} = |\mathbf{d}_{2m}| + |\mathbf{d}_{3m}| = 2(\bar{d} - 2M + D_t). \quad (5.76)$$

Together with (5.74), we have

$$\bar{d} + Z_1 = 3\bar{d} - 2M + D_t, \quad (5.77)$$

which can be verified to be no greater than N for both cases in (5.75). Therefore, the full decodability condition (5.65) is satisfied; and hence the DoF value given in (5.75) for $p = 1$ is achievable.

Case II: $p = 2$. By substituting with $p = 2$, (5.70) reduces to

$$\bar{d} = \begin{cases} \frac{2N+2M-D_t}{5}, & D_t \geq D^* \\ \frac{2M}{3}, & D_t < D^*, \end{cases} \quad (5.78)$$

where $D^* = 2N - \frac{4M}{3}$.

First, we consider the case $D_t \geq D^*$ so that the objective is to transmit $\bar{d} = \frac{2N+2M-D_t}{5}$ data symbols for each user. In this case, we need to make use of the technique called “subspace alignment chains” introduced in [171], where ideally, each transmitted signal subspace is aligned with another interference subspace at *both* undesired receivers and an alignment chain is formed. As given in [171], with $\tilde{M}$ transmit and $\tilde{N}$ receive antennas, the length of the alignment chain is equal to $\lceil \frac{\tilde{M}}{\tilde{N}-\tilde{M}} \rceil = p$. For the design of $\mathbf{E}_{km}$ of our interest, we have $\tilde{M} = D_t - M$ and $\tilde{N} = N + D_t - 2M$ as discussed previously. Following the notations used in [171], we use $\mathbf{E}_{km(s)}^i$, where $k, i \in \{1, 2, 3\}$, to denote the s th q -dimensional subspace at transmitter k that participates in the chain originated from transmitter i , where q is a design parameter. For the current case considered, we let

$$q = \frac{|\mathbf{d}_{km}|}{2} = \frac{\bar{d} - 2M + D_t}{2} = \frac{N - 4M + 2D_t}{5}. \quad (5.79)$$

We first consider the alignment chain that originates from transmitter 1, i.e., $i = 1$. With $p = 2$, the chain length is 2, or equivalently, two subspaces participate in the interference alignment for each chain. In this case, the interference alignment must be achieved at receiver 2, and the alignment chain can be represented as

$$\mathbf{E}_{1m(1)}^1 \xleftrightarrow{\text{Rx } 2} \mathbf{E}_{3m(1)}^1. \quad (5.80)$$

The alignment condition (5.80) is achievable by designing the q -dimensional matrices $\mathbf{E}_{1m(1)}^1$ and $\mathbf{E}_{3m(1)}^1$ such that

$$\mathbf{P}_{21}\mathbf{E}_{1m(1)}^1 = \mathbf{P}_{23}\mathbf{E}_{3m(1)}^1, \quad (5.81)$$

where $\mathbf{P}_{21}$ and $\mathbf{P}_{23}$ are the resulting interfering matrices to receiver 2, both of size $(N + D_t - 2M) \times (D_t - M)$, as given in (5.22). (5.81) can be equivalently written

as

$$\underbrace{\begin{bmatrix} \mathbf{P}_{21} & -\mathbf{P}_{23} \end{bmatrix}}_{\mathbf{A}_2} \underbrace{\begin{bmatrix} \mathbf{E}_{1m(1)}^1 \\ \mathbf{E}_{3m(1)}^1 \end{bmatrix}}_{\mathbf{G}_2} = \mathbf{0} \quad (5.82)$$

Since $\mathbf{A}_2$ is an $(N + D_t - 2M) \times 2(D_t - M)$ matrix, the dimension of its null space is $D_t - N$, which can be verified to be no smaller than q under $p = 2$ and $D_t \geq D^*$. Therefore, the columns of $\mathbf{G}_2$ can be chosen to be the q basis vectors for the null space of $\mathbf{A}_2$, based on which the matrices $\mathbf{E}_{1m(1)}^1$ and $\mathbf{E}_{3m(1)}^1$ can be obtained.

With a similar approach, we can design the precoding matrices to achieve two other alignment chains that originate from transmitter 2 and 3, which are represented as

$$\mathbf{E}_{2m(1)}^2 \xleftrightarrow{\text{Rx } 3} \mathbf{E}_{1m(1)}^2, \quad (5.83)$$

$$\mathbf{E}_{3m(1)}^3 \xleftrightarrow{\text{Rx } 1} \mathbf{E}_{2m(1)}^3. \quad (5.84)$$

Lastly, the precoding matrices $\mathbf{E}_{km} \in \mathbb{C}^{(D_t-M) \times |\mathbf{d}_{km}|}$ for the middle antennas of the three users are given by

$$\begin{aligned} \mathbf{E}_{1m} &= \begin{bmatrix} \mathbf{E}_{1m(1)}^1 & \mathbf{E}_{1m(1)}^2 \end{bmatrix} \\ \mathbf{E}_{2m} &= \begin{bmatrix} \mathbf{E}_{2m(1)}^2 & \mathbf{E}_{2m(1)}^3 \end{bmatrix} \\ \mathbf{E}_{3m} &= \begin{bmatrix} \mathbf{E}_{3m(1)}^1 & \mathbf{E}_{3m(1)}^3 \end{bmatrix}. \end{aligned} \quad (5.85)$$

With similar arguments as that in [171], it can be shown that the precoding vectors obtained above are linearly independent and hence the resulting $\mathbf{E}_{km}, k = 1, 2, 3$, is of full column rank. The interference dimension Z_{1m} in (5.74) can be calculated as

$$Z_{1m} = |\mathbf{d}_{2m}| + |\mathbf{d}_{3m}| - q = \frac{3N - 12M + 6D_t}{5}, \quad (5.86)$$

where the minus term is due to the interference alignment achieved by two q -dimensional subspaces at receiver 1. Together with (5.74) and $\bar{d} = \frac{2N+2M-D_t}{5}$, we have $\bar{d} + Z_1 = N$. Therefore, the full decodability condition (5.65) is satisfied with equality; and hence the DoF given in (5.78) for the case $D_t \geq D^*$ is achievable.

Next, we consider the case $D_t < D^*$ in (5.78) so that the objective is to

transmit $\bar{d} = \frac{2M}{3}$ data symbols for each user. First, the q' -dimensional precoding matrices are designed similarly to achieve interference alignment given in (5.80), (5.83) and (5.84). The difference is that in this case, we set $q' = D_t - N$, which is equal to the dimension of the null space of $\mathbf{A}_2$ defined in (5.82). Since for each user, two such precoding matrices are used as shown in (5.85), the number of data symbols participating in interference alignment for each user is given by $2q' = 2(D_t - N)$, which is less than $|\mathbf{d}_{km}| = \frac{2M}{3} - 2M + D_t$ as required by (5.72) under $D_t < D^*$. Therefore, for each user k , we need to transmit another $\hat{q}$ data symbols with precoding matrices denoted as $\hat{\mathbf{E}}_{km}$, where

$$\begin{aligned}\hat{q} &= |\mathbf{d}_{km}| - 2q' \\ &= \frac{2M}{3} - (2M - D_t) - 2(D_t - N) \\ &= \left(2N - \frac{4M}{3}\right) - D_t.\end{aligned}$$

Furthermore, $\hat{\mathbf{E}}_{km}$ are randomly generated. Then the precoding matrices $\mathbf{E}_{km}$ for the middle antennas are obtained as

$$\begin{aligned}\mathbf{E}_{1m} &= \begin{bmatrix} \mathbf{E}_{1m(1)}^1 & \mathbf{E}_{1m(1)}^2 & \hat{\mathbf{E}}_{1m} \end{bmatrix} \\ \mathbf{E}_{2m} &= \begin{bmatrix} \mathbf{E}_{2m(1)}^2 & \mathbf{E}_{2m(1)}^3 & \hat{\mathbf{E}}_{2m} \end{bmatrix} \\ \mathbf{E}_{3m} &= \begin{bmatrix} \mathbf{E}_{3m(1)}^1 & \mathbf{E}_{3m(1)}^3 & \hat{\mathbf{E}}_{3m} \end{bmatrix}\end{aligned}\tag{5.87}$$

It can be verified that the resulting $\mathbf{E}_{km}$ is of full column rank. The interference dimension Z_{1m} at the middle antennas of receiver 1 is given by

$$Z_{1m} = |\mathbf{d}_{2m}| + |\mathbf{d}_{3m}| - q' = D_t + N - \frac{8M}{3}.$$

Together with (5.74), we have

$$\bar{d} + Z_1 = \frac{2M}{3} + (2M - D_t) + \left(D_t + N - \frac{8M}{3}\right) = N.$$

Therefore, the decodability condition (5.65) is satisfied with equality; and hence $\bar{d} = \frac{2M}{3}$ is achievable.

So far we have shown the achievability of $\bar{d} = \min\left(\frac{pN+2M-D_t}{2p+1}, \frac{pM}{2p-1}\right)$ for $p = 1$ and $p = 2$. The achievability for general p values is presented in Appendix G.

5.7 Numerical Results

In this section, numerical examples are provided for the proposed techniques. We assume that each terminal is equipped with a uniform linear array (ULA) with adjacent elements separated by distance Δ measured in wavelength. The channel matrix $\mathbf{H}_{ki}$ between transmitter i and receiver k can be modeled as [185]

$$\mathbf{H}_{ki} = \frac{1}{\sqrt{L_{ki}}} \sum_{l=1}^{L_{ki}} \mathbf{a}_R(\phi_{ki}^l) \mathbf{a}_T^H(\theta_{ki}^l), \quad k, i = 1, 2, 3, \quad (5.88)$$

where L_{ki} represents the number of signal paths; ϕ_{ki}^l and θ_{ki}^l denote the angle of arrival (AoA) and angle of departure (AoD) for the l th path between transmitter i and receiver k , respectively. Moreover, $\mathbf{a}_T$ and $\mathbf{a}_R$ are the transmit and receive array response, respectively, which are given by

$$\mathbf{a}_T(\theta) = \left[1, e^{j2\pi\Delta \sin \theta}, \dots, e^{j2\pi\Delta(M_T-1) \sin \theta} \right]^T, \quad (5.89)$$

$$\mathbf{a}_R(\phi) = \left[1, e^{j2\pi\Delta \sin \phi}, \dots, e^{j2\pi\Delta(M_R-1) \sin \phi} \right]^T. \quad (5.90)$$

Since $\mathbf{H}_{ki}$ in (5.88) is given by a summation of L_{ki} rank-1 matrices, it is clear that in the poor scattering environment with $L_{ki} < \min(M_T, M_R)$, the channel matrix $\mathbf{H}_{ki}$ will be rank deficient.

We consider a 3-user system with $M_T = 2$ and $M_R = 4$, and all direct links have two paths $L_{kk} = 2$ and the interfering links have single path only $L_{ki} = 1, \forall k \neq i$. With the AoAs and AoDs randomly generated, this corresponds to a rank-deficient MIMO-IC with $D_0 = 2$ and $D_1 = D_2 = 1$. Based on Theorem 5.1, the DoF per user is $\bar{d} = 2$, i.e., each receiver is expected to be able to detect two data streams sent by their respective transmitters. As a numerical illustration, we consider a particular channel realization for a set of randomly generated AoAs and AoDs given by

$$\Phi^1 = \begin{bmatrix} 0.05 & 3.47 & 2.47 \\ 4.57 & 4.66 & 0.85 \\ 3.53 & 5.18 & 1.23 \end{bmatrix}, \quad \Theta^1 = \begin{bmatrix} 4.53 & 2.41 & 0.93 \\ 4.21 & 3.09 & 3.05 \\ 0.38 & 4.86 & 0.45 \end{bmatrix} \quad (5.91)$$

$$\phi^2 = \begin{bmatrix} 4.95 & 2.47 & 1.48 \end{bmatrix}, \quad \theta^2 = \begin{bmatrix} 0.51 & 0.50 & 5.83 \end{bmatrix}, \quad (5.92)$$

where the (k, i) th entry of Φ^1 and Θ^1 represent the AoA and AoD of the first path for the link between transmitter i and receiver k , respectively, and the k th entry of ϕ^2 and θ^2 are the AoA and AoD of the second path for the direct link $\mathbf{H}_{kk}$. With the scheme presented in Section 5.6, the precoding matrices $\mathbf{E}_k$ in Figure 5.10 can be any randomly generated 2×2 diagonal matrices, and $\mathbf{T}_k$ can be found as

$$\begin{aligned} \mathbf{T}_1 &= \begin{bmatrix} 0.37 - 0.33i & -0.48 + 0.13i \\ -0.45 - 0.21i & -0.50 - 0.07i \end{bmatrix}, \quad \mathbf{T}_2 = \begin{bmatrix} -0.40 - 0.31i & -0.50 + 0.01i \\ -0.46 + 0.18i & -0.50 - 0.01i \end{bmatrix}, \\ \mathbf{T}_3 &= \begin{bmatrix} 0.49 - 0.10i & -0.46 - 0.20i \\ -0.50 - 0.05i & -0.49 + 0.11i \end{bmatrix}, \end{aligned} \quad (5.93)$$

Furthermore, with the following linear transformation applied at receiver 1:

$$\mathbf{R}_1 = \begin{bmatrix} -0.34 - 0.08i & 0.11 + 0.34i & 0.31 - 0.18i & -0.31 - 0.16i \\ -0.46 & -0.02 + 0.54i & -0.30 + 0.28i & 0.41 + 0.41i \\ 0.55 & -0.02 + 0.45i & 0.28 - 0.50i & 0.37 + 0.19i \\ 0.29 + 0.20i & -0.08 + 0.35i & -0.31 + 0.18i & -0.27 - 0.22i \end{bmatrix}, \quad (5.94)$$

it can be verified that the second and the third antennas of $\tilde{\mathbf{y}}_1$ only contain the two desired data symbols sent from transmitter 1, and hence can be used to fully decode the desired symbols. Similarly, receiver 2 and 3 can both detect their desired data symbols after applying $\mathbf{R}_2$ and $\mathbf{R}_3$, respectively.

5.8 Chapter Summary

In this chapter, we have provided the DoF characterization for the 3-user rank-deficient MIMO-IC parameterized by $(M_T, M_R, D_0, D_1, D_2)$. By exploiting the rank-deficiency of the interfering channel matrices and applying invertible linear transformations, we first convert the original fully-connected rank-deficient MIMO-IC into an equivalent partially-connected full-rank MIMO-IC, based on which the existing techniques developed for full-rank channels can be incorporated to aid for the outer and inner bounds derivations. The outer bound is obtained by the genie-aided signaling technique, where the appropriate genie signals to be provided to the receivers are determined based on the developed equivalent channel model. To achieve the optimal DoF, a two-layered linear processing scheme that hybrids zero forcing and interference alignment has been proposed.

