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**TIME-VARYING EMISSION OF ELECTRONS
AND ITS RELEVANT EM PHENOMENON**

LIU YANGJIE

**SCHOOL OF ELECTRICAL AND
ELECTRONIC ENGINEERING**

2013

Dedicated to those who would like to take time to discuss with me on any topic.

-Michael Lutherski

Time-Varying Emission of Electrons and its Relevant EM Phenomenon

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A thesis submitted to the Nanyang Technological University
in partial fulfillment of the requirement for the degree of
Doctor of Philosophy

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Dedicated to those who would like to take time to discuss with me on any topic.

– *Michael Luthervski*

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SUMMARY(ABSTRACT)

The main-body of this thesis or dissertation attempts to consider two aspects of electron emission scenario: one on increasing electron emission counts by injecting time-varying electron flow under short-pulse condition and the other on induced electromagnetic fields by electron flow within space.

(1). Time-varying profile of injection electron flow from cathode side offers possibilities to transcend conventional time-uniform space-charge limit under short-pulse condition, as pointed out by Á. Valfells *et al.* in *Phys. Plasmas* **9**, 2377(2002) and monotonously increasing functions of time-profile could be such a case. A laser-excited emission mechanism is computationally introduced here, illustrating that for ultrahigh laser field space charge effect becomes notable and DC electric field can be an efficient parameter to tune the space-charge limit. Particle-In-Cell(PIC) simulation is also performed to demonstrate agreement with theoretical analysis.

(2). Another phenomenon related to electron flow is that evanescent wave produced by mechanical motion of an electron bunch can be accumulated under certain artificial circumstances, inhomogeneous or anisotropic space (realizable via metamaterial) such as Maxwell's fisheye structure. Other than the former electrostatics problem, electrodynamics has to be formulated to derive solution in frequency domain from vector Helmholtz equation. Dyadic Green's function and method of scattering superposition are chosen to give analytic electromagnetic field solution for transition radiation from Maxwell's fisheye. This calculation may pave the way to investigate novel scheme of brand band light source, transferring from electron's kinetics to radiation.

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LIST OF SYMBOLS

k	Constant in Coulomb's law	4
E	Energy of an electron(in unit of electron·volt)	22
E	Electric field (amplitude)	23
V	energy potential(in unit of electron·volt)	22
m	elementary mass for a single electron	20
e	elementary charge for a single electron	20
ϕ	static potential(in unit of Volt)	20
V_g	diode voltage in Volt	29
D	gap distance for the diode	20
β_m	the magnitude of the current density (in unit of <i>Ampere</i> / m^2)	59
j_m	time-dependent profile of no dimension	59

CHAPTER 1

INTRODUCTION AND MOTIVATION

1.1 Introduction

Electron and photon have been recognized as so hugely disparate two categories of microscopic particles from the perspective of their boson and fermion classifications, respectively. Opposite to the macroscopic structure size of condense-matter materials, classical electrodynamics describes collective behaviors of groups of photons in a form of wave – so-called electromagnetic waves; while electron behaves prone to a microscopic particle although wavelike properties is not absent from its interaction with material lattice. These two microscopic particles are manipulated in miscellaneous methodologies to benefit human beings, starting from power microwave devices to electromagnetic wave ones, such as classical laser and free electron laser, etc. Among all these methodologies, one thing which never ceases to occur is the energy transfer between both particles.

A typical energy-transfer from photons to electrons could be laser-excitation, in which light imparts its energy to electron reservoir(say, metal material) so that electrons may find their own way out to emit. And a reverse process also transferring energy may be Čerenkov radiation or transition radiation, both of which accumulate evanescent wave along electron's motion trajectory into powerful radiation—or namely motion-induced radiation. In this thesis, the author shall focus on these two

phenomena above, both of which fall into concept of self-fields due to electrons. For the first aspect(Chap. 2), we shall calculate time-dependent emission current density of a short-pulse electron flow, due to laser-excitation under different physical parameters to explore methods to promote higher emission quantity than former short-pulse space-charge limit [3]. Also we use simulation tools to explore possible options to choose time-dependent function for electron's short flow(Chap. 3). For the other aspect, we choose an inhomogeneous optical media(in Chap. 4, Maxwell's fisheye sphere) and calculate its motion-induced radiation from electron's flow by a well-known method of dyadic Green function. Before coming to the term, let us revisit the main theoretical background of these two problems to warm up, from two perspectives of electrostatics and electrodynamics.

1.1.1 Electrostatic framework for electron emissions

For the electron emission problem, the usual basic configuration on setup is made up from cathode, anode and the diode space between former two in Fig. 1.1. We start from basic concept of plasma to explain the physics behind the electron beam current flow in vacuum tube.

Concept Definition of plasma

First and foremost, plasma in physics refers to an ionized gas, “fourth state of matter”. It is relevant to mention other three common states to make some distinguishes between. A liquid is possible to form upon sufficient heating of a solid, when crystal lattice structure, or any other equivalent breaks apart into a weaker bond which still relate neighboring particles but enables composite particles to flow under the shape of container. While a gas forms when a liquid receives enough heat so that vaporization competes over the re-condensation. Such a gas can be heated even further so that the vibration of atoms aggregates into collisions between, and knock off electrons away from ions: this is how plasma is generated. Plasma in Greek, $\pi\lambda\acute{\alpha}\sigma\mu\alpha$ [4], means moldable substance or jelly, coined by Irving Langmuir, was imagined as a reminiscent of a blues piece, *It Must Be Jelly Cause Jam Don't Shake Like That* [1]*.

*The author must admit traditional Chinese translation of plasma in Taiwan, “Dianjiang”, is an excellent instance to impart physics essence to succinct Chinese characters.

1.1 Introduction

Langmuir had his reason to separately categorize this ionized gas into a novel kind of “plasma” because the charge separation becoming distant between electrons and ions gives rise to dominant electric fields and charged particle flow gives rise to current and relevant magnetic fields, all of which begin to interplay around collectively and thus create complexity and beauty which the other three categories of materials usually fall short of.

Electron Beam Flows in a Vacuum Tube

A simple, or maybe the simplest form of plasma is a bunch of electrons injected from cathode-anode configuration. Suppose a pair of planar cathode-anode materials are configured to incorporate a vacuum tube in Fig. 1.1. Under miscellaneous perturbation circumstances, cathode material (e.g. metal) is able to emit copious electrons off its surface and anode absorbs them back again. Therefore, instead, a specific electric potential ϕ (in unit of V[olt]) is exerted internally between cathode and anode, thus directional electron flow away from cathode is able to form toward anode accordingly.

Figure 1.1: Configuration of cathode and anode structure to emit DC electron beam flows. From **Fig. 1.1**, p4, Ref. [1].

Chapter 1. INTRODUCTION AND MOTIVATION

Part of this thesis(Chaps. 2 and 3) originates from electrostatics and concerns about calculations to determine whether electron flow has saturated to an extreme extent which may surpass traditional space-charge limit. So it is worth a little volume to predefine the ground stone of electrostatics. Physics discipline is definitely of experimental sciences, which urges people to always wait for experimental results to guide theorists to pursue the possibly correct path, instead of claiming any other speculations. Electrostatics establishes all its skyscrapers out of a simple experimental fact – Coulomb’s law: (applying to charges in vacuum or in media of negligible susceptibility). The electric field $\vec{E}$ at point $\vec{x}$ due to a point charge q_1 at point $\vec{x}_1$ can be written directly:

$$\vec{E}(\vec{x}) = kq_1 \frac{\vec{x} - \vec{x}_1}{|\vec{x} - \vec{x}_1|^3}. \quad (1.1)$$

The constant k differs in different yet convertible systems of unit. This thesis shall stick to International Standard(SI) unit system unless otherwise circumstances call upon(cf. Chap. 4). However, the system of Gaussian cgs(cm-g-s) units is not yet obscure thanks to its formulation convenience in theoretical physics. For instance, in electrostatic units(esu), $k = 1$ and unit charge(statcoulomb) is chosen so that a force of one dyne is exerted 1 cm away. An electric field thus-measured is in unit of statvolts/cm. While in SI units, $k = \frac{1}{4\pi\epsilon_0} = 10^{-7}c^2$, $\epsilon_0 \approx 8.854 \times 10^{-12}\text{F/m}$ is vacuum permittivity. The SI unit of electric charge is C[olumnb] and electric field is in V/m. One may still occasionally encounter conversions above in formulation nowadays.

When more than one charges q_i located at positions $\vec{x}_i, i \in \mathbf{N}^+$ (positive integer set) exerts their forces to an arbitrary point $\vec{x}$, equivalent electric field $\vec{E}$ conforms to linear superposition principle,

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_i q_i \frac{\vec{x} - \vec{x}_i}{|\vec{x} - \vec{x}_i|^3}. \quad (1.2)$$

An intrinsic flaw of this mathematical representation might be that electric field loses its appropriate definition when observation point aligns with charge position($\vec{x} = \vec{x}'$). This flaw is possibly boiled away when it collapses to a more realistic geometric model of charge density, however, renormalization theory claims to repair it rigorously but this is not to author’s purpose to mention it. Instead, the author thinks this scenario

1.1 Introduction

enchants one to ponder over self-energy of a charge particle. One possibility of the self-energy for a charged particle lies in the framework of classical electrodynamics. Let us consider that an electron moves, which should be compatible because relative motion always could become true as long as we choose a suitable reference frame. De facto, in this scenario, the evanescent wave induced by the motion of electrons, in the second part of this thesis, can also be such an example to demonstrate self-fields from a particle(EM fields) although this self-energy framework is incomplete—we shall neglect the back-reaction on the electron from its own EM fields for simplicity.

Still get back to the term of electrostatics. The discrete sum of any charged particles can be extrapolated into a continuum integral when charges form a proper continuum of charge density distribution $\rho(\vec{x}')$:

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} d\vec{x}'. \quad (1.3)$$

From Coulomb's law, we are able to deduce another integral result named as Gauss's law as follows. S is a closed surface in conventional Euclidean sense, and $\vec{n}$ denotes a unit normal to the surface S pointing outward at any point of surface. For discrete set of point charges, Gauss's law stands in form of

$$\oint_S \vec{E} \cdot \vec{n} dS = \frac{\sum_i q_i}{\epsilon_0} = \frac{1}{\epsilon_0} \int_V \rho(\vec{x}) d\vec{x}, \quad (1.4)$$

where the sum is over only those charges inside the surface S and V the volume enclosed by surface S . An integral formulation of the law of electrostatics, Gauss's law is convertible to a differential form by using Gauss's theorem in mathematics(Jackson called it the divergence theorem [5]),

$$\oint_S \vec{E} \cdot \vec{n} dS = \int_V \nabla \cdot \vec{E} \cdot d\vec{x} = \frac{1}{\epsilon_0} \int_V \rho(\vec{x}) d\vec{x}, \Rightarrow \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}. \quad (1.5)$$

However, this differential equation is not enough to specify an electric field $\vec{E}$. Therefore some mathematics comes into interplay as a technique to rescue us. Looking a bit into generalized Coulomb's law in continuum form Eq. (1.3), we find that

$$\vec{E}(\vec{x}) = \frac{-1}{4\pi\epsilon_0} \nabla \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d\vec{x}'. \quad (1.6)$$

Chapter 1. INTRODUCTION AND MOTIVATION

Since the curl of the gradient of any well-behaved scalar field vanishes, i.e.

$$\nabla \times \vec{E} \equiv 0, \quad (1.7)$$

which specify the irrotational property of electric field. As a matter of this, a scalar potential field is defined by the equation

$$\vec{E} = -\nabla\Phi, \quad (1.8)$$

and is able to represent the aforementioned conservative vector field $\vec{E}^\dagger$, of which the minus sign is taken as a traditional convention to favor some description of physical picture. Naturally, the scalar potential is defined in terms of charge density all over the whole space(our universe, in principle) except an arbitrary constant:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d\vec{x}' + \text{Const.} \quad (1.9)$$

Gauss's law in integration form (1.4) allows one to write down a partial result. When a surface S with its unit normal $\vec{n}$ directed from side 1 to 2 of the surface S itself, carries a surface-charge density $\sigma(\vec{x})$ and electric fields are denoted as $\vec{E}_1$ and $\vec{E}_2$ on either side, the partial result writes

$$(\vec{E}_2 - \vec{E}_1) \cdot \vec{n} = \frac{\sigma}{\epsilon_0}. \quad (1.10)$$

This difference form tells one that a discontinuity of normal component of electric field $\vec{E}$ occurs in crossing a surface of charge density σ and also resulting from that.

Poisson Equation

If electrostatic problems we face always occurred in free space without any boundary surfaces except locally discrete or continuous distribution of charges, scalar potential calculated from Eq. (1.9) would be as omnipotent all-around a solution to any electrostatic problem and the following texts especially on Poisson equation would be useless instead then. Unfortunately, many, if not dominant engineering problems

[†]An important property of conservative vector field is that any closed path integral in it vanishes $\oint \vec{E} \cdot d\vec{l} \equiv 0$.

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on electrostatics involve finite regions of space, which estranges one from Eq. (1.9) and tempts one to investigate instead Poisson equation to solve related problems.

Now we shall introduce Poisson equation as a convenient method to solve electrostatic problems. Following aforementioned differential form of Gauss's law,

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad ((1.5))$$

curl-free property of electric field

$$\nabla \times \vec{E} \equiv 0, \quad ((1.7))$$

and definition of scalar potential Φ

$$\vec{E} = -\nabla\Phi, \quad ((1.8))$$

we are able to combine them into a second-order partial differential equation for scalar function $\Phi(\vec{x})$,

$$\nabla^2\Phi(\vec{x}) \equiv \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} \Phi(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0}, \quad (1.11)$$

which is often called the *Poisson equation*. Sometimes people write Laplacian[‡] $\nabla^2 \equiv \nabla \cdot \nabla$ as Δ , but the author would take ∇^2 because this capital Greek letter Δ may cause confusion with differential form of a variable if not manipulated properly. Put in geometrical or fluid language, the Laplacian $\nabla^2 f(\mathbf{p})$ of a function f at a point $\mathbf{p}$, denotes the rate at which the function value over spheres centered at point $\mathbf{p}$, deviates from the function value $f(\mathbf{p})$ as the radius of the sphere grows, except for a constant depending upon the dimension [7]. A homogeneous counterpart of Poisson equation, carrying no charge density, is called *Laplace equation*. Although they differentiate each other by whether charge density vanishes respectively, attacking methods of

[‡]The Laplacian in Eq. (B.4) includes merely Cartesian coordinate, and yet generally under arbitrary curved manifold requires description by tool of tensor, which is out of this thesis' scope [6]. However, such Laplacian operator does not differ between various coordinate, which is an accepted postulate in physical principle. For instance, $\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) \equiv -4\pi\delta(\vec{x}-\vec{x}')$ no matter how complicated curvilinear coordinate is adopted beneath the Laplacian sign.

Poisson and Laplace equations still bear much resemblance from conventional methods to solve differential equations. And this point also extrapolates to framework of electromagnetics, enabling one to derive solutions to non-homogeneous (sometime written as *inhomogeneous*) form of Maxwell's equations with a source term of current density, which will be narrated in next subsection 1.1.2.

Poisson equation is at least compatible with Coulomb's law since scalar potential defined by charge density, Eq. (1.9) satisfies the former equation (cf. page 35 of Jackson's book [5]). Thus we are able to incorporate all aspects of an electrostatics problem into Poisson equation, which calls upon directly solving a second-order partial differential equation with appropriate boundary conditions. It is physically plausible and yet indeed provable that each of three boundary conditions specified guarantees the uniqueness of solution to Poisson (or Laplace) equation, Dirichlet boundary condition, Neumann boundary condition (i.e. specification of electric field on surface) and a mixed one between the former two over a whole closed surface. An overspecification of them, for example, both Dirichlet and Neumann conditions generally lead to a Poisson equation with no existent solution. One should take care when constructing its boundary conditions.

Formal solution to Poisson equation

(a) Green function method to solve Poisson equation

Normally one is met with a Poisson or Laplace equation in a finite volume V with either Dirichlet or Neumann boundary conditions on bounding closed surface S . In this subsection, the author shall introduce two methods to solve it: Green function's method and variational approach. A typical or say rather elegant way to treat nonhomogeneous differential equation is called Green function (attributed to George Green, 1824) method. Since Poisson equation indeed falls into this category mathematically, it surely bears such an advantage to make use of Green function. First, one finds that scalar potential $\psi(\vec{x}) = \frac{1}{|\vec{x} - \vec{x}'|}$ satisfy such a Laplacian statement:

$$\nabla'^2 \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) = -4\pi\delta(\vec{x} - \vec{x}'), \quad (1.12)$$

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where $\vec{x}$ indicates the observation point and $\vec{x}'$ the charge position. Now in order not to lose generality, define

$$\nabla'^2 G(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}'), \quad (1.13)$$

where

$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + F(\vec{x}, \vec{x}'), \quad (1.14)$$

while additional freedom F makes Laplace(homogeneous) equation inside volume V stand according to general solution to nonhomogeneous differential equation. Apart from its mathematical role as a solution of Laplace equation inside a volume V , physically the function F can also be interpreted from the image charge effect $1/|\vec{x} - \vec{x}'|$ induced by some boundary condition. This shall be revealed later after Green function is determined.

Second we recall famous Green's second identity or Green's theorem, for arbitrary scalar fields $\phi(\vec{x})$ and $\psi(\vec{x})$, integral relation

$$\int_V dV (\phi \nabla^2 \psi - \psi \nabla^2 \phi) = \oint_S dS (\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}), \quad (1.15)$$

always stands. We let $\phi = \Phi$, $\psi = G(\vec{x}, \vec{x}')$ then a new statement comes into play,

$$\Phi(\vec{x}) = \begin{cases} \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{x}') G(\vec{x}, \vec{x}') d^3x' + \\ \quad \frac{1}{4\pi} \oint_S d^2x' \left[G(\vec{x}, \vec{x}') \frac{\partial \Phi}{\partial n'} - \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} \right], & \vec{x} \in V \\ 0. & \vec{x} \notin V \end{cases} \quad (1.16)$$

This is consistent with interpretation $\sigma = \epsilon_0 \partial \Phi / \partial n'$ thus outside bounding surface, vanished charge leads to null potential. This statement is at best an integral relation satisfied by Φ . With this concept of Green function in mind, one is possible to choose $F(\vec{x}, \vec{x}')$ to eliminate either one of the two surface integral appearing abundant in it because only part of surface integrals in Eq. (1.16) is adequate to combine with Dirichlet or Neumann boundary condition. For Dirichlet condition,

$$G_D(\vec{x}, \vec{x}') = 0. \quad \forall \vec{x}' \in S \quad (1.17)$$

For Neumann condition,

$$\frac{\partial G_N}{\partial n'}(\vec{x}, \vec{x}') = -\frac{4\pi}{S} \cdot \forall \vec{x}' \in S \quad (1.18)$$

Therefore Green function satisfies rather simple boundary conditions irrelevant to detailed form of boundary conditions prescribed for a concrete electrostatic problem. Nonetheless, derivation of Green function sometimes becomes quite tedious due to its dependence on geometrical shape of surface S . Another property of Green function is symmetry property that source and observation points are swappable, $G(\vec{x}, \vec{x}') = G(\vec{x}', \vec{x})$ for Dirichlet or Neumann conditions (the later condition demands a separate requirement specifically).

As to additional freedom function $F(\vec{x}, \vec{x}')$, it is a solution of Laplace equation inside volume V and physically represents a potential of some charge external to V , which simultaneously has to align with homogeneous boundary conditions of vanished quantity combined with some external charge source $\vec{x}'$. Therefore, the method of images should be such a simple instance to satisfy Eqs. (1.17) and (1.18).

In electrostatics, method of images is widely acknowledged to account for induced potential efficaciously contributed by one or several point charges in presence of boundary surfaces specified at fixed potentials. In the simplest situation, an electron emitter of an infinite plane can be in equivalent effect understood as creating virtual image charges at opposite positions of any real point charges existent in space between cathode and anode constructed to emit electron flow in experiment. The virtual image charges serve as replacement of conductor plane (cathode) herein during process of attacking such an electrostatic problem.

We shall hold here for Green function, leave more concrete illustration of it to Subsection 1.1.2 *Electromagnetics* because as seen very soon, wave equation is able to be solved by Green function as well. Chap. 4 are such a concrete example.

Variational Approach

Variational approach retrospects its origin to a very far-reaching concept that any physical system in equilibrium *should* possess a minimal amount of energy functional. It should apply to electrostatics because electrostatics assumes to describe

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circumstances when a system being investigated relaxes to its final or at least quasi-final status, which should have reached its equilibrium state. Therefore, any solution to electrostatics should merely contain information due to equilibrium rather than any other temporal or transient phenomenon. The author shall solely cite the energy-like functional Jackson used [5] to derive a Poisson-like equation again to serve the purpose to justify and illuminate functional method to solve Poisson equation. Define a functional

$$I[\psi] = \frac{1}{2} \int_V \nabla\psi \cdot \nabla\psi d^3x - \int_V g\psi d^3x, \quad (1.19)$$

in which ψ is well-behaved inside V and on its surface $S = \partial V$, and $g(\vec{x})$ a specified "source" function without singularities within V . As usual, when function $\psi(\vec{x})$ infinitesimally within V varies to $\psi(\vec{x}) + \delta\psi$, one examines the first-order change in the functional,

$$\delta I[\psi] \approx \int_V \nabla\psi \cdot \nabla(\delta\psi) d^3x - \int_V g\delta\psi d^3x, \quad (1.20)$$

$$= \int_V (-\nabla^2\psi - g)\delta\psi d^3x + \oint_S \delta\psi \frac{\partial\psi}{\partial n} d^2x. \quad (1.21)$$

Because variational function is infinitesimal on S to induce infinitesimal variance, we can infer as follows,

$$\left. \begin{array}{l} \delta\psi = 0, \\ \delta I[\psi] = 0 \end{array} \right\} \Rightarrow \nabla^2\psi = -g. \quad (1.22)$$

Because the approximation above is accepted at expense of dropping some term semipositive definite, one is able to see functional $I[\psi]$ achieves its stationary minimum if the last Poisson-like equation stands with departures $\delta\psi$ vanished on the boundary. Thus Poisson equation is recovered from variational approach with Dirichlet boundary condition ($\psi \rightarrow \Phi, g \rightarrow \rho/\epsilon_0$ so $\delta\Phi = 0$). As a matter of this, we are equipped with a pragmatic tool to approximate solution to Poisson equation. We may choose an arbitrary trial function $\psi(\vec{x})$ to satisfy given boundary conditions, in which absolutely some intuition is called in to play a part. Then we vary parameters in this trial function to seek an extremum (herein minimum de facto). In this method are we able to try out possible solutions to Poisson equation to some extent of tolerance to error. However, we are reminded that this approach actually does a better job in estimating eigenvalues rather than estimating a solution itself.

Space Charge concept

Space charge is a concept in which excess electric charge is treated as a continuum of charge distributed over a region of space (either a volume or an area) rather than distinct point-like charges. This model typically applies when charge carriers have been emitted from some region of a solid—the cloud of emitted carriers can form a space charge region if they are sufficiently spread out, or the charged atoms or molecules left behind in the solid can form a space charge region. Space charge usually only occurs in dielectric media (including vacuum) because in a conductive medium the charge tends to be rapidly neutralized or screened. The sign of the space charge can be either negative or positive. This situation is perhaps most familiar in the area near a metal object when it is heated to incandescence in a vacuum. This effect was first observed by Thomas Edison in light bulb filaments, where it is sometimes called the Edison effect, but space charge is a more general significant phenomenon in many vacuum and solid-state electronic devices [8]. In Chaps. 2 and 3, we shall treat the space charge effect in electron's emission from solving Poisson equation.

1.1.2 Electrodynamics framework for current density

Electrostatics for electron's emission is built upon certain conditions when electron's motion still does not deviate away from electrostatics problem. More generally, one needs to migrate to its temporal evolution to consider electrodynamics, and usually a full vectorial form of derivation is required to fully incorporate vector fields in 3-dimensional space. In electrodynamics, a basic mathematical challenge appears in a vectorial field representation instead of any scalar potential field. However, it is possible to introduce another vector potential in conventional symbol $\vec{A}$ together with the scalar potential Φ (not yet outdated) to cover a fully equivalent set of Maxwell equations in a smaller number(4) of second-order equations[§].

[§]It is interesting to notice that all dominant dynamical equations are almost, if not all, written in *second-order* differential equations, which may imply the limit of human horizon of perspective in analogy to three-dimensional Euclidean space shaping and constraining human's imagination.

