Development of the nodal based discontinuous deformation analysis and its engineering applications

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**NANYANG
TECHNOLOGICAL
UNIVERSITY**

**DEVELOPMENT OF THE NODAL BASED
DISCONTINUOUS DEFORMATION ANALYSIS AND ITS
ENGINEERING APPLICATIONS**

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SCHOOL OF CIVIL AND ENVIRONMENTAL ENGINEERING

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**A thesis submitted to Nanyang Technological University
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Abstract

The Discontinuous Deformation Analysis (DDA) and the Finite Element Method (FEM) belong to two different numerical approaches, the discontinuum-based method and the continuum-based method. However, these two methods share many common characteristics: they are both displacement-based implicit methods and their global equations can be derived by the same total potential energy minimization procedure. The Nodal-based Discontinuous Deformation Analysis (NDDA) introduced in this thesis is a coupled method of the DDA and the FEM. By incorporating the finite element mesh into the discrete block, the unique block kinematics of the DDA is inherited and the stress field within the block is refined.

The NDDA method is able to model the occurrence of tensile and shear cracks inside an intact block by treating each FEM mesh line as a potential crack. When the given fracturing criterion is satisfied, a double-noded interface is inserted along the FEM mesh line to separate the two elements sharing that mesh line. In order to get a realistic fracturing mode, a “crack propagation procedure”, including the determination of inter-element stress, the failure criterion, the fracture sequence based on the failure time and the node split algorithm, is implemented with detailed procedures. The crack propagation capability provides a possible route for the NDDA method to model the transition from the continuum material to a discontinuum domain, as well as to extend its application to the failure analysis of rock mass.

In order to numerically simulate the rock/rockbolt interaction, a finite difference based rockbolt model is implemented in the NDDA framework. In this model, the rockbolt is simulated as a bar element which does not possess independent degrees of freedom. The axial deformation of the rockbolt is obtained by interpolating the rock deformation while the relative movement between the rock mass and the rockbolt in the normal direction is restricted. In case of the rockbolt crossing a discontinuity, an interface bolt segmentation is incorporated. Various kinds of rockbolts can be modeled by assigning different tangential stiffness to the grouting material.

The validation and the application of the NDDA method are carried out by applying it to the slope stability analysis and the toppling analysis. Several examples are used to validate the NDDA method, by comparing the NDDA results with the analytical solutions, the physical observations and other available numerical approaches.

The contact analysis is a critical part in the discrete element modeling. An alternative contact mechanism, the Augmented Lagrangian Method (ALM) is introduced into the standard DDA method. The ALM can be regarded as a combination of the Lagrange

Multiplier Method and the Penalty method, because a penalty number P (representing the stiffness of the contact spring) and a Lagrange multiplier λ^* (representing the contact force) are used in the ALM. Both the efficiency and the accuracy of the ALM are discussed through five validation examples. As the NDDA method inherits all the unique features of the DDA method, it is also possible to implement the ALM into the NDDA method.

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Chapter 1. Introduction

1.1 Background

Rock mechanics is defined as “the theoretical and applied science of the mechanical behavior of rock and rock masses; it is the branch of mechanics concerned with the response of rock and rock masses to the force fields of their physical environment” by American National Academe of Science (Hoek, 1966).

When engineering activities start on rock, the force fields of rock’s physical environment will be changed, and the response of rocks and rock masses subject to engineering activity will become the main concern of the Engineer. Numerical approach is one of the important tools in the prediction of the engineering responds of the rocks and rock masses. With the increasing needs in the design analysis and retrospective analysis, numerical approach has been developed for decades.

To numerically solve a continuous problem, the domain of interest is usually subdivided into a finite number of sub-domains, by using simpler mathematical expressions with finite degrees of freedom, the sub-domains can easily be approximated. These sub-domains must satisfy both the governing differential equations of the problem and the continuity condition at their interfaces with adjacent elements. The discretization of a continuum is in fact an approximation of a continuous system with infinite degrees of freedom by a discrete system with finite degrees of freedom (Jing, 2003). Herein, numerical approaches based on this concept are termed as continuum-based numerical methods.

However, from a geological structure viewpoint, rock blocks are usually regarded as a system of blocks cut by planes in space. The geological planes, which serve as discontinuities, could be faults, joints, or cracks. Major structure planes such as faults and large joints are the most important discontinuous planes for rock systems. The geometrical distribution and physical properties of discontinuities have significant effect on the mechanical behavior of a jointed rock system (Chappel, 1974; Hoek, 1981). As the continuum methods are based on the assumption of the displacement continuity across elements, they may not be suitable for problems in which large deformations and displacements may occur. To overcome this limitation, discontinuum-based methods are developed. In this approach, the continuity is only maintained within the element and its displacement function is considered to be discontinuous across element boundaries.

Actually, the concepts of continuum and dicontinuum are not absolute but relative and problem-specific, especially depending on the problem scales. Due to the differences in the underlying material assumptions, different numerical methods have been developed

for continuous and discontinuous systems. The most commonly applied numerical methods for rock mechanics and rock engineering are sorted into three categories (Jing, 2003):

- (1) Continuum methods---Finite Difference Method (FDM), Finite Element Method (FEM), Boundary Element Method (BEM);
- (2) Discontinuum methods---Distinct Element Method (DEM); Discrete Fracture Network (DFN) method, Discontinuous Deformation Analysis (DDA);
- (3) Hybrid continuum/discontinuum models---hybrid FEM/BEM, hybrid DEM/BEM, hybrid FEM/DEM.

Among all the numerical methods, the FEM has been the most popular numerical method in engineering sciences, including rock mechanics and rock engineering. First introduced (Clough, 1960) for plane stress problems, the method gained rapid development and promoted in many scientific and engineering fields in the past half century.

The FEM is based on the concept of domain discretization, which is dividing the domain into a finite number of sub-domains (elements) of standard shapes defined by a fixed number of nodes at the vertices and/or on the sides. The step after domain discretization is local approximation, the FEM uses the basic assumption that the unknown function over each element can be approximated through a trial function of its nodal values of the system unknowns in a polynomial. After the local algebraic equations representing the behavior of the elements are generated; the local elemental equations are then assembled into a global system of algebraic equations according to the topologic relations between the nodes and elements. After imposing the proper initial force and boundary conditions, the global equations are solved.

Different from the continuous based FEM, the DEM is a numerical method falls within the general classification of discontinuum analysis technique. Beginning with the initial presentation by Cundall (1971), the formulation and development of the DEM have progressed for nearly 40 years. In the DEM, the rock mass is represented as an assembly of discrete blocks and the joints are viewed as interfaces between distinct bodies (i.e., the discontinuity is treated as boundary condition). The contact forces and displacements at the interfaces of a stressed assembly of blocks are found through a series of calculations which trace the movements of the blocks. It is an explicit method using a time-stepping algorithm. At the beginning of each time step, block contacts are determined kinematically and the contact stiffness is added at the contact points, thus the contact force can be obtained and regarded as additional external forces of the system at the current step, with a mathematical damping to dissipate the extra kinetic energy, the solution process will be driven by the unbalanced forces.

Introduced by Shi, the DDA method was firstly a backward model, which can compute the strains and deformations of a block system (Shi, 1984), and then extended to a new numerical model to perform the complete deformation analysis of a block system (Shi, 1988). Although the DDA is a discontinuum-based method in which blocks are bounded by pre-existing discontinuities, unlike the explicit approach of DEM (Cundall, 1971), DDA is an implicit method and chooses the displacements as variables, equilibrium equations of DDA are established by minimizing the total potential energy, the same as FEM does (Jing, 2003).

In the DDA method, a complete first order displacement approximation is used, thus every point inside a block has same stresses and strains. The assumption of constant stresses within a block is reasonable when blocks are small and there are no major blocks in the system. However, when a more complicated stress field or a more deformable block boundary is desired, the enhancements to the assumption are necessary. Improvement of this limitation can be sorted into three categories: 1) sub-block system; 2) higher order displacement function; 3) coupling with finite element mesh. Among them, coupling with finite element mesh is the most attractive one because it is easy to apply it into the code without losing any advantages of the original DDA as well as including many mature FEM techniques. Shyu (1993) firstly presented an NDDA method which added a finite element mesh into each DDA block. The combination greatly increased the deformation ability of the DDA blocks and refined stress field within the block. Chang (1994) included the nonlinear inelastic material behavior in each element, thus the global stability and the failure modes of the block system can be studied by the NDDA method. Bao (2010) improved shyu's work with a crack propagation ability, which made NDDA a more promising method.

However, although the concept of coupling the DDA with the finite element mesh has been proposed for decades, the validation and the application of this method are limited. While the theoretical derivation and programming realization is the first step to develop a new method, the verification should be the next step before more complicated application could be carried out.

At the same time, through inspection of the failure of the block it is found that, due to the stress jump between triangular elements, the crack path obtained from the NDDA method is formed by a group of small tiny cracks, not a main crack observed in the laboratory test. More sophisticated method is needed to be implemented in the NDDA method to get the accurate inter-element stresses, thus obtain a more realistic failure mode.

Rockbolts are routinely used as one of the most important rock reinforcement in civil and mining engineering. The principle of the rockbolts is to help the rock to support itself through suspending, beam building and arch building. In the NDDA method, the rockbolt

model is oversimplified as a spring connecting two blocks, which is unrealistic and restrict the capacity of the method for real rock engineering application. In this case, development of realistic rockbolt models in the NDDA method is necessary.

The penalty method used in the DDA method has proved to be effective in various numerical models and discussed by Vignjevic (1999), Felippa (1978) and Gu (2002) et al. However, the method has several limitations. The accuracy of the solution is highly dependent on the choice of the penalty value and the optimal value is difficult to be explicitly found beforehand; second, the penalty approach satisfies the contact constraints only approximately (Bao, 2012). To improve the contact precision in the DDA, an Augmented Lagrangian Method (ALM) is introduced, which can make use of the advantages of both Lagrange multiplier method and the penalty method. Although the employment of ALM in the DDA can be found in previous researches, few details on the computational efficiency of it had been provided.

1.2 Objective and scope

The main goal of this thesis is to develop and extend the NDDA method in the rock engineering. The scope of the work includes:

- To review the theories, recent developments and applications of the DDA method and the NDDA method.
- To implement the double minimization procedure introduced by Ciancio (2006) into the NDDA method to get a more accurate inter-element forces as well as recover the nodal stress state. A fracture sequence, including the mesh updating and the crack inserting, is proposed to obtain a more realistic fracture pattern.
- To develop an advanced rockbolt model in the NDDA framework, different kinds of rockbolts can be realized and different failure modes are incorporated. Comparison of the performance of different kinds of rockbolts is carried out.
- To implement an alternative contact constraint mechanism—ALM into the standard DDA method. In this case, both the advantages of Lagrange multiplier method and the penalty method can be incorporated.
- To verify the NDDA method with simple but meaningful problem and extended the application of the new method to the rock engineering, i.e. slope sliding and toppling.

1.3 Organization of the thesis

This thesis includes 8 chapters. The organization of the thesis is as follow:

In chapter 1, the research background, as well as the objective and scope of the study are provided.

In chapter 2, firstly the basic concepts and theories of Discontinuous Deformation Analysis (DDA) are introduced. The development and the application of the DDA are then reviewed. A detailed introduction about the Nodal-based Discontinuous Deformation Analysis (NDDA) is presented and the comparison between the NDDA method and other numerical methods is carried out. Both the advantages and disadvantages of the NDDA method are concluded.

In chapter 3, the application of the NDDA method in the slope stability analysis is carried out. The limit equilibrium methods are used as a comparison.

In chapter 4, a double-minimization procedure is introduced into the NDDA method to improve the accuracy of the stresses evaluated at the finite element mesh lines and recover the nodal stress state. A fracture sequence is proposed to obtain a more realistic fracture pattern. At the same time, the differences between the NDDA method and the sub-block DDA method are discussed in terms of accuracy, computational efficiency, and convergent speed.

In chapter 5, the NDDA method with the stress recovery is applied in the toppling analysis. In the NDDA method, discrete fractures can be readily accommodated and the new fractures are able to grow if necessary. The simulation results show that the NDDA method is able to predict the relative movement occurs along the pre-defined discontinuities as well as capture the right failure mode. In case of no pre-existing discontinuities, the NDDA method is able to predict the rupture surface inside the rock mass accurately.

In chapter 6, the rockbolt model is developed in the NDDA method. Different types of rockbolts could be realized in the proposed numerical model and the possible failure modes, namely tensile failure of bolt shank, debonding along the bolt/rock interface and the failure of the face plate are available in the analysis procedure.

In chapter 7, the contact mechanics in the DDA method is discussed, the supplements and the modifications are carried out for advancing the ability of the DDA method. Five examples are given to validate the ALM enhanced DDA program.

Finally, conclusions as well as the recommendations are given in Chapter 8.

Chapter 2. Literature Review

The purpose of this chapter is to present the techniques, advances, problems and likely future developments in numerical modeling in rock mechanics and rock engineering. In section 2.1, the basic theory of the DDA method is introduced. In section 2.2, the development and the validation of the DDA method is reviewed. Section 2.3 presents the basic theory of a coupling method—NDDA, and section 2.4 compares the NDDA method with other numerical methods commonly used in the rock mechanics and rock engineering. Section 2.5 discusses the advantages and the limitations of the current NDDA method.

2.1 Basic theory of the DDA method

The DDA method was originated as a back-analysis algorithm to determine a best fit to a deformed configuration of a block system from field measured displacements and deformations (Shi, 1988). It was later extended to perform the complete deformation analysis of a block system. The DDA method is a generalization of the FEM and the DEM. It is similar to the DEM in that it considers the jointed rock mass as an assemblage of discrete blocks, but differs from DEM in its solution scheme. The DDA method is displacement-based implicit method, by using the penalty method to simulate the interaction between the blocks and minimizing the total potential energy of the system, the basic functions and the system of simultaneous equations can be obtained.

2.1.1 Displacement function in the DDA method

In the DDA method, each block has six degrees of freedom, $(u_0, v_0, r_0, \varepsilon_x, \varepsilon_y, \gamma_{xy})$ in which (u_0, v_0, r_0) are rigid body motion terms and $(\varepsilon_x, \varepsilon_y, \gamma_{xy})$ are constant strain terms. By using a complete first-order displacement function, a homogeneously deformable block system can be formed.

The complete first order approximation of block displacements has the following forms (Shi, 1988):

$$\begin{aligned} u &= a_1 + a_2x + a_3y \\ v &= b_1 + b_2x + b_3y \end{aligned} \quad (2-1)$$

Where (u, v) are the displacements of point (x, y) . At the point (x_0, y_0) the displacements are (u_0, v_0) :

$$\begin{aligned} u_0 &= a_1 + a_2x_0 + a_3y_0 \\ v_0 &= b_1 + b_2x_0 + b_3y_0 \end{aligned} \quad (2-2)$$

Subtracting Eq. (2-2) from Eq. (2-1):

$$\begin{aligned} u &= a_2(x - x_0) + a_3(y - y_0) + u_0 \\ v &= b_2(x - x_0) + b_3(y - y_0) + v_0 \end{aligned} \quad (2-3)$$

In Eq. (2-3), (a_2, a_3, b_2, b_3) can be obtained from the following geometrical equation and rotation relationship:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} = a_2 \\ \varepsilon_y &= \frac{\partial v}{\partial y} = b_3 \end{aligned} \quad (2-4)$$

$$\begin{aligned} \frac{1}{2}\gamma_{xy} &= \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{1}{2}(b_2 + a_3) \\ r_0 &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(b_2 - a_3) \end{aligned} \quad (2-5)$$

Inserting these four parameters into Eq. (2-3):

$$\begin{aligned} u &= \varepsilon_x(x - x_0) + \left(\frac{1}{2}\gamma_{xy} - r_0\right)(y - y_0) + u_0 \\ v &= \left(\frac{1}{2}\gamma_{xy} + r_0\right)(x - x_0) + \varepsilon_y(y - y_0) + v_0 \end{aligned} \quad (2-6)$$

As the block in the DDA method has constant stress and strain, the displacement (u, v) of any point (x, y) within a block can be represented in a matrix form:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 & -(y - y_0) & (x - x_0) & 0 & \frac{(y - y_0)}{2} \\ 0 & 1 & (x - x_0) & 0 & (y - y_0) & \frac{(x - x_0)}{2} \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ r_0 \\ \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \quad (2-7)$$

Eq. (2-7) can be written in a more generalized form:

$$\begin{pmatrix} u \\ v \end{pmatrix} = [T_i][D_i] = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16} \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} & t_{26} \end{pmatrix} \begin{pmatrix} d_{1i} \\ d_{2i} \\ d_{3i} \\ d_{4i} \\ d_{5i} \\ d_{6i} \end{pmatrix} \quad (2-8)$$

where $[D_i]$ represents the displacement variables of block i . When the strains are known and the displacements at the center of rotation are given, Eq. (2-7) enables the calculation of the displacements at any point (x, y) within the block i .

2.1.2 Simultaneous Equations of the DDA method

Although the DDA method belongs to the family of discontinuous-based method, it is more like the FEM as it's a displacement-based implicit method. The process of yielding the global equations is very similar to the FEM.

Firstly, all the potential energies of elastic strains, initial stresses, point loading, line loading, volume loading, viscosity and inertia force are computed. By the second law of thermodynamics, a mechanical system under loading must move or deform in a direction which produces the minimum total energy of the whole system. For a block system, the total energy Π_i can be expressed as

$$\Pi_i = [f]^T [d_B] + \frac{1}{2} [\varepsilon]^T [\sigma] dA \quad (2-9)$$

where $[f]^T$ is the loading on the block i distributed to the six deformation variables; $[d_B]$ is the displacement of the block i ; $[\varepsilon]$ and $[\sigma]$ are the strain and stress of the block i , respectively.

Secondly, the derivatives of individual potential energy with respect to deformation variables are computed and the corresponding sub-matrices are formed separately.

$$\frac{\partial \Pi_i}{\partial d_{ri}} = 0, \quad r = 1, \dots, 6 \quad (2-10)$$

where d_{ri} is the deformation variable of the block i .

