Dynamics models of flexible aircraft

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DYNAMICS MODELS OF FLEXIBLE AIRCRAFT

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Dedicated to Amma, Appa and Kotte

Abstract

Recent trends in aircraft design owing to requirements of high performance and weight reduction have resulted in aircraft possessing significant structural flexibility. Under these conditions, flight vehicles become vulnerable to unfavourable interactions between the structural dynamics of flexible components, aerodynamics and the overall motion of the body. In order to utilize automatic flight control systems to mitigate these effects, integrated flight models that account for the internal dependencies in the system must be developed. To this end, this work presents the derivation of flight models describing the dynamics of flexible aircraft. The equations are expressed in compact matrix-vector form, suitable for integration with state-space control systems. The dynamics of elastic bodies are first discussed, followed by numerical simulation results of a slender beam undergoing a spin-up maneuver. Finally, the equations governing the dynamics of flexible aircraft are derived and simplifications to obtain lower complexity models are discussed. The formulation is demonstrated through an example aircraft and some preliminary simulation results are presented. It is hoped that the integrated dynamics models derived in this work can be used for control design, flight simulation, parameter estimation and other studies, which are traditionally performed on six degree-of-freedom rigid-body flight models.

Keywords: Flexible Aircraft, Equations of Motion

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LIST OF SYMBOLS

$\mathbb{B}$	Body fixed basis : $(\hat{x}_b, \hat{y}_b, \hat{z}_b)$, with its origin denoted by O
$\mathbb{E}$	Inertial basis : $(\hat{E}_x, \hat{E}_y, \hat{E}_z)$
$\vec{R}_O$	Vector from the origin of $\mathbb{E}$ to that of $\mathbb{B}$
$\vec{r}$	Position vector to a mass element on the elastic body <i>in the undeformed configuration</i> from O
$\mathbf{r}$	$\vec{r}$ resolved in $\mathbb{B}$ (column vector)
$\mathbf{v}^\times$	Cross product operator corresponding to the column vector $\mathbf{v}$
$\mathbf{T}_{BI}$	Transformation matrix from the inertial to the body-fixed basis
$\Theta = [\psi \ \theta \ \phi]^T$	Symbolic vector of Euler angles
$\omega = [p \ q \ r]^T$	Angular velocity of the body fixed reference frame relative to the inertial frame, expressed in $\mathbb{B}$
$\mathbf{E}$	Matrix relating $(\dot{\psi}, \dot{\theta}, \dot{\phi})$ and (p, q, r)
$\mathbf{w}$	Inertial velocity of a mass element expressed in $\mathbb{B}$
$\mathbf{V} = [u \ v \ w]^T$	Inertial velocity of O expressed in $\mathbb{B}$
$\mathbf{u}_e(\mathbf{r}, t)$	Elastic deformation at a location $\mathbf{r}$ at time t , expressed in $\mathbb{B}$
$\psi_\alpha(\mathbf{r}), q_\alpha(t)$	α^{th} assumed mode and associated generalized coordinate
T, U	Kinetic and Potential energy of a system
L	Lagrangian function for the system ($= T - U$)
$\mathbf{q}(t)$	Column vector of generalized coordinates ($= [q_1(t) \dots q_N(t)]^T$)
m	Mass of the system
$\mathbf{J}$	Inertia matrix of the system, expressed in $\mathbb{B}$

$f(\mathbf{r}, t)$	Distributed force density vector acting at a point $\mathbf{r}$ at time t , expressed in $\mathbb{B}$
$\mathbf{F}$	Net force acting on the system, expressed in $\mathbb{B}$
$\mathbf{M}$	Net moment about O acting on the system, expressed in $\mathbb{B}$
f_α	Generalized force associated with the α^{th} generalized coordinate q_α
$\mathbf{K}$	Stiffness matrix of the system, with $K_{\alpha\beta}$ being a typical term
$\mathbf{C}$	Damping matrix of the system, with $C_{\alpha\beta}$ being a typical term
EA	Axial stiffness
EI	Flexural stiffness
A	Cross sectional area of the body
l	Length of the body
m_R	Mass of the rigid subsystem
$\mathbf{J}_R$	Inertia matrix of the rigid subsystem, expressed in $\mathbb{B}$
$\mathbf{J}_W$	Inertia matrix of the wing, expressed in $\mathbb{B}$
$\mathbf{u}_{bending}(\mathbf{r}, t)$	Bending deformation at a point $\mathbf{r}$ at time t
$\mathbf{u}_{torsion}(\mathbf{r}, t)$	Torsional deformation at a point $\mathbf{r}$ at time t
$\psi_{\alpha_b}(\mathbf{r}), q_{\alpha_b}(t)$	α^{th} assumed mode in bending and associated generalized coordinate
$\psi_{\alpha_t}(\mathbf{r}), q_{\alpha_t}(t)$	α^{th} assumed mode in torsion and associated generalized coordinate
$\xi(\mathbf{r}, t)$	Local bending deformation at $\mathbf{r}$ at time t
$\mathbf{r}_{ea}$	Point on the elastic axis closest to mass element at $\mathbf{r}$
ρ	Offset of mass element from elastic axis
$\theta_t(\mathbf{r}_{ea}, t)$	Local twist angle about elastic axis at $\mathbf{r}_{ea}$ at time t
σ	Proportionality constant for structural damping.
$c(y)$	Wing chord at a spanwise station
s	Wing semi-span
$\mathbf{V}_{3/4}$	Velocity of the $\frac{3c}{4}$ point at a spanwise station, expressed in $\mathbb{B}$
$\mathbf{S}$	Section fixed basis : $(\hat{S}_x, \hat{S}_y, \hat{S}_z)$, fixed at a spanwise station on the wing. Parallel to $\mathbb{B}$ when the wing is untwisted.

$\mathbf{T}_{SB}$	Transformation matrix from body to section-fixed basis
$\mathbf{V}'_{3/4}$	$\mathbf{V}_{3/4}$ expressed in the section-fixed basis $\mathbb{S}$
α	Local angle of attack at a span station
α_{eff}	Effective angle of attack at a span station
l, d	Lift and Drag of aerodynamic strip respectively.
$\mathbf{F}_{Aero}$	Local aerodynamic forces expressed in $\mathbb{B}$
α_{∞}	Freestream angle of attack
β_{∞}	Freestream angle of sideslip
$\mathbf{F}_{\delta_{\star}}$	Aerodynamic force generated by control surface deflection, $\star =$ elevator(e),aileron(a),rudder (r)
$\mathbf{r}_{\delta_{\star}}$	Location of aerodynamic center of control surface, $\star =$ elevator(e),aileron(a),rudder(r)

CHAPTER 1

INTRODUCTION

1.1 Motivation

Recent design trends in flight vehicles, owing to requirements of high performance and weight reduction, have led to flight vehicles having high structural flexibility. These trends are especially visible among newer commercial aircraft such as the Boeing 787, high performance gliders like Schempp-Hirth Arcus T (Figure 1.1) and also among Unmanned Aerial Vehicles (UAVs) of the High Altitude Long Endurance (HALE) class like NASA Helios (Figure 1.2). The dynamics of such aircraft possessing significant structural flexibility differ from relatively rigid aircraft. Moreover, the low elastic frequencies of the former make them vulnerable to unfavourable interaction with rigid body dynamics.

Figure 1.1: Boeing 787 (Left) (©www.Boeing.com), Schempp-Hirth Arcus T (Right) (©Akira Uekawa)

The importance of this issue was demonstrated when the Helios prototype crashed on 26 June 2003 (Figure 1.2). The subsequent investigation [1] into the mishap revealed that the aircraft

Figure 1.2: NASA Helios (©www.NASA.gov)

attained a high dihedral configuration resulting in an unstable pitch oscillation, eventually leading to the failure of the aircraft structure. One of the root causes identified and the corresponding recommendation made was (see page 89 of [1]) :

Root cause : Lack of adequate analysis methods led to an inaccurate risk assessment of the effects of configuration changes leading to an inappropriate decision to fly an aircraft configuration highly sensitive to disturbances.

Recommendation : Develop more advanced, multidisciplinary (structures, aeroelastic, aerodynamics, atmospheric, materials, propulsion, controls, etc) “**time-domain**” analysis methods appropriate to highly flexible, “morphing” vehicles.)

Aircraft structures are not perfectly rigid and consequently, they deform under applied loads. The applied loads may result from a combination of aerodynamic, inertia and elastic effects. Time-varying loads cause the resulting deformation to be time-dependent. However, if the aircraft has considerable stiffness such that the deformation due to applied loads are small and the elastic and rigid-body frequencies are well separated, then the problem of aircraft dynamics can be decoupled into rigid-body aircraft dynamics (Flight Dynamics [2], [3]) and Aeroelasticity [4], [5]. In recent years, the trend towards lighter aircraft structures has resulted in reduction of aircraft structural stiffness and hence, increased the possibility of coupling between the overall motion and structural dynamics of aircraft. This was noted in a comprehensive review paper by Livne and Weisshaar [6] as follows :

... When the frequency separation between the rigid-body and elastic motions becomes small, the traditional fields of aeroelasticity and rigid-body flight mechanics cannot be

separated, and the flight mechanics of the deformable airplane, covering motion in all degrees of freedom, must be treated in an integrated manner....With the coupling of flight mechanics and aeroelastic behavior interactions in analysis comes the need for an integrated approach to design. Synthesis of flight control systems must take all modes of dynamic behavior into account. ...

Consequently, modeling the dynamics of flexible aircraft, from the perspective of investigating the dynamics as well as for control purposes, has received increasing attention over the past few decades. However, modeling such flight vehicles is complex due to the inherent coupling present in the system. A schematic illustrating this fact is shown in Figure 1.3. The solid lines represent the paths found in conventional rigid body dynamics, while the dashed lines indicate the paths due to flexibility effects. The effects of flexibility can be understood as adding two additional loops to the system - one inner loop between the rigid body dynamics and structural dynamics and one outer loop between the structural dynamics and aerodynamics. Changes in the inertial properties (mass distribution, mass moment of inertia, center of mass) due to deformation influence the overall motion of the body. Additionally, the inertial effects due to rigid body motion act as a forcing function to the structural dynamics. Finally, the deformation effects (deformed configuration and deformation rates) on the aerodynamic surfaces (wings, tail etc) influence the aerodynamic forces and moments produced.

A variety of approaches have been adopted in order to derive the Equations of Motion (EOM) of flexible aircraft, with a corresponding assortment of assumptions, methodologies and complexities. In general, the approach to modeling of flexible flight vehicles differs in the dynamic models of the structure and the models of forcing terms (primarily forces and moments arising from aerodynamics) driving them.

(a) **Aerodynamic models** : In the literature on flexible aircraft dynamics, the most frequently used model to compute aerodynamics forces and moments is the so-called ‘quasi-steady aerodynamic strip theory’ (usually referred to as simply ‘strip theory’) where the lifting surface is divided into strips and the aerodynamics forces and moments of each strip is

Figure 1.3: Flexible aircraft dynamics

computed using 2D thin airfoil theory and then summed up over the aircraft. This method is attractive since it allows for fast computation with reasonable accuracy. Other techniques based on potential flow theory such as lifting line theory and vortex panel methods have also been used. At the other end of spectrum, sophisticated Computational Fluid Dynamics (CFD) simulations have been used, but these come at significant computation cost and effort.

(b) **Structural dynamics** : In conventional rigid-body flight dynamics, the EOM are expressed in terms of components along a body-fixed coordinate system with its origin at the aircraft center of mass. Since aircraft instruments such as Accelerometers and Gyroscopes measure accelerations and rotation rates in a body-fixed coordinate system, expressing the equations of flight dynamics in the same coordinate system provides a natural way to include sensor measurements. However, in the case of deformable aircraft, the location of the center of mass as well as the inertia properties are not constant due to time-varying deformation. This presents difficulties in writing the equations in terms of a coordinate system fixed at a point on the body. Moreover, the EOM for flexible flight vehicles are more easily derived

through Lagrangian mechanics, rather than through Newton-Euler mechanics traditionally used in flight dynamics. Lagrange’s equations, when expressed in terms of *quasi-coordinates*, provide an attractive method of directly deriving the EOM in a body-fixed coordinate system. Here, *Quasi-Coordinates* refer to coordinates whose time rates cannot be integrated to obtain parameters that specify the configuration of a body (see Greenwood [7]).