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Basic formulation of electrodynamics

Rest on the fact or postulate of absence of free magnetic poles

$$\nabla \cdot \vec{B} = 0, \quad (1.23)$$

one can define magnetic induction $\vec{B}$ in terms of the vector potential $\vec{A}$:

$$\vec{B} = \nabla \times \vec{A}. \quad (1.24)$$

Then Faraday's law

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \quad (1.25)$$

holds as

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0. \quad (1.26)$$

An irrotational vector field in the parentheses above is capable to be represented as the gradient of a certain scalar function Φ (we pretend temporarily that this is irrelevant to primitive scalar potential Φ , which will surprise us very soon). Thus electric field $\vec{E}$ becomes functional of both potentials we just defined:

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}. \quad (1.27)$$

The rest two non-homogeneous differential in Maxwell equations are thus transformed into principle equations of potentials $\vec{A}$ and Φ ,

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = \frac{\rho}{\epsilon_0}, \quad (1.28)$$

$$\nabla^2\vec{A} - \frac{\partial^2\vec{A}}{c^2\partial t^2} - \nabla\left(\nabla \cdot \vec{A} + \frac{\partial\Phi}{c^2\partial t}\right) = -\mu_0\vec{J}. \quad (1.29)$$

The author shall skip the clichéd gauge transformation available in Jackson's book(p240) [5], and instead repeat merely one of gauges of potentials: *Coulomb*, *radiation* or *transverse gauge*:

$$\nabla \cdot \vec{A} = 0. \quad (1.30)$$

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Therefore principle equations of potentials Eqs. (1.28) are simplified to drop divergence term of vector potential,

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}, \quad (1.31)$$

$$\nabla^2 \vec{A} - \frac{\partial^2 \vec{A}}{c^2 \partial t^2} - \nabla \frac{\partial \Phi}{c^2 \partial t} = -\mu_0 \vec{J}. \quad (1.32)$$

The first principle equation of actually scalar potential Φ only now surprisingly retreats into our familiar Poisson equation with definition of scalar potential

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d\vec{x}', \quad (1.33)$$

with arbitrary constant set as null for simplicity, denotes that Φ is truly our acquaintance, *Coulomb* potential Φ in electrostatics! Whereas the second principle equation of both potentials is possible to be reformed to a neater form, making use of irrotational property of gradient of any scalar. Any vector field such as current density can be decomposed into two terms,

$$\vec{J} = \vec{J}_l + \vec{J}_t, \quad (1.34)$$

where subscript l indicates longitudinal (irrotational)

$$\nabla \times \vec{J}_l = 0, \quad (1.35)$$

and subscript t transverse (solenoidal)

$$\nabla \cdot \vec{J}_t = 0. \quad (1.36)$$

Starting from the vector identity

$$\nabla \times (\nabla \times \vec{J}) = \nabla(\nabla \cdot \vec{J}) - \nabla^2 \vec{J}, \quad (1.37)$$

as well as the important Green function Eq. (1.12), one can write current density

$$\vec{J} = -\frac{1}{4\pi} \int d^3 \vec{x}' \nabla^2 \frac{\vec{J}(\vec{x}', t)}{|\vec{x} - \vec{x}'|}. \quad (1.38)$$

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Hence it is possible to determine longitudinal and transverse current as

$$\vec{J}_l = -\frac{1}{4\pi} \nabla \int d^3x' \frac{\nabla' \cdot \vec{J}}{|\vec{x} - \vec{x}'|}, \quad (1.39)$$

$$\vec{J}_t = \frac{1}{4\pi} \nabla \times \nabla \times \int d^3x' \frac{\vec{J}}{|\vec{x} - \vec{x}'|}. \quad (1.40)$$

Thus Eq. (1.33) and continuity equation lead to the fact that longitudinal current links to scalar potential

$$\vec{J}_l = \frac{1}{c^2 \mu_0} \nabla \partial_t \Phi(\vec{x}, t). \quad (1.41)$$

Scalar potential could be removed from the second principle equation Eq. (1.32), which becomes thus

$$\nabla^2 \vec{A} - \frac{\partial^2 \vec{A}}{c^2 \partial t^2} = -\mu_0 \vec{J}_t. \quad (1.42)$$

Notice here that although Coulomb radiation existent instantaneously everywhere in space seems to contradict the context of electromagnetics, physical fields still propagates in speed of light.

Ray optics and varying refractive index

Although well recognized as almighty analytic tools to describe electromagnetic fields in principle, yet Maxwell equations do not enable people to solve all the practical problems. One of the possible approximations relies on such a fact: when wavelength of light is far below characteristic length scale of interaction, finite length of wavelength is approximated into negligible and concept of light ray comes into play. It is understood as a limit of situation when wavelength approaches infinitesimal $\lambda \rightarrow 0$ (but not exact zero). In such a situation, diffraction effect tends to nullify itself because light beam has been postulated as constantly infinitesimal narrow, which is de facto the concept of ray optics or geometrical optics. Essence to treat propagation problem of light ray, is the famous Fermat's principle or the principle of least of time. We shall narrate it and extend to the method of Lagrangian form to specially trace light ray within Maxwell's fisheye.

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Analogy to principle of least action in classical mechanics, Fermat principle serves as an equivalent cornerstone in ray optics. Just like classical mechanics, variational method gives rise to the method of Lagrangian. Fermat principle is narrated as that rays of light traverse the path of stationary optical length with respect to variations of the path in form of

$$\delta \int_P^Q n(x, y, z) ds = 0, \quad (1.43)$$

in which starting point P and end point Q determine a certain path, n means refractive index function upon space. It is worth pointing that Fermat principle is stationary, which indicates extrema: maxima or minima. Usually or in most common cases, this refers to maxima but not necessarily guarantees so. It is believed that even saddle point may show up in some certain-specified cases. Since Lagrangian $\mathcal{L}$ is defined as integrand of time instead of space in Fermat's principle Eq. (1.43):

$$\delta \int_{t_1}^{t_2} \mathcal{L} dt = 0. \quad (1.44)$$

It is easy to see Lagrangian can be accordingly defined as

$$L(x, y, \dot{x} = \frac{dx}{dz}, \dot{y} = \frac{dy}{dz}, z) = n(x, y, z) \left[1 + \dot{x}^2 + \dot{y}^2 \right]^{\frac{1}{2}}. \quad (1.45)$$

Although arbitrary, z direction is predominantly chosen close to the propagation direction of light ray and often aligns with symmetry axis of the optical system. According to Lagrangian equations,

$$\frac{d}{dz} \frac{\partial L}{\partial \dot{x}} = - \frac{\partial L}{\partial x}, \quad (1.46)$$

$$\frac{d}{dz} \frac{\partial L}{\partial \dot{y}} = - \frac{\partial L}{\partial y}, \quad (1.47)$$

one derives the vector equation of light ray,

$$\frac{d}{ds} \left(n \frac{d\vec{r}}{ds} \right) = \nabla n(\vec{r}). \quad (1.48)$$

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If unit vector along tangent direction of light ray is defined as

$$\hat{\vec{s}} = \frac{d\vec{r}}{ds}, \quad (1.49)$$

we have another form of Eq. (1.48):

$$\frac{d}{ds}(n\hat{\vec{s}}) = \nabla n. \quad (1.50)$$

Now we derive light rays in Maxwell's fisheye structure [9]

$$n(r) = \frac{n_0}{1 + \frac{r^2}{R_1^2}}, \quad (1.51)$$

Since this is a radius-symmetrical structure, gradient of refractive index is along radius direction,

$$\nabla n \sim \hat{\vec{r}}, \Rightarrow \hat{\vec{r}} \times \frac{d}{ds}(n\hat{\vec{s}}) = 0. \quad (1.52)$$

One is able to write

$$\frac{d}{ds}(n\hat{\vec{s}} \times \hat{\vec{r}}) = 0, \quad (1.53)$$

thus all light rays are confined within a plane, so we choose it as x-y plane. Under normal spherical coordinate, along light path $\hat{\vec{s}}$, arc length

$$\begin{aligned} ds &= [dr^2 + r^2 \sin^2 \theta d\theta^2 + r^2 d\phi^2]^{\frac{1}{2}} \\ &= [dr^2 + r^2 d\phi^2]^{\frac{1}{2}} \\ &= [1 + r^2 \dot{\phi}^2]^{1/2} dr, \end{aligned} \quad (1.54)$$

where the dot above symbol indicates differential to radius r . Notice that in spherical coordinate, light rays are confined within plane $\theta = \pi/2$, so only two variables r and ϕ are independent. Similar to Eqs. (1.46),

$$L = n(r)(1 + r^2 \dot{\phi}^2)^{\frac{1}{2}}, \quad (1.55)$$

$$\frac{d}{d\phi} \frac{\partial L}{\partial \dot{\phi}} = -\frac{\partial L}{\partial \phi} = 0, \Rightarrow \frac{n(r)r^2 \dot{\phi}}{[1 + r^2 \dot{\phi}^2]^{1/2}} = C_1, \quad (1.56)$$

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where C_1 is an integral constant independent of ϕ (and thus independent of r , since otherwise ϕ would implicitly contain r dependence). In a special case of Maxwell's fisheye, integration of ϕ gives light ray path in this medium,

$$\sin(\phi + C_2) = \sin(\phi_0 + C_2) \frac{(r^2 - R_1^2)r_0}{(r_0^2 - R_1^2)r}. \quad (1.57)$$

This formula also indicates that all light rays originating from (r_0, ϕ_0) will intersect again at $(\frac{R_1^2}{r_0}, \phi_0 + \pi)$, making a perfect image via circular arc [9]. An alternative-yet-equivalent derivation can be as follows. Let Φ indicate the eikonal, (ϕ is replaced into θ to avoid vagueness)

$$|\nabla\Phi|^2 = n^2(\vec{r}), \left(\frac{\partial\Phi}{\partial r}\right)^2 + \left(\frac{\partial\Phi}{r\partial\theta}\right)^2 = n^2(r). \quad (1.58)$$

Because no radial dependence is observed on the right hand side above, we assume

$$\frac{\partial\Phi}{\partial\theta} = m, \quad (1.59)$$

and take the trick to write

$$n(r) \frac{r\partial\theta}{\partial L} = \frac{m}{r}, n(r) \frac{\partial r}{\partial L} = \sqrt{n^2(r) - \frac{m^2}{r^2}}. \quad (1.60)$$

Therefore,

$$\theta - \theta_0 = \int_{r_0}^r \frac{m/r}{\sqrt{n^2(r)r^2 - m^2}} dr \quad (1.61)$$

$$= \sin^{-1} \left(\frac{K}{\sqrt{1 - 4K^2}} \frac{\rho^2 - 1}{\rho} \right) - \sin^{-1} \left(\frac{K}{\sqrt{1 - 4K^2}} \frac{\rho_0^2 - 1}{\rho_0} \right), \quad (1.62)$$

where $K = \frac{m}{R_1 n_0}$ and $\rho = \frac{r}{R_1}$ [10].

In Chap. 4, dyadic Green function method will be used to derive electromagnetic fields from motion of electron bunches. Dyadic Green function is essentially dyadic operator which transfers current density vector into sought field vectors. The detailed derivation is shown in respective chapters for the relevant concrete geometrical

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structure. A stable numerical method to trace the light trajectory is to use the Hamilton's equations in Sec. 4.2: $\dot{\vec{r}} = \partial\omega/\partial\vec{k}$ and $\dot{\vec{k}} = -\partial\omega/\partial\vec{r}$, in which $\vec{k}$ indicates momentum, ω the Hamiltonian, the dot above the derivative of time [11, 6].

1.2 Literature Review and Motivation

This section covers the review and motivation for the problems investigated in this thesis. We shall start from basic physical understanding of our problems relevant and then try to indicate our motivation based on the literature review of the Space-Charge Limited current problems and the radiation from electrons' motion.

1.2.1 Time-dependent space-charge-effect for field emission

According to Coulomb's law Eq. (1.3), arbitrary portion of electric charge density ρ , instead of distinct point-like charges in vacuum (for simplicity we discuss only vacuum although dielectric material is also possible to extend to in semiconductor-device physics, but not conductors which tends to rapidly neutralize or screen other charges) induces a form of electrostatic field defined as

$$d\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \rho(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} d\vec{x}'. \quad (1.63)$$

Moreover, this electric field is linear thus accountable from integral upon space as long as charge density $\rho(\vec{x})$ is justified. This effect is often named as *space charge effect*. Physically, space charge effect from some charges indicates repulsion action towards other charges, or holistic speaking the self-field of electric charges (in common cases, exclusively electrons). In high current diode engineering, this effect refers to a specific problem to seek the maximum current density(space-charge-limited current density) under certain diode conditions, say, electrons are emitted from some region of a solid called cathode (in opposition with anode, another side made of certain solid-state material of diode to accept electrons). Since electrons are represented in form of charge density $\rho(\vec{x})$, we can interpret them as a cloud of electrons. The cloud of emitted charged carriers can form a space charge region if they are sufficiently spread out, or the charged atoms or molecules left behind in the solid can form a space

charge region, which will not be touched in this thesis. This situation is perhaps also familiar in the area near a metal object when heated to incandescence in a vacuum. This effect was first observed by Thomas Edison in light bulb filaments, where it is sometimes called the Edison effect[8].

Now some physical picture is given for space-charge effect. Imagine first only limited electrons are existent within space D under voltage ϕ between cathode and anode of a diode. Since electron quantity is small, we are able to inject or emit more electrons from cathode but there is still room for more electrons. Repeat this process until we find no room for emitted electrons. This *gedankenexperiment* or thought experiment reminds us that when electrons are accumulated in the diode space in such a compact way that it leaves no room for any further addition of electrons, we achieve a space-charge limit. Child and Langmuir ever investigated this problem in framework of electrostatics via derivation from Poisson equation respectively in 1911 and 1913. The author puts this smart derivation in Appendix A1, which gives the maximum current density

$$J_{\text{CL}} = \frac{4\varepsilon_0}{9D^2} \sqrt{\frac{2e}{m}} \phi(x = D)^{3/2}, \quad (1.64)$$

where ε_0 , e and m are vacuum permittivity, particle charge and mass, respectively [12, 13]. Child-Langmuir(CL) law serves as a basic theoretical prediction for application of high current diodes. However, this space-charge-limited current is steady, restricted as temporal constant when the electrons are filled in the diode gap all the time. Concerns shall arise when these assumptions are broken. For instance, it can not be applied directly to account for the space charge effect in the ultrafast laser induced electron emission process because the pulse duration of the electron emission is less than the electron transit time—so called *short pulse condition*. This short pulse effect was studied in a short pulse model for the CL law by Á. Valfells *et al.* in 2002 [3] given by

$$J_{\text{crit}} = 2 \frac{1 - \sqrt{1 - \frac{3}{4} X_{\text{CL}}^2}}{X_{\text{CL}}^3} J_{\text{CL}}, \quad (1.65)$$

1.2 Literature Review and Motivation

where $X_{\text{CL}} = \tau_p/\tau (< 1)$ is the normalized pulse length and $\tau = 3D\sqrt{m/2eV_g}$ is the transit time at SCL condition. It anticipates a much larger current density than Child-Langmuir formula. In order to derive this critical current density formula for a short-pulse electron flow *constant in time*, only virtual cathode beam front is considered and internal repulsion force within pulse spatially is sidestepped smartly.

There are also quantum extension [14, 15] of space-charge effect for the collective behaviors of electrons in quantum diode [16]. This space-charge limit can be also extended to electrodynamic effect: the displacement current effect(Ampère's law) instead of the self-magnetic field(Faraday's law)^{||} increases the space-charge limit to some extent [17]. A most recent work found the Langmuir-Blodgett solutions for the space charge limited current density, for both cylindrical and spherical diodes, by the method of accurate transit time model [18]. However, these are out of the scope of this thesis, we shall restrict ourselves in one-dimensional classical electrostatics.

However, both formulae above only apply to temporally-steady status of electron flow and time-dependence of injection current density has not been touched. There are questions to ponder whether a time-varying injection electron flow will change the space-charge limited(SCL) current. In Chaps 2 and 3, we investigate how time-varying effect demonstrates its advantage to transcend traditional space-charge limit aforementioned.

A natural and physical example of time-varying electron emitter could be field emitter excited from ultrafast laser pump [19, 20, 21, 22, 23, 24, 25, 26]. In Chap. 2, an iterative calculation method to obtain time-dependent short-pulse space-charge-limited current density is developed, based upon a non-equilibrium model to describe ultrafast laser-excited electron emission from metallic under an externally-applied DC field [23], confirmed by a three-dimensional particle-in-cell(PIC) simulation [27]. Our calculation aims to show that for ultrahigh laser field amplitude the space charge effect becomes notable and electric DC field can be an efficient parameter to tune the space-charge limit.

Before turning to the abstraction of the next chapter, some introduction to the non-equilibrium model to treat electron emission from metal surface [23] is made herein as a preparation. Following Rethfeld's relaxation-time approach [28], consider

^{||}Be alert that it is the fields in Coulomb's and Faraday's laws that reduce to space-charge field in electrostatics, not in Ampère's law!

Chapter 1. INTRODUCTION AND MOTIVATION

a simple case when electron gas and phonon gas interacts with each other within metal, whose distribution functions $f(k)$ and $g(q)$ respectively, both obey Boltzmann's equations.

$$\frac{\partial f(k)}{\partial t} = \frac{\partial f(k)}{\partial t}\bigg|_{e-e} + \frac{\partial f(k)}{\partial t}\bigg|_{e-p} + \frac{\partial f(k)}{\partial t}\bigg|_{absorb}, \quad (1.66)$$

$$\frac{\partial g(q)}{\partial t} = \frac{\partial g(q)}{\partial t}\bigg|_{p-e}. \quad (1.67)$$

All the rest interaction to electrons and phonons are implicitly neglected. Before illumination by laser, they obey Fermi-Dirac and Bose-Einstein distributions, respectively. Knowing all the coefficients inside the ordinary differential equation for $f(k, t)$, it is at ease to numerically solve it by integration. The tunneling probability $T(E)$ for electrons of energy E to emit out of metal surface, is replied on the approximation below,

$$T(E) = \left[1 + \exp \left(\frac{4\pi}{h} \sqrt{2me} \int_{x_1}^{x_2} dx \sqrt{V(x) - E} \right) \right]^{-1}, \quad (1.68)$$

in which unit of eV is used and x_1, x_2 refer to the two roots of equation

$$V(x) - E = 0. \quad (1.69)$$

Notice that when energy E lies above $V(x)$, the traditional Murphy-Good method is taken to approximate the tunnelling probability in linear extrapolation [29](p34). One is now ready to turn to Chap. 2.

In Chap. 3, time-dependence of injection current density is imparted more freedom to explore methods to promote space-charge limit further. Knowing a specific laser-excitation mechanism involves time varying injection electron flows but may also induce other factors which might perplex or even shadow our observation to seek time varying profile of electron flows, we take a simple way to walk around interference from laser-excitation: simply remove it. We simplify our model to only involve time-dependent injection current density $j(t)$. We extend the short-pulse space-charge limited(SCL) electron flow model [3] by having time-dependent injection of electron flow. By trying different profiles of time-variant for the injected current density, we wish to speculate on how to promote space-charge limit under time varying condition.

1.2 Literature Review and Motivation

M. E. Griswold *et al.* in 2010 [30], offered an upper bound of time-averaged current density analytically, speculating that such an electron flow will surpass what classical CL law predicts [30, 31, 32]. They impose the electric field in the diode written as

$$E(x) = E_b + \frac{1}{2\epsilon_0} \left[\int_0^x \rho(x') dx' - \int_x^D \rho(x') dx' \right], \quad (1.70)$$

in which E_b is a constant of integration which later turns out to represent the electric field from the image charge on the electrode plates **. To derive the value of E_b , the voltage drop across the diode was enforced as V_g [let $V_g = \phi(D) - \phi(0)$]:

$$V_g = \int_0^D E(x) dx, \quad (1.71)$$

which gives

$$E_b = \frac{V}{D} - \frac{Q}{\epsilon_0} \left(1 - \frac{x_q}{D}\right) + \frac{1}{2\epsilon_0} \int_0^D \rho(x') dx', \quad (1.72)$$

in which

$$Q \equiv \int_0^D \rho(x') dx', \quad (1.73)$$

$$x_q \equiv \frac{1}{Q} \int_0^D x \rho(x') dx'. \quad (1.74)$$

Henceforth, the electric field in the diode Eq. (1.70) becomes

$$E(x) = \frac{V_g}{D} - \frac{Q}{\epsilon_0} \left(1 - \frac{x_q}{D}\right) + \frac{1}{\epsilon_0} \int_0^x dx' \rho(x'). \quad (1.75)$$

It is thus of particular interest to point out the equality

$$2E_b = E(x=0) + E(x=D). \quad (1.76)$$

The upper bound to time-averaged space-charge limit under long time (large compared with transit time across the diode) was conjectured from an iterative process similar to our numerical algorithm in Chap. 2, as high as no more than $2.45J_{CL}$ but no affirmative proof was given yet. Therein, a time-varying electric field

**M. E. Griswold, *private communication*, 2014

at the cathode $E_K(t)$ (boundary condition) was allowed. This conjecture originated from the limit(notice every E s appears as nonnegative quantity)

$$E_K \equiv E(x=0) \geq 0, \quad (1.77)$$

(pushing electrons away from the cathode) and the upper limit originated from the relation

$$< J_{CL} > \leq \frac{Q_{\max}}{\tau_{\min}}, \quad (1.78)$$

where $\tau_{\min}$ indicates the minimum value of transit time of the fastest possible situation. Notice that Eq. (1.77) means

$$\frac{V_g}{D} \geq \frac{Q}{\epsilon_0} \left(1 - \frac{x_q}{D}\right), \quad (1.79)$$

implying the upper limit could be approached by putting more electrons further towards the anode. This is the first constraint for the upper limit on the average current density. The second comes from the expansion of an element of charge due to its own space charge repulsion:

$$\rho(x) \leq \rho_{\max}(x) = \frac{2m\epsilon_0}{et_{\min}^2(x)}. \quad (1.80)$$

Both constraints finally lead to the picture shown by Figure 2.9(cf. also Figure 2.7) in Griswold's PhD thesis [32], which appeared to contradict the implication from the first constraint (1.79). This point might indicate that during the compromised balance one of the two finally dominates.

Hereby in Chap. 3, we numerically attempt to test injection electron flows in different time-profiles, in order to search for an optimal profile function, which may enable them to transcend Valfells' formula under short-pulse situation, by means of particle-in-cell(PIC) simulation to consider *time-dependent* effect for short-pulse electron flow in one-dimensional case. We find in general, an increasing current ($j(t)$) will enforce an enhancement as compared to decreasing profiles.

1.2 Literature Review and Motivation

1.2.2 Radiation from motion of electron bunches

The aforementioned picture of electrostatics is based on the assumption that Poisson equation stands. However, it could be otherwise – when electrostatics fails and time-dependent electrodynamics comes into play, for example the displacement current effect [17, 33]. In the framework of electrodynamics, travelling wave shall occur under certain circumstances of current density source(e.g. electrons) and therefore the radiation induced by the moving electron source also deserves some academic interest. This method of calculating radiation from the source of a certain current density could further benefit the physical understanding behind emission process.

Let us start this topic of motion-induced radiation from the electrostatics picture in previous subsection. Consider a single electron static at one spatial point in vacuum. Coulomb law, reducible from Poisson’s equation, seems to describe all electric field environment around it except its source point, and nothing wavy appears herein. Now let us suppose, like in mostly natural situations, the electron moves, at a constant predefined velocity v for simplicity. Without even resorting to Maxwell’s equations, we know, definitely some trace of electromagnetic accompanying fields, of course wavy stuff will generate because moving electron distorts the concentric electric field lines into thicker ones in its vicinity. But normally this is just evanescent wave which carries almost no energy thus no radiation at all; unless some critical conditions stand, can radiation occur in such a physical situation. One simple instance can be *Čerenkov radiation*(CR), which occurs, because electron travels faster than light velocity in the medium(vacuum is excluded, limited by special relativity) so that wavefronts from past moving electron accumulate coherently into travelling waves. Another instance, *transition radiation*(TR) also bears the similar physical essence because interface between two media actually force electron lose partial momentum which transfers into transition radiation energy, since both kinds of radiation energy comes from kinetic power of electron.

On the other hand, in current optics community, an emergent *Transformation Optics* strategy [34, 35], developed from traditional photonic crystal [36], provides such a possibility, which enables manipulating light flow almost arbitrarily, for instance to invisibilize objects hidden inside an *ad-hoc* device–cloak, of inhomogeneous permittivity and permeability(transformation optics is nothing novel, which de facto

by heterogeneity and anisotropy distorts electromagnetic space to a certain extent). This great idea [37] also inspired a novel field – metamaterial, which refers to structures smaller than the wavelength of light [6]. This metamaterial awakes again people’s desire to manipulate and shape light, resulting in many developments in engineering world [38]. The corner stone of all the emergent facilities provided by metamaterial is the invariance form of Maxwell’s equation (Maxwell’s equations stand in curved coordinates), an old mathematical fact known for long [39].