Finally, the global equations are yielded by adding the sub-matrices to the matrices of the global equation at the corresponding position. Assuming there are n blocks in the defined block system, the simultaneous global equilibrium equations have the form:

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \cdots & k_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{n1} & k_{n2} & k_{n3} & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{Bmatrix} \quad (2-11)$$

which can be written in a more compact form:

$$KD = F \quad (2-12)$$

Eq. (2-11) is the system of equations of motion used in the DDA method. The matrix $[K]$ is called the “global stiffness matrix” in which $[K_{ii}]$ depends on the material properties of the block i and represents the sum of contributing sub-matrices for the block i , $[K_{ij}]$ ($i \neq j$) is defined by the contact between the block i and the block j , which represents the sum of contributing sub-matrices of contacts between the block i and the block j and other inter-block actions such as bolting. The vector $[D_i]$ represents displacement variables of the block i and the vector $[F_i]$ represents resultant general forces acting on the block i .

In Eq. (2-11), the term $[K_{ii}]$ which lies on the main diagonal of the matrix comes from the differentiations:

$$\frac{\partial^2 \Pi}{\partial d_{ri} \partial d_{si}}, \quad r, s = 1, \dots, 6 \quad (2-13)$$

where d_{ri} , d_{si} are the deformation variables of the block i .

$[K_{ij}]$ comes from the differentiations:

$$\frac{\partial^2 \Pi}{\partial d_{ri} \partial d_{sj}}, \quad r, s = 1, \dots, 6 \quad (2-14)$$

where d_{ri} is the deformation variable of the block i , d_{sj} is the deformation variable of the block j .

$[F_i]$ comes from the differentiations:

$$-\frac{\partial \Pi(0)}{\partial d_{ri}}, \quad r = 1, \dots, 6 \quad (2-15)$$

2.1.3 DDA contact mechanism

In order to connect the individual blocks into a block system and extend the single block into the rock mass formulation, the constraining equations and inequalities should be introduced and then a method imposing these equations on the global equilibrium equation should be established.

In the DDA method, there are three types of contacts, namely, edge-to-edge contact, corner-to-edge contact, and corner-to-corner contact, as shown in Figure 2-1. Among these contact modes, the corner-to-edge contact is the fundamental contact, since the edge-to-edge contact can be represented as a combination of two corner-to-edge contacts while the corner-to-corner contact can be degenerated as a corner-to-edge contact.

The contact searching algorithm in the DDA method follows the orientation theory. First the envelope space of each block is found. The envelope space is the smallest rectangular with sides parallel to the coordinate axes that can cover the block. The distance criteria is

then used to find the possible contact pairs and in the end, the angle criteria is used to select the real contact pairs from all the possible contact pairs as well as specify the type of each contact pair.

Figure 2-1 Three contacts between two adjacent blocks

After all the contact pairs are detected, a penalty-based contact system is employed to deal with contacts between blocks, and the Coulomb's friction law is employed to control the contact modes. The physical meaning of a penalty parameter is the stiffness of the contact spring. Normally, two kinds of contact spring, normal contact spring and shear contact spring, are placed between contacting surfaces of a frictional contact. The contact sub-matrix is then computed and added to the global equation systems at the corresponding position, and Eq. (2-11) is solved. Since the continually updated positions and shapes of blocks will introduce new contacts and interactive forces within each time step, the normal and shear contact springs which represent contacts between blocks, are repeatedly added or removed. This is the so called "open-close (O-C)" iterations in the DDA. The converging condition of O-C iterations is that the contact status of each contact keeps unchanged from last iteration.

2.1.4 Time integration scheme used in the DDA method

In the DDA method, an implicit integration scheme which requires an iterative solver to solve a set of linear equations at each step is used. Two types of iterative solver, namely, directly equation solver and the Successive Over-Relaxation (SOR) equation solver are used in the DDA method.

The main advantage of the implicit integration is that it is unconditionally stable and no numerical damping is needed. In this case, a relatively large time step can be used, especially for the problem with large block displacement.

Though it may not be a major problem with the development of the 64 bit computer operating, the memory required for the storage of the global matrix coefficient may restrict the size of the problem, and the iteration solver may be time consuming especially for problems requiring large number of time steps.

2.2 Recent developments and applications of the DDA method

Over the last decades, many researchers have involved in the study of the DDA method and dedicated a great deal of effort to validate and enhance the method.

2.2.1 Development of the DDA method for rigid body rotation

Considering a block with centroid (x_0, y_0) , the rigid body motion at the centroid is (u_0, v_0, r_0) and the strain at the centroid is $(\varepsilon_x, \varepsilon_y, \gamma_{xy})$, constantly within the block. The displacements at any point (x, y) within the block are:

$$\begin{aligned} u &= \varepsilon_x(x - x_0) + \frac{1}{2}\gamma_{xy}(y - y_0) + u_0 + (x - x_0)(\cos r_0 - 1) - (y - y_0)\sin r_0 \\ v &= \frac{1}{2}\gamma_{xy}(x - x_0) + \varepsilon_y(y - y_0) + v_0 + (x - x_0)\sin r_0 + (y - y_0)(\cos r_0 - 1) \end{aligned} \quad (2-16)$$

As the first-order displacement function is used in the DDA method, the displacements at point (x, y) are then be approximated as:

$$\begin{aligned} u &= \varepsilon_x(x - x_0) + \left(\frac{1}{2}\gamma_{xy} - r_0\right)(y - y_0) + u_0 \\ v &= \left(\frac{1}{2}\gamma_{xy} + r_0\right)(x - x_0) + \varepsilon_y(y - y_0) + v_0 \end{aligned} \quad (2-17)$$

This approximate function is accurate enough when the rigid body rotation is relatively small; however, when the rigid body rotation is large, the expansion of the block will appear.

MacLaughlin and Sitar (1996) proposed a modification of using the second order term to account for the effect of the rigid body rotation. And this implementation is suitable when the angular rotation is less than 0.4 degree within a time-step.

Koo and Chern (1998) proposed a post-correction procedure. In their rigid block version DDA, in order to reduce the number of unknowns and thus reduce the computation burden. Each block was assumed to have only three degrees of freedoms, (u_0, v_0, r_0) . Simulation results showed that their rigid body version DDA was suitable for the engineering problems with low stress, small deformation and large displacement.

Another possible approach to deal with the uncontrolled block distortion due to rigid body rotation is to use an iterative method (Cheng and Zhang, 2000). In this approach, the displacement function $[T_i]$ used is

$$\begin{bmatrix} 1 & 0 & -(x-x_0)\frac{\sin r_0}{1+\cos r_0}-(y-y_0) & (x-x_0) & 0 & (y-y_0)/2 \\ 0 & 1 & -(y-y_0)\frac{\sin r_0}{1+\cos r_0}-(x-x_0) & 0 & (y-y_0) & (x-x_0)/2 \end{bmatrix} \quad (2-18)$$

and the displacement vector $[D_i]$ is

$$[D_i] = \{u_0 \quad v_0 \quad \sin r_0 \quad \varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}\}^T \quad (2-19)$$

At each time-step, r_0 is the value obtained in the previous time-step. This approach can obtain exact representation of strain term if the angular acceleration is zero or remains unchanged.

2.2.2 Development of particulate system in the DDA method

In the 2D simulation, particles are modeled as circular, rigid disks. Ke and Bray (1994) presented a complete formulation of the DDA method for a particulate system (DDAD). In the DDAD, each disk is assumed to be rigid with three degrees of freedom, (u_0, v_0, r_0) , in which (u_0, v_0) are rigid body translations and (r_0) is the rigid body rotation. The first order approximation of the disc movement and the equilibrium equations of the particulate system are then deducted in the similar way of the DDA method. Three kinds of contact modes, namely, disk to disk contact, disk to boundary edge contact and disk to boundary corner contact are considered. Different from the DDA method, in the DDAD, Ke and Bray adopted a dashpot at the contact position to simulate the physical attenuation of vibration.

In case of accumulated shear movement for locked contacts appeared in the particulate system (Ke, 1993), Sullivan et al. (2003) adopted a cumulative spring displacement instead of the original incremental displacement to calculate the shear contact force at each contact.

In order to represent the non-spherical shape of granular media with disk while keeping the simplicity of the computation, a disk cluster which contains a group of rigid individual disks connected permanently was developed (Thomas and Bray, 1999). Validation examples show that the disk cluster is able to provide a close representation of granular materials.

2.2.3 The sub-block concept in the DDA method

Ke (1993) first proposed an extended DDA code to incorporate ‘artificial’ joints for numerically simulating fracture propagation. The artificial joint concept consists of (1) cutting blocks into sub-blocks, and (2) binding the sub-blocks within a block with ‘cement’ to provide an incipient joint with tensile strength. The formulation is implemented by

extending the original Mohr–Coulomb contact law (friction, cohesion) to allow a tensile force to exist at the boundary of each sub-block.

Lin (1995, 1996) presented a sub-block system in the DDA method. The continuity condition of the sub-block system is preserved along the sub-block contacts and no relative displacement is allowed along the sub-block contact interface to satisfy the displacement compatibility. With the sub-block system, a better stress and strain distribution within each block can then be obtained.

2.2.4 Coupling the DDA method with finite element mesh

The main purpose of coupling the DDA block with finite element mesh is to improve the deformation ability as well as refine the distribution of stress field of the block.

Shyu (1993) first introduced the term “Nodal based Discontinuous Deformation Analysis (NDDA)”, which was a coupling method of the DDA method and the FEM. It used the nodal displacements as the basic unknowns. The conventional triangular element and four-node isoparametric element were modeled. Validation examples showed that the enhanced DDA worked well. However, the computer code was limited to linearly elastic deformations and the flexibility of the block boundary will introduce more contact judge, which will surely affect the computation efficiency.

Chang (1994) developed material nonlinearity formulation in the framework of the NDDA method. The piecewise linear stress-strain curve is used to solve the problem of material nonlinear step by step and principal strain was used as the criterion for the adjustment of Young’s Modulus and Poisson’s ratio.

Bao (2010) introduced the Mohr-Coulomb failure criterion in the NDDA method and represented the grid lines of the triangular elements as virtual joints, which made the occurring of fracture in an intact block when the failure criterion is violated became possible.

2.2.5 Implementation of alternative contact mechanism in the DDA method

The Augmented Lagrangian method (ALM) was first implemented into the original DDA program by Lin (1994, 1995, 1996) with a variable penalty number function. In each time step and all iterations within that time step, the optimal penalty number was found for all contacts automatically. This method was also been applied into the standard DDA method by Ning et al. (2010) and verified with two static problems. However, the static examples cannot show whether the modified program can still work properly in dynamic problems where contacts are created or diminished. Bao et al. (2012) provided a detailed

implementation of the ALM in the DDA program and compared it with the standard DDA on the computational efficiency and contact precision.

Other contact mechanism, such as the contact based on “soft spring” concept, was also implemented in the DDA method by some researchers (Cheng, 1998, Khan, 2010). By incorporating a simple joint model with the normal and shear stiffness instead of the penalty, the soft spring approach allows the DDA method to converge quickly.

2.2.6 Development of the rockbolt model in the DDA method

The rockbolt model was first presented by Shi (1988) and then implemented in the DDA program by Yeung (1991). It was a simple model in which the bolt was represented by a spring connecting two points in the block system, the finite tensile strength for the bolt and the shearing across the bolt were not considered.

Clapp (2003) developed a fully grouted rock bolt element in the DDA method. The main improvements of his fully grouted rock bolt model were that: 1) the linear stretching was corrected by improving the bolt formulation to allow accumulation of resisting force 2) the missing of the grout component in the original bolt model was accounted by the improvement of the material property values; 3) the bolt/block interaction was improved by bolt segmentation.

Moosavi and Grayeli (2006) coupled the fully grouted bolt algorithm with the DDA method to enable the realistic model of behavior of a blocky rock mass reinforced by the fully grouted rockbolt and the interaction between the bolt and the host rock.

2.2.7 Development of the 3D-DDA

Since most geotechnical engineering problems are three-dimensional, the two dimensional representation can only be a crude approximation. The development of the 3D-DDA method began in 2002 (Shi, 2002). Similar with the 2D-DDA, the first order displacement approximation of the 3D-DDA takes the following form (Hwang et al. 2004):

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = T_i D_i \quad (2-20)$$

where T_i is a 3*12 matrix corresponded with the block geometry and D_i is a 12×1 matrix corresponded with the block deformation(Hwang et al. 2004):

$$T_i = \begin{bmatrix} 1 & 0 & 0 & 0 & (z - z_s) & -(y - y_s) & (x - x_s) & 0 & 0 & (y - y_s) / 2 & 0 & (z - z_s) / 2 \\ 0 & 1 & 0 & -(z - z_s) & 0 & (x - x_s) & 0 & (y - y_s) & 0 & (x - x_s) / 2 & (z - z_s) / 2 & 0 \\ 0 & 0 & 1 & (y - y_s) & (x - x_s) & 0 & 0 & 0 & (z - z_s) & 0 & (y - y_s) / 2 & (x - x_s) / 2 \end{bmatrix} \quad (2-21)$$

$$D_i^T = \{u_0 \quad v_0 \quad w_0 \quad r_{x0} \quad r_{y0} \quad r_{z0} \quad \varepsilon_x \quad \varepsilon_y \quad \varepsilon_z \quad \gamma_{yz} \quad \gamma_{zx} \quad \gamma_{xy}\}$$

There are six contact types involved in the 3D-DDA, namely, vertex-to-vertex contact, vertex-to-edge contact, vertex-to-face contact, edge-to-edge contact, edge-to-face contact and face-to-face contact.

One of the most important aspects in the 3D-DDA method is the contact judgment algorithm. Jiang and Yeung (2004) presented a point-to-face contact model in the 3D-DDA method, in this algorithm, first the contact type was determined using the common-plane (C-P) method developed by Cundall (1988), and then the contact was decomposed to one or more point-to-face contacts. Wu et al. (2005) then introduced a vertex-to-face contact searching algorithm for the 3D-DDA using the topology. Yeung et al. (2007) also presented a model of edge-to-edge contact for the 3D-DDA based on the C-P method, in his paper, the detailed information of the choice of entrance faces and the criteria for interpenetration were also discussed. Wu (2008) pointed out that the number of iterations finding the correct C-P for a contact between two blocks using conventional algorithm depends on the accuracy of the initial guess of the C-P, and developed an algorithm calculating contact patterns among blocks directly based on their geometric circumstances.

Other developments of the 3D-DDA method including development of the spherical rigid block for the rock simulation (Yang, 2004), coupling the 3D-DDA method with the FEM (Beyabanaki et al. 2009) and using higher-order 3D-DDA method (Beyabanaki et al. 2010).

2.2.8 Other developments of the DDA method

Other developments of the DDA method in the past decades include employing higher order displacement functions in the DDA method to provide a more accurate distribution of stress and strain in a block (Koo et al. 1995, 1996); coupling the hydro-mechanical analysis in the DDA method to simulate the fluid flow procedure in the fracture rock mass (Kim et al. 1999, Jing et al. 2001, Chen et al. 2012); implementation of the Non-reflection boundary in the DDA method (Gu and Zhao, 2009) and developing of an alternative scheme for the corner-corner contact to solve the contact indeterminate state involved in the corner-corner contact (Bao and Zhao 2009, 2012) and so on.

2.2.9 Dynamic stability analysis of the DDA method

Hatzor and Feintuch (2001) validated the dynamic displacement of the DDA method by comparing with the analytical solution. The analytical integration of sinusoidal functions

was performed first to find the velocity and displacement functions with time, and the DDA simulation of a single block on an inclined surface subjected to the same acceleration function was carried out then. The agreement between the DDA method and the analytical solution was found to be quite good.

Hatzor et al. (2002) then validated the dynamic behavior of the DDA method by using the shaking table experiments. During the physical experiments, the displacement of a single block on an incline subject to dynamic loading is tested. The control parameters, including the dynamic factor, the spring stiffness, the maximum time-displacement ratio and the time step size were estimated. Finally they suggested that the choice of spring stiffness should be in conjunction with the time step size, and for accurate prediction of dynamic displace, artificial numerical damping is required. The DDA method is then applied to a real jointed rock slope subject to shaking load (Hatzor et al. 2004). Also, a kinetic damping in the order of 2% is suggested, in agreement with their previous work.

Ning and Zhao (2011) also investigated the block dynamic sliding by the DDA method. Two seismic loading input methods, namely, applying the seismic acceleration as constraint displacement time histories to the base and applying the seismic acceleration as volume forces to the sliding block were tested.

2.2.10 Fracture propagation simulation by the DDA method

Lin (1995) proposed a block fracturing algorithm through block centroids in the DDA method to allow the tensile and shear fracturing of intact blocks. The Mohr-Coulomb fracturing criterion with a tension cut-off was adopted, and the direction of the crack propagation was depended on the stress state of the block. No energy dissipation was considered during the fracturing. Validation examples demonstrated that the fracturing algorithm combined with the sub-block concept made the DDA method a powerful tool for modeling fracture propagation in block rock mass.

Pearce et al. (2000) applied the DDA method for the modeling of the concrete fracture. The stress state is checked against the fracture initiation criteria and the propagation direction is along the predetermined discontinuities. A fracture energy control scheme equivalent to that used in the Rigid Body-Spring Model (RBSM) was introduced in the DDA framework to release the contact gradually and associate it with some energy dissipation.

Ning (2010) presented an advanced discretization approach in the DDA framework in which the continuous rock domain was divided into small blocks by artificial joints first. The artificial joints were assigned to have the same strength with the intact block and the displacement compatibility across the artificial joints was satisfied by the bonding effect of the joint strength. The crack propagated along the pre-defined artificial joints and by

incorporating the maximum tensile strength criterion and the Mohr-coulomb criterion, both tensile and shear failures were considered.

2.2.11 Stability analysis of the rock cavern

Yueng (1993) applied the DDA method in the stability analysis of a horizontally-layered, cross-jointed mine roof. During the analysis, the effects of loading, joint friction angle and rock bolting were investigated.