In order to derive the EOM for flexible bodies moving in space, the so-called ‘Mean Axes’ (also called ‘Tisserand condition’) reference frame was proposed in which the reference frame is not fixed to a material point on the body and ‘floats’ along with it in such a way that the deformation has no linear and angular momentum with respect to the frame. This idea was initially applied to spacecraft with flexible appendages and then later extended to flexible aircraft. Writing the deformation field in terms of the free-free vibration modes of the structure and placing the reference frame at the instantaneous center of mass satisfied the ‘Mean Axes’ condition and led to simplifications in the expression for Kinetic Energy of the system. An elaborate discussion of such reference frames is provided in the paper by Canavin and Likins [8].

Aside from the choice of a coordinate system, various methods have been employed to address the issue of describing deformation of the aircraft. The assumed modes method [9] is widely used for this purpose due to its simplicity and ease of use, coupled with reasonable accuracy. For flight vehicles of the HALE class that tend to sustain large deformations, nonlinear beam models with spatial discretization using the Finite Element Method [9] have been employed. In more recent work, ideas from Multibody dynamics [10] have also been used. In the following section, we discuss the application of the various modeling methods as previously described to modeling the dynamics of flexible aircraft.

1.2 Flight Dynamics of Flexible Aircraft

This section presents a review of relevant literature pertaining to the flight dynamics of flexible aircraft, particularly those dealing with the modeling of such flight vehicles. One of the earliest investigations on this problem was performed by Milne [11]. In a two part report, the EOM for a flexible aircraft was derived using the ‘Mean Axes’ formulation, which was then used to evaluate trim and longitudinal stability characteristics of a slender aircraft. Assuming the availability of modal vibration data (free-free vibration modes and frequencies), Waszak and Schmidt [12] derived the EOM of a flexible aircraft by using the Mean Axes reference frame and Lagrange’s equations. Strip theory was used to compute the aerodynamic forces. This resulted in a set of coupled nonlinear ordinary differential equations identical to traditional flight dynamics equations, augmented with a set of uncoupled linear ordinary differential equations in the modal coordinates. The coupling between structural vibration and the overall motion was only through the generalized aerodynamic forces and moments. This formulation was attractive as the effects of flexibility could be incorporated into standard flight dynamics models with relatively less modifications. Schmidt and Raney [13] applied this methodology to study flying qualities of flexible aircraft using motion based simulators and reported that flexibility effects have significant effects on flying qualities.

Silvestre and Paglione [14] derive EOM of a flexible aircraft by using the mean axis formulation and Lagrange’s equations. The aerodynamic loads are captured in terms of coefficients that depend on rigid body motion and aircraft deformation. To enable the integration of a flexible aircraft model with existing FEM-based structural models, Reschke [15] derived the EOM of a flexible aircraft described by lumped masses and inertia properties starting from the so-called quasi-coordinate form of Lagrange’s equations [16] and enforcing mean axis constraints.

As mentioned previously, a number of investigators have opted to use a body-fixed coordinate system to derive the dynamics equations for a flexible aircraft. Patil et al [17] describe the

modeling and simulation of a HALE class aircraft. A nonlinear beam model for the wing and an unsteady aerodynamics model is used. This work was extended by Patil and Hodges [18] to analyze the flight dynamics of a flying wing configuration. They reported that the flight dynamics characteristics (short period and phugoid modes) for a flexible flying wing are different from the rigid case. Moreover, they note the disappearance of the short period mode at certain payload conditions. A similar methodology was adopted by Shearer and Cesnik [19] for the dynamic analysis of HALE class of aircraft and they conclude that nonlinear structural modeling is important for analyzing this class of aircraft. A parametric study of the effect of design parameters such as fuselage length and mass distribution, horizontal tail area etc on the flight dynamics of a HALE class aircraft was carried out by Chang et al [20].

An evaluation of aerodynamics and structural models used in flight dynamics of very flexible aircraft was presented by Palacios et al [21]. Since the focus is on wings undergoing large deformations, the structural models evaluated are displacement-based, strain-based and intrinsic beam finite element models. Unsteady thin airfoil theory and Vortex Lattice Method are evaluated for aerodynamics modeling. Using the model developed previously [18], trim and stability analysis of a high aspect ratio flying wing was presented by Raghavan and Patil [22]. They report that using the deformed shape at trim (accounting for static aeroelastic deformation) was better than the undeformed aircraft shape for predicting flight dynamics behaviour of a flexible aircraft at trim. Su and Cesnik [23] extended the work of [19] by accounting for change in torsional stiffness due to wing bending and studied the gust response of a flying wing. The effect of wake-tail interference on the dynamics of flexible aircraft was studied by Murua et al [24]. The wings, tail and fuselage are modeled as nonlinear beams, while the unsteady Vortex Lattice Method [25] is used for the aerodynamics. The authors conclude that the wake-tail interference may be significant in this class of aircraft.

Meirovitch and Tuzcu [26] present a comprehensive unified theory for dynamics and control of flexible aircraft. Starting from the quasi-coordinate form of Lagrange's equations and using strip theory for computing aerodynamic loads, they arrive at a set of ordinary differen-

tial equations (ODEs) for the overall motion of the aircraft and a set of partial differential equations (PDEs) describing the deformation. Spatial discretization of the PDEs using the assumed modes method results in a set of ODEs in the generalized coordinates describing the structural dynamics. The same authors present simulation results of level flight and steady turn maneuvers in [27] using the formulation developed previously. The authors then apply the framework developed earlier to study the effect of flexibility on the stability of flying aircraft [28]. Vehicle stability is evaluated using three cases - (a) flexibility effects are ignored (standard rigid body dynamics), (b) - elastic deformation is included, but rigid body motion is suppressed and (c) both rigid body and elastic deformation are included, and control gains from assuming the aircraft as rigid is used to stabilize the flexible aircraft. The authors report that it is not advisable to neglect flexibility effects when designing control systems. A paper criticizing the use of the ‘Mean Axes’ formulation was published by the same authors [29]. They argue that the use of ‘Mean Axes’ was justified in the absence of powerful computers, when simplifications were necessary to obtain analytical solutions. In addition, they state that use of ‘Mean Axes’ could be justified for flexible spacecraft rather than aircraft due to the absence of complex aerodynamic forces in the former’s operating environment. Some examples from literature that used ‘Mean Axes’ simplifications without enforcing the associated constraints are also discussed. Finally, they observe that properly enforcing such constraints results in complicating the equations and hence render the perceived advantages for application to atmospheric flight vehicles to be questionable.

Grant et al [30] evaluated the suitability of flexible aircraft models for the purposes of real-time simulation. The models considered were the ‘Mean Axes’ formulation [12] and the body-fixed quasi-coordinate approach [26]. Their preliminary results indicated that both models produced similar results. However, the increased accuracy of the body-fixed axes approach over the Mean-Axis approach comes at the cost of longer computation time. A similar exercise was performed by Li et al [31], wherein the ‘Mean Axes’ formulation was compared with the body-fixed quasi-coordinate approach proposed by D’Eleuterio and Barfoot [32]. Baluch and van Tooren [33] extended the work of [26] by considering anisotropic

composite materials and removing the restriction of using the elastic axis as the reference for specifying deformation.

Among other work using the body-fixed coordinate system approach, Avanzini et al [34] proposed a mixed Newtonian-Lagrangian method for deriving EOM for a flexible aircraft. The overall motion of the aircraft is governed by standard rigid-body flight dynamics equations, while the structural dynamics due to deformation are obtained using standard Lagrange's equations. Strip theory was used to compute aerodynamic loads. This approach, though attractive for its simplicity, ignores the various coupling terms between the structural dynamics and overall body motion. Nguyen et al [35] used a Newtonian framework for deriving the dynamics of a flexible aircraft with wing under combined bending and torsion. The authors argue that this approach provides better physical insight as compared to energy based formulations, with the aeroelastic angle being computed as a function of aeroelastic deformation and rigid-body motion. The Finite Element Method is used to discretize the wing while the aerodynamics is computed using the Vortex Lattice Method.

In an interesting recent paper, Zhao and Ren [34] presented the application of ideas from multi-body dynamics [10] to the dynamics of flexible aircraft. The fuselage, fin, rudder and horizontal tail are modeled as interconnected rigid bodies with constraints. The wings, flaps and ailerons are modeled as rigid segments connected by flexible beams. An unsteady aerodynamics model is used for calculation of aerodynamic loads. Some discussion on the presented model undergoing flight maneuvers is provided. A feedback control system was developed to stabilize the aircraft and prevent flutter.

1.3 Contributions

As is clear from the discussion in the previous sections, there are a number of approaches to model the dynamics of flexible aircraft, which differ in validity and complexity. However,

there are few available formulations that provide readily usable and intuitive flight models of typical flexible aircraft, for use during conceptual design studies. In order to address this issue, the goal of this work is to derive models of flexible aircraft suitable for use during the conceptual design phase when detailed aircraft information (mass distribution, undamped vibration modes etc) are not available. The aircraft dynamics models so derived should capture the essential features of the physical system, while keeping the model size reasonable. In view of these requirements, the ‘Mean Axes’ formulation discussed previously is not considered since it ignores the inertial coupling effects between the overall dynamics and the structural vibration of the aircraft. The modeling approach and notation follows the second order approach of D’Eleuterio and Barfoot [32], wherein a body-fixed coordinate system is used along with Lagrange’s equations in quasi-coordinates to obtain the equations of motion. Although nonlinear beam models are more accurate than those based on linear elasticity, for the class of aircraft considered in this work (aircraft that do not undergo large deformations), aircraft structural models based on linear elasticity provide the necessary fidelity for our purposes. Finally, two dimensional quasi-steady aerodynamic strip theory is used to model the aerodynamics. It is hoped that the integrated aircraft dynamics models derived in this work can be used for control design, flight simulation, parameter identification and other studies, which are traditionally performed on six degree-of-freedom rigid-body flight models.

CHAPTER 2

DYNAMICS OF ELASTIC BODIES

2.1 Introduction

In this chapter, the equations of motion for a flexible body moving in space are derived, based on the methodology proposed by D’Eleuterio and Barfoot [32]. The flexible body is assumed to undergo elastic deformation under the influence of external loads. The earth is assumed to be flat, non-rotating and an inertial frame of reference. As discussed in the literature review, we choose to attach a coordinate system to a material point O on the elastic body $\mathcal{B}$ (referred to as a ‘body-fixed coordinate system’). Lagrange’s equations for a flexible body in terms of quasi-coordinates are used to arrive at a set of second order, coupled nonlinear equations. Finally, the equations are expressed in compact matrix-vector form suitable for numerical simulation. The following section provides a brief overview of the various coordinate systems and notation used in the formulation. In addition, transformation matrices relating these coordinate systems and operators used during the derivation are defined.

2.2 Preliminaries

A vector quantity is denoted by $\vec{V}$. A column vector representing the components of $\vec{V}$ resolved in a specified coordinate system $\mathbb{C}$ is denoted by $\mathbf{V}_{\mathbb{C}}$. It is to be noted that the basis in which a vector is resolved is not explicitly stated in the following derivation as all vector quantities are resolved in the body fixed coordinate system.

Consider the flexible body $\mathcal{B}$ moving in space as shown in Figure 2.1. Let $\mathbb{E} : (\hat{E}_x, \hat{E}_y, \hat{E}_z)$ represent the basis of a coordinate system fixed to the earth. The body-fixed coordinate system fixed at a material point O has the basis $\mathbb{B} : (\hat{x}_b, \hat{y}_b, \hat{z}_b)$. $\vec{R}_O$ denotes the position vector from the origin of $\mathbb{E}$ to $\mathbb{B}$. Let $\vec{r}$ denote the position of a mass element on the body *in the undeformed configuration* from O . The three dimensional elastic deformation at $\vec{r}$ at a time instant t is given by $\vec{u}_e(\vec{r}, t)$.