Chapter 6

Modified Block Diagonalization for Multi-Cell Cooperative Processing

6.1 Introduction and Literature Review

The transceiver designs discussed in the previous chapters are mainly based on the interference channel model, where each receiver only obtains the desired information from its own transmitter and the signals from all other transmitters constitute the detrimental interference. Recent studies on multi-cell cooperative processing (MCP) [1, 18–21], in which the data symbols for each user are jointly transmitted from the multiple cooperating base stations (BSs), have revealed that the spectral efficiency can be significantly enhanced due to the utilization of the interfering links to carry desired symbols as well.

MCP requires the exchange of both CSI and data symbols by the cooperating BSs. This may be achieved via the high-capacity low-latency backhaul links. For large cellular networks where full BS cooperation is impractical to implement, some clustered BS cooperation strategies can be applied [32]. In the ideal case with infinite-capacity and delay-free backhaul links, the downlink MCP-enabled network can be modeled as a broadcast channel with per-group-of-antennas power constraint [22–24]. To reduce the backhaul load due to the extensive CSI and symbol exchanges, a decentralized MCP framework has been proposed in [186]. With *local* instantaneous or statistical CSI, the achievable rate region has been

characterized for MCP-enabled network in [187]. Distributed non-linear and linear precoder designs have been introduced for MCP in [188]. MCP has also been studied in the scenario with limited backhaul links [25–29, 189]. In [30], under the assumption of imperfect CSI sharing, a precoding method based on the *team decisional* framework was proposed.

For broadcast channel with sum-power or per-group-of-antenna power constraints, significant research attentions have been devoted to linear precoding schemes due to their ability to achieve a good balance between complexity and performance [41–46]. One such linear scheme of particular interest is block diagonalization (BD) [47–52], where the inter-user interference is completely eliminated by restricting the precoding matrix for each mobile station (MS) to be orthogonal to the channels associated with all other MSs. The initial study on BD mostly focused on single-cell systems with sum-power constraint [47]. Later it was extended to multi-cell networks under per-BS power constraint [51, 52]. In [52], the BD precoding for weighted sum-rate maximization was formulated as a convex optimization problem, from which a semi-closed form solution was derived.

While BD performs quite well at high SNR, the performance degrades in the low-to-medium SNR regime due to the transmit power boost problem, which is caused by the hard constraint of zero inter-user interference. The transmit power boost problem is analogous to the well-known noise enhancement problem associated with the zero-forcing receiver filters, i.e., to achieve the same performance in the low-to-medium SNR regime, higher transmit power is required for schemes with zero inter-user interference constraint as compared with the schemes with certain interference toleration. In the low-to-medium SNR regime where the noise power is comparable with the interference power level, both noise and interference should be taken into account for effective transmissions, whereas the former is obviously ignored by the BD scheme.

One straightforward solution to improve the low-to-medium SNR performance of BD is the minimum mean square error (MMSE) transmission. Although the MMSE channel inversion (MMSE-CI) scheme overcomes the drawback of zero forcing precoding [190], it is still confined to the case with single receiver antenna. For the multi-user MIMO scenario where the receivers are equipped with multiple antennas, the MSE is coupled with both the transmitter and receiver designs, where the optimal solution is unknown. A number of heuristic algorithms have been proposed. In [21], the precoder is decomposed into a preliminary matrix and

a power allocation matrix, where the preliminary matrix is designed to have the MMSE structure to balance the noise and interference effects. In [191], the sum-MSE is minimized by iteratively optimizing the transmitter and receiver filters. However, only a local optimal solution can be obtained due to the non-convexity of the problem. Since these algorithms only give sub-optimal MMSE solutions, simulation results show that their high-SNR performance is degraded as compared with the BD scheme, which is undesirable.

In this chapter, a modified BD scheme is proposed to improve the sum-rate performance over the original BD scheme in the low-to-medium SNR regime while maintaining its good performance at high SNR. This is achieved by the so-called “effective-SNR-enhancement” technique, i.e., by relaxing the hard constraint of zero inter-user interference imposed by the BD scheme and properly controlling the interference leakage via a rate-constrained power minimization problem, the effective noise power can be suppressed and hence the effective SNR is enhanced. Therefore, the transmit power boost problem associated with the original BD scheme is mitigated and a sum-rate gain is attained.

6.2 System Model and Problem Formulation

Consider a downlink MCP-enabled network with K_t BSs, all of which are connected via low-latency high-capacity backhaul links. Denote the number of antennas at each BS as N_t and the total number of transmit antennas of all the BSs as $M = K_t N_t$. Denote by K_r the number of scheduled MSs, each of which is equipped with N_r antennas.¹ We assume that at each time slot, the number of scheduled MSs K_r satisfies $K_r N_r = M$ so that inter-user interference can be completely eliminated by applying techniques such as BD precoding.² Denote by $\mathbf{d}_k \in \mathbb{C}^{M \times 1}$ the information-bearing symbols for the k th mobile station (denoted as MS_k). We assume that circularly symmetric complex Gaussian codebook is used for each MS so that $\mathbf{d}_k$ is distributed as $\mathbf{d}_k \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}), \forall k$. With MCP, the data symbols $\mathbf{d}_k$ for each MS are exchanged among the cooperating BSs via the backhaul links, and are then jointly transmitted from the K_t BSs. As such, the equivalent system can be modeled by a broadcast channel, but with per-group-of-antennas (i.e., per-BS)

¹The techniques developed in this chapter can be straightforwardly extended to the setup where each terminal is equipped with different number of antennas.

²The downlink scheduling problems for ZF/BD precoding in MISO/MIMO broadcast channel have been discussed in e.g., [192, 193].

power constraint due to the geographically separated BSs. We further assume that inter-user interference is treated as noise at the BSs, i.e., interference pre-cancellation via the non-linear DPC scheme is not performed at the BSs.³ The received baseband signal at MS_k is then given by

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{W}_k \mathbf{d}_k + \sum_{i=1, i \neq k}^{K_r} \mathbf{H}_k \mathbf{W}_i \mathbf{d}_i + \mathbf{n}_k, \quad k = 1, \dots, K_r, \quad (6.1)$$

where $\mathbf{H}_k = [\mathbf{H}_{k1} \ \mathbf{H}_{k2} \ \dots \ \mathbf{H}_{kK_t}] \in \mathbb{C}^{N_r \times M}$ denotes the aggregate channel of MS_k from all the cooperating BSs, with $\mathbf{H}_{kj} \in \mathbb{C}^{N_r \times N_t}$ denoting the channel from base station j (BS_j) to MS_k; We assume the matrix $\mathbf{H}_k$ is of full row rank due to independent fading. $\mathbf{W}_k \in \mathbb{C}^{M \times M}$ is the linear precoding matrix for MS_k, with each column corresponding to one data stream. Depending on the optimization result, it is possible that the number of data streams for MS_k is less than M , in which case, the corresponding columns of $\mathbf{W}_k$ are set to zero vectors. Furthermore, $\mathbf{n}_k \in \mathbb{C}^{N_r \times 1}$ denotes i.i.d. zero-mean CSCG additive noise with variance σ^2 , i.e., $\mathbf{n}_k \sim \mathcal{CN}(\mathbf{0}, \sigma^2 \mathbf{I})$, $\forall k$. Under the assumption that the inter-user interference is treated as noise at each MS, the achievable rate R_k for MS_k is given by [162]

$$R_k = \log \frac{\left| \sigma^2 \mathbf{I} + \sum_{i=1}^{K_r} \mathbf{H}_k \mathbf{W}_i \mathbf{W}_i^H \mathbf{H}_k^H \right|}{\left| \sigma^2 \mathbf{I} + \sum_{i=1, i \neq k}^{K_r} \mathbf{H}_k \mathbf{W}_i \mathbf{W}_i^H \mathbf{H}_k^H \right|}} \quad (6.2)$$

$$= \log \left| \mathbf{I} + \left(\sigma^2 \mathbf{I} + \sum_{i=1, i \neq k}^{K_r} \mathbf{H}_k \mathbf{W}_i \mathbf{W}_i^H \mathbf{H}_k^H \right)^{-1} \mathbf{H}_k \mathbf{W}_k \mathbf{W}_k^H \mathbf{H}_k^H \right|, \quad \forall k. \quad (6.3)$$

In the remaining part of this chapter, we assume that the noise power is normalized to unity, i.e., $\sigma^2 = 1$.⁴ Denote the transmit covariance matrix for MS_k as $\mathbf{S}_k = \mathbb{E}(\mathbf{W}_k \mathbf{d}_k \mathbf{d}_k^H \mathbf{W}_k^H) = \mathbf{W}_k \mathbf{W}_k^H$, where $\mathbf{S}_k$ is positive semidefinite, i.e., $\mathbf{S}_k \succeq \mathbf{0}$.

³As has been discussed in Section 4.3, under the assumption that interference pre-cancellation is not performed at the cooperating BSs, the achievable rates of broadcast channels can be improved by employing improper Gaussian signaling (via widely linear precoding) over the conventional proper Gaussian signaling (via strictly linear precoding). However, for the purpose of illustrating the main idea of the modified BD scheme, we will focus on the conventional proper Gaussian signaling (linear precoding) in this chapter.

⁴As evidenced from the rate expression (6.2), if $\sigma^2 \neq 1$, a normalization can always be made by defining a new set of channels $\tilde{\mathbf{H}}_k = \mathbf{H}_k / \sqrt{\sigma^2}$, $\forall k$.

For BS_j , define a binary matrix $\mathbf{B}_j$ as

$$\mathbf{B}_j \triangleq \text{Diag}\left\{ \underbrace{0, \dots, 0}_{(j-1)N_t}, \underbrace{1, \dots, 1}_{N_t}, \underbrace{0, \dots, 0}_{(K_t-j)N_t} \right\}. \quad (6.4)$$

Without loss of generality, we assume that all BSs have the same power constraint P . Then the per-BS power constraint can be specified as

$$\sum_{k=1}^{K_r} \text{Tr}(\mathbf{B}_j \mathbf{S}_k) \leq P, \quad j = 1, \dots, K_t. \quad (6.5)$$

The sum-rate maximization problem under per-BS power constraint can then be formulated as

$$\begin{aligned} & \max_{\{R_k\}, \{\mathbf{S}_k\}} \sum_{k=1}^{K_r} R_k \\ & \text{s.t. } R_k \leq \log \frac{\left| \mathbf{I} + \sum_{i=1}^{K_r} \mathbf{H}_k \mathbf{S}_i \mathbf{H}_k^H \right|}{\left| \mathbf{I} + \sum_{i=1, i \neq k}^{K_r} \mathbf{H}_k \mathbf{S}_i \mathbf{H}_k^H \right|}, \quad \forall k \\ & \sum_{k=1}^{K_r} \text{Tr}(\mathbf{B}_j \mathbf{S}_k) \leq P, \quad \forall j \\ & \mathbf{S}_k \succeq \mathbf{0}, \quad \forall k. \end{aligned} \quad (6.6)$$

In (6.6), we have formulated the optimization problem over the transmit covariance matrices by substituting $\mathbf{S}_i = \mathbf{W}_i \mathbf{W}_i^H$ into (6.2). With the optimal covariance matrix $\mathbf{S}_k^*$ obtained for MS_k , the corresponding precoding matrix $\mathbf{W}_k^*$ can be readily obtained via methods such as eigenvalue decomposition, i.e.,

$$\mathbf{W}_k^* = \mathbf{U}_k \mathbf{D}_k^{1/2}, \quad (6.7)$$

where $\mathbf{U}_k$ and $\mathbf{D}_k$ are obtained by the eigenvalue decomposition $\mathbf{S}_k^* = \mathbf{U}_k \mathbf{D}_k \mathbf{U}_k^H$.

Unfortunately, problem (6.6) is non-convex due to the non-convex rate constraint. In fact, it was noted that sum-rate maximization problems are NP-hard in general [39, 40, 107]. As a result, it is difficult to find the globally optimal solution efficiently. Iterative algorithms based on solving a sequence of convex optimization problems such as [43, 45, 46] can be applied to find sub-optimal solutions to (6.6). In this chapter, we will focus on the simple BD precoding scheme for which a convex formulation is known, and our proposed modified BD scheme, where only

two convex optimization problems need to be solved.

With BD precoding, the inter-user interference is eliminated by designing the precoders $\{\mathbf{W}_k\}$ such that $\mathbf{H}_i \mathbf{W}_k = \mathbf{0}$, $\forall i \neq k$, or equivalently

$$\mathbf{H}_i \mathbf{S}_k \mathbf{H}_i^H = \mathbf{0}, \quad \forall i \neq k. \quad (6.8)$$

As a result, the sum-rate maximization problem (6.6) with BD precoding reduces to [52]

$$\begin{aligned} \max_{\{\mathbf{S}_k\}} & \sum_{k=1}^{K_r} \log \left| \mathbf{I} + \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H \right| \\ \text{s.t.} & \mathbf{H}_i \mathbf{S}_k \mathbf{H}_i^H = \mathbf{0}, \quad \forall i \neq k \\ & \sum_{k=1}^{K_r} \text{Tr}(\mathbf{B}_j \mathbf{S}_k) \leq P, \quad \forall j \\ & \mathbf{S}_k \succeq \mathbf{0}, \quad \forall k. \end{aligned} \quad (6.9)$$

(6.9) is a convex optimization problem and hence can be efficiently solved with standard convex optimization techniques, e.g., the interior-point method [65].

6.3 Modified BD Precoding

BD precoding performs well at high SNR since the inter-user interference is completely eliminated. Nevertheless, in the low-to-medium SNR regime, its performance is poor due to the hard constraint of zero inter-user interference. We therefore propose a modified BD scheme, in which inter-user interference is allowed and properly controlled. The benefit is that the effective noise power can be suppressed in the low-to-medium SNR regime, which thus mitigates the power boost problem associated with the original BD scheme.

Denote by $\{R_k^{\text{BD}}\}$ the rate-tuple achieved by the optimal BD precoding obtained by solving problem (6.9). To obtain an improved sum-rate performance, we

first consider the following optimization problem:

$$\begin{aligned}
& \max_{\rho, \{\mathbf{S}_k\}} \quad -\rho \\
& \text{s.t.} \quad \log \frac{\left| \mathbf{I} + \sum_{i=1}^{K_r} \mathbf{H}_k \mathbf{S}_i \mathbf{H}_k^H \right|}{\left| \mathbf{I} + \sum_{i=1, i \neq k}^{K_r} \mathbf{H}_k \mathbf{S}_i \mathbf{H}_k^H \right|} \geq R_k^{\text{BD}}, \quad \forall k \\
& \quad \sum_{k=1}^{K_r} \text{Tr}(\mathbf{B}_j \mathbf{S}_k) \leq \rho P, \quad \forall j \\
& \quad \mathbf{S}_k \succeq \mathbf{0}, \quad \forall k.
\end{aligned} \tag{6.10}$$

Problem (6.10) minimizes a common power factor ρ for all BSs, while ensuring that each user achieves a minimum rate R_k^{BD} that is achievable by the BD scheme. In contrast to the zero-interference constraint imposed by the BD scheme in (6.9), interference leakage is allowed in problem (6.10) as long as the achievable rate is not below R_k^{BD} .