Invariance form of Maxwell’s equations guarantees light to be shaped by curved electromagnetic space in order to behave according to man’s design. However from perspective of electron, it does not bend its trajectory opposite to light and instead experience curved space to induce radiation. This inspired an EM detection method for cloak service proposed by Zhang *et al.* [40]. In Zhang’s paper, for a nontrivial anisotropic space for photon (electromagnetic space perceived by photons, generally tensors of electric permittivity and magnetic permeability), special manipulation enables some kind of cloak effect — electron does not perceive its motion space distorted as photon thus feels its distortion in optical space [40], which is possible for cloak sensing.

Moreover any *curved electromagnetic space* becomes a possible energy-pumping candidate to give rise to radiation from motion of charges.

This curved EM space could be in various cases in general principle. Traditional photonic crystal in periodic structure also enables radiation from motion of charges [41]. Similar to photonic crystal, researchers structured a special tunnel to permit electrons travel through periodically-alternating material, diffraction radiation also sparked its light over a broad angular range [42]. This transfer from kinetic energy of electron bunches to radiation also bears resemblance to transition radiation because electrons interact with inhomogeneous dielectric parameters $\epsilon(\vec{r})$ during their passage.

In Chap. 4, the focus is given to motion-induced radiation through a simple curved geometry (heterogeneity) of light: *Maxwell’s fish-eye*, which in principle provides unlimited resolution as a perfect imaging lens [43]. The reason we choose this refraction distribution is that Maxwell fisheye profile is the simplest curved space known for light and thus serves as a good toy-model to investigate general EM properties from curved space. This curvature of electromagnetic space hides itself

1.2 Literature Review and Motivation

from photons' perspective but manifest itself to electrons because they perceive the space mechanically due to its distinct feature as a particle of mass comparing to photon, which is massless[37].

Here I finish the Intro. Sec.

CHAPTER 2

SPACE CHARGE EFFECT OF TIME-DEPENDENT EMISSION CURRENT EXCITED FROM ULTRAFAST LASER

2.1 Introduction

Significant efforts have been recently paid on research for femtosecond laser-metal interaction to emit electrons [19, 20, 21, 22, 23, 24, 25, 26] for miscellaneous purposes to provide ultrafast time-resolved information about the underlying dynamics in physics, chemistry and biology [44]. In most studies, space charge effect from repulsion force of electron cloud, has been ignored, which may be important at the high current regime operating at the high laser fields [45, 46].

Therefore when electrons emit out of any form of cathode, a question often neglected is whether space charge limit could be reached in such a process and a further one arises: whether this affect laser-tunneling mechanism.

Before coming to term of this question above, it necessitates to retrospect the concept of space charge limit in electrostatics. Space charge limited (SCL) law describes the maximum steady-state current density to transport across a diode of

2.1 Introduction

distance D under a voltage ϕ between cathode and anode, i.e. Child-Langmuir law

$$J_{\text{CL}} = \frac{4\varepsilon_0}{9D^2} \sqrt{\frac{2e}{m}} \phi^{3/2}, \quad (2.1)$$

where ε_0 , e and m are vacuum permittivity, particle charge and mass, respectively [12, 13].

SCL limited electron flow occurs when the total charges of the emitted electrons are sufficient to suppress the electric field at the cathode to zero, when the electrostatic potential distribution function $\phi(x)$ is

$$\phi(x) = V_g \left(\frac{x}{D} \right)^{4/3}. \quad (2.2)$$

as shown in Fig. 2.1. However, the CL law is only valid for steady-state electron flow

Figure 2.1: (Color online) Electrostatic potential distribution function $\phi(x)$ of both empty spacing (blue) and full of steady space-charge-limited electron flow (red). Notice that the red curve is not symmetrical between cathode and anode, which indicates the asymmetry of the diode voltage distribution. Inset: illustration of the physical model of the diode system. D is the distance of the diode, and $V_g = \phi(x = D)$ indicates the voltage drop across the diode.

with a constant emitting current density J . It can not be applied directly to account for the space charge effect in the ultrafast laser induced electron emission process

because the pulse duration of the electron emission is less than the electron transit time. This short pulse effect had been studied in a short pulse model for the CL law by Á. Valfells *et al.* in 2002 [3] given by

$$J_{\text{crit}} = 2 \frac{1 - \sqrt{1 - \frac{3}{4} X_{\text{CL}}^2}}{X_{\text{CL}}^3} J_{\text{CL}}, \quad (2.3)$$

where $X_{\text{CL}} = \tau_p / \tau (< 1)$ is the normalized pulse length and $\tau = 3D\sqrt{m/2eV_g}$

Figure 2.2: (Color online) A higher-than-CL-law space-charge-limit current density J_{crit} (Eq. (2.3), red curve) for short pulse electron beam of a temporal length τ_p . Illustration for single sheet model [3] is shown in dashed blue curve. Parameters: normalized pulse length $X_{\text{CL}} = \tau_p / (3D\sqrt{\frac{m}{2e\phi}})$, D gap distance, m, e, V electron mass and charge, and ϕ anode voltage, respectively. Generated through **Mathematica** [47].

is the transit time at SCL condition. It anticipates a much larger current density than Child-Langmuir formula. In order to derive this critical current density formula for a short-pulse electron flow *constant in time*, only virtual cathode beam front is considered and internal repulsive force within pulse spatially is sidestepped smartly*; however, when injection current density becomes time-dependent (varying with time), repulsion between internal electrons reshape its own spatial profile as a result of

*This by no means ignores, it just avoids.

2.2 A Numerical Method to Treat Time-Dependent Space Charge Effect

nonuniform temporal profile of the electron beam, and thus this trick of considering only beam front [3] fails to validate itself. The constant assumption in time is not valid for ultrafast laser induced photo-field electron emission due to the non-equilibrium heating at a time scale even shorter than the laser pulse length [23]. Here, from $t = 0$ (beginning of the laser pulse) to $t = \tau_p$ (the end of the pulse), the injected current density from the cathode at $x = 0$ is a function of time, given by $J(t)$. To solve this problem, we need to find the spatial variation of the injected electron density into the gap, $J(x)$ for $0 \leq x \leq s$ at $t = \tau_p$, when $s(< D)$ is the position of the beam front. Once this variation $J(x)$ is obtained, we may use the similar approach in Ref. [3] to calculate the space charge limited electron flow for the ultrafast-laser-induced emission.

Despite other extensive studies that have been done to extend this 1D classical CL law and Valfells' formula [48, 49, 15, 17], we attempt to figure out the importance of space-charge effect during ultrafast laser-induced electron emission in this chapter. We thus develop a model to calculate the saturation (due to space charge effects) of ultrafast laser-excited electron emission from a metallic surface under an applied electric (dc) field [23][15].

Thus in this Chapter, we develop a numerical algorithm to calculate the time-dependent space-charge-limited current density of a short-pulse electron flow, based on our previous work on a non-equilibrium model to describe ultrafast laser-excited electron emission from a metallic surface under an applied electric (DC) field [23].

2.2 An Iterative Numerical Method to Treat Time-Dependent Space Charge Effect

Consider a metal-vacuum interface under an ultrafast laser pulse as well as an external constant electric field F_{dc} , which is applied across a distance of D from cathode ($x = 0$) and anode ($x = D$). Thus electric potential distribution is $\phi_{dc}(0 \leq x \leq D) = F_{dc}x$. Classical tunneling mechanism of conduction electrons in metal deem that, they first encounter a modified energy barrier potential due to electrostatic force and then possess a certain probability to penetrate it, i.e. a laser-excited tunneling process [23, 50, 28, 19, 20, 22, 21]. Whereas existent emitted electrons in spacing

D reaching as far as distance $s(0 \leq x \leq s < D)$ resist electrons emitting out from metal and serve wholly as an additional space charge electric potential distribution—space charge potential profile. This physically induces a new potential term $\Delta\phi(x)$ in original electrostatic potential distribution function across cathode and anode (original potential displayed in Fig. 2.3), thus

Figure 2.3: Original linear electrostatic potential distribution function $\phi(0 \leq x \leq D) = F_{dc}x$. Parameters: occupation length of emitted electrons $s = 0.2196\text{nm}$, electrodes spacing $D = 1\text{cm}$, electrostatic field $F_{dc} = 1 \times 10^6\text{V/m}$.

$$\phi(0 \leq x \leq D) = F_{dc}x + \Delta\phi(x), \quad (2.4)$$

$$\Delta\phi(s < x \leq D) \neq 0! \quad (2.5)$$

where s means the longest distance emitted electrons are able to transit. The value of s can be calculated by solving the equation of motion for $x(t)$ given by

$$\frac{d^2}{dt^2}x(t) = \frac{e}{m}[F_{dc} + \frac{d}{dx}\Delta\phi(x)], \quad (2.6)$$

[†]with initial conditions: $x(t = 0) = 0$ and $x'(t = 0) = v_0 \approx 0$. Here, the initial velocity v_0 is kept to be reasonably small (but not equal to 0) to avoid the difficulty in the numerical integration of Poisson equation (see below) at $x = 0$. In general, we have $v_0\tau_p \ll s$ in our calculation. Obviously, assumption that electrons keeps travelling

[†]Positive symbols J, e etc. are used to avoid confusion.

2.2 A Numerical Method to Treat Time-Dependent Space Charge Effect

forward is taken to allow invertibility of $x(t)$ and spatial profile of electron beam is assumed unchanged during period of travelling forward (a numerical iterative method is taken later in the last paragraph of this Section to restore such-induced error).

According to a modified Wentzel-Kramers-Brillouin (WKB) method, electron tunneling probability $T(E)$ is calculated through energy barrier profile $V(0 \leq x \leq D) = V_f + W - \frac{e^2}{16\pi\epsilon_0 x} - exF_{dc}$ [51, 52], in which V_f stands for Fermi energy value of emitter material and W its work function. To take into consideration that emitted electrons exerts a space charge field so as to obstruct more electron from emitting out of metal, a perturbation term of space-charge-effect induced potential, $-\Delta\phi(x)$ is added, making the potential barrier near the cathode surface

$$V(x) = V_f + W - \frac{e^2}{16\pi\epsilon_0 x} - exF_{dc} - e\Delta\phi, \quad (2.7)$$

Under this modified energy barrier profile $V(0 \leq x \leq D)$, time-dependent tunneled current density function $J(t)$ is calculated from

$$J(t) = \frac{em}{2\pi^2\hbar^3} \int_0^\infty dW T(W) \int_W^\infty f(E, t) dE, \quad (2.8)$$

in which W is a dummy variable in integration and $f(E, t)$ is non-equilibrium electron distribution function, $\hbar$ is reduced Plank constant [23]. Accordingly using Eq.(2.8), the emitted charge density (per unit area) within the laser pulse-width is calculated by

$$\sigma_{EC} = \int_0^{\tau_p} J(t) dt. \quad (2.9)$$

Now we show our mechanism to consider the space-charge effect in the process of laser-tunneled electron emission. To determine $\Delta\phi(x)$ in the region of $0 \leq x \leq s$, we solve the Poisson equation after Eq. (2.4) is substituted,

$$\frac{d^2}{dx^2} \Delta\phi(x) = \frac{J(x)}{\epsilon_0 v(x)}, \quad (2.10)$$

Chapter 2. Space Charge Effect of Time-dependent Emission from Laser

where $v(x)$ is the velocity profile of the electron flow that can be obtained by using the energy conservation,

$$\frac{mv^2(x)}{2} = e[F_{dc}x + \Delta\phi(x)]. \quad (2.11)$$

Here, $J(x)$ is the current density profile at the end of the pulse, and it is related to the time-dependent emission velocity [cf. Eq.(2.8)] by

$$J(x) = J[\tau_p - t(x)], \quad (2.12)$$

where $t(x)$ is the inverse function of $x(t)$ solved in Eq.(2.6). This approximation implies that our algorithm focuses on the end of electron pulse $t = \tau_p$. Combining Eqs.(2.10) to (2.12), we have

$$\frac{d^2}{dx^2}\Delta\phi(x) = \frac{J[\tau_p - t(x)]}{\epsilon_0 \sqrt{\frac{2e}{m}(F_{dc}x + \Delta\phi(x))}}. \quad (2.13)$$

The boundary conditions for solving Eq.(B.4) are the zero space potential at the cathode ($x = 0$) and the continuous electric field at the beam front ($x = s$), which are respectively,

$$\Delta\phi(0) = 0, \quad (2.14)$$

and

$$\frac{d}{dx}\Delta\phi(x)|_{x=s} = -\frac{\Delta\phi(s)}{D-s}. \quad (2.15)$$

Note Eq.(2.15) is reduced from $d\phi(x)/dx|_{x=s} = (V_g - \phi(s))/(D-s)$. The last boundary condition Eq. (2.15) denotes that electrostatic field continues at electron beam end position $x = s$ [3]. Finally, the electrostatic potential in the vacuum region of $s < x \leq D$ (in front of the electron beam) is

$$\Delta\phi(s \leq x \leq D) = \frac{\Delta\phi(x=s)}{D-s}(D-x). \quad (2.16)$$

Solving $\Delta\phi(x)$ appears essential in all this algorithm to make use of energy potential modification Eq. (2.7) and to obtain electron position function from

2.2 A Numerical Method to Treat Time-Dependent Space Charge Effect

Eq. (2.6). However, straightforward derivation from Eqs. (2.7), (2.8), (2.13), (2.6) and (2.17) to obtain an analytic Child-Langmuir-law-like formula turns out to failure due to complexity of Poisson equation Eq. (2.13): the author finds no affordable analytic method to solve $\Delta\phi(x)$ from Poisson's equation Eq. (2.13), which is involved in both sides of Eq. (2.13) simultaneously making it not a well-posed problem. This feature of Eq. (2.13) perplexes its solution because directly solving it numerically results in a positive $\Delta\phi(x)$ distribution which is omissible here, violating physical intuition—we perceive electron bunch induces negative electrical potential term here additionally.

Not only to sidestep this unnecessary mathematical trouble, but also more significantly to restore induced errors due to aforementioned neglected repulsion force among electrons in Eq. (2.12); henceforth, a *numerical* algorithm is constructed to perform the calculation iteratively, illustrated by a flow chart in Fig. 2.4: On one hand, postulate that in the first iterative unit, $\Delta\phi(x) = 0$ in Eq. (2.7) to obtain the first tunneled electron current density $J_1(t)$ from Eq. (2.8), which only serves as a roughly-guessed approximate estimated value of an exact one of $J(t)$. On the other hand, $\Delta\phi = 0$ is substituted into Eq. (2.6) to obtain electron position function $x(t)$ and hence its inverse function $t(x)$. So far, the first iterative unit is completed and solution of $\Delta\phi(x) \triangleq \Delta\phi_1(x)$ will be taken into Eqs. (2.7), (2.13) and (2.6) again in the next iterative one to obtain a second tunneled electron current density $J_2(t)$, which modifies value of first tunneled electron current density $J_1(t)$ to some extent due to difference between $\Delta\phi_1(x)$ and 0. Afterwards, this unit above is iteratively repeated to modify $J_2(t), J_3(t), \dots J_i(t)$, into $J_3(t), J_4(t), \dots J_{i+1}(t), \dots$; in which $J_i(t)$ means the i th density function as described above, until an evaluation condition (we observe the convergence of current density of final timepoint, $|J_{i+1}(t = \tau_p) - J_i(t = \tau_p)|/J_i(t = \tau_p) \leq 1.5 \times 10^{-4\dagger}$ as evaluation reference) is satisfied that the i th electron count is stabilized within a small tolerance range. It is acceptable to conclude that other physical parameters such as $\phi_i(x)$ and $J_i(t)$ are close enough to their true values; for instance, space charge potential distribution $\Delta\phi(x)$ is demonstrated in Figs. 2.5[§] and 2.6 .

[†]A code in **Mathematica** was written to perform the calculation.

[§]In Eq. (11) of Ref. [3], the positive sign before the second term should be minus due to the restrict $\phi(\xi) < V$.

Figure 2.4: Flow chart (by the TikZ and PGF Packages under $\text{\LaTeX}$ 6.1) of our iterative algorithm to calculate emitted electron counts under space charge effect. First roughly guess that in the first iterative loop, the first space charge potential profile $\Delta\phi(x) = 0$ to obtain the first tunneled density of electron count; then afterwards use Newton's second law to trace the inverse of electron position function versus time, $t(x)$. So far, the first iterative unit is completed and the first solution of Poisson's equation, $\Delta\phi_1(x)$ will be adopted in the next calculation loop to obtain a second tunneled electron number and so on recursively in the following loops, until an evaluation criteria is satisfied so that the final density of electron count is converged within an acceptable range.

2.2 A Numerical Method to Treat Time-Dependent Space Charge Effect

Figure 2.5: (a) Space charge potential distribution ($\Delta\phi(0 \leq x \leq D)$) within the whole spacing D between cathode and anode, with abscissa axis in logarithm scale; (b) Space charge potential distribution ($\Delta\phi(0 \leq x \leq s)$) within distance s as far as electrons reach, dashed black curve indicates $\phi_V(x > s)$ for a uniform short pulse (2.3)[3] and dotted magenta $\phi_V(x) + \Delta\phi(x)|_{x>s}$ superimposed; (c) Space charge potential distribution beyond distance s ($\Delta\phi(s \leq x \leq D)$) where no electrons exist and it behaves uniform electrostatic field. (a)-(c) Physical parameters: occupation distance of emitted electrons $s = 0.2196\text{nm}$, electrodes spacing $D = 1\text{cm}$, ultrafast laser temporal length $\tau_p = 50\text{fs}$, external electric field strength $F_{dc} = 1 \times 10^6\text{V/m}$, laser field amplitude $F_L = 5\text{GV/m}$.

Figure 2.6: Space charge potential distribution ($\Delta\phi(0 \leq x \leq D)$) within the whole spacing D between cathode and anode, both empty spacing (blue) and full of steady space-charge-limited electron flow (red); physical parameters: occupation distance of emitted electrons $s = 0.2196\text{nm}$, electrodes spacing $D = 1\text{cm}$, ultrafast laser temporal length $\tau_p = 50\text{fs}$, external electric field strength $F_{dc} = 1 \times 10^6\text{V/m}$, laser field amplitude $F_L = 5\text{GV/m}$, $X_{CL} = 9.87814 \times 10^{-5}$.

Therefore when iterative algorithm obtains its converged value, reshaping effect to spatial profile of transporting electron beam has been taken into account already. It is demonstrated in Fig. 2.7 that after each time of calculation unit, electron count number from Eq. (2.9) becomes stabilized gradually and this iterative algorithm produces valid data efficiently within less than 10 repetitive units.

Remark: In the algorithm above, the time-dependence in $J_i(t)$ is included in each solving process of Poisson equation, and then transferred into spatial dependence(approximation) of charge density $\rho(x)$ on the right hand of Poisson equation, as Eq. (2.13) shows.

2.2 A Numerical Method to Treat Time-Dependent Space Charge Effect

Figure 2.7: Electron count of each iterative calculation unit. Physical parameters: ultrafast laser temporal length $\tau_p = 50\text{fs}$, external electric field strength $F_{dc} = 1\text{V/nm}$, laser field amplitude $F_L = 5\text{GV/m}$.

It is important to note that this approach chose to overlook the space charge field within the short time scale less than τ_p but actually did not ignore it. The approach is similar to the short pulse model with constant current density J [3], which had been confirmed with particle-in-cell (PIC) code. We will also show our numerical results and the comparison with PIC simulation in the next subsection. Once the convergence is reached, we can determine the space charge field $\Delta\phi(x)$, and compute the emission number according to Eq. (2.9) under space-charge effect, as compared to our previous work that ignored it completely [23] [cf. Figs. 2.8 and 2.9 below].

To investigate space-charge effect for our particular time-dependent injection current density, $J(t)$ [23], emitted electron count number is calculated from our numerical method. In Fig. 2.8, electron count densities with space-charge effect (red square symbols) are generally lower than those without space-charge effect black dots. This variance less than one order of magnitude denotes that space-charge limit has not been achieved in our mechanism. We shall discuss more in the following sections.

Space Charge Limit

The aforementioned iterative algorithm calculates laser-induced emission charge density with space-charge effect comparing with the density without considering space-charge effect. Before presenting the results, we are also interested to calculate the saturation of the SCL current density at high fields due to the suppression of the *total* electric field towards zero at the cathode. Thus the space charge potential $\Delta\phi(x)$ must be large enough to suppress the applied DC field F_{dc} , given by

$$F_{\text{dc}} = -\frac{d}{dx}\Delta\phi|_{x=0}. \quad (2.17)$$

This formulation above is analogical to derivations of both Child-Langmuir law and short pulse critical injection current density formula Eq. (2.3), however, has considered time-dependence of short-pulse injection current density $J(t)$. An attempt to obtain an easy corollary for time-dependent injection current density $J(t)$ goes invalid since Eqs. (8)

$$E(x=s) = -\frac{J_0\tau_p}{\epsilon_0}, \quad (2.18)$$

and (10) of Ref. [3]

$$E(x=s) = -\frac{4\phi(x=s)}{3s}, \quad (2.19)$$

where $E(x=s)$ stands for electrostatic field at interface $x=s$ between electron flow and vacuum, lose their equalities and no quantitative relations are possible to establish directly; Eq. (2.3) fails as a result of nonuniform electron flow. Thus the space-charge-limited current density of final timepoint, $J_{\text{end}} \triangleq J(t=\tau_p)$ is expectedly higher than uniform space-charge-limited current density, J_{crit} , trivially. It is worth pointing that de facto, average current density is also possible to transcend traditional space-charge limit by Valfell's formula Eq. (2.3), which will be covered in next two sections.

When the above condition (2.17) is fulfilled, we define a critical (or SCL) current density (upper limit in emission yield) ,

$$J_{\text{SCL}}(t) = f J_0(t), \quad (2.20)$$

2.3 Results and Discussions to Compare with PIC 3D Simulation

where f is an enhancement factor over the time-dependent ultrafast laser emission model (without space charge effects) from Ref. [23], indicating that this f factor enhances electron yield to adequate extent to nullify electric field at cathode down to zero, as Eq. (2.17) stands. Here, f can be determined by

$$f = -\frac{F_{\text{dc}}}{\frac{d}{dx}\Delta\phi_0(x=0)}, \quad (2.21)$$

where the $\Delta\phi_0$ is the space charge potential obtained by solving the Poisson equation using $J_0(t)$.

2.3 Results and Discussions to Compare with Analytic Calculation and Particle-in-Cell(PIC) 3D Simulation

In this section, we attempt utilizing an approximate analytic method and 3D PIC simulation to supplement our numerical algorithm. In the simulation, we inject a time-dependent electron current density based on our model [23], and we determine the SCL current when reflection of electrons are detected [53, 17]. The size of emitter for 3D PIC is a square emitter of $100D \times 100D$. Need to note that this analytic method is not self-standing and has to be utilized under iterative loop of numerical method thus we only list it here for readers' reference. Because our model considers only one-dimensional variance, we deem 3D simulation expensive, and 3D simulation tends to induce incompatible data with our calculation results. So in next section, two dimensional simulation will be adopted to favor better our model of one-dimensional variance due to time-varying dependence of injection electron pulse.

(1) To contrast with and confirm our numerical results, an approximate analytic method is also derived as below. If current density function $J(t)$ is fitted as

$$J(t) = c_0 + c_1\frac{t}{\tau_p} + c_2\left(\frac{t}{\tau_p}\right)^2 + c_3\left(\frac{t}{\tau_p}\right)^3, \quad (2.22)$$

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then as the end of Sec. 2.2 narrates, space charge limit is reached when $J(t)$ multiplies a factor f ,

$$f = \frac{60\varepsilon_0 F_{dc}}{e\tau_p(40c_0 + 25c_1 + 18c_2 + 14c_3)}. \quad (2.23)$$

Its electron charge density results are marked in dark green open circle of Fig. 2.8, slightly higher than numerical density because this analytic method ignores $\Delta\phi(x)$ in Eqs. (2.13) and (2.6). If one assumes

$$\Delta\phi(x) = 0, \quad (2.24)$$

then Eqs. (2.13) and (2.6) simplify into

$$\frac{d^2}{dt^2}x(t) = \frac{e}{m}F_{dc}, \quad (2.25)$$

$$\frac{d^2}{dx^2}\Delta\phi(x) = \frac{J[\tau_p - t(x)]}{\sqrt{\frac{2e}{m}F_{dc}x}}. \quad (2.26)$$

Thus $\Delta\phi(x < s)$ is solved to insert into Eq. (2.21) to obtain Eq. (2.23) via Eq. (2.17) finally.

(2) To correct this error, another formula for current density factor f , more accurate,

$$f = -\frac{\varepsilon_0 F_{dc} D}{[\int_0^x \int_0^\xi \rho(\zeta) d\zeta d\xi + (D - s) \int_0^x \rho(\zeta) d\zeta]}, \quad (2.27)$$

is derived from Eqs. (2.13) and (2.17)—similar to (1) above, exactly the same as our numerical results shown in blue open down-triangle of Fig. 2.8. The exact agreement results from the fact that this factor is implicitly included in our numerical algorithm described in previous Sec.