The skew-symmetric matrix corresponding to the cross product operator is defined as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^\times \triangleq \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \quad (2.1)$$

Figure 2.1: Flexible body moving in space

The transformation matrix from the inertial to the body-fixed coordinate system is denoted

by $\mathbf{T}_{BI}$. For a (3,2,1) Euler angle sequence, this is given by [36] as

$$\mathbf{T}_{BI} = \begin{bmatrix} \cos \theta \cos \psi & \cos \theta \sin \psi & -\sin \theta \\ \sin \theta \sin \phi \cos \psi - \sin \psi \cos \phi & \sin \psi \sin \theta \sin \phi + \cos \psi \cos \phi & \sin \phi \cos \theta \\ \sin \theta \cos \phi \cos \psi + \sin \psi \sin \phi & \sin \psi \sin \theta \cos \phi - \cos \psi \sin \phi & \cos \phi \cos \theta \end{bmatrix} \quad (2.2)$$

Finally, the matrix relating the Euler angle rates $(\dot{\psi}, \dot{\theta}, \dot{\phi})$ and the angular velocity components (p, q, r) is given by [36] as

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{E} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \quad ; \quad \mathbf{E} = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \sin \phi \cos \theta \\ 0 & -\sin \phi & \cos \phi \cos \theta \end{bmatrix} \quad (2.3)$$

2.3 Equations of Motion

Kinetic Energy

Consider the flexible body $\mathcal{B}$ as shown in Figure 2.1. The inertial position of any mass element can be written as

$$\vec{R} = \vec{R}_O + \vec{r} + \vec{u}_e(\vec{r}, t) \quad (2.4)$$

The inertial velocity of a mass element is then given as

$$\dot{\vec{R}} = \dot{\vec{R}}_O + \dot{\vec{r}} + \dot{\vec{u}}_e(\vec{r}, t) \quad (2.5)$$

Resolving all vectors in the body-fixed coordinate system, the inertial velocity of a mass element becomes

$$\mathbf{w}(\mathbf{r}, t) = \mathbf{V}(t) + \dot{\mathbf{u}}_e(\mathbf{r}, t) + (\boldsymbol{\omega}(t)^\times)[\mathbf{r} + \mathbf{u}_e(\mathbf{r}, t)] \quad (2.6)$$

The deformation field of the body at a given time instant is discretized as

$$\mathbf{u}_e(\mathbf{r}, t) = \sum_{\alpha=1}^N \boldsymbol{\psi}_\alpha(\mathbf{r}) q_\alpha(t) = \boldsymbol{\Psi}(\mathbf{r}) \mathbf{q}(t) \quad (2.7)$$

where, $\boldsymbol{\Psi}(\mathbf{r})$ represents a $[3 \times N]$ matrix of known space-dependent functions satisfying the cantilever conditions at O , while $q_\alpha(t)$ are the unknown time-dependent generalized coordinates.

The kinetic energy of the body is given by

$$T = \int_{\mathcal{B}} \mathbf{w}^T \mathbf{w} dm \quad (2.8)$$

where $dm = \rho(\mathbf{r})dV$, with $\rho(\mathbf{r})$ being the mass density and dV being the element volume.

Using (2.6) and (2.7) in (2.8), we have

$$T = \frac{1}{2} m \mathbf{V}^T \mathbf{V} + \frac{1}{2} \boldsymbol{\omega}^T \hat{\mathbf{J}} \boldsymbol{\omega} + \frac{1}{2} M_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta - \mathbf{V}^T \hat{\mathbf{c}}^\times \boldsymbol{\omega} + \mathbf{V}^T \mathbf{p}_\alpha \dot{q}_\alpha + \boldsymbol{\omega}^T \hat{\mathbf{h}}_\alpha \dot{q}_\alpha \quad (2.9)$$

where Einstein's summation convention implying summation on repeated indices has been employed for conciseness. The terms appearing in (2.9) have the meaning

$$\hat{\mathbf{c}} = \mathbf{c}^{(0)} + \mathbf{c}^{(1)}$$

$$\mathbf{c}^{(0)} = \int_{\mathcal{B}} \mathbf{r} dm \quad ; \quad \mathbf{c}^{(1)} = \mathbf{p}_\alpha q_\alpha \quad ; \quad \mathbf{p}_\alpha = \int_{\mathcal{B}} \boldsymbol{\psi}_\alpha(\mathbf{r}) dm$$

$$\hat{\mathbf{J}} = \mathbf{J}^{(0)} + \mathbf{J}^{(1)} + \mathbf{J}^{(2)}$$

$$\mathbf{J}^{(0)} = - \int_{\mathcal{B}} \mathbf{r}^\times \mathbf{r}^\times dm \quad ; \quad \mathbf{J}^{(1)} = (\boldsymbol{\Gamma}_\beta^T + \boldsymbol{\Gamma}_\beta) q_\beta \quad ; \quad \mathbf{J}^{(2)} = \boldsymbol{\Pi}_{\beta\gamma} q_\beta q_\gamma$$

with

$$\mathbf{\Gamma}_\beta = - \int_{\mathcal{B}} \boldsymbol{\psi}_\beta^\times(\mathbf{r}) \mathbf{r}^\times dm \quad ; \quad \mathbf{\Pi}_{\beta\gamma} = - \int_{\mathcal{B}} \boldsymbol{\psi}_\beta^\times(\mathbf{r}) \boldsymbol{\psi}_\gamma^\times(\mathbf{r}) dm$$

Then,

$$\hat{\mathbf{h}}_\alpha = \mathbf{h}_\alpha^{(0)} + \mathbf{h}_\alpha^{(1)}$$

with

$$\mathbf{h}_\alpha^{(0)} = \int_{\mathcal{B}} \mathbf{r}^\times \boldsymbol{\psi}_\alpha(\mathbf{r}) dm \quad ; \quad \mathbf{h}_\alpha^{(1)} = \boldsymbol{\nu}_{\alpha\beta} q_\beta \quad ; \quad \boldsymbol{\nu}_{\alpha\beta} = - \int_{\mathcal{B}} \boldsymbol{\psi}_\alpha^\times(\mathbf{r}) \boldsymbol{\psi}_\beta(\mathbf{r}) dm$$

Lastly,

$$M_{\alpha\beta} = \int_{\mathcal{B}} \boldsymbol{\psi}_\alpha^T(\mathbf{r}) \boldsymbol{\psi}_\beta(\mathbf{r}) dm$$

In the above expressions, $(\hat{\cdot})$ over a term indicates its dependence on the elastic deformation through q_α . Also, the superscripts (0), (1) and (2) indicate zeroeth, first and second degree dependence on the generalized coordinates.

Potential Energy

The gravitational force is considered to be an applied force so that the potential energy stored in the body is completely due to strain energy. Since the potential energy expression depends on the specific kind of deformation undergone by the body, it is written in a general functional form as

$$U = U(\mathbf{u}_e) = U(\mathbf{q}(t)) \quad \text{where} \quad \mathbf{q}(t) = [q_1(t) \quad q_2(t) \quad \cdots \quad q_N(t)]^T \quad (2.10)$$

Lagrange's Equations

The Lagrangian function is given by $L = T - U$. As mentioned previously, we will use the quasi-coordinate form of Lagrange's equations since they result in the final equations being conveniently expressed in a body-fixed coordinate system. From Meirovitch and Tuzcu [29],

these equations are given as,

$$\begin{aligned}
\frac{d}{dt} \left(\frac{\partial L}{\partial \mathbf{V}} \right) + (\boldsymbol{\omega}^\times) \frac{\partial L}{\partial \mathbf{V}} - (\mathbf{T}_{BI}) \frac{\partial L}{\partial \mathbf{R}_O} &= \mathbf{F} \\
\frac{d}{dt} \left(\frac{\partial L}{\partial \boldsymbol{\omega}} \right) + (\mathbf{V}^\times) \frac{\partial L}{\partial \mathbf{V}} + (\boldsymbol{\omega}^\times) \frac{\partial L}{\partial \boldsymbol{\omega}} - (\mathbf{E}^{-T}) \frac{\partial L}{\partial \boldsymbol{\theta}} &= \mathbf{M} \\
\frac{d}{dt} \left(\frac{\partial \bar{T}}{\partial \dot{\mathbf{q}}} \right) - \left(\frac{\partial \bar{T}}{\partial \mathbf{q}} \right) + \frac{\partial \bar{\mathcal{F}}}{\partial \dot{\mathbf{q}}} + \mathcal{L} \mathbf{q} &= \bar{\mathbf{Q}}
\end{aligned} \tag{2.11}$$

where $\mathbf{F}$, $\mathbf{M}$ and $\bar{\mathbf{Q}}$ are the total force, moment and generalized force vectors respectively. Note that the $\mathbf{F}$ and $\mathbf{M}$ are of size $[3 \times 1]$, while $\bar{\mathbf{Q}}$ is of size $[N \times 1]$. $\bar{T}$ is the kinetic energy density function, $\bar{\mathcal{F}}$ represents the Rayleigh dissipation function density and $\mathcal{L}$ represents a matrix of stiffness differential operators.

Following Meirovitch and Tuzcu [29], using (2.7) in (2.11) and recognizing that L is not a function of either $\mathbf{R}_O$ or $\boldsymbol{\Theta} = [\phi \quad \theta \quad \psi]^T$, (2.11) can be simplified to

$$\begin{aligned}
\frac{d}{dt} \left(\frac{\partial L}{\partial \mathbf{V}} \right) + (\boldsymbol{\omega}^\times) \frac{\partial L}{\partial \mathbf{V}} &= \mathbf{F} \\
\frac{d}{dt} \left(\frac{\partial L}{\partial \boldsymbol{\omega}} \right) + (\mathbf{V}^\times) \frac{\partial L}{\partial \mathbf{V}} + (\boldsymbol{\omega}^\times) \frac{\partial L}{\partial \boldsymbol{\omega}} &= \mathbf{M} \\
\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} + \frac{\partial \mathcal{F}}{\partial \dot{\mathbf{q}}} &= \mathbf{Q}
\end{aligned} \tag{2.12}$$

with

$$\mathbf{Q} = [Q_1(t) \quad \cdots \quad Q_N(t)]^T \text{ and}$$

$\mathcal{F}$ is the Rayleigh dissipation function given by $\mathcal{F} = \frac{1}{2} \dot{\mathbf{q}} \mathbf{C} \dot{\mathbf{q}}$

Using (2.9) and (2.10), the terms in (2.12) are given by

$$\begin{aligned}
\frac{\partial L}{\partial \mathbf{V}} &= m\mathbf{V} + \mathbf{p}_\alpha \dot{q}_\alpha - \hat{\mathbf{c}}^\times \boldsymbol{\omega} \\
\frac{\partial L}{\partial \boldsymbol{\omega}} &= \hat{\mathbf{c}}^\times \mathbf{V} + \hat{\mathbf{h}}_\alpha \dot{q}_\alpha + \hat{\mathbf{J}} \boldsymbol{\omega} \\
\frac{d}{dt} \left(\frac{\partial L}{\partial \mathbf{V}} \right) &= m\dot{\mathbf{V}} + \mathbf{p}_\alpha \ddot{q}_\alpha - \hat{\mathbf{c}}^\times \dot{\boldsymbol{\omega}} - \mathbf{p}_\alpha^\times \dot{q}_\alpha \boldsymbol{\omega} \\
\frac{d}{dt} \left(\frac{\partial L}{\partial \boldsymbol{\omega}} \right) &= \hat{\mathbf{c}}^\times \dot{\mathbf{V}} + \mathbf{p}_\alpha^\times \dot{q}_\alpha \mathbf{V} + \dot{\hat{\mathbf{h}}}_\alpha \dot{q}_\alpha + \hat{\mathbf{h}}_\alpha \ddot{q}_\alpha + \hat{\mathbf{J}} \dot{\boldsymbol{\omega}} + \dot{\hat{\mathbf{J}}} \boldsymbol{\omega} \\
\frac{\partial L}{\partial q_\alpha} &= \mathbf{V}^T \boldsymbol{\omega}^\times \mathbf{p}_\alpha + \boldsymbol{\omega}^T \boldsymbol{\nu}_{\beta\alpha} \dot{q}_\beta + \frac{1}{2} \boldsymbol{\omega}^T \boldsymbol{\Delta}_\alpha \boldsymbol{\omega} + \frac{\partial U}{\partial q_\alpha} \\
\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \right) &= \mathbf{p}_\alpha^T \dot{\mathbf{V}} + M_{\alpha\beta} \ddot{q}_\beta + \hat{\mathbf{h}}_\alpha^T \dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^T \boldsymbol{\nu}_{\alpha\beta} \dot{q}_\beta
\end{aligned} \tag{2.13}$$

where $\boldsymbol{\Delta}_\alpha = \hat{\boldsymbol{\Gamma}}_\alpha + \hat{\boldsymbol{\Gamma}}_\alpha^T$ with $\hat{\boldsymbol{\Gamma}}_\alpha = \boldsymbol{\Gamma}_\alpha + \boldsymbol{\Pi}_{\alpha\beta} q_\beta$

