Electrokinetic effects in closed microchannels

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Electrokinetic Effects in Closed Microchannels

Submitted by

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Abstract

In the field of BioMEMS, electroosmosis phenomena have always been an integral part of its research due to its abundant applications. For instance, electroosmosis has been utilized as a powerful pumping means to precisely transport and manipulate liquid samples in microfluidic devices used for chemical analyses and biomedical diagnoses. Compared with conventional mechanical pumping modes, the electroosmosis-based pumping has numerous advantages including ease of fabrication and control, no movable mechanical parts, and high repeatability and reliability.

Generally, the use of electroosmosis involves DC (direct current) electric field to generate a steady-state electroosmotic flow. With an AC (alternating current) electric field, electroosmosis still occurs, but the electroosmotic flow becomes time and frequency dependent. During recent years, AC driven electroosmotic flows have drawn a great attention of researchers due to its numerous applications, such as measuring the electromobility of colloidal particles and enhancing mixing processes in microfluidic devices.

The frequency-dependent electroosmotic flow in closed-end cylindrical and rectangular microchannels is analyzed in this study. An AC electroosmotic flow field is obtained analytically by solving the Navier-Stokes equation using the Green's function formulation in combination with a complex variable approach. The Onsager's principle of reciprocity is demonstrated to be valid for AC electroosmotic

flow. The effect of the frequency-dependent AC electric field on the oscillating electroosmotic flow is studied. The induced pressure gradient is analyzed under effects of the channel dimension and the frequency of the electric field. Based on the Stokes second problem formulation, the solution of the slip velocity approximation is presented for comparison with the results obtained from the analytical solution developed in this study. In addition, the expression for the electroosmotic vorticity field is derived, and the characteristic of the vorticity field in AC electroosmotic flow is discussed.

The oscillating electroosmotic flow is found to be strongly dependent on the frequency of the applied sinusoidal electric field: the higher the frequency, the smaller the perturbed flow region and the higher the amplitude of the induced backpressure gradient. In the slip velocity approximation, results show that the velocity and vorticity distributions in the center region of the channel are similar to the analytical scheme. However, discrepancy is observed adjacent to the channel wall.

Furthermore, a numerical study on the steady-state developing electroosmotic flow in closed-end cylindrical microchannels is presented. To simulate the flow, a mathematical model, which includes the Poisson-Boltzmann equation describing the electrical potential distribution and the modified two-dimensional Navier-Stokes equations governing the velocity field, is presented. The governing equations are discretized using the control volume integration method, and the staggered grid system is utilized to solve the Navier-Stokes equations. The pressure-velocity coupling in the Navier-Stokes equation is resolved using SIMPLER (Semi-Implicit

Abstract

Method for Pressure Linked Equations Revised) algorithm. The results of the spatial development of the flow and the pressure distributions along the microchannel are presented. Parametric studies of the effects of the channel size and the electric field strength on electroosmotic flows in closed-end microchannels are conducted. It is found that the length of the developing region is only dependent on the radius of the channel, and the induced backpressure gradient, though is present in the entire channel, varies only in the developing region. In addition, the numerical simulations have been validated with the analytical solutions in the fully-developed region.

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Nomenclature

a	= cylindrical channel radius or rectangular half channel height [m]
b	= rectangular half channel width [m]
L	= length of microchannel [m]
G	= Green's function
u	= fluid axial velocity [m/s]
v	= fluid radial velocity [m/s]
$\vec{\omega}$	= vorticity [rad/s]
U	= reference velocity [m/s]
P	= pressure [N/m ²]
ω	= oscillation angle frequency [rad/s]
t	= time [s]
f	= oscillation frequency [Hz]
f^*	= characteristic frequency [Hz]
D_h	= hydraulic diameter of rectangular channel [m]
ρ	= density of fluid [kg/m ³]
μ	= dynamic viscosity of fluid [N-s/m ²]
ν	= kinematics viscosity of fluid [m ² /s]
Re_0	= reference Reynold's number (based on reference velocity)
E_0	= strength of electric field [V/m]
P_0	= amplitude of induced back-pressure [Pa]
k_b	= Boltzmann constant [J/K]

Nomenclature

T	= absolute temperature [K]
z_v	= valence of electrolyte solution
n_0	= bulk concentration of ions [m^{-3}]
e_0	= elementary charge = 1.602×10^{-19} Coulombs
ζ	= reference electrical potential (zeta potential) [V]
κ	= Debye-Hückel parameter [m^{-1}]
ψ	= local electrical potential [V]
ρ_e	= electric net charge density [C/m^3]
ϵ_r	= dielectric constant of the solution
ϵ_0	= permittivity of the vacuum [F/m]
$\bar{H}$	= dimensionless parameter represents a measure of the ability to convert the electrical energy to the fluid kinetic energy
$\bar{u}$	= dimensionless axial velocity
$\bar{v}$	= dimensionless radial velocity
$\bar{r}$	= dimensionless radius
X	= dimensionless x – axis
Y	= dimensionless y – axis
Z	= dimensionless z – axis
$\bar{P}_0$	= dimensionless amplitude of induced backpressure
$\bar{t}$	= dimensionless time
Ψ	= dimensionless electrical potential
$\bar{E}_0$	= dimensionless strength of electric field
$\bar{\zeta}$	= dimensionless zeta potential

CHAPTER 1

Introduction

1.1 Background

Biomedical field has attracted the greatest interest from all sectors in recent years. The study of gene therapy, DNA recombinant technique and bioinformatics has been contributed by various interdisciplinary fields, ranging from mechanical engineering, electrical engineering, chemical engineering to biochemistry. In fact, with the rapid growth of technology, a new interdisciplinary area – BioMEMS (Biological-Micro-Electro-Mechanical System) has emerged from the advancement of nanotechnology in the application of biomedical field. With the emergence of BioMEMS, it has become possible to conduct high-speed performance of various medical diagnoses in biochips [1-2]. This system is also known as lab-on-a-chip system. The term lab-on-a-chip is used simply because the processes, which are normally done in laboratory using different equipment coupled with manual labors, can be integrated and automated into a single chip. It consumes extremely low volumes of sample material, reduces power consumption, reduces cost of manufacture and increases the speed of reaction [3].

Several types of biochips have been developed. One of them is microfluidic biochip. Microfluidic chip involves moving fluids through microchannels, and mostly the movement of fluids through the microchannel is driven by electroosmotic pumping system, which makes use of the electrical properties of the fluid [1]. Electroosmosis

refers to the movement of the fluid relative to a stationary charged solid surface under the application of a tangential electric field [4]. For instance, electroosmosis has been utilized as a powerful pumping means to precisely transport and manipulate liquid samples in microfluidic devices used for chemical analyses and biomedical diagnoses [1-2]. Compared to conventional mechanical pumping modes, the electroosmosis-based pumping has numerous advantages including ease of fabrication and control, no movable mechanical parts, and high repeatability and reliability [3]. However, the only drawback is that the liquid used in the channel must contain charges (ions), such that it can be pumped by electric field. Nevertheless, this is not a major problem, since most of the working fluids in biological and medical applications are charged, to some extent, in wide range of conductivity. On the other hand, conventional mechanical fluid propulsion system in microchannel can pump all type of fluid (charged or uncharged). However, it possesses some drawbacks, such as the manufacturing and maintenance of the micro-pump.

Due to the pressure building ability of electroosmosis phenomena, many studies, both experimental and theoretical, AC (Alternating Current) and DC (Direct Current), have been carried out. However, in the literature, most of these studies dealt with flow in open-end channels. The study of electroosmotic flow in closed-end channels not only is relevant to the development of the new microelectrophoretic approach [5] but it also provides insight into the development of novel electrokinetic micropumps and microactuators [6]. Therefore, the motivation of this study is to investigate both developing and fully developed electroosmotic flow in closed-end microchannels.

1.2 Objectives

This project aims to:

- a. obtain closed form analytical solutions of frequency dependent electroosmotic flow in closed-end cylindrical and rectangular microchannels,
- b. investigate the flow responses under the effect of the channel sizes and the frequencies of the externally applied electric field,
- c. develop a slip velocity approximation approach to compare with the analytical solutions obtained in this study, and
- d. develop a numerical simulation to study the developing flow in closed-end cylindrical microchannels.

1.3 Organization of report

This report consists of six chapters. Chapter 1 presents the background introduction to the application of electroosmotic flow, and the importance of the electroosmotic pump for the microfluidic chips in the biomedical area. Following the background, the motivation and the objectives of this study are also presented in this chapter.

In order to provide basic understanding of the electroosmotic flow field discussed throughout this report, basic theories on Electrical Double Layer, the Poisson-Boltzmann equation, and the Navier-Stokes equation are presented in Chapter 2. The literatures related to this study are also presented in this chapter.

Chapter 3 and Chapter 4 present studies on frequency dependent electroosmotic flow in closed-end cylindrical and rectangular microchannels, respectively. The detail mathematical derivations on the closed form solutions to the velocity and vorticity fields using the Green's function formulation are presented. An analytical expression of the induced backpressure is obtained. Further, for the purpose of comparison, an analytical solution to the slip velocity approximation is presented. The results and discussion pertaining to the flow are also presented in these chapters.

Chapter 5 discusses a numerical study on the steady-state developing electroosmotic flow in closed-end cylindrical microchannels. A mathematical model, which describes the electrical distribution and the velocity field, is presented. The numerical scheme used to discretize the governing equations is briefly discussed. The results of the spatial development of the flow and the pressure distributions along the microchannel are presented. In addition, the prediction from the numerical simulation is compared with the analytical solutions in the fully-developed region for the purpose of validation.

Finally, Chapter 6 concludes the report and proposals for future work are presented.

CHAPTER 2

Literature Review

2.1 Theory Overview

2.1.1 Electrical Double Layer

Most solid surfaces acquire a certain amount of electrostatic charges when they are in contact with polar liquids, and hence a difference in potential is developed across the interface between the two phases. The charged interface attracts ions of opposite charge (counter-ions) and repels ions of like charge (co-ions) in the liquid (e.g., electrolyte solution). The arrangement of charges that occurs near the interface leads to the development of an electrical double layer (EDL) [4]. The EDL consists of two layers: compact and diffuse layer. These two layers are separated by a plane called *Stern plane*. The compact layer, usually less than one nanometer, exists immediately next to the solid surface, where counter-ions are strongly attracted to the solid surface and hence are immobile. The diffuse layer of the EDL is the layer from the compact layer to the uniform bulk liquid, and its thickness ranges from several to hundred nanometers, depending on liquid properties. Within the diffuselayer, ions are mobile and the net charge density is not zero. Application of an electric field tangentially along the charged surface would exert an electric body force on ions, resulting in a net migration of the ions in the EDL region. This movement of ions will then pull the surrounding fluid with them to produce a bulk electroosmotic flow.

Electroosmotic process can be understood in terms of the surface potential at the surface of shear (approximately stem plane), known as the zeta potential, ζ . The shear surface itself is somewhat arbitrary but is characterized as the plane at which the mobile portion of the diffuse layer can slip or flow past the charged surface. The shear plane is where the no-slip boundary condition is assumed to apply.

2.1.2 The Poisson-Boltzmann Equation

The electrical potential, ψ at any point can be related to the electric net charge density, ρ_e by employing the theory of electrostatics, which is also known as Poisson's equation [4]:

$$\nabla^2 \psi = \frac{-\rho_e}{\epsilon_r \epsilon_0} \quad (2.1)$$

where ϵ_r is the dielectric constant of the solution and ϵ_0 is the permittivity of the vacuum. The electric charge density is given by:

$$\rho_e = \sum n_i z_i e_0 \quad (2.2)$$

where n_i is the ionic concentration of the i -th species, z_i is the valence of type- i ions, e_0 is the fundamental electric charge (1.602×10^{-19} Coulombs).

The distribution of the ionic concentration can be described using the Boltzmann distribution equation [4]:

$$n_i = n_0 \exp\left(\frac{-z_i e_0}{k_b T} \psi\right) \quad (2.3)$$

where n_0 is the ionic concentration in the bulk solution, k_b is the Boltzmann constant, and T is the absolute temperature

It turns out to be possible to treat most electrolytes as though they were symmetric ($z_+ = -z_- = z_v$), with the valency of coions, equal to the valency of counterions. In that case, Eq. (2.2) can be written as:

$$\begin{aligned}\rho_e &= z_v e_0 (n_+ - n_-) \\ &= z_v e_0 n_0 \exp\left(\frac{-z_v e_0}{k_b T} \psi\right) - z_v e_0 n_0 \exp\left(\frac{z_v e_0}{k_b T} \psi\right) \\ \rho_e &= -2z_v e_0 n_0 \sinh\left(\frac{z_v e_0}{k_b T} \psi\right)\end{aligned}\quad (2.4)$$

Substituting Eq. (2.4) into Eq. (2.1) results in the Poisson-Boltzmann equation which governs the electrical potential distribution in the microchannel.

$$\nabla^2 \psi = \frac{2z_v e_0 n_0}{\epsilon_r \epsilon_0} \sinh\left(\frac{z_v e_0}{k_b T} \psi\right) \quad (2.5)$$

2.1.3 The Navier-Stokes Equation

The equation of motion governing the flow of an incompressible Newtonian fluid is given by the Navier-Stokes equation. The Navier-Stokes equation may be written in various equivalent forms by using certain identities from the theory of vector calculus. One of the simplified forms is given by [7]:

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho(\vec{V} \cdot \nabla) \vec{V} = -\nabla P + \vec{F} + \mu \nabla^2 \vec{V} \quad (2.6)$$

where $\vec{V}$ is the velocity vector, P is the pressure, $\vec{F}$ is the body force applied to the fluid, μ is the dynamic viscosity of fluid, and ρ is the density of fluid.

2.2 Literature Review

In the literature, both theoretical and numerical studies have been reported on the electroosmotic flow in microchannels. In this section, the review of the literature on the steady-state and time and/or frequency dependent electroosmotic flow will be presented. In addition, studies on the developing electroosmotic flow will also be reported.

2.2.1 Steady-state electroosmotic flow

Many studies on electrokinetic effects on liquid flow in fine capillaries of various geometric cross sections have been reported since 1960s. Burgeen and Nakache [8] conducted an analytical study of electroosmotic flow in very fine slit parallel plate.

In the following year, a theoretical study of electrokinetic flow in narrow cylindrical capillaries was presented by Rice and Whitehead [9]. They discussed the dependence of the usual electrokinetic phenomena on the electrokinetic radius, which of interest is the prediction of a maximum in the electroviscous effect. However, due to the usage of the Debye-Huckel approximation in solving the Poisson-Boltzmann equation, their discussions were limited to only low zeta potentials, of say less than 25 mV.

The limitation of Rice and Whitehead model was broken through 10 years later by Levine *et al.* [10], who modeled electrokinetic flow in cylindrical capillaries for high zeta potentials. In their work, they developed an analytical approximation similar to the method used by Philip and Wooding [11] to solve the Poisson-

Boltzman equation of the cylindrical capillary, and the results differ slightly from those by numerical method.

Still in cylindrical capillary, Keh and Liu [12] analyzed the steady electroosmotic flow in a long uniform circular capillary bearing a layer of solvent-permeable and ion-penetrable adsorbed polyelectrolytes on its inside wall. By assuming a uniform density distribution on the polyelectrolyte segments and solving the linearized Poisson-Boltzmann equation, they obtained the electrical potential and space charge density distribution.

The study of steady-state electroosmotic flow had always been dealing on flow in simple geometries, such as parallel plates and cylindrical capillaries, until the advancement of micromachining technology which fabricates microchannels with cross section closer to a rectangular shape.

Yang *et al.* [13-14] analyzed the effects of the EDL at the solid-liquid interface on the pressure driven liquid flow and heat transfer through a rectangular microchannel. The electric double layer field in the cross-section of the rectangular microchannel was determined by solving a non-linear, two-dimensional Poisson-Boltzmann equation. By using the Green's function formulation, an exact solution for the steady-state, fully-developed velocity profile was obtained. The heat transfer and fluid flow characteristics with/without consideration of the electrokinetic effects were evaluated. The results clearly show that, for aqueous solutions of low ionic concentrations and a solid surface of high zeta potential, the liquid flow and heat

transfer in rectangular microchannels were significantly influenced by the presence of the electric double layer field and the induced electrokinetic flow.

In addition to rectangular geometry, the analyses of steady-state electrokinetic flow were also carried out in other geometries, such as elliptical pore [15] and annulus [16].

Surface electrokinetic heterogeneity occurs in a variety of electrokinetic characterization and microfluidic applications. Herr *et al.* [17], Ren and Li [18], and Gleeson [19-20] investigated the characteristics of the electroosmotic flow in a cylindrical microchannel with non-uniform zeta potential. Their theoretical and numerical results show the distorted electroosmotic velocity profiles resulted from the axial variation of the zeta potential. Also, the influences of the unequal section size and the direction of the zeta potential change on the velocity profile, the induced pressure distribution, and the volumetric flow rate were discussed.

In a recent study, Erickson and Li [21] studied the effects of surface electrokinetic heterogeneity on the electroosmotic flow and mixing efficiency of a T-shaped micromixer using 3D finite-element based numerical simulation. In this study, it was investigated that surface heterogeneity was shown to enhance species mixing, and the greatest improvements were found when the zeta potential of the heterogeneous surface was of opposite sign to that of the homogeneous surface. It was also found that for a fixed zeta potential, the length of the heterogeneous patch is the most important factor on the mixing efficiency. However, it was also shown that general

method of improving efficiency, such as decreasing applied voltage and channel size, can be enhanced by the introduction of surface heterogeneity, in some cases resulting in a 70% reduction in the required mixing length. In the same year, Erickson and Li [22] also investigated that the presence of a heterogeneous patch induces a circulating flow pattern perpendicular to the main flow axis, which was caused by an electrostatic potential difference between the double layer field over the homogeneous surface and heterogeneous patch. The strength of this circulation was found to be proportional to both the Reynolds number and the double layer thickness. At higher Reynolds number ($Re = 10$) convective effects were found to be significant to the double layer distribution and led to a deviation from the classical Poisson-Boltzmann distribution observed at lower Reynolds numbers.

2.2.2 Time dependent electroosmotic flow

Generally, the use of electroosmosis involves a DC (direct current) electric field to generate a steady-state electroosmotic flow. With an applied AC (alternating current) electric field, electroosmosis still occurs, but the electroosmotic flow becomes time and frequency dependent. During recent years, frequency-dependent AC electroosmosis has drawn a great attention of researchers.

Söderman and Jönsson [23] developed a theoretical framework to describe the temporal and spatial resolutions of the electroosmotic flow in both planar and cylindrical geometries.

In view of the problem associated with electroosmosis in the conventional DC microelectrophoretic method (e.g., the difficulty in precisely locating the stationary level inside the microelectrophoretic cell), Minor *et al.* [5] proposed a new fast method for measuring the electromobility of colloidal particles using an AC electric field. They showed that it is possible to apply an AC electric field with a specified frequency so that electroosmosis is suppressed, whereas particles are still able to follow the field according to their DC mobility. Using slip velocity approach, they carried out an analysis of the dynamic aspects of electroosmosis and electrophoresis in a parallel slit.