Problem (6.10) is non-convex in general due to the non-convex rate constraint. However, in the special case of single antenna at each MS ($N_r = 1$), (6.10) can be transformed into an equivalent SOCP problem [41, 114], which can be efficiently solved as shown in Section 6.3.2. As for the general case with $N_r \geq 2$, no convex formulation of (6.10) is known, and hence we will pursue efficient approximate solutions as will be given in Section 6.3.3. Before presenting the algorithms for solving (6.10), we first discuss in the next subsection how the solution to (6.10) can be used to obtain an improved precoder design over the original BD scheme with better low-to-medium SNR performance.

6.3.1 Effective SNR Enhancement

Theorem 6.1. *Problem (6.10) is guaranteed to be feasible and the solution $\{\rho^{\text{opt}}, \{\mathbf{S}_k^{\text{opt}}\}\}$ satisfies $\rho^{\text{opt}} \leq 1$.*

Proof. It can be verified that $\{\rho = 1, \{\mathbf{S}_k^{\text{BD}}\}\}$ is feasible for problem (6.10), where $\{\mathbf{S}_k^{\text{BD}}\}$ is the optimal transmit covariance matrices with BD scheme obtained by solving (6.9). Therefore, Theorem 6.1 follows. $\square$

Although the transmit covariance matrices $\{\mathbf{S}_k^{\text{opt}}\}$ obtained by solving (6.10) do not directly increase the users' rates over that obtained by the BD scheme, the minimized power factor ρ^{opt} makes it possible to effectively suppress the noise power and hence to enhance the effective SNR. This is achieved by using a new set of transmit covariance matrices given by $\mathbf{S}_k^{\text{new}} = \mathbf{S}_k^{\text{opt}}/\rho^{\text{opt}}$, $\forall k$. Since the optimal solution $\{\rho^{\text{opt}}, \{\mathbf{S}_k^{\text{opt}}\}\}$ to (6.10) satisfies the power constraint in problem (6.10), it can be easily obtained that the power constraint of the original sum-rate maximization problem (6.6) is satisfied by $\{\mathbf{S}_k^{\text{new}}\}$. Furthermore, the achievable rate for MS_k with $\{\mathbf{S}_k^{\text{new}} = \mathbf{S}_k^{\text{opt}}/\rho^{\text{opt}}\}$ satisfies

$$R_k^{\text{new}} = \log \frac{\left| \mathbf{I} + \sum_{i=1}^{K_r} \mathbf{H}_k (\mathbf{S}_i^{\text{opt}}/\rho^{\text{opt}}) \mathbf{H}_k^H \right|}{\left| \mathbf{I} + \sum_{i=1, i \neq k}^{K_r} \mathbf{H}_k (\mathbf{S}_i^{\text{opt}}/\rho^{\text{opt}}) \mathbf{H}_k^H \right|}} \quad (6.11)$$

$$= \log \left| \mathbf{I} + \left(\rho^{\text{opt}} \mathbf{I} + \sum_{i=1, i \neq k}^{K_r} \mathbf{H}_k \mathbf{S}_i^{\text{opt}} \mathbf{H}_k^H \right)^{-1} \mathbf{H}_k \mathbf{S}_k^{\text{opt}} \mathbf{H}_k^H \right| \quad (6.12)$$

$$\geq \log \left| \mathbf{I} + \left(\mathbf{I} + \sum_{i=1, i \neq k}^{K_r} \mathbf{H}_k \mathbf{S}_i^{\text{opt}} \mathbf{H}_k^H \right)^{-1} \mathbf{H}_k \mathbf{S}_k^{\text{opt}} \mathbf{H}_k^H \right| \quad (6.13)$$

$$= \log \frac{\left| \mathbf{I} + \sum_{i=1}^{K_r} \mathbf{H}_k \mathbf{S}_i^{\text{opt}} \mathbf{H}_k^H \right|}{\left| \mathbf{I} + \sum_{i=1, i \neq k}^{K_r} \mathbf{H}_k \mathbf{S}_i^{\text{opt}} \mathbf{H}_k^H \right|}} \quad (6.14)$$

$$\geq R_k^{\text{BD}} \quad (6.15)$$

where the equality (6.12) follows from the identity $\log \frac{|\mathbf{A}|}{|\mathbf{B}|} = \log |\mathbf{B}^{-1} \mathbf{A}|$, and the inequality (6.13) is true since the expression in (6.12) is a decreasing function of ρ , and the fact that $\rho^{\text{opt}} \leq 1$ as given by Theorem 6.1.

The above relationship shows that the newly obtained transmit covariance matrices $\{\mathbf{S}_k^{\text{new}}\}$ will perform at least no worse than the original BD scheme. Furthermore, it can be interpreted based on (6.12) that R_k^{new} is the rate achievable by applying the transmit covariance matrices $\{\mathbf{S}_k^{\text{opt}}\}$ in an environment with a noise power ρ^{opt} , instead of 1 as in the original system. Since $\rho^{\text{opt}} \leq 1$, this implies an effective SNR enhancement by $10 \log_{10} (1/\rho^{\text{opt}})$ dB is achieved by applying the new covariance matrices $\{\mathbf{S}_k^{\text{new}}\}$ over $\{\mathbf{S}_k^{\text{opt}}\}$. Furthermore, since $\{\mathbf{S}_k^{\text{opt}}\}$ gives at least the same performance as the BD scheme due to the rate constraint imposed by problem (6.10), an effective SNR enhancement of $10 \log_{10} (1/\rho^{\text{opt}})$ dB over the BD scheme is also guaranteed by the newly proposed transmit covariance matrices $\{\mathbf{S}_k^{\text{new}}\}$. Such an enhancement of SNR mitigates the transmit power boost problem

associated with the BD scheme in the low-to-medium SNR regime, and hence the sum-rate performance can be improved.

Next, the power minimization problem (6.10) is solved by distinguishing two cases: $N_r = 1$ and $N_r \geq 2$.

6.3.2 Power Minimization for Single-Antenna Receivers

When each MS has a single antenna ($N_r = 1$), and hence will receive one data stream only, the BD scheme reduces to zero forcing precoding [20, 190]. Denote the channel vector and the transmit beamforming vector of user k as $\mathbf{h}_k \in \mathbb{C}^{1 \times M}$ and $\mathbf{w}_k \in \mathbb{C}^{M \times 1}$, respectively. By substituting with $\mathbf{S}_k = \mathbf{w}_k \mathbf{w}_k^H$, problem (6.10) can be simplified as

$$\begin{aligned} \min_{\rho, \{\mathbf{w}_k\}} \quad & \rho \\ \text{s.t.} \quad & \log \left(1 + \frac{|\mathbf{h}_k \mathbf{w}_k|^2}{1 + \sum_{i=1, i \neq k}^{K_r} |\mathbf{h}_k \mathbf{w}_i|^2} \right) \geq R_k^{\text{BD}}, \quad \forall k \\ & \sum_{k=1}^{K_r} \|\mathbf{w}_k^{[j]}\|^2 \leq \rho P, \quad \forall j, \end{aligned} \quad (6.16)$$

where $\mathbf{w}_k^{[j]} \in \mathbb{C}^{N_t \times 1}$ denotes the beamforming vector for MS _{k} applied by BS _{j} . Denote by γ_k^{BD} the SINR of MS _{k} corresponding to the rate R_k^{BD} , i.e. $\gamma_k^{\text{BD}} = 2^{R_k^{\text{BD}}} - 1$. Then the rate constraint in (6.16) can be expressed as

$$1 + \sum_{i=1}^{K_r} |\mathbf{h}_k \mathbf{w}_i|^2 \leq \left(1 + \frac{1}{\gamma_k} \right) |\mathbf{h}_k \mathbf{w}_k|^2. \quad (6.17)$$

Since a phase rotation to the beamforming vector $\mathbf{w}_k$ does not change the achievable rate for user k , without loss of generality, we assume that $\mathbf{h}_k \mathbf{w}_k$ is a non-negative real number for all k . As a result, (6.16) can be equivalently written as

$$\begin{aligned} \min_{\tilde{\rho}, \{\mathbf{w}_k\}} \quad & \tilde{\rho} \\ \text{s.t.} \quad & \left\| \begin{bmatrix} \mathbf{h}_k \mathbf{W} \\ 1 \end{bmatrix}^T \right\|_2 \leq \sqrt{1 + \frac{1}{\gamma_k} (\mathbf{h}_k \mathbf{w}_k)}, \quad \forall k \\ & \Im\{\mathbf{h}_k \mathbf{w}_k\} = 0, \quad \forall k \\ & \left\| \text{vec}(\mathbf{M}_j) \right\|_2 \leq \tilde{\rho} \sqrt{P}, \quad \forall j, \end{aligned} \quad (6.18)$$

where $\rho = \tilde{\rho}^2$, $\mathbf{W} \triangleq [\mathbf{w}_1 \ \mathbf{w}_2 \ \dots \ \mathbf{w}_{K_r}] \in \mathbb{C}^{M \times K_r}$, and $\mathbf{M}_j \triangleq [\mathbf{w}_1^{[j]} \ \mathbf{w}_2^{[j]} \ \dots \ \mathbf{w}_{K_r}^{[j]}] \in \mathbb{C}^{N_t \times K_r}$. (6.18) is an SOCP problem, which can be efficiently solved [65].

6.3.3 Power Minimization for Multi-Antenna Receivers

When each MS is equipped with multiple antennas ($N_r \geq 2$), no convex formulation for the power minimization problem (6.10) is known. In this subsection, we propose an efficient algorithm to find an approximate solution to (6.10) based on convexifying the non-convex rate constraint in (6.10). Intuitively, to achieve a good performance, the interference leakage should be small as compared with the noise and the desired signal power, i.e., $\|\mathbf{H}_k(\sum_{i \neq k}^{K_r} \mathbf{S}_i)\mathbf{H}_k^H\|_F$ should be a small number. Therefore, the following first-order Taylor approximation of the log-determinant function is made

$$\log \left| \mathbf{I} + \mathbf{H}_k \left(\sum_{i \neq k}^{K_r} \mathbf{S}_i \right) \mathbf{H}_k^H \right| \approx \text{Tr} \left[\mathbf{H}_k \left(\sum_{i \neq k}^{K_r} \mathbf{S}_i \right) \mathbf{H}_k^H \right] = \text{Tr} \left[\mathbf{H}_k^H \mathbf{H}_k \left(\sum_{i \neq k}^{K_r} \mathbf{S}_i \right) \right] \quad (6.19)$$

$$\begin{aligned} & \log \left| \mathbf{I} + \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H + \mathbf{H}_k \left(\sum_{i \neq k}^{K_r} \mathbf{S}_i \right) \mathbf{H}_k^H \right| \\ & \approx \log \left| \mathbf{I} + \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H \right| + \text{Tr} \left[\left(\mathbf{I} + \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H \right)^{-1} \mathbf{H}_k \left(\sum_{i \neq k}^{K_r} \mathbf{S}_i \right) \mathbf{H}_k^H \right] \\ & \approx \log \left| \mathbf{I} + \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H \right| + \text{Tr} \left[\mathbf{H}_k^H \left(\mathbf{I} + \mathbf{H}_k \mathbf{S}_k^{\text{BD}} \mathbf{H}_k^H \right)^{-1} \mathbf{H}_k \left(\sum_{i \neq k}^{K_r} \mathbf{S}_i \right) \right], \end{aligned} \quad (6.20)$$

where in both (6.19) and (6.20), we have used that the gradient of the log-determinant function $\log |\mathbf{X}|$ is $\mathbf{X}^{-1}$ [194, 195]. The approximation (6.19) is performed around the point $\mathbf{X} = \mathbf{I}$, and (6.20) is performed around the point $\mathbf{X} = \mathbf{I} + \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H$, both taking the interference term $\mathbf{H}_k(\sum_{i \neq k}^{K_r} \mathbf{S}_i)\mathbf{H}_k^H$ as the differential incremental. Furthermore, in (6.20), the gradient of the log-determinant function at the point $\mathbf{I} + \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H$ has been approximated by $(\mathbf{I} + \mathbf{H}_k \mathbf{S}_k^{\text{BD}} \mathbf{H}_k^H)^{-1}$. With

(6.19) and (6.20), problem (6.10) is now approximated as

$$\begin{aligned}
& \max_{\rho, \{\mathbf{S}_k\}} \quad -\rho \\
& \text{s.t.} \quad \log|\mathbf{I} + \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H| \geq \text{Tr}\left(\mathbf{F}_k \sum_{i \neq k}^{K_r} \mathbf{S}_i\right) + R_k^{\text{BD}}, \quad \forall k \\
& \sum_{k=1}^{K_r} \text{Tr}(\mathbf{B}_j \mathbf{S}_k) \leq \rho P, \quad \forall j \\
& \mathbf{S}_k \succeq \mathbf{0}, \quad \forall k,
\end{aligned} \tag{6.21}$$

where $\mathbf{F}_k = \mathbf{H}_k^H [\mathbf{I} - (\mathbf{I} + \mathbf{H}_k \mathbf{S}_k^{\text{BD}} \mathbf{H}_k^H)^{-1}] \mathbf{H}_k$. It is easy to verify that $\{\rho = 1, \{\mathbf{S}_k^{\text{BD}}\}\}$ is feasible for problem (6.21); therefore, the optimal power factor ρ^{**} of (6.21) satisfies $\rho^{**} \leq 1$. (6.21) is a convex optimization problem, which can be solved by the standard technique such as the interior-point method [65]. Next, we propose a dual decomposition based approach to obtain a semi-closed form solution of (6.21).