Substituting the entries in (2.13) in (2.12), and using the fact that $\mathbf{a}^\times \mathbf{b} = -\mathbf{b}^\times \mathbf{a}$, we obtain the equations of motion of a flexible body moving in space as

$$\begin{aligned}
m\dot{\mathbf{V}} - \hat{\mathbf{c}}^\times \dot{\boldsymbol{\omega}} + \mathbf{p}_\alpha \ddot{q}_\alpha + m\boldsymbol{\omega}^\times \mathbf{V} - \boldsymbol{\omega}^\times \hat{\mathbf{c}}^\times \boldsymbol{\omega} + 2\boldsymbol{\omega}^\times \mathbf{p}_\alpha \dot{q}_\alpha &= \mathbf{F} \\
\hat{\mathbf{c}}^\times \dot{\mathbf{V}} + \hat{\mathbf{J}} \dot{\boldsymbol{\omega}} + \dot{\hat{\mathbf{h}}}_\alpha \ddot{q}_\alpha + \hat{\mathbf{c}}^\times \boldsymbol{\omega}^\times \mathbf{V} + \hat{\mathbf{J}}_* \boldsymbol{\omega} + \hat{\mathbf{h}}_{\alpha*} \dot{q}_\alpha &= \mathbf{M} \\
\mathbf{p}_\alpha^T \dot{\mathbf{V}} + \hat{\mathbf{h}}_\alpha^T \dot{\boldsymbol{\omega}} + M_{\alpha\beta} \ddot{q}_\beta + \mathbf{p}_\alpha^T \boldsymbol{\omega}^\times \mathbf{V} - \frac{1}{2} \boldsymbol{\omega}^T \boldsymbol{\Delta}_\alpha \boldsymbol{\omega} + 2\boldsymbol{\nu}_{\alpha\beta}^T \boldsymbol{\omega} \dot{q}_\beta + K_{\alpha\beta} q_\beta + C_{\alpha\beta} \dot{q}_\beta &= f_\alpha \\
Q_\alpha(t) = f_\alpha(t) = \int_{\mathcal{B}} \boldsymbol{\psi}_\alpha^T(\mathbf{r}) \mathbf{f}(\mathbf{r}, t) dV &\quad ; \quad \alpha = 1, 2, \dots, N
\end{aligned} \tag{2.14}$$

where

$$\hat{\mathbf{J}}_* = \boldsymbol{\omega}^\times \hat{\mathbf{J}} + \dot{\hat{\mathbf{J}}} \quad ; \quad \hat{\mathbf{h}}_{\alpha*} = \boldsymbol{\omega}^\times \hat{\mathbf{h}}_\alpha + \dot{\hat{\mathbf{h}}}_\alpha$$

$\mathbf{f}(\mathbf{r}, t)$ is the distributed force density vector acting on $\mathcal{B}$, while $C_{\alpha\beta}$ and $K_{\alpha\beta}$ represent terms of the damping and stiffness matrices respectively.

Lastly, the kinematic equations accompanying (2.14) are given by

$$\dot{\mathbf{R}}_O = (\mathbf{T}_{IB}) \mathbf{V} \quad ; \quad \mathbf{T}_{IB} = (\mathbf{T}_{BI})^T \quad (2.15)$$

$$\dot{\boldsymbol{\Theta}} = (\mathbf{E}^{-1}) \boldsymbol{\omega} \quad ; \quad \boldsymbol{\Theta} = [\phi \quad \theta \quad \psi]^T \quad (2.16)$$

Notes

- The first two equations in (2.14) have summation on α while the third equation has summation on β .
- The first two equations give us a set of six $(3 + 3)$ scalar equations while the third equation contributes a set of N equations, resulting in a total of $(6 + N)$ scalar second order nonlinear coupled ordinary differential equations.
- It may be observed that the second equation given in (2.14) differs from that provided by D'Eleuterio and Barfoot [32]. The former includes terms such as $\dot{\mathbf{J}}\boldsymbol{\omega}$, $\boldsymbol{\omega}^\times \hat{\mathbf{h}}_\alpha \dot{q}_\alpha$ and $\dot{\hat{\mathbf{h}}}_\alpha \dot{q}_\alpha$, which influence the rotational dynamics of the body. These terms are not present in the corresponding equation given by D'Eleuterio and Barfoot [32]. In addition, the fifth term of the third equation in (2.14) would simplify to the corresponding term given by [32] if $\hat{\boldsymbol{\Gamma}}_\alpha = \hat{\boldsymbol{\Gamma}}_\alpha^T$, which doesn't appear to be the case.
- In view of the preceding comments, the equations (2.14) can be considered to represent the corrected version of the equations given by [32].

Writing the equations (2.14) in a compact matrix-vector form, we have

$$\mathbf{A}\ddot{\mathbf{X}} = \mathbf{D} - \mathbf{B}\dot{\mathbf{X}} - \mathbf{G}\mathbf{X} \quad (2.17)$$

where,

$$\mathbf{A} = \begin{bmatrix} \mathbf{M}_{RR} & \mathbf{M}_{RE} \\ \mathbf{M}_{ER} & \mathbf{M}_{EE} \end{bmatrix} \quad ; \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{RR} & \mathbf{B}_{RE} \\ \mathbf{B}_{ER} & \mathbf{B}_{EE} \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} \mathcal{O}_{6 \times 6} & \mathcal{O}_{6 \times N} \\ \mathcal{O}_{N \times 6} & \mathbf{K} \end{bmatrix} \quad ; \quad \mathbf{D} = [\mathbf{F} \quad \mathbf{M} \quad f_1 \quad \dots \quad f_N]^T$$

with,

$$\mathbf{M}_{RR} = \begin{bmatrix} m\mathcal{I}_{3 \times 3} & -\hat{\mathbf{c}}^\times \\ \hat{\mathbf{c}}^\times & \hat{\mathbf{J}} \end{bmatrix} \quad ; \quad \mathbf{M}_{RE} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \dots & \mathbf{p}_N \\ \hat{\mathbf{h}}_1 & \hat{\mathbf{h}}_2 & \dots & \hat{\mathbf{h}}_N \end{bmatrix}$$

$$\mathbf{M}_{ER} = (\mathbf{M}_{RE})^T \quad ; \quad \mathbf{M}_{EE} = \begin{bmatrix} M_{11} & \dots & M_{1N} \\ \vdots & \ddots & \vdots \\ M_{N1} & \dots & M_{NN} \end{bmatrix}$$

and

$$\mathbf{B}_{RR} = \begin{bmatrix} m\boldsymbol{\omega}^\times & -\boldsymbol{\omega}^\times \hat{\mathbf{c}}^\times \\ \hat{\mathbf{c}}^\times \boldsymbol{\omega}^\times & \hat{\mathbf{J}}_* \end{bmatrix} \quad ; \quad \mathbf{B}_{RE} = \begin{bmatrix} 2\boldsymbol{\omega}^\times \mathbf{p}_1 & \dots & 2\boldsymbol{\omega}^\times \mathbf{p}_N \\ \hat{\mathbf{h}}_{1*} & \dots & \hat{\mathbf{h}}_{N*} \end{bmatrix}$$

$$\mathbf{B}_{ER} = \begin{bmatrix} \mathbf{p}_1^T \boldsymbol{\omega}^\times & -\frac{1}{2} \boldsymbol{\omega}^T \boldsymbol{\Delta}_1 \\ \vdots & \vdots \\ \mathbf{p}_N^T \boldsymbol{\omega}^\times & -\frac{1}{2} \boldsymbol{\omega}^T \boldsymbol{\Delta}_N \end{bmatrix} \quad ; \quad \mathbf{B}_{EE} = \begin{bmatrix} 2\boldsymbol{\nu}_{11}^T \boldsymbol{\omega} + C_{11} & \dots & 2\boldsymbol{\nu}_{1N}^T \boldsymbol{\omega} + C_{1N} \\ \vdots & \ddots & \vdots \\ 2\boldsymbol{\nu}_{N1}^T \boldsymbol{\omega} + C_{N1} & \dots & 2\boldsymbol{\nu}_{NN}^T \boldsymbol{\omega} + C_{NN} \end{bmatrix}$$

and,

$$\mathbf{K} = \begin{bmatrix} K_{11} & \dots & K_{1N} \\ \vdots & \ddots & \vdots \\ K_{N1} & \dots & K_{NN} \end{bmatrix} \quad ; \quad \text{Finally, we have } \dot{\mathbf{X}} = [\mathbf{V} \quad \boldsymbol{\omega} \quad \dot{\mathbf{q}}]^T$$

2.4 Summary

This chapter presented the derivation of the equations of motion governing the general motion of a flexible body moving in space. The displacement field was discretized using the assumed modes method, and Lagrange's equations in quasi-coordinates were used to arrive at the equations, expressed in a body-fixed coordinate system. The equations take into account all the coupling effects between the elastic and motion variables. Finally, the equations are cast into compact matrix-vector form suitable for numerical simulation. In the next chapters, this formulation will be applied to the spin-up maneuver of a slender beam and subsequently to the dynamics of flexible aircraft.

CHAPTER 3

NUMERICAL SIMULATION

3.1 Spin-up of a slender beam

In this chapter, the methodology presented in Chapter 2 is applied to the dynamic simulation of a free-free slender beam undergoing a spin-up maneuver starting from rest. The example discussed in this chapter has been used by other researchers (Hanagud and Sarkar [37]) and hence provides a basis for validation of the formulation and its implementation.

A schematic of the problem is shown in Figure 3.1. Since the beam is assumed to be slender, the deformation of the beam is given by the displacement of the neutral axis of the beam. The effects of shear deformation and rotary inertia are ignored in this formulation. The strain energy function is determined using nonlinear strain-displacement relations to accommodate large deformations.

Figure 3.1: Spin-up of a slender beam

In its undeformed state, the neutral axis of the beam is along $\hat{x}_b$ (denoted by the dashed line in Figure 3.1), with x denoting the position of a mass element P. The location of the mass element upon deformation is denoted by P'. u_1, u_2 and u_3 are the components of the displacement of P resolved along $\hat{x}_b, \hat{y}_b$ and $\hat{z}_b$ respectively. Following the discussion in Chapter 2, the components of displacement are discretized as

$$\begin{aligned} u_1(x, t) &= \sum_{\alpha=1}^{n_1} \psi_{1,\alpha}(x) q_{1,\alpha}(t) \\ u_2(x, t) &= \sum_{\beta=1}^{n_2} \psi_{2,\beta}(x) q_{2,\beta}(t) \\ u_3(x, t) &= \sum_{\gamma=1}^{n_3} \psi_{3,\gamma}(x) q_{3,\gamma}(t) \end{aligned} \quad (3.1)$$

where $N = n_1 + n_2 + n_3$ is the total number of assumed modes used.