Yeung and Leong (1997) applied the DDA method in the stability analysis of the tunneling, the effects of tunnel depth and joint attributes, including the joint orientation, joint spacing and joint friction angle were considered. The conclusions obtained from the DDA simulation generally agree with the existing knowledge of rock mass behavior, indicated the adequacy of the DDA method.

Tsesarsky and Hatzor (2005) also applied the DDA method to study the effect of the joint spacing and friction on the tunnel roof deflection in blocky rock masses.

Wu et al. (2005) applied the DDA method to simulate the mechanical behavior of the inclined layered joined rock masses during tunneling. The simulation result demonstrated that the DDA method is able to reproduce the stress arching developed in the trap door test.

2.2.12 Application of the DDA method to the landslides

Sitar and MacLaughlin (1997, 2001, 2005) applied the DDA method to study the kinematics effect on the movement of landslides and the results shown that the kinematics played an important role on the failure mode of the slope and the predicted velocity of sliding.

Doolin et al. (2002) applied the DDA method to sliding block and did the sensitivity analyses. Results show that the perturbation in the initial time steps of the simulation has the significant effect on the residual and relative error in the displacement of frictional sliding block.

2.2.13 Other applications of the DDA method

Huang (1997) proposed a new type of retaining wall and used the DDA method to study the behavior of the H-block wall system. The adequacy of the DDA method was confirmed by field test.

Gu and Zhao (2009) applied the DDA method in the modeling of the wave propagation problems in rock mass and did the parametric studies on the influencing factors.

Zhao et al. (2007) and Ning (2010) applied the DDA method in the rock blasting simulation. The example showed that the DDA method is able to model the blast-induced rock mass failure as well as the formation of the blasting piles.

2.3 Basic theory of the NDDA method

As reviewed in the previous section, in the DDA method, a complete first order approximation of the displacement field is used, thus every point inside a block has the same stresses and strains. The assumption of constant stresses within a block is reasonable when blocks are small and the stress field is relatively smooth. However, when a more complicated stress field or a more deformable block boundary is desired, the stress accuracy from the DDA method can be poor, incapable to model the true block behavior, thus there is a need to improve the stress evaluation in the DDA method. One possible approach is to introduce artificial joints and sub-blocks into the DDA block system (Koo and Chern, 1998; Ke, 1994; Cheng and Zhang, 2002). By dividing an intact block into many sub-blocks and setting a very strong penalty value to the artificial joints, a refined stress field within the block can be obtained. However, the sub-block approach will dramatically increase the contact analysis due to the increased number of blocks, thus the computation efficiency will be poor. Furthermore, even with a strong penalty value used, a small amount of normal and shear deformations of contact springs are inevitable, that a block refined by sub-blocks will be “softer” in a certain direction or in a certain zone, depending on the distribution of the internal artificial joints.

The second possible approach is to incorporate a higher order displacement function into the DDA method. With the development of second order (Koo et al. 1995) or third order displacement functions (Koo and Chern, 1995), a more accurate distribution of stress and strain in a block can be reached. More computational effort is needed as the number of unknowns for each block is increased from 6 for the standard DDA to 12 and 20 for the second order and the third order displacement functions. Though the higher order DDA is suitable for many problems, it may be inadequate when the size of the block is very large or has an odd shape, or when the stress and the strain fields vary rapidly.

Another alternative approach is to couple the DDA method with the finite element method (Shyu, 1993; Bao, 2010) which was firstly introduced by Shyu (1993) under the name ‘nodal-based DDA (NDDA)’. The NDDA method couples the DDA method’s unique block kinematics with the well-developed finite element mesh inside the block, after the combination, the DDA method’s block kinematics remains unchanged and the stress fields of blocks are refined.

The feasibility of coupling the DDA method and FEM method are (Bao, 2010):

- (1) Both methods are displacement-based methods;

- (2) Both methods use implicit algorithm;
- (3) Both methods establish the equilibrium equations by minimizing the total potential energy;
- (4) Both methods add stiffness, loading, and inertia sub-matrices to the global stiffness matrix of the simultaneous governing equations.

With these common properties, it is natural to add a finite element mesh in each block to improve the deformation ability and refine the stress distribution inside the block.

A general description of the NDDA method and formulation of important components is given in the following sections and the triangular element is used for illustration. It should be noticed that among all the components, the part of contact matrices is the most important part which distinguishes the NDDA from the FEM.

2.3.1 Displacement functions

In the original DDA method, basic unknowns are six displacements of a block, $\{u_0, v_0, r_0, \varepsilon_x, \varepsilon_y, \gamma_{xy}\}$, which are independent of the block shape. In the NDDA, the nodal displacements are the basic unknowns. Hence, the degrees of freedom of a block are now depending on the number of nodes it has. The more nodes a block has, the better deformation ability it will have.

Figure 2-2 Triangular element

Consider a triangular element as illustrated in Figure 2-2. The displacement field (u, v) of this triangular element can be represented as:

$$\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y \\ v &= \alpha_4 + \alpha_5 x + \alpha_6 y \end{aligned} \quad (2-22)$$

In this equation, (x, y) is the coordinate of any point inside the element and $\alpha_{i,(i=1\ldots6)}$ are the constants corresponded with the element geometry.

The six constants $\alpha_{i,(i=1\ldots6)}$ can be evaluated by solving the two sets of three simultaneous equations, for example, along the global x -axis direction we have the following equations:

$$\begin{aligned} u_i &= \alpha_1 + \alpha_2 x_i + \alpha_3 y_i \\ u_j &= \alpha_1 + \alpha_2 x_j + \alpha_3 y_j \\ u_m &= \alpha_1 + \alpha_2 x_m + \alpha_3 y_m \end{aligned} \quad (2-23)$$

in which (x_i, y_i) , (x_j, y_j) , (x_m, y_m) are the coordinates of the element vertex and u_i , u_j , u_m are the x -direction displacements of each element vertex, respectively.

Solving the six constants $\alpha_{i,(i=1\ldots6)}$, and substituting the values of $\alpha_{i,(i=1\ldots6)}$ into Eq. (2-22), we finally obtain:

$$\begin{aligned} u &= \frac{1}{2\Delta} [(a_i + b_i x + c_i y)u_i + (a_j + b_j x + c_j y)u_j + (a_m + b_m x + c_m y)u_m] \\ v &= \frac{1}{2\Delta} [(a_i + b_i x + c_i y)v_i + (a_j + b_j x + c_j y)v_j + (a_m + b_m x + c_m y)v_m] \end{aligned} \quad (2-24)$$

in which

$$\begin{aligned} a_i &= x_j y_m - x_m y_j \\ b_i &= y_j - y_m \\ c_i &= x_m - x_j \end{aligned} \quad (2-25)$$

with the other coefficients obtained by a cyclic permutation of subscripts in the order i, j, m , and where Δ is the area of Δ_{ijm} obtained by

$$2\Delta = \det \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} \quad (2-26)$$

The six components of element vertex displacements can be represented as a vector:

$$a = \begin{Bmatrix} d_i \\ d_j \\ d_m \end{Bmatrix} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} \quad (2-27)$$

and the displacement field of the triangular element can then be expressed in a standard form:

$$u = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} \quad (2-28)$$

where

$$\begin{aligned} N_i &= \frac{a_i + b_i x + c_i y}{2\Delta} \\ N_j &= \frac{a_j + b_j x + c_j y}{2\Delta} \\ N_m &= \frac{a_m + b_m x + c_m y}{2\Delta} \end{aligned} \quad (2-29)$$

2.3.2 Simultaneous Equations

The process of yielding the global equilibrium equations for the NDDA method is similar with that of the DDA method.

For the block system, the total energy Π_i can be expressed as:

$$\Pi_i = [f]^T [d_B] + \frac{1}{2} [\varepsilon]^T [\sigma] dA \quad (2-30)$$

where $[f]^T$ is the external load acted on element node, $[d_B]$ is the displacement of the element node; $[\varepsilon]$ and $[\sigma]$ are the strain and stress of the element, respectively.

The derivatives of individual potential energy with respect to deformation variables are then computed and the corresponding sub-matrices are formed separately.

$$\begin{cases} \frac{\partial \Pi_i}{\partial u_i} = 0 \\ \frac{\partial \Pi_i}{\partial v_i} = 0 \end{cases} \quad (2-31)$$

where u_i and v_i are the deformation variable of the node i .

Assuming there are n nodes in the system, the global equilibrium equations will have similar form of the global equilibrium equations in the DDA method:

$$KD=F \quad (2-32)$$

where K is the global stiffness matrix, a $2n \times 2n$ matrix. D is the unknown displacement vector and F is the equivalent force vector. Eq. (2-32) can then be written in a submatrix form,

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \cdots & k_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{n1} & k_{n2} & k_{n3} & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{Bmatrix} \quad (2-33)$$

in which each K_{ii} is a 2×2 submatrix which comes from the differentiation of $\frac{\partial^2 \Pi_i}{\partial u_i \partial v_i}$;

d_i is a 2×1 submatrix and denotes the displacements unknowns $\{u_i, v_i\}^T$ of node i ; f_i is a 2×1 submatrix and denotes the equivalent nodal forces $\{f_{ix}, f_{iy}\}^T$ applied on node i .

2.3.3 Elastic Sub-matrices

The strain energy Π_e^i of an element i is defined as

$$\Pi_e^i = \frac{1}{2} \iint \varepsilon_i^T \sigma_i dx dy + \iint \varepsilon_i^T \sigma_i^0 dx dy \quad (2-34)$$

where σ_i^0 is the initial stress of element i , and the integration is over the entire area of element i .

Assuming the elements are linear elastic for each time-step, and the relationship between stresses and strains can be expressed in a linear form, for plane stress problem:

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \quad (2-35)$$

where E and ν are the Young's modulus and Poisson's ratio of the element, respectively. For the plane strain problem, the stress-strain relationship can be simply represented by replace the E , ν in the Eq. (2-35) as $E/(1-\nu^2)$ and $\nu/(1-\nu)$, respectively.

The strain at any point within the element is

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{pmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = SU \quad (2-36)$$

in which S is defined as

$$S = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{pmatrix}$$

Substituting Eq. (2-28) into the Eq. (2-36), we have

$$\varepsilon = SNa = [B_i \quad B_j \quad B_m] \begin{Bmatrix} d_i \\ d_j \\ d_m \end{Bmatrix} = Ba \quad (2-37)$$

in which

$$\begin{aligned}
B = SN &= \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_j}{\partial x} & 0 & \frac{\partial N_m}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & \frac{\partial N_j}{\partial y} & 0 & \frac{\partial N_m}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_j}{\partial y} & \frac{\partial N_m}{\partial x} & \frac{\partial N_m}{\partial y} \end{bmatrix} \\
&= \frac{1}{2\Delta} \begin{bmatrix} b_i & 0 & b_j & 0 & b_m & 0 \\ 0 & c_i & 0 & c_j & 0 & c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{bmatrix}
\end{aligned} \tag{2-38}$$

By substituting Eq. (2-35), the Eq. (2-34) can be rewritten as

$$\Pi_e^i = \frac{1}{2} \iint \varepsilon_i^T E_i \varepsilon_i dx dy + \iint \varepsilon_i^T \sigma_i^0 dx dy \tag{2-39}$$

Substituting Eq. (2-37) into Eq. (2-39), we have

$$\begin{aligned}
\Pi_e^i &= \frac{1}{2} \iint a_i^T B_i^T E_i B_i a_i dx dy + \iint a_i^T B_i^T \sigma_i^0 dx dy \\
&= \frac{1}{2} a_i^T B_i^T E_i B_i a_i \iint dx dy + a_i^T B_i^T \sigma_i^0 \iint dx dy \\
&= \frac{A_0}{2} a_i^T B_i^T E_i B_i a_i + A_0 a_i^T B_i^T \sigma_i^0
\end{aligned} \tag{2-40}$$

where A_0 is the initial area of element i .

Minimizing the strain energy Π_e^i with respect to the element displacement, the local stiffness submatrix and the equivalent nodal force vector can be obtained

$$K_e^i = \frac{\partial^2 \Pi_e^i}{\partial a_i \partial a_i} = \frac{A_0}{2} \frac{\partial^2 (a_i^T B_i^T E_i B_i a_i)}{\partial a_i \partial a_i} = A_0 B_i^T E_i B_i \tag{2-41}$$

$$f_e^i = - \left. \frac{\partial \Pi_e^i}{\partial a_i} \right|_0 = -A_0 \left. \frac{\partial (a_i^T B_i^T \sigma_i^0)}{\partial a_i} \right|_0 = -A_0 B_i^T \sigma_i^0 \tag{2-42}$$

K_e^i is a 6×6 sub-matrix and is added to the corresponding position of global stiffness matrix $[K]$; f_e^i is a 6×1 sub-matrix and is added to the corresponding row of the force vector f .

2.3.4 Equivalent nodal forces

a) Point load

Assuming there is a point loading force $P = \{p_x, p_y\}^T$ on point (x, y) within the element i , as illustrated in Figure 2-3. The displacements of point (x_i, y_i) are

$$U = N_i a_i \quad (2-43)$$

The potential energy of the point loading P is

$$\Pi_p^i = -(p_x u + p_y v) = -U^T P = -a_i^T N_i^T P \quad (2-44)$$

Minimizing Π_p^i , the local force submatrix f_i can be obtained

$$f_i = \begin{Bmatrix} f_a \\ f_b \\ f_c \end{Bmatrix} = - \frac{\partial \Pi_p^i}{\partial a_i} \Big|_0 = \frac{\partial (a_i^T N_i^T P)}{\partial a_i} \Big|_0 = N_i^T P \quad (2-45)$$

each component f_i ($i=a, b, c$) is a 2×1 submatrix which is added to the corresponding rows of the right hand side of global equation system.

Figure 2-3 Point load in an element

b) Distributed linear load along element boundary

Assuming the load is distributed linearly along on the element boundary $\bar{ij}$, as shown in Figure 2-4. The equation of the loading line is:

$$\begin{aligned} x &= (x_j - x_i)t + x_i \\ y &= (y_j - y_i)t + y_i \quad 0 \leq t \leq 1 \end{aligned} \quad (2-46)$$

where (x_i, y_i) , (x_j, y_j) are the coordination of the node i and node j , respectively.

Figure 2-4: Distributed line load on block boundary

The length of this line segment is:

$$l = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2} \quad (2-47)$$

The potential energy of the line loading $(F_x(t), F_y(t))$ is given by

$$\begin{aligned} \Pi_l &= - \int_0^1 (u \quad v) \begin{pmatrix} F_x(t) \\ F_y(t) \end{pmatrix} l dt \\ &= - [D_l]^T \int_0^1 [N_l]^T \begin{pmatrix} F_x(t) \\ F_y(t) \end{pmatrix} l dt \\ &= - (u_i \quad v_i \quad u_j \quad v_j) \int_0^1 \begin{bmatrix} 1-t & 0 & t & 0 \\ 0 & 1-t & 0 & t \end{bmatrix}^T \begin{pmatrix} F_x(t) \\ F_y(t) \end{pmatrix} l dt \end{aligned} \quad (2-48)$$

and the force sub-matrix f_i is then obtained by minimizing the potential energy Π_l with respect to the nodal displacements

$$f_l = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix} = - \frac{\partial^2 \Pi_l}{\partial d} \bigg|_0 = \int_0^1 [N_l]^T \begin{pmatrix} F_x(t) \\ F_y(t) \end{pmatrix} l dt \quad (2-49)$$

when the line load is uniformly distributed along the boundary $\bar{ij}$, in which

$$\begin{pmatrix} F_x(t) \\ F_y(t) \end{pmatrix} = \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \quad (2-50)$$

Then Eq. (2-50) can be written as

$$f_l = \int_0^l \begin{bmatrix} 1-t & 0 \\ 0 & 1-t \\ t & 0 \\ 0 & t \end{bmatrix}^T \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} dl = \frac{l}{2} \begin{Bmatrix} f_x \\ f_y \\ f_x \\ f_y \end{Bmatrix} \quad (2-51)$$

c) Distributed body forces

Assuming the constant body force $b = \{b_x, b_y\}^T$ is applied on the element i , the potential energy of this constant body force is given by

$$\begin{aligned} \Pi_{bd}^i &= -\iint U_i^T b dx dy = -\iint a_i^T N_i^T b dx dy \\ &= -a_i^T \left(\iint N_i^T dx dy \right) b \end{aligned} \quad (2-52)$$

since

$$\iint N_i^T dx dy = \frac{A_0}{3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (2-53)$$

substituting Eq. (2-53) into Eq. (2-52), we have

$$\Pi_{bd}^i = -a_i^T \frac{A_0}{3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = -\frac{A_0}{3} a_i^T \begin{Bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{Bmatrix} \quad (2-54)$$

The force submatrix f_{bd}^i is obtained by take the derivation of Π_{bd}^i with respective to the element displacement variables

$$f_{bd}^i = \begin{Bmatrix} f_a \\ f_b \\ f_c \end{Bmatrix} = - \frac{\partial \Pi_b^i}{\partial b_i} \bigg|_0 = \frac{A_0}{3} \begin{Bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{Bmatrix} \quad (2-55)$$

and is added to the corresponding rows of the force vector of global equations.