Denote the set of dual variables of problem (6.21) by $\{\lambda_k \geq 0\}$ and $\{\mu_j \geq 0\}$, which are associated with the first and second constraints of (6.21), respectively. Then the Lagrangian function of (6.21) can be written as

$$\begin{aligned}
& L(\rho, \{\mathbf{S}_k\}, \{\lambda_k\}, \{\mu_j\}) \\
& = -\rho + \sum_{k=1}^{K_r} \lambda_k \left[\log|\mathbf{I} + \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H| - \text{Tr}\left(\mathbf{F}_k \sum_{i \neq k}^{K_r} \mathbf{S}_i\right) - R_k^{\text{BD}} \right] + \sum_{j=1}^{K_t} \mu_j \left[\rho P - \sum_{k=1}^{K_r} \text{Tr}(\mathbf{B}_j \mathbf{S}_k) \right] \\
& = \rho \left(P \sum_{j=1}^{K_t} \mu_j - 1 \right) + \sum_{k=1}^{K_r} \left[\lambda_k \log|\mathbf{I} + \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H| - \text{Tr}(\mathbf{C}_k \mathbf{S}_k) - \lambda_k R_k^{\text{BD}} \right],
\end{aligned} \tag{6.22}$$

where $\mathbf{C}_k \triangleq \sum_{j=1}^{K_t} \mu_j \mathbf{B}_j + \sum_{i=1, i \neq k}^{K_r} \lambda_i \mathbf{F}_i$. The Lagrangian dual objective function is then given as

$$\begin{aligned}
g(\{\lambda_k\}, \{\mu_j\}) & = \max_{\mathbf{S}_k \succeq \mathbf{0}, \forall k} \max_{\rho} L(\rho, \{\mathbf{S}_k\}, \{\lambda_k\}, \{\mu_j\}) \\
& = \begin{cases} \max_{\mathbf{S}_k \succeq \mathbf{0}, \forall k} \tilde{L}(\{\mathbf{S}_k\}, \{\lambda_k\}, \{\mu_j\}), & \text{if } \sum_{j=1}^{K_t} \mu_j = 1/P \\ \infty, & \text{otherwise} \end{cases}
\end{aligned} \tag{6.23}$$

where $\tilde{L}(\{\mathbf{S}_k\}, \{\lambda_k\}, \{\mu_j\}) \triangleq \sum_{k=1}^{K_r} \left[\lambda_k \log|\mathbf{I} + \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H| - \text{Tr}(\mathbf{C}_k \mathbf{S}_k) - \lambda_k R_k^{\text{BD}} \right]$. Note that since $L(\rho, \{\mathbf{S}_k\}, \{\lambda_k\}, \{\mu_j\})$ is an affine function of ρ , the dual function $g(\{\lambda_k\}, \{\mu_j\})$ is finite only when $\sum_{j=1}^{K_t} \mu_j = 1/P$, as given in (6.23). Since the dual variables should be chosen such that the Lagrangian dual function is bounded, this

imposes an equality constraint on the dual optimization problem, which is

$$\begin{aligned}
& \min_{\{\lambda_k\}, \{\mu_j \geq 0\}} g(\{\lambda_k\}, \{\mu_j\}) \\
& \text{s.t.} \quad \sum_{j=1}^{K_t} \mu_j = 1/P \\
& \quad \mu_j \geq 0, \quad j = 1, \dots, K_t \\
& \quad \lambda_k \geq 0, \quad k = 1, \dots, K_r.
\end{aligned} \tag{6.24}$$

Assume that the Slater's condition is satisfied by the convex problem (6.21) [65], then the duality gap between the optimal objective function value of (6.21) and that of its dual (6.24) is zero. Thus, the optimal solution can be obtained by simultaneously updating the primal variables, $\{\mathbf{S}_k\}$, and the dual variables $\{\lambda_k\}$ and $\{\mu_j\}$. For a given set of feasible dual variables $\{\lambda_k\}$ and $\{\mu_j\}$, the optimal primal variables $\{\mathbf{S}_k^*\}$ can be obtained by solving the maximization problem (6.23). With $\{\mathbf{S}_k^*\}$ obtained, the dual variables $\{\lambda_k\}$ and $\{\mu_j\}$ can be updated with subgradient-based method [70–73].

6.3.3.1 Primal Update

We firstly focus on solving $\{\mathbf{S}_k^*\}$ with a given set of dual variables $\{\lambda_k\}$ and $\{\mu_j\}$. It can be observed from (6.23) that the maximization of $\tilde{L}(\{\mathbf{S}_k\}, \{\lambda_k\}, \{\mu_j\})$ over $\{\mathbf{S}_k\}$ can be decoupled into K_r parallel sub-problems, each solving for one $\mathbf{S}_k$. By discarding the irrelevant terms, the sub-problem for solving $\mathbf{S}_k$, given $\{\lambda_k\}$ and $\{\mu_j\}$, is written as

$$\max_{\mathbf{S}_k \succeq \mathbf{0}} \lambda_k \log |\mathbf{I} + \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H| - \text{Tr}(\mathbf{C}_k \mathbf{S}_k). \tag{6.25}$$

Lemma 6.2. *For problem (6.25) to have a bounded objective value, the dual variables $\{\lambda_k\}$ and $\{\mu_j\}$ should have values such that $\mathbf{C}_k \triangleq \sum_{j=1}^{K_t} \mu_j \mathbf{B}_j + \sum_{i=1, i \neq k}^{K_r} \lambda_i \mathbf{F}_i$ is positive definite, i.e., $\mathbf{C}_k \succ \mathbf{0}$.*

Proof. Please refer to Appendix H. □

With Lemma 6.2, $\mathbf{C}_k$ can be decomposed as $\mathbf{C}_k = \mathbf{C}_k^{1/2} \mathbf{C}_k^{1/2}$, where $\mathbf{C}_k^{1/2}$ is the Hermitian square root of $\mathbf{C}_k$, which is nonsingular. Furthermore, $\text{Tr}(\mathbf{C}_k \mathbf{S}_k) =$

$\text{Tr}(\mathbf{C}_k^{1/2} \mathbf{S}_k \mathbf{C}_k^{1/2})$. Define $\tilde{\mathbf{S}}_k \triangleq \mathbf{C}_k^{1/2} \mathbf{S}_k \mathbf{C}_k^{1/2}$, then $\mathbf{S}_k = \mathbf{C}_k^{-1/2} \tilde{\mathbf{S}}_k \mathbf{C}_k^{-1/2}$. Problem (6.25) is then equivalent to

$$\max_{\tilde{\mathbf{S}}_k \succeq \mathbf{0}} \lambda_k \log |\mathbf{I} + \mathbf{H}_k \mathbf{C}_k^{-1/2} \tilde{\mathbf{S}}_k \mathbf{C}_k^{-1/2} \mathbf{H}_k^H| - \text{Tr}(\tilde{\mathbf{S}}_k). \quad (6.26)$$

Express the (reduced) SVD of $\mathbf{H}_k \mathbf{C}_k^{-1/2}$ as $\mathbf{H}_k \mathbf{C}_k^{-1/2} = \bar{\mathbf{U}}_k \bar{\mathbf{\Sigma}}_k \bar{\mathbf{V}}_k^H$, where $\bar{\mathbf{U}}_k \in \mathbb{C}^{N_r \times N_r}$, $\bar{\mathbf{V}}_k \in \mathbb{C}^{M \times N_r}$ and $\bar{\mathbf{\Sigma}}_k = \text{Diag}(\bar{\sigma}_{k,1}, \dots, \bar{\sigma}_{k,N_r})$. Then (6.26) is equivalent to

$$\max_{\tilde{\mathbf{S}}_k \succeq \mathbf{0}} \lambda_k \log |\mathbf{I} + \bar{\mathbf{V}}_k^H \tilde{\mathbf{S}}_k \bar{\mathbf{V}}_k \bar{\mathbf{\Sigma}}_k^2| - \text{Tr}(\tilde{\mathbf{S}}_k). \quad (6.27)$$

Applying the Hadamard's inequality [162], the optimal solution to problem (6.27) and hence that to (6.26) is given as $\tilde{\mathbf{S}}_k^* = \bar{\mathbf{V}}_k \mathbf{D}_k \bar{\mathbf{V}}_k^H$, where $\mathbf{D}_k = \text{Diag}(d_{k,1}, \dots, d_{k,N_r})$ with $d_{k,s}$ obtained by standard water-filling solution [162]

$$d_{k,s} = \left(\lambda_k - \frac{1}{\bar{\sigma}_{k,s}^2} \right)^+, \quad s = 1, \dots, N_r, \quad (6.28)$$

where $(x)^+ \triangleq \max(0, x)$. Then the solution to (6.25) is given as

$$\mathbf{S}_k^* = \mathbf{C}_k^{-1/2} \bar{\mathbf{V}}_k \mathbf{D}_k \bar{\mathbf{V}}_k^H \mathbf{C}_k^{-1/2}. \quad (6.29)$$

The expression in (6.29) gives the optimal solution to problem (6.21) for any given dual variables $\{\lambda_k\}$ and $\{\mu_j\}$. The optimal solution $\{\mathbf{S}_k^{**}\}$ to (6.21) is achieved when the dual variables are optimal, i.e.,

$$\mathbf{S}_k^{**} = \mathbf{C}_k^{*-1/2} \bar{\mathbf{V}}_k \mathbf{D}_k \bar{\mathbf{V}}_k^H \mathbf{C}_k^{*-1/2} \quad (6.30)$$

where $\mathbf{C}_k^*$ corresponds to the optimal dual variables $\{\lambda_k^*\}$ and $\{\mu_j^*\}$, which can be obtained via subgradient-based method as will be discussed in the following subsection.

Remark 6.3. Note that since $\bar{\mathbf{V}}_k \in \mathbb{C}^{M \times N_r}$, we have $\text{rank}\{\mathbf{S}_k^{**}\} \leq N_r$. As can be seen from (6.7), the rank of the transmit covariance matrix $\mathbf{S}_k^{**}$ determines the number of data streams for user k . In other words, with the proposed solution, each user will receive no more than N_r data streams, which is expected since there are only N_r antennas at the receiver.

With $\{\mathbf{S}_k^{**}\}$ obtained, the power reduction factor ρ^{**} to (6.21) can be calculated as

$$\rho^{**} = \max_{j=1, \dots, K_t} \frac{1}{P} \sum_{k=1}^{K_r} \text{Tr}(\mathbf{B}_j \mathbf{S}_k^{**}). \quad (6.31)$$

6.3.3.2 Dual Update

We will now consider the dual optimization problem (6.24), where the dual variables $\{\lambda_k\}$ and $\{\mu_j\}$ can be updated by applying the subgradient method based on the optimal solution $\{\mathbf{S}_k^*\}$ to (6.23). To remove the equality constraint in the dual problem (6.24), we substitute with $\mu_{K_t} = \frac{1}{P} - \sum_{j=1}^{K_t-1} \mu_j$ so that the number of dual variables is reduced by 1. With (6.22), the resulting dual objective function, now denoted as $\tilde{g}(\{\lambda_k\}, \{\mu_j\}_{j=1}^{K_t-1})$, is given by

$$\begin{aligned} \tilde{g}(\{\lambda_k\}, \{\mu_j\}_{j=1}^{K_t-1}) = \max_{\mathbf{S}_k \succeq \mathbf{0}, \forall k} & \left\{ \sum_{k=1}^{K_r} \lambda_k [\log |\mathbf{I} + \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H| - \text{Tr}(\mathbf{F}_k \sum_{i \neq k}^{K_r} \mathbf{S}_i) - R_k^{BD}] \right. \\ & \left. + \sum_{j=1}^{K_t-1} \mu_j [(\mathbf{B}_{K_t} - \mathbf{B}_j) \sum_{k=1}^{K_r} \mathbf{S}_k] - \frac{1}{P} \sum_{k=1}^{K_r} \text{Tr}(\mathbf{B}_{K_t} \mathbf{S}_k) \right\} \end{aligned} \quad (6.32)$$

Then problem (6.24) is equivalent to

$$\begin{aligned} \min_{\{\lambda_k\}, \{\mu_j\}_{j=1}^{K_t-1}} & \quad \tilde{g}(\{\lambda_k\}, \{\mu_j\}_{j=1}^{K_t-1}) \\ \text{s.t.} & \quad \sum_{j=1}^{K_t-1} \mu_j \leq 1/P \\ & \quad \mu_j \geq 0, \quad j = 1, \dots, K_t - 1 \\ & \quad \lambda_k \geq 0, \quad k = 1, \dots, K_r. \end{aligned} \quad (6.33)$$

The subgradient of (6.33) is obtained by using the following Lemma.

Lemma 6.4. *With the primal solution $\{\mathbf{S}_k^*\}$ obtained by (6.29) for a given set of dual variables $\{\lambda_k\}$ and $\{\mu_j\}$, the subgradient of $\tilde{g}(\{\lambda_k\}, \{\mu_j\}_{j=1}^{K_t-1})$ is given by*

$$s_{\lambda_k} = \log |\mathbf{I} + \mathbf{H}_k \mathbf{S}_k^* \mathbf{H}_k^H| - \text{Tr}(\mathbf{F}_k \sum_{i \neq k}^{K_r} \mathbf{S}_i^*) - R_k^{BD}, \quad k = 1, \dots, K_r \quad (6.34)$$

$$s_{\mu_j} = \sum_{k=1}^{K_r} \text{Tr}[(\mathbf{B}_{K_t} - \mathbf{B}_j) \mathbf{S}_k^*], \quad j = 1, \dots, K_t - 1 \quad (6.35)$$

Proof. Please refer to Appendix I. □

With the subgradient determined, the dual variables for problem (6.33) can be updated with the subgradient-based methods, such as the ellipsoid method [74].

6.3.3.3 Primal-Dual Method for Power Minimization

The proposed algorithm for solving the power minimization problem (6.21) is summarized as follows:

Algorithm 9 Primal-Dual Method for Solving (6.21)

- 1: Initialize $\lambda_k \geq 0$, $k = 1, \dots, K_r$, $\mu_j \geq 0$, $j = 1, \dots, K_t - 1$, and $\sum_{j=1}^{K_t-1} \mu_j \leq 1/P$.
 - 2: **repeat**
 - 3: With $\{\lambda_k\}$ and $\{\mu_j\}_{j=1}^{K_t}$, where $\mu_{K_t} = 1/P - \sum_{j=1}^{K_t-1} \mu_j$, solve for $\{\mathbf{S}_k^*\}$ using (6.29).
 - 4: Compute the subgradient of $\tilde{g}(\{\lambda_k\}, \{\mu_j\}_{j=1}^{K_t-1})$ with Lemma 6.4, then update $\{\lambda_k\}$ and $\{\mu_j\}_{j=1}^{K_t-1}$ accordingly based on the ellipsoid method [74].
 - 5: **until** $\{\lambda_k\}_{k=1}^{K_r}$ and $\{\mu_j\}_{j=1}^{K_t-1}$ converge to a prescribed accuracy.
 - 6: Then $\{\mathbf{S}_k^*\}$ approaches to the optimal solution $\{\mathbf{S}_k^{**}\}$. Obtain the power factor ρ^{**} using (6.31).
-

6.3.4 Algorithm Summary

Based on the previous discussions, the proposed modified BD scheme is summarized in Algorithm 10.

Algorithm 10 The Proposed Modified BD Scheme

- 1: Obtain the optimal BD precoding by solving the convex optimization problem (6.9), and denote the achieved rates by $\{R_k^{\text{BD}}\}$.
 - 2: If $N_r = 1$, solve the power minimization problem (6.10) via the SOCP problem (6.18); otherwise, obtain an approximate solution to (6.10) with Algorithm 9. Denote the solution as $\{\mathbf{S}_k^{**}\}$ and ρ^{**} .
 - 3: Then the proposed transmit covariance matrices are given by $\{\mathbf{S}_k^{\text{prop}} = \mathbf{S}_k^{**}/\rho^{**}\}$.
-

FIGURE 6.1: Convergence behavior of Algorithm 9. The dashed line shows the achievable sum-rate with the optimal BD scheme.

Remark 6.5. It can be seen from Algorithm 10 that the proposed modified BD scheme requires solving two convex optimization problems only, and hence is quite efficient.