Following [32, 37], the strain energy function U is given as

$$U = \frac{1}{2} \int_0^l \left[EA u_1'^2 + EI u_2''^2 + EI u_3''^2 + EA u_1' \left(u_2'^2 + u_3'^2 \right) + \frac{1}{4} EA \left(u_2'^4 + 2u_2'^2 u_3'^2 + u_3'^4 \right) \right] dx \quad (3.2)$$

where $(\cdot)' = \partial(\cdot)/\partial x$.

In the present simulation, the eigenfunctions of a fixed-free rod uniform rod undergoing longitudinal vibrations are used as the assumed modes for u_1 (axial deformation), while the assumed modes for u_2 and u_3 (transverse deformations) are taken to be the eigenfunctions of a cantilever beam undergoing transverse vibrations. It is to be noted that the eigenfunctions satisfy the cantilever condition at O . The number of assumed modes for u_1, u_2 and u_3 are taken as 1, 3 and 2 respectively. Some issues regarding the choice of assumed modes are discussed in the Section 3.2. Hence, we have

$$\psi_{1,1}(x) = \sin \pi x / 2l$$

$$\psi_{2,1}(x) = \psi_{3,1}(x) = \cosh 1.87x - \cos 1.87x - 0.73 (\sinh 1.87x - \sin 1.87x)$$

$$\begin{aligned}
\psi_{2,2}(x) &= \psi_{3,2}(x) = \cosh 4.69x - \cos 4.69x - 1.02 (\sinh 4.69x - \sin 4.69x) \\
\psi_{2,3}(x) &= \cosh 7.86x - \cos 7.86x - 0.999 (\sinh 7.86x - \sin 7.86x)
\end{aligned} \tag{3.3}$$

In this example, the effect of damping has been ignored. The elastic force vector associated with the generalized coordinates is computed as

$$\frac{\partial U}{\partial \mathbf{q}} = \mathbf{K}(\mathbf{q})\mathbf{q} \tag{3.4}$$

where $\mathbf{K}(\mathbf{q})$ is the nonlinear stiffness matrix arising due to the nonlinear terms in the strain energy equation.

The geometric and material parameters of the beam are the same as [37] and are as follows:

$$E = 6.895 \times 10^{10} \text{ N/m}^2, I = 2.031 \times 10^{-7} \text{ m}^4, m = 12 \text{ kg}, l = 10 \text{ m} \text{ and } A = 4.601 \times 10^{-4} \text{ m}^2.$$

The beam is initially at rest and a torque is applied about the $\hat{z}_b$ axis. The torque profile is specified as

$$M_3(t) = \begin{cases} 40 \left(1 - \cos \frac{2\pi t}{15}\right) & 0 \leq t \leq 15 \\ 0 & t > 15 \end{cases} \tag{3.5}$$

The torque profile given in (3.5) is shown in Figure 3.2. The prescribed torque is computed using a rigid model of the beam such that the angular velocity profile obtained (Figure 3.3) is the same as that of Hanagud and Sarkar [37].

Figure 3.2: Applied torque profile

Figure 3.3: Angular velocity profile

3.2 Results and Discussion

The equations of motion were solved numerically using `ode15s` in MATLAB, with ‘MaxOrder’ being set to 1*. During the implementation of the equations, it was noted that the number of assumed modes chosen along each direction affected the rank of the generalized mass matrix $\mathbf{A}$ (Eq 2.17). For instance, using 1 assumed mode along u_1 and 2 modes along u_2, u_3 resulted in a non-invertible $\mathbf{A}$. This becomes an issue since $\mathbf{A}$ has to be inverted, at least in principle, to obtain the solution. Since the applied torque is expected to have a more significant impact on the deformation along u_2 , an additional mode is chosen along this direction, which also preserves the rank of $\mathbf{A}$.

The axial and transverse displacement of the tip of the beam during the spin-up are shown in Figure 3.4 and Figure 3.5 respectively.

Figure 3.4: Axial displacement at tip

Figure 3.5: Transverse displacement at tip

From Figure 3.4, it can be seen that the beam initially undergoes a shortening in length before stretching back and settling into its steady-state elongation. This effect (called ‘fore-shortening’) is due to the coupling between the axial deformation and the bending deformation of the beam in the strain energy expression. From the tip transverse displacement plot shown in Figure 3.5, it can be observed that the beam initially deforms in a direction opposite to the applied torque due to inertial effects and reaches a maximum value when the applied

*‘MaxOrder’ is set to 1 using the `odeset` command to ensure stable computation. See MATLAB ODE documentation for more details.

torque is the highest, after which it moves back to zero as the applied torque goes to zero. The results in Figure 3.4 and 3.5 show the same trend with those presented by Hanagud and Sarkar [37].

3.3 Summary

In this chapter, the formulation developed in Chapter 2 is applied to the problem of a slender beam undergoing a spin-up maneuver. The results obtained show reasonable agreement with previous investigations, and is able to capture the nonlinear effects in the problem. However, numerical discrepancies do exist and explanations for them need to be studied. Nevertheless, the formulation of Chapter 2 provides an attractive method to describe the dynamics of more complex flexible bodies such as flexible aircraft, which is discussed in subsequent chapters.

CHAPTER 4

DYNAMICS MODELS OF FLEXIBLE AIRCRAFT

4.1 Introduction

In this chapter, the equations of motion for a flexible aircraft are derived, following the methodology presented in Chapter 2. The aircraft is assumed to be composed of a rigid fuselage and tail, with a flexible wing undergoing linear elastic bending and torsional deformation. The aircraft undergoes rigid-body motion (translation and rotation) with respect to a flat, non-rotating Earth, which is considered to be an inertial frame of reference.

The dynamics equations of motion for flexible aircraft primarily involve the description of three aspects : aerodynamics, structural dynamics and the overall dynamics of the body. Previous efforts in this direction have resulted in numerous models, varying in complexity and range of applicability. However, few of the previously formulations are specifically intended for use during the conceptual design phase, when accurate information about aircraft properties (mass distribution, vibration modes, aerodynamic data etc) are not available. In the following sections, an attempt is made to address this issue by providing a variety of readily usable dynamics models for flexible aircraft, following a consistent notational framework. The equations are first developed in their most general form, followed by a discussion of various possible simplifications. Finally, simplified models of flexible aircraft are presented.

Figure 4.1: Flexible aircraft moving in space

4.2 Equations of Motion - Model A

This section presents the derivation of the equations of motion of a flexible aircraft. Let $\mathbb{E} : (\hat{E}_x, \hat{E}_y, \hat{E}_z)$ represent an inertial coordinate system. A body-fixed coordinate system $\mathbb{B}$, having the basis $(\hat{x}_b, \hat{y}_b, \hat{z}_b)$ and located at the center of mass of the *rigid* fuselage-tail combination is employed to specify the deformation of the wing. The $\hat{y}_b$ axis is assumed to be coincident with the elastic axis of the wing, so that the bending and torsional deformation are decoupled. Following the process as given in Chapter 2, all vectors are expressed in $\mathbb{B}$. These aspects are illustrated in Figure 4.1.

Kinetic Energy

Consider the flexible aircraft as shown in Figure 4.1. The total kinetic energy of the system is the sum of contributions from the rigid and flexible components and is written as

$$T = T_{rigid} + T_{flexible} \quad (4.1)$$

Since the fuselage and tail of the flexible aircraft system are considered to be rigid, the rigid body component of the total kinetic energy can be split into translational and rotational kinetic energy as

$$T_{rigid} = \underbrace{\frac{1}{2}m_R \mathbf{V}^T \mathbf{V}}_{\text{Translational rigid-body kinetic energy}} + \underbrace{\frac{1}{2}\boldsymbol{\omega}^T \mathbf{J}_R \boldsymbol{\omega}}_{\text{Rotational rigid-body kinetic energy}} \quad (4.2)$$

The elastic deformation of a mass element of the wing located at $\mathbf{r}$ is specified by $\mathbf{u}_e(\mathbf{r}, t)$. The inertial velocity of a mass element of the wing is then given as

$$\mathbf{w}(\mathbf{r}, t) = \mathbf{V}(t) + \dot{\mathbf{u}}_e(\mathbf{r}, t) + (\boldsymbol{\omega}(t)^\times)[\mathbf{r} + \mathbf{u}_e(\mathbf{r}, t)] \quad (4.3)$$

Ignoring the fuselage width, the deformation field of the wing is discretized as

$$\mathbf{u}_e(\mathbf{r}, t) = \sum_{\alpha=1}^N \boldsymbol{\psi}_\alpha(\mathbf{r}) q_\alpha(t) = \boldsymbol{\Psi}(\mathbf{r}) \mathbf{q}(t) \quad (4.4)$$

where, $\boldsymbol{\Psi}(\mathbf{r})$ represents a $[3 \times N]$ matrix of known space-dependent functions satisfying the cantilever conditions at the origin of $\mathbb{B}$, while $q_\alpha(t)$ are the unknown time-dependent generalized coordinates. The shape functions used for describing the wing deformation are discussed in the structural model (Section 4.5).

The contribution to the total kinetic energy from the flexible components is given by

$$T_{flexible} = \frac{1}{2} \int_{wing} \mathbf{w}^T \mathbf{w} dm \quad (4.5)$$

Using (4.3) and (4.4) in (4.5) the kinetic energy of flexible components (wing) becomes

$$T_{flexible} = \frac{1}{2}m_W \mathbf{V}^T \mathbf{V} + \frac{1}{2}\boldsymbol{\omega}^T \hat{\mathbf{J}}_W \boldsymbol{\omega} + \frac{1}{2}M_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta - \mathbf{V}^T \hat{\mathbf{c}}^\times \boldsymbol{\omega} + \mathbf{V}^T \mathbf{p}_\alpha \dot{q}_\alpha + \boldsymbol{\omega}^T \hat{\mathbf{h}}_\alpha \dot{q}_\alpha \quad (4.6)$$

Substituting (4.2) and (4.6) in (4.1), the expression for total kinetic energy becomes

$$T = \frac{1}{2}m \mathbf{V}^T \mathbf{V} + \frac{1}{2}\boldsymbol{\omega}^T \hat{\mathbf{J}} \boldsymbol{\omega} + \frac{1}{2}M_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta - \mathbf{V}^T \hat{\mathbf{c}}^\times \boldsymbol{\omega} + \mathbf{V}^T \mathbf{p}_\alpha \dot{q}_\alpha + \boldsymbol{\omega}^T \hat{\mathbf{h}}_\alpha \dot{q}_\alpha \quad (4.7)$$

where $m = m_R + m_W$ is the total mass and $\hat{\mathbf{J}} = \mathbf{J}_R + \mathbf{J}_W$ is the total mass moment of inertia of the flexible aircraft system respectively. This form of kinetic energy which takes into account all the coupling between rigid-body motion variables and elastic variables is referred to as KE-Model A. The final equations resulting from the use of this form of kinetic energy will be referred to as Model A.

Potential Energy

The potential energy U depends on the specific deformation undergone by the body. However, since we are assuming the wing to undergo linear elastic deformation, U can be written as

$$U = \frac{1}{2} \mathbf{q}^T \mathbf{K} \mathbf{q} \quad (4.8)$$

The construction of the stiffness matrix $\mathbf{K}$ is discussed in the structural model (Section 4.5).

Lagrange's Equations

Following the same methodology as in Chapter 2, Lagrange's equations in quasi-coordinates are used to arrive at the equations of motion. Since the procedure is the same as discussed in Section 2.3, we only mention the final form of the equations here. It is to be noted that the equations given here represent the general case of a flexible aircraft moving in space.

$$\mathbf{A}(\mathbf{X}) \ddot{\mathbf{X}} = \mathbf{D}(t, \mathbf{X}, \dot{\mathbf{X}}) - \mathbf{B}(\mathbf{X}, \dot{\mathbf{X}}) \dot{\mathbf{X}} - \mathbf{G} \mathbf{X} \quad (4.9)$$

where the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{G}$ have the meaning given in (2.17). $\mathbf{D} = [\mathbf{F} \quad \mathbf{M} \quad f_1 \quad \dots \quad f_N]^T$, where $\mathbf{F}$ and $\mathbf{M}$ represent the total force and moment about O acting on the flexible aircraft respectively, and f_α is the generalized force associated with q_α . $\dot{\mathbf{X}} = [\mathbf{V} \quad \boldsymbol{\omega} \quad \dot{q}_1 \quad \dots \quad \dot{q}_N]^T$, where $\mathbf{V} = [u \quad v \quad w]^T$ and $\boldsymbol{\omega} = [p \quad q \quad r]^T$ represent the conventional translational and rotational velocities used in flight dynamics.