(3) A particle-in-cell(PIC) finite-difference-time-domain (FDTD) numerical software (VORPAL) [54] is run in three dimensions, where space charge limit effect is considered to be reached when reflection electron momentum is observed [53, 17]. It is worth noting that our numerical mechanism assumes that electron flow has uniform space distribution in transverse directions perpendicular to flow direction (along x axis), which overlooks some transverse plasma expansion effect (like electron repulsion) of electron flow which behaves in PIC simulation. As a matter of fact,

2.4 Results and Discussions to Compare with PIC 2D Simulation

VORPAL 3D simulation data *do* behave slightly greater than our numerical results in magenta left triangle of Fig. 2.8(c).

Using this time-dependent method to treat an ultrafast laser-excited metal interface, tunnelled current density under different laser field amplitude F_L is investigated. As Fig. 2.8(a-b) shows, when space charge field is considered, electron count and thus current density becomes notably less than ignored near laser amplitude $F_L \sim 5\text{GV/m}$, and tends to saturate afterwards.

Moreover, from Fig. 2.8(c), space-charge limited electron count (both analytic and numerical ones) still overweighs our calculated emitted count density under laser-excitation; therefore, space-charge limit has not been reached under our parameters chosen. We shall change parameters in next section to observe the situation when space-charge limit has been reached under our light-excitation mechanism.

2.4 Results and Discussions to Compare with Particle-in-Cell(PIC) 2D Simulation

As mentioned in former section, it is more wise to resort to 2-D simulation of Particle-In-Cell because our consideration only touches one-dimensional variance along propagation direction of electron pulse flow. Here in this section, we will show two-dimensional PIC simulation results—our numerical algorithm still stick to one-dimensional case(1D).

Again, our numerical mechanism assumes that electron flow has uniform space distribution (a hundred fold of gap spacing D) in transverse directions perpendicular to flow direction (along x axis), which overlooks some transverse plasma expansion effect (like electron repulsion) of electron flow which behaves in 2D-PIC simulation. In the simulation, we inject a time-dependent electron current density based on our model [23], and we determine the SCL current when reflection of electrons are detected [53, 17]. In this Sec., we have a large emitting area that the electron flow has an uniform space distribution in the transverse directions in order to compare with our 1D model.

Using this time-dependent method to treat an ultrafast laser-excited metal interface [23], tunneled current density under different laser field amplitude F_L is

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investigated. In Fig. 2.9, we present the emitting charge density σ_{EC} as a function of laser field F_L with the following parameters: $\tau_p = 50$ fs, $F_{dc} = 1$ GV/m, $D = 1\mu\text{m}$ and $\Phi = 4.4$ eV.

From Fig. 2.9, the comparison shows that our model (open square) is slightly lower than the PIC simulation results (triangle) as shown in inset of Fig. 2.9. In the figure, we see that at the low laser field $F_L < 5$ GV/m, σ_{EC} values calculated by models with (red squares) and without space-charge effect (black circles) are nearly identical, which indicates space charge effects are not important in this range of F_L . Around $F_L = 5$ to 10 GV/m, we see a smooth transition into the SCL regime (open square) [see the inset of Fig. 2.9]. Another observation is that both values are larger than what Valfells' formula (time-uniform injection current case) predicts, demonstrating possibility to achieve a higher upper limit of space charge in time-dependent current case [30].

We also list minima of solutions to space-charge potential $\Delta\phi(0 \leq x \leq s)$ in Tab. 2.1 to demonstrate that as laser field and thus emission density increases, space-charge potential increases by order of magnitude as well to become more important in this laser-excited emission process. Therefore space-charge effect is positively relevant to emission density. We also plot space-charge potentials and emission charge densities under different laser field values in Fig. 2.10. Comparing panels (1a-2a) and (1b-2b) in Fig. 2.10, we find as laser field increases, resultant space-charge potential becomes larger in order of amplitude, and thus emission charge density increases to approach space-charge limit (also in wine empty square symbol in Fig. 2.9). This is also where space-charge field becomes dominant since symbols with and without space-charge effect (SC) diverges near 5GV/m of laser field. We see that at the low laser field $F_L < 5$ GV/m, σ_{EC} values calculated by models with (red squares) and without space-charge effect (black circles) are nearly identical, which indicates space charge effects are not important in this range of F_L . Around $F_L = 5$ to 10 GV/m, we see a smooth transition into the SCL regime (open square) [cf. inset of Fig. 2.9].

More data are plotted in Figs. 2.11, Figs. 2.12 and 2.13 to investigate physical insight on parameters such as pulse durations, electric DC field strengths and work functions. We study the dependence of our results by varying the work function Φ , dc field F_{dc} and laser pulse length τ_p . In Fig. 2.11, we present the cases at $\tau = 10$ fs for $F_{dc} = 1$ MV/m, 1 GV/m and 3 GV/m (top to bottom). On each panel of Figs.

2.4 Results and Discussions to Compare with PIC 2D Simulation

Laser field $F_L : [\text{GV/m}]$	minima of $\Delta\phi(0 \leq x \leq s)$
0.1	-4.3×10^{-4}
0.2	-4.3×10^{-4}
0.5	-5.1×10^{-4}
1	-1.54×10^{-3}
2	-4.5×10^{-2}
5	-4.5
10	-160

Table 2.1: Table of minima of $\Delta\phi(0 \leq x \leq s)$ under different laser field values, displaying a trend that space-charge effect is positively relevant to emission density.

2.11 and 2.13, we show two calculations at different work functions $\Phi = 2.2$ and 4.4 eV. Figs. 2.11 and 2.13 are presented similarly to Fig. 2.11, except at longer laser pulse $\tau_p = 50, 100\text{fs}$. By comparing 9 panels of Figs. 2.11 and 2.13, we have several observations as follows.

In general, it is easy to reach the SCL regime (critical value of F_L is small) for small Φ , and low F_{dc} while τ_p does not vary the SCL regime much.

(1) Similar to Fig. 2.9, as laser field increases from 0.1 until 100GV/m, quantity of electron emission first increases and gradually enters plateau stage near 10GV/m even drops by a small extent. The reason of this plateau behavior comes from laser-induced tunnel physics itself and demonstrates that a saturation effect of multi-photon interaction with metal, beyond scope of this paper. But this plateau part can be strongly clamped by space-charge effect, without which it shall be easily neglected.

(2) As electric DC field increases, emission is greatly promoted because tunneled probability increases shown in Eq.(2.7). For example, when DC field increases by 3 orders of magnitude from Figs. 2.11(a) to (b), space charge limit increases by 3 order also and effective upper point of laser tuning (laser field strength) is shifted from ~ 5 to ~ 20 GV/m but this trend is manifest smaller, from ~ 2 to ~ 5 GV/m in Fig. 2.13(a) and (b).

(3) Extension of ultrashort laser pulse duration τ_p , from 10fs to 100fs, allows one to observe an increasing trend of the maximum electron count near the end of pulse duration, due to monotonously increasing behavior of laser-induced emission mechanism [23], shown in Figs. 2.11 2.12 and 2.13. However, space charge limits (the electron count in the pulse under space-charge limited condition) of two pulse durations($\tau = 10, 100$ fs) still remain almost indistinguishable from each other as shown in Table 2.2, in agreement with Valfells' formula Eq. (2.3) to suggest a lower limit for longer pulse duration: the total charge in the pulse is given by

$$\begin{aligned} Q(X_{CL}) &= T_{CL} X_{CL} 2 \frac{1 - \sqrt{1 - 0.75 X_{CL}^2}}{X_{CL}^3} J_{CL} = 2 \frac{1 - \sqrt{1 - 0.75 X_{CL}^2}}{X_{CL}^2} T_{CL} J_{CL} \\ &= 2 \frac{1 - \sqrt{1 - 0.75 X_{CL}^2}}{X_{CL}^3} Q_{CL} = Q_{\text{mod}}(X_{CL}) Q_{CL}, \end{aligned} \quad (2.28)$$

where Q_{CL} is the charge in the diode under the Child-Langmuir condition. In all the cases investigated in Chapter 2, X_{CL} values never exceeds 0.011 and thus one expects no significant increase in electron count (essentially $Q_{\text{mod}} \approx 0.75$ when the pulse length increases, as seen in Fig. 2.14, i.e. the longer time duration cancels the enhancement from space-charge current density.

2.4 Results and Discussions to Compare with PIC 2D Simulation

F_L (Unit: V/m)	10fs	100fs
1×10^6	(5GV/m) 5.61451E13	(10GV/m) 5.59664E13
1×10^9	(2GV/m) 6.49976E14	(10GV/m) 5.59130E16
3×10^9	(10GV/m) 1.67923E17	(20GV/m) 1.67642E17

Table 2.2: Table of maxima of calculated electron count (in unit of $\#/\text{m}^2$) for all red curves in both Figs. 2.11 and 2.13. The laser field amplitude where the maximum occurs in each curve is indicated in the parentheses in front of each maximum.

F_{dc} (Unit: GV/m)	$1\mu\text{m}$	1cm
1	1.37947E16	1.37296E16
3	5.34563E16	5.15580E16

Table 2.3: Table of calculated electron count (in unit of $\#/\text{m}^2$) extracted from red curves in Figs. 2.8, 2.9 and 2.12. The laser field amplitude $F_L = 5\text{GV/m}$, short pulse duration $\tau_p = 50\text{fs}$.

Hence, a relatively long laser-tunneling process may not aid in giving a larger space-charge in such a physical environment unless $X \sim 0.2$. Another observation is that the upper bound point for laser field to control emission less than space-charge limit (c.f. (2)) becomes lower for longer pulse duration of electron flow.

(4) A trial to consider effect of work function within laser-induced emission mechanism, half of tungsten's work function ($4.4/2 = 2.2\text{eV}$) is adopted to generate a pronouncedly larger emission quantity, but space charge limit is maintained *almost* un-shifted at all (all convergent as dash dot lines in Figs. 2.11, 2.12 and 2.13). But the upper point when laser field is large enough to reach space-charge limit (c.f. (2) too) is smaller when work function is lowered, which actually restricts capability range of laser tuning to emit electrons.

(5) Assume that the vacuum field is kept constant, when vacuum distance D is varied smaller than original parameter $D = 1\text{cm}$, electron count number does maintain unaffected unless it is varied smaller than electron beam front position s according to the data contrast in Table 2.3 below. This could also be justified from Eq. (2.28):

$$Q(X_{CL}) = Q_{\text{mod}}(X_{CL})Q_{CL} \sim \frac{3}{4} \frac{\phi(x=D)}{D} = \frac{3}{4} F_{dc}. \quad (2.29)$$

Therefore as our figures and tables clearly demonstrate, when space charge field is considered, electron count number becomes noticeable when the laser amplitude varies from low to dominate, and tends to saturate afterwards. Moreover, space-charge limited electron count always limits and clamps the emitted electron counts, which shall never exceed the former. Therefore, space-charge effect must be concerned under our time-dependent laser induced electron emitter mechanism, when laser field value is large enough to induce current density large enough.

2.5 Conclusion

To summarize this paper, we develop a numerical algorithm to calculate a time-dependent short-pulse space-charge-limited current density, based on our previous work on a non-equilibrium model to describe ultrafast laser-excited electron emission, confirmed by commercial PIC software. Smooth transition from the emission-based to SCL regime is obtained. The calculated results are compared with PIC simulation and the short pulse SCL current model with constant injection current [3]. This model is restricted to one dimensional treatment that the exact comparison with experimental results using a sharp metallic tip (under ultrafast laser excitation) is beyond the scope of this paper.

We conclude that space charge effect has to be considered during this emission process and calculate space charge limit current values, and laser field, pulse duration and metal work function are all possible parameters to tune the threshold value to reach the space-charge-limited emission, among which the electric DC field is the most sensitive one.

A natural extension is to investigate space charge effect under other miscellaneous emission mechanism, e.g. a monotonously decreasing injection function dependent upon time, or more generously, investigate possibility of time-dependent emission current function which might overweigh uniform SCL current of traditional time-uniform injection case, which will be discussed in Chap. 3. It is also worth noting that our space charge effect calculation method obtains electrical potential from electrostatics and thus ignores electron-motion-induced electrical potential, which as a matter of fact, will inspire much further interest.

2.5 Conclusion

Figure 2.8: (Color online) (a) Emission count densities of electrons with (red square) and without space charge effect (black dot) and space-charge-limited electron count ($\tau_p = 50\text{fs}$, $F_{dc} = 3\text{GV/m}$) (dark-yellow open circle: analytic results from Eq. (2.23); wine open square: numerical results; blue open down triangle: analytic results from Eq. (2.27)); magenta left triangle: VORPAL simulation (3D) results; (b) Enlargement of Electron count number contrast between $F_L = 5 \sim 100\text{GV/m}$; (c) Space-charge limited count number calculated from our algorithm compared with 3D PIC simulation. Produced from Origin 8.6-32bit.

Figure 2.9: (Color online) Laser-induced emission charge densities σ_{EC} as a function of laser fields F_L for cases with (red solid square) and without space charge effect (black dot) ($\tau_p = 50\text{fs}$, $F_{dc} = 1\text{GV/m}$, $D = 1\mu\text{m}$, $\Phi = 4.4\text{eV}$). The comparisons are respectively, the space-charge(SC) limited density (wine open square), from Eq. (2.20) in our mechanism, space-charge limited electron count from VORPAL simulation (2D) (in magenta left triangle), space-charge limit charge density from Valfells' formula (in violet dashed line). (Inset) Enlargement of Electron charge densities in space-charge limit for data from VORPAL, our mechanism Eq. (2.20) and Valfells' formula.

2.5 Conclusion

Figure 2.10: The space charge potential profile ($\Delta\phi(0 \leq x \leq s)$) within distance s as far as electron flow reaches at the end of short-pulse emission $t = \tau_p$, for two different laser amplitudes $F_L = 2$ (1a) and 5 GV/m (2a) ($F_{dc} = 1$ GV/m). The corresponding time dependent emitting current density $J(t)$ are plotted in (1b) and (2b). The parameters used are same as those in Fig. 2.9.

Figure 2.11: (Color online) Laser-induced electron charge densities with (red dot with line), without space charge effect (black dashed line) and space-charge limited electron count number (blue dashed dot line) ($\tau_p = 10\text{fs}$, $D = 1\text{cm}$, $\Phi = 4.4\text{eV}$) under applied field value $F_{\text{dc}} = 1 \times 10^6$, 1×10^9 and 3×10^9 V/m from our iterative mechanism. To show how tunneled electron number changes under lower work function ($\Phi = 2.2\text{eV}$), laser-induced densities with (olive diamond with line), without space charge effect (magenta dashed line) and space-charge limited density (navy dashed dot line) are also plotted.

2.5 Conclusion

Figure 2.12: The same caption as for Fig. 2.12 but under longer ultrafast laser duration ($\tau_p = 50\text{fs}$).

Figure 2.13: The same caption as for Fig. 2.11 but under longer ultrafast laser duration ($\tau_p = 100fs$).

2.5 Conclusion

Figure 2.14: No significant increase of space-charge limit (charge) can be obtained as the pulse length X_{CL} increases for $X_{\text{XL}} < 10^{-2}$.

CHAPTER 3

TIME-DEPENDENT SPACE-CHARGE LIMITED ELECTRON FLOW

3.1 Motivation

As reviewed in previous chapters, space charge limited current serves as a basic theoretical prediction for application of high current diodes and other vacuum electronics devices in field of non-neutral plasma physics. However, both equations of Child-Langmuir law and Valfells' formula [3] only apply to temporally-steady status of electron flow and time-dependence of injection current density has not yet been touched thoroughly. While in 2010, M. E. Griswold *et al.*, offered an upper bound of time-averaged current density analytically, speculating that such an electron flow will surpass what classical CL law predicts [30, 55]. Hereby in this paper, we numerically attempt to test injection electron flows in different time-profiles, in order to search for an optimal profile function, which may enable them to transcend Valfells' formula under short-pulse situation, by means of commercial simulation(PIC) to consider *time-dependent* effect for short-pulse electron flow in one-dimensional case. We find monotonously decreasing functions of time-profile may offer possibility to transcend time-uniform space-charge limit.

We also utilize self-derived iterative calculation method in the former Chap. 2 to consider *time-dependent* effect for short-pulse electron flow in one-dimensional case, in

3.1 Motivation

order to search for an optimal profile function, which may enable them to transcend Valfells' formula under short-pulse situation. But this algorithm results in great conflict with our PIC simulation, so we speculate that our home-made calculation may lose some capture of time-variant injection current thus list the result in Appendix B.

Space charge limited (SCL) law gives the maximum current density allowed for steady-state electrons emitted from cathode to transport across vacuum gap D under a voltage ϕ across the diode. For a vacuum diode with a gap spacing of D and a dc voltage of V_g , it is known as the one-dimensional (1D) Child-Langmuir (CL) law [12, 13], given by

$$J_{\text{CL}} = \frac{4\epsilon_0}{9D^2} \sqrt{\frac{2e}{m}} V_g^{3/2}, \quad (3.1)$$

where ϵ_0 , e and m is, respectively, vacuum permittivity, electron charge and electron mass. Space charge limited electron flow occurs when the charge of the emitted electrons is sufficient to suppress the electric field at the cathode to zero, and the electrostatic potential distribution function $\phi(x)$ is

$$\phi(x) = V_g \left(\frac{x}{D} \right)^{4/3}. \quad (3.2)$$

as shown in Fig. 2.1, Chap. 2.

However, this formula turns into incompetence, for steady electron beam whose transit time across the whole spacing is much longer than temporal length of short-pulse electron beam, τ_p ; hence the formula for short-pulse critical injection current density (denoted in black line of Fig. 3.2) under space-charge limit,

$$J_s(X_{\text{CL}}) = 2 \frac{1 - \sqrt{1 - \frac{3}{4} X_{\text{CL}}^2}}{X_{\text{CL}}^3} J_{\text{CL}}, \quad (3.3)$$

was proposed by Á. Valfells *et al.* in 2002 [3], where $X_{\text{CL}} = \tau_p / (3D\sqrt{m/2e\phi})$ is a normalized pulse length. It anticipates a much larger current density than Child-Langmuir law does, when the time-independent electron flow is injected into the gap with a constant current density of J_s over a pulse length of $0 \leq t \leq \tau_p$.

There are extensions to this 1D classical CL law to multidimensional uniform models [56, 57, 48], to quantum models [14, 58, 59], and to short pulse models [3,

15]. There are also related studies on THz sources [60], Coulomb blockade [55], 2D electromagnetic effects [17], and novel scaling based on surface electric field [18]. Note in most studies, the electron beam injection into the gap is assumed to be uniform and the amount of current is time-independent. The non-uniformity was studied in several 2D models [61, 62, 63].

The effects of having a time-dependent injection was discussed by M. E. Griswold *et al.* [30], and it was speculated there might be an upper bound of time-averaged SCL current density for a given time-dependent current, which can be higher than the time-independent case such as CL law. However the exact time-dependent profile remains unclear.

In this chapter, we are interested to study the characteristics of different temporal profiles in order to have time average SCL current density as compared to the short pulse time-independent SCL model [3]. In particular, we assume the electron is injected into the gap following some prescribed time profiles, and use the particle-in-cell (PIC) (in 1D limit) to obtain the time-average SCL current density, based on our previous work to describe ultrafast laser-excited electron emission from a metallic surface [23, 64, 65].

3.2 The Problem of Time-Dependent Space-Charge Effect

Here in this chapter we shall revisit our previous algorithm to calculate time-dependent space-charge-limited current density of a short-pulse electron flow. Consider a gap of spacing D with an external external constant electric field $F_{dc} = V_g/D$. Instead of considering tunneling process to emit electrons [64, 66], here we suppose the electrons are injected from the cathode (at $x = 0$) with an arbitrary time-dependent current density of $J(t)$ over a pulse length $\tau_p \leq \tau = 3D\sqrt{m/2eV_g}$, where τ is the electron transit time at the SCL condition.

Whereas existent emitted electrons in spacing D reaching as far as distance $s(0 \leq x \leq s < D)$ resist electrons emitting out from metal and serve wholly as an additional space charge electric potential distribution. This physically induces a new potential term $\Delta\phi(x)$ in original electrostatic potential distribution function across

3.2 The Problem of Time-Dependent Space-Charge Effect

cathode and anode, thus

$$\phi(0 \leq x \leq D) = F_{\text{dc}}x + \Delta\phi(x), \quad (3.4)$$

where s means the longest distance emitted electrons are able to transit.

Here, we assume that $J(t)$ can be expressed as

$$J(t) = \beta_m \times j_m(t), \quad (3.5)$$

where β_m is the magnitude of the current density, and $j_m(0 \leq \bar{t} = t/\tau_p \leq 1)$ is a particular time-dependent profile assumed in the model (for $m = 0 \sim 8$), which are (see Fig. 3.2 below)

$$j_0(\bar{t}) = 1, \quad (3.6)$$

$$j_1(\bar{t}) = 2\bar{t}, \quad (3.7)$$

$$j_2(\bar{t}) = 2(1 - \bar{t}), \quad (3.8)$$

$$j_3(\bar{t}) = H\left(\frac{1}{2} - \bar{t}\right) \cdot 4\bar{t} + H\left(\bar{t} - \frac{1}{2}\right) \cdot 4(1 - \bar{t}), \quad (3.9)$$

$$j_4(\bar{t}) = 3\bar{t}^2, \quad (3.10)$$

$$j_5(\bar{t}) = 4\bar{t}^3, \quad (3.11)$$

$$j_6(\bar{t}) = 5\bar{t}^4, \quad (3.12)$$

$$j_7(\bar{t}) = 11\bar{t}^{10}, \quad (3.13)$$

$$j_8(\bar{t}) = 22\bar{t}^{21}, \quad (3.14)$$

Here, $j_m(\bar{t})$ has been normalized ($\int_0^1 j_m(\bar{t}) d\bar{t} = 1$), and $H(t)$ is the Heaviside step function.

We have to point out here, during the whole temporal process of electron flow, maximum value of occupation length of electron flow, defined as s_{max} can never surpass distance between cathode and anode, which justifies the reason our computation in this paper is only valid within short-pulse condition of electron flow; since otherwise when it does, the front of electron pulse will be absorbed at anode, this holistic calculation of maximum injection current density become meaningless. An attempt to obtain an easy corollary for time-dependent injection current density

Figure 3.1: Four normalized time profiles $j_m(t)$ ($m = 1, 2, 3, 4$) (Eqs. (3.7)–(3.10)) which are selected to insert into Eq. (3.5) to test space-charge limit in VORPAL simulation. Notice that profiles $j_m(t)$ ($m = 5 \sim 8$) (Eqs. (3.11)–(3.14)) are omitted in this figure because of their higher amplitudes.

$J(t)$ goes invalid since that Eqs. (8)

$$E(x = s) = -\frac{J_0 \tau_p}{\epsilon_0}, \quad (3.15)$$

and (10) of Ref. [3]

$$E(x = s) = -\frac{4\phi(x = s)}{3s}, \quad (3.16)$$

lose their equalities and no quantitative relations are possible to establish directly, where $E(x = s)$ stands for electrostatic field at interface $x = s$ between electron flow and vacuum. Henceforth, Eq. (3.3) fails as a result of nonuniform electron flow. Thus the space-charge-limited current density of final temporal point, $J_{\text{end}} \triangleq J(t = \tau_p)$ is expected higher than uniform space-charge-limited current density, J_{crit} , trivially.

3.3 Particle-in-Cell(PIC) Simulation and its setup

To check whether if any representative the time-varying current densities $J_m(t)$ for ($m = 1 \sim 8$) is able to be higher than the constant profile $m = 0$ case, we use a Particle-In-cell (PIC) code (VORPAL4.2) [54, 67] to determine the occurrence of SCL current at the end of the short pulse. The SCL current is determined by increasing the magnitude of the injected current β_m until the reflection of electrons during the pulse-length($0 < t \leq \tau_p$) is observed [53, 17].

Here we briefly record the procedures to set up PIC calculation by VORPAL to explore time-dependent problem in space-charge limit. In order to evade additional problem of plasma expansion in multidimensional space(as 2D or 3D cases), we only compute on one-dimensional problem to seek utter essence of time-dependent electron flow in promoting space-charge limit. (Other factors could be added in when this work validates itself but that is beyond our purpose here.) Then we use global uniform grid(*uniCartGrid* in Python script) to divide equally 1D space and assign values of electro-static potentials for both walls(cathode and anode) as Dirichlet boundary conditions. For any particle emitted within simulation grid [67], a particle source is defined exactly according to physics properties of electrons. A bunch of electrons are defined emitted at rest from cathode surface(eg. $x = 0$ *emitSurface* in script writing). Time-dependent injection electron density is also included in this part to vary electron number upon time emitted from cathode surface. To avoid disasters arising due to missing particles running out of simulation grid, two particle-absorbers adjacent to both ends of grid have to be given to absorb any particle emitted out of demarcation borders. Notice our method to emit electrons from cathode surface is accurate to the extent of current density rather than of particle density in space, adopted in conventional PIC calculation. This emit-flux-mechanism saves indirect conversion from current density to particle density, giving rise to accuracy extent.