6.4 Numerical results

In this section, numerical results are presented to evaluate the performance of the proposed algorithm. Since the noise power has been normalized to unity, the system SNR is defined as $\text{SNR} \triangleq P$, where P is the maximum available power for each BS.

6.4.1 Convergence of the Proposed Algorithm

The convergence behavior of Algorithm 9 is illustrated with a setup specified by $[K_t \ N_t \ K_r \ N_r] = [3 \ 2 \ 3 \ 2]$ at $\text{SNR} = 10$ dB. The initial values of the dual variables are assigned with $\mu_j = \frac{1}{P K_t}$, $j = 1, \dots, K_t$, and $\lambda_k = 0.1$, $k = 1, \dots, K_r$. Algorithm 9 is terminated when the volume of the ellipsoid containing the dual

FIGURE 6.2: Sum-rate comparison for a network with $[K_t \ N_t \ K_r \ N_r] = [3 \ 2 \ 3 \ 2]$.

variables is sufficiently small, i.e., when $\sqrt{\mathbf{s}^T \mathbf{E} \mathbf{s}} \leq 10^{-6}$, where $\mathbf{s}$ is the subgradient vector, and $\mathbf{E}$ is the positive definite matrix whose eigenvectors define the principal directions of the ellipsoid. A sequence of transmit covariance matrices $\{\mathbf{S}_k^*\}$ as given in (6.30) are generated by Algorithm 9 and the corresponding power reduction factor ρ^* is given by (6.31). The achieved sum rates with the scaled transmit covariance matrices $\{\mathbf{S}_k^*/\rho^*\}$ are plotted in Figure 6.1. As a comparison, the sum rate achievable with the optimal BD scheme is also plotted using the dashed line. It is observed that the algorithm eventually converges to a fixed sum rate with significantly improved performance over the optimal BD scheme. As pointed out in [52], the convergence speed of Algorithm 9 depends on the total number of dual variables $K_r + K_t - 1$. With the ellipsoid method, it is known that the complexity is of the order $\mathcal{O}[(K_r + K_t - 1)^2]$.

6.4.2 Sum-Rate Comparison

In this subsection, the achievable sum rate of the proposed modified BD scheme is compared against the optimal BD precoding obtained by solving (6.9), as well as two MMSE-based precoding schemes [21, 191], denoted respectively as “MMSE Zhang” and “MMSE Shi” in Figure 6.2. A network with $[K_t \ N_t \ K_r \ N_r] = [3 \ 2 \ 3 \ 2]$

FIGURE 6.3: Approximated effective SNR enhancement $1/\rho^{**}$ in dB, $[K_t \ N_t \ K_r \ N_r] = [3 \ 2 \ 3 \ 2]$.

is considered. The average sum rate over 10000 random channel realizations is plotted, where the channel coefficients are drawn from i.i.d CSCG random variables with zero mean and unit variance. It is observed that although the two MMSE-based schemes perform better than the BD scheme in the low-to-medium SNR regime, they fail to achieve the same sum-rate scaling as the BD scheme at high SNR. In contrast, the proposed modified BD precoding outperforms the optimal BD scheme across all SNR values, with more pronounced gain in the low-to-medium SNR regime.

6.4.3 Effective SNR Enhancement

To illustrate the enhancement of the effective SNR achieved by the proposed scheme, the average value of $1/\rho^{**}$ in dB is plotted in Figure 6.3, where ρ^{**} is the optimal value of the power minimization problem (6.21). As has been discussed in Section 6.3.1, $1/\rho^{**}$ provides an approximation of the effective SNR enhancement by the proposed modified BD precoding over the original BD scheme. As can be seen from Figure 6.3, a positive SNR enhancement is always achieved and the gain

is more pronounced in the low-to-medium SNR regime. Such results are consistent with the sum-rate performance observed in Figure 6.2.

6.5 Chapter Summary

In this chapter, we have proposed a modified BD scheme for MCP-enabled downlink networks. It has been shown that by relaxing the hard zero-interference constraint imposed by the original BD scheme, the effective noise power can be suppressed. This is achieved by solving a rate-constrained power minimization problem and applying properly scaled transmit covariance matrices at each transmitter. This technique provides a method to mitigate the transmit power boost problem associated with the original BD scheme, so that its low-to-medium SNR performance is improved and the high-SNR performance is preserved. Numerical results have been provided to verify the performance of the proposed scheme.

Chapter 7

Conclusions and Future Work

7.1 Conclusions

This thesis has optimized the transceiver designs in interference channels and MCP-enabled networks for rate improvement and DoF optimality.

For Gaussian interference channel with interference treated as noise, a novel achievable rate expression has been derived, which is applicable to the more general *improper* or circularly *asymmetric* complex Gaussian signaling. It has been found that compared with the conventional proper Gaussian signaling, the user's achievable rate with improper complex Gaussian signaling has one additional term, which is a function of both the users' transmit covariance and *pseudo-covariance* matrices. Such a separable rate structure provides a method to improve the users' achievable rates over the conventional proper Gaussian signaling, by optimizing the pseudo-covariance matrices to make the additional rate term strictly positive. The technique of *widely linear precoding* has been introduced to generate improper Gaussian signals for any given valid pair of covariance and pseudo-covariance matrices. Efficient algorithms have been proposed to optimize the transmit covariance and pseudo-covariance matrices for the two-user SISO-IC and K -user MISO-IC to achieve Pareto-optimal rates with improper Gaussian signaling. For K -user MISO-IC, the optimality of rank-1 pseudo-covariance matrices has been established. Furthermore, each of such rank-1 pseudo-covariance matrices can be parameterized by one single variable; hence the number of variables for pseudo-covariance optimization is significantly reduced.

The Gaussian interference channel has also been studied from the DoF perspective. For the 3-user rank-deficient MIMO-IC specified by $(M_T, M_R, D_0, D_1, D_2)$, the complete DoF characterization has been obtained. The outer bound is derived with the genie-aided signaling technique based on the developed equivalent layered channel model. For the achievability, a linear scheme is proposed. When the interfering links are weak, i.e., $D_1 + D_2 \leq \min(M_T, M_R)$, it has been found that zero forcing is sufficient to achieve the optimal DoF; whereas when $D_1 + D_2 > \min(M_T, M_R)$, a combination of zero forcing and interference alignment is required for DoF optimality. The DoF characterization obtained in this thesis provides a unification on several existing results in the literature.

Under the more aggressive MCP-enabled network, a modified BD precoding with improved rate performance in the low-to-medium SNR regime has been proposed. In contrast to the original BD precoding where the hard constraint of zero inter-user interference is imposed, in the modified scheme, inter-user interference is allowed and carefully controlled via a rate-constrained power minimization problem. It has been shown that by properly scaling the obtained transmit covariance matrices, the effective noise power can be reduced and hence the effective SNR is enhanced. This thus mitigates the transmit power boost problem of the original BD scheme and hence a rate gain is attained in the low-to-medium SNR regime.

7.2 Future Work

Some future research directions that may extend the results presented in this thesis are given in the following.

Regarding to improper complex Gaussian signaling, the following studies are promising:

- improper signaling with the more practical finite discrete inputs [196–200]; instead of the continuous Gaussian signaling discussed in this thesis.
- the achievable rate region optimization for the more general K -user MIMO-IC.
- covariance and pseudo-covariance optimization for (weighted) sum-rate maximization.

- improper Gaussian signaling by including symbol extensions and interference alignment.
- transceiver design for improper Gaussian signaling with imperfect CSI, for which the transmitters either only have statistical channel knowledge [201], or have estimates of the truce channels with bounded estimation errors [124, 202].

With respect to the DoF characterization of rank-deficient MIMO-IC, some interesting problems are:

- the DoF characterization for the 3-user rank-deficient MIMO-IC with more general settings, i.e., with arbitrary number of antennas at each terminal and arbitrary ranks for each channel matrix.
- the DoF study of rank-deficient MIMO-IC with more than 3 users.
- the DoF characterization for 3-user rank-deficient MIMO-IC with *correlated* channel matrices, an interference model that finds its application in obtaining the achievable rate regions for multi-hop networks with network coding [181].
- the DoF study of rank-deficient MIMO-IC under limited feedback [203].

Author's Publications

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- [6] X. Xu, Y. Zeng, and Y. L. Guan, "Performance Analysis of Finite-Length Spatial-Temporal Network Coding," submitted to *IEEE Communications Letters*.

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- [1] Y. Zeng, C. M. Yetis, E. Gunawan, Y. L. Guan, and R. Zhang, "Improper Gaussian signaling for the K-user SISO interference channel," in *Proc. IEEE*

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Appendix A

Proof of Lemma 3.10

The derivations for (3.88) and (3.89) follow similar arguments. For brevity, we only show that of (3.88) as follows:

$$\begin{aligned} |\tilde{C}_{y_k}|^2 + \sigma^4 &\leq (|\tilde{C}_{y_k}| + \sigma^2)^2 = \left(|h_{k1}^2 \tilde{C}_{x_1} + h_{k2}^2 \tilde{C}_{x_2}| + \sigma^2 \right)^2 \\ &\stackrel{(a)}{\leq} \left(|h_{k1}|^2 |\tilde{C}_{x_1}| + |h_{k2}|^2 |\tilde{C}_{x_2}| + \sigma^2 \right)^2 \\ &\stackrel{(b)}{\leq} \left(|h_{k1}|^2 C_{x_1} + |h_{k2}|^2 C_{x_2} + \sigma^2 \right)^2 = C_{y_k}^2, \end{aligned}$$

where (a) follows from the triangle inequality, and (b) is true due to the second constraint in (3.87).

Appendix B

Proof of Theorem 3.11

For notational convenience, in this appendix, we use X_1 and X_2 to represent $\tilde{C}_{x_1}$ and $\tilde{C}_{x_2}$, respectively. First, the following proposition shows that to solve (3.109), we may consider exterior solutions only, i.e., the solutions at which at least one of the inequality constraints in (3.109) is active.

Proposition B.1. *If $\{X_1, X_2\}$ is feasible to (3.109) with $|X_1| < C_{x_1}^*$ and $|X_2| < C_{x_2}^*$, then there exists another feasible solution $\{X'_1, X'_2\}$ with $|X'_1| = C_{x_1}^*$ or $|X'_2| = C_{x_2}^*$.*

Proof. Let $\tau \triangleq \min \left\{ \frac{C_{x_1}^*}{|X_1|}, \frac{C_{x_2}^*}{|X_2|} \right\}$. Then $\tau > 1$. Define $X'_1 = \tau X_1$ and $X'_2 = \tau X_2$. Then the last two constraints in (3.109) are satisfied by X'_1 and X'_2 , i.e., $|X'_1| \leq C_{x_1}^*$ and $|X'_2| \leq C_{x_2}^*$. Furthermore, at least one of them is satisfied with equality. The first constraint of (3.109) is also satisfied by X'_1 and X'_2 since

$$\begin{aligned} a_1|h_{11}^2X'_1 + h_{12}^2X'_2|^2 + b_1 &= \tau^2 a_1|h_{11}^2X_1 + h_{12}^2X_2|^2 + b_1 \\ &\stackrel{(a)}{\leq} \tau^2(a_1|h_{11}^2X_1 + h_{12}^2X_2|^2 + b_1) \stackrel{(b)}{\leq} \tau^2|X_2|^2 = |X'_2|^2, \end{aligned}$$

where (a) follows from $\tau > 1$ and $b_1 \geq 0$, and (b) is true since $\{X_1, X_2\}$ is feasible to (3.109). Similarly, the second constraint of (3.109) is also satisfied by X'_1 and X'_2 . Therefore, $\{X'_1, X'_2\}$ is a feasible solution to (3.109) with at least one of the inequality constraints being active. $\square$

Next, we derive Theorem 3.11 using the Karush-Kuhn-Tucker (KKT) conditions, which are necessary optimality conditions for the constrained optimization problem (3.109) [65]. For notational convenience, denote the inequality constraints in (3.109) by $f_1 \leq 0, f_2 \leq 0, h_1 \leq 0$ and $h_2 \leq 0$, respectively. Denote $\lambda_1, \lambda_2, \mu_1, \mu_2$ as the corresponding dual variables, respectively. The Lagrangian function of

(3.109) is given by

$$\begin{aligned}
L(X_1, X_2, \lambda_1, \lambda_2, \mu_1, \mu_2) = & \lambda_1 \left\{ a_1 (|h_{11}|^4 X_1^* X_1 + |h_{12}|^4 X_2^* X_2 \right. \\
& + 2\Re\{h_{11}^{2*} h_{12}^2 X_1^* X_2\}) + b_1 - X_2^* X_2 \Big\} + \lambda_2 \left\{ a_2 (|h_{21}|^4 X_1^* X_1 \right. \\
& + |h_{22}|^4 X_2^* X_2 + 2\Re\{h_{21}^{2*} h_{22}^2 X_1^* X_2\}) + b_2 - X_1^* X_1 \Big\} \\
& + \mu_1 (X_1^* X_1 - C_{x_1}^{*2}) + \mu_2 (X_2^* X_2 - C_{x_2}^{*2}).
\end{aligned}$$

For $\{X_1, X_2\}$ to be a solution to (3.109), the following KKT conditions must be satisfied:

1. Dual feasibility: $\lambda_1 \geq 0, \lambda_2 \geq 0, \mu_1 \geq 0, \mu_2 \geq 0$.
2. Zero derivative: The derivatives of the Lagrangian function (B.1) are zero [195]:

$$\begin{aligned}
\frac{\partial L}{\partial X_2^*} = 0 \Rightarrow & -X_2 \underbrace{[(a_1 |h_{12}|^4 - 1)\lambda_1 + \lambda_2 a_2 |h_{22}|^4 + \mu_2]}_{\triangleq c_2} \\
= & X_1 (\lambda_1 \underbrace{a_1 h_{12}^{2*} h_{11}^2}_{\triangleq V_1} + \lambda_2 \underbrace{a_2 h_{22}^{2*} h_{21}^2}_{\triangleq V_2})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial X_1^*} = 0 \Rightarrow & X_1 \underbrace{[(a_2 |h_{21}|^4 - 1)\lambda_2 + \lambda_1 a_1 |h_{11}|^4 + \mu_1]}_{\triangleq c_1} \\
= & -X_2 (\lambda_1 \underbrace{a_1 h_{11}^{2*} h_{12}^2}_{=V_1^*} + \lambda_2 \underbrace{a_2 h_{21}^{2*} h_{22}^2}_{=V_2^*}),
\end{aligned}$$

$\Downarrow$

$$-X_2 c_2 = X_1 (\lambda_1 V_1 + \lambda_2 V_2), \tag{B.1}$$

$$X_1 c_1 = -X_2 (\lambda_1 V_1^* + \lambda_2 V_2^*). \tag{B.2}$$

3. Complementary slackness: $\lambda_1 f_1 = 0, \lambda_2 f_2 = 0, \mu_1 h_1 = 0, \mu_2 h_2 = 0$.

As have been discussed previously, without loss of generality, X_1 can be assumed to be a nonnegative real number. The following cases are then considered to derive the possible phases θ of X_2 :

- Case I: $f_1 = 0$ and $f_2 \neq 0$. Then from the complementary slackness condition, we have $\lambda_1 > 0$ and $\lambda_2 = 0$. Substituting them into (B.2), we have

$$X_2 = -\frac{X_1(\lambda_1 a_1 |h_{11}|^4 + \mu_1)}{\lambda_1 |V_1|^2} V_1.$$

Since $\lambda_1 \geq 0$, $\mu_1 \geq 0$, $a_1 \geq 0$ and $X_1 \geq 0$, the phase θ of X_2 equals to that of V_1 rotated by π , which is $\pi + 2(\phi_{11} - \phi_{12})$ since $V_1 = a_1 h_{12}^{2*} h_{11}^2$.