4.3 Equations of Motion - Model B

Consider the velocity field expression (4.3). For relatively stiff aircraft experiencing moderate angular velocities, we have $\boldsymbol{\omega}^\times \mathbf{u}_e \ll \boldsymbol{\omega}^\times \mathbf{r}$, the velocity field then becomes

$$\mathbf{w}(\mathbf{r}, t) = \mathbf{V}(t) + \dot{\mathbf{u}}_e(\mathbf{r}, t) + (\boldsymbol{\omega}(t)^\times) \mathbf{r} \quad (4.10)$$

Using (4.10) in (4.5), adding the resulting expression to (4.2), the total kinetic energy for Model B becomes

$$T = \frac{1}{2} m \mathbf{V}^T \mathbf{V} + \frac{1}{2} \boldsymbol{\omega}^T \mathbf{J} \boldsymbol{\omega} + \frac{1}{2} M_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta - \mathbf{V}^T \mathbf{c}^\times \boldsymbol{\omega} + \mathbf{V}^T \mathbf{p}_\alpha \dot{q}_\alpha + \boldsymbol{\omega}^T \mathbf{h}_\alpha \dot{q}_\alpha \quad (4.11)$$

where $m = m_W + m_R$, $\mathbf{J} = \mathbf{J}_R + \mathbf{J}_W^{(0)}$, $\mathbf{c} = \mathbf{c}^{(0)}$ and $\mathbf{h}_\alpha = \mathbf{h}_\alpha^{(0)}$, with terms having the superscript (0) implying the same meaning as given in Chapter 2. This simplified form of the kinetic energy is referred to as KE-Model B. The terms required to apply Lagrange's equations are computed as follows

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{V}} &= m \mathbf{V} - \mathbf{c}^\times \boldsymbol{\omega} + \mathbf{p}_\alpha \dot{q}_\alpha \\ \frac{\partial L}{\partial \boldsymbol{\omega}} &= \mathbf{c}^\times \mathbf{V} + \mathbf{h}_\alpha \dot{q}_\alpha + \mathbf{J} \boldsymbol{\omega} \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \right) &= \mathbf{p}_\alpha^T \dot{\mathbf{V}} + \mathbf{h}_\alpha^T \dot{\boldsymbol{\omega}} + M_{\alpha\beta} \ddot{q}_\beta \end{aligned} \quad (4.12)$$

Substituting (4.12) in the Lagrange's equations in quasi-coordinates (2.12), the equations of motion of Model B are obtained as

$$\begin{aligned} m \dot{\mathbf{V}} - \mathbf{c}^\times \dot{\boldsymbol{\omega}} + \mathbf{p}_\alpha \ddot{q}_\alpha + m \boldsymbol{\omega}^\times \mathbf{V} - \boldsymbol{\omega}^\times \mathbf{c}^\times \boldsymbol{\omega} + \boldsymbol{\omega}^\times \mathbf{p}_\alpha \dot{q}_\alpha &= \mathbf{F} \\ \mathbf{c}^\times \dot{\mathbf{V}} + \mathbf{J} \dot{\boldsymbol{\omega}} + \mathbf{h}_\alpha \ddot{q}_\alpha + \mathbf{c}^\times \boldsymbol{\omega}^\times \mathbf{V} + \boldsymbol{\omega}^\times \mathbf{J} \boldsymbol{\omega} + \{ \mathbf{V}^\times \mathbf{p}_\alpha + \boldsymbol{\omega}^\times \mathbf{h}_\alpha \} \dot{q}_\alpha &= \mathbf{M} \\ \mathbf{p}_\alpha^T \dot{\mathbf{V}} + \mathbf{h}_\alpha^T \dot{\boldsymbol{\omega}} + M_{\alpha\beta} \ddot{q}_\beta + K_{\alpha\beta} q_\beta &= f_\alpha \end{aligned} \quad (4.13)$$

Writing (4.13) in matrix-vector notation, we have

$$\mathbf{A}\ddot{\mathbf{X}} = \mathbf{D}(t, \mathbf{X}, \dot{\mathbf{X}}) - \mathbf{B}(\dot{\mathbf{X}})\dot{\mathbf{X}} - \mathbf{G}\mathbf{X} \quad (4.14)$$

where, keeping in mind that $\mathbf{J} = \mathbf{J}_R + \mathbf{J}_W^{(0)}$, $\mathbf{c} = \mathbf{c}^{(0)}$ and $\mathbf{h}_\alpha = \mathbf{h}_\alpha^{(0)}$ are now constant terms,

$$\mathbf{A} = \begin{bmatrix} \mathbf{M}_{RR} & \mathbf{M}_{RE} \\ \mathbf{M}_{ER} & \mathbf{M}_{EE} \end{bmatrix} \quad ; \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{RR} & \mathbf{B}_{RE} \\ \mathbf{B}_{ER} & \mathbf{B}_{EE} \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} \mathcal{O}_{6 \times 6} & \mathcal{O}_{6 \times N} \\ \mathcal{O}_{N \times 6} & \mathbf{K} \end{bmatrix} \quad ; \quad \mathbf{D} = [\mathbf{F} \quad \mathbf{M} \quad f_1 \quad \dots \quad f_N]^T$$

with,

$$\mathbf{M}_{RR} = \begin{bmatrix} m\mathcal{I}_{3 \times 3} & -\mathbf{c}^\times \\ \mathbf{c}^\times & \mathbf{J} \end{bmatrix} \quad ; \quad \mathbf{M}_{RE} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \dots & \mathbf{p}_N \\ \mathbf{h}_1 & \mathbf{h}_2 & \dots & \mathbf{h}_N \end{bmatrix}$$

$$\mathbf{M}_{ER} = (\mathbf{M}_{RE})^T \quad ; \quad \mathbf{M}_{EE} = \begin{bmatrix} M_{11} & \dots & M_{1N} \\ \vdots & \ddots & \vdots \\ M_{N1} & \dots & M_{NN} \end{bmatrix}$$

and

$$\mathbf{B}_{RR} = \begin{bmatrix} m\boldsymbol{\omega}^\times & -\boldsymbol{\omega}^\times \mathbf{c}^\times \\ \mathbf{c}^\times \boldsymbol{\omega}^\times & \boldsymbol{\omega}^\times \mathbf{J} \end{bmatrix} \quad ; \quad \mathbf{B}_{RE} = \begin{bmatrix} \boldsymbol{\omega}^\times \mathbf{p}_1 & \dots & \boldsymbol{\omega}^\times \mathbf{p}_N \\ \{\mathbf{V}^\times \mathbf{p}_1 + \boldsymbol{\omega}^\times \mathbf{h}_1\} & \dots & \{\mathbf{V}^\times \mathbf{p}_N + \boldsymbol{\omega}^\times \mathbf{h}_N\} \end{bmatrix}$$

$$\mathbf{B}_{ER} = [\mathcal{O}_{N \times 6}] \quad ; \quad \mathbf{B}_{EE} = \begin{bmatrix} C_{11} & \dots & C_{1N} \\ \vdots & \ddots & \vdots \\ C_{N1} & \dots & C_{NN} \end{bmatrix}$$

It is important to note that, for Model B, the generalized mass matrix $\mathbf{A}$ in (4.14) is a constant matrix. Since the solution of the equations require inversion of $\mathbf{A}$, this would significantly speed up the computation process.

4.4 Rigid-Body Aircraft Model

The rigid-body aircraft dynamics model conventionally used in flight dynamics can be obtained by setting the generalized coordinates and the shape function matrix to zero, i.e., $\mathbf{q}(t) = \mathbf{\Psi}(\mathbf{r}) = \mathbf{0}$. Hence, (2.14) becomes

$$\begin{aligned} m\dot{\mathbf{V}} + m\boldsymbol{\omega}^\times \mathbf{V} &= \mathbf{F} \\ \mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^\times \mathbf{J}\boldsymbol{\omega} &= \mathbf{M} \end{aligned} \quad (4.15)$$

where $\mathbf{J} = \mathbf{J}_R + \mathbf{J}_W^{(0)}$.

4.5 Structural Model

As mentioned previously, the flexible aircraft is assumed to be composed of a rigid fuselage-tail combination and a flexible wing undergoing bending and torsional deformation. The wing is modeled as a Euler-Bernoulli beam cantilevered at the fuselage, undergoing bending in the $\hat{z}_b$ direction and torsion about the elastic axis, which is assumed to lie along $\hat{y}_b$. The deformation at any point on the wing can be understood as a sum of bending displacement of the elastic axis and a twist deformation about the elastic axis. The thickness of the wing is ignored in this discussion.

As illustrated in Figure 4.2, the deformation of a mass element originally at $\mathbf{r}$ at an instant of time is given by

$$\mathbf{u}_e(\mathbf{r}, t) = \mathbf{u}_{e_{bending}}(\mathbf{r}, t) + \mathbf{u}_{e_{torsion}}(\mathbf{r}, t) \quad (4.16)$$

The bending deformation is discretized as

$$\mathbf{u}_{e_{bending}}(\mathbf{r}, t) = \begin{bmatrix} 0 \\ 0 \\ \xi(\mathbf{r}, t) \end{bmatrix} = \sum_{\alpha=1}^{N_b} \boldsymbol{\psi}_{\alpha_b}(\mathbf{r}) q_{\alpha_b}(t) \quad (4.17)$$

Figure 4.2: Bending and Twist deformation at a spanwise station on the wing (airfoil thickness shown is only for illustration)

The eigenfunctions of a uniform cantilever beam under transverse vibration are used as the assumed modes for bending displacement ($\psi_{\alpha_b}(\mathbf{r})$).

In a similar fashion, the torsional deformation is discretized as

$$\mathbf{u}_{\text{torsion}}(\mathbf{r}, t) = \sum_{\beta=1}^{N_t} \psi_{\beta_t}(\mathbf{r}) q_{\beta_t}(t) \quad (4.18)$$

At a given spanwise station, the displacement of a mass element due to twist depends on its offset from the axis of twist (i.e., the elastic axis). This is illustrated in Figure 4.2 by the green arrows.

Let $\vec{\rho}$ be the vector normal to the elastic axis to the mass element. Then, from Figure 4.3,

$$\vec{r} = \vec{r}_{ea} + \vec{\rho} \Rightarrow \vec{\rho} = \vec{r} - \vec{r}_{ea} \quad (4.19)$$

Then, we have $\mathbf{r} = [x \ y \ 0]^T$, $\mathbf{r}_{ea} = [0 \ y \ 0]^T$ and $\boldsymbol{\rho} = [x \ 0 \ 0]^T$.