3.4 Results and Discussions

For the reference case at $m = 0$, we have $J_0 = \beta_0$, which is a constant. Note we expect this $m = 0$ case should recover to the time-independent model given by Eq. (3.3), which gives $\beta_0 = J_s$ (see Fig. 3.2 below). In Fig. 3.2, we first compare the

numerical results (normalized to the 1D CL law Eq. (3.3)) between β_0 ($m = 0$ case) and J_s at $D = 1$ cm, $F_{dc} = 3$ GV/m (or $V_g = 10$ kV) as a function of X_{CL} up to < 1 , where the electron transit time is about $\tau = 9.24$ ps. The comparison shows pretty good agreement with β_0 slightly higher (10 %) than those predicted by Eq. (3.3) at small X_{CL} .

Figure 3.2: The normalized (in terms of J_{CL}) short pulse SCL current density (for time-independent injection, $m = 0$ case): $\beta_0 J_s / J_{CL}$ (symbol) and J_s / J_{CL} (line) from Eq. (3.3) as a function of normalized pulse length X_{CL} . The parameters used are $D = 1$ cm, $V_g = 30$ MV, and SCL transit time $\tau_{CL} = 9.24$ ps.

In Fig. 3.3, we plot the values of β_m (for $m = 0 \sim 8$) in terms of J_s as a function of X_{CL} with the same set of parameters used in Fig. 3.2 except that $V_g = 10$ kV. The comparison shows that the highest enhancement is $m = 8$ case. The finding indicates that an increasing time-varying of $J(t)$ will have a higher time-average current magnitude ($\beta_8 > \beta_7 > \beta_6 > \beta_5 > \beta_4, \beta_1 > \beta_2, \beta_3$). In all cases, the enhancement is increased for small X_{CL} , and the enhancement decreases with large

3.4 Results and Discussions

Figure 3.3: (Color Online) The time average SCL current density: β_m ($m = 1 \sim 8$) normalized with respected to J_s from Eq. (3.3) as a function of normalized pulse length $X_{CL} < 1$. The dashed line is equal to 1. The parameters used are $D = 1$ cm, $V_g = 10$ kV.

X_{CL} approaching 1. Note, the reference case $m = 0$ case shows good agreement within 10% as indicated before in Fig. 3.2.

Figure 3.4: The time dependence of the difference between the electric field on the cathode and anode, $E_K(t) - E_A(t)$ for $m = 0$ case: (black square) and $m = 4$ case (red dot), which shows, respectively, the t and t^3 scaling. The parameters used are $D = 1$ cm, $V_g = 30$ MV, and $\tau_p = 10$ fs.

From 1D Gauss's law, and conservation relation of charges, we know that the difference of the electric field between the cathode and anode is proportional to integral of current density $J(t)$, given by

$$\begin{aligned}
 E_K(t) - E_A(t) &= \int_0^D \frac{\partial E(x)}{\partial x} dx \\
 &= \frac{1}{\epsilon_0} \int_0^D \rho(x) dx = \frac{1}{\epsilon_0} \int_0^t J(t') dt',
 \end{aligned} \tag{3.17}$$

3.4 Results and Discussions

where $E_K(t)$ and $E_A(t)$ is respectively the electric field at cathode and anode [68]. To demonstrate the time dependence given by the equation, we retrieved the time-varying electric fields on the cathode and anode from the PIC simulation for two cases: $m = 0$ and 4^* . The results are plotted in Fig. 3.4, which indicates that $[E_K(t) - E_A(t)] \sim t$ (for $m = 0$ case) and t^3 (for $m = 4$ case) as expected from the Eq. (3.17).

In Fig. 3.5, we also plot the histogram for the number of electron counts inside the gap (near to the cathode) at the end of the pulse ($t = \tau_p$) for $m = 0$ case ((a)) and $m = 4$ case ((b)). From the figures, we see that $m = 4$ case has a relatively fast decreasing function of the electron distribution as compared to $m = 0$ case. This finding suggests that the higher time-average SCL current density in $m = 4 \sim 8$ case may be due the narrower spatial distribution of the electron near the cathode.

Finally, it is important to note that all results presented here are valid for only for short pulse regime $X_{CL} < 1$ where the electron pulse length is less than the transit time. At $X_{CL} \geq 1$, the electrons will be collected by the anode and the SCL limited condition have been determined at a much larger time scale used in the simulation approaching steady state condition. Thus the results presented for those values closed to $X_{CL} \approx 1$ will require careful investigation in future.

3.4.1 Additional PIC simulation to check

For the former results from simulation, the space-charge limit is determined when the short-pulse just start to reflect at the end of the temporal pulse τ_p . If one further checks to look into the electron flow behavior after end of the pulse (when electrons are still transporting inside the diode gap), the values of β_m (for $m = 0 \sim 8$) in terms of J_s as a function of X_{CL} are plotted in Fig. 3.6(a). From this plot, we observe the space-charge limit for time-dependent injection current is greatly compromised if we consider space-charge effect after the end of injection pulse. From a rearranged

*An observation following is that relative ratio of β values between any two temporal profiles should align with their ratio in the differences $(E_K - E_A)|_{t=\tau_p}$. Furthermore, under a single sheet model of electrons [3], for $m = 0$ case, $(E_K - E_A)|_{t=\tau_p} \approx \frac{\phi(x=D)}{D}$ when $X_{CL} \ll 1$. This contradiction results from the negative field at the cathode $E_K(x = 0^+)$ but the author does not figure out how to clarify this except to question the equivalence between Poisson equation and Gauss's law in one-dimensional situation. This is what remains to be discovered.

Figure 3.5: (a) (b) Histograms of electron count number (arbitrary unit) versus x space between cathode and anode at the pulse end of short electron flow ($t = \tau_p$), from VorpallView 4.0.0 [2]. (a) for $j_0(t)$; (b) for $j_4(t)$. (the same parameters as Fig. 3.4).

3.5 Conclusion

plot Fig. 3.6(b), for the long pulse cases, space-charge limit is even lower than time-constant limit($m=0$). For shorter pulse as $\tau_p = 0.5ps$, close to 1.2 times time-constant limit(Valfells' formula) could be achieved with several time-dependent profiles($m = 1 \sim 5$). Besides, a general trend to degrade to $m = 0$ case when m increases to high values($m = 8$). This confirms the speculation that high-order time-dependent profile approaches time-constant one from the perspective of space-charge-limited current.

3.5 Conclusion

In this paper, we have presented PIC simulation results to study the effect of having time-dependent injection of short electron pulse into a diode to reach the space charge limited (SCL) condition. It is found that it is possible to have a higher time-average SCL current flow for a given time-varying injection profile, as compared to the time-independent short pulse SCL current flow determined before [3]. Among the 8 cases that we have studied, we found that having an increasing time-profile of current will provide such enhancement $\sim 20\%$ depending on the types of profiles used. This finding also supports the theoretical conjecture by Griswold [30] that there might be an upper bound of time-dependent SCL current flow, where an upper bound to time-averaged space-charge limit was conjectured from an iterative process similar to our numerical algorithm in Chapter 2, as high as as no more than $2.45J_{CL}$ but no affirmative proof was given yet. Therein, a time-varying electric field at the cathode $E_K(t)$ was allowed. This conjecture originated from the limit

$$E_K \geq 0, \quad (3.18)$$

(pushing electrons back into the cathode[†]) which also occurred in our simulation. It is of interest to notice that this 2.45 aligns with our PIC simulation of space-charge limit. However, their upper limit was speculated over times larger than transit time across the diode. Moreover, our additional PIC simulation in the previous subsection convinces us that in our physical setup the true space-charge limit occurs only when the whole pulse of electrons flows across beyond the anode. Therefore the space-charge

[†]Inconsistency appears in the conjecture [30] compared with their amended conjecture [31]. But in this Chapter, $\bar{E}(x) = -\partial\phi(x)/\partial x$.

Figure 3.6: (Color Online) (a) The time average SCL current density: β_m ($m = 1 \sim 8$) normalized with respected to J_s from Eq. (3.3) as a function of normalized pulse length $X_{CL} < 1$. The dashed line is equal to 1. The parameters are the same as Fig. 3.3 (b) rearrangement of the β_m vs. different time-profiles indicated by m integers.

3.5 Conclusion

limit number $\beta_m < 1.2$ should be less than Griswold's conjecture [30] according to our amended PIC simulation.

It is reminded that the previous experimental result [3](cf. FIGs. 3 and 4 therein) illustrated also an exceeding behavior (also by $\sim 20\%$) than Child-Langmuir limit, when the virtual cathode oscillation extends the pulse and results in a dip inside (de facto close to the order of τ_{CL}). However, this case does not belong to short-pulse situation and shall not blink away the novelty of our results.

Our results may pave the way in using laser to excite a specific time-varying current at high current regime so that it can surpass the SCL condition in order to have larger charge numbers per pulse for various applications.

CHAPTER 4

RADIATION FROM MAXWELL'S FISHEYE SPHERE

4.1 Introduction of radiation from the motion of electrons

As introduced in the Introduction Chapter, the aforementioned picture of electrostatics could be complicated – when the time-dependent electrodynamics has to come into play. In the framework of electrodynamics, travelling wave shall occur under certain circumstances for current density source(e.g. electrons). The calculation of radiation from the source of a certain current density could further unfold some novel physics in emission process.

Transfer from evanescent wave into propagating wave is an interesting optical problem [69], which is still relatively unexplored due to the distinct feature of the evanescent wave. Evanescent wave may appear along with moving electrons, and does not carry power unless electron moves fast to transcend light velocity in medium, which forms exactly Čerenkov radiation *. Čerenkov radiation occurs because fast

*The local wave front from the motion of charged particles is not a surface wave since no surfaces are involved in geometry but should not disallow one to consider it as evanescent wave. The disturbance decays away when the particles move slowly; and when fast enough, it is left in the wake of the particle to radiate as a coherent shockwave.

4.1 Introduction of radiation from the motion of electrons

motion of electrons leave their accompanying evanescent fields lagged behind in space, which coherently accumulate into energetic radiation, similar to the sonic boom created by an object travelling faster than sound. This dissipative motion-induced radiation requires particles to move faster than light in medium or to encounter velocity transition to pump energy; otherwise, motion-induced evanescent field is not able to be accumulated into physical energy transfer. However, a different radiation called transition radiation gets around this superluminal restraint. Instead, transition radiation occurs as long as electrons move through inhomogeneous media, such as a boundary between two different materials. It is worth mentioning here that Čerenkov radiation from photonic crystals [41] is a special case because in photonic crystal loss of translational invariance allows emission of radiation without any threshold velocity of electrons, which intrinsically couples with transition radiation. A third radiation that also accumulates evanescent wave into radiation from electrons' motion is the diffraction radiation, in which electrons generate radiation in passage near structured surface without penetrating it. Recently a metamaterial light source driven by a free electron beam bunch [42] has attracted attention in academia. In this light-well source, free electrons interact with periodically-alternating environment as they travel through tunnels inside, and by effect of diffraction radiation, a continuum of radiation over a broad angular range occurs. This transfer from kinetic energy of electron bunches to radiation also bears resemblance to transition radiation because electrons interact with inhomogeneous dielectric parameters $\epsilon(\vec{r})$ during their passage. This proof-of-concept not only opens up new doors for vacuum electronics devices such as coherent radiation sources, also inspires one to ponder whether other inhomogeneous environment can also induce similar radiation phenomenon. Maxwell's fisheye could be such a simple case.

Whereas on the other hand an emergent Transformation Optics strategy [70, 6] provides human with capability to manipulate light flow almost arbitrarily, for instance to hide objects inside a cloak device made of inhomogeneous and anisotropic permittivity and permeability. The invariant form of Maxwell's equations, under various coordinates(including even non-Euclidean), guarantees light to be shaped by curved electromagnetic space in order to behave according to man's design. But from the perspective of the electron, it does not bend its trajectory similarly to light and instead experience curved space to induce radiation [40]. Therefore such a curved

electromagnetic space becomes an energy-pumping candidate to give rise to motion-induced radiation of particles with aid of exponential development of nanofabrication technology [71, 72]. Inspired by this asymmetry of coordinate transformation of cloak, Zhang *et al.* invented a method to detect cloak by observing radiation of a fast-moving electron bunch going through the cloak device [40]. Here in this Chap., we derive another transition radiation of moving electron bunch going through a differently curved geometry of light: Maxwell's fish-eye, which in principle provides unlimited resolution as a perfect imaging lens [43], and report our calculation on radiation from passage of electrons traversing Maxwell's fisheye sphere, which consists of both Čerenkov radiation and transition radiation.

The rest of this Chap. is structured as follows. In Sec. 4.2, the basic equations of electromagnetic fields due to moving charges are posed and the analytic solutions to electromagnetic fields are derived by dyadic Green's function. In Sec. 4.3, main calculation results are shown including electric fields on xz plane where electron's trajectory is confined and near field radiation of this process in a certain frequency 300THz. We also analyze and discuss temporal evolution of electron's radiation. In Sec. 4.5, we summarize this report with potential application of our work.

4.2 Question of radiation from Maxwell's fisheye sphere

In this chapter we shall pose our problem and try to give some qualitative prediction from perspective of transformed uniform space. The Maxwell's fisheye sphere, whose permittivity function is defined as

$$\epsilon_r(r) = \frac{4}{[1 + (\frac{r}{R_1})^2]^2}, r \in [0, R_1], \quad (4.1)$$

invented by great Maxwell, focuses light-rays emitted from a point source located at the surface to another focus antipodally. For outer space of Maxwell's lens, simply vacuum permittivity is adopted (suppose the whole space we investigate is un-magnetic, $\mu_r(0 < r < \infty) = 1$). As the surface of a sphere can be viewed as a form of curved space (with a constant curvature), the fish eye profile can be mapped onto

4.2 Question of radiation from Maxwell's fisheye sphere

Figure 4.1: (Color Online) Physical space (top panel) can be transformed into virtual space (lower panel) which is curved space under stereographic projection. In both panels green lines indicate light trajectory while magenta electron's trajectory. (Top panel): Physical space of xz plane with permittivity distribution $\epsilon_r(r)$ marked in blue color (lake in **Mathematica**), which can be divided into two parts demarcated by black circle line: inner part $0 \leq r \leq R_1$ is Maxwell's fisheye and outer part $r > R_1$ is uniform space ($\epsilon_r(r) = 1$). In colorbar, brighter color indicates the higher value of permittivity. The magenta line indicates the trajectory of electron bunch along straight line $x = R_1/2, y = 0, z = 0$. (Lower panel) Virtual uniform space, composed of two parts corresponding to physical space. One is invariant planar space marked in light blue, $Y = 0, X^2 + Z^2 \geq R_1^2$; the other lower half spherical surface in uniform space in pink (mapped from Maxwell's fisheye sphere on xz plane zone, $Y = 0, X^2 + Z^2 < R_1^2$ shown in top panel of Fig. 4.1, according to stereographic projection). Note: green and magenta curve intersects at point $x = 1/2, z = \sqrt{3}/2, y = 0$. This is solely coincidence due to the parameters we choose.

the surface of a hypersphere(4D manifold) in virtual uniform space from Luneburg's visualization [34]. For simplicity, suppose we generate a pulse of electron beam $\bar{J}(\bar{r})$ moving along a straight line in the z direction inside a Maxwell's fisheye sphere,

$$\bar{J}(\bar{r}, t) = \hat{z}qv\delta(x - x_0)\delta(y - y_0)\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(z-vt)^2}{2\sigma^2}}. \quad (4.2)$$

Our goal is to obtain solution to inhomogeneous wave equations of electromagnetic fields from dyadic Green's function method analytically [73, 74, 75]. In this Chap., we take $q = 1000e$, $R_1 = 2\mu\text{m}$, $x_0 = 1\mu\text{m}$, $y_0 = 0$, $\sigma = 50\text{nm}$, $v = 0.9c$ (e indicates elementary charge and c light velocity in vacuum). We use 1000 electrons to represent an electron bunch.

This is a problem to solve inhomogeneous partial differential equations in mathematics. We first write Maxwell's equations in frequency domain according to Fourier transform. Second, we separate the whole current density or the source going through whole infinite space into two parts. The solution in uniform space(outside Maxwell's fisheye), is calculated through traditional Green's function(cf. Sec. C.1 (iv) in Appendix C). The other one in curved space(inside Maxwell's fisheye) shall be solved via dyadic Green's function narrated in Sec. C.1 (i-iii) in Appendix C. The electromagnetic fields in curved space (inside sphere of Maxwell's fisheye) can be written as a vector mapped from current density vector by operation of dyadic Green's function(herein this dyadic operator works to map one vector to another by algebraic manipulation).

$$\bar{E}(\bar{r}) = i\omega\mu_0 \iiint \bar{\bar{G}}_e(\bar{r}, \bar{r}') \cdot \bar{J}(\bar{r}')dV', \quad (4.3)$$

$$\bar{H}(\bar{r}) = \iiint \nabla \times \bar{\bar{G}}_e(\bar{r}, \bar{r}') \cdot \bar{J}(\bar{r}')dV'. \quad (4.4)$$

The sum of both field solutions above make up the complete solution of electromagnetic fields we seek. After that, inverse Fourier transform provides us with temporal EM fields:

$$\bar{E}(\bar{r}, t) = \int_{-\infty}^{\infty} d\omega \bar{E}(\bar{r}; \omega)e^{-i\omega t}, \quad (4.5)$$

4.2 Question of radiation from Maxwell's fisheye sphere

Readers shall refer to detail of the complete derivation of dyad used here in Appendix C. We shall only indicate results of dyads in Eq (4.3) here:

$$\bar{\bar{G}}_e^{(11)} = \bar{\bar{G}}_{eo}(\bar{r}, \bar{r}') + \bar{\bar{G}}_{es}^{(11)}(\bar{r}, \bar{r}'), r < R_1 \quad (4.6)$$

and

$$\bar{\bar{G}}_e^{(21)} = \bar{\bar{G}}_{es}^{(21)}(\bar{r}, \bar{r}'), r > R_1. \quad (4.7)$$

$$\bar{\bar{G}}_{eo} = -\frac{1}{k^2} \hat{r} \hat{r}' \delta(\bar{r} - \bar{r}') + \quad (4.8)$$

$$\begin{aligned} & \frac{ik}{4\pi} \sum_{m,n} C_{mn} \begin{cases} \bar{M}^{(m1)}(k) \bar{M}'^{(m)}(k) + \bar{N}^{(e1)}(k) \bar{N}'^{(e)}(k), r > r', \\ \bar{M}^{(m)}(k) \bar{M}'^{(m1)}(k) + \bar{N}^{(e)}(k) \bar{N}'^{(e1)}(k), r < r'. \end{cases} \\ \bar{\bar{G}}_e^{(11)} = & \frac{ik}{4\pi} \sum_{n=1}^{\infty} \sum_{m=0}^n C_{mn} [A_n \bar{M}^{(m)}(k) \bar{M}'^{(m)}(k) + B_n \bar{N}^{(e)}(k) \bar{N}'^{(e)}(k)], \end{aligned} \quad (4.9)$$

$$\bar{\bar{G}}_e^{(21)} = \frac{ik}{4\pi} \sum_{n=1}^{\infty} \sum_{m=0}^n C_{mn} [C_n \bar{M}^{(1)}(k) \bar{M}'^{(m)}(k) + D_n \bar{N}^{(1)}(k) \bar{N}'^{(e)}(k)]. \quad (4.10)$$

Before any further calculation, it is suggestive to analyze how this curved geometry manipulates different dynamics of light and electron both in physical space and virtual uniform space. In top panel of Fig. 4.1, permittivity distribution on xz plane is demonstrated in bluecolor. According to Hamilton's equations $\dot{\bar{r}} = \partial\omega/\partial\bar{k}$ and $\dot{\bar{k}} = -\partial\omega/\partial\bar{r}$, in which $\bar{k}$ indicates momentum, ω the Hamiltonian [11, 6], our coupled ordinary differential equations become,

$$\dot{x} = \frac{k_x}{kn(x, y)}, \quad (4.11)$$

$$\dot{y} = \frac{k_y}{kn(x, y)}, \quad (4.12)$$

$$\dot{k}_x = \frac{k_x}{n^2 \sqrt{x^2 + y^2}} \frac{dn(x, y)}{d\sqrt{x^2 + y^2}}, \quad (4.13)$$

$$\dot{k}_y = \frac{k_y}{n^2 \sqrt{x^2 + y^2}} \frac{dn(x, y)}{d\sqrt{x^2 + y^2}}, \quad (4.14)$$

in which constant light speed c is absorbed into scaled time t to simplify the equations, initial momentum (k_{x0}, k_{y0}) can be set as unitary vector $(\cos \theta, \sin \theta)$.

From this Hamilton's equations, light trajectory in xoz plane can be traced to follow an arc of a circle within Maxwell's fisheye in green curve, in physical space in top panel of Fig. 4.1; whereas motion of charge does not bend its trajectory at all in dashed magenta line according to our predetermined current density function (4.2). Now we transform physical space to virtual uniform space to contrast dynamics of light and charge. Since it is generally difficult to plot surface of a 4-dimensional hypersphere from our human perspective, we reduce this to a simpler counterpart without losing its curved property: surface of a 3-dimensional sphere(lower panel of Fig. 4.1) and limit our view to the cross section in the xz plane.

The one-one map in Fig. 4.1 comes from the inverse stereographic projection

$$X = \frac{2x}{1 + x^2 + y^2}, \quad (4.15)$$

$$Y = \frac{2y}{1 + x^2 + y^2}, \quad (4.16)$$

$$Z = \frac{x^2 + y^2 - 1}{1 + x^2 + y^2}, \quad (4.17)$$

which transforms xz plane into a three-dimensional virtual (uniform)space [†][6]. Mapped in uniform space, light follows geodesics(the green curve) on lower half (corresponding to interior Maxwell's fisheye)spherical surface. However, the electrons deviate their trajectory from geodesics of spherical surface in uniform space if we transform their trajectory to uniform space according to stereographic projection. Instead, electrons propagate on lower half spherical surface differently inside Maxwell's fisheye sphere, shown in dashed magenta circle in lower panel of Fig. 4.1. From the bent trajectory of charge in virtual uniform space, we can explain the reason why electron radiate in our calculation. In this stretched uniform space, electrons actually experience a curved path on lower spherical sphere. This predicts that the electrons will generate a synchrotron radiation within Maxwell's fisheye sphere, in agreement with our calculation below.

[†]U. Leonhardt, *private communication*, June 2013, Guangzhou

4.3 Calculation results of radiation from nonmagnetic Maxwell's fisheye sphere

First to demonstrate unique electromagnetic geometry of Maxwell's fisheye structure, we position a Hertzian dipole at the middle point of the current line, $x = R_1/2 = 1\mu m, y = 0, z = 0$, indicated by a black arrow in Fig. 4.2. In Fig. 4.2, z component of electric field on plane $y = 0$ is plotted for single frequency $300\text{THz}(\lambda = 1\mu m)$. We can see that in Maxwell's fisheye structure in Fig. 4.2(a), light generates from dipole point and follows a curved path which becomes denser in inner part than in outer. This can be explained by the fact that permittivity (or equivalently refractive index) gradually reaches higher in inner part than in outer. The dashed line in white indicates the discontinuity at circle $r = r'$, which is due to the piecewise form of dyadic Green function $\bar{\bar{G}}_{eo}$ itself in Eq. (C.21) in Sec. C.1. Outside the Fisheye, electric field distribution in Fig. 4.2(a) becomes curvilinear smooth contours, different from scattered contours in (b) (permittivity $\epsilon_r = 4$ is uniformly distributed inside the blue circle for a contrast). The smooth contours which squeeze inside fisheye implies that interior sphere of Maxwell's fisheye could in principle be used to design light absorber.

Second, we present the magnitude of electric field on xz plane with time. In Fig. 4.4 we present 6 snapshots of magnitudes of the electric field resulting from electron's motion into Maxwell's fisheye sphere. Green left arrows indicate the position of moving electrons, which are moved by distance of $0.5\mu m$ to the right of their actual positions to avoid shading radiation. We define our time scale coordinate so that electrons pass the middle point $x = 1\mu m, y = z = 0$ at time $t = 0$.