- Case II: $f_1 \neq 0$ and $f_2 = 0$. Then $\lambda_1 = 0$ and $\lambda_2 > 0$. Similarly, by using (B.1), we have $\theta = \pi + 2(\phi_{21} - \phi_{22})$.
- Case III: $f_1 \neq 0$ and $f_2 \neq 0$, then $\lambda_1 = 0$ and $\lambda_2 = 0$. By substituting them into (B.1) and (B.2), we have $\mu_2 X_2 = 0$ and $\mu_1 X_1 = 0$. If $X_2 = 0$, then θ can be any arbitrary value. If $X_1 = 0$, then $X_2 = 0$ is implied due to the second constraint in (3.109) and again, θ is arbitrary. Therefore, we may focus on $\mu_1 = 0$ and $\mu_2 = 0$ only. However, Proposition B.1 suggests that we may consider the exterior solutions only, i.e., either $h_1 = 0$ or $h_2 = 0$ is satisfied. Thus, $\mu_1 > 0$ or $\mu_2 > 0$ can be assumed. Therefore, case III can be ignored without loss of optimality.
- Case IV: $f_1 = 0$ and $f_2 = 0$, then θ belongs to the solution set for the equations given in (3.111) and (3.112), which are obtained by satisfying the constraints f_1 and f_2 with equality. (3.111) and (3.112) correspond to $|X_1| = C_{x_1}^*$ and $|X_2| = C_{x_2}^*$, respectively, which can be assumed without loss of generality due to Proposition B.1.

This completes the proof of Theorem 3.11.

Appendix C

Solving $\Theta_{\mathcal{A}}$ and $\Theta_{\mathcal{B}}$ in Theorem 3.11

In this appendix, we show the steps to solve $\Theta_{\mathcal{A}}$ of Theorem 3.11, while the solution for $\Theta_{\mathcal{B}}$ can be obtained similarly and thus the details are omitted for brevity. Note that the unknown variables in (3.111) are θ and t . After some manipulations, (3.111) can be written as

$$t \cos \eta + d_1 t^2 + d_2 = 0 \quad (\text{C.1})$$

$$t \cos(\eta + \omega) + d_3 t^2 + d_4 = 0, \quad (\text{C.2})$$

where

$$\omega \triangleq 2(\phi_{22} + \phi_{11} - \phi_{12} - \phi_{21}), \quad \eta \triangleq \theta + 2(\phi_{12} - \phi_{11}), \quad (\text{C.3})$$

$$\begin{aligned} d_1 &\triangleq \frac{a_1 |h_{12}|^4 - 1}{2a_1 |h_{11}|^2 |h_{12}|^2 C_{x_1}^*}, \quad d_2 \triangleq \frac{a_1 |h_{11}|^4 C_{x_1}^{*2} + b_1}{2a_1 |h_{11}|^2 |h_{12}|^2 C_{x_1}^*}, \\ d_3 &\triangleq \frac{|h_{22}|^2}{2|h_{21}|^2 C_{x_1}^*}, \quad d_4 = \frac{(a_2 |h_{21}|^4 - 1) C_{x_1}^{*2} + b_2}{2a_2 |h_{21}|^2 |h_{22}|^2 C_{x_1}^*}. \end{aligned}$$

From (C.2), we have

$$\begin{aligned} t \sin \eta \sin \omega &= t \cos \eta \cos \omega + d_3 t^2 + d_4 \Rightarrow \\ t^2 (1 - \cos^2 \eta) \sin^2 \omega &= (t \cos \eta \cos \omega + d_3 t^2 + d_4)^2 \end{aligned} \quad (\text{C.4})$$

Solving $\cos \eta$ from (C.1), we have

$$\cos \eta = -(d_1 t^2 + d_2)/t. \quad (\text{C.5})$$

Substituting (C.5) into (C.4) gives the following fourth order polynomial equation with respect to t :

$$[t^2 - (d_1 t^2 + d_2)^2] \sin^2 \omega = [(d_3 - d_1 \cos \omega) t^2 + d_4 - d_2 \cos \omega]^2.$$

Since the above equation only has terms involving t^2 , it can be transformed to the following quadratic equation by setting $z = t^2$, i.e.,

$$e_1 z^2 + e_2 z + e_3 = 0, \tag{C.6}$$

where

$$\begin{aligned} e_1 &= d_3^2 + d_1^2 - 2d_1 d_3 \cos \omega, \\ e_2 &= 2(d_1 d_2 + d_3 d_4) - 2(d_1 d_4 + d_2 d_3) \cos \omega - \sin^2 \omega, \\ e_3 &= d_2^2 + d_4^2 - 2d_2 d_4 \cos \omega. \end{aligned}$$

Then z can be easily solved. Since $z = t^2$ and t is the magnitude of $\tilde{C}_{x_2}$, only the solutions of z that are real and satisfy $0 \leq z \leq C_{x_2}^{\star 2}$ need to be kept, based on which the values for t are obtained. For those values of t satisfying $|(d_1 t^2 + d_2)/t| \leq 1$, we can get the value for η based on (C.5), i.e., $\eta = \arccos[-(d_1 t^2 + d_2)/t]$ or $\eta = 2\pi - \arccos[-(d_1 t^2 + d_2)/t]$. Then θ can be obtained from (C.3). If no such solutions exist, then $\Theta_{\mathcal{A}}$ is set to empty. It can be seen that the cardinality of the set $\Theta_{\mathcal{A}}$ is upper bounded by 4.

Appendix D

An Upper Bound R_U for (3.108)

In this appendix, we derive an upper bound R_U for the optimal solution R^* to the pseudo-covariance optimization problem (3.108). Based on our discussions in Section 4.2, the target rate R should be feasible to problem (3.109). From the first and fourth constraints of (3.109), it can be seen that the following inequalities should be satisfied:

$$b_1 \leq a_1 |h_{11}^2 \tilde{C}_{x_1} + h_{12}^2 \tilde{C}_{x_2}|^2 + b_1 \leq |\tilde{C}_{x_2}|^2 \leq C_{x_2}^{*2}, \quad (\text{D.1})$$

where $b_1 = \frac{(1-1/\beta_1)C_{s_1}^2}{|h_{12}|^4}$ and $\beta_1 = 2^{2\alpha_1(R-r^*)}$. By substituting b_1 into (D.1), we have

$$\frac{1}{\beta_1} \geq 1 - \frac{|h_{12}|^4 C_{x_2}^{*2}}{C_{s_1}^2} = \frac{\sigma^4 + 2\sigma^2 |h_{12}|^2 C_{x_2}^*}{(|h_{12}|^2 C_{x_2}^* + \sigma^2)^2},$$

where for the last equality, we have used the identity $C_{s_1} = |h_{12}|^2 C_{x_2}^* + \sigma^2$. Then we have

$$\beta_1 \leq 1 + \frac{|h_{12}|^4 C_{x_2}^{*2}}{\sigma^2(\sigma^2 + 2|h_{12}|^2 C_{x_2}^*)}. \quad (\text{D.2})$$

By substituting β_1 into (D.2), we obtain

$$R \leq r^* + \frac{1}{2\alpha_1} \log \left(1 + \frac{|h_{12}|^4 C_{x_2}^{*2}}{\sigma^2(\sigma^2 + 2|h_{12}|^2 C_{x_2}^*)} \right). \quad (\text{D.3})$$

A similar upper bound can be obtained based on the second and third constraints of (3.109). Thus an upper bound R_U for the optimal value R^* of (3.108) is given

by

$$R_U = r^\star + \min \left\{ \frac{1}{2\alpha_1} \log \left(1 + \frac{|h_{12}|^4 C_{x_2}^{\star 2}}{\sigma^2(\sigma^2 + 2|h_{12}|^2 C_{x_2}^\star)} \right), \right. \\ \left. \frac{1}{2\alpha_2} \log \left(1 + \frac{|h_{21}|^4 C_{x_1}^{\star 2}}{\sigma^2(\sigma^2 + 2|h_{21}|^2 C_{x_1}^\star)} \right) \right\}. \quad (\text{D.4})$$

Appendix E

Proof of Lemma 4.2

The following result will be used for proving Lemma 4.2.

Lemma E.1. [65] *If $\mathbf{X}$ is Hermitian and is partitioned as $\mathbf{X} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^H & \mathbf{C} \end{bmatrix}$, then $\mathbf{X} \succeq \mathbf{0}$ if and only if the following three conditions are satisfied:*

$$(a) \mathbf{A} \succeq \mathbf{0}; (b) (\mathbf{I} - \mathbf{A}\mathbf{A}^\dagger)\mathbf{B} = \mathbf{0}; (c) \mathbf{C} - \mathbf{B}^H\mathbf{A}^\dagger\mathbf{B} \succeq \mathbf{0},$$

where $(\cdot)^\dagger$ represents the Moore-Penrose pseudo-inverse.

We now re-express the positive semidefinite constraint in (4.14) using the above three conditions in Lemma E.1. First, it is clear that (a) is guaranteed in (4.14). Next, to use the condition in (b), we express the singular value decomposition (SVD) of the rank-1 covariance matrix $\mathbf{C}_{\mathbf{x}_k}^*$ as

$$\mathbf{C}_{\mathbf{x}_k}^* = \mathbf{t}_k \mathbf{t}_k^H = \begin{bmatrix} \tilde{\mathbf{t}}_k & \mathbf{U}_k \end{bmatrix} \begin{bmatrix} \|\mathbf{t}_k\|^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{t}}_k^H \\ \mathbf{U}_k^H \end{bmatrix},$$

where $\tilde{\mathbf{t}}_k = \mathbf{t}_k / \|\mathbf{t}_k\|$, and $\mathbf{U}_k \in \mathbb{C}^{M \times (M-1)}$ satisfies $\mathbf{U}_k^H \mathbf{U}_k = \mathbf{I}_{M-1}$, and $\tilde{\mathbf{t}}_k^H \mathbf{U}_k = \mathbf{0}$. Then the Moore-Penrose pseudo-inverse of $\mathbf{C}_{\mathbf{x}_k}^*$ can be obtained as

$$(\mathbf{C}_{\mathbf{x}_k}^*)^\dagger = \|\mathbf{t}_k\|^{-2} \tilde{\mathbf{t}}_k \tilde{\mathbf{t}}_k^H.$$

Applying the condition in (b) of Lemma E.1 to the positive semidefinite constraint in (4.14) yields

$$\begin{aligned} (\mathbf{I} - \mathbf{C}_{\mathbf{x}_k}^* (\mathbf{C}_{\mathbf{x}_k}^*)^\dagger) \tilde{\mathbf{C}}_{\mathbf{x}_k} = \mathbf{0} &\iff (\mathbf{I} - \tilde{\mathbf{t}}_k \tilde{\mathbf{t}}_k^H) \tilde{\mathbf{C}}_{\mathbf{x}_k} = \mathbf{0} \\ &\iff \tilde{\mathbf{C}}_{\mathbf{x}_k} = \tilde{\mathbf{t}}_k \mathbf{v}_k^H \end{aligned} \tag{E.1}$$

$$\iff \tilde{\mathbf{C}}_{\mathbf{x}_k} = \gamma_k \tilde{\mathbf{t}}_k \tilde{\mathbf{v}}_k^H, \tag{E.2}$$

where $\mathbf{v}_k \in \mathbb{C}^{M \times 1}$ is an arbitrary vector with its Euclidian norm denoted by γ_k and normalized direction by $\tilde{\mathbf{v}}_k$. To show both the “if” and “only if” conditions in (E.1), we first note that $\tilde{\mathbf{C}}_{\mathbf{x}_k} = \tilde{\mathbf{t}}_k \mathbf{v}_k^H$ is a solution of $(\mathbf{I} - \tilde{\mathbf{t}}_k \tilde{\mathbf{t}}_k^H) \tilde{\mathbf{C}}_{\mathbf{x}_k} = \mathbf{0}$. In addition, any solution $\tilde{\mathbf{C}}_{\mathbf{x}_k}$ of the equation $(\mathbf{I} - \tilde{\mathbf{t}}_k \tilde{\mathbf{t}}_k^H) \tilde{\mathbf{C}}_{\mathbf{x}_k} = \mathbf{0}$ must satisfy $\tilde{\mathbf{C}}_{\mathbf{x}_k} = \tilde{\mathbf{t}}_k (\tilde{\mathbf{C}}_{\mathbf{x}_k}^H \tilde{\mathbf{t}}_k)^H$, which confirms the existence of a vector $\mathbf{v}_k = \tilde{\mathbf{C}}_{\mathbf{x}_k}^H \tilde{\mathbf{t}}_k$ such that $\tilde{\mathbf{C}}_{\mathbf{x}_k} = \tilde{\mathbf{t}}_k \mathbf{v}_k^H$.

Since $\tilde{\mathbf{C}}_{\mathbf{x}_k}$ is a pseudo-covariance matrix, which must be symmetric, we have

$$\tilde{\mathbf{C}}_{\mathbf{x}_k} = \gamma_k \tilde{\mathbf{t}}_k \tilde{\mathbf{v}}_k^H = \gamma_k \tilde{\mathbf{v}}_k^* \tilde{\mathbf{t}}_k^T = \tilde{\mathbf{C}}_{\mathbf{x}_k}^T.$$

By expressing $\tilde{\mathbf{C}}_{\mathbf{x}_k} \tilde{\mathbf{C}}_{\mathbf{x}_k}^H$ using two alternative forms, we have

$$\begin{aligned} \tilde{\mathbf{C}}_{\mathbf{x}_k} \tilde{\mathbf{C}}_{\mathbf{x}_k}^H &= \gamma_k^2 \tilde{\mathbf{t}}_k \tilde{\mathbf{t}}_k^H = \gamma_k^2 \tilde{\mathbf{v}}_k^* (\tilde{\mathbf{v}}_k^*)^H \\ \iff \tilde{\mathbf{t}}_k &= e^{i\theta_k} \tilde{\mathbf{v}}_k^*, \text{ or } \tilde{\mathbf{v}}_k = e^{i\theta_k} \tilde{\mathbf{t}}_k^*, \end{aligned} \quad (\text{E.3})$$

where $\theta_k \in [0, 2\pi)$. By substituting $\mathbf{v}_k$ into (E.2), we have

$$\tilde{\mathbf{C}}_{\mathbf{x}_k} = \gamma_k e^{-i\theta_k} \tilde{\mathbf{t}}_k \tilde{\mathbf{t}}_k^T = Z_k \tilde{\mathbf{t}}_k \tilde{\mathbf{t}}_k^T, \quad (\text{E.4})$$

where we have defined the complex variable $Z_k = \gamma_k e^{-i\theta_k}$. Furthermore, the condition in (c) of Lemma E.1 implies that for the positive semidefinite constraint in (4.14) to be satisfied, we need to have

$$\begin{aligned} (\mathbf{C}_{\mathbf{x}_k}^*)^* - \tilde{\mathbf{C}}_{\mathbf{x}_k}^* (\mathbf{C}_{\mathbf{x}_k}^*)^\dagger \tilde{\mathbf{C}}_{\mathbf{x}_k} &\succeq \mathbf{0} \\ \iff (\|\mathbf{t}_k\|^2 - |Z_k|^2 \|\mathbf{t}_k\|^{-2}) \tilde{\mathbf{t}}_k^* \tilde{\mathbf{t}}_k^T &\succeq \mathbf{0} \\ \iff |Z_k| &\leq \|\mathbf{t}_k\|^2. \end{aligned} \quad (\text{E.5})$$

This completes the proof of Lemma 4.2.