Figure 4.3: Offset of mass element from elastic axis

The twist deformation field, $\theta_t = \theta_t(\mathbf{r}_{ea}, t)$, is discretized as

$$\theta_t(\mathbf{r}_{ea}, t) = \sum_{\beta=1}^{N_t} \phi_{\beta_t}(\mathbf{r}_{ea}) q_{\beta_t}(t) \quad (4.20)$$

Using the small angle approximation, the assumed modes for torsional deformation (4.18) are written as

$$\boldsymbol{\psi}_{\beta_t}(\mathbf{r}) = \begin{bmatrix} 0 \\ 0 \\ -x\phi_{\beta_t}(\mathbf{r}_{ea}) \end{bmatrix} \quad (4.21)$$

The eigenfunctions of a uniform rod under torsional vibration are chosen as the assumed mode shapes to discretize the twist (ϕ_{β_t}). Finally, the bending and torsional deformation fields are assembled to give the general deformation field as

$$\mathbf{u}_e(\mathbf{r}, t) = \left[\begin{array}{ccc|ccc} \boldsymbol{\psi}_{1_b} & \dots & \boldsymbol{\psi}_{N_{bb}} & \boldsymbol{\psi}_{1_t} & \dots & \boldsymbol{\psi}_{N_{tt}} \end{array} \right] \begin{bmatrix} q_{1_b} \\ \vdots \\ q_{N_{bb}} \\ \dots \\ q_{1_t} \\ \vdots \\ q_{N_{tt}} \end{bmatrix} = \boldsymbol{\Psi}(\mathbf{r})\mathbf{q}(t) \quad (4.22)$$

Stiffness matrix

As mentioned previously, the potential energy of the system is completely due to strain energy since gravity is treated as an applied force. The strain energy of the system contains contributions from bending and torsional deformations, and is given as

$$U = \frac{1}{2} \int_0^l \left[EI \xi''^2 + GJ \theta_t'^2 \right] dy \quad (4.23)$$

where $(\cdot)' = \partial(\cdot)/\partial y$. The elements of the stiffness matrix can be obtained by $K_{\alpha\beta} = \frac{\partial^2 U}{\partial q_\alpha \partial q_\beta}$.

Damping matrix

Structural damping enters the equations of motion through the $\mathbf{B}_{EE}$ submatrix of $\mathbf{B}$ in (4.9) and (4.14). Damping models are difficult to estimate and require extensive experiments. For the purpose of simplifying the analysis, we assume proportional structural damping in this work. However, more precise damping models may be readily substituted if they are available. The damping matrix is given as

$$\mathbf{C} = \sigma \mathbf{K} \quad (4.24)$$

where σ is the proportionality constant.

4.6 Forces, Moments and Generalized Forces

The applied loads acting on the flexible aircraft originate from gravity, thrust and aerodynamics. Furthermore, these loads can occur either as distributed (acting over the whole body, e.g. aerodynamic loads generated by the wing) or discrete forces (acting at specific locations on the body, e.g. engine thrust). The applied loads affect the dynamics through the $\mathbf{D}$ matrix in (4.9) and (4.14).

Following Meirovitch and Tuzcu [29], we employ the principle of virtual work to compute the forces, moments and generalized forces. The virtual work done by a system of forces is given

by

$$\delta W = \int_{body} \left[\mathbf{f}^T(\mathbf{r}, t) + \sum_i \mathbf{F}_i^T(t) \delta(\mathbf{r} - \mathbf{r}_i) \right] \delta \mathbf{R}^* dV \quad (4.25)$$

where $\mathbf{f}(\mathbf{r}, t)$ represents the distributed force density vector and $\mathbf{F}_i(t)$ represents a discrete force acting at location $\mathbf{r}_i$. $\delta \mathbf{R}^*$ is the virtual displacement vector of a point on the body. Note that discrete forces are considered to be distributed by using the the spatial dirac delta function.

Analogous to the velocity expression (4.3), the virtual displacement vector can be expressed in terms of the motion variables as

$$\delta \mathbf{R}^* = \delta \mathbf{R}_O^* + [\mathbf{r}^\times + \mathbf{u}_e^\times]^T \delta \boldsymbol{\Theta}^* + \delta(\mathbf{u}_e) \quad (4.26)$$

In (4.25) and (4.26), the ‘*’ superscript indicates that the vector has been expressed in the body-fixed basis $\mathbb{B}$. Using (4.26) in (4.25), the elements of the $\mathbf{D}$ matrix can then be computed as

$$\mathbf{F} = \frac{\partial(\delta W)}{\partial(\delta \mathbf{R}_O^*)} \quad ; \quad \mathbf{M} = \frac{\partial(\delta W)}{\partial(\delta \boldsymbol{\Theta}^*)} \quad ; \quad \mathbf{f} = [f_1 \quad f_2 \quad \cdots \quad f_N]^T = \frac{\partial(\delta W)}{\partial(\delta \mathbf{q})} \quad (4.27)$$

An illustration of the force system on the flexible aircraft is provided in Figure 4.4. They are as follows:

- Gravity (shown in brown) : Discrete gravitational force due to rigid fuselage-tail ($\mathbf{F}_{R_G}$), distributed graviational force density acting on the wing ($\mathbf{f}_G$), shown on a typical strip.
- Thrust (shown in blue) : Discrete thrust force acting along $\hat{x}_b$ ($\mathbf{F}_T$).
- Aerodynamics (shown in orange) : Discrete aerodynamic force due to rigid fuselage-tail ($\mathbf{F}_{R_A}$), discrete forces due to control surface deflections ($\mathbf{F}_{\delta_\star}$, $\star = \text{elevator}(e)$, $\text{aileron}(a)$, $\text{rudder}(r)$) and distributed aerodynamic force density over the wing, treated as a number of discrete forces by using quasi-steady two dimensional strip theory ($\mathbf{F}_{A_S}$).

Figure 4.4: Force system on a flexible aircraft

4.6.1 Gravity and Thrust Forces

Following the process mentioned previously, the cumulative contributions of the thrust and gravity forces to $\mathbf{D}$ can be written as

$$\begin{bmatrix} m_W \\ \hat{\mathbf{c}}^\times \\ \mathbf{P}^T \end{bmatrix} \mathbf{g} + \begin{bmatrix} m_R \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{g} + \begin{bmatrix} \mathbf{F}_T \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad (4.28)$$

where, $\mathbf{g} = \mathbf{T}_{BI} [0 \quad 0 \quad 9.81]^T \text{ N/m}^2$ and $\mathbf{P} = [\mathbf{p}_1 \dots \mathbf{p}_N]$.

4.6.2 Aerodynamics

The aerodynamic loads acting on the flexible aircraft arise from distributed aerodynamic loads generated by the wing, aerodynamic forces generated by the rigid fuselage-tail (treated as a discrete force) and discrete aerodynamic forces due to control surface deflections.

In order to compute the aerodynamic forces generated by the wing, we use two-dimensional

airfoil strip theory. The formulation is restricted to steady state aerodynamics and unsteady effects are not considered. As shown in Figure 4.4, the wing is divided into a number of chord-wise strips and two dimensional airfoil aerodynamics is applied to each strip to compute the local aerodynamic loads, which are then summed to obtain the total aerodynamic loads acting on the flexible aircraft. In addition, to account for spanwise variation of lift distribution, the lift coefficient is expressed as a function of the span location (see Wright and Cooper [38], page 80) as

$$C_L(y) = 1 - (y/s)^2 \quad (4.29)$$

Consider Figure 4.4. At a given strip, the aerodynamic center is assumed to be located at the quarter-chord position in the midline of the strip. The local velocity of the strip midline at $\frac{3}{4}c$ is used to compute the local induced angle of attack of the strip. It is to be noted that the angle of sideslip is ignored when computing the strip aerodynamic forces since the strip aerodynamic forces are predominantly due to angle of attack. The (inertial) velocity of the $\frac{3}{4}c$ location, expressed in $\mathbb{B}$ is given as

$$\mathbf{V}_{3/4} = \mathbf{V} + \boldsymbol{\omega}^\times [\mathbf{r}_{3/4} + \mathbf{u}_e(\mathbf{r}_{3/4}, t)] + \dot{\mathbf{u}}_e(\mathbf{r}_{3/4}, t) \quad (4.30)$$

Equation (4.30) can be expressed in a basis fixed to the section using

$$\mathbf{V}'_{3/4} = (\mathbf{T}_{SB}) \mathbf{V}_{3/4} \quad (4.31)$$

where $\mathbf{T}_{SB} = \mathbf{T}_{SB}(\theta_t)$ is the transformation matrix from the body fixed basis $\mathbb{B}$ to a section fixed basis $\mathbb{S}$ and θ_t being the local twist angle, which is discretized as mentioned previously in (4.20). The effective local angle of attack is computed as

$$\alpha_{eff} = \tan^{-1} \left(\frac{\left\{ \mathbf{V}'_{3/4} \right\}_3}{\left\{ \mathbf{V}'_{3/4} \right\}_1} \right) \quad (4.32)$$

The aerodynamic forces at a section, expressed in $\mathbb{B}$ are given by

$$\mathbf{F}_{Aero}(\mathbf{r}, t) = \begin{bmatrix} l \sin \alpha - d \cos \alpha \\ 0 \\ -l \cos \alpha - d \sin \alpha \end{bmatrix} \quad (4.33)$$

where $\alpha = \tan^{-1} \left(\frac{\{\mathbf{V}_{3/4}\}_3}{\{\mathbf{V}_{3/4}\}_1} \right)$.

The lift (l) and drag (d) forces are given by

$$l = \frac{1}{2} \rho \|\mathbf{V}_{3/4}\|^2 C_L(\alpha_{eff}) c \Delta C_L(y) \quad ; \quad d = \frac{1}{2} \rho \|\mathbf{V}_{3/4}\|^2 C_D(\alpha_{eff}) c \Delta \quad (4.34)$$

where c and Δ represent the average strip chord and width respectively, and C_L and C_D represent the 2D lift and drag coefficients of wing airfoil.

As mentioned before, the aerodynamic force generated by a strip acts the aerodynamic center, which is assumed to be at the quarter-chord location on the midline of the strip. Using the principle of virtual work (equations (4.26) and (4.27)) the contribution of an aerodynamic strip to $\mathbf{D}$ can be computed as

$$\begin{bmatrix} \mathbf{F}_{Aero} \\ (\mathbf{r} + \mathbf{u}_e(\mathbf{r}, t))^\times \mathbf{F}_{Aero} \\ \boldsymbol{\Psi}^T(\mathbf{r}) \mathbf{F}_{Aero} \end{bmatrix} \quad (4.35)$$

where $\mathbf{r}$ refers to the aerodynamic center of the strip.

Considering the control surfaces, the flexible aircraft is assumed to have ailerons, elevator and rudder as the flight control surfaces. As can be observed from Figure 4.4, the discrete forces generated by ailerons ($\mathbf{F}_{\delta_a}$) are influenced by local wing deformation. These forces are included into the the aerodynamic forces generated by the wing by considering the strips containing ailerons as strips with trailing edge flaps. In contrast, aerodynamic forces generated by the elevator and rudder act on rigid parts of the flexible aircraft and can be treated more

directly. Following a process similar to that used for the wing, the elevator ($\mathbf{F}_{\delta_e}$) and rudder ($\mathbf{F}_{\delta_r}$) forces can be computed by first computing the velocity of the aerodynamic center of the respective control surface and then inferring the local flow angles. For sake of simplicity, interference effects are ignored while computing $\mathbf{F}_{\delta_e}$ and $\mathbf{F}_{\delta_r}$.

The discrete aerodynamic force due to the rigid fuselage-tail ($\mathbf{F}_{R_A}$) is assumed to act at the origin of $\mathbb{B}$. The fuselage-tail is assumed to only contribute to drag and side forces and its contribution to lift is ignored. Following standard rigid-body flight dynamics, the freestream angle of attack and angle of sideslip are used to compute the drag and side forces, which are then written in the body-fixed basis. $\mathbf{F}_{R_A}$ can be computed as

$$\mathbf{F}_{R_A} = \begin{bmatrix} -D \cos \alpha_\infty \cos \beta_\infty \\ S - D \sin \beta_\infty \\ -D \cos \beta_\infty \sin \alpha_\infty \end{bmatrix} \quad (4.36)$$

where α_∞ and β_∞ refer to freestream angle of attack and sideslip respectively. The side (S) and drag (D) forces can be computed using coefficient buildup, table lookups and other standard methods used in flight dynamics.

Finally, the cumulative contributions of ($\mathbf{F}_{R_A}$) and control forces ($\mathbf{F}_{\delta_\star}$, $\star = e, r$) to $\mathbf{D}$ can be written as

$$\begin{bmatrix} \mathbf{F}_{R_A} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \sum_{\star} \begin{bmatrix} \mathbf{F}_{\delta_\star} \\ \mathbf{r}_{\delta_\star}^\times \mathbf{F}_{\delta_\star} \\ \mathbf{0} \end{bmatrix} \quad \star = e, r \quad (4.37)$$

with $\mathbf{r}_{\delta_\star}$ indicating the aerodynamic center of the pertinent control surface.