2.3.5 Inertia Sub-matrices

Denoting the (u, v) as the time dependent displacement of any point (x, y) of inside the element i and m as the mass per unit area, the force of inertia per unit area can be represented as:

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = -m \begin{pmatrix} \frac{d^2 u}{dt^2} \\ \frac{d^2 v}{dt^2} \end{pmatrix} \quad (2-56)$$

and

$$\begin{Bmatrix} \frac{d^2 u}{dt^2} \\ \frac{d^2 v}{dt^2} \end{Bmatrix} = \frac{d^2}{dt^2} \begin{Bmatrix} u \\ v \end{Bmatrix} = \frac{d^2 (N_i a_i)}{dt^2} = N_i \frac{d^2 a_i}{dt^2} \quad (2-58)$$

Considering a_i as the displacements of element i in current time step, a_{i0} as the displacements of the previous time step and δ as the time interval, the second order differentiation of the element displacement can be represented as

$$\begin{aligned} \frac{d^2 a(t)}{dt^2} &= \frac{1}{\delta^2} (a(t) - 2a(t - \delta) + a(t - 2\delta)) \\ &= \frac{1}{\delta^2} \{ [a(t) - a(t - \delta)] - [a(t - \delta) - a(t - 2\delta)] \} \end{aligned} \quad (2-59)$$

$$a = a(t) - a(t - \delta) \quad (2-60)$$

$$a_0 = a(t - \delta) - a(t - 2\delta) \quad (2-61)$$

$$\frac{d^2 a_i}{dt^2} = \frac{1}{\delta^2} (a_i - a_{i0}) \quad (2-62)$$

The potential energy of the inertia force of the element i can be represented as

$$\Pi_m^i = m \iint a_i^T N_i^T N_i \frac{d^2 a_i}{dt^2} dx dy \quad (2-63)$$

Substitute Eq. (2-62) into Eq. (2-63), then

$$\Pi_m^i = \frac{m}{\delta^2} \iint a_i^T N_i^T N_i (a_i - a_{i0}) dx dy \quad (2-64)$$

The derivation of Π_m^i is computed to minimize the potential energy

$$K_m^i = \frac{\partial^2 \Pi_m^i}{\partial a_i \partial a_i} = \frac{2m}{\delta^2} \iint N_i^T N_i dx dy \quad (2-65)$$

where K_m^i is a 6×6 sub-matrix and is added to the corresponding position of global stiffness matrix K .

$$\begin{aligned} f_m^i &= - \left. \frac{\partial \Pi_m^i}{\partial a_i} \right|_0 = - \frac{m}{\delta^2} \iint N_i^T N_i (2a_{i0} - a_{i0}) dx dy \\ &= - \frac{m}{\delta^2} \left(\iint N_i^T N_i dx dy \right) a_{i0} \end{aligned} \quad (2-66)$$

forms a 6×1 sub-matrix and is added to the corresponding row of the force vector f .

2.3.6 Displacement Constraints

In the NDDA method, there are three kinds of boundary conditions: free boundary, fixed boundary and rolling boundary along a direction. For the free boundary, no constraint is needed. For the fixed boundary, constraints along both x and y directions are needed. For the rolling boundary along a direction, one constraint corresponded with a specified direction is needed. In this case, only the sub-matrix about the constraint along a direction is needed to be deduced.

Assuming the direction angle of the rolling boundary line is α , then the direction normal to the rolling boundary line is $\alpha + 90^\circ$. δ is the displacement of support along the normal direction of the rolling boundary line. Denote the normal vector of the slope as

$$n = \begin{Bmatrix} l_x \\ l_y \end{Bmatrix} \quad (2-67)$$

as the direction angle of the rolling boundary line is α , we have

$$\begin{aligned} l_x &= \cos(\alpha + 90^\circ) = -\sin \alpha \\ l_y &= \sin(\alpha + 90^\circ) = \cos \alpha \end{aligned} \quad (2-68)$$

A very stiff spring is used in the constrained direction to achieve the equivalent effect. Denoting the stiffness of the constrain spring as K and (u_i, v_i) as the displacement of the constrained point, the spring displacement is

$$d = (\delta - (l_x u + l_y v)) \quad (2-69)$$

And the spring force is:

$$f = -kd = -k(\delta - (l_x u_i + l_y v_i)) \quad (2-70)$$

The strain energy of this spring is:

$$\begin{aligned} \Pi_c &= \frac{k}{2} d^2 = \frac{k}{2} ((l_x u_i + l_y v_i) - \delta)^2 \\ &= \frac{k}{2} (u_i \quad v_i) \begin{pmatrix} l_x \\ l_y \end{pmatrix} \begin{pmatrix} l_x & l_y \end{pmatrix} \begin{pmatrix} u_i \\ v_i \end{pmatrix} - k\delta (u_i \quad v_i) \begin{pmatrix} l_x \\ l_y \end{pmatrix} + \frac{k}{2} \delta^2 \\ &= \frac{k}{2} d_i^T n n^T d_i - k\delta n^T d_i + \frac{k}{2} \delta^2 \end{aligned} \quad (2-71)$$

Then the sub-matrices can be obtained by taking the derivatives of the strain energy of the constraint spring

$$k_c = \frac{\partial^2 \Pi_c}{\partial d_i \partial d_i} = k n n^T \quad (2-72)$$

where k_c is a 2×2 sub-matrix which is added to the intersection of i -th row and i -th column in the global stiffness matrix.

Taking the derivatives of the strain energy of the displacements, we have

$$f_c = -\frac{\partial \Pi_c(0)}{\partial d_{ri}} = k\delta n \quad (2-73)$$

which is a 2×1 vector and is added to the i -th row of force vector in the global equation.

2.3.7 Contact sub-matrices

In the DDA method, a penalty-based contact system is employed to deal with contacts between blocks, and the Coulomb's friction law is employed to control the contact modes. In the NDDA method, the same contact system is adopted. There are three types of contacts, namely, vertex-to-vertex contact, vertex-to-edge contact and edge-to-edge

contact. Also, the vertex-to-edge contact is the basic contact type. In the NDDA method, only boundary nodes will be considered in the contact detection, as shown in Figure 2-5.

Figure 2-5 Vertex-to-edge contact model (After Bao, 2010)

Figure 2-6 Normal and shear contact displacements (After Bao, 2010)

In the penalty method, the contact force is assumed to be proportional to the penetration distance. Suppose the coordinates of vertex i are (x_i, y_i) and the coordinates of its corresponding contact edge $\overline{jm}$ ends at (x_j, y_j) and (x_m, y_m) , respectively. The displacements of node i, j, m are

$$\begin{aligned} d_i &= \{u_i \ v_i\}^T \\ d_j &= \{u_j \ v_j\}^T \end{aligned} \quad (2-74)$$

$$d_m = \{u_m \ v_m\}^T$$

a) Normal contact spring sub-matrix

Once a contact is detected, a very stiff spring is introduced along the normal direction of the contact edge to prevent the inter-penetration. The normal spring displacement at the end of current time step is

$$d_n = \frac{\Delta}{l} = \frac{1}{l} \begin{vmatrix} 1 & x_i + u_i & y_i + v_i \\ 1 & x_j + u_j & y_j + v_j \\ 1 & x_m + u_m & y_m + v_m \end{vmatrix} \quad (2-75)$$

where Δ is the area of Δ_{ijm} at the end of current time step and l is the length of edge $\overline{jm}$.

$$l = \sqrt{(x_j + u_j - x_m - u_m)^2 + (y_j + v_j - y_m - v_m)^2} \\ \approx \sqrt{(x_j - x_m)^2 + (y_j - y_m)^2} \quad (2-76)$$

$$\Delta = \begin{vmatrix} 1 & x_i + u_i & y_i + v_i \\ 1 & x_j + u_j & y_j + v_j \\ 1 & x_m + u_m & y_m + v_m \end{vmatrix} \\ = \frac{1}{l} \left(\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} + \begin{vmatrix} 1 & u_i & y_i \\ 1 & u_j & y_j \\ 1 & u_m & y_m \end{vmatrix} + \begin{vmatrix} 1 & x_i & v_i \\ 1 & x_j & v_j \\ 1 & x_m & v_m \end{vmatrix} + \begin{vmatrix} 1 & u_i & v_i \\ 1 & u_j & v_j \\ 1 & u_m & v_m \end{vmatrix} \right) \quad (2-77)$$

the last term of the above equation can be neglected as it is a second order infinitesimal. Denote A_0 as the area of Δ_{ijm} at the beginning of current time step

$$A_0 = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} \quad (2-78)$$

and

$$n = \begin{Bmatrix} n_i \\ n_j \\ n_m \end{Bmatrix} = \begin{Bmatrix} y_j - y_m \\ x_m - x_j \\ y_m - y_i \\ x_i - x_m \\ y_i - y_j \\ x_j - x_i \end{Bmatrix} \quad (2-79)$$

Eq. (2-77) can be rewritten as

$$\begin{aligned} \Delta \approx & A_0 + u_i(y_j - y_m) + v_i(x_m - x_j) \\ & + u_j(y_m - y_i) + v_j(x_i - x_m) \\ & + u_m(y_i - y_j) + v_m(x_j - x_i) \end{aligned} \quad (2-80)$$

$$\begin{aligned} \Delta \approx & A_0 + \begin{Bmatrix} y_j - y_m & x_m - x_j \end{Bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} \\ & + \begin{Bmatrix} y_m - y_i & x_i - x_m \end{Bmatrix} \begin{Bmatrix} u_j \\ v_j \end{Bmatrix} \\ & + \begin{Bmatrix} y_i - y_j & x_j - x_i \end{Bmatrix} \begin{Bmatrix} u_m \\ v_m \end{Bmatrix} \end{aligned} \quad (2-81)$$

$$\Delta \approx A_0 + n^T d \quad (2-82)$$

and the normal spring displacement is

$$d_n = \frac{\Delta}{l} \approx \frac{1}{l} (A_0 + n^T d) \quad (2-83)$$

Assuming the stiffness of the normal contact spring is k_n , then the potential energy of the contact spring is

$$\Pi_{kn} = \frac{k_n}{2} d_n^2 \quad (2-84)$$

Substituting Eq. (2-83) into Eq. (2-84), we have

$$\begin{aligned} \Pi_{kn} &= \frac{k_n}{2l^2} (A_0 + n^T d)^2 \\ &= \frac{k_n}{2l^2} (d^T n n^T d + 2A_0 n^T d + A_0^2) \end{aligned} \quad (2-85)$$

The contact sub-matrices can be obtained by taking the derivatives of the potential energy Π_{kn}

$$K_{kn} = \begin{bmatrix} k_{ii} & k_{ij} & k_{im} \\ k_{ji} & k_{jj} & k_{jm} \\ k_{mi} & k_{mj} & k_{mm} \end{bmatrix} = \frac{\partial^2 \Pi_{kn}}{\partial d \partial d} = \frac{k_n}{l^2} n n^T \quad (2-86)$$

in which each component is a 2×2 sub-matrix and is added to the global coefficient matrix in the corresponding position.

Also, minimize Π_{kn} by taking the derivatives at $d=0$

$$f_{kn} = \begin{Bmatrix} f_i \\ f_j \\ f_m \end{Bmatrix} = -\frac{\partial \Pi_{kn}(0)}{\partial d} = -\frac{k_n A_0}{l^2} n \quad (2-87)$$

the force sub-matrix is calculated and added to the right side of the global equilibrium equation.

b) Shear contact spring sub-matrix

A contact position may be locked if the shear resistance is larger than the shear force along the contact edge. In this case, a shear spring is added to prevent the displacement along the contact edge. Assuming the positive direction along edge $\overline{jm}$ is from j to m and the target point to be locked on $\overline{jm}$ is i'' , the movement of the i relative to i'' along the direction of $\overline{jm}$ is fixed by the shear spring, which is given by

$$\begin{aligned} d_i &= \frac{\vec{i}i \times \overline{jm}}{l} \\ &= \frac{1}{l} \left\{ (x_i + u_i - x_{i'} - u_{i'} \quad y_i + v_i - y_{i'} - v_{i'}) \right\} \begin{Bmatrix} x_m + u_m - x_j - u_j \\ y_m + v_m - y_j - v_j \end{Bmatrix} \\ &= \frac{1}{l} \left\{ (x_i + u_i - x_{i'} - u_{i'} \quad y_i + v_i - y_{i'} - v_{i'}) \right\} \begin{Bmatrix} x_m - x_j \\ y_m - y_j \end{Bmatrix} \\ &\quad + \frac{1}{l} \left\{ (x_i + u_i - x_{i'} - u_{i'} \quad y_i + v_i - y_{i'} - v_{i'}) \right\} \begin{Bmatrix} u_m - u_j \\ v_m - v_j \end{Bmatrix} \end{aligned} \quad (2-88)$$

The second order infinitesimal term

$$\{u_i - u_{i'} \quad v_i - v_{i'}\} \begin{Bmatrix} u_m - u_j \\ v_m - v_j \end{Bmatrix}$$

and

$$\begin{Bmatrix} x_i - x_{i'} & y_i - y_{i'} \end{Bmatrix} \begin{Bmatrix} u_m - u_j \\ v_m - v_j \end{Bmatrix}$$

can be neglected and the Eq. (2-88) can then be rewritten as

$$d_t \approx \frac{1}{l} \left\{ (x_i - x_{i'} + u_i - u_{i'} \quad y_i - y_{i'} + v_i - v_{i'}) \right\} \begin{Bmatrix} x_m - x_j \\ y_m - y_j \end{Bmatrix} \quad (2-89)$$

The position and the displacement of the projection point i' can be presented in terms of the position and the displacement of point j and m

$$\begin{aligned} x_{i'} &= (1-t)x_j + tx_m \\ y_{i'} &= (1-t)y_j + ty_m \\ u_{i'} &= (1-t)u_j + tu_m \\ v_{i'} &= (1-t)v_j + tv_m \end{aligned} \quad (2-90)$$

where $0 \leq t \leq 1$, and

$$t = \frac{\overrightarrow{ji'} \times \overrightarrow{jm}}{l^2} = \frac{1}{l^2} \begin{Bmatrix} x_i - x_j & y_i - y_j \end{Bmatrix} \begin{Bmatrix} x_m - x_j \\ y_m - y_j \end{Bmatrix} \quad (2-91)$$

Eq. (2-89) can be rewritten as

$$\begin{aligned} d_t &\approx \frac{S_0}{l} + \frac{1}{l} \begin{Bmatrix} x_m - x_j & y_m - y_j \end{Bmatrix} \begin{Bmatrix} u_i - u_{i'} \\ v_i - v_{i'} \end{Bmatrix} \\ &= \frac{S_0}{l} + \frac{1}{l} \begin{Bmatrix} x_m - x_j & y_m - y_j \end{Bmatrix} \left(\begin{Bmatrix} u_i \\ v_i \end{Bmatrix} - (1-t) \begin{Bmatrix} u_j \\ v_j \end{Bmatrix} - t \begin{Bmatrix} u_m \\ v_m \end{Bmatrix} \right) \end{aligned} \quad (2-92)$$

where

$$S_0 = \begin{Bmatrix} x_i - x_{i'} & y_i - y_{i'} \end{Bmatrix} \begin{Bmatrix} x_m - x_j \\ y_m - y_j \end{Bmatrix} \quad (2-93)$$

Denote

$$t = \begin{Bmatrix} x_m - x_j \\ y_m - y_j \\ (t-1)(x_m - x_j) \\ (t-1)(y_m - y_j) \\ -t(x_m - x_i) \\ -t(y_m - y_j) \end{Bmatrix}$$

then

$$d_t \approx \frac{1}{l}(S_0 + t^T d) \quad (2-94)$$

The potential energy of the shear contact spring is

$$\begin{aligned} \Pi_{kt} &= \frac{k_t}{2} d_t^2 \\ &= \frac{k_t}{2l^2} (S_0 + t^T d)^2 \\ &= \frac{k_t}{2l^2} (d^T t t^T d + 2S_0 t^T d + s_0^2) \end{aligned} \quad (2-95)$$

The shear contact sub-matrices can be obtained by taking the derivatives of the potential energy Π_{kt}

$$K_{kt} = \begin{bmatrix} k_{ii} & k_{ij} & k_{im} \\ k_{ji} & k_{jj} & k_{jm} \\ k_{mi} & k_{mj} & k_{mm} \end{bmatrix} = \frac{\partial^2 \Pi_{kt}}{\partial d \partial d} = \frac{k_t}{l^2} t t^T \quad (2-96)$$

in which each component is a 2×2 sub-matrix and is added to the global coefficient matrix in the corresponding position.

Also, minimize Π_{kt} by taking the derivatives at $d=0$

$$f_{kt} = \begin{Bmatrix} f_i \\ f_j \\ f_m \end{Bmatrix} = -\frac{\partial \Pi_{kt}(0)}{\partial d} = -\frac{k_n S_0}{l^2} t \quad (2-97)$$

the force sub-matrix is calculated and added to the right side of the global equilibrium equation.

c) Friction sub-matrix

In the NDDA method, the Coulomb friction law is used as the governing criterion. When the sliding is allowed according to the Coulomb friction law, the shear contact spring is removed from the contact position and the friction force is calculated to accounts for the energy consumption during the sliding. The friction force in the NDDA method is treated as an external loading which is obtained from the explicit method:

$$F = -k_n d_n \tan \varphi + c \quad (2-98)$$

where c and φ are the cohesion and friction angle of the contact face, respectively. d_n is the penetration normal to the contact face and k_n is the normal spring stiffness. $k_n d_n$ accounts for the normal force on the contact face, with the positive direction in tension.

Assume the slide displacement along the contact edge $\overline{jm}$ is d_t (see Figure 2-6), the work done by friction force is

$$\Pi_f = -F d_t = -k_n d_n d_t \tan \varphi + c d_t \quad (2-99)$$

Substituting Eq. (2-83) and Eq. (2-94) to (2-99), we have

$$\Pi_f = \frac{k_n}{l^2} (S_0 + t^T d) (A_0 + n^T d) - c (S_0 + t^T d) \quad (2-100)$$

The stiffness sub-matrix and the force sub-matrix can be obtained by taking the derivative of the potential energy Π_f

$$K_f = \begin{bmatrix} k_{ii} & k_{ij} & k_{im} \\ k_{ji} & k_{jj} & k_{jm} \\ k_{mi} & k_{mj} & k_{mm} \end{bmatrix} = \frac{\partial^2 \Pi_f}{\partial d \partial d} = \frac{k_n \tan \varphi}{l^2} t n^T \quad (2-101)$$

$$f_f = \begin{Bmatrix} f_i \\ f_j \\ f_m \end{Bmatrix} = -\frac{\partial \Pi_f(0)}{\partial d} = -\frac{k_n \tan \varphi}{l^2} (S_0 n + A_0 t) + c t \quad (2-102)$$

2.4 Comparison between the NDDA method and other numerical methods

As a hybrid method, the NDDA method shares the similarities with the DDA method and the FEM. However, the distinct features which differentiate the NDDA method from other numerical methods are also obvious. In this section, the NDDA method is compared with the FEM, the DDA method and the DEM.