Appendix F

Proof of Theorem 4.4

Since the inequalities in (4.28) and (4.29) of Theorem 4.4 can be proved similarly, for brevity, we only show the proof of (4.28) in this appendix. First, if $\mathbf{W}_k = \mathbf{w}_k \mathbf{w}_k^H = \mathbf{0}$, i.e., $\mathbf{w}_k$ is a zero-vector, then (4.28) is satisfied since

$$C_{s_k}^2 = \left(\sum_{j \neq k} |\mathbf{h}_{kj} \mathbf{t}_j|^2 + \sigma^2 \right)^2 \geq \sigma^4 \quad (\text{F.1})$$

is true. Therefore, in the following, we assume without loss of generality that at least one of the elements in $\mathbf{w}_k$ is non-zero. Then we consider the following optimization problem:

$$\begin{aligned} \max_{\mathbf{Z} \succeq \mathbf{0}} \quad & \text{Tr}(\mathbf{W}_k \mathbf{Z}) = \text{Tr}(\mathbf{w}_k \mathbf{w}_k^H \mathbf{Z}) \\ \text{s.t.} \quad & \text{Tr}(\mathbf{E}_k \mathbf{Z}) \leq \|\mathbf{t}_k\|^4, \quad \forall k. \end{aligned} \quad (\text{F.2})$$

In order to show (4.28) in Theorem 4.4, it is sufficient to prove that the optimal objective value of (F.2) is no greater than $1 - C_{s_k}^{-2} \sigma^4$. With (F.1), this is clearly true if the optimal solution $\mathbf{Z}^*$ to (F.2) is a zero matrix. Therefore, in the following, we assume $\mathbf{Z}^* \neq \mathbf{0}$. First, we give the following lemma.

Lemma F.1. *There exists a rank-1 optimal solution to (F.2).*

Proof. Lemma F.1 can be proved by applying the result given in Lemma I of [50], or that in Lemma 1.6 of [35]. For the problem of our interest where $\mathbf{E}_k$ is a matrix with all zero-elements except that the (k, k) -th entry equals to one, we provide an alternative proof for the same result based on the KKT optimality conditions to provide more insight. Without loss of generality, by rearranging terms if necessary, we can partition $\mathbf{w}_k \in \mathbb{C}^K$ as

$$\mathbf{w}_k = \begin{bmatrix} \bar{\mathbf{w}}_k \\ \mathbf{0} \end{bmatrix}, \quad (\text{F.3})$$

where $\bar{\mathbf{w}}_k \in \mathbb{C}^{K'}$ consists of all *non-zero* elements in $\mathbf{w}_k$, where $1 \leq K' \leq K$. Accordingly, we partition $\mathbf{Z} \in \mathbb{C}^{K \times K}$ as

$$\mathbf{Z} = \begin{bmatrix} \bar{\mathbf{Z}} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{bmatrix}, \quad (\text{F.4})$$

where the dimensions of $\bar{\mathbf{Z}}$, $\mathbf{Z}_{12}$, $\mathbf{Z}_{21}$ and $\mathbf{Z}_{22}$ are given by $K' \times K'$, $K' \times (K - K')$, $(K - K') \times K'$, and $(K - K') \times (K - K')$, respectively. Then we have

$$\mathbf{w}_k^H \mathbf{Z} \mathbf{w}_k = \bar{\mathbf{w}}_k^H \bar{\mathbf{Z}} \bar{\mathbf{w}}_k. \quad (\text{F.5})$$

In other words, the matrices $\mathbf{Z}_{12}$, $\mathbf{Z}_{21}$ and $\mathbf{Z}_{22}$ are not related to the objective value of (F.2). Furthermore, since $\mathbf{E}_k$ is diagonal, without loss of optimality, $\mathbf{Z}_{12}$, $\mathbf{Z}_{21}$ and $\mathbf{Z}_{22}$ can all be set to zero matrices. Then (F.2) can be recast as

$$\begin{aligned} \max_{\bar{\mathbf{Z}}} & \text{Tr}(\bar{\mathbf{w}}_k \bar{\mathbf{w}}_k^H \bar{\mathbf{Z}}) \\ \text{s.t.} & \text{Tr}(\bar{\mathbf{E}}_k \bar{\mathbf{Z}}) \leq \|\mathbf{t}_k\|^4, \quad k = 1, \dots, K', \\ & \bar{\mathbf{Z}} \succeq \mathbf{0}, \end{aligned} \quad (\text{F.6})$$

where $\bar{\mathbf{E}}_k \in \mathbb{C}^{K' \times K'}$ is a matrix with all zero-elements except that the (k, k) -th entry equals to one. Next, we show that the optimal solution to (F.6) is rank-1 by investigating its KKT conditions. Specifically, let $\bar{\mathbf{Z}}^*$ be the optimal solution to (F.6), $\{\lambda_k^* \geq 0\}$ and $\bar{\mathbf{Y}}^* \succeq \mathbf{0}$ be the optimal dual solutions associated with the two constraints in (F.6). According to the KKT conditions, one can verify that the following conditions must be satisfied:

$$\bar{\mathbf{Y}}^* = \mathbf{\Lambda} - \bar{\mathbf{w}}_k \bar{\mathbf{w}}_k^H \succeq \mathbf{0}, \quad (\text{F.7})$$

$$\bar{\mathbf{Y}}^* \bar{\mathbf{Z}}^* = \mathbf{0}, \quad (\text{F.8})$$

where $\mathbf{\Lambda} = \sum_{k=1}^{K'} \lambda_k^* \bar{\mathbf{E}}_k = \text{Diag}\{\lambda_1^*, \dots, \lambda_{K'}^*\}$. From (F.7), we have

$$\lambda_k^* > 0, \quad k = 1, \dots, K', \quad (\text{F.9})$$

which can be shown as follows by contradiction. Suppose that there exists an index $l \in \{1, \dots, K'\}$ with $\lambda_l^* = 0$. Then we define a vector $\mathbf{v} = \delta \mathbf{e}_l$, where $\delta > 0$ and $\mathbf{e}_l$ is the l th column of the identity matrix $\mathbf{I}_{K'}$. We thus have

$$\mathbf{v}^H \bar{\mathbf{Y}}^* \mathbf{v} = -\delta^2 |\mathbf{e}_l^H \bar{\mathbf{w}}_k|^2 = -\delta^2 |[\bar{\mathbf{w}}_k]_l|^2 < 0, \quad (\text{F.10})$$

where the last inequality follows since all the elements in $\bar{\mathbf{w}}_k$ are non-zero. (F.10) clearly contradicts with the positive semidefinite constraint of $\bar{\mathbf{Y}}^*$ in (F.7), and hence (F.9) must be true. Then we have $\mathbf{\Lambda} = \mathbf{\Lambda}^{\frac{1}{2}} \mathbf{\Lambda}^{\frac{1}{2}}$, with $\mathbf{\Lambda}^{\frac{1}{2}}$ being invertible.

Thus, we have

$$\begin{aligned}\text{rank}(\bar{\mathbf{Y}}^*) &= \text{rank}\left(\Lambda^{\frac{1}{2}}[\mathbf{I}_{K'} - \Lambda^{-\frac{1}{2}}\bar{\mathbf{w}}_k\bar{\mathbf{w}}_k^H\Lambda^{-\frac{1}{2}}]\Lambda^{\frac{1}{2}}\right) \\ &= \text{rank}\left(\mathbf{I}_{K'} - \Lambda^{-\frac{1}{2}}\bar{\mathbf{w}}_k\bar{\mathbf{w}}_k^H\Lambda^{-\frac{1}{2}}\right) \\ &\geq K' - 1.\end{aligned}\tag{F.11}$$

It follows from (F.8) and (F.11) that $\bar{\mathbf{Z}}^*$ must be rank-1 since

$$0 < \text{rank}(\bar{\mathbf{Z}}^*) \leq \text{Nullity}(\bar{\mathbf{Y}}^*) = K' - \text{rank}(\bar{\mathbf{Y}}^*) \leq 1.\tag{F.12}$$

Given (F.4) and the zero matrices for $\mathbf{Z}_{12}$, $\mathbf{Z}_{21}$ and $\mathbf{Z}_{22}$, the solution $\mathbf{Z}^*$ to (F.2) must be of rank-1 since

$$\text{rank}(\mathbf{Z}^*) = \text{rank}\left(\begin{bmatrix} \bar{\mathbf{Z}}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\right) = \text{rank}(\bar{\mathbf{Z}}^*) = 1.\tag{F.13}$$

This completes the proof of Lemma F.1. $\square$

With Lemma F.1, problem (F.2) can be re-expressed as

$$\begin{aligned}\max_{\mathbf{z}} \quad & \mathbf{z}^H \mathbf{W}_k \mathbf{z} \\ \text{s.t.} \quad & |\mathbf{e}_k^H \mathbf{z}|^2 \leq \|\mathbf{t}_k\|^4, \quad \forall k.\end{aligned}$$

Using (4.16) and (4.21), we have

$$\begin{aligned}\mathbf{z}^H \mathbf{W}_k \mathbf{z} &= C_{s_k}^{-2} \left| \sum_{j \neq k} (\mathbf{h}_{kj} \tilde{\mathbf{t}}_j)^2 Z_j \right|^2 \\ &\leq C_{s_k}^{-2} \left(\sum_{j \neq k} |\mathbf{h}_{kj} \tilde{\mathbf{t}}_j|^2 |Z_j| \right)^2\end{aligned}\tag{F.14}$$

$$\leq C_{s_k}^{-2} \left(\sum_{j \neq k} |\mathbf{h}_{kj} \tilde{\mathbf{t}}_j|^2 \|\mathbf{t}_j\|^2 \right)^2\tag{F.15}$$

$$\leq C_{s_k}^{-2} \left[\left(\sum_{j \neq k} |\mathbf{h}_{kj} \mathbf{t}_j|^2 + \sigma^2 \right)^2 - \sigma^4 \right]\tag{F.16}$$

$$= 1 - C_{s_k}^{-2} \sigma^4.\tag{F.17}$$

where (F.14) is due to the triangle inequality; (F.15) is due to the constraint $|Z_k| \leq \|\mathbf{t}_k\|^2, \forall k$; and (F.17) is true since $C_{s_k} = \sum_{j \neq k} |\mathbf{h}_{kj} \mathbf{t}_j|^2 + \sigma^2$. The above result shows that the optimal objective value of (F.14), and hence that of (F.2), is no larger than $1 - C_{s_k}^{-2} \sigma^4$, as desired.

This completes the proof of Theorem 4.4.

Appendix G

Achievability of $\bar{d} = \min \left(\frac{pN+2M-D_t}{2p+1}, \frac{pM}{2p-1} \right)$

In this appnedix, we present the achievability of $\bar{d} = \min \left(\frac{pN+2M-D_t}{2p+1}, \frac{pM}{2p-1} \right)$ as given in (5.70) for general p values.

Achievability of $\bar{d} = \frac{pN+2M-D_t}{2p+1}$

We first consider the case when $D_t \geq D^*$ so that $\bar{d} = \frac{pN+2M-D_t}{2p+1}$. Then from (5.72), we have $|\mathbf{d}_{km}| = \frac{p(N-4M+2D_t)}{2p+1}$, which is the number of independent symbols need to be transmitted by each user via the middle $D_t - M$ antennas with precoding matrix $\mathbf{E}_{km}$. With the technique of subspace alignment chains similar to the case $p = 2$, we use $\mathbf{E}_{km(s)}^i$, where $k, i \in \{1, 2, 3\}$, to denote the s th q -dimensional subspace at transmitter k that participates in the chain originated from transmitter i . Here, we let $q = \frac{|\mathbf{d}_{km}|}{p} = \frac{N-4M+2D_t}{2p+1}$.

We first consider the alignment chain that originates from transmitter 1. With $p = \lceil \frac{\tilde{M}}{\tilde{N}-\tilde{M}} \rceil$, where $\tilde{M} = D_t - M$ and $\tilde{N} = N + D_t - 2M$ representing the number of transmit and receive antennas in the middle block as shown in Fig. 5.5, it follows from [171] that the chain length is equal to p . Following similar notations used in [171], we let $\underline{p} = (p \bmod 3)$, $\bar{p} = \lceil \frac{p}{3} \rceil$, $a(1) = 0$, $a(2) = 3$ and $a(0) = 2$. Then the alignment chain that originates from transmitter 1 can be represented as

$$\begin{aligned} & \mathbf{E}_{1m(1)}^1 \xleftrightarrow{\text{Rx } 2} \mathbf{E}_{3m(1)}^1 \xleftrightarrow{\text{Rx } 1} \mathbf{E}_{2m(1)}^1 \xleftrightarrow{\text{Rx } 3} \mathbf{E}_{1m(2)}^1 \xleftrightarrow{\text{Rx } 2} \mathbf{E}_{3m(2)}^1 \\ & \dots \mathbf{E}_{a(p-1)m(\bar{p}-1)}^1 \xleftrightarrow{\text{Rx } a(p+1)} \mathbf{E}_{a(p)m(\bar{p})}^1. \end{aligned} \quad (\text{G.1})$$

The alignment chain in (G.1) can be equivalently written as

$$\mathbf{A}\mathbf{G} = \mathbf{0}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{P}_{21} & -\mathbf{P}_{23} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{13} & -\mathbf{P}_{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_{32} & -\mathbf{P}_{31} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \ddots & \vdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{P}_{a(\underline{p+1})a(\underline{p-1})} & -\mathbf{P}_{a(\underline{p+1})a(\underline{p})} \end{bmatrix}$$

and $\mathbf{G} = \begin{bmatrix} \mathbf{E}_{1m(1)}^1 \\ \mathbf{E}_{3m(1)}^1 \\ \vdots \\ \mathbf{E}_{a(\underline{p})m(\underline{p})}^1 \end{bmatrix}$.