4.7 Limitations

Although this chapter provides a general framework for investigating the dynamics of flexible aircraft, it may be worthwhile to keep in mind the limitations of the proposed models. On the aerodynamics side, the proposed models are restricted to steady aerodynamics and moderate angles of attack. Hence, unsteady effects and stall models have not been included in the formulation. On the structural side, the use of Euler-Bernoulli beams and uniform rods restricts us to linear elastic deformations.

4.8 Summary

This chapter presented the derivation of flight models of flexible aircraft. The aircraft was modeled as a rigid fuselage-tail combination along with a flexible wing undergoing bending and torsion. First, Lagrange's equations in quasi-coordinates were used to arrive at the general dynamics equations. Next, a simplified model, consisting of a constant generalized mass matrix is obtained. The standard rigid body flight dynamics equations are obtained by setting the quantities related to deformation to zero. The equations, conveniently expressed in a body-fixed coordinate system, are written in compact matrix-vector form suitable for simulation and control design. The structural modeling of the flexible aircraft using the assumed modes method was presented. Finally, computation of the applied forces, moments and generalized forces using the principle of virtual work was described.

CHAPTER 5

RESULTS AND DISCUSSIONS

5.1 Simulation Setup

In this chapter, the methodology presented in Chapter 4 is applied to an example aircraft. The aircraft properties are first described, followed by a discussion of the trim condition pertaining to steady wings-level flight. Next, the various flight models are subjected to control inputs and differences in the responses are analyzed. Finally, some comments about the flight dynamics of flexible aircraft are made.

The aircraft used in the present work is the the same used by Patil et al [17] and is shown in Figure 5.1 along with the body-fixed basis $\mathbb{B}$. The material and physical properties of the aircraft are provided in Table 5.1. A mid-wing aircraft configuration is assumed so that the center of mass of the rigid fuselage-tail coincides with the elastic axis of the wing. Further, all the aerodynamic surfaces are assumed to have NACA 0012 as the airfoil, and the horizontal and vertical tails are assumed to be of an all-moving configuration. Aerodynamic parameters required for simulation are computed using Digital DATCOM+ [39]^{*} and XFLR5 [40][†].

Since our interest is to investigate the interactions between structural dynamics and rigid-

^{*}Digital DATCOM+, produced by Holy Cows Inc., is an updated version of the USAF Digital DATCOM software used for computing static stability, control and dynamic derivative characteristics of fixed-wing aircraft.

[†]XFLR5 is an open-source analysis tool for airfoils and wings operating at low Reynolds numbers.

body motion, lower frequency vibration modes are more relevant to our purpose . Hence, we restrict ourselves to two symmetric bending modes (ψ_{1_b}, ψ_{2_b}) in the $\hat{z}_b$ direction and two symmetric torsional modes (ψ_{1_t}, ψ_{2_t}). The proportionality constant for structural damping (σ) is chosen as 0.001. Finally, we choose 16 aerodynamic strips on the wing for the aerodynamics. All the aircraft models are implemented in MATLAB. The equations of motion for flexible aircraft were solved numerically using the `ode15s` solver, with ‘MaxOrder’ being set to 1, while the rigid-body model was solved using `ode45`.

Parameter	Value
Wing half-span	16 m
Wing chord	1 m
Mass per unit length	0.75 Kg/m
Spanwise elastic axis	50% of chord
Bending rigidity	$2 \times 10^4 \text{ Nm}^2$
Torsional rigidity	$1 \times 10^4 \text{ Nm}^2$
Wing and tail airfoil	NACA 0012
Mass of fuselage-tail	50.6 Kg
Mass Moment of Inertia of fuselage-tail	200 Kg·m ²
Length of tail boom	10 m
Tail half-span	2.5 m
Tail chord	0.5 m
Maximum Thrust	330 N

Table 5.1: Aircraft Properties

Parameter	Value
Altitude	6 Km
Speed	25 m/s
Air density	0.66 Kg/m ³
ω	0
$\dot{\mathbf{q}}$	0
ϕ, ψ	0
δ_a, δ_r	0
$\mathbf{v}$	0

Table 5.2: Flight condition

Figure 5.1: Flexible Aircraft (Patil et al [17])

5.2 Trim Conditions

Computation of the trim conditions forms an important step in flight dynamics analysis. It enables investigation of aircraft dynamics around an equilibrium condition, as well as providing an initial condition for numerical simulation. In the present work, the aircraft is trimmed to wings-level steady flight at a specified altitude. The flight conditions at trim are specified in Table 5.2. Given this flight condition, we look for the freestream angle of attack (α_∞), elevator deflection (δ_e), throttle setting as a percentage of maximum thrust (δ_t) and values for the generalized elastic coordinates $\mathbf{q}$ (for flexible aircraft models) such that the sum of all forces, moments and generalized forces acting on the aircraft goes to zero, with the aircraft flying at constant altitude and speed.

Figure 5.2: Static aeroelastic deformation of wing at trim

The trim conditions are computed using the `fsolve` function in MATLAB, and summarized in Table 5.3. As can be expected, the trim values obtained for a rigid aircraft differs slightly from that for flexible aircraft. The static aeroelastic deformation of the wing at trim is shown in Figure 5.2, with the undeformed wing shown in gray colour. Incidentally, the trim configuration remains the same for both **Model A** and **Model B** as differences in the generalized mass matrix could be expected to appear in the dynamics, rather than during

static equilibrium.

Parameter	Rigid	Flexible
$\theta(rad)$	0.026057	0.022594
$\delta_e(rad)$	-0.020421	-0.016829
δ_t	0.42255	0.42133
$\mathbf{q}$	–	$[-0.16344 \ -0.0086154 \ 0.0043494 \ 0.0004297]^T$

Table 5.3: Trim values

5.3 Response to Control Inputs

In this section, responses of various aircraft models when subjected to control inputs is discussed. It is to be noted that these results are preliminary and more work needs to be done before definite conclusions can be drawn. Prior to applying control inputs, all models are trimmed to wings-level steady flight as discussed in Section 5.2. A doublet input over and above the trim value with an amplitude of $\pm 5\%$ of the trim value is applied to the elevator, while all other controls are held at their trim values. The control time histories are shown in Figure 5.3.

Figure 5.3: Control Time Histories

5.3.1 Rigid vs Flexible Aircraft

From table 5.3, it is observed that rigid and flexible aircraft settle to slightly different trim values. Since our intention is to compare the deviation from trim rather than absolute values, we plot $\Delta \mathbf{x}$, where $\Delta \mathbf{x} = \mathbf{x} - \mathbf{x}_{trim}$, $\mathbf{x}$ being some variable. It should also be mentioned that ‘flexible aircraft’ here refers to the full flexible aircraft model (**Model A**).

The body axes translational and angular velocity components are shown in Figure 5.4. From the figure, it is observed that the lateral and directional dynamics variables (v, p, r) are essentially identical with the differences being very small as expected, since the elevator input would only influence the longitudinal dynamics. The longitudinal dynamics variables (u, w, q) show that both models experience an initial nose down moment, which results in an increase in forward speed, followed by a nose up moment due to the second part of the doublet. The trends for the two models are very similar trends except for small differences in the peak values attained.

Figure 5.4: Body axes translational and angular velocity components

Figure 5.5 shows the time histories of position and attitude variables. As can be observed from the figure, the rigid and flexible aircraft models are essentially identical except for the slight variation in the altitude during and after the elevator input.

Figure 5.5: Position variables and Euler Angles

From the preceding discussion, it is clear that the rigid and flexible aircraft produce very similar responses for the input considered here. It is difficult to reach any conclusion about the differences in the models from this study alone. The models need to be explored more fully by considering different aircraft and using varied control inputs. This is considered a part of future effort.

5.3.2 Flexible Aircraft : Model A vs Model B

In this section, we compare the responses of Model A (Eq 4.9) and Model B (Eq 4.14) to control inputs. As before, an elevator doublet is applied as the control input. Figures 5.6 to 5.10 depict the components of body axes translational velocities, body axes angular velocities, generalized coordinates, attitude variables (Euler Angles) and inertial position

variables respectively.

Figure 5.6: Body axes velocity components

Figure 5.7: Body axes angular velocity components

Figure 5.8: Generalized elastic coordinates

Figure 5.9: Euler Angles

As can be observed from the figures, both Model A and Model B produce very similar responses. This may be attributed to the fact that the angular velocity of the aircraft and the

Figure 5.10: Aircraft position variables

deformations may not be large enough to effect significant changes in the response. Also, since **Model A** includes a time-varying generalized mass matrix while **Model B** has a constant generalized mass matrix, it can be inferred that the deformations are small enough so that changes in inertial properties due to deformation effects do not have significant impact on the overall dynamics. This aspect can be noted in Figure 5.9 and 5.10, where the aircraft attitude and position variables are shown. Although more work needs to be done to better understand the models, at this stage, it can be noted that, for aircraft experiencing low angular velocities and small deformations, using **Model B** may be good enough for control design purposes, with the added advantage of a constant generalized mass matrix.

5.4 Summary

In this chapter, the formulation developed in the previous chapters was applied to an example aircraft. Next, trim conditions pertaining to steady wings-level flight for rigid and flexible models was discussed. The aircraft models were then subjected to control input in the form

of an elevator doublet. Some preliminary simulation results were presented, comparing the responses of rigid and flexible aircraft, and of the two flexible aircraft models.

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

6.1 Conclusions

This work presents a methodology for modeling the dynamics of flexible aircraft. The motivation behind the problem as well as a review of related work was presented in the first chapter. Specifically, a need for readily available formulations of various fidelities incorporating all the important system couplings was identified. The second chapter discussed the dynamics of elastic bodies moving in space. The equations of motion were derived using the *quasi-coordinate* form of Lagrange's equations, and retain all the coupling terms between elastic variables and rigid-body motion. These equations include the terms missing from those presented by D'Eleuterio and Barfoot [32]. Moreover, the dynamics variables are expressed in the more natural body-fixed coordinate system instead of an inertial basis. In the next chapter, the developed formulation was applied to the case of a slender beam undergoing a spin-up maneuver. The results obtained indicate that the formulation was able to capture the nonlinear effects in the problem.

The fourth chapter deals with the formulation of equations governing the dynamics of flexible aircraft, which is the main focus of this work. The methodology of chapter 2 was applied to derive the equations of motion of flexible aircraft, modeled as a rigid fuselage-tail combination with the wings undergoing bending and torsion. A general model was first derived, followed by simplifications to obtain a lower complexity model. The structural model of the wing was

described in detail, followed by a discussion of the force system acting on the aircraft and application of the principle of virtual work to obtain the applied forces, moments and generalized forces. Finally, the equations were reduced to the standard rigid-body equations of flight dynamics by setting quantities related to deformation to zero. The formulation developed previously was applied to an example aircraft. Trim conditions for wings-level steady flight was described, followed by a discussion of aircraft responses to control inputs. Some preliminary simulation results were presented.

In summary, the primary contributions of this thesis are the derivation of corrected equations of motion for the general motion of flexible bodies and the development of intuitive, integrated models for the dynamics of typical flexible aircraft. These flight models can be used for control design, flight simulation, parameter estimation and other studies, which are traditionally performed on six degree-of-freedom rigid-body flight models.

6.2 Future directions

The aircraft models derived in this work provide a starting point for dynamic analysis of typical flexible aircraft. Since this work only used the elevator control input, a natural next step would be to introduce multiple control inputs (ailerons, rudder, flaps etc) and observe the dynamics. On the theoretical side, linearization and eigen analysis could provide more insight into the underlying dynamics of flexible aircraft, and relative importance of the various coupling terms present in the equations of motion. In order to explore gust response and aeroelastic phenomena such as flutter, the aerodynamics would need to include unsteady effects and stall models. Finally, nonlinear dynamics of flexible aircraft is a relatively new area and has the potential to provide interesting insights into the dynamics of such flight vehicles.

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