2.4.1 NDDA vs. FEM

FEM has been the most popular numerical method in engineering sciences, including rock mechanics and rock engineering. First introduced by Clough (1975) for plane stress problems, the method gained rapidly development and promoted in many scientific and engineering fields in the past half century.

A typical analysis of the FEM involves (1) dividing the problem domain into a finite number of sub-domains with standard shapes, followed by (2) using a set of element equations to approximately represent each sub-domain of the original problem, and (3) assembling all sets of local elemental equations into a global system of algebraic equations and imposing the proper initial force and boundary conditions for the final solution.

The NDDA method is similar with the FEM in aspects that (1) both methods discretize the problem domain into a finite number of elements; (2) both methods approximate the unknown function over each element through a trial function of its nodal values and (3) both methods are the implicit based method and establish the global equilibrium equations by minimizing the total potential energy.

On the other hand, the displacement compatibility in the FEM is maintained by keeping constant element-node connectivity topology and consistent orders of the trial function between the neighboring elements. While in the NDDA method, displacement compatibility is replaced by the contact conditions between blocks. The independent blocks have connections with other blocks only when they are contacting with each other, and stiff springs are added at the contact positions to enforce contact constraints, similar with that in the DDA method.

2.4.2 NDDA vs. DDA

The DDA method belongs to the family of the discrete element methods. It originated from a back analysis algorithm for determining a best fit strains and deformations of a block system from measured displacements and deformations, and later extended to a new numerical model to perform the complete deformation analysis of a block system.

The DDA method is an implicit method which chooses the displacements as variables and establishes equilibrium equations by minimizing the total potential energy and solves equilibrium equations in the manner of matrix analysis of structure. The formulation of the DDA method is strictly based on the rules of classical mechanics, displacements, deformations and strains are permitted for each block, and sliding, opening and closing of block interfaces are permitted for the entire block system.

The unique features of the DDA can be generalized as follows (Ke, 1993): (1) complete block kinematics and its numerical realization; (2) perfect first order displacement

approximation; (3) strict postulate of equilibrium; (4) correct energy consumption; (5) friction loss along contacts is the sole source of energy consumption without adding any artificial damping; (6) easy incorporate parameters and mechanical relations.

The NDDA method inherits almost all the above mentioned features of the DDA method. The differences between the NDDA method and the DDA method are:

- 1) In the original DDA method, basic unknowns are six displacements of a block, $\{u_0, v_0, r_0, \varepsilon_x, \varepsilon_y, \gamma_{xy}\}$, which are independent of the block shape. In the NDDA, the basic unknowns are the nodal displacements.
- 2) In the DDA method, the stress and strain distributed uniformly within one block as the first order approximation is used. While in the NDDA method, by introducing the finite element mesh in the single block, the stress field within the block is refined and the deformability of the block is improved.
- 3) Unless the sub-block method is used, the DDA method is unable to deal with the fracture propagation simulation. In the NDDA method, by introducing the Mohr-Coulomb failure criterion as the governing mechanism and representing the grid lines of the triangular elements as virtual joints, the occurrence of fracture in an intact block when the failure criterion is violated becomes possible (Bao, 2010).

2.4.3 NDDA vs. DEM

The Distinct Element Method (DEM) is a numerical method falls within the general classification of discontinuum analysis technique. Originally conceived as a tool of modeling progressive failure of rock slopes (Cundall, 1971), the formulation and development of the distinct element method has progressed for nearly 40 years.

The DEM method presents the rock mass as an assemblage of discrete blocks and views the joints as interfaces between the blocks, similar as that in the DDA method and the NDDA method.

However, the DEM is an explicit method which calculates the state of the system at the later time step from the state of the system at the current time step. The calculations performed in the DEM including the application of a force-displacement law to obtain contact forces from known displacements and the application of Newton's law of motion to obtain the velocities and displacements of each block from known forces acting on it. Different from the implicit methods such as the DDA method and the NDDA method, the explicit method is more robust because it is easy to implement and no extra computation is required. On the other hand, in order to ensure the constant velocities and accelerations within one time step, the sufficiently small time step must be used in the DEM. At the same time, to reach the equilibrium state as quickly as possible, a mechanical damping is

used in DEM. While in the implicit DDA/NDDA method, as the global equilibrium equation is unconditional stable, a relatively large time step can be used and no numerical damping is needed.

The other difference between the DEM and the DDA/NDDA method is the contact constraint mechanism. In both the DDA method and the NDDA method, the penalty constraint is used at each contact position. However, in the DEM, the joint behavior model based on the physical properties of the joint is adopted. With the coulomb slip model as the basic constitutive model, other more comprehensive model, such as the displacement-weakening model, the Barton-Bandis joint model (Barton, 1985) are also available in DEM.

2.4.4 NDDA vs. NMM

First presented by Shi (1991, 1995), the numerical manifold method (NMM) is a general method in simulating the material response to external and internal changes in loads. The approximation in the NMM is established based on covers. There are two types of covers in the NMM: the mathematical covers for the entire problem domain and the physical covers for the intersection of each mathematical cover. Each physical cover is assigned a local function and all the local functions are pasted together by partition of unity (PU).

The NMM can be degraded to the standard FEM if constants are adopted as the cover function for each physical cover, the nodal shape function in the FEM are adopted at the PU functions for the mathematical covers and the physical features do not intersect the finite elements. At the same time, the NMM can be degraded to the DDA method if one block is discretized into one manifold element with linear displacement function.

Compare with the discontinuum based method, the NMM can provide a more accurate stress and deformation analysis inside the block, and compared with the continuum based method, the NMM can handle discontinuities more easily. However, as other PU based method, the method requires some “tracking algorithm” to identify the crack path line and is limited on the simulation of the branch or bridge of multi cracks. On contrast, in the NDDA method, each mesh line is treated as a potential crack, though the crack is restricted to develop along the initial mesh line, it propagates in a fully automatic and fully integrated way.

2.4.5 NDDA vs. combined FEM/DEM

The combined finite-discrete element method (FEM/DEM) is a state-of-the-art numerical technique pioneered in the mid 1990s (Munjiza 2004, 2010). In this method, each discrete element is discretized into finite elements to describe the shape of the discrete element, contacts conditions among the discrete elements and deformability of each discrete element. The finite elements are employed to model continuum behavior while the

discrete elements are used to analyze discontinuous behavior, progressive fracturing is allowed to take place when the fracture criterion is satisfied, thereby forming new discrete elements which may be composed of one or more deformable finite elements.

Same as the NDDA method, to transform from continuum to discontinuum, fracture and fragmentation play an important role in the combined FEM/DEM. However, in the combined FEM/DEM, the crack analysis is based on the Griffith theory while in the NDDA method, the governing criterion is the Mohr-Coulomb's theory. Also, different from the implicit based NDDA method, in the combined FEM/DEM, an explicit time integration is applied, and the motion of the discrete elements is driven by the unbalanced force.

2.5 Advantages and limitations of the NDDA method

2.5.1 Advantages of the NDDA method

As reviewed in the previous section, in the NDDA method, the DDA's unique block kinematics is combined with the well-developed finite element mesh. Compared with the traditional continuum based method, the advantages of the NDDA method include the easy modeling of the discontinuities, such as fractures and joints in the rock slope can be easily incorporated into the numerical model, and the interaction between individual blocks can be investigated simply and directly in the NDDA method. Compared with the discontinuum based method, the stress field within the blocks and the load-deformation behavior of the block system can be described more accurately in the NDDA method. What's more, by introducing the proper failure criterion at the mesh lines, the new fractures are able to grow if required, thus the transformation from the continuum to discontinuum becomes possible in the NDDA method.

2.5.2 Limitations of the NDDA method

Although the concept of coupling the DDA method with the finite element mesh was introduced since the 1990s, the validation and the application of this method are limited. While most researchers concentrated on the theoretical derivation and programming realization, the importance of verifying a new method should not be ignored. Start with the simple but meaningful problems, the application of the NDDA method on the more complicated cases has been stressed in this thesis.

Though the NDDA method is able to simulate the occurrence of the fracture in an intact block by introducing the Mohr-Coulomb failure criterion as the governing mechanism and representing the grid lines of the triangular elements as virtual joints (Bao, 2010), through inspection of the failure mode it is found that, the crack path obtained from the NDDA method is formed by a group of small tiny cracks, not a main crack observed in the

laboratory test. This is due to the stress jump between triangular elements. In this case, more sophisticated methods need to be implemented in the NDDA method to get the accurate inter-element stresses, thus obtained a more realistic failure mode.

Rockbolts have a history of one hundred years and is now becoming one of the most important rock reinforcement in civil engineering works such as hard rock mining, tunneling, rock slope stability and so on. The rockbolt system can be installed simply and quickly to give an immediately as well as a permanent support to the rock system. In the NDDA method, the rockbolt model is not established, which restrict the capacity of the method for real rock engineering application. In this case, development of realistic rockbolt models in the NDDA method is necessary.

Chapter 3. The NDDA method in slope stability analysis

Rock slope stability analysis is used to support the safe and functional design of rock slopes. Due to the different types of potential rock slope failure, three types of rock slope movement can be defined, namely falls, topples and slides.

The start of a rockfall movement includes two components: block detaching and an initial throwing condition; and the trajectory of the rockfall movement in the air is a parabola. When a free falling block impacts at the slope surface, the movement of this rock block can be transferred to bouncing, rolling and sliding.

Goodman and Bray (1976) defined the toppling failure as the down-slope overturning, and this type of failure is often developed in slopes with well developed steeply dipping discontinuities. In chapter 5, toppling failure will be discussed in detail, including the mechanism, analytical solution and the application of the NDDA method of the toppling failure.

Sliding failure is defined as the movement of rock blocks along one or more surfaces. When the rock mass is heavily jointed or broken, it will tend to follow a circular failure surface similar with soil, and the purpose of slope stability analysis of this kind of slope is to locate the critical failure surface. But in more general case, the rock mass behavior is controlled by the geological discontinuities, and the failure surface can be planar, stepped or polygonal. In this chapter, sliding failure controlled by the geological discontinuities, including the failure mechanism, the limit equilibrium analysis and the application of NDDA on the analysis of sliding failure will be studied thoroughly.

3.1 Conventional method of slope stability analysis

3.1.1 Kinematic analysis

Kinematic analysis refers to the study of the motion of bodies without consideration of its mass and external force, such as stereographic projection techniques (Goodman, 1976) and the block theory (Goodman and Shi, 1985). It is a useful way to help the slope designer to predict the potential failure modes in the early stage of a project.

Priest (1980) assessed kinematic feasibility, sliding direction and the volume of rock blocks defined by the intersection of planar discontinuities and rock face using the inclined hemisphere projection method. Hoek and Bray (1981) summarized the usage of the kinematic analysis in predicting the basic failure modes, namely: planar sliding, wedge sliding and toppling. Yoon et al. (2002) applied the kinematic analysis in sliding failure of the multi-faced slopes with two or more discontinuities planar with different strikes. Mote et al. (2004) performed the kinematic analysis within the GIS framework and successfully addressed slope stability for a canyon landfill in southern California.

Block theory, first introduced by Goodman and Shi (1985), made the fairly complicated discontinuous problem of block system simplified based on the kinematic analysis. In the block theory, blocks are generated from intersections between discontinuities. The blocks are classified into infinite and finite blocks. The finite blocks are then divided into removable block and non-removable block. “key-block” is referred as the block both unstable and removable, and identifying the “key block” is the most important part in using block theory in the rock slope stability analysis. As the failure modes predicted by the block theory involve translation only, Goodman and Mauldon (1990) supplemented the rotational stability analysis of blocks considering both geometric conditions and kinematic condition for rotation. Jeongi-gi and Kulatilake (2001) used the kinematic and the block theory together to analyze the ship lock slope of the three gorges dam. Kulatilake et al. (2011) conducted the kinematic and block theory analysis to evaluate the rock slope stability for Yujiang River dam, and calculated the maximum safe slope angles (MSSA) for different slope surface.

3.1.2 Slope stability classification

As the engineering properties of a jointed rock mass cannot be described by a single parameter, various parameters with different significance have to be taken together to rate the rock mass. Rock mass classification systems belong to the empirical estimation of rock mass behavior with the consideration of different affecting factors. They were first proposed for tunneling and underground excavation and then extended to rock slopes and foundations assessments. The rock mass rating (RMR) system, which is first proposed by Bieniawski (1973) and the tunneling quality index (Q) system, which is first proposed by Barton et al. (1974), are the most commonly used rock mass system in rock engineering. And among several rock mass classification systems, the rock mass rating (RMR) (Bieniawski, 1989) and its extensions, i.e. slope mass rating (SMR) (Romana, 1993) and slope rock mass rating (SRMR) (Robertson, 1988) are most commonly used in the rock slope assessment.

3.1.3 Limit equilibrium methods

The limit equilibrium methods which originated from an empirical background and developed to be more rigorous by many researchers (Bishop 1955; Morgenster & Price 1965; Spencer 1967; Janbu 1968; Sarma 1979) later are the most commonly used method in the slope stability analysis. In the limit equilibrium methods, the rock mass is assumed to be rigid and the forces acting on the possible failure surface are assumed in equilibrium state. The major difference between various methods is the assumption about the inter-slice forces. Among all the limit equilibrium methods, the Morgenster-Price method (Morgenster & Price 1965) is one of the most rigorous methods as it satisfies all the conditions of equilibrium and can be applied for any shape of slip surface. In recent years,

despite the popularity of limit equilibrium methods in engineering practice, the shortcomings, such as the statistically indeterminacy caused by the insufficient number of equilibrium equations, the unrealistic assumption of one factor of safety for all slices and the missing physics have been discussed in more depths (Ching 1983; Krahn 2003).

In most limit equilibrium methods, the sliding mass is divided into vertical slices; this may be more suitable when the individual particles of rock mass are very small compared with the sliding mass, for example, heavily weathered rocks and heavily fractured rocks. However, for most rock slopes, the behavior of rock mass is controlled by geological discontinuities such as joints and fractures which divide the intact rock mass into individual blocks. In this case, the inclined discontinuities need to be considered in the limit equilibrium method and the failure surface will likely be composed by one or more discontinuities. Sarma (1979) introduced the concept of “Oblique slices”, which may be the first and the simplest method in the stability analysis of rock slope with inclined geological discontinuities.

3.2 Validation of the NDDA method in the slope sliding problem

In order to apply the NDDA method in the more complicated slope stability analysis, the simple problems should be used first to validate the method. Same as the DDA method, the NDDA method uses the inertial forces to model the motion of the blocks for both dynamic and static analyses. For the static analysis, the initial velocities are set as zero at the beginning of each time step; while for the dynamic analysis, the initial velocities are inherited from the previous time step. One typical problem encountered in the DDA and the NDDA analyses, as pointed out by Maclaughlin (1997), is that the suddenly turn-on of the gravity at the beginning of the first time step will produce the unwanted displacements and velocity terms due to the elastic deformation of the blocks and the penetration of the contact spring. For the dynamic analysis, the resulted velocity terms will propagate throughout the analysis; and for the static analysis, though the effect of the velocity terms can be eliminated, the displacement is inevitable.

3.2.1 Single planar failure

A very simple example of a rock slope on a single planar surface is presented here. The geometry of the example is shown in Figure 3-1.

3.2.1.1 Analytical solution

Consider a rock slope with slope angle β and slope height H . The unit weight of the rock mass is γ . Suppose a discontinuity surface dipping towards the slope face with angle α , the cohesion and the friction angle of the discontinuity surface are c and ϕ , respectively. The sliding force can be calculated as:

$$f = w \times \sin \beta \quad (3-1)$$

where w is the weight of the wedge.

Figure 3-1 Geometry of the NDDA example

The resistant force can be calculated as:

$$R = cA + w \times \cos \beta \times \tan \varphi \quad (3-2)$$

where A is the base area of the wedge.

The factor of safety for single plane failure without the consideration of water pressure can be expressed as:

$$F = \frac{R}{f} = \frac{cA + w \times \cos \beta \times \tan \varphi}{w \times \sin \beta} \quad (3-3)$$

3.2.1.2 NDDA simulation

Table 3-1 Input parameter of NDDA simulation

Static	yes
Time step size(s)	0.05
Mass per unit(Kg/m ³)	2551
Poisson's ratio	0.49
Spring stiffness(N/m)	6e ¹⁰
Young's modulus(N/m)	6e ⁹
Max disp. ratio	0.001
Joint friction angle °	29.9-30
Joint cohesion(Pa)	0
Joint tensile strength	0

In the NDDA model, there are totally 122 triangular elements inside the blocks, 4 elements in the upper block and 118 elements in the base block, as shown in Figure 3-2. The measurement point is placed at the toe of the upper block. Table 3-1 lists the input parameters of the NDDA simulation and two different joint friction angles, 29.9° and 30° are tested in the simulation. The critical friction angle obtained from the analytical solution is 30°. Figure 3-3(a) shows the displacement time histories of the measurement

point when the joint friction angle equals to 29.9° and 30° , respectively. As it can be seen, the critical friction angle obtained from NDDA simulation is 30° , agrees with the analytical solution.

Figure 3-2 Mesh detail of the problem

Theoretically, the block should be complete stable in case of the joint friction angle equals to 30° ; while due to the effect of “gravity turn-on”, a displacement appears at the beginning of the simulation. At the same time, as the penalty method satisfies the contact constraint only approximately, a slight compression of the shear contact spring is required to balance the block. However, the magnitude of this displacement is very small, and it is easy to distinguish it from the smooth sliding in case of the joint friction angle equals to 29.9° .

Figure 3-3 Time displacement history of measurement point with different friction angle: (a) NDDA result; (b) DDA result.

