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Xia, Bing Xing

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Numerical Analysis of Some Multiscale and Stochastic Partial Differential Equations

Xia Bing Xing

Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
Singapore

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Abstract

Multiscale partial differential equations (PDEs) and stochastic PDEs arise from many technological and engineering situations, such as composite materials, ground water flow and oil recovery. For multiscale PDEs, as the scales differ from each other by several orders of magnitude, classical finite element (FE) methods are prohibitively expensive as the mesh width has to be of the order of the smallest scale for the approximating solution to represent correctly the exact solution of the multiscale equation. For stochastic PDEs, the cost of computing the statistical properties of the solution is high. The complexity of these problems may surpass the current available computing power. Studying new computational and approximation methods that can solve these problems within acceptable computational time, using reasonable computational resources without sacrificing accuracy is one of the central topics in applied mathematics today. This thesis aims to make novel contributions to this timely scientific challenge. We study novel computational methods for multiscale wave equations, multiscale elasticity equations and multiscale elastic wave equations. We also devote a part of this thesis to studying approximation for random and parametric elasticity equations.

For the multiscale equations, we assume that they depend on n separable microscopic scales. We study the homogenization limit when the scales converge to 0. Employing the concept of multiscale convergence (see Nguetseng [44], Allaire [3], Allaire and Briane [4]), extending to time dependent functions for the wave and elastic wave equations, we deduce an $(n+1)d$ dimensional problem that contains all the macroscopic and microscopic information on the multiscale equation. The full tensor product finite element approach is natural to solve the high dimensional multiscale homogenized equations but is too expensive practically. We develop a sparse tensor finite element approach that achieves essentially equal accuracy as the full tensor product finite elements but requires essentially equal number of degrees of freedom as for solving a problem in $\mathbb{R}^d$. For the wave equation and the elastic wave equation considered in Chapter 2 and Chapter 4, we study the spatial semidiscretized and the fully discretized problems following the approach of Dupont [20] for the high dimensional multiscale homogenized equation. After deducing error estimates for general FE spaces, we construct the full tensor FE and sparse tensor FE approximations and deduce error estimates in terms of the FE mesh size and the time step size. For these error estimates, we show that the regularity required hold under mild conditions for the coefficients, the forcing and the domain. Restricting our consideration to a zero wave displacement, we deduce a

numerical corrector from the FE solutions. For the two-scale case, we deduce a new homogenization error, which together with the FE error implies a rate of convergence for the numerical corrector. Multiscale elasticity equation is studied in Chapter 3. We study FE approximations for the multiscale homogenized equation by using full and sparse FE approximations. We again show that the rate of convergence for the sparse tensor FEs is essentially equal to that for the full tensor FEs but using only an essentially equal number of degrees of freedom as for solving a single scale elliptic PDE in $\mathbb{R}^d$. Numerical correctors are constructed from the FE solutions. An analytic homogenization rate of convergence is deduced for the two-scale case from which we deduce a rate of convergence for the numerical corrector. For all of these problems, we solve numerically several two and three scale problems in one and two dimensions. The numerical results confirm our analysis.

Random elasticity equations are studied in Chapter 6. We consider linear elasticity equations whose elastic moduli are random in a domain $D \subset \mathbb{R}^d$. We assume that the coefficients are written as a series of random variables which are uniformly distributed in a bounded interval, and that the coefficient functions of the expansion satisfy a decaying condition so that their $L^\infty(D)$ norms are $\ell^p(\mathbb{N})$ summable. We first consider the displacement formulation. We study the semidiscretized Galerkin problem by projecting to the spaces of those functions obtained by truncating the generalized polynomial chaos (GPC) expansion at a finite level. Using the method developed by Cohen et al. [18], we deduce the rate of convergence for the best N term approximation from the ℓ^p summability of the GPC expansion's coefficients. We then study the Hellinger-Reissner mixed formulation that allows computing the stress tensor directly. We deduce conditions on the stochastic/parametric moduli for the uniform inf-sup condition of the semidiscrete Galerkin approximation problem to exist that is independent of the domain D , and in particular of the Korn's inequality's constant. We also deduce a condition on the elastic moduli's expansion for the ℓ^p summability of the coefficients of the GPC expansion of the stress tensor and the displacement vector, that are independent of the domain; from this best N term approximations for the stress and the displacement are derived. For nearly incompressible materials, we obtain explicit rates of convergence for the best N term approximation for the case of mixed Dirichlet and Neumann boundary conditions that is independent of the Lamé constant λ when it converges to ∞ , using a mixed formulation.

Introduction

Multiscale partial differential equations (PDEs) and stochastic PDEs arise from many technological and engineering situations, such as composite materials, ground water flow and oil recovery. For multiscale PDEs, as the scales differ from each other by several orders of magnitude, classical finite element (FE) methods are prohibitively expensive as the mesh width has to be of the order of the smallest scale for the approximating solution to represent correctly the exact solution of the multiscale equation. For stochastic PDEs, the cost of computing the statistical properties of the solution is high. The complexity of these problems may surpass the current available computing power. Studying new computational and approximation methods that can solve these problems within acceptable computational time, using reasonable computational resources without sacrificing accuracy is one of the central topics in applied mathematics today. This thesis aims to make novel contributions to this timely scientific challenge.

For multiscale PDEs, a direct numerical simulation requires resolving all the fine scales and is prohibitively expensive. The homogenization theory studies the limit when the fine scales converge to zero; and this limit approximates the solution to the original multiscale problems macroscopically. For periodic homogenization, in order to compute the homogenized coefficients, we need to solve the cell problems. The total cost of doing this in terms of computer memory and floating point operations can be very large if the problem is locally periodic or the problem depends on multiple microscopic scales. For general problems, G -convergence and Γ -convergence are employed (see Jikov et al. [37]). These methods are abstract and identification of the homogenization limits of the multiscale problems can be difficult. There has been significant effort in developing efficient numerical procedures for solving multiscale problems with acceptable complexity and high accuracy; see for examples, the Multiscale Finite Element Methods [35] and the Heterogeneous Multiscale Method [22], see also the books by Efendiev and Hou [23] and E [21]. For computing the homogenized solution of the problem, the number of degrees of freedom these methods required is significantly less than that for a direct simulation, but the computation cost still grows superlinearly when the discretizing mesh decreases.

For periodic problems, the multiscale convergence method deduces a macroscopic variational problem for the solution of the homogenized problem and the scale interacting terms that describe the oscillating behaviour of the multiscale solution (see Nguetseng [44], Allaire [3] and Allaire and Briane [4]). Solving this multiscale homogenized problem, we get all the necessary information. However, this problem is posed in a high dimensional tensorized space. For elliptic problems in $\mathbb{R}^d$ that depend on n microscopic scales, the limiting problem depends on $(n + 1)d$ dimensional variables. The full tensor FE method is natural for this problem, but it can be prohibitively costly. Hoang and Schwab [32] develop the sparse tensor product FE method for multiscale elliptic problems. It achieves essentially equal accuracy as the full tensor product FEs, but requires only an essentially equal number of degrees of freedom as for solving an elliptic equation in $\mathbb{R}^d$. The number of degrees of freedom required grows log-linearly when the discretizing mesh decreases while the rate of convergence is essentially equal to that for the full tensor product finite elements, whose complexity grows superlinearly when the approximating mesh size decreases. Apart from the log term, the rate of convergence and the complexity we obtain is uniform with respect to the number of the scales. Further, as shown in Hoang and Schwab [31], the resulting system can be solved by the conjugate gradient method that requires only a log-linear number of floating point operations with respect to the number of degrees of freedom. The method has been successfully applied to linear elliptic multiscale equations in [27], [31], and multiscale monotone equations in [30]. We note that the Heterogeneous Multiscale Method was applied to solve the high dimensional multiscale homogenized equation in Ohlberger [46].

In this thesis, we develop the sparse tensor FE approach for the multiscale homogenized equations of wave, elasticity and elastic wave problems. We then construct numerical correctors from the FE solutions. For two-scale problems, we deduce an explicit rate of convergence for these correctors. They are the sum of the homogenization and the FE rates of convergence.

In Chapter 2, we consider multiscale wave equations whose coefficients depend on n separable microscopic scales. Employing multiscale convergence (extending to time dependent functions), we derive the multiscale homogenized equation for the problem. Adopting the approach of Dupont [20], we analyze the spatial semidiscrete problem and the fully discrete problem with respect to both the spatial and temporal variables x and t using a general FE spaces. We then construct the full and sparse tensor FE spaces and apply them to the general frameworks. We derive error estimates for these approximations when the solution is sufficiently regular, and show that the sparse tensor FE approximation achieves a rate of convergence which is essentially equal to that for the full tensor FE approximation but uses only an essentially equal number of degrees of freedom as for solving a wave equation in $\mathbb{R}^d$. We show that the regularity required to obtain these rates of convergence can be achieved with only reasonable requirements on the coefficients and the domains. For multiscale wave equations, a corrector similar to that for elliptic problems does not always hold as the energy of the multiscale wave equation only converges to the energy of the homogenized equation

when the initial condition satisfies special properties, see Brahim-Otsmane et al. [11]. However, restricting our consideration to a zero initial wave displacement, we derive a numerical corrector from the FE solutions of the multiscale homogenized equation. For two-scale problems, we deduce a homogenization rate of convergence which together with the FE rate of convergence, implies a rate of convergence for this numerical corrector. Finally we present numerical examples for two and three scale problems in one and two dimensions that confirm our analysis.

Chapter 3 studies sparse tensor FE methods for multiscale elasticity equations. We analyze the full and sparse tensor product FE approach to the multiscale homogenized equation obtained from multiscale homogenization. We derive explicit FE error estimates for these methods and again demonstrate that when the solution is sufficiently regular the error for the sparse tensor product FE approach is essentially equal to that for the full tensor product FEs using only an essentially equal number of degrees of freedom as for solving an equation in $\mathbb{R}^d$. We then construct a numerical corrector from these FE solutions. Adopting the approach of Jikov et al. [37] for elliptic equations, we derive a homogenization rate of convergence for the two-scale case which implies a rate of convergence for the numerical corrector. Employing the sparse tensor FEs, we solve two and three scale elasticity problems in two dimensions. The numerical results illustrate our analysis.

In Chapter 4, we consider elastic wave equations whose moduli depend on n separable microscopic scales and are periodic with respect to each scale in a domain D in $\mathbb{R}^d$. By using multiscale convergence, we deduce an $(n+1)d$ dimensional limiting problem. We develop a sparse tensor finite element approach that achieves essentially equal accuracy as the full tensor product finite elements but with an essentially equal number of degrees of freedom as for solving a problem in $\mathbb{R}^d$. Restricting our consideration to the case of a zero initial wave displacement, we deduce an analytic corrector, with a homogenization error estimate in the case of two scales. From this a numerical corrector is constructed from the finite element solutions of the high dimensional multiscale homogenized problem with an error of the order of the sum of the homogenization and the finite element errors. For the general multiscale problems, we construct a numerical corrector from the finite element solutions but without an explicit error. Numerical examples on two and three scale elasticity wave equations in two dimensions confirm our results.

In Chapter 6, we consider linear elasticity equations whose elastic moduli depend linearly on a countable set of random variables in a domain $D \subset \mathbb{R}^d$. This is the case, for example, when they are written as a Karhnen-Love expansion when the covariance is known. Such problems describe physical processes in random media such as composite materials (see for example, Willis [54], Milton [43]). Computing statistical quantities of the solutions of PDEs with random coefficients is highly complicated. This attracts much attention at the moment due to its importance in technology and engineering. There have been many attempts using various mathematical methods and techniques to tackle these problems: collocation methods, stochastic finite element methods, spectral methods, but the challenge is still enormous. In this thesis,

we assume that the random variables in the expansion of the coefficient are uniformly distributed in a bounded interval, and that the coefficient functions of the expansion satisfy a decaying condition so that their $L^\infty(D)$ norms are $\ell^p(\mathbb{N})$ summable. This decaying condition holds in many cases where the covariance is piecewise sufficiently smooth. This is justified in Schwab and Todor [51], Bieri et al. [8] for scalar coefficients and Hoang and Schwab [34] for tensor coefficients. We note that our method is able to tackle more general distributions, though for simplicity we only consider a uniform distribution. We do not consider the case of non-uniformly distributed random variables but for this case, results similar to those of Cohen et al. [18] can be deduced from our framework. The stochastic problem is casted into a deterministic parametric elasticity equation whose solution represents the stochastic solution's law. We first consider the displacement formulation. We study the best N term approximation of the generalized polynomial chaos (GPC) expansion of the solution. By considering the semidiscretized Galerkin problem via projecting to the spaces of those functions obtained through truncating the GPC expansion at a finite level, the rate of convergence is deduced from the ℓ^p summability of the GPC expansion's coefficients. We use the method developed by Cohen et al. [18] for bounding the GPC coefficients by first establishing bounds for derivatives of these coefficients with respect to the parameters. We then study the best N term approximation for the Hellinger-Reissner mixed formulation of the elasticity problem which allows for computing the stress directly. We establish conditions on the expansion of the elastic moduli so that the semidiscretized problems satisfy a uniform inf-sup condition. The conditions are independent of the domain, in particular, independent of the constant in the Korn's inequality. For nearly incompressible materials, we obtain explicit rates of convergence for the best N term approximation for the case of mixed Dirichlet and Neumann boundary conditions that is independent of the Lamé constant λ when it converges to ∞ , using a mixed formulation of the elasticity equation.

Multiscale Wave Equation

2.1 Introduction

We develop in this chapter the high dimensional approach for locally periodic wave equations that depend on multiple separated microscopic scales. The method is based on the sparse high dimensional finite element method developed in Hoang and Schwab [31] for the high dimensional multiscale homogenized problem which is obtained via multiscale convergence. It is posed on a tensorized domain and depends on $n + 1$ variables, one for each scale that the problem depends on. We compute the solution of the homogenized equation and the scale interacting terms. As shown in Brahim-Otsmane et al. [11], it is more complicated to construct correctors for multiscale wave equations than for elliptic equations as the energy of the multiscale wave equation does not always converge to that of the homogenized equation. Restricting our consideration to the case where the initial condition g^0 in (2.5) equals 0, we construct a numerical corrector from the scale interacting terms for general multiscale problems. For two-scale problems, we deduce explicitly a homogenization rate of convergence, which together with the finite element rate of convergence, form an explicit rate of convergence for the numerical corrector. We note that for the general initial conditions, these scale interacting terms always form a part of the corrector, albeit another term which requires solving a separate multiscale problem is needed (see Brahim-Otsmane et al. [11]).

There have been some attempts at solving multiscale wave equations that describe wave propagation in heterogeneous media. Owhadi and Zhang [48] consider a general method that solves d microscopic scale equations and uses their solutions to build a multiscale basis for solving the general problem. Multiscale Finite Element Method (Hou and Wu [35]) is employed for wave equations that depend on a continuous spectrum of scales using a limited amount of global information in Jiang et al. [36]. Finite difference method is employed by Engquist et al. in [24] where the solution at large time that shows a dispersive effect is also computed. The Heterogeneous Multiscale Method (E and Engquist [22]) is employed for solving multiscale wave equations by Abdulle and Grote in [2]. However, in these papers the number of degrees of freedom

grows superlinearly when the error decreases. In this chapter, we achieve a log-linear complexity at each time step for locally periodic multiscale wave equations, using sparse tensor FEs for the high dimensional limiting multiscale homogenized problem.

The chapter is organized as follows. In the next section, we define the multiscale wave equation and recall the concept of multiscale convergence applying to time dependent problems; and we then deduce the multiscale limit of the wave equation. General consideration for the finite element method for the high-dimensional limiting equation deduced in Section 2.3 is considered in Section 2.4 where we work with both the spatially semidiscrete and the fully discrete problems. For the convergence of these approximating schemes, and for deducing explicit error estimates, we need regularity of the high dimensional solution. This is deduced in Section 2.5. In Section 2.6, we consider the full tensor product and the sparse tensor product approximations. When the solution satisfies the regularity deduced in Section 2.5, we deduce explicit rates of convergence for the approximations, and we show that the number of degrees of freedom for the sparse tensor product approximation is essentially (within a log term) equal to that for solving a single equation that depends on only the slow variable, but achieves essentially equal accuracy. To get the rate of convergence for the corrector, we need a homogenization rate of convergence for equation (2.5). To the best of our knowledge, such a rate has not been established. When the initial condition g^0 in (2.5) equals 0, using the technique of Jikov et al. [37] for elliptic equations we deduce the $\varepsilon^{1/2}$ rate of convergence, although we need to overcome some technical difficulty associated with the wave equation. For multiscale equations, this rate is not available. However, we can deduce a general analytic corrector. These are done in Section 2.7. In Section 2.8, we construct numerical correctors using the finite element solutions for multiscale problems, and for two-scale problems, we deduce an explicit rate of convergence in terms of both the homogenization rate of convergence and the finite element error. Finally, in Section 2.9, we presents some numerical examples for two and three scale problems in one or two dimensional spaces that confirm our analysis.

Throughout the chapter, by c we denote a generic constant that does not depend on the scales and the approximating parameters. By ∇ without indicating the variable, we mean the gradient with respect to x . For a Banach space X , we denote by $L^p(0, T; X)$ the Bochner spaces of functions from $[0, T]$ to X and $L^\infty(0, T; X)$ the space of bounded functions from $[0, T]$ to X . The notation $\#$ denotes spaces of periodic functions, in particular, $H_\#^k(Y)$ denotes the Sobolev spaces of periodic functions in the unit cube $Y \subset \mathbb{R}^d$ and $C_\#^k(Y)$ denotes the space of periodic k time differentiable functions in Y .

2.2 Problem setting

Let $D \subset \mathbb{R}^d$ be a bounded domain. Let $Y = (0, 1)^d$ be the unit cube in $\mathbb{R}^d$. For an integer n that denotes the number of microscopic scales that the problem depends on, let $Y_1, Y_2, \dots, Y_n$ be n copies of Y ; and let $\mathbf{Y}$ be the set of all nd dimensional vectors $\mathbf{y} = (y_1, y_2, \dots, y_n)$ such that $y_p \in Y_p$ for $p = 1, \dots, n$. We denote by A a mapping

from the product space $D \times \mathbf{Y}$ to the space $\mathbb{R}_{sym}^{d \times d}$ of real symmetric matrices such that for all $x \in D$ and $\mathbf{y} \in \mathbf{Y}$, $A(x, \mathbf{y})$ satisfies the boundedness condition

$$|A(x, \mathbf{y})\xi \cdot \zeta| \leq \beta|\xi||\zeta|, \quad (2.1)$$

and the coercivity condition

$$\alpha|\xi|^2 \leq A(x, \mathbf{y})\xi \cdot \xi, \quad (2.2)$$

for all vectors $\xi, \zeta \in \mathbb{R}^d$, where α and β are positive constants. We assume that the coefficient A is periodic with respect to y_p for $p = 1, \dots, n$ and is sufficiently smooth:

$$A(x, \mathbf{y}) \in C^1(D, C_{\#}^1(Y_1, \dots, C_{\#}^1(Y_n) \dots)). \quad (2.3)$$

Let $\varepsilon > 0$ be a small quantity. Let $\varepsilon_1(\varepsilon), \dots, \varepsilon_n(\varepsilon)$ be n positive functions of ε that denote the n microscopic scales that the problem depends on. These scales converge to 0 when $\varepsilon \rightarrow 0$. We assume that they satisfy the scale separation assumption, i.e., for $p = 1, \dots, n-1$,

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon_{p+1}(\varepsilon)}{\varepsilon_p(\varepsilon)} = 0. \quad (2.4)$$

Without loss of generality, we let $\varepsilon_1 = \varepsilon$. The multiscale coefficient $A^\varepsilon : D \rightarrow \mathbb{R}_{sym}^{d \times d}$ is defined as

$$A^\varepsilon(x) = A\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}\right).$$

We denote by $V = H_0^1(D)$ and $H = L^2(D)$. Let $g^0 \in V$ and $g^1 \in H$. For $T > 0$, let $f(t, x) \in L^2(0, T; H)$. We consider the wave equation in D with the Dirichlet boundary condition: Find $u^\varepsilon(t, x) \in L^2(0, T; V) \cap H^1(0, T; H)$ such that

$$\begin{aligned} \frac{\partial^2 u^\varepsilon}{\partial t^2} - \nabla \cdot (A^\varepsilon(x) \nabla u^\varepsilon) &= f, \\ u^\varepsilon(0, \cdot) &= g^0, \quad u_t^\varepsilon(0, \cdot) = g^1, \\ u^\varepsilon(t, x) &= 0, \quad \text{for } x \in \partial D. \end{aligned} \quad (2.5)$$

Denoting by $(\cdot, \cdot)$ the inner product in H , extended to the duality pairing $\langle V', V \rangle$, we consider the weak form of (2.5) as

$$\left(\frac{\partial^2 u^\varepsilon}{\partial t^2}, \phi \right) + \int_D A^\varepsilon(x) \nabla u^\varepsilon(t, x) \cdot \nabla \phi(x) dx = \int_D f(t, x) \phi(x) dx, \quad \forall \phi \in V. \quad (2.6)$$

From conditions (2.1) and (2.2), we have the following existence and uniqueness result.

Proposition 2.2.1. *Problem (2.6) has a unique solution $u^\varepsilon \in L^2(0, T; V) \cap H^1(0, T; H)$ such that*

$$\|u^\varepsilon\|_{L^2(0, T; V) \cap H^1(0, T; H)} \leq c(\|f\|_{L^2(0, T; H)} + \|g^0\|_V + \|g^1\|_H), \quad (2.7)$$

where the constant c only depends on α , β and T .

The proof of this proposition is standard; see for example Wloka [55]. Indeed, it can be shown that $u^\varepsilon \in C([0, T]; V) \cap C([0, T]; H)$. We study in this chapter the asymptotic behaviour of u^ε when $\varepsilon \rightarrow 0$.

2.3 Multiscale homogenized equation

We establish the behaviour of u^ε when $\varepsilon \rightarrow 0$ via the multiscale convergence method. We first extend the standard notions of Nguetseng [44], Allaire [3] and Allaire and Briane [4] for functions that depend on time variable t .

2.3.1 Multiscale convergence

We first formulate the concept of multiscale convergence for functions that depend on t . The definition is a simple extension of that by Allaire and Briane [4].

Definition 2.3.1. *A sequence of function $\{w^\varepsilon\}_\varepsilon \subset L^2(0, T; H)$ is said to $(n+1)$ -scale converge to a function $w^0 \in L^2(0, T; D \times \mathbf{Y})$ if for all smooth functions $\phi(t, x, \mathbf{y})$ which are Y_p periodic with respect to y_p :*

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^T \int_D \int_{\mathbf{Y}} w^\varepsilon(t, x) \phi(t, x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}) d\mathbf{y} dx dt \\ &= \int_0^T \int_D \int_{\mathbf{Y}} w^0(t, x, \mathbf{y}) \phi(t, x, \mathbf{y}) d\mathbf{y} dx dt. \end{aligned}$$

This definition makes sense because of the following result.

Proposition 2.3.2. *From a bounded sequence $\{w^\varepsilon\}_\varepsilon \subset L^2(0, T; H)$, there is a subsequence (not renumbered) that $(n+1)$ -scale converges to a function $w^0 \in L^2(0, T; D \times \mathbf{Y})$.*

For a bounded sequence $\{w^\varepsilon\}_\varepsilon \subset L^2(0, T; V)$ we have the result:

Proposition 2.3.3. *Let $\{w^\varepsilon\}_\varepsilon \subset L^2(0, T; V)$. There is a subsequence (not renumbered), a function $w^0 \in L^2(0, T; V)$ and n functions $w^p \in L^2((0, T) \times D \times Y_1 \times \dots \times Y_{p-1}, H_\#^1(Y_p)/\mathbb{R})$ such that $w^\varepsilon \rightharpoonup w^0$ in $L^2(0, T; V)$ and*

$$\nabla w^\varepsilon \xrightarrow{(n+1)\text{-scale}} \nabla_x w^0 + \nabla_{y_1} w^1 + \dots + \nabla_{y_n} w^n.$$

The proof of Proposition 2.3.2 and 2.3.3 are almost identical to those in Allaire [3] and Allaire and Briane [4].

2.3.2 Multiscale limit of the wave equation

We study the multiscale convergence of the solution to the multiscale wave equation (2.6). For conciseness of presentation, for $p = 1, \dots, n$, we denote by

$$V_p := L^2(D \times Y_1 \times \dots \times Y_{p-1}, H_\#^1(Y_p)/\mathbb{R}),$$

and by

$$\mathbf{V} := V \times V_1 \times \dots \times V_n.$$

The space $\mathbf{V}$ is equipped with the norm: for $\mathbf{v} = (v^0, v^1, \dots, v^n) \in \mathbf{V}$

$$\begin{aligned}\|\mathbf{v}\|_{\mathbf{V}} &= \|v^0\|_V + \sum_{p=1}^n \|v^p\|_{V_p} \\ &= \|\nabla v^0\|_H + \sum_{p=1}^n \|\nabla_{y_p} v^p\|_{L^2(D \times Y_1 \times \dots \times Y_p)}.\end{aligned}$$

We then have

$$L^2(0, T; V_p) = L^2((0, T) \times D \times Y_1 \times \dots \times Y_{p-1}, H_{\#}^1(Y_p)),$$

and

$$L^2(0, T; \mathbf{V}) = \prod_{p=1}^n L^2((0, T) \times D \times Y_1 \times \dots \times Y_{p-1}, H_{\#}^1(Y_p)).$$

With the bilinear form $B : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$ defined by

$$\begin{aligned}B(\mathbf{w}, \mathbf{v}) &= \int_D \int_{\mathbf{Y}} A(x, \mathbf{y}) (\nabla w^0 + \nabla_{y_1} w^1 + \dots + \nabla_{y_n} w^n) \\ &\quad \cdot (\nabla v^0 + \nabla_{y_1} v^1 + \dots + \nabla_{y_n} v^n) d\mathbf{y} dx\end{aligned}$$

for all $\mathbf{w} = (w^0, w^1, \dots, w^n), \mathbf{v} = (v^0, v^1, \dots, v^n)$ in $\mathbf{V}$, we have the following result.

Theorem 2.3.4. *The solution u^ε of (2.6) converges weakly to u^0 in $L^2(0, T; V)$ and ∇u^ε ($n+1$)-scale converges to $\nabla u^0 + \nabla_{y_1} u^1 + \dots + \nabla_{y_n} u^n$ where $u^p \in L^2(0, T; V_p)$; $\mathbf{u} = (u^0, u^1, \dots, u^n)$ is the unique solution of the problem*

$$\left(\frac{\partial^2 u^0(t)}{\partial t^2}, v^0(t) \right) + B(\mathbf{u}, \mathbf{v}) = \int_D f(t, x) v^0(t, x) dx \quad (2.8)$$

for all $\mathbf{v} = (v^0, v^1, \dots, v^n) \in \mathbf{V}$, with the initial condition

$$u^0(0, x) = g^0(x), \quad \frac{\partial u^0}{\partial t}(0, x) = g^1(x). \quad (2.9)$$

Proof From (2.7), we deduce that from Proposition 2.3.2 and Proposition 2.3.3 there exists $\mathbf{u} \in \mathbf{V}$ such that $u^\varepsilon \rightharpoonup u^0$ in $L^2(0, T; V)$ and ∇u^ε ($n+1$)-scale converges to $\nabla u^0 + \nabla_{y_1} u^1 + \dots + \nabla_{y_n} u^n$. Let $\psi^0 \in C_0^\infty(D)$; and for $p = 1, \dots, n$, let $\psi^p \in C_0^\infty(D, C_{\#}^\infty(Y_1, \dots, C_{\#}^\infty(Y_{p-1}, C_{\#}^\infty(Y_p))))$ and $w(t) \in C_0^\infty(0, T)$. Let the test function ϕ in (2.6) be

$$\phi(t, x) = \left(\psi^0(x) + \sum_{p=1}^n \varepsilon_p \psi^p \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p} \right) \right) w(t).$$

We then obtain

$$\begin{aligned}
& \int_0^T \left(u^0(t), \psi^0 + \sum_{p=1}^n \varepsilon_p \psi^p(\cdot, \frac{\cdot}{\varepsilon_1}, \dots, \frac{\cdot}{\varepsilon_p}) \right) \frac{d^2 w(t)}{dt^2} dt \\
& + \int_0^T \int_D A^\varepsilon(x) \nabla u^\varepsilon(x) \cdot \left(\nabla \psi^0 + \sum_{p=1}^n \varepsilon_p \left(\nabla_x \psi^p(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p}) \right. \right. \\
& \quad \left. \left. + \sum_{q=1}^p \frac{1}{\varepsilon_q} \nabla_{y_q} \psi^p(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p}) \right) \right) w(t) dx dt \\
& = \int_0^T \int_D f(t, x) \left(\psi^0(x) + \sum_{p=1}^n \varepsilon_p \psi^p\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p}\right) \right) w(t) dx dt.
\end{aligned}$$

From (2.4), in the limit $\varepsilon \rightarrow 0$, we obtain

$$\int_0^T \left(u^0(t, \cdot), \psi^0(\cdot) \right) \frac{d^2 w(t)}{dt^2} dt + \int_0^T B(\mathbf{u}, \boldsymbol{\psi}) w(t) dt = \int_0^T \int_D f(t, x) \psi^0(x) w(t) dx dt.$$

As this holds for all $w(t) \in C_0^\infty(0, T)$, we deduce

$$\left(\frac{d^2 u^0}{dt^2}, \psi^0 \right) + B(\mathbf{u}, \boldsymbol{\psi}) = \int_D f \psi^0 dx$$

in the sense of distribution, where

$$\boldsymbol{\psi} = (\psi^0, \psi^1, \dots, \psi^n).$$

Using a density argument, we deduce (2.8). Initial conditions (2.9) come from the fact that

$$\frac{\partial u^\varepsilon}{\partial t} \rightharpoonup \frac{\partial u^0}{\partial t} \text{ in } L^2(0, T; H) \quad \text{and} \quad \frac{\partial^2 u^\varepsilon}{\partial t^2} \rightharpoonup \frac{\partial^2 u^0}{\partial t^2} \text{ in } L^2(0, T; V').$$

For all $\phi \in C^\infty([0, T]; C_0^\infty(D))^d$ with $\phi(T, x) = 0$, we have

$$\begin{aligned}
& \int_0^T \int_D \frac{\partial u^\varepsilon}{\partial t} \phi(t, x) dx dt \rightarrow \int_0^T \int_D \frac{\partial u^0}{\partial t} \phi(t, x) dx dt \\
& = - \int_D u^0(0, x) \phi(0, x) dx - \int_0^T \int_D u^0 \frac{\partial \phi}{\partial t} dx dt.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_0^T \int_D \frac{\partial u^\varepsilon}{\partial t} \phi(t, x) dt dx & = - \int_D u^\varepsilon(0, x) \phi(0, x) dx - \int_0^T \int_D u^\varepsilon \frac{\partial \phi}{\partial t} dx dt \\
& \rightarrow - \int_D g^0 \phi(0, x) dx - \int_0^T \int_D u^0 \frac{\partial \phi}{\partial t} dx dt.
\end{aligned}$$

Therefore $u^0(0, \cdot) = g^0$. Similarly, we have that $u_t^0(0, \cdot) = g^1$.

We now show that problem (2.8) has a unique solution, i.e. problem (2.8) with zero initial conditions and with $f = 0$ has zero solution. Following the same procedure as in Wloka [55] Theorem 29.1, fixing $s \in [0, T]$, we define for each $p = 0, 1, \dots, n$

$$w^p(t) = \begin{cases} -\int_t^s u^p(\sigma) d\sigma & \text{for } t < s \\ 0 & \text{for } t \geq s. \end{cases}$$

Let $\mathbf{w} = (w^0, w^1, \dots, w^n)$. We have

$$\frac{d}{dt} \left(\frac{\partial u^0(t)}{\partial t}, w^0(t) \right) = \left(\frac{\partial^2 u^0(t)}{\partial t^2}, w^0(t) \right) + \left(\frac{\partial u^0(t)}{\partial t}, \frac{\partial w^0(t)}{\partial t} \right).$$

Letting $\mathbf{v} = \mathbf{w} = (w^0, w^1, \dots, w^n)$ in (2.8), we have

$$\int_0^s \left(B(\mathbf{u}(t), \mathbf{w}(t)) + \frac{d}{dt} \left(\frac{\partial u^0(t)}{\partial t}, w^0(t) \right) - \left(\frac{\partial u^0(t)}{\partial t}, \frac{\partial w^0(t)}{\partial t} \right) \right) dt = 0. \quad (2.10)$$

As

$$B(\mathbf{u}(t), \mathbf{w}(t)) = \frac{1}{2} \frac{d}{dt} B(\mathbf{w}(t), \mathbf{w}(t))$$

and

$$\left(\frac{\partial u^0}{\partial t}(t), \frac{\partial w^0}{\partial t}(t) \right) = \frac{1}{2} \frac{d}{dt} (u^0(t), u^0(t)),$$

from the fact that $\partial u^0 / \partial t(0) = 0$, $w^0(s) = 0$, $\mathbf{w}(s) = 0$ and $u^0(0) = 0$, we have from (2.10)

$$\begin{aligned} & \int_0^s \left(B(\mathbf{u}(t), \mathbf{w}(t)) + \frac{d}{dt} \left(\frac{\partial u^0(t)}{\partial t}, w^0(t) \right) - \left(\frac{\partial u^0(t)}{\partial t}, \frac{\partial w^0(t)}{\partial t} \right) \right) dt \\ &= \int_0^s \frac{1}{2} \frac{d}{dt} B(\mathbf{w}(t), \mathbf{w}(t)) dt + \int_0^s \frac{d}{dt} \left(\frac{\partial u^0(t)}{\partial t}, w^0(t) \right) dt \\ & \quad - \int_0^s \left(\frac{\partial u^0(t)}{\partial t}, \frac{\partial w^0(t)}{\partial t} \right) dt \\ &= \frac{1}{2} \left(B(\mathbf{w}(s), \mathbf{w}(s)) - B(\mathbf{w}(0), \mathbf{w}(0)) \right) + \left(\frac{\partial u^0(s)}{\partial t}, w^0(s) \right) \\ & \quad - \left(\frac{\partial u^0(0)}{\partial t}, w^0(0) \right) - \frac{1}{2} \left((u^0(s), u^0(s)) - (u^0(0), u^0(0)) \right) \\ &= -\frac{1}{2} B(\mathbf{w}(0), \mathbf{w}(0)) - \frac{1}{2} \|u^0(s)\|_H^2 \\ &= 0. \end{aligned}$$

Thus

$$B(\mathbf{w}(0), \mathbf{w}(0)) + \|u^0(s)\|_H^2 = 0.$$

This implies that $u^0(s) = 0$ and $\int_0^s \mathbf{u}(\sigma) d\sigma = 0$ for all s . From this, we deduce that $\mathbf{u}(s) = 0$. $\square$

To illustrate how the solution u^ε of the multiscale problem (2.5) can be approximated by the solution $\mathbf{u}$ of problem (2.8), we mention the following result for the case of two scales ($n = 1$). The proof for it can be found, for example, in Brahim-Otsmane et al. [11].

Proposition 2.3.5. *For $n = 1$, assume that the initial condition $g^0 = 0$. Assume further that the solution $w^p(x, y)$ of cell problems (2.55) are in $C(D, W^{1,\infty}(Y))$. The following corrector result holds*

$$\lim_{\varepsilon \rightarrow 0} \left\| \nabla u^\varepsilon(t, \cdot) - (\nabla u^0(t, \cdot) + \nabla_y u^1(t, \cdot, \frac{\cdot}{\varepsilon})) \right\|_{L^\infty(0, T; H)} = 0. \quad (2.11)$$

We will study in more details correctors for the homogenization problem (2.8) in Section 2.7.

2.4 Finite element discretization

We consider in this section the finite element discretization of the high dimensional limiting equation (2.8). To study the general framework, we assume a general nested sequence of finite element spaces for approximating $\mathbf{u} \in \mathbf{V}$ without specifying these spaces explicitly. We will study detailed discretization with particular finite element spaces in the later sections. To approximate $u^0 \in V$, we consider a hierarchy of finite element spaces of V

$$V_0^1 \subset V_0^2 \subset \dots \subset V$$

such that $\bigcup_{L=1}^\infty V_0^L$ is dense in V ; and for $p = 1, \dots, n$, to approximate u^p , we consider the finite element spaces of V_p

$$V_p^1 \subset V_p^2 \subset \dots \subset V_p$$

such that $\bigcup_{L=1}^\infty V_p^L$ is dense in V_p . We first consider the spatially semidiscrete problem of (2.8).

2.4.1 Spatially semidiscrete problem

With the spaces $V_0^L \subset V$ and $V_p^L \subset V_p$ above, we define by

$$\mathbf{V}^L = V_0^L \times V_1^L \times \dots \times V_n^L \subset \mathbf{V}.$$

We consider the spatially semidiscrete approximating problem : Find $\mathbf{u}^L = (u^{0L}, u^{1L}, \dots, u^{nL}) \in \mathbf{V}^L$ such that

$$\left(\frac{\partial^2 u^{0L}}{\partial t^2}, v^{0L} \right) + B(\mathbf{u}^L, \mathbf{v}^L) = (f, v^{0L}) \quad (2.12)$$

for all $\mathbf{v}^L = (v^{0L}, v^{1L}, \dots, v^{nL}) \in \mathbf{V}^L$, with the initial condition $u^{0L}(0, \cdot) = g^{0L} \in V_0^L$ and $\frac{du^{0L}}{dt}(0, \cdot) = g^{1L} \in V_0^L$. The initial conditions g^{0L} and g^{1L} are assumed to be chosen so that they approximate g^0 and g^1 in V and H respectively, e.g, as the orthogonal projections of g^0 and g^1 to V_0^L .

Proposition 2.4.1. *Problem (2.12) has a unique solution.*

Proof Let $\mathbf{M}$ be the Gram matrix for the basis of V_0^L . For the bilinear form $B(\mathbf{u}^L, \mathbf{v}^L)$, let $\mathbf{A}_{00}$ be the matrix describing the interaction of the basis function of V_0^L , let $\mathbf{A}_{01}$ be the matrix describing the interaction of the basis functions of V_0^L with the basis functions of $V_1^L \times V_2^L \times \dots \times V_n^L$, and $\mathbf{A}_{11}$ be the matrix describing the interaction of the basis functions of $V_1^L \times V_2^L \times \dots \times V_n^L$ with themselves. Let $\mathbf{F}$ be the column matrix describing the interaction of f and the basis functions of V_0^L . Let $\mathbf{c}_0$ be the coefficient vector in the expansion of u^{0L} with respect to the basis of V_0^L . Let $\mathbf{c}_1$ be the coefficient vector in the expansion of $(u^{1L}, \dots, u^{nL})$ with respect to the basis of $V_1^L \times \dots \times V_n^L$. We have

$$\begin{aligned} \mathbf{M} \frac{d^2 \mathbf{c}_0}{dt^2} + \mathbf{A}_{00} \mathbf{c}_0 + \mathbf{A}_{01} \mathbf{c}_1 &= \mathbf{F}, \\ \mathbf{A}_{01}^\top \mathbf{c}_0 + \mathbf{A}_{11} \mathbf{c}_1 &= 0. \end{aligned}$$

As $\mathbf{A}_{11}$ is positively definite, $\mathbf{A}_{11}^{-1}$ is well defined and we have

$$\mathbf{M} \frac{d^2 \mathbf{c}_0}{dt^2} + (\mathbf{A}_{00} - \mathbf{A}_{01} \mathbf{A}_{11}^{-1} \mathbf{A}_{01}^\top) \mathbf{c}_0 = \mathbf{F}.$$

Since $\det \mathbf{M} \neq 0$, this equation has a unique solution. $\square$

We analyze problem (2.12) following the approach of Dupont [20]. For each $t \in [0, T]$, let $\mathbf{w}^L(t) = (w^{0L}, w^{1L}, \dots, w^{nL}) \in \mathbf{V}^L$ be the solution of the problem

$$B(\mathbf{w}^L(t) - \mathbf{u}(t), \mathbf{v}^L) = 0 \quad (2.13)$$

for all $\mathbf{v}^L = (v^{0L}, \dots, v^{nL}) \in \mathbf{V}^L$.

For $p = 0, \dots, n$, let

$$\eta^{pL} = w^{pL} - u^p. \quad (2.14)$$

We denote by $\boldsymbol{\eta}^L = (\eta^{0L}, \eta^{1L}, \dots, \eta^{nL}) \in \mathbf{V}$. We then have from (2.8) and (2.13)

$$B(\mathbf{w}^L, \mathbf{v}^L) = \left(f - \frac{\partial^2 u^0}{\partial t^2}, v^{0L} \right) \quad (2.15)$$

in the distribution sense for all $\mathbf{v}^L = (v^{0L}, v^{1L}, \dots, v^{nL}) \in \mathbf{V}^L$. We have the following estimates for $\|\boldsymbol{\eta}^L\|_{\mathbf{V}}$.

Lemma 2.4.2. *For the solution $\mathbf{w}^L$ of (2.13),*

$$\|\boldsymbol{\eta}^L(t)\|_{\mathbf{V}} \leq c \inf_{\mathbf{v}^L \in \mathbf{V}^L} \|\mathbf{u}(t) - \mathbf{v}^L\|_{\mathbf{V}}. \quad (2.16)$$

Proof From (2.13), take $\mathbf{v}^L = \mathbf{w}^L$ we have

$$B(\mathbf{w}^L - \mathbf{u}, \mathbf{w}^L - \mathbf{u}) = B(\mathbf{w}^L - \mathbf{u}, \mathbf{v}^L - \mathbf{u})$$

for all $\mathbf{v}^L \in \mathbf{V}^L$. From (2.1) and (2.2),

$$\alpha \|\mathbf{w}^L - \mathbf{u}\|_{\mathbf{V}}^2 \leq B(\mathbf{w}^L - \mathbf{u}, \mathbf{w}^L - \mathbf{u}) = B(\mathbf{w}^L - \mathbf{u}, \mathbf{v}^L - \mathbf{u}) \leq \beta \|\mathbf{w}^L - \mathbf{u}\|_{\mathbf{V}} \|\mathbf{v}^L - \mathbf{u}\|_{\mathbf{V}}.$$

We get the conclusion. $\square$

For the development of the finite element convergence in the later sections, we develop the following estimates for the time derivatives of $\mathbf{w}^L$.

Lemma 2.4.3. *If $\frac{\partial \mathbf{u}}{\partial t} \in L^\infty(0, T; \mathbf{V})$ then*

$$\left\| \frac{\partial \boldsymbol{\eta}^L}{\partial t} \right\|_{L^\infty(0, T; \mathbf{V})} \leq c \sup_{t \in [0, T]} \inf_{\mathbf{v}^L \in \mathbf{V}^L} \left\| \frac{\partial \mathbf{u}}{\partial t}(t) - \mathbf{v}^L \right\|_{\mathbf{V}}.$$

If $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2(0, T; \mathbf{V})$ then

$$\left\| \frac{\partial^2 \boldsymbol{\eta}^L}{\partial t^2} \right\|_{L^2(0, T; \mathbf{V})} \leq c \inf_{\mathbf{v}^L \in L^2(0, T; \mathbf{V}^L)} \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathbf{v}^L \right\|_{L^2(0, T; \mathbf{V})}.$$

Proof If $\frac{\partial \mathbf{u}}{\partial t} \in L^\infty(0, T; \mathbf{V})$, from equation (2.13) we have

$$B\left(\frac{\partial(\mathbf{w}^L - \mathbf{u})}{\partial t}, \mathbf{v}^L\right) = 0 \quad (2.17)$$

for all $\mathbf{v}^L \in \mathbf{V}^L$. We then proceed as in the proof of Lemma 2.4.2. The proof for the case $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2(0, T; \mathbf{V})$ is similar. $\square$

Let $\boldsymbol{\zeta}^L = (\zeta^{0L}, \zeta^{1L}, \dots, \zeta^{nL}) \in \mathbf{V}^L$ be

$$\boldsymbol{\zeta}^L = \mathbf{u}^L - \mathbf{w}^L. \quad (2.18)$$

We have by taking the difference of (2.15) and (2.12)

$$\left(\frac{\partial^2 \zeta^{0L}}{\partial t^2}, v^{0L}\right) + B(\boldsymbol{\zeta}^L, \mathbf{v}^L) = -\left(\frac{\partial^2 \eta^{0L}}{\partial t^2}, v^{0L}\right) \quad (2.19)$$

for all $\mathbf{v}^L = (v^{0L}, v^{1L}, \dots, v^{nL}) \in \mathbf{V}^L$. We then have the following estimate.

Lemma 2.4.4. *Assume that $\mathbf{u} \in H^1(0, T; \mathbf{V})$ and $\frac{\partial^2 \eta^{0L}}{\partial t^2} \in L^2(0, T; H)$. Then*

$$\begin{aligned} & \left\| \frac{\partial \zeta^{0L}}{\partial t}(t) \right\|_{L^\infty(0, T; H)} + \|\boldsymbol{\zeta}^L(t)\|_{L^\infty(0, T; \mathbf{V})} \\ & \leq c \left[\|\nabla \zeta^{0L}(0)\|_H + \left\| \frac{\partial \zeta^{0L}}{\partial t}(0) \right\|_H + \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0, T; H)} \right]. \end{aligned} \quad (2.20)$$

Proof As $\mathbf{u} \in H^1(0, T; \mathbf{V})$, $\frac{\partial \mathbf{u}}{\partial t}$ is well defined and so are $\frac{\partial \mathbf{w}^L}{\partial t}$ and $\frac{\partial \zeta^L}{\partial t}$. Letting $\mathbf{v}^L = \frac{\partial \zeta^L}{\partial t}$ in (2.19)

$$\left(\frac{\partial^2 \zeta^{0L}}{\partial t^2}, \frac{\partial \zeta^{0L}}{\partial t} \right) + B(\zeta^L, \frac{\partial \zeta^L}{\partial t}) = - \left(\frac{\partial^2 \eta^{0L}}{\partial t^2}, \frac{\partial \zeta^{0L}}{\partial t} \right),$$

then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left[\left(\frac{\partial \zeta^{0L}}{\partial t}, \frac{\partial \zeta^{0L}}{\partial t} \right) + B(\zeta^L, \zeta^L) \right] &= - \left(\frac{\partial^2 \eta^{0L}}{\partial t^2}, \frac{\partial \zeta^{0L}}{\partial t} \right) \\ &\leq \frac{2}{\gamma} \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_H^2 + 2\gamma \left\| \frac{\partial \zeta^{0L}}{\partial t} \right\|_H^2, \end{aligned}$$

where $\gamma > 0$ is chosen to be sufficiently small. Integrating on $[0, T]$, from (2.1) and (2.2) we have

$$\begin{aligned} &\sup_{t \in [0, T]} \left\{ \left\| \frac{\partial \zeta^{0L}}{\partial t}(t) \right\|_H^2 + \alpha \|\zeta^L(t)\|_{\mathbf{V}}^2 \right\} \\ &\leq \frac{2}{\gamma} \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0, T; H)}^2 + 2\gamma T \sup_{t \in [0, T]} \left\| \frac{\partial \zeta^{0L}}{\partial t}(t) \right\|_H^2 + \beta \|\zeta^L(0)\|_{\mathbf{V}}^2 + \left\| \frac{\partial \zeta^{0L}}{\partial t}(0) \right\|_H^2. \end{aligned}$$

Let $v^{0L} = 0$ and $v^{pL} = \zeta^{pL}(0)$ in (2.19), we have,

$$\begin{aligned} &\int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) (\nabla \zeta^{0L}(0) + \nabla_{y_1} \zeta^{1L}(0) + \dots + \nabla_{y_n} \zeta^{nL}(0)) \cdot \\ &\quad (\nabla_{y_1} \zeta^{1L}(0) + \dots + \nabla_{y_n} \zeta^{nL}(0)) d\mathbf{y} dx = 0. \end{aligned}$$

From this, (2.1) and (2.2), we deduce that

$$\alpha \|\nabla_{y_1} \zeta^{1L}(0) + \dots + \nabla_{y_n} \zeta^{nL}(0)\|_{L^2(D \times \mathbf{Y})} \leq \beta \|\nabla \zeta^{0L}(0)\|_H.$$

Letting $\gamma = \frac{1}{4T}$, we get the conclusion. $\square$

We then have the following convergence result for the semidiscrete problem (2.12).

Proposition 2.4.5. Assume that $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2(0, T; \mathbf{V})$, and that

$$\lim_{L \rightarrow \infty} \|g^{0L} - g^0\|_V = 0, \quad \text{and} \quad \lim_{L \rightarrow \infty} \|g^{1L} - g^1\|_H = 0. \quad (2.21)$$

Then

$$\lim_{L \rightarrow \infty} \left\{ \left\| \frac{\partial(u^{0L} - u^0)}{\partial t} \right\|_{L^\infty(0, T; H)} + \|\mathbf{u}^L - \mathbf{u}\|_{L^\infty(0, T; \mathbf{V})} \right\} = 0.$$

Proof From Lemma 2.4.4, as $\mathbf{u}^L - \mathbf{u} = \boldsymbol{\zeta}^L + \boldsymbol{\eta}^L$, we have

$$\begin{aligned} & \left\| \frac{\partial(u^{0L} - u^0)}{\partial t} \right\|_{L^\infty(0,T;H)} + \|\mathbf{u}^L - \mathbf{u}\|_{L^\infty(0,T;\mathbf{V})} \\ & \leq c \left[\|\nabla \zeta^{0L}(0)\|_H + \left\| \frac{\partial \zeta^{0L}}{\partial t}(0) \right\|_H + \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0,T;H)} \right] \\ & \quad + \left\| \frac{\partial \eta^{0L}}{\partial t} \right\|_{L^\infty(0,T;H)} + c \|\boldsymbol{\eta}^L\|_{L^\infty(0,T;\mathbf{V})}. \end{aligned} \quad (2.22)$$

We show that $\lim_{L \rightarrow \infty} \|\boldsymbol{\eta}^L\|_{L^\infty(0,T;\mathbf{V})} = 0$. As $\mathbf{u} \in C([0, T], \mathbf{V})$ it is uniformly continuous as a function from $[0, T]$ to $\mathbf{V}$. For $\epsilon > 0$, there is a piecewise constant (with respect to t) function $\mathbf{u}' \in L^\infty(0, T; \mathbf{V})$ such that $\|\mathbf{u} - \mathbf{u}'\|_{L^\infty(0,T;\mathbf{V})} < \epsilon$. As $\mathbf{u}'(t)$ obtains only a finite number of $\mathbf{V}$ -values, there is an L and $\mathbf{v}' \in L^\infty(0, T; \mathbf{V})$ such that $\|\mathbf{u}' - \mathbf{v}'\|_{L^\infty(0,T;\mathbf{V})} < \epsilon$. Thus

$$\lim_{L \rightarrow \infty} \sup_{t \in [0, T]} \inf_{\mathbf{v}^L \in \mathbf{V}^L} \|\mathbf{u}(t) - \mathbf{v}^L\|_{\mathbf{V}} = 0.$$

The result then follows from Lemma 2.4.2.

Similarly, we have

$$\lim_{L \rightarrow \infty} \left\| \frac{\partial \eta^{0L}}{\partial t} \right\|_{L^2(0,T;H)} = 0, \quad \text{and} \quad \lim_{L \rightarrow \infty} \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0,T;H)} = 0.$$

Further, we have that

$$\|\nabla \zeta^{0L}(0)\|_H \leq \|\nabla u^{0L}(0) - \nabla u^0(0)\|_H + \|\nabla u^0(0) - \nabla w^{0L}(0)\|_H,$$

which converges to 0 when $L \rightarrow \infty$ due to (2.21) and Lemma 2.4.2. Similarly, we have

$$\lim_{L \rightarrow \infty} \left\| \frac{\partial \zeta^{0L}}{\partial t}(0) \right\|_H = 0.$$

From these, we get the conclusion. $\square$

2.4.2 Fully discrete problem

In this subsection we consider time discretization of the Galerkin approximating problem (2.12). We follow the approach and notations of Dupont [20] for a stable time discretizing scheme.

Let M be a positive integer and let $\Delta t = T/M$. Let $t_m = m\Delta t$ and for a Banach space X , for any functions $r(t, x) \in C([0, T]; X)$ we denote by $r^m = r(t_m, \cdot)$. Following Dupont [20] we employ the following notations

$$\begin{aligned} r^{m+1/2} &= (1/2)(r^{m+1} + r^m), & r^{m,\theta} &= \theta r^{m+1} + (1 - 2\theta)r^m + \theta r^{m-1}, \\ \partial_t r^{m+1/2} &= (r^{m+1} - r^m)/\Delta t, & \partial_t^2 r^m &= (r^{m+1} - 2r^m + r^{m-1})/(\Delta t)^2, \\ \delta_t r^m &= (r^{m+1} - r^{m-1})/(2\Delta t). \end{aligned}$$

Assuming that $f \in C([0, T]; H)$, for problem (2.8), with the finite element space $\mathbf{V}^L$ in the previous subsection, we define the following fully discrete problem: for $m = 1, \dots, M$, find $\mathbf{u}^{L;m} = (u^{0L;m}, u^{1L;m}, \dots, u^{nL;m}) \in \mathbf{V}^L$ such that for $m = 1, \dots, M-1$,

$$(\partial_t^2 u^{0L;m}, v^{0L}) + B(\mathbf{u}^{L;m,1/4}, \mathbf{v}^L) = (f^{m,1/4}, v^{0L}) \quad (2.23)$$

for all $\mathbf{v}^L = (v^{0L}, v^{1L}, \dots, v^{nL}) \in \mathbf{V}^L$.

As in Dupont [20], we choose $\theta = 1/4$ as the scheme is stable and with the minimal time truncation error when the solution is sufficiently smooth.

For a continuous function $r : [0, T] \rightarrow X$, we denote by

$$\|r\|_{\tilde{L}^\infty(0,T;X)} = \max_{0 \leq m < M} \|r^{m+1/2}\|_X.$$

We also denote by

$$\|\partial_t r\|_{\tilde{L}^\infty(0,T;X)} = \max_{0 \leq m < M} \|\partial_t r^{m+1/2}\|_X.$$

For the solution of problem (2.23), we define $\zeta^{L;m}$ as

$$\zeta^{L;m} = \mathbf{u}^{L;m} - \mathbf{w}^{L;m}, \quad (2.24)$$

where $\mathbf{w}^L$ is defined in (2.13). We then have the following result.

Lemma 2.4.6. *Assume that $\mathbf{u} \in H^1(0, T; \mathbf{V})$ and $\partial^2 \eta^{0L} / \partial t^2 \in L^2(0, T; H)$. If $\partial^3 u^0 / \partial t^3 \in L^2(0, T; H)$ then there exists a constant c (independent of Δt and $\mathbf{u}$) such that*

$$\begin{aligned} & \|\partial_t \zeta^{0L}\|_{\tilde{L}^\infty(0,T;H)} + \|\zeta^L\|_{\tilde{L}^\infty(0,T;\mathbf{V})} \\ & \leq c \left[\|\zeta^{L;1/2}\|_{\mathbf{V}} + \|\partial_t \zeta^{0L;1/2}\|_H + \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0,T;H)} + \left\| \frac{\partial^3 u^0}{\partial t^3} \right\|_{L^2(0,T;H)} \Delta t \right]; \end{aligned}$$

and if $\partial^4 u^0 / \partial t^4 \in L^2(0, T; H)$ then there exists a constant c such that

$$\begin{aligned} & \|\partial_t \zeta^{0L}\|_{\tilde{L}^\infty(0,T;H)} + \|\zeta^L\|_{\tilde{L}^\infty(0,T;\mathbf{V})} \\ & \leq c \left[\|\zeta^{L;1/2}\|_{\mathbf{V}} + \|\partial_t \zeta^{0L;1/2}\|_H + \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0,T;H)} + \left\| \frac{\partial^4 u^0}{\partial t^4} \right\|_{L^2(0,T;H)} (\Delta t)^2 \right]. \end{aligned}$$

Proof Averaging equation (2.15) at time steps t_{m+1} , t_m and t_{m-1} with weights $1/4$, $1/2$ and $1/4$

$$B(\mathbf{w}^{L;m,1/4}, \mathbf{v}^L) = \left(f^{m,1/4} - \frac{\partial^2 u^{0L;m,1/4}}{\partial t^2}, v^{0L} \right). \quad (2.25)$$

Subtracting equation (2.23) by (2.25), we have

$$(\partial_t^2 u^{0L;m}, v^{0L}) + B(\zeta^{L;m,1/4}, \mathbf{v}^L) = \left(\frac{\partial^2 u^{0L;m,1/4}}{\partial t^2}, v^{0L} \right).$$

We therefore have

$$(\partial_t^2 \zeta^{0L;m}, v^{0L}) + B(\zeta^{L;m,1/4}, \mathbf{v}^L) = \left(-\partial_t^2 w^{0L;m} + \frac{\partial^2 u^{0L;m,1/4}}{\partial t^2}, v^{0L} \right).$$

Since $w^{0L} = u^0 + \eta^{0L}$, we deduce that

$$(\partial_t^2 \zeta^{0L;m}, v^{0L}) + B(\zeta^{L;m,1/4}, \mathbf{v}^L) = \left(-\partial_t^2 \eta^{0L;m} - \partial_t^2 u^{0;m} + \frac{\partial^2 u^{0;m,1/4}}{\partial t^2}, v^{0L} \right). \quad (2.26)$$

Choose $\mathbf{v}^L = \delta_t \zeta^{L;m}$, (2.26) implies

$$(\partial_t^2 \zeta^{0L;m}, \delta_t \zeta^{0L;m}) + B(\zeta^{L;m,1/4}, \delta_t \zeta^{L;m}) = \left(-\partial_t^2 \eta^{0L;m} - \partial_t^2 u^{0;m} + \frac{\partial^2 u^{0;m,1/4}}{\partial t^2}, \delta_t \zeta^{0L;m} \right).$$

We have the following relationships

$$\begin{aligned} \partial_t^2 r^m &= \frac{1}{\Delta t} (\partial_t r^{m+1/2} - \partial_t r^{m-1/2}), \quad r^{m,1/4} = \frac{1}{2}(r^{m+1/2} + r^{m-1/2}), \\ \delta_t r^m &= \frac{1}{2} (\partial_t r^{m+1/2} + \partial_t r^{m-1/2}) = \frac{1}{\Delta t} (r^{m+1/2} - r^{m-1/2}). \end{aligned}$$

Let $s^m = \frac{\partial^2 u^{0;m,1/4}}{\partial t^2} - \partial_t^2 u^{0;m}$, we get

$$\begin{aligned} & \frac{1}{2\Delta t} \left[(\partial_t \zeta^{0L;m+1/2} - \partial_t \zeta^{0L;m-1/2}, \partial_t \zeta^{0L;m+1/2} + \partial_t \zeta^{0L;m-1/2}) \right. \\ & \quad \left. + B(\zeta^{L;m+1/2} + \zeta^{L;m-1/2}, \zeta^{L;m+1/2} - \zeta^{L;m-1/2}) \right] \\ &= \frac{1}{2} (s^m - \partial_t^2 \eta^{0L;m}, \partial_t \zeta^{0L;m+1/2} + \partial_t \zeta^{0L;m-1/2}). \end{aligned}$$

As $B(\cdot, \cdot)$ is a symmetric bilinear form, we deduce

$$\begin{aligned} & \frac{1}{2\Delta t} \left[\|\partial_t \zeta^{0L;m+1/2}\|_H^2 - \|\partial_t \zeta^{0L;m-1/2}\|_H^2 \right. \\ & \quad \left. + B(\zeta^{L;m+1/2}, \zeta^{L;m+1/2}) - B(\zeta^{L;m-1/2}, \zeta^{L;m-1/2}) \right] \\ & \leq \frac{1}{2} \|s^m - \partial_t^2 \eta^{0L;m}\|_H \|\partial_t \zeta^{0L;m+1/2} + \partial_t \zeta^{0L;m-1/2}\|_H \\ & \leq \frac{1}{2\gamma} \left(\|s^m\|_H^2 + \|\partial_t^2 \eta^{0L;m}\|_H^2 \right) + \frac{\gamma}{2} \left(\|\partial_t \zeta^{0L;m+1/2}\|_H^2 + \|\partial_t \zeta^{0L;m-1/2}\|_H^2 \right) \end{aligned}$$

for all $\gamma > 0$. For $1 \leq j < M$, summing this up over $m = 1, \dots, j$, we deduce that

$$\begin{aligned} & \left[\|\partial_t \zeta^{0L;j+1/2}\|_H^2 - \|\partial_t \zeta^{0L;1/2}\|_H^2 + B(\zeta^{L;j+1/2}, \zeta^{L;j+1/2}) - B(\zeta^{L;1/2}, \zeta^{L;1/2}) \right] \\ & \leq \frac{1}{\gamma} \Delta t \sum_{m=1}^j \left(\|s^m\|_H^2 + \|\partial_t^2 \eta^{0L;m}\|_H^2 \right) + 2\gamma \Delta t M \max_{1 \leq m < M} \|\partial_t \zeta^{0L;m+1/2}\|_H^2 \\ & \quad + \gamma \Delta t \|\partial_t \zeta^{0L;1/2}\|_H^2. \end{aligned}$$

Choosing $\gamma = \frac{1}{4\Delta t M}$, we deduce that

$$\begin{aligned} & \max_{1 \leq j < M} \left[\|\partial_t \zeta^{0L;j+1/2}\|_H^2 + B(\zeta^{L;j+1/2}, \zeta^{L;j+1/2}) \right] \\ & \leq c\Delta t \sum_{m=1}^M \left(\|s^m\|_H^2 + \|\partial_t^2 \eta^{0L;m}\|_H^2 \right) + c\|\partial_t \zeta^{0L;1/2}\|_H^2. \end{aligned} \quad (2.27)$$

Following Dupont [20], using the integral formula of the remainder of Taylor expansion, we write

$$\partial_t^2 \eta^{0L;m} = (\Delta t)^{-2} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|) \frac{\partial^2 \eta^{0L}}{\partial t^2}(t_m + \tau) d\tau;$$

this implies that

$$\sum_{m=1}^M \|\partial_t^2 \eta^{0L;m}\|_H^2 \Delta t \leq \frac{4}{3} \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0,T;H)}^2.$$

For s^m , we have

$$\begin{aligned} s^m &= \frac{1}{4} \int_0^{\Delta t} \left(1 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^3 u^0}{\partial t^3}(t_m + \tau) d\tau \\ &\quad - \frac{1}{4} \int_{-\Delta t}^0 \left(1 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^3 u^0}{\partial t^3}(t_m + \tau) d\tau. \end{aligned}$$

Therefore

$$\|s^m\|_H^2 \leq c\Delta t \int_{t_{m-1}}^{t_m} \left\| \frac{\partial^3 u^0}{\partial t^3}(\tau) \right\|_H^2 d\tau.$$

From this, applying (2.1) and (2.2) to (2.27), we get

$$\begin{aligned} & \|\partial_t \zeta^{0L}\|_{\tilde{L}^\infty(0,T;H)} + \|\zeta^L\|_{\tilde{L}^\infty(0,T;\mathbf{V})} \\ & \leq c \left[(\Delta t)^2 \left\| \frac{\partial^3 u^0}{\partial t^3} \right\|_{L^2(0,T;H)} + \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0,T;H)} + \|\partial_t \zeta^{0L;1/2}\|_H + \|\zeta^{L;1/2}\|_{\mathbf{V}} \right]. \end{aligned}$$

If $\partial^4 u^0 / \partial t^4 \in L^2(0, T; H)$, we have

$$s^m = \frac{1}{12} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|) \left(3 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^4 u^0}{\partial t^4}(t_m + \tau) d\tau.$$

From this, we get

$$\|s^m\|_H^2 \leq c(\Delta t)^3 \int_{t_{m-1}}^{t_{m+1}} \left\| \frac{\partial^4 u^0}{\partial t^4}(\tau) \right\|_H^2 d\tau.$$

Applying (2.1) and (2.2) to (2.27), we get

$$\begin{aligned} & \|\partial_t \zeta^{0L}\|_{\tilde{L}^\infty(0,T;H)} + \|\zeta^L\|_{\tilde{L}^\infty(0,T;\mathbf{V})} \\ & \leq c \left[(\Delta t)^4 \left\| \frac{\partial^4 u^0}{\partial t^4} \right\|_{L^2(0,T;H)} + \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0,T;H)} + \|\partial_t \zeta^{0L;1/2}\|_H + \|\zeta^{L;1/2}\|_{\mathbf{V}} \right]. \end{aligned}$$

□

Lemma 2.4.7. *There is a constant c independent of the finite element space $\mathbf{V}^L$ and Δt such that*

$$\|\boldsymbol{\eta}^L\|_{\tilde{L}^\infty(0,T;\mathbf{V})} \leq c \sup_{0 \leq m < M} \inf_{\mathbf{v}^L \in \mathbf{V}^L} \|\mathbf{u}^{m+1/2} - \mathbf{v}^L\|_{\mathbf{V}}. \quad (2.28)$$

Further, if $\partial u^0 / \partial t \in L^\infty(0, T; V)$, then there exists a constant c such that

$$\|\partial_t \eta^{0L}\|_{\tilde{L}^\infty(0,T;V)} \leq c \sup_{0 \leq m < M} \inf_{\mathbf{v}^L \in \mathbf{V}^L} \|\partial_t \mathbf{u}^{m+1/2} - \mathbf{v}^L\|_{\mathbf{V}}.$$

Proof Averaging (2.13) at t_{m+1} and t_m with weights $1/2$ and $1/2$, we obtain

$$B(\mathbf{w}^{L;m+1/2} - \mathbf{u}^{m+1/2}, \mathbf{v}^L) = 0$$

for all $\mathbf{v}^L \in \mathbf{V}^L$. We therefore have

$$B(\mathbf{w}^{L;m+1/2} - \mathbf{u}^{m+1/2}, \mathbf{w}^{L;m+1/2} - \mathbf{u}^{m+1/2}) = B(\mathbf{w}^{L;m+1/2} - \mathbf{u}^{m+1/2}, \mathbf{v}^L - \mathbf{u}^{m+1/2})$$

for all $\mathbf{v}^L \in \mathbf{V}^L$. Using (2.1) and (2.2), we have

$$\|\mathbf{w}^{L;m+1/2} - \mathbf{u}^{m+1/2}\|_{\mathbf{V}} \leq c \|\mathbf{v}^L - \mathbf{u}^{m+1/2}\|_{\mathbf{V}}$$

for all $\mathbf{v}^L \in \mathbf{V}^L$. From this, we get the conclusion. The proof of $\|\partial_t \eta^{0L}\|_{\tilde{L}^\infty(0,T;V)}$ is similar. $\square$

Proposition 2.4.8. *Assume that $\mathbf{u} \in H^1(0, T; \mathbf{V})$ and $\partial^2 \eta^{0L} / \partial t^2 \in L^2(0, T; H)$. If $\partial^3 u^0 / \partial t^3 \in L^2(0, T; H)$, then there exists a constant c such that*

$$\begin{aligned} & \|\partial_t u^{0L} - \partial_t u^0\|_{\tilde{L}^\infty(0,T;H)} + \|\mathbf{u}^L - \mathbf{u}\|_{\tilde{L}^\infty(0,T;\mathbf{V})} \\ & \leq c \left[\Delta t \left\| \frac{\partial^3 u^0}{\partial t^3} \right\|_{L^2(0,T;H)} + \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0,T;H)} + \|\partial_t \zeta^{0L;1/2}\|_H + \|\zeta^{L;1/2}\|_{\mathbf{V}} \right] \\ & \quad + \|\partial_t \eta^{0L}\|_{\tilde{L}^\infty(0,T;H)} + \|\boldsymbol{\eta}^L\|_{\tilde{L}^\infty(0,T;\mathbf{V})}. \end{aligned} \quad (2.29)$$

If $\partial^4 u^0 / \partial t^4 \in L^2(0, T; H)$, then there exists a constant c such that

$$\begin{aligned} & \|\partial_t u^{0L} - \partial_t u^0\|_{\tilde{L}^\infty(0,T;H)} + \|\mathbf{u}^L - \mathbf{u}\|_{\tilde{L}^\infty(0,T;\mathbf{V})} \\ & \leq c \left[(\Delta t)^2 \left\| \frac{\partial^4 u^0}{\partial t^4} \right\|_{L^2(0,T;H)} + \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0,T;H)} + \|\partial_t \zeta^{0L;1/2}\|_H + \|\zeta^{L;1/2}\|_{\mathbf{V}} \right] \\ & \quad + \|\partial_t \eta^{0L}\|_{\tilde{L}^\infty(0,T;H)} + \|\boldsymbol{\eta}^L\|_{\tilde{L}^\infty(0,T;\mathbf{V})}. \end{aligned} \quad (2.30)$$

In particular, if $\partial^2 \mathbf{u} / \partial t^2 \in L^2(0, T; \mathbf{V})$ and $\partial^3 u^0 / \partial t^3 \in L^2(0, T; H)$, if $\mathbf{u}^{0L}$ and $\mathbf{u}^{1L}$ are chosen so that

$$\lim_{L \rightarrow \infty} \|\partial_t \zeta^{0L;1/2}\|_H = 0, \quad \lim_{L \rightarrow \infty} \|\zeta^{L;1/2}\|_{\mathbf{V}} = 0, \quad (2.31)$$

then

$$\lim_{\substack{L \rightarrow \infty \\ \Delta t \rightarrow 0}} \|\partial_t u^{0L} - \partial_t u^0\|_{\tilde{L}^\infty(0,T;H)} + \|\mathbf{u}^L - \mathbf{u}\|_{\tilde{L}^\infty(0,T;\mathbf{V})} = 0. \quad (2.32)$$

Proof The proof follows from the definition of ζ^L and w^L and Lemma 2.4.6. The limit (2.32) then follows from Lemma 2.4.2 and Lemma 2.4.7. We note that as $u \in H^2(0, T; \mathbf{V})$, from Lemma 2.4.3

$$\lim_{L \rightarrow \infty} \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0, T; H)} = 0.$$

Further, as $\frac{\partial u}{\partial t} \in C([0, T]; \mathbf{V})$ we have

$$\sup_{0 \leq m < M} \inf_{v^L \in \mathbf{V}^L} \|\partial_t u^{m+1/2} - v^L\|_{\mathbf{V}} \leq \sup_{0 \leq t \leq T} \inf_{v^L \in \mathbf{V}^L} \left\| \frac{\partial u}{\partial t} - v^L \right\|_{\mathbf{V}} \rightarrow 0 \text{ when } L \rightarrow \infty.$$

The proof for this uses the fact that $\frac{\partial u}{\partial t}$ is uniformly continuous. The argument is similar to that in the proof of Proposition 2.4.5. $\square$

2.5 Regularity of the solution

To obtain the estimates and convergence in the previous section and the rates of convergence in Section 2.6, it is essential that the solution u of (2.8) possesses sufficient temporal and spatial regularity. We establish in this section the smoothness condition on the data that guarantees the regularity needed.