Since $\mathbf{A}$ is a $(p-1)\tilde{N} \times p\tilde{M}$ matrix, the dimension of its null space is given by $\dim(\mathcal{N}(\mathbf{A})) = p\tilde{M} - (p-1)\tilde{N} = D_t - (p-1)N + (p-2)M$. With $D_t \geq D^*$, it can be verified that $q \leq \dim(\mathcal{N}(\mathbf{A}))$ is always satisfied. Therefore, the q linearly independent columns of $\mathbf{G}$ are chosen from the null space of $\mathbf{A}$, from which we can obtain $\mathbf{E}_{1m(1)}^1, \dots, \mathbf{E}_{a(\underline{p})m(\underline{p})}^1$.

Similarly, we can design the precoding matrices for two other q -dimensional subspace alignment chains with chain length p that originate from transmitter 2 and transmitter 3, respectively. The precoding matrix $\mathbf{E}_{km}$ for the middle antennas is then given by

$$\mathbf{E}_{km} = [\mathbf{E}_{km(1)}^1, \mathbf{E}_{km(2)}^1, \dots, \mathbf{E}_{km(1)}^2, \mathbf{E}_{km(2)}^2, \dots, \mathbf{E}_{km(1)}^3, \mathbf{E}_{km(2)}^3, \dots].$$

Since there are pq vectors participating in each alignment chain, in total $3pq$ symbols are transmitted from the middle antennas by all the three users. Due to symmetry, each user transmits pq symbols by the the middle antennas, which is equal to $|\mathbf{d}_{km}|$, as desired. Following similar arguments as that in [171], it can be shown that the columns in $\mathbf{E}_{km}$ are linearly independent and $\mathbf{E}_{km}$ is full column rank.

With q -dimensional subspace alignment chain of length p , it follows that $(p-1)q$ -dimensional interfering signals are aligned at receiver 1. Therefore, we have

$$\begin{aligned} Z_{1m} &= |\mathbf{d}_{2m}| + |\mathbf{d}_{3m}| - (p-1)q = (p+1)q \\ &= \frac{(p+1)(N - 4M + 2D_t)}{2p+1}. \end{aligned}$$

Together with (5.74) and $\bar{d} = \frac{pN+2M-D_t}{2p+1}$, we have $\bar{d} + Z_1 = N$. Therefore, the full decodability condition (5.65) is satisfied with equality, hence $\bar{d} = \frac{pN+2M-D_t}{2p+1}$ is achievable.

Achievability of $\bar{d} = \frac{pM}{2p-1}$

With $D_t < D^*$, we show that each user is able to transmit $\frac{pM}{2p-1}$ independent data streams. From (5.72), the number of symbols $|\mathbf{d}_{km}|$ need to be transmitted by the $D_t - M$ middle antennas is given by $|\mathbf{d}_{km}| = D_t - \frac{(3p-2)M}{2p-1}$. To this end, we need to use two groups of alignment chains, one with chain length p and subspace dimension q' , and the other with chain length $p - 1$ and subspace dimension $\hat{q}$. The design of the first group of alignment chains follows exactly the same manner as the previous case, except that now the number of columns of $\mathbf{G}$ is set to be $q' = \dim(\mathcal{N}(\mathbf{A})) = D_t - (p-1)N + (p-2)M$. The number of symbols transmitted by the first group of alignment chains from the middle antennas is then given by pq' , which is less than $|\mathbf{d}_{km}|$ under the condition $D_t < D^*$. Therefore, we need to use another group of alignment chains of length $p - 1$. For this group of alignment chains, we use $\hat{\mathbf{E}}_{km(s)}^i$, where $k, i \in \{1, 2, 3\}$, to denote the s th $\hat{q}$ -dimensional subspace at transmitter k that participates in the chain originated from transmitter i . To ensure $|\mathbf{d}_{km}|$ data symbols are sent by the middle antennas, $\hat{q}$ is set as

$$\hat{q} = \frac{|\mathbf{d}_{km}| - q'p}{p-1} = D_t - D^*.$$

Then the alignment chain originated from transmitter 1 can be represented as

$$\begin{aligned} \hat{\mathbf{E}}_{1m(1)}^1 &\xleftrightarrow{\text{Rx } 2} \hat{\mathbf{E}}_{3m(1)}^1 \xleftrightarrow{\text{Rx } 1} \hat{\mathbf{E}}_{2m(1)}^1 \xleftrightarrow{\text{Rx } 3} \hat{\mathbf{E}}_{1m(2)}^1 \\ \dots \hat{\mathbf{E}}_{a(p-2)m(p-2)}^1 &\xleftrightarrow{\text{Rx } a(p)} \hat{\mathbf{E}}_{a(p-1)m(p-1)}^1, \end{aligned} \quad (\text{G.2})$$

or equivalently

$$\hat{\mathbf{A}}\hat{\mathbf{G}} = \mathbf{0}, \quad (\text{G.3})$$

where $\hat{\mathbf{A}}$ is a sub-matrix of $\mathbf{A}$ given by

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{P}_{21} & -\mathbf{P}_{23} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{13} & -\mathbf{P}_{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_{32} & -\mathbf{P}_{31} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \ddots & \vdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{P}_{a(p)a(p-2)} & -\mathbf{P}_{a(p)a(p-1)} \end{bmatrix}$$

and $\hat{\mathbf{G}} = \begin{bmatrix} \hat{\mathbf{E}}_{1m(1)}^1 \\ \hat{\mathbf{E}}_{3m(1)}^1 \\ \vdots \\ \hat{\mathbf{E}}_{a(p-1)m(p-1)}^1 \end{bmatrix}.$

The size of $\hat{\mathbf{A}}$ is $(p-2)\tilde{N} \times (p-1)\tilde{M}$ and thus the dimension of its null space is given by $\dim(\mathcal{N}(\hat{\mathbf{A}})) = (p-1)\tilde{M} - (p-2)\tilde{N} = D_t - (p-2)N + (p-3)M$. Note that since the submatrix of $\mathbf{G}$ formed by its first $(p-1)\tilde{M}$ rows also satisfy (G.3). To ensure the linear independence among the precoding vectors, the $\hat{q}$ columns of $\hat{\mathbf{G}}$ are selected from the null space of $\hat{\mathbf{A}}$, but not in the subspace spanned by the columns of the aforementioned submatrix of $\mathbf{G}$. The dimension is

$$\dim(\mathcal{N}(\hat{\mathbf{A}})) - \dim(\mathcal{N}(\mathbf{A})) = N - M.$$

It can be verified that $\hat{q} \leq N - M$, therefore, the $\hat{q}$ columns of $\hat{\mathbf{G}}$ can be obtained. Similarly, the precoding vectors $\hat{\mathbf{E}}_{k(s)}^2$ and $\hat{\mathbf{E}}_{k(s)}^3$ for the other two alignment chains originating from transmitter 2 and 3 can be found. The overall precoding matrix $\mathbf{E}_k$ for the middle antennas of user k is then obtained accordingly. For the two groups of alignment chains discussed above, the number of aligned interference dimensions at each receiver are respectively given by $q'(p-1)$ and $\hat{q}(p-2)$. Therefore, we have With proposed precoder, at receiver 1, we have achieved

$$\begin{aligned} Z_{1m} &= |\mathbf{d}_{2m}| + |\mathbf{d}_{3m}| - q'(p-1) - \hat{q}(p-2) \\ &= N - (2M - D_t) - \frac{pM}{2p-1}. \end{aligned} \tag{G.4}$$

Then together with (5.74), we have $\bar{d} + Z_1 = N$. In other words, the full decodability condition (5.65) is satisfied with equality. Therefore, $\bar{d} = \frac{pM}{2p-1}$ is achievable.

Appendix H

Proof of Lemma 6.2

First, we show that at the optimal solution to problem (6.21), the first inequality constraint is active for all k . Suppose, to the contrary, that at the optimal solution $\{\rho^*, \{\mathbf{S}_k^*\}\}$, there exists a user k such that $\log|\mathbf{I} + \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H| > \text{Tr}(\mathbf{F}_k \sum_{i \neq k}^{K_r} \mathbf{S}_i) + R_k^{\text{BD}}$. Then the transmit power to user k can be strictly decreased with all the constraints in problem (6.21) still satisfied. This in turn makes it possible to reduce the transmit power for other users. As a consequence, the power factor ρ can be further reduced, which contradicts that ρ^{opt} is the optimal solution.

Then based on the complementary slackness condition [65], we can assume that the optimal dual variables $\{\lambda_k^*\}$ are positive. Therefore, we can assume that $\lambda_k > 0$ in (6.25). We then prove Lemma 6.2 by contradiction. Since $\mathbf{C}_k$ is a Hermitian matrix, all of its eigenvalues are real. Suppose that $\mathbf{C}_k$ has a non-positive eigenvalue, i.e., $\exists \alpha \leq 0$ and a normalized vector $\mathbf{q}$, with $\mathbf{q}^H \mathbf{q} = 1$ such that $\mathbf{C}_k \mathbf{q} = \alpha \mathbf{q}$. Then let $\mathbf{S}_k = t \mathbf{q} \mathbf{q}^H$ with $t \geq 0$ and by substituting it into the objective function of (6.25), we have

$$\lambda_k \log |\mathbf{I} + t \mathbf{H}_k \mathbf{q} \mathbf{q}^H \mathbf{H}_k^H| - \text{Tr}(t \mathbf{C}_k \mathbf{q} \mathbf{q}^H) = \lambda_k \log(1 + t \|\mathbf{H}_k \mathbf{q}\|^2) - \alpha t. \quad (\text{H.1})$$

Since $\alpha \leq 0$, as $t \rightarrow \infty$, the value of (H.1) becomes unbounded provided that $\mathbf{H}_k \mathbf{q} \neq \mathbf{0}$, which is true with probability one with independent channel realizations and the fact that $\mathbf{C}_k$ does not depend on $\mathbf{H}_k$. Therefore, we conclude that in order to have a bounded objective value for (6.25), all eigenvalues of $\mathbf{C}_k$ should be positive, and hence $\mathbf{C}_k \succ \mathbf{0}$. As a result, Lemma 6.2 follows.

Appendix I

Proof of Lemma 6.4

A vector $\mathbf{s}$ is a subgradient of function $\tilde{g}(\mathbf{x})$ at point $\mathbf{x}$ if

$$\tilde{g}(\bar{\mathbf{x}}) \geq \tilde{g}(\mathbf{x}) + \mathbf{s}^T(\bar{\mathbf{x}} - \mathbf{x}), \forall \bar{\mathbf{x}}, \quad (\text{I.1})$$

Or the vector formed by $\{s_{\lambda_k}\}$ and $\{s_{\mu_j}\}$ is a subgradient of $\tilde{g}(\{\lambda_k\}, \{\mu_j\}_{j=1}^{K_t-1})$ if $\forall \bar{\lambda}_k, \bar{\mu}_j$, we have

$$\tilde{g}(\{\bar{\lambda}_k\}, \{\bar{\mu}_j\}_{j=1}^{K_t-1}) \geq \tilde{g}(\{\lambda_k\}, \{\mu_j\}_{j=1}^{K_t-1}) + \sum_{k=1}^{K_r} s_{\lambda_k}(\bar{\lambda}_k - \lambda_k) + \sum_{j=1}^{K_t-1} s_{\mu_j}(\bar{\mu}_j - \mu_j). \quad (\text{I.2})$$

For the given $\{\lambda_k\}$ and $\{\mu_j\}_{j=1}^{K_t-1}$, denote by $\{\mathbf{S}_k^*\}$ be the transmit covariance matrices that achieves the maximum dual function value $\tilde{g}(\{\lambda_k\}, \{\mu_j\}_{j=1}^{K_t-1})$. Then $\forall \bar{\lambda}_k, \bar{\mu}_j$, we have

$$\begin{aligned} \tilde{g}(\{\bar{\lambda}_k\}, \{\bar{\mu}_j\}_{j=1}^{K_t-1}) &\stackrel{(a)}{=} \max_{\mathbf{S}_k \succeq \mathbf{0}, \forall k} \left\{ \sum_{k=1}^{K_r} \bar{\lambda}_k [\log |\mathbf{I} + \mathbf{H}_k \mathbf{S}_k \mathbf{H}_k^H| - \text{Tr}(\mathbf{F}_k \sum_{i \neq k}^{K_r} \mathbf{S}_i) - R_k^{BD}] \right. \\ &\quad \left. + \sum_{j=1}^{K_t-1} \bar{\mu}_j \sum_{k=1}^{K_r} \text{Tr}[(\mathbf{B}_{K_t} - \mathbf{B}_j) \mathbf{S}_k] - \frac{1}{P} \sum_{k=1}^{K_r} \text{Tr}(\mathbf{B}_{K_t} \mathbf{S}_k) \right\} \\ &\stackrel{(b)}{\geq} \sum_{k=1}^{K_r} \bar{\lambda}_k [\log |\mathbf{I} + \mathbf{H}_k \mathbf{S}_k^* \mathbf{H}_k^H| - \text{Tr}(\mathbf{F}_k \sum_{i \neq k}^{K_r} \mathbf{S}_i^*) - R_k^{BD}] \\ &\quad + \sum_{j=1}^{K_t-1} \bar{\mu}_j \sum_{k=1}^{K_r} \text{Tr}[(\mathbf{B}_{K_t} - \mathbf{B}_j) \mathbf{S}_k^*] - \frac{1}{P} \sum_{k=1}^{K_r} \text{Tr}(\mathbf{B}_{K_t} \mathbf{S}_k^*) \\ &\stackrel{(c)}{=} \tilde{g}(\{\lambda_k\}, \{\mu_j\}_{j=1}^{K_t-1}) + \sum_{k=1}^{K_r} s_{\lambda_k}(\bar{\lambda}_k - \lambda_k) + \sum_{j=1}^{K_t-1} s_{\mu_j}(\bar{\mu}_j - \mu_j), \end{aligned}$$

where (a) follows from (6.32), (b) is true since $\tilde{g}(\{\bar{\lambda}_k\}, \{\bar{\mu}_j\}_{j=1}^{K_t-1})$ is the maximum value over all $\mathbf{S}_k \succeq \mathbf{0}$ for the given dual variables $\{\bar{\lambda}_k\}$ and $\{\bar{\mu}_j\}_{j=1}^{K_t-1}$, and s_{λ_k}

and s_{μ_j} are given by (6.34) and (6.35), respectively; and (c) is obtained by using $\bar{\lambda}_k = (\bar{\lambda}_k - \lambda_k) + \lambda_k$, $\bar{\mu}_j = (\bar{\mu}_j - \mu_j) + \mu_j$ and the fact that $\{\mathbf{S}_k^*\}$ achieves the maximum value $\tilde{g}(\{\lambda_k\}, \{\mu_j\}_{j=1}^{K_t-1})$. Together with (I.2), Lemma 6.4 follows.