This problem is also simulated by the DDA method for comparison. The input parameters are the same as that listed in Table 3-1, except that the joint friction angle ranges from 29.9° to 30.2° are tested. The simulation results are depicted in Figure 3-3(b). It is interesting to note that when the joint friction angle equals to 30° , the displacement of the upper block rises in a vibration. The possible reason for this phenomenon is that, in the

DDA method, only two pairs of vertex-edge contacts are involved in the contact detection, the contact status may not be reliable at the critical condition. While in the NDDA method, several vertex-edge contacts are involved in the contact detection instead of two pairs of vertex-edge contacts, thus the contact status is more stable. When the joint friction angle equals to 30.1° , the status of the two contact pairs are “slide”, not “locked”; as a result, though very slowly, the upper block moves. The critical friction angle obtained from the DDA method is 30.2° , though small displacement appears due to the effect of “gravity turn-on” and “shear spring compression”. It is important to note that for any numerical method, the choice of the input parameters will have significant effect on the simulation results. Though the NDDA methods originated from DDA, the optimum input parameters may not be the same for both methods. To prevent bias, the author tries to use same input parameters when comparing the DDA method and the NDDA method.

3.2.2 Dynamic sliding of a single block

Dynamic sliding analysis is of great importance in the assessment of the slope stability when a surcharge load resulting from earthquake or blasting is involved.

In order to apply the dynamic load into the numerical simulation, three methods are available. Method 1 is to apply time-dependant acceleration on the upper block; method 2 is to apply time-dependant acceleration on the base block; method 3 is to apply time-dependant displacement on the base block (Wu, 2010).

3.2.2.1 Analytical solution

Consider the time-dependent acceleration applied on the upper block in the horizontal and the vertical directions as a_x and a_y , respectively; when the upper block slides downwards, the force equilibrium equations in the directions parallel and normal to the incline are (Ning and Zhao, 2013):

$$\begin{aligned} ma_x \cos \alpha + m(g - a_y) \sin \alpha - f &= ma_2 \\ N + ma_x \sin \alpha &= m(g - a_y) \cos \alpha \end{aligned} \quad (3-4)$$

where α is the incline angle of the base, g is the acceleration of gravity, m is the mass of the upper block, N is the normal force applied by the inclined base, a_2 is the acceleration of the upper block in the direction parallel to the inclined base and f is the friction force between the inclined base and the upper block.

According to the friction law, in case of the upper block slides downward, the friction force equals to

$$f = N \tan \varphi \quad (3-5)$$

where φ is the friction angle of the sliding surface.

Substitute Eq. (3-5) into Eq. (3-4), the acceleration of the upper block in the direction parallel to the inclined base can be calculated as (Ning and Zhao, 2013):

$$a_2 = a_2^d = a_x(\cos \alpha + \sin \alpha \tan \varphi) + (g - a_y)(\sin \alpha - \cos \alpha \tan \varphi) \quad (3-6)$$

In case of the upper block slides upward, the friction force equals to

$$f = -N \tan \varphi \quad (3-7)$$

and the acceleration of the upper block in the direction parallel to the inclined base becomes (Ning and Zhao, 2013):

$$a_2 = a_2^u = a_x(\cos \alpha - \sin \alpha \tan \varphi) + (g - a_y)(\sin \alpha - \cos \alpha \tan \varphi) \quad (3-8)$$

If the upper block stops on the inclined base, the acceleration of the upper block in the direction parallel to the inclined base and the friction force can be derived as (Ning and Zhao, 2013)

$$\begin{aligned} \text{if } a_2^d > 0, \quad a_2 &= a_2^d, \quad f = N \tan \varphi \\ \text{if } a_2^u < 0, \quad a_2 &= a_2^u, \quad f = -N \tan \varphi \\ \text{if } a_2^d \leq 0 \leq a_2^u, \quad a_2 &= 0, \quad f = -ma_x \cos \alpha + m(g - a_y) \end{aligned} \quad (3-9)$$

The supporting force applied by the inclined base is

$$N = -ma_x \sin \alpha + m(g - a_y) \cos \alpha \quad (3-10)$$

Giving the acceleration of the upper block, the velocity and displacement of the upper block along the inclined base can be calculated in an iterative way

$$\begin{aligned} v_{n+1} &= v_n + a_n \Delta t \\ d_{n+1} &= d_n + v_n \Delta t + a_n \Delta t^2 / 2 \end{aligned} \quad (3-11)$$

where Δt is the time step interval, a_n is the acceleration at the n^{th} time step; v_n and v_{n+1} are the velocities of the n^{th} and the $n+1^{th}$ time step, respectively; d_n and d_{n+1} are the displacements of the n^{th} and the $n+1^{th}$ time step, respectively. The downward direction along the inclined base is denoted as positive.

3.2.2.2 Numerical simulation

Figure 3-4 shows a block on a single incline. The dimension of the upper block is 10 meters in length and 5 meters in height, the basement has a 20 degree incline angle and the length of the incline surface is 100 meters. In this example, the time-dependant horizontal acceleration $a_x = g \sin t$ and vertical acceleration $a_y = 0.5g \sin t$ are applied at the center of the upper block. In the analytical solution, the time step interval in Eq. (3-11) is chosen as 0.0025s. This time step interval is able to predict the performance of the block system with good accuracy because when the time step interval decreased from 0.0025s to

0.00025s, the displacement of the upper block at $t=10$ s increased from 77.9851m to 77.9997m, the difference is only 0.019%. At the same time, both the NDDA and the DDA methods are used to study this problem, and the input parameters of the numerical simulation are listed in Table 3-2. The mesh detail used in the NDDA analysis is shown in Figure 3-5. The time histories of the displacement and the horizontal velocity of the upper block obtained from the analytical solution (ANA), the NDDA method and the DDA method are shown in Figure 3-6. Both the NDDA and the DDA results match the analytical solution well, indicates that both methods are able to predict the block performance in case of dynamic loading.

Figure 3-4 Block on the single incline

Figure 3-5 Mesh detail of the problem

Table 3-2 Input parameter of the numerical simulation

Dynamic	yes
Time step size(s)	0.0025
Total time step	4000
Mass per unit(KG/m ³)	2500
Young's modulus(GP)	2
Poisson ratio	0.25
Max disp. ratio	0.001
Joint friction angle	35
Joint cohesion	0
Joint tensile strength	0
SOR factor	1.4

As it can be seen, the status of the upper block changes with time when the dynamic loading is applied. Special attention should be paid to the critical points, at which the

status of the upper block changes from stable to sliding, or from sliding to stable. The block velocity within the two time range, 0 to 0.5s and 4.5 to 6.75s are picked up to show the performance of the NDDA method and the DDA method at the critical points.

Figure 3-6 Displacement and the velocity of the upper block obtained from different methods: (a) displacement time histories; (b) horizontal velocity time histories.

Figure 3-7 Horizontal velocities obtained from different methods: (a) from 0-0.5s; (b) from 4.5-6.75s.

At the initial stage, the block rests on the inclined base and the theoretical horizontal velocity is zero. However, the velocities obtained from both the NDDA method and the DDA method show fluctuations at this stage, as shown in Figure 3-7(a). For both methods, the horizontal velocity fluctuates around zero and the amplitude decreases with time. At about 0.25s, the status of the upper block changes from stable to sliding, and at this critical point, the velocity obtained from the DDA method fluctuates much fiercely than that of the NDDA method. At about 4.64s, the status of the upper block changes from sliding to stable and the velocities obtained from both methods show great fluctuations. As the block

rests on the inclined base then, the theoretical velocity should be zero again. Similar as Figure 3-7(a), the horizontal velocities obtained from both methods fluctuate around zero and the amplitudes of the fluctuation decrease with time. At about 6.52s, the status of the upper block changes from stable to sliding. Again, the velocity obtained from the DDA method fluctuates a lot while the velocity obtained from the NDDA method matches well with the analytical solution.

The relative error, which is defined as

$$relative\ error = \left| \frac{numerical\ result - analytical\ result}{analytical\ result} \right| \times 100\% \quad (3-12)$$

is used in comparing the performance of the DDA method and the NDDA method in more detail. The relative error in case of the displacement or the velocity equals to zero will not be calculated because zero cannot act as a denominator. At the same time, a cutoff of 25% is applied because when the displacement or the velocity is a tiny value, even a small absolute error will result in an extremely large relative error.

The relative errors of the displacement and the velocity are shown in Figure 3-8. The relative displacement error obtained from the DDA method is smaller than that obtained from the NDDA method at the beginning, with the increase of the block displacement, the relative displacement error of both methods reduce to below 1%. In most time, the relative velocity error obtained from the NDDA method is smaller than that obtained from the DDA method; and for both methods, the large relative velocity error appears near the critical point at which the status of the upper block changes.

Figure 3-8 Relative errors obtained from two methods: (a) relative error of the displacement; (b) relative error of the velocity.

3.3 Application of the NDDA method in slope sliding problem

If more discontinuities are involved in the sliding failure, the polygonal plane failure is observed. In this kind of failure, the sliding surface is a curvilinear shape composed by several plane segments with the similar strike of the slope face. Due to the consideration of kinematic, Kovari et al. (1978) pointed out that a number of internal shear surfaces will develop and the directions of the internal surfaces are controlled by the geological and structural situation, i.e. the existing planes of weakness. The geometry of the polygonal sliding is shown in Figure 3-9.

Figure 3-9 The polygonal sliding surface (after Kovari and Fritetz, 1984)

3.3.1 Introduction of the Sarma's method

In case of the slope controlled by the geological discontinuities, the Sarma's method (1979), which uses the concept of “un-vertical slices”, may be the most suitable one in the limit equilibrium methods.

Figure 3-10 Force acting on the slice in Sarma's method

Consider the force acting on the i_{th} block as shown in Figure 3-10. W_i is the weight of block i ; N_i , T_i are the resultant normal and shear force acting on the base of the block; E_i , X_i are the resultant normal and shear force acting on the left side of the block; E_{i+1} , X_{i+1} are the resultant normal and shear force acting on the right side of the block; α_i is the inclined angle of the base of the block; and β_i is inclination of the left side of the block

measured clockwise from the positive y axis. K_e is a fictitious horizontal acceleration acting on the centroid of the block.

The equilibrium equations of the vertical and horizontal directions can be expressed as:

$$\begin{aligned} N_i \cos \alpha_i + T_i \sin \alpha_i &= W_i + X_{i+1} \cos \beta_{i+1} - X_i \cos \beta_i - E_{i+1} \sin \beta_{i+1} + E_i \sin \beta_i \\ T_i \cos \alpha_i - N_i \sin \alpha_i &= K_e W_i + X_{i+1} \sin \beta_{i+1} - X_i \sin \beta_i + E_{i+1} \cos \beta_{i+1} - E_i \cos \beta_i \end{aligned} \quad (3-13)$$

In order to make the problem statistically determinate, Sarma assumed that both the base and the interface of each block satisfies Mohr-Coulomb failure criterion when the slope is at the critical state. For i_{th} block, the friction angles on the base and the left side of the block are φ_{Bi} and φ_{Si} , the cohesions on the base and the left side of the block are c_{Bi} and c_{Si} . The critical acceleration of the slope can be expressed as (Hoek, 1987):

$$K_c = \frac{a_n + a_{n-1}e_n + a_{n-2}e_n e_{n-1} + \dots + a_1 e_n e_{n-1} \dots e_3 e_2}{p_n + p_{n-1}e_n + p_{n-2}e_n e_{n-1} + \dots + p_1 e_n e_{n-1} \dots e_3 e_2} \quad (3-14)$$

Where

$$\begin{aligned} a_i &= \frac{W_i \sin(\varphi_{Bi} - \alpha_i) + c_{Bi} b_i \sec \alpha_i + c_{Si+1} d_{i+1} \sin(\varphi_{Bi} - \alpha_i - \beta_{i+1}) - C_{Si} d_i \sin(\varphi_{Bi} - \alpha_i - \beta_i)}{\cos(\varphi_{Bi} - \alpha_i + \varphi_{Si+1} - \beta_{i+1}) \sec \varphi_{Si+1}} \\ p_i &= \frac{W_i \cos(\varphi_{Bi} - \alpha_i)}{\cos(\varphi_{Bi} - \alpha_i + \varphi_{Si+1} - \beta_{i+1}) \sec \varphi_{Si+1}} \\ e_i &= \frac{\cos(\varphi_{Bi} - \alpha_i + \varphi_{Si} - \beta_{i+1}) \sec \varphi_{Si}}{\cos(\varphi_{Bi} - \alpha_i + \varphi_{Si+1} - \beta_{i+1}) \sec \varphi_{Si+1}} \end{aligned}$$

If $K_e=0$, the slope is at the limit equilibrium state. If $K_e<0$, the status of the slope is unstable. If $K_e>0$, the status of the slope is stable, and the factor of safety can be found by reducing the shear strength parameters to

$$\begin{aligned} \tan \varphi_e &= \frac{\tan \varphi}{F} \\ c_e &= \frac{c}{F} \end{aligned} \quad (3-15)$$

until the acceleration K_e equals to zero.

3.3.2 Example one-slope problem in open pit coalmine

This example is about a slope problem in a large open pit coalmine cited from Hoek (1987). The geometry of the problem is illustrated in Figure 3-11. In this case, the thin coal seam is overlain by soft tuff, and the sliding surface occurs along the coal seam and breaking out through the soft tuff at the toe. The material properties of the coal seam and the soft tuff are obtained from the laboratory test and the back-analysis of previous

failures. Table 3-3 lists the coordinates of the slope profile as well as the strength parameters along sliding surface of the coal seam and the soft tuff. The DDA, NDDA and Sarma's method are used in the study. In the NDDA simulation, there are totally 1184 triangular elements in the numerical model, as shown in Figure 3-12. The input parameters of the DDA and NDDA simulations are listed in Table 3-4. As the limit equilibrium method is derived based on the rigid block assumption, the Poisson's ratio used in the numerical simulation is set as 0.49 to preserve the area of the blocks in two-dimensional analysis and the Young's modulus of the numerical simulation is set at 6GPa to reduce the deformation of the block. The penalty value used in the simulation is 60GPa, ten times of the Young's modulus of the blocks. Because the limit equilibrium method is derived based on the static equations, the static analysis is conducted in the simulation.

For numerical simulation, two measurement points A and B, are placed at the toe and the top of the slope, as shown in Figure 3-11. The static safety factor is obtained by keeping the horizontal acceleration Kc as zero while reducing the shear strength parameters by a known reduction factor. Reduction factor $F=1$ is first tested, and increased gradually until the slope reaches its limited equilibrium status. The increment reduction factor of each time is 0.01. At each time step, the horizontal displacements at the measurement points are used to judge the status of the slope: if the displacement time histories of the measurement points keep stable, the status of the slope is regarded as stable; otherwise, if the displacements of the measurement points keep increasing with time, the status of the slope is regarded as unstable.

Table 3-3 Slope geometry and the strength parameters of an open pit coal

Side no.	1	2	3	4	5	6	7	8	9
XT(m)	4	17	29	50	68	140	165	178	204
YT(m)	17	26	26	25	37	88	90	99	103
XB(m)	4	17	29	50	80	155	173	186	204
YB(m)	17	12	10	8	11	65	80	89	103
Friction(°)	0	30	30	30	18	18	18	18	0
Cohesion(KPa)	0	20	20	20	0	0	0	0	0
Block	1	2	3	4	5	6	7	8	
$\gamma(\text{KN/m}^3)$	21	21	21	21	21	21	21	21	21
Friction(°)	30	30	30	30	30	30	30	30	30
Cohesion(KPa)	20	20	20	20	20	20	20	20	20

The factors of safety obtained from different methods are listed in Table 3-5, as it can be seen, the DDA method and the NDDA method result in similar factor of safeties while the Sarma's method results in a much smaller one. The difference may be explained by the fact that, in the limit equilibrium methods, the analysis is based on the static equilibrium and only the initial configuration of the slope is considered, while in the DDA and the

NDDA method, the small displacement appeared during the simulation may change the initial configuration of the slope as well as cause the load of the active slice to be transferred to the lower slope, thus change the stability of the slope.

Table 3-4 Input parameters of the example

Static	yes
Time step size(s)	0.05
Poisson's ratio	0.49
Spring stiffness(GPa)	60
Young's modulus(GPa)	6
Max disp. ratio	0.01

Figure 3-11 Configuration of the open pit mine slope

Figure 3-12 Mesh detail of the slope

Table 3-5 Calculated factor of safety for different methods

F_{Sarma}	F_{NDDA}	F_{DDA}
2.07	2.77	2.73

The F - K_c curve which reflects the relationship between the horizontal acceleration and the factor of safety can be obtained by indicating the reduction factor F while varying the horizontal acceleration K_c until the slope reaches its limit equilibrium status. In both the DDA and the NDDA simulation, the horizontal acceleration is considered by applying the

corresponding horizontal body force on each block. The comparison between the $F-Kc$ curve obtained from the Sarma's method, the DDA method and the NDDA method is shown in Figure 3-13. For a same horizontal acceleration, the DDA and the NDDA method result in a similar factor of safety while the Sarma's method results in a different one. When F is smaller than 1.2, normal stresses obtained by Sarma's method will appear to be negative on some block's base, and the calculated acceleration Kc is unacceptable in this case. This phenomenon is a human deficiency in making unreasonable assumption, which will never occur in the DDA and the NDDA simulation. In another word, for any given strength reduction factor, the DDA and the NDDA method are able to give the corresponding horizontal acceleration while the Sarma's method cannot guarantee it.

Figure 3-13 Safety factor F versus horizontal acceleration Kc

In the Sarma's method, if the slope is at the limit equilibrium status, the normal and shear force acting on the side of rock blocks and the slope base can be obtained using a progressive solution suggested by Hoek (1987) when the starting point E_I is set as zero:

$$\begin{aligned}
 E_{i+1} &= a_i - p_i K + E_i e_i \\
 X_i &= E_i \tan \varphi_{si} + c_{si} d_i \\
 N_i &= \frac{(W_i + X_{i+1} \cos \beta_{i+1} - X_i \cos \beta_i - E_{i+1} \sin \beta_{i+1} + E_i \sin \delta_i - R_i \sin \alpha_i)}{\cos(\varphi_{b,i} - \alpha_i)} \\
 T_i &= N_i \tan \varphi_{b,i} + c_{b,i} b_i \sec \alpha_i
 \end{aligned} \tag{3-16}$$

The normal stresses derived from the above equations are also used as a check of the acceptability of the solution. For an acceptable solution, all normal stresses must be positive.

Sarma also suggested a final check about the moment equilibrium conditions of each blocks:

$$N_i l_i - \frac{X_{i+1} b_i \cos(\beta_{i+1} + \alpha_i)}{\cos \alpha_i} - E_i Z_i + E_{i-1} (Z_{i+1} + \frac{b_i \sin(\beta_{i+1} + \alpha_i)}{\cos \alpha_i}) - W_i (x_{Gi} - X_{bi}) + K_c W_i (y_{Gi} - Y_{bi}) = 0 \quad (3-17)$$

This equation considers the moments about the lower left corner of each block; l_i and E_i in the equation stand for the distance from the lower left corner to the acting point N_i and the acting point E_i , respectively. The calculation starts from the first block at the toe of the slope, and Z_1 equals to zero in this condition. Assuming a value of l_1 , the moment arm Z_2 can then be solved. Following this step, all the values of Z_i can be obtained. For an acceptable solution, the values of Z_i should lie within the boundary of the slice.

Figure 3-14 shows the comparison between the normal and shear force acting at the side of rock blocks and the slope base obtained from the Sarma's method and the NDDA method when the factor of safety F_s equals to 2.07 and K_c equals to zero.

Figure 3-14(a) and Figure 3-14(b) show the distribution patterns of the normal and shear force acting along the slope base obtained from the NDDA and Sarma's method. The values of normal forces obtained by the two methods are different for most of the blocks; however, the distribution patterns are similar. Also, there exists a difference between the shear forces obtained from the two methods. For most of the blocks, the shear forces obtained from the Sarma's method are larger than that obtained from the NDDA method as all the base segments reach their shear resistant strength in the Sarma's method. But it is not always the case because the shear resistant is depended not only on the joint strength but also on the normal force. It is possible for the NDDA method to get a larger shear force if the normal force obtained from the NDDA method is larger, such as that in block no.2.

Figure 3-14(c) and Figure 3-14(d) show the distribution pattern of the normal and shear thrust forces between blocks obtained from the two methods. The maximum normal thrust force appears at the interface of block no.4 and block no.5 and at the same time, the difference between the two methods also reaches its maximum value. The discrepancy can be explained by the effect of different slope status and the block size. For shear thrust force, the values obtained from NDDA method are lower than that obtained from Sarma's method as the Sarma's method assumes that the limited equilibrium conditions prevail on the slope base and block interfaces at the same time. Also, it is noticed that the shear thrust forces of the interface between block no.6 and block no.7 and the interface between block no.7 and block no.8 take different direction compared with other interfaces, indicates that the relative movement of two adjacent blocks may be different as Sarma's assumption. In Sarma's method, it is assumed that the left block moves upward relative to the right block. However, it is possible that the left block move downward relative to the

right one in some cases. On contrary, the NDDA method can identify the direction of the relative movement automatically because of the complete block kinematics.

Figure 3-14 The normal and shear force acting on the side of rock blocks and the slope base for both methods when $F=2.07$, $Kc=0$: (a) normal force at the base; (b) Shear force at the base; (c) normal thrust force; (d) shear thrust force.

In addition, the comparison between the two methods under the same slope status is carried out. Figure 3-15 shows the normal and shear force acting on the side of rock blocks and the slope base obtained by the Sarma's method and the NDDA method at the limited equilibrium status when the safety factor $F=2.77$, the corresponding horizontal acceleration for the two methods are $kc=-0.121$ and $kc=0$, respectively.

Figure 3-15(a) and Figure 3-15(b) show the distribution patterns of the normal and shear forces acting along the slope base. The distribution patterns of the normal forces obtained by the two methods are similar. During the NDDA simulation, small displacement will appear; this small displacement may cause the blocks to contact with their adjacent base

segments instead of their original base segments as Sarma's method assumed, and the change of contact conditions may result in the change of the normal forces at base.

Figure 3-15 The normal and shear force acting on the side of rock blocks and the slope base for both methods in the limited equilibrium condition: (a) normal force at the base; (b). Shear force at the base; (c) normal thrust force; (d) shear thrust force.

Figure 3-15(c) and Figure 3-15(d) show the distribution pattern of the normal and shear thrust forces between blocks. The differences between the normal thrust forces obtained from the two methods are obvious. The possible reason is that even the limit equilibrium conditions are satisfied for both methods, the applied horizontal acceleration K_c is different; and the bigger the volume of the block, the bigger the effect of the K_c ; thus, the bigger the difference between the two methods. However, though the values of the normal thrust forces obtained by the two methods differ a lot, the distribution patterns obtained by the two methods are quite similar. For shear thrust force, the absolute values obtained

from NDDA method are lower than that obtained from Sarma's method, indicates that even the slope is at the initiation of the failure, the limit equilibrium condition does not occur along the block interface in the NDDA simulation.

Figure 3-16 Normal force at the base: (a) total force; (b) distributed force

Figure 3-16 (a) shows the total normal force at each base segment with different reduction factor F . The general distribution pattern of the total normal force is similar. In order to further investigate the block kinematics during the strength reduction procedure, the normal force distribution along the slope base with different reduction factor F obtained from NDDA method is presented in Figure 3-16(b). The Y axis is the normal force converted from the resultant force acting at each element node, while the X axis is the x-coordinate of each element node lay on the slope base.

As it can be observed from Figure 3-16 (a) and Figure 3-16 (b):

- 1) When F equals to 1, all the blocks stay stable and contact with its original base segment. The normal force distribution along each block base segment is almost uniformly.
- 2) When F equals to 1.5, the small displacement during the simulation carry the lower left corners of all the blocks, except slice no.1 and no.7, to contact to their left adjacent base segment instead of its original base segment, which will lead the element nodes near to the lower left corner to have the tendency of detaching from the slope base. As illustrated in Figure 3-17. In this case the lower left corner takes the maximum normal force and the element node next to the lower left boundary takes the minimum normal force, and the normal force of the element node increases from the right adjacent node of the lower left corner to the left adjacent node of the lower right corner.
- 3) When F equals to 2 and 2.5, the general distribution pattern of the normal force for each block is similar with that obtained in the case $F=1.5$. However, the effects of

the block kinematics and slope geometry are more severe due to larger strength reduction factor. The normal force concentrated more at the lower left corner, and the element nodes near to the lower left corner get zero normal force due to the fully detachment from the slope base. In order to support the block, the element nodes far from the lower left corner need to supply larger normal force to make up the small normal force supplied at the element node near to the lower left corner, thus the increment of the normal force from the element node near to the lower left corner to the element node far from the lower left corner is more obviously than the previous case.

- 4) When F equals to 2.77, the slope is at the verge of potential failure initiation. The lower left corner of all the blocks, except block no.1 and block no.7, contacts with its left adjacent base segment, indicates that all the blocks move towards left relative to the slope base. The lower right corner of all the blocks contact with its original base segment, which indicates that the failure mode of all the blocks is sliding.

Figure 3-17 Changes of the contact status during the block sliding

In the limit equilibrium methods, one of the possible approaches to make the problem statically determinate is to make an assumption regarding the position of the line of thrust; however, numerical difficulties often arise in the solution of differential equations when using this approach (Ching and Fredlund, 1983). While in the NDDA method, by converting the contact information into the stress information, the stress distribution along the inter-slice can be obtained, thus the position of the action can then be calculated using the moment equilibrium equation.

Figure 3-18 shows the distribution of the interslice normal force and the corresponding position of line of thrust with different reduction factor F . In the earth pressure theory, it is expected that the distribution pattern of the inter-slice normal stress forms a triangle shape and the line of thrust should fall at the lower one-third point of the slice side. However, the NDDA simulation result shows that in this example, the distribution pattern of the inter-slice normal force does not follow the earth pressure theory, it differs dramatically

with different reduction factor F , and the position of the line of thrust changes simultaneously.

(a) $F=1$

(b) $F=1.5$

(c) $F=2$

(d) $F=2.5$

Figure 3-18 Distribution of the inter-slice normal force and the corresponding line of thrust with different strength reduction factor: (a) $F=1$; (b) $F=1.5$; (c) $F=2$; (d) $F=2.5$; (e) $F=2.77$.

Figure 3-19 Ratio between the interslice shear force and the shear resistant

Another possible approach for the statically indeterminate problem is to make an assumption regarding to the relation between the inter-slice normal force and the inter-slice shear force. For example, in the Sarma's method, it is assumed that the Mohr-Coulomb failure criterion occur along the interfaces. Figure 3-19 shows the ratio between the shear thrust force and the corresponding resistance of shear strength obtained from the NDDA simulation. The ratios obtained from NDDA method are all smaller than 1, indicates that in this example, the limit equilibrium condition is not occur along the interfaces when the failure initiates.

3.3.3 Example two—a rock slope excavated in an open pit mine

The example presented here is originated from Hoek (1983). In order to save the computation time, the dimension of the rock slope is reduced. The geometry of the problem is illustrated in Figure 3-20, the vertical height is 16m and the overall angle of the slope is about 30° , and two measurement points are placed at the toe and the top of the

slope. As the upper portion of the slope is comprised by sands, gravels and clays while the lower part of the slope is formed by shale, the joint properties are different. The fault which separated slope to the overburden and the lower part is assumed to have the shear strength of 15° friction angle and zero cohesion. The detailed information about the slope profile as well as the strength parameters along the sliding surface are listed in Table 3-6.

Figure 3-20 Configuration of the open pit mine slope

Figure 3-21 Mesh detail of the slope

Table 3-6 Slope geometry and the strength parameters of an open pit coal

Side no.	1	2	3	4	5	6	7	8	9
XT(m)	0.8	5.4	6.8	11.52	12.48	18	23.2	26.4	30.6
YT(m)	2	6	6	10	10	14	16.4	18	18
XB(m)	0.8	3.28	5.6	10.96	12	23.2	25.4	28.4	30.6
YB(m)	2	2.4	2.72	4.6	4.92	10.6	12.44	15.2	18
Friction(°)	0	30	30	30	30	15	18	18	0
Cohesion(KPa)	0	0	0	0	0	0	0	0	0
Block	1	2	3	4	5	6	7	8	
Unit weight(Kg/m ³)	23	23	23	23	23	23	23	23	
Friction(°)	30	30	30	30	30	18	18	18	
Cohesion(KPa)	0	0	0	0	0	0	0	0	

In this example, there are totally 1207 triangular elements involved in the NDDA numerical, and the mesh detail is shown in Figure 3-21. The input parameters are the same as that used in the previous example. Table 3-7 lists the factor of safety obtained from DDA method, the NDDA method and the Sarma's method. Similar as the previous example, the NDDA method and the DDA method result in very close factor of safety. However, the Sarma's method results in a higher one in this case.

Table 3-7 Static factor of safety obtained from different methods

F_{Sarma}	F_{NDDA}	F_{DDA}
1.43	1.32	1.31

Figure 3-22 Safety factor F versus horizontal acceleration K_c

Figure 3-22 shows the comparison between the F - K_c curves obtained from these three methods. Similar as the first example, for a same horizontal acceleration, the NDDA and the DDA method have similar result while the Sarma's method has a different factor of safety. When the reduction factor is small, the horizontal acceleration obtained from Sarma's method is larger than that obtained from the DDA and the NDDA method. When the reduction factor is relatively large, the acceleration obtained from the Sarma's method is smaller than that obtained from the DDA and the NDDA method.

Figure 3-23 shows the comparison between normal and shear force acting at the side of rock blocks and the slope base obtained when the safety factor F_s equals to 1.32. The horizontal acceleration K_c for the NDDA and the Sarma's method is 0 and 0.096, respectively.

Figure 3-23(a) and Figure 3-23(b) show the distribution patterns of the normal and shear forces acting along the slope base. Also, the distribution patterns of the normal forces

obtained by the two methods are similar. As no cohesion is involved in the joint strength, the shear force should be proportional to the normal force, thus for each method the obtained distribution pattern of the normal force and the shear force is very similar.

Figure 3-23(c) and Figure 3-23(d) shows the distribution pattern of the normal and shear thrust forces between blocks. The Sarma's method results in very large interslice normal force at the interface between block no.4 and block no.5 and the interface between block no.5 and block no.6. This can be explained by the effect of the horizontal acceleration and the block size. For shear thrust force, the absolute values obtained from the NDDA method are lower than that obtained from Sarma's method, and the directions may be positive and negative.

Figure 3-23 The normal and shear force acting on the side of rock blocks and the slope base for $F=2.07$, $K_c=0$: (a) normal force at the base; (b) Shear force at the base; (c) normal thrust force; (d) shear thrust force.

Figure 3-24(a) shows the total normal force at each base segment when the reduction factor $F=1$ and $F=1.32$, respectively. Same as the first example, the obtained total normal force of each base segment is similar with different reduction factor F . The detailed information about normal force distribution along the slope base is presented in Figure 3-24(b). As it can be seen, when F equals to 1, all the blocks stay stable, the total normal force distribution along each block base segment is almost uniformly; when the slope is at the verge of potential failure initiation, the small displacement appeared during the simulation carry the lower left corners of some blocks to contact with their left adjacent base segment instead of their original base segment, at the same time, the element nodes near the lower left corner detached from the slope base and the normal force concentrated at the corner of the block. For block no.6, only the lower left corner contacted with the slope base, indicates that the status of the block no.6 is sliding and toppling.

Figure 3-24 Normal force at the base: (a) total force; (b) distributed force

Figure 3-25 shows the distribution of the interslice normal force and the corresponding position of line of thrust with different reduction factor F . In this example, the distribution pattern of the interslice normal force does not follow the earth pressure theory, and it differs dramatically with different reduction factor F , and as a result, the position of the line of thrust changes simultaneously.

Figure 3-26 shows the ratio between the shear thrust force and the corresponding resistance of shear strength obtained from the NDDA simulation. Again, the ratios obtained from the NDDA method are all smaller than 1, indicating that in this example, the limit equilibrium condition is not occur along the interfaces when the failure initiates.

Figure 3-25 Distribution of the inter-slice normal force and the corresponding line of thrust with different strength reduction factor: (a) $F=1$; (b) $F=1.32$.

Figure 3-26 Ratio between the interslice shear force and the shear resistance

Actually, it is very difficult to compare the Sarma's method and the NDDA method in a straight forward way, because it is difficult to find a pair of factor of safety F and horizontal acceleration K_c which satisfies the limit equilibrium conditions of both the Sarma's method and the NDDA method. In the Sarma's method, the limit equilibrium

condition is one of the underlying assumptions, while in the NDDA method, the complete block kinematics is considered, and the limit equilibrium condition is only one of the results of the analysis. Under the same factor of safety, the horizontal acceleration obtained by NDDA method is quite different from that obtained by the Sarma's method, thus restrict the comparison between the two methods.

However, we can still get many useful information by comparing the two methods when: 1) F and K_c values are the same and the slope status is different; and 2) F value and the status of the slope are the same and the K_c value is different. From the above comparison, it is found that different from the Sarma's assumption, during the NDDA simulation: 1) the initial configuration of the problem may be changed; 2) even the slope is at the initiation of the failure, the limited equilibrium condition may not occur along the block interface; 3) the possible movement of the left block may be upward or downward relative to the right block; 4) each block of the slope may experiences different types of failure mode, i.e. sliding, toppling, or the combination of sliding and toppling though the general failure mode of the whole slope is sliding.

3.4 Conclusion

Stability analysis is used to support the safe and functional design of rock slopes; it is one of the classical and challenging problems in rock mechanics. In this chapter, the review of the conventional rock slope stability analysis methods is carried out, and the NDDA method is used to model slope stability problems. The advantages of using the NDDA method in the slope stability analysis include: 1) the geometry of discontinuities, such as fractures and joints, can be readily incorporated into the numerical model; 2) the stress field within the block and the load-deformation behavior of the rock slope can be accurately described and; 3) a solution that satisfies all the equilibrium conditions can be obtained without any assumptions regarding the interslice forces. The numerical results show that the NDDA method provides different factors of safety as compared with the Sarma's method. More detailed information, including the distribution pattern of the interslice normal forces, the position of the "line of thrust", the ratio between of the interslice shear force and the corresponding shear resistance, and the distribution pattern of the normal force along the potential slip surface can be obtained from the NDDA simulation.

Chapter 4. Stress recovery and block fracturing analysis in the NDDA method

A very important topic that drew many attentions in rock mass modeling is how to numerically simulate the crack propagation. The Linear Elastic Fracture Mechanics concept (Bazant 1996, 2002) can only be applied when the fracture process zone in front of the crack tip is sufficient small compared with the problem dimension, and the mesh needs to be updated at every time step during the fracturing process. The cohesive-zone model is one of the simplest models to consider the effect of the fracture process zone, where an interface elements equipped with a cohesive zone model can be inserted into the finite element mesh (Rots, 1991; Schelleken, 1992). However, this method needs to estimate the crack path in advance, which is difficult in most cases. Recently, based on the partition of unity property of finite element (Melenk et al. 1996), Extended Finite Element Method (XFEM) (Dolbow et al., 2001; Nicolas and Belytschko, 2002.) and the Generalized Finite Element method (GFEM) (Strouboulis et al., 2000) were developed for crack propagation problems. By incorporating enrichment function to approximate discontinuous displacement field, these methods can simulate the growth of cracks without remeshing.

In contrast to the continuum methods, the NDDA method can also be used to simulate cracks and their growth directly. In the NDDA method, each mesh line is regarded as a potential crack, and when the given fracturing criterion is satisfied, a double-node interface is inserted along that mesh line to separate the two elements sharing that mesh line (Bao 2010; Bao and Zhao, 2013). However, our experience shows that the crack path obtained from the NDDA method is formed by a group of small tiny cracks, not a main crack observed in the laboratory test, which is due to the stress jump between triangular elements. In order to get a realistic failure mode, accurate inter-element stresses are needed. In this chapter, a double-minimization procedure presented by Ciancio (2006) is introduced into the NDDA method to determine the inter-element forces and traction, so as to obtain a more realistic fracture mode in the NDDA. Another by-product of the proposed method is that, by evaluating the stress-tensor at the element nodes, a more accurate nodal stress state can be obtained. In other word, the proposed method can also be applied as a post-processing stress recovery procedure.

In this chapter, firstly the NDDA program framework is presented, and an introduction on the double minimization procedure which is used to get the inter-element forces and recover the nodal stress state is given. Section 4.3 presents three static examples to calibrate the NDDA method with double minimization procedure as well as compare the NDDA method with the sub-block DDA method. Section 4.4 introduces the fracture sequence, including the mesh updating and the crack inserting procedure in the NDDA

method, and in Section 4.5, three examples are presented to demonstrate the ability of the NDDA method in dealing with the brittle fracture. And finally, the main conclusions will be drawn in Section 4.6.

4.1 NDDA program framework

In the NDDA method, the dynamic behavior of the block is represented numerically by a time stepping algorithm. In each time step, the global equilibrium equation is yielded by minimizing the total potential energy, and the displacements are obtained by solving the global equilibrium equation. Large displacements and deformations are the accumulation of small displacements and deformations of each time step. The flowchart of the NDDA calculation procedure is shown in Figure 4-1. The main work in this chapter can be concluded as the “crack propagation procedure” shown in this flowchart. The inter-element forces are computed first based on the double minimization procedure, and the obtained inter-element forces are then checked against the failure criterion to see whether the crack will occur in this time step. And at the end, a crack inserting procedure, including node splitting and stiffness matrix updating, will be carried out if the failure criterion is met.

Figure 4-1 Flowchart of the NDDA method

4.2 Inter-element force and the nodal stress recovery based on the double minimization procedure

In the NDDA framework, the element nodes can be categorized into two groups: the nodes inside the block are marked as internal node while the nodes at the boundary (block boundary and the crack boundary) are marked as external node. The calculation procedure for the inter-element forces of different kinds of nodes is discussed in the following sections.

4.2.1 Internal Nodes

Suppose there are n elements sharing node i , as shown in Figure 4-2. The nodal force $\mathbf{f}$ at the node i can be regarded as the sum of nodal force of each element sharing this node.

Figure 4-2 Equilibrated nodal force $\mathbf{f}$ and inter-element force $\mathbf{r}$ at internal node

Considering the nodal force at the node N as $\mathbf{f}_n$, the inter-element force between element N and element $N+1$ as $\mathbf{r}_n$, the equilibrium equation at each node can then be written as (Ciancio, 2006)

$$\begin{cases} f_1 - r_n + r_1 = 0; \\ f_2 - r_1 + r_2 = 0; \\ f_3 - r_2 + r_3 = 0; \\ \dots \\ \dots \\ \dots \\ f_n - r_{n-1} + r_n = 0 \end{cases} \quad (4-1)$$

The nodal force vector f_k for an element in an equilibrium state can be obtained using the equation

$$f = \int_a B^T \sigma dA \quad (4-2)$$

where B is the strain matrix; σ is the stress field in the element and A is the area of the element.

As the nodal forces $f_1, f_2, \dots, f_n$ converging at point i must be in equilibrium $\sum_{k=1}^N f_k = 0$, the N equations in Eq. (4-1) are not independent. One extra equation is needed in order to solve N unknowns in the $N-1$ equations.

Here, r_1 is chose to be the arbitrary vector with $r=r_1$, then Eq. (4-1) can be expressed as

$$\begin{cases} r_2 = r - f_2; \\ r_3 = r_2 - f_3 = r - f_2 - f_3; \\ r_4 = r_3 - f_4 = r - f_2 - f_3 - f_4; \\ \dots \\ \dots \\ \dots \\ r_n = r_{n-1} - f_n = r - f_2 - f_3 - \dots - f_n \end{cases} \quad (4-3)$$

Assuming X' and Y' are the local reference system in k^{th} inter-face while X and Y are the global reference system, as illustrated in Figure 4-3. The stress traction vector on the inter-face can be written as

Figure 4-3 Plane element in global and local reference system

$$\sigma^{(k)} = \begin{pmatrix} \sigma^{(k)} \\ \tau^{(k)} \end{pmatrix} = \frac{1}{\Omega^{(k)}} Q^{(k)} \begin{pmatrix} r_x^{(k)} \\ r_y^{(k)} \end{pmatrix} \quad (4-4)$$

The concept in Eq. (4-4) is that the inter-element forces are constant distributed over the contribution area, here, $\Omega^{(k)}$ is the relative contribution area of the element side, for three nodes triangular element, the nodal area is equal to half of the edge length. $Q^{(k)}$ is the rotation matrix to transform the global reference system to the local reference system.

$$Q = \begin{pmatrix} n \\ t \end{pmatrix} = \begin{pmatrix} -\sin \alpha & \cos \alpha \\ \cos \alpha & \sin \alpha \end{pmatrix} \quad (4-5)$$

where α being the anticlockwise angle from X to X'.

The normal and shear stresses $\sigma_{x'}$, $\sigma_{y'}$ on the inclined edge can then be expressed by the global stress tensor and the rotation matrix.

$$\begin{aligned} \sigma_{x'} &= nTn^T = (\cos \alpha \sin \alpha) \begin{pmatrix} \sigma_x & \tau \\ \tau & \sigma_y \end{pmatrix} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha + \tau_{xy} \sin 2\alpha \\ \sigma_{y'} &= tTn^T = (-\sin \alpha \cos \alpha) \begin{pmatrix} \sigma_x & \tau \\ \tau & \sigma_y \end{pmatrix} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\alpha + \tau_{xy} \cos 2\alpha \end{aligned} \quad (4-6)$$

where T is the global stress tensor and its matrix form is

$$T = \begin{pmatrix} \sigma_x & \tau \\ \tau & \sigma_y \end{pmatrix} \quad (4-7)$$

The vector $\sigma^{(k)}$ for each interface k can be expressed as a function of $\mathbf{r}$ by substituting Eq. (4-3) which is the expression of the inter-element force $\mathbf{r}_k$ into the Eq. (4-4) which is the expression of the stress vector. The difference between the stress vector and the projection of T on the interface k is calculated as an intrinsic error to determine the arbitrary value $\mathbf{r}$ (Ciancio, 2006):

$$\phi = \sum_{k=1}^N [(\sigma^{(k)}\{\mathbf{r}\} - n^{(k)}Tn^{(k)T})^2 + (\tau^{(k)}\{\mathbf{r}\} - t^{(k)}Tn^{(k)T})^2] \quad (4-8)$$

If the value of T is known, the only unknown variable in Eq. (4-8) will becomes $\sigma^{(k)}\{\mathbf{r}\}$. Thus, a minimization of ϕ with respect to $\mathbf{r}$ would be carried out to solve the arbitrary value $\mathbf{r}$. The equation system can be written as a compact form (Ciancio, 2006):

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \quad (4-9)$$

In Eq. (4-9), D is the matrix related with the geometrical data, $\mathbf{d}$ is the vector related with the numerical result attained from the NDDA analysis. The expression of matrix D and vector $\mathbf{d}$ are as follows (Ciancio, 2006):

$$\begin{aligned} D &= B - L^T A^{-1} L \\ \mathbf{d} &= \mathbf{b} - L^T A^{-1} \mathbf{e} \end{aligned} \quad (4-10)$$

where

$$\begin{aligned} B &= \sum_{k=1}^N \begin{pmatrix} \Omega^{(k)^{-2}} & 0 \\ 0 & \Omega^{(k)^{-2}} \end{pmatrix} \\ A &= \sum_{k=1}^N \begin{pmatrix} \sin^2 \alpha_k & 0 & -\sin \alpha_k \cos \alpha_k \\ 0 & \cos^2 \alpha_k & -\sin \alpha_k \cos \alpha_k \\ -\sin \alpha_k \cos \alpha_k & -\sin \alpha_k \cos \alpha_k & 1 \end{pmatrix} \\ L &= \sum_{k=1}^N \frac{1}{\Omega^{(k)}} \begin{pmatrix} -\sin \alpha^{(k)} & 0 \\ 0 & \cos \alpha^{(k)} \\ \cos \alpha^{(k)} & -\sin \alpha^{(k)} \end{pmatrix} \\ \mathbf{b} &= \sum_{k=2}^N \Omega^{(k)^{-2}} \begin{pmatrix} \sum_{i=2}^k f_x^{(i)} \\ \sum_{i=2}^k f_y^{(i)} \end{pmatrix} \\ \mathbf{e} &= \sum_{k=2}^N \frac{1}{\Omega^{(k)}} \begin{pmatrix} -\sin \alpha^{(k)} & 0 \\ 0 & \cos \alpha^{(k)} \\ \cos \alpha^{(k)} & -\sin \alpha^{(k)} \end{pmatrix} \begin{pmatrix} \sum_{i=2}^k f_x^{(i)} \\ \sum_{i=2}^k f_y^{(i)} \end{pmatrix} \end{aligned} \quad (4-11)$$

Solving Eq. (4-9), we can then get $\mathbf{r}$. Substitute Eq. (4-3) using the result of Eq. (4-9), we can then get the inter-element force $\mathbf{r}_2, \mathbf{r}_3, \dots, \mathbf{r}_n$.

As $\mathbf{r}$ is solved, value of $\sigma^{(k)}\{\mathbf{r}\}$ can be determined next, the only unknown variable in Eq. (4-8) is T . A minimization of ϕ with respect to $\mathbf{T}$ is performed to get the relationship between $\sigma^{(k)}\{\mathbf{r}\}$ and σ . This relationship can be written as a compact form

$$A\sigma = c \quad (4-12)$$

Where

$$A = \sum_{k=1}^N \begin{pmatrix} \sin^2 \alpha_k & 0 & -\sin \alpha_k \cos \alpha_k \\ 0 & \cos^2 \alpha_k & -\sin \alpha_k \cos \alpha_k \\ -\sin \alpha_k \cos \alpha_k & -\sin \alpha_k \cos \alpha_k & 1 \end{pmatrix} \quad (4-13)$$

$$\sigma = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \text{ and } c = \sum_{k=1}^N \begin{pmatrix} \sigma^k \cos^2 \alpha_k - \tau^k \sin \alpha_k \cos \alpha_k \\ -2\sigma^k \sin \alpha_k \cos \alpha_k - \tau^k (\sin^2 \alpha_k - \cos^2 \alpha_k) \\ \sigma^k \sin^2 \alpha_k + \tau^k \sin \alpha_k \cos \alpha_k \end{pmatrix}$$

Then, recovered stress σ can be obtained by solving this equation.

4.2.2 Nodes at the boundary

For nodes along the boundary, the minimization method is still suitable in getting the inter-element forces and only a minor change need to be done.

Figure 4-4 Equilibrated nodal force f and inter-element force r at the boundary node

The equilibrium equation can be written as

$$\begin{cases} f_1 + r_1 = F^{(k)}; \\ f_2 - r_1 + r_2 = 0; \\ f_3 - r_2 + r_3 = 0; \\ \dots \\ \dots \\ \dots \\ f_n - r_{n-1} = 0 \end{cases} \quad (4-14)$$

Here, the nodal force on node k is represented by f_k and the inter-element force between element k and $k+1$ is represented by r_k .

Actually, equation system (4-15) has $N-1$ equations and $N-1$ unknowns as the nodal forces $f_1, f_2, \dots, f_n$ converging at point i must be in equilibrium with the applied forces:

$$\sum_{k=1}^N f_k = \sum_{K=1}^N F^{(K)}, \text{ and no extra equation is needed to solve the inter-element forces.}$$

$$\begin{cases} r_1 = F^{(k)} - f_1; \\ r_2 = F^{(k)} - f_1 - f_2; \\ r_3 = F^{(k)} - f_1 - f_2 - f_3; \\ \dots \\ \dots \\ \dots \\ r_{n-1} = F^{(k)} - f_1 - f_2 - f_3 - \dots - f_{n-1} \end{cases} \quad (4-15)$$

Then, stress traction vector on the inter-element face can then be calculated by substituting Eq. (4-15) in Eq. (4-4). Same as the inside nodes, the recovered stress for the boundary nodes can be obtained by minimizing Eq. (4-8) with respect to σ_t and σ_n . In this case:

$$\begin{aligned} \sigma_t &= \frac{\sum_{k=1}^N [\sigma^{(k)} \sin^2(\beta - \alpha^{(k)}) + \tau^{(k)} \cos(\beta - \alpha^{(k)}) \sin(\beta - \alpha^{(k)})]}{\sum_{k=1}^N \sin^2(\beta - \alpha^{(k)})} \\ \sigma_n &= \frac{\sum_{k=1}^N [\sigma^{(k)} \cos^2(\beta - \alpha^{(k)}) + \tau^{(k)} \cos(\beta - \alpha^{(k)}) \sin(\beta - \alpha^{(k)})]}{\sum_{k=1}^N \cos^2(\beta - \alpha^{(k)})} \end{aligned} \quad (4-16)$$

4.3 Validation of the nodal stress recovery

In this section, several examples are presented to demonstrate the effectiveness of the stress recovery method introduced in the previous section. At the same time, a discussion about the effect of the time step interval on the computational efficiency for the DDA and the NDDA method is carried out.

The choice of time step size is very sensitive to the problem magnitude, block size, block number and block moving speed. Although both the DDA method and the NDDA method are the implicit based methods which are not restricted by the time step size, as stated in a recent research (Khan, 2010), a larger time step size does not necessarily reduce the computational time of the whole analysis since it may increase the O-C iteration and even make it difficult to converge.

In this section, two time interval sizes are tested and the numbers of time step iteration, open-close iteration and successive over relaxation (SOR) are recorded for each example. The relationship between the three iterations is illustrated in Figure 4-5: the time step iteration is the largest loop which includes contact detection, global stiffness matrix

building, O-C iteration, and system status updating; the SOR iteration is the smallest loop for solving the system equations, which is considered to be the most time consuming procedure; the O-C iteration is the most important procedure as it checks the contact status and includes the SOR iterations, the more O-C iteration in a time step, the more computational time will be taken.

Figure 4-5 Schematic illustration of the relationship of three main loops

4.3.1 Infinite plate with a circular hole under uniaxial tension

Considered an infinite elastic medium with a circular hole subjected to a uniform far-field stress σ , as depicted in Figure 4-6. The dimension of the plane is 20m×20m and the diameter of the circular hole is 2m.

Figure 4-6 Geometry of an infinite plane with a circular hole

The analytical solution provided by Timoshenko (1951) is:

$$\begin{aligned}
\sigma_r &= \frac{\sigma}{2} \left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma}{2} \left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right) \cos 2\theta \\
\sigma_\theta &= \frac{\sigma}{2} \left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma}{2} \left(1 + \frac{3a^4}{r^4}\right) \sin 2\theta \\
\sigma_{r\theta} &= -\frac{\sigma}{2} \left(1 - \frac{3a^4}{r^4} - \frac{2a^2}{r^2}\right) \sin 2\theta
\end{aligned} \tag{4-17}$$

where σ is the far-field stress, a is the radius of the circular hole, r and θ are polar coordinates, as shown in Figure 4-6.

The analytical solution can be expressed by Cartesian coordinate system using the coordinate transformation (Timoshenko, 1951)

$$\begin{aligned}
\sigma_x &= \sigma_r \cos^2 \theta + \sigma_\theta \sin^2 \theta + 2\sigma_{r\theta} \sin \theta \cos \theta \\
\sigma_y &= \sigma_r \sin^2 \theta + \sigma_\theta \cos^2 \theta - 2\sigma_{r\theta} \sin \theta \cos \theta \\
\tau_{xy} &= (\sigma_r - \sigma_\theta) \sin \theta \cos \theta + 2\sigma_{r\theta} (\cos^2 \theta - \sin^2 \theta)
\end{aligned} \tag{4-18}$$

The original DDA method is invalid to describe the stress distribution for this kind of problem because the first order polynomial is used as the displacement function in the original DDA, which restricts the block to constant stresses. In order to solve a continue problem by DDA, a sub-block method (Ke, 1993; Lin, 1995) is used. In the sub-block method, the entire block is first discretized into sub-blocks, and then very stiff springs will be placed at the sub-block interface to satisfy the displacement compatibility by allowing no relative displacement along the sub-block interface.

Due to the symmetry, only one quadrant of the problem needs to be modeled. The domain size is 10m×10m with the radius of the circular hole as 1m. The problem domain is discretized into 768 triangular elements and the mesh detail is showed in Figure 4-7. A far-field uniaxial tension $\sigma=1\text{KPa}$ is applied at the upper boundary. The symmetrical axis is fixed in x-direction only and the boundaries at the right hand side and the left are free surface. 25 measurement points are set along the 315 °direction.

Table 4-1 Material properties of the infinite plane

Rock sample	Poisson's ration	0.25
	Mass per unit (Kg/m ³)	2500
	Young's modulus(GPa)	20
Artificial joint	Friction angle	70
	Cohesion strength (MPa)	30
	Tensile strength (MPa)	15

For comparison purpose, four methods are employed: the standard DDA with sub-blocks (DDA), the standard NDDA (NDDA-S), the NDDA with stress recovery scheme (NDDA-R), and analytical method (ANA). The material properties of the block are shown in

Table 4-1. During the simulation, the upper boundary stress is applied in a manner that it increases linearly from 0 to 1KPa at the first 0.5s and then kept as constant at the level of 1KPa.

Figure 4-7 Numerical simulation model of the infinite plane with a circular hole

Two time interval size, 0.05s and 0.005s, are tested. For each time interval size, the maximum displacement ratio is set as one fifth of the time interval and the contact spring stiffness for both methods is set as 10 times of the Young's modulus of the material. As the double-minimization procedure is only a post-processing step to obtain a more accurate stress state and inter-element forces