We first deduce the homogenized problem for u^0 from (2.8). We have from (2.8) that

$$u^n = w_l^n \left(\frac{\partial u^0}{\partial x_l} + \frac{\partial u^1}{\partial y_{1,l}} + \dots + \frac{\partial u^{n-1}}{\partial y_{(n-1),l}} \right),$$

where $w_l^n \in L^2(D \times Y_1 \times \dots \times Y_{n-1}, H_{\#}^1(Y_n)/\mathbb{R})$ satisfies the problem

$$\int_D \int_{Y_1} \dots \int_{Y_n} A(e_l + \nabla_{y_n} w_l^n) \cdot \nabla_{y_n} \phi^n dy dx = 0$$

for all $\phi^n \in L^2(D \times Y_1 \times \dots \times Y_{n-1}, H_{\#}^1(Y_n)/\mathbb{R})$; e_l is the l th unit vector in $\mathbb{R}^d$. Using this, we have from (2.8) that

$$\begin{aligned} \int_D \int_{Y_1} \dots \int_{Y_{n-1}} \int_{Y_n} A(I + \nabla_{y_n} w^n) \left(\nabla u^0 + \sum_{p=1}^{n-1} \nabla_{y_p} u^p \right) \\ \cdot \nabla_{y_{n-1}} \phi^{n-1} dy_1 \dots dy_{n-1} dy_n dx = 0, \end{aligned}$$

where w^n denotes the vector $(w_1^n, \dots, w_d^n)$. Denoting by

$$A^{n-1}(x, y_1, \dots, y_{n-1}) = \int_{Y_n} A(I + \nabla_{y_n} w^n) \cdot (I + \nabla_{y_n} w^n) dy_n,$$

we have that

$$u^{n-1} = w_l^{n-1} \left(\frac{\partial u^0}{\partial x_l} + \frac{\partial u^1}{\partial y_{1,l}} + \dots + \frac{\partial u^{n-2}}{\partial y_{(n-2),l}} \right),$$

where $w_l^{n-1} \in L^2(D \times Y_1 \times \dots \times Y_{n-1}, H_{\#}^1(Y_n)/\mathbb{R})$ satisfies the problem

$$\int_D \int_{Y_1} \dots \int_{Y_{n-1}} A^{n-1}(e_l + \nabla_{y_{n-1}} w_l^{n-1}) \cdot \nabla_{y_{n-1}} \phi^{n-1} dy_1 \dots dy_{n-1} dx = 0,$$

for $\phi^{n-1} \in L^2(D \times Y_1 \times \dots \times Y_{n-2}, H_{\#}^1(Y_{n-1})/\mathbb{R})$. Letting $A^n = A$, we have recursively

$$u^p = w_l^p \left(\frac{\partial u^0}{\partial x_l} + \dots + \frac{\partial u^{p-1}}{\partial y_{(p-1),l}} \right),$$

where $w_l^p \in L^2(D \times Y_1 \times \dots \times Y_{p-1}, H_{\#}^1(Y_p)/\mathbb{R})$ satisfies the problem

$$\int_D \int_{Y_1} \dots \int_{Y_p} A^p(e_l + \nabla_{y_p} w_l^p) \cdot \nabla_{y_p} \phi^p dx dy_1 \dots dy_p = 0, \quad (2.33)$$

for $\phi^p \in L^2(D \times Y_1 \times \dots \times Y_{p-1}, H_{\#}^1(Y_p))$. The matrix A^p is defined as

$$A^p(x, y_1, \dots, y_p) = \int_{Y_{p+1}} A^{p+1}(I + \nabla_{y_{p+1}} w^{p+1}) \cdot (I + \nabla_{y_{p+1}} w^{p+1}) dy_{p+1} \quad (2.34)$$

for $p < n$. Continuing this process, we finally get the homogenized coefficient $A^0(x)$ as

$$A^0(x) = \int_{Y_1} A^1(x, y_1) (I + \nabla_{y_1} w^1) \cdot (I + \nabla_{y_1} w^1) dy_1.$$

The homogenized problem is then

$$\frac{\partial^2 u^0}{\partial t^2} - \nabla \cdot (A^0(x) \nabla u^0) = f(t, x), \quad u^0(0, \cdot) = g^0, \quad \frac{\partial u^0}{\partial t}(0, \cdot) = g^1. \quad (2.35)$$

The solution $\mathbf{u}$ of (2.8) is written in terms of u^0 as

$$u^p = w^p \cdot (I + \nabla_{y_{p-1}} w^{p-1}) \dots (I + \nabla_{y_1} w^1) \nabla u^0. \quad (2.36)$$

As $A \in C^1(D, C_{\#}^1(Y_1, (\dots C_{\#}^1(Y_n) \dots)))$, we deduce that $w_l^n \in C^1(D, C_{\#}^1(Y_1, \dots H_{\#}^1(Y_n) \dots))$. From which we deduce recursively that $A^p \in C^1(D, C^1(Y_1, \dots, C^1(Y_p) \dots))$ and $A^0 \in C^1(D)$. For the convergence in Propositions 2.4.5 and 2.4.8, we need regularity of $\mathbf{u}$ with respect to t . We now establish these.

Proposition 2.5.1. *Assume that*

$$\begin{aligned} f &\in H^2(0, T; H), \\ g^1 &\in V, \\ f(0) - \nabla \cdot (A^0 \nabla g^0) &\in V, \\ \frac{\partial f}{\partial t}(0) - \nabla \cdot (A^0 \nabla g^1) &\in H, \end{aligned} \quad (2.37)$$

then

$$\frac{\partial^2 u^0}{\partial t^2} \in L^2(0, T; V), \quad \text{and} \quad \frac{\partial^3 u^0}{\partial t^3} \in L^2(0, T; H). \quad (2.38)$$

Assume further that

$$\begin{aligned} f &\in H^3(0, T; H), \\ g^1 &\in V, \\ f(0) - \nabla \cdot (A^0 \nabla g^0) &\in V, \\ \frac{\partial f}{\partial t}(0) - \nabla \cdot (A^0 \nabla g^1) &\in V, \\ \frac{\partial^2 f}{\partial t^2}(0) + \nabla \cdot (A^0 \nabla (f(0) - \nabla \cdot (A^0 \nabla g^0))) &\in H, \end{aligned} \quad (2.39)$$

then

$$\frac{\partial^3 u^0}{\partial t^3} \in L^2(0, T; V), \quad \text{and} \quad \frac{\partial^4 u^0}{\partial t^4} \in L^2(0, T; H). \quad (2.40)$$

Proof The proof follows directly from the regularity of the wave equation. Equations (2.37) and (2.39) are the compatibility conditions that guarantee the regularity (2.38) and (2.40) (see, e.g. Wloka [55] Chapter 5). From (2.35), we have that

$$\frac{\partial^2}{\partial t^2} \left(\frac{\partial u^0}{\partial t} \right) - \nabla \cdot (A^0 \nabla \frac{\partial u^0}{\partial t}) = \frac{\partial f}{\partial t}, \quad (2.41)$$

with the compatibility initial condition

$$\frac{\partial u^0}{\partial t}(0, \cdot) = g^1(\cdot) \in V, \quad \frac{\partial}{\partial t} \frac{\partial u^0}{\partial t}(0, \cdot) = f(0, \cdot) - \nabla(A^0(\cdot) \nabla g^0(\cdot)) \in H.$$

We have further that

$$\frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 u^0}{\partial t^2} \right) - \nabla \cdot (A^0 \nabla \frac{\partial^2 u^0}{\partial t^2}) = \frac{\partial^2 f}{\partial t^2}, \quad (2.42)$$

with the compatibility initial condition

$$\frac{\partial^2 u^0}{\partial t^2}(0, \cdot) = f(0, \cdot) - \nabla(A^0(\cdot) \nabla g^0(\cdot)) \in V$$

and

$$\frac{\partial}{\partial t} \frac{\partial^2 u^0}{\partial t^2}(0, \cdot) = \frac{\partial f}{\partial t}(0, \cdot) - \nabla(A^0(\cdot) \nabla g^1(\cdot)) \in H.$$

We then deduce (2.38). Regularity result (2.40) is deduced similarly. □

Remark 2.5.2. In Section 2.5 where we consider the corrector for the homogenized problem (2.35), we restrict our consideration to the case where $g^0 = 0$. In this case, condition (2.37c) holds when $f(0) \in V$ and (2.37d) holds when $g^1 \in V \cap H^2(D)$. Condition (2.39b) holds when $g^1 \in H_0^3(D)$ and $\partial f / \partial t(0) \in V$, and condition (2.39c) holds when $\partial^2 f / \partial t^2(0) \in H$ and $f(0) \in H^2(D)$.

For the finite element rates of convergence in Section 2.6, we need spatial regularity of $\mathbf{u}$. To obtain the explicit rates of convergence, we define the regularity spaces $\bar{\mathcal{H}}$ and $\hat{\mathcal{H}}$ as in Hoang and Schwab [31].

We define by $\bar{\mathcal{H}}_p$ the spaces of functions $w(x, y_1, \dots, y_p) \in L^2(D \times Y_1 \times \dots \times Y_{p-1}, H^2_\#(Y_p))$ such that $w \in L^2(Y_1 \times \dots \times Y_p, H^1(D))$ and $w \in L^2(D \times \prod_{j \neq k} Y_j, H^1_\#(Y_k))$ for all $k = 1, \dots, p-1$. The space $\bar{\mathcal{H}}_p$ is equipped with the norm

$$\begin{aligned} \|w\|_{\bar{\mathcal{H}}_p} &= \|w\|_{L^2(D \times Y_1 \times \dots \times Y_{p-1}, H^2_\#(Y_p))} + \|w\|_{L^2(Y_1 \times \dots \times Y_p, H^1(D))} \\ &\quad + \sum_{k=1}^{p-1} \|w\|_{L^2(D \times \prod_{j \neq k} Y_j, H^1_\#(Y_k))}. \end{aligned}$$

We define by $\bar{\mathcal{H}}$ the space

$$\bar{\mathcal{H}} = H^2(D) \otimes \bar{\mathcal{H}}_1 \otimes \dots \otimes \bar{\mathcal{H}}_p, \quad (2.43)$$

which is equipped with the norm

$$\|\mathbf{w}\|_{\bar{\mathcal{H}}} = \|w^0\|_{H^2(D)} + \sum_{p=1}^n \|w^p\|_{\bar{\mathcal{H}}_p},$$

where $\mathbf{w} = (w^0, w^1, \dots, w^n) \in \bar{\mathcal{H}}$.

For sparse tensor product approximation in Subsection 2.6.3, we define the regularity space $\hat{\mathcal{H}} \subset \bar{\mathcal{H}}$. First we define the spaces $\hat{\mathcal{H}}_p$ of functions $w(x, y_1, \dots, y_p)$ that are Y_j -periodic with respect to y_j for $j = 1, \dots, p$ such that, for all $(\alpha_0, \dots, \alpha_p) \in (\mathbb{R}^d)^{p+1}$ with $0 \leq |\alpha_j| \leq 1$ for $j = 0, \dots, p-1$ and $|\alpha_p| \leq 2$,

$$\frac{\partial^{|\alpha_0|+\dots+|\alpha_p|} w}{\partial^{\alpha_0} x \partial^{\alpha_1} y_1 \dots \partial^{\alpha_p} y_p} \in L^2(D \times Y_1 \times \dots \times Y_p).$$

We then equip the space $\hat{\mathcal{H}}_p$ with the norm

$$\|\mathbf{w}\|_{\hat{\mathcal{H}}_p} = \sum_{\substack{0 \leq |\alpha_p| \leq 2 \\ 0 \leq |\alpha_j| \leq 1 (0 \leq j \leq p-1)}} \left\| \frac{\partial^{|\alpha_0|+\dots+|\alpha_p|} w}{\partial^{\alpha_0} x \partial^{\alpha_1} y_1 \dots \partial^{\alpha_p} y_p} \right\|_{L^2(D \times Y_1 \times \dots \times Y_p)}.$$

We define the space

$$\hat{\mathcal{H}} = \{\mathbf{w} = (w^0, w^1, \dots, w^n) \in \mathbf{V} : w^0 \in H^2(D), w^p \in \hat{\mathcal{H}}_p, p = 1, \dots, n\} \quad (2.44)$$

which is equipped with the norm

$$\|\mathbf{w}\|_{\hat{\mathcal{H}}} = \|w^0\|_{H^2(D)} + \sum_{p=1}^n \|w^p\|_{\hat{\mathcal{H}}_p}.$$

Clearly $\hat{\mathcal{H}} \subset \bar{\mathcal{H}}$. We have the following results.

Proposition 2.5.3. *Assume that $f \in H^1(0, T; H)$, $g^1 \in V$ and $g^0 \in V \cap H^2(D)$ and the domain D is convex. Then $\mathbf{u} \in L^\infty(0, T; \hat{\mathcal{H}})$.*

Proof As $A(x, \mathbf{y}) \in C^1(D, C^1(Y_1, \dots, C^1(Y_p) \dots))$, $A^0(\cdot) \in C^1(D)$. From (2.41), we deduce that

$$\frac{\partial u^0}{\partial t} \in L^\infty(0, T; V) \quad \text{and} \quad \frac{\partial^2 u^0}{\partial t^2} \in L^\infty(0, T; H).$$

Thus

$$-\nabla \cdot (A^0 \nabla u^0) = f - \frac{\partial^2 u^0}{\partial t^2} \in L^\infty(0, T; H). \quad (2.45)$$

Due to the convexity of the domain D , from Theorem 3.2.1.2 of Grisvard [28], we deduce that $u^0 \in L^\infty(0, T; H^2(D))$. From (2.33), we have that

$$-\nabla_{y_p}(A(x, y_1, \dots, y_p) \nabla_{y_p} w_l^p) = \nabla_{y_p}(A(x, y_1, \dots, y_p) e_p).$$

From (2.3), we then have

$$w_l^p \in C^1(D, C^1(Y_1, \dots, H_\#^2(Y_p) \dots)). \quad (2.46)$$

Therefore, from (2.36) we deduce that $u^p \in \hat{\mathcal{H}}_p$. $\square$

2.6 Rates of convergence for the discrete problems

In this section, we deduce the explicit rates of convergence in terms of the discretizing mesh size. As u^p is posed in the product domain $D \times Y_1 \times \dots \times Y_p$, it is natural to consider the finite element approximating spaces V_p^L in the previous section as tensor product spaces. In particular, we consider V_p^L as the full tensor and the sparse tensor product finite element approximating spaces. We therefore first define a hierarchy of finite element spaces from which the tensor product spaces are constructed.

2.6.1 Hierarchical finite element spaces

Let D be a polyhedron with plane sides. We divide D into simplices recursively. The set of simplices $\mathcal{T}^l$ of mesh size $h_l = O(2^{-l})$ is obtained by dividing each simplex in $\mathcal{T}^{l-1}$ into 4 congruent triangles for $d = 2$ or 8 tetrahedra for $d = 3$. The cube Y is divided into sets of simplices $\mathcal{T}_\#^l$ which are periodically distributed in a similar manner. We define the following finite element spaces

$$\begin{aligned} V^l &= \{u \in H^1(D) : u|_K \in \mathcal{P}_1(K) \quad \forall K \in \mathcal{T}^l\}, \\ V_0^l &= \{u \in H_0^1(D) : u|_K \in \mathcal{P}_1(K) \quad \forall K \in \mathcal{T}^l\}, \\ V_\#^l &= \{u \in H_\#^1(Y) : u|_K \in \mathcal{P}_1(K) \quad \forall K \in \mathcal{T}_\#^l\}, \end{aligned}$$

where $\mathcal{P}_1(K)$ denotes the set of linear polynomials in K . We recall the following approximating properties for functions with sufficient regularity (see, e.g. Ciarlet [15]). For $s \geq 0$

$$\begin{aligned} \inf_{v \in V^l} \|u - v\|_{H^1(D)} &\leq ch_l^s \|u\|_{H^{1+s}(D)}, \\ \inf_{v \in V^l} \|u - v\|_{L^2(D)} &\leq ch_l^{1+s} \|u\|_{H^{1+s}(D)}, \end{aligned}$$

for all $u \in H^{1+s}(D)$, and

$$\inf_{v \in V_0^l} \|u - v\|_{H^1(D)} \leq ch_l^s \|u\|_{H^{1+s}(D)},$$

for $u \in H^{1+s}(D) \cap V$. For periodic functions on Y we have

$$\begin{aligned} \inf_{v \in V_{\#}^l} \|u - v\|_{H^1(Y)} &\leq ch_l^s \|u\|_{H^{1+s}(Y)}, \\ \inf_{v \in V_{\#}^l} \|u - v\|_{L^2(Y)} &\leq ch_l^{1+s} \|u\|_{H^{1+s}(Y)}, \end{aligned}$$

for all periodic functions $u \in H_{\#}^{1+s}(Y)$.

2.6.2 Full tensor product finite elements

As $u^p \in L^2(D \times Y_1 \times \dots \times Y_{p-1}, H_{\#}^1(Y_p)) \cong L^2(D) \otimes L^2(Y_1) \otimes \dots \otimes L^2(Y_{p-1}) \otimes H_{\#}^1(Y_p)$, we choose $\bar{V}_p^L$ as

$$\bar{V}_p^L = V_0^L \otimes \underbrace{V_{\#}^L \otimes \dots \otimes V_{\#}^L}_{p \text{ times}}$$

to approximate u^p . For these spaces, when w is in the regularity space $\bar{\mathcal{H}}_p$ defined in Section 2.5, we have the following approximating properties.

Lemma 2.6.1. *For $w \in \bar{\mathcal{H}}_p$,*

$$\inf_{v \in \bar{V}_p^L} \|w - v\|_{L^2(D \times Y_1 \times \dots \times Y_{p-1}, H^1(Y_p))} \leq ch_L \|w\|_{\bar{\mathcal{H}}_p}.$$

We refer to Hoang and Schwab [31] for a proof. We define the full tensor FE space as

$$\bar{\mathbf{V}}^L = V_0^L \times V_1^L \times \dots \times V_n^L. \quad (2.47)$$

For functions $w \in \bar{\mathcal{H}}$ defined in (2.43), we deduce that

$$\inf_{v \in \bar{\mathbf{V}}^L} \|\mathbf{w} - \mathbf{v}\|_{\mathbf{V}} \leq ch_L \|\mathbf{w}\|_{\bar{\mathcal{H}}}.$$

The spatially semidiscrete problem (2.12) is now written as: Find $\bar{\mathbf{u}}^L = (\bar{u}^{0L}, \bar{u}^{1L}, \dots, \bar{u}^{nL})$ such that

$$\left(\frac{\partial^2 \bar{u}^{0L}}{\partial t^2}, \bar{v}^{0L} \right) + B(\bar{\mathbf{u}}^L, \bar{\mathbf{v}}^L) = (f, \bar{v}^{0L}), \quad \forall \bar{\mathbf{v}}^L \in \bar{\mathbf{V}}^L, \quad (2.48)$$

with $\bar{u}^{0L}(0) = g^{0L}$ and $\frac{\partial \bar{u}^{0L}}{\partial t}(0) = g^{1L}$. Equation (2.18) now becomes

$$\zeta^L = \bar{\mathbf{u}}^L - \mathbf{w}^L.$$

We then have the following result for the continuous time approximation.

Proposition 2.6.2. *Assume that condition (2.37) holds and the domain D is convex and that g^{0L} and g^{1L} are chosen so that*

$$\|g^{0L} - g^0\|_V \leq ch_L, \quad \text{and} \quad \|g^{1L} - g^1\|_H \leq ch_L, \quad (2.49)$$

then

$$\left\| \frac{\partial(\bar{u}^{0L} - u^0)}{\partial t} \right\|_{L^\infty(0,T;H)} + \|\bar{\mathbf{u}}^L - \mathbf{u}\|_{L^\infty(0,T;\mathbf{V})} \leq ch_L.$$

Proof We show that the right hand side of (2.22) is smaller than ch_L where c is independent of L . With condition (2.37), from (2.42), we deduce that $\frac{\partial^2 u^0}{\partial t^2} \in L^2(0, T; V)$. From (2.36) and (2.46), we have $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2(0, T; \mathbf{V})$. From (2.45), we have that $u^0 \in L^\infty(0, T; H^2(D))$ so $\mathbf{u} \in L^\infty(0, T; \bar{\mathcal{H}})$.

From Lemmas 2.4.2 and 2.6.1, we deduce that for all $t \in [0, T]$

$$\|\boldsymbol{\eta}^L(t)\|_{\mathbf{V}} \leq ch_L \|\mathbf{u}(t)\|_{\bar{\mathcal{H}}}.$$

We have further that

$$\|\nabla \zeta^{0L}(0)\|_H \leq \|\nabla u^{0L}(0) - \nabla u^0(0)\|_H + \|\nabla u^0(0) - \nabla w^{0L}\|_H \leq ch_L,$$

due to (2.49) and $\|\boldsymbol{\eta}(0)\|_{\mathbf{V}} \leq ch_L$. We now show that

$$\left\| \frac{\partial \eta^{0L}}{\partial t} \right\|_{L^\infty(0,T;H)} = \left\| \frac{\partial u^0}{\partial t} - \frac{\partial w^{0L}}{\partial t} \right\|_{L^\infty(0,T;H)} \leq ch_L,$$

where c is independent of t . We consider $\phi \in \mathbf{V}$ as the unique solution of the elliptic problem

$$B(\phi, \mathbf{v}) = \int_D \frac{\partial(u^0 - w^{0L})}{\partial t}(t, x) v^0(x) dx, \quad (2.50)$$

for all $\mathbf{v} = (v^0, v^1, \dots, v^n) \in \mathbf{V}$. This problem has a unique solution as $\frac{\partial(u^0 - w^{0L})}{\partial t}(t, \cdot) \in H$. We have that ϕ^0 is the solution of the elliptic problem

$$-\nabla \cdot (A^0(x) \nabla \phi^0) = \frac{\partial(u^0 - w^{0L})}{\partial t}.$$

As D is convex,

$$\|\phi^0\|_{H^2(D)} \leq c \left\| \frac{\partial(u^0 - w^{0L})}{\partial t}(t, \cdot) \right\|_H.$$

Further, similarly to (2.36), we have

$$\phi^p = w^p \cdot (I + \nabla_{y_{p-1}} w^{p-1}) \dots (I + \nabla_{y_1} w^1) \nabla \phi^0,$$

which implies

$$\|\phi\|_{\tilde{\mathbf{H}}} \leq c \left\| \frac{\partial(u^0 - w^{0L})}{\partial t}(t, \cdot) \right\|_H.$$

Let $\mathbf{v} = \frac{\partial(\mathbf{u} - \mathbf{w}^L)}{\partial t}(t) \in \mathbf{V}$. We have

$$B\left(\phi, \frac{\partial(\mathbf{u} - \mathbf{w}^L)}{\partial t}(t)\right) = \left\| \frac{\partial(u^0 - w^{0L})}{\partial t}(t, \cdot) \right\|_H^2.$$

From (2.17), due to the symmetry of $B(\cdot, \cdot)$, for all $\mathbf{v}^L \in V^L$

$$B\left(\phi - \mathbf{v}^L, \frac{\partial(\mathbf{u} - \mathbf{w}^L)}{\partial t}(t)\right) = \left\| \frac{\partial(u^0 - w^{0L})}{\partial t}(t, \cdot) \right\|_H^2.$$

We therefore have

$$\begin{aligned} & \left\| \frac{\partial(u^0 - w^{0L})}{\partial t}(t, \cdot) \right\|_H^2 \\ & \leq \beta \inf_{\mathbf{v}^L \in \mathbf{V}^L} \|\phi - \mathbf{v}^L\|_{\mathbf{V}} \left\| \frac{\partial(\mathbf{u} - \mathbf{w}^L)}{\partial t} \right\|_{\mathbf{V}} \\ & \leq ch_L \|\phi\|_{\tilde{\mathbf{H}}} \left\| \frac{\partial(\mathbf{u} - \mathbf{w}^L)}{\partial t} \right\|_{\mathbf{V}} \\ & \leq ch_L \left\| \frac{\partial(u^0 - w^{0L})}{\partial t}(t, \cdot) \right\|_H \left\| \frac{\partial(\mathbf{u} - \mathbf{w}^L)}{\partial t}(t) \right\|_{\mathbf{V}}. \end{aligned}$$

From Lemma 2.4.3, as $\frac{\partial \mathbf{u}}{\partial t} \in L^\infty(0, T; \mathbf{V})$, we deduce that $\left\| \frac{\partial(\mathbf{u} - \mathbf{w}^L)}{\partial t}(t) \right\|_{\mathbf{V}}$ is bounded uniformly for all t . Hence

$$\left\| \frac{\partial(u^0 - w^{0L})}{\partial t}(t, \cdot) \right\|_H \leq ch_L.$$

In particular, for $t = 0$, we have

$$\left\| \frac{\partial \zeta^{0L}}{\partial t}(0, \cdot) \right\|_H \leq \|g^{1L} - g^1\|_H + \left\| \frac{\partial u^0}{\partial t}(0) - \frac{\partial w^{0L}}{\partial t}(0) \right\|_H \leq ch_L.$$

Similarly, by considering the problem

$$B(\phi, \mathbf{v}) = \int_D \frac{\partial^2(u^0 - w^{0L})}{\partial t^2}(t, x) v^0(x) dx,$$

using the fact that $\left\| \frac{\partial^2(\mathbf{u} - \mathbf{w}^L)}{\partial t^2} \right\|_{L^2(0,T;\mathbf{V})}$ is bounded with respect to L , we deduce that

$$\left\| \frac{\partial^2(u^0 - w^{0L})}{\partial t^2} \right\|_{L^2(0,T;H)} \leq ch_L.$$

From this, we get the conclusion. $\square$

The fully discrete problem is now written as: find $\bar{\mathbf{u}}^{L;m} = (\bar{u}^{0L;m}, \bar{u}^{1L;m}, \dots, \bar{u}^{nL;m}) \in \bar{\mathbf{V}}^L$ such that

$$(\partial_t^2 \bar{u}^{0L;m}, \bar{v}^{0L}) + B(\bar{\mathbf{u}}^{L;m,1/4}, \bar{\mathbf{v}}^L) = (f^{m,1/4}, \bar{v}^{0L}), \quad (2.51)$$

for all $\bar{\mathbf{v}}^L = (\bar{v}^{0L}, \bar{v}^{1L}, \dots, \bar{v}^{nL}) \in \bar{\mathbf{V}}^L$.

We have the following result.

Proposition 2.6.3. *Assume that condition (2.37) holds and that D is a convex domain. If*

$$\|\partial_t \zeta^{0L;1/2}\|_H \leq c(\Delta t + h_L), \quad \|\zeta^{L;1/2}\|_{\mathbf{V}} \leq c(\Delta t + h_L),$$

then

$$\|\partial_t \bar{u}^{0L} - \partial_t u^0\|_{\tilde{L}^\infty(0,T;H)} + \|\bar{\mathbf{u}}^L - \mathbf{u}\|_{\tilde{L}^\infty(0,T;\mathbf{V})} \leq c(\Delta t + h_L).$$

If (2.39) holds, and

$$\|\partial_t \zeta^{0L;1/2}\|_H \leq c((\Delta t)^2 + h_L), \quad \|\zeta^{L;1/2}\|_{\mathbf{V}} \leq c((\Delta t)^2 + h_L),$$

then

$$\|\partial_t \bar{u}^{0L} - \partial_t u^0\|_{\tilde{L}^\infty(0,T;H)} + \|\bar{\mathbf{u}}^L - \mathbf{u}\|_{\tilde{L}^\infty(0,T;\mathbf{V})} \leq c((\Delta t)^2 + h_L).$$

Proof We use (2.29) and (2.30). As shown in the proof of Proposition 2.6.2, we have

$$\left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0,T;H)} \leq ch_L.$$

From Lemma 2.4.2 and the regularity $\mathbf{u} \in L^\infty(0,T;\bar{\mathcal{H}})$, we have

$$\|\boldsymbol{\eta}^L\|_{\tilde{L}^\infty(0,T;\mathbf{V})} \leq ch_L.$$

It remains to show that $\|\partial_t \eta^{0L}\|_{\tilde{L}^\infty(0,T;H)} \leq ch_L$. We proceed similarly as for the proof of Proposition 2.6.2. We consider $\boldsymbol{\phi} \in \mathbf{V}$ as the solution of the problem

$$B(\boldsymbol{\phi}, \mathbf{v}) = \int_D \frac{\eta^{0L;m+1} - \eta^{0L;m}}{\Delta t} v^0 dx$$

for all $\mathbf{v} = (v^0, v^1, \dots, v^n) \in \mathbf{V}$. As $\frac{\eta^{0L;m+1} - \eta^{0L;m}}{\Delta t} \in H$, we deduce that $\boldsymbol{\phi} \in \bar{\mathcal{H}}$ and that

$$\|\boldsymbol{\phi}\|_{\bar{\mathcal{H}}} \leq c \left\| \frac{\eta^{0L;m+1} - \eta^{0L;m}}{\Delta t} \right\|_H.$$

Let $\mathbf{v} = \frac{\boldsymbol{\eta}^{L;m+1} - \boldsymbol{\eta}^{L;m}}{\Delta t}$, we have that

$$B\left(\boldsymbol{\phi}, \frac{\boldsymbol{\eta}^{L;m+1} - \boldsymbol{\eta}^{L;m}}{\Delta t}\right) = \int_D \left(\frac{\eta^{0L;m+1} - \eta^{0L;m}}{\Delta t}\right)^2 dx.$$

From (2.15), we have

$$B\left(\boldsymbol{\phi} - \mathbf{v}^L, \frac{\boldsymbol{\eta}^{L;m+1} - \boldsymbol{\eta}^{L;m}}{\Delta t}\right) = \int_D \left(\frac{\eta^{0L;m+1} - \eta^{0L;m}}{\Delta t}\right)^2 dx$$

for all $\mathbf{v}^L \in \mathbf{V}^L$. We deduce that

$$\begin{aligned} \left\| \frac{\eta^{0L;m+1} - \eta^{0L;m}}{\Delta t} \right\|_H^2 &\leq \beta \inf_{\mathbf{v}^L \in \mathbf{V}^L} \|\boldsymbol{\phi} - \mathbf{v}^L\|_{\mathbf{V}} \left\| \frac{\boldsymbol{\eta}^{L;m+1} - \boldsymbol{\eta}^{L;m}}{\Delta t} \right\|_{\mathbf{V}} \\ &\leq ch_L \|\boldsymbol{\phi}\|_{\bar{\mathbf{H}}} \left\| \frac{\boldsymbol{\eta}^{L;m+1} - \boldsymbol{\eta}^{L;m}}{\Delta t} \right\|_{\mathbf{V}} \\ &\leq ch_L \left\| \frac{\eta^{0L;m+1} - \eta^{0L;m}}{\Delta t} \right\|_{\mathbf{V}} \left\| \frac{\boldsymbol{\eta}^{L;m+1} - \boldsymbol{\eta}^{L;m}}{\Delta t} \right\|_{\mathbf{V}}. \end{aligned}$$

As $\mathbf{u} \in H^2(0, T; \mathbf{V})$ so $\boldsymbol{\eta}^L \in H^2(0, T; \mathbf{V})$. We have that

$$\left\| \frac{\boldsymbol{\eta}^{L;m+1} - \boldsymbol{\eta}^{L;m}}{\Delta t} \right\|_{\mathbf{V}} \leq c \sup_{t \in [0, T]} \left\| \frac{\partial \boldsymbol{\eta}^L}{\partial t} \right\|_{\mathbf{V}},$$

which is uniformly bounded for all Δt due to Lemma 2.4.3. From this we have

$$\left\| \frac{\eta^{0L;m+1} - \eta^{0L;m}}{\Delta t} \right\|_H \leq ch_L.$$

We then get the conclusion □

2.6.3 Sparse tensor product finite elements

The dimension of the full tensor product space $\bar{\mathbf{V}}^L$ is $O(2^{ndL})$ which is prohibitively large when n and L are large for solving problems (2.48) and (2.51). In this section, we develop the sparse tensor product finite element spaces with much reduced dimension but with essentially equal accuracy for the spatially semidiscrete problem and the fully discrete problem. We first define the following orthogonal projections in the norm of $L^2(D)$, $L^2(Y)$ and $H_{\#}^1(Y)$ as in Hoang and Schwab [31].

$$P^{l0} : L^2(D) \rightarrow V^l, \quad P_{\#}^{l0} : L^2(Y) \rightarrow V_{\#}^l, \quad P_{\#}^{l1} : H_{\#}^1(Y) \rightarrow V_{\#}^l.$$

The increment spaces are defined as

$$W^l = (P^{l0} - P^{(l-1)0})V^l, \quad W_{\#}^{l0} = (P_{\#}^{l0} - P_{\#}^{(l-1)0})V_{\#}^l, \quad W_{\#}^{l1} = (P_{\#}^{l1} - P_{\#}^{(l-1)1})V_{\#}^l.$$

We then have

$$V^l = \bigoplus_{0 \leq j \leq l} W^j, \quad V_{\#}^l = \bigotimes_{0 \leq j \leq l} W_{\#}^{j0} = \bigotimes_{0 \leq j \leq l} W_{\#}^{j1}.$$

The full tensor product space $\bar{V}_p^L$ can be written as

$$\bar{V}_p^L = \bigoplus_{\substack{0 \leq j_k \leq L \\ k=0,1,\dots,p}} W^{j_0} \otimes W_{\#}^{j_1 0} \otimes \dots \otimes W_{\#}^{j_{p-1} 0} \otimes W_{\#}^{j_p 1}.$$

We define the sparse tensor product spaces as

$$\hat{V}_p^L = \bigoplus_{0 \leq j_0 + j_1 + \dots + j_p \leq L} W^{j_0} \otimes W_{\#}^{j_1 0} \otimes \dots \otimes W_{\#}^{j_{p-1} 0} \otimes W_{\#}^{j_p 1},$$

and the space

$$\hat{\mathbf{V}}^L = \{(\hat{u}^{0L}, \{\hat{u}^{pL}\}) : \hat{u}^{0L} \in \hat{V}_0^L, \hat{u}^{pL} \in \hat{V}_p^L\}.$$

For functions in the regularity spaces $\hat{\mathcal{H}}_p$ defined in Section 2.5, we have the following estimate.

Lemma 2.6.4. *For $w \in \hat{\mathcal{H}}_p$:*

$$\inf_{v \in \hat{V}_p^L} \|w - v\|_{V_p} \leq c(L+1)^{p/2} h_L \|w\|_{\hat{\mathcal{H}}_p}.$$

The proof of this lemma can be found in Hoang and Schwab [31].

The spatially semidiscrete problem (2.12) now becomes:

Find $\hat{\mathbf{u}}^L = (\hat{u}^{0L}, \hat{u}^{1L}, \dots, \hat{u}^{nL})$ such that

$$\left(\frac{\partial^2 \hat{u}^{0L}}{\partial t^2}, \hat{v}^{0L} \right) + B(\hat{\mathbf{u}}^L, \hat{\mathbf{v}}^L) = (f, \hat{v}^{0L}), \quad \forall \hat{\mathbf{v}}^L \in \mathbf{V}^L \quad (2.52)$$

with $\hat{u}^{0L} = g^{0L}$ and $\frac{\partial \hat{u}^{0L}}{\partial t} = g^{1L}$. We then have the following result for the semidiscrete problem.

Proposition 2.6.5. *Assume that condition (2.37) holds and the domain D is convex and that g^{0L} and g^{1L} are chosen so that*

$$\|g^{0L} - g^0\|_V \leq cL^{n/2} h_L, \quad \text{and} \quad \|g^{1L} - g^1\|_H \leq cL^{n/2} h_L, \quad (2.53)$$

then

$$\left\| \frac{\partial(\hat{u}^{0L} - u^0)}{\partial t} \right\|_{L^\infty(0,T;H)} + \|\hat{\mathbf{u}}^L - \mathbf{u}\|_{L^\infty(0,T;\mathbf{V})} \leq cL^{n/2} h_L.$$

The proof of the proposition is identical to that for Proposition 2.6.2. Indeed, the regularity $\phi^0 \in H^2(D)$ implies that $\phi \in \hat{\mathcal{H}}$ and that

$$\|\phi\|_{\hat{\mathcal{H}}} \leq c \left\| \frac{\partial(u^0 - w^{0L})}{\partial t}(t) \right\|_H.$$

From this, by a similar argument as in the proof of Proposition 2.6.2, we get

$$\left\| \frac{\partial(u^0 - w^{0L})}{\partial t}(t) \right\|_H \leq cL^{n/2}h_L.$$

□

The fully discrete problem (2.23) becomes: find $\hat{\mathbf{u}}^{L;m} = (\hat{u}^{0L;m}, \hat{u}^{1L;m}, \dots, \hat{u}^{nL;m}) \in \hat{\mathbf{V}}^L$ such that

$$(\partial_t^2 \hat{u}^{0L;m}, \hat{v}^{0L}) + B(\hat{\mathbf{u}}^{L;m,1/4}, \hat{\mathbf{v}}^L) = (f^{m,1/4}, \hat{v}^{0L}) \quad (2.54)$$

for all $\hat{\mathbf{v}}^L = (\hat{v}^{0L}, \hat{v}^{1L}, \dots, \hat{v}^{nL}) \in \hat{\mathbf{V}}^L$. We have the following result.

Proposition 2.6.6. *Assume that condition (2.37) holds and that D is a convex domain. Assume further that*

$$\|\partial_t \zeta^{0L;1/2}\|_H \leq c(\Delta t + L^{n/2}h_L), \quad \|\zeta^{L;1/2}\|_{\mathbf{V}} \leq c(\Delta t + L^{n/2}h_L).$$

Then

$$\|\partial_t \hat{u}^{0L} - \partial_t u^0\|_{\tilde{L}^\infty(0,T;H)} + \|\hat{\mathbf{u}}^L - \mathbf{u}\|_{\tilde{L}^\infty(0,T;\mathbf{V})} \leq c(\Delta t + L^{n/2}h_L).$$

If condition (2.39) holds and

$$\|\partial_t \zeta^{0L;1/2}\|_H \leq c((\Delta t)^2 + L^{n/2}h_L), \quad \|\zeta^{L;1/2}\|_{\mathbf{V}} \leq c((\Delta t)^2 + L^{n/2}h_L),$$

then

$$\|\partial_t \hat{u}^{0L} - \partial_t u^0\|_{\tilde{L}^\infty(0,T;H)} + \|\hat{\mathbf{u}}^L - \mathbf{u}\|_{\tilde{L}^\infty(0,T;\mathbf{V})} \leq c((\Delta t)^2 + L^{n/2}h_L).$$

The proof of this proposition is similar to that for Proposition 2.6.3.

The dimension of the sparse tensor product space $\hat{\mathbf{V}}^L$ is $O(h_L^{-d} |\log h_L|^n)$ which is much smaller than the dimension of $\bar{\mathbf{V}}^L$. The errors we obtain in Proposition 2.6.6 are essentially equal to those obtained in Proposition 2.6.3. We will make some remarks on the choice of $\hat{\mathbf{u}}^{L;0}$ and $\hat{\mathbf{u}}^{L;1}$ in Section 2.10.

2.7 Corrector for the homogenization problem

For two-scale elliptic problems (where we denote the coefficient by $A(x, y)$), it is well known that with sufficient regularity for the solutions $w^p(x, y)$ of the cell problems (2.55), there is always a corrector for the weak convergence $u^\varepsilon \rightharpoonup u^0$ in $H_0^1(D)$ (see, e.g. Bensoussan et al. [7]). However, a similar corrector result for the wave equation (2.5) does not always hold as the energy of the multiscale problem (2.5) only converges to the energy of the homogenized equation when the initial condition g^0 satisfies special conditions. Brahim-Otsmane et al. [11] proved a corrector result for the wave equation (2.5) with another term v^ε which satisfies the multiscale wave equation (2.5) with zero right hand side and special initial conditions.

In this section, we restrict our attention to the case with the initial condition $g^0 = 0$. This allows us to deduce corrector results similar to those for elliptic equations, and to deduce a homogenization rate of convergence for the two-scale case. These results will be used to deduce the convergence of the finite element solution $\mathbf{u}^L$ in the physical variable x .

2.7.1 Corrector for two-scale problem

For two-scale problems ($n = 1$), we deduce in this section a homogenization rate of convergence.

For $p = 1, \dots, d$, we let $w^p(x, y) \in L^2(D; H_{\#}^1(Y)/\mathbb{R})$ be the solution of the cell problem

$$-\nabla_y \cdot (A(x, y)(e_p + \nabla_y w^p(x, y))) = 0, \quad (2.55)$$

where e_p is the p th unit vector in $\mathbb{R}^d$. The homogenized coefficient $A^0(x)$ is defined as

$$A^0(x) = \int_Y A(x, y)(\mathbf{I} + \nabla_y w) \cdot (\mathbf{I} + \nabla_y w) dy, \quad (2.56)$$

where $\mathbf{I}$ is the identity matrix and $w = (w_1, w_2, \dots, w_d) \in L^2(D, H_{\#}^1(Y))^d$. Since $A(x, y) \in (C^1(D, C_{\#}^1(Y)))^{d \times d}$, $w^p(x, y) \in C^1(D, H_{\#}^1(Y))$ and therefore $A^0(x) \in (C^1(D))_{sym}^{d \times d}$. We then have the following result on the homogenization rate of convergence.

Proposition 2.7.1. *For two-scale problems, assume that*

$$g^0 = 0, \quad g^1 \in H^2(D) \cap V, \quad f \in H^2(0, T; H), \quad f(0) \in V$$

and D is a convex domain. There exists a constant c that does not depend on ε such that

$$\left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u^0}{\partial t} \right\|_{L^\infty(0, T; H)} + \|\nabla u^\varepsilon(\cdot) - [\nabla u^0(\cdot) + \nabla_y u^1(\cdot, \frac{\cdot}{\varepsilon})]\|_{L^\infty(0, T; H)} \leq c\varepsilon^{1/2}. \quad (2.57)$$

Proof First we establish the regularity

$$u^0 \in H^2(0, T; V) \cap L^\infty(0, T; H^2(D)), \quad \frac{\partial u^0}{\partial t} \in L^\infty(0, T; V \cap H^2(D)). \quad (2.58)$$

As $g^0 = 0$, condition (2.37) holds. From (2.42), we have $\frac{\partial^2 u^0}{\partial t^2} \in L^\infty(0, T; V)$ and $\frac{\partial^3 u^0}{\partial t^3} \in L^\infty(0, T; H)$. From (2.41), we have

$$-\nabla \cdot (A^0 \nabla \frac{\partial u^0}{\partial t}) = \frac{\partial f}{\partial t} - \frac{\partial^3 u^0}{\partial t^3} \in L^\infty(0, T; H).$$

We therefore deduce that $\frac{\partial u^0}{\partial t} \in L^\infty(0, T; H^2(D))$.

For u^ε , we have that

$$\frac{\partial^2}{\partial t^2} \left(\frac{\partial u^\varepsilon}{\partial t} \right) - \nabla \cdot (A^\varepsilon \nabla \frac{\partial u^\varepsilon}{\partial t}) = \frac{\partial f}{\partial t}, \quad (2.59)$$

with the compatibility condition

$$\frac{\partial u^\varepsilon}{\partial t}(0, \cdot) = g^1 \in V, \quad \frac{\partial}{\partial t} \frac{\partial u^\varepsilon}{\partial t}(0, \cdot) = f(0, \cdot) - \nabla \cdot (A^\varepsilon(\cdot) \nabla g^0(\cdot)) = f(0, \cdot) \in H.$$

Thus

$$\frac{\partial u^\varepsilon}{\partial t} \in L^2(0, T; V) \cap H^1(0, T; H). \quad (2.60)$$

Let

$$u^{1\varepsilon}(t, x) = u^0(t, x) + \varepsilon u^1(t, x, \frac{x}{\varepsilon}).$$

We first show that

$$\|\nabla \cdot (A^\varepsilon \nabla u^{1\varepsilon}) - \nabla \cdot (A^0 \nabla u^0)\|_{L^\infty(0, T; H^{-1}(D))} \leq c\varepsilon.$$

For elliptic problems, Jikov et al. [37] show this for the case where $A(x, y) = A(y)$ and for smooth u^0 . However, the result also holds for the case where $A(x, y)$ depends also on x and for the weaker regularity $u^0 \in H^2(D)$. This is done in details in Hoang and Schwab [34].

We note that

$$\begin{aligned} (A^\varepsilon \nabla u^{1\varepsilon})_i(t, x) &= A_{ij}^0(x) \frac{\partial u^0}{\partial x_j}(t, x) + g_i^j(x, \frac{x}{\varepsilon}) \frac{\partial u^0}{\partial x_j}(t, x) \\ &\quad + \varepsilon w^k(x, \frac{x}{\varepsilon}) A_{ij}(x, \frac{x}{\varepsilon}) \frac{\partial^2 u^0}{\partial x_j \partial x_k}(t, x) \\ &\quad + \varepsilon A_{ik}(x, \frac{x}{\varepsilon}) \frac{\partial w^j}{\partial x_k}(x, \frac{x}{\varepsilon}) \frac{\partial u^0}{\partial x_j}(t, x), \end{aligned}$$

where

$$g_i^j(x, y) = A_{ij}(x, y) + A_{ik}(x, y) \frac{\partial w^j(x, y)}{\partial y_k} - A_{ij}^0(x). \quad (2.61)$$

For every $x \in D$, we have that

$$\begin{aligned} &\int_Y g_i^j(x, y) dy \\ &= \int_Y A_{ij}(x, y) + A_{ik}(x, y) \frac{\partial w^j(x, y)}{\partial y_k} dy - A_{ij}^0(x) \\ &= \int_Y A_{ik}(x, y) e_k^j + A_{ik}(x, y) \frac{\partial w^j(x, y)}{\partial y_k} dy - A_{ij}^0(x) \\ &= \int_Y A_{ik}(x, y) \left(e_k^j + \frac{\partial w^j(x, y)}{\partial y_k} \right) dy - A_{ij}^0(x) \\ &= \int_Y A_{ik}(x, y) \left(e_k^j + \frac{\partial w^j(x, y)}{\partial y_k} \right) e_l^i dy - A_{ij}^0(x), \end{aligned}$$

where $e_k^j = 1$ if $k = j$ and $e_k^j = 0$ otherwise. From (2.55),

$$\int_D \int_Y A(x, y) (e^j + \nabla_y w^j(x, y)) : \nabla_y \phi dy dx = 0, \quad \forall \phi \in L^2(D, H_\#^1(Y))^d. \quad (2.62)$$

We take $\phi = w^i(x, y)$ such that

$$\int_Y A(x, y)(e^j + \nabla_y w^j) : \nabla_y w^i dy = 0. \quad (2.63)$$

From (2.56),

$$A^0(x) = \int_Y A(x, y)(\mathbf{I} + \nabla w) \cdot (\mathbf{I} + \nabla w) dy. \quad (2.64)$$

Subtract (2.63) from (2.64), we get

$$A_{ij}^0(x) = \int_Y A_{ik}(x, y) \left(e_k^j + \frac{\partial w^j(x, y)}{\partial y_k} \right) e_i^j dy.$$

Thus we have

$$\int_Y g_i^j(x, y) dy = 0. \quad (2.65)$$

Again from (2.61)

$$\begin{aligned} \frac{\partial}{\partial y_j} g_i^j(x, y) &= \frac{\partial}{\partial y_j} \left(A_{ij}(x, y) + A_{ik}(x, y) \frac{\partial w^j(x, y)}{\partial y_k} \right) \\ &= \frac{\partial}{\partial y_j} \left(A_{ik} \left(e_k^j + \frac{\partial w^j(x, y)}{\partial y_k} \right) \right). \end{aligned}$$

By (2.62), for all $\phi \in L^2(D, H_{\#}^1(Y)/\mathbb{R})^d$

$$\begin{aligned} &\int_Y \frac{\partial}{\partial y_j} g_i^j(x, y) \phi(y) dy \\ &= \int_Y \frac{\partial}{\partial y_j} \left(A_{ik} \left(e_k^j + \frac{\partial w^j(x, y)}{\partial y_k} \right) \right) \phi(y) dy \\ &= \int_Y A_{ik} \left(e_k^j + \frac{\partial w^j(x, y)}{\partial y_k} \right) : \frac{\partial \phi}{\partial y_j} dy \\ &= 0. \end{aligned}$$

Since $\phi(y)$ is arbitrarily chosen

$$\frac{\partial}{\partial y_j} g_i^j(x, y) = 0. \quad (2.66)$$

Adapting the proof of Jikov et al. [37] for the case $A(x, y) = A(y)$, Hoang and Schwab [34] show that there are functions $\alpha_{ij}^k(x, y)$ which are Y -periodic with respect to y such that $\alpha_{ij}^k = -\alpha_{ji}^k$ and

$$g_i^k(x, y) = \frac{\partial}{\partial y_j} \alpha_{ij}^k(x, y),$$

and that when $A(x, y) \in (C^1(\bar{D}, C_{\#}^1(\bar{Y})))^{d \times d}$, $\alpha_{ij}^k(x, y) \in C^1(\bar{D}, C_{\#}(\bar{Y}))$. We therefore have

$$(A^\varepsilon \nabla u^{1\varepsilon}(t, x) - A^0(x) \nabla u^0(t, x))_i = \varepsilon \frac{\partial}{\partial x_j} \left(\alpha_{ij}^k(x, \frac{x}{\varepsilon}) \frac{\partial u^0}{\partial x_k}(t, x) \right) + (r^\varepsilon)_i(t, x),$$

where

$$\begin{aligned} (r^\varepsilon)_i(t, x) &= -\varepsilon \frac{\partial \alpha_{ij}^k(x, y)}{\partial x_j} \Big|_{y=x/\varepsilon} \frac{\partial u^0(t, x)}{\partial x_k}(t, x) - \varepsilon \alpha_{ij}^k(x, \frac{x}{\varepsilon}) \frac{\partial^2 u^0}{\partial x_k \partial x_j}(t, x) \\ &\quad + \varepsilon w^k(x, \frac{x}{\varepsilon}) A_{ij}(x, \frac{x}{\varepsilon}) \frac{\partial^2 u^0}{\partial x_k \partial x_j}(t, x) \\ &\quad + \varepsilon A_{ik}(x, \frac{x}{\varepsilon}) \frac{\partial w^j}{\partial x_k}(x, \frac{x}{\varepsilon}) \frac{\partial u^0}{\partial x_j}(t, x). \end{aligned}$$

As $\alpha_{ij}^k(x, y) \in C^1(\bar{D}, C(\bar{Y}))$, we deduce from (2.58) that $\|(r^\varepsilon)_i(\cdot)\|_{L^\infty(0, T; H)} \leq c\varepsilon$. Using $\alpha_{ij}^k = -\alpha_{ji}^k$ we get

$$\|\nabla \cdot (A^\varepsilon \nabla u^{1\varepsilon}(t, x)) - \nabla \cdot (A^0 \nabla u^0(t, x))\|_{L^\infty(0, T; H^{-1}(D))} \leq c\varepsilon. \quad (2.67)$$

Let $\tau^\varepsilon \in C_0^\infty(D)$ be such that $\tau^\varepsilon = 1$ outside an ε neighbourhood of ∂D and $\varepsilon |\nabla \tau^\varepsilon(x)| \leq c$ for all $\varepsilon > 0$. Let

$$\begin{aligned} w^{1\varepsilon}(t, x) &= u^0(t, x) + \varepsilon \tau^\varepsilon(x) w^k(x, \frac{x}{\varepsilon}) \frac{\partial u^0}{\partial x_k}(t, x) \\ &= u^{1\varepsilon}(t, x) - \varepsilon (1 - \tau^\varepsilon(x)) w^k(x, \frac{x}{\varepsilon}) \frac{\partial u^0}{\partial x_k}(t, x). \end{aligned}$$

We have that

$$\begin{aligned} \nabla(u^{1\varepsilon} - w^{1\varepsilon})(t, x) &= -\varepsilon \nabla \tau^\varepsilon(x) w^k(x, \frac{x}{\varepsilon}) \frac{\partial u^0}{\partial x_k}(t, x) \\ &\quad + (1 - \tau^\varepsilon(x)) \nabla_y w^k(x, \frac{x}{\varepsilon}) \frac{\partial u^0}{\partial x_k}(t, x) \\ &\quad + \varepsilon (1 - \tau^\varepsilon(x)) \nabla w^k(x, \frac{x}{\varepsilon}) \frac{\partial u^0}{\partial x_k}(t, x) \\ &\quad + \varepsilon (1 - \tau^\varepsilon(x)) w^k(x, \frac{x}{\varepsilon}) \nabla \frac{\partial u^0}{\partial x_k}(t, x). \end{aligned} \quad (2.68)$$

Thus

$$\begin{aligned} \nabla \cdot (A^\varepsilon (\nabla(u^{1\varepsilon} - w^{1\varepsilon}))) &= \varepsilon \nabla \cdot \left(A^\varepsilon(x) (1 - \tau^\varepsilon(x)) \nabla w^k(x, \frac{x}{\varepsilon}) \frac{\partial u^0}{\partial x_k}(t, x) \right) \\ &\quad + \varepsilon \nabla \cdot \left(A^\varepsilon(x) (1 - \tau^\varepsilon(x)) w^k(x, \frac{x}{\varepsilon}) \nabla \frac{\partial u^0}{\partial x_k}(t, x) \right) \\ &\quad + \nabla \cdot \left(A^\varepsilon(x) \left(-\varepsilon \nabla \tau^\varepsilon(x) w^k(x, \frac{x}{\varepsilon}) + (1 - \tau^\varepsilon(x)) \nabla_y w^k(x, \frac{x}{\varepsilon}) \right) \frac{\partial u^0}{\partial x_k}(t, x) \right). \end{aligned} \quad (2.69)$$

Since $\frac{\partial^2 u^0}{\partial t^2} \in L^\infty(0, T; V)$, we deduce from (2.68)

$$\left\| \frac{\partial^2 w^{1\varepsilon}}{\partial t^2} - \frac{\partial^2 u^0}{\partial t^2} \right\|_{L^\infty(0, T; H)} \leq c\varepsilon. \quad (2.70)$$

Using

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} - \nabla \cdot (A^\varepsilon \nabla u^\varepsilon) = \frac{\partial^2 u^0}{\partial t^2} - \nabla \cdot (A^0 \nabla u^0),$$

from (2.67), (2.69) and (2.70), we deduce that

$$\begin{aligned} & \frac{\partial^2(u^\varepsilon - w^{1\varepsilon})}{\partial t^2} - \nabla \cdot (A^\varepsilon \nabla(u^\varepsilon - w^{1\varepsilon})) \\ &= \frac{\partial^2 u^0}{\partial t^2} - \frac{\partial^2 w^{1\varepsilon}}{\partial t^2} + \nabla \cdot (A^\varepsilon \nabla(w^{1\varepsilon} - u^{1\varepsilon})) + \nabla \cdot (A^\varepsilon \nabla u^{1\varepsilon}) - \nabla \cdot (A^0 \nabla u^0) \\ &= \frac{\partial^2 u^0}{\partial t^2} - \frac{\partial w^{1\varepsilon}}{\partial t^2} + \nabla \cdot (A^\varepsilon \nabla u^{1\varepsilon}) - \nabla \cdot (A^0 \nabla u^0) \\ &\quad - \varepsilon \nabla \cdot \left(A^\varepsilon (1 - \tau^\varepsilon) \nabla w^k(\cdot, \frac{\cdot}{\varepsilon}) \frac{\partial u^0}{\partial x_k} \right) \\ &\quad - \varepsilon \nabla \cdot \left(A^\varepsilon (1 - \tau^\varepsilon) w^k(\cdot, \frac{\cdot}{\varepsilon}) \nabla \frac{\partial u^0}{\partial x_k} \right) \\ &\quad - \nabla \cdot \left(A^\varepsilon \left(-\varepsilon \nabla \tau^\varepsilon w^k(\cdot, \frac{\cdot}{\varepsilon}) + (1 - \tau^\varepsilon) \nabla_y w^k(\cdot, \frac{\cdot}{\varepsilon}) \right) \frac{\partial u^0}{\partial x_k} \right) \\ &= X - \nabla \cdot \left(A^\varepsilon \left(-\varepsilon \nabla \tau^\varepsilon w^k(\cdot, \frac{\cdot}{\varepsilon}) + (1 - \tau^\varepsilon) \nabla_y w^k(\cdot, \frac{\cdot}{\varepsilon}) \right) \frac{\partial u^0}{\partial x_k} \right), \end{aligned}$$

where X denotes the sum of the first six terms. From (2.58), (2.67) and (2.70), we deduce that $\|X\|_{L^\infty(0, T; H^{-1}(D))} \leq c\varepsilon$. Since $\frac{\partial u^0}{\partial t} \in L^\infty(0, T; V \cap H^2(D))$, we have that $\frac{\partial w^{1\varepsilon}}{\partial t} \in L^\infty(0, T; V)$ where $\sup_{t \in [0, T]} \|\frac{\partial w^{1\varepsilon}}{\partial t}\|_V < c$ for c independent of ε . Therefore

$$\begin{aligned} & \int_0^t \left(\frac{\partial^2(u^\varepsilon - w^{1\varepsilon})}{\partial s^2}(s, \cdot), \frac{\partial(u^\varepsilon - w^{1\varepsilon})}{\partial s}(s, \cdot) \right) ds + \\ & \quad \int_0^t \int_D A^\varepsilon(x) \nabla(u^\varepsilon(s, x) - w^{1\varepsilon}(s, x)) \cdot \frac{\partial}{\partial s} \nabla(u^\varepsilon(s, x) - w^{1\varepsilon}(s, x)) dx ds \\ &= \int_0^t \left(X(s, \cdot), \frac{\partial(u^\varepsilon - w^{1\varepsilon})}{\partial s}(s, \cdot) \right) ds + \\ & \quad \int_0^t \int_D A^\varepsilon(x) \left(-\varepsilon \nabla \tau^\varepsilon(x) w^k(x, \frac{x}{\varepsilon}) + (1 - \tau^\varepsilon(x)) \nabla_y w^k(x, \frac{x}{\varepsilon}) \right) \cdot \\ & \quad \frac{\partial u^0}{\partial x_k}(s, x) \frac{\partial}{\partial s} \nabla(u^\varepsilon(s, x) - w^{1\varepsilon}(s, x)) dx ds \end{aligned}$$

$$\begin{aligned}
&\leq c\|X\|_{L^2(0,T;H^{-1}(D))} \left(\left\| \frac{\partial u^\varepsilon}{\partial s} \right\|_{L^2(0,T;V)} + \left\| \frac{\partial w^{1\varepsilon}}{\partial s} \right\|_{L^2(0,T;V)} \right) + \\
&\quad \int_0^t \int_D A^\varepsilon(x) \left(-\varepsilon \nabla \tau^\varepsilon(x) w^k(x, \frac{x}{\varepsilon}) + (1 - \tau^\varepsilon(x)) \nabla_y w^k(x, \frac{x}{\varepsilon}) \right) \cdot \\
&\quad \left[\frac{\partial}{\partial s} \left(\frac{\partial u^0}{\partial x_k}(s, x) \nabla(u^\varepsilon(s, x) - w^{1\varepsilon}(s, x)) \right) \right. \\
&\quad \left. - \frac{\partial^2 u^0}{\partial s \partial x_k}(s, x) \nabla(u^\varepsilon(s, x) - w^{1\varepsilon}(s, x)) \right] dx ds.
\end{aligned}$$

Let $\hat{D}^\varepsilon \subset D$ be an ε neighbourhood of ∂D outside which $\tau^\varepsilon(x) = 1$. As $u^\varepsilon(0, x) = 0$ and $w^{1\varepsilon}(0, x) = 0$, and

$$\left\| \frac{\partial u^\varepsilon}{\partial t}(0) - \frac{\partial w^{1\varepsilon}}{\partial t}(0) \right\|_H = \left\| \frac{\partial u^0}{\partial t}(0) - \frac{\partial w^{1\varepsilon}}{\partial t}(0) \right\|_H \leq c\varepsilon \left\| \frac{\partial}{\partial t} \nabla u^0(0) \right\|_H,$$

we deduce that

$$\begin{aligned}
&\left\| \frac{\partial(u^\varepsilon - w^{1\varepsilon})}{\partial t}(t, \cdot) \right\|_H^2 + \|\nabla(u^\varepsilon(t, x) - w^{1\varepsilon}(t, x))\|_H^2 \\
&\leq c\varepsilon + c \left\| \frac{\partial u^0}{\partial x_k}(t, \cdot) \right\|_{L^2(\hat{D}^\varepsilon)} \|\nabla(u^\varepsilon(t, \cdot) - w^{1\varepsilon}(t, \cdot))\|_H \\
&\quad + c \int_0^T \left\| \frac{\partial^2 u^0}{\partial t \partial x_k}(t, \cdot) \right\|_{L^2(\hat{D}^\varepsilon)} dt \sup_{0 \leq t \leq T} \|\nabla(u^\varepsilon(t, \cdot) - w^{1\varepsilon}(t, \cdot))\|_H.
\end{aligned}$$

Following Hoang and Schwab [34], as ∂D is Lipschitz, for all functions $\phi \in C^\infty(D)$ we have $\|\phi\|_{L^2(\hat{D}^\varepsilon)}^2 \leq c\varepsilon^2 \|\phi\|_{H^1(D)}^2 + c\varepsilon \|\phi\|_{L^2(\partial D)}^2$, so

$$\|\phi\|_{L^2(\hat{D}^\varepsilon)} \leq c\varepsilon^{1/2} \|\phi\|_{H^1(D)} \quad \forall \phi \in H^1(D). \quad (2.71)$$

We therefore have

$$\left\| \frac{\partial u^0}{\partial x_k}(t, \cdot) \right\|_{L^2(\hat{D}^\varepsilon)} \leq c\varepsilon^{1/2} \left\| \frac{\partial u^0}{\partial x_k}(t, \cdot) \right\|_{H^1(D)}$$

and

$$\left\| \frac{\partial^2 u^0}{\partial t \partial x_k}(t, \cdot) \right\|_{L^2(\hat{D}^\varepsilon)} \leq c\varepsilon^{1/2} \left\| \frac{\partial^2 u^0}{\partial t \partial x_k}(t, \cdot) \right\|_{H^1(D)}.$$

From (2.58), we deduce that for all $t \in [0, T]$

$$\begin{aligned}
\left\| \frac{\partial(u^\varepsilon - w^{1\varepsilon})}{\partial t}(t) \right\|_H^2 + \|\nabla(u^\varepsilon - w^{1\varepsilon})(t)\|_H^2 &\leq c\varepsilon + c\varepsilon^{1/2} \|\nabla(u^\varepsilon - w^{1\varepsilon})(t)\|_H \\
&\quad + c\varepsilon^{1/2} \sup_{0 \leq s \leq T} \|\nabla(u^\varepsilon - w^{1\varepsilon})(s)\|_H,
\end{aligned}$$

which implies

$$\left\| \frac{\partial(u^\varepsilon - w^{1\varepsilon})}{\partial t}(t) \right\|_H + \|\nabla(u^\varepsilon - w^{1\varepsilon})(t)\|_H \leq c\varepsilon^{1/2}.$$

From (2.68) and (2.58), we have that

$$\left\| \frac{\partial(u^{1\varepsilon} - w^{1\varepsilon})}{\partial t} \right\|_{L^\infty(0,T;H)} \leq c\varepsilon.$$

Further, from (2.68) and (2.58), by using (2.71), we get

$$\|\nabla(u^{1\varepsilon} - w^{1\varepsilon})\|_{L^\infty(0,T;H)} \leq c\varepsilon^{1/2}.$$

We then get the conclusion. $\square$

2.7.2 Corrector for multiscale problems

For general multiscale problems, we are not aware of any analytic homogenization rate of convergence, even for elliptic problems. However, for the case where the scales are such that $\varepsilon_p/\varepsilon_{p+1}$ is an integer for all $p = 1, \dots, n-1$, a corrector for the elliptic problem is deduced in Hoang and Schwab [31] following the earlier work by Cioranescu et al. [17]. We now deduce a corresponding corrector for the general multiscale wave equation (2.5). We will use the following operators where $[x/\varepsilon]$ denotes the integer part of x/ε with respect to Y and $\{x/\varepsilon\} = x/\varepsilon - [x/\varepsilon]$.

Definition 2.7.2. The $(n+1)$ -scale “unfolding” operator $\mathcal{T}_n^\varepsilon : L^1(D) \rightarrow L^1(D \times \mathbf{Y})$ is defined as

$$\mathcal{T}_n^\varepsilon(\phi)(x, \mathbf{y}) = \phi\left(\varepsilon_1 \left[\frac{x}{\varepsilon_1}\right] + \varepsilon_2 \left[\frac{y_1}{\varepsilon_2/\varepsilon_1}\right] + \dots + \varepsilon_n \left[\frac{y_{n-1}}{\varepsilon_n/\varepsilon_{n-1}}\right] + \varepsilon_n y_n\right),$$

where the function $\phi \in L^1(D \times Y)$ is understood as 0 outside D .

Let D^ε be the 2ε neighbourhood of D , we have

$$\int_D \phi dx = \int_{D^\varepsilon} \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(\phi) d\mathbf{y} dx. \quad (2.72)$$

A simple proof for this can be found in, e.g. Cioranescu et al. [17]. It can be shown that

$$\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon) \rightharpoonup \nabla u^0 + \nabla_{y_1} u^1 + \dots + \nabla_{y_n} u^n \text{ in } L^2(D \times \mathbf{Y}). \quad (2.73)$$

Following [17], we define the “folding” operator $\mathcal{U}_n^\varepsilon$ as follows.

Definition 2.7.3. For $\Phi \in L^1(D \times \mathbf{Y})$ (understood to be zero when $x \notin D$) and for $\varepsilon > 0$ sufficiently small, the “folding” operator $\mathcal{U}_n^\varepsilon(\Phi) \in L^1(D)$ is defined as

$$\begin{aligned} \mathcal{U}_n^\varepsilon(\Phi)(x) = \int_{Y_1} \dots \int_{Y_n} \Phi\left(\varepsilon_1 \left[\frac{x}{\varepsilon_1}\right] + \varepsilon_1 t_1, \frac{\varepsilon_2}{\varepsilon_1} \left[\frac{\varepsilon_1}{\varepsilon_2} \left\{\frac{x}{\varepsilon_1}\right\}\right] + \frac{\varepsilon_2}{\varepsilon_1} t_2, \dots, \right. \\ \left. \frac{\varepsilon_n}{\varepsilon_{n-1}} \left[\frac{\varepsilon_{n-1}}{\varepsilon_n} \left\{\frac{x}{\varepsilon_{n-1}}\right\}\right] + \frac{\varepsilon_n}{\varepsilon_{n-1}} t_n, \left\{\frac{x}{\varepsilon_n}\right\}\right) dt_n \dots dt_1. \end{aligned} \quad (2.74)$$

It can be shown that for $\Phi \in L^1(D \times \mathbf{Y})$ which are understood as 0 when $x \notin D$

$$\int_{D^\varepsilon} \mathcal{U}_n^\varepsilon(\Phi)(x) dx = \int_D \int_{\mathbf{Y}} \Phi d\mathbf{y} dx$$

(see Cioranescu et al. [17]).