Electrons generate transition radiation as they enter the Maxwell's fisheye sphere (in Fig. 4.4(a), only the radiation accompanying electrons are real radiation and other spots in front of them are considered as spurious artefact) because they see different medium upon boundary of fisheye. During the whole process when electrons traverse Maxwell's fisheye, because permittivity varies from low to high and low again along the electron path, transition radiation is generated continuously but not uniformly in time due to inhomogeneous permittivity experienced. We may also understand the transition radiation from perspective of virtual space. In virtual space in lower panel of Fig. 4.1, electrons move along magenta circle upon

Figure 4.2: (Color Online) Electric fields in z direction, $E_z(x, z)$ on xz plane from Hertzian dipole at point $x = R_1/2 = 1\mu\text{m}, y = 0, z = 0$ (marked as a small black circle) for (a) Maxwell's fisheye sphere and (b) uniform sphere ($\epsilon_r(r < R_1 = 4)$). The huge-value points along blue circle in (a) are thought to be numerical defect of hypergeometric function. The dashed lines in white in (a-b) indicate where some discontinuity at circle $r = r'$ is due to piecewise form of dyadic Green function $\bar{\bar{G}}_{eo}$ itself in Eq. (C.21) in Sec. C.1 of Appendix C.

4.3 Calculation results of radiation from nonmagnetic Maxwell's fisheye sphere

spherical surface, which is equivalent to a synchrotron radiation from accelerated electrons in uniform medium. This acceleration includes both bending trajectory and varying velocity. Since electrons move at a fixed speed in physical space, they move at varying speed along a curved path in virtual space. The radiation thus is a synchrotron radiation nonuniform in time. This referral to synchrotron radiation was made based only on the equivalent trajectory of electrons in virtual space: it is curved(cf. *Remark on synchrotron radiation in virtual space* below). We observe stronger radiation mainly from the middle part (Fig. 4.4 (b-d)) of the trajectory within fisheye. To investigate the reason why stronger radiation mainly occurs within the middle part, we still need to transform back to the physical space. Now in physical space, permittivity distribution along the electron path in the fisheye informs us the maximum permittivity gradient lies near fisheye's boundary(cf. red curve in Fig. 4.8), which would suggest stronger transition radiation near the boundary instead. However, this disagrees with our calculation results. The disagreement can be explained from the perspective of Čerenkov radiation. In dominant part of electron's path inside fisheye, permittivity is greater than $(c/v)^2(= 1.23)$ (cf. blue curve in Fig. 4.8), therefore Čerenkov radiation contributes its power and make the radiation stronger especially in middle part with higher permittivity. This higher permittivity in middle part could also explain why radiation pulse slows down compared to electrons during time $t = 0 \sim 1\text{fs}$ since light speed slows in high permittivity region. As electrons move out of the fisheye, transition radiation bounces backward and leftward(cf. Fig. 4.4(e-f)). And there is, however radiation leftover inside fisheye, although electrons are then absent.

Radiation patterns corresponding to a single frequency of the whole frequency band, for instance 300THz, are also demonstrated in Fig. 4.6. The enclosing sphere to calculate the near field pattern is defined as two fold of Fisheye radius $2R_1$ and far field $20R_1$ (formula of radiation pattern attached in Appendix A(v)). For near field pattern, main lobe of radiation is pointed toward motion direction of electrons, z direction. Other two first side lobes in paris are symmetric along y coordinate, which combines with main lobe to form a lotus pattern. However, from the contour lines on the ground plane in Fig. 4.6(a), radiation pattern is slightly biased on negative x coordinate. For far-field pattern, main lobe points to z and -x directions, which is a pronounced feature in far field. Only one pair of side lobes are visible in far field. We

Figure 4.3: (Color Online) (Color Online) Six snapshots of temporal electric field magnitudes $|E(y=0, x, z, t)|$ on xz plane for motion-induced radiation from Maxwell's fisheye sphere. Dashed pink lines indicate the trajectory of electrons. Green left arrows indicate the positions of electrons, which are moved by distance of $0.5\mu\text{m}$ to the right of their actual positions to avoid shading. We define our time scale coordinate so that electrons pass middle point $x = 1\mu\text{m}, y = z = 0$ at time $t = 0$. Parameters: $R_1 = 2\mu\text{m}, q = 1000e, v = 0.9c$.

4.3 Calculation results of radiation from nonmagnetic Maxwell's fisheye sphere

Figure 4.4: The diagnostics of the electron beam dynamics $|\bar{J}(\bar{r}, t)| = J_z \propto e^{-\frac{(z-vt)^2}{2\sigma^2}}$ in companion with Fig. 4.4. Notice that here the back reaction to electron of radiation was neglected.

attribute this bias of pattern to broken symmetry from biased position of injected electron bunch, given by Eq. (4.2). In these two pattern shown, we see patterns are both symmetrical along y direction.

Figure 4.5: Permittivity function $\epsilon_r(z/R_1)$ and its derivative function $\frac{d\epsilon_r(z/R_1)}{dz/R_1}$ of z coordinate along charge trajectory ($x = 1\mu m, y = 0, -\sqrt{3}/2 < z < \sqrt{3}/2$) investigated.

4.4 Radiation from the Impedance-matched profile of the full Maxwell fisheye

The geometric picture of spherical light trajectory in bottom panel of Fig. 4.1 is only an *approximation* under this nonmagnetic rescaling in Eq. (4.1). The exact spherical light trajectory should result from the full Maxwell fisheye $\epsilon_r = \mu_r = 2/(1 + (r/R_1)^2)$. Hence this Subsec. is spared to investigate the radiation from full Maxwell fisheye.

4.4.1 A. Question of radiation from the full Maxwell's fisheye

In this Sec. we shall pose our problem and try to give some qualitative prediction from the perspective of transformed uniform space. The Maxwell's fisheye lens of refractive index profile

$$n(r) = \frac{2}{1 + (\frac{r}{R_1})^2}, r \in [0, \infty), \quad (4.18)$$

invented by Maxwell himself, focuses light-rays emitted from a source point to its image point antipodally. In this Sec. , we investigate the case of full Maxwell fisheye

4.4 Radiation from the Impedance-matched profile of the full Maxwell fisheye

Figure 4.6: (Color Online) Near field and far field spatial radiation pattern at 300THz from moving charged particle through Maxwell's fisheye sphere. Enclosing sphere of calculation is double size of fisheye radius $2R_1 = 4\mu\text{m}$ and $20R_1 = 40\mu\text{m}$. Contours of radiation pattern on xy plane are projected onto ground plane to show asymmetry in x coordinates due to the broken symmetry of injection position. Note: `surf` plot in MATLAB is supplemented by a command of data interpolation(`interp2`).

profile: the impedance-matched full Maxwell fisheye, $\epsilon = \mu = n(0 < r < \infty)$. As the surface of a sphere is a curved space(with constant curvature), the fish eye profile $n(r)$ can be mapped onto the surface of a hypersphere(4D manifold) in virtual uniform space from Luneburg's visualization [34].

The Green function for electromagnetic fields in this full Maxwell fisheye has been solved recently[76, 77]. The Green tensor for electric field should be,

$$G(\bar{r}, \bar{r}', k) = \frac{\nabla \times n(r, r') \nabla \otimes \nabla' D(r, r') \times \bar{\nabla}'}{n(r)n(r')k^2} - \frac{\delta(\bar{r} - \bar{r}')\mathbb{1}}{n(r)k^2}, \quad (4.19)$$

in which r and r_0 are taken in the relative unit of inner sphere radius R_1 and light speed $c = 1$ to avoid unnecessary factors(See Appendix C.3 for its explicit form), however one requires to convert the unit after obtaining the physical quantities. Thus electric field in the impedance-matched Maxwell's fisheye can be written the same as equation (4.5).

Notice this geometric picture of electron moving on a sphere applies *exactly only* under the condition of *impedance-match*, $\epsilon = \mu = n(0 < r < \infty)$. While the *nonmagnetic* case, $\epsilon = n^2(r), \mu = 1$, within Maxwell's fisheye, is derived under the eikonal approximation which allows one to vary permittivity and permeability so that refractive index maintains unchanged [78], however at the price of varying the wavefront of electromagnetic fields to a certain extent(this 3D case could not guarantee a trick to keep the polarization under eikonal approximation instead of the 2D case in Ref. [43]). Therefore the geometric picture in Fig. 4.1(b) is only an *approximation* under this nonmagnetic rescaling in Eq. (4.1). However, we shall demonstrate in the next subsection, that the two cases both induce a combination of Čerenkov radiation and transition radiation, and thus our approximation remains valid with regard to the perspective of radiation.

4.4.2 B. Main results

Second, we present the magnitude of electric field on xz plane with time. In Fig. 4.7(impedance-matched case), we present another set of 6 snapshots of magnitudes of the electric field resulting from electron's motion into Maxwell's fisheye sphere. Green left arrows indicate the position of moving electrons, which are

4.4 Radiation from the Impedance-matched profile of the full Maxwell fisheye

moved by distance of $0.5\mu m$ to the right of their actual positions to avoid shading radiation. We define our time scale coordinate so that electrons pass the middle point $x = 1\mu m, y = z = 0$ at time $t = 0$.

The analyse of impedance-matched full Maxwell fisheye in Fig. 4.7 is similar but not identical to that of Fig. 4.4. Stronger radiation is also mainly observed in the middle part(panels(b-d) of Fig. 4.7), because Čerenkov radiation(cf. light orange curve in Fig. 4.8) also overweighs and make the whole radiation stronger in the middle part by its higher refractive index(notice that in impedance-matched case, refractive index $n(r) = \epsilon_r(r)$). Transition radiation also occurs since the refractive index is continuously varying along the electrons' path in physical space. However, a *unique* feature of the radiation from impedance-matched Maxwell fisheye is, the electric field accompanying electrons (we call it Electron's Spot(ES) for shortness) carries front and back lobes generally, which switch from the front to the back with time (cf. (a), (c) and (e) in Fig. 4.7). In front of or behind field spots, there are other weaker field spots which resembles in shape with ES, except containing a hole in its interior. This type of weak spot keeps rotating away to the left side of electrons' trajectory until it rotates back to align with the electrons' position, when the electrons fill the hole of the weak spot to emanate real radiation from their motion(cf. the gif file attached). No scattering light is observed in this case when the electrons move out the the boundary of R_1 because impedance-match condition remains valid along this spherical boundary.

Radiation patterns of near and far field corresponding to a single frequency 300THz of the whole frequency band, are demonstrated for this full Maxwell fisheye profile investigated. In Fig. 4.9, the patterns are distinctly different from those for the nonmagnetic sphere case—the near field one carries a radiation power pointing inwards(minus valued) along the minus-x direction and less side lobes are observed. This negative power flow of radiation can be explained by the rotational motion of the weak field spot with a hole by indicating that more power of radiation flows inward than outward at the minus x side of the boundary $r = R_1$. All four radiation patterns in Figs. 4.6 and 4.9 demonstrate that the main lobe of radiation is pointed toward motion direction of electrons, z direction and maintains a symmetrical feature along y coordinate as well as a biased broken symmetry along x coordinate. We attribute this

Figure 4.7: (Color online) Six snapshots of temporal electric field magnitudes $|E(y = 0, x, z, t)|$ on xz plane for motion-induced radiation in the full Maxwell's fisheye medium (impedance matched case). Also see the `.gif` file for the transient effect in Supplementary information of Thesis. All the parameters and figure configuration is the same as Fig. 4.4.

4.4 Radiation from the Impedance-matched profile of the full Maxwell fisheye

Figure 4.8: (Color online) For Maxwell fisheye in nonmagnetic and impedance-matched cases, permittivity function $\epsilon_r(z/R_1)$ and its derivative function $d\epsilon_r(z/R_1)/(dz/R_1)$ of z coordinate along charge trajectory ($x = 1\mu m, y = 0, -\sqrt{3}/2 < z < \sqrt{3}/2$) investigated. The two arrows indicate the two points where Čerenkov radiation starts and stops, i.e. $n(r) = c/v$. The interval where Čerenkov radiation lasts is colored lighter in permittivity function lines.

bias of pattern to broken symmetry from biased position of injected electron bunch, given by equation (4.2).

4.4.3 C. Additional simulation on the radiation with a different beam quality

A concern arises that if the improvement of the beam quality can considerably enhance the efficiency or change the generation processes of the EM radiation. To address this, the main simulation results were conducted for a lower velocity of electron bunch, $v_0 = 0.6c$ with all other parameters unvaried. These results are demonstrated in Figs. 4.10 and 4.11.

Figure 4.9: (Color online) (a) Near field and (b) far field radiation patterns at 300THz from moving charged particle through the full Maxwell's fisheye for impedance-matched case. Enclosing sphere of calculation is double size of fisheye radius $R_1 = 2\mu\text{m}$ and $20R_1 = 40\mu\text{m}$. Other configurations of the figure are the same as Fig. 4.6.

4.4 Radiation from the Impedance-matched profile of the full Maxwell fisheye

Figure 4.10: (Color online)(a-f) Six snapshots of temporal electric field magnitudes $|E(y=0, x, z, t)|$ on xz plane for motion-induced radiation from the full impedance-matched Maxwell's fisheye profile. Green left arrows indicate the positions of electrons, which are moved by distance of $0.5\mu\text{m}$ to the right of their actual positions to avoid shading. We define our time scale coordinate so that electrons pass middle point $x = 1\mu\text{m}, y = z = 0$ at time $t = 0$. Parameters: $R_1 = 2\mu\text{m}, q = 1000e, v = 0.6c$.

Figure 4.11: (Color Online) Derivative of electric field $d|E(y = 0, x, z, t = 0)|/dz$ on xz plane for the full impedance-matched Maxwell's fisheye profile. Red left arrows indicate the positions of electrons, which are moved by distance of $0.5\mu m$ to the right of their actual positions to avoid shading. We define our time scale coordinate so that electrons pass middle point $x = 1\mu m, y = z = 0$ at time $t = 0$. Parameters: $R_1 = 2\mu m, q = 1000e, v = 0.6c$.

4.5 Discussion

The near-field radiation pattern in Fig. 4.6, which points to the motion direction of electrons, demonstrates a complicated yet symmetrical lotus pattern due to its curved geometry for light and electrons both(cf. bottom panel of Fig. 4.1). It is therefore interesting to wonder other possible radiation patterns due to other possibly curved geometry of light. From Figs. 4.2, Maxwell's fisheye structure(of inhomogeneous refractive indices) manipulates light to transit smoothly from uniform space into curved space instead of just scattering the light with uniform permittivity. The different radiation patterns of two kinds of Maxwell fisheye medium, also inspires one to wonder other possible radiation patterns due to other possible but more complicated curved geometry of light.

However, solely from electron's circular trajectory on the lower half sphere in virtual uniform space for impedance-matched Maxwell fisheye profile, we are able to see how electrons perceive this curvature of EM space thus radiate transition radiation. Therefore the radiation from otherwise curved space will bend the electron's trajectory in uniform space to be along other curves (instead of spherical circles), and radiation pattern could contain even more side-lobes. This observation inspires one to engineer certain radiation patterns by varying permittivity and permeability in space and even add in anisotropy similar to engineering directional beam in Refs[79, 80].

Remark: In this Chapter material dispersion is not considered for the simplicity of Green function because the emphasis is put on the spatial profile of refractive index which dominantly shapes the electromagnetic wave. Any imposed dispersion relation of $\epsilon(r, \omega)$ and $\mu(r, \omega)$ could further complicate the derivation of the relevant Green function. Thus this calculation is only valid within a narrow frequency band where our parameter profile applies.

Remark on synchrotron radiation in virtual space: For the radiation from an electron on a sphere, it is theoretically feasible to calculate from Lienard-Wiechart potential in the virtual 3D space where electrons move on the lower half sphere in Fig. 4.1(bottom), a three-vector (let's call it $E_2(X, Y, Z)$) in X, Y, Z direction. However, according to the best of the author's knowledge, it is difficult to compare this electric field from an electron on a sphere with the field in physical space(because of our setup,

this field(let's call it $E_1(x, y = 0, z)$) has vanished y component thus is a two-vector in x and z directions. Because spatial transformation from a 2D(physical) space to a 3D(virtual) space does not necessarily works between E_1 and E_2 as we defined above, it may not be reasonable to check stereographic projection between the two fields.

It is also important to discuss the relativistic effect arising from this computation. Firstly the author treated the electron trajectory as predefined and no back-reaction (force upon the electrons) of radiation from electron's motion was considered. One can imagine that there is some external force(say a capacitor) to keep the electrons on the predefined path. So electrons would experience no deceleration in our case. If, however, the external imaginary force is not existent, our relativistic velocity $0.9c$ of electrons might well be beneficial, because the relative amount of kinetic energy lost in the radiation will be small thus negligible compared to their relativistic kinetic energy. For the reason above, no relativistic term was put in the computation in Chapter 4. It is justified that Maxwell equation is compatible with special relativity. Without resort to relativity, this computation we used is only based on Maxwell equations. So observers were imagined to be everywhere on the plane of xz to observe the induced electric field in Fig. 4.3. However, current computation was not as that strict as special relativity goes—it compromised some relativistic effect during observation—but that is not so important to weaken the radiation point of this Chapter. Otherwise, we'll have to derive the whole thing in four-vector, which is at least unnecessary in current form of results[‡].

4.6 Methods

In this section, we explain our methods to treat electromagnetic fields for radiation from the Maxwell's fisheye profiles. This is a problem to solve inhomogeneous partial differential equations in mathematics. We first write Maxwell's equations in frequency domain according to Fourier transform. Second, we separate the whole current density or the source going through whole infinite space into two parts. The solution in uniform space(outside Maxwell's fisheye sphere), is also calculated through dyadic Green's function. Interested readers shall see more detail in Appendix C4. The

[‡]Kelvin Ooi and Scott Robertson, *private communication*, Nov 2013 and Mar 2014.

4.6 Methods

other one inside Maxwell's fisheye sphere shall be solved via dyadic Green's function narrated in Appendix C1-C3. The sum of both field solutions above make up the complete solution of electromagnetic fields. After that, inverse Fourier transform provides us solutions in time domain:

$$\bar{E}(\bar{r}, t) = \int_{-\infty}^{\infty} d\omega \bar{E}(\bar{r}; \omega) e^{-i\omega t}, \quad (4.20)$$

Famous Fast-Fourier Transform can be a numerical solution to this infinite transform [81, 82].

The explicit expression of Green tensor for the full Maxwell fisheye in impedance-matched case is also given in Appendix C3.

4.6.1 To manipulate Fourier transform numerically

A conventional method to calculate Fourier-transform(FT) and its inverse Fourier-transform(IFT) is Discrete Fourier Transform(DFT) and its numerical trick Fast Fourier Transform(FFT) [83].

To accord with Zhang's derivation, inverse Fourier transform is defined as

$$\bar{E}(\bar{r}, t) = \int_{-\infty}^{\infty} d\omega \bar{E}(\bar{r}; \omega) e^{-i\omega t}, \quad (4.21)$$

consistent with classical physics Fourier Parameters $\{a = -1, b = 1\}$ in `Mathematica` version 7.0.

First suppose we focus on a frequency interval of $[-a/2, a/2]$. And divide it into N equal subintervals of $\beta = a/N$. So

$$\omega_k = (k - 1 - \frac{N}{2})\beta, 1 \leq k \leq N; \quad (4.22)$$

$$t_j = 2\pi \frac{j - 1 - \frac{N}{2}}{a} = 2\pi \frac{j - 1 - \frac{N}{2}}{N\beta}, 1 \leq k \leq N. \quad (4.23)$$

This integer label comes from purpose to align with inverse Fast Fourier transform in `MATLAB 7.11.0(R2010b)`. Hence, an infinite integral first gives way to a definite

integral over $[-a/2, a/2]$; and further simplified, goes to finite series.

$$\int_{-\infty}^{\infty} d\omega E(\omega) e^{-i\omega t} \approx \int_{-\frac{a}{2}}^{\frac{a}{2}} d\omega E(\omega) e^{-i\omega t} \quad (4.24)$$

$$\approx \sum_{k=1}^N E(\omega_k) e^{-i\omega_k t_j \beta} \quad (4.25)$$

$$= \beta \sum E(\omega_k) e^{-i(k-1-\frac{N}{2})2\pi\beta(j-1-\frac{N}{2})/(N\beta)} \quad (4.26)$$

$$= \beta \sum E(\omega_k) e^{i2\pi\frac{N}{2}\beta(j-1-\frac{N}{2})/(N\beta)} e^{-i(k-1)2\pi\beta(j-1-\frac{N}{2})/(N\beta)} \quad (4.27)$$

$$= \frac{a}{N} e^{i2\pi\frac{N}{2}\beta(j-1-\frac{N}{2})/(N\beta)} \sum E(\omega_k) e^{-i(k-1)2\pi\beta(j-1-\frac{N}{2})/(N\beta)} \quad (4.28)$$

$$= \frac{a}{N} e^{i\pi(j-1-\frac{N}{2})} \sum E(\omega_k) e^{-i(k-1)2\pi(j-1-\frac{N}{2})/N} \quad (4.29)$$

$$= \frac{a}{N} (-1)^{j-1-\frac{N}{2}} \sum E(\omega_k) e^{i(k-1)\pi} e^{-i(k-1)2\pi(j-1)/N} \quad (4.30)$$

$$= \frac{a}{N} (-1)^{j-1-\frac{N}{2}} \sum_{k=1}^N E(\omega_k) (-1)^{k-1} \omega_N^{-i(k-1)(j-1)} \quad (4.31)$$

$$(\omega_N \triangleq e^{i2\pi/N}) \quad (4.32)$$

$$= a(-1)^{j-1-\frac{N}{2}} \text{ifft}(E(\omega_k)(-1)^{k-1}). \quad (4.33)$$

In my MATLAB code, $a = \frac{2\pi}{0.5 \times 10^{-15}} \text{rad/s}$. To realize time interval of $\Delta t = 0.5 \text{fs}$, so $tNum = N = 46$. Time range spans from $-11.5, -11, \dots$ until 11fs . Notice that this concrete series is not the only possible scheme.

4.7 Conclusion of calculation on radiation from Maxwell's fisheye sphere

In conclusion, this chapter investigates radiation from electron's motion through curved space—Maxwell's fisheye sphere. We derive radiation from Maxwell's fisheye sphere using the dyadic Green's function method, and finds that it combines both Čerenkov radiation and transition radiation. We also compare physical space and virtual uniform space to explain the reason electron's motion induces radiation in such a curved space geometry. Our calculation may point to novel methods to manufacture light source from electron's kinetic energy. We believe it is useful to explore possible

4.7 Conclusion of calculation on radiation from Maxwell's fisheye sphere

radiation characteristics from engineering permittivity and permeability. It is worth mentioning that electron-induced surface plasmon is also possible to occur when swift electrons interact with metal structure [84]. This is out of scope of this chapter, but also shares physics of transfer from evanescent wave to radiation.

CHAPTER 5

SUMMARY AND FUTURE WORKS

5.1 Summary

In summary, this thesis consider two aspects of electron emission scenario: one on increasing electron emission counts by injecting time-varying electron flow under short-pulse condition and the other on induced electromagnetic fields by electron flow within curved electromagnetic space.

In Chap. 2, we develop a numerical algorithm to calculate a time-dependent short-pulse space-charge-limited current density, based on our previous work on a non-equilibrium model to describe ultrafast laser-excited electron emission from a metallic surface under an applied electric field. This is confirmed by commercial PIC software. We conclude that space charge effect has to been considered during this emission process and calculate space charge limit current values, and laser field, pulse duration and metal work function are all insensitive parameters except that electric DC field is an efficient parameter to tune range of space-charge limit current.

In Chap. 3, we test injection electron flows in different time-profiles, in order to search for an optimal profile function, which may enable them to transcend Valfells' formula under short-pulse situation, by means of commercial simulation(PIC) to consider *time-dependent* effect for short-pulse electron flow in one-dimensional case.

5.2 Future works

We find monotonously increasing time-profile may offer possibility to transcend time-uniform space-charge limit.

While in Chap. 4, to unfold novel physical understanding in emission process, we derive a transition radiation of moving electron bunch going through a simple curved geometry of light: Maxwell's fish-eye, which in principle provides unlimited resolution as a perfect imaging lens [43], and report our calculation on radiation from passage of electrons traversing Maxwell's fisheye sphere, which consists of both Čerenkov radiation and transition radiation.

5.2 Future works

Some possible extensions from summary of my works are given as follows:

- Our previous time-dependent field emission model only consider up to three photons to interact with phonon. Thus extension to interaction of multi-photon and phonon is to account for saturation of electron emission of high laser amplitude regime. This may be related to laser-induced tunnel mechanism which could be further explored.
- For the old equation to surpass the traditional space-charge limit via time-varying conditions whether on boundaries or on injection currents, it is possible to peep into the case when a virtual cathode might appear again

$$E_K > 0. \tag{5.1}$$

But the math knowledge shall be much more complicated than in this Thesis.