Ramos *et al.* [24] experimentally studied the forces acting on particles of macromolecular sizes in microelectrode arrays. They performed calculations for various forces affecting the AC electrokinetic behavior of particles. In a subsequent work, Ramos *et al.* [25] presented a simple model describing the frequency dependence of the AC electroosmotic flow velocity in planar microelectrodes.

Barragán and Bauza [26] studied the influence of an AC sinusoidal perturbation on the electroosmotic flow through a cation-exchange membrane. Their experimental results showed that the presence of the AC perturbation affects the electroosmotic flow through the frequency of the AC signal and the stirring condition.

Considering slow mixing processes occurred in microfluidic devices due to low Reynolds number dominated laminar flows, Oddy *et al.* [27] demonstrated an electrokinetic chaotic micromixer using fluctuating electric field. They developed an

electrokinetic process to rapidly stir micro- and nano- liter volume solutions by initiating a flow instability, which was observed in sinusoidally oscillating, electroosmotic channel flows. The steadily oscillating electroosmotic velocity profile was presented by using slip velocity boundary condition. A preliminary description of the electrokinetic instability and the design and fabrication of two micromixing devices capable of rapidly stirring two fluid streams was presented in their work. In their experiment, instability of electroosmotic flow driven by sinusoidally alternating electric fields was observed.

Dutta and Beskok [28] presented an analytical model for the time-periodic electroosmotic flow in a parallel slit, and compared their analytical solution of electroosmotic flow field with that of the Stokes second problem. The slip velocity model was first proposed by Overbeek [29] who showed that for microchannels with relatively thin EDL thickness, the flow field outside the EDL is an irrotational flow with a slip velocity boundary condition determined by the well-known Smoluchowski equation.

Santiago [30] studied the effects of fluid inertial and pressure on the transient electroosmotic flow in a parallel plate using slip velocity approximation, and the classical Debye-Hückel linear approximation was used to solve the Poisson-Boltzmann equation. The validity of this assumption is investigated by analyzing the velocity field in a pressure-driven, two-dimensional flow channel with an impulsively started electric field. He also discussed the inertial forces, surface conditions, and pressure-gradient conditions for a full-field similarity between the

electric and velocity fields in electroosmotic flows. Simple relations describing the generation of vorticity in electroosmotic flow were derived using a wall-local, streamline coordinate system in his work.

Without assumption of the thin EDL thickness, Keh and Tseng [31] obtained an analytical solution for the transient electrokinetic flow in a capillary tube and a capillary slit. They studied the transient response of flow due to a step change in the electric field and pressure gradient by solving the linearized Poisson-Boltzmann equation. Closed-form formulas for the transient flow rate, electroosmotic velocity, electric current, and streaming potential in the capillary were also derived as functions of relevant parameters. They demonstrated that the behavior of the transient electrokinetic flow in a capillary tube and in a capillary slit is similar; however, the rate of evolution of the flow in a tube with time is faster by a factor of about 2 than that in a slit with its half thickness equal to the tube radius.

Recently, Kang *et al.* [32] analyzed the dynamic aspects of electroosmotic flow in a cylindrical capillary. By solving the complete Poisson-Boltzmann equation for arbitrary zeta-potentials, the analytical solution for electrostatic potential was derived. The analytical solution of transient electroosmotic flow filed in response to the application of time-dependent electric field was obtained using the Green's function method. Using sinusoidal alternating (AC) electric fields, the effect of frequency-dependent oscillation on the electroosmotic flow was discussed. It was found that the oscillating electroosmotic flow is strongly dependent on the modulation frequency of the applied sinusoidally alternating electric field. The slip velocity approximation

was also used to compare with the analytical method used in this work. It was demonstrated that at the center region of the capillary, the two methods give nearly the same results for transient velocity distributions while adjacent to the charged channel wall the discrepancy was observed.

Bhattacharrya *et al.* [33] addressed the problem of AC electroosmosis in an infinitely extended circular microchannel based on the Debye-Hückel approximation. The analytical solution was obtained using a complex variable approach. It was found that an increasing normalized frequency results in a reduction of EDL effects, leading to (i) a volumetric flow rate in the case of streaming potential approaching that predicted by the theory without EDL effects, and (ii) reduction in the volumetric flow rate in the case of electroosmosis.

The discussion associated with AC electroosmotic flow in a cylindrical microcapillary was also addressed by Reppert and Morgan [34]. In their study, the flow is artificially divided into two regimes: a near-wall regime and a bulk-fluid regime. In the near-wall regime, their analysis neglected the fluid inertia effect; while in the bulk-fluid regime, the slip velocity approach was used. Using Fourier transformation method, they solved the Navier-Stokes equation with the aid of complex phasor notation.

Yang *et al.* [35] has analyzed the problem of laminar flow in microchannels with rectangular cross-section subjected to a time-dependent sinusoidal pressure gradient and sinusoidal electric field using a complex variable approach. It was shown that the

electroviscous effects were not only dependent on the Debye length, but also on the oscillation frequency and the microchannel width. Parametric studies of electroosmosis demonstrate that the flow rate decreases due to an increase in frequency. It was also found that the amplitude of the flow rate of electroosmotic pumping decreases with an increasing frequency for a given width of microchannel. At a given frequency, the amplitude approached an asymptotic value with an increasing width of the microchannel.

2.2.3 Developing electroosmotic flow

The studies reported in the previous sections were all reported only on electroosmotic flow under the fully-developed assumption. Very few analyses have been reported on the developing electroosmotic flows. Without neglecting the presence of Joule heating effects, Tang *et al.* [36] developed a mathematical model to describe the Joule heating and its effects on electroosmotic flow and mass species transport in microchannels. The proposed model includes the Poisson equation, the modified Navier-Stokes equation, and the conjugate energy equation (for the liquid solution and the capillary wall). Specifically, the ionic concentration distributions were modeled using (i) the general Nernst-Planck equation and (ii) the simple Boltzmann distribution. The relevant governing equations were coupled through the temperature-dependent solution dielectric constant, viscosity, and thermal conductivity, and they were numerically solved using a finite-volume-based CFD technique. The applicability of the Nernst-Planck equation and the Boltzmann distribution in the electroosmotic flow with Joule heating was discussed. The results of the time and spatial development for both the electroosmotic flow field and the

Joule heating induced temperature field were presented. They found that the presence of the Joule heating can result in significantly different electroosmotic flow and mass species transport characteristics.

More recently, Yang *et al.* [37] presented an analytical and numerical investigation of electroosmotic flow, taking into account the end effects associated with the inlets and outlets of the open-end microchannels. In their work, an empirical relation between the entrance length and the Reynolds number was established via parametric studies. The results indicate that the electroosmotic flow contractions and expansions which occur in the inlet and outlet regions induce streamwise pressure gradients. Thus, non-flat velocity profiles within the microchannel are obtained.

CHAPTER 3**Frequency Dependent Electroosmotic Flow in a Closed-End Cylindrical Microchannel****3.1 Introduction**

This chapter presents an analysis of the frequency dependent AC electroosmotic flow in a closed-end cylindrical microchannel. Since the two ends of the channel are closed, the AC electroosmosis exhibits a different frequency response due to the superposition of the two viscous waves generated by the electroosmotic flow and the induced counter-flow. In the present study, the Navier-Stokes equation governing the AC electroosmosis is solved analytically using the Green's function formulation. The exact solutions for the frequency responses of the electroosmotic flow and the induced pressure gradient are obtained. Also, the solution to the complete Poisson-Boltzmann equation governing the electrical potential distribution is demonstrated. Further, the validity of the Onsager's principle of reciprocity is provided. For the purpose of comparison, a solution of AC electroosmosis under the slip velocity approximation based on the Stokes second problem is obtained. In addition, the vorticity field of the AC electroosmotic flow generated in a closed-end microchannel is discussed.

The characteristics of electroosmotic flow field and the assumptions made in the following formulation are as follow:

- The body force, $\vec{F}$, is a Coulombic force acting on the net charge density ρ_e in the EDL region caused by applying the alternating electric field in the direction parallel to the flow.
- The liquid is assumed to be incompressible, Newtonian, symmetric electrolyte of constant density, ρ , and viscosity, μ .
- The channel wall is uniformly charged with a zeta potential, ζ .
- The flow is along axial direction, laminar and is fully developed.
- The temperature is assumed to be uniform throughout the flow and Joule heating effects are neglected.
- Pressure varies only along the axial direction.

3.2 Problem Formulation

Figure 3.1: Geometry of a cylindrical microchannel and schematic showing an electroosmotic flow velocity profile and a backpressure velocity profile superimposed

Consider an electroosmotic flow in a closed-end cylindrical microchannel of radius a , as shown in Fig. 3.1. Since the two ends of the channel are closed, there exists an induced backpressure which generates counter flow to ensure zero net flow rate. Using the assumptions and the characteristic of the flow stated in Section 3.1, the motion of liquid in a laminar, fully-developed frequency-dependent electroosmotic flow through a cylindrical microchannel of radius a can be described by the modified Navier-Stokes equation [7]:

$$\rho \frac{\partial u(r,t)}{\partial t} = -\frac{dP(t)}{dz} + E(t)\rho_e(r) + \mu \left[\frac{\partial^2 u(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial u(r,t)}{\partial r} \right] \quad (3.1)$$

where $u(r,t)$ is the transient velocity field, $P(t)$ is the induced back-pressure, $E(t)$ is the externally applied electric field, and $\rho_e(r)$ is the local net charge density due to the presence of the electric double layer (EDL).

The externally applied electric field and the induced backpressure are sinusoidal with an alternating frequency of ω . Mathematically, they can be expressed as

$$E(t) = E_0 e^{i\omega t} \quad P(t) = P_0 e^{i(\omega t + \phi)} \quad (3.2)$$

where E_0 is the amplitude of the electric field, i is the unit of imaginary number, P_0 is the amplitude of the backpressure, and ϕ is the phase angle due to fluid inertia.

Introducing the following dimensionless parameters:

$$\bar{r} = \frac{r}{a} \quad \bar{u} = \frac{u}{U} \quad \bar{t} = \frac{\nu}{a^2} t \quad \bar{P}_0 = \frac{P_0}{\rho U^2} \quad Z = \frac{z}{a \text{Re}_0} \quad (3.3a)$$

$$\Psi = \frac{z_v e_0 \psi}{k_b T} \quad \bar{E}_0 = \frac{E_0 a \text{Re}_0}{\zeta} \quad \bar{H} = \frac{2 z_v e_0 n_0 \zeta}{\rho U^2} \quad (3.3b)$$

where U is the reference velocity, $\nu = \frac{\mu}{\rho}$ is the liquid kinematic viscosity, $Re_0 = \frac{aU}{\nu}$

is the reference Reynolds number, and $\bar{H}$ represents a measure of the ability to convert the electrical energy, $z_v e_0 n_0 \zeta$ to the fluid kinetic energy, $\rho U^2/2$.

Substituting Eqs. (2.4) and (3.2) into Eq. (3.1) and using the non-dimensional parameters, we can obtain

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{u}}{\partial \bar{r}} - \frac{d\bar{P}_0}{dZ} e^{i\left(\frac{\omega a^2 \bar{t}}{\nu} + \phi\right)} - \bar{H} \bar{E}_0 e^{\frac{i\omega a^2 \bar{t}}{\nu}} \sinh \Psi(\bar{r}) \quad (3.4)$$

Equation (3.4) is subjected to the following initial and boundary conditions:

$$\bar{u}(\bar{r}, \bar{t})|_{\bar{t}=0} = 0 \quad (3.5a)$$

$$\left. \frac{\partial \bar{u}}{\partial \bar{r}} \right|_{\bar{r}=0} = 0 \quad \bar{u}|_{\bar{r}=1} = 0 \quad (3.5b)$$

The analytical solution to Eq. (3.4), subjected to the initial and boundary conditions described in Eq. (3.5), can be obtained by using the Green's function formulation approach [38] and is given as

$$\bar{u}(\bar{r}, \bar{t}) = REAL \left\{ - \int_{\tau=0}^{\bar{t}} d\tau \int_{r'=0}^1 r' G(\bar{r}, \bar{t} | r', \tau) \left[\frac{d\bar{P}_0}{dZ} e^{i\left(\frac{\omega a^2 \tau}{\nu} + \phi\right)} + \bar{H} \bar{E}_0 e^{\frac{i\omega a^2 \tau}{\nu}} \sinh \Psi(r') \right] dr' \right\}$$

where $G(\bar{r}, \bar{t} | r', \tau)$ is the Green's function which can be obtained using the separation of variables method [39].

3.3 Green's Function Formulation

The Green's function for solving Eq. (3.6) can be found by firstly solving the corresponding homogeneous equation of Eq. (3.4), which is:

$$\frac{\partial^2 \phi}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \phi}{\partial \bar{r}} = \frac{\partial \phi}{\partial \bar{t}}$$

and the corresponding initial and boundary conditions are given by:

$$\phi(\bar{r}, \bar{t})|_{\bar{t}=0} = R(\bar{r}) \quad \left. \frac{\partial \phi}{\partial \bar{r}} \right|_{\bar{r}=0} = 0 \quad \phi|_{\bar{r}=1} = 0 \quad (3.8)$$

Using the separation of variables method, the solution of Eq. (3.7) is

$$\phi = F(\bar{r})G(\bar{t}) \quad (3.9)$$

Substituting Eq. (3.9) into Eq. (3.7):

$$\frac{F''}{F} + \frac{1}{\bar{r}} \frac{F'}{F} = \frac{\dot{G}}{G} = \text{const} = -\beta^2 \quad (3.10)$$

$F(\bar{r})$ can be obtained from:

$$\frac{F''}{F} + \frac{1}{\bar{r}} \frac{F'}{F} = -\beta^2 \quad \Rightarrow \quad \frac{d^2 F}{d\bar{r}^2} + \frac{1}{\bar{r}} \frac{dF}{d\bar{r}} + \beta^2 F = 0$$

$$F(\bar{r}) = c_1 J_0(\beta \bar{r}) + c_2 Y_0(\beta \bar{r})$$

Using the boundary conditions:

$$\left. \frac{dF}{d\bar{r}} \right|_{\bar{r}=0} = 0 \rightarrow c_2 = 0$$

$$F(\bar{r}=1) = 0 \rightarrow c_1 J_0(\beta) = 0$$

β_m is the positive roots of $J_0(\beta_m) = 0$. Hence,

$$F_m(\bar{r}) = \sum_{m=1}^{\infty} a_m J_0(\beta_m \bar{r}) \quad (3.11)$$

Finally, the solution of $G(\bar{t})$ can be obtained from:

$$\frac{\dot{G}}{G} = -\beta^2 \Rightarrow \dot{G} + \beta^2 G = 0$$

$$G(\bar{t}) = e^{-\beta^2 \bar{t}} \quad (3.12)$$

Thus, by substituting Eqs. (3.11) and (3.12) into Eq. (3.9), we can obtain the solution of $\phi(\bar{r}, \bar{t})$ given by

$$\phi(\bar{r}, \bar{t}) = \sum_{m=1}^{\infty} a_m e^{-\beta_m^2 \bar{t}} J_0(\beta_m \bar{r}) \quad (3.13)$$

The constant a_m can be obtained by utilizing the initial condition.

$$\phi(r', 0) = R(r') = \sum_{m=1}^{\infty} a_m J_0(\beta_m \bar{r}) \Rightarrow a_m = \frac{2}{J_1^2(\beta_m)} \int_{r'=0}^1 r' R(r') J_0(\beta_m r') dr'$$

Substituting a_m to Eq. (3.13) gives

$$\phi(\bar{r}, \bar{t}) = \sum_{m=1}^{\infty} \frac{2}{J_1^2(\beta_m)} \int_{r'=0}^1 r' e^{-\beta_m^2 \bar{t}} J_0(\beta_m r') J_0(\beta_m \bar{r}) R(r') dr' \quad (3.14)$$

Similarly, using the Green's function formulation, the solution to Eq. (3.7) is

$$\phi(\bar{r}, \bar{t}) = \int_{r'=0}^1 r' G(\bar{r}, \bar{t} | r', \tau) \Big|_{\tau=0} R(r') dr' \quad (3.15)$$

Thus by comparing Eqs. (3.14) and (3.15), we can obtain the Green's function given by:

$$G(\bar{r}, \bar{t} | r', \tau) \Big|_{\tau=0} = 2 \sum_{m=1}^{\infty} \frac{e^{-\beta_m^2 \bar{t}} J_0(\beta_m r') J_0(\beta_m \bar{r})}{J_1^2(\beta_m)}$$

or

$$G(\bar{r}, \bar{t} | r', \tau) = 2 \sum_{m=1}^{\infty} \frac{e^{-\beta_m^2 (\bar{t}-\tau)} J_0(\beta_m r') J_0(\beta_m \bar{r})}{J_1^2(\beta_m)} \quad (3.16)$$

3.4 Velocity field in Closed-end Cylindrical Microchannels

Having obtained the Green's function of the governing equation of motion, we can proceed to solve Eq. (3.6). Substituting Eq. (3.16) into Eq. (3.6), we can obtain:

$$\bar{u}(\bar{r}, \bar{t}) = \text{REAL} \left[\begin{aligned} & -2 \frac{d\bar{P}_0}{dZ} \sum_{m=1}^{\infty} \frac{J_0(\beta_m \bar{r})}{J_1(\beta_m)} \frac{e^{i\left(\frac{\omega a^2}{\nu} \bar{t} + \phi\right)} - e^{-\beta_m^2 \bar{t} + i\phi}}{\beta_m \left(\beta_m^2 + i \frac{\omega a^2}{\nu} \right)} \\ & - 2 \bar{H} \bar{E}_0 \sum_{m=1}^{\infty} C_m \frac{J_0(\beta_m \bar{r})}{J_1^2(\beta_m)} \frac{e^{i\frac{\omega a^2}{\nu} \bar{t}} - e^{-\beta_m^2 \bar{t}}}{\beta_m^2 + i \frac{\omega a^2}{\nu}} \end{aligned} \right] \quad (3.17)$$

where
$$C_m = \int_{r'=0}^1 r' J_0(\beta_m r') \sinh \Psi(r') dr' \quad (3.18)$$

Equation (3.17) shows that electroosmotic flow in a closed-end cylindrical microchannel is a linear superposition of: (i) the flow due to the applied electric field, and (ii) the flow due to the induced backpressure gradient. In this study, steadily oscillating electroosmotic flow under an AC electrical field is being considered. In this situation, the exponential term on the right-hand side of Eq. (3.17) decays, and hence Eq. (3.17) reduces to

$$\bar{u}(\bar{r}, \bar{t}) = \text{REAL} \left[-2 \frac{d\bar{P}(\bar{t})}{dZ} \sum_{m=1}^{\infty} \frac{J_0(\beta_m \bar{r})}{\beta_m J_1(\beta_m) \left(\beta_m^2 + i \frac{\omega a^2}{\nu} \right)} - 2 \bar{H} \bar{E}(\bar{t}) \sum_{m=1}^{\infty} \frac{J_0(\beta_m \bar{r})}{J_1^2(\beta_m)} \frac{C_m}{\beta_m^2 + i \frac{\omega a^2}{\nu}} \right] \quad (3.19)$$

where
$$\frac{d\bar{P}(\bar{t})}{dZ} = \frac{d\bar{P}_0}{dZ} e^{i\left(\frac{\omega a^2}{\nu} \bar{t} + \phi\right)} \quad \text{and} \quad \bar{E}(\bar{t}) = \bar{E}_0 e^{i\frac{\omega a^2}{\nu} \bar{t}} \quad (3.20)$$

For a closed-end electroosmotic system, the net flow rate is equal to zero, or mathematically the system fulfills the following condition,

$$\int_{\bar{r}=0}^1 \bar{r} \bar{u}(\bar{r}, \bar{t}) d\bar{r} = 0 \quad (3.21)$$

Substituting Eq. (3.19) into Eq. (3.21) and performing the integration, we can show that the induced backpressure gradient can be expressed as

$$\frac{d\bar{P}(\bar{t})}{dZ} = -\bar{H}\bar{E}(\bar{t}) \frac{\sum_{m=1}^{\infty} \frac{C_m}{\beta_m J_1(\beta_m) \left(\beta_m^2 + i \frac{\omega a^2}{\nu} \right)}}{\sum_{m=1}^{\infty} \frac{1}{\beta_m^2 \left(\beta_m^2 + i \frac{\omega a^2}{\nu} \right)}} \quad (3.22)$$

According to Eq. (3.22), the induced pressure gradient is linearly proportional to the strength of the applied electric field, and the parameter $\bar{H}$. Moreover, in a limiting situation as $\omega \rightarrow \infty$, Eq. (3.22) can be reduced to

$$\frac{d\bar{P}_0}{dZ} = -\bar{H}\bar{E}_0 \frac{\sum_{m=1}^{\infty} \frac{C_m}{\beta_m J_1(\beta_m)}}{\sum_{m=1}^{\infty} \frac{1}{\beta_m^2}} \quad (3.23)$$

Equation (3.23) suggests that the amplitude of the induced pressure gradient under extremely high applied frequencies is independent on the frequency of the applied electric field since the flow only occurs within the EDL region.