Assume $\frac{\varepsilon_{p-1}}{\varepsilon_p}$ is an integer, we have

$$\mathcal{U}_n^\varepsilon(\mathcal{T}_n^\varepsilon(\phi)) = \phi. \quad (2.75)$$

The proof of this is as follows

$$\begin{aligned} & \mathcal{U}_n^\varepsilon(\mathcal{T}_n^\varepsilon(\phi)) \\ &= \int_{\mathbf{Y}} \phi \left(\varepsilon_1 \left[\frac{\varepsilon_1 \left\lfloor \frac{x}{\varepsilon_1} \right\rfloor + \varepsilon_1 t_1}{\varepsilon_1} \right] + \varepsilon_2 \left[\frac{\frac{\varepsilon_2}{\varepsilon_1} \left\lfloor \frac{\varepsilon_1}{\varepsilon_2} \left\{ \frac{x}{\varepsilon_1} \right\} \right\rfloor + \frac{\varepsilon_2}{\varepsilon_1} t_2}{\varepsilon_2/\varepsilon_1} \right] + \dots \right. \\ & \quad \left. + \varepsilon_n \left[\frac{\frac{\varepsilon_n}{\varepsilon_{n-1}} \left\lfloor \frac{\varepsilon_{n-1}}{\varepsilon_n} \left\{ \frac{x}{\varepsilon_{n-1}} \right\} \right\rfloor + \frac{\varepsilon_n}{\varepsilon_{n-1}} t_n}{\varepsilon_n/\varepsilon_{n-1}} \right] + \varepsilon_n \left\{ \frac{x}{\varepsilon_n} \right\} \right) d\mathbf{t} \\ &= \int_{\mathbf{Y}} \phi \left(\varepsilon_1 \left[\left\lfloor \frac{x}{\varepsilon_1} \right\rfloor + t_1 \right] + \varepsilon_2 \left[\left\lfloor \frac{\varepsilon_1}{\varepsilon_2} \left\{ \frac{x}{\varepsilon_1} \right\} \right\rfloor + t_2 \right] + \dots \right. \\ & \quad \left. + \varepsilon_n \left[\left\lfloor \frac{\varepsilon_{n-1}}{\varepsilon_n} \left\{ \frac{x}{\varepsilon_{n-1}} \right\} \right\rfloor + t_n \right] + \varepsilon_n \left\{ \frac{x}{\varepsilon_n} \right\} \right) d\mathbf{t} \\ &= \int_{\mathbf{Y}} \phi \left(\varepsilon_1 \left\lfloor \frac{x}{\varepsilon_1} \right\rfloor + \varepsilon_2 \left\lfloor \frac{\varepsilon_1}{\varepsilon_2} \left\{ \frac{x}{\varepsilon_1} \right\} \right\rfloor + \dots \right. \\ & \quad \left. + \varepsilon_n \left\lfloor \frac{\varepsilon_{n-1}}{\varepsilon_n} \left\{ \frac{x}{\varepsilon_{n-1}} \right\} \right\rfloor + \varepsilon_n \left\{ \frac{x}{\varepsilon_n} \right\} \right) d\mathbf{t} \\ &= \int_{\mathbf{Y}} \phi \left(\varepsilon_1 \left\lfloor \frac{x}{\varepsilon_1} \right\rfloor + \varepsilon_2 \left[\frac{\varepsilon_1}{\varepsilon_2} \frac{x}{\varepsilon_1} - \frac{\varepsilon_1}{\varepsilon_2} \left\lfloor \frac{x}{\varepsilon_1} \right\rfloor \right] + \dots \right. \\ & \quad \left. + \varepsilon_n \left[\frac{\varepsilon_{n-1}}{\varepsilon_n} \frac{x}{\varepsilon_{n-1}} - \frac{\varepsilon_{n-1}}{\varepsilon_n} \left\lfloor \frac{x}{\varepsilon_{n-1}} \right\rfloor \right] + \varepsilon_n \left\{ \frac{x}{\varepsilon_n} \right\} \right) d\mathbf{t} \\ &= \int_{\mathbf{Y}} \phi \left(\varepsilon_1 \left\lfloor \frac{x}{\varepsilon_1} \right\rfloor + \varepsilon_2 \left[\frac{\varepsilon_1}{\varepsilon_2} \frac{x}{\varepsilon_1} - \frac{\varepsilon_1}{\varepsilon_2} \left\lfloor \frac{x}{\varepsilon_1} \right\rfloor \right] + \dots \right. \\ & \quad \left. + \varepsilon_n \left[\frac{\varepsilon_{n-1}}{\varepsilon_n} \frac{x}{\varepsilon_{n-1}} - \frac{\varepsilon_{n-1}}{\varepsilon_n} \left\lfloor \frac{x}{\varepsilon_{n-1}} \right\rfloor \right] + \varepsilon_n \left\{ \frac{x}{\varepsilon_n} \right\} \right) d\mathbf{t} \\ &= \phi \left(\varepsilon_1 \left\lfloor \frac{x}{\varepsilon_1} \right\rfloor - \varepsilon_1 \left\lfloor \frac{x}{\varepsilon_1} \right\rfloor + \varepsilon_2 \left\lfloor \frac{x}{\varepsilon_2} \right\rfloor - \dots - \varepsilon_{n-1} \left\lfloor \frac{x}{\varepsilon_{n-1}} \right\rfloor + \varepsilon_n \left\lfloor \frac{x}{\varepsilon_n} \right\rfloor + \varepsilon_n \left\{ \frac{x}{\varepsilon_n} \right\} \right) \\ &= \phi \left(\varepsilon_n \left(\left\lfloor \frac{x}{\varepsilon_n} \right\rfloor + \left\{ \frac{x}{\varepsilon_n} \right\} \right) \right) \\ &= \phi(x), \end{aligned}$$

where $\mathbf{t} = (t_1, \dots, t_n)$ and $0 \leq t_i \leq 1$.

We have the following results.

Proposition 2.7.4. *For the multiscale wave equation (2.5), assuming that $g^0 = 0$, $g^1 \in V \cap H^2(D)$ and $f \in H^1(0, T; H)$, we have*

$$\lim_{\varepsilon \rightarrow 0} \left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t} \right\|_{L^\infty(0, T; H)} + \left\| \nabla u^\varepsilon(t) - \mathcal{U}_n^\varepsilon(\nabla u^0(t) + \nabla_{y_1} u^1(t) + \dots + \nabla_{y_n} u^n(t)) \right\|_{L^\infty(0, T; H)} = 0.$$

Proof We consider the expression

$$\begin{aligned} E^\varepsilon(t) &= \int_D \left(\frac{\partial u^\varepsilon}{\partial t}(t) - \frac{\partial u^0}{\partial t}(t) \right)^2 dx \\ &+ \int_D \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(A^\varepsilon)(\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon(t)) - (\nabla u^0(t) + \nabla_{y_1} u^1(t) + \dots + \nabla_{y_n} u^n(t))) \cdot \\ &(\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon(t)) - (\nabla u^0(t) + \nabla_{y_1} u^1(t) + \dots + \nabla_{y_n} u^n(t))) d\mathbf{y} dx. \end{aligned}$$

From (2.72) and the conservation of energy of wave equations (see Lions and Margenes [40], also Brahim-Otsmane et al. [11]) we have

$$\begin{aligned} \int_D \left(\frac{\partial u^\varepsilon}{\partial t} \right)^2 dx + \int_{D^\varepsilon} \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(A^\varepsilon) \mathcal{T}_n^\varepsilon(\nabla u^\varepsilon(t)) \cdot \mathcal{T}_n^\varepsilon(\nabla u^\varepsilon(t)) d\mathbf{y} dx \\ = \int_D \left(\frac{\partial u^\varepsilon}{\partial t} \right)^2 dx + \int_D A^\varepsilon \nabla u^\varepsilon(t) \cdot \nabla u^\varepsilon(t) dx \\ = \int_D (g^1)^2 dx + 2 \int_0^t \int_D f \frac{\partial u^\varepsilon}{\partial t} dx dt \\ \rightarrow \int_D (g^1)^2 dx + 2 \int_0^t \int_D f \frac{\partial u^0}{\partial t} dx dt. \end{aligned}$$

From this and (2.73), we have that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} E^\varepsilon(t) &= 2 \int_D (g^1)^2 dx + 2 \int_0^t \int_D f \frac{\partial u^0}{\partial t} dx dt - \\ &2 \left[\int_D \left(\frac{\partial u^0}{\partial t} \right)^2 dx + \int_D \int_{\mathbf{Y}} A(x, \mathbf{y}) (\nabla u^0(t) + \nabla_{y_1} u^1(t) + \dots + \nabla_{y_n} u^n(t)) \cdot \right. \\ &\quad \left. (\nabla u^0(t) + \nabla_{y_1} u^1(t) + \dots + \nabla_{y_n} u^n(t)) d\mathbf{y} dx \right]. \end{aligned}$$

Since A^0 is the homogenized coefficient of the multiscale coefficient $A(x, \mathbf{y})$, we have

$$\begin{aligned} \int_D \int_{\mathbf{Y}} A(x, \mathbf{y}) (\nabla u^0(t) + \nabla_{y_1} u^1(t) + \dots + \nabla_{y_n} u^n(t)) \cdot \\ (\nabla u^0(t) + \nabla_{y_1} u^1(t) + \dots + \nabla_{y_n} u^n(t)) d\mathbf{y} dx \\ = \int_D A^0 \nabla u^0 \cdot \nabla u^0 dx. \end{aligned}$$

From (2.35), we deduce that

$$\int_D \left(\frac{\partial u^0}{\partial t} \right)^2 dx + \int_D A^0 \nabla u^0 \cdot \nabla u^0 dx = \int_D (g^1)^2 dx + 2 \int_0^t \int_D f \frac{\partial u^0}{\partial t} dx dt.$$

Thus $\lim_{\varepsilon \rightarrow 0} E^\varepsilon(t) = 0$.

Next we show that the convergence is uniform. From (2.59), we find that $\frac{\partial u^\varepsilon}{\partial t}$ is uniformly bounded in $H^1(0, T; H) \cap L^2(0, T; V)$ with respect to ε and $\frac{\partial u^0}{\partial t}$ is in $H^1(0, T; H) \cap L^2(0, T; V)$. Also from (2.41), $\frac{\partial u^0}{\partial t} \in L^2(0, T; V)$ so $\frac{\partial u}{\partial t} \in L^2(0, T; \mathbf{V})$.

We show that $E^\varepsilon(t)$ is equicontinuous with respect to ε . Considering the first term, we have for $s < t$

$$\begin{aligned}
& \left| \int_D \left[\left(\frac{\partial u^\varepsilon}{\partial t}(t) - \frac{\partial u^0}{\partial t}(t) \right)^2 - \left(\frac{\partial u^\varepsilon}{\partial t}(s) - \frac{\partial u^0}{\partial t}(s) \right)^2 \right] dx \right| = \\
& \left| \int_D \left(\left(\frac{\partial u^\varepsilon}{\partial t}(t) - \frac{\partial u^\varepsilon}{\partial t}(s) \right) - \left(\frac{\partial u^0}{\partial t}(t) - \frac{\partial u^0}{\partial t}(s) \right) \right) \right. \\
& \quad \left. \left(\frac{\partial u^\varepsilon}{\partial t}(t) + \frac{\partial u^\varepsilon}{\partial t}(s) - \frac{\partial u^0}{\partial t}(t) - \frac{\partial u^0}{\partial t}(s) \right) dx \right| \\
& \leq \left(\left\| \frac{\partial u^\varepsilon}{\partial t}(t) - \frac{\partial u^\varepsilon}{\partial t}(s) \right\|_H + \left\| \frac{\partial u^0}{\partial t}(t) - \frac{\partial u^0}{\partial t}(s) \right\|_H \right) \\
& \quad \left(\left\| \frac{\partial u^\varepsilon}{\partial t}(t) \right\|_H + \left\| \frac{\partial u^\varepsilon}{\partial t}(s) \right\|_H + \left\| \frac{\partial u^0}{\partial t}(t) \right\|_H + \left\| \frac{\partial u^0}{\partial t}(s) \right\|_H \right).
\end{aligned}$$

From the regularity $\frac{\partial u^\varepsilon}{\partial t} \in H^1(0, T; H)$, we have that $\|\frac{\partial u^\varepsilon}{\partial t}(t)\|_H$ is uniformly bounded for all t . Similarly, $\|\frac{\partial u^0}{\partial t}(t)\|_H$ is uniformly bounded for all t . Therefore

$$\begin{aligned}
& \left| \int_D \left[\left(\frac{\partial u^\varepsilon}{\partial t}(t) - \frac{\partial u^0}{\partial t}(t) \right)^2 - \left(\frac{\partial u^\varepsilon}{\partial t}(s) - \frac{\partial u^0}{\partial t}(s) \right)^2 \right] dx \right| \\
& \leq c \left(\left\| \int_s^t \frac{\partial^2 u^\varepsilon}{\partial t^2}(\tau) d\tau \right\|_H + \left\| \int_s^t \frac{\partial^2 u^0}{\partial t^2}(\tau) d\tau \right\|_H \right) \\
& \leq c \int_s^t \left(\left\| \frac{\partial^2 u^\varepsilon}{\partial t^2}(\tau) \right\|_H + \left\| \frac{\partial^2 u^0}{\partial t^2}(\tau) \right\|_H \right) d\tau \\
& \leq c(t-s)^{1/2} \left(\int_s^t \left(\left\| \frac{\partial^2 u^\varepsilon}{\partial t^2}(\tau) \right\|_H^2 + \left\| \frac{\partial^2 u^0}{\partial t^2}(\tau) \right\|_H^2 \right) d\tau \right)^{1/2} \leq c(t-s)^{1/2}
\end{aligned}$$

due to $\|\frac{\partial u^\varepsilon}{\partial t}\|_{H^1(0, T; H)}$ is uniformly bounded with respect to ε and $\frac{\partial u^0}{\partial t} \in H^1(0, T; H)$.

We can deal with the second term in the expression of $E^\varepsilon(t)$ in a similar manner, using the fact that u^ε is uniformly bounded in $H^1(0, T; V)$ and $\mathbf{u} \in H^1(0, T; \mathbf{V})$. Using (2.75), we have

$$\begin{aligned}
& \int_D |\nabla(u^\varepsilon) - \mathcal{U}_n^\varepsilon(\nabla(u^0) + \nabla_{y_1}(u^1) + \dots + \nabla_{y_n}(u^n))(x)|^2 dx \\
& = \int_D \left| \mathcal{U}_n^\varepsilon(\mathcal{T}_n^\varepsilon(\nabla(u^\varepsilon)) - (\nabla(u^0) + \nabla_{y_1}(u^1) + \dots + \nabla_{y_n}(u^n)))(x) \right|^2 dx \\
& \leq \int_D \mathcal{U}_n^\varepsilon(\mathcal{T}_n^\varepsilon((\nabla(u^\varepsilon)) - (\nabla(u^0) + \nabla_{y_1}(u^1) + \dots + \nabla_{y_n}(u^n)))^2) dx \\
& \leq \int_D \int_{\mathbf{Y}} (\mathcal{T}_n^\varepsilon(\nabla(u^\varepsilon)) - (\nabla(u^0) + \nabla_{y_1}(u^1) + \dots + \nabla_{y_n}(u^n)))^2 d\mathbf{y} dx.
\end{aligned}$$

We then get the conclusion from the Arzelà-Ascoli theorem. $\square$

2.8 Convergence in the physical variable

We now derive numerical correctors from the finite element approximation of problem (2.8) for the multiscale solution u^ε .

2.8.1 Convergence in the physical variables for two-scale problem

In this section, we establish the convergence in the physical variables of the solutions of problems (2.12), (2.23) for both the full and sparse tensor approximations. Even though $\|\nabla_y u^{1L}(t, x, y)\|_{L^2((0,T) \times D \times Y)^d}$ is bounded uniformly for all L , the term $\|\nabla_y u^{1L}(t, x, x/\varepsilon)\|_{L^2((0,T) \times D)^d}$ may not be bounded. Therefore substituting the finite element solution $u^{1L}(t, x, x/\varepsilon)$ into (2.57) may not give us a bound uniformly for all ε . In order to get an approximation for u^ε in the $H^1(D)$ norm, we need to post process the finite element solution. The operator $\mathcal{U}_n^\varepsilon$ defined in (2.74) now becomes

$$\mathcal{U}^\varepsilon(\Phi)(x) = \int_Y \Phi\left(\varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}\right) dz. \quad (2.76)$$

Let D^ε be a 2ε neighbourhood of D . Regarding Φ as zero when x is outside D , we have

$$\int_{D^\varepsilon} \mathcal{U}^\varepsilon(\Phi)(x) dx = \int_D \int_Y \Phi(x, y) dx dy. \quad (2.77)$$

We then have the following result.

Lemma 2.8.1. *Assume that $g^1 \in V$, $f \in H^1(0, T; H)$, and the domain D is convex. Then*

$$\sup_{t \in [0, T]} \int_D \left| \nabla_y u^1(t, x, \frac{x}{\varepsilon}) - \mathcal{U}^\varepsilon(\nabla_y u^1(t, \cdot, \cdot))(x) \right|^2 dx \leq c\varepsilon^2.$$

Proof We note from (2.41) that $\partial u^0 / \partial t \in L^\infty(0, T; H)$. Therefore, from (2.35), we have

$$-\nabla \cdot (A^0 \nabla u^0) = f - \frac{\partial^2 u^0}{\partial t^2} \in L^\infty(0, T; H).$$

Thus $u^0 \in L^\infty(0, T; H^2(D))$. Since $A \in (C^1(D, C_\#^1(Y)))^{d \times d}$, $w^l(x, y) \in C^1(D, C_\#^1(Y))$. The proof then follows from that of Lemma 5.5 of Hoang and Schwab [34]. $\square$

We then have the following result on the semidiscrete approximation (2.12).

Theorem 2.8.2. *Assume that $f \in H^2(0, T; H)$, $g^1 \in V \cap H^2(D)$, $f(0) \in V$ and the domain D is convex, and the initial conditions are chosen so that $g^{0L} = 0$ and*

$\|g^{1L} - g^1\|_H \leq ch_L$. Then the solution of the semidiscrete problem (2.48) for the full tensor product FE spaces satisfies

$$\left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial \bar{u}^{0L}}{\partial t} \right\|_H + \|\nabla u^\varepsilon - [\nabla \bar{u}^{0L} + \mathcal{U}^\varepsilon(\nabla_y \bar{u}^{1L}(t, \cdot, \cdot))]\|_{L^\infty(0,T;H)} \leq c(h_L + \varepsilon^{1/2}).$$

and the solution of the problem (2.52) for the sparse tensor product FE spaces satisfies

$$\left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial \hat{u}^{0L}}{\partial t} \right\|_H + \|\nabla u^\varepsilon - [\nabla \hat{u}^{0L} + \mathcal{U}^\varepsilon(\nabla_y \hat{u}^{1L}(t, \cdot, \cdot))]\|_{L^\infty(0,T;H)} \leq c(h_L |\log h_L|^{1/2} + \varepsilon^{1/2}).$$

Proof With the condition of the theorem, condition (2.37) holds. The regularity required for Propositions 2.6.2 and 2.6.5 and 2.7.1 hold. For all $\Phi \in L^2(D \times Y)$, we have

$$(\mathcal{U}^\varepsilon(\Phi)(x))^2 \leq \mathcal{U}^\varepsilon(\Phi^2)(x).$$

Therefore

$$\begin{aligned} & \|\mathcal{U}^\varepsilon(\nabla_y u^1)(\cdot) - \mathcal{U}^\varepsilon(\nabla_y u^{1L})(\cdot)\|_{L^2(D)} \\ &= \int_D |\mathcal{U}^\varepsilon(\nabla_y u^1)(x) - \mathcal{U}^\varepsilon(\nabla_y u^{1L})(x)|^2 dx \\ &\leq \int_{D^\varepsilon} |\mathcal{U}^\varepsilon(\nabla_y u^1 - \nabla_y u^{1L})(x)|^2 dx \\ &\leq \int_{D^\varepsilon} \mathcal{U}^\varepsilon(|\nabla_y u^1 - \nabla_y u^{1L}|^2)(x) dx \\ &= \int_D \int_Y |\nabla_y u^1 - \nabla_y u^{1L}|^2 dy dx \\ &= \|\nabla_y u^1(x, y) - \nabla_y u^{1L}(x, y)\|_{L^2(D \times Y)} \\ &\leq ch_L. \end{aligned}$$

From Propositions 2.6.2, 2.6.5 and 2.7.1 we get the conclusion. $\square$

For the fully discrete problems (2.51), (2.54) we have the following result.

Theorem 2.8.3. Assume that $g^0 = 0$ and the domain D is convex. If

$$f \in H^2(0, T; H), \quad g^1 \in V \cap H^2(D), \quad f(0) \in V \quad (2.78)$$

and

$$\|\partial_t \zeta^{0L;1/2}\|_H \leq c(\Delta t + h_L), \quad \|\zeta^{L;1/2}\|_{\mathbf{V}} \leq c(\Delta t + h_L),$$

then for the full tensor product approximating problem (2.51) we have

$$\begin{aligned} & \left\| \partial_t \bar{u}^{0L} - \frac{\partial u^\varepsilon}{\partial t} \right\|_{\tilde{L}^\infty(0,T;H)} + \|\nabla u^\varepsilon - [\nabla \bar{u}^{0L} + \mathcal{U}^\varepsilon(\nabla_y \bar{u}^{1L}(\cdot, \cdot, \cdot))]\|_{\tilde{L}^\infty(0,T;H)} \\ & \leq c(\Delta t + h_L + \varepsilon^{1/2}); \end{aligned} \quad (2.79)$$

and if

$$\|\partial_t \zeta^{0L;1/2}\|_H \leq c(\Delta t + h_L L^{1/2}), \quad \|\zeta^{L;1/2}\|_{\mathbf{V}} \leq c(\Delta t + h_L L^{1/2}),$$

then for the sparse tensor product approximating problem (2.54) we have

$$\begin{aligned} & \|\partial_t \hat{u}^{0L} - \frac{\partial u^\varepsilon}{\partial t}\|_{\tilde{L}^\infty(0,T;H)} + \|\nabla u^\varepsilon - [\nabla \hat{u}^{0L} + \mathcal{U}^\varepsilon(\nabla_y \hat{u}^{1L}(\cdot, \cdot, \cdot))]\|_{\tilde{L}^\infty(0,T;H)} \\ & \leq c(\Delta t + h_L L^{1/2} + \varepsilon^{1/2}). \end{aligned} \quad (2.80)$$

If we assume further that

$$f \in H^3(0, T; H), \quad g^1 \in H_0^3(D) \text{ and } f(0) \in H^2(D), \quad (2.81)$$

and

$$\|\partial_t \zeta^{0L;1/2}\|_H \leq c((\Delta t)^2 + h_L), \quad \|\zeta^{L;1/2}\|_{\mathbf{V}} \leq c((\Delta t)^2 + h_L),$$

then for the full tensor approximation problem (2.51) we have

$$\begin{aligned} & \Delta t \|\partial_t \bar{u}^{0L} - \frac{\partial u^\varepsilon}{\partial t}\|_{\tilde{L}^\infty(0,T;H)} + \|\nabla u^\varepsilon - [\nabla \bar{u}^{0L} + \mathcal{U}^\varepsilon(\nabla_y \bar{u}^{1L}(\cdot, \cdot, \cdot))]\|_{\tilde{L}^\infty(0,T;H)} \\ & \leq c((\Delta t)^2 + h_L + \varepsilon^{1/2}); \end{aligned} \quad (2.82)$$

and if

$$\|\partial_t \zeta^{0L;1/2}\|_H \leq c((\Delta t)^2 + h_L L^{1/2}), \quad \|\zeta^{L;1/2}\|_{\mathbf{V}} \leq c((\Delta t)^2 + h_L L^{1/2}),$$

for the sparse tensor product approximating problem (2.54)

$$\begin{aligned} & \Delta t \|\partial_t \hat{u}^{0L} - \frac{\partial u^\varepsilon}{\partial t}\|_{\tilde{L}^\infty(0,T;H)} + \|\nabla u^\varepsilon - [\nabla \hat{u}^{0L} + \mathcal{U}^\varepsilon(\nabla_y \hat{u}^{1L}(\cdot, \cdot, \cdot))]\|_{\tilde{L}^\infty(0,T;H)} \\ & \leq c((\Delta t)^2 + h_L L^{1/2} + \varepsilon^{1/2}). \end{aligned} \quad (2.83)$$

Proof With $A(x, y) \in C^1(\bar{D}, C^1(\bar{Y}))$, $g^0 = 0$ and the domain D is convex, and with either (2.78) or (2.81), the conditions of Proposition 2.7.1 holds. We note that

$$\frac{1}{\Delta t}(u^{0;m+1} - u^{0;m}) - \frac{\partial u^0}{\partial t}(t_m) = \frac{\partial u^0}{\partial t}(\tau) - \frac{\partial u^0}{\partial t}(t_m) = \int_{t_m}^{\tau} \frac{\partial^2 u^0}{\partial t^2}(\tau') d\tau',$$

for $t_m \leq \tau \leq t_{m+1}$. From (2.41), we deduce that $\frac{\partial^2 u^0}{\partial t^2} \in L^\infty(0, T; H)$. Therefore

$$\sup_{0 \leq m < M} \|\partial_t u^{0;m+1/2} - \frac{\partial u^0}{\partial t}(t_m)\|_H \leq c\Delta t.$$

With (2.78), (2.37) holds. From Propositions 2.6.3 and 2.6.6, we deduce (2.79) and (2.80). The two estimates (2.82) and (2.83) are deduced similarly as (2.81) leads to (2.39). $\square$

2.8.2 Convergence in the physical variables in the multiscale case

For general multiscale problems, we deduce in this section the convergence of the solution of both the semidiscrete problem (2.12) and the fully discrete problem (2.23). As we do not have an explicit homogenization rate of convergence in this case we will not distinguish the two cases of full and sparse tensor products. We thus use our general notations for general finite element spaces in Section 2.4. The results in this section hold for both full and sparse tensor product spaces. For problem (2.12), we have the following result.

Theorem 2.8.4. *Assume that $g^0 = 0$, $g^1 \in V$, $f \in H^2(0, T; H)$ and $f(0) \in V$, and $g^{0L} = 0$, $\|g^{1L} - g^1\|_H \rightarrow 0$ when $L \rightarrow \infty$. Then for the solution of the semidiscrete problem (2.12) we have*

$$\lim_{\substack{L \rightarrow \infty \\ \varepsilon \rightarrow 0}} \left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u^{0L}}{\partial t} \right\|_{L^\infty(0, T; H)} + \|\nabla u^\varepsilon - \mathcal{U}_n^\varepsilon(\nabla u^{0L} + \nabla_{y_1} u^{1L} + \dots + \nabla_{y_n} u^{nL})\|_{L^\infty(0, T; H)} = 0.$$

Proof With the condition of the theorem, condition (2.37) holds. We therefore have $\frac{\partial^2 u^0}{\partial t^2} \in L^2(0, T; V)$. Thus from (2.36), $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2(0, T; \mathbf{V})$.

We have further that

$$\begin{aligned} & \|\mathcal{U}_n^\varepsilon(\nabla u^0(t) + \nabla_{y_1} u^1(t) + \dots + \nabla_{y_n} u^n(t)) \\ & \quad - \mathcal{U}_n^\varepsilon(\nabla u^{0L}(t) + \nabla_{y_1} u^{1L}(t) + \dots + \nabla_{y_n} u^{nL}(t))\|_H \\ & \leq \|\mathbf{u}(t) - \mathbf{u}^L(t)\|_{\mathbf{V}}. \end{aligned}$$

The result then follows from Propositions 2.4.5 and 2.7.4. $\square$

Similarly, we have the following result for the fully discrete problem (2.23).

Theorem 2.8.5. *Assume that $g^0 = 0$, $g^1 \in V$, $f \in H^2(0, T; H)$ and $f(0) \in V$, and $g^{0L} = 0$. If $\mathbf{u}^{L;0}$ and $\mathbf{u}^{L;1}$ are chosen so that*

$$\lim_{L \rightarrow \infty} \|\partial_t \zeta^{0L;1/2}\|_H = 0, \quad \lim_{L \rightarrow \infty} \|\zeta^{L;1/2}\|_{\mathbf{V}} = 0,$$

then

$$\begin{aligned} & \lim_{\substack{L \rightarrow \infty \\ \Delta t \rightarrow 0 \\ \varepsilon \rightarrow 0}} \sup_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t u^{0L; m+1/2} \right\|_H + \\ & \quad \|\nabla u^\varepsilon - \mathcal{U}_n^\varepsilon(\nabla u^{0L} + \nabla_{y_1} u^{1L} + \dots + \nabla_{y_n} u^{nL})\|_{\tilde{L}^\infty(0, T; H)} = 0. \end{aligned}$$

Proof From the condition of the theorem, condition (2.37) holds. Convergence (2.32) holds. From Propositions 2.4.8 and 2.7.4 we get the result. $\square$

2.9 Numerical results

We present in this section some numerical examples for two and three scale problems that confirm our analysis above. We follow the approach and notations of Dupont [20] and solve the fully discretized Galerkin approximating problem (2.23). At each time step m , we solve for $\mathbf{u}_{m+1}^L$ using full or sparse tensor finite element product spaces as developed in Subsections 2.6.2 and 2.6.3. The discretized problem is formulated as a linear system of $\mathbf{c}_m$ that represents components of $\mathbf{u}_m^L$ in the basis for $\mathbf{V}^L$. By $c_{0,m}$, we mean a $\dim \mathbf{V}^L \times 1$ column vector whose first $\dim V_0^L$ components are those of $u_{0,m}^L$ in the basis for V_0^L . The finite element problem is then constructed as

$$\begin{aligned} \frac{1}{(\Delta t)^2} \mathbf{M} c_{0,m+1} + \frac{1}{4} \mathbf{S} \mathbf{c}_{m+1} &= \frac{1}{4} \mathbf{F}_{m+1} + \frac{1}{2} \mathbf{F}_m + \frac{1}{4} \mathbf{F}_{m-1} \\ &+ \frac{2}{(\Delta t)^2} \mathbf{M} c_{0,m} - \frac{1}{(\Delta t)^2} \mathbf{M} c_{0,m-1} - \frac{1}{2} \mathbf{S} \mathbf{c}_m - \frac{1}{4} \mathbf{S} \mathbf{c}_{m-1}. \end{aligned}$$

In the equation above, $\mathbf{M}$ is the Gram matrix for the basis vectors in V_0^L in the H norm extending to 0 to form a $\dim \mathbf{V}^L \times \dim \mathbf{V}^L$ matrix. The stiffness matrix $\mathbf{S}$ is derived from $\mathbf{A}_{00}$, $\mathbf{A}_{01}$, $\mathbf{A}_{11}$ in Proposition 2.4.1. On the right hand side, $\mathbf{F}_m$ is the time dependent column vector whose first $\dim V_0^L$ components form the column vector that describes the interaction of $f(t_m)$ and the basis functions of V_0^L in the H norm and all other components are 0. The problem is then solved by conjugate gradient method with full and sparse tensor FE basis in Subsection 2.9.2 and sparse tensor FE basis in Subsection 2.9.3 and 2.9.4.

To construct the sparse tensor finite element product spaces, it is convenient to use the wavelet preconditioning. We thus briefly present the assumptions on the finite element spaces on D and Y in the next subsection.

2.9.1 Multilevel FE spaces and preconditioning

Following Hoang and Schwab [31], we make the following assumptions on the existence of a multilevel hierarchy of finite element basis functions.

Assumption 2.9.1. (i) For each $j \in \mathbb{N}_0^d$, there exists a set of indices $I^j \subset \mathbb{N}_0^d$ and a set of basis functions $\phi^{jk} \in H^1(D)$, $k \in I^j$, such that $V^l = \text{span}\{\phi^{jk} : |j|_\infty \leq l\}$. There are constants $c_2 > c_1 > 0$ such that if $\phi = \sum_{|j|_\infty \leq l, k \in I^j} \phi^{jk} c_{jk} \in V^l$, then the norm equivalence holds:

$$c_1 \sum_{\substack{|j|_\infty \leq l \\ k \in I^j}} 2^{2s|j|_\infty} |c_{jk}|^2 \leq \|\phi\|_{H^s(D)}^2 \leq c_2 \sum_{\substack{|j|_\infty \leq l \\ k \in I^j}} 2^{2s|j|_\infty} |c_{jk}|^2, \quad s = 0, 1.$$

(ii) For the space $L^2(Y)$, for each $j \in \mathbb{N}_0^d$, there exists a set of indices $I_0^j \subset \mathbb{N}_0^d$ and a set of basis functions $\phi_0^{jk} \in L^2(Y)$, $k \in I_0^j$, such that $V_\#^l = \text{span}\{\phi_0^{jk} : |j|_\infty \leq l\}$.

There are constants $c_4 > c_3 > 0$ such that if $\phi = \sum_{|j|_\infty \leq l, k \in I^j} \phi_0^{jk} c_{jk} \in V_\#^l$, then

$$c_3 \sum_{\substack{|j|_\infty \leq l \\ k \in I_0^j}} |c_{jk}|^2 \leq \|\phi\|_{L^2(Y)}^2 \leq c_4 \sum_{\substack{|j|_\infty \leq l \\ k \in I_0^j}} |c_{jk}|^2.$$

(iii) For each $j \in \mathbb{N}_0^d$, there exists a set of indices $I_1^j \subset \mathbb{N}_0^d$ and a set of basis functions $\phi_1^{jk} \in H_\#^1(Y)/\mathbb{R}$, $k \in I_1^j$, such that $\{\phi_1^{jk} : |j|_\infty \leq l\}$ is a basis of $V_\#^l/\mathbb{R}$. There are constants $c_6 > c_5 > 0$ such that if $\phi = \sum_{|j|_\infty \leq l, k \in I_1^j} \phi_1^{jk} c_{jk}$, then

$$c_5 \sum_{\substack{|j|_\infty \leq l \\ k \in I_1^j}} |c_{jk}|^2 \leq \|\phi\|_{H_\#^1(Y)/\mathbb{R}}^2 \leq c_6 \sum_{\substack{|j|_\infty \leq l \\ k \in I_1^j}} |c_{jk}|^2.$$

We consider the following examples of hierarchical FE bases in one dimension.

Example

(i) For $D = (0, 1)$ we define the hierarchical base for $L^2(D)$ as follows. At level 0, we choose three piecewise linear functions: ψ^{01} obtains values $(1, 0)$ at $(0, 1/2)$ and is 0 in $(1/2, 1)$, ψ^{02} is continuous piecewise linear and obtains values $(0, 1, 0)$ at $(0, 1/2, 1)$, and ψ^{03} obtains values $(0, 1)$ at $(1/2, 1)$ and is 0 in $(0, 1/2)$. For each level, we choose piecewise linear function ψ that takes values $(0, -1, 2, -1, 0)$ at $(0, 1/2, 1, 3/2, 2)$. The left boundary function ψ^{left} takes values $(-2, 2, -1, 0)$ at $(0, 1/2, 1, 3/2)$, and the right boundary function ψ^{right} takes values $(0, -1, 2, -2)$ at $(1/2, 1, 3/2, 2)$. For levels $j \geq 1$, the index set $I^j = \{1, 2, \dots, 2^j\}$. The wavelet functions are defined as $\psi^{j1}(x) = 2^{-j/2} \psi^{left}(2^j x)$, $\psi^{jk}(x) = 2^{-j/2} \psi(2^j x - k + 3/2)$ for $k = 2, \dots, 2^j - 1$ and $\psi^{j2^j} = \psi^{right}(2^j x - 2^j + 2)$. This base satisfy Assumption 2.9.1 (i).

(ii) For $Y = (0, 1)$, we define the base of $H_\#^1(Y)/\mathbb{R}$ as in (i) but exclude ψ^{01} , ψ^{03} . At other levels, all the functions ψ^{left} and ψ^{right} are replaced by the piecewise linear functions that take values $(0, 2, -1, 0)$ at $(0, 1/2, 1, 3/2)$ and values $(0, -1, 2, 0)$ at $(1/2, 1, 3/2, 2)$ respectively. The wavelet functions $\phi^{jk}(y) = 2^{-j/2} \phi(2^j y - k + 1)$ satisfy Assumption 2.9.1 (iii).

In higher dimensions, the basis functions in $(0, 1)^d$ can be constructed by taking the tensor products of the basis functions constructed in (i). They satisfy Assumption 2.9.1 after appropriate scaling, see [27].

The basis functions used in (i) are illustrated in Figure 2.1 and Figure 2.2.

2.9.2 One-dimensional two-scale problem

We first consider a one-dimensional two-scale problem where $A(x, y) = (2/3)(1+x)(1+\cos^2 2\pi y)$ in the domain $D = (0, 1)$. The multiscale problem is

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} - \frac{\partial}{\partial x} \left(\frac{2}{3}(1+x) \left(1 + \cos^2 \left(2\pi \frac{x}{\varepsilon} \right) \right) \frac{\partial u^\varepsilon}{\partial x} \right) = t^3 + 9t \left(x - \frac{\log(1+x)}{\log 2} \right) \quad \text{in } D,$$

$$u^\varepsilon(t, 0) = u^\varepsilon(t, 1) = 0.$$

Figure 2.1: The basis ψ^{01} , ψ^{02} and ψ^{03} for $L^2(0, 1)$. Figure 2.2: The basis ψ^{11} , ψ^{12} and ψ^{13} for $L^2(0, 1)$.

The two-scale limiting homogenized equation has the exact homogenized solution

$$u^0(t, x) = \frac{3}{2\sqrt{2}} \left(x - \frac{\log(1+x)}{\log 2} \right) t^3,$$

and the scale interacting term $u^1(t, x, y)$ is determined with an additive constant c

$$u^1(t, x, y) = \frac{3}{2\sqrt{2}} \left(1 - \frac{1}{(1+x)\log 2} \right) \left(\frac{1}{2\pi} \tan^{-1} \left(\frac{\tan 2\pi y}{\sqrt{2}} \right) - y + c \right) t^3.$$

In Figure 2.3, we plot the error $\|\mathbf{u} - \mathbf{u}^L\|_{\mathbf{V}}$ versus the mesh size h . We choose $\Delta t = h_L^{1/2}/2$ or the closest value so that $1/\Delta t$ is an integer. In particular, we plot for $(h_L, \Delta t) = (1/4, 1/4), (1/8, 1/6), (1/16, 1/8), (1/32, 1/12)$ and $(1/64, 1/16)$. From Figure 2.3, we observe that the full and sparse tensor product FE errors are essentially similar.

2.9.3 One-dimensional three-scale problem

We then consider a one-dimensional three-scale problem where $A(x, y_1, y_2) = (4/9)(1+x)(1 + \cos^2 2\pi y_1)(1 + \cos^2 2\pi y_2)$. The problem is

$$\begin{aligned} \frac{\partial^2 u^\varepsilon}{\partial t^2} - \frac{\partial}{\partial x} \left(\frac{4}{9}(1+x) \left(1 + \cos^2 \left(2\pi \frac{x}{\varepsilon_1} \right) \right) \left(1 + \cos^2 \left(2\pi \frac{x}{\varepsilon_2} \right) \right) \frac{\partial u^\varepsilon}{\partial x} \right) \\ = t^3 + \frac{27t}{4} \left(x - \frac{\log(1+x)}{\log 2} \right) \quad \text{in } D, \end{aligned}$$

Figure 2.3: The full and sparse tensor errors $\|\mathbf{u} - \mathbf{u}^L\|_{\mathbf{V}}$ for 1D two-scale problem

$$u^\varepsilon(t, 0) = u^\varepsilon(t, 1) = 0.$$

The three-scale limiting homogenized problem has the exact homogenized solution

$$u^0(t, x) = \frac{9}{8} \left(x - \frac{\log(1+x)}{\log 2} \right) t^3,$$

and the scale interacting term $u^1(t, x, y)$, and $u^2(t, x, y_1, y_2)$ are determined with additive constants c_1 and c_2

$$u^1(t, x, y_1) = \frac{9}{8} \left(1 - \frac{1}{(1+x) \log 2} \right) \left(\frac{1}{2\pi} \tan^{-1} \left(\frac{\tan 2\pi y_1}{\sqrt{2}} \right) - y_1 + c_1 \right) t^3,$$

$$u^2(t, x, y_1, y_2) = \frac{9}{8} \left(1 - \frac{1}{(1+x) \log 2} \right) \left(\frac{1}{2\pi} \tan^{-1} \left(\frac{\tan 2\pi y_2}{\sqrt{2}} \right) - y_2 + c_2 \right) \cdot \frac{\sqrt{2}}{1 + \cos^2 2\pi y_1} t^3.$$

In Figure 2.4 the error $\|\mathbf{u} - \mathbf{u}^L\|_{\mathbf{V}}$ is plotted versus the mesh size for the sparse tensor product FEs. Mesh sizes h and time steps Δt are the same as in the two-scale problem above. The result confirms our analysis.

2.9.4 Two-dimensional two-scale problem

Finally we consider in $D = (0, 1) \times (0, 1)$ the two-scale problem

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} - \nabla_x \cdot (A^\varepsilon(x) \nabla_x u^\varepsilon) = f(t, x) \quad \text{in } D,$$

Figure 2.4: The sparse tensor error $\|\mathbf{u} - \mathbf{u}^L\|_{\mathbf{V}}$ for 1D three-scale problem

$$u^\varepsilon(t, \cdot) = 0 \quad \text{on } \partial D.$$

where

$$A^\varepsilon(x) = (1 + x')(1 + x'') \left(1 + \cos^2 \left(2\pi \frac{x'}{\varepsilon} \right) \right) \left(1 + \cos^2 \left(2\pi \frac{x''}{\varepsilon} \right) \right),$$

$$f(t, x) = 6t(x' - x'^2)(x'' - x''^2) - \nabla_x \cdot \left(\frac{3}{\sqrt{2}} (1 + x')(1 + x'') \nabla_x ((x' - x'^2)(x'' - x''^2)) \right) t^3,$$

where $x = (x', x'')$. Following Jikov et al. [37] page 17, the homogenized coefficient can be computed exactly. This problem has the exact homogenized solution

$$u^0(t, x) = (x' - x'^2)(x'' - x''^2)t^3,$$

and the scale interacting terms

$$\begin{aligned} u^1(t, x, y) &= (1 - 2x')(x'' - x''^2) \left(\frac{1}{2\pi} \tan^{-1} \left(\frac{\tan 2\pi y'}{\sqrt{2}} \right) - y' + c_1 \right) t^3 \\ &+ (x' - x'^2)(1 - 2x'') \left(\frac{1}{2\pi} \tan^{-1} \left(\frac{\tan 2\pi y''}{\sqrt{2}} \right) - y'' + c_2 \right) t^3. \end{aligned}$$

In Figure 2.5 the error $\|\mathbf{u} - \mathbf{u}^L\|_{\mathbf{V}}$ is plotted versus the mesh size for the sparse tensor product for $(\Delta t, h) = (1/4, 1/4), (1/8, 1/6)$, and $(1/16, 1/8)$. The numerical result once more confirms our analysis.

For this problem, we plot in the following figures the finite element approximation for ∇u^ε in Theorem 2.8.5

$$\nabla_x \hat{u}^{0L} + \mathcal{U}^\varepsilon (\nabla_y \hat{u}^{1L}),$$

and compare it to the approximation

$$\nabla_x u^0(x) + \nabla_y u^1(x, \frac{x}{\varepsilon}),$$

Figure 2.5: The sparse tensor error $\|\mathbf{u} - \mathbf{u}^L\|_{\mathbf{V}}$ for 2D two-scale problem

in Proposition 2.7.1.

In Figure 2.6 and Figure 2.7, the graph on the left is the analytical approximation $\frac{\partial u^0}{\partial x_i}(x) + \frac{\partial u^1}{\partial y_i}(x, \frac{x}{\varepsilon})$ and the graph the right is its numerical approximation $\frac{\partial \hat{u}^{0L}}{\partial x_i}(x) + \mathcal{U}^\varepsilon \left(\frac{\partial \hat{u}^{1L}}{\partial y_i} \right)(x)$. We choose $h = 1/8$ and $\varepsilon = 1/64$ to achieve the optimal convergence rate by $\varepsilon = h^2$. We see that the two functions are similar even with a rather large mesh size $h = 1/8$.

2.10 Choice of $\hat{\mathbf{u}}^{L;0}$ and $\hat{\mathbf{u}}^{L;1}$

We make some remarks on the choice of $\hat{\mathbf{u}}^{L;0}$ and $\hat{\mathbf{u}}^{L;1}$ so that the conditions of Proposition 2.6.6 are satisfied. We assume conditions (2.39) and that the domain is convex. We find $\hat{u}^{1L} \in \hat{V}_1^L, \dots, \hat{u}^{nL} \in \hat{V}_n^L$ such that

$$\int_D \int_{\mathbf{Y}} A(x, \mathbf{y}) (\nabla g^0 + \nabla_{y_1} \hat{u}^{1L} + \dots + \nabla_{y_n} \hat{u}^{nL}) \cdot (\nabla_{y_1} \hat{v}^{1L} + \dots + \nabla_{y_n} \hat{v}^{nL}) dx d\mathbf{y} = 0,$$

for all $\hat{v}^{1L} \in \hat{V}_1^L, \dots, \hat{v}^{nL} \in \hat{V}_n^L$. We have

$$\|u^1(0) - \hat{u}^{1L}\|_{V_1} + \dots + \|u^n(0) - \hat{u}^{nL}\|_{V_n} \leq ch_L L^{n/2}.$$

We find $\hat{u}^{0L} \in V^L$ by projection such that $\|\hat{u}^{0L} - u^0\|_V \leq ch_L$ and set $\hat{\mathbf{u}}^{L;0} = (\hat{u}^{0L}, \hat{u}^{1L}, \dots, \hat{u}^{nL})$. The function $\hat{u}_t^{1L} \in \hat{V}_1^L, \dots, \hat{u}_t^{nL} \in \hat{V}_n^L$ are found by solving

$$\int_D \int_{\mathbf{Y}} A(x, \mathbf{y}) (\nabla g^1 + \nabla_{y_1} \hat{u}_t^{1L} + \dots + \nabla_{y_n} \hat{u}_t^{nL}) \cdot (\nabla_{y_1} \hat{v}^{1L} + \dots + \nabla_{y_n} \hat{v}^{nL}) dx d\mathbf{y} = 0$$

for all $\hat{v}^{1L} \in \hat{V}_1^L, \dots, \hat{v}^{nL} \in \hat{V}_n^L$. From (2.39), $\frac{\partial u^0}{\partial t}(0) \in H^2(D)$ so $\frac{\partial u^p}{\partial t} \in \hat{\mathcal{H}}_p$. We then have

$$\|\hat{u}_t^{1L} - \frac{\partial u^1}{\partial t}(0)\|_{V_1} + \dots + \|\hat{u}_t^{nL} - \frac{\partial u^n}{\partial t}(0)\|_{V_n} \leq ch_L L^{n/2}.$$

Figure 2.6: The analytical and numerical approximation to $\frac{\partial u^\varepsilon}{\partial x_1}$

Figure 2.7: The analytical and numerical approximation to $\frac{\partial u^\varepsilon}{\partial x_2}$

Letting $g^{1L} \in V^L$ be such that $\|g^{1L} - g^1\|_V \leq ch_L$, we set $\hat{u}_t^{0L} = g^{1L}$, $\hat{\mathbf{u}}_t^{L;0} = (\hat{u}_t^{0L}, \hat{u}_t^{1L}, \dots, \hat{u}_t^{nL})$. We find $\hat{u}_{tt}^{0L} \in V^L$ such that $\|\hat{u}_{tt}^{0L} - \frac{\partial^2 u^0}{\partial t^2}\|_H \leq ch_L$ and $\|\hat{u}_{tt}^{0L}\|_V \leq \|\frac{\partial^2 u^0}{\partial t^2}\|_V$ (condition (2.39) implies that $\frac{\partial^2 u^0}{\partial t^2} \in L^\infty(0, T; V)$). We choose $\hat{\mathbf{u}}^{L;1}$ as

$$\hat{\mathbf{u}}^{L;1} = \hat{\mathbf{u}}^{L;0} + \Delta t \hat{\mathbf{u}}_t^{L;0} + \frac{1}{2}(\Delta t)^2 \hat{\mathbf{u}}_{tt}^{L;0},$$

where $\hat{\mathbf{u}}_{tt}^{L;0} = (\hat{u}_{tt}^{0L}, 0, \dots, 0)$. We have

$$\begin{aligned} \partial_t \zeta^{0L;1/2} &= \frac{1}{\Delta t}(\hat{u}^{0L;1} - \hat{u}^{0L;0} - (w^{0L;1} - w^{0L;0})) \\ &= \hat{u}_t^{L;0} + \frac{1}{2}\Delta t \hat{u}_{tt}^{L;0} - \frac{\partial w^{L;0}}{\partial t}(0) - \frac{1}{2}\Delta t \frac{\partial^2 w^{L;0}}{\partial t^2}(0) + X, \end{aligned}$$

where $\|X\|_H \leq c(\Delta t)^2$. We note that

$$\|\hat{u}_t^{L;0} - \frac{\partial w^{L;0}}{\partial t}(0)\|_H \leq \|\hat{u}_t^{L;0} - \frac{\partial u^0}{\partial t}(0)\|_H + \|\frac{\partial u^0}{\partial t}(0) - \frac{\partial w^{L;0}}{\partial t}(0)\|_H \leq ch_L L^{n/2},$$

due to Lemma 2.4.3. Similarly, we have

$$\|\hat{u}_{tt}^{L;0} - \frac{\partial^2 w^{L;0}}{\partial t^2}(0)\|_H \leq \|\hat{u}_{tt}^{L;0} - \frac{\partial^2 u^0}{\partial t^2}(0)\|_H + \|\frac{\partial^2 u^0}{\partial t^2}(0) - \frac{\partial^2 w^{L;0}}{\partial t^2}(0)\|_H \leq ch_L L^{n/2}.$$

Therefore $\|\partial_t \zeta^{0L;1/2}\|_H \leq c((\Delta t)^2 + h_L L^{n/2})$. Further

$$\begin{aligned} \|\zeta^{L;1/2}\|_{\mathbf{V}} &\leq \frac{1}{2}(\|\zeta^{L;0}\|_{\mathbf{V}} + \|\zeta^{L;1}\|_{\mathbf{V}}) \\ &\leq \frac{1}{2}\|\hat{\mathbf{u}}^{L;0} - \mathbf{u}^0\|_{\mathbf{V}} + \frac{1}{2}\|\mathbf{u}^0 - \mathbf{w}^{L;0}\|_{\mathbf{V}} + \frac{1}{2}\left\|\hat{\mathbf{u}}^{L;0} + \Delta t \hat{\mathbf{u}}_t^{L;0} - \mathbf{u}^0 - \Delta t \frac{\partial \mathbf{u}}{\partial t}(0)\right\|_{\mathbf{V}} + \\ &\quad \frac{1}{2}\left\|\mathbf{u}^0 + \Delta t \frac{\partial \mathbf{u}}{\partial t}(0) - \mathbf{w}^{L;0} - \Delta t \frac{\partial \mathbf{w}^L}{\partial t}(0)\right\|_{\mathbf{V}} + c(\Delta t)^2 \\ &\leq c((\Delta t)^2 + h_L L^{n/2}). \end{aligned}$$

If only (2.37) holds, we only require $O(\Delta t) + h_L L^{n/2}$ error, and therefore do not need to consider $\frac{\partial^2 u^0}{\partial t^2}(0)$. We note that

$$\begin{aligned} \frac{\partial^2 u^0}{\partial t^2}(0) &= f(0) - \nabla \cdot \int_Y A(x, y)(\nabla u^0(0) + \nabla_{y_1} u^1(0) + \dots + \nabla_{y_n} u^n(0)) d\mathbf{y} \\ &= f(0) - \int_Y \nabla A(x, y)(\nabla u^0(0) + \nabla_{y_1} u^1(0) + \dots + \nabla_{y_n} u^n(0)) d\mathbf{y} \\ &\quad - \int_Y A(x, y) \nabla((\nabla u^0(0) + \nabla_{y_1} u^1(0) + \dots + \nabla_{y_n} u^n(0))) d\mathbf{y}. \end{aligned}$$

The spatial derivative of $\frac{\partial^2 u^0}{\partial t^2}(0)$ can be computed as follows. We note that for almost all $x \in D$, for all $\phi^1 \in V_1, \dots, \phi^n \in V_n$, we have

$$\int_Y A(x, \mathbf{y})(\nabla u^0(0) + \nabla_{y_1} u^1(0) + \dots + \nabla_{y_n} u^n(0)) \cdot (\nabla_{y_1} \phi^1 + \dots + \nabla_{y_n} \phi^n) d\mathbf{y} = 0.$$

Therefore

$$\begin{aligned} & \int_{\mathbf{Y}} A(x, \mathbf{y}) \frac{\partial}{\partial x_k} (\nabla u^0(0) + \nabla_{y_1} u^1(0) + \dots + \nabla_{y_n} u^n(0)) \cdot (\nabla_{y_1} \phi^1 + \dots + \nabla_{y_n} \phi^n) d\mathbf{y} \\ &= - \int_{\mathbf{Y}} \frac{\partial A(x, \mathbf{y})}{\partial x_k} (\nabla u^0(0) + \nabla_{y_1} u^1(0) + \dots + \nabla_{y_n} u^n(0)) \cdot (\nabla_{y_1} \phi^1 + \dots + \nabla_{y_n} \phi^n) d\mathbf{y}. \end{aligned}$$

Thus for all $\phi^1 \in V_1, \dots, \phi^n \in V_n$

$$\begin{aligned} & \int_D \int_{\mathbf{Y}} A(x, \mathbf{y}) \left(\nabla \frac{\partial}{\partial x_k} u^0(0) + \nabla_{y_1} \frac{\partial}{\partial x_k} u^1(0) + \dots + \nabla_{y_n} \frac{\partial}{\partial x_k} u^n(0) \right) \cdot \\ & \quad (\nabla_{y_1} \phi^1 + \dots + \nabla_{y_n} \phi^n) d\mathbf{y} dx \\ &= - \int_D \int_{\mathbf{Y}} \frac{\partial A(x, \mathbf{y})}{\partial x_k} (\nabla u^0(0) + \nabla_{y_1} u^1(0) + \dots + \nabla_{y_n} u^n(0)) \cdot \\ & \quad (\nabla_{y_1} \phi^1 + \dots + \nabla_{y_n} \phi^n) d\mathbf{y} dx. \end{aligned}$$

Once the approximation $\hat{\mathbf{u}}^{L;0}$ has been computed, we can then compute an approximation for $\frac{\partial \mathbf{u}}{\partial x_k}(0)$ in $\mathbf{V}$. Other spatial derivatives of $\mathbf{u}^0$ can be computed similarly. From this, we can approximate $\frac{\partial^2 u^0}{\partial t^2}$ and find $\hat{u}_{tt}^{L;0}$.

If $g^0 = 0$, the problem becomes much easier as $\mathbf{u}^0 = 0$. Note that if $g^0 \neq 0$, in order for condition (2.39c) to hold, $g^0 \in H_0^3(D)$ and for (2.39e) to hold, $A^0(x)$ needs to be in $W^{2,\infty}(D)$, i.e. $A(x, \mathbf{y})$ needs to be in $W^{2,\infty}$ with respect to x .

Multiscale Elasticity Equation

3.1 Introduction

In this chapter, we consider multiscale elasticity equations which are locally periodic with respect to each microscopic scale, by using the high dimensional approach originally developed by Hoang and Schwab [31]. The method solves the multiscale homogenized equation that is obtained from multiscale convergence to find the solution to the homogenized equation and the corrector terms at the same time. By using sparse finite element discretization with mesh size h , when the solution is sufficiently smooth, we achieve an accuracy $O(h)$ for the solution. The computational complexity of the method in terms of the number of degrees of freedom grows as $O(h^{-d})$ in d -dimensions (with the possibility of a logarithmic multiplying factor), which is comparable to that for solving a single elliptic PDE in $\mathbb{R}^d$. We develop in this chapter the theory for multiscale linear elasticity equations whose solutions are vector functions and illustrate the results by solving numerically two and three scale problems in two dimensions. For related numerical analysis and computation research for two-scale linear elasticity, we refer to Abdulle [1] for the analysis of the Heterogeneous Multiscale Method and the references therein.

The chapter is organized as follows. In Section 3.2, we present the setting up of the multiscale elasticity equation. Section 3.3 applies the multiscale convergence theory to the linear elasticity equation and derives the high dimensional limiting equation which provides the solution to the homogenized problem, which represents the multiscale solution macroscopically, and the scale interaction terms which encode the microscale information of the solution. We consider finite element discretization of the high dimensional multiscale homogenized problem in Section 3.4 where we employ the full tensor and sparse tensor product finite element approximations. We deduce in this section explicit finite element error estimates when the solution possesses sufficient regularity. Unlike scalar elliptic problems, solution of an elasticity equation with smooth coefficient in a convex polygon $D \in \mathbb{R}^d$ may not belong to $H^2(D)^d$. To obtain sufficient regularity for the FE error, we consider a smooth domain D for which we discretize

a polygon that is inscribed inside it. However, this only affects the solution u^0 of the homogenized equation which only depends on $x \in \mathbb{R}^d$, and only requires standard FE approximation. For the scale interacting terms u^p ($p = 1, \dots, n$) for which we need tensor product FE spaces to approximate, we only require an approximation in the $L^2(D)$ norm with respect to x . This can be achieved by using a hierarchy of FE spaces in a polygon that contains the smooth domain D , and use the extension properties of D to get an explicit error. However, when D is a convex polygon in two dimensions, if the homogenized problem is isotropic and homogeneous (i.e. the multiscale coefficient $a(x, y_1, \dots, y_n)$ does not depend on y), then u^0 is in $H^2(D)^d$ and our theory applies (see Remark 3.5.2). The required regularity for the error of sparse tensor product FE approximations to hold is shown in Section 3.5.

In Section 3.6, we construct numerical approximations and correctors for the solution to the multiscale problem. To achieve an explicit error estimate, we need an analytic homogenization rate of convergence. This is deduced for the two-scale case in Section 3.6.1. The rate of convergence deduced in Oleinik et al. [47] requires the regularity $u^0 \in H^3(D)^d$. Adapting the proof of Jikov et al. [37] for elliptic problems, we deduce the same rate of convergence requiring only $u^0 \in H^2(D)^d$, employing some technical results developed by Hoang and Schwab [34]. Therefore for the two-scale problem, we deduce an approximation for the multiscale solution u^ε in terms of the finite element solution of the multiscale homogenized problem with an error in terms of the FE mesh size and the microscale. For more than two-scale problems, we are not aware of any homogenization error estimates. Nevertheless an approximation for the solution to the multiscale problem (without an explicit error) can be found. However, the derivation is more complicated than for the elliptic problem in Hoang and Schwab [31] as the elasticity equation only gives an approximation for the elastic strain tensor. To get an approximation for the solution in the $H^1(D)$ norm, extra manipulation of the corrector operator $\mathcal{U}_n^\varepsilon$ (defined in (2.74)) is needed in order to use Korn's inequality. This is done in Section 3.6.3. In Section 3.7, we solve two and three scale problems in two dimensions to illustrate our theory. We consider Hill's materials which are isotropic, whose shear modulus μ is not oscillating. In this case the homogenized coefficients can be computed explicitly without solving the cell problems. Jikov et al. [37] present this for two-scale problems. The result is not immediately generalized to three-scale problems. We show this for three-scale problems in Section 3.8.

Throughout the chapter, we follow the notations of Chapter 2. For the space of symmetric matrices $\mathbb{R}_{sym}^{d \times d}$, $\cdot$ denotes the inner product. By ε without indicating the variable, we denote the strain tensor with respect to x .

3.2 Problem setting

Let a_{ijkl} be a fourth order tensor value function from the product space $D \times \mathbf{Y}$ to the space $\mathbb{R}^{d^4}$ which is periodic with respect to y_p with the period Y_p .

Assumption 3.2.1. *The fourth order tensor a_{ijkl} satisfies the symmetry, boundedness*

and coerciveness condition such that for all i, j, k, l ,

$$a_{ijkl} = a_{klij} = a_{jikl} = a_{ijlk} \quad (3.1)$$

and there exist positive constants α and β so that for all $(x, \mathbf{y}) \in D \times \mathbf{Y}$,

$$|a_{ijkl}(x, \mathbf{y}) \xi_{kl} \zeta_{ij}| \leq \beta |\xi| |\zeta|, \quad (3.2)$$

and

$$\alpha |\xi|^2 \leq a_{ijkl}(x, \mathbf{y}) \xi_{kl} \xi_{ij}, \quad (3.3)$$

for all symmetric tensors ξ_{ij}, ζ_{ij} in $\mathbb{R}_{sym}^{d^2}$.

We assume that $a \in C^1(\bar{D}, C_{\#}^1(\bar{Y}_1, \dots, C_{\#}^1(\bar{Y}_n) \dots))$. With ε_p defined in (2.4), the multiscale stiffness tensor $a_{ijkl}^{\varepsilon} : D \rightarrow \mathbb{R}^{d^4}$ is defined as

$$a_{ijkl}^{\varepsilon}(x) = a_{ijkl}\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}\right).$$

Let $f = (f_1(x), \dots, f_d(x)) \in H^{-1}(D)^d$. As in Chapter 2, we denote by V the space $H_0^1(D)$ and by H the space $L^2(D)$. We consider the multiscale elasticity equation in D with the Dirichlet boundary condition: Find $u^{\varepsilon}(x, t) \in V^d$ such that

$$-\frac{\partial}{\partial x_j}(a_{ijkl}^{\varepsilon}(x) \epsilon_{kl}(u^{\varepsilon})) = f_i, \quad (3.4)$$

where for each function $v \in V^d$, we define ϵ , the strain tensor, by

$$\epsilon_{ij}(v) = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

By Korn's inequality, there exists c that only depends on D so that

$$\|v\|_{V^d} \leq c \|\epsilon(v)\|_{H^{d \times d}}. \quad (3.5)$$

In variational form, this problem is written as

$$\int_D a_{ijkl}^{\varepsilon}(x) \epsilon_{kl}(u^{\varepsilon}(x)) : \epsilon_{ij}(\phi(x)) dx = \int_D f_i(x) \phi_i(x) dx, \quad \forall \phi \in V^d. \quad (3.6)$$

Proposition 3.2.2. *Assume that the multiscale stiffness tensor satisfies the boundedness and coerciveness condition in (3.2) and (3.3) and $f \in (V')^d$, problem (3.6) possesses a unique solution with the following bound*

$$\|u^{\varepsilon}\|_{V^d} \leq c \|f\|_{H^{-1}(D)^d} \quad (3.7)$$

for a constant c that only depends on the domain D .

Proof. Lax-Milgram lemma together with (3.2) and (3.3) imply that (3.6) possesses a unique solution. From Korn's inequality (3.5), we deduce that

$$\|u^{\varepsilon}\|_{V^d} \leq c_1 \|\epsilon(u^{\varepsilon})\|_{H^{d \times d}} \leq c_2 \|f\|_{H^{-1}(D)^d}$$

for a constant c_2 that only depends on the domain D , because the constants α and β in (3.2) and (3.3) do not depend on ε . $\square$

3.3 Multiscale homogenized equation

In this section, we analyze the behaviour of u^ε when $\varepsilon \rightarrow 0$ via multiscale convergence method in Nguetseng [44], Allaire [3] and Allaire and Briane [4].

3.3.1 Multiscale convergence

The multiscale convergence of a sequence is defined as follows.

Definition 3.3.1. *A sequence of functions $\{w^\varepsilon\}_\varepsilon \subset H$ is said to $(n+1)$ -scale converge to a function $w^0 \in L^2(D \times \mathbf{Y})$ if for all smooth functions $\phi(x, \mathbf{y})$ which are Y_p periodic with respect to y_p :*

$$\lim_{\varepsilon \rightarrow 0} \int_D \int_{\mathbf{Y}} w^\varepsilon(x) \phi(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}) d\mathbf{y} dx = \int_D \int_{\mathbf{Y}} w^0(x, \mathbf{y}) \phi(x, \mathbf{y}) d\mathbf{y} dx.$$

This definition makes sense due to the following result.

Proposition 3.3.2. *From a bounded sequence $\{w^\varepsilon\}_\varepsilon$ in H , there is a subsequence that $(n+1)$ -scale converges to a function $w^0 \in L^2(D \times \mathbf{Y})$.*

For a bounded sequence $\{w^\varepsilon\}_\varepsilon \subset V$ we have the result:

Proposition 3.3.3. *Let $\{w^\varepsilon\}_\varepsilon$ be a bounded sequence in V . There is a subsequence (not renumbered), a function $w^0 \in V$ and n functions $w^p \in L^2(D \times Y_1 \times \dots \times Y_{p-1}, H^1_\#(Y_p)/\mathbb{R})$ ($p = 1, \dots, n$) such that $w^\varepsilon \rightharpoonup w^0$ in $H^1_0(D)$ and*

$$\nabla w^\varepsilon \xrightarrow{(n+1)\text{-scale}} \nabla w^0 + \nabla_{y_1} w^1 + \dots + \nabla_{y_n} w^n.$$

The proofs of Propositions 3.3.2 and 3.3.3 can be found in Allaire and Briane [4].

3.3.2 Multiscale limit of the elasticity equation

We study in this subsection the multiscale convergence of the solution to equation (3.6). With V_p as in Section 2.3.2, we define

$$\mathbf{V}^d := V^d \times V_1^d \times \dots \times V_n^d,$$

which is equipped with the norm

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{V}^d} &= \|u^0\|_{V^d} + \sum_{p=1}^n \|u^p\|_{V_p^d} \\ &= \sum_{i=1}^d \left(\|\nabla_x u_i^0\|_{H^d} + \sum_{p=1}^n \|\nabla_{y_p} u_i^p\|_{L^2(D \times Y_1 \times \dots \times Y_p)^d} \right) \end{aligned}$$

for $\mathbf{u} = (u^0, u^1, \dots, u^n) \in \mathbf{V}^d$. For $\mathbf{v} = (v_0, v_1, \dots, v_n) \in \mathbf{V}^d$, we denote by

$$\epsilon_{ij}(\mathbf{v}) = \frac{1}{2} \left[\left(\frac{\partial v_i^0}{\partial x_j} + \frac{\partial v_i^1}{\partial y_{1,j}} + \dots + \frac{\partial v_i^n}{\partial y_{n,j}} \right) + \left(\frac{\partial v_j^0}{\partial x_i} + \frac{\partial v_j^1}{\partial y_{1,i}} + \dots + \frac{\partial v_j^n}{\partial y_{n,i}} \right) \right].$$

We then have the Korn's inequality for $\mathbf{v} \in \mathbf{V}^d$.

Proposition 3.3.4. *For $\mathbf{v} \in \mathbf{V}^d$, there exists c such that*

$$\|\epsilon(\mathbf{v})\|_{L^2(D \times Y_1 \times \dots \times Y_n)^{d \times d}} \geq c \|\mathbf{v}\|_{\mathbf{V}^d}. \quad (3.8)$$

Proof We have that

$$\|\epsilon(\mathbf{v})\|_{L^2(D \times Y_1 \times \dots \times Y_n)^{d \times d}}^2 = \|\epsilon(v_0)\|_{H^{d \times d}}^2 + \sum_{m=1}^n \|\epsilon_{y_m}(v_m)\|_{L^2(D \times Y_1 \times \dots \times Y_m)^{d \times d}}^2.$$

Inequality (3.8) then follows from Korn's inequality for functions in V^d and for periodic functions (see, for example, [47]). $\square$

We define the bilinear form $B : \mathbf{V}^d \times \mathbf{V}^d \rightarrow \mathbb{R}$ by

$$B(\mathbf{u}, \mathbf{v}) = \int_D \int_{\mathbf{Y}} a_{ijkl}(x, \mathbf{y}) \epsilon_{kl}(\mathbf{u}) \epsilon_{ij}(\mathbf{v}) d\mathbf{y} dx$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}^d$. We then have the following result.

Theorem 3.3.5. *There exists $\mathbf{u} = (u^0, u^1, \dots, u^n) \in \mathbf{V}^d$ such that the solution u^ε of (3.6) converges weakly to u^0 in $\mathbf{V}^d$ and for all $i = 1, \dots, d$, ∇u_i^ε ($n+1$)-scale converges to $\nabla u_i^0 + \nabla_{y_1} u_i^1 + \dots + \nabla_{y_n} u_i^n$; $\mathbf{u}$ is the unique solution of the following problem*

$$B(\mathbf{u}, \mathbf{v}) = \int_D f \cdot v^0 dx, \quad (3.9)$$

for all $\mathbf{v} \in \mathbf{V}^d$.

Proof We deduce from (3.7) that u^ε is bounded in V^d so that there exists $u^0 \in V^d$ such that $u^\varepsilon \rightharpoonup u^0$ in V^d . By Proposition 3.3.3 there are $u^p \in V_p^d$ ($p = 1, \dots, n$) such that for each $i = 1, \dots, d$,

$$\nabla_x u_i^\varepsilon \xrightarrow{(n+1)\text{-scale}} \nabla u_i^0 + \nabla_{y_1} u_i^1 + \dots + \nabla_{y_n} u_i^n,$$

so

$$\epsilon_{ij}(u^\varepsilon) \xrightarrow{(n+1)\text{-scale}} \epsilon_{ij}(\mathbf{u}).$$

Let $\phi^0 \in (C_0^\infty(D))^d$; and for $p = 1, \dots, n$, let $\phi^p \in (C_0^\infty(D \times Y_1 \times \dots \times Y_{p-1}, C_\#^\infty(Y_p)))^d$. Let the test function $\phi \in (C_0^\infty(D))^d$ in (3.6) be such that for $i = 1, \dots, d$

$$\phi_i(x) = \phi_i^0(x) + \sum_{p=1}^n \varepsilon_p \phi_i^p\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p}\right).$$

We then obtain from (3.7) that,

$$\begin{aligned}
& \frac{1}{2} \int_D a_{ijkl}^\varepsilon(x) \epsilon_{kl}(u^\varepsilon) \left[\frac{\partial \phi_i^0}{\partial x_j} + \frac{\partial \phi_j^0}{\partial x_i} \right. \\
& + \sum_{p=1}^n \left(\varepsilon_p \frac{\partial \phi_i^p}{\partial x_j} \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p} \right) + \sum_{q=1}^p \frac{\varepsilon_p}{\varepsilon_q} \frac{\partial \phi_i^p}{\partial y_{q,j}} \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p} \right) \right) \\
& \left. + \sum_{p=1}^n \left(\varepsilon_p \frac{\partial \phi_j^p}{\partial x_i} \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p} \right) + \sum_{q=1}^p \frac{\varepsilon_p}{\varepsilon_q} \frac{\partial \phi_j^p}{\partial y_{q,i}} \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p} \right) \right) \right] dx \\
& = \int_D f_i(t, x) \left(\phi_i^0(x) + \sum_{p=1}^n \varepsilon_p \phi_i^p \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p} \right) \right) dx.
\end{aligned}$$

From

$$\lim_{\varepsilon \rightarrow 0} \sum_{q=1}^p \frac{\varepsilon_p}{\varepsilon_q} \frac{\partial \phi_i^p}{\partial y_{q,j}} \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p} \right) = \frac{\partial \phi_i^p}{\partial y_{p,j}} \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p} \right)$$

and

$$\lim_{\varepsilon \rightarrow 0} \phi_i^0(x) + \sum_{p=1}^n \varepsilon_p \phi_i^p \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p} \right) = \phi_i^0(x).$$

We have

$$B(\mathbf{u}, \phi) = \int_D f \cdot \phi^0 dx.$$

By using a density argument, we deduce problem (3.9). By (3.2), (3.3) and Korn's inequality (3.8), we deduce from Lax-Milgram lemma that the solution of (3.9) is unique. $\square$

The solution u^ε of the multiscale problem (3.4) can be approximated by the solution $\mathbf{u}$ of problem (3.9). For example, in the case of two scales ($n = 1$) we have the following proposition.

Proposition 3.3.6. *For $n = 1$, assuming that u^0 and u^1 are smooth, we have*

$$\lim_{\varepsilon \rightarrow 0} \left\| \nabla u^\varepsilon(\cdot) - (\nabla u^0(\cdot) + \nabla_y u^1(\cdot, \frac{\cdot}{\varepsilon})) \right\|_{L^2(D)^{d \times d}} = 0.$$

We will study in more details correctors for homogenization of the problem (3.4) in Section 3.6.

3.4 Finite element discretization

Following definition of $\bar{V}_p^L$ and $\hat{V}_p^L$ in Section 2.6, we construct the full and sparse tensor product finite element spaces as

$$(\bar{\mathbf{V}}^L)^d = \{(\bar{u}^{0L}, \{\bar{u}^{pL}\}) : \bar{u}^{0L} \in (V_0^L)^d, \bar{u}^{pL} \in (\bar{V}_p^L)^d\},$$

and

$$(\hat{\mathbf{V}}^L)^d = \{(\hat{u}^{0L}, \{\hat{u}^{pL}\}) : \hat{u}^{0L} \in (V_0^L)^d, \hat{u}^{pL} \in (\hat{V}_p^L)^d\}.$$

For $\bar{\mathcal{H}}_p$ and $\hat{\mathcal{H}}_p$ from Section 2.5, we define by $\bar{\mathcal{H}}^d$ and $\hat{\mathcal{H}}^d$ the spaces

$$\bar{\mathcal{H}}^d = \{\mathbf{w} = (w^0, w^1, \dots, w^n) \in \mathbf{V}^d : w^0 \in H^2(D)^d, w^p \in \bar{\mathcal{H}}_p^d, p = 1, \dots, n\}, \quad (3.10)$$

and

$$\hat{\mathcal{H}}^d = \{\mathbf{w} = (w^0, w^1, \dots, w^n) \in \mathbf{V}^d : w^0 \in H^2(D)^d, w^p \in \hat{\mathcal{H}}_p^d, p = 1, \dots, n\}. \quad (3.11)$$

3.4.1 Full tensor product finite elements

We consider the full tensor product approximating problem: Find $\bar{\mathbf{u}}^L = (\bar{u}^{0L}, \bar{u}^{1L}, \dots, \bar{u}^{nL}) \in (\bar{\mathbf{V}}^L)^d$ such that

$$B(\bar{\mathbf{u}}^L, \bar{\mathbf{v}}^L) = \int_D f \cdot \bar{v}^{0L} dx \quad \forall \bar{\mathbf{v}}^L = (\bar{v}^{0L}, \bar{v}^{1L}, \dots, \bar{v}^{nL}) \in (\bar{\mathbf{V}}^L)^d. \quad (3.12)$$

We then have the following rate of convergence for the full tensor approximating problem (3.12).

Proposition 3.4.1. *Assume that the solution $\mathbf{u}$ of problem (3.9) belongs to $\bar{\mathcal{H}}^d$, then for the FE approximating problem (3.12), we have*

$$\|\mathbf{u} - \bar{\mathbf{u}}^L\|_{\mathbf{V}^d} \leq ch_L \|\mathbf{u}\|_{\bar{\mathcal{H}}^d}.$$

Proof Using the argument of the Cea's lemma and Korn's inequality (3.8), we get $c_1 \|\mathbf{u} - \bar{\mathbf{u}}^L\|_{\mathbf{V}^d}^2 \leq B(\mathbf{u} - \bar{\mathbf{u}}^L, \mathbf{u} - \bar{\mathbf{u}}^L) = B(\mathbf{u} - \bar{\mathbf{u}}^L, \mathbf{u} - \bar{\mathbf{v}}^L) \leq c_2 \|\mathbf{u} - \bar{\mathbf{u}}^L\|_{\mathbf{V}^d} \|\mathbf{u} - \bar{\mathbf{v}}^L\|_{\mathbf{V}^d}$, thus

$$\|\mathbf{u} - \bar{\mathbf{u}}^L\|_{\mathbf{V}^d} \leq c \inf_{\bar{\mathbf{v}}^L \in (\bar{\mathbf{V}}^L)^d} \|\mathbf{u} - \bar{\mathbf{v}}^L\|_{\mathbf{V}^d}.$$

We then get the conclusion from Lemma 2.6.1. $\square$

3.4.2 Sparse tensor product finite elements

Following notations for sparse tensor product spaces in Section 2.6, we then consider the sparse tensor product FE approximating problem: Find $\hat{\mathbf{u}}^L \in (\hat{\mathbf{V}}^L)^d$ such that

$$B(\hat{\mathbf{u}}^L, \hat{\mathbf{v}}^L) = \int_D f(x) \cdot \hat{v}^{0L}(x) dx \quad (3.13)$$

for all $\hat{\mathbf{v}}^L = (\hat{v}^{0L}, \hat{v}^{1L}, \dots, \hat{v}^{nL}) \in (\hat{\mathbf{V}}^L)^d$.

Proposition 3.4.2. *Assume that the solution $\mathbf{u}$ of (3.9) belongs to $\hat{\mathcal{H}}^d$. Then the solution $\hat{\mathbf{u}}^L$ of the finite element approximating problem (3.13) satisfies*

$$\|\mathbf{u} - \hat{\mathbf{u}}^L\|_{\mathbf{V}^d} \leq cL^{n/2} h_L \|\mathbf{u}\|_{\hat{\mathcal{H}}^d}. \quad (3.14)$$

The proof of this proposition uses Cea's lemma and Lemma 2.6.4.

3.4.3 Sparse tensor finite element for non-polygonal domains

In polygonal domains, the solution u^0 of the homogenized problem (3.22) in general does not belong to $H^2(D)^d$. It only belongs to some weighted Sobolev spaces, namely the Kondratiev spaces (see, e.g. Mazzucato and Nistor [41]). To get the best rate of convergence, we need to use mesh-refinement near the vertices of the domains, which we do not consider in this chapter. For sufficiently smooth domains D , the required regularity of the solution $\mathbf{u}$ for the rate of convergence of the sparse tensor product FE holds; we proved this in Section 3.5. The discretization of these domains need some modification from that in the previous two subsections. We follow the procedure in Hoang [30].

Assume that D is a convex domain with a smooth boundary in $\mathbb{R}^d$. We discretize a convex polygonal domain D^l that is inscribed into it. The affine parts of the boundary of D^l is of the size $O(2^{-l})$ (see Frehse and Ranacher [26] and Strang and Fix [52]). We construct the family $\mathcal{T}^l$ of simplices in D as follows. We first divide D into simplices each has a curvilinear boundary which is a subset of ∂D . We then construct $\mathcal{T}^l$ by dividing each simplex in $\mathcal{T}^{l-1}$ into 4 congruent triangles for $d = 2$ and into 8 tetrahedra for $d = 3$. Each simplex in $\mathcal{T}^{l-1}$ at the boundary with a curvilinear boundary segment is divided into 4 or 8 curvilinear simplices. We let D^l be the convex polygon whose vertices are those of the simplices in $\mathcal{T}^l$ that are on ∂D . The simplices in D_l are the restriction of those in $\mathcal{T}^l$ into D_l . We then define

$$V_0^l = \{u \in H_0^1(D)^d, \quad u \in V^l\}$$

with the approximation property

$$\inf_{v \in V_0^l} \|u - v\|_{H^1(D_l)^d} \leq ch_l \|u\|_{H^2(D_l)^d},$$

where the mesh size $h_l = O(2^{-l})$.

For approximating $u^p, p = 1, \dots, n$, we consider a polygonal domain D^* that contains D and construct the hierarchical spaces for approximating $L^2(D^*)$ as in the previous subsections. For $p = 1, \dots, n$, the sparse FE spaces for approximating functions in $L^2(D \times Y_1 \times \dots \times Y_{p-1}, H_{\#}^1(Y_p))$ is the restriction of those for approximating $L^2(D^* \times Y_1 \times \dots \times Y_{p-1}, H_{\#}^1(Y_p))$ to $D \times Y_1 \times \dots \times Y_p$. As the domain D is smooth, there is a bounded continuous map from $H^1(D)$ to $H^1(D^*)$. Therefore Lemma 2.6.4 holds. For a domain D^L , the sparse tensor approximating problem now becomes: find $\hat{\mathbf{u}}^L \in (\hat{\mathbf{V}}^L)^d$ such that

$$\int_{D^L} \int_{\mathbf{Y}} a_{ijkl}(x, \mathbf{y}) \epsilon_{kl}(\hat{\mathbf{u}}^L) \epsilon_{ij}(\hat{\mathbf{v}}^L) d\mathbf{y} dx = \int_{D^L} f \cdot \hat{v}^{0L} dx \quad (3.15)$$

for all $\hat{\mathbf{v}}^L \in (\hat{\mathbf{V}}^L)^d$. Applying Cea's lemma on (3.15), estimate (3.14) holds for the polygonal domain D^L .

3.5 Regularity of the solution

We prove in this section that when the domain and the forcing are sufficiently regular, the regularity required on the solution for obtaining the above error estimate for the sparse tensor product approximation holds. We first deduce the homogenized equation from (3.9).

For integers $1 \leq r, s \leq d$, let $e^{rs} \in \mathbb{R}_{sym}^{d \times d}$ be a second order tensor such that

$$e_{kl}^{rs} = \frac{1}{2}(\delta_{rk}\delta_{sl} + \delta_{rl}\delta_{sk}). \quad (3.16)$$

Let $a^n(x, \mathbf{y}_n) = a(x, \mathbf{y})$. We define the p th level homogenized coefficient $a^p(x, \mathbf{y}_p)$ recursively as follows. Let $N^{p,rs}(x, \mathbf{y}_p) \in L^2(D \times Y_1 \times \dots \times Y_{p-1}, H_{\#}^1(Y_p)/\mathbb{R})^d$ be the solution of the cell problem (see, e.g. Sanchez-Palencia [49])

$$\int_D \int_{\mathbf{Y}_p} a^p(x, \mathbf{y}_p) \epsilon_{y_p}(N^{p,rs}) : \epsilon_{y_p}(\phi) d\mathbf{y}_p dx = - \int_D \int_{\mathbf{Y}_p} a_{ijrs}^p \frac{\partial \phi_i}{\partial y_{p,i}} d\mathbf{y}_p dx. \quad (3.17)$$

Since $a_{ijkl}^p = a_{ijlk}^p$

$$\begin{aligned} & \int_D \int_{\mathbf{Y}_p} a_{ijrs}^p \frac{\partial \phi_i}{\partial y_{p,j}} d\mathbf{y}_p dx \\ &= \frac{1}{2} \left(\int_D \int_{\mathbf{Y}_p} a_{ijrs}^p \frac{\partial \phi_j}{\partial y_{p,i}} d\mathbf{y}_p dx + \int_D \int_{\mathbf{Y}_p} a_{ijrs}^p \frac{\partial \phi_i}{\partial y_{p,j}} d\mathbf{y}_p dx \right) \\ &= \int_D \int_{\mathbf{Y}_p} a_{ijrs}^p \epsilon_{y_p ij}(\phi) d\mathbf{y}_p dx. \end{aligned}$$

By definition of e^{rs} in (3.16),

$$\int_D \int_{\mathbf{Y}_p} a_{ijrs}^p \epsilon_{y_p ij}(\phi) d\mathbf{y}_p dx = \int_D \int_{\mathbf{Y}_p} a_{ijkl}^p e_{kl}^{rs} \epsilon_{y_p ij}(\phi) d\mathbf{y}_p dx.$$

Thus the cell problem becomes

$$\int_D \int_{\mathbf{Y}_p} a^p(x, \mathbf{y}_p) \epsilon_{y_p}(N^{p,rs}) : \epsilon_{y_p}(\phi) d\mathbf{y}_p dx = - \int_D \int_{\mathbf{Y}_p} a_{ijkl}^p e_{kl}^{rs} \epsilon_{y_p ij}(\phi) d\mathbf{y}_p dx,$$

i.e.

$$\int_D \int_{Y_1} \dots \int_{Y_p} a^p(x, \mathbf{y}_p) (e^{rs} + \epsilon_{y_p}(N^{p,rs})) : \epsilon_{y_p}(\phi) d\mathbf{y}_p dx = 0, \quad (3.18)$$

for all $\phi \in L^2(D \times Y_1 \times \dots \times Y_{p-1}, H_{\#}^1(Y_p)/\mathbb{R})^d$. Multiplying both sides by $(\epsilon_x(u^0(x)) + \dots + (\epsilon_{y_{p-1}})_{rs}(u^{p-1}(x, \mathbf{y}_{p-1})))$, (3.18) implies

$$\begin{aligned} & \int_D \int_{Y_1} \dots \int_{Y_p} a^p(x, \mathbf{y}_p) \left((\epsilon_x(u^0(x)) + \dots + \epsilon_{y_{p-1}}(u^{p-1}(x, \mathbf{y}_{p-1}))) \right. \\ & \left. + \epsilon_{y_p}(N^{p,rs}) \left((\epsilon_x(u^0(x)) + \dots + (\epsilon_{y_{p-1}})_{rs}(u^{p-1}(x, \mathbf{y}_{p-1}))) \right) \right) : \epsilon_{y_p}(\phi) d\mathbf{y}_p dx = 0. \end{aligned}$$

So the solution u^p is of the form

$$u^p = N^{p,rs}(x, \mathbf{y}_p)((\epsilon_x)_{rs}(u^0(x)) + \dots + (\epsilon_{y_{p-1}})_{rs}(u^{p-1}(x, \mathbf{y}_{p-1}))). \quad (3.19)$$

Using this formula for u^{p-1} , we have

$$\begin{aligned} u^p &= N^{p,r_{p-1}s_{p-1}}(x, \mathbf{y}_p) \left(\delta_{r_{p-1}r_{p-2}} \delta_{s_{p-1}s_{p-2}} \right. \\ &\quad \left. + (\epsilon_{y_{p-1}})_{r_{p-1}s_{p-1}} (N^{p-1,r_{p-2}s_{p-2}}(x, \mathbf{y}_{p-1})) \right) \cdot \\ &\quad \left((\epsilon_x)_{rs}(u^0(x)) + \dots + (\epsilon_{y_{p-2}})_{rs}(u^{p-2}(x, \mathbf{y}_{p-2})) \right), \end{aligned}$$

where $\delta_{r,s} = 1$ if $r = s$ and $\delta_{r,s} = 0$ if $r \neq s$. Recursively we show that

$$\begin{aligned} u^p &= N^{p,r_{p-1}s_{p-1}}(x, \mathbf{y}_p) \left(\delta_{r_{p-1}r_{p-2}} \delta_{s_{p-1}s_{p-2}} \right. \\ &\quad \left. + (\epsilon_{y_{p-1}})_{r_{p-1}s_{p-1}} (N^{p-1,r_{p-2}s_{p-2}}(x, \mathbf{y}_{p-1})) \right) \cdot \\ &\quad \left(\delta_{r_{p-2}r_{p-3}} \delta_{s_{p-2}s_{p-3}} + (\epsilon_{y_{p-2}})_{r_{p-2}s_{p-2}} (N^{p-2,r_{p-3}s_{p-3}}(x, \mathbf{y}_{p-2})) \right) \dots \\ &\quad \left(\delta_{r_1r_0} \delta_{s_1s_0} + (\epsilon_{y_1})_{r_1s_1} (N^{1,r_0s_0}(x, y_1)) \right) \epsilon_{r_0s_0}(u^0(x)), \end{aligned} \quad (3.20)$$

where in the last expression, summation is taken over $r_i, s_i = 1, \dots, d$ for $i = 1, \dots, p-1$. The coefficient $a^{p-1}(x, \mathbf{y}_{p-1})$ is defined recursively from $a^p(x, \mathbf{y}_p)$ as

$$(a^{p-1})_{ijkl}(x, \mathbf{y}_{p-1}) = \int_{Y_p} a^p(x, \mathbf{y}_p) (e^{kl} + \epsilon_{y_p}(N^{p,kl})) : (e^{ij} + \epsilon_{y_p}(N^{p,ij})) d\mathbf{y}_p; \quad (3.21)$$

and $a^0(x)$ is the homogenized coefficient. The homogenized equation is

$$-\frac{\partial}{\partial x_j} (a_{ijkl}^0 \epsilon_{kl}(u^0)) = f_i. \quad (3.22)$$

We then have the following regularity result.

Proposition 3.5.1. *Assume that the boundary ∂D belongs to the C^1 class and $f \in L^2(D)^d$. Then $u^0 \in H^2(D)^d$ and $u^p \in \hat{\mathcal{H}}_p^d$.*

Proof Using integration by parts, we have from (3.18) that

$$\int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \epsilon_{y_n}(N^{n,rs}) : \epsilon(\phi) d\mathbf{y} dx = \frac{1}{2} \int_D \int_{\mathbf{Y}} \frac{\partial}{\partial y_j} (a_{ijkl} e_{kl}^{rs}) \phi_i + \frac{\partial}{\partial y_i} (a_{ijkl} e_{kl}^{rs}) \phi_j dx d\mathbf{y}$$

for all $\phi \in L^2(D \times Y_1 \times Y_2 \times \dots \times Y_{n-1}, H_{\#}^1(Y))^d$.

With $x, y_1, \dots, y_{p-1}$ play the roles of parameters, we deduce that $N^{n,rs}(x, y_1, \dots, y_{p-1}, \cdot) \in H^2(Y_n)^d$. As $a \in (C^1(\bar{D}, C_{\#}^1(Y_1, \dots, C_{\#}^1(Y_n), \dots)))^{d^4}$, we have that $N^{n,rs} \in (C^1(\bar{D}, C_{\#}^1(Y_1, \dots, C_{\#}^1(Y_{n-1}, H^2(Y_n), \dots)))^d$. Therefore a^{n-1} as defined in (3.21) for

$p = n - 1$ is in $(C^1(\bar{D}, C_{\#}^1(Y_1, \dots, C_{\#}^1(Y_{n-1}) \dots)))^{d^4}$. Inductively, we then have $N^{p,rs} \in (C^1(D, C_{\#}^1(Y_1, \dots, C_{\#}^1(Y_{p-1}, H^2(Y_p)) \dots)))^d$ and $a^p \in (C^1(\bar{D}, C_{\#}^1(Y_1, \dots, C_{\#}^1(Y_p) \dots)))^{d^4}$ for all $p = 1, \dots, n$. From elliptic regularity theory, see e.g. Wloka [55] or Ciarlet [15], we deduce from (3.22) that $u^0 \in H^2(D)^d$. Thus from (3.20), for $p = 1$, we have

$$u^1 = N^{1,rs}(x, y_1) \epsilon_{rs}(u^0) \in H^1(D, H^2(Y_1))^d.$$

For $p = 2$, we have

$$u^2(x, y_1, y_2) = N^{2,rs}(x, y_1, y_2) \left((\epsilon_x)_{rs}(u^0(x)) + \epsilon_{y_1}(u^1(x, y_1)) \right).$$

As $N^{2,rs}(x, y_1, y_2) \in (C^1(\bar{D}, C_{\#}^1(Y_1, H^2(Y_2))))^d$, $(\epsilon_x)_{rs}(u^0(x)) \in H^1(D)^{d \times d}$ and $\epsilon_{y_1}(u^1(x, y_1)) \in H^1(D, H^1(Y_1))^{d \times d}$, we deduce that $u^2 \in \hat{\mathcal{H}}_2^d$. Inductively we have that $u^p \in \hat{\mathcal{H}}_p^d$ for all $p = 1, 2, \dots, n$. $\square$

Remark 3.5.2. *If the homogenized problem (3.22) is isotropic and a^0 is independent of x , e.g. when $a = a(y_1, y_2, \dots, y_n)$, for a convex polygonal domain D in two dimensions, $u^0 \in H^2(D)^d$. The proof for this can be found in, e.g. Brenner and Sung [13], Bacuta and Bramble [5]. Therefore $u^p \in \hat{\mathcal{H}}_p^d$ for all p .*

3.6 Homogenization error and convergence in the physical variables

We deduce an approximation for the solution u^ε of the multiscale elasticity equation (3.4) from the finite element solution of the multiscale homogenized problem (3.9) in this section. In order to get an explicit rate of convergence for this approximation, we need an analytic homogenization error estimate for problem (3.4). For two-scale elasticity equations, the available rate of convergence $O(\varepsilon^{1/2})$, as deduced by Oleinik et al. [47], requires the regularity $f \in H^1(D)^d$ for the forcing so that the homogenized solution u^0 is in $H^3(D)^d$ which is stronger than the regularity $f \in L^2(D)^d$ and $u^0 \in H^2(D)^d$ required for the elliptic problem in Jikov et al. [37]. In this section, we present a proof to obtain the $O(\varepsilon^{1/2})$ rate of convergence that only requires $u^0 \in H^2(D)^d$.

3.6.1 Homogenization error for two-scale problem

For the two-scale problem, we denote the coefficient $a(x, y_1)$ as $a(x, y)$. The cell problem (3.18) is written as

$$\int_D \int_Y a(x, y) (e^{rs} + \epsilon_y(N^{rs})) : \epsilon_y(\phi) dy dx = 0, \quad \forall \phi \in L^2(D, H_{\#}^1(Y))^d. \quad (3.23)$$

The homogenized elastic moduli is given by

$$a_{ijkl}^0(x) = \int_Y a(x, y) (e^{kl} + \epsilon_y(N^{kl})) : (e^{ij} + \epsilon_y(N^{ij})) dy. \quad (3.24)$$

The function $u^1(x, y)$ is then written as

$$u^1(x, y) = N^{rs}(x, y)\epsilon_{rs}(u^0(x)).$$

Proposition 3.6.1. *Assume that $u^0 \in H^2(D)^d$, $N^{rs} \in (C^1(\bar{D}; H_{\#}^2(Y)))^d$ and the boundary ∂D is Lipschitz, then*

$$\left\| u^\varepsilon(\cdot) - u^0(\cdot) - \varepsilon u^1\left(\cdot, \frac{\cdot}{\varepsilon}\right) \right\|_{H^1(D)^d} \leq c\varepsilon^{1/2}. \quad (3.25)$$

Proof We define

$$\begin{aligned} u^{1\varepsilon}(x) &= u^0(x) + \varepsilon u^1\left(x, \frac{x}{\varepsilon}\right) \\ &= u^0(x) + \varepsilon N^{rs}\left(x, \frac{x}{\varepsilon}\right)\epsilon_{rs}(u^0(x)). \end{aligned}$$

We show that for each $i, j = 1, \dots, d$

$$\left\| \frac{\partial}{\partial x_j} \left(a_{ijkl}^\varepsilon(x)\epsilon_{kl}(u^{1\varepsilon}(x)) - a_{ijkl}^0\epsilon_{kl}(u^0) \right) \right\|_{H^{-1}(D)^d} \leq c\varepsilon.$$

We note that

$$\epsilon\left(\phi\left(x, \frac{x}{\varepsilon}\right)\right) = \epsilon_x\left(\phi\left(x, \frac{x}{\varepsilon}\right)\right) + \frac{1}{\varepsilon}\epsilon_y\left(\phi\left(x, \frac{x}{\varepsilon}\right)\right),$$

so

$$\begin{aligned} \epsilon_{kl}(u^{1\varepsilon}(x)) &= \epsilon_{kl}(u^0(x)) + \epsilon_{ykl}\left(N^{rs}\left(x, \frac{x}{\varepsilon}\right)\right)\epsilon_{rs}(u^0(x)) \\ &+ \varepsilon\epsilon_{xkl}\left(N^{rs}\left(x, \frac{x}{\varepsilon}\right)\right)\epsilon_{rs}(u^0(x)) + \varepsilon\left(N_l^{rs}\left(x, \frac{x}{\varepsilon}\right)\frac{\partial\epsilon_{rs}(u^0(x))}{\partial x_k} + N_k^{rs}\left(x, \frac{x}{\varepsilon}\right)\frac{\partial\epsilon_{rs}(u^0(x))}{\partial x_l}\right). \end{aligned}$$

Therefore

$$\begin{aligned} &a_{ijkl}^\varepsilon(x)\epsilon_{kl}(u^{1\varepsilon}(x)) \\ &= a_{ijkl}^0(x)\epsilon_{kl}(u^0(x)) + a_{ijkl}^\varepsilon(x)\epsilon_{kl}(u^{1\varepsilon}(x)) - a_{ijkl}^0(x)\epsilon_{kl}(u^0(x)) \\ &= a_{ijkl}^0(x)\epsilon_{kl}(u^0(x)) + \left(a_{ijkl}\left(x, \frac{x}{\varepsilon}\right)\epsilon_{kl}(u^0(x)) \right. \\ &\quad \left. + a_{ijkl}\left(x, \frac{x}{\varepsilon}\right)\epsilon_{ykl}\left(N^{rs}\left(x, \frac{x}{\varepsilon}\right)\right)\epsilon_{rs}(u^0(x)) - a_{ijkl}^0(x)\epsilon_{kl}(u^0(x)) \right) \\ &\quad + \varepsilon a_{ijkl}\left(x, \frac{x}{\varepsilon}\right)\left(\epsilon_{xkl}\left(N^{rs}\left(x, \frac{x}{\varepsilon}\right)\right)\epsilon_{rs}(u^0(x)) \right. \\ &\quad \left. + \left(N_l^{rs}\left(x, \frac{x}{\varepsilon}\right)\frac{\partial\epsilon_{rs}(u^0(x))}{\partial x_k} + N_k^{rs}\left(x, \frac{x}{\varepsilon}\right)\frac{\partial\epsilon_{rs}(u^0(x))}{\partial x_l} \right) \right) \\ &= a_{ijkl}^0(x)\epsilon_{kl}(u^0(x)) + g_{ijkl}\left(x, \frac{x}{\varepsilon}\right)\epsilon_{kl}(u^0(x)) + \varepsilon I(x), \end{aligned}$$

where

$$g_{ijkl}(x, y) = a_{ijkl}(x, y) + a_{ijmn}(x, y)\epsilon_{ymn}(N^{kl}(x, y)) - a_{ijkl}^0(x) \quad (3.26)$$

and

$$\begin{aligned} I(x) = a_{ijkl}^\varepsilon(x) & \left(\epsilon_{xkl}(N^{rs}(x, \frac{x}{\varepsilon}))\epsilon_{rs}(u^0(x)) \right. \\ & \left. + (N_l^{rs}(x, \frac{x}{\varepsilon})\frac{\partial \epsilon_{rs}(u^0(x))}{\partial x_k} + N_k^{rs}(x, \frac{x}{\varepsilon})\frac{\partial \epsilon_{rs}(u^0(x))}{\partial x_l}) \right). \end{aligned}$$

From the definition of $g_{ijkl}(x, y)$ in (3.26),

$$\begin{aligned} & \int_Y g_{ijkl}(x, y) dy \\ = & \int_Y a_{ijkl}(x, y) + a_{ijmn}(x, y)\epsilon_{ymn}(N^{kl}(x, y)) dy - a_{ijkl}^0(x) \\ = & \int_Y a_{ijmn}(x, y)e_{mn}^{kl} + a_{ijmn}(x, y)\epsilon_{ymn}(N^{kl}(x, y)) dy - a_{ijkl}^0(x) \\ = & \int_Y a_{ijmn}(x, y) (e_{mn}^{kl} + \epsilon_{ymn}(N^{kl}(x, y))) dy - a_{ijkl}^0(x) \\ = & \int_Y a_{pqmn}(x, y) (e_{mn}^{kl} + \epsilon_{ymn}(N^{kl}(x, y))) e_{pq}^{ij} dy - a_{ijkl}^0(x). \end{aligned}$$

From (3.23),

$$\int_D \int_Y a(x, y)(e^{rs} + \epsilon_y(N^{rs})) : \epsilon_y(\phi) dy dx = 0, \quad \forall \phi \in L^2(D, H_{\#}^1(Y))^d.$$

We take $\phi = N^{ij}(x, y)$ such that

$$\int_Y a(x, y)(e^{kl} + \epsilon_y(N^{kl})) : \epsilon_y(N^{ij}(x, y)) dy = 0. \quad (3.27)$$

From (3.24),

$$a^0(x) = \int_Y a(x, y)(e^{kl} + \epsilon_y(N^{kl})) : (e^{ij} + \epsilon_y(N^{ij})) dy. \quad (3.28)$$

Subtract (3.27) from (3.28), we get

$$a_{ijkl}^0(x) = \int_Y a_{pqmn}(x, y) (e_{mn}^{kl} + \epsilon_{ymn}(N^{kl}(x, y))) e_{pq}^{ij} dy.$$

Thus we have

$$\int_Y g_{ijkl}(x, y) dy = 0. \quad (3.29)$$

Again from (3.26)

$$\begin{aligned}\frac{\partial}{\partial y_j} g_{ijkl}(x, y) &= \frac{\partial}{\partial y_j} (a_{ijkl}(x, y) + a_{ijmn}(x, y) \epsilon_{ymn}(N^{kl}(x, y))) \\ &= \frac{\partial}{\partial y_j} (a_{ijmn}(e^{kl} + \epsilon_y(N^{kl}))) .\end{aligned}$$

By (3.23) and symmetry of a_{ijkl} that for all $\phi \in L^2(D, H_{\#}^1(Y)/\mathbb{R})^d$

$$\begin{aligned}& \int_Y \frac{\partial}{\partial y_j} g_{ijkl}(x, y) \cdot \phi(y) dy \\ &= \int_Y \frac{\partial}{\partial y_j} a_{ijmn}(e^{kl} + \epsilon_y(N^{kl})) \cdot \phi(y) dy \\ &= \int_Y a_{ijmn}(e^{kl} + \epsilon_y(N^{kl})) : \epsilon_y(\phi) dy \\ &= 0.\end{aligned}$$

Since $\phi(y)$ is arbitrarily chosen

$$\frac{\partial}{\partial y_j} g_{ijkl}(x, y) = 0. \quad (3.30)$$

Therefore, from the result on page 7 of Jikov et al. [37], we deduce that there are functions $\alpha_{jr}^{ikl}(x, y) \in H_{\#}^1(Y)$ for each fixed $x \in D$, such that $\alpha_{jr}^{ikl} = -\alpha_{rj}^{ikl}$ and

$$g_{ijkl}(x, y) = \frac{\partial}{\partial y_r} \alpha_{jr}^{ikl}(x, y).$$

Therefore

$$a_{ijkl}^{\varepsilon}(x) \epsilon_{kl}(u^{1\varepsilon}(x)) - a_{ijkl}^0(x) \epsilon_{kl}(u^0(x)) = \varepsilon \frac{\partial}{\partial x_r} \left(\alpha_{jr}^{ikl}(x, \frac{x}{\varepsilon}) \epsilon_{kl}(u^0(x)) \right) + \varepsilon r_{\varepsilon}(x),$$

where

$$r_{\varepsilon}(x) = -\frac{\partial \alpha_{jr}^{ikl}}{\partial x_r}(x, \frac{x}{\varepsilon}) \epsilon_{kl}(u^0(x)) - \alpha_{jr}^{ikl}(x, \frac{x}{\varepsilon}) \frac{\partial}{\partial x_r} \epsilon_{kl}(u^0(x)) + I(x).$$

Thus

$$\begin{aligned}& \left\langle \frac{\partial}{\partial x_j} (a_{ijkl}^{\varepsilon}(x) \epsilon_{kl}(u^{1\varepsilon}(x)) - a_{ijkl}^0(x) \epsilon_{kl}(u^0(x))), \phi \right\rangle_{V', V} \\ &= \varepsilon \int_D \alpha_{jr}^{ikl}(x, \frac{x}{\varepsilon}) \epsilon_{kl}(u^0(x)) \frac{\partial^2 \phi}{\partial x_j \partial x_r}(x) - \varepsilon \int_D r_{\varepsilon}(x) \frac{\partial \phi}{\partial x_j}(x) dx.\end{aligned}$$

As $N^{rs} \in (C^1(\bar{D}, H_{\#}^2(Y)))^d$, $g_{ijkl} \in (C^1(\bar{D}, H^1(Y)))^d$. From the construction of α_{jr}^{ikl} on page 7 of Jikov et al. [37], (see also Hoang and Schwab [34]), we deduce that $\alpha_{jr}^{ikl} \in C^1(\bar{D}, H_{\#}^2(\bar{Y}))$, ($d = 2, 3$). By the boundedness of a_{ijkl}^{ε} and $u^0 \in H^2(D)^d$, we

therefore deduce that $r_\varepsilon(x)$ is uniformly bounded in $L^2(D)$. By the symmetry that $\alpha_{jr}^{ikl} = -\alpha_{rj}^{ikl}$, we have

$$\begin{aligned}
& \int_D \alpha_{jr}^{ikl}(x, \frac{x}{\varepsilon}) \epsilon_{kl}(u^0(x)) \frac{\partial^2 \phi}{\partial x_j \partial x_r}(x) dx \\
&= \frac{1}{2} \left(\int_D \alpha_{jr}^{ikl}(x, \frac{x}{\varepsilon}) \epsilon_{kl}(u^0(x)) \frac{\partial^2 \phi}{\partial x_j \partial x_r}(x) dx \right. \\
&\quad \left. + \int_D \alpha_{rj}^{ikl}(x, \frac{x}{\varepsilon}) \epsilon_{kl}(u^0(x)) \frac{\partial^2 \phi}{\partial x_r \partial x_j}(x) dx \right) \\
&= \frac{1}{2} \int_D \left(\alpha_{jr}^{ikl}(x, \frac{x}{\varepsilon}) + \alpha_{rj}^{ikl}(x, \frac{x}{\varepsilon}) \right) \epsilon_{kl}(u^0(x)) \frac{\partial^2 \phi}{\partial x_j \partial x_r}(x) dx \\
&= 0.
\end{aligned}$$

These imply

$$\begin{aligned}
& \left\| \frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon \epsilon_{kl}(u^{1\varepsilon}) - a_{ijkl}^0 \epsilon_{kl}(u^0)) \right\|_{H^{-1}(D)^d} \\
&\leq \varepsilon \left\| \frac{\partial}{\partial x_j} \left(\alpha_{jr}^{ikl}(x, \frac{x}{\varepsilon}) \epsilon_{kl}(u^0(x)) \right) \right\|_{H^1(D)^d} \\
&\leq c\varepsilon.
\end{aligned}$$

From (3.4) and (3.22)

$$\frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon \epsilon_{kl}(u^\varepsilon)) = \frac{\partial}{\partial x_j} (a_{ijkl}^0 \epsilon_{kl}(u^0)) = -f_i,$$

we deduce

$$\left\| \frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon(x) \epsilon_{kl}(u^{1\varepsilon} - u^\varepsilon)) \right\|_{H^{-1}(D)^d} \leq c\varepsilon. \quad (3.31)$$

Let $\tau^\varepsilon(x)$ be a function in $\mathcal{D}(D)$ and $\tau^\varepsilon(x) = 1$ outside an ε neighbourhood of ∂D such that $\varepsilon |\nabla \tau^\varepsilon(x)| < c$ for all ε where c is independent of ε . We consider the function

$$w^{1\varepsilon}(x) = u^0(x) + \varepsilon \tau^\varepsilon(x) N^{rs}(x, \frac{x}{\varepsilon}) \epsilon_{rs}(u^0(x)).$$

We then have

$$u^{1\varepsilon} - w^{1\varepsilon} = \varepsilon(1 - \tau^\varepsilon(x)) N^{rs}(x, \frac{x}{\varepsilon}) \epsilon_{rs}(u^0(x)).$$

Therefore for $i, j = 1, \dots, d$,

$$\begin{aligned}
\frac{\partial}{\partial x_j} (u^{1\varepsilon} - w^{1\varepsilon})_i &= - \varepsilon \frac{\partial \tau^\varepsilon}{\partial x_j}(x) N_i^{rs}(x, \frac{x}{\varepsilon}) \epsilon_{rs}(u^0(x)) \\
&\quad + \varepsilon(1 - \tau^\varepsilon(x)) \frac{\partial N_i^{rs}}{\partial x_j}(x, \frac{x}{\varepsilon}) \epsilon_{rs}(u^0(x)) \\
&\quad + (1 - \tau^\varepsilon(x)) \frac{\partial N_i^{rs}}{\partial y_j}(x, \frac{x}{\varepsilon}) \epsilon_{rs}(u^0(x)) \\
&\quad + \varepsilon(1 - \tau^\varepsilon(x)) N_i^{rs}(x, \frac{x}{\varepsilon}) \frac{\partial}{\partial x_j} \epsilon_{rs}(u^0(x)).
\end{aligned}$$

Using the inequality $\|\phi\|_{L^2(\hat{D}^\varepsilon)} \leq c\varepsilon^{1/2}\|\phi\|_{H^1(D)}$ from Chapter 2 where $\hat{D}^\varepsilon \subset D$ is the ε neighbourhood of ∂D outside which $\tau^\varepsilon(x) = 1$ (see Section 2.7.1), we have

$$\|\epsilon_{rs}(u^0)\|_{L^2(\hat{D}^\varepsilon)^{d \times d}} \leq c\varepsilon^{1/2}\|u^0\|_{H^2(D)^d} \quad (3.32)$$

and $\|\varepsilon \frac{\partial \tau^\varepsilon}{\partial x_j}(x) \epsilon_{rs}(u^0(x))\|_{L^2(D)^{d \times d}}$ is bounded. We have from the proof of Proposition 3.5.1 that $N_i^{rs} \in C^1(\bar{D}, H^2(Y))^d \subset C^1(\bar{D}, C(\bar{Y}))$, which implies both N_i^{rs} and $\partial N_i^{rs}/\partial x_j$ are bounded pointwise $\forall i, j$. Thus the L^2 -norm of first term is bounded by $c\varepsilon^{1/2}$. We have from the assumption of this proposition that $u_0 \in H^2(D)^d$, thus $\|\epsilon_{rs}(u^0(x))\|_{L^2(D)}$ and $\|\partial \epsilon_{rs}(u^0(x))/\partial x_j\|_{L^2(D)}$ are bounded, so the second and the fourth term are bounded by $c\varepsilon^{1/2}$. From (3.32), the L^2 -norm of the third term is also bounded by $c\varepsilon^{1/2}$. From these, for all $i, j = 1, \dots, d$,

$$\left\| \frac{\partial}{\partial x_j} (u^{1\varepsilon} - w^{1\varepsilon})_i \right\|_{L^2(D)} \leq c\varepsilon^{1/2}. \quad (3.33)$$

Therefore by the boundedness of a_{ijkl}^ε

$$\left\| \frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon(x) \epsilon_{kl}(u^{1\varepsilon} - w^{1\varepsilon})) \right\|_{H^{-1}(D)^d} \leq c\varepsilon^{1/2}.$$

From this and (3.31), we have from the triangle inequality

$$\left\| \frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon(x) \epsilon_{kl}(u^\varepsilon - w^{1\varepsilon})) \right\|_{H^{-1}(D)^d} \leq c\varepsilon^{1/2}.$$

Therefore

$$\int_D a_{ijkl}^\varepsilon(x) \epsilon_{kl}(u^\varepsilon(x) - w^{1\varepsilon}(x)) \epsilon_{ij}(u^\varepsilon(x) - w^{1\varepsilon}(x)) dx \leq c\varepsilon^{1/2} \|u^\varepsilon - w^{1\varepsilon}\|_{V^d}.$$

We then deduce from Korn's inequality and (3.3)

$$\|u^\varepsilon - w^{1\varepsilon}\|_{V^d} \leq c\varepsilon^{1/2}.$$

This together with (3.33) implies

$$\|u^\varepsilon - u^{1\varepsilon}\|_{H^1(D)^d} \leq c\varepsilon^{1/2}.$$

□

3.6.2 Convergence in the physical variables for two-scale problem

We employ the operator $\mathcal{U}^\varepsilon : L^1(D \times Y) \rightarrow L^1(D)$ defined in (2.74) in Chapter 2 Section 2.7.2.

Lemma 3.6.2. *Assume that $u^0 \in H^2(D)^d$ and $N^{rs} \in (C^1(\bar{D}, C^1(\bar{Y})))^d$ for all $r, s = 1, \dots, d$, then*

$$\int_D \left| \nabla_y u^1(x, \frac{x}{\varepsilon}) - \mathcal{U}^\varepsilon(\nabla_y u^1)(x) \right|^2 dx \leq c\varepsilon^2,$$

where the constant c is independent of ε .

The lemma is easy to prove when u^1 is smooth. Here we establish this lemma when only limited regularity for u^0 is required so that we do not need a smooth forcing, and in the case where the homogenized equation is isotropic and homogeneous, we only need D to be a convex domain for the condition $u^0 \in H^2(D)^d$ to be satisfied (see Remark 3.5.2). The proof of this lemma follows from that of Lemma 5.5 in Hoang and Schwab [34]. The required regularity for N^{rs} is satisfied due to the following Lemma.

Lemma 3.6.3. *Assume that $a_{ijkl} \in C^2(\bar{D}, C^2(\bar{Y}))$, for all $r, s = 1, \dots, d$, $N^{rs} \in (C^2(\bar{D}, C^1(\bar{Y})))^d$.*

Proof We denote by $z^{rs} \in \mathbb{R}^d$ the vector such that

$$z_i^{rs} = \frac{1}{2}(y_s \delta_{ri} + y_r \delta_{si}).$$

We then have

$$\epsilon_y(z^{rs}) = \frac{1}{2}\epsilon_y(y_s \delta_{ri} + y_r \delta_{si}) = \frac{1}{2}(\delta_{sj} \delta_{ri} + \delta_{rj} \delta_{si}) = e^{rs}.$$

We therefore have from (3.23) that fixing $x \in D$

$$\int_Y a(x, y) \epsilon_y(N^{rs}(x, y) + z^{rs}) : \epsilon_y(\phi) dy = 0 \quad (3.34)$$

for all $\phi \in H_{\#}^1(Y)^d$ and thus for all $\phi \in \mathcal{D}(\mathbb{R}^d)^d$ with a sufficiently small support. Using partition of unity, we deduce that (3.34) holds for all $\phi \in \mathcal{D}(\mathbb{R}^d)^d$. From the regularity theory of coercive systems of partial differential equations (see, e.g. Theorem 4.16 in McLean [42]), we deduce that

$$\|N^{rs}(x, \cdot) + z^{rs}\|_{H^3(Y)^d} \leq c \|N^{rs}(x, \cdot) + z^{rs}\|_{H^1(Y)^d},$$

where the constant c only depends on the $C^1(\bar{Y})$ norm of $a(x, \cdot)$. Therefore $N^{rs}(x, \cdot)$ is uniformly bounded in $H^3(Y)^d$ and thus in $C^1(\bar{Y})^d$ for $d = 2, 3$. By considering (3.23) for x and $x + h$ for a vector h whose all components are zero except one, we have

$$\int_Y a(x, y) (e^{rs} + \epsilon_y(N^{rs}(x, \cdot))) : \epsilon_y(\phi) dy = 0,$$

and

$$\int_Y a(x + h, y) (e^{rs} + \epsilon_y(N^{rs}(x + h, \cdot))) : \epsilon_y(\phi) dy = 0.$$

Taking the difference, we have

$$\begin{aligned} & \int_Y a(x+h, y) \epsilon_y (N^{rs}(x+h, y) - N^{rs}(x, y)) : \epsilon_y(\phi) dy dx \\ & + \int_Y (a(x+h, y) - a(x, y)) (e^{rs} + \epsilon_y(N^{rs}(x, y))) : \epsilon_y(\phi) dy dx = 0. \end{aligned}$$

By taking limit $h \rightarrow 0$, we deduce that for each $i = 1, \dots, d$ there exists $\frac{\partial N^{rs}}{\partial x_i}(x, \cdot) \in H_{\#}^1(Y)^{d \times d}$ which satisfies

$$\begin{aligned} & \int_Y a(x, y) \epsilon_y \left(\frac{\partial N^{rs}}{\partial x_i} \right) : \epsilon_y(\phi) dy = - \int_Y \frac{\partial a}{\partial x_i} (e^{rs} + \epsilon_y(N^{rs})) : \epsilon_y(\phi) dy \\ & = \frac{1}{2} \int_Y \left[\frac{\partial}{\partial y_j} \left(\frac{\partial a_{ijkl}}{\partial x_i} (x, y) (e_{kl}^{rs} + \epsilon_{ykl}(N^{rs})) \right) \phi_i(y) \right. \\ & \quad \left. + \frac{\partial}{\partial y_i} \left(\frac{\partial a_{ijkl}}{\partial x_j} (x, y) (e_{kl}^{rs} + \epsilon_{ykl}(N^{rs})) \right) \phi_j(y) \right] dy \end{aligned}$$

for all $\phi \in H_{\#}^1(Y)^d$. Since $a \in (C^2(\bar{D}, C^2(\bar{Y})))^{d^4}$ and $N^{rs}(x, \cdot) \in H^3(Y)^d$

$$\frac{\partial}{\partial y_j} \left(\frac{\partial a_{ijkl}}{\partial x_i} (x, y) (e_{kl}^{rs} + \epsilon_{ykl}(N^{rs})) \right) \in H^1(Y)^d$$

for all i, j, k and l . Thus we have that $\frac{\partial N^{rs}}{\partial x_i} \in H^3(Y)^d \subset C^1(\bar{Y})^d$ from Sobolev embedding theorem, e.g. in Ciarlet [15]. Therefore $N^{rs} \in (C^2(\bar{D}, C^1(\bar{Y})))^d$. $\square$

We then have the following approximation result, where D^L is the polygonal domain inscribed in a smooth convex domain D in which we discretize the problem as in Section 3.4.3.

Theorem 3.6.4. *Assume that the elastic moduli $a(x, y) \in (C^2(\bar{D}, C^2(\bar{Y})))^{d^4}$, D is a smooth convex domain and $f \in L^2(D)^d$. With the full tensor product FE approximation, we have*

$$\|\nabla u^\varepsilon(\cdot) - [\nabla \bar{u}^{0L}(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \bar{u}^{1L})(\cdot)]\|_{L^2(D^L)^{d \times d}} \leq c(\varepsilon^{1/2} + ch_L).$$

With the sparse tensor product approximation, we have

$$\|\nabla u^\varepsilon(\cdot) - [\nabla \hat{u}^{0L}(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \hat{u}^{1L})(\cdot)]\|_{L^2(D^L)^{d \times d}} \leq c(\varepsilon^{1/2} + ch_L L^{1/2}).$$

Proof From Lemma 3.6.3, conditions for Proposition 3.6.1, Lemma 3.6.2 are satisfied. From Proposition 3.6.1

$$\begin{aligned} & \|\nabla_x u^\varepsilon(x) - \nabla_x u^0(x) - \varepsilon \nabla_x N^{rs}(x, y) \epsilon(u^0(x)) - \varepsilon N^{rs}(x) \nabla_x \epsilon_{rs}(u^0(x)) \\ & - \nabla_y N^{rs}(x, y) \epsilon_{rs}(u^0(x))\|_{L^2(D^L)^{d \times d}} \leq c\varepsilon^{1/2} \end{aligned}$$

by $N^{rs} \in (C^2(\bar{D}, C^1(\bar{Y})))^{d \times d}$ and $u^0 \in H^2(D)^d$

$$\|\nabla_x u^\varepsilon(x) - \nabla_x u^0(x) - \nabla_y N^{rs}(x, y) \epsilon_{rs}(u^0(x))\|_{L^2(D^L)^{d \times d}} \leq c\varepsilon^{1/2}.$$

Since $u^1(x, y) = N^{rs}(x, y)\epsilon_{rs}(u^0(x))$

$$\|\nabla_x u^\varepsilon(x) - \nabla_x u^0(x) - \nabla_y u^1(x, x/\varepsilon)\|_{L^2(D^L)^{d \times d}} \leq c\varepsilon^{1/2}.$$

From this and Lemma 3.6.2, we have from triangle inequality

$$\|\nabla u^\varepsilon(x) - \nabla u^0(x) - \mathcal{U}^\varepsilon(\nabla_y u^1)(x)\|_{L^2(D^L)^{d \times d}} \leq c\varepsilon^{1/2}. \quad (3.35)$$

From Cauchy-Schwartz inequality, we have

$$(\mathcal{U}^\varepsilon(\Phi)(x))^2 \leq \mathcal{U}^\varepsilon(\Phi^2)(x), \quad (3.36)$$

for all $x \in D$. From (2.78) and (3.36), we have

$$\|\mathcal{U}^\varepsilon(\nabla_y u^1)(\cdot) - \mathcal{U}^\varepsilon(\nabla_y \bar{u}^{1L})(\cdot)\|_{L^2(D^L)^{d \times d}} \leq ch_L.$$

The above estimate and (3.35) imply the error estimate for full tensor finite element approximation. The estimate for sparse tensor approximation is deduced similarly. $\square$

Remark 3.6.5. *Following Remark 3.5.2, Theorem 3.6.4 holds in the $L^2(D)^{d \times d}$ norm when the homogenized elastic moduli a^0 is isotropic and homogeneous, and D is a convex polygon in two dimensions.*

3.6.3 Convergence in the physical variables for the multiscale case

As for the scalar multiscale elliptic problems, for more than two scales, we are not aware of an explicit homogenization rate of convergence so an explicit rate of convergence in terms of the microscopic scale and the finite element mesh similar to those in Proposition 3.6.4 is not available. However, similar to the multiscale elliptic problems considered in Hoang and Schwab [31], we can deduce a numerical corrector for u^ε in terms of the FE solutions $\bar{\mathbf{u}}^L$ and $\hat{\mathbf{u}}^L$ for the multiscale elasticity problems for the case where $\varepsilon_p/\varepsilon_{p+1}$ is an integer for all $p = 1, \dots, n-1$. As in Section 2.8.2, we will use the general notation $\mathbf{u}^L$ for the approximating solution obtained by using general finite element spaces. The operator $\mathcal{T}_n^\varepsilon(\phi)(x, \mathbf{y})$ is define in Definition 2.7.2. We have

$$\mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon)) \rightharpoonup \epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n) \quad (3.37)$$

in $L^2(D \times Y_1 \times \dots \times Y_n)^{d \times d}$.

Further, letting D^{ε_1} be the $2\varepsilon_1$ neighbourhood of D , we have

$$\int_D \phi dx = \int_{D^{\varepsilon_1}} \int_{Y_1} \dots \int_{Y_n} \mathcal{T}_n^\varepsilon(\phi) dy_n \dots dy_1 dx. \quad (3.38)$$

for all $\phi \in L^1(D)$. The proofs for these facts can be found in Cioranescu et al. [17]. To deduce the numerical corrector, we need the following lemma.

Lemma 3.6.6. *The solution u^ε of (3.4) and the solution $\mathbf{u} = (u^0, u^1, \dots, u^n)$ of problem (3.9) satisfy*

$$\lim_{\varepsilon \rightarrow 0} \|\epsilon(u^\varepsilon) - \mathcal{U}_n^\varepsilon(\epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n))\|_{L^2(D)^{d \times d}} = 0. \quad (3.39)$$

Further, if $u^0 \in H^2(D)^d$ and $a \in C^2(D, C^2(\bar{Y}_1, \dots, C^2(\bar{Y}_n) \dots))$, then

$$\lim_{\varepsilon \rightarrow 0} \|\nabla u^\varepsilon - \mathcal{U}_n^\varepsilon(\nabla u^0 + \nabla_{y_1} u^1 + \dots + \nabla_{y_n} u^n)\|_{L^2(D)^{d \times d}} = 0. \quad (3.40)$$

Proof We consider the expression

$$\begin{aligned} & \int_D \int_{Y_1} \dots \int_{Y_n} \mathcal{T}_n^\varepsilon(a^\varepsilon) (\mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon)) - (\epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n))) : \\ & \quad (\mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon)) - (\epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n))) dy_n \dots dy_1 dx. \\ = & \int_D \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(a^\varepsilon) \mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon)) : \mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon)) d\mathbf{y} dx \\ & - 2 \int_D \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(a^\varepsilon) \mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon)) : (\epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n)) d\mathbf{y} dx \\ & + \int_D \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(a^\varepsilon) (\epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n)) : \\ & \quad (\epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n)) d\mathbf{y} dx. \end{aligned}$$

We note that $\mathcal{T}_n^\varepsilon(a^\varepsilon) \rightarrow a(x, y_1, \dots, y_n)$ pointwise when $\varepsilon \rightarrow 0$. From (3.6), (3.37) and (3.38) we deduce that each of the integral converges to $\int_D f u^0 dx$ thus this expression converges to 0 when $\varepsilon \rightarrow 0$. From this together with (3.3), we have the strong convergence

$$\mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon)) \rightarrow \epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n) \quad (3.41)$$

in $L^2(D \times Y_1 \times \dots \times Y_n)^{d \times d}$. Using $\mathcal{U}_n^\varepsilon(\mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon))) = \epsilon(u^\varepsilon)$, we then have from (3.36) and (2.75)

$$\begin{aligned} & \int_D |\epsilon(u^\varepsilon) - \mathcal{U}_n^\varepsilon(\epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n))(x)|^2 dx \\ = & \int_D \left| \mathcal{U}_n^\varepsilon \left(\mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon)) - (\epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n)) \right) (x) \right|^2 dx \\ \leq & \int_D \mathcal{U}_n^\varepsilon \left(\mathcal{T}_n^\varepsilon((\epsilon(u^\varepsilon)) - (\epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n)))^2 \right) dx \\ \leq & \int_D \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon)) - (\epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n))^2 d\mathbf{y} dx. \end{aligned}$$

We get the conclusion of (3.39).

If $u^0 \in H^2(D)^d$ then $\epsilon(u^0) \in H^1(D)^{d \times d}$. From the proof of Lemma 5.5 in Hoang and Schwab [34] we have that

$$\int_D |\epsilon(u^0(x)) - \mathcal{U}_n^\varepsilon(\epsilon(u^0))(x)|^2 dx \leq c\varepsilon^2. \quad (3.42)$$

As $a \in C^2(D, C^2(\bar{Y}_1, \dots, C^2(\bar{Y}_n) \dots))^{d^4}$, a proof similar to that of Lemma 3.6.3 shows that $N^{p,rs} \in C^1(\bar{D}, C^1(\bar{Y}_1, \dots, C^1(\bar{Y}_p) \dots))^d$ for all $p = 1, \dots, n$ and all $r, s = 1, \dots, d$, so $\epsilon_{y_p}(N^{p,rs})$ is uniformly continuous with respect to $x, y_1, \dots, y_p$. Thus

$$\limsup_{\varepsilon \rightarrow 0} \sup_{x \in D} \left| \epsilon_{y_p} \left(N^{p,rs} \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p} \right) \right) - \mathcal{U}_n^\varepsilon(\epsilon_{y_p}(N^{p,rs}))(x) \right| = 0.$$

From (3.20), we write $\epsilon_{y_p}(u^p) = M^{p,r_0 s_0} \epsilon_{r_0 s_0}(u^0(x))$ where

$$\begin{aligned} M^{p,r_0 s_0}(x, \mathbf{y}_p) &= \epsilon_{y_p}(N^{p,r_{p-1} s_{p-1}})(x, \mathbf{y}_p) \left(\delta_{s_{p-1} s_{p-2}} \delta_{r_{p-1} r_{p-2}} \right. \\ &\quad \left. + \epsilon_{y_{p-1} r_{p-1} s_{p-1}}(N^{p-1,r_{p-2} s_{p-2}}(x, \mathbf{y}_{p-1})) \right) \cdot \\ &\quad \left(\delta_{r_{p-2} r_{p-3}} \delta_{s_{p-2} s_{p-3}} + \epsilon_{y_{p-2} r_{p-2} s_{p-2}}(N^{p-2,r_{p-3} s_{p-3}}(x, \mathbf{y}_{p-2})) \right) \dots \\ &\quad \left(\delta_{s_1 s_0} \delta_{r_1 r_0} + \epsilon_{y_1 r_1 s_1}(N^{1,r_0 s_0}(x, y_1)) \right). \end{aligned}$$

We have that $M^{r_0 s_0} \in C(\bar{D} \times \bar{Y}_1 \times \dots \times \bar{Y}_p)^d$ and that

$$\limsup_{\varepsilon \rightarrow 0} \sup_{x \in D} |\mathcal{U}_n^\varepsilon(M^{r_0 s_0})(x) - M^{r_0 s_0}(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p})| = 0.$$

We therefore deduce that

$$\begin{aligned} &\int_D \left| M^{p,r_0 s_0} \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p} \right) \epsilon_{r_0 s_0}(u^0(x)) - \mathcal{U}_n^\varepsilon(M^{p,r_0 s_0} \epsilon_{r_0 s_0}(u^0))(x) \right|^2 \\ &= 2 \sup_{x, y_1, \dots, y_p} (M^{p,r_0 s_0})^2 \int_D |\epsilon_{r_0 s_0}(u^0(x)) - \mathcal{U}_n^\varepsilon(\epsilon_{r_0 s_0}(u^0))(x)|^2 dx + \\ &\quad 2 \sup_{x \in D} \left| M \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p} \right) - \mathcal{U}_n^\varepsilon(M)(x) \right|^2 \int_D (\epsilon_{r_0 s_0}(u^0)(x))^2 dx, \end{aligned}$$

which converges to 0 when $\varepsilon \rightarrow 0$, i.e.

$$\lim_{\varepsilon \rightarrow 0} \int_D \left| \epsilon_{y_p}(u^p(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p})) - \mathcal{U}_n^\varepsilon(\epsilon_{y_p}(u^p))(x) \right|^2 dx = 0. \quad (3.43)$$

From (3.43), we have

$$\lim_{\varepsilon \rightarrow 0} \|\epsilon(u^\varepsilon) - \epsilon_x(u^0(\cdot) + \varepsilon_1 u^1(\cdot, \frac{\cdot}{\varepsilon_1}) + \dots + \varepsilon_n u^n(\cdot, \frac{\cdot}{\varepsilon_1}, \dots, \frac{\cdot}{\varepsilon_n}))\|_{L^2(D)^{d \times d}} = 0. \quad (3.44)$$

From Korn's inequality, for functions in $H^1(D)^d$ (see Oleinik et al. [47]), we have

$$\begin{aligned} &\|u^\varepsilon(\cdot) - [u^0(\cdot) + \varepsilon_1 u^1(\cdot, \frac{\cdot}{\varepsilon_1}) + \dots + \varepsilon_n u^n(\cdot, \frac{\cdot}{\varepsilon_1}, \dots, \frac{\cdot}{\varepsilon_n})]\|_{H^1(D)^d} \\ &\leq \|u^\varepsilon(\cdot) - [u^0(\cdot) + \varepsilon_1 u^1(\cdot, \frac{\cdot}{\varepsilon_1}) + \dots + \varepsilon_n u^n(\cdot, \frac{\cdot}{\varepsilon_1}, \dots, \frac{\cdot}{\varepsilon_n})]\|_{L^2(D)^d} \\ &\quad + \|\epsilon(u^\varepsilon) - \epsilon_x(u^0(\cdot) + \varepsilon_1 u^1(\cdot, \frac{\cdot}{\varepsilon_1}) + \dots + \varepsilon_n u^n(\cdot, \frac{\cdot}{\varepsilon_1}, \dots, \frac{\cdot}{\varepsilon_n}))\|_{L^2(D)^{d \times d}}, \end{aligned}$$

which converges to 0 when $\varepsilon \rightarrow 0$ as $u^\varepsilon \rightarrow u^0$ in $L^2(D)^d$ and (3.44). Using a similar argument as for showing (3.43), we deduce

$$\lim_{\varepsilon \rightarrow 0} \int_D \left| \nabla_{y_p} u^p(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p}) - \mathcal{U}_n^\varepsilon(\nabla_{y_p} u^p)(x) \right|^2 dx = 0. \quad (3.45)$$

Summing this over all p and using triangle inequality with the above inequality, we get (3.40). $\square$

We then deduce the following approximation for the solution u^ε to the multiscale elasticity problem (3.4) in terms of the finite element solution of (3.9).

Theorem 3.6.7. *Assume that $f \in L^2(D)^d$ and the domain D is a smooth convex domain. With the full and sparse tensor finite element product approximations, we have*

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ L \rightarrow \infty}} \|\nabla u^\varepsilon(\cdot) - \mathcal{U}_n^\varepsilon(\nabla u^{0L} + \nabla_{y_1} u^{1L} + \dots + \nabla_{y_n} u^{nL})\|_{L^2(D^L)^{d \times d}} = 0.$$

Proof We first note that

$$\begin{aligned} & \|\mathcal{U}_n^\varepsilon(\nabla u^0 + \dots + \nabla_{y_n} u^n) - \mathcal{U}_n^\varepsilon(\nabla u^{0L} + \dots + \nabla_{y_n} u^{nL})\|_{L^2(D^L)^{d \times d}}^2 \\ & \leq \int_{D^L} \mathcal{U}_n^\varepsilon(|(\nabla u^0 + \dots + \nabla_{y_n} u^n) - (\nabla u^{0L} + \dots + \nabla_{y_n} u^{nL})|^2)(x) dx \\ & \leq \int_{D^L} \int_{Y_1} \dots \int_{Y_n} |(\nabla u^0 + \dots + \nabla_{y_n} u^n) - (\nabla u^{0L} + \dots + \nabla_{y_n} u^{nL})|^2 dy_n \dots dy_1 dx \end{aligned}$$

which converges to 0 when $\varepsilon \rightarrow 0$. From this and (3.40) we deduce the result by

$$\begin{aligned} & \lim_{\substack{\varepsilon \rightarrow 0 \\ L \rightarrow \infty}} \|\nabla u^\varepsilon(\cdot) - \mathcal{U}_n^\varepsilon(\nabla u^{0L} + \nabla_{y_1} u^{1L} + \dots + \nabla_{y_n} u^{nL})\|_{L^2(D^L)^{d \times d}} \\ & \leq \lim_{\substack{\varepsilon \rightarrow 0 \\ L \rightarrow \infty}} \|\nabla u^\varepsilon(\cdot) - \mathcal{U}_n^\varepsilon(\nabla u^0 + \nabla_{y_1} u^1 + \dots + \nabla_{y_n} u^n)\|_{L^2(D^L)^{d \times d}} \\ & \quad + \|\mathcal{U}_n^\varepsilon(\nabla u^0 + \dots + \nabla_{y_n} u^n) - \mathcal{U}_n^\varepsilon(\nabla u^{0L} + \dots + \nabla_{y_n} u^{nL})\|_{L^2(D^L)^{d \times d}}. \end{aligned}$$

$\square$

Remark 3.6.8. *Following Remark 3.5.2, Theorem 3.6.7 holds when the homogenized elastic moduli a^0 is isotropic and homogeneous, and D is a convex polygon in two dimensions.*

3.7 Numerical Results

We present in this section some numerical examples for two and three scale problems that confirm our analysis above. The sparse tensor finite element product space is constructed using the wavelet preconditioning, as in the Example in Section 2.9.

3.7.1 Two-dimensional two-scale problem

We consider the isotropic case where the elastic moduli are of the form

$$a_{ijkl}(x, y) = \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda(x, y)\delta_{ij}\delta_{kl},$$

where $\mu(x)$ is independent of y . This is a generalization of the Hill's materials considered in Jikov et al. [37]. Indeed, Jikov et al. [37] consider the case where μ is a constant and $\lambda(x, y) = \lambda(y)$. In this case, the homogenized elastic moduli are also isotropic and can be computed exactly without solving the cell problem. Following exactly the argument of Jikov et al. [37], we can show that when $\mu = \mu(x)$ and $\lambda = \lambda(x, y)$, the effective elastic moduli is isotropic and is of the form

$$a_{ijkl}^0(x) = \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d}\delta_{ij}\delta_{kl}) + K^0(x)\delta_{ij}\delta_{kl}. \quad (3.46)$$

The homogenized bulk modulus is determined by

$$K^0(x) = -\beta(x) + \langle (\beta(x) + K(x, y))^{-1} \rangle^{-1}, \quad (3.47)$$

where

$$K(x, y) = \lambda(x, y) + \frac{2\mu(x)}{d}, \quad \beta(x) = \frac{2(d-1)}{d}\mu(x),$$

and $\langle \cdot \rangle$ denotes the integral average over Y . We consider the multiscale problem (3.4) where $\mu(x) = (1 + x')(1 + x'')$ and $\lambda(x, y) = x'x'' \cos^2(2\pi y') \cos^2(2\pi y'')$ in the domain $D = (0, 1) \times (0, 1)$. The multiscale elasticity problem is

$$-\frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon(x) \epsilon_{kl}(u^\varepsilon(x))) = f_i(x), \quad (3.48)$$

$$u^\varepsilon(x) = 0 \text{ on } \partial D,$$

where f_1 and f_2 are calculated numerically by the homogenized problem,

$$f_1(x) = -\frac{\partial}{\partial x_j} (a_{1jkl}^0(x) \epsilon_{kl}(u^0(x))),$$

$$f_2(x) = -\frac{\partial}{\partial x_j} (a_{2jkl}^0(x) \epsilon_{kl}(u^0(x))).$$

where the homogenized solution is

$$u_1^0(x) = u_2^0(x) = \sin(\pi x') \sin(\pi x''). \quad (3.49)$$

In Figure 3.1, the energy error $B(\mathbf{u} - \mathbf{u}^L, \mathbf{u} - \mathbf{u}^L)$ is plotted versus mesh size h . The straight line with slope 2 confirms our analysis on the sparse tensor products.

Figure 3.1: Energy error for sparse tensor product FEs for 2D two-scale problem

3.7.2 Two-dimensional three-scale problem

We then consider a two-dimensional three-scale problem with an isotropic material where

$$\mu(x) = (1 + x')(1 + x'')$$

$$\lambda(x, y_1, y_2) = x'x'' \cos^2(2\pi y'_1) \cos^2(2\pi y''_1) \cos^2(2\pi y'_2) \cos^2(2\pi y''_2)$$

in the domain $D = (0, 1) \times (0, 1)$. Following the procedure of Jikov et al. [37] (see the Section 3.8), the homogenized coefficient is determined by (3.46) where

$$K^0(x) = \lambda^0(x) + \frac{2\mu(x)}{d} = -\beta(x) + \frac{1}{\int_{Y_1} \int_{Y_2} \frac{dy_1 dy_2}{\beta(x) + K(x, y_1, y_2)}}. \quad (3.50)$$

and

$$\beta(x) = \frac{2(d-1)}{d} \mu(x),$$

$$K(x, y_1, y_2) = \lambda(x, y_1, y_2) + \frac{2\mu(x)}{d}.$$

The multiscale elasticity problem is

$$-\frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon(x) \epsilon_{kl}(u^\varepsilon(x))) = f_i(x), \quad u^\varepsilon(x) = 0 \text{ on } \partial D. \quad (3.51)$$

where f_1 and f_2 are

$$f_1(x) = -\frac{\partial}{\partial x_j} (a_{1jkl}^0(x) \epsilon_{kl}(u^0(x))),$$

$$f_2(x) = -\frac{\partial}{\partial x_j} (a_{2jkl}^0(x) \epsilon_{kl}(u^0(x))).$$

The homogenized solution is

$$u_1^0(x) = u_2^0(x) = \sin(\pi x') \sin(\pi x'').$$

Figure 3.2: Energy error for sparse tensor product FEs for 2D three-scale problem

In Figure 3.2, the energy error $B(\mathbf{u} - \mathbf{u}^L, \mathbf{u} - \mathbf{u}^L)$ is plotted with mesh sizes $h = 1/2, 1/4, 1/8$. The straight line with slope 2 again confirms our analysis.

3.8 Homogenized effective bulk modulus

We now prove the homogenized effective bulk modulus $K^0(x)$ for three-scale Hill's material given in (3.50). The formula for the two-scale case is given in Jikov et al. [37]. Even though they only consider the case where μ is a constant, their approach can be extended to the case where μ depends on x but does not depend on y . The result by Jikov does not extend immediately to the three-scale case. For a three-scale problem, let $M^{1,rs} \in L^2(D, H_{\#}^1(Y_1)/\mathbb{R})$, $M^{2,rs} \in L^2(D \times Y_1, H_{\#}^1(Y_2)/\mathbb{R})$ be the solution of the problem

$$\int_D \int_{Y_1} \int_{Y_2} a(x, y_1, y_2) (e^{rs} + \epsilon_{y_1}(M^{1,rs}) + \epsilon_{y_2}(M^{2,rs})) : (\epsilon_{y_1}(\phi^1) + \epsilon_{y_2}(\phi^2)) dy_2 dy_1 dx = 0 \quad (3.52)$$

for all $\phi^1 \in L^2(D, H_{\#}^1(Y_1)/\mathbb{R})^d$ and $\phi^2 \in L^2(D \times Y_1, H_{\#}^1(Y_2)/\mathbb{R})^d$. Then $u^1 = M^{1,rs} e_{rs}(u^0)$ and $u^2 = M^{2,rs} e_{rs}(u^0)$. The homogenized elastic moduli is determined by

$$a_{ijrs}^0(x) = \int_{Y_1} \int_{Y_2} a_{ijkl}(e_{kl}^{rs} + \epsilon_{y_1 kl}(M^{1,rs}) + \epsilon_{y_2 kl}(M^{2,rs})) dy_2 dy_1. \quad (3.53)$$

We consider the case where $a(x, y_1, y_2)$ is isotropic and is of the form

$$a_{ijkl}(x, y_1, y_2) = \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda(x, y_1, y_2)\delta_{ij}\delta_{kl}.$$

From (3.53), we deduce

$$a_{ijrs}^0(x) = \mu(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}) + \langle \lambda \rangle \delta_{ij}\delta_{rs} + \delta_{ij} \int_{Y_1} \int_{Y_2} \lambda (\operatorname{div}_{y_1} M^{1,rs} + \operatorname{div}_{y_2} M^{2,rs}) dy_2 dy_1. \quad (3.54)$$

When $r \neq s$, it is easy to see that $M^{1,rs} = 0$ and $M^{2,rs} = 0$ is the solution of (3.52). We now consider the case where $r = s$. Problem (3.52) is written as

$$\begin{aligned} & 2\mu \int_D \int_{Y_1} \int_{Y_2} (\epsilon_{y_1 ij}(M^{1,ss}) + \epsilon_{y_2 ij}(M^{2,ss})) (\epsilon_{y_1 ij}(\phi^1) + \epsilon_{y_2 ij}(\phi^2)) dy_2 dy_1 dx + \\ & \int_D \int_{Y_1} \int_{Y_2} \lambda (\operatorname{div}_{y_1} M^{1,ss} + \operatorname{div}_{y_2} M^{2,ss}) (\operatorname{div}_{y_1}(\phi^1) + \operatorname{div}_{y_2}(\phi^2)) dy_2 dy_1 dx + \\ & \int_D \int_{Y_1} \int_{Y_2} \lambda (\operatorname{div}_{y_1} \phi^1 + \operatorname{div}_{y_2} \phi^2) dy_2 dy_1 dx = 0. \end{aligned}$$

From this we deduce that $M^{1,ss} = M^1$ and $M^{2,ss} = M^2$ for all s .

Let $\phi^1 = 0$ in (3.52). We get

$$-\mu \Delta_{y_2} M_i^2 - \mu \frac{\partial}{\partial y_{2,i}} \operatorname{div}_{y_2} M_i^2 - \frac{\partial \lambda}{\partial y_{2,i}} \operatorname{div}_{y_1} M_i^1 - \frac{\partial}{\partial y_{2,i}} (\lambda \operatorname{div}_{y_2} M_i^2) = \frac{\partial \lambda}{\partial y_{2,i}}.$$

Following Jikov et al. [37], we consider $M^2 = \nabla_{y_2} h_2$ for a function h_2 which is periodic with respect to y_2 . We then deduce that

$$-\mu \frac{\partial}{\partial y_{2,i}} \Delta h_2 - \frac{\partial}{\partial y_{2,i}} ((\mu + \lambda) \Delta h_2) - \frac{\partial \lambda}{\partial y_{2,i}} \operatorname{div}_{y_1} M_i^1 = \frac{\partial \lambda}{\partial y_{2,i}}.$$

Thus there is a function $c_2(x, y_1)$ such that

$$-(2\mu + \lambda) \Delta h_2 - \lambda \operatorname{div}_{y_1} M^1 = \lambda + c_2(x, y_1).$$

Consider $M^1 = \nabla_{y_1} h_1$ where h_1 is periodic with respect to y_1 , we have

$$-(2\mu + \lambda) \Delta h_2 - \lambda \Delta h_1 = \lambda + c_2(x, y_1). \quad (3.55)$$

Let $\phi^2 = 0$ in (3.52). We have

$$\begin{aligned} & -\mu \Delta_{y_1} M_i^1 - \mu \frac{\partial}{\partial y_{1,i}} \operatorname{div}_{y_1} M_i^1 - \frac{\partial}{\partial y_{1,i}} \left(\left(\int_{Y_2} \lambda dy_2 \right) \operatorname{div}_{y_1} M_i^1 \right) \\ & = \frac{\partial}{\partial y_{1,i}} \left(\int_{Y_2} \lambda \operatorname{div}_{y_2} M_i^2 dy_2 \right) + \frac{\partial}{\partial y_{1,i}} \left(\int_{Y_2} \lambda dy_2 \right). \end{aligned}$$

With $M^1 = \nabla_{y_1} h_1$, we have

$$-\left(2\mu + \int_{Y_2} \lambda dy_2 \right) \Delta h_1 = \int_{Y_2} \lambda \Delta_{y_2} h_2 dy_2 + \int_{Y_2} \lambda dy_2 + c_1(x). \quad (3.56)$$

From (3.54), we have

$$a_{ijrs}^0(x) = \mu(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}) + \langle \lambda \rangle \delta_{ij}\delta_{rs} + \delta_{rs}\delta_{ij} \int_{Y_1} \int_{Y_2} \lambda(\Delta_{y_1}h_1 + \Delta_{y_2}h_2)dy_2dy_1.$$

We therefore need to find $\int_{Y_1} \int_{Y_2} \lambda(\Delta_{y_1}h_1 + \Delta_{y_2}h_2)dy_2dy_1$. Integrating the two sides of (3.55) with respect to y_2 , we get

$$\int_{Y_2} \lambda(\Delta_{y_2}h_2 + \Delta_{y_1}h_1)dy_2 = - \int_{Y_2} \lambda dy_2 - c_2(x, y_1). \quad (3.57)$$

Thus, from (3.56), we have

$$-2\mu\Delta_{y_1}h_1 = -c_2(x, y_1) + c_1(x). \quad (3.58)$$

Integrating both sides with respect to y_1 , we get

$$c_1(x) = \int_{Y_1} c_2(x, y_1)dy_1.$$

From (3.55),

$$\Delta h_2 = -\frac{\lambda}{2\mu + \lambda}\Delta_{y_1}h_1 - \frac{\lambda}{2\mu + \lambda} - \frac{c_2(x, y_1)}{2\mu + \lambda}. \quad (3.59)$$

We therefore have

$$\int_{Y_2} \frac{\lambda}{2\mu + \lambda} dy_2 \Delta_{y_1}h_1 + \int_{Y_2} \frac{\lambda}{2\mu + \lambda} dy_2 + \int_{Y_2} \frac{1}{2\mu + \lambda} dy_2 c_2(x, y_1) = 0.$$

As $\Delta_{y_1}h_1 = (c_1(x) - c_2(x, y_1))/(2\mu)$, we have

$$- \left(\int_{Y_2} \frac{\lambda}{2\mu + \lambda} dy_2 \right) \frac{c_1}{2\mu} + \frac{1}{2\mu} c_2 + \int_{Y_2} \frac{\lambda}{2\mu + \lambda} dy_2 = 0. \quad (3.60)$$

Integrating this with respect to y_1 , we have

$$- \left(\int_{Y_1} \int_{Y_2} \frac{\lambda}{2\mu + \lambda} dy_2 dy_1 \right) \frac{c_1}{2\mu} + \frac{c_1}{2\mu} + \int_{Y_1} \int_{Y_2} \frac{\lambda}{2\mu + \lambda} dy_2 dy_1 = 0.$$

Therefore

$$\begin{aligned} c_1(x) &= - \frac{2\mu \int_{Y_1} \int_{Y_2} \frac{\lambda}{2\mu + \lambda} dy_2 dy_1}{1 - \int_{Y_1} \int_{Y_2} \frac{\lambda}{2\mu + \lambda} dy_2 dy_1} \\ &= - \frac{\int_{Y_1} \int_{Y_2} \frac{\lambda}{2\mu + \lambda} dy_2 dy_1}{\int_{Y_1} \int_{Y_2} \frac{1}{2\mu + \lambda} dy_2 dy_1} \\ &= - \frac{1}{\int_{Y_1} \int_{Y_2} \frac{1}{2\mu + \lambda} dy_2 dy_1} + 2\mu. \end{aligned}$$

The function $c_2(x, y_1)$ is then found from (3.60). Then (3.58) possesses a unique solution h_1 in $H_{\#}^1(Y_1)/\mathbb{R}$ for each x , (3.59) possesses unique solution h_2 in $H_{\#}^1(Y_2)/\mathbb{R}$ for each x and y_1 . Assuming that λ and μ are sufficiently smooth, $M^1 = \nabla_{y_1} h_1$ and $M^2 = \nabla_{y_2} h_2$ is the unique solution of (3.52). We therefore have from (3.57)

$$\int_{Y_1} \int_{Y_2} \lambda (\Delta_{y_2} h_2 + \Delta_{y_1} h_1) dy_2 dy_1 = -\langle \lambda \rangle - 2\mu + \frac{1}{\int_{Y_1} \int_{Y_2} \frac{1}{2\mu + \lambda} dy_2 dy_1}.$$

We then get (3.50).

Multiscale Elastic Wave Equation

4.1 Introduction

In this chapter, we consider an elastic wave equation whose moduli depend on n separable microscopic scales and are periodic with respect to each scale in a domain D in $\mathbb{R}^d$. By using the multiscale convergence (see [44],[3],[4]), we deduce an $(n+1)d$ dimensional problem for the solution of the homogenized equation, which describes the multiscale solution macroscopically, and the scale interacting terms, which encode the microscopic information of the solution. We apply the full and the sparse tensor FE methods for the vector solution of the multiscale elastic wave equation. We find FE approximations for the solution of the homogenized equation and the scale interacting terms and use them to derive an approximation for the solution of the original multiscale equation. Following our analysis in Chapter 2 and restricting our consideration to the case of a zero initial wave displacement, a corrector similar to that for the wave equation in Chapter 2 can be deduced. We can, therefore, derive an approximation in the $H^1(D)$ norm for the solution of the multiscale equation in terms of the finite element approximation of the multiscale homogenized problem. In the two-scale case, due to the availability of a homogenization rate of convergence, an error for this approximation is deduced. It is the sum of the homogenization and the FE errors. As in Chapter 3, the homogenization rate of convergence deduced in this chapter requires less regularity than that for stationary two-scale elasticity equations in Oleinik et al. [47]. For problems that depend on more than two microscopic scales, such a homogenization rate of convergence is not available. Thus the numerical corrector for the solution of the multiscale elastic wave equation is derived without an explicit error estimate.

The chapter is organized as follows. In the next section, we provide the setting up of the elastic wave equation that depends on multiple separable microscopic scales. In Section 4.3, we extend the theory of multiscale convergence to functions that depend on time and specify the multiscale homogenized limit of the multiscale elastic wave equation. In Section 4.4, we establish FE approximations for the high dimensional limiting multiscale homogenized problem. We first consider the semidiscrete problems

where the equation is discretized with respect only to the spatial variable. We then consider the fully discrete problem where the temporal variable is also discretized. To obtain the rate of convergence for the general FE discretization in Section 4.4, and for the full and sparse tensor product FEs in Section 4.6, we need regularity for the solution of the multiscale homogenized problem. We establish these in Section 4.5. We construct the full and sparse tensor product FEs in Section 4.6. The dimension of the full tensor product FE can be prohibitively large. However, we show that the sparse tensor product FE approximation achieves essentially the same level of accuracy as for the full tensor product FEs but with a much reduced number of degrees of freedom that is essentially equal to that required for solving an equation in $\mathbb{R}^d$. Assuming that the initial displacement is zero, we establish analytic correctors for the homogenized problem in Section 4.7. For the two-scale case, we deduce an explicit rate of convergence. For general multiscale problems, such an explicit homogenization rate of convergence is not available. However, we prove a corrector result when the ratio of any two consecutive microscopic scales $\frac{\varepsilon_{p-1}}{\varepsilon_p}(\varepsilon_{p-1} > \varepsilon_p)$ is an integer. In Section 4.8, we derive numerical correctors for the solution of the original multiscale equation using the FE approximations of the multiscale homogenized equation. We deduce an explicit rate of convergence for these correctors for the two-scale problem, which is the sum of the FE error and the analytic homogenization error. Numerical correctors for multiscale problems are deduced without an error estimate due to the unavailability of an analytic homogenization error. Finally in Section 4.9, we present numerical results for two-scale and three-scale problems in two dimensions that confirm our analysis. In Section 4.10 we derive the exact term u^1 that we use to compare our FE solution to in the two-scale example in Section 4.9.

Throughout this chapter, we follow the notations and the definitions in Chapter 2 and Chapter 3.

4.2 Problem setting

As in Chapter 2, we denote by V the space $H_0^1(D)$ and by H the space $L^2(D)$. For $T > 0$, let $f = (f_1, \dots, f_d) \in L^2(0, T; H^d)$. We consider the multiscale elastic wave equation in D with the Dirichlet boundary condition: Find $u^\varepsilon(t, x) \in L^2(0, T; V^d) \cap H^1(0, T; H^d)$ such that for each $i = 1, \dots, d$

$$\begin{aligned} \frac{\partial^2 u_i^\varepsilon}{\partial t^2} - \frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon(x) \epsilon_{kl}(u^\varepsilon)) &= f_i, \\ u^\varepsilon(0, \cdot) &= g^0, \quad u_t^\varepsilon(0, \cdot) = g^1, \\ u^\varepsilon(t, x) &= 0, \quad \text{for } x \in \partial D, \end{aligned} \tag{4.1}$$

where the multiscale elastic moduli are defined in Section 3.2, $g^0 \in V^d$ and $g^1 \in H^d$

Problem (4.1) is written in the weak form as

$$\left(\frac{\partial^2 u_i^\varepsilon(t, \cdot)}{\partial t^2}, \phi_i \right) + \int_D a_{ijkl}^\varepsilon(x) \epsilon_{kl}(u^\varepsilon(t, x)) \epsilon_{ij}(\phi(x)) dx = \int_D f_i(t, x) \phi_i(x) dx, \quad \forall \phi \in V^d. \quad (4.2)$$

Applying the existence and uniqueness theory of hyperbolic equations for the Gelfand triple $V^d \subset H^d \subset (V^d)'$ (see Wloka [55]), we conclude that (4.2) has a unique solution that satisfies

$$\|u^\varepsilon\|_{L^2(0,T;V^d) \cap H^1(0,T;H^d)} \leq c(\|f\|_{L^2(0,T;H^d)} + \|g^0\|_{V^d} + \|g^1\|_{H^d}), \quad (4.3)$$

where the constant c only depends on α, β, T and the domain D . In particular it does not depend on ε .

4.3 Multiscale homogenized equation

In this section, we study the asymptotic behaviour of u^ε when $\varepsilon \rightarrow 0$ by multiscale homogenization. We formulate the concept of multiscale convergence following Subsection 2.3.1.

4.3.1 Multiscale homogenization of the elastic wave equation

We study the multiscale convergence of the solution to equation (4.2). We define the bilinear form $B : \mathbf{V}^d \times \mathbf{V}^d \rightarrow \mathbb{R}$ by

$$B(\mathbf{u}, \mathbf{v}) = \int_D \int_{\mathbf{Y}} a_{ijkl}(x, \mathbf{y}) \epsilon_{kl}(\mathbf{u}) \epsilon_{ij}(\mathbf{v}) d\mathbf{y} dx$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}^d$, where $\mathbf{V}^d$ is defined in Section 3.3.2. We have the following result.

Theorem 4.3.1. *There is a function $\mathbf{u} = (u^0, u^1, \dots, u^n) \in L^2(0, T; \mathbf{V}^d)$ such that the solution u^ε of (4.2) converges weakly to u^0 in $L^2(0, T; V^d)$ and for all $i, j = 1, \dots, d$, $\epsilon_{ij}(u^\varepsilon)$ $(n+1)$ -scale converges to $\epsilon_{ij}(\mathbf{u})$. Further $\mathbf{u}$ is the unique solution of the following problem*

$$\left(\frac{\partial^2 u^0(t, \cdot)}{\partial t^2}, v^0 \right) + B(\mathbf{u}, \mathbf{v}) = \int_D f(t, x) \cdot v^0(x) dx \quad (4.4)$$

for all $\mathbf{v} \in \mathbf{V}^d$, with the initial condition

$$u^0(0, x) = g^0(x), \quad \frac{\partial u^0}{\partial t}(0, x) = g^1(x). \quad (4.5)$$

Proof of this theorem is similar to the proof of Theorem 2.3.4 extended to d -dimensional vector functions. The solution u^ε of the multiscale problem (4.1) can be approximated by the solution of the high dimensional multiscale homogenized problem (4.4) as illustrated in the following result. We will consider this issue in details in Section 4.7.

Proposition 4.3.2. *For the two-scale problem, i.e. $n = 1$, assume that the initial condition $g^0 = 0$. Assume further that the solutions $N^{rs}(x, y)$ of cell problems in (4.24) are in $C(\bar{D}, W^{1,\infty}(Y))^d$ for all $r, s = 1, \dots, d$. The following corrector result holds*

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \left\| \nabla u^\varepsilon(t, \cdot) - \left[\nabla u^0(t, \cdot) + \nabla_y u^1(t, \cdot, \frac{\cdot}{\varepsilon}) \right] \right\|_{H^{d \times d}} = 0.$$

4.4 Finite element discretization

In the following subsections, we consider spatially semidiscrete problems and fully discrete problems with respect to both space and time.

4.4.1 Spatially semidiscrete problem

With the spaces V_0^L and V_p^L in Section 2.4, we define by

$$(\mathbf{V}^L)^d = (V_0^L)^d \times (V_1^L)^d \times \dots \times (V_n^L)^d \subset \mathbf{V}^d. \quad (4.6)$$

The spatially semi-discretized approximating problem is defined as: Find $\mathbf{u}^L = (u^{0L}, u^{1L}, \dots, u^{nL}) \in L^2(0, T; (\mathbf{V}^L)^d)$ such that

$$\left(\frac{\partial^2 u^{0L}}{\partial t^2}, v^{0L} \right) + B(\mathbf{u}^L, \mathbf{v}^L) = (f, v^{0L}) \quad (4.7)$$

for all $\mathbf{v}^L = (v^{0L}, v^{1L}, \dots, v^{nL}) \in (\mathbf{V}^L)^d$, with the initial conditions $u^{0L}(0, \cdot) = g^{0L} \in (V_0^L)^d$ and $\frac{du^{0L}}{dt}(0, \cdot) = g^{1L} \in (V_0^L)^d$. The discretized initial conditions g^{0L} and g^{1L} are chosen to approximate g^0 and g^1 in V^d and in H^d respectively. The following proposition is shown by following the procedure in the proof of Proposition 2.4.1.

Proposition 4.4.1. *Problem (4.7) has a unique solution.*

We prove the convergence of the semidiscrete problem (4.7) following the approach of Dupont [20]. For each $t \in [0, T]$, let $\mathbf{w}^L(t) = (w^{0L}(t), w^{1L}(t), \dots, w^{nL}(t)) \in (\mathbf{V}^L)^d$ be the solution of the problem

$$B(\mathbf{w}^L(t) - \mathbf{u}(t), \mathbf{v}^L) = 0 \quad (4.8)$$

for all $\mathbf{v}^L = (v^{0L}, v^{1L}, \dots, v^{nL}) \in (\mathbf{V}^L)^d$.

We define $\boldsymbol{\eta}^L(t) = (\eta^{0L}(t), \eta^{1L}(t), \dots, \eta^{nL}(t)) \in (\mathbf{V}^L)^d$, such that

$$\boldsymbol{\eta}^L(t) = \mathbf{w}^L(t) - \mathbf{u}(t). \quad (4.9)$$

From (4.4) and (4.8) we get

$$B(\mathbf{w}^L, \mathbf{v}^L) = \left(f - \frac{\partial^2 u^0}{\partial t^2}, v^{0L} \right) \quad (4.10)$$

for all $\mathbf{v}^L = (v^{0L}, v^{1L}, \dots, v^{nL}) \in (\mathbf{V}^L)^d$.

We have the following estimates.

Lemma 4.4.2. *We have*

$$\|\boldsymbol{\eta}^L(t)\|_{\mathbf{V}^d} \leq c \inf_{\mathbf{v}^L \in (\mathbf{V}^L)^d} \|\mathbf{u}(t) - \mathbf{v}^L\|_{\mathbf{V}^d}. \quad (4.11)$$

Proof We have from (4.8) that

$$B(\mathbf{w}^L - \mathbf{u}, \mathbf{w}^L - \mathbf{u}) = B(\mathbf{w}^L - \mathbf{u}, \mathbf{v}^L - \mathbf{u})$$

for all $\mathbf{v}^L \in (\mathbf{V}^L)^d$. The conclusion follows from Korn's inequality. $\square$

For the error estimates of the FE methods in the next sections, we need to bound the time derivatives of $\boldsymbol{\eta}^L$ in the approximation norms. We thus show the following results.

Lemma 4.4.3. *If $\frac{\partial \mathbf{u}}{\partial t} \in L^\infty(0, T; \mathbf{V}^d)$ then*

$$\left\| \frac{\partial \boldsymbol{\eta}^L}{\partial t} \right\|_{L^\infty(0, T; \mathbf{V}^d)} \leq c \sup_{t \in [0, T]} \inf_{\mathbf{v}^L \in (\mathbf{V}^L)^d} \left\| \frac{\partial \mathbf{u}}{\partial t}(t) - \mathbf{v}^L \right\|_{\mathbf{V}^d}.$$

If $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2(0, T; \mathbf{V}^d)$ then

$$\left\| \frac{\partial^2 \boldsymbol{\eta}^L}{\partial t^2} \right\|_{L^2(0, T; \mathbf{V}^d)} \leq c \inf_{\mathbf{v}^L \in L^2(0, T; (\mathbf{V}^L)^d)} \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathbf{v}^L \right\|_{L^2(0, T; \mathbf{V}^d)}.$$

Proof From equation (4.8) we have

$$B\left(\frac{\partial(\mathbf{w}^L - \mathbf{u})}{\partial t}(t), \mathbf{v}^L\right) = 0 \quad (4.12)$$

for all $\mathbf{v}^L \in (\mathbf{V}^L)^d$. Thus

$$\left\| \frac{\partial(\mathbf{w}^L - \mathbf{u})}{\partial t}(t) \right\|_{\mathbf{V}^d} \leq c \inf_{\mathbf{v}^L \in (\mathbf{V}^L)^d} \left\| \frac{\partial \mathbf{u}}{\partial t}(t) - \mathbf{v}^L \right\|_{\mathbf{V}^d}.$$

From this we get the first conclusion. The second inequality is proved in a similar manner. $\square$

Letting $\boldsymbol{\zeta}^L(t) = (\zeta^{0L}(t), \zeta^{1L}(t), \dots, \zeta^{nL}(t))$ be such that

$$\boldsymbol{\zeta}^L(t) = \mathbf{u}^L(t) - \mathbf{w}^L(t), \quad (4.13)$$

we have from (4.7) and (4.10) that

$$\left(\frac{\partial^2 \zeta^{0L}}{\partial t^2}, v^{0L} \right) + B(\boldsymbol{\zeta}^L, \mathbf{v}^L) = - \left(\frac{\partial^2 \eta^{0L}}{\partial t^2}, v^{0L} \right) \quad (4.14)$$

for all $\mathbf{v}^L \in (\mathbf{V}^L)^d$.

We have the following bounds for $\boldsymbol{\zeta}^L$.

Lemma 4.4.4. *If the solution $\mathbf{u}$ of (4.4) belongs to $H^1(0, T; \mathbf{V}^d)$ and $\frac{\partial^2 \eta^{0L}}{\partial t^2}$ belongs to $L^2(0, T; H^d)$ then*

$$\begin{aligned} & \left\| \frac{\partial \zeta^{0L}}{\partial t} \right\|_{L^\infty(0, T; H^d)} + \|\zeta^L\|_{L^\infty(0, T; \mathbf{V}^d)} \\ & \leq c \left[\|\nabla \zeta^{0L}(0)\|_{H^{d \times d}} + \left\| \frac{\partial \zeta^{0L}}{\partial t}(0) \right\|_{H^d} + \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0, T; H^d)} \right]. \end{aligned} \quad (4.15)$$

Proof Letting $\mathbf{v}^L = \frac{\partial \zeta^L}{\partial t}$ in (4.14), we have

$$\left(\frac{\partial^2 \zeta^{0L}}{\partial t^2}, \frac{\partial \zeta^{0L}}{\partial t} \right) + B(\zeta^L, \frac{\partial \zeta^L}{\partial t}) = - \left(\frac{\partial^2 \eta^{0L}}{\partial t^2}, \frac{\partial \zeta^{0L}}{\partial t} \right).$$

Thus

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left[\left(\frac{\partial \zeta^{0L}}{\partial t}, \frac{\partial \zeta^{0L}}{\partial t} \right) + B(\zeta^L, \zeta^L) \right] &= - \left(\frac{\partial^2 \eta^{0L}}{\partial t^2}, \frac{\partial \zeta^{0L}}{\partial t} \right) \\ &\leq \frac{1}{2\gamma} \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{H^d}^2 + \frac{\gamma}{2} \left\| \frac{\partial \zeta^{0L}}{\partial t} \right\|_{H^d}^2, \end{aligned}$$

where γ can be any positive constants. Integrating over $[0, t]$, from the boundedness and coerciveness conditions of $B(\zeta^L, \zeta^L)$, we have

$$\begin{aligned} & \sup_{t \in [0, T]} \left\{ \left\| \frac{\partial \zeta^{0L}}{\partial t}(t) \right\|_{H^d}^2 + \alpha \|\epsilon(\zeta^{0L}(t)) + \epsilon_{y_1}(\zeta^{1L}(t)) + \dots + \epsilon_{y_n}(\zeta^{nL}(t))\|_{L^2(D \times \mathbf{Y})^{d \times d}}^2 \right\} \\ & \leq \frac{1}{\gamma} \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0, T; H^d)}^2 + \gamma T \sup_{t \in [0, T]} \left\| \frac{\partial \zeta^{0L}}{\partial t}(t) \right\|_{H^d}^2 \\ & \quad + \beta \|\epsilon(\zeta^{0L}(0)) + \epsilon_{y_1}(\zeta^{1L}(0)) + \dots + \epsilon_{y_n}(\zeta^{nL}(0))\|_{L^2(D \times \mathbf{Y})^{d \times d}}^2 + \left\| \frac{\partial \zeta^{0L}}{\partial t}(0) \right\|_{H^d}^2. \end{aligned}$$

Let $v^{0L} = 0$ and $v^{pL} = \zeta^{pL}(0)$ for $p = 1, \dots, n$ in (4.14), we have,

$$\begin{aligned} & \int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) (\epsilon(\zeta^{0L}(0)) + \epsilon_{y_1}(\zeta^{1L}(0)) + \dots + \epsilon_{y_n}(\zeta^{nL}(0))) \\ & \quad : (\epsilon_{y_1}(\zeta^{1L}(0)) + \dots + \epsilon_{y_n}(\zeta^{nL}(0))) d\mathbf{y} dx = 0. \end{aligned}$$

We therefore have that

$$\alpha \|\epsilon_{y_1}(\zeta^{1L}(0)) + \dots + \epsilon_{y_n}(\zeta^{nL}(0))\|_{L^2(D \times \mathbf{Y})^{d \times d}} \leq \beta \|\epsilon(\zeta^{0L}(0))\|_{H^{d \times d}}.$$

Letting γ be sufficiently small, using Korn's inequality we get (4.15). $\square$

Thus the following convergence result for the solution $\mathbf{u}^L$ of the approximating problem (4.7) holds.

Proposition 4.4.5. *Assume that $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2(0, T; \mathbf{V}^d)$, and that*

$$\lim_{L \rightarrow \infty} \|g^{0L} - g^0\|_{V^d} = 0 \quad \text{and} \quad \lim_{L \rightarrow \infty} \|g^{1L} - g^1\|_{H^d} = 0. \quad (4.16)$$

Then

$$\lim_{L \rightarrow \infty} \left\{ \left\| \frac{\partial(u^{0L} - u^0)}{\partial t} \right\|_{L^\infty(0, T; H^d)} + \|\mathbf{u}^L - \mathbf{u}\|_{L^\infty(0, T; \mathbf{V}^d)} \right\} = 0.$$

The proof of this proposition is shown by results from Lemma 4.4.2, Lemma 4.4.3 and Lemma 4.4.4. It extends the proof of Proposition 2.4.5 to multiscale elastic wave equations.

4.4.2 Fully discrete problem

Following the scheme of Dupont [20] as in Section 2.4.2, we consider time discretization of problem (4.7). Let the time step $\Delta t = T/M$ where M is a positive integer. Assuming that $f \in C([0, T]; H^d)$ we consider the following fully discrete problem: for $m = 1, \dots, M-1$, find $\mathbf{u}^{L;m} = (u^{0L;m}, u^{1L;m}, \dots, u^{nL;m}) \in (\mathbf{V}^L)^d$ such that

$$(\partial_t^2 u^{0L;m}, v^{0L}) + B(\mathbf{u}^{L;m, 1/4}, \mathbf{v}^L) = (f^{m, 1/4}, v^{0L}) \quad (4.17)$$

for all $\mathbf{v}^L \in (\mathbf{V}^L)^d$.

For $\mathbf{u}^{L;m}$ defined in (4.17) and $\mathbf{w}^L$ defined in (4.8) we define $\boldsymbol{\zeta}^{L;m}$ as

$$\boldsymbol{\zeta}^{L;m} = \mathbf{u}^{L;m} - \mathbf{w}^{L;m}. \quad (4.18)$$

We then have the following lemma.

Lemma 4.4.6. *Assume that $\mathbf{u} \in H^1(0, T; \mathbf{V}^d)$ and $\partial^2 \eta^{0L} / \partial t^2 \in L^2(0, T; H^d)$. If $\partial^3 u^0 / \partial t^3 \in L^2(0, T; H^d)$ then there exists a constant c (independent of L , Δt and $\mathbf{u}$) such that*

$$\begin{aligned} & \|\partial_t \zeta^{0L}\|_{\tilde{L}^\infty(0, T; H^d)} + \|\boldsymbol{\zeta}^L\|_{\tilde{L}^\infty(0, T; \mathbf{V}^d)} \\ & \leq c \left[\|\boldsymbol{\zeta}^{L; 1/2}\|_{\mathbf{V}^d} + \|\partial_t \zeta^{0L; 1/2}\|_{H^d} + \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0, T; H^d)} + \left\| \frac{\partial^3 u^0}{\partial t^3} \right\|_{L^2(0, T; H^d)} \Delta t \right]; \end{aligned}$$

and if $\partial^4 u^0 / \partial t^4 \in L^2(0, T; H^d)$ then there exists a constant c such that

$$\begin{aligned} & \|\partial_t \zeta^{0L}\|_{\tilde{L}^\infty(0, T; H^d)} + \|\boldsymbol{\zeta}^L\|_{\tilde{L}^\infty(0, T; \mathbf{V}^d)} \\ & \leq c \left[\|\boldsymbol{\zeta}^{L; 1/2}\|_{\mathbf{V}^d} + \|\partial_t \zeta^{0L; 1/2}\|_{H^d} + \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0, T; H^d)} + \left\| \frac{\partial^4 u^0}{\partial t^4} \right\|_{L^2(0, T; H^d)} (\Delta t)^2 \right]. \end{aligned}$$

Proof of this lemma is similar to that of the proof of Lemma 2.4.6.

Lemma 4.4.7. *There are constants c independent of L and Δt such that*

$$\|\boldsymbol{\eta}^L\|_{\tilde{L}^\infty(0,T;\mathbf{V}^d)} \leq c \sup_{0 \leq m < M} \inf_{\mathbf{v}^L \in (\mathbf{V}^L)^d} \|\mathbf{u}^{m+1/2} - \mathbf{v}^L\|_{\mathbf{V}^d} \quad (4.19)$$

and

$$\|\partial_t \boldsymbol{\eta}^L\|_{\tilde{L}^\infty(0,T;\mathbf{V}^d)} \leq c \sup_{0 \leq m < M} \inf_{\mathbf{v}^L \in (\mathbf{V}^L)^d} \|\partial_t \mathbf{u}^{m+1/2} - \mathbf{v}^L\|_{\mathbf{V}^d}.$$

Proof of this lemma is similar to the proof of Lemma 2.4.7.

Proposition 4.4.8. *Assume that $\mathbf{u} \in H^1(0,T;\mathbf{V}^d)$ and $\partial^2 \eta^{0L}/\partial t^2 \in L^2(0,T;H^d)$. If $\partial^3 u^0/\partial t^3 \in L^2(0,T;H^d)$, then*

$$\begin{aligned} & \|\partial_t u^{0L} - \partial_t u^0\|_{\tilde{L}^\infty(0,T;H^d)} + \|\mathbf{u}^L - \mathbf{u}\|_{\tilde{L}^\infty(0,T;\mathbf{V}^d)} \\ & \leq c \left[\Delta t \left\| \frac{\partial^3 u^0}{\partial t^3} \right\|_{L^2(0,T;H^d)} + \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0,T;H^d)} + \|\partial_t \zeta^{0L;1/2}\|_{H^d} + \|\zeta^{L;1/2}\|_{\mathbf{V}^d} \right] \\ & \quad + \|\partial_t \eta^{0L}\|_{\tilde{L}^\infty(0,T;H^d)} + \|\boldsymbol{\eta}^L\|_{\tilde{L}^\infty(0,T;\mathbf{V}^d)}, \end{aligned} \quad (4.20)$$

where c is independent of L and Δt . If $\partial^4 u^0/\partial t^4 \in L^2(0,T;H^d)$, then

$$\begin{aligned} & \|\partial_t u^{0L} - \partial_t u^0\|_{\tilde{L}^\infty(0,T;H^d)} + \|\mathbf{u}^L - \mathbf{u}\|_{\tilde{L}^\infty(0,T;\mathbf{V}^d)} \\ & \leq c \left[(\Delta t)^2 \left\| \frac{\partial^4 u^0}{\partial t^4} \right\|_{L^2(0,T;H^d)} + \left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0,T;H^d)} + \|\partial_t \zeta^{0L;1/2}\|_{H^d} + \|\zeta^{L;1/2}\|_{\mathbf{V}^d} \right] \\ & \quad + \|\partial_t \eta^{0L}\|_{\tilde{L}^\infty(0,T;H^d)} + \|\boldsymbol{\eta}^L\|_{\tilde{L}^\infty(0,T;\mathbf{V}^d)}. \end{aligned} \quad (4.21)$$

As a consequence, if $\partial^2 \mathbf{u}/\partial t^2 \in L^2(0,T;\mathbf{V}^d)$ and $\partial^3 u^0/\partial t^3 \in L^2(0,T;H^d)$, and when $\mathbf{u}^{0L;0}$ and $\mathbf{u}^{0L;1}$ are chosen so that

$$\lim_{L \rightarrow \infty} \|\partial_t \zeta^{0L;1/2}\|_{H^d} = 0, \quad \lim_{L \rightarrow \infty} \|\zeta^{L;1/2}\|_{\mathbf{V}^d} = 0, \quad (4.22)$$

then

$$\lim_{\substack{L \rightarrow \infty \\ \Delta t \rightarrow 0}} \|\partial_t u^{0L} - \partial_t u^0\|_{\tilde{L}^\infty(0,T;H^d)} + \|\mathbf{u}^L - \mathbf{u}\|_{\tilde{L}^\infty(0,T;\mathbf{V}^d)} = 0. \quad (4.23)$$

Proof of this Proposition is similar to that of Proposition 2.4.8. We derive (4.20) and (4.20) from Lemma 4.4.6. The limit (4.23) then follows from Lemma 4.4.2 and Lemma 4.4.7.

4.5 Regularity for the solution

We prove in this section that when the domain and the forcing are sufficiently regular, the regularity required on the solution of (4.4) for obtaining the error estimates in the previous section, and for the error estimates for the full and sparse tensor product FE

approximations considered in Section 4.6 hold. We extend the derivation in Section 3.5 to deduce the homogenized equation from (4.4).

The solution u^p is written in terms of the solutions of the cell problems as

$$\begin{aligned} u_p = & N^{p, r_{p-1} s_{p-1}}(x, \mathbf{y}_p) (\delta_{r_{p-1} r_{p-2}} \delta_{s_{p-1} s_{p-2}} + \epsilon_{y_{p-1} r_{p-1} s_{p-1}} (N^{p-1, r_{p-2} s_{p-2}}(x, \mathbf{y}_{p-1}))) \\ & (\delta_{r_{p-2} r_{p-3}} \delta_{s_{p-2} s_{p-3}} + \epsilon_{y_{p-2} r_{p-2} s_{p-2}} (N^{p-2, r_{p-3} s_{p-3}}(x, \mathbf{y}_{p-2}))) \dots \\ & (\delta_{r_1 r_0} \delta_{s_1 s_0} + \epsilon_{y_1 r_1 s_1} (N^{1, r_0 s_0}(x, y_1))) \epsilon_{r_0 s_0} (u^0(t, x)). \end{aligned} \quad (4.24)$$

The homogenized elastic wave equation is

$$\frac{\partial^2 u_i^0}{\partial t^2} - \frac{\partial}{\partial x_j} (a_{ijkl}^0 \epsilon_{kl}(u^0)) = f_i \quad (4.25)$$

for all $i = 1, \dots, d$.

We then have the following regularity result.

Proposition 4.5.1. *If the following conditions hold*

$$\begin{aligned} f & \in H^2(0, T; H^d), \\ g^1 & \in V^d, \\ f(0) + \nabla \cdot (a^0(\cdot) \epsilon(g^0)) & \in V^d, \\ \frac{\partial f}{\partial t}(0) + \nabla \cdot (a^0(\cdot) \epsilon(g^1)) & \in H^d, \end{aligned} \quad (4.26)$$

then

$$\frac{\partial^2 u^0}{\partial t^2} \in L^2(0, T; V^d), \quad \text{and} \quad \frac{\partial^3 u^0}{\partial t^3} \in L^2(0, T; H^d). \quad (4.27)$$

If the following conditions hold

$$\begin{aligned} f & \in H^3(0, T; H^d), \\ g^1 & \in V^d, \\ f(0) + \nabla \cdot (a^0(\cdot) \epsilon(g^0)) & \in V^d, \\ \frac{\partial f}{\partial t}(0) + \nabla \cdot (a^0(\cdot) \epsilon(g^1)) & \in V^d, \\ \frac{\partial^2 f}{\partial t^2}(0) + \nabla \cdot (a^0(\cdot) \epsilon(f(0) + \nabla \cdot (a^0(\cdot) \epsilon(g^0)))) & \in H^d, \end{aligned} \quad (4.28)$$

then

$$\frac{\partial^3 u^0}{\partial t^3} \in L^2(0, T; V^d), \quad \text{and} \quad \frac{\partial^4 u^0}{\partial t^4} \in L^2(0, T; H^d). \quad (4.29)$$

Proof We use the regularity theory of general wave equations (Wloka [55] Chapter 5) where (4.26) and (4.28) are the compatibility conditions. Under (4.26), we have that

$$\frac{\partial^2}{\partial t^2} \left(\frac{\partial u^0}{\partial t} \right) - \nabla \cdot \left(a^0 \epsilon \left(\frac{\partial u^0}{\partial t} \right) \right) = \frac{\partial f}{\partial t}, \quad (4.30)$$

with the compatibility initial condition

$$\frac{\partial u^0}{\partial t}(0, \cdot) = g^1(\cdot) \in V^d, \quad \frac{\partial}{\partial t} \frac{\partial u^0}{\partial t}(0, \cdot) = f(0, \cdot) + \nabla \cdot (a^0(\cdot) \epsilon(g^0(\cdot))) \in H^d.$$

We also have

$$\frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 u^0}{\partial t^2} \right) - \nabla \cdot \left(a^0 \epsilon \left(\frac{\partial^2 u^0}{\partial t^2} \right) \right) = \frac{\partial^2 f}{\partial t^2}, \quad (4.31)$$

with the compatibility initial condition

$$\frac{\partial^2 u^0}{\partial t^2}(0, \cdot) = f(0, \cdot) + \nabla \cdot (a^0(\cdot) \epsilon(g^0(\cdot))) \in V^d$$

and

$$\frac{\partial}{\partial t} \frac{\partial^2 u^0}{\partial t^2}(0, \cdot) = \frac{\partial f}{\partial t}(0, \cdot) + \nabla \cdot (a^0(\cdot) \epsilon(g^1(\cdot))) \in H^d.$$

Therefore (4.27) holds. The regularity (4.29) is proved similarly. $\square$

We follow the definition of the full tensor regularity space $\bar{\mathcal{H}}^d$ in (3.10) and the sparse tensor regularity space $\hat{\mathcal{H}}^d$ in (3.11). The convergence rates for the full and sparse tensor product finite element spaces will be obtained in the following section. We make the following regularity assumption for the solution of an elasticity equation on D with the elastic moduli a^0 .

Assumption 4.5.2. *For the elasticity equation*

$$-\frac{\partial}{\partial x_j} (a_{ijkl}^0 \epsilon_{kl}(U)) = F_i, \quad U \in H_0^1(D)^d, \quad (4.32)$$

if $F = (F_1, F_2, \dots, F_d) \in L^2(D)^d$ then $U \in H^2(D)^d$ and

$$\|U\|_{H^2(D)^d} \leq c \|F\|_{L^2(D)^d},$$

where c is independent of F .

Remark 4.5.3. As $a \in (C^1(\bar{D}, C_{\#}^1(\bar{Y}_1, \dots, C_{\#}^1(\bar{Y}_n), \dots)))^{d^4}$, we have that $N^{n,rs} \in (C^1(\bar{D}, C_{\#}^1(\bar{Y}_1, \dots, C_{\#}^1(\bar{Y}_{n-1}, H^2(Y_n)) \dots)))^d$. Therefore a^{n-1} as defined in (3.21) for $p = n$ is in $(C^1(\bar{D}, C_{\#}^1(\bar{Y}_1, \dots, C_{\#}^1(\bar{Y}_{n-1}) \dots)))^{d^4}$. Inductively, we then have $a^p \in (C^1(\bar{D}, C_{\#}^1(\bar{Y}_1, \dots, C_{\#}^1(\bar{Y}_p) \dots)))^{d^4}$ and $N^{p,rs} \in (C^1(\bar{D}, C_{\#}^1(\bar{Y}_1, \dots, C_{\#}^1(\bar{Y}_{p-1}, H^2(Y_p)) \dots)))^d$ for all $p = 1, \dots, n$. Thus $a_{ijkl}^0 \in C^1(\bar{D})$.

From elliptic regularity theory, see e.g. Wloka [55] or Ciarlet [15], if the boundary ∂D belongs to the C^1 class, we deduce that the solution U of (4.32) belongs to $H^2(D)^d$.

When the homogenized elastic moduli a^0 in (4.25) is homogeneous, i.e. they do not depend on x (this is the case when a does not depend on x), and is isotropic, then if D is a two dimensional convex polygon, the solution U of (4.32) belongs to $H^2(D)^2$ (see for example [13] and [5]) and thus $u^0 \in L^\infty(0, T; H^2(D)^2)$.

We have the following regularity result for the solution $\mathbf{u}$ of (4.4).

Proposition 4.5.4. *Under Assumption 4.5.2, if $f \in H^1(0, T; H^d)$, $g^1 \in V^d$ and $g^0 \in V^d \cap H^2(D)^d$, then $\mathbf{u} \in L^\infty(0, T; \hat{\mathcal{H}}^d)$.*

Proof We have from (4.30) that

$$\frac{\partial u^0}{\partial t} \in L^\infty(0, T; V^d) \quad \text{and} \quad \frac{\partial^2 u^0}{\partial t^2} \in L^\infty(0, T; H^d)$$

(see [55] page 436) which implies

$$-\nabla \cdot (a^0 \epsilon(u^0)) = f - \frac{\partial^2 u^0}{\partial t^2} \in L^\infty(0, T; H^d). \quad (4.33)$$

From Assumption 4.5.2 we deduce that $u^0 \in L^\infty(0, T; H^2(D)^d)$. From (3.18), we have that

$$-\nabla_{y_p}(a(x, y_1, \dots, y_p) \epsilon_{y_p}(N^{p,rs})) = \nabla_{y_p}(a(x, y_1, \dots, y_p) e^{rs}).$$

As $a^p \in C^1(\bar{D}, C_\#^1(\bar{Y}_1, \dots, C_\#^1(\bar{Y}_p) \dots))^{d^4}$ (see Remark 4.5.3), we then have

$$N^{p,rs} \in C^1(\bar{D}, C^1(\bar{Y}_1, \dots, H_\#^2(Y_p)/\mathbb{R} \dots))^d. \quad (4.34)$$

Thus from (3.20), for $p = 1$, we have

$$u^1(t, x, y_1) = N^{1,rs}(x, y_1) \epsilon_{rs}(u^0(t, x)) \in L^\infty(0, T; H^1(D, H^2(Y_1))^d).$$

For $p = 2$, we have

$$u^2(t, x, y_1, y_2) = N^{2,rs}(x, y_1, y_2) (\epsilon_{rs}(u^0(t, x)) + \epsilon_{y_1 rs}(u^1(t, x, y_1))).$$

As $N^{2,rs}(x, y_1, y_2) \in (C^1(\bar{D}, C_\#^1(\bar{Y}_1, H^2(Y_2))))^d$, $\epsilon_{rs}(u^0(t, x)) \in L^\infty(0, T; H^1(D)^{d \times d})$ and $\epsilon_{y_1 rs}(u^1(t, x, y_1)) \in L^\infty(0, T; H^1(D, H^1(Y_1))^{d \times d})$, we deduce that $u_2 \in L^\infty(0, T; \hat{\mathcal{H}}_2^d)$. Inductively we have that $u_p \in L^\infty(0, T; \hat{\mathcal{H}}_p^d)$ for all $p = 1, 2, \dots, n$. $\square$

Remark 4.5.5. *If u^0 only belongs to a weaker regularity space $L^\infty(0, T; H^{1+s}(D))$, then u belongs to a weaker regularity space in which u^p only possesses $H^s(D)$ regularity with respect to x . The full and sparse tensor product finite elements will have weaker rates of convergence. We will discuss the situation when Assumption 4.5.2 does not hold in Remark 4.6.6.*

4.6 Rates of convergence for the discretized problems

4.6.1 Full tensor product finite elements

Following the definition of $\bar{V}_p^L$ in Section 2.6.2, we define the full tensor finite element space as

$$(\bar{\mathbf{V}}^L)^d = (V_0^L)^d \times (\bar{V}_1^L)^d \times \dots \times (\bar{V}_n^L)^d.$$

The semidiscrete finite element problem is now defined as: find $\bar{\mathbf{u}}^L = (\bar{u}^{0L}, \bar{u}^{1L}, \dots, \bar{u}^{nL}) \in L^\infty(0, T; (\bar{\mathbf{V}}^L)^d)$ such that

$$\left(\frac{\partial^2 \bar{u}^{0L}}{\partial t^2}, \bar{v}^{0L} \right) + B(\bar{\mathbf{u}}^L, \bar{\mathbf{v}}^L) = (f, \bar{v}^{0L}), \quad \forall \bar{\mathbf{v}}^L \in (\bar{\mathbf{V}}^L)^d \quad (4.35)$$

with $\bar{u}^{0L}(0) = g^{0L}$ and $\frac{\partial \bar{u}^{0L}}{\partial t}(0) = g^{1L}$.

We then have the following result for the continuous time approximation.

Proposition 4.6.1. *Assume that condition (4.26) and Assumption 4.5.2 hold and that g^{0L} and g^{1L} are chosen so that*

$$\|g^{0L} - g^0\|_{V^d} \leq ch_L, \quad \text{and} \quad \|g^{1L} - g^1\|_{H^d} \leq ch_L, \quad (4.36)$$

then

$$\left\| \frac{\partial(\bar{u}^{0L} - u^0)}{\partial t} \right\|_{L^\infty(0, T; H^d)} + \|\bar{\mathbf{u}}^L - \mathbf{u}\|_{L^\infty(0, T; \mathbf{V}^d)} \leq ch_L.$$

Proof From (4.31), we deduce that $\frac{\partial^2 u^0}{\partial t^2} \in L^2(0, T; V^d)$ which together with (4.24) and (4.34) implies $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2(0, T; \mathbf{V}^d)$. With Assumption 4.5.2 we have $\mathbf{u} \in L^\infty(0, T; \bar{\mathcal{H}}^d)$. We therefore deduce from Lemmas 4.4.2 and 2.6.1 that for all $t \in [0, T]$

$$\|\boldsymbol{\eta}^L(t)\|_{\mathbf{V}^d} \leq ch_L \|\mathbf{u}(t)\|_{\bar{\mathcal{H}}^d}.$$

We then get from (4.36) and $\|\boldsymbol{\eta}^L(0)\|_{\mathbf{V}^d} \leq ch_L$ that

$$\|\nabla \zeta^{0L}(0)\|_{H^{d \times d}} \leq \|\nabla u^{0L}(0) - \nabla u^0(0)\|_{H^{d \times d}} + \|\nabla u^0(0) - \nabla w^{0L}\|_{H^{d \times d}} \leq ch_L.$$

We now show that

$$\left\| \frac{\partial \eta^{0L}}{\partial t} \right\|_{L^\infty(0, T; H^d)} = \left\| \frac{\partial u^0}{\partial t} - \frac{\partial w^{0L}}{\partial t} \right\|_{L^\infty(0, T; H^d)} \leq ch_L,$$

where c is independent of L .

From the compatibility condition (4.26), we deduce that $\frac{\partial^3 u^0}{\partial t^3} \in L^\infty(0, T; H^d)$. From (4.30) we deduce

$$-\nabla \cdot (a^0 \epsilon(\frac{\partial u^0}{\partial t})) = \frac{\partial f}{\partial t} - \frac{\partial^3 u^0}{\partial t^3} \in L^\infty(0, T; H^d).$$

Therefore from Assumption 4.5.2 we deduce that $\frac{\partial u^0}{\partial t} \in L^\infty(0, T; H^2(D))$. Thus $\frac{\partial \mathbf{u}}{\partial t} \in L^\infty(0, T; \bar{\mathcal{H}}^d)$. Thus from Lemma 4.4.3, we get

$$\left\| \frac{\partial \eta^{0L}}{\partial t} \right\|_{L^\infty(0, T; H^d)} \leq ch_L. \quad (4.37)$$

Now we show that

$$\left\| \frac{\partial^2(u^0 - w^{0L})}{\partial t^2}(t, \cdot) \right\|_{L^2(0, T; H^d)} \leq ch_L.$$

This can be shown in the same manner as for (4.37) if $\frac{\partial^2 u_0}{\partial t^2} \in L^2(0, T; H^2(D))$. However, the condition (4.26) that we assume in this Proposition does not imply this (though (4.28), which we do not assume here, guarantees this condition). However, it can be shown in the following way.

For each $t \in [0, T]$, as $\frac{\partial^2(u^0 - w^{0L})}{\partial t^2}(t, \cdot) \in H^d$ there is a unique solution $\phi \in \mathbf{V}^d$ of the elliptic problem

$$B(\phi, \mathbf{v}) = \int_D \frac{\partial^2(u^0 - w^{0L})}{\partial t^2}(t, x) \cdot v^0(x) dx, \quad (4.38)$$

for all $\mathbf{v} \in \mathbf{V}^d$. The function ϕ^0 is the solution of the elliptic problem

$$-\frac{\partial}{\partial x_j}(a_{ijkl}^0(x)\epsilon_{kl}(\phi^0)) = \frac{\partial^2(u_i^0 - w_i^{0L})}{\partial t^2}$$

for all $i = 1, \dots, d$. From Assumption 4.5.2

$$\|\phi^0(t, \cdot)\|_{H^2(D)^d} \leq c \left\| \frac{\partial^2(u^0 - w^{0L})}{\partial t^2}(t, \cdot) \right\|_{H^d}.$$

As in (4.24), we can write

$$\begin{aligned} \phi^p &= N^{p, r_{p-1} s_{p-1}}(x, \mathbf{y}_p) (\delta_{r_{p-1} r_{p-2}} \delta_{s_{p-1} s_{p-2}} + \epsilon_{y_{p-1} r_{p-1} s_{p-1}} (N^{p-1, r_{p-2} s_{p-2}}(x, \mathbf{y}_{p-1}))) \\ &\quad (\delta_{r_{p-2} r_{p-3}} \delta_{s_{p-2} s_{p-3}} + \epsilon_{y_{p-2} r_{p-2} s_{p-2}} (N^{p-2, r_{p-3} s_{p-3}}(x, \mathbf{y}_{p-2}))) \dots \\ &\quad (\delta_{r_1 r_0} \delta_{s_1 s_0} + \epsilon_{y_1 r_1 s_1} (N^{1, r_0 s_0}(x, y_1))) \epsilon_{r_0 s_0}(\phi^0(x)). \end{aligned} \quad (4.39)$$

From this we have

$$\|\phi(t, \cdot)\|_{\tilde{\mathbf{H}}^d} \leq c \left\| \frac{\partial^2(u^0 - w^{0L})}{\partial t^2}(t, \cdot) \right\|_{H^d}.$$

Letting $\mathbf{v} = \frac{\partial^2(\mathbf{u} - \mathbf{w}^L)}{\partial t^2}(t, \cdot) \in \mathbf{V}^d$ we get

$$B\left(\phi, \frac{\partial^2(\mathbf{u} - \mathbf{w}^L)}{\partial t^2}(t, \cdot)\right) = \left\| \frac{\partial^2(u^0 - w^{0L})}{\partial t^2}(t, \cdot) \right\|_{H^d}^2.$$

From (4.8), due to the symmetry of B , for all $\mathbf{v}^L \in \bar{\mathbf{V}}^L$

$$B\left(\phi - \mathbf{v}^L, \frac{\partial^2(\mathbf{u} - \mathbf{w}^L)}{\partial t^2}(t, \cdot)\right) = \left\| \frac{\partial^2(u^0 - w^{0L})}{\partial t^2}(t, \cdot) \right\|_{H^d}^2.$$

Therefore

$$\begin{aligned} \left\| \frac{\partial^2(u^0 - w^{0L})}{\partial t^2}(t, \cdot) \right\|_{H^d}^2 &\leq c \inf_{\mathbf{v}^L \in (\bar{\mathbf{V}}^L)^d} \|\phi - \mathbf{v}^L\|_{\mathbf{V}}^d \left\| \frac{\partial^2(\mathbf{u} - \mathbf{w}^L)}{\partial t^2}(t, \cdot) \right\|_{\mathbf{V}^d} \\ &\leq ch_L \|\phi(t, \cdot)\|_{\tilde{\mathbf{H}}^d} \left\| \frac{\partial^2(\mathbf{u} - \mathbf{w}^L)}{\partial t^2}(t, \cdot) \right\|_{\mathbf{V}^d} \\ &\leq ch_L \left\| \frac{\partial^2(u^0 - w^{0L})}{\partial t^2}(t, \cdot) \right\|_{H^d} \left\| \frac{\partial^2(\mathbf{u} - \mathbf{w}^L)}{\partial t^2}(t, \cdot) \right\|_{\mathbf{V}^d}. \end{aligned}$$

Using the uniform boundedness of $\|\frac{\partial^2(\mathbf{u}-\mathbf{w}^L)}{\partial t^2}(t)\|_{L^2(0,T;\mathbf{V}^d)}$ for all L , we get

$$\left\| \frac{\partial^2(u^0 - w^{0L})}{\partial t^2}(t, \cdot) \right\|_{L^2(0,T;H^d)} \leq ch_L.$$

When $t = 0$, we have

$$\left\| \frac{\partial \zeta^{0L}}{\partial t}(0, \cdot) \right\|_{H^d} \leq \|g^{1L} - g^1\|_{H^d} + \left\| \frac{\partial u^0}{\partial t}(0) - \frac{\partial w^{0L}}{\partial t}(0) \right\|_{H^d} \leq ch_L.$$

The theorem is thus proved. $\square$

Remark 4.6.2. By following a similar procedure as for showing that $\|\partial^2(u^0 - w^{0L})/\partial t^2\|_{L^2(0,T;H^d)} \leq ch_L$ we can show that $\|\partial(u^0 - w^{0L})/\partial t\|_{L^\infty(0,T;H^d)} \leq ch_L^2$ which is better than (4.37). However, this will not improve the overall estimates for the finite element error.

We now consider the fully discrete problem: Find $\bar{\mathbf{u}}^{L;m} = (\bar{u}^{0L;m}, \bar{u}^{1L;m}, \dots, \bar{u}^{nL;m}) \in (\bar{\mathbf{V}}^L)^d$ such that

$$(\partial_t^2 \bar{u}^{0L;m}, \bar{v}^{0L}) + B(\bar{\mathbf{u}}^{L;m,1/4}, \bar{\mathbf{v}}^L) = (f^{m,1/4}, \bar{v}^{0L}) \quad (4.40)$$

for all $\bar{\mathbf{v}}^L = (\bar{v}^{0L}, \bar{v}^{1L}, \dots, \bar{v}^{nL}) \in (\bar{\mathbf{V}}^L)^d$.

We have the following error estimates.

Proposition 4.6.3. Assume that condition (4.26) and Assumption 4.5.2 hold. If

$$\|\partial_t \zeta^{0L;1/2}\|_{H^d} \leq c(\Delta t + h_L), \quad \|\zeta^{L;1/2}\|_{\mathbf{V}^d} \leq c(\Delta t + h_L),$$

then

$$\|\partial_t \bar{u}^{0L} - \partial_t u^0\|_{\tilde{L}^\infty(0,T;H^d)} + \|\bar{\mathbf{u}}^L - \mathbf{u}\|_{\tilde{L}^\infty(0,T;\mathbf{V}^d)} \leq c(\Delta t + h_L).$$

If (4.28) holds, and

$$\|\partial_t \zeta^{0L;1/2}\|_{H^d} \leq c((\Delta t)^2 + h_L), \quad \|\zeta^{L;1/2}\|_{\mathbf{V}^d} \leq c((\Delta t)^2 + h_L),$$

then

$$\|\partial_t \bar{u}^{0L} - \partial_t u^0\|_{\tilde{L}^\infty(0,T;H^d)} + \|\bar{\mathbf{u}}^L - \mathbf{u}\|_{\tilde{L}^\infty(0,T;\mathbf{V}^d)} \leq c((\Delta t)^2 + h_L).$$

Proof We bound the right hand sides of (4.20) and (4.21). We first note from the proof of Proposition 4.6.1 that

$$\left\| \frac{\partial^2 \eta^{0L}}{\partial t^2} \right\|_{L^2(0,T;H^d)} \leq ch_L.$$

With the regularity $\mathbf{u} \in L^\infty(0,T;\bar{\mathcal{H}}^d)$ we deduce from Lemma 4.4.2 that

$$\|\boldsymbol{\eta}^L\|_{\tilde{L}^\infty(0,T;\mathbf{V}^d)} \leq ch_L.$$

We now show that $\|\partial_t \eta^{0L}\|_{\tilde{L}^\infty(0,T;H^d)} \leq ch_L$. As $\mathbf{u} \in H^2(0,T;\mathbf{V}^d)$, $\boldsymbol{\eta}^L \in H^2(0,T;\mathbf{V}^d) \subset C^1([0,T];\mathbf{V}^d)$ so

$$\left\| \frac{\eta^{0L;m+1} - \eta^{0L;m}}{\Delta t} \right\|_{H^d} = \left\| \frac{\partial \eta^{0L}}{\partial t}(\tau) \right\|_{H^d}$$

for a $\tau \in (t_m, t_{m+1})$. Therefore

$$\|\partial_t \eta^{0L}\|_{\tilde{L}^\infty(0,T;H^d)} \leq \left\| \frac{\partial \boldsymbol{\eta}^L}{\partial t} \right\|_{L^\infty(0,T;\mathbf{V}^d)} \leq \left\| \frac{\partial \mathbf{w}^L}{\partial t} - \frac{\partial \mathbf{u}}{\partial t} \right\|_{L^\infty(0,T;\mathbf{V}^d)} \leq ch_L$$

as $\frac{\partial \mathbf{u}}{\partial t} \in L^\infty(0,T;\bar{\mathcal{H}}^d)$ (which is due to $\frac{\partial u^0}{\partial t} \in L^\infty(0,T;H^2(D))$).

We then get the conclusion. $\square$

4.6.2 Sparse tensor product finite elements

We employ in this subsection the sparse tensor product finite element spaces $\hat{V}_p^L$ as in Section 2.6.3. The sparse FE space for solving problem (4.4) is

$$(\hat{\mathbf{V}}^L)^d = \{(\hat{u}^{0L}, \{\hat{u}^{pL}\}) : \hat{u}^{0L} \in (V_0^L)^d, \hat{u}^{pL} \in (\hat{V}_p^L)^d\}.$$

We consider the spatially semidiscrete problem: Find $\hat{\mathbf{u}}^L = (\hat{u}^{0L}, \hat{u}^{1L}, \dots, \hat{u}^{nL})$ such that

$$\left(\frac{\partial^2 \hat{u}^{0L}}{\partial t^2}, \hat{v}^{0L} \right) + B(\hat{\mathbf{u}}^L, \hat{\mathbf{v}}^L) = (f, \hat{v}^{0L}), \quad \forall \hat{\mathbf{v}}^L \in (\hat{\mathbf{V}}^L)^d \quad (4.41)$$

with the initial conditions $\hat{u}^{0L}(0) = g^{0L}$ and $\frac{\partial \hat{u}^{0L}}{\partial t}(0) = g^{1L}$. We have the following rate of convergence:

Proposition 4.6.4. *Assume that condition (4.26) and Assumption 4.5.2 hold, and that g^{0L} and g^{1L} are chosen so that*

$$\|g^{0L} - g^0\|_{V^d} \leq cL^{n/2}h_L, \quad \text{and} \quad \|g^{1L} - g^1\|_{H^d} \leq cL^{n/2}h_L, \quad (4.42)$$

then

$$\left\| \frac{\partial(\hat{u}^{0L} - u^0)}{\partial t} \right\|_{L^\infty(0,T;H^d)} + \|\hat{\mathbf{u}}^L - \mathbf{u}\|_{L^\infty(0,T;\mathbf{V}^d)} \leq cL^{n/2}h_L.$$

To prove this proposition, we follow the approach of the proof of Proposition 4.6.1, and use the estimates

$$\left\| \frac{\partial(u^0 - w^{0L})}{\partial t} \right\|_{L^\infty(0,T;H^d)} \leq ch_L L^{n/2} \quad \text{and} \quad \left\| \frac{\partial^2(u^0 - w^{0L})}{\partial t^2} \right\|_{L^2(0,T;H^d)} \leq ch_L L^{n/2}.$$

The fully discrete problem (4.17) becomes: find $\hat{\mathbf{u}}^{L;m} = (\hat{u}^{0L;m}, \hat{u}^{1L;m}, \dots, \hat{u}^{nL;m}) \in (\hat{\mathbf{V}}^L)^d$ such that

$$(\partial_t^2 \hat{u}^{0L;m}, \hat{v}^{0L}) + B(\hat{\mathbf{u}}^{L;m,1/4}, \hat{\mathbf{v}}^L) = (f^{m,1/4}, \hat{v}^{0L}) \quad (4.43)$$

for all $\hat{\mathbf{v}}^L = (\hat{v}^{0L}, \hat{v}^{1L}, \dots, \hat{v}^{nL}) \in (\hat{\mathbf{V}}^L)^d$.

We have the following error estimates.

Proposition 4.6.5. *Assume that condition (4.26) and Assumption 4.5.2 hold. If*

$$\|\partial_t \zeta^{0L;1/2}\|_{H^d} \leq c(\Delta t + h_L L^{n/2}), \quad \|\zeta^{L;1/2}\|_{\mathbf{V}^d} \leq c(\Delta t + h_L L^{n/2}),$$

then

$$\|\partial_t \hat{u}^{0L} - \partial_t u^0\|_{\tilde{L}^\infty(0,T;H^d)} + \|\hat{\mathbf{u}}^L - \hat{\mathbf{u}}\|_{\tilde{L}^\infty(0,T;\mathbf{V}^d)} \leq c(\Delta t + h_L L^{n/2}).$$

If condition (4.28) holds and

$$\|\partial_t \zeta^{0L;1/2}\|_{H^d} \leq c((\Delta t)^2 + h_L L^{n/2}), \quad \|\zeta^{L;1/2}\|_{\mathbf{V}^d} \leq c((\Delta t)^2 + h_L L^{n/2}),$$

then

$$\|\partial_t \hat{u}^{0L} - \partial_t u^0\|_{\tilde{L}^\infty(0,T;H^d)} + \|\hat{\mathbf{u}}^L - \mathbf{u}\|_{\tilde{L}^\infty(0,T;\mathbf{V}^d)} \leq c((\Delta t)^2 + h_L L^{n/2}).$$

The proof of this proposition is similar to that for Proposition 4.6.3.

Remark 4.6.6. *We now discuss the case where Assumption 4.5.2 does not hold. If the solution U of (4.32) only belongs to a space $H^{1+s}(D)^d$ for $0 < s < 1$, then $u^0 \in L^\infty(0, T; H^{1+s}(D))$. The functions w^p only possess $H^s(D)$ regularity with respect to x . We thus obtain weaker rates of convergence for both the full and sparse tensor finite elements above, which are $O(h_L^s)$ and $O(h_L^s L^{n/2})$ respectively. Indeed, we only use Assumption 4.5.2 for the regularity of u^0 and the solution of problem (4.38). Without regularity for the solution ϕ of this problem, an estimate for $\|\partial^2(u^0 - w^{0L})/\partial t^2\|_{L^2(0,T;H^d)}$ can be obtained if we have sufficient regularity for $\partial^2 \mathbf{u}/\partial t^2$. Hence, without Assumption 4.5.2, we can obtain the error estimates by assuming directly regularity for u^0 and $\partial \mathbf{u}/\partial t$, ...; and again, if these functions only possess $H^{1+s}(D)$ regularity with respect to x , then the error is only of the order $O(h^s)$ and $O(h^s L^{n/2})$ for the full and the sparse tensor product FE elements respectively.*

4.6.3 Sparse tensor finite element for smooth domains

For a convex smooth domain D , following Section 3.4.3, we discretize a polygon D^L that is inscribed in D . The semidiscretized problem (4.41) and the fully discretized problem (4.43) are restricted to a convex polygon domain D^L . All the analysis above holds as problem (4.4) holds for $\mathbf{v} = (v^0, v^1, \dots, v^n)$ with $v^0 \in H_0^1(D^L)$ and $v^p \in L^2(D^L \times Y_1 \times \dots \times Y_{p-1}, H_{\#}^1(Y_p))$ by extending them to 0 for $x \in D \setminus D^L$.

4.7 Corrector for the homogenization problem

It is well known that for multiscale wave equations, a corrector result similar to that for elliptic equations as constructed in [7] or [37] does not always hold, as the energy of the multiscale problem (4.1) only converges to the energy of the homogenized equation when the initial condition g^0 satisfies special conditions. This phenomenon is analyzed in details by Brahim-Otsmane et al. in [11].

As for the multiscale wave problem in Chapter 2, we restrict our attention to the case where the initial condition $g^0 = 0$ in order to construct corrector results similar to those for elliptic equations, and to deduce a homogenization rate of convergence for the two-scale case.

4.7.1 Corrector for two-scale problem

With the zero initial condition $g^0 = 0$, we can deduce an explicit homogenization error estimate for the two-scale ($n = 1$) elastic wave equation. For conciseness, in the two-scale case we denote y_1 as y and the coefficient $a(x, y_1)$ as $a(x, y)$. The functions $N^{rs}(x, y)$ are defined in (3.23) and the homogenized elastic moduli are given by (3.24). The function $u^1(t, x, y)$ is then written as

$$u^1(t, x, y) = N^{rs}(x, y)\epsilon_{rs}(u^0(t, x)).$$

We then deduce the following result on the homogenization rate of convergence.

Proposition 4.7.1. *Assume Assumptions 4.5.2, $a \in C^2(\bar{D}, C^2(\bar{Y}))^{d^4}$ and that*

$$g^0 = 0, \quad g^1 \in H^2(D)^d \cap V^d, \quad f \in H^2(0, T; H^d), \quad f(0) \in V^d.$$

There exists a constant c independent of ε such that

$$\left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u^0}{\partial t} \right\|_{L^\infty(0, T; H^d)} + \|\nabla u^\varepsilon - [\nabla u^0 + \nabla_y u^1(\cdot, \cdot, \frac{\cdot}{\varepsilon})]\|_{L^\infty(0, T; H^{d \times d})} \leq c\varepsilon^{1/2}.$$

Proof Using the condition of the proposition, we show that

$$u^0 \in H^2(0, T; V^d) \cap L^\infty(0, T; H^2(D)^d), \quad \frac{\partial u^0}{\partial t} \in L^\infty(0, T; V^d \cap H^2(D)^d). \quad (4.44)$$

The compatibility condition (4.26) holds when $g^0 = 0$. Therefore $\frac{\partial^2 u^0}{\partial t^2} \in L^\infty(0, T; V^d)$ and $\frac{\partial^3 u^0}{\partial t^3} \in L^\infty(0, T; H^d)$. From (4.30), we have

$$-\nabla \cdot (a^0 \epsilon(\frac{\partial u^0}{\partial t})) = \frac{\partial f}{\partial t} - \frac{\partial^3 u^0}{\partial t^3} \in L^\infty(0, T; H^d).$$

From Assumption 4.5.2 we have $\frac{\partial u^0}{\partial t} \in L^\infty(0, T; H^2(D)^d)$.

With the proposition's conditions, we can establish regularity for u^ε . We have that

$$\frac{\partial^2}{\partial t^2} \left(\frac{\partial u^\varepsilon}{\partial t} \right) - \nabla \cdot (a_{ijkl}^\varepsilon \epsilon_{kl}(\frac{\partial u^\varepsilon}{\partial t})) = \frac{\partial f}{\partial t}, \quad (4.45)$$

with the compatibility condition

$$\frac{\partial u^\varepsilon}{\partial t}(0) = g^1 \in V^d, \quad \frac{\partial}{\partial t} \frac{\partial u^\varepsilon}{\partial t}(0) = f(0) - \nabla \cdot (a_{ijkl}^\varepsilon(\cdot) \epsilon_{kl}(g^0(\cdot))) = f(0) \in H^d.$$

From this

$$\frac{\partial u^\varepsilon}{\partial t} \in L^2(0, T; V^d) \cap H^1(0, T; H^d). \quad (4.46)$$

Let

$$u^{1\varepsilon}(t, x) = u^0(t, x) + \varepsilon u^1(t, x, \frac{x}{\varepsilon}).$$

Following the proof of Proposition 3.6.1, we have from (3.31)

$$\left\| \frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon \epsilon_{kl}(u^{1\varepsilon}) - a_{ijkl}^0 \epsilon_{kl}(u^0)) \right\|_{L^\infty(0, T; H^{-1}(D)^d)} \leq c\varepsilon.$$

Let $\tau^\varepsilon \in C_0^\infty(D)$ be such that $\tau^\varepsilon = 1$ outside an ε neighbourhood of ∂D and $\varepsilon |\nabla \tau^\varepsilon(x)| \leq c$ for all $x \in D$ and for all $\varepsilon > 0$. Let

$$\begin{aligned} w^{1\varepsilon}(t, x) &= u^0(t, x) + \varepsilon \tau^\varepsilon(x) N^{rs}(x, \frac{x}{\varepsilon}) \epsilon_{rs}(u^0(t, x)) \\ &= u^{1\varepsilon}(t, x) - \varepsilon (1 - \tau^\varepsilon(x)) N^{rs}(x, \frac{x}{\varepsilon}) \epsilon_{rs}(u^0(t, x)). \end{aligned}$$

For $k, l = 1, \dots, d$, we have

$$\begin{aligned} & \frac{\partial}{\partial x_k} (u_l^{1\varepsilon} - w_l^{1\varepsilon})(t, x) \\ &= -\varepsilon \frac{\partial \tau^\varepsilon}{\partial x_k}(x) N_l^{rs}(x, \frac{x}{\varepsilon}) \epsilon_{rs}(u^0(t, x)) + (1 - \tau^\varepsilon(x)) \frac{\partial N_l^{rs}}{\partial y_k}(x, \frac{x}{\varepsilon}) \epsilon_{rs}(u^0(t, x)) + \\ & \quad \varepsilon (1 - \tau^\varepsilon(x)) \frac{\partial N_l^{rs}}{\partial x_k}(x, \frac{x}{\varepsilon}) \epsilon_{rs}(u^0(t, x)) + \varepsilon (1 - \tau^\varepsilon(x)) N_l^{rs}(x, \frac{x}{\varepsilon}) \frac{\partial}{\partial x_k} \epsilon_{rs}(u^0(t, x)). \end{aligned} \quad (4.47)$$

From this

$$\begin{aligned} & \frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon \epsilon_{kl}(u^{1\varepsilon} - w^{1\varepsilon})) \\ &= \frac{1}{2} \varepsilon \frac{\partial}{\partial x_j} \left[a_{ijkl}^\varepsilon(x) (1 - \tau^\varepsilon(x)) \left(\frac{\partial N_l^{rs}}{\partial x_k}(x, \frac{x}{\varepsilon}) + \frac{\partial N_k^{rs}}{\partial x_l}(x, \frac{x}{\varepsilon}) \right) \epsilon_{rs}(u^0(t, x)) \right] + \\ & \quad \frac{1}{2} \varepsilon \frac{\partial}{\partial x_j} \left[a_{ijkl}^\varepsilon(x) (1 - \tau^\varepsilon(x)) \left(N_l^{rs}(x, \frac{x}{\varepsilon}) \frac{\partial}{\partial x_k} \epsilon_{rs}(u^0(t, x)) + \right. \right. \\ & \quad \left. \left. N_k^{rs}(x, \frac{x}{\varepsilon}) \frac{\partial}{\partial x_l} \epsilon_{rs}(u^0(t, x)) \right) \right] + \\ & \quad \frac{1}{2} \frac{\partial}{\partial x_j} \left[a_{ijkl}^\varepsilon(x) \left(-\varepsilon \frac{\partial \tau^\varepsilon}{\partial x_k}(x) N_l^{rs}(x, \frac{x}{\varepsilon}) - \varepsilon \frac{\partial \tau^\varepsilon}{\partial x_l}(x) N_k^{rs}(x, \frac{x}{\varepsilon}) + \right. \right. \\ & \quad \left. \left. (1 - \tau^\varepsilon(x)) \left(\frac{\partial N_k^{rs}}{\partial y_l}(x, \frac{x}{\varepsilon}) + \frac{\partial N_l^{rs}}{\partial y_k}(x, \frac{x}{\varepsilon}) \right) \right) \epsilon_{rs}(u^0(t, x)) \right]. \end{aligned} \quad (4.48)$$

Using $\frac{\partial^2 u^0}{\partial t^2} \in L^\infty(0, T; V^d)$, we deduce from (4.47) that

$$\left\| \frac{\partial^2 w^{1\varepsilon}}{\partial t^2} - \frac{\partial^2 u^0}{\partial t^2} \right\|_{L^\infty(0, T; H^d)} \leq c\varepsilon. \quad (4.49)$$

From

$$\frac{\partial^2 u_i^\varepsilon}{\partial t^2} - \frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon \epsilon_{kl}(u^\varepsilon)) = \frac{\partial^2 u_i^0}{\partial t^2} - \frac{\partial}{\partial x_j} (a_{ijkl}^0 \epsilon_{kl}(u^0)),$$

(4.48) and (4.49), we deduce that

$$\begin{aligned} & \frac{\partial^2 (u_i^\varepsilon - w_i^{1\varepsilon})}{\partial t^2} - \frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon \epsilon_{kl}(u^\varepsilon - w^{1\varepsilon})) \\ &= \frac{\partial^2 u_i^0}{\partial t^2} - \frac{\partial^2 w_i^{1\varepsilon}}{\partial t^2} + \frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon \epsilon_{kl}(w^{1\varepsilon} - u^{1\varepsilon})) \\ & \quad + \frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon \epsilon_{kl}(u^{1\varepsilon})) - \frac{\partial}{\partial x_j} (a_{ijkl}^0 \epsilon_{kl}(u^0)) \\ &= \frac{\partial^2 u_i^0}{\partial t^2} - \frac{\partial^2 w_i^{1\varepsilon}}{\partial t^2} + \frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon \epsilon_{kl}(u^{1\varepsilon})) - \frac{\partial}{\partial x_j} (a_{ijkl}^0 \epsilon_{kl}(u^0)) \\ & \quad - \frac{1}{2} \varepsilon \frac{\partial}{\partial x_j} \left[a_{ijkl}^\varepsilon(x) (1 - \tau^\varepsilon(x)) \left(\frac{\partial N_l^{rs}}{\partial x_k} \left(x, \frac{x}{\varepsilon} \right) + \frac{\partial N_k^{rs}}{\partial x_l} \left(x, \frac{x}{\varepsilon} \right) \right) \epsilon_{rs}(u^0(t, x)) \right] \\ & \quad - \frac{1}{2} \varepsilon \frac{\partial}{\partial x_j} \left[a_{ijkl}^\varepsilon(x) (1 - \tau^\varepsilon(x)) \left(N_l^{rs} \left(x, \frac{x}{\varepsilon} \right) \frac{\partial}{\partial x_k} \epsilon_{rs}(u^0(t, x)) \right. \right. \\ & \quad \quad \left. \left. + N_k^{rs} \left(x, \frac{x}{\varepsilon} \right) \frac{\partial}{\partial x_l} \epsilon_{rs}(u^0(t, x)) \right) \right] \\ & \quad - \frac{1}{2} \frac{\partial}{\partial x_j} \left[a_{ijkl}^\varepsilon(x) \left(-\varepsilon \frac{\partial \tau^\varepsilon}{\partial x_k}(x) N_l^{rs} \left(x, \frac{x}{\varepsilon} \right) - \varepsilon \frac{\partial \tau^\varepsilon}{\partial x_l}(x) N_k^{rs} \left(x, \frac{x}{\varepsilon} \right) \right. \right. \\ & \quad \quad \left. \left. + (1 - \tau^\varepsilon(x)) \left(\frac{\partial N_k^{rs}}{\partial y_l} \left(x, \frac{x}{\varepsilon} \right) + \frac{\partial N_l^{rs}}{\partial y_k} \left(x, \frac{x}{\varepsilon} \right) \right) \right) \epsilon_{rs}(u^0(t, x)) \right]. \\ &= X - \frac{1}{2} \frac{\partial}{\partial x_j} \left[a_{ijkl}^\varepsilon(x) \left(-\varepsilon \frac{\partial \tau^\varepsilon}{\partial x_k}(x) N_l^{rs} \left(x, \frac{x}{\varepsilon} \right) - \varepsilon \frac{\partial \tau^\varepsilon}{\partial x_l}(x) N_k^{rs} \left(x, \frac{x}{\varepsilon} \right) \right. \right. \\ & \quad \left. \left. + (1 - \tau^\varepsilon(x)) \left(\frac{\partial N_k^{rs}}{\partial y_l} \left(x, \frac{x}{\varepsilon} \right) + \frac{\partial N_l^{rs}}{\partial y_k} \left(x, \frac{x}{\varepsilon} \right) \right) \right) \epsilon_{rs}(u^0(t, x)) \right], \end{aligned}$$

where X denotes the sum of the first six terms. From (4.49) and (4.44), we deduce that $\|X\|_{L^\infty(0,T;H^{-1}(D)^d)} \leq c\varepsilon$. Using $\frac{\partial u^0}{\partial t} \in L^\infty(0,T;V^d \cap H^2(D)^d)$, we deduce $\frac{\partial w^{1\varepsilon}}{\partial t} \in$

$L^\infty(0, T; V^d)$ where $\sup_{t \in [0, T]} \|\frac{\partial w^{1\varepsilon}}{\partial t}(t)\|_{V^d} < c$ for c independent of ε . Therefore

$$\begin{aligned}
& \int_0^t \left(\frac{\partial^2(u^\varepsilon - w^{1\varepsilon})}{\partial t'^2}(t', \cdot), \frac{\partial(u^\varepsilon - w^{1\varepsilon})}{\partial t'}(t', \cdot) \right) dt' + \\
& \quad \int_0^t \int_D a_{ijkl}^\varepsilon(x) \epsilon_{kl}(u^\varepsilon(t', x) - w^{1\varepsilon}(t', x)) \epsilon_{ij} \left(\frac{\partial}{\partial t'}(u^\varepsilon(t', x) - w^{1\varepsilon}(t', x)) \right) dx dt' \\
& = \int_0^t \left(X(t', \cdot), \frac{\partial(u^\varepsilon - w^{1\varepsilon})}{\partial t'}(t', \cdot) \right) dt' + \\
& \quad \frac{1}{2} \int_0^t \int_D a_{ijkl}^\varepsilon(x) \left(-\varepsilon \frac{\partial \tau^\varepsilon}{\partial x_k}(x) N_l^{rs}(x, \frac{x}{\varepsilon}) - \varepsilon \frac{\partial \tau^\varepsilon}{\partial x_l}(x) N_k^{rs}(x, \frac{x}{\varepsilon}) + \right. \\
& \quad \left. (1 - \tau^\varepsilon(x)) \left(\frac{\partial N_k^{rs}}{\partial y_l}(x, \frac{x}{\varepsilon}) + \frac{\partial N_l^{rs}}{\partial y_k}(x, \frac{x}{\varepsilon}) \right) \right) \epsilon_{rs}(u^0(t', x)) \\
& \quad \epsilon_{ij} \left(\frac{\partial}{\partial t'}(u^\varepsilon(t', x) - w^{1\varepsilon}(t', x)) \right) dx dt' \\
& \leq c \|X\|_{L^2(0, T; H^{-1}(D)^d)} \left(\left\| \frac{\partial u^\varepsilon}{\partial t'} \right\|_{L^2(0, T; V^d)} + \left\| \frac{\partial w^{1\varepsilon}}{\partial t'} \right\|_{L^2(0, T; V^d)} \right) + \\
& \quad \frac{1}{2} \int_0^t \int_D a_{ijkl}^\varepsilon(x) \left(-\varepsilon \frac{\partial \tau^\varepsilon}{\partial x_k}(x) N_l^{rs}(x, \frac{x}{\varepsilon}) - \varepsilon \frac{\partial \tau^\varepsilon}{\partial x_l}(x) N_k^{rs}(x, \frac{x}{\varepsilon}) + \right. \\
& \quad \left. (1 - \tau^\varepsilon(x)) \left(\frac{\partial N_k^{rs}}{\partial y_l}(x, \frac{x}{\varepsilon}) + \frac{\partial N_l^{rs}}{\partial y_k}(x, \frac{x}{\varepsilon}) \right) \right) \cdot \\
& \quad \left[\frac{\partial}{\partial t'} \left(\epsilon_{rs}(u^0(t', x)) \epsilon_{ij}(u^\varepsilon(t', x) - w^{1\varepsilon}(t', x)) \right) - \right. \\
& \quad \left. \frac{\partial}{\partial t'} \epsilon_{rs}(u^0(t', x)) \epsilon_{ij}(u^\varepsilon(t', x) - w^{1\varepsilon}(t', x)) \right] dx dt'.
\end{aligned}$$

Let $\hat{D}^\varepsilon \subset D$ be the ε neighbourhood of ∂D outside which $\tau^\varepsilon(x) = 1$. As $u^\varepsilon(0, x) = 0$ and $w^{1\varepsilon}(0, x) = 0$, and

$$\left\| \frac{\partial u^\varepsilon}{\partial t}(0) - \frac{\partial w^{1\varepsilon}}{\partial t}(0) \right\|_{H^d} = \left\| \frac{\partial u^0}{\partial t}(0, \cdot) - \frac{\partial w^{1\varepsilon}}{\partial t} \right\|_{H^d} \leq c\varepsilon \left\| \frac{\partial}{\partial t} \epsilon(u^0(0, \cdot)) \right\|_{H^d},$$

we deduce that

$$\begin{aligned}
& \left\| \frac{\partial(u^\varepsilon - w^{1\varepsilon})}{\partial t}(t, \cdot) \right\|_{H^d}^2 + \|\epsilon(u^\varepsilon(t, \cdot) - w^{1\varepsilon}(t, \cdot))\|_{H^{d \times d}}^2 \\
& \leq c\varepsilon + c \max_{r,s} \|\epsilon_{rs}(u^0(t, \cdot))\|_{L^2(\hat{D}^\varepsilon)} \|u^\varepsilon(t, \cdot) - w^{1\varepsilon}(t, \cdot)\|_{V^d} \\
& \quad + c \max_{r,s} \int_0^T \left\| \frac{\partial}{\partial t} \epsilon_{rs}(u^0(t, \cdot)) \right\|_{L^2(\hat{D}^\varepsilon)} dt \sup_{0 \leq t \leq T} \|u^\varepsilon(t, \cdot) - w^{1\varepsilon}(t, \cdot)\|_{V^d}.
\end{aligned}$$

Hoang and Schwab [34] show that for all functions $\phi \in C^\infty(\bar{D})$

$$\|\phi\|_{L^2(\hat{D}^\varepsilon)}^2 \leq c\varepsilon^2 \|\phi\|_{H^1(D)}^2 + c\varepsilon \|\phi\|_{L^2(\partial D)}^2,$$

so

$$\|\phi\|_{L^2(\hat{D}^\varepsilon)} \leq c\varepsilon^{1/2} \|\phi\|_{H^1(D)} \quad \forall \phi \in H^1(D). \quad (4.50)$$

We therefore have for each $k = 1, \dots, d$

$$\left\| \frac{\partial u^0}{\partial x_k}(t, \cdot) \right\|_{L^2(\hat{D}^\varepsilon)^d} \leq c\varepsilon^{1/2} \left\| \frac{\partial u^0}{\partial x_k}(t, \cdot) \right\|_{H^1(D)^d}$$

and

$$\left\| \frac{\partial^2 u^0}{\partial t \partial x_k}(t, \cdot) \right\|_{L^2(\hat{D}^\varepsilon)^d} \leq c\varepsilon^{1/2} \left\| \frac{\partial^2 u^0}{\partial t \partial x_k}(t, \cdot) \right\|_{H^1(D)^d}.$$

We thus deduce from (4.44) and Korn's inequality that for all $t \in [0, T]$

$$\left\| \frac{\partial(u^\varepsilon - w^{1\varepsilon})}{\partial t}(t) \right\|_{H^d}^2 + \|(u^\varepsilon - w^{1\varepsilon})(t)\|_{V^d}^2 \leq c\varepsilon^{1/2} \sup_{0 \leq t \leq T} \|(u^\varepsilon - w^{1\varepsilon})(t)\|_{V^d},$$

which implies that for all $t \in [0, T]$

$$\left\| \frac{\partial(u^\varepsilon - w^{1\varepsilon})}{\partial t}(t) \right\|_{H^d} + \|(u^\varepsilon - w^{1\varepsilon})(t)\|_{V^d} \leq c\varepsilon^{1/2}.$$

From (4.47) and (4.44),

$$\left\| \frac{\partial(u^{1\varepsilon} - w^{1\varepsilon})}{\partial t} \right\|_{L^\infty(0, T; H^d)} \leq c\varepsilon.$$

Using (4.48), (4.44) and (4.50), we deduce

$$\|u^{1\varepsilon} - w^{1\varepsilon}\|_{L^\infty(0, T; V^d)} \leq c\varepsilon^{1/2}.$$

The conclusion follows. $\square$

4.7.2 Corrector for multiscale problem

A homogenization rate of convergence is not available for general multiscale problems. However, as for multiscale elliptic problems in Hoang and Schwab [31] when $\varepsilon_p/\varepsilon_{p+1}$ is an integer for all $p = 1, \dots, n-1$, we can deduce a corrector for the multiscale elastic wave equation.

Let D^ε be the 2ε neighbourhood of D . We use the “unfolding” operator and “folding” operators from Definition 2.7.2 and 2.7.3.

We have the following corrector result.

Proposition 4.7.2. *Assume that $g^0 = 0$, $g^1 \in V^d \cap H^2(D)^d$ and $f \in H^1(0, T; H^d)$. We have*

$$\lim_{\varepsilon \rightarrow 0} \left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u^0}{\partial t} \right\|_{L^\infty(0, T; H^d)} = 0$$

and

$$\lim_{\varepsilon \rightarrow 0} \|\epsilon(u^\varepsilon) - \mathcal{U}_n^\varepsilon(\epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n))\|_{L^\infty(0,T;H^{d \times d})} = 0. \quad (4.51)$$

Further, if $a \in C^2(\bar{D}, C^2(\bar{Y}_1, \dots, C^2(\bar{Y}_n, \dots))^{d^4})$ and if $u^0 \in H^2(D)^d$ then

$$\lim_{\varepsilon \rightarrow 0} \|\nabla u^\varepsilon - \mathcal{U}_n^\varepsilon(\nabla u^0 + \nabla_{y_1} u^1 + \dots + \nabla_{y_n} u^n)\|_{H^{d \times d}} = 0. \quad (4.52)$$

Proof We consider the expression

$$\begin{aligned} E^\varepsilon(t) &= \int_D \left(\frac{\partial u^\varepsilon}{\partial t}(t) - \frac{\partial u^0}{\partial t}(t), \frac{\partial u^\varepsilon}{\partial t}(t) - \frac{\partial u^0}{\partial t}(t) \right) dx + \\ &\quad \int_D \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(a^\varepsilon)(\mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon(t)) - (\epsilon(u^0(t)) + \epsilon_{y_1}(u^1(t)) + \dots + \epsilon_{y_n}(u^n(t)))) : \\ &\quad (\mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon(t)) - (\epsilon(u^0(t)) + \epsilon_{y_1}(u^1(t)) + \dots + \epsilon_{y_n}(u^n(t)))) d\mathbf{y} dx. \end{aligned}$$

From (2.72) and the conservation of energy of wave equations (see Lions and Magenes [40] Lemma 8.3 of Chapter 3, also Brahim-Otsmane et al. [11]) we have

$$\begin{aligned} &\int_D \left(\frac{\partial u^\varepsilon}{\partial t}, \frac{\partial u^\varepsilon}{\partial t} \right) dx + \int_{D^\varepsilon} \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(a^\varepsilon) \mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon(t))) : \mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon(t))) d\mathbf{y} dx \\ &= \int_D \left(\frac{\partial u^\varepsilon}{\partial t}, \frac{\partial u^\varepsilon}{\partial t} \right) dx + \int_D a^\varepsilon \epsilon(u^\varepsilon(t)) : \epsilon(u^\varepsilon(t)) dx \\ &= \int_D (g^1, g^1) dx + 2 \int_0^t \int_D f \cdot \frac{\partial u^\varepsilon}{\partial t} dx dt \\ &\rightarrow \int_D (g^1, g^1) dx + 2 \int_0^t \int_D f \cdot \frac{\partial u^0}{\partial t} dx dt. \end{aligned}$$

Using the convergence property (2.73), we deduce

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} E^\varepsilon(t) &= \int_D (g^1, g^1) dx + 2 \int_0^t \int_D \left(f, \frac{\partial u^0}{\partial t} \right) dx dt - \\ &\quad \left[\int_D \left(\frac{\partial u^0}{\partial t}, \frac{\partial u^0}{\partial t} \right) dx + \int_D \int_{\mathbf{Y}} a(x, \mathbf{y})(\epsilon(u^0(t)) + \epsilon_{y_1}(u^1(t)) + \dots + \epsilon_{y_n}(u^n(t))) : \right. \\ &\quad \left. (\epsilon(u^0(t)) + \epsilon_{y_1}(u^1(t)) + \dots + \epsilon_{y_n}(u^n(t))) d\mathbf{y} dx \right]. \end{aligned}$$

From

$$\begin{aligned} &\int_D \int_{\mathbf{Y}} a(x, \mathbf{y})(\epsilon(u^0(t)) + \epsilon_{y_1}(u^1(t)) + \dots + \epsilon_{y_n}(u^n(t))) : \\ &\quad (\epsilon(u^0(t)) + \epsilon_{y_1}(u^1(t)) + \dots + \epsilon_{y_n}(u^n(t))) d\mathbf{y} dx \\ &= \int_D a^0 \epsilon(u^0(t)) : \epsilon(u^0(t)) dx \end{aligned}$$

and the energy conservation of the homogenized equation

$$\int_D \left(\frac{\partial u^0}{\partial t}, \frac{\partial u^0}{\partial t} \right) dx + \int_D a^0 \epsilon(u^0(t)) : \epsilon(u^0(t)) dx = \int_D (g^1, g^1) dx + 2 \int_0^t \int_D \left(f, \frac{\partial u^0}{\partial t} \right) dx dt,$$

we deduce $\lim_{\varepsilon \rightarrow 0} E^\varepsilon(t) = 0$. The proof of equicontinuity for $E^\varepsilon(t)$ is similar to that in Proposition 2.7.4.

For the second term, we have from (3.41) that

$$\lim_{\varepsilon \rightarrow 0} \|\mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon)) - (\epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n))\|_{L^\infty(0,T;H^{d \times d})} = 0.$$

Using the fact that $(\mathcal{U}_n^\varepsilon(\Phi))^2 \leq \mathcal{U}_n^\varepsilon(\Phi^2)$ we have

$$\begin{aligned} & \int_D |\mathcal{U}_n^\varepsilon(\mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon)) - (\epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n)))(x)|^2 dx \\ & \leq \int_D \mathcal{U}_n^\varepsilon(|\mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon)) - (\epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n))|^2) dx \\ & \leq \int_D \int_{Y_1} \dots \int_{Y_n} |\mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon)) - (\epsilon(u^0) + \epsilon_{y_1}(u^1) + \dots + \epsilon_{y_n}(u^n))|^2 dy_n dy_{n-1} \dots dy_1 dx. \end{aligned}$$

From $\mathcal{U}_n^\varepsilon(\mathcal{T}_n^\varepsilon(\epsilon(u^\varepsilon))) = \epsilon(u^\varepsilon)$ we get (4.51). Now we show that (4.52) holds.

Since $u^0 \in L^\infty(0, T; H^2(D)^d)$, $\epsilon(u^0) \in L^\infty(0, T; H^1(D)^{d \times d})$. From the proof of Lemma 5.5 in Hoang and Schwab [34] we have that

$$\int_D |\epsilon(u^0(t, x)) - \mathcal{U}_n^\varepsilon(\epsilon(u^0))(t, x)|^2 dx \leq c\varepsilon^2 \quad (4.53)$$

uniformly for all $t \in [0, T]$.

Due to the smoothness of the elastic moduli a , $N^{p,rs} \in (C^2(\bar{D}, C^1(\bar{Y}_1, \dots, C^1(\bar{Y}_p) \dots)))^d$ for all $p = 1, \dots, n$ and all $r, s = 1, \dots, d$ (the proof of this is a simple extension of the proof of Lemma 3.6.2 for the case of two scales), so $\epsilon_{y_p}(N^{p,rs})$ is uniformly continuous with respect to $x, y_1, \dots, y_p$. Thus

$$\limsup_{\varepsilon \rightarrow 0} \sup_{x \in D} \left| \epsilon_{y_p} \left(N^{p,rs} \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p} \right) \right) - \mathcal{U}_n^\varepsilon(\epsilon_{y_p}(N^{p,rs}))(x) \right| = 0.$$

Following the proof of Lemma 3.6.6, we have from (3.45) that

$$\lim_{\varepsilon \rightarrow 0} \int_D \left| \nabla_{y_p} u^p(t, x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_p}) - \mathcal{U}_n^\varepsilon(\nabla_{y_p} u^p)(t, x) \right|^2 dx = 0$$

uniformly for all $t \in [0, T]$. Therefore (4.52) holds. $\square$

4.8 Convergence in the physical variable x

Following the approach by Hoang and Schwab [31], we find numerical correctors for the solution of the multiscale elastic wave equation (4.1) from the solutions of the full tensor and the sparse tensor product approximating problems. We first establish this for the two-scale problem for which an explicit convergence rate can be obtained. We then construct a numerical corrector for the multiscale problems.

4.8.1 Convergence in the physical variable for two-scale problem

For the two-scale problem, we consider correctors for u^ε from the finite element solutions in this section. For all results in this and in the next section, we assume that D is a smooth convex domain and D^L is the polygon inscribed in D in which we solve the discrete problem as in Section 4.6.3.

Lemma 4.8.1. *Assume that $g^1 \in V^d$, $f \in H^1(0, T; H^d)$, Assumptions 4.5.2 and $a \in C^2(\bar{D}, C^2(\bar{Y}_1, \dots, C^2(\bar{Y}_n) \dots))^{d^4}$. Then*

$$\sup_{t \in [0, T]} \int_D \left| \nabla_y u^1(t, x, \frac{x}{\varepsilon}) - \mathcal{U}^\varepsilon(\nabla_y u^1(t, \cdot, \cdot))(x) \right|^2 dx \leq c\varepsilon^2,$$

where c is independent of ε and t .

Proof We note from (4.30) that $\partial^2 u^0 / \partial t^2 \in L^\infty(0, T; H^d)$. Therefore, from (4.25), we have

$$-\frac{\partial}{\partial x_j}(a_{ijkl}^0 \epsilon_{kl}(u^0)) = f_i - \frac{\partial^2 u_i^0}{\partial t^2} \in L^\infty(0, T; H^d).$$

Thus $u^0 \in L^\infty(0, T; H^2(D)^d)$. The proof then follows from Lemma 3.6.2 and Lemma 5.5 of Hoang and Schwab [34]. $\square$

We then have the following convergence rates for the semidiscrete approximation (4.7).

Theorem 4.8.2. *Assume that $f \in H^2(0, T; H^d)$, $g^0 = 0$, $g^1 \in V^d \cap H^2(D)^d$, $f(0) \in V^d$, and $a \in C^2(\bar{D}, (C^2(\bar{Y})))^{d^4}$, and the initial conditions are chosen so that $g^{0L} = 0$ and $\|g^{1L} - g^1\|_{H^d} \leq ch_L$. Then the solution of the semidiscrete problem (4.35) for the full tensor product FE spaces satisfies*

$$\left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial \bar{u}^{0L}}{\partial t} \right\|_{L^\infty(0, T; L^2(D^L)^d)} + \|\nabla u^\varepsilon - [\nabla \bar{u}^{0L} + \mathcal{U}^\varepsilon(\nabla_y \bar{u}^{1L})]\|_{L^\infty(0, T; L^2(D^L)^{d \times d})} \leq c(h_L + \varepsilon^{1/2}), \quad (4.54)$$

and the solution of the problem (4.41) for the sparse tensor product FE spaces satisfies

$$\left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial \hat{u}^{0L}}{\partial t} \right\|_{L^\infty(0, T; L^2(D^L)^d)} + \|\nabla u^\varepsilon - [\nabla \hat{u}^{0L} + \mathcal{U}^\varepsilon(\nabla_y \hat{u}^{1L})]\|_{L^\infty(0, T; L^2(D^L)^{d \times d})} \leq c(h_L L^{1/2} + \varepsilon^{1/2}).$$

Proof Due to condition (4.26), the regularity required for Propositions 4.6.1, 4.6.4 and 4.7.1 hold. Using the fact that for all $\Phi \in L^2(D \times Y)$,

$$(\mathcal{U}^\varepsilon(\Phi)(x))^2 \leq \mathcal{U}^\varepsilon(\Phi^2)(x).$$

Therefore for $t \in [0, T]$

$$\|\mathcal{U}^\varepsilon(\nabla_y u^1(t)) - \mathcal{U}^\varepsilon(\nabla_y \bar{u}^{1L}(t))\|_{H^{d \times d}} \leq \|\nabla_y u^1(t) - \nabla_y \bar{u}^{1L}(t)\|_{L^2(D \times Y)^{d \times d}}.$$

The conclusion for the full tensor product approximation follows from Propositions 4.6.1, 4.6.4 and 4.7.1. The result for sparse tensor product is proved similarly. $\square$

For the fully discrete problems (4.40), (4.43) we have the following convergence rates.

Theorem 4.8.3. *Assume that $g^0 = 0$ and that $a \in C^2(\bar{D}, (C^2(\bar{Y})))^{d^4}$. Assume further that*

$$f \in H^2(0, T; H^d), \quad g^1 \in V^d \cap H^2(D)^d, \quad f(0) \in V^d. \quad (4.55)$$

If

$$\|\partial_t \zeta^{0L;1/2}\|_{H^d} \leq c(\Delta t + h_L), \quad \|\zeta^{L;1/2}\|_{V^d} \leq c(\Delta t + h_L),$$

then for the full tensor product approximating problem (4.40)

$$\begin{aligned} & \max_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t \bar{u}^{0L;m+1/2} \right\|_{L^2(D^L)^d} \\ & \quad + \|\nabla u^\varepsilon - [\nabla \bar{u}^{0L} + \mathcal{U}^\varepsilon(\nabla_y \bar{u}^{1L}(\cdot, \cdot, \cdot))]\|_{\tilde{L}^\infty(0, T; L^2(D^L)^{d \times d})} \\ & \leq c(\Delta t + h_L + \varepsilon^{1/2}); \end{aligned} \quad (4.56)$$

and if

$$\|\partial_t \zeta^{0L;1/2}\|_{H^d} \leq c(\Delta t + h_L L^{n/2}), \quad \|\zeta^{L;1/2}\|_{V^d} \leq c(\Delta t + h_L L^{n/2}),$$

then for the sparse tensor product approximating problem (4.43)

$$\begin{aligned} & \max_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t \hat{u}^{0L;m+1/2} \right\|_{L^2(D^L)^d} \\ & \quad + \|\nabla u^\varepsilon - [\nabla \hat{u}^{0L} + \mathcal{U}^\varepsilon(\nabla_y \hat{u}^{1L}(\cdot, \cdot, \cdot))]\|_{\tilde{L}^\infty(0, T; L^2(D^L)^{d \times d})} \\ & \leq c(\Delta t + h_L L^{n/2} + \varepsilon^{1/2}). \end{aligned} \quad (4.57)$$

Assume that

$$f \in H^3(0, T; H^d), \quad g^1 \in H_0^3(D)^d \text{ and } f(0) \in H^2(D)^d. \quad (4.58)$$

If

$$\|\partial_t \zeta^{0L;1/2}\|_{H^d} \leq c((\Delta t)^2 + h_L), \quad \|\zeta^{L;1/2}\|_{V^d} \leq c((\Delta t)^2 + h_L),$$

then for the full tensor product approximating problem (4.40)

$$\begin{aligned} & \Delta t \max_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t \bar{u}^{L0;m+1/2} \right\|_{L^2(D^L)^d} \\ & \quad + \|\nabla u^\varepsilon - [\nabla \bar{u}^{0L} + \mathcal{U}^\varepsilon(\nabla_y \bar{u}^{1L}(\cdot, \cdot, \cdot))]\|_{\tilde{L}^\infty(0, T; L^2(D^L)^{d \times d})} \\ & \leq c((\Delta t)^2 + h_L + \varepsilon^{1/2}); \end{aligned} \quad (4.59)$$

if

$$\|\partial_t \zeta^{0L;1/2}\|_{H^d} \leq c((\Delta t)^2 + h_L L^{n/2}), \quad \|\zeta^{L;1/2}\|_{V^d} \leq c((\Delta t)^2 + h_L L^{n/2}),$$

then for the sparse tensor product approximating problem (4.43)

$$\begin{aligned} & \Delta t \max_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t \hat{u}^{0L;m+1/2} \right\|_{L^2(D^L)^d} \\ & \quad + \left\| \nabla u^\varepsilon - [\nabla \hat{u}^{0L} + \mathcal{U}^\varepsilon(\nabla_y \hat{u}^{1L}(\cdot, \cdot, \cdot))] \right\|_{\tilde{L}^\infty(0,T;L^2(D^L)^{d \times d})} \\ & \leq c((\Delta t)^2 + h_L L^{n/2} + \varepsilon^{1/2}). \end{aligned} \quad (4.60)$$

Proof With $g^0 = 0$ and Assumption 4.5.2, and with either (4.55) or (4.58), the conditions of Proposition 4.7.1 holds. Using

$$\frac{1}{\Delta t}(u^{0,m+1} - u^{0,m}) - \frac{\partial u^0}{\partial t}(t_m) = \frac{\partial u^0}{\partial t}(\tau) - \frac{\partial u^0}{\partial t}(t_m) = \int_{t_m}^{\tau} \frac{\partial^2 u^0}{\partial t^2}(\tau') d\tau',$$

for $t_m \leq \tau \leq t_{m+1}$ and the fact that $\frac{\partial^2 u^0}{\partial t^2} \in L^\infty(0, T; H^d)$ which follows from (4.30), we deduce

$$\sup_{0 \leq m < M} \left\| \partial_t u^{0,m+1/2} - \frac{\partial u^0}{\partial t}(t_m) \right\|_{H^d} \leq c \Delta t.$$

From Propositions 4.6.3 and 4.6.5, we deduce (4.56) and (4.57). As (4.58), together with the fact that $a^0 \in C^2(\bar{D})$ (this is due to $a \in C^2(\bar{D}, C^2(\bar{Y}))^{d^4}$) imply (4.28), we can deduce (4.59) and (4.60) similarly. $\square$

4.8.2 Convergence in the physical variable in the multiscale case

We construct in this section numerical correctors for the general multiscale problems. As an explicit homogenization rate of convergence is not available, we do not have a convergence rate for the numerical corrector. We thus use our general notations for general finite element spaces in Section 4.4 and do not consider separately the two cases of full and sparse tensor product FE spaces. For the semidiscrete approximation problem (4.7), we have the following result.

Theorem 4.8.4. *Assume $a \in C^2(\bar{D}, C^2(\bar{Y}_1, \dots, C^2(\bar{Y}_n) \dots))^{d^4}$, $\varepsilon_{p-1}/\varepsilon_p$ is an integer for all $p = 2, \dots, n$ and $g^0 = 0$, $g^1 \in V^d$, $f \in H^2(0, T; H^d)$, $f(0) \in V^d$, $g^{0L} = 0$, and $\|g^{1L} - g^1\|_{H^d} \rightarrow 0$ when $L \rightarrow \infty$. Then for the solution of the semidiscrete problem (4.7) we have*

$$\begin{aligned} & \lim_{\substack{L \rightarrow \infty \\ \varepsilon \rightarrow 0}} \left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u^{0L}}{\partial t} \right\|_{L^\infty(0,T;L^2(D^L)^d)} + \\ & \quad \left\| \nabla u^\varepsilon - \mathcal{U}_n^\varepsilon(\nabla u^{0L} + \nabla_{y_1} u^{1L} + \dots + \nabla_{y_n} u^{nL}) \right\|_{L^\infty(0,T;L^2(D^L)^{d \times d})} = 0. \end{aligned}$$

Proof Due to (4.26), $\frac{\partial^2 u^0}{\partial t^2} \in L^2(0, T; V^d)$. From (3.20), we get $\frac{\partial^2 u}{\partial t^2} \in L^2(0, T; \mathbf{V}^d)$.

From

$$\begin{aligned} & \|\mathcal{U}_n^\varepsilon(\nabla u^0(t) + \nabla_{y_1} u^1(t) + \dots + \nabla_{y_n} u^n(t)) \\ & \quad - \mathcal{U}_n^\varepsilon(\nabla u^{0L}(t) + \nabla_{y_1} u^{1L}(t) + \dots + \nabla_{y_n} u^{nL}(t))\|_{H^{d \times d}} \\ & \leq \|u(t) - u^L(t)\|_{\mathbf{V}^d}. \end{aligned}$$

We get the convergence from Propositions 4.4.5 and 4.7.2. $\square$

For the fully discrete problem (4.17) we have the following result.

Theorem 4.8.5. *Assume $a \in C^2(\bar{D}, C^2(\bar{Y}_1, \dots, C^2(\bar{Y}_n) \dots))^{d^4}$, $\varepsilon_{p-1}/\varepsilon_p$ is an integer for all $p = 2, \dots, n$, $g^0 = 0$, $g^1 \in V^d$, $f \in H^2(0, T; H^d)$ and $f(0) \in V^d$. If*

$$\lim_{L \rightarrow \infty} \|\partial_t \zeta^{0L;1/2}\|_{H^d} = 0, \quad \lim_{L \rightarrow \infty} \|\zeta^{L;1/2}\|_{\mathbf{V}^d} = 0,$$

then

$$\begin{aligned} & \lim_{\substack{L \rightarrow \infty \\ \Delta t \rightarrow 0 \\ \varepsilon \rightarrow 0}} \sup_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t u^{0L; m+1/2} \right\|_{L^2(D^L)^d} + \\ & \quad \|\nabla u^\varepsilon - \mathcal{U}_n^\varepsilon(\nabla u^{0L} + \nabla_{y_1} u^{1L} + \dots + \nabla_{y_n} u^{nL})\|_{\tilde{L}^\infty(0, T; L^2(D^L)^{d \times d})} = 0. \end{aligned}$$

Proof Due to (4.26), (4.23) holds. We then get the result from Proposition 4.7.2. $\square$

Remark 4.8.6. *If a^0 is homogeneous and isotropic and D is a convex two dimensional domain, the results in Theorem 4.8.2, 4.8.3, 4.8.4 and 4.8.5 hold when D^L is replaced by D .*

4.9 Numerical results

We show in this section some numerical results for two and three scale elastic wave equations in two dimensions, using sparse tensor product FE approximation. These results confirm the analysis in the above sections. The multilevel hierarchy of finite element basis functions are constructed in Section 2.9.1.

4.9.1 Two-dimensional two-scale problem

We first consider a two-dimensional two-scale problem with isotropic elastic moduli

$$a_{ijkl}(x, y) = \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda(x, y)\delta_{ij}\delta_{kl}.$$

For a Hill's material already mentioned in Section 3.7, the homogenized moduli a^0 can also be computed by following exactly the same approach in [37] and are given by

$$a_{ijkl}^0(x, y) = \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d}\delta_{ij}\delta_{kl}) + K^0(x)\delta_{ij}\delta_{kl}.$$

The homogenized bulk modulus is determined by

$$K^0(x) = -\beta(x) + \langle (\beta(x) + K(x, y))^{-1} \rangle^{-1},$$

where

$$K(x, y) = \lambda(x, y) + \frac{2\mu(x)}{d}, \quad \beta(x) = \frac{2(d-1)}{d}\mu(x),$$

and $\langle \cdot \rangle$ denotes the integral average over Y . We consider the multiscale problem where $\mu(x) = (1+x')(1+x'')$ and $\lambda(x, y) = (1+x')(1+x'') \left(\frac{4}{(1+\sin(2\pi y'))(1+\sin(2\pi y''))} - 2 \right)$ in the domain $D = (0, 1)^2$. The problem is

$$\frac{\partial^2 u_i^\varepsilon(t, x)}{\partial t^2} - \frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon(x) \epsilon_{kl}(u^\varepsilon(t, x))) = f_i(t, x), \quad u^\varepsilon(t, x) = 0 \text{ for } x \in \partial D \quad (4.61)$$

$$u^\varepsilon(t, 0) = u^\varepsilon(t, 1) = 0,$$

where f is calculated numerically from the homogenized solution $u^0(t, x)$,

$$f_i(t, x) = \frac{\partial^2 u_i^0(t, x)}{\partial t^2} - \frac{\partial}{\partial x_j} (a_{ijkl}^0(x) \epsilon_{kl}(u^0(t, x))).$$

We assume that the exact homogenized solution is

$$u^0(t, x) = \begin{pmatrix} (\sin(\pi x') \sin(\pi x'')) t^3 \\ (\sin(\pi x') \sin(\pi x'')) t^3 \end{pmatrix}.$$

Computing the exact u^1 is a nontrivial task. For this particular choice of $\lambda(x, y)$, this can be done. We present it in Section 4.10. The term $u^1(t, x, y)$ is (within an additive function of x)

$$u^1(t, x, y) = \begin{pmatrix} -(\frac{1}{4} \cos(2\pi y') + \frac{1}{16} \cos(2\pi y') \sin(2\pi y'')) \sin(\pi(x' + x'')) t^3 \\ -(\frac{1}{4} \cos(2\pi y'') + \frac{1}{16} \sin(2\pi y') \cos(2\pi y'')) \sin(\pi(x' + x'')) t^3 \end{pmatrix}.$$

In Figure 4.1, we plot the error $\|\mathbf{u} - \mathbf{u}^L\|_{\mathbf{V}^d} = \|u^0 - u^{0L}\|_{V^d} + \|u^1 - u^{1L}\|_{V_1^d}$ versus the mesh size h at time $t = 1$ for the sparse tensor product FEs. We choose $\Delta t = h_L^{1/2}/2$ or the closest value so that $1/\Delta t$ is an integer. In particular, we plot for $(h_L, \Delta t) = (1/4, 1/4)$, $(1/8, 1/6)$ and $(1/16, 1/8)$. Figure 4.1 confirms our analysis.

4.9.2 Two-dimensional three-scale problem

We then consider a two-dimensional three-scale problem with the Hill's material where

$$\mu(x) = (1+x')(1+x''),$$

$$\lambda(x, y) = (1+x')(1+x'') \cos^2(2\pi y'_1) \cos^2(2\pi y''_1) \cos^2(2\pi y'_2) \cos^2(2\pi y''_2).$$

Figure 4.1: The error $\|\mathbf{u} - \mathbf{u}^L\|_{\mathbf{V}}$ for sparse tensor product FEs for 2D two-scale problem

The multiscale problem is

$$\frac{\partial^2 u_i^\varepsilon(t, x)}{\partial t^2} - \frac{\partial}{\partial x_j} (a_{ijkl}^\varepsilon(x) \epsilon_{kl}(u^\varepsilon(t, x))) = f_i(t, x), \quad u^\varepsilon(t, x) = 0 \text{ on } \partial D \quad (4.62)$$

$$u^\varepsilon(t, 0) = u^\varepsilon(t, 1) = 0.$$

As shown in Section 3.8, the homogenized bulk modulus is determined by

$$K^0(x) = -\beta(x) + \frac{1}{\int_{Y_1} \int_{Y_2} \frac{1}{\beta(x) + K(x, y_1, y_2)} dy_2 dy_1},$$

where

$$K(x, y_1, y_2) = \lambda(x, y_1, y_2) + \frac{2\mu(x)}{d}, \quad \beta(x) = \frac{2(d-1)}{d}\mu(x).$$

We assume that the solution of the homogenized equation is given by

$$u^0(t, x) = \begin{pmatrix} (\sin(\pi x') \sin(\pi x'')) t^3 \\ (\sin(\pi x') \sin(\pi x'')) t^3 \end{pmatrix}.$$

The forcing term f is calculated numerically by the homogenized solution $u^0(t, x)$,

$$f_i(t, x) = \frac{\partial^2 u_i^0(t, x)}{\partial t^2} - \frac{\partial}{\partial x_j} (a_{ijkl}^0(x) \epsilon_{kl}(u^0(t, x))).$$

In Figure 4.2, the error $\|u^0 - u^{0L}\|_{\mathbf{V}}$ is plotted with mesh sizes and time steps $(h_L, \Delta t) = (1/2, 1/3)$, $(1/4, 1/4)$ and $(1/8, 1/6)$ using sparse tensor product FE spaces (where we have chosen Δt as in the two-scale problem). The straight line with slope 1 again confirms our analysis.

For this problem, we do not have the exact solution u^1 and u^2 so we do not have a plot for the errors $\|u^1 - u^{1L}\|_{L^2(D, H_{\#}^1(Y_1)/\mathbb{R})^d}$ and $\|u^2 - u^{2L}\|_{L^2(D \times Y_1, H_{\#}^1(Y_2)/\mathbb{R})^d}$.

Figure 4.2: The error $\|\mathbf{u} - \mathbf{u}^L\|_{\mathbf{V}}$ for sparse tensor product FEs for 2D three-scale problem

4.10 Exact solution u^1 for the two-scale problem

We derive in this section the exact term u^1 for the two-scale example in the previous section. Following the procedure of Jikov et. al. [37], the scale interacting term $u^1(x, y)$ is found by solving the cell problem for two scales

$$\int a(\epsilon^{rs} + N^{rs}(u)) : \epsilon(\phi) dy = 0, \quad (4.63)$$

where

$$u^1(t, x, y) = \sum_{r,s} N^{rs} \epsilon_{rs}(u^0).$$

The homogenized solution is

$$u_1^0 = u_2^0 = \sin(\pi x') \sin(\pi x'') t^3.$$

Following Jikov et al. [37], when $r \neq s$, $N^{rs} = 0$ is the solution of (4.63) and when $r = s$, we consider function $h(x, y)$ which is periodic with respect to y such that $N^{11} = N^{22} = \nabla h$; $h(x, y)$ satisfies the problem

$$\Delta_y h = \frac{\langle \frac{K}{\beta+K} \rangle - \frac{1}{\langle \frac{1}{\beta+K} \rangle} - K}{\beta + K}, \quad (4.64)$$

where

$$K(x, y) = \lambda(x, y) + \frac{2\mu(x)}{d}, \quad \beta(x) = \frac{2(d-1)}{d} \mu(x).$$

We choose β and K so that we can find an exact solution for (4.64).

Let $\beta(x) = (1 + x')(1 + x'')$ and $K(x, y) = \beta(x) \bar{K}(y) > 0$.

Then (4.64) is simplified to

$$\Delta_y h = \frac{1}{\langle \frac{1}{1+\bar{K}} \rangle} \cdot \frac{1}{1+\bar{K}} - 1.$$

Let $\bar{\bar{K}}(y) = \frac{1}{1+\bar{K}(y)}$ and $\bar{\bar{K}}(y) = a + b \cdot g(y)$ such that $\langle g(y) \rangle = 0$.
Then

$$\Delta_y h = \frac{b}{a} g(y),$$

which may be solved exactly. Thus if we choose

$$\bar{\bar{K}}(y) = \frac{1}{4} \left(1 + \frac{1}{2} \sin(2\pi y') \right) \left(1 + \frac{1}{2} \sin(2\pi y'') \right),$$

which implies

$$K(x, y) = (1 + x')(1 + x'') \left(\frac{4}{(1 + \frac{1}{2} \sin(2\pi y'))(1 + \frac{1}{2} \sin(2\pi y''))} - 1 \right),$$

then the scale interacting term $u^1(t, x, y)$ is

$$u^1(t, x, y) = \begin{pmatrix} - \left(\frac{1}{4} \cos(2\pi y') + \frac{1}{16} \cos(2\pi y') \sin(2\pi y'') \right) \sin(\pi(x' + x'')) t^3 \\ - \left(\frac{1}{4} \cos(2\pi y'') + \frac{1}{16} \sin(2\pi y') \cos(2\pi y'') \right) \sin(\pi(x' + x'')) t^3 \end{pmatrix}.$$

Stochastic Elasticity Equations

5.1 Introduction

We consider linear elasticity equations whose elastic moduli are random in a domain $D \subset \mathbb{R}^d$. In many engineering problems, the physical properties of composite materials are uncertain thus we need to assume some degrees of randomness in the elastic moduli. In this chapter, we assume that the elastic moduli depend linearly on a countable set of random variables. Following the work by Cohen et al. [18], Bieri et al. [8] for scalar coefficients and Hoang and Schwab [34] for tensor coefficients, we assume further that these random variables are uniformly distributed in a bounded interval, and that the coefficient functions of the expansion satisfy a decaying condition so that their $L^\infty(D)$ norms are $\ell^p(\mathbb{N})$ summable. This decaying condition holds in many cases where the covariance is piecewise sufficiently smooth. For the case of non-uniformly distributed random variables, results similar to those of Cohen et al. [18] can be deduced from our framework though we do not consider this situation. Solution of the stochastic problem is approximated by solving a deterministic parametric elasticity equation where we choose the best N terms of the generalized polynomial chaos (GPC) expansion of the solution. In particular, we consider the semidiscretized Galerkin problem by projecting to the spaces of those functions obtained by truncating the GPC expansion at a finite level. Using the method developed by Cohen et al. [18], we deduce the rate of convergence from the ℓ^p summability of the GPC expansion's coefficients.

We first consider the displacement formulation. The results for this case are extensions of those by Cohen et al. [18] to the vector solution of the elasticity equation. We then consider the mixed Hellinger-Reissner formulation for both stochastic and deterministic parametric equations. This formula allows for approximating the stress directly via the best N terms of its GPC expansion, rather than deducing it from an approximation of the stochastic/parametric displacement. However, the Hellinger-Reissner formula uses the inverse of the elastic moduli. Only in some special cases, a

Kahnen-Love expansion of the inverse tensor can be deduced from the known covariance of the elastic moduli. For our general assumption of a linear dependence of the moduli on a countable set of random variables, working with the Hellinger-Reissner formula we cannot employ the special linear form of the coefficients. This makes deducing bounds for the GPC expansion coefficients very problematic. Also, the linear form of the coefficients simplifies significantly the discretized problem as shown in Bieri et al. [8]. The Hellinger-Reissner formula cannot benefit from this simplification. We therefore consider an equivalent mixed problem for which we use the elastic moduli rather than their inverse. Proving a uniform inf-sup condition for the semidiscretized Galerkin problem with respect to all the levels of truncation of the GPC expansion is much more complicated for this form, as it is not a saddle point problem. This requires careful handling of the various estimates for the inf-sup inequalities. The constant of the inf-sup condition depends on the constant of the Korn's inequality. However, we deduce a condition on the stochastic/parametric moduli for the uniform inf-sup condition to exist that is independent of the domain D , and does not depend on the Korn's inequality's constant. We also deduce a condition on the elastic moduli's expansion for the ℓ^p summability of the GPC coefficients that is independent of the domain. We note that a general parametric inf-sup formulation for control problems is studied by Kunoth and Schwab [39] where the parametric bilinear form is a perturbation of a bilinear form that satisfies an inf-sup condition. The perturbation is expressed as a parametric infinite expansion of bilinear forms where each satisfies a boundedness condition. However, applying this framework to our problem will require the conditions on the coefficients to depend on the particular domain on which the equation is posed. We give a condition that is domain independent.

For isotropic materials, the multiplying constant of the best N term rate for the displacement formula depends on the ratio of the Lam constants. Hence for nearly incompressible materials, for example, when the random elastic moduli are perturbations of those of a nearly incompressible material, to achieve a prescribed level of accuracy, the number of terms needed increases as this ratio gets larger. This is similar to the locking effect (see, for example, Brenner and Scott [12] and Braess [10]). We therefore consider a mixed problem to overcome this. The traditional form involves the inverse of the Lam constant λ . Again, to employ the linearity of λ with respect to the parameters, we consider a modified mixed form that uses λ rather than $1/\lambda$. This form requires technical handling of the inf-sup formula to show the uniform inf-sup condition for the semidiscretized Galerkin approximation. We also deduce sufficient conditions for the ℓ^p summability of the GPC expansion's coefficients from this form. This allows for an explicit rate of convergence for the best N term approximation to be obtained for the case of mixed Dirichlet and Neumann boundary conditions that is independent of the Lam constant λ when it converges to ∞ .

The chapter is organized as follows. In Section 5.2, we formulate the stochastic linear elasticity problem and introduce necessary conditions on the coefficients of the moduli's expansion to guarantee the uniform boundedness and coerciveness for all realizations. We then formulate the displacement formula, the Hellinger-Reissner mixed

formula and the mixed formula for nearly incompressible materials. Section 5.3 considers the best N term GPC approximation for the displacement formula. We reduce the problem to a deterministic parametric one and write the solution as an expansion with respect to multivariate polynomials of the parameters. We then study the semidiscretized approximation with respect to the parameters and deduce the best N term rate of convergence. The Hellinger-Reissner mixed formulation and its equivalent form which uses the elastic modulus tensor rather than its inverse are studied in Section 5.4. For each formulation, we establish conditions for the inf-sup condition to hold uniformly for all levels of truncation of the GPC expansion. From this we deduce bounds for the GPC coefficients and the best N term convergence rate. Section 5.5 considers mixed problems for nearly incompressible materials. We concentrate on the case of mixed Dirichlet and Neumann boundary conditions. We prove that the inf-sup condition for the semidiscrete Galerkin approximation is uniform with respect to the ratio of the Lamé constant. We then establish a condition for the ℓ^p summability of the coefficients of the solution's GPC expansion. We have not been able to resolve fully the case of a pure Dirichlet boundary condition.

We do not consider regularity of the GPC coefficients which is necessary for obtaining convergence rates of finite element spatial discretization. We refer to Nistor and Schwab [45] for regularity of these coefficients in Babuska-Kondrat'ev spaces on domains such as polygons or polyhedral.

Throughout the chapter, by c and C we denote various constants which do not depend on the approximating parameters. For a vector $\nu = (\nu_1, \nu_2, \dots) \in \mathbb{N}_0^{\mathbb{N}}$ where only a finite number of ν_j are non zero, by $\nu!$ we denote $\nu_1! \nu_2! \dots$, and for these vectors ν , for $d \in \mathbb{R}^{\mathbb{N}}$, we denote by d^ν the product $d_1^{\nu_1} d_2^{\nu_2} \dots$ with the convention that $0^0 = 1$. For a measure space (U, Θ, ρ) and a Banach space X , we denote by $L^p(U, \rho; X)$ the usual Bochner space of p summable maps from U to X (see Wloka [55]).

5.2 Problem setting

5.2.1 Stochastic elasticity problem

Let D be a bounded domain in $\mathbb{R}^d$ with a Lipschitz boundary. Let Γ be a subset of the boundary ∂D of D with a positive measure. We denote by V the subspace of $H^1(D)$ of functions whose traces on Γ equal 0; V can be defined as the closure of the set of functions in $C^\infty(D)$ that equal zero in a neighbourhood of Γ . When $\Gamma = \partial D$, $V = H_0^1(D)$. We denote by $\mathbf{V} = V^d$. We consider the following stochastic elasticity problem with Dirichlet boundary condition on Γ and Neumann boundary condition on $\partial D \setminus \Gamma$:

$$-\frac{\partial}{\partial x_j}(a_{ijkl}\epsilon_{kl}(u)) = f_i, \quad u = 0 \text{ on } \Gamma, \quad a_{ijkl}\epsilon_{kl}(u)n_j = 0 \text{ on } \partial D \setminus \Gamma \quad (5.1)$$

for $i = 1, \dots, d$ where the random elastic moduli $a_{ijkl}(x, \omega)$ satisfy the symmetry conditions $a_{ijkl} = a_{klij} = a_{jikl} = a_{ijlk}$ and the elastic strain $\epsilon(u)$ is defined as

$$\epsilon_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right);$$

and n_j denotes the j th component of the outward normal vector to ∂D . The realization ω belongs to a probability space $(\Omega, \Sigma, \mathbb{P})$ where Σ is the σ -algebra and $\mathbb{P}$ is the probability measure. We assume that the problem is uniformly elliptic for all $\omega \in \Omega$, i.e. there are positive constants α and β so that for all $x \in D$ and all symmetric tensors $\xi, \zeta \in \mathbb{R}_{sym}^{d \times d}$,

$$\alpha |\xi|^2 \leq a_{ijkl}(x, \omega) \xi_{ij} \xi_{kl}, \quad a_{ijkl}(x, \omega) \xi_{ij} \zeta_{kl} \leq \beta |\xi| |\zeta|. \quad (5.2)$$

The forcing $f(x) \in \mathbf{V}'$ is assumed to be deterministic, though we can extend our subsequent analysis to the case of a stochastic forcing without difficulty.

To specify the distribution of the elastic moduli a_{ijkl} , we assume the following structure. There exist functions $\Psi_m : D \rightarrow \mathbb{R}^{d^4}$ and scalar random variables $y_m : \Omega \rightarrow [-1, 1]$ which are uniformly distributed and pairwise independent such that

$$a_{ijkl}(x, \omega) = \bar{a}_{ijkl}(x) + \sum_{m=1}^{\infty} (\Psi_m)_{ijkl}(x) y_m(\omega) \quad (5.3)$$

or in a concise form,

$$a(x, \omega) = \bar{a}(x) + \sum_{m=1}^{\infty} \Psi_m(x) y_m(\omega), \quad (5.4)$$

where the fourth order tensor $\bar{a}$ is the expectation of a . The fourth order tensors Ψ_m are assumed to be symmetric, i.e.

$$(\Psi_m)_{ijkl} = (\Psi_m)_{klij} = (\Psi_m)_{jikl} = (\Psi_m)_{ijlk}.$$

For the uniform ellipticity condition (5.2) to hold, we make the following assumptions.

Assumption 5.2.1. *There exists a sequence $(\beta_m)_m \in \ell^1(\mathbb{N})$ such that $\forall \xi, \zeta \in \mathbb{R}_{sym}^{d \times d}$*

$$|\Psi_m \xi : \zeta| \leq \beta_m |\xi| |\zeta|. \quad (5.5)$$

Assumption 5.2.2. *The fourth order tensor $\bar{a}$ is uniformly bounded and coercive for all $x \in D$ i.e. there are positive constants α_0 and β_0 such that $\forall \xi, \zeta \in \mathbb{R}_{sym}^{d \times d}$*

$$\alpha_0 |\xi|^2 \leq |\bar{a}(x) \xi : \xi|, \quad |\bar{a}(x) \xi : \zeta| \leq \beta_0 |\xi| |\zeta|. \quad (5.6)$$

To ensure the coerciveness of a_{ijkl} , we have the further assumption.

Assumption 5.2.3. *There exists a positive constant κ such that*

$$\sum_{m=1}^{\infty} \beta_m < \frac{\kappa}{1 + \kappa} \alpha_0. \quad (5.7)$$

The constants α and β in (5.2) can be chosen as

$$\alpha = \frac{\alpha_0}{1 + \kappa}, \quad \beta = \beta_0 + \frac{\kappa}{1 + \kappa} \alpha_0.$$

It then follows that

$$\frac{1}{\alpha} \sum_{m=1}^{\infty} \beta_m < \kappa. \quad (5.8)$$

In variational form, problem (5.1) is written as

$$\int_D a_{ijkl} \epsilon_{kl}(u) \epsilon_{ij}(v) dx = \int_D f_i v_i dx \quad \forall v \in \mathbf{V}, \quad (5.9)$$

or in a concise form

$$\int_D a \epsilon(u) : \epsilon(v) dx = \int_D f \cdot v dx.$$

From this and Korn's inequality (Korn's inequality for functions with zero trace on a subset Γ of the boundary ∂D can be found, e.g. in Oleinik et al. [47] page 21), we deduce that for all $\omega \in \Omega$

$$\|u(\cdot, \omega)\|_{\mathbf{V}} \leq c \|f\|_{\mathbf{V}'}$$

for a constant c independent of ω .

In practice, we are often interested in computing the stress accurately. We therefore consider also the stochastic mixed formulation of (5.1) to compute the stress $\sigma(x, \omega) = a(x, \omega) \epsilon(u(x, \omega))$ directly.

Let $H = L^2(D)$ and $\mathcal{H} = (L^2(D))_{sym}^{d \times d}$ and $(\cdot, \cdot)$ denote the inner product in H and $\mathcal{H}$. We consider the stochastic mixed problem: Find $\{\sigma, u\} \in \mathcal{H} \times \mathbf{V}$ so that

$$\begin{cases} (a^{-1} \sigma, \tau) - (\tau, \epsilon(u)) = 0, & \forall \tau \in \mathcal{H} \\ -(\sigma, \epsilon(v)) = -(f, v), & \forall v \in \mathbf{V}. \end{cases} \quad (5.10)$$

With the particular structure of the moduli a in (5.3), it is more convenient to work with a directly than with the compliance tensor a^{-1} . We thus consider the equivalent mixed problem: Find $\{\sigma, u\} \in \mathcal{H} \times \mathbf{V}$ such that

$$\begin{cases} (\sigma, \tau) - (\tau, a \epsilon(u)) = 0, & \forall \tau \in \mathcal{H} \\ -(\sigma, \epsilon(v)) = -(f, v), & \forall v \in \mathbf{V}. \end{cases} \quad (5.11)$$

For isotropic materials, the moduli a is written as

$$a_{ijkl}(x, \omega) = \mu(x, \omega) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \lambda(x, \omega) \delta_{ij} \delta_{kl}. \quad (5.12)$$

We assume a similar structure for $\mu(\cdot, \omega)$ and $\lambda(\cdot, \omega)$ to (5.3), i.e.

$$\mu(x, \omega) = \bar{\mu}(x) + \sum_m \mu_m(x) y_m(\omega), \quad (5.13)$$

and

$$\lambda(x, \omega) = \bar{\lambda}(x) + \sum_m \lambda_m(x) y_m(\omega). \quad (5.14)$$

Let $\gamma_m = \sup_{x \in D} |\mu_m(x)|$ and $\delta_m = \sup_{x \in D} |\lambda_m(x)|$. Let $\bar{\mu}_{\min} = \inf_{x \in D} \bar{\mu}(x)$ and $\bar{\lambda}_{\min} = \inf_{x \in D} \bar{\lambda}(x)$. For the uniform coerciveness and boundedness of the tensor a_{ijkl} in (5.12), we make the following assumption for the moduli $\mu(x, \omega)$ and $\lambda(x, \omega)$.

Assumption 5.2.4. *There exists a constant $\kappa > 0$ such that the functions $\mu_m(x)$ and $\lambda_m(x)$ in (5.13) and (5.14) satisfy*

$$\sum_{m=1}^{\infty} \gamma_m \leq \frac{\kappa}{1 + \kappa} \bar{\mu}_{\min},$$

and

$$\sum_{m=1}^{\infty} \delta_m \leq \frac{\kappa}{1 + \kappa} \bar{\lambda}_{\min},$$

where $\bar{\mu}_{\min} > 0$ and $\bar{\lambda}_{\min} > 0$.

As demonstrated below, the best N term approximation for problem (5.1) depends on the ratio of the Lamé constants $\mu(x, \omega)$ and $\lambda(x, \omega)$, and thus requires a very large number of terms for the approximation to achieve a prescribed level of accuracy for nearly incompressible materials. To overcome this, we consider the following mixed problem: Find $\{u, p\} \in \mathbf{V} \times H$ such that

$$\begin{cases} 2(\mu(\cdot, \omega) \epsilon(u(\cdot, \omega)), \epsilon(v(\cdot))) + (\operatorname{div} v(\cdot), p(\cdot, \omega)) = (f, v), & \forall v \in \mathbf{V} \\ (\operatorname{div} u(\cdot, \omega), q(\cdot)) - (\frac{1}{\lambda(\cdot, \omega)} p(\cdot, \omega), q(\cdot)) = 0, & \forall q \in H. \end{cases} \quad (5.15)$$

To employ the linear dependence of $\lambda(x, \omega)$ on y_m , we will consider the equivalent mixed problem

$$\begin{cases} 2(\mu(\cdot, \omega) \epsilon(u(\cdot, \omega)), \epsilon(v(\cdot))) + (\operatorname{div} v(\cdot), p(\cdot, \omega)) = (f, v), & \forall v \in \mathbf{V} \\ (\frac{\lambda(\cdot, \omega)}{\bar{\lambda}_{\min}} \operatorname{div} u(\cdot, \omega), q(\cdot)) - (\frac{1}{\bar{\lambda}_{\min}} p(\cdot, \omega), q(\cdot)) = 0, & \forall q \in H. \end{cases} \quad (5.16)$$

5.2.2 Probability space

To define the law of the solution $u(\cdot, \omega) \in \mathbf{V}$ of problem (5.1), the solution $\{\sigma(\cdot, \omega), u(\cdot, \omega)\} \in \mathcal{H} \times \mathbf{V}$ of problems (5.10) and (5.11), and the solution $\{u(\cdot, \omega), p(\cdot, \omega)\} \in \mathbf{V} \times H$ of (5.15) and (5.16), we define a probability measure on the space $U = [-1, 1]^{\mathbb{N}}$. The σ -algebra Θ in U is defined by $\Theta = \otimes_{m \geq 1} \mathcal{B}([-1, 1])$ where $\mathcal{B}([-1, 1])$ denotes the Borel σ -algebra on the interval $[-1, 1]$. On the measurable space (U, Θ) , we define the product measure

$$d\rho(y) := \otimes_{m \geq 1} dy_m / 2. \quad (5.17)$$

Since $\frac{1}{2}dy_m$ is a probability measure on $[-1,1]$, so is $d\rho(y)$ on (U, Θ) and hence (U, Θ, ρ) is a probability space. As the random variables y_m are uniformly distributed and pairwise independent, for any set of the form $S = \prod_{m=1}^{\infty} S_m$ with $S_m \in \mathcal{B}([-1,1])$, we have

$$\rho(S) = \prod_{m=1}^{\infty} P\{\omega : y_m(\omega) \in S_m\}. \quad (5.18)$$

5.3 Displacement problem

We consider the displacement problem (5.1) in this section. We first cast it into the form of a deterministic problem that depends on a countable number of parameters. We then study the GPC expansion of the solution and its best N term convergence rate.

5.3.1 Deterministic parametric problem

For each $y = (y_1, y_2, \dots) \in U = [-1,1]^{\mathbb{N}}$, we define the deterministic parametric elastic tensor as

$$a_{ijkl}(x, y) = \bar{a}_{ijkl}(x) + \sum_{m=1}^{\infty} (\Psi_m)_{ijkl}(x) y_m. \quad (5.19)$$

Due to Assumptions 5.2.1 and 5.2.2, we have

$$\alpha|\xi|^2 \leq a_{ijkl}(x, y)\xi_{ij}\xi_{kl}, \quad \text{and} \quad |a_{ijkl}(x, y)\xi_{ij}\zeta_{kl}| \leq \beta|\xi||\zeta|,$$

for all ξ and ζ in $\mathbb{R}_{sym}^{d \times d}$. We consider the deterministic parametric elasticity problem

$$-\frac{\partial}{\partial x_j}(a_{ijkl}(\cdot, y)\epsilon_{kl}(u(\cdot, y))) = f_i, \quad u \in \mathbf{V} \quad (5.20)$$

for $i = 1, \dots, d$ with the Dirichlet boundary condition on Γ and Neumann boundary condition on $\partial D \setminus \Gamma$ which in variational form becomes: Find $u(\cdot, y) \in \mathbf{V}$ such that

$$b(y; u, v) = \int_D a_{ijkl}(x, y)\epsilon_{kl}(u(x, y))\epsilon_{ij}(v(x))dx = \int_D f_i v_i dx \quad \forall v \in \mathbf{V}. \quad (5.21)$$

Lax-Milgram lemma and Korn's inequality imply that for each $y \in U$ this problem has a unique solution which satisfies

$$\|u(\cdot, y)\|_{\mathbf{V}} \leq c\|f\|_{\mathbf{V}'} \quad (5.22)$$

where c does not depend on y .

We have the following measurability result:

Lemma 5.3.1. *The function u , as a map from $U \subset \ell^\infty(\mathbb{N})$ to $\mathbf{V}$ is Lipschitz. In particular, it is measurable with respect to the σ -algebra Θ .*

Proof. The proof of this lemma is similar to that for parabolic problems considered in Hoang and Schwab [32]. For y and y' in U , we have

$$\int_D a(x, y) \epsilon(u(x, y) - u(x, y')) : \epsilon(v(x)) dx = \int_D (a(x, y') - a(x, y)) \epsilon(u(x, y')) : \epsilon(v(x)) dx$$

for all $v \in \mathbf{V}$. From this we deduce that

$$\|u(\cdot, y) - u(\cdot, y')\|_{\mathbf{V}} \leq c \|a(\cdot, y') - a(\cdot, y)\|_{L^\infty(D)^{d^4}} \|\epsilon(u(\cdot, y'))\|_{\mathbf{H}} \leq c \sup_m |y_m - y'_m|$$

due to (5.22). The measurability of u as a map from U to $\mathbf{V}$ then follows. $\square$

As the solution of the deterministic problem (5.21) is uniformly bounded, we deduce that $u \in \underline{\mathbf{V}} := L^2(U, \rho; \mathbf{V})$. We then define the bilinear form $B : \underline{\mathbf{V}} \times \underline{\mathbf{V}} \rightarrow \mathbb{R}$ as

$$B(u, v) = \int_U b(y; u, v) d\rho(y).$$

We define further the linear form $F : \underline{\mathbf{V}} \rightarrow \mathbb{R}$ as

$$F(v) = \int_U \int_D f_i(x) v_i(x, y) dx d\rho(y).$$

We consider the following problem: Find $u \in \underline{\mathbf{V}}$ such that

$$B(u, v) = F(v), \quad \forall v \in \underline{\mathbf{V}}. \quad (5.23)$$

The following result holds:

Proposition 5.3.2. *Problem (5.23) possesses a unique solution $u \in \underline{\mathbf{V}}$. The solution $u(\cdot, \omega)$ of problem (5.9) is determined as*

$$u(\cdot, \omega) = u(\cdot, y)|_{y=y(\omega)}. \quad (5.24)$$

Proof. The uniqueness of a solution of problem (5.23) follows from Lax-Milgram lemma. An argument similar to that of Proposition 1.6 of [33] shows that for almost all $y \in U$, (5.21) holds. Let the test function in (5.23) be of the form $v(x)w(y)$ for $v \in \mathbf{V}$ and $w \in L^2(U, \rho)$. We have

$$\int_U b(y; u(\cdot, y), v(\cdot)) w(y) d\rho(y) = \int_U \int_D f_i(x) v_i(x) w(y) dx d\rho(y).$$

As this holds for all $w \in L^2(U, \rho)$, for almost all $y \in U$

$$b(y; u(\cdot, y), v(\cdot)) = \int_D f_i(x) v_i(x) dx.$$

Letting v run over a countable dense set of $\mathbf{V}$, we deduce that for almost all $y \in U$, (5.21) holds. Assertion (5.24) then follows. $\square$

5.3.2 Semidiscrete Galerkin approximation

5.3.2.1 Polynomial spaces in U

We define $\mathcal{F}$ as the countable set of sequences $\nu = (\nu_j)_{j \geq 1}$ where ν_j are nonnegative integers such that only finitely many ν_j are non-zero. Let $L_n(t)$ be the n th Legendre polynomial normalized so that

$$\int_{-1}^1 |L_n(t)|^2 \frac{dt}{2} = 1.$$

For $\nu \in \mathcal{F}$, we consider the tensorized Legendre polynomials

$$L_\nu(y) = \prod_{j \geq 1} L_{\nu_j}(y_j).$$

The tensorized Legendre polynomials L_ν form a complete orthonormal basis for $L^2(U, \rho)$. Thus, as $u \in L^2(U, \rho; \mathbf{V})$, we can write u as

$$u = \sum_{\nu \in \mathcal{F}} u_\nu L_\nu \quad (5.25)$$

where

$$u_\nu = \int_U u(\cdot, y) L_\nu(y) d\rho(y) \in \mathbf{V}. \quad (5.26)$$

5.3.2.2 Semidiscretization in y

For every set $\Lambda \subset \mathcal{F}$ of finite cardinality we define the following subspace $\underline{\mathbf{V}}_\Lambda$ of $\underline{\mathbf{V}}$ as

$$\underline{\mathbf{V}}_\Lambda = \{v_\Lambda \in \underline{\mathbf{V}} : v_\Lambda(x, y) = \sum_{\nu \in \Lambda} v_\nu(x) L_\nu(y), v_\nu \in \mathbf{V}\} \subset \underline{\mathbf{V}}.$$

We consider the semidiscrete Galerkin projection u_Λ of u onto $\underline{\mathbf{V}}_\Lambda$ such that

$$B(u_\Lambda, v_\Lambda) = F(v_\Lambda), \quad \forall v_\Lambda \in \underline{\mathbf{V}}_\Lambda. \quad (5.27)$$

Proposition 5.3.3. *Problem (5.27) has a unique solution u_Λ . The following quasi-optimal error estimate holds*

$$\|u - u_\Lambda\|_{\underline{\mathbf{V}}} \leq C \inf_{v_\Lambda \in \underline{\mathbf{V}}_\Lambda} \|u - v_\Lambda\|_{\underline{\mathbf{V}}}, \quad (5.28)$$

where C does not depend on Λ .