- Extension to transition radiation problem could be considered in the context of diffraction radiation. In diffraction radiation, free electrons interact with periodically-alternating environment as they travel through, and by effect of diffraction radiation, a continuum of radiation over a broad angular range occurs. This transfer from kinetic energy of electron bunches to radiation also bears resemblance to transition radiation because electrons interact with inhomogeneous dielectric parameters $\epsilon(\vec{r})$ during their passage. It is possible to

derive or calculate other diffraction radiation fields and radiation patterns in a simpler way than dyadic Green function we adopt in this Chap. 4.

It is worth mentioning that electron-induced surface plasmon is also possible to occur when swift electrons interact with metal structure [84]. This also shares physics of transfer from evanescent wave to radiation, *viz.* Smith-Purcell radiation [85]. Therefore it is also possible to link the concept of evanescent wave and surface plasmon in certain metal structures. Moreover, recent new work [86, 87] shows the possibility to link surface plasmon by a nonlocal effect from perspective of hydrodynamical Drude model to account for nonlocal, spatial dispersive interactions between electrons in metal structures. Using this nonlocal effect to calculate EM wave from motion of charges near metal grating structures could be a possible instance to explore.

An interesting combination of behaviors of electron and photon lies in Čerenkov radiation (CR), which refers to continuous frequency-spectrum electromagnetic wave when a charged particle passes through insulator with a velocity larger than light speed in that medium. Jin-Kyu So *et al.* [88] has shown metamaterial serves as a tunable dielectric medium which supports threshold-free CR with the presence of convection electrons. Extension to derive Smith-Purcell radiation from energetic electron bunch on metamaterial(one-dimensional array of slits perforated on a metallic film hereby) is possible for outlook of application in which measuring dispersion-relation of guided EM mode and detection of electromagnetic space distortion by convection electron.

- An even further possible investigation can be from combination of transport and dynamics behavior of both electron and photon. For the electrons, if we tend to consider energy dissipation, actual electron velocity will tend to decrease due to energy dissipation into radiation. Electron radiation loss can be coupled with Maxwell's equations, to accommodate radiation loss problem, which from another perspective, becomes an optical force issue. Exerted by optical force induced from itself, electron will even receive acceleration in other directions and thus, diverge from original straight trajectory under this curved electromagnetic space. This will turn out the analogy to elliptical tracks of celestial bodies, where gravity serves to distort space-time.

APPENDIX A

DERIVATION OF UNIFORM SPACE-CHARGE-LIMITED CURRENT (CHILD-LANGMUIR LAW)

A.1 Review on derivation of Child-Langmuir law

In this appendix, the author repeats derivation of Child-Langmuir law, which is not valid in treating time-variant problem of injection current density $J(t)$. However, this derivation below shows that analytic solution to it requires some unknown technique to treat term $\phi(x)$ neatly. We shall abandon this route and resort to numerical method as Chap. 2 reports.

$$\rho(x) = \frac{J}{v(x)} = \frac{J}{\sqrt{\frac{2e}{m}(F_{dc}x + \Delta\phi(x))}} > 0, \quad (\text{A.1})$$

supposing current density J is positive too for simplicity. Therefore Poisson equation is written in following form,

$$\frac{\partial^2}{\partial x^2} \phi(x) = \frac{\rho(x)}{\epsilon_0} > 0 \quad (\text{A.2})$$

$$= \frac{J(x)}{v(x)\epsilon_0} = \frac{J}{\sqrt{\frac{2q\phi(x)}{m}}} \quad (\text{A.3})$$

$$\frac{d\phi'(x)}{d\phi} \frac{d\phi(x)}{dx} = \frac{J}{\epsilon_0 \sqrt{\frac{2q\phi(x)}{m}}} \quad (\text{A.4})$$

$$\epsilon_0 \frac{1}{2} d\phi'^2(x) = \frac{4J}{\sqrt{2q/m}} d\sqrt{\phi(x)} \quad (\text{A.5})$$

Since

$$\phi(0) = \phi'(0) = 0, \quad (\text{A.6})$$

$$\frac{1}{2} \phi'^2(x) = \frac{2J}{\epsilon_0 \sqrt{2q/m}} \sqrt{\phi(x)} \quad (\text{A.7})$$

$$[\phi'(x)]^2 = \frac{4J}{\epsilon_0 \sqrt{2q/m}} \sqrt{\phi(x)} \quad (\text{A.8})$$

$$\therefore \phi'(x) = 2 \left(\frac{J}{\epsilon_0 \sqrt{2q/m}} \right)^{1/2} \phi^{1/4}(x). \quad (\text{A.9})$$

We take the positive solution to interpret it physically.

We can deduce that

$$J = \frac{4}{9\epsilon_0} \sqrt{\frac{2q}{m}} \frac{\phi^{3/2}}{D^2} \quad \square. \quad (\text{A.10})$$

$$V(x)/\phi = \left(\frac{x}{D} \right)^{\frac{4}{3}}, \quad (\text{A.11})$$

which is exact Child-Langmuir law.

Just for the record, the author utilizing **Mathematica** 7.0, find that solution after direct integration is

$$[6\sqrt{A}c_1^2\phi^{\frac{1}{4}} - 6Ac_1\sqrt{\phi} + 8A^{3/2}\phi^{3/4} - 3c_1^3\ln[c_1 + 2\sqrt{A}\phi^{1/4}] + c_2 = x. \quad (\text{A.12})$$

A.1 Review on derivation of Child-Langmuir law

Boundary conditions give

$$0 = -3c_1^3 \ln[c_1] + c_2 \quad (\text{A.13})$$

$$D = [6\sqrt{A}c_1^2V_g^{\frac{1}{4}} - 6Ac_1\sqrt{V_g} + 8A^{3/2}V_g^{3/4} - 3c_1^3 \ln[c_1 + 2\sqrt{A}V_g^{1/4}] + c_2. \quad (\text{A.14})$$

In the equations above, A denotes coefficient in front of $\phi(x)$, c_1, c_2 the integration constants, $V_g \equiv \phi(x=0)$ the cross electric potential in Volt. It is predictable that these solutions will be different from Child-Langmuir law, which implies that at least mathematics involves more possibility than this restricted physical situation of space-charge limit. Shall more relaxed from this situation, other possible solution to this equation is also possible to be interpreted as other physical situation.

APPENDIX B

AN ITERATIVE NUMERICAL METHOD TO TREAT TIME-VARIANT SPACE CHARGE EFFECT

In Chap. 1 we use our numerical algorithm to seek maximum injection current under laser-excitation condition. While in Chap. 2, we remove laser-excitation condition, and focus on seeking optimal time-variant profile to achieve maximum injection charge. During working on this part, we modified our algorithm in Chap. 1 and compared resultant maximum charges with Particle-In-Cell simulation, which turned out not in good agreement. Therefore this appendix is spared to report the negative result. In this appendix, we list the result to seek space-charge limit under time-variant injection condition. We will first revisit our algorithm and then provide the calculation result compared with Particle-In-Cell(PIC) simulation(VORPAL). We draw the conclusion that this numerical method requires further correction under time-variant injection condition.

B.1 An Iterative Numerical Method to Treat Time-Dependent Space Charge Effect

Here in this chapter we shall report our algorithm to calculate time-dependent space-charge-limited current density of a short-pulse electron flow based on calculation method in Chap. 1. Consider a metal-vacuum interface under an external constant electric field F_{dc} , which is applied across a distance of D from cathode ($x = 0$) and

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anode ($x = D$). Thus electric potential distribution writes

$$\phi(0 \leq x \leq D) = F_{dc}x + \Delta\phi(x), = \begin{cases} F_{dc}x + \Delta\phi(x), & 0 \leq x \leq s; \\ F_{dc}x, & s < x \leq D, \end{cases} \quad (\text{B.1})$$

where s means the longest distance emitted electrons are able to transit.

Now miscellaneous time-dependent functions of emitted current density $J(t)$ are enforced to emit out from the planar cathode, in order to seek optimal function form to achieve higher value than Valfells' formula. Accordingly, the whole electron count number density (per unit emitter area) N_{EC} is defined as

$$N_{EC} = \int_0^{\tau_p} \frac{J(t)}{e} dt. \quad (\text{B.2})$$

Also to compare with Valfells' formula, a reference value f_{ref} shall be determined as

$$f_{ref} = \frac{J_{crit}\tau_p}{e \int_0^{\tau_p} j_m(t) dt}, \quad (\text{B.3})$$

to be used later to decide whether our time-dependent current are able to surmount short-pulse one of space-charge limit.

For a specific short-pulse electron of flow of time-dependent function $j_m(t)$, in order to find the corresponding limit current, we seek the factor f before $j_m(t)$ in $J(t)$'s expression by traversing beneath possible intervals as follows.

B.1.1 Solving Poisson's equation

In order to obtain additional space charge potential distribution $\Delta\phi(0 \leq x \leq s)$, Poisson's equation

$$\begin{aligned} \phi''(x) = \Delta\phi''(x) &= -\frac{\rho(x)}{\varepsilon_0} \\ &= -\frac{\bar{J}(x)}{\varepsilon_0 v(x)} \\ &= -\frac{J[\tau_p - t(x)]}{\sqrt{\frac{2e}{m}\phi(x)}} \\ &= -\frac{f j_m[\tau_p - t(x)]}{\sqrt{\frac{2e}{m}(F_{dc}x + \Delta\phi(x))}}, \end{aligned} \quad (\text{B.4})$$

demands solving, in which $\rho(x)$ is electron density function, $\bar{J}(x)$ is current density distribution function and $v(x)$ is electron velocity function and Eq. (B.1) is substituted; boundary conditions are

$$\Delta\phi(0) = 0, \quad (\text{B.5})$$

and

$$F_{\text{dc}} + \Delta\phi'(s) = \frac{F_{\text{dc}}D - [F_{\text{dc}}s + \Delta\phi(s)]}{D - s}. \quad (\text{B.6})$$

While the rest part of electric potential $\Delta\phi(s \leq x \leq D)$ is naturally linear function,

$$\Delta\phi(s \leq x \leq D) = -\frac{\Delta\phi(x=s)}{D-s}(x-D). \quad (\text{B.7})$$

The last boundary condition Eq. (B.6) denote that electrostatic field continues at electron beam end position $x=s$ [3]. To derive Eq. (B.4), electron density $\bar{J}(x)$ and velocity $v(x)$ are determined as follows,

$$\bar{J}(x) = J[\tau_p - t(x)] = fj_m[\tau_p - t(x)], \quad (\text{B.8})$$

and

$$v(x) = \sqrt{\frac{2e}{m}(F_{\text{dc}}x + \Delta\phi(x))}, \quad (\text{B.9})$$

in which $t(x)$ is inverse of electron position function $x(t)$; obviously, assumption that electrons keeps travelling forward is taken to allow existence of reverse function of $x(t)$ and spatial profile of electron beam is assumed maintained during travelling forward (a numerical iterative method is adopted later to restore such an induced error).

B.1.2 Tracing electron's position upon time

According to Newton's second law of motion, the second derivative of electron position function $x(t)$ is proportional to the first derivative of electric potential distribution function $\phi(x)$, i.e.

$$\begin{aligned} \frac{d^2}{dt^2}x(t) &= a(x) \\ &= \frac{e}{m} \frac{d}{dx}\phi(x) \\ &= \frac{e}{m} \frac{d}{dx}[\phi_{\text{dc}}(x) + \Delta\phi(x)] \\ &= \frac{e}{m} [F_{\text{dc}} + \frac{d}{dx}\Delta\phi(x)]; \end{aligned} \quad (\text{B.10})$$

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in which $a(x)$ is acceleration function dependent on position and static electric field force formula is used, and of which initial conditions are $x(t=0) = 0$ and $x'(t=0) = v(t=0) = 0$, which becomes pointing to anode as soon as electron see direct electric field F_{dc} in between. Notably, in order to avoid numerical integration difficulty of right-hand side of (B.4) (an infinity $\sim x^{-1/2}$ is encountered in integrand at $x = 0^+$), a reasonably small positive velocity $v(t=0 \Leftrightarrow x=0)$ can be assumed providing that

$$v(t=0)\tau_p \ll s_{\max}, \quad (\text{B.11})$$

in which $s_{\max}$ means the maximum value of occupation length of emitted electrons, s . We have to point out here, maximum value of occupation length $s_{\max}$ can never surpass distance between cathode and anode, which justifies the reason our numerical algorithm herein is only valid within short-pulse condition of electron flow; since otherwise when it does, the front of electron pulse will be absorbed at anode, this holistic calculation of maximum injection current density become meaningless. This is also the reason such situations are beyond our scope in the Particle-In-Cell simulation part.

B.1.3 Iterative algorithm to solve Poisson's equation

Solving $\Delta\phi(x)$ appears essential in all this algorithm to obtain electron position function from (B.10). However, as already pointed, straightforward derivation from Eqs.(3.5), (B.4), (B.10) and (2.17) to obtain an analytic Child-Langmuir-law-like formula turns out to failure due to inhomogeneous Poisson's equation, Eq. (B.4): no analytic method to solve $\Delta\phi(x)$ from Eq. (B.4), which is involved in its both sides simultaneously. (This feature of Eq.(B.4) perplexes its solution because directly solving it numerically results in a positive $\Delta\phi(x)$ distribution, definitely violating physical intuition—we perceive electron bunch induces negative electrical potential term here additionally.)

Not only to sidestep this unnecessary mathematical trouble, but also more significantly to restore induced errors due to aforementioned neglected repulsion force among electrons in Eq. (B.8), a *numerical* algorithm is taken in such an iterative way: first, $\Delta\phi = 0$ is substituted into Eq. (B.10) to obtain electron position function $x(t)$ and hence its inverse function $t(x)$, and also record the value of acceleration at cathode plane, $a_1|_{x=0}$. Then start with a factor f value and the first solution of $\Delta\phi(x) \triangleq \Delta\phi_1(x)$ will be taken into Eqs. (B.4) and (2.6) again in the next iterative one to obtain a second acceleration at cathode $a_2|_{x=0}$, which modifies value of first acceleration a_1 to some extent due to difference between $\Delta\phi_1(x)$ and 0. Afterwards, this unit above is iteratively repeated to modify $a_2, a_3, \dots a_i$, into $a_3, a_4, \dots a_{i+1}, \dots$; in

all of which, the i th acceleration at cathode, $a_i|_{x=0}$ defined as

$$a_i|_{x=0} = \frac{e}{m} \left(F_{\text{dc}} + \frac{d}{dx} \Delta\phi|_{x=0} \right). \quad (\text{B.12})$$

These iterative loops continue until an evaluation condition (we observe the convergence of acceleration at cathode, $|a_{i+1} - a_i|/a_i \leq 1. \times 10^{-6}$ as evaluation reference) is satisfied that the i th electron position function upon time is stabilized within a small tolerance range. It is acceptable to conclude that other physical parameters such as $\phi_i(x)$ are close enough to their true values; for instance, space charge potential distribution $\Delta\phi(x)$ is demonstrated in Fig. 2.5. So far, the one whole loop for a certain factor value has been completed and we traverse factor from a lower-than-unity one (e.g. 0.5) until acceleration at cathode in Eq. (B.12) becomes non-positive, reaching space-charge limit. The entire algorithm has been depicted in flow chart of Fig. B.1.

Therefore when iterative algorithm obtains its stabilized value, the stabilization criterion, acceleration value, depicted in Fig. B.2 becomes stabilized gradually and produces valid data efficiently within less than 10 repetitive units.

B.2 Comparison with Particle-in-Cell(PIC) Simulation

In order to check our mechanism, we conduct a PIC finite-difference-time-domain (FDTD) numerical software (VORPAL)[54], run in one dimension, where space charge limit effect is considered to be reached when reflection electron momentum is observed [53, 17].

Despite of the trend to achieve higher space-charge limit at shorter temporal range from PIC simulation reported in Chap. 3, we find here all scaled factors f/f_{ref} converge with Valfells formula, well below data of simulation. This may demonstrate our numerical method fails to capture time-variant behavior and thus requires further correction.

B.2 Comparison with Particle-in-Cell(PIC) Simulation

Figure B.1: Flow chart (by the TikZ and PGF Packages under $\text{\LaTeX}$ 6.1) of our iterative algorithm to calculate emitted density of electron count under space charge effect. First roughly guess that in the first iterative loop, the first space charge electron potential profile $\Delta\phi(x) = 0$ to obtain the first tunneled density of electron count; then afterwards use Newton's second law to trace the inverse of electron position function versus time, $t(x)$. So far, the first iterative unit is completed and the first solution of Poisson's equation, $\Delta\phi_1(x)$ will be adopted in the next calculation loop to obtain a second tunneled electron number and so on recursively in the following loops, until an evaluation criteria is satisfied so that the final density of electron count is stabilized within acceptable range.

Figure B.2: (a) Relative acceleration values at cathode plane, $a_i|_{x=0}/\frac{eF_{dc}}{m}$ and (b) relative discrepancy between acceleration values, $(a_{i+1} - a_i)/a_i|_{x=0}$ in neighboring loops of each iterative calculation unit. Physical parameters: temporal length of electron pulse $\tau_p = 1.0\text{ps}$, external electric field strength $F_{dc} = 3\text{V/nm}$, $D = 1\text{cm}$ and $J(t) = 0.95f_{ref}t/\tau_p$.

B.2 Comparison with Particle-in-Cell(PIC) Simulation

Figure B.3: Scaled factor f/f_{ref} of our numerical and simulation results, plotted versus temporal duration of electron pulse. Its ordinate values are scaled to unit of short-pulse space-charge limit current of Valfells' formula ($D = 1\text{cm}$, $F_{\text{dc}} = 3\text{GV/m}$).

APPENDIX C

DERIVATION TO SOLVE WAVE EQUATION FOR MAXWELL'S FISHEYE SPHERE

In Chap. 4 we treat the problem to seek electromagnetic fields due to motions of an electron bunch by method of dyadic Green function. We list the relevant dyadic Green function in Sec. 4.2 but have not indicated the concrete formula of dyad $\bar{\bar{G}}_e(\bar{r}, \bar{r}')$. In this appendix, we divide the long derivation into two parts. The first is to derive dyadic Green's function generally for all inhomogeneous medium whose permittivity varies by only radius, in Sec. C.1. The other is to give scalar wave solution which is used to generate building blocks of dyadic Green function specially for Maxwell's fisheye medium($\epsilon_r(r) = \frac{4}{[1+(\frac{r}{R_1})^2]^2}, r \in [0, R_1]$), in Sec. C.2.

C.1 Dyadic Green function method to solve electromagnetic fields in inhomogeneous medium

For a source problem for electromagnetic fields, two *inhomogeneous* wave equations below require solving,

$$\nabla \times \nabla \bar{E} - k^2 \epsilon_r(r) \bar{E} = i\omega\mu_0 \bar{J} \quad (C.1)$$

$$\nabla \times \left[\frac{1}{\epsilon_r(r)} \nabla \times \bar{H} \right] - k^2 \bar{H} = \nabla \times \frac{\bar{J}}{\epsilon_r(\bar{r})}. \quad (C.2)$$

Main steps of dyadic Green's function to solve electromagnetic field with source are as follows,

C.1 Dyadic Green function method to solve electromagnetic fields in inhomogeneous medium

(i) Define four inhomogeneous spherical vector eigenwaves $\bar{M}(\bar{r})$ and $\bar{N}(\bar{r})$ from two homogeneous wave equations without source below:

$$\nabla^2 \Psi + k^2 \epsilon_r(r) \Psi = 0 \quad (\text{C.3})$$

$$\nabla^2 \Phi - \frac{1}{\epsilon_r(r)} \frac{d\epsilon_r(r)}{dr} \frac{\partial \Phi}{\partial r} + k^2 \epsilon_r(r) \Phi = 0. \quad (\text{C.4})$$

For any defined source function like (4.2), four spherical vector eigen-waves,

$$\bar{M}^{(m)} = \nabla \times (\Psi \bar{R}) \quad (\text{C.5})$$

$$\bar{N}^{(m)} = \frac{1}{k} \nabla \times \nabla \times (\Psi \bar{R}) \quad (\text{C.6})$$

$$\bar{M}^{(e)} = \nabla \times (\Phi \bar{R}) \quad (\text{C.7})$$

$$\bar{N}^{(e)} = \frac{1}{k \epsilon_r(r)} \nabla \times \nabla \times (\Phi \bar{R}), \quad (\text{C.8})$$

are induced to solve (C.1) and (C.2), providing that the generating functions, scalar quantities Ψ and Φ satisfy following two differential equations (C.3) and (C.4). Here piloting vector is defined as $\bar{R} = r\hat{r}$.

The symmetrical relations of these four vector eigenwaves are

$$\bar{M}^{(m)} = \frac{1}{k \epsilon_r(r)} \nabla \times \bar{N}^{(m)} \quad (\text{C.9})$$

$$\bar{N}^{(m)} = \frac{1}{k} \nabla \times \bar{M}^{(m)} \quad (\text{C.10})$$

$$\bar{M}^{(e)} = \frac{1}{k} \nabla \times \bar{N}^{(e)} \quad (\text{C.11})$$

$$\bar{N}^{(e)} = \frac{1}{k \epsilon_r(r)} \nabla \times \bar{M}^{(e)} \quad (\text{C.12})$$

Note that both $\bar{M}^{(m)}$ and $\bar{N}^{(e)}$ are solutions for homogeneous vector wave equation of electric field $\bar{E}$,

$$\nabla \times \nabla \bar{E} - k^2 \epsilon_r(r) \bar{E} = 0, \quad (\text{C.13})$$

while both $\bar{M}^{(e)}$ and $\bar{N}^{(m)}$ are solutions for vector wave equation of magnetic field $\bar{H}$,

$$\nabla \times \left[\frac{1}{\epsilon_r(r)} \nabla \times \bar{H} \right] - k^2 \bar{H} = 0. \quad (\text{C.14})$$

Thus with traditional vector spherical harmonics method in convenient spherical coordinates, by method of separation of radial and transverse components, scalar

eigenwaves can be written explicitly

$$\Psi_n = \frac{1}{r} S_n(kr) P_n^m(\cos \theta) \exp(im\phi) \quad (\text{C.15})$$

$$\Phi_n = \frac{1}{r} T_n(kr) P_n^m(\cos \theta) \exp(im\phi), \quad (\text{C.16})$$

where radial components $S_n(kr)$ and $T_n(kr)$ satisfy ordinary differential equations

$$\frac{d^2 S_n}{dr^2} + [k^2 \epsilon_r(r) - \frac{n(n+1)}{r^2}] S_n = 0, \quad (\text{C.17})$$

$$\epsilon_r(r) \frac{d}{dr} \left[\frac{1}{\epsilon_r(r)} \frac{dT_n}{dr} \right] + [k^2 \epsilon_r(r) - \frac{n(n+1)}{r^2}] T_n = 0, \quad (\text{C.18})$$

according to Eqs. (C.3) and (C.4).