In another limiting situation when $\omega = 0$, the electric field becomes a constant DC field with a magnitude of E_0 and thus Eq. (3.22) becomes

$$\frac{d\bar{P}_0}{dZ} = -\bar{H}\bar{E}_0 \frac{\sum_{m=1}^{\infty} \frac{C_m}{\beta_m^3 J_1(\beta_m)}}{\sum_{m=1}^{\infty} \frac{1}{\beta_m^4}} \quad (3.24)$$

And the corresponding flow field becomes

$$\bar{u}(\bar{r}) = -2 \frac{d\bar{P}_0}{dZ} \sum_{m=1}^{\infty} \frac{J_0(\beta_m \bar{r})}{\beta_m^3 J_1(\beta_m)} - 2\bar{H}\bar{E}_0 \sum_{m=1}^{\infty} C_m \frac{J_0(\beta_m \bar{r})}{\beta_m^2 J_1^2(\beta_m)} \quad (3.25)$$

3.5 Electrical Potential Distribution

In the previous section, it has been discussed that the electrical potential distribution can be described by the Poisson-Boltzmann equation as shown in Eq. (2.5). In cylindrical coordinate, Eq. (2.5) becomes

$$\frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} = \frac{2z_v e_0 n_0}{\epsilon_r \epsilon_0} \sinh\left(\frac{z_v e_0 \psi}{k_b T}\right) \quad (3.26)$$

This equation can be non-dimensionalized by using the previous defined dimensionless parameters in Eq. (3.3). Thus Poisson-Boltzmann becomes:

$$\frac{d^2\Psi}{d\bar{r}^2} + \frac{1}{\bar{r}} \frac{d\Psi}{d\bar{r}} = (\kappa a)^2 \sinh \Psi \quad (3.27)$$

where κ , the Debye-Hückel parameter, is defined as:

$$\kappa = \left(\frac{2z_v^2 e_0^2 n_0}{\epsilon_r \epsilon_0 k_b T} \right)^{\frac{1}{2}} \quad (3.28)$$

Equation (3.27) is subjected to the following boundary conditions:

$$\left. \frac{d\Psi}{d\bar{r}} \right|_{\bar{r}=0} = 0 \quad \Psi(\bar{r}=1) = \bar{\zeta} \quad (3.29)$$

where $\bar{\zeta}$ is the non-dimensional zeta potential which is defined $\bar{\zeta} = \frac{z_v e_0 \zeta}{k_b T}$.

Kang *et al.* [32] successfully solved Eq. (3.27) using the analytical scheme proposed by Philip and Wooding [11] to approximate the hyperbolic sine function as:

$$\sinh \Psi \approx \begin{cases} -\frac{1}{2} \exp(-\Psi) & \Psi < -1 \\ \Psi & -1 \leq \Psi \leq 1 \\ \frac{1}{2} \exp(\Psi) & \Psi > 1 \end{cases} \quad (3.30)$$

Using this method, the entire cylindrical region is artificially divided into two regimes as shown in Fig. 3.2 so that at the junction ($\bar{r} = \bar{r}^*$), we have $\Psi = 1$.

Figure 3.2: Regimes defined in a cylindrical cross section for solving the Poisson-Boltzmann equation

Using this hypothesis, Eq. (3.27) is transformed into the following equations:

Low potential regime L $(0 \leq \bar{r} \leq \bar{r}^*)$

$$\frac{d^2\Psi_L}{d\bar{r}^2} + \frac{1}{\bar{r}} \frac{d\Psi_L}{d\bar{r}} = \Psi_L \quad (3.31)$$

High potential regime H $(\bar{r}^* \leq \bar{r} \leq 1)$

$$\frac{d^2\Psi_H}{d\bar{r}^2} + \frac{1}{\bar{r}} \frac{d\Psi_H}{d\bar{r}} = \frac{1}{2} \exp(\Psi_H) \quad (3.32)$$

Equations (3.31) and (3.32) are subjected to the boundary conditions specified in Eq.

(3.29) and the following coupling boundary conditions:

$$\left. \frac{d\Psi_L}{d\bar{r}} \right|_{\bar{r}=\bar{r}^*} = \left. \frac{d\Psi_H}{d\bar{r}} \right|_{\bar{r}=\bar{r}^*} \quad \Psi_L(\bar{r}^*) = \Psi_H(\bar{r}^*) = 1 \quad (3.33)$$

The analytical solution of Eqs. (3.31) and (3.32) can be given as

$$\Psi_L(\bar{r}) = \frac{I_0(\kappa a \bar{r})}{I_0(\kappa a \bar{r}^*)} \quad (0 \leq \bar{r} \leq \bar{r}^*) \quad (3.34)$$

where I_0 is the zero-order modified Bessel function of the first kind, and

$$\Psi_H(\bar{r}) = \ln \left\{ \frac{-C}{(\kappa a \bar{r})^2 \cos^2 \left[\cos^{-1} \sqrt{\frac{-C}{e^{\frac{\bar{r}}{\bar{r}^*}} (\kappa a)^2}} + \frac{1}{2} \sqrt{-C} \ln(\bar{r}) \right]} \right\} \quad (\bar{r}^* \leq \bar{r} \leq 1) \quad (3.35)$$

where C is the integration constant, determined by

$$C = \left[2 + \kappa a \bar{r}^* \frac{I_1(\kappa a \bar{r}^*)}{I_0(\kappa a \bar{r}^*)} \right]^2 - (\kappa a \bar{r}^*)^2 e \quad (3.36)$$

Such an analytical scheme was found to differ only slightly from the result of numerical integration of the complete Poisson-Boltzmann equation [32].

3.6 Onsager's Principle of Reciprocity

Equation (3.19) shows that the flow field in a closed-end microchannel is linearly proportional to the driving electric field and the induced backpressure gradient. The linear relationship of the flow field, the driving electric field and the induced backpressure gradient suggests a possible constitution of the Onsager principle of reciprocity familiar in the theory of non-equilibrium thermodynamics. In the literature, the Onsager reciprocal relations for electrokinetic flows *under a steady-state situation* were demonstrated by Burgreen and Nakache [8] for a parallel slit and by Rice and Whitehead [9], Levine *et al.* [10] and Keh and Liu [12] for a cylindrical capillary. Recently, Bhattacharrya *et al.* [33], Reppert and Morgan [34] and Yang *et al.* [35] also demonstrated the validity of Onsager reciprocal relations in oscillating laminar electrokinetic flow in open-end circular and rectangular microchannels, respectively. The validity of the Onsager's reciprocity relations for AC oscillating electroosmotic flows in a closed-end cylindrical microchannel is demonstrated here.

The flow rate $Q(t)$ in the microchannel can be obtained by

$$Q(t) = 2\pi \int_0^a r u(r, t) dr \quad (3.37a)$$

Due to the symmetry, the flow rate is further expressed as

$$Q(t) = 2\pi U a^2 \int_0^1 \bar{r} \bar{u}(\bar{r}, \bar{t}) d\bar{r} \quad (3.37b)$$

Substituting Eq. (3.19) into Eq. (3.37b), and making use of Eq. (3.3), we can express the flow rate in terms of the dimensional induced pressure gradient and the dimensional electric field as

$$Q(t) = J_{11} \frac{dP(t)}{dz} + J_{12} E(t) \quad (3.38)$$

where the Onsager transport coefficients are

$$J_{11} = -\frac{4\pi a^4}{\mu} \sum_{m=1}^{\infty} \frac{1}{\beta_m^2 \left(\beta_m^2 + \frac{i\omega a^2}{\nu} \right)} \quad (3.39a)$$

$$J_{12} = -\frac{8\pi z_v e_0 n_0 a^4}{\mu} \sum_{m=1}^{\infty} \frac{C_m}{\beta_m J_1(\beta_m) \left(\beta_m^2 + \frac{i\omega a^2}{\nu} \right)} \quad (3.39b)$$

In general, the axial component of the electric current density is expressed as [40]

$$i_z(t) = u z_v e_0 (n_+ - n_-) - z_v e_0 \left(D_+ \frac{\partial n_+}{\partial z} + D_- \frac{\partial n_-}{\partial z} \right) + \lambda_0 E_z \quad (3.40)$$

where λ_0 is the electric conductivity of the fluid, and $D_{\pm}$ are the mass diffusivity of cations and anions, respectively. Since no ionic concentration gradient is built up along the axial direction, the second term on the right-hand side of Eq. (3.40) drops off. Therefore, the electric current across a straight cylindrical channel can be expressed as

$$I(t) = 2\pi U a^2 \int_0^1 \bar{r} \rho_e u(\bar{r}, t) d\bar{r} + \pi \lambda_0 E_0 a^2 \quad (3.41)$$

Substituting Eq. (2.4) and Eq. (3.19) into Eq. (3.41) and performing the integration, we can obtain the electrical current expressed in the Onsager form as

$$I(t) = J_{21} \frac{dP(t)}{dz} + J_{22} E(t) \quad (3.42)$$

where the Onsager transport coefficients are

$$J_{21} = \frac{8\pi z_v e_0 n_0 a^4}{\mu} \sum_{m=1}^{\infty} \frac{C_m}{\beta_m J_1(\beta_m) \left(\beta_m^2 + \frac{i\omega a^2}{\nu} \right)} \quad (3.43a)$$

$$J_{22} = \frac{16\pi z_v^2 e_0^2 n_0^2 a^4}{\mu} \sum_{m=1}^{\infty} \frac{C_m^2}{J_1^2(\beta_m) \left(\beta_m^2 + \frac{i\omega a^2}{\nu} \right)} + \pi \lambda_0 a^2 \quad (3.43b)$$

From Eqs. (3.39) and (3.43), it is noted that $J_{12} = -J_{21}$, which suggests the validity of the Onsager principle of reciprocity for steadily oscillating electroosmotic flow in a closed-end cylindrical microchannel.

3.7 Slip Velocity Approximation

Instead of applying an electric field to drive the flow, the channel wall is set into motion to drive the flow. Conceptually, the velocity profile induced by the application of a sinusoidally alternating electric field is similar to that induced by a mechanical oscillating motion of the channel wall which moves under the same frequency as the electric field. In the literature, the fluid flow induced by moving the boundary of the wall is referred to as the Stokes second problem [41]. The idea of introducing the slip velocity approximation is to neglect the thin EDL region such that the entire flow is driven by the hydrodynamic shear stresses originated from the liquid viscosity. Here, the thin EDL region can be regarded as an oscillating moving channel wall. The outer slip boundary velocity is thus determined by the Smoluchowski equation,

$$U_s(t) = -\frac{\varepsilon_0 \varepsilon_r \zeta E_0 e^{i\omega t}}{\mu} \quad (3.44)$$

The oscillating velocity field under a steadily alternating AC moving boundary can be obtained by solving the following transient Stokes equation,

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{u}}{\partial \bar{r}} - \frac{d\bar{P}_0}{dZ} e^{i\left(\frac{\omega a^2 \bar{t}}{\nu} + \phi\right)} \quad (3.45)$$

which is subjected to the following boundary conditions

$$\left. \frac{\partial \bar{u}}{\partial \bar{r}} \right|_{\bar{r}=0} = 0 \quad \bar{u}|_{\bar{r}=1} = \bar{U}_o e^{\frac{i\omega a^2 \bar{t}}{\nu}} \quad (3.46)$$

$$\bar{U}_o = -\frac{\varepsilon_0 \varepsilon_r \zeta E_0}{\mu U}$$

Letting $\bar{u}(\bar{r}, \bar{t}) = v(\bar{r}, \bar{t}) + w(\bar{r}, \bar{t})$, the solution of Eq. (3.45) can be obtained by solving two separated partial differential equations (PDE) of v and w as shown below:

The first PDE is

$$\frac{\partial v}{\partial \bar{t}} = \frac{\partial^2 v}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial v}{\partial \bar{r}} \quad (3.47)$$

with boundary conditions:

$$\left. \frac{\partial v}{\partial \bar{r}} \right|_{\bar{r}=0} = 0 \quad v|_{\bar{r}=1} = \bar{U}_o e^{\frac{i\omega a^2 \bar{t}}{\nu}} \quad (3.48)$$

and the second PDE,

$$\frac{\partial^2 w}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial w}{\partial \bar{r}} - \frac{d\bar{P}_0}{dZ} e^{i\left(\frac{\omega a^2 \bar{t}}{\nu} + \phi\right)} = \frac{\partial w}{\partial \bar{t}} \quad (3.49)$$

with boundary conditions:

$$\left. \frac{\partial w}{\partial \bar{r}} \right|_{\bar{r}=0} = 0 \quad w|_{\bar{r}=1} = 0 \quad (3.50)$$

Equation (3.47) can be solved by using the separation of variables method,

$v = F(\bar{r})H(\bar{t})$ and assuming $H(\bar{t}) = e^{\frac{i\omega a^2}{\nu}\bar{t}}$, we can show that

$$v(\bar{r}, \bar{t}) = REAL \left[\frac{\bar{U}_o I_0 \left(\sqrt{\frac{i\omega a^2}{\nu}} \bar{r} \right) e^{\frac{i\omega a^2}{\nu} \bar{t}}}{I_0 \left(\sqrt{\frac{i\omega a^2}{\nu}} \right)} \right] \quad (3.51)$$

where i is the unit of imaginary number, and "REAL" represents the real part of the solution.

Using the Green's function method, we can show that the solution of Eq. (3.49) is

$$w(\bar{r}, \bar{t}) = REAL \left[-2 \frac{d\bar{P}(\bar{t})}{dZ} \sum_{m=1}^{\infty} \frac{J_0(\beta_m \bar{r})}{\beta_m J_1(\beta_m) \left(\beta_m^2 + i \frac{\omega a^2}{\nu} \right)} \right] \quad (3.52)$$

where $\frac{d\bar{P}(\bar{t})}{dZ} = \frac{d\bar{P}_0}{dZ} e^{i \left(\frac{\omega a^2}{\nu} \bar{t} + \phi \right)}$.

Hence, the complete solution of Eq. (3.45) is

$$\bar{u}(\bar{r}, \bar{t}) = REAL \left[\frac{\bar{U}_o I_0 \left(\sqrt{\frac{i\omega a^2}{\nu}} \bar{r} \right) e^{\frac{i\omega a^2}{\nu} \bar{t}}}{I_0 \left(\sqrt{\frac{i\omega a^2}{\nu}} \right)} - 2 \frac{d\bar{P}(\bar{t})}{dZ} \sum_{m=1}^{\infty} \frac{J_0(\beta_m \bar{r})}{\beta_m J_1(\beta_m) \left(\beta_m^2 + i \frac{\omega a^2}{\nu} \right)} \right] \quad (3.53)$$

and the pressure gradient expression can be obtained as

$$\frac{d\bar{P}(\bar{t})}{dZ} = \frac{\bar{U}_o I_1 \left(\sqrt{\frac{i\omega a^2}{\nu}} \right) e^{\frac{i\omega a^2}{\nu} \bar{t}}}{2 \sqrt{\frac{i\omega a^2}{\nu}} I_0 \left(\sqrt{\frac{i\omega a^2}{\nu}} \right) \sum_{m=1}^{\infty} \frac{1}{\beta_m^2 \left(\beta_m^2 + i \frac{\omega a^2}{\nu} \right)}} \quad (3.54)$$

3.8 Electroosmotic Vorticity Field

In this section, the vorticity field for AC oscillating electroosmotic flows in a closed-end cylindrical microchannel is presented. Vorticity, $\vec{\omega}$ is defined as the curl of the velocity vector, $\vec{V}$. Mathematically, it is expressed as [42]

$$\vec{\omega} = \nabla \times \vec{V} \quad (3.55)$$

According to Eq. (3.55), vorticity is the gradient of the velocity field. Therefore, by studying the vorticity field, we can predict the shear stress distribution of the flow across the cross-section of the channel, as the latter also involves the computation of the velocity field gradient. In addition, vorticity can be used as an indication to assess whether a flow is rotational or irrotational, thus it enables a better visualization of the flow.

For a fully-developed electroosmotic flow, there is no velocity component in the radial or angular direction, and hence Eq. (3.55) reduces to

$$\vec{\omega} = -\frac{du(r,t)}{dr} \hat{\theta} = -\frac{U}{a} \frac{d\bar{u}(\bar{r},\bar{t})}{d\bar{r}} \hat{\theta} \quad (3.56)$$

Substituting Eq. (3.19) to Eq. (3.56) leads to

$$\vec{\omega} = REAL \left[-\frac{2U}{a} \frac{d\bar{P}(\bar{t})}{dZ} \sum_{m=1}^{\infty} \frac{J_1(\beta_m \bar{r})}{J_1(\beta_m)} \frac{1}{\left(\beta_m^2 + i \frac{\omega a^2}{\nu} \right)} - \frac{2U}{a} \bar{H}\bar{E}(\bar{t}) \sum_{m=1}^{\infty} C_m \beta_m \frac{J_1(\beta_m \bar{r})}{J_1^2(\beta_m)} \frac{1}{\beta_m^2 + i \frac{\omega a^2}{\nu}} \right] \hat{\theta} \quad (3.57)$$

Alternatively, we can also use the slip velocity approximation to obtain the vorticity by substituting Eq. (3.53) into Eq. (3.56)

$$\vec{\omega} = REAL \left[\frac{\frac{2U}{a} \frac{d\bar{P}(\bar{r})}{dZ} \sum_{m=1}^{\infty} \frac{J_1(\beta_m \bar{r})}{J_1(\beta_m) \left(\beta_m^2 + i \frac{\omega a^2}{\nu} \right)} + \frac{U \bar{U}_o}{a} \frac{\sqrt{\frac{i\omega a^2}{\nu}} I_1 \left(\sqrt{\frac{i\omega a^2}{\nu}} \bar{r} \right) e^{\frac{i\omega a^2}{\nu} \bar{r}}}{I_0 \left(\sqrt{\frac{i\omega a^2}{\nu}} \right)} \right] \hat{\theta} \quad (3.58)$$

3.9 Results and Discussion

In all the calculations, NaCl solution with valence z_v of unity is used as the electrolyte solution. The density, ρ , viscosity, μ , and dielectric constant, ϵ_r of the electrolyte are taken to be the same as those of water, and these are 1000 kg/m^3 , $9 \times 10^{-4} \text{ N-s/m}^2$, and 80, respectively. The temperature T of the electrolyte is taken as 293 K. The reference velocity U is chosen as 1 mm/s. The strength of the electric field E_0 is 10000 V/m. The reference channel radius, a is $10 \mu\text{m}$.

3.9.1 Oscillating Velocity Distribution

For a cylindrical channel of radius $a = 10 \mu\text{m}$, the characteristic time of the electroosmotic flow can be estimated from Eq. (3.17), $t^* = \frac{\rho a^2}{\mu \beta_1^2} = 19.2 \mu\text{s}$. The

corresponding eigen-frequency of the system is $f^* = \frac{1}{t^*} = 52.05 \text{ kHz}$. In this work,

three different frequencies of the applied electric field, $0.1 f^*$, f^* and $10 f^*$ ($f = \omega/2\pi$)

are considered to study the characteristics of the oscillating electroosmotic flow in closed-end channels. The results of both the analytical solution (represented by the solid lines) and the slip velocity approximation (denoted by the dashed lines) are summarized in Figs. 3.3a-c which present the steadily oscillating velocity distributions in a symmetric half of the cylindrical channel for five different characteristic moments: $\omega t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ and π .

(a)

(b)

(c)

Figure 3.3: Oscillating electroosmotic velocity distributions for three different frequencies of AC electric field, (a) $0.1f^*$, (b) f^* and (c) $10f^*$

Upon an application of the electric field, the flow begins to develop in the region close to the channel wall, and the flow velocity increases rapidly from zero at the wall to its peak value within the EDL region. At the bulk liquid region (i.e., the region away from EDL), there exists a reverse flow. This scenario reveals the unique feature of the electroosmotic flow in a closed-end channel; within the EDL region, it is driven by the electrical body force resulted from the interaction of the electric field and the net charge density; and outside the EDL region, the flow is driven by the induced pressure gradient to counter-balance the flow at the EDL region so that the net flow rate is zero.

When the frequency of the electric field is low (e.g., $0.1 f^*$ as shown in Fig. 3.3a), the flow extends to the entire cross section of the channel and the flow frequency response follows closely that of the alternating electric field. As the applied frequency increases, the perturbed flow region becomes smaller, and the flow response in the central region of the channel lags behind the alternating electric field (e.g., f^* as shown in Fig. 3.3b). While at a high applied frequency (e.g., $10 f^*$ as shown in Fig. 3.3c), the flow in the central channel remains virtually motionless regardless of the fast oscillating flow occurring within a thin EDL region close to the channel wall. This phenomenon is logically anticipated because as the frequency goes very high, the electric field changes its direction so fast that due to the inertia of the fluid, the fluid cannot response fast enough to develop the flow across the entire channel. To provide an insight into the observed scenario, we refer to the frequency-dependent Stokes penetration depth, δ_s , representing a typical length scale of the oscillatory laminar viscous flows in response to a harmonic external excitation (e.g.,

AC electric field or oscillating boundary). According to Minor *et al.* [5] δ_s is defined as

$$\delta_s = \sqrt{\frac{\mu}{\rho \omega}} = \sqrt{\frac{\mu}{2\pi \rho f}} \quad (3.59)$$

Equation (3.59) shows that the penetration depth (i.e., the perturbed flow region) is inversely proportional to the square root of the frequency of the applied electric field. The slip velocity approximation is rooted from an analogy to the Stokes second problem, i.e., the oscillating viscous flow induced by a sinusoidally oscillating infinite flat plate [41]. It is demonstrated in Figs. 3.3a-c that the velocity distributions at the central region of the channel, both given by the analytical solution (developed in this work) and the slip velocity approximation method are very close to each other. However in the region adjacent to the channel wall (i.e., the EDL region), the two methods predict different velocity distributions. The discrepancy lies in the neglect of the detailed flow characteristics in the EDL region under the slip velocity approximation and the no-slip boundary condition applied in the analytical scheme. It should be pointed out here that according to Kang *et al.* [32], in open-end microcapillaries, both the analytical solution and the slip velocity approximation can give an identical solution in the region outside the EDL. While in closed-end microcapillaries in which the net flow rate is zero, if the analytical solution gives a different result from the slip velocity approximation in the EDL region, a difference between these methods also exists outside the EDL region. Such a difference varies depending on the relative ratio of the EDL thickness to the capillary radius. It is anticipated that the smaller the κa , the larger the difference.

3.9.2 Induced Pressure Gradient

The induced oscillating pressure gradient is shown in Fig. 3.4. It is noted that the induced pressure gradient response is always leading the applied sinusoidal electric field. The phase lead becomes larger as the frequency increases due to the effect of the fluid inertia (more comprehensive discussion on phase angle is provided in Chapter 4). Moreover, in contrast to the dependence of the fluid velocity on frequency, a high backpressure is generated as the frequency increases. This scenario is also demonstrated in Fig. 3.5, which presents the induced backpressure gradient versus the frequency of applied electric field. It is interesting to observe that the amplitude of the oscillating pressure gradient remains almost constant when the applied frequency is less than the characteristic frequency, f^* . Under this situation, the flow extends to the entire cross-section of the channel, and the induced backpressure gradient is insensitive to the applied frequency. However, as the frequency becomes higher than f^* , the amplitude of the oscillating backpressure gradient increases significantly. The fast increase of the pressure gradient with frequency is attributed to the compression of the perturbed flow region in the channel. As discussed earlier, when the applied frequency is high, the Stokes penetration depth, δ_s , decreases and the liquid is excited only within a small region near the channel wall while the liquid at the large portion of the channel central region remains almost stationary. Thus the perturbed flow region behaves as if the flow under compression in a reduced-size channel. Hence, high induced backpressure gradient is needed to counter-balance the electroosmotic flow, fulfilling the condition of zero flow rate in closed-end microchannels.

Figure 3.4: Induced oscillating backpressure gradient under AC electric field

Figure 3.5: Amplitude of induced pressure gradient under AC electric field versus the ratio of applied frequency to characteristic frequency (f/f^*)

The effect of channel size on the induced pressure gradient under a steady-state DC electric field is presented in Fig. 3.6. The horizontal coordinate is the actual channel radius with respect to the reference radius of the cylindrical channel of $a = 10 \mu m$. It is shown that the pressure gradient monotonically decreases as the channel size increases. This observation may be understood as follows: According to the Smoluchowski equation, the maximum magnitude of the fluid velocity is independent of the channel size; this is an important feature of the electroosmotic flow. However, the hydrodynamic drag increases as the channel size decreases. Hence, to counter-balance the electroosmotic flow, it requires higher backpressure for smaller-sized microchannels.

Figure 3.6: Steady-state induced pressure gradient under DC electric field versus channel's radius with respect to a reference radius (a) of $10 \mu m$.

3.9.3 Vorticity Field

Figures 3.7a, 3.8a and 3.9a show the oscillating vorticity field predicted from the slip velocity approximation in a half cross-section of a closed-end cylindrical microchannel for three different frequencies, $0.1f^*$, f^* and $10f^*$, respectively. Since the electroosmotic vorticity is defined as the radial gradient of the electroosmotic velocity (see Eq. (3.56)), the dependence of the vorticity field on the applied frequency is very similar to that of the velocity field.

Figure 3.7a: Oscillating electroosmotic vorticity distributions under the slip velocity approximation for a low applied frequency, $f = 0.1f^*$.

Figure 3.7b: Comparison of the oscillating vorticity distributions in the region near the channel wall for a low applied frequency, $f = 0.1 f^*$.

Figure 3.8a: Oscillating electroosmotic vorticity distributions under the slip velocity approximation for a low applied frequency, $f = f^*$.

Figure 3.8b: Comparison of the oscillating vorticity distributions in the region near the channel wall for a low applied frequency, $f = f^*$.

Figure 3.9a: Oscillating electroosmotic vorticity distributions under the slip velocity approximation for a low applied frequency, $f = 10 f^*$.

Figure 3.9b: Comparison of the oscillating vorticity distributions in the region near the channel wall for a low applied frequency, $f = 10f^*$.

It can be seen that at a low applied frequency (Fig. 3.7a), the vorticity field is generated throughout the entire channel, indicating the flow is rotational everywhere. As the applied frequency increases, the rotational flow region is reduced. For instance, Fig. 3.9a shows the vorticity field occurs only within about one third of the region close to the channel wall.

For the purpose of comparison, Figs. 3.7b, 3.8b, and 3.9b show the vorticity fields computed from the analytical solution derived in this study, corresponding to the same conditions of Figs. 3.7a, 3.8a, and 3.9a, respectively. However, only a small portion nearby the channel wall is presented because, similar to the velocity fields, both the analytical solution and the slip velocity approximation give almost the same

vorticity fields in the rest of the region of the channel. It is interesting to observe that within the EDL region, the slip velocity approximation not only underestimates the vorticity, but also gives a misleading result that a higher applied frequency gives a stronger the vorticity. However, the analytical solution predicts that the vorticity magnitude decreases insignificantly as the frequency increases and can be considered as independent of the applied frequency. Moreover, the vorticity direction predicted by the slip velocity approach is opposite to the analytical scheme. This is easy to follow because the no-slip boundary condition is used for all the cases in the analytical solution and the vorticity is usually generated from the solid wall. Furthermore, the fluid inertia within the EDL region can usually be neglected for electroosmotic flow [34] so that both the velocity and vorticity response can follow the applied frequency. In addition, Figs. 3.7b, 3.8b, and 3.9b also show that both the analytical solution and the slip velocity approximation can predict the same trend and magnitude of the vorticity fields outside the EDL region. For example, both methods show that the vorticity is stronger at a higher applied frequency. This conforms to experimental observations by Oddy *et al.* [27].

3.10 Summary

A theoretical analysis for an oscillating AC electroosmotic flow and the vorticity fields in a closed-end cylindrical microchannel is presented. The Onsager's principle of reciprocity for transient and AC electroosmotic flows is demonstrated. A comparison of the electroosmotic flow and vorticity fields obtained from the analytical solution developed in this study and the slip velocity approximation is provided. The analyses show that electroosmotic flow in closed-end channels is the

superposition of an electroosmotic flow and a reverse pressure driven flow due to the induced pressure gradient. Moreover, it is shown that the AC electroosmotic flow is strongly dependent on the frequency of the applied sinusoidal electric field. It is identified that it is the frequency of electric field that determines the thickness of the Stokes penetration layer, and thus governs the extent of the oscillating flow and vorticity fields. Due to the fluid inertia, there is always a phase lead for the oscillating backpressure gradient built-up in a closed-end channel. Additionally, it is demonstrated that the induced pressure gradient monotonically decreases as the channel size increases. It is also shown that there is a similarity of the velocity and vorticity fields between the slip velocity approximation and the present analytical solution in the bulk fluid region, while in the EDL region, the discrepancy is observed.

CHAPTER 4

Frequency Dependent Electroosmotic Flow in a Closed-End Rectangular Microchannel

4.1 Introduction

This chapter addresses a theoretical study for frequency-dependent AC electroosmotic flows in a closed-end rectangular microchannel. Using the Green's function formulation, an analytical solution for the velocity field is derived. Since the two ends of the channel are closed, there exists an induced backpressure which generates counter flow to ensure zero net flow. The EDL potential distribution is obtained by solving the Poisson-Boltzmann equation under the Debye-Hückel approximation. The transient evolutions of velocity fields under the application of AC electric fields are addressed. Effects of the external applied frequency on the velocity fields and the induced backpressure are presented. Moreover, an insight to the vorticity field generated in the microchannel is shown in this study. In addition, a solution under the slip velocity approximation approach for both the velocity and the vorticity fields is also presented to provide a comparison with the analytical approach developed in this work.

4.2 Problem Formulation

Figure 4.1: Geometry of the rectangular microchannel

In this study, the liquid driven by electroosmosis in a rectangular microchannel of length L , height $2a$ and width $2b$ (Fig. 1) is assumed to be incompressible, Newtonian, symmetric electrolyte of constant density, ρ , and viscosity, μ . The channel wall is uniformly charged with a zeta potential, ζ . The motion of the liquid in a laminar, fully-developed electroosmotic flow through an infinitely extended rectangular microchannel can be described by a simplified Navier-Stokes equation given in Eq. (4.1).

$$\rho \frac{\partial u(x, y, t)}{\partial t} = -\frac{dP(t)}{dz} + E(t)\rho_e(x, y) + \mu \left(\frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} \right) \quad (4.1)$$

where $u(x, y, t)$ is the transient velocity field, $P(t)$ is the induced pressure that generates the backflow in a closed-end channel, and $\rho_e(x, y)$ is the local net charge density due to the presence of the EDL.

The external sinusoidal electric field and the induced sinusoidal backpressure can be expressed as

$$E(t) = E_0 e^{i\omega t} \quad P(t) = P_0 e^{i(\omega t + \phi)} \quad (4.2)$$

where E_0 is the amplitude of the applied electric field strength, ω is the alternating frequency, i is the unit of the imaginary number, P_0 is the magnitude of the pressure, and ϕ is the phase angle due to fluid inertia.

Introducing the following dimensionless parameters:

$$\bar{u} = \frac{u}{U} \quad \bar{t} = \frac{\mu}{\rho D_h^2} t \quad X = \frac{x}{D_h} \quad Y = \frac{y}{D_h} \quad Z = \frac{z}{D_h \text{Re}_0} \quad (4.3a)$$

$$\bar{P}_0 = \frac{P_0}{\rho U^2} \quad \bar{E}_0 = \frac{E_0 D_h \text{Re}_0}{\zeta} \quad \Psi = \frac{z_v e_0 \psi}{k_b T} \quad \bar{H} = \frac{2 z_v e_0 n_0 \zeta}{\rho U^2} \quad (4.3b)$$

where U is the reference velocity, $D_h = \frac{4ab}{a+b}$ is the hydraulic diameter of the

rectangular channel, $\nu = \frac{\mu}{\rho}$ is the liquid kinematics viscosity, and $\text{Re}_0 = \frac{D_h U}{\nu}$ is

the reference Reynolds number.

Substituting Eqs. (2.4) and (4.2) into Eq. (4.1), we can obtain

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial X^2} + \frac{\partial^2 \bar{u}}{\partial Y^2} - \frac{d\bar{P}_0}{dZ} e^{i\left(\frac{\omega D_h^2 \bar{t}}{\nu} + \phi\right)} - \bar{H} \bar{E}_0 e^{\frac{i\omega D_h^2 \bar{t}}{\nu}} \sinh \Psi(X, Y) \quad (4.4)$$

The initial and boundary conditions applicable to Eq. (4.4) are

$$\bar{u}(X, Y, \bar{t})|_{\bar{t}=0} = 0 \quad (4.5a)$$

$$\left. \frac{\partial \bar{u}}{\partial X} \right|_{X=0} = 0 \quad \left. \frac{\partial \bar{u}}{\partial Y} \right|_{Y=0} = 0 \quad (4.5b)$$

$$\bar{u}|_{X=b/D_h} = 0 \quad \bar{u}|_{Y=a/D_h} = 0 \quad (4.5c)$$

An analytical solution for the dynamic electroosmotic flow in a closed-end rectangular microchannel under a time-dependent electric field will be obtained by

using the Green's function approach. According to the Green's function method [38], the solution of Eq. (4.4) subjected to the initial and boundary conditions given in Eq.

(4.5) can be explicitly expressed as

$$\bar{u}(X, Y, \bar{t}) = REAL \left[- \int_{\tau=0}^{\bar{t}} d\tau \int_{X'=0}^{b/D_h} \int_{Y'=0}^{a/D_h} G(X, Y, \bar{t} | X', Y', \tau) \times \left(\frac{d\bar{P}_0}{dZ} e^{i \left(\frac{\omega D_h^2 \tau}{\nu} + \phi \right)} + \bar{H}\bar{E}_0 e^{\frac{i\omega D_h^2 \tau}{\nu}} \sinh \Psi(X', Y') \right) dX' dY' \right] \quad (4.6)$$

Here, the Green's function $G(X, Y, \bar{t} | X', Y', \tau)$ can be obtained by using the separation of variables method [39].

4.3 Green's Function Formulation

The Green's function needed to solve Eq. (4.6) can be obtained by solving the corresponding homogeneous equation of Eq. (4.4), which is:

$$\frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial Y^2} = \frac{\partial \phi}{\partial \bar{t}}$$

with the boundary conditions of:

$$\left. \frac{\partial \phi}{\partial X} \right|_{X=0} = 0 \quad \left. \frac{\partial \phi}{\partial Y} \right|_{Y=0} = 0 \quad \phi \left(\frac{b}{D_h}, Y \right) = 0 \quad \phi \left(X, \frac{a}{D_h} \right) = 0 \quad (4.8a)$$

and the initial condition:

$$\phi(X, Y, \bar{t})|_{\bar{t}=0} = R(X, Y) \quad (4.8b)$$

Using the separation of variables method, the solution of Eq. (4.7) is

$$\phi = F(X)G(Y)H(\bar{t}) \quad (4.9)$$

Substituting Eq. (4.9) into Eq. (4.7) gives

$$\frac{F''}{F} + \frac{G''}{G} = \frac{\dot{H}}{H} = \text{const} \quad (4.10)$$

The solutions for $F(X)$ and $G(Y)$ are:

$\frac{F''}{F} = -\alpha^2 \Rightarrow F'' + \alpha^2 F = 0$ $F = a_1 \cos \alpha X + a_2 \sin \alpha X$ $F' = -a_1 \alpha \sin \alpha X + a_2 \alpha \cos \alpha X$ $F'(0) = 0 \rightarrow a_2 = 0$ $F\left(\frac{b}{D_h}\right) = a_1 \cos \frac{\alpha b}{D_h} = 0$ $\alpha_m = \left(\frac{2m-1}{2}\right) \frac{D_h}{b} \pi$ $F_m(X) = \sum_{m=1}^{\infty} a_m \cos \left[\left(\frac{2m-1}{2b} \right) D_h \pi X \right] \quad (4.11)$	$\frac{G''}{G} = -\beta^2 \Rightarrow G'' + \beta^2 G = 0$ $G = a_1 \cos \beta X + a_2 \sin \beta X$ $G' = -a_1 \beta \sin \beta X + a_2 \beta \cos \beta X$ $G'(0) = 0 \rightarrow a_2 = 0$ $G\left(\frac{a}{D_h}\right) = a_1 \cos \frac{\beta a}{D_h} = 0$ $\beta_n = \left(\frac{2n-1}{2}\right) \frac{D_h}{a} \pi$ $G_n(Y) = \sum_{n=1}^{\infty} a_n \cos \left[\left(\frac{2n-1}{2a} \right) D_h \pi Y \right] \quad (4.12)$
--	--

And finally, the solution of $H(\bar{t})$ is

$$\frac{\dot{H}}{H} = -\alpha^2 - \beta^2 \Rightarrow \dot{H} + (\alpha^2 + \beta^2)H = 0$$

$$H(\bar{t}) = \exp \left[-\frac{D_h^2 \pi^2}{4} \left(\frac{(2m-1)^2}{b^2} + \frac{(2n-1)^2}{a^2} \right) \bar{t} \right] \quad (4.13)$$

Hence, the solution of $\phi(X, Y, \bar{t})$ is:

$$\phi(X, Y, \bar{t}) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \exp \left[-\frac{D_h^2 \pi^2}{4} \left(\frac{(2m-1)^2}{b^2} + \frac{(2n-1)^2}{a^2} \right) \bar{t} \right] \times \cos \left[\left(\frac{2m-1}{2} \right) \frac{D_h}{b} \pi X \right] \cos \left[\left(\frac{2n-1}{2} \right) \frac{D_h}{a} \pi Y \right] \quad (4.14)$$

Using the initial condition, we have

$$\phi(X', Y', 0) = R(X', Y') = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \cos\left[\left(\frac{2m-1}{2}\right) \frac{D_h}{b} \pi X'\right] \cos\left[\left(\frac{2n-1}{2}\right) \frac{D_h}{a} \pi Y'\right]$$

Let $K_m = \sum_{n=1}^{\infty} B_{mn} \cos\left[\left(\frac{2n-1}{2}\right) \frac{D_h}{a} \pi Y'\right]$ and simplifying, one can obtain:

$$R(X', Y') = \sum_{m=1}^{\infty} K_m \cos\left[\left(\frac{2m-1}{2}\right) \frac{D_h}{b} \pi X'\right]$$

$$K_m = \frac{2D_h}{b} \int_0^{b/D_h} R(X', Y') \cos\left[\left(\frac{2m-1}{2}\right) \frac{D_h}{b} \pi X'\right] dX'$$

and

$$B_{mn} = \frac{2D_h}{a} \int_0^{a/D_h} K_m \cos\left[\left(\frac{2n-1}{2}\right) \frac{D_h}{a} \pi Y'\right] dY'$$

$$B_{mn} = \frac{4D_h^2}{ab} \int_{Y'=0}^{a/D_h} \int_{X'=0}^{b/D_h} R(X', Y') \cos\left[\left(\frac{2m-1}{2}\right) \frac{D_h}{b} \pi X'\right] \cos\left[\left(\frac{2n-1}{2}\right) \frac{D_h}{a} \pi Y'\right] dX' dY'$$

Substituting B_{mn} to Eq. (4.14) leads to

$$\begin{aligned} \phi(X, Y, \bar{t}) &= \frac{4D_h^2}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \exp\left[-\frac{D_h^2 \pi^2}{4} \left(\frac{(2m-1)^2}{b^2} + \frac{(2n-1)^2}{a^2}\right) \bar{t}\right] \\ &\times \int_{Y'=0}^{a/D_h} \int_{X'=0}^{b/D_h} R(X', Y') \cos\left[\left(\frac{2m-1}{2}\right) \frac{D_h}{b} \pi X'\right] \cos\left[\left(\frac{2n-1}{2}\right) \frac{D_h}{a} \pi Y'\right] dX' dY' \\ &\times \cos\left[\left(\frac{2m-1}{2}\right) \frac{D_h}{b} \pi X\right] \cos\left[\left(\frac{2n-1}{2}\right) \frac{D_h}{a} \pi Y\right] \end{aligned} \quad (4.15)$$

Using the Green's function formulation, the solution of Eq. (4.7) is given by

$$\phi(X, Y, \bar{t}) = \int_{X'=0}^{b/D_h} \int_{Y'=0}^{a/D_h} G(X, Y, \bar{t} | X', Y', \tau) \Big|_{\tau=0} R(X', Y') dX' dY' \quad (4.16)$$

Hence, by comparing Eq. (4.15) and Eq. (4.16), we can obtain the expression for the Green's function which is given by

$$\begin{aligned}
 G(X, Y, \bar{t} | X', Y', \tau) = & 4 \frac{D_h^2}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \exp \left[-\frac{D_h^2 \pi^2}{4} \left(\frac{(2m-1)^2}{b^2} + \frac{(2n-1)^2}{a^2} \right) (\bar{t} - \tau) \right] \\
 & \times \cos \left[\left(\frac{2m-1}{2} \right) \frac{D_h}{b} \pi X \right] \cos \left[\left(\frac{2m-1}{2} \right) \frac{D_h}{b} \pi X' \right] \\
 & \times \cos \left[\left(\frac{2n-1}{2} \right) \frac{D_h}{a} \pi Y \right] \cos \left[\left(\frac{2n-1}{2} \right) \frac{D_h}{a} \pi Y' \right]
 \end{aligned} \tag{4.17}$$

4.4 Velocity field in Closed-end Rectangular Microchannels

Having obtained the Green's function of the governing equation of motion, we can proceed to solve Eq. (4.6). Substituting Eq.(4.17) into Eq. (4.6), we can obtain the non-dimensional fluid velocity distribution in the microchannel

$$\begin{aligned}
 \bar{u}(X, Y, \bar{t}) = REAL \left[\right. & -\frac{64}{\pi^2 D_h^2} \frac{d\bar{P}_0}{dZ} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} \frac{e^{i \left(\frac{\omega D_h^2 \bar{t}}{\nu} + \phi \right)} - e^{-\left[\frac{D_h^2}{4} R_{mn} \bar{t} - i \phi \right]}}{(2m-1)(2n-1) \left[R_{mn} + i \frac{4\omega}{\nu} \right]} \\
 & \times \cos \left[\left(\frac{2m-1}{2} \right) \frac{D_h}{b} \pi X \right] \cos \left[\left(\frac{2n-1}{2} \right) \frac{D_h}{a} \pi Y \right] \\
 & - \frac{16}{ab} \bar{H} \bar{E}_0 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \frac{e^{\frac{i \omega D_h^2 \bar{t}}{\nu}} - e^{-\left[\frac{D_h^2}{4} R_{mn} \bar{t} \right]}}{\left[R_{mn} + i \frac{4\omega}{\nu} \right]} \\
 & \times \cos \left[\left(\frac{2m-1}{2} \right) \frac{D_h}{b} \pi X \right] \cos \left[\left(\frac{2n-1}{2} \right) \frac{D_h}{a} \pi Y \right] \left. \right] \tag{4.18}
 \end{aligned}$$

where

$$C_{mn} = \int_{X'=0}^{b/D_h} \int_{Y'=0}^{a/D_h} \cos\left[\left(\frac{2m-1}{2}\right)\frac{D_h}{b}\pi X'\right] \cos\left[\left(\frac{2n-1}{2}\right)\frac{D_h}{a}\pi Y'\right] \sinh \Psi(X', Y') dX' dY' \quad (4.19)$$

$$R_{mn} = \pi^2 \left[\frac{(2m-1)^2}{b^2} + \frac{(2n-1)^2}{a^2} \right] \quad (4.20)$$

Equation (4.18) shows that the flow in a closed-end rectangular microchamel actuated by an applied oscillating electric field consists of simple linear superposition of two parts: (i) the flow due to the applied electric field, and (ii) the flow due to the induced backpressure gradient. In this study, the steadily oscillating electroosmotic flow field under an AC electric field is being considered. In this situation, the exponential term on the right-hand side of Eq. (4.18) decays, and hence Eq. (4.18) is reduced to

$$\bar{u}(X, Y, \bar{t}) = REAL \left[\begin{aligned} & -\frac{64}{\pi^2 D_h^2} \frac{d\bar{P}(\bar{t})}{dZ} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n}}{(2m-1)(2n-1) \left(R_{mn} + i \frac{4\omega}{\nu} \right)} \\ & \times \cos\left[\left(\frac{2m-1}{2}\right)\frac{D_h}{b}\pi X\right] \cos\left[\left(\frac{2n-1}{2}\right)\frac{D_h}{a}\pi Y\right] \\ & -\frac{16}{ab} \bar{H}\bar{E}(\bar{t}) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{mn}}{\left(R_{mn} + i \frac{4\omega}{\nu} \right)} \\ & \times \cos\left[\left(\frac{2m-1}{2}\right)\frac{D_h}{b}\pi X\right] \cos\left[\left(\frac{2n-1}{2}\right)\frac{D_h}{a}\pi Y\right] \end{aligned} \right] \quad (4.21)$$

where

$$\frac{d\bar{P}(\bar{t})}{dZ} = \frac{d\bar{P}_0}{dZ} e^{i\left(\frac{\omega D_h^2 \bar{t}}{\nu} + \phi\right)} \quad \bar{E}(\bar{t}) = \bar{E}_0 e^{\frac{i\omega D_h^2 \bar{t}}{\nu}} \quad (4.22)$$

For a closed-end rectangular microchannel, the net flow rate equals to zero.

Mathematically, it fulfills the following condition,

$$\int_{-b/D_h}^{b/D_h} \int_{-a/D_h}^{a/D_h} \bar{u}(X, Y, \bar{t}) dXdY = 0 \quad (4.23)$$

Thus, the induced pressure gradient expression for a closed-end system can be obtained as

$$\frac{d\bar{P}(\bar{t})}{dZ} = -\frac{\pi^2 D_h^2 \bar{H}\bar{E}(\bar{t})}{4ab} \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n} C_{mn}}{(2m-1)(2n-1) \left(R_{mn} + i \frac{4\omega}{v} \right)}}{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(2m-1)^2 (2n-1)^2 \left(R_{mn} + i \frac{4\omega}{v} \right)}} \quad (4.24)$$

Note that Eq. (4.24) is expressed in the complex variable form. By separating the Real and Imaginary part of Eq. (4.24), we can obtain the amplitude and the phase angle of the induced backpressure which is given by:

$$\frac{d\bar{P}(\bar{t})}{dZ} = -\frac{\pi^2 D_h^2 \bar{H}\bar{E}(\bar{t})}{4ab} \left[\frac{\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} N_{mn} R_{mn} \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} R_{mn} \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} N_{mn} \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} D_{mn} \right)}{\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} R_{mn} \right)^2 + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} D_{mn} \right)^2} + i \frac{\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} N_{mn} R_{mn} \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} D_{mn} \right) - \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} N_{mn} \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} R_{mn} \right)}{\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} R_{mn} \right)^2 + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} D_{mn} \right)^2} \right] \quad (4.25)$$

where

$$N_{mn} = \frac{(-1)^{m+n} C_{mn}}{(2m-1)(2n-1) \left(R_{mn}^2 + \frac{16\omega^2}{v^2} \right)} \quad D_{mn} = \frac{1}{(2m-1)^2 (2n-1)^2 \left(R_{mn}^2 + \frac{16\omega^2}{v^2} \right)} \quad (4.26)$$

Substituting Eq. (4.22) into Eq. (4.25) and applying the Euler's formula in complex analysis, we can obtain

$$\frac{d\bar{P}_0}{dZ} \cos \phi = -\frac{\pi^2 D_h^2 \bar{H} \bar{E}_0}{4ab} \frac{\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} N_{mn} R_{mn} \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} R_{mn} \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} N_{mn} \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} D_{mn} \right)}{\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} R_{mn} \right)^2 + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} D_{mn} \right)^2} \quad (4.27)$$

$$\frac{d\bar{P}_0}{dZ} \sin \phi = -\frac{\pi^2 D_h^2 \bar{H} \bar{E}_0}{4ab} \frac{\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} N_{mn} R_{mn} \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} D_{mn} \right) - \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} N_{mn} \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} R_{mn} \right)}{\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} R_{mn} \right)^2 + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} D_{mn} \right)^2} \quad (4.28)$$

Manipulating Eqs. (4.27) and (4.28), we can obtain the amplitude, $\frac{dP_0}{dZ}$, and the phase angle, ϕ , of the induce backpressure given by:

$$\frac{d\bar{P}_0}{dZ} = \frac{\pi^2 D_h^2 \bar{H} \bar{E}_0}{4ab} \sqrt{\left[\frac{\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} N_{mn} R_{mn} \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} R_{mn} \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} N_{mn} \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} D_{mn} \right)}{\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} R_{mn} \right)^2 + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} D_{mn} \right)^2} \right]^2 + \left[\frac{\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} N_{mn} R_{mn} \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} D_{mn} \right) - \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} N_{mn} \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} R_{mn} \right)}{\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} R_{mn} \right)^2 + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} D_{mn} \right)^2} \right]^2} \quad (4.29)$$

$$\phi = \tan^{-1} \left[\frac{\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} N_{mn} R_{mn} \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} D_{mn} \right) - \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} N_{mn} \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} R_{mn} \right)}{\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} N_{mn} R_{mn} \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} R_{mn} \right) + \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} N_{mn} \right) \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\omega}{v} D_{mn} \right)} \right] \quad (4.30)$$

Taking advantage of the exact solutions obtained in this study, we examine Eq. (4.24) and find that the induced pressure gradient is proportional to the strength of applied

electric field, and the dimensionless parameter, $\bar{H}$. According to the definition for $\bar{H}$ in Eq. (4.3b), $\bar{H}$ physically denotes a measure of the ability to convert the electrical energy, $z_v e_0 n_0 \zeta$ to the fluid kinetic energy, $\rho U^2/2$. Furthermore, it is interesting to discuss another limiting situation that as $\omega \rightarrow \infty$, Eq. (4.24) is reduced to

$$\frac{d\bar{P}_0}{dZ} = -\frac{\pi^2 D_h^2 \bar{H} \bar{E}_0}{4ab} \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n} C_{mn}}{(2m-1)(2n-1)}}{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(2m-1)^2 (2n-1)^2}} \quad (4.31)$$

Also, as $\omega \rightarrow \infty$, Eq. (4.30) suggest that $\phi \rightarrow 0$. Similar to the case presented in Chapter 3 for cylindrical microchannels, Eq. (4.31) suggests that the induced pressure gradient under extremely high applied frequencies is independent of the external field frequency because the flow occurs only within the EDL region.

In another limiting situation, that is $\omega = 0$, the electric field then becomes a constant DC field with a magnitude of E_0 and the phase angle, ϕ reduces to zero and thus Eq. (4.24) reduces to

$$\frac{d\bar{P}_0}{dZ} = -\frac{\pi^2 D_h^2 \bar{H} \bar{E}_0}{4ab} \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n} C_{mn}}{(2m-1)(2n-1)R_{mn}}}{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(2m-1)^2 (2n-1)^2 R_{mn}}} \quad (4.32)$$

and the corresponding velocity distribution becomes

$$\begin{aligned} \bar{u}(X, Y) = & -\frac{64}{\pi^2 D_h^2} \frac{d\bar{P}_0}{dZ} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n}}{(2m-1)(2n-1)R_{mn}} \cos\left[\left(\frac{2m-1}{2}\right) \frac{D_h}{b} \pi X\right] \cos\left[\left(\frac{2n-1}{2}\right) \frac{D_h}{a} \pi Y\right] \\ & -\frac{16}{ab} \bar{H} \bar{E}_0 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{mn}}{R_{mn}} \cos\left[\left(\frac{2m-1}{2}\right) \frac{D_h}{b} \pi X\right] \cos\left[\left(\frac{2n-1}{2}\right) \frac{D_h}{a} \pi Y\right] \end{aligned} \quad (4.33)$$

4.5 Electrical Potential Distribution

Kang *et al.* [32] has shown that for microchannels with fully-developed electroosmotic flows under a time and frequency dependent electric field, the EDL potential distribution can still be described by the well-known Poisson-Boltzmann equation provided that the frequency of the electric field is not very high, say less than 1 MHz. The Poisson-Boltzmann equation in a rectangular domain is expressed as

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{2z_v e_0 n_0}{\epsilon_r \epsilon_0} \sinh\left(\frac{z_v e_0 \psi}{k_b T}\right) \quad (4.34)$$

By using the previously defined dimensionless parameters, we can non-dimensionalize Eq. (4.34) as

$$\frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Y^2} = K^2 \sinh \Psi \quad (4.35)$$

where $\kappa = \left(\frac{2z_v^2 e_0^2 n_0}{\epsilon_r \epsilon_0 k_b T}\right)^{1/2}$ is the Debye-Hückel parameter ($1/\kappa$ is normally referred to

as the EDL thickness), and $K = \kappa D_h$. Assuming that the zeta potential is small and

using the Debye-Huckel linear approximation, $\sinh \Psi \approx \Psi$, we can linearize Eq. (4.35)

to

$$\frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Y^2} = K^2 \Psi \quad (4.36)$$

Equation (4.36) is subjected to the following boundary conditions:

$$\left.\frac{\partial \Psi}{\partial X}\right|_{X=0} = 0 \quad \left.\frac{\partial \Psi}{\partial Y}\right|_{Y=0} = 0 \quad \Psi|_{X=b/D_h} = \bar{\zeta} \quad \Psi|_{Y=a/D_h} = \bar{\zeta} \quad (4.37)$$

where $\bar{\zeta}$ is the non-dimensional zeta potential, defined as $\bar{\zeta} = \frac{z_v e_0 \zeta}{k_b T}$.

Using the separation of variables method, we can solve Eq. (4.36), and the solution of Eq. (4.36) is expressed as [13-14,43-44]

$$\begin{aligned} \Psi(X, Y) = & 4\bar{\zeta} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cosh \left[\left(1 + \frac{(2n-1)^2 \pi^2 D_h^2}{4K^2 b^2} \right)^{1/2} KY \right] \cos \left[\frac{(2n-1)D_h}{2b} \pi X \right]}{(2n-1)\pi \cosh \left[\left(1 + \frac{(2n-1)^2 \pi^2 D_h^2}{4K^2 b^2} \right)^{1/2} \frac{Ka}{D_h} \right]} \\ & + 4\bar{\zeta} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \cosh \left[\left(1 + \frac{(2m-1)^2 \pi^2 D_h^2}{4K^2 a^2} \right)^{1/2} KX \right] \cos \left[\frac{(2m-1)D_h}{2a} \pi Y \right]}{(2m-1)\pi \cosh \left[\left(1 + \frac{(2m-1)^2 \pi^2 D_h^2}{4K^2 a^2} \right)^{1/2} \frac{Kb}{D_h} \right]} \end{aligned} \quad (4.38)$$

Certainly, the Debye-Hückel linear approximation used for solving the Poisson-Boltzmann equation is valid only when the zeta potential is small (usually less than 25mV). However, if one wants to study the flow behavior of a highly charged system where only the outer region of the diffuse double layer is important, the linear approximation can give a good prediction even when the zeta potential is as high as 100mV [4]; especially so when $\kappa D_h \gg 1$.

4.6 Solving the Double Integration (C_{mn})

Having obtained the electrical potential distribution, $\Psi(X, Y)$, and using the Debye-Hückel linear approximation, $\sinh \Psi \approx Y$, we can carry out the double integration, C_{mn} in Eq. (4.19) analytically, and obtain

$$C_{mn} = \int_{X'=0}^{b/D_h} \int_{Y'=0}^{a/D_h} \cos \left[\left(\frac{2m-1}{2} \right) \frac{D_h}{b} \pi X' \right] \cos \left[\left(\frac{2n-1}{2} \right) \frac{D_h}{a} \pi Y' \right] \Psi(X', Y') dX' dY' \quad (4.39)$$

Substituting Eq. (4.38) into Eq. (4.39), we can show that

$$\begin{aligned}
 C_{mn} = & 2\bar{\zeta} \frac{(-1)^{m+n} b}{(2m-1)\pi D_h \left(\frac{\left(1 + \frac{(2m-1)^2 \pi^2 D_h^2}{4K^2 b^2}\right) K^2}{\left(\frac{2n-1}{2}\right) \frac{D_h}{a} \pi} + \left(\frac{2n-1}{2}\right) \frac{D_h}{a} \pi \right)} \\
 & + 2\bar{\zeta} \frac{(-1)^{m+n} a}{(2n-1)\pi D_h \left(\frac{\left(1 + \frac{(2n-1)^2 \pi^2 D_h^2}{4K^2 a^2}\right) K^2}{\left(\frac{2m-1}{2}\right) \frac{D_h}{b} \pi} + \left(\frac{2m-1}{2}\right) \frac{D_h}{b} \pi \right)}
 \end{aligned} \tag{4.40}$$

4.7 Onsager's Principle of Reciprocity

Equation (4.21) shows that the flow in a closed-end rectangular microchannel is resulted from the applied electric field and the induced backpressure gradient. Further examination of Eq. (4.21) indicates the phenomenological relationships between the flow field and the driving electric field and the pressure gradient are linear, suggesting a possible constitution of the Onsager principle of reciprocity familiar in the theory of non-equilibrium thermodynamics. In the following, we will show the validity of the Onsager reciprocal relations for both transient electroosmotic flow and AC oscillating electroosmotic flow in a closed-end rectangular microchannel.

The flow rate $Q(t)$ in the rectangular microchannel can be obtained by:

$$Q(t) = \int_{-a-b}^a \int_{-b}^b u(x, y, t) dx dy \tag{4.41a}$$

Because of the symmetry, the flow rate can be further expressed in the form:

$$Q(t) = 4UD_h^2 \int_0^{a/D_h} \int_0^{b/D_h} \bar{u}(X, Y, \bar{t}) dXdY \quad (4.41b)$$

Substituting Eq. (4.21) into Eq. (4.41b), and making use of Eq. (4.3b) to convert the dimensionless pressure and electric field into the dimensional ones, we can obtain the flow rate as:

$$Q(t) = J_{11} \frac{dP(t)}{dz} + J_{12} E(t) \quad (4.42)$$

where the Onsager transport coefficients are

$$J_{11} = -\frac{1024ab}{\pi^4 \mu} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(2m-1)^2 (2n-1)^2 \left(R_{mn} + i \frac{4\omega}{\nu} \right)} \quad (4.43a)$$

$$J_{12} = -\frac{512z_v e_0 n_0 D_h^2}{\pi^2 \mu} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n} C_{mn}}{(2m-1)(2n-1) \left(R_{mn} + i \frac{4\omega}{\nu} \right)} \quad (4.43b)$$

As mentioned in Chapter 3, in general, the axial component of the electric current density is expressed as [40]

$$i_z(t) = uz_v e_0 (n_+ - n_-) - z_v e_0 \left(D_+ \frac{\partial n_+}{\partial z} + D_- \frac{\partial n_-}{\partial z} \right) + \lambda_0 E_z \quad (4.44)$$

where λ_0 is the electric conductivity of the fluid, and $D_{\pm}$ are the mass diffusivity of cations and anions, respectively. Since no ionic concentration gradient is built up along the axial direction during the flow, the second term on the right-hand of Eq. (4.44) drops off. The electric current across a straight rectangular channel can be expressed as

$$I(t) = 4 \int_0^a \int_0^b \rho_e u(x, y, t) dx dy + 4 a b \lambda_0 E_0 \quad (4.45)$$

With the Debye-Hückle linear approximation, substituting Eq. (2.4) into Eq. (4.45) gives

$$I(t) = -8 z_v e_0 n_0 U D_h^2 \int_0^{a/D_h} \int_0^{b/D_h} \Psi(X, Y) \bar{u}(X, Y, \bar{t}) dX dY + 4 \lambda_0 E_0 ab \quad (4.46)$$

Recall that the velocity field, $\bar{u}(X, Y, \bar{t})$ and the electrical potential distribution, $\Psi(X, Y)$ are given by Eq. (4.21) and Eq. (4.38), respectively. We then can show that the electric current can be written in the Onsager form by substituting Eqs. (4.21) and (4.38) into Eq. (4.46)

$$I(t) = J_{21} \frac{dP(t)}{dz} + J_{22} E(t) \quad (4.47)$$

where the Onsager transport coefficients are

$$J_{21} = \frac{512 z_v e_0 n_0 D_h^2}{\pi^2 \mu} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n} C_{mn}}{(2m-1)(2n-1) \left(R_{mn} + i \frac{4\omega}{v} \right)} \quad (4.48a)$$

$$J_{22} = \frac{256 z_v^2 e_0^2 n_0^2 D_h^4}{ab \mu} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{mn}^2}{\left(R_{mn} + i \frac{4\omega}{v} \right)} + 4 \lambda_0 ab \quad (4.48b)$$

From Eq. (4.43) and Eq.(4.48), it can be noted that $J_{12} = -J_{21}$, suggesting the validity of the Onsager principle of reciprocity for oscillating electroosmotic flow in a closed-end rectangular microchannel.

4.8 Slip Velocity Approximation

Conceptually, the steadily oscillating velocity profile induced by the application of a sinusoidal alternating electric field of a certain frequency is similar to the fluid flow induced by an oscillating motion of the channel wall of that same frequency. Fluid flow induced by moving boundary of the wall is also known as the Stokes second problem [41]. The idea of introducing the slip velocity approximation is to neglect the thin EDL region such that the entire flow is driven by the hydrodynamic shear stresses originated from the liquid viscosity. Here, the thin EDL region can be regarded as an oscillating moving channel wall. The outer slip boundary velocity is determined by the Smoluchowski equation, $U_s(t) = -\frac{\varepsilon_0 \varepsilon_r \zeta E_0 e^{i\omega t}}{\mu}$. Then the oscillating velocity profile can be obtained by solving the following transient Stokes equation, in a dimensionless form

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial X^2} + \frac{\partial^2 \bar{u}}{\partial Y^2} - \frac{d\bar{P}_0}{dZ} e^{i\left(\frac{\omega D_h^2 \bar{t}}{\nu} + \phi\right)} \quad (4.49)$$

which is subjected to the boundary conditions:

$$\left. \frac{\partial \bar{u}}{\partial X} \right|_{X=0} = 0 \quad \bar{u}\left(\frac{b}{D_h}, Y, \bar{t}\right) = \bar{U}_o \exp\left(\frac{i\omega D_h^2}{\nu} \bar{t}\right) \quad (4.50a)$$

$$\left. \frac{\partial \bar{u}}{\partial Y} \right|_{Y=0} = 0 \quad \bar{u}\left(X, \frac{a}{D_h}, \bar{t}\right) = \bar{U}_o \exp\left(\frac{i\omega D_h^2}{\nu} \bar{t}\right) \quad (4.50b)$$

$$\text{where } \bar{U}_o = -\frac{\varepsilon_0 \varepsilon_r \zeta E_0}{\mu U}.$$

Letting $\bar{u}(X, Y, \bar{t}) = v(X, Y, \bar{t}) + w(X, Y, \bar{t})$, the solution of Eq. (4.49) can be obtained by solving the two new partial differential equations of v and w as shown below:

The first PDE is

$$\frac{\partial v}{\partial \bar{t}} = \frac{\partial^2 v}{\partial X^2} + \frac{\partial^2 v}{\partial Y^2} \quad (4.51)$$

with the boundary conditions:

$$\left. \frac{\partial v}{\partial X} \right|_{X=0} = 0 \quad \left. \frac{\partial v}{\partial Y} \right|_{Y=0} = 0$$

$$v(b/D_h, Y, \bar{t}) = \bar{U}_o \exp\left(\frac{i\omega D_h^2 \bar{t}}{\nu}\right)$$

$$v(X, a/D_h, \bar{t}) = \bar{U}_o \exp\left(\frac{i\omega D_h^2 \bar{t}}{\nu}\right)$$

The second PDE is

$$\frac{\partial^2 w}{\partial X^2} + \frac{\partial^2 w}{\partial Y^2} - \frac{d\bar{P}_0}{dZ} e^{i\left(\frac{\omega D_h^2 \bar{t}}{\nu} + \phi\right)} = \frac{\partial w}{\partial \bar{t}} \quad (4.52)$$

with the boundary conditions:

$$\left. \frac{\partial w}{\partial X} \right|_{X=0} = 0 \quad \left. \frac{\partial w}{\partial Y} \right|_{Y=0} = 0$$

$$w(b/D_h, Y, \bar{t}) = 0$$

$$w(X, a/D_h, \bar{t}) = 0$$

Eq. (4.51) can be solved by using the separation of variables method,

$v = F_1(X)F_2(Y)H(\bar{t})$. Assuming $H(\bar{t}) = \exp\left(\frac{i\omega D_h^2 \bar{t}}{\nu}\right)$, we can show that

$$v(X, Y, \bar{t}) = REAL \left[\begin{aligned} & \frac{2D_h \bar{U}(\bar{t})}{b} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \cos(\alpha_m X) \cosh\left[\left(\alpha_m^2 + \frac{i\omega D_h^2}{\nu}\right)^{1/2} Y\right]}{\alpha_m \cosh\left[\left(\alpha_m^2 + \frac{i\omega D_h^2}{\nu}\right)^{1/2} \frac{a}{D_h}\right]} \\ & + \frac{2D_h \bar{U}(\bar{t})}{a} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(\beta_n Y) \cosh\left[\left(\beta_n^2 + \frac{i\omega D_h^2}{\nu}\right)^{1/2} X\right]}{\beta_n \cosh\left[\left(\beta_n^2 + \frac{i\omega D_h^2}{\nu}\right)^{1/2} \frac{b}{D_h}\right]} \end{aligned} \right] \quad (4.53)$$

where $\alpha_m = \left(\frac{2m-1}{2}\right) \frac{D_h}{b} \pi$, $\beta_n = \left(\frac{2n-1}{2}\right) \frac{D_h}{a} \pi$. i is the unit of imaginary number,

$\bar{U}(\bar{t}) = \bar{U}_o \exp\left(i \frac{\omega D_h^2 \bar{t}}{\nu}\right)$ and "REAL" represents the real part of the solution.

Using the Green's function formulation, we can show that the solution of Eq. (4.52)

is

$$w(X, Y, \bar{t}) = \text{REAL} \left[-\frac{64}{\pi^2 D_h^2} \frac{d\bar{P}(\bar{t})}{dZ} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n} \cos(\alpha_m X) \cos(\beta_n Y)}{(2m-1)(2n-1) \left[\pi^2 \left(\frac{(2m-1)^2}{b^2} + \frac{(2n-1)^2}{a^2} \right) + i \frac{4\omega}{\nu} \right]} \right] \quad (4.54)$$

$$\text{where } \frac{d\bar{P}(\bar{t})}{dZ} = \frac{d\bar{P}_0}{dZ} e^{i \left(\frac{\omega D_h^2 \bar{t}}{\nu} + \phi \right)}$$

Hence, the total solution of Eq. (4.49) is:

$$\bar{u}(X, Y, \bar{t}) = \text{REAL} \left[\begin{aligned} & \frac{2D_h \bar{U}(\bar{t})}{b} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \cos(\alpha_m X) \cosh \left[\left(\alpha_m^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2} Y \right]}{\alpha_m \cosh \left[\left(\alpha_m^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2} \frac{a}{D_h} \right]} \\ & + \frac{2D_h \bar{U}(\bar{t})}{a} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(\beta_n Y) \cosh \left[\left(\beta_n^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2} X \right]}{\beta_n \cosh \left[\left(\beta_n^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2} \frac{b}{D_h} \right]} \\ & - \frac{64}{\pi^2 D_h^2} \frac{d\bar{P}(\bar{t})}{dZ} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n} \cos(\alpha_m X) \cos(\beta_n Y)}{(2m-1)(2n-1) \left[\pi^2 \left(\frac{(2m-1)^2}{b^2} + \frac{(2n-1)^2}{a^2} \right) + i \frac{4\omega}{\nu} \right]} \end{aligned} \right] \quad (4.55)$$

For a closed channel system, $\bar{u}(X, Y, \bar{t})$ should fulfill Eq. (4.23) and the pressure gradient expression can be obtained as

$$\frac{d\bar{P}(\bar{t})}{dZ} = \frac{2D_h \bar{U}(\bar{t}) \left\{ \sum_{m=1}^{\infty} \frac{\tanh \left[\left(\alpha_m^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2} \frac{a}{D_h} \right]}{b\alpha_m^2 \left(\alpha_m^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2}} + \sum_{n=1}^{\infty} \frac{\tanh \left[\left(\beta_n^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2} \frac{b}{D_h} \right]}{a\beta_n^2 \left(\beta_n^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2}} \right\}}{\frac{256ab}{\pi^4 D_h^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(2m-1)^2 (2n-1)^2 \left[\pi^2 \left(\frac{(2m-1)^2}{b^2} + \frac{(2n-1)^2}{a^2} \right) + i \frac{4\omega}{\nu} \right]}} \quad (4.56)$$

4.9 Electroosmotic Vorticity Field

In this section, the vorticity field for AC oscillating electroosmotic flows in a closed-end cylindrical microchannel is presented. As presented in Chapter 3, vorticity, $\vec{\omega}$ is defined as the curl of the velocity vector, $\vec{V}$. For a flow through a rectangular channel, the vorticity field can be expressed as

$$\vec{\omega} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \hat{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \hat{k} \quad (4.57)$$

For a fully-developed electroosmotic flow, $V_x = V_y = 0$ and $V_z = u(x, y, t)$, hence Eq.

(4.57) reduces to

$$\vec{\omega} = \frac{\partial u}{\partial y} \hat{i} - \frac{\partial u}{\partial x} \hat{j} = \frac{U}{D_h} \left(\frac{\partial \bar{u}}{\partial Y} \hat{i} - \frac{\partial \bar{u}}{\partial X} \hat{j} \right) \quad (4.58)$$

where $\hat{i}$ and $\hat{j}$ are the unit vectors with respect to x and y directions, respectively.

Substituting Eq. (4.21) to Eq. (4.58) leads to

$$\vec{\omega} = \text{REAL}(\varpi_x \hat{i} + \varpi_y \hat{j}) \quad (4.59)$$

where

$$\begin{aligned} \varpi_x = & \frac{32U}{\pi a D_h^2} \frac{d\bar{P}(\bar{t})}{dZ} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n}}{(2m-1) \left[R_{mn} + i \frac{4\omega}{\nu} \right]} \\ & \times \cos \left[\left(\frac{2m-1}{2} \right) \frac{D_h}{b} \pi X \right] \sin \left[\left(\frac{2n-1}{2} \right) \frac{D_h}{a} \pi Y \right] \\ & + \frac{8\pi U}{a^2 b} \bar{H}\bar{E}(\bar{t}) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{mn}(2n-1)}{\left[R_{mn} + i \frac{4\omega}{\nu} \right]} \\ & \times \cos \left[\left(\frac{2m-1}{2} \right) \frac{D_h}{b} \pi X \right] \sin \left[\left(\frac{2n-1}{2} \right) \frac{D_h}{a} \pi Y \right] \end{aligned} \quad (4.60a)$$

$$\begin{aligned} \varpi_y = & -\frac{32U}{\pi b D_h^2} \frac{d\bar{P}(\bar{t})}{dZ} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n}}{(2n-1) \left[R_{mn} + i \frac{4\omega}{\nu} \right]} \\ & \times \sin \left[\left(\frac{2m-1}{2} \right) \frac{D_h}{b} \pi X \right] \cos \left[\left(\frac{2n-1}{2} \right) \frac{D_h}{a} \pi Y \right] \\ & - \frac{8\pi U}{ab^2} \bar{H}\bar{E}(\bar{t}) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{mn}(2m-1)}{\left[R_{mn} + i \frac{4\omega}{\nu} \right]} \\ & \times \sin \left[\left(\frac{2m-1}{2} \right) \frac{D_h}{b} \pi X \right] \cos \left[\left(\frac{2n-1}{2} \right) \frac{D_h}{a} \pi Y \right] \end{aligned} \quad (4.60b)$$

Under the slip velocity approximation approach, the vorticity field can be obtained by substituting Eq. (4.55) into Eq. (4.58), and it can be expressed by Eq. (4.59), where the vorticity components are given by:

$$\begin{aligned}
\varpi_x = & \frac{2U\bar{U}(\bar{t})}{b} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \cos(\alpha_m X) \sinh \left[\left(\alpha_m^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2} Y \right] \left(\alpha_m^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2}}{\alpha_m \cosh \left[\left(\alpha_m^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2} \frac{a}{D_h} \right]} \\
& - \frac{2U\bar{U}(\bar{t})}{a} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(\beta_n Y) \cosh \left[\left(\beta_n^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2} X \right]}{\cosh \left[\left(\beta_n^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2} \frac{b}{D_h} \right]} \\
& + \frac{64U}{\pi^2 D_h^3} \frac{d\bar{P}(\bar{t})}{dZ} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n} \cos(\alpha_m X) \sin(\beta_n Y) \beta_n}{(2m-1)(2n-1) \left[\pi^2 \left(\frac{(2m-1)^2}{b^2} + \frac{(2n-1)^2}{a^2} \right) + i \frac{4\omega}{\nu} \right]}
\end{aligned} \tag{4.61a}$$

$$\begin{aligned}
\varpi_y = & \frac{2U\bar{U}(\bar{t})}{b} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \sin(\alpha_m X) \cosh \left[\left(\alpha_m^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2} Y \right]}{\cosh \left[\left(\alpha_m^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2} \frac{a}{D_h} \right]} \\
& - \frac{2U\bar{U}(\bar{t})}{a} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(\beta_n Y) \sinh \left[\left(\beta_n^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2} X \right] \left(\beta_n^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2}}{\beta_n \cosh \left[\left(\beta_n^2 + \frac{i\omega D_h^2}{\nu} \right)^{1/2} \frac{b}{D_h} \right]} \\
& - \frac{64U}{\pi^2 D_h^3} \frac{d\bar{P}(\bar{t})}{dZ} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n} \sin(\alpha_m X) \cos(\beta_n Y) \alpha_m}{(2m-1)(2n-1) \left[\pi^2 \left(\frac{(2m-1)^2}{b^2} + \frac{(2n-1)^2}{a^2} \right) + i \frac{4\omega}{\nu} \right]}
\end{aligned} \tag{4.61b}$$

4.10 Results and Discussion

In a typical electroosmotic flow through an open-end channel, upon the application of the electric field, the flow begins to develop in the region close to the channel wall and the flow velocity increases rapidly from zero at the wall to its peak value within the EDL region. This scenario reveals the unique feature of the electroosmotic flow; it is driven by the electrical body force resulted from the interaction of the electric field and the net charge density. Such driving force exists only within the EDL region where the non-zero electrical charge density is present. The flow outside the EDL region is due to viscous shear diffusion. However, different from the electroosmotic flow in an open-end microchannel, there exists a reverse flow in the bulk liquid region in a closed-end microchannel. Such a reverse flow is generated by the induced backpressure to counter-balance the electroosmotic flow so that the net flow rate is zero.

In all the calculations, NaCl solution with valence z_v of unity is used as the electrolyte solution. The dielectric constant, ϵ_r , density, ρ , and viscosity, μ of the electrolyte are taken to be the same as those of water, and these are 80, 1000 kg/m^3 , and $9 \times 10^{-4} \text{ N-s/m}^2$, respectively. The temperature T of the electrolyte is taken as 293 K. The reference velocity U is chosen as 1 mm/s. The strength of the electric field E_0 is fixed as 10000 V/m. The reference geometric size of the channel is $2a \times 2b$ which is $40 \mu\text{m} \times 100 \mu\text{m}$.

4.10.1 Oscillating Velocity Distribution

For a rectangular channel cross section of $2a \times 2b = 40 \times 100 \mu m$, the characteristic time of the electroosmotic flow can be estimated from Eq. (4.18),

$t^* = \frac{4\rho}{\mu\pi^2} \left(\frac{a^2 b^2}{a^2 + b^2} \right) = 155.3 \mu s$. The corresponding eigen-frequency of the system is

$f^* = \frac{1}{t^*} = 6.44 \text{ kHz}$. In this work, three different frequencies of the applied electric

field, $0.1 f^*$, f^* and $10 f^*$ ($f = \omega/2\pi$) are considered to study the characteristics of the oscillating electroosmotic flow in a closed-end channel. The results are summarized in Figs. 4.2A-C which present the steadily oscillating velocity distributions in a symmetric quarter of the rectangular channel for five different characteristic

moments: $\omega t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ and π . Furthermore, the results of the electroosmotic

flow field under the slip velocity approximation are presented in Figs. 4.3A-C for comparison with the analytical method developed in this study.

Figure 4.2: Steadily oscillating velocity distributions along the dimensionless channel width and height for different frequencies of the external field: (A) $0.1 f^*$, (B) f^* and (C) $10 f^*$.

Results are obtained from the analytical scheme developed in this study.

Figure 4.3: Steadily oscillating velocity distributions along the dimensionless channel width and height for different frequencies of the external field: (A) $0.1 f^*$, (B) f^* and (C) $10 f^*$. Results are obtained from the slip velocity approximation.

In addition, to provide a quantitative comparison of the characteristics of the electroosmotic flow fields under the AC electric field for three different frequencies, two-dimensional plots corresponding to the cases presented in Figs. 4.2A-C and 4.3A-C are shown in Figs. 4.4A-C, in which the results obtained from both the analytical scheme (represented by the solid lines) and the slip velocity approximation (denoted by the dotted lines) are compared.

(A)

(B)

(C)

Figure 4.4: Two-Dimensional plots of steadily oscillating velocity distributions along the dimensionless channel height (at $x/D_h = 0$) for three different frequencies of the external field, (A) $0.1 f^*$, (B) f^* and (C) $10 f^*$.

As discussed earlier, when an electric field is applied, the electroosmotic flow begins within the EDL region near the charged channel wall. Because of the generation of the induced backpressure gradient in the closed-end channel, fluid at the central region flows in the opposite direction of that in the EDL region. The physical phenomena of electroosmotic flow in closed-end rectangular microchannels are very similar to that of cylindrical microchannels. As the frequency of the electric field is low (e.g., $0.1 f^*$ as shown in Figs. 4.2A and 4.3A), the flow extends through the entire channel and the changes in the flow frequency follows closely that of the alternating electric field. As the applied frequency increases, the perturbed flow region becomes smaller, and the flow response in the central region of the channel lags behind that of the fast alternating electric field (e.g., f^* as shown in Figs. 4.2B and 4.3B). While at a high applied frequency (e.g., $10 f^*$ as shown in Figs. 4.2C and 4.3C), the flow in the central channel remains virtually motionless regardless of the fast oscillating flow occurring within the EDL region which is close to the channel wall. The detailed discussion of this phenomenon is presented in section 3.9.1 of this report.

Figures 4.4A-C demonstrate that the slip velocity approximation method gives good estimation of velocity distributions at the central region of the channel, while discrepancy is observed in the EDL region. (Please refer to section 3.9.1 for a detailed discussion on slip velocity approximation)

4.10.2 Induced Pressure Gradient

The induced steadily oscillating pressure gradient in a closed-end rectangular microchannel is shown in Fig. 4.5. It can be noted that the induced pressure gradient response is always ahead of the applied sinusoidal electric field. The phase lead becomes larger as the frequency increases due to the effect of fluid inertia. As mentioned earlier, the response of the fluid at the central region of the channel is always slower compared to the fluid near the channel wall. Therefore to ensure a zero net flow rate, the induced pressure gradient has to be always ahead of the externally applied electric field. Moreover, in contrast to the dependence of fluid velocity on frequency, high backpressure is generated as the frequency goes up. This scenario is also clearly demonstrated in Fig. 4.6 which presents the amplitude of the induced backpressure gradient versus the frequency of applied electric field. (The explanation on this phenomenon is presented in section 3.9.2)

Figure 4.5: Induced pressure gradient under AC electric field versus characteristic moments from 0 to 4π .

Figure 4.6: Amplitude of induced pressure gradient under AC electric field versus the ratio of applied frequency to characteristic frequency (f/f^*).

Figure 4.7 presents the phase angle of the induced backpressure gradient to the externally applied electric field. It can be seen that as the external field frequency increases, the phase lead becomes larger when the frequency is smaller than $10 f^*$. However as the external frequency becomes larger than $10 f^*$, the phase lead starts to decrease. As discussed earlier, the phase lead phenomenon is caused by the inertia of the liquid. However, when the applied frequency becomes very large, there is virtually no flow at the central region of the channel, and thus results in a very small value of phase angle.

Figure 4.7: Phase angle of induced pressure gradient under AC electric field versus the ratio of applied frequency to characteristic frequency (f/f^*).

Figure 4.8: Steady-state induced pressure gradient under DC electric field versus dimensionless channel's hydraulic diameter with respect to a reference geometric size of $40 \times 100 \mu\text{m}$.

Figure 4.8 presents the effect of rectangular channel size (with fixed aspect ratio) on the induced pressure gradient under steady-state. The graph is plotted using log-log axis. The horizontal axis is a ratio of the actual channel hydraulic diameters to the reference hydraulic diameter of the rectangular channel of $2a \times 2b = 40 \times 100 \mu m$. It is shown in Fig. 4.8 that the pressure gradient decreases as the channel size increases. This observation is due to the fact that as the channel size increases, the cross-sectional area of the center core region increases, the driving pressure differential needed to effect the counter-flow in this central region decreases and hence the required backpressure gradient reduces. An alternative explanation to this observation is presented in section 3.9.2.

4.10.3 Vorticity Field

A representative three-dimensional vorticity plots for the case of low frequency ($0.1 f^*$) at characteristic moment of $\pi/2$ for ϖ_x , ϖ_y , and the total vorticity magnitude is presented in Figs. 4.9A, B and C respectively. It can be observed from Fig. 4.9A that along the center line $y/D_h = 0$ and at the wall $x/D_h = b/D_h$, the magnitude of $\varpi_x = 0$. Similarly, $\varpi_y = 0$ at $x/D_h = 0$ and $y/D_h = a/D_h$. This phenomenon is expected and can be explained from the analytical solutions obtained in Eq. (4.60), which defines the vorticity as the gradient of the electroosmotic velocity. It can also be seen that ϖ_x and ϖ_y are not symmetrical. The degree of symmetry is determined by the aspect ratio of the geometry. In a square channel (i.e., $a = b$), ϖ_x and ϖ_y become symmetrical.

Figure 4.9: Steadily oscillating vorticity distributions ($f = 0.1 f^*$, $\omega t = \pi/2$) along the dimensionless channel width and height. Results are obtained from the analytical scheme developed in this study. (A) vorticity in the x-direction, ϖ_x ; (B) vorticity in the y-direction, ϖ_y ; (C) the total vorticity magnitude, $|\vec{\varpi}| = \sqrt{\varpi_x^2 + \varpi_y^2}$

The three-dimensional vorticity plots ϖ_x , ϖ_y , and the total vorticity magnitude under the slip velocity approximation and the analytical scheme developed in this work are presented in Figs. 4.10-4.12 and Figs. 4.13-4.15 respectively.

Figure 4.10: Steadily oscillating vorticity distributions in the x-direction, ϖ_x along the dimensionless channel width and height for different frequencies of the external field: (A) $0.1 f^*$, (B) f^* and (C) $10 f^*$. Results are obtained from the slip velocity approach.

Figure 4.12: Steadily oscillating the total vorticity magnitude distribution, $|\vec{\omega}| = \sqrt{\omega_x^2 + \omega_y^2}$ along the dimensionless channel width and height for different frequencies of the external field: (A) $0.1 f^*$, (B) f^* and (C) $10 f^*$. Results are obtained from the slip velocity approach.

Figure 4.14: Steadily oscillating velocity distributions in the y-direction, w_y , along the dimensionless channel width and height for different frequencies of the external field: (A) $0.1 f^*$, (B) f^* and (C) $10 f^*$. Results are obtained from the analytical approach.

Figure 4.15: Steadily oscillating the total vorticity magnitude distribution, $|\vec{\omega}| = \sqrt{\omega_x^2 + \omega_y^2}$ along the dimensionless channel width and height for different frequencies of the external field: (A) $0.1 f^*$, (B) f^* and (C) $10 f^*$. Results are obtained from the analytical approach.

(a)

(b)

(c)

Figure 4.16: Oscillating electroosmotic vorticity distributions, ϖ_x , along the dimensionless channel height (at $x/D_h = 0$) for three different frequencies of the external field under the slip velocity approximation. (a) $f = 0.1f^*$; (b) $f = f^*$; (c) $f = 10f^*$

Figures 4.16a-c show the two-dimensional oscillating vorticity field, ϖ_x , predicted from the slip velocity approximation along the dimensionless channel height (at $x/D_h = 0$) for three different frequencies, $0.1 f^*$, f^* and $10 f^*$, respectively. Since the electroosmotic vorticity is defined as the gradient of the electroosmotic velocity (see Eq. (4.58)), the dependence of the vorticity field on the applied frequency is very similar to that of the velocity field. It can be seen that at a low applied frequency (Fig. 4.16a), the vorticity field is generated throughout the entire channel, indicating that the flow is rotational everywhere. As the applied frequency increases, the rotational flow region is reduced. For instance, Fig. 4.16c shows the vorticity field occurs only

within about one third of the region close to the channel wall. It should also be noted that the vorticity is stronger at a higher applied frequency. For the purpose of comparison, Figs. 4.17a-c show the vorticity fields computed from the analytical solution derived in this study, corresponding to the same conditions of Figs. 4.16a-c, respectively. However, only a small portion nearby the channel wall is presented here because, similar to the velocity fields, both the analytical solution and the slip velocity approximation give the same vorticity fields in the rest of the region of the channel. A comprehensive discussion on this phenomenon is provided in section 3.9.3.

(a)

Figure 4.17: Oscillating electroosmotic vorticity distributions, ϖ_x , along the dimensionless channel height (at $x/D_h = 0$) in the region near the channel wall for three different frequencies of the external field under the analytical scheme developed in this work

(a) $f = 0.1f^*$; (b) $f = f^*$; (c) $f = 10f^*$

4.11 Summary

The theory of AC electroosmotic flow in a closed-end rectangular microchannel is presented. The Onsager's principle of reciprocity of the AC electroosmotic flows is demonstrated. The analyses show that the flow in a closed-end channel is the superposition of the electroosmosis in the EDL region and the reverse pressure-driven flow outside the EDL region (i.e., the bulk fluid region). It is shown that the AC electroosmotic flow is strongly dependent on the frequency of the applied sinusoidal electric field. It is identified that it is the frequency of electric field that determines the thickness of the Stokes penetration layer, and thus governs the extent of the oscillating flow and the vorticity fields. Furthermore, it is found that the amplitude of the induced back pressure increases with increasing the frequency of applied electric field. The induced pressure gradient response is always ahead of the externally applied electric field and the phase lead increases as the frequency increases until the frequency becomes $10f^*$. Beyond this frequency, the phase lead starts to decrease. In addition, it is shown that there is a similarity of the velocity and vorticity fields between the slip velocity approximation and the present analytical solution in the bulk fluid region, while in the EDL region, the discrepancy is observed. This leads to the applicability of the second Stokes solution to the oscillating electroosmotic flow by use of the instantaneous Smoluchowski velocity as appropriate slip boundary condition when $\kappa D_h \gg 1$.

CHAPTER 5

Developing Electroosmotic Flow in Closed-End Microchannels

5.1 Introduction

This chapter discusses a numerical study on the steady-state developing electroosmotic flow in closed-end cylindrical microchannels. To simulate the flow, a mathematical model, which includes the Poisson-Boltzmann equation describing the EDL distribution and the modified two-dimensional Navier-Stokes equations governing the velocity field, is presented. The governing equations are discretized using the control volume integration method, and the staggered grid system is employed to solve the Navier-Stokes equations. The results of the spatial development of the flow and the pressure distributions along the microchannel are presented. Parametric studies of the effects of the channel size and the electric field strength on electroosmotic flows in closed-end microchannels are conducted. In addition, the numerical simulations have been validated with the analytical solutions in the fully-developed region.

5.2 Problem Formulation

Figure 5.1: Geometry of cylindrical microchannel with radius a and length L

Consider a steady-state, laminar, developing electroosmotic flow in a closed-end cylindrical microchannel of radius a and length L as shown in Fig. 5.1. The channel wall is uniformly charged with a zeta potential, ζ . The liquid in the channel is assumed to be Newtonian, incompressible and with a symmetric electrolyte with constant hydrodynamic, thermal and electrical properties. In this section, the mathematical model describing the governing equations is presented. The temperature is assumed to be uniform throughout the flow and Joule heating effects are neglected.

5.2.1 Electrical Potential Distribution

As described earlier in Chapter 2, when a solid surface is in contact with a polar medium, the surface usually acquires electrostatic charges. Due to the interactions of the polar medium and the solid surface, the EDL is formed. Since both ends of the channel are closed, the distribution of the electrical potential is two-dimensional. However, in this study, both ends of the channel are modeled as electrodes and we assume that there is no EDL formation at the ends of the channel. Therefore, the

EDL potential distribution can be treated to be one-dimensional along the radial direction, $\psi(r)$, and is governed by the Poisson-Boltzmann equation given by:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) = \frac{2z_v e_0 n_0}{\varepsilon_r \varepsilon_0} \sinh \left(\frac{z_v e_0 \psi}{k_b T} \right) \quad (5.1)$$

where ε_r is the dielectric constant of the solution and ε_0 is the permittivity of the vacuum, z_v is the valence of the solution, e_0 is the elementary charge, n_0 is the ionic concentration in the bulk solution, k_b is the Boltzmann constant, and T is the absolute temperature.

Equation (5.1) can be non-dimensionalized by introducing the following dimensionless parameters:

$$\bar{r} = \frac{r}{a} \quad \Psi = \frac{z_v e_0 \psi}{k_b T} \quad (5.2)$$

Substituting Eq. (5.2) into Eq. (5.1), and using the Debye-Hückel parameter

$$\kappa = \left(\frac{2z_v^2 e_0^2 n_0}{\varepsilon_r \varepsilon_0 k_b T} \right)^{1/2} \quad (1/\kappa \text{ is normally referred to as the EDL thickness}), \text{ equation (5.3) is}$$

obtained.

$$\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\bar{r} \frac{\partial \Psi}{\partial \bar{r}} \right) = (\kappa a)^2 \sinh \Psi \quad (5.3)$$

Equation (5.3) is subjected to the following boundary conditions:

$$\left. \frac{\partial \Psi}{\partial \bar{r}} \right|_{\bar{r}=0} = 0 \quad \Psi|_{\bar{r}=1} = \bar{\zeta} \quad (5.4)$$

where $\bar{\zeta}$ is the non-dimensional zeta potential which is defined as $\bar{\zeta} = \frac{z_v e_0 \zeta}{k_b T}$.

5.2.2 Equation of Motion

The motion of liquid in a laminar, incompressible, steady electroosmotic flow through a closed-end cylindrical microchannel can be described by the modified Navier-Stokes equations given by

$$\rho \left(v \frac{\partial u}{\partial r} + u \frac{\partial u}{\partial z} \right) = -\frac{\partial P}{\partial z} + E_0 \rho_e + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} \right] \quad (5.5)$$

$$\rho \left(v \frac{\partial v}{\partial r} + u \frac{\partial v}{\partial z} \right) = -\frac{\partial P}{\partial r} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} \right] \quad (5.6)$$

where ρ is the liquid density, u and v are the velocity of EOF in the axial and radial directions, respectively, P is the induced backpressure, E_0 is the strength of the applied electric field, ρ_e is the local net charge density due to the presence of EDL and μ is the dynamic viscosity of the liquid.

These governing equations can be non-dimensionalized by using the previously defined parameters given in Eq. (5.2) and introducing the following dimensionless parameters

$$\bar{z} = \frac{z}{a} \quad \bar{u} = \frac{u}{U} \quad \bar{v} = \frac{v}{U} \quad \text{Re}_0 = \frac{\rho a U}{\mu} \quad (5.7a)$$

$$\bar{P}_0 = \frac{P - P_{ref}}{\rho U^2} \quad \bar{E}_0 = \frac{E_0 a}{\zeta} \quad \bar{H} = \frac{2 z_v e_0 n_0 \zeta}{\rho U^2} \quad (5.7b)$$

where U is the reference velocity, Re_0 is the reference Reynolds number, P_{ref} is the reference pressure, and $\bar{H}$ represents a measure of the ability to convert the electrical energy $z_v e_0 n_0 \zeta$, to the fluid kinetic energy, $\rho U^2/2$.

Substituting Eqs. (2.4), (5.2) and (5.7) into Eqs. (5.5) and (5.6) , we can obtain the dimensionless form of the governing equations.

$$\bar{v} \frac{\partial \bar{u}}{\partial \bar{r}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{z}} = -\frac{\partial \bar{P}_0}{\partial \bar{z}} - \bar{H}\bar{E}_0 \sinh(\Psi) + \frac{1}{\text{Re}_0} \left[\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\bar{r} \frac{\partial \bar{u}}{\partial \bar{r}} \right) + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2} \right] \quad (5.8)$$

$$\bar{v} \frac{\partial \bar{v}}{\partial \bar{r}} + \bar{u} \frac{\partial \bar{v}}{\partial \bar{z}} = -\frac{\partial \bar{P}_0}{\partial \bar{r}} + \frac{1}{\text{Re}_0} \left[\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\bar{r} \frac{\partial \bar{v}}{\partial \bar{r}} \right) - \frac{\bar{v}}{\bar{r}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{z}^2} \right] \quad (5.9)$$

Equations (5.8) and (5.9) are subjected to the following boundary conditions:

$$\left. \frac{\partial \bar{u}}{\partial \bar{r}} \right|_{\bar{r}=0} = 0 \quad \bar{u} \Big|_{\bar{r}=1} = 0 \quad \left. \frac{\partial \bar{v}}{\partial \bar{r}} \right|_{\bar{r}=0} = 0 \quad \bar{v} \Big|_{\bar{r}=1} = 0 \quad (5.10)$$

and for the two-closed ends, these are,

$$\bar{u} \Big|_{\bar{z}=0} = 0 \quad \bar{u} \Big|_{\bar{z}=\frac{L}{a}} = 0 \quad \bar{v} \Big|_{\bar{z}=0} = 0 \quad \bar{v} \Big|_{\bar{z}=\frac{L}{a}} = 0 \quad (5.11)$$

Following the momentum equations, the continuity equation must also be satisfied. For a two dimensional axi-symmetry flow, the continuity equation in non-dimensional form is given as

$$\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} (\bar{r} \bar{v}) + \frac{\partial \bar{u}}{\partial \bar{z}} = 0 \quad (5.12)$$

5.3 Numerical Method

In this study, the numerical scheme used to solve the governing equations is based on the finite volume method of Patankar [45]. The pressure-velocity coupling in the Navier-Stokes equation is resolved using the Semi-Implicit Method for Pressure Linked Equations Revised (SIMPLER) algorithm. In the calculation, the discrete algebraic equations are solved by employing the line-by-line Tridiagonal Matrix

Algorithm scheme. The nonlinear source terms are linearized using the Taylor's series expansion. A grid-independent test is carried out to ensure that the chosen grid system can produce grid-independent solutions. In the following section, a brief discussion on SIMPLER algorithm is presented.

5.3.1 General Differential Equation

SIMPLER algorithm can be used to solve fluid flow and heat transfer problems, since the governing laws of these processes are expressed in terms of conservation equations. All these equations can be expressed in a general solution, known as the general convective-diffusive differential equation, expressed in Cartesian-tensor form given by

$$\frac{\partial}{\partial t}(\rho\phi) + \frac{\partial}{\partial x_i}(\rho u_i \phi) = \frac{\partial}{\partial x_i} \left(\Gamma \frac{\partial \phi}{\partial x_i} \right) + S \quad (5.13)$$

where ϕ denotes a general dependent variable, Γ represents the diffusion coefficient and S stands for the source term. The subscript i denotes the Einstein's rule of summation.

The terms on the left hand side are the unsteady term and the convection term while the right hand side denotes the diffusion term and the source term. The variable ϕ can stand for various quantities, such as a velocity component, temperature, etc. For each definition of the variable, an appropriate expression for Γ and S will be specified.

The source term is often a function of the dependent variable ϕ . For this purpose, it is desirable to express a linear dependence of S on ϕ . Hence linearization is a must

when the source term is not a linear function of ϕ . Here, the source term can be expressed as:

$$S = S_C + S_P \phi \tag{5.14}$$

where S_C is the constant part of S and S_P is the coefficient of ϕ . To ensure convergence of the source term linearization, the positive value of S_P must always be ensured.

5.3.2 Discretization

In the FVM, the domain of interest is divided into a number of control volumes (CVs). Figure 5.2 shows the basic construction of the control volumes and the grid in axisymmetric system. A grid point is located at the centre of every CV. Integration of the governing equations is performed over every CV to derive algebraic equations relating the value of ϕ of that CV to those of the neighbouring CVs. The derived algebraic equations express the conservation principle for the CV.

Figure 5.2: Control volumes

The shaded region in Fig. 5.2 represents a control volume. An enlarge drawing of a typical control volume is presented in Fig. 5.3. For the CV labeled P, the neighbouring nodes are denoted as W, E, N and S. This CV has four boundaries, denoted by e, w, n and s (with areas of A_w , A_e , A_s , and A_n respectively). Δz and Δr are the distances between CV boundaries. δz and δr are the distances between grid points.

Figure 5.3: A typical control volume and its neighbouring nodes.

Performing integration over each control volume gives the discretized equations in a general form as

$$a_P \phi_P = a_E \phi_E + a_W \phi_W + a_N \phi_N + a_S \phi_S + b \tag{5.15}$$

or in general, it can be written as

$$a_P \phi_P = \sum a_{nb} \phi_{nb} + b \tag{5.16}$$

where a is the coefficient for ϕ at the corresponding nodes, b is the source term, and the subscript nb denotes a neighbour grid of P and the summation is taken over the four neighbours.

Further theory and concept regarding SIMPLER algorithm can be obtained from cited reference [45].

5.4 Results and Discussions

In a conventional electroosmotic flow through an open-end microchannel, the flow starts to develop at the EDL region as the electrical body force mobilizes the ions at this region. Due to the hydrodynamic viscous effect, this movement of ions will drag the surrounding fluid with them, resulting in a bulk electroosmotic flow, which in its steady and fully developed state resembles a plug flow velocity profile as shown in Fig. 5.4a. In the case of an electroosmotic flow through a closed-end microchannel, as the fluid layer at the EDL region flow towards the end of the channel where the fluid velocity becomes zero, the fluid gradually slows down and thus induces pressure distributions in the channel. This causes the fluid to travel in the radial direction towards the centre region of the channel and flow through the centre core of the cross-section area to ensure zero net flow. The overall schematic of an electroosmotic flow through a closed-end microchannel is presented as the dashed arrow shown in Fig. 5.4b. The figure also shows the velocity profile for the flow in the fully developed region, where the two peaks near the wall are the flow due to electroosmosis, while at the center core region, the flow is due to the induced backpressure.

Figure 5.4: Schematic showing a fully-developed velocity profile of electroosmotic flow in (a) an open-end microchannel; (b) closed-end microchannel

In all the calculations, NaCl solution with valence z_v of unity is used as the electrolyte solution. The density, ρ , viscosity, μ , and dielectric constant, ϵ_r of the electrolyte are taken to be the same as those of water, and these are 1000 kg/m^3 , $9 \times 10^{-4} \text{ Ns/m}^2$, and 80, respectively. The temperature, T of the electrolyte is taken as 293 K. The reference velocity, U and the strength of the electric field, E_0 are chosen as 1 mm/s and 100 V/cm, respectively. The reference microchannel radius, a , length of the channel, L , and zeta potential are taken to be $10 \text{ }\mu\text{m}$, $100 \text{ }\mu\text{m}$ and -50 mV . The aspect ratio of the channel, L/a ($= 10$) is kept constant in all the calculations.

Figure 5.5: Velocity vector distributions along the dimensionless channel radius and length.

(a) Flow field throughout the microchannel; (b) Flow field near the end of the microchannel. $0 \leq \bar{z} \leq 1.5$

The velocity vector field of the closed-end electroosmotic flow in the upper half of the cylindrical microchannel is shown in Fig. 5.5a. A closer view of the velocity field near the channel's end is presented in Fig. 5.5b. Furthermore, to provide quantitative comparison of the electroosmotic flow fields at different axial locations, the axial and radial velocity plots along the radial direction corresponding to Fig. 5.5 are shown in Figs. 5.6 and 5.7, respectively. In addition, the result obtained from the analytical scheme for the fully-developed region (represented by circles) and the numerical scheme (denoted by solid lines) is compared in Fig. 5.6. The detailed mathematical derivation of the analytical scheme is presented in Chapter 3.

As mentioned earlier, upon the application of the electric field, the flow begins to develop in the axial direction at the region close to the channel wall. As the fluid approaches the end of the channel, the magnitude of the axial velocity reduces and the fluid begins to flow in the radial direction towards the center region of the channel. At the same time, the fluid attempts to flow in the reverse direction to that in the EDL region. This fluid will flow towards the leading end of the channel, where the pressure is lower. As it reaches the leading end of the channel, the fluid will flow in the radial direction towards the wall surface to complete a continuous flow. This scenario reveals the unique feature of the electroosmotic flow in a closed-end channel; within the EDL region, it is driven by the electrical body force resulted from the interaction of the electric field and the net charge density; and outside the EDL region, the flow is driven by the induced pressure gradient, to counter-balance the flow at the EDL region so that the net flow rate is zero. This phenomenon is clearly depicted in Fig. 5.5.

Figure 5.6: Axial velocity distributions along the dimensionless channel radius for various axial locations

Figure 5.7: Radial velocity distributions along the dimensionless channel radius for various axial locations

Figure 5.6 specifically presents the development of axial velocity from the end of the channel to the center region of the channel where the flow is fully developed. It can be seen that the axial velocity at the end equals to zero and as it moves away from the channel end, the velocity develops until it becomes fully developed in the centre length of the channel. Moreover, it is observed that the axial velocity field at an arbitrary distance z from the left-end is exactly the same as the axial velocity field at a distance z away from the right-end, or mathematically: $u(z, r) = u(L - z, r)$. Or simply, the axial velocity is found to be symmetrical along $z = L/2$. This is logically anticipated as in addition to the fact that the geometry of the channel is symmetrical and the boundary conditions at the ends of the channel are identical, electroosmotic flow is driven by the electrical body force which is acting at the EDL in the axial direction. Furthermore, the fully developed region is observed to exist at a location approximately 1.5 times of the channel's radius from either end of the channel. In addition, the analytical solution for the axial velocity at the fully developed region matches the results obtained under the numerical scheme developed in this work.

The development of the radial velocity field along the channel length is presented in Fig. 5.7. It is shown that the magnitude of the radial velocity is only prevalent near the ends of the channel, in the fully developed region, the radial velocity vanishes. This velocity distribution is expected because in the fully-developed region, the flow is one-dimensional, where only the axial velocity exists. It is also noticed that the radial velocity field at an arbitrary distance z from the left-end is equal in magnitude but opposite in direction to the same distance z measured from the right-end of the

channel. Mathematically, $v(z, r) = -v(L - z, r)$. As discussed earlier, in order to ensure a zero net flow rate across the channel, the fluid at the center region of the channel must flow in the reverse direction to that of the flow in the EDL near the channel wall. At one end of the channel, the fluid has to flow in the radial direction towards the center region of the channel. At the other end, the fluid has to flow in the radial direction towards the channel wall. In order to satisfy the conservation of mass, the magnitude of the flow flowing towards the wall has to be the same as the one flowing towards the center.

Figure 5.8a shows the dimensionless induced pressure contour throughout the microchannel. It can be seen that at the fully developed region, the induced pressure variation is present along the axial direction. The pressure distribution is two-dimensional only at the developing region, i.e. the region near the two ends of the channel. Figure 5.8b presents a quantitative comparison between the pressure distribution along the centerline of the microchannel (denoted by a dashed line) and along the channel wall (denoted by a solid line). Near the ends of the channel, the pressure at the centerline and the wall surface is not the same. These pressure differences are responsible for the creation of the radial flow. At a distance approximately 1.5 times the radius of the channel from either end of the channel, the solid line and the dashed line show the same distribution. This figure confirms that the fully-developed region occurs approximately at a distance 1.5 times of the channel's radius.

Figure 5.8a: Pressure contour along the dimensionless channel radius and length

Figure 5.8b: Pressure distribution along the dimensionless channel length at the centerline and at the channel wall

Furthermore, parametric studies have been conducted to study the effects of the channel radius and strengths of applied electric field on the flow. Figures 5.9 and 5.10 present (a) the axial and (b) the radial velocity distributions at various locations along the channel for the channel radius of 1 μm and 100 μm , respectively. The aspect ratio of the channel is kept constant, i.e. $L / a = 10$. Although the general velocity profile differs as the radius is changed, it is observed that the fully developed region exists at a location approximately $1.5 a$ away from either end of the channel.

Figure 5.11 shows the effect of the electric field strength on the axial velocity distribution at various locations along the channel. It can be seen that the strength of the electric field does not affect the shape of the velocity profile, and it only alters the absolute magnitude of the velocity. This is anticipated since according to the Smoluchowski equation [4], the velocity amplitude is linearly proportional to the strength of electric field.

(a)

(b)

Figure 5.9: Velocity distributions along the dimensionless channel radius for various axial locations with the channel radius, $a = 1 \mu\text{m}$, length, $L = 10 \mu\text{m}$. (a) Axial velocity field; (b) Radial velocity field.

Figure 5.10: Velocity distributions along the dimensionless channel radius for various axial locations with the channel radius, $a = 100 \mu\text{m}$, length, $L = 1 \text{ mm}$. (a) Axial velocity field; (b) Radial velocity field.

Figure 5.11: Velocity field under application of different strength of electric field for various axial locations with the channel radius, $a = 10 \mu\text{m}$, length, $L = 100 \mu\text{m}$. (a) Axial velocity field; (b) Radial velocity field.

It is interesting to observe that a change in the channel radius or the electric field strength does not contribute to any changes in the length of the developing region of the flow. This means that as long as the channel's length is at least 3 times more than the radius, there exists a fully developed region located at a distance approximately $1.5a$ away from either end of the channel. Contrary to the work presented by Yang et al. [37] for EOF through open-end channels, the effect of the strength of applied electric field in EOF through closed-end channels contributes only to the change in the magnitude of the velocity while the profile remains unchanged. Physically, the developing region of EOF in an open-end micro-channel is affected by the magnitude of the flow, because the boundary layers on the wall have not merged together. However, in a closed-end microchannel, the boundary layer everywhere along the channel has already merged due to the conservation of mass; that is the net flow rate across the cross section must be equal to zero. The definition of developing region in this work refers to how far the end effects become prevalent to the analysis before the fully-developed assumption is valid.

5.5 Summary

A numerical analysis of steady-state developing electroosmotic flows in closed-end cylindrical micro-channels is discussed. In this study, a mathematical model has been presented to evaluate the flow near the two ends of the channel. An analytical solution for the velocity field in the fully-developed region is compared with the numerical solution. The results obtained by the analytical solution shows to be in excellent agreement with the results obtained from the numerical model developed in this work. The analyses show that an electroosmotic flow in a closed-end channel is

the superposition of an electroosmotic flow in open-end channel and a reverse pressure driven flow due to the induced backpressure gradient. In the developing region near the channel ends, the pressure distribution is two-dimensional, and drives the radial component of the flow. In the fully-developed region, the pressure distribution is present only along the axial direction. Moreover, it is found that the strength of externally applied electric field does not contribute to the overall distribution of the flow but only the magnitude. It is noticed that the fully-developed region always exists at a distance $1.5a$ away from either ends of the channel.

CHAPTER 6

Conclusion and Recommendations

6.1 Concluding Remarks

Complete closed form analytical solutions to the fully-developed, laminar, frequency dependent electroosmosis velocity and vorticity flow field in closed-end cylindrical and rectangular microchannels under the application of alternating electric field have been obtained using the Green's function approach. The analytical solutions are used to investigate the effects of the channel size and modulation frequency on the transient responses of the electroosmotic flow. The analyses show that the flow in closed-end channels is the superposition of the electroosmosis in the EDL region and the reverse pressure-driven flow outside the EDL region (i.e., the bulk fluid region). The analyses also show that there are general phenomena similarities between the flow in cylindrical and rectangular geometries.

The oscillating electroosmotic flow is found to be strongly dependent on the frequency of the applied sinusoidal electric field. The results demonstrate that the higher the frequency, the smaller the perturbed flow region. Additionally, it is found that the amplitude of the induced back pressure remains almost a constant value when the applied frequency is less than the characteristic frequency, f^* . In this situation, the flow can extend to the entire cross-section of the channel, and the induced backpressure gradient is insensitive to the applied frequency. However, as

the frequency becomes larger than f^* , the amplitude of the oscillating backpressure gradient increases significantly. The induced pressure gradient response is always ahead of the externally applied electric field and the phase lead increases with frequency until the frequency becomes $10f^*$. Beyond this frequency, the phase lead starts to decrease.

The results obtained using the slip velocity approximation are used as comparison to the results obtained from the analytical method. The results show that the velocity and vorticity distributions in the center region of the channel are similar for both methods; while adjacent to the channel wall the discrepancy is observed. This leads to the applicability of the second Stokes solution to the oscillating electroosmotic flow by use of the instantaneous Smoluchowski velocity as the appropriate slip boundary condition, especially when the channel is relatively large.

From the vorticity field analysis, it can be concluded that electroosmosis flow in closed-end microchannels is rotational and the vorticity strength in the EDL region is found to be independent of the frequency of the externally applied electric field.

A numerical analysis for steady-state developing electroosmotic flows in closed-end cylindrical microchannels is also carried out. The numerical model has been successfully employed to evaluate the flow characteristics near the ends of the channel. The analyses show that at the developing region near the channel end, the pressure distribution is two-dimensional which drives the radial component of the flow. In the fully-developed region, the pressure distribution is present only along the

axial direction. Moreover, it is found that the strength of the externally applied electric field does not contribute to the general distribution of the overall flow field but the magnitude of the flow. It is noticed that the fully-developed flow region always exists at a distance $1.5a$ away from both ends of the channel.

6.2 Recommendations for Future Studies

Based on the results presented in this study, some recommendations are made for possible future studies in order to achieve a better and more complete understanding of electroosmotic flow in closed-end microchannels.

6.2.1 Time- and frequency-dependent developing electroosmotic flow in closed-end microchannels

In this study, the spatial development of a steady-state developing flow in closed-end microchannels has been presented and discussed. However, the transient development of the flow has yet to be studied. It would be interesting to observe the temporal development of the flow during the initial state of the application of the electric field. In addition, it is suggested to develop a numerical model of the flow which is driven by an alternating electric field to observe the time- and frequency-dependent velocity and pressure distribution at the ends of the channel.

6.2.2 Electroosmotic flow in closed-end geometry with varying cross-sectional area

The pressure generation ability of electroosmotic flow in a closed-end channel is very useful in many applications. It has provoked many thoughts on how to design a channel that can produce a higher force. It is suggested that future studies can be made to examine the flow in microchannels of varying cross-sectional area, which can either be a nozzle like microchannel or a T-shape geometry shown in Figure 6.1. The purpose of this study is to investigate the pressure distribution and the possible vortices generation in the flow.

Figure 6.1: Geometry of a T-shape microchannel

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