Proof. As $\underline{\mathbf{V}}_\Lambda$ is a Hilbert space, the uniqueness of a solution follows from Lax-Milgram lemma. The quasi-optimal error estimate follows from Cea's lemma. $\square$

Proposition 5.3.4. *For the solution u_Λ of problem (5.27), we have*

$$\|u - u_\Lambda\|_{\underline{\mathbf{V}}} \leq C \left(\sum_{\nu \notin \Lambda} \|u_\nu\|_{\mathbf{V}}^2 \right)^{1/2}. \quad (5.29)$$

Proof. The estimate is obtained from Proposition 5.3.3 and Parseval's identity by letting $v_\Lambda = \sum_{\nu \in \Lambda} u_\nu L_\nu(y)$. $\square$

5.3.3 Best N term approximation

To deduce an explicit estimate for the quasi-optimal error of the semidiscretization in y , we need summability properties for the sequence $(\|u_\nu\|_{\mathbf{V}})_\nu$. We therefore deduce bounds for $\|u_\nu\|_{\mathbf{V}}$.

5.3.3.1 Bounds for $\|u_\nu\|_{\mathbf{V}}$

Let

$$d_m = \frac{\beta_m}{\alpha}$$

and $d = (d_1, d_2, \dots) \in \mathbb{R}^{\mathbb{N}}$.

Proposition 5.3.5. *The following bound for the derivatives of u with respect to y holds uniformly for all $y \in U$*

$$\|\partial_y^\nu u(\cdot, y)\|_{\mathbf{V}} \leq c|\nu|!d^\nu,$$

where c is a constant independent of ν .

Proof. First we show by induction with respect to $|\nu|$ that there exists a constant C_0 such that

$$\|\epsilon(\partial_y^\nu u)\|_{\mathbf{H}} \leq C_0|\nu|!d^\nu. \quad (5.30)$$

From (5.21) and Korn's inequality, we deduce

$$\alpha\|\epsilon(u)\|_{\mathbf{H}}^2 \leq \|f\|_{\mathbf{V}'}\|u\|_{\mathbf{V}} \leq c\|f\|_{\mathbf{V}'}\|\epsilon(u)\|_{\mathbf{H}}.$$

Therefore

$$\|\epsilon(u)\|_{\mathbf{H}} \leq \frac{c\|f\|_{\mathbf{V}'}}{\alpha} = C_0.$$

We now present a brief proof that problem (5.21) can be differentiated with respect to y_m for each $m \in \mathbb{N}$. For $\delta \in \mathbb{R}$, let $y' \in U$ such that $y'_m = y_m + \delta$ and $y'_n = y_n$ for $n \neq m$. From (5.21) we have that

$$\begin{aligned} & \int_D a_{ijkl}(x, y) \epsilon_{kl}(u(x, y') - u(x, y)) \epsilon_{ij}(v(x)) dx \\ &= \int_D (a_{ijkl}(x, y) - a_{ijkl}(x, y')) \epsilon_{kl}(u(x, y')) \epsilon_{ij}(v(x)) dx \\ &= -\delta \int_D (\Psi_m(x))_{ijkl} \epsilon_{kl}(u(x, y')) \epsilon_{ij}(v(x)) dx. \end{aligned}$$

We then deduce that

$$\|\epsilon(u(\cdot, y') - u(\cdot, y))\|_{\mathbf{H}} \leq \frac{\delta\beta_m}{\alpha} \|\epsilon_{kl}(u(\cdot, y'))\|_{\mathbf{H}}.$$

This together with (5.22) and Korn's inequality implies

$$\|u(\cdot, y') - u(\cdot, y)\|_{\mathbf{V}} \leq c \frac{\delta\beta_m}{\alpha}.$$

Let $\eta \in V$ be the solution of problem:

$$\int_D a_{ijkl}(x, y) \epsilon_{kl}(\eta) \epsilon_{ij}(v(x)) dx = - \int_D (\Psi_m(x))_{ijkl} \epsilon_{kl}(u(x, y)) \epsilon_{ij}(v(x)) dx \quad \forall v \in V. \quad (5.31)$$

Let $w = (u(\cdot, y') - u(\cdot, y))/\delta$. We then have

$$\int_D a_{ijkl}(x, y) \epsilon_{kl}(w - \eta) \epsilon_{ij}(v(x)) dx = - \int_D (\Psi_m(x))_{ijkl} \epsilon_{kl}(u(x, y') - u(x, y)) \epsilon_{ij}(v(x)) dx.$$

Thus

$$\|w - \eta\|_{\mathbf{V}} \leq c \|u(\cdot, y') - u(\cdot, y)\|_{\mathbf{V}} \rightarrow 0$$

when $\delta \rightarrow 0$. Therefore $\eta = \partial_{y_m} u$. Let e_m be the sequence in $\mathcal{F}$ whose components are all 0 except the m th component which equals 1. By differentiating (5.31) further, following the inductive argument presented in the proof of Theorem 4.3 in Cohen et al. [18] we deduce that for all $v \in \mathbf{V}$:

$$\int_D a(x, y) \epsilon(\partial_y^\nu u(x, y)) : \epsilon(v(x)) dx = - \sum_{m, \nu_m \neq 0} \nu_m \int_D \Psi_m(x) \epsilon(\partial_y^{\nu - e_m} u(x, y)) : \epsilon(v(x)) dx. \quad (5.32)$$

From this we have

$$\alpha \|\epsilon(\partial_y^\nu u(\cdot, y))\|_{\mathcal{H}}^2 \leq \sum_m \nu_m \beta_m \|\epsilon(\partial_y^{\nu - e_m} u(\cdot, y))\|_{\mathcal{H}} \|\epsilon(\partial_y^\nu u(\cdot, y))\|_{\mathcal{H}}.$$

Therefore

$$\|\epsilon(\partial_y^\nu u(\cdot, y))\|_{\mathcal{H}} \leq \sum_m \nu_m \frac{\beta_m}{\alpha} \|\epsilon(\partial_y^{\nu - e_m} u(\cdot, y))\|_{\mathcal{H}}.$$

Assuming that (5.30) holds for $\nu - e_m$, i.e.

$$\|\epsilon(\partial_y^{\nu - e_m} u)\|_{\mathcal{H}} \leq C_0(|\nu| - 1)! d^{\nu - e_m},$$

we have

$$\|\epsilon(\partial_y^\nu u(\cdot, y))\|_{\mathcal{H}} \leq C_0 \sum_m \nu_m d_m (|\nu| - 1)! d^{\nu - e_m} = C_0 |\nu|! d^\nu.$$

We then get the conclusion from Korn's inequality. $\square$

From Proposition 5.3.5, we deduce the following bound for the coefficients of the GPC expansion.

Proposition 5.3.6. *The coefficients u_ν of the expansion (5.25) satisfy*

$$\|u_\nu\|_{\mathbf{V}} \leq c \frac{|\nu|!}{\nu!} \bar{d}^\nu$$

where $\bar{d}_m = d_m/\sqrt{3}$ and $\bar{d} = (\bar{d}_1, \bar{d}_2, \dots)$.

Proof. Following the procedure in Section 6 of Cohen et al. [18], we deduce from (5.26) that

$$\|u_\nu\|_{\mathbf{V}} \leq \frac{1}{(\sqrt{3})^{|\nu|} \nu!} \|\partial_y^\nu u\|_{L^\infty(U; \mathbf{V})}.$$

The result then follows. $\square$

5.3.3.2 Best N term convergence rate

To deduce an explicit rate of convergence for the semidiscrete Galerkin approximation (5.29), we need summability of $(\|u_\nu\|_{\mathbf{V}})_\nu$. To do this, we impose the following assumption.

Assumption 5.3.7. *There is a constant $0 < p < 1$ so that $\sum_{m=1}^{\infty} \beta_m^p < \infty$.*

We use the following result, which is due to Cohen et al. [18].

Lemma 5.3.8. *The sequence $(\frac{|\nu|!}{\nu!} \bar{d}^\nu)_{\nu \in \mathcal{F}}$ belongs to $\ell^p(\mathcal{F})$ if and only if $\|\bar{d}\|_{\ell^1(\mathbb{N})} < 1$ and $\bar{d} \in \ell^p(\mathbb{N})$.*

From this and (5.8) we have the following result.

Proposition 5.3.9. *Under Assumption 5.2.3 with $\kappa < \sqrt{3}$ and Assumption 5.3.7, $(\|u_\nu\|)_\nu \in \ell^p(\mathcal{F})$.*

To get the rate of convergence for the best N term approximation, we will use the following lemma which is due to Stechkin.

Lemma 5.3.10. *If $(b_n)_{n \geq 1}$ is a non-increasing sequence of positive numbers, then for $0 < p < q \leq \infty$ and any integer N :*

$$\left(\sum_{n > N} b_n^q \right)^{\frac{1}{q}} \leq N^{\frac{1}{q} - \frac{1}{p}} \left(\sum_{n \geq 1} b_n^p \right)^{1/p}.$$

Let $r = 1/p - 1/2$. Using Lemma 5.3.10, we deduce the following rate of convergence for the semidiscretized error estimate (5.29) for the best N term approximation.

Theorem 5.3.11. *Under Assumption 5.2.3 with $\kappa < \sqrt{3}$ and Assumption 5.3.7, for any $N \in \mathbb{N}$, there is a set $\Lambda \subset \mathcal{F}$ of cardinality not larger than N such that the following rate of convergence for the semidiscretized error estimate (5.29) holds:*

$$\|u - u_\Lambda\|_{\mathbf{V}} \leq CN^{-r} \|(u_\nu)_\nu\|_{\ell^p(\mathcal{F})}, \quad (5.33)$$

where C is the constant in (5.29).

Proof. We choose $\Lambda \subset \mathcal{F}$ as the set of indices ν corresponding to the N terms in the GPC expansion of u with the largest norms $\|u_\nu\|_{\mathbf{V}}$. The result then follows from (5.29) and Lemma 5.3.10. $\square$

5.4 Mixed problems

5.4.1 Deterministic parametric mixed problem

With the parametric elastic moduli in (5.19), for each $y \in U$ we consider the deterministic parametric mixed problems:

$$\begin{cases} (a^{-1}(\cdot, y)\sigma, \tau) - (\tau, \epsilon(u)) = 0, & \forall \tau \in \mathcal{H} \\ -(\sigma, \epsilon(v)) = -(f, v), & \forall v \in \mathbf{V} \end{cases} \quad (5.34)$$

and

$$\begin{cases} (\sigma, \tau) - (\tau, a(\cdot, y)\epsilon(u)) = 0, & \forall \tau \in \mathcal{H} \\ -(\sigma, \epsilon(v)) = -(f, v), & \forall v \in \mathbf{V}. \end{cases} \quad (5.35)$$

Let $\mathbf{X} = \mathcal{H} \times \mathbf{V}$ with the norm

$$\|\{\tau, v\}\|_{\mathbf{X}} = \|\tau\|_{\mathcal{H}} + \|v\|_{\mathbf{V}}.$$

We define the deterministic parametric bilinear forms $b_1, b_2 : \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{R}$ by

$$b_1(y; \{\sigma, u\}, \{\tau, v\}) = (a^{-1}(\cdot, y)\sigma, \tau) - (\tau, \epsilon(u)) - (\sigma, \epsilon(v)), \quad (5.36)$$

$$b_2(y; \{\sigma, u\}, \{\tau, v\}) = (\sigma, \tau) - (\tau, a(\cdot, y)\epsilon(u)) - (\sigma, \epsilon(v)). \quad (5.37)$$

Problems (5.34) and (5.35) can be reformulated as: Find $\{\sigma, u\} \in \mathbf{X}$ so that

$$b_1(y; \{\sigma, u\}, \{\tau, v\}) = -(f, v) \quad \forall \{\tau, v\} \in \mathbf{X};$$

and: Find $\{\sigma, u\} \in \mathbf{X}$ so that

$$b_2(y; \{\sigma, u\}, \{\tau, v\}) = -(f, v) \quad \forall \{\tau, v\} \in \mathbf{X}.$$

Proposition 5.4.1. *Problems (5.10), (5.11), (5.34) and (5.35) have a unique solution. Further, problems (5.10) and (5.11) are equivalent, so are problems (5.34) and (5.35).*

Proof. From (5.2), all the eigenvalues of $a(x, y)$ as a map that maps $\xi \in \mathbb{R}_{sym}^{d \times d}$ to $a(x, y)\xi \in \mathbb{R}_{sym}^{d \times d}$ are smaller than or equal to β , so all the eigenvalues of $a^{-1}(x, y)$ as a map that maps $\xi \in \mathbb{R}_{sym}^{d \times d}$ to $a^{-1}(x, y)\xi \in \mathbb{R}_{sym}^{d \times d}$ are greater than or equal to $1/\beta$. From this, for all $\xi \in \mathcal{H}$

$$(a^{-1}(\cdot, y)\xi, \xi) \geq \frac{1}{\beta} \|\xi\|_{\mathcal{H}}^2.$$

On the other hand, from Korn's inequality, there is a constant c that only depends on the domain D so that

$$\inf_{u \in \mathbf{V}} \sup_{\tau \in \mathcal{H}} \frac{(\tau, \epsilon(u))}{\|\tau\|_{\mathcal{H}} \|u\|_{\mathbf{V}}} \geq \inf_{u \in \mathbf{V}} \frac{\|\epsilon(u)\|_{\mathcal{H}}}{\|u\|_{\mathbf{V}}} \geq c.$$

We then have from the standard inf – sup condition for saddle point problems (see, e.g. Theorem 4.3 in Braess [10]) that there is a constant c that only depends on α , β and the domain D such that

$$\inf_{\{\sigma, u\} \in \mathbf{X}} \sup_{\{\tau, v\} \in \mathbf{X}} \frac{b_1(y; \{\sigma, u\}, \{\tau, v\})}{\|\{\sigma, u\}\|_{\mathbf{X}} \|\{\tau, v\}\|_{\mathbf{X}}} \geq c. \quad (5.38)$$

We note further that for $\{0, 0\} \neq \{\tau, v\} \in \mathbf{X}$, letting $\sigma = \tau$ and $u = -v$, we deduce that $b_1(y, \{\sigma, u\}, \{\tau, v\})$ cannot be zero for all $\{\sigma, u\} \in \mathbf{X}$. Thus problem (5.34) has a unique solution.

For the form b_2 , we have

$$b_2(y; \{\sigma, u\}, \{\tau, v\}) = b_1(y; \{\sigma, u\}, \{a\tau, v\}), \quad (5.39)$$

so

$$\frac{b_2(y; \{\sigma, u\}, \{\tau, v\})}{\|\{\sigma, u\}\|_{\mathbf{X}} \|\{\tau, v\}\|_{\mathbf{X}}} = \frac{b_1(y; \{\sigma, u\}, \{a\tau, v\})}{\|\{\sigma, u\}\|_{\mathbf{X}} \|\{\tau, v\}\|_{\mathbf{X}}} = \frac{b_1(y; \{\sigma, u\}, \{a\tau, v\})}{\|\{\sigma, u\}\|_{\mathbf{X}} \|\{a\tau, v\}\|_{\mathbf{X}}} \frac{\|\{a\tau, v\}\|_{\mathbf{X}}}{\|\{\tau, v\}\|_{\mathbf{X}}}.$$

From (5.2), all the eigenvalues of $a(x, y)$ as a map that maps $\xi \in \mathbb{R}_{sym}^{d \times d}$ to $a(x, y)\xi \in \mathbb{R}_{sym}^{d \times d}$ are greater than or equal to α . Thus, for $\tau \in \mathcal{H}$, $\|a\tau\|_{\mathcal{H}} \geq \alpha \|\tau\|_{\mathcal{H}}$. As every $\tau \in \mathcal{H}$ can be written as $a\tau'$ for $\tau' \in \mathcal{H}$, we conclude that there is a constant c independent of y such that

$$\inf_{\{\sigma, u\} \in \mathbf{X}} \sup_{\{\tau, v\} \in \mathbf{X}} \frac{b_2(y; \{\sigma, u\}, \{\tau, v\})}{\|\{\sigma, u\}\|_{\mathbf{X}} \|\{\tau, v\}\|_{\mathbf{X}}} \geq c. \quad (5.40)$$

This implies that problem (5.35) has a unique solution. Similarly, (5.11) also has a unique solution. Problems (5.34) and (5.35) are equivalent as every $\tau \in \mathcal{H}$ can be written in the form $a\tau'$ for $\tau' \in \mathcal{H}$. □

Lemma 5.4.2. *The mapping $\{\sigma(\cdot, y), u(\cdot, y)\}$ from $U \subset \ell^\infty(\mathbb{N})$ to $\mathbf{X}$ is Lipschitz and is, therefore, measurable with respect to the σ -algebra Θ .*

Proof. We consider the solution $\{\sigma(\cdot, y), u(\cdot, y)\}$ and $\{\sigma(\cdot, y'), u(\cdot, y')\}$ of (5.35) for y and y' in U respectively. We have

$$\begin{cases} (\sigma(\cdot, y) - \sigma(\cdot, y'), \tau) - (\tau, a(\cdot, y)(\epsilon(u(\cdot, y)) - \epsilon(u(\cdot, y')))) = \\ \quad (\tau, (a(\cdot, y) - a(\cdot, y'))\epsilon(u(\cdot, y'))), \quad \forall \tau \in \mathcal{H}, \\ (\sigma(\cdot, y) - \sigma(\cdot, y'), \epsilon(v)) = 0, \quad \forall v \in \mathbf{V}. \end{cases}$$

From (5.40), we have

$$\begin{aligned} & \|\sigma(\cdot, y) - \sigma(\cdot, y')\|_{\mathcal{H}} + \|u(\cdot, y) - u(\cdot, y')\|_{\mathbf{V}} \\ & \leq c \| (a(\cdot, y) - a(\cdot, y'))\epsilon(u(\cdot, y')) \|_{\mathcal{H}} \\ & \leq c \|a(\cdot, y) - a(\cdot, y')\|_{L^\infty(D)^{d^4}} \leq c \sup_m |y_m - y'_m| \end{aligned}$$

due to (5.22). Thus $\{\sigma(\cdot, y), u(\cdot, y)\}$ is a Lipschitz map from U to $\mathbf{X}$ where U is equipped with the topology induced by $\ell^\infty(\mathbb{N})$ and is, therefore, measurable. $\square$

We therefore have:

Proposition 5.4.3. *The solution $\{\sigma(\cdot, \omega), u(\cdot, \omega)\}$ of problems (5.10) and (5.11) is determined from the deterministic parametric solution $\{\sigma(\cdot, y), u(\cdot, y)\}$ of (5.34) and (5.35) respectively by*

$$\{\sigma(\cdot, \omega), u(\cdot, \omega)\}|_{y=y(\omega)}.$$

We denote by $\underline{\mathbf{V}} = L^2(U, \rho; \mathbf{V})$, $\underline{\mathbf{H}} = L^2(U, \rho; \mathbf{H})$ and $\underline{\mathbf{X}} = \underline{\mathbf{H}} \times \underline{\mathbf{V}} = L^2(U, \rho, \mathbf{X})$. As $\|\{\sigma(\cdot, y), u(\cdot, y)\}\|_{\mathbf{X}}$ is uniformly bounded for all $y \in U$, we deduce that $\{\sigma(\cdot, \cdot), u(\cdot, \cdot)\} \in \underline{\mathbf{X}}$. We thus write

$$\sigma = \sum_{\nu \in \mathcal{F}} \sigma_\nu L_\nu, \quad \text{and} \quad u = \sum_{\nu \in \mathcal{F}} u_\nu L_\nu, \quad \{\sigma_\nu, u_\nu\} \in \mathbf{X}$$

where

$$\sigma_\nu = \int_U \sigma(\cdot, y) L_\nu(y) d\rho(y), \quad u_\nu = \int_U u(\cdot, y) L_\nu(y) d\rho(y).$$

We now consider the following mixed problems:

$$\begin{cases} \int_U [(a^{-1}(\cdot, y)\sigma(\cdot, y), \tau(\cdot, y)) - (\tau(\cdot, y), \epsilon(u(\cdot, y)))] d\rho(y) = 0, & \forall \tau \in \underline{\mathbf{H}} \\ - \int_U (\sigma(\cdot, y), \epsilon(v(\cdot, y))) d\rho(y) = - \int_U (f, v(\cdot, y)) d\rho(y), & \forall v \in \underline{\mathbf{V}} \end{cases} \quad (5.41)$$

and

$$\begin{cases} \int_U [(\sigma(\cdot, y), \tau(\cdot, y)) - (\tau(\cdot, y), a(\cdot, y)\epsilon(u(\cdot, y)))] d\rho(y) = 0, & \forall \tau \in \underline{\mathbf{H}} \\ - \int_U (\sigma(\cdot, y), \epsilon(v(\cdot, y))) d\rho(y) = - \int_U (f, v(\cdot, y)) d\rho(y), & \forall v \in \underline{\mathbf{V}}. \end{cases} \quad (5.42)$$

We define the bilinear forms $B_1, B_2: \underline{\mathbf{X}} \times \underline{\mathbf{X}} \rightarrow \mathbb{R}$ as

$$B_1(\{\sigma, u\}, \{\tau, v\}) = \int_U b_1(y; \{\sigma(\cdot, y), u(\cdot, y)\}, \{\tau(\cdot, y), v(\cdot, y)\}) d\rho(y), \quad (5.43)$$

$$B_2(\{\sigma, u\}, \{\tau, v\}) = \int_U b_2(y; \{\sigma(\cdot, y), u(\cdot, y)\}, \{\tau(\cdot, y), v(\cdot, y)\}) d\rho(y). \quad (5.44)$$

We further define the linear form $F: \underline{\mathbf{X}} \rightarrow \mathbb{R}$ as

$$F(\{\tau, v\}) = - \int_U (f, v(\cdot, y)) d\rho(y). \quad (5.45)$$

Problems (5.41) and (5.42) can be reformulated as: Find $\{\sigma, u\} \in \underline{\mathbf{X}}$ such that

$$B_1(\{\sigma, u\}, \{\tau, v\}) = F(\{\tau, v\}) \quad \forall \{\tau, v\} \in \underline{\mathbf{X}}, \quad (5.46)$$

and: Find $\{\sigma, u\} \in \underline{\mathbf{X}}$ such that

$$B_2(\{\sigma, u\}, \{\tau, v\}) = F(\{\tau, v\}) \quad \forall \{\tau, v\} \in \underline{\mathbf{X}}, \quad (5.47)$$

respectively.

Proposition 5.4.4. *Problems (5.41) and (5.42) are equivalent and have a unique solution.*

Proof. The existence and uniqueness of a solution of problem (5.41) can be established from standard procedures for saddle point problems as for (5.34).

We define the bilinear form $\bar{a} : \underline{\mathbf{H}} \times \underline{\mathbf{H}} \rightarrow \mathbb{R}$ as

$$\bar{a}(\sigma, \tau) = \int_U (a^{-1}(\cdot, y) \sigma(\cdot, y), \tau(\cdot, y)) d\rho(y)$$

and the linear form $\bar{b} : \underline{\mathbf{X}} \rightarrow \mathbb{R}$ as

$$\bar{b}(\tau, v) = - \int_U (\tau(\cdot, y), \epsilon(v(\cdot, y))) d\rho(y).$$

As the eigenvalues of the map $a^{-1}(x, y) : \tau \mapsto a^{-1}(x, y)\tau$ are larger than or equal to $1/\beta$, $(a^{-1}\eta, \eta) \geq |\eta|^2/\beta$ for all $\eta \in \mathbb{R}_{sym}^{d \times d}$. From this, we deduce

$$\inf_{\sigma \in \underline{\mathbf{H}}} \sup_{\tau \in \underline{\mathbf{H}}} \frac{\bar{a}(\sigma, \tau)}{\|\sigma\|_{\underline{\mathbf{H}}} \|\tau\|_{\underline{\mathbf{H}}}} \geq \frac{1}{\beta}.$$

On the other hand, we have

$$\inf_{v \in \underline{\mathbf{V}}} \sup_{\tau \in \underline{\mathbf{H}}} \frac{\bar{b}(\tau, v)}{\|\tau\|_{\underline{\mathbf{H}}} \|v\|_{\underline{\mathbf{V}}}} \geq \inf_{v \in \underline{\mathbf{V}}} \frac{\int_U (\epsilon(v(\cdot, y)), \epsilon(v(\cdot, y))) d\rho(y)}{\|\epsilon(v)\|_{\underline{\mathbf{H}}} \|v\|_{\underline{\mathbf{V}}}} \geq c$$

due to Korn's inequality. We therefore get the uniform inf-sup condition for the bilinear form B_1 .

The existence and uniqueness of a solution of problem (5.42) is shown from the existence and uniqueness of (5.41) and the fact that every $\tau \in \underline{\mathbf{H}}$ can be written as $a\tau'$ where $\tau' \in \underline{\mathbf{H}}$.

□

5.4.2 Semidiscrete Galerkin approximation

For a set $\Lambda \subset \mathcal{F}$ of finite cardinality, we define the following spaces

$$\underline{\mathbf{V}}_\Lambda = \{v_\Lambda \in \underline{\mathbf{V}} : v_\Lambda(x, y) = \sum_{\nu \in \Lambda} v_\nu(x) L_\nu(y), v_\nu \in \underline{\mathbf{V}}\},$$

$$\underline{\mathcal{H}}_\Lambda = \{\tau_\Lambda \in \underline{\mathcal{H}} : \tau_\Lambda(x, y) = \sum_{\nu \in \Lambda} \tau_\nu(x) L_\nu(y), \tau_\nu \in \underline{\mathcal{H}}\},$$

and

$$\underline{\mathbf{X}}_\Lambda = \{\{\tau_\Lambda, v_\Lambda\} : \tau_\Lambda \in \underline{\mathcal{H}}_\Lambda, v_\Lambda \in \underline{\mathbf{V}}_\Lambda\}.$$

We consider the semidiscrete Galerkin approximation for (5.46): Find $\{\sigma_\Lambda, u_\Lambda\} \in \underline{\mathbf{X}}_\Lambda$ so that

$$B_1(\{\sigma_\Lambda, u_\Lambda\}, \{\tau_\Lambda, v_\Lambda\}) = F(\{\tau_\Lambda, v_\Lambda\}) \quad \forall \{\tau_\Lambda, v_\Lambda\} \in \underline{\mathbf{X}}_\Lambda, \quad (5.48)$$

and the semidiscrete Galerkin approximation for (5.47): Find $\{\sigma_\Lambda, u_\Lambda\} \in \underline{\mathbf{X}}_\Lambda$ so that

$$B_2(\{\sigma_\Lambda, u_\Lambda\}, \{\tau_\Lambda, v_\Lambda\}) = F(\{\tau_\Lambda, v_\Lambda\}) \quad \forall \{\tau_\Lambda, v_\Lambda\} \in \underline{\mathbf{X}}_\Lambda. \quad (5.49)$$

We have the following proposition.

Proposition 5.4.5. *Under condition (5.2), problem (5.48) has a unique solution.*

Proof. A proof similar to that for the inf-sup condition of the bilinear form b_1 in Proposition 5.4.1 shows that there is a positive constant c independent of Λ such that

$$\inf_{\{\sigma_\Lambda, u_\Lambda\} \in \underline{\mathbf{X}}_\Lambda} \sup_{\{\tau_\Lambda, v_\Lambda\} \in \underline{\mathbf{X}}_\Lambda} \frac{B_1(\{\sigma_\Lambda, u_\Lambda\}, \{\tau_\Lambda, v_\Lambda\})}{\|\{\sigma_\Lambda, u_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda} \|\{\tau_\Lambda, v_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda}} \geq c. \quad (5.50)$$

Problem (5.48) thus has a unique solution. $\square$

Remark 5.4.6. *To show the equivalence of (5.43) and (5.44), we use the fact that every function $\tau \in \underline{\mathcal{H}}$ can be written as $a\tau'$ for $\tau' \in \underline{\mathcal{H}}$. However, for (5.48) and (5.49) this is not the case as if $\tau \in \underline{\mathcal{H}}_\Lambda$, $a^{-1}\tau$ is generally not in $\underline{\mathcal{H}}_\Lambda$ so (5.48) and (5.49) are not equivalent. Therefore the inf-sup condition for problem (5.48) does not imply the inf-sup condition for problem (5.49).*

For the existence and uniqueness of a solution of problem (5.49), we make the following assumption.

Assumption 5.4.7. *There exists a positive constant $\tilde{\kappa}$ such that the constants β_m , α_0 and β_0 in Assumptions 5.2.1 and 5.2.2 satisfy*

$$\sum_{m=1}^{\infty} \beta_m \leq \frac{\tilde{\kappa}}{1 + \tilde{\kappa}} \frac{\alpha_0}{\alpha_0 + \beta_0} \alpha_0,$$

Remark 5.4.8. *Since*

$$\frac{\alpha_0}{\alpha_0 + \beta_0} < 1,$$

Assumption 5.4.7 implies Assumption 5.2.3.

We then have the following proposition.

Proposition 5.4.9. *Under Assumption 5.4.7, problem (5.49) has a unique solution.*

Proof. We define the spaces

$$\mathcal{H}^\epsilon = \{\zeta \in \mathcal{H} : \zeta = \epsilon(v) \text{ for } v \in \mathbf{V}\}, \quad (5.51)$$

and

$$\underline{\mathcal{H}}_\Lambda^\epsilon = \{\zeta_\Lambda \in \underline{\mathcal{H}} : \zeta_\Lambda(x, y) = \sum_{\nu \in \Lambda} \epsilon(v_\nu(x)) L_\nu(y), \quad v_\nu \in \mathbf{V}\}. \quad (5.52)$$

As $\mathcal{H}^\epsilon$ is a closed subspace of $\mathcal{H}$, $\underline{\mathcal{H}}_\Lambda^\epsilon$ is a closed subspace of $\underline{\mathcal{H}}_\Lambda$. Let $\underline{\mathbf{X}}_\Lambda^\epsilon = \underline{\mathcal{H}}_\Lambda \times \underline{\mathcal{H}}_\Lambda^\epsilon$.

We define the bilinear form $\bar{a}_\Lambda : \underline{\mathcal{H}}_\Lambda \times \underline{\mathcal{H}}_\Lambda \rightarrow \mathbb{R}$ as

$$\bar{a}_\Lambda(\sigma_\Lambda, \tau_\Lambda) = \int_U (\bar{a}^{-1}(\cdot) \sigma_\Lambda(\cdot, y), \tau_\Lambda(\cdot, y)) d\rho(y)$$

where $\bar{a}$ is the first term in the expansion (5.3), and the linear form $\bar{b}_\Lambda^\epsilon : \underline{\mathbf{X}}_\Lambda^\epsilon \rightarrow \mathbb{R}$ as

$$\bar{b}_\Lambda^\epsilon(\tau_\Lambda, \zeta_\Lambda) = - \int_U (\tau_\Lambda(\cdot, y), \zeta_\Lambda(\cdot, y)) d\rho(y).$$

The bilinear form $\bar{B}^\epsilon : \underline{\mathbf{X}}_\Lambda^\epsilon \times \underline{\mathbf{X}}_\Lambda^\epsilon \rightarrow \mathbb{R}$ is defined by

$$\bar{B}^\epsilon(\{\sigma_\Lambda, \xi_\Lambda\}, \{\tau_\Lambda, \zeta_\Lambda\}) = \bar{a}_\Lambda(\sigma_\Lambda, \tau_\Lambda) + \bar{b}_\Lambda^\epsilon(\tau_\Lambda, \xi_\Lambda) + \bar{b}_\Lambda^\epsilon(\sigma_\Lambda, \zeta_\Lambda).$$

We define further the bilinear form $B_2^\epsilon : \underline{\mathbf{X}}_\Lambda^\epsilon \times \underline{\mathbf{X}}_\Lambda^\epsilon \rightarrow \mathbb{R}$ as

$$B_2^\epsilon(\{\sigma_\Lambda, \xi_\Lambda\}, \{\tau_\Lambda, \zeta_\Lambda\}) = \bar{a}_\Lambda(\sigma_\Lambda, \tau_\Lambda) + \bar{b}_\Lambda^\epsilon(\tau_\Lambda, \xi_\Lambda) + \bar{b}_\Lambda^\epsilon(\sigma_\Lambda, \zeta_\Lambda) - \int_U \left(\tau_\Lambda(\cdot, y), \sum_{m=1}^{\infty} y_m \bar{a}(\cdot)^{-1} \Psi_m(\cdot, y) \xi_\Lambda(\cdot, y) \right) d\rho(y).$$

Let $\bar{b}_\Lambda^{\epsilon'} : \underline{\mathcal{H}}_\Lambda \rightarrow (\underline{\mathcal{H}}_\Lambda^\epsilon)'$ be defined by

$$\bar{b}_\Lambda^{\epsilon'}(\tau_\Lambda)(\zeta_\Lambda) = \bar{b}_\Lambda^\epsilon(\tau_\Lambda, \zeta_\Lambda)$$

and let $\text{Ker}(\bar{b}_\Lambda^{\epsilon'})$ denote its kernel. For $\sigma_\Lambda(x, y) = \sum_{\nu \in \Lambda} \sigma_\nu(x) L_\nu(y) \in \text{Ker}(\bar{b}_\Lambda^{\epsilon'})$, we have

$$\sup_{\tau_\Lambda \in \text{Ker}(\bar{b}_\Lambda^{\epsilon'})} \frac{\bar{a}_\Lambda(\sigma_\Lambda, \tau_\Lambda)}{\|\sigma_\Lambda\|_{\underline{\mathcal{H}}} \|\tau_\Lambda\|_{\underline{\mathcal{H}}}} \geq \frac{\bar{a}_\Lambda(\sigma_\Lambda, \sigma_\Lambda)}{\|\sigma_\Lambda\|_{\underline{\mathcal{H}}}^2} = \frac{\sum_{\nu \in \Lambda} (\bar{a}^{-1}(\cdot) \sigma_\nu(\cdot), \sigma_\nu(\cdot))}{\sum_{\nu \in \Lambda} \|\sigma_\nu\|_{\underline{\mathcal{H}}}^2}.$$

From Assumption 5.2.2, the eigenvalues of the map $\bar{a}(x) : \eta \mapsto \bar{a}(x)\eta$ from $\mathbb{R}^{d \times d}$ to $\mathbb{R}_{sym}^{d \times d}$ are smaller than or equal to β_0 . Therefore the eigenvalues of the map $\bar{a}^{-1}(x) : \eta \mapsto \bar{a}^{-1}(x)\eta$ are greater than or equal to $\frac{1}{\beta_0}$. We thus conclude that $(\bar{a}^{-1}\eta, \eta) \geq \|\eta\|_{\mathbb{R}^{d \times d}}^2 / \beta_0$ for all $\eta \in \mathbb{R}_{sym}^{d \times d}$. Thus for $\sigma_\Lambda \in \text{Ker}(\bar{b}_\Lambda^{\epsilon'})$,

$$\sup_{\tau_\Lambda \in \text{Ker}(\bar{b}_\Lambda^{\epsilon'})} \frac{\bar{a}_\Lambda(\sigma_\Lambda, \tau_\Lambda)}{\|\sigma_\Lambda\|_{\underline{\mathbf{H}}} \|\tau_\Lambda\|_{\underline{\mathbf{H}}}} \geq \frac{\bar{a}_\Lambda(\sigma_\Lambda, \sigma_\Lambda)}{\|\sigma_\Lambda\|_{\underline{\mathbf{H}}}^2} \geq \frac{1}{\beta_0}.$$

We show the proposition following the idea of the proof of Proposition 2.36 in Ern and Guermond [25].

Let $\{\sigma_\Lambda, \xi_\Lambda\} \in \underline{\mathbf{X}}_\Lambda^\epsilon$. Let $\hat{\sigma}_\Lambda$ be the orthogonal projection of σ_Λ to $\underline{\mathbf{H}}_\Lambda^\epsilon$. We have that

$$\sup_{\zeta_\Lambda \in \underline{\mathbf{H}}_\Lambda^\epsilon} \frac{\bar{B}^\epsilon(\{\hat{\sigma}_\Lambda, \xi_\Lambda\}, \{0, \zeta_\Lambda\})}{\|\zeta_\Lambda\|_{\underline{\mathbf{H}}}} = \sup_{\zeta_\Lambda \in \underline{\mathbf{H}}_\Lambda^\epsilon} \frac{\bar{b}_\Lambda^\epsilon(\hat{\sigma}_\Lambda, \zeta_\Lambda)}{\|\zeta_\Lambda\|_{\underline{\mathbf{H}}}} \geq \frac{\bar{b}_\Lambda^\epsilon(\hat{\sigma}_\Lambda, -\hat{\sigma}_\Lambda)}{\|\hat{\sigma}_\Lambda\|_{\underline{\mathbf{H}}}} = \|\hat{\sigma}_\Lambda\|_{\underline{\mathbf{H}}}.$$

As $\bar{b}_\Lambda^\epsilon(\sigma_\Lambda, \zeta_\Lambda) = \bar{b}_\Lambda^\epsilon(\hat{\sigma}_\Lambda, \zeta_\Lambda)$,

$$\|\hat{\sigma}_\Lambda\|_{\underline{\mathbf{H}}} \leq \sup_{\{\tau_\Lambda, \zeta_\Lambda\} \in \underline{\mathbf{X}}_\Lambda^\epsilon} \frac{B_2^\epsilon(\{\sigma_\Lambda, \xi_\Lambda\}, \{\tau_\Lambda, \zeta_\Lambda\})}{\|\{\tau_\Lambda, \zeta_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda^\epsilon}}. \quad (5.53)$$

Since $\sigma_\Lambda - \hat{\sigma}_\Lambda \in \text{Ker}(\bar{b}_\Lambda^{\epsilon'})$, we have

$$\begin{aligned} \frac{1}{\beta_0} \|\sigma_\Lambda - \hat{\sigma}_\Lambda\|_{\underline{\mathbf{H}}} &\leq \sup_{\tau_\Lambda \in \text{Ker}(\bar{b}_\Lambda^{\epsilon'})} \frac{\bar{a}_\Lambda(\sigma_\Lambda - \hat{\sigma}_\Lambda, \tau_\Lambda)}{\|\tau_\Lambda\|_{\underline{\mathbf{H}}}} \\ &= \sup_{\tau_\Lambda \in \text{Ker}(\bar{b}_\Lambda^{\epsilon'})} \frac{\bar{a}_\Lambda(\sigma_\Lambda - \hat{\sigma}_\Lambda, \tau_\Lambda) + \bar{b}_\Lambda^\epsilon(\tau_\Lambda, \xi_\Lambda) + \bar{b}_\Lambda^\epsilon(\sigma_\Lambda, 0)}{\|(\tau_\Lambda, 0)\|_{\underline{\mathbf{X}}_\Lambda^\epsilon}} \\ &\leq \sup_{\{\tau_\Lambda, \zeta_\Lambda\} \in \underline{\mathbf{X}}_\Lambda^\epsilon} \frac{\bar{B}^\epsilon(\{\sigma_\Lambda, \xi_\Lambda\}, \{\tau_\Lambda, \zeta_\Lambda\})}{\|\{\tau_\Lambda, \zeta_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda^\epsilon}} + \|\bar{a}_\Lambda\|_{\underline{\mathbf{H}}_\Lambda \times \underline{\mathbf{H}}_\Lambda \rightarrow \mathbb{R}} \|\hat{\sigma}_\Lambda\|_{\underline{\mathbf{H}}} \end{aligned}$$

for all $\xi_\Lambda \in \underline{\mathbf{H}}_\Lambda^\epsilon$. From the fact that

$$\bar{B}^\epsilon(\{\sigma_\Lambda, \xi_\Lambda\}, \{\tau_\Lambda, \zeta_\Lambda\}) \leq B_2^\epsilon(\{\sigma_\Lambda, \xi_\Lambda\}, \{\tau_\Lambda, \zeta_\Lambda\}) + \frac{1}{\alpha_0} \left(\sum_{m=1}^{\infty} \beta_m \right) \|\tau_\Lambda\|_{\underline{\mathbf{H}}} \|\xi_\Lambda\|_{\underline{\mathbf{H}}},$$

we deduce that

$$\|\sigma_\Lambda - \hat{\sigma}_\Lambda\|_{\underline{\mathbf{H}}} \leq \beta_0 \sup_{\{\tau_\Lambda, \zeta_\Lambda\} \in \underline{\mathbf{X}}_\Lambda^\epsilon} \frac{B_2^\epsilon(\{\sigma_\Lambda, \xi_\Lambda\}, \{\tau_\Lambda, \zeta_\Lambda\})}{\|\{\tau_\Lambda, \zeta_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda^\epsilon}} + \frac{\beta_0}{\alpha_0} \left(\sum_{m=1}^{\infty} \beta_m \right) \|\xi_\Lambda\|_{\underline{\mathbf{H}}} + \frac{\beta_0}{\alpha_0} \|\hat{\sigma}_\Lambda\|_{\underline{\mathbf{H}}}.$$

Thus together with (5.53) we obtain

$$\begin{aligned} \|\sigma_\Lambda\|_{\underline{\mathbf{H}}} &\leq \|\sigma_\Lambda - \hat{\sigma}_\Lambda\|_{\underline{\mathbf{H}}} + \|\hat{\sigma}_\Lambda\|_{\underline{\mathbf{H}}} \\ &\leq \left(\beta_0 + 1 + \frac{\beta_0}{\alpha_0} \right) \sup_{\{\tau_\Lambda, \zeta_\Lambda\} \in \underline{\mathbf{X}}_\Lambda^\epsilon} \frac{B_2^\epsilon(\{\sigma_\Lambda, \xi_\Lambda\}, \{\tau_\Lambda, \zeta_\Lambda\})}{\|\{\tau_\Lambda, \zeta_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda^\epsilon}} \\ &\quad + \frac{\beta_0}{\alpha_0} \left(\sum_{m=1}^{\infty} \beta_m \right) \|\xi_\Lambda\|_{\underline{\mathbf{H}}}. \end{aligned} \quad (5.54)$$

We have further that for $\xi_\Lambda \in \underline{\mathcal{H}}_\Lambda^\epsilon$

$$\begin{aligned}
\|\xi_\Lambda\|_{\underline{\mathcal{H}}} &\leq \sup_{\tau_\Lambda \in \underline{\mathcal{H}}_\Lambda} \frac{\bar{b}_\Lambda^\epsilon(\tau_\Lambda, \xi_\Lambda)}{\|\tau_\Lambda\|_{\underline{\mathcal{H}}}} \\
&\leq \sup_{\tau_\Lambda \in \underline{\mathcal{H}}_\Lambda} \frac{\bar{a}_\Lambda(\sigma_\Lambda, \tau_\Lambda) + \bar{b}_\Lambda^\epsilon(\tau_\Lambda, \xi_\Lambda) + \bar{b}_\Lambda^\epsilon(\sigma_\Lambda, 0)}{\|\{\tau_\Lambda, 0\}\|_{\underline{\mathbf{X}}_\Lambda^\epsilon}} + \sup_{\tau_\Lambda \in \underline{\mathcal{H}}_\Lambda} \frac{\bar{a}_\Lambda(\sigma_\Lambda, \tau_\Lambda)}{\|\tau_\Lambda\|_{\underline{\mathcal{H}}}} \\
&\leq \sup_{\{\tau_\Lambda, \zeta_\Lambda\} \in \underline{\mathbf{X}}_\Lambda^\epsilon} \frac{\bar{B}^\epsilon(\{\sigma_\Lambda, \xi_\Lambda\}, \{\tau_\Lambda, \zeta_\Lambda\})}{\|\{\tau_\Lambda, \zeta_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda^\epsilon}} + \|\bar{a}_\Lambda\|_{\underline{\mathcal{H}}_\Lambda \times \underline{\mathcal{H}}_\Lambda \rightarrow \mathbb{R}} \|\sigma_\Lambda\|_{\underline{\mathcal{H}}} \\
&\leq \sup_{\{\tau_\Lambda, \zeta_\Lambda\} \in \underline{\mathbf{X}}_\Lambda^\epsilon} \frac{B_2^\epsilon(\{\sigma_\Lambda, \xi_\Lambda\}, \{\tau_\Lambda, \zeta_\Lambda\})}{\|\{\tau_\Lambda, \zeta_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda^\epsilon}} + \frac{1}{\alpha_0} \left(\sum_{m=1}^{\infty} \beta_m \right) \|\xi_\Lambda\|_{\underline{\mathcal{H}}} + \\
&\quad \frac{1}{\alpha_0} \left(\beta_0 + 1 + \frac{\beta_0}{\alpha_0} \right) \sup_{\{\tau_\Lambda, \zeta_\Lambda\} \in \underline{\mathbf{X}}_\Lambda^\epsilon} \frac{B_2^\epsilon(\{\sigma_\Lambda, \xi_\Lambda\}, \{\tau_\Lambda, \zeta_\Lambda\})}{\|\{\tau_\Lambda, \zeta_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda^\epsilon}} + \frac{\beta_0}{\alpha_0^2} \left(\sum_{m=1}^{\infty} \beta_m \right) \|\xi_\Lambda\|_{\underline{\mathcal{H}}} \\
&\leq \left(1 + \frac{1}{\alpha_0} \left(\beta_0 + 1 + \frac{\beta_0}{\alpha_0} \right) \right) \sup_{\{\tau_\Lambda, \zeta_\Lambda\} \in \underline{\mathbf{X}}_\Lambda^\epsilon} \frac{B_2^\epsilon(\{\sigma_\Lambda, \xi_\Lambda\}, \{\tau_\Lambda, \zeta_\Lambda\})}{\|\{\tau_\Lambda, \zeta_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda^\epsilon}} + \\
&\quad \left(\frac{1}{\alpha_0} + \frac{\beta_0}{\alpha_0^2} \right) \left(\sum_{m=1}^{\infty} \beta_m \right) \|\xi_\Lambda\|_{\underline{\mathcal{H}}}.
\end{aligned}$$

From Assumption 5.4.7 we deduce that

$$\frac{1}{1 + \tilde{\kappa}} \|\xi_\Lambda\|_{\underline{\mathcal{H}}} \leq \left(1 + \frac{1}{\alpha_0} \left(\beta_0 + 1 + \frac{\beta_0}{\alpha_0} \right) \right) \sup_{\{\tau_\Lambda, \zeta_\Lambda\} \in \underline{\mathbf{X}}_\Lambda^\epsilon} \frac{B_2^\epsilon(\{\sigma_\Lambda, \xi_\Lambda\}, \{\tau_\Lambda, \zeta_\Lambda\})}{\|\{\tau_\Lambda, \zeta_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda^\epsilon}}. \quad (5.55)$$

From (5.54) and (5.55) we deduce that there is a constant c that is independent of Λ such that

$$\inf_{\{\sigma_\Lambda, \xi_\Lambda\} \in \underline{\mathbf{X}}_\Lambda^\epsilon} \sup_{\{\tau_\Lambda, \zeta_\Lambda\} \in \underline{\mathbf{X}}_\Lambda^\epsilon} \frac{B_2^\epsilon(\{\sigma_\Lambda, \xi_\Lambda\}, \{\tau_\Lambda, \zeta_\Lambda\})}{\|\{\sigma_\Lambda, \xi_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda^\epsilon} \|\{\tau_\Lambda, \zeta_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda^\epsilon}} \geq c.$$

We note that

$$B_2(\{\sigma_\Lambda, u_\Lambda\}, \{\tau_\Lambda, v_\Lambda\}) = B_2^\epsilon(\{\sigma_\Lambda, \epsilon(u_\Lambda)\}, \{\bar{a}\tau_\Lambda, \epsilon(v_\Lambda)\}).$$

Further

$$\begin{aligned}
&\frac{B_2(\{\sigma_\Lambda, u_\Lambda\}, \{\tau_\Lambda, v_\Lambda\})}{\|\{\sigma_\Lambda, u_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda} \|\{\tau_\Lambda, v_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda}} \\
&= \frac{B_2^\epsilon(\{\sigma_\Lambda, \epsilon(u_\Lambda)\}, \{\bar{a}\tau_\Lambda, \epsilon(v_\Lambda)\})}{\|\{\sigma_\Lambda, \epsilon(u_\Lambda)\}\|_{\underline{\mathbf{X}}_\Lambda^\epsilon} \|\{\bar{a}\tau_\Lambda, \epsilon(v_\Lambda)\}\|_{\underline{\mathbf{X}}_\Lambda^\epsilon}} \cdot \frac{\|\{\sigma_\Lambda, \epsilon(u_\Lambda)\}\|_{\underline{\mathbf{X}}_\Lambda^\epsilon} \|\{\bar{a}\tau_\Lambda, \epsilon(v_\Lambda)\}\|_{\underline{\mathbf{X}}_\Lambda^\epsilon}}{\|\{\sigma_\Lambda, u_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda} \|\{\tau_\Lambda, v_\Lambda\}\|_{\underline{\mathbf{X}}_\Lambda}}.
\end{aligned}$$

From Korn's inequality we deduce the uniform with respect to Λ inf-sup condition for B_2 . The proposition is then proved. $\square$

As the lower bounds of the inf – sup conditions for both problems (5.48) and (5.49) do not depend on the index set Λ , we deduce that:

Proposition 5.4.10. *For the Galerkin approximation (5.48) of problem (5.46) under condition (5.2) there is a constant C which does not depend on Λ such that*

$$\begin{aligned} \|\sigma - \sigma_\Lambda\|_{\underline{\mathbf{H}}} + \|u - u_\Lambda\|_{\underline{\mathbf{V}}} &\leq C \inf_{\{\tau_\Lambda, v_\Lambda\} \in \underline{\mathbf{X}}_\Lambda} (\|\sigma - \tau_\Lambda\|_{\underline{\mathbf{H}}} + \|u - v_\Lambda\|_{\underline{\mathbf{V}}}) \\ &\leq C \left[\left(\sum_{\nu \notin \Lambda} \|\sigma_\nu\|_{\underline{\mathbf{H}}}^2 \right)^{1/2} + \left(\sum_{\nu \notin \Lambda} \|u_\nu\|_{\underline{\mathbf{V}}}^2 \right)^{1/2} \right]. \end{aligned} \quad (5.56)$$

Proposition 5.4.11. *For the Galerkin approximation (5.49) of problem (5.47), under Assumption 5.4.7, there is a constant C which does not depend on Λ such that*

$$\begin{aligned} \|\sigma - \sigma_\Lambda\|_{\underline{\mathbf{H}}} + \|u - u_\Lambda\|_{\underline{\mathbf{V}}} &\leq C \inf_{\{\tau_\Lambda, v_\Lambda\} \in \underline{\mathbf{X}}_\Lambda} (\|\sigma - \tau_\Lambda\|_{\underline{\mathbf{H}}} + \|u - v_\Lambda\|_{\underline{\mathbf{V}}}) \\ &\leq C \left[\left(\sum_{\nu \notin \Lambda} \|\sigma_\nu\|_{\underline{\mathbf{H}}}^2 \right)^{1/2} + \left(\sum_{\nu \notin \Lambda} \|u_\nu\|_{\underline{\mathbf{V}}}^2 \right)^{1/2} \right]. \end{aligned} \quad (5.57)$$

5.4.3 Best N term convergence rate

5.4.3.1 Bounds for $\|u_\nu\|_{\underline{\mathbf{V}}}$ and $\|\sigma_\nu\|_{\underline{\mathbf{H}}}$

We deduce in this section explicit bounds for $\|u_\nu\|_{\underline{\mathbf{V}}}$ and $\|\sigma_\nu\|_{\underline{\mathbf{H}}}$ which will facilitate the establishment of a rate of convergence in terms of the cardinality of the set Λ for the semi-discrete Galerkin approximation in the previous section. We have the following result.

Proposition 5.4.12. *Suppose that Assumption 5.4.7 holds. Let*

$$\bar{d}_m = \frac{(1/\alpha_0 + \beta_0/\alpha_0^2)\beta_m}{1 - (1/\alpha_0 + \beta_0/\alpha_0^2)(\sum_{m=1}^{\infty} \beta_m)},$$

and $\bar{d} = (\bar{d}_1, \bar{d}_2, \dots)$. *There is a constant C which does not depend on ν such that the solution $\{\sigma, u\} \in \underline{\mathbf{X}}$ of problems (5.46) and (5.47) satisfies*

$$\|\partial_y^\nu \sigma\|_{\underline{\mathbf{H}}} + \|\partial_y^\nu u\|_{\underline{\mathbf{V}}} \leq C |\nu|! \bar{d}^\nu.$$

Proof. As the bilinear form $B_2 : \underline{\mathbf{X}} \times \underline{\mathbf{X}} \rightarrow \mathbb{R}$ satisfies the inf-sup condition, from (5.47), we deduce that there is a constant c independent of y such that

$$\|\{\sigma(\cdot, y), u(\cdot, y)\}\|_{\underline{\mathbf{X}}} \leq c \|f\|_{\underline{\mathbf{V}}'}.$$

We define the space $\mathbf{X}^\epsilon = \underline{\mathbf{H}} \times \underline{\mathbf{H}}^\epsilon$ equipped with the norm

$$\|\{\tau, \zeta\}\|_{\mathbf{X}^\epsilon} = \|\tau\|_{\underline{\mathbf{H}}} + \|\zeta\|_{\underline{\mathbf{H}}}.$$

We then have

$$\|\{\sigma(\cdot, y), \epsilon(u(\cdot, y))\}\|_{\mathbf{X}^\epsilon} \leq c \|f\|_{\underline{\mathbf{V}}'}.$$

The derivative with respect to y of the solution (σ, u) of problem (5.35) satisfies the problem:

$$\begin{aligned} (\partial_y^\nu \sigma(\cdot, y), \tau) - (\tau, a(\cdot, y) \epsilon(\partial_y^\nu u(\cdot, y))) &= \sum_{m=1}^{\infty} \nu_m (\tau, \Psi_m \epsilon(\partial_y^{\nu-e_m} u(\cdot, y))), \\ -(\partial_y^\nu \sigma(\cdot, y), \epsilon(v)) &= 0 \end{aligned} \quad (5.58)$$

for all $\{\tau, v\} \in \mathbf{X}$. The proof for (5.58) is similar to that for (5.32) using the inductive argument of [18]. We rewrite equations (5.58) as:

$$\begin{aligned} (\bar{a}^{-1} \partial_y^\nu \sigma(\cdot, y), \tau) - (\tau, \epsilon(\partial_y^\nu (u(\cdot, y)))) &= \sum_{m=1}^{\infty} \nu_m (\tau, \bar{a}^{-1} \Psi_m \epsilon(\partial_y^{\nu-e_m} u(\cdot, y))) \\ &+ \sum_{m=1}^{\infty} (\tau, y_m \bar{a}^{-1} \Psi_m \epsilon(\partial_y^\nu (u(\cdot, y)))), \\ -(\partial_y^\nu \sigma(\cdot, y), \zeta) &= 0 \end{aligned} \quad (5.59)$$

for all $\{\tau, \zeta\} \in \mathbf{X}^\epsilon$. We note that

$$\inf_{\sigma \in \mathcal{H}} \sup_{\tau \in \mathcal{H}} \frac{(\bar{a}^{-1} \sigma, \tau)}{\|\sigma\|_{\mathcal{H}} \|\tau\|_{\mathcal{H}}} \geq \frac{1}{\beta_0},$$

and

$$\inf_{\zeta \in \mathcal{H}^\epsilon} \sup_{\tau \in \mathcal{H}} \frac{(\tau, \zeta)}{\|\tau\|_{\mathcal{H}} \|\zeta\|_{\mathcal{H}}} \geq 1.$$

Further, for $\zeta, \xi \in \mathcal{H}$, $|(\bar{a}^{-1} \xi, \zeta)| \leq (1/\alpha_0) \|\xi\|_{\mathcal{H}} \|\zeta\|_{\mathcal{H}}$. From standard estimates for saddle point problems (see, e.g. Theorem 2.34 of [25]) we have

$$\begin{aligned} \|\epsilon(\partial_y^\nu u(\cdot, y))\|_{\mathcal{H}} &\leq \left(1 + \frac{\beta_0}{\alpha_0}\right) \left(\frac{1}{\alpha_0} \sum_{m=1}^{\infty} \nu_m \beta_m \|\epsilon(\partial_y^{\nu-e_m} u(\cdot, y))\|_{\mathcal{H}}\right. \\ &\quad \left. + \frac{1}{\alpha_0} \left(\sum_{m=1}^{\infty} \beta_m\right) \|\epsilon(\partial_y^\nu u(\cdot, y))\|_{\mathcal{H}}\right). \end{aligned}$$

We therefore have

$$\left(1 - \left(\frac{1}{\alpha_0} + \frac{\beta_0}{\alpha_0^2}\right) \sum_{m=1}^{\infty} \beta_m\right) \|\epsilon(\partial_y^\nu u(\cdot, y))\|_{\mathcal{H}} \leq \left(\frac{1}{\alpha_0} + \frac{\beta_0}{\alpha_0^2}\right) \sum_{m=1}^{\infty} \nu_m \beta_m \|\epsilon(\partial_y^{\nu-e_m} u(\cdot, y))\|_{\mathcal{H}},$$

i.e.

$$\|\epsilon(\partial_y^\nu u(\cdot, y))\|_{\mathcal{H}} \leq \sum_{m=1}^{\infty} \nu_m \bar{d}_m \|\epsilon(\partial_y^{\nu-e_m} u(\cdot, y))\|_{\mathcal{H}}.$$

We prove the proposition by induction. Assume that $\|\epsilon(\partial_y^{\nu-e_m} u(\cdot, y))\|_{\mathcal{H}} \leq c(|\nu| - 1)! \bar{d}^{\nu-e_m}$ for all $y \in U$. Then

$$\|\epsilon(\partial_y^\nu u(\cdot, y))\|_{\mathcal{H}} \leq c \sum_{m=1}^{\infty} \nu_m (|\nu| - 1)! \bar{d}_m \bar{d}^{\nu-e_m} = c |\nu|! \bar{d}^\nu.$$

Korn's inequality implies that there is a constant $C > 0$ such that

$$\|\partial_y^\nu u\|_{\mathbf{V}} \leq C|\nu|!\bar{d}^\nu.$$

Moreover, standard estimates for saddle point problems and (5.59) yield

$$\begin{aligned} \|\partial_y^\nu \sigma(\cdot, y)\|_{\mathbf{H}} &\leq \beta_0 \left(\frac{1}{\alpha_0} \sum_{m=1}^{\infty} \nu_m \beta_m \|\epsilon(\partial_y^{\nu-e_m} u(\cdot, y))\|_{\mathbf{H}} + \right. \\ &\quad \left. \frac{1}{\alpha_0} \left(\sum_{m=1}^{\infty} \beta_m \right) \|\epsilon(\partial_y^\nu u(\cdot, y))\|_{\mathbf{H}} \right) \\ &\leq \frac{\beta_0}{\alpha_0} \frac{1 - (1/\alpha_0 + \beta_0/\alpha_0^2) \sum_{m=1}^{\infty} \beta_m}{1/\alpha_0 + \beta_0/\alpha_0^2} \sum_{m=1}^{\infty} \nu_m \bar{d}_m c(|\nu| - 1)! \bar{d}^{\nu-e_m} \\ &\quad + \frac{\beta_0}{\alpha_0} \left(\sum_{m=1}^{\infty} \beta_m \right) c|\nu|! \bar{d}^\nu \\ &\leq C|\nu|! \bar{d}^\nu. \end{aligned}$$

□

We then deduce the following bounds for the coefficients of the GPC expansion.

Proposition 5.4.13. *Let $\bar{\bar{d}}_m = \bar{d}_m/\sqrt{3}$ and $\bar{\bar{d}} = (\bar{\bar{d}}_1, \bar{\bar{d}}_2, \dots) \in \mathbb{R}^{\mathbb{N}}$. Under Assumption 5.4.7, we have*

$$\|\{\sigma_\nu, u_\nu\}\|_{\mathbf{X}} \leq C \frac{|\nu|!}{\nu!} \bar{\bar{d}}^\nu.$$

The proof of this proposition is similar to that of Proposition 5.3.6.

5.4.3.2 Best N term convergence rate

We have the following summability property for the coefficients of the GPC expansions of σ and u .

Proposition 5.4.14. *Let Assumptions 5.3.7 and 5.4.7 be fulfilled with $\tilde{\kappa} < \sqrt{3}$, then $(\|\{\sigma_\nu, u_\nu\}\|_{\mathbf{X}})_\nu \in \ell^p(\mathcal{F})$.*

Proof. From Lemma 5.3.8 and Assumption 5.3.7, $\left(\frac{|\nu|!}{\nu!} \bar{\bar{d}}^\nu\right)_{\nu \in \mathcal{F}} \in \ell^p(\mathcal{F})$ if and only if $\sum_{m=1}^{\infty} \bar{\bar{d}}_m < 1$. This holds if

$$\left(\frac{1}{\alpha_0} + \frac{\beta_0}{\alpha_0^2}\right) \sum_{m=1}^{\infty} \beta_m < \sqrt{3} - \sqrt{3} \left(\frac{1}{\alpha_0} + \frac{\beta_0}{\alpha_0^2}\right) \sum_{m=1}^{\infty} \beta_m$$

so

$$\sum_{m=1}^{\infty} \beta_m < \frac{\sqrt{3}}{1 + \sqrt{3}} \frac{\alpha_0^2}{\alpha_0 + \beta_0}.$$

This is satisfied when $\tilde{\kappa} < \sqrt{3}$.

□

Let $r = 1/p - 1/2$. We then have the following rate of convergence for the semidiscrete Galerkin approximation (5.48) and (5.49).

Theorem 5.4.15. *Let Assumptions 5.2.1, 5.3.7 and 5.4.7 be satisfied with a constant $\tilde{\kappa} < \sqrt{3}$. Then for each integer N , there is a set $\Lambda \subset \mathcal{F}$ of cardinality not larger than N such that the solution of problems (5.46) and (5.47) and the semidiscrete Galerkin approximations (5.48) and (5.49) respectively satisfy*

$$\|\sigma - \sigma_\Lambda\|_{\underline{\mathbf{H}}} + \|u - u_\Lambda\|_{\underline{\mathbf{V}}} \leq cN^{-r}, \quad (5.60)$$

where c does not depend on Λ and N .

The proof of this theorem follows from Lemma 5.3.10 and Proposition 5.4.14.

5.5 Nearly incompressible material and mixed problem

The constant C in the semidiscretized error (5.29) has the form $c\beta/\alpha$. We note that

$$\frac{\beta}{\alpha} = \frac{\beta_0 + \kappa/(1 + \kappa)\alpha_0}{\alpha_0/(1 + \kappa)} \geq \frac{\beta_0}{\alpha_0}.$$

From (5.54), the constant C in the estimate (5.57) is larger than $c\beta_0/\alpha_0$. Thus when β_0/α_0 is very large, the multiplying constant in the best N term error estimates (5.33) and (5.60) is very large. To achieve a prescribed level of accuracy, we need a very large number of terms to estimate. In this section, we consider mixed problems (5.15) and (5.16) for which the best N term error estimate does not depend on the ratio β_0/α_0 when it goes to ∞ . We restrict our consideration to the case where the measure of $\partial D \setminus \Gamma$ is positive. The difficulty with the case of pure Dirichlet boundary condition ($\Gamma = \partial D$) is explained in Remark 5.5.2.

5.5.1 Deterministic parametric mixed problem

For each $y \in U$, we consider the parametric Lamé constants

$$\mu(x, y) = \bar{\mu}(x) + \sum_{m=1}^{\infty} \mu_m(x) y_m,$$

and

$$\lambda(x, y) = \bar{\lambda}(x) + \sum_{m=1}^{\infty} \lambda_m(x) y_m.$$

From Assumption 5.2.4, for all $x \in D$ and $y \in U$ we have

$$\begin{aligned} \mu_{\min} &:= \frac{1}{1 + \kappa} \bar{\mu}_{\min} \leq \mu(x, y) \leq \sup_{x \in D} \bar{\mu}(x) + \frac{\kappa}{1 + \kappa} \bar{\mu}_{\min} := \mu_{\max}, \\ \lambda_{\min} &:= \frac{1}{1 + \kappa} \bar{\lambda}_{\min} \leq \lambda(x, y) \leq \sup_{x \in D} \bar{\lambda}(x) + \frac{\kappa}{1 + \kappa} \bar{\lambda}_{\min} := \lambda_{\max}. \end{aligned} \quad (5.61)$$

Let $\mathbf{Y} = \mathbf{V} \times H$. We equip $\{u, p\} \in \mathbf{Y}$ with the norm

$$\|\{u, p\}\|_{\mathbf{Y}} = \|u\|_{\mathbf{V}} + \|p\|_H.$$

For mixed problems (5.15) and (5.16), we consider the corresponding parametric problems: Find $\{u(\cdot, y), p(\cdot, y)\} \in \mathbf{Y}$ such that

$$\begin{cases} 2(\mu(\cdot, y)\epsilon(u(\cdot, y)), \epsilon(v(\cdot))) + (\operatorname{div} v(\cdot), p(\cdot, y)) = (f(\cdot), v(\cdot)), & \forall v \in \mathbf{V} \\ (\operatorname{div} u(\cdot, y), q(\cdot)) - (\frac{1}{\lambda(\cdot, y)} p(\cdot, y), q(\cdot)) = 0, & \forall q \in H \end{cases} \quad (5.62)$$

and: Find $\{u(\cdot, y), p(\cdot, y)\} \in \mathbf{Y}$ such that

$$\begin{cases} 2(\mu(\cdot, y)\epsilon(u(\cdot, y)), \epsilon(v(\cdot))) + (\operatorname{div} v(\cdot), p(\cdot, y)) = (f(\cdot), v(\cdot)), & \forall v \in \mathbf{V} \\ (\frac{\lambda}{\bar{\lambda}_{\min}}(\cdot, y)\operatorname{div} u(\cdot, y), q(\cdot)) - (\frac{1}{\bar{\lambda}_{\min}} p(\cdot, y), q(\cdot)) = 0, & \forall q \in H. \end{cases} \quad (5.63)$$

We define the bilinear forms $b_3, b_4 : \mathbf{Y} \times \mathbf{Y} \rightarrow \mathbb{R}$ as

$$\begin{aligned} b_3(y; \{u, p\}, \{v, q\}) &= 2(\mu(\cdot, y)\epsilon(u(\cdot, y)), \epsilon(v(\cdot))) + (\operatorname{div} v(\cdot), p(\cdot, y)) \\ &\quad + (\operatorname{div} u(\cdot, y), q(\cdot)) - (\frac{1}{\lambda(\cdot, y)} p(\cdot, y), q(\cdot)), \end{aligned} \quad (5.64)$$

and

$$\begin{aligned} b_4(y; \{u, p\}, \{v, q\}) &= 2(\mu(\cdot, y)\epsilon(u(\cdot, y)), \epsilon(v(\cdot))) + (\operatorname{div} v(\cdot), p(\cdot, y)) \\ &\quad + (\frac{\lambda}{\bar{\lambda}_{\min}}(\cdot, y)\operatorname{div} u(\cdot, y), q(\cdot)) - (\frac{1}{\bar{\lambda}_{\min}} p(\cdot, y), q(\cdot)). \end{aligned} \quad (5.65)$$

Problems (5.62) and (5.63) can be written as: Find $\{u, p\} \in \mathbf{Y}$ such that

$$b_3(y; \{u, p\}, \{v, q\}) = (f, v), \quad \forall \{v, q\} \in \mathbf{Y}, \quad (5.66)$$

and: Find $\{u, p\} \in \mathbf{Y}$ such that

$$b_4(y; \{u, p\}, \{v, q\}) = (f, v), \quad \forall \{v, q\} \in \mathbf{Y}. \quad (5.67)$$

Proposition 5.5.1. *Problems (5.62) and (5.63) are equivalent. There are positive constants χ_3 and χ_4 such that for all $y \in U$:*

$$\inf_{\{u, p\} \in \mathbf{Y}} \sup_{\{v, q\} \in \mathbf{Y}} \frac{b_3(y; \{u, p\}, \{v, q\})}{\|\{u, p\}\|_{\mathbf{Y}} \|\{v, q\}\|_{\mathbf{Y}}} \geq \chi_3, \quad (5.68)$$

and

$$\inf_{\{u, p\} \in \mathbf{Y}} \sup_{\{v, q\} \in \mathbf{Y}} \frac{b_4(y; \{u, p\}, \{v, q\})}{\|\{u, p\}\|_{\mathbf{Y}} \|\{v, q\}\|_{\mathbf{Y}}} \geq \chi_4 \quad (5.69)$$

so that (5.62) and (5.63) have a unique solution. Problems (5.15) and (5.16) are thus equivalent and have a unique solution. Further, if $\bar{\lambda}_{\min} > \vartheta$ and $\kappa < \kappa_0$ then χ_3 and χ_4 can be chosen so that they only depend on ϑ and κ_0 .

Proof. Due to Assumption 5.2.4, $\lambda/\bar{\lambda}_{\min}$ is uniformly bounded above and below with respect to x and y so for all $q \in H$, $\bar{\lambda}_{\min}q/\lambda \in H$. Problems (5.62) and (5.63) are thus equivalent.

We note that the bilinear form mapping $\{u, v\} \in \mathbf{V} \times \mathbf{V}$ to $(\mu(\cdot, y)\epsilon(u), \epsilon(v))$ is uniformly coercive for all $y \in U$.

Due to $\text{meas}(\partial D \setminus \Gamma) > 0$, there is a constant c_0 that depends on D such that for each $q \in H$, there is $v \in \mathbf{V}$ that satisfies

$$\text{div} v = q \text{ and } \|v\|_{\mathbf{V}} \leq \|q\|_H / c_0 \quad (5.70)$$

(see, e.g., Ern and Guermond [25] Lemma 4.9 page 181). Therefore

$$\inf_{q \in H} \sup_{v \in \mathbf{V}} \frac{(\text{div} v, q)}{\|v\|_{\mathbf{V}} \|q\|_H} \geq c_0$$

so following the proof of Braess [9] for saddle point problems with a penalty term, the inf-sup condition for b_3 holds uniformly with respect to y for the following norm of $\{v, q\}$:

$$\|v\|_{\mathbf{V}} + \|q\|_H + \left\| \frac{q}{\lambda^{1/2}} \right\|_H. \quad (5.71)$$

Since $\lambda(x, y) \geq \bar{\lambda}_{\min}/(1 + \kappa)$ for all $x \in D, y \in U$, this norm is equivalent to $\|\{v, q\}\|_{\mathbf{Y}}$ uniformly with respect to $y \in U$ so the inf-sup condition for b_3 holds uniformly for all $y \in U$ with respect to the norm $\|\cdot\|_{\mathbf{Y}}$ defined above. This norm equivalence is uniform with respect to λ if $\bar{\lambda}_{\min} > \vartheta$ and $\kappa < \kappa_0$.

We note that for each $\{u, p\} \in \mathbf{Y}$

$$\sup_{\{v, q\} \in \mathbf{Y}} \frac{b_4(y; \{u, p\}, \{v, q\})}{\|\{u, p\}\|_{\mathbf{Y}} \|\{v, q\}\|_{\mathbf{Y}}} = \sup_{\{v, q\} \in \mathbf{Y}} \frac{b_3(y; \{u, p\}, \{v, \frac{\lambda}{\bar{\lambda}_{\min}} q\})}{\|\{u, p\}\|_{\mathbf{Y}} \|\{v, \frac{\lambda}{\bar{\lambda}_{\min}} q\}\|_{\mathbf{Y}}} \frac{\|\{v, \frac{\lambda}{\bar{\lambda}_{\min}} q\}\|_{\mathbf{Y}}}{\|\{v, q\}\|_{\mathbf{Y}}}. \quad (5.72)$$

As $\|\frac{\lambda}{\bar{\lambda}_{\min}} q\|_H \geq \frac{1}{1 + \kappa} \|q\|_H$, (5.68) yields

$$\sup_{\{v, q\} \in \mathbf{Y}} \frac{b_4(y; \{u, p\}, \{v, q\})}{\|\{u, p\}\|_{\mathbf{Y}} \|\{v, q\}\|_{\mathbf{Y}}} \geq \frac{1}{1 + \kappa} \sup_{\{v, q\} \in \mathbf{Y}} \frac{b_3(y; \{u, p\}, \{v, q\})}{\|\{u, p\}\|_{\mathbf{Y}} \|\{v, q\}\|_{\mathbf{Y}}} \geq \frac{1}{1 + \kappa} \chi_3$$

for all $y \in U$.

Due to Lemma 5.5.3 below, the solution $\{u(\cdot, \omega), p(\cdot, \omega)\}$ of problems (5.15) and (5.16) is obtained from the solution $\{u(\cdot, y), p(\cdot, y)\}$ of (5.62) and (5.63) respectively by inserting $y_j = y_j(\omega)$. We deduce that problems (5.15) and (5.16) are equivalent and have a unique solution. $\square$

From this, we deduce that $\|\{u(\cdot, y), p(\cdot, y)\}\|_{\mathbf{Y}}$ is uniformly bounded for all $y \in U$.

Remark 5.5.2. For the case of pure Dirichlet boundary condition (i.e. $\Gamma = \partial D$), instead of considering (5.71), we need

$$|||\{v, q\}||| := \|v\|_{\mathbf{V}} + \|q\|_{H/\mathbb{R}} + \left\| \frac{q}{\lambda^{1/2}} \right\|_H.$$

We note that as $\bar{\lambda}_{\min}/(1 + \kappa) \leq \lambda(x, y) \leq \bar{\lambda}_{\max}(1 + \kappa/(1 + \kappa))$,

$$\begin{aligned} & \|v\|_{\mathbf{V}} + \left(\frac{1 + \kappa}{(1 + 2\kappa)\bar{\lambda}_{\max}} \right)^{1/2} \|q\|_H \\ & \leq \|v\|_{\mathbf{V}} + \|q\|_{H/\mathbb{R}} + \left\| \frac{q}{\lambda^{1/2}} \right\|_H \\ & \leq \|v\|_{\mathbf{V}} + \|q\|_{H/\mathbb{R}} + \left(\frac{1 + \kappa}{\bar{\lambda}_{\min}} \right)^{1/2} \|q\|_H, \end{aligned}$$

so $|||\{v, q\}|||$ is equivalent to $\|\{u, v\}\|_{\mathbf{Y}}$ for each fixed $\bar{\lambda}$ but the equivalence is not uniform when $\bar{\lambda}_{\max} \rightarrow \infty$. However, following Braess [9], with respect to this $|||\cdot|||$ norm the inf-sup condition for problem (5.62) still holds uniformly with respect to λ . The purpose of considering (5.63) is to employ the linear dependence of λ on y_m . To deduce the inf-sup condition for b_4 , similar to (5.72), we would have to bound the ratio

$$\frac{\|v\|_{\mathbf{V}} + \left\| \frac{\lambda}{\bar{\lambda}_{\min}} q \right\|_{H/\mathbb{R}} + \left\| \frac{\lambda^{1/2}}{\bar{\lambda}_{\min}} q \right\|_H}{\|v\|_{\mathbf{V}} + \|q\|_{H/\mathbb{R}} + \left\| \frac{q}{\lambda^{1/2}} \right\|_H},$$

which does not admit a uniform lower bound with respect to λ due to the non-equivalence of the norms $\|\lambda q / \bar{\lambda}_{\min}\|_{H/\mathbb{R}}$ and $\|q\|_{H/\mathbb{R}}$ when $\lambda / \bar{\lambda}_{\min}$ is not a constant. We face similar difficulty when proving the uniform with respect to $\bar{\lambda}_{\min}$ inf-sup condition for the Galerkin problem (5.81) for the case of a Dirichlet boundary condition. We therefore restrict our consideration in this section to the case where the measure of $\partial D \setminus \Gamma$ is positive.

We have the following result on the measurability of $\{u(\cdot, y), p(\cdot, y)\}$ as a map from U to $\mathbf{Y}$.

Lemma 5.5.3. For problems (5.62) and (5.63), $\{u(\cdot, y), p(\cdot, y)\}$ as a map from $U \subset \ell^\infty(\mathbb{N})$ to $\mathbf{Y}$ is Lipschitz and is therefore measurable.

Proof. Let $\{u(\cdot, y), p(\cdot, y)\}$ and $\{u(\cdot, y'), p(\cdot, y')\}$ be the solution of (5.67) for $y, y' \in U$. Then $\{u(\cdot, y) - u(\cdot, y'), p(\cdot, y) - p(\cdot, y')\} \in \mathbf{Y}$ satisfies the problem

$$\begin{aligned} & 2(\mu(\cdot, y)(\epsilon(u(\cdot, y)) - \epsilon(u(\cdot, y'))), \epsilon(v)) + (\operatorname{div} v, p(\cdot, y) - p(\cdot, y')) + \\ & \left(\frac{\lambda}{\bar{\lambda}_{\min}}(\cdot, y)(\operatorname{div} u(\cdot, y) - \operatorname{div} u(\cdot, y')), q \right) - \left(\frac{1}{\bar{\lambda}_{\min}}(p(\cdot, y) - p(\cdot, y')), q \right) = \\ & 2((\mu(\cdot, y') - \mu(\cdot, y))\epsilon(u(\cdot, y')), \epsilon(v)) + \left(\left(\frac{\lambda}{\bar{\lambda}_{\min}}(\cdot, y') - \frac{\lambda}{\bar{\lambda}_{\min}}(\cdot, y) \right) \operatorname{div} u(y'), q \right). \end{aligned}$$

From this we deduce that there is a constant c independent of y and y' such that

$$\|\{u(\cdot, y) - u(\cdot, y'), p(\cdot, y) - p(\cdot, y')\}\|_{\mathbf{Y}} \leq c(\|\mu(\cdot, y) - \mu(\cdot, y')\|_{L^\infty(D)} + \|\lambda(\cdot, y) - \lambda(\cdot, y')\|_{L^\infty(D)}).$$

The measurability of $\{u(\cdot, y), p(\cdot, y)\}$ as a map from U to $\mathbf{Y}$ then follows. $\square$

From this and the uniform boundedness of $\|\{u(\cdot, y), p(\cdot, y)\}\|_{\mathbf{Y}}$, we conclude that $\{u(\cdot, \cdot), p(\cdot, \cdot)\} \in \underline{\mathbf{Y}} := L^2(U, \rho; \mathbf{Y})$. Therefore we can write

$$u = \sum_{\nu \in \mathcal{F}} u_\nu L_\nu, \quad \text{and} \quad p = \sum_{\nu \in \mathcal{F}} p_\nu L_\nu, \quad (5.73)$$

where $u_\nu \in \mathbf{V}$ and $p_\nu \in H$. Let $\underline{H} = L^2(U, \rho; H)$. We then consider the mixed problems: Find $\{u(\cdot, \cdot), p(\cdot, \cdot)\} \in \underline{\mathbf{Y}}$ such that

$$\begin{cases} \int_U [2(\mu(\cdot, y)\epsilon(u(\cdot, y)), \epsilon(v(\cdot, y))) + (\operatorname{div} v(\cdot, y), p(\cdot, y))] d\rho(y) = \\ \int_U (f(\cdot), v(\cdot, y)) d\rho(y), \quad \forall v \in \underline{\mathbf{V}} \\ \int_U \left[(\operatorname{div} u(\cdot, y), q(\cdot, y)) - \left(\frac{1}{\lambda(\cdot, y)} p(\cdot, y), q(\cdot, y) \right) \right] d\rho(y) = 0, \quad \forall q \in \underline{H} \end{cases} \quad (5.74)$$

and: Find $\{u(\cdot, \cdot), p(\cdot, \cdot)\} \in \underline{\mathbf{Y}}$ such that

$$\begin{cases} \int_U [2(\mu(\cdot, y)\epsilon(u(\cdot, y)), \epsilon(v(\cdot, y))) + (\operatorname{div} v(\cdot, y), p(\cdot, y))] d\rho(y) = \\ \int_U (f(\cdot), v(\cdot, y)) d\rho(y), \quad \forall v \in \underline{\mathbf{V}} \\ \int_U \left[\left(\frac{\lambda}{\lambda_{\min}}(\cdot, y) \operatorname{div} u(\cdot, y), q(\cdot, y) \right) - \left(\frac{1}{\lambda_{\min}} p(\cdot, y), q(\cdot, y) \right) \right] d\rho(y) = 0, \quad \forall q \in \underline{H}. \end{cases} \quad (5.75)$$

We define the bilinear forms $B_3, B_4 : \underline{\mathbf{Y}} \times \underline{\mathbf{Y}} \rightarrow \mathbb{R}$ as

$$B_3(\{u, p\}, \{v, q\}) = \int_U b_3(y; \{u(\cdot, y), p(\cdot, y)\}, \{v(\cdot, y), q(\cdot, y)\}) d\rho(y), \quad (5.76)$$

and

$$B_4(\{u, p\}, \{v, q\}) = \int_U b_4(y; \{u(\cdot, y), p(\cdot, y)\}, \{v(\cdot, y), q(\cdot, y)\}) d\rho(y). \quad (5.77)$$

For the linear form $F : \underline{\mathbf{Y}} \rightarrow \mathbb{R}$ defined as

$$F(\{v, q\}) = \int_U (f, v(\cdot, y)) d\rho(y),$$

we rewrite problems (5.74) and (5.75) as: Find $\{u, p\} \in \underline{\mathbf{Y}}$ such that

$$B_3(\{u, p\}, \{v, q\}) = F(\{v, q\}) \quad \forall \{v, q\} \in \underline{\mathbf{Y}}, \quad (5.78)$$

and: Find $\{u, p\} \in \underline{\mathbf{Y}}$ such that

$$B_4(\{u, p\}, \{v, q\}) = F(\{v, q\}) \quad \forall \{v, q\} \in \underline{\mathbf{Y}}, \quad (5.79)$$

respectively. We have the following existence and uniqueness theorem.

Proposition 5.5.4. *Problems (5.78) and (5.79) are equivalent and have a unique solution.*

The proof of this proposition is similar to that for Proposition 5.5.1.

5.5.2 Semidiscrete Galerkin approximation

For a subset $\Lambda \subset \mathcal{F}$ of finite cardinality, we denote by

$$\underline{\mathbf{V}}_\Lambda = \{ v_\Lambda \in \underline{\mathbf{V}} : v_\Lambda(x, y) = \sum_{\nu \in \Lambda} v_\nu(x) L_\nu(y), \quad v_\nu \in \mathbf{V} \},$$

$$\underline{H}_\Lambda = \{ q_\Lambda \in \underline{H} : q_\Lambda(x, y) = \sum_{\nu \in \Lambda} q_\nu(x) L_\nu(y), \quad q_\nu \in H \},$$

and

$$\underline{\mathbf{Y}}_\Lambda = \{ \{v_\Lambda, q_\Lambda\} : v_\Lambda \in \underline{\mathbf{V}}_\Lambda, \quad q_\Lambda \in \underline{H}_\Lambda \}.$$

We consider the semidiscrete Galerkin approximation for (5.78): Find $\{u_\Lambda, p_\Lambda\} \in \underline{\mathbf{Y}}_\Lambda$ so that

$$B_3(\{u_\Lambda, p_\Lambda\}, \{v_\Lambda, q_\Lambda\}) = F(\{v_\Lambda, q_\Lambda\}), \quad \forall \{v_\Lambda, q_\Lambda\} \in \underline{\mathbf{Y}}_\Lambda, \quad (5.80)$$

and the semidiscrete Galerkin approximation for (5.79): Find $\{u_\Lambda, p_\Lambda\} \in \underline{\mathbf{Y}}_\Lambda$ so that

$$B_4(\{u_\Lambda, p_\Lambda\}, \{v_\Lambda, q_\Lambda\}) = F(\{v_\Lambda, q_\Lambda\}), \quad \forall \{v_\Lambda, q_\Lambda\} \in \underline{\mathbf{Y}}_\Lambda. \quad (5.81)$$

We have the following result.

Proposition 5.5.5. *There exists a constant χ'_3 independent of Λ so that*

$$\inf_{\{u_\Lambda, p_\Lambda\} \in \underline{\mathbf{Y}}_\Lambda} \sup_{\{v_\Lambda, q_\Lambda\} \in \underline{\mathbf{Y}}_\Lambda} \frac{B_3(\{u_\Lambda, p_\Lambda\}, \{v_\Lambda, q_\Lambda\})}{\|\{u_\Lambda, p_\Lambda\}\|_{\underline{\mathbf{Y}}} \|\{v_\Lambda, q_\Lambda\}\|_{\underline{\mathbf{Y}}}} \geq \chi'_3. \quad (5.82)$$

Further if $B_3(\{u_\Lambda, p_\Lambda\}, \{v_\Lambda, q_\Lambda\}) = 0$ for all $\{u_\Lambda, p_\Lambda\} \in \underline{\mathbf{Y}}_\Lambda$ then $v_\Lambda = 0$ and $q_\Lambda = 0$. Problem (5.80) thus has a unique solution.

The proof of the inf-sup condition (5.82) follows from the standard proof for saddle point problems with penalty term (see, e.g. Braess [10] or Braess [9]).

Remark 5.5.6. *For each $y \in U$, we show that (5.62) and (5.63) are equivalent by noting that for each $q \in H$, $\lambda q / \bar{\lambda}_{\min} \in H$, and similarly for the equivalence of (5.78) and (5.79). However, when $q \in \underline{H}_\Lambda$, $\lambda q / \bar{\lambda}_{\min}$ is not necessarily in $\underline{H}_\Lambda$ so problems (5.80) and (5.81) are generally not equivalent. Therefore the inf-sup condition (5.82) does not imply an inf-sup condition for (5.81).*

For the semidiscretized Galerkin approximation problem (5.81), we have the following results on the inf-sup condition.

Proposition 5.5.7. *Assume Assumption 5.2.4 holds with $\kappa \leq \kappa_0$ where $\kappa_0 > 0$ is a fixed constant. There is a constant ϑ_1 which depends on κ_0 , μ and the domain D such that if $\bar{\lambda}_{\min} > \vartheta_1$, then there is a positive constant χ'_4 which does not depend on λ and Λ such that*

$$\inf_{\{u_\Lambda, p_\Lambda\} \in \underline{\mathbf{Y}}_\Lambda} \sup_{\{v_\Lambda, q_\Lambda\} \in \underline{\mathbf{Y}}_\Lambda} \frac{B_4(\{u_\Lambda, p_\Lambda\}, \{v_\Lambda, q_\Lambda\})}{\|\{u_\Lambda, p_\Lambda\}\|_{\underline{\mathbf{Y}}} \|\{v_\Lambda, q_\Lambda\}\|_{\underline{\mathbf{Y}}}} \geq \chi'_4. \quad (5.83)$$

Further, if $B_4(\{u_\Lambda, p_\Lambda\}, \{v_\Lambda, q_\Lambda\}) = 0 \ \forall \{u_\Lambda, p_\Lambda\} \in \underline{\mathbf{Y}}_\Lambda$, then $v_\Lambda = 0$ and $q_\Lambda = 0$. Problem (5.81) thus has a unique solution.

Proof. We follow the approach for saddle point problems with penalty in Braess [9]. Let $q = \sum_{\nu \in \Lambda} q_\nu L_\nu \in \underline{H}_\Lambda$. Let $v_\nu \in \mathbf{V}$ be so that $\operatorname{div} v_\nu = q_\nu$ and $\|v_\nu\|_{\mathbf{V}} \leq \|q_\nu\|_{\underline{H}}/c_0$ where c_0 is the constant in (5.70). Let $v_\Lambda = \sum_{\nu \in \Lambda} v_\nu L_\nu$. Then $\|v_\Lambda\|_{\underline{\mathbf{V}}} \leq \|q_\Lambda\|_{\underline{H}}/c_0$. Thus

$$\sup_{v_\Lambda \in \underline{\mathbf{V}}_\Lambda} \frac{\int_U (\operatorname{div} v_\Lambda, q_\Lambda) d\rho(y)}{\|v_\Lambda\|_{\underline{\mathbf{V}}} \|q_\Lambda\|_{\underline{H}}} \geq c_0.$$

From (5.61), there is a constant μ^* such that for all $y \in U$, $u, v \in \mathbf{V}$,

$$|(\mu(\cdot, y)\epsilon(u), \epsilon(v))| \leq \mu^* \|u\|_{\mathbf{V}} \|v\|_{\mathbf{V}}.$$

Let $\{u_\Lambda, p_\Lambda\} \in \underline{\mathbf{Y}}_\Lambda$. We consider first the case where

$$\|u_\Lambda\|_{\underline{\mathbf{V}}} \leq \frac{c_0 \|p_\Lambda\|_{\underline{H}}}{4\mu^*}. \quad (5.84)$$

We then have

$$\begin{aligned} c_0 \|p_\Lambda\|_{\underline{H}} &\leq \sup_{v_\Lambda \in \underline{\mathbf{V}}_\Lambda} \frac{\int_U (\operatorname{div} v_\Lambda, p_\Lambda) d\rho(y)}{\|v_\Lambda\|_{\underline{\mathbf{V}}}} \\ &= \sup_{v_\Lambda \in \underline{\mathbf{V}}_\Lambda} \frac{B_4(\{u_\Lambda, p_\Lambda\}, \{v_\Lambda, 0\})}{\|v_\Lambda\|_{\underline{\mathbf{V}}}} - \frac{2 \int_U (\mu\epsilon(u_\Lambda), \epsilon(v_\Lambda)) d\rho(y)}{\|v_\Lambda\|_{\underline{\mathbf{V}}}} \\ &\leq \sup_{\{v_\Lambda, q_\Lambda\} \in \underline{\mathbf{Y}}_\Lambda} \frac{B_4(\{u_\Lambda, p_\Lambda\}, \{v_\Lambda, q_\Lambda\})}{\|\{v_\Lambda, q_\Lambda\}\|_{\underline{\mathbf{Y}}}} + 2\mu^* \|u_\Lambda\|_{\underline{\mathbf{V}}} \\ &\leq \sup_{\{v_\Lambda, q_\Lambda\} \in \underline{\mathbf{Y}}_\Lambda} \frac{B_4(\{u_\Lambda, p_\Lambda\}, \{v_\Lambda, q_\Lambda\})}{\|\{v_\Lambda, q_\Lambda\}\|_{\underline{\mathbf{Y}}}} + \frac{c_0}{2} \|p_\Lambda\|_{\underline{H}}. \end{aligned}$$

Therefore

$$\sup_{\{v_\Lambda, q_\Lambda\} \in \underline{\mathbf{Y}}_\Lambda} \frac{B_4(\{u_\Lambda, p_\Lambda\}, \{v_\Lambda, q_\Lambda\})}{\|\{v_\Lambda, q_\Lambda\}\|_{\underline{\mathbf{Y}}}} \geq \frac{c_0}{2} \|p_\Lambda\|_{\underline{H}} \geq \min\left(\frac{c_0}{4}, \mu^*\right) \|\{u_\Lambda, p_\Lambda\}\|_{\underline{\mathbf{Y}}}$$

where at the end we once more use (5.84). We now consider the case where

$$\|u_\Lambda\|_{\underline{\mathbf{V}}} > \frac{c_0 \|p_\Lambda\|_{\underline{H}}}{4\mu^*}. \quad (5.85)$$

From Assumption 5.2.4 and Korn's inequality, we deduce that there is a positive constant c_1 such that for all $u_\Lambda \in \underline{\mathbf{V}}_\Lambda$,

$$2(\mu\epsilon(u_\Lambda), \epsilon(u_\Lambda)) \geq c_1 \|u_\Lambda\|_{\underline{\mathbf{V}}}^2.$$

Let $q_\Lambda = t \operatorname{div} u_\Lambda$ for a constant $t > 0$ to be chosen later. We have

$$\begin{aligned} B_4(\{u_\Lambda, p_\Lambda\}, \{u_\Lambda, q_\Lambda\}) &\geq c_1 \|u_\Lambda\|_{\underline{\mathbf{V}}}^2 - \|p_\Lambda\|_{\underline{H}} \|\operatorname{div} u_\Lambda\|_{\underline{H}} + \frac{t}{1+\kappa} \|\operatorname{div} u_\Lambda\|_{\underline{H}}^2 - \\ &\quad \frac{t}{\bar{\lambda}_{\min}} \|p_\Lambda\|_{\underline{H}} \|\operatorname{div} u_\Lambda\|_{\underline{H}}. \end{aligned}$$

Assuming that $\bar{\lambda}_{\min} \geq t$, we then have:

$$B_4(\{u_\Lambda, p_\Lambda\}, \{u_\Lambda, q_\Lambda\}) \geq c_1 \|u_\Lambda\|_{\underline{\mathbf{V}}}^2 + \frac{t}{1+\kappa} \|\operatorname{div} u_\Lambda\|_{\underline{H}}^2 - 2\|p_\Lambda\|_{\underline{H}} \|\operatorname{div} u_\Lambda\|_{\underline{H}}.$$

From (5.85), there is a constant c_3 independent of λ such that

$$\begin{aligned} B_4(\{u_\Lambda, p_\Lambda\}, \{u_\Lambda, q_\Lambda\}) &\geq c_1 \|u_\Lambda\|_{\underline{\mathbf{V}}}^2 + \frac{t}{1+\kappa} \|\operatorname{div} u_\Lambda\|_{\underline{H}}^2 - 2c_3 \|u_\Lambda\|_{\underline{\mathbf{V}}} \|\operatorname{div} u_\Lambda\|_{\underline{H}} \\ &\geq c_1 \|u_\Lambda\|_{\underline{\mathbf{V}}}^2 + \frac{t}{1+\kappa} \|\operatorname{div} u_\Lambda\|_{\underline{H}}^2 - c_3 c_4 \|u_\Lambda\|_{\underline{\mathbf{V}}}^2 - \frac{c_3}{c_4} \|\operatorname{div} u_\Lambda\|_{\underline{H}}^2 \end{aligned}$$

for all $c_4 > 0$. Letting $c_4 = c_1/(2c_3)$ and $\bar{\lambda}_{\min} > \vartheta_1 = t = (1+\kappa)3c_3^2/c_1$, we deduce that for $c_5 = \min(c_1/2, c_3^2/(t^2 c_1))$

$$B_4(\{u_\Lambda, p_\Lambda\}, \{u_\Lambda, q_\Lambda\}) \geq c_5 (\|u_\Lambda\|_{\underline{\mathbf{V}}} + \|q_\Lambda\|_{\underline{H}})^2.$$

Thus there is a constant $c_6 > 0$ such that

$$\frac{B_4(\{u_\Lambda, p_\Lambda\}, \{u_\Lambda, q_\Lambda\})}{\|u_\Lambda\|_{\underline{\mathbf{V}}} + \|q_\Lambda\|_{\underline{H}}} \geq c_5 \|u_\Lambda\|_{\underline{\mathbf{V}}} \geq c_6 (\|u_\Lambda\|_{\underline{\mathbf{V}}} + \|p_\Lambda\|_{\underline{H}}).$$

We then get the conclusion. $\square$

5.5.3 Best N term approximation

5.5.3.1 Bounds for $\|\{u_\nu, p_\nu\}\|_{\mathbf{Y}}$

To deduce a rate for the best N term approximation for u and p , we first deduce bounds for $\|\{u_\nu, p_\nu\}\|_{\mathbf{Y}}$. We have the following results.

Proposition 5.5.8. *Assume that $\kappa < \kappa_0$. There are positive constants $\vartheta_2, C_1, C_2, C_3, C_4$ and C_5 which depend on κ_0, μ and the domain D such that if $\bar{\lambda}_{\min} > \vartheta_2$, for $0 < \varsigma < 1$*

$$\begin{aligned} \|\epsilon(\partial_y^\nu u)\|_{\mathbf{H}} + \frac{1}{\lambda_{\min}^\varsigma} \|\lambda \operatorname{div} \partial_y^\nu u\|_{\mathbf{H}} &\leq \left(1 + \frac{C_1}{\lambda_{\min}} + \frac{C_4}{\lambda_{\min}^\varsigma}\right) \sum_{m=1}^{\infty} \nu_m \frac{\gamma_m}{\mu_{\min}} \|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathbf{H}} \\ &\quad + \left(1 + \frac{C_2}{\lambda_{\min}^{1-\varsigma}} + \frac{C_3}{\lambda_{\min}^{2-\varsigma}} + \frac{C_5}{\lambda_{\min}}\right) \sum_{m=1}^{\infty} \nu_m \frac{\delta_m}{\lambda_{\min}} \frac{1}{\lambda_{\min}^\varsigma} \|\lambda \operatorname{div} \partial_y^{\nu-e_m} u\|_{\mathbf{H}}. \end{aligned}$$

Proof. For all $y \in U$, the derivative $\{\partial_y^\nu u, \partial_y^\nu p\} \in \mathbf{Y}$ satisfies the problem

$$\begin{aligned} & 2(\mu(\cdot, y)\epsilon(\partial_y^\nu u), \epsilon(v)) + (\operatorname{div} v, \partial_y^\nu p) + \left(\frac{\lambda(\cdot, y)}{\bar{\lambda}_{\min}} \operatorname{div} \partial_y^\nu u, q\right) - \left(\frac{1}{\bar{\lambda}_{\min}} \partial_y^\nu p, q\right) \\ = & -2 \sum_{m=1}^{\infty} \nu_m (\mu_m \epsilon(\partial_y^{\nu-e_m} u), \epsilon(v)) - \sum_{m=1}^{\infty} \nu_m \left(\frac{\lambda_m}{\bar{\lambda}_{\min}} \operatorname{div} \partial_y^{\nu-e_m} u, q\right), \end{aligned}$$

for all $\{v, q\} \in \mathbf{Y}$. The proof for this is similar to that for (5.32) using the inductive argument of Cohen, DeVore and Schwab [18]. To prove the proposition, we deduce in the following estimates for $\partial_y^\nu u(\cdot, y)$ and $\partial_y^\nu p(\cdot, y)$ for each $y \in U$ though the bounds are independent of y . As every function $q \in H$ can be written as $\bar{\lambda}_{\min} q' / \lambda$ for a function $q' \in H$, we can rewrite the problem in the saddle point form as

$$2(\mu(\cdot, y)\epsilon(\partial_y^\nu u), \epsilon(v)) + (\operatorname{div} v, \partial_y^\nu p) = -2 \sum_{m=1}^{\infty} \nu_m (\mu_m \epsilon(\partial_y^{\nu-e_m} u), \epsilon(v)) \quad (5.86)$$

$$(\operatorname{div} \partial_y^\nu u, q) = \left(\frac{1}{\lambda} \partial_y^\nu p - \sum_{m=1}^{\infty} \nu_m \frac{\lambda_m}{\lambda} \operatorname{div} \partial_y^{\nu-e_m} u, q \right). \quad (5.87)$$

We denote by

$$g = \frac{1}{\lambda} \partial_y^\nu p - \sum_{m=1}^{\infty} \nu_m \frac{\lambda_m}{\lambda} \operatorname{div} \partial_y^{\nu-e_m} u \in H.$$

Let $u_g \in \mathbf{V}$ be such that $\operatorname{div} u_g = g$ and

$$c_0 \|u_g\|_{\mathbf{V}} \leq \|g\|_H \quad (5.88)$$

where c_0 is the constant in (5.70). Then for all $v \in \mathbf{V}$ such that $\operatorname{div} v = 0$, we have

$$2(\mu(\cdot, y)(\epsilon(\partial_y^\nu u) - \epsilon(u_g)), \epsilon(v)) = -2 \sum_{m=1}^{\infty} \nu_m (\mu_m \epsilon(\partial_y^{\nu-e_m} u), \epsilon(v)) - 2(\mu(\cdot, y)\epsilon(u_g), \epsilon(v)).$$

As $(\operatorname{div} \partial_y^\nu u, q) = (g, q) = (\operatorname{div} u_g, q)$, letting $v = \partial_y^\nu u - u_g$ we obtain

$$\|\epsilon(\partial_y^\nu u) - \epsilon(u_g)\|_{\mathbf{H}} \leq \sum_{m=1}^{\infty} \nu_m \frac{\gamma_m}{\mu_{\min}} \|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathbf{H}} + \frac{\mu_{\max}}{\mu_{\min}} \|\epsilon(u_g)\|_{\mathbf{H}}$$

which implies

$$\|\epsilon(\partial_y^\nu u)\|_{\mathbf{H}} \leq \sum_{m=1}^{\infty} \nu_m \frac{\gamma_m}{\mu_{\min}} \|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathbf{H}} + \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right) \|\epsilon(u_g)\|_{\mathbf{H}}.$$

Let $c_7 = c_7(d)$ be a constant such that $\|\epsilon(v)\|_{\mathcal{H}} \leq c_7 \|v\|_{\mathbf{V}}$ for all $v \in \mathbf{V}$. We have

$$\begin{aligned}
& \|\epsilon(\partial_y^\nu u)\|_{\mathcal{H}} \\
& \leq \sum_{m=1}^{\infty} \nu_m \frac{\gamma_m}{\mu_{\min}} \|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathcal{H}} + \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right) \frac{c_7}{c_0} \|g\|_H \\
& \leq \sum_{m=1}^{\infty} \nu_m \frac{\gamma_m}{\mu_{\min}} \|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathcal{H}} \\
& \quad + \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right) \frac{c_7}{c_0 \lambda_{\min}} \left(\|\partial_y^\nu p\|_H + \sum_{m=1}^{\infty} \nu_m \delta_m \|\operatorname{div} \partial_y^{\nu-e_m} u\|_H \right) \\
& \leq \sum_{m=1}^{\infty} \nu_m \frac{\gamma_m}{\mu_{\min}} \|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathcal{H}} \\
& \quad + \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right) \frac{c_7}{c_0 \lambda_{\min}} \left(\|\partial_y^\nu p\|_H + \sum_{m=1}^{\infty} \nu_m \frac{\delta_m}{\lambda_{\min}} \|\lambda \operatorname{div} \partial_y^{\nu-e_m} u\|_H \right). \tag{5.89}
\end{aligned}$$

Choosing $v \in \mathbf{V}$ such that $\operatorname{div} v = \partial_y^\nu p$ and $c_0 \|v\|_{\mathbf{V}} \leq \|\partial_y^\nu p\|_H$ as in (5.70). From (5.86) we have that

$$\begin{aligned}
\|\partial_y^\nu p\|_H^2 & \leq 2\mu_{\max} \|\epsilon(\partial_y^\nu u)\|_{\mathcal{H}} \|\epsilon(v)\|_{\mathcal{H}} + 2 \sum_{m=1}^{\infty} \nu_m \gamma_m \|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathcal{H}} \|\epsilon(v)\|_{\mathcal{H}} \\
& \leq 2\mu_{\max} \frac{c_7}{c_0} \|\epsilon(\partial_y^\nu u)\|_{\mathcal{H}} \|\partial_y^\nu p\|_H + 2 \sum_{m=1}^{\infty} \nu_m \gamma_m \frac{c_7}{c_0} \|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathcal{H}} \|\partial_y^\nu p\|_H.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\partial_y^\nu p\|_H & \leq 2\mu_{\max} \frac{c_7}{c_0} \|\epsilon(\partial_y^\nu u)\|_{\mathcal{H}} + 2 \sum_{m=1}^{\infty} \nu_m \gamma_m \frac{c_7}{c_0} \|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathcal{H}} \\
& \leq 2 \frac{c_7}{c_0} \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right) \sum_{m=1}^{\infty} \nu_m \gamma_m \|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathcal{H}} + 2\mu_{\max} \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right) \frac{c_7^2}{c_0^2 \lambda_{\min}} \|\partial_y^\nu p\|_H \\
& \quad + 2\mu_{\max} \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right) \frac{c_7^2}{c_0^2 \lambda_{\min}} \sum_{m=1}^{\infty} \nu_m \frac{\delta_m}{\lambda_{\min}} \|\lambda \operatorname{div} \partial_y^{\nu-e_m} u\|_H. \tag{5.90}
\end{aligned}$$

Assuming that

$$\bar{\lambda}_{\min} > \vartheta_2 := 4\mu_{\max}(1 + \kappa_0) \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right) \frac{c_7^2}{c_0^2}$$

then

$$\lambda_{\min} = \frac{\bar{\lambda}_{\min}}{1 + \kappa} > 4\mu_{\max} \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right) \frac{c_7^2}{c_0^2}.$$

We then deduce that

$$\begin{aligned} \|\partial_y^\nu p\|_H &\leq 4 \frac{c_7}{c_0} \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right) \sum_{m=1}^{\infty} \nu_m \gamma_m \|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathcal{H}} \\ &+ 4\mu_{\max} \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right) \frac{c_7^2}{c_0^2 \lambda_{\min}} \sum_{m=1}^{\infty} \nu_m \frac{\delta_m}{\lambda_{\min}} \|\lambda \operatorname{div} \partial_y^{\nu-e_m} u\|_H. \end{aligned} \quad (5.91)$$

From (5.89) and (5.91), we deduce that

$$\begin{aligned} \|\epsilon(\partial_y^\nu u)\|_{\mathcal{H}} &\leq \left(1 + 4 \frac{c_7^2 \mu_{\min}}{c_0^2 \lambda_{\min}} \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right)^2\right) \sum_{m=1}^{\infty} \nu_m \frac{\gamma_m}{\mu_{\min}} \|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathcal{H}} \\ &+ \frac{c_7}{c_0 \lambda_{\min}} \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right) \left(1 + \frac{4\mu_{\max} c_7^2}{c_0^2 \lambda_{\min}} \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right)\right) \\ &\quad \cdot \sum_{m=1}^{\infty} \nu_m \frac{\delta_m}{\lambda_{\min}} \|\lambda \operatorname{div} \partial_y^{\nu-e_m} u\|_H \\ &= \left(1 + \frac{C_1}{\lambda_{\min}}\right) \sum_{m=1}^{\infty} \nu_m \frac{\gamma_m}{\mu_{\min}} \|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathcal{H}} \\ &+ \left(\frac{C_2}{\lambda_{\min}} + \frac{C_3}{\lambda_{\min}^2}\right) \sum_{m=1}^{\infty} \nu_m \frac{\delta_m}{\lambda_{\min}} \|\lambda \operatorname{div} \partial_y^{\nu-e_m} u\|_H \end{aligned} \quad (5.92)$$

where

$$C_1 = 4 \frac{c_7^2 \mu_{\min}}{c_0^2} \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right)^2, \quad C_2 = \frac{c_7}{c_0} \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right), \quad C_3 = \frac{4\mu_{\max} c_7^3}{c_0^3} \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right)^2$$

which do not depend on $\lambda_{\min}$. On the other hand, we have from (5.87) that

$$\begin{aligned} \|\lambda \operatorname{div} \partial_y^\nu u\|_H &\leq \|\partial_y^\nu p\|_H + \sum_{m=1}^{\infty} \nu_m \frac{\delta_m}{\lambda_{\min}} \|\lambda \operatorname{div} \partial_y^{\nu-e_m} u\|_H \\ &\leq 4 \frac{c_7 \mu_{\min}}{c_0} \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right) \sum_{m=1}^{\infty} \nu_m \frac{\gamma_m}{\mu_{\min}} \|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathcal{H}} \\ &+ \left(1 + \frac{4\mu_{\max} c_7^2}{c_0^2 \lambda_{\min}} \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right)\right) \sum_{m=1}^{\infty} \nu_m \frac{\delta_m}{\lambda_{\min}} \|\lambda \operatorname{div} \partial_y^{\nu-e_m} u\|_H \\ &= C_4 \sum_{m=1}^{\infty} \nu_m \frac{\gamma_m}{\mu_{\min}} \|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathcal{H}} + \left(1 + \frac{C_5}{\lambda_{\min}}\right) \\ &\quad \cdot \sum_{m=1}^{\infty} \nu_m \frac{\delta_m}{\lambda_{\min}} \|\lambda \operatorname{div} \partial_y^{\nu-e_m} u\|_H \end{aligned} \quad (5.93)$$

where

$$C_4 = 4 \frac{c_7 \mu_{\min}}{c_0} \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right), \quad C_5 = \frac{4 \mu_{\max} c_7^2}{c_0^2} \left(1 + \frac{\mu_{\max}}{\mu_{\min}}\right)$$

which do not depend on $\lambda_{\min}$. From (5.92) and (5.93), for $0 < \varsigma < 1$ we have

$$\begin{aligned} \|\epsilon(\partial_y^\nu u)\|_{\mathbf{H}} + \frac{1}{\lambda_{\min}^\varsigma} \|\lambda \operatorname{div} \partial_y^\nu u\|_H &\leq \left(1 + \frac{C_1}{\lambda_{\min}} + \frac{C_4}{\lambda_{\min}^\varsigma}\right) \sum_{m=1}^{\infty} \nu_m \frac{\gamma_m}{\mu_{\min}} \|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathbf{H}} + \\ &+ \left(1 + \frac{C_2}{\lambda_{\min}^{1-\varsigma}} + \frac{C_3}{\lambda_{\min}^{2-\varsigma}} + \frac{C_5}{\lambda_{\min}}\right) \sum_{m=1}^{\infty} \nu_m \frac{\delta_m}{\lambda_{\min}} \frac{1}{\lambda_{\min}^\varsigma} \|\lambda \operatorname{div} \partial_y^{\nu-e_m} u\|_H. \end{aligned}$$

We get the conclusion. $\square$

With the notations of Proposition 5.5.8, let

$$\hat{d}_m = \max \left\{ \frac{\gamma_m}{\mu_{\min}} \left(1 + \frac{C_1}{\lambda_{\min}} + \frac{C_4}{\lambda_{\min}^\varsigma}\right), \frac{\delta_m}{\lambda_{\min}} \left(1 + \frac{C_2}{\lambda_{\min}^{1-\varsigma}} + \frac{C_3}{\lambda_{\min}^{2-\varsigma}} + \frac{C_5}{\lambda_{\min}}\right) \right\}$$

and $\hat{d} = (\hat{d}_1, \hat{d}_2, \dots)$. We have

$$\|\epsilon(\partial_y^\nu u)\|_{\mathbf{H}} + \frac{1}{\lambda_{\min}^\varsigma} \|\lambda \operatorname{div} \partial_y^\nu u\|_H \leq \sum_{m=1}^{\infty} \nu_m \hat{d}_m \left(\|\epsilon(\partial_y^{\nu-e_m} u)\|_{\mathbf{H}} + \frac{1}{\lambda_{\min}^\varsigma} \|\lambda \operatorname{div} \partial_y^{\nu-e_m} u\|_H \right). \quad (5.94)$$

From this we deduce the following bounds.

Proposition 5.5.9. *Assume $\kappa < \kappa_0$ and $\bar{\lambda}_{\min} > \vartheta_2$ where κ_0 and ϑ_2 are the constants in Proposition 5.5.8. There is a constant C_0 which is independent of ν such that the solution $\{u, p\} \in \underline{\mathbf{Y}}$ of problems (5.62) and (5.63) satisfies*

$$\|\partial_y^\nu u\|_{\mathbf{V}} + \|\partial_y^\nu p\|_H \leq C_0 |\nu|! \hat{d}^\nu.$$

Proof. The proof of this proposition is by induction, similar to those of Propositions 5.3.5 and 5.4.12. We have from (5.68) that there is a constant c independent of λ such that

$$\|u\|_{\mathbf{V}} + \|p\|_H \leq c \|f\|_{\mathbf{V}'}.$$

As $p = \lambda \operatorname{div} u$ and $\lambda_{\min} = \bar{\lambda}_{\min}/(1+\kappa) > \vartheta_2/(1+\kappa_0)$, there is a constant C^* independent of λ such that

$$\|\epsilon(u)\|_{\mathbf{H}} + \frac{1}{\lambda_{\min}^\varsigma} \|\lambda \operatorname{div} u\|_H \leq C^*. \quad (5.95)$$

We then deduce inductively from (5.94) that

$$\|\epsilon(\partial_y^\nu u)\|_{\mathbf{H}} + \frac{1}{\lambda_{\min}^\varsigma} \|\lambda \operatorname{div} \partial_y^\nu u\|_H \leq C^* |\nu|! \hat{d}^\nu.$$

From Korn's inequality, we get the bound for $\|\partial_y^\nu u\|_{\mathbf{V}}$. The bound for $\|\partial_y^\nu p\|_H$ is deduced from the bound for $\|\partial_y^\nu u\|_{\mathbf{V}}$ and equation (5.90). $\square$

From this we have

Proposition 5.5.10. *The coefficients u_ν and p_ν of the expansion (5.73) satisfy*

$$\|u_\nu\|_{\mathbf{V}} + \|p_\nu\|_H \leq c \frac{|\nu|!}{\nu!} \hat{d}^\nu, \quad (5.96)$$

where $\hat{d}_m = \hat{d}_m/\sqrt{3}$ and $\hat{d} = (\hat{d}_1, \hat{d}_2, \dots)$.

The proof of this proposition is similar to that for Proposition 5.3.6.

5.5.3.2 Best N term convergence rate

To deduce the explicit rate of convergence, we make the following assumption.

Assumption 5.5.11. *The constants γ_m and δ_m in Assumption 5.2.4 satisfy: $\sum_{m=1}^{\infty} \gamma_m^p < \infty$ and $\sum_{m=1}^{\infty} \delta_m^p < \infty$.*

We then have the following result.

Proposition 5.5.12. *Under Assumption 5.5.11, for $\varepsilon > 0$ there is a constant ϑ_3 that depends on ε , μ and the domain D such that if $\bar{\lambda}_{\min} > \vartheta_3$ then $(\|u_\nu\|_{\mathbf{V}})_\nu$ and $(\|p_\nu\|_H)_\nu$ are in $\ell^p(\mathcal{F})$ if $\kappa < \sqrt{3}/(1 + \varepsilon)$ and $\sum_{m=1}^{\infty} \max\left\{\frac{\gamma_m}{\bar{\mu}_{\min}}, \frac{\delta_m}{\bar{\lambda}_{\min}}\right\} \leq \frac{\kappa}{1 + \kappa}$.*

Proof. From Lemma 5.3.8, $(\|u_\nu\|_{\mathbf{V}})_\nu$ and $(\|p_\nu\|_H)_\nu$ are in $\ell^p(\mathcal{F})$ if $\sum_{m=1}^{\infty} \hat{d}_m < 1$, i.e. $\sum_{m=1}^{\infty} \hat{d}_m < \sqrt{3}$. With $\mu_{\min} = \bar{\mu}_{\min}/(1 + \kappa)$ and $\lambda_{\min} = \bar{\lambda}_{\min}/(1 + \kappa)$, we have that

$$\begin{aligned} & \sum_{m=1}^{\infty} \hat{d}_m \\ &= \sum_{m=1}^{\infty} \max\left\{\frac{\gamma_m}{\mu_{\min}}\left(1 + \frac{C_1}{\lambda_{\min}} + \frac{C_4}{\lambda_{\min}^\varsigma}\right), \frac{\delta_m}{\lambda_{\min}}\left(1 + \frac{C_2}{\lambda_{\min}^{1-\varsigma}} + \frac{C_3}{\lambda_{\min}^{2-\varsigma}} + \frac{C_5}{\lambda_{\min}}\right)\right\} \\ &\leq \kappa \max\left\{\left(1 + \frac{C_1}{\lambda_{\min}} + \frac{C_4}{\lambda_{\min}^\varsigma}\right), \left(1 + \frac{C_2}{\lambda_{\min}^{1-\varsigma}} + \frac{C_3}{\lambda_{\min}^{2-\varsigma}} + \frac{C_5}{\lambda_{\min}}\right)\right\}. \end{aligned}$$

Therefore, when $\kappa < \sqrt{3}/(1 + \varepsilon)$ for $\varepsilon > 0$, for $0 < \varsigma < 1$, if $\bar{\lambda}_{\min} > \vartheta_3(\varepsilon)$ then $\lambda_{\min} = \bar{\lambda}_{\min}/(1 + \kappa)$ satisfy

$$\max\left\{\frac{C_1}{\lambda_{\min}} + \frac{C_4}{\lambda_{\min}^\varsigma}, \frac{C_2}{\lambda_{\min}^{1-\varsigma}} + \frac{C_3}{\lambda_{\min}^{2-\varsigma}} + \frac{C_5}{\lambda_{\min}}\right\} < \varepsilon.$$

Condition $\kappa < \sqrt{3}/(1 + \varepsilon)$ implies that $\sum_{m=1}^{\infty} \hat{d}_m < \sqrt{3}$. From this, we get the conclusion. $\square$

Remark 5.5.13. We consider a sequence $\{\kappa_m\}_m \subset \mathbb{R}$ such that $\sum_{m=1}^{\infty} \kappa_m^p < \infty$. For all λ that satisfy $\delta_m/\bar{\lambda}_{\min} \leq \kappa_m$ we have

$$\hat{d}_m \leq \max \left\{ \frac{\gamma_m}{\mu_{\min}} \left(1 + \frac{C_1}{\lambda_{\min}} + \frac{C_4}{\lambda_{\min}^{\varsigma}} \right), \kappa_m(1 + \kappa) \left(1 + \frac{C_2}{\lambda_{\min}^{1-\varsigma}} + \frac{C_3}{\lambda_{\min}^{2-\varsigma}} + \frac{C_5}{\lambda_{\min}} \right) \right\}.$$

Thus

$$\sum_{m=1}^{\infty} \hat{d}_m^p \leq C \sum_{m=1}^{\infty} (\gamma_m^p + \kappa_m^p),$$

where C is independent of λ . Let $\eta = (1 - \kappa(1 + \varepsilon))/\sqrt{3}/2$ and J be the smallest integer such that $C \sum_{m>J} (\gamma_m^p + \kappa_m^p) < \eta$. From Theorem 7.2 of Cohen et al. [18], we have that $\|(|\nu|! \hat{d}^{\nu}/\nu!)_{\nu}\|_{\ell^p(\mathcal{F})}$ is bounded by a bound that only depends on J and η , and is independent of λ .

Further, if $\bar{\lambda}_{\min} \geq \vartheta_3$ and $\kappa < \sqrt{3}/(1 + \varepsilon)$, $\lambda_{\min} = \bar{\lambda}_{\min}/(1 + \kappa) \geq \vartheta_3(1 + \varepsilon)/(1 + \varepsilon + \sqrt{3})$ so the constant C^* in (5.95) can be chosen to depend only on ϑ_3 and ε . Therefore $\|(|u_{\nu}|_{\mathbf{V}})\|_{\ell^p(\mathbb{N})}$ and $\|(|p_{\nu}|_H)\|_{\ell^p(\mathbb{N})}$ are bounded by a constant that only depends on ϑ_3 and ε .

Remark 5.5.14. Although physically we are not interested in the quantity $\frac{1}{\lambda_{\min}^{\varsigma}} \|\lambda \operatorname{div} \partial_y^{\nu} u\|_H$ in Proposition 5.5.8, it contributes to determining the bound (5.96) for u_{ν} and p_{ν} . It is necessary to use $\varsigma > 0$ to deduce the condition $\kappa < \sqrt{3}/(1 + \varepsilon)$ in Proposition 5.5.12. If $\varsigma = 0$, we will need

$$\kappa \left(1 + \frac{C_1}{\lambda_{\min}} + C_4 \right) < \sqrt{3}.$$

For this to hold when $\lambda_{\min}$ goes to ∞ , we would require the stronger condition that $\kappa < \sqrt{3}/(1 + C_4 + \varepsilon)$.

Let $r = 1/p - 1/2$. Let $\bar{\lambda}_{\max} = \sup_{x \in D} \bar{\lambda}(x)$. We then deduce the best N term convergence rate for the approximations (5.80) and (5.81).

Theorem 5.5.15. Assume that $\sum_{m=1}^{\infty} \max \left\{ \frac{\gamma_m}{\bar{\mu}_{\min}}, \frac{\delta_m}{\bar{\lambda}_{\min}} \right\} \leq \frac{\kappa}{1 + \kappa}$ with $\kappa < \sqrt{3}/(1 + \varepsilon)$, $\varepsilon > 0$. Under Assumption 5.5.11, there is $\vartheta > 0$ that depends on ε , μ such that if $\bar{\lambda}_{\min} > \vartheta$ then for each integer N we can choose a set $\Lambda \subset \mathcal{F}$ with cardinality not larger than N such that the solution $\{u, p\}$ of problems (5.62) and (5.63) and their approximations $\{u_{\Lambda}, p_{\Lambda}\}$ in (5.80) and (5.81) respectively satisfy

$$\|u - u_{\Lambda}\|_{\mathbf{V}} + \|p - p_{\Lambda}\|_{\underline{H}} \leq CN^{-r}$$

where C depends only on $\bar{\lambda}_{\max}/\bar{\lambda}_{\min}$, $\|(|u_{\nu}|_{\mathbf{V}})\|_{\ell^p(\mathcal{F})}$ and $\|(|p_{\nu}|_H)\|_{\ell^p(\mathcal{F})}$, and in particular it does not depend on λ when $\lambda_{\min} \rightarrow \infty$ in such a fashion that $\bar{\lambda}_{\max}/\bar{\lambda}_{\min}$ remains bounded, and $\delta_m/\bar{\lambda}_{\min} \leq \kappa_m$ for a fixed sequence $\{\kappa_m\}_m \subset \mathbb{R}$ that satisfies $\sum_{m=1}^{\infty} \kappa_m^p < \infty$.

Proof. Let $\vartheta = \max\{\vartheta_1, \vartheta_2, \vartheta_3\}$ where ϑ_1, ϑ_2 and ϑ_3 are the constants in Propositions 5.5.7, 5.5.8 and 5.5.12 respectively. We choose Λ as the set corresponding to the N largest bounds $c|\nu|!\hat{d}^\nu/\nu!$ in (5.96). From Proposition 5.5.7, we deduce

$$\|u - u_\Lambda\|_{\underline{\mathbf{Y}}} + \|p - p_\Lambda\|_{\underline{H}} \leq \left(1 + \frac{\|B_4\|_{\underline{\mathbf{Y}} \times \underline{\mathbf{Y}} \rightarrow \mathbb{R}}}{\chi'_4}\right) \left(\left\|\sum_{\nu \notin \Lambda} u_\nu L_\nu\right\|_{\underline{\mathbf{Y}}} + \left\|\sum_{\nu \notin \Lambda} p_\nu L_\nu\right\|_{\underline{H}}\right)$$

where χ'_4 is the constant in (5.83). From (5.65), we have that

$$|B_4(\{u_\Lambda, p_\Lambda\}, \{v_\Lambda, q_\Lambda\})| \leq (2\mu^* + 1 + \frac{\lambda_{\max}}{\bar{\lambda}_{\min}} + \frac{1}{\bar{\lambda}_{\min}}) \|\{u_\Lambda, p_\Lambda\}\|_{\underline{\mathbf{Y}}} \|\{v_\Lambda, q_\Lambda\}\|_{\underline{\mathbf{Y}}}.$$

Thus

$$\begin{aligned} & \|u - u_\Lambda\|_{\underline{\mathbf{Y}}} + \|p - p_\Lambda\|_{\underline{H}} \\ & \leq \left(1 + \frac{1}{\chi'_4} (2\mu^* + 1 + \frac{\bar{\lambda}_{\max}}{\bar{\lambda}_{\min}} + \frac{\kappa}{1 + \kappa} + \frac{1}{\vartheta})\right) \left(\left\|\sum_{\nu \notin \Lambda} u_\nu L_\nu\right\|_{\underline{\mathbf{Y}}} + \left\|\sum_{\nu \notin \Lambda} p_\nu L_\nu\right\|_{\underline{H}}\right) \\ & \leq C \left[\left(\sum_{\nu \notin \Lambda} \|u_\nu\|_{\underline{\mathbf{V}}}^2\right)^{1/2} + \left(\sum_{\nu \notin \Lambda} \|p_\nu\|_{\underline{H}}^2\right)^{1/2} \right] \end{aligned}$$

where C only depends on $\bar{\lambda}_{\max}/\bar{\lambda}_{\min}$ and can be chosen to be independent of λ when $\bar{\lambda}_{\min}$ goes to ∞ in a fashion such that $\bar{\lambda}_{\max}/\bar{\lambda}_{\min}$ remains bounded. The conclusion then follows from Lemma 5.3.10 and Remark 5.5.13 on the uniform boundedness with respect to λ of $\|(|\nu|!\hat{d}^\nu/\nu!)_\nu\|_{\ell^p(\mathcal{F})}$. $\square$

Remark 5.5.16. In Proposition 5.5.12 and Theorem 5.5.15, if we only assume Assumption 5.2.4 without the condition $\sum_{m=1}^{\infty} \max\{\frac{\gamma_m}{\bar{\mu}_{\min}}, \frac{\delta_m}{\bar{\lambda}_{\min}}\} \leq \frac{\kappa}{1 + \kappa}$, then as

$\sum_{m=1}^{\infty} \max\{\frac{\gamma_m}{\mu_{\min}}, \frac{\delta_m}{\lambda_{\min}}\} \leq \sum_{m=1}^{\infty} \frac{\gamma_m}{\mu_{\min}} + \frac{\delta_m}{\lambda_{\min}} \leq 2\kappa$, the condition for κ should be replaced by $\kappa < \sqrt{3}/(2(1 + \varepsilon))$.

Summary and Future Work

6.1 Summary

In this dissertation, we studied mathematical and numerical analysis of Multiscale Wave Equation, Multiscale Elastic Equation, Multiscale Elastic Wave Equation and Stochastic Elastic Equation.

In Chapter 2, we study the linear wave equation in a locally periodic medium. We develop the multiscale homogenized equation for multiscale wave equation. Solving it we get all the macroscopic and microscopic information. We then develop full and sparse tensor product finite element methods for the equation. We show that the sparse tensor FE method achieves essentially optimal complexity: the number of degrees of freedom is essentially equal to that for the full tensor product FE with an essentially equal accuracy. Rates of convergence are derived for both full and sparse tensor finite elements. We show that the regularity for these error estimates to hold can be obtained. A new homogenization error for the multiscale wave equations is derived for the two-scale case, and a new corrector for the multiscale case is derived. We finally construct numerical correctors from the FE solutions, with explicit error estimates in the two-scale case. The theory is supported by numerical examples for one and two dimensional problems with two and three scales.

In Chapter 3, we follow the approach in Chapter 2 to solve the multiscale elasticity problem. We develop full and sparse tensor FE methods for the multiscale homogenized elasticity equation. We then show a homogenization rate of convergence for the two-scale case which require less regularity than the existing method in the literature. Numerical correctors are then constructed from the FE solution. Numerical examples in two dimensions support our analysis.

In Chapter 4, we combine the developments for the multiscale wave equation and the multiscale elasticity equation for the multiscale elastic wave equation. We get explicit error estimates in two-scale case. Numerical examples in two dimensions are consistent with the theory.

In Chapter 5, we construct the best N terms GPC approximations for the stochastic

elasticity equations whose elastic tensors are a linear expansion of random variables. First we consider the displacement problems. Explicit bounds for the GPC coefficients are deduced. From the summability for the coefficients, we deduced the best N term convergence rate. We then consider the Hellinger-Reissner mixed formulation that allows for computing the stress tensor directly. To employ the linear dependence of the elastic tensor on the random variables, we consider an equivalent form and deduce bounds for the GPC coefficients and establish the summability of the GPC coefficients. From these we get the best N term convergence rate. For nearly incompressible materials, we need a very large number of terms to achieve the prescribed level of accuracy. We consider a mixed problem and deduce bounds and summability of the GPC coefficients so that the best N term convergence rate does not depend on the ratio of the Lamé moduli.

6.2 Future work

In our analysis, we need to approximate u_1 in H^1 in order to construct numerical corrector for u^ϵ . Baker [6] uses a different test function that leads to an L^2 estimate which requires less regularity than our results. However, it is not clear how the term u_1 can be approximated from this method as we only have time derivative for u_0 . It is not straightforward to extend Baker's method to our problem. Perhaps we need to use different test functions for $u^0, u^1, \dots$ and u^n in order to get an H^1 error estimates for all of these terms. This will be a significant new project for the future.

Another possible research project in the future is to consider sparse tensor finite element method for the multiscale homogenized problems of all the multiscale wave, elasticity and elastic wave equations when the regularity required in this thesis does not hold. This may require mesh refinement towards the points of singularity. Note that mesh refinement for wave equations that depend on a single macroscopic scale is still the current subject of research. It would be interesting to consider mesh refinement for our multiscale wave equation.

For the multiscale wave equations and the multiscale elastic wave equations, we restricted the initial wave displacement to 0 to deduce a homogenization rate of convergence in two scales. For the general initial conditions, the corrector contains another highly oscillating term which requires solving another multiscale problem. It is not clear right now how to solve these problems efficiently. This will be another separate project for the future.

In Chapter 5, we did not develop numerical procedures for approximating the stochastic elasticity equation. In particular, the best N terms are not difficult to identify according to the bounds. The resulting semidiscrete Galerkin problems for the best N terms needs to be approximated adaptively by finite elements where the mesh size needs not be the same for all the terms but can be chosen according to the bounds to achieve the optimal complexity.

6.3 Author's Publications

B. Xia and V. H. Hoang. *Best N term approximations for stochastic linear elastic equations*. Mathematical Models and Methods in Applied Sciences, 24(3): 513-552, 2013.

B. Xia and V. H. Hoang. *High dimensional finite elements for multiscale wave equation*. SIAM Multiscale Modelling and Simulation, 2014. (Accepted).

B. Xia and V. H. Hoang. *High dimensional finite element method for multiscale linear elasticity*. (In Review).

B. Xia and V. H. Hoang. *Sparse tensor finite elements for elastic wave equation with multiple scale*. (In Review).

Bibliography

- [1] A. Abdulle. *Analysis of a heterogeneous multiscale FEM for problems in elasticity*. Mathematical Models & Methods in Applied Sciences, 16(4):615–635, 2006.
- [2] A. Abdulle and M. J. Grote. *Finite element heterogeneous multiscale method for the wave equation*. Multiscale Modeling and Simulation, 9(2):766–792, 2011.
- [3] G. Allaire. *Homogenization and two-scale convergence*. SIAM Journal on Mathematical Analysis, 23(6):1482–1518, 1992.
- [4] G. Allaire and M. Briance. *Multiscale convergence and reiterated homogenization*. Proceedings of the Royal Society of Edinburgh. Section A. Mathematics, 126(2):297–342, 1996.
- [5] C. Bacuta and J. H. Bramble. *Regularity estimates for solutions of the equations of linear elasticity in convex plane polygonal domains*. Zeitschrift für angewandte Mathematik und Physik ZAMP, 54(5):874–878, 2003.
- [6] G. A. Baker. *Error estimates for finite element methods for second order hyperbolic equations*. SIAM J. Numer. Anal. 13, no. 4, 564–576, 1976.
- [7] A. Bensoussan, J. L. Lions and G. Papanicolaou. *Asymptotic analysis for periodic structures*. North-Holland Publishing Co., 1978.
- [8] M. Bieri, R. Andreev and Ch. Schwab. *Sparse tensor discretization of elliptic sPDEs*. SIAM Journal on Scientific Computing, 31(6):4281–4304, 2009.
- [9] D. Braess. *Stability of saddle point problems with penalty*. M²AN, 30:731–742, 1992.
- [10] D. Braess. *Finite Elements*. Cambridge University Press, 2007.
- [11] S. Brahim-Otsmane, G. A. Francfort and F. Murat. *Correctors for the homogenization of the wave and heat equations*. Journal de Mathématiques Pures et Appliquées. Neuvième Série (9), 71(3):197–231, 1992.
- [12] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods*. Texts in Applied Mathematics, 15, 2002.

- [13] S. C. Brenner and L. Y. Sung. *Linear finite element methods for planar linear elasticity*. Mathematics of Computation, 59(200):321-338, 1992.
- [14] H.-J. Bungartz and M. Griebel. *Sparse grids*. Acta Numerica, 13:1-123, 2004.
- [15] P. Ciarlet. *The finite element method for elliptic problems*. North-Holland Publishing Co., 1978.
- [16] D. Cioranescu, A. Damlamian and G. Griso. *Periodic unfolding and homogenization*. Comptes Rendus Mathématique. Académie des Sciences. Paris, 335(1):99-104, 2002.
- [17] D. Cioranescu, A. Damlamian and G. Griso. *The periodic unfolding method in homogenization*. SIAM Journal on Mathematical Analysis, 40(4):1585-1620, 2008.
- [18] A. Cohen, R. DeVore and Ch. Schwab. *Convergence Rates of Best N -term Galerkin Approximations for a Class of Elliptic $spDE$ s*. Foundations of Computational Mathematics, 10(6):615-646, 2010.
- [19] Q. L. Dong and L. Q. Cao. *Multiscale asymptotic expansions and numerical algorithms for the wave equations of second order with rapidly oscillating coefficients*. Applied Numerical Mathematics, 59(12):3008-3032, 2009.
- [20] T. Dupont. *L^2 -estimates for Galerkin methods for second order hyperbolic equations*. SIAM Journal on Numerical Analysis, 10:880-889, 1973.
- [21] W. E. *Principles of Multiscale Modeling*. Cambridge University Press, 2011.
- [22] W. E and B. Engquist. *The heterogeneous multiscale methods*. Communications in Mathematical Sciences, 1(1):87-132, 2003.
- [23] Y. Efendiev and T. Y. Hou. *Multiscale Finite Element Methods: Theory and Applications*. Springer-Verlag New York Inc., 2009.
- [24] B. Engquist, H. Holst and O. Runborg. *Multi-scale methods for wave propagation in heterogeneous media*. Communications in Mathematical Sciences, 9(1):33-56, 2011.
- [25] A. Ern and J. L. Guermond . *Theory and Practice of Finite Elements*. Applied Mathematical Sciences, 159:xiv+524, 2004.
- [26] J. Frehse and R. Rannacher. *Asymptotic L^∞ -Error Estimates for Linear Finite Element Approximations of Quasilinear Boundary Value Problems*. SIAM Journal on Numerical Analysis, 15(2):418-431, 1978.
- [27] M. Griebel and P. Oswald. *Tensor product type subspace splittings and multilevel iterative methods for anisotropic problems*. Advances in Computational Mathematics, 4(1-2):171-206, 1995.
- [28] P. Grisvard. *Elliptic problems in nonsmooth domains*. Pitman (Advanced Publishing Program), 1985.

- [29] H. Harbrecht and Ch. Schwab. *Sparse tensor finite elements for elliptic multiple scale problems*. Computer Methods in Applied Mechanics and Engineering, 200:3100–3110, 2011.
- [30] V. H. Hoang. *Sparse Finite Element Method for Periodic Multiscale Nonlinear Monotone Problems*. Multiscale Modeling & Simulation, 7(3):1042–1072, 2009.
- [31] V. H. Hoang and Ch. Schwab. *High-dimensional finite elements for elliptic problems with multiple scales*. Multiscale Modeling & Simulation, 3(1):168–194, 2005.
- [32] V. H. Hoang and Ch. Schwab. *Sparse Tensor Galerkin Discretizations for parametric and random parabolic PDEs I: Analytic Regularity and gpc Approximation*. Report 2010-10, Seminar for Applied Mathematics, ETH Zürich, 2010.
- [33] V. H. Hoang and Ch. Schwab. *Regularity and generalized polynomial chaos approximation of parametric and random 2nd order hyperbolic partial differential equation*. Analysis and Applications, 10:295–326, 2012.
- [34] V. H. Hoang and Ch. Schwab. *Analytic regularity and polynomial approximation of stochastic, parametric elliptic multiscale partial differential equations*. Analysis and Applications, 11, 2013.
- [35] T. Y. Hou and X.-H. Wu. *A multiscale finite element method for elliptic problems in composite materials and porous media*. Journal of Computational Physics, 134(1):169–189, 1997.
- [36] L. Jiang, Y. Efendiev and V. Ginting. *Analysis of global multiscale finite element methods for wave equations with continuum spatial scales*. Applied Numerical Mathematics, 60(8):862–876, 2010.
- [37] V. V. Jikov, S. M. Kozlov and O. A. Oleĭnik. *Homogenization of differential operators and integral functionals*. Springer-Verlag, 1994.
- [38] O. Korostyshevskaya and S. E. Minkoff. *A matrix analysis of operator-based upscaling for the wave equation*. SIAM Journal on Numerical Analysis, 44(2):586–612, 2006.
- [39] A. Kunoth and Ch. Schwab. *Analytic regularity and GPC approximation for control problems constrained by linear parametric elliptic and parabolic PDEs*. Report 2011-54, Seminar for Applied Mathematics, ETH Zürich, 2011.
- [40] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I*. Springer-Verlag, 1972.
- [41] A. Mazzucato and V. Nistor. *Well-posedness and Regularity for the Elasticity Equation with Mixed Boundary Conditions on Polyhedral Domains and Domains with Cracks*. Archive for Rational Mechanics and Analysis, 195(1):25–73, 2010.
- [42] W. Mclean. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, 2000.

- [43] G. W. Milton. *The Theory of Composites*. Cambridge University Press, 2002.
- [44] G. Nguetseng. *A general convergence result for a functional related to the theory of homogenization*. SIAM Journal on Mathematical Analysis, 20(3):608–623, 1989.
- [45] V. Nistor and Ch. Schwab. *High order Galerkin approximation for parametric second order elliptic partial differential equations*. Report 2012-21 , Seminar for Applied Mathematics, ETH Zürich, 2012.
- [46] M. Ohlberger. *A Posteriori Error Estimates for the Heterogeneous Multiscale Finite Element Method for Elliptic Homogenization Problems*. Multiscale Modeling and Simulation 4:1, 88-114, 2005.
- [47] O. A. Oleinik, A. S. Shamaev and G. A. Yosifian. *Mathematical Problems in Elasticity and Homogenization*. Elsevier, 1992.
- [48] H. Owhadi and L. Zhang. *Numerical homogenization of the acoustic wave equations with a continuum of scales*. Computer Methods in Applied Mechanics and Engineering, 198(3-4):397–406, 2008.
- [49] E. Sanchez-Palencia. *Non-Homogeneous Media and Vibration Theory*. Springer Berlin Heidelberg, 1980.
- [50] F. Santosa and W. Symes. *A dispersive effective medium for wave propagation in periodic composites*. SIAM Journal of Applied Mathematics, 51:984-1005, 1991.
- [51] Ch. Schwab and T. A. Todor. *Karhunen-Loève approximation of random fields by generalized fast multipole methods*. Journal of Computational Physics, 217(1):100–122, 2006.
- [52] G. Strang and G. J. Fix. *An Analysis of the Finite Element Method*. Wellesley Cambridge Pr, 2008.
- [53] T. Vdovina, S. E. Minkoff and O. Korostyshevskaya. *Operator upscaling for the acoustic wave equation*. Multiscale Modeling and Simulation, 4(4):1305–1338, 2005.
- [54] J. R. Willis. *Variational and related methods for the overall properties of composites*. Advances in Applied Mechanics, 21:1-78, 1981.
- [55] J. Wloka. *Partial differential equations*. Cambridge University Press, 1987.
- [56] B. Xia and V. H. Hoang. *Best N term approximations for stochastic linear elastic equations*. Mathematical Models and Methods in Applied Sciences, 24(3): 513-552, 2013.
- [57] B. Xia and V. H. Hoang. *High dimensional finite elements for multiscale wave equation*. SIAM Multiscale Modelling and Simulation, 2014. (Accepted).
- [58] B. Xia and V. H. Hoang. *High dimensional finite element method for multiscale linear elasticity*. (In Review).

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- [59] B. Xia and V. H. Hoang. *Sparse tensor finite elements for elastic wave equation with multiple scale*. (In Review).