(ii) Let us now denote the interior Maxwell's fisheye as region 1, and the exterior as region 2. The source coordinate of the dyadic Green's function is in region 1. Linear combinations of vector eigenfunctions $\bar{M}$ and $\bar{N}$ are juxtaposed into dyadic Green's function $\bar{\bar{G}}_e(\bar{r}, \bar{r}')$ in which the two labels in superscripts of Green's functions denotes the region in which the image coordinate is situated. In this part (ii), we only consider Green function when the source lies in region 1, so number 1 always appears as the second label herein. Now we have

$$\bar{\bar{G}}_e^{(11)} = \bar{\bar{G}}_{eo}(\bar{r}, \bar{r}') + \bar{\bar{G}}_{es}^{(11)}(\bar{r}, \bar{r}'), r < R_1 \quad (\text{C.19})$$

and

$$\bar{\bar{G}}_e^{(21)} = \bar{\bar{G}}_{es}^{(21)}(\bar{r}, \bar{r}'), r > R_1. \quad (\text{C.20})$$

$$\bar{\bar{G}}_{eo} = -\frac{1}{k^2} \hat{r} \hat{r} \delta(\bar{r} - \bar{r}') + \quad (\text{C.21})$$

$$\frac{ik}{4\pi} \sum_{m,n} C_{mn} \begin{cases} \bar{M}^{(m1)}(k) \bar{M}'^{(m)}(k) + \bar{N}^{(e1)}(k) \bar{N}'^{(e)}(k), r > r', \\ \bar{M}^{(m)}(k) \bar{M}'^{(m1)}(k) + \bar{N}^{(e)}(k) \bar{N}'^{(e1)}(k), r < r'. \end{cases}$$

$$\bar{\bar{G}}_e^{(11)} = \frac{ik}{4\pi} \sum_{n=1}^{\infty} \sum_{m=0}^n C_{mn} [A_n \bar{M}^{(m)}(k) \bar{M}'^{(m)}(k) + B_n \bar{N}^{(e)}(k) \bar{N}'^{(e)}(k)], \quad (\text{C.22})$$

$$\bar{\bar{G}}_e^{(21)} = \frac{ik}{4\pi} \sum_{n=1}^{\infty} \sum_{m=0}^n C_{mn} [C_n \bar{M}^{(1)}(k) \bar{M}'^{(m)}(k) + D_n \bar{N}^{(1)}(k) \bar{N}'^{(e)}(k)]. \quad (\text{C.23})$$

$$C_{mn} = \frac{2n+1}{n(n+1)} (-1)^m. \quad (\text{C.24})$$

C.1 Dyadic Green function method to solve electromagnetic fields in inhomogeneous medium

This coefficient is different from Tai's definition because we use $e^{im\phi}$ instead of $\begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}$. The dyadic Green functions above are written in analogy with their counterparts in uniform space. The dyadic Green's function for uniform space write,

$$\bar{\bar{G}}_{eo} = -\frac{1}{k^2} \hat{r} \hat{r} \delta(\bar{r} - \bar{r}') + \frac{ik}{4\pi} \sum_{m,n} C_{mn} \begin{cases} \bar{M}^{(1)}(k) \bar{M}'(k) + \bar{N}^{(1)}(k) \bar{N}'^{(0)}(k), & r > r', \\ \bar{M}(k) \bar{M}'^{(1)}(k) + \bar{N}(k) \bar{N}'^{(1)}(k), & r < r'. \end{cases} \quad (\text{C.25})$$

$$\bar{\bar{G}}_e^{(11)} = \frac{ik}{4\pi} \sum_{n=1}^{\infty} \sum_{m=0}^n C_{mn} [A_n \bar{M}(k) \bar{M}'(k) + B_n \bar{N}(k) \bar{N}'(k)], \quad (\text{C.26})$$

$$\bar{\bar{G}}_e^{(21)} = \frac{ik}{4\pi} \sum_{n=1}^{\infty} \sum_{m=0}^n C_{mn} [C_n \bar{M}^{(1)}(k) \bar{M}'(k) + D_n \bar{N}^{(1)}(k) \bar{N}'(k)]; \quad (\text{C.27})$$

whose building blocks, eigen-wave functions are defined as below:

$$\bar{M} = \nabla \times [j_n(kr) P_n^m(r) e^{im\phi} \bar{R}] \quad (\text{C.28})$$

$$\bar{N} = \frac{1}{k} \nabla \times \bar{M}, \quad (\text{C.29})$$

$$\bar{M}^{(1)} = \nabla \times [h_n^{(1)}(kr) P_n^m(r) e^{im\phi} \bar{R}] \quad (\text{C.30})$$

$$\bar{N}^{(1)} = \frac{1}{k} \nabla \times \bar{M}^{(1)}. \quad (\text{C.31})$$

For magnetic fields $\vec{H}$,

$$\bar{\bar{G}}_m^{(11)} = \bar{\bar{G}}_{mo}(\bar{r}, \bar{r}') + \bar{\bar{G}}_{ms}^{(11)}(\bar{r}, \bar{r}'), r < R_1 \quad (\text{C.32})$$

and

$$\bar{\bar{G}}_m^{(21)} = \bar{\bar{G}}_{ms}^{(21)}(\bar{r}, \bar{r}'), r > R_1. \quad (C.33)$$

$$\bar{\bar{G}}_m = \nabla \times \bar{\bar{G}}_e : \quad (C.34)$$

$$\bar{\bar{G}}_{mo} = \frac{ik^2}{4\pi} \sum_{m,n} C_{mn} \begin{cases} \bar{N}^{(m1)}(k) \bar{M}'^{(m)}(k) + \bar{M}^{(e1)}(k) \bar{N}'^{(e)}(k), & r > r', \\ \bar{N}^{(m)}(k) \bar{M}'^{(m1)}(k) + \bar{M}^{(e)}(k) \bar{N}'^{(e1)}(k), & r < r'. \end{cases} \quad (C.35)$$

$$\bar{\bar{G}}_{ms}^{(11)} = \frac{ik^2}{4\pi} \sum_{n=1}^{\infty} \sum_{m=0}^n C_{mn} [A_n \bar{N}^{(m)}(k) \bar{M}'^{(m)}(k) + B_n \bar{M}^{(e)}(k) \bar{N}'^{(e)}(k)], \quad (C.36)$$

$$\bar{\bar{G}}_{ms}^{(21)} = \frac{ik^2}{4\pi} \sum_{n=1}^{\infty} \sum_{m=0}^n C_{mn} [C_n \bar{N}^{(1)}(k) \bar{M}'^{(m)}(k) + D_n \bar{M}^{(1)}(k) \bar{N}'^{(e)}(k)]. \quad (C.37)$$

According to method of scattering superposition, coefficients A_n , B_n , C_n and D_n are determined from boundary conditions at lens boundary $r = R_1$ (take the first branch of $r = R_1 \geq r'$):

$$\hat{r} \times \bar{E} = \hat{r} \times \bar{E} \implies \hat{r} \times \bar{\bar{G}}_e^{(11)} = \hat{r} \times \bar{\bar{G}}_e^{(21)} \quad (C.38)$$

$$\hat{r} \times \bar{H} = \hat{r} \times \bar{H}, \nabla \times \bar{E} = i\omega\mu_0 \bar{H} \implies \hat{r} \times \nabla \times \bar{\bar{G}}_e^{(11)} = \hat{r} \times \nabla \times \bar{\bar{G}}_e^{(21)}. \quad (C.39)$$

Therefore a set of algebraic equations are deduced,

$$S^{(1)} + A_n S = C_n Q, \quad (C.40)$$

$$T^{(1)} + B_n T = D_n Q, \quad (C.41)$$

$$S'^{(1)} + A_n S' = C_n Q', \quad (C.42)$$

$$T'^{(1)} + B_n T' = D_n Q'; \quad (C.43)$$

where

$$S^{(1)} = k S_n^{(1)}(k \rho_a), \quad (C.44)$$

$$S = k S_n(\rho_a), \quad (C.45)$$

$$Q = \rho_a h_n^{(1)}(\rho_a), \quad (C.46)$$

$$Q' = \frac{d}{d\rho_a} S_n(\rho_a), \quad (C.47)$$

$$T = k T_n(\rho_a), \quad (C.48)$$

$$T' = \frac{d}{d\rho_a} k T_n(\rho_a), \quad (C.49)$$

$$\rho_a = k r_1. \quad (C.50)$$

C.1 Dyadic Green function method to solve electromagnetic fields in inhomogeneous medium

Once we solve the four coefficients, we know the dyadic Green functions when source is located in region 1.

(iii) Multiplied by current source term $\bar{J}(\omega; \bar{r})$, dyadic Green's function $\bar{\bar{G}}_e(\bar{r}, \bar{r}')$ gives electromagnetic field solution $\bar{E}(\omega; \bar{r})$ explicitly. For a line-section of current density $\bar{J}(\bar{r}, \omega)$ in (4.2), only the line section penetrating the Maxwell's lens inside (zone 1) is to be treated using the dyadic Green functions in Eqs. (C.19) and (C.20), and we write

$$\bar{E}(\bar{r}) = i\omega\mu_0 \iiint \bar{\bar{G}}_e(\bar{r}, \bar{r}') \cdot \bar{J}(\bar{r}') dV', \quad (\text{C.51})$$

$$\bar{H}(\bar{r}) = \iiint \nabla \times \bar{\bar{G}}_e(\bar{r}, \bar{r}') \cdot \bar{J}(\bar{r}') dV'. \quad (\text{C.52})$$

Let us write current density $\bar{J}(\bar{r}, t)$ in Eq. (4.2) as $\bar{J}_z = \hat{z}J_3$ into spherical coordinates (curvilinear orthogonal coordinates [34, 89, 74]). The identity to be used here writes $d\bar{r} = dx^i \hat{e}_i = dx^j \hat{e}_j = \frac{\partial x^{i'}}{\partial x^j} dx^j \hat{e}_{i'} \Rightarrow \frac{\partial x^{i'}}{\partial x^j} \hat{e}_{i'} = \hat{e}_j$, so $\hat{z}' = \hat{r}' \cos \theta' - \hat{\theta}' \sin \theta'$. To further implement line integration along z coordinate, integrand above

$$\bar{\bar{G}}_e(\bar{r}, \bar{r}') \cdot \bar{J}(\bar{r}') = \bar{\bar{G}}_e(\bar{r}, \bar{r}') \cdot \hat{z}' J_3 = \bar{\bar{G}}_e(\bar{r}, \bar{r}') \cdot (\hat{r}' \cos \theta' - \hat{\theta}' \sin \theta') J_3, \quad (\text{C.53})$$

which finally has to be converted into Cartesian coordinates to implement the line integral. In such a step, a vector in spherical coordinates (r, θ, ϕ) converts into one in Cartesian coordinates (x, y, z) according to

$$x = r \sin \theta \cos \phi, \quad (\text{C.54})$$

$$y = r \sin \theta \sin \phi, \quad (\text{C.55})$$

$$z = r \cos \theta. \quad (\text{C.56})$$

(iv) The remaining infinite long line of current density in uniform space, we shall resort to traditional dyadic Green's method. Current density in frequency domain

$$\bar{J}(\bar{r}, \omega) = \hat{z} \frac{q}{2\pi} \delta(x - x_0) \delta(y - y_0) \exp\left(-\frac{\sigma^2 \omega^2}{2v^2} + i \frac{\omega z}{v}\right), \quad (\text{C.57})$$

can be written in form of

$$\begin{aligned} \bar{J}(\bar{r}, \omega) = & \left\{ \bar{J}(\bar{r}, \omega) - \bar{J}(\bar{r}, \omega) [u(z - z_1) - u(z - z_2)] \right\} + \\ & \bar{J}(\bar{r}, \omega) [u(z - z_1) - u(z - z_2)] = \bar{J}_1 + \bar{J}_2, \end{aligned} \quad (\text{C.58})$$

where

$$\bar{J}_1 = \bar{J}(\bar{r}, \omega) - \bar{J}(\bar{r}, \omega)[u(z - z_1) - u(z - z_2)], \quad (\text{C.59})$$

$$\bar{J}_2 = \bar{J}(\bar{r}, \omega)[u(z - z_1) - u(z - z_2)], \quad (\text{C.60})$$

in which $u(\cdot)$ represents step function and z_1, z_2 indicates z coordinates of points where injection current line intersects Maxwell's fisheye. The terms $\bar{J}_1$ and $\bar{J}_2$ is respectively positioned in the interior and exterior part of Maxwell's fisheye sphere, therefore they should be treated with different dyadic Green's functions due to different EM space structures. According to Zhang's work [40] and its supplementary material, the line-section $\bar{J}_1$ outside of Maxwell's fisheye can be considered as a difference between an infinite current line and a definite one, both of which can be treated straightforwardly [75]. Electric field solution for the infinite current line can be written in form of

$$\bar{E}_1(\bar{r}, \omega) = -\frac{q \exp\left(-\frac{\sigma^2 \omega^2}{2v^2} + i\frac{\omega z}{v}\right)}{8\pi\omega\epsilon_0} \left[\hat{z} \left(\frac{\omega^2}{c^2} - \frac{\omega^2}{v^2} \right) H_0^{(1)}(k_\rho \rho) - \hat{\rho} \frac{i\omega k_\rho}{v} H_1^{(1)}(k_\rho \rho) \right], \quad (\text{C.61})$$

in which

$$k_\rho = \sqrt{k^2 - \frac{\omega^2}{v^2}}, \quad (\text{C.62})$$

and $H_i^{(1)}(\cdot)$ means i th order Hankel function of the first kind. For $\bar{E}_1(\bar{r}, \omega)$ above, the cylindrical system's origin is set as point $(x_0, y_0, 0)$.

Electric field solution for definite line segment inside uniform sphere R_1 can be obtained from equation

$$\bar{E}_2(\bar{r}, \omega) = i\omega\mu_0 \left[\bar{I} + \frac{1}{k^2} \nabla \nabla \right] \cdot \iiint \frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \bar{J}_2(\bar{r}') dV'. \quad (\text{C.63})$$

Therefore the electric field we seek is,

$$\bar{E}(\bar{r}, \omega) = \bar{E}_1(\bar{r}, \omega) - \bar{E}_2(\bar{r}, \omega) + \bar{E}_3(\bar{r}, \omega), \quad (\text{C.64})$$

in which $\bar{E}_3(\bar{r}, \omega)$ is calculated from Eq. (C.51) in (iii) we have derived.

(v) It is straightforward to write radiation energy spectrum density for radiation from Maxwell's fisheye. The enclosing sphere used for calculation is defined as $r = 2R_1 = 4\mu\text{m}$. We use $\Re$ and $\Im$ to denote real and imaginary parts respectively.

$$\begin{aligned} \frac{dW}{d\omega} &= \int_0^{2\pi} \int_0^\pi r^2 \sin\theta d\theta d\phi \cdot 4\pi \\ &[\Re \bar{E}(\omega) \times \Re \bar{H}(\omega) + \Im \bar{E}(\omega) \times \Im \bar{H}(\omega)] \cdot \hat{r}. \end{aligned} \quad (\text{C.65})$$

C.2 Scalar wave solution for nonmagnetic Maxwell's fisheye sphere

The radiation pattern formula is the integrand of the integral above in Eq.(C.65).

C.2 Scalar wave solution for nonmagnetic Maxwell's fisheye sphere

This appendix reports how we solve scalar Helmholtz Eqs. (C.17) and (C.18). For the Maxwell's fisheye lens, given permittivity function defined as

$$\epsilon_r(r) = \frac{4}{\left[1 + \left(\frac{r}{R_1}\right)^2\right]^2}, r \in [0, R_1], \quad (\text{C.66})$$

Radius r ranges from 0 to R_1 . Tai transformed radial variable in this way [90, 74],

$$\rho = kr, \rho_a = kR_1, \xi = -\left(\frac{\rho}{\rho_a}\right)^2 \quad (\text{C.67})$$

$$S_n(\xi) = \xi^{\frac{n+1}{2}} (\xi - 1)^\mu U_n(\xi) \quad (\text{C.68})$$

$$T_n(\xi) = \xi^{\frac{n+1}{2}} (\xi - 1)^{\mu-1} V_n(\xi), \quad (\text{C.69})$$

where

$$\mu = \frac{1}{2}(1 + \sqrt{1 + 4\rho_a^2}) \quad (\text{C.70})$$

thus new functions $U_n(\xi)$ and $V_n(\xi)$ form two hypergeometric equations,

$$\xi(\xi - 1) \frac{d^2 U_n}{d\xi^2} + [(2\mu + \beta)\xi - \beta] \frac{dU_n}{d\xi} + \alpha U_n = 0 \quad (\text{C.71})$$

$$\xi(\xi - 1) \frac{d^2 V_n}{d\xi^2} + [(2\mu + \beta)\xi - \beta] \frac{dV_n}{d\xi} + \left(\alpha - \frac{1}{2}\right) V_n = 0. \quad (\text{C.72})$$

whose solutions are hypergeometric function ${}_2F_1(a, b; c; \xi)$ and $\xi_2^{1-c} F_1(1 + a - c, 1 + b - c; 2 - c; \xi)$. Comparing relevant coefficients of Eqs. (C.71) and (C.72) with those of normal form of hypergeometric equation,

$$a + b + 1 = 2\mu + \beta \quad (\text{C.73})$$

$$c = \beta \quad (\text{C.74})$$

$$ab = \alpha \quad \text{or} \quad \alpha - \frac{1}{2}. \quad (\text{C.75})$$

Hence solutions to Eqs. (C.71) and (C.72) read, regular solution

$${}_2F_1(a, b; c; z) \triangleq F(a, b, c, z) = F(a(2\mu + \beta, \alpha), b(2\mu + \beta, \alpha), \beta, \xi), \quad (\text{C.76})$$

and the solution singular at origin

$$\xi_2^{1-c} F_1(1 + a - c, 1 + b - c; 2 - c; \xi) \triangleq \xi^{1-c} F^{(1)}(1 + a - c, 1 + b - c; 2 - c; \xi), \quad (\text{C.77})$$

so we expect to determine four inhomogeneous spherical vector eigenwaves, $\bar{M}^{(m)}, \bar{N}^{(m)}, \bar{M}^{(e)}, \bar{N}^{(e)}$ and $\bar{M}^{(m1)}, \bar{N}^{(m1)}, \bar{M}^{(e1)}, \bar{N}^{(e1)}$ from Eqs. (C.9), (C.10), (C.11) and (C.12).

Then dyadic Green's function $\bar{\bar{G}}_e(\bar{r}, \bar{r}')$ in Maxwell's fisheye sphere is therefore determined, which determines electric fields in the fisheye sphere according to equations in Appendix C.1(i-iii).

C.3 Explicit form of Green tensor for the full impedance-matched Maxwell fisheye

(i) Before writing out its explicit form of Green tensor, the electric field pattern from a Hertzian electric dipole inside the full impedance-matched Maxwell fisheye is plotted in Fig. S1(a) to compare with that of uniform space in Fig. S1(b). We adopt the same parameters as Fig. 2(a) in the maintext and observe a similar continuously-varying behavior of electric field.

(ii) The Green tensor for electromagnetic fields in the full Maxwell fisheye has been solved recently[76, 77], in which the Green tensor for electric field should be,

$$G(\bar{r}, \bar{r}', k) = \frac{\nabla \times n(\gamma) \nabla \otimes \nabla' D(\gamma) \times \bar{\nabla}'}{n(r)n(r')k^2} - \frac{\delta(\bar{r} - \bar{r}') \mathbb{1}}{n(r)k^2}, \quad (\text{C.78})$$

$$n(r) = \frac{2}{1 + r^2}, \quad (\text{C.79})$$

$$D(\gamma, \omega) = \left(\gamma + \frac{1}{\gamma} \right) \frac{\sin(2\omega \operatorname{arccot} \gamma)}{(4\pi)^2 \sin \pi \omega}, \quad (\text{C.80})$$

$$\gamma = \frac{|\vec{r} - \vec{r}'|}{\sqrt{1 + 2\vec{r} \cdot \vec{r}' + |\vec{r}|^2 |\vec{r}'|^2}}. \quad (\text{C.81})$$

in which the length quantities r and r' are taken in the relative unit of inner sphere radius R_1 and light speed $c = 1$ to avoid unnecessary factors, however one requires to convert the unit after obtaining the physical quantities. Thus electric field in the full Maxwell's fisheye sphere can be written the same as equation (C.51). The only non-trivial part for this Green tensor is the nominator of its first term, let us call it

C.3 Explicit Green tensor for the full impedance-matched Maxwell fisheye

Figure C.1: (Color online) Electric fields in z direction, $E_z(x, z)$ on xz plane from Hertzian dipole at point $x = R_1/2 = 1\mu m, y = 0, z = 0$ (marked as a small black circle) for (a) the full Maxwell's fisheye sphere in impedance-match case and (b) (Replica of Figure 4. 2(b) in maintext) uniform sphere ($\epsilon_r(r < R_1) = 4$).

K . If we define (x, y, z are written as x_1, x_2, x_3 to facilitate tensor symbol)

$$D_{ij} = \partial_{x_i} \partial_{x'_j} D(r, r') (i = 1, 2, 3), \quad (C.82)$$

$$K(\bar{r}, \bar{r}', k) = \nabla \times n(r, r') \nabla \otimes \nabla' D(r, r') \times \overleftarrow{\nabla}' \quad (C.83)$$

$$= \nabla \times n(r, r') D_{ij} \hat{x}_i \hat{x}'_j \times \overleftarrow{\nabla}' \quad (C.84)$$

$$= \nabla \times n(r, r') \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix} \times \overleftarrow{\nabla}' \quad (C.85)$$

$$= [acd][bef] \frac{\partial^2 n(r, r')}{\partial x_c \partial x'_e} \frac{\partial^2 D(r, r')}{\partial x_d \partial x'_f} \hat{x}_a \otimes \hat{x}'_b, \quad (C.86)$$

$$K \cdot \hat{z}' = \begin{bmatrix} D_{32} \partial_{x'} \partial_y n(r, r') - D_{22} \partial_{x'} \partial_z n(r, r') \\ D_{12} \partial_{x'} \partial_z n(r, r') - D_{32} \partial_{x'} \partial_x n(r, r') \\ D_{22} \partial_{x'} \partial_x n(r, r') - D_{12} \partial_{x'} \partial_y n(r, r') \end{bmatrix} - \quad (C.87)$$

$$\begin{bmatrix} D_{31} \partial_{y'} \partial_y n(r, r') - D_{21} \partial_{y'} \partial_z n(r, r') \\ D_{11} \partial_{y'} \partial_z n(r, r') - D_{31} \partial_{y'} \partial_x n(r, r') \\ D_{21} \partial_{y'} \partial_x n(r, r') - D_{11} \partial_{y'} \partial_y n(r, r') \end{bmatrix}. \quad (C.88)$$

To arrive at this step, the rule that a curl of a gradient vanishes is used, as well as a little tensor algebra [91]. Be aware that this vector could be simplified further (we retreat to Leonhardt's symbols) [76]

$$K \cdot \hat{z}_0 = \left[\begin{array}{c} r'_{32} \partial_y r' - r'_{22} \partial_z r' \\ r'_{12} \partial_z r' - r'_{32} \partial_x r' \\ r'_{22} \partial_x r' - r'_{12} \partial_y r' \end{array} \right] \partial_{x_0} r' - \left[\begin{array}{c} r'_{31} \partial_y r' - r'_{21} \partial_z r' \\ r'_{11} \partial_z r' - r'_{31} \partial_x r' \\ r'_{21} \partial_x r' - r'_{11} \partial_y r' \end{array} \right] \partial_{y_0} r' \cdot [n^{(2)}(r') \cdot D^{(1)}(r') + n^{(1)}(r') \cdot D^{(2)}(r')]. \quad (C.89)$$

Once we know how to write the explicit Green tensor, electric field is known according to equation (C.51).

Although from the temporal perspective, it is also possible write time-dependent Green tensor[77] in this case, however, to obtain electric field solution, explicit derivation shows that the only trouble is to solve a transcend equation, which happens to be much more messy than in frequency domain. That is why we adopt Green tensor in frequency domain.

(iii) A movie file(**srep-Liu-gif.gif**) attached in the Thesis disk is also attached to display electric field patterns with time for impedance-matched Maxwell fisheye medium.

LIST OF AUTHOR'S PUBLICATION

Journal Papers

1. Liu Yangjie and L. K. Ang*, “Transition from ultrafast laser photo-electron emission to space charge limited current in a 1D gap”, *J. Phys. D: Appl. Phys.*, 47 125502(2014); <http://iopscience.iop.org/0022-3727/47/12/125502/>, <http://arxiv.org/abs/1307.5596v2>.
2. Liu Yangjie* and L. K. Ang, “Motion-induced radiation from electrons moving in Maxwell’s fish-eye”, *Scientific Reports*, 3, 3065(2013); <http://dx.doi.org/10.1038/srep03065>.

Conference Papers

1. Liu Yangjie, Wu Lin and L. K. Ang, “Saturation of Ultrafast Laser Excited Electron Emission from Metallic Nanotip at High Current Regime”, *East Asian Postgraduate Workshop on Nanoscience and Technology, Hong Kong*, Poster 109 (2011).
2. Liu Yangjie and L. K. Ang, “Time-Dependent Space-Charge-Limited Electron Flow”, *39th IEEE International Conference on Plasma Science (ICOPS2012)*, *Edinburgh, UK*, Session 3P: Computational Plasma Physics (poster session 2), 36.
3. Liu Yangjie and L. K. Ang, “Motion-Induced Radiation of Charged Particles in Curved Electromagnetic Space”, *39th IEEE International Conference on*

Plasma Science (ICOPS2012), Edinburgh, UK, Session 3P-37: Computational Plasma Physics.

4. Liu Yangjie and L. K. Ang, “Analysis of Motion-Induced Radiation of Charged Particles”, *Photonics Global Conference (PGC) 2012, Singapore*, poster, Paper ID: c12a528 (DOI: 10.1109/pgc.2012.6458032).
5. Liu Yangjie and L. K. Ang, “Space Charge Effect of Time-dependent Emission Current Excited from Ultrafast Laser ”, *14th IEEE International Vacuum Electronics Conference (IVEC) 2013, Paris, France*, Poster Session I, 22 May 2013.
6. Liu Yangjie and L. K. Ang, “Motion-Induced Radiation of Charged Particles”, *7th International Conference on Materials for Advanced Technologies 30 June-5 July 2013, Suntec Singapore*, accepted as poster, withdrawn, Paper ID: ICMAT13-A-0578.
7. L. K. Ang and Liu Yangjie, “Transition Radiation from Motion of Electrons in Curved Space”, *the IEEE Pulsed Power and Plasma Science Conference (PPPS2013), San Francisco, US*, Session P4-45.
8. Liu Yangjie and L. K. Ang, “Transition Radiation from Maxwell’s Fisheye”, *The 34th PIERS(Progress In Electromagnetics Research Symposium) 12-15 August, 2013, Stockholm, Sweden*, accepted as oral talk, withdrawn, Key: 130319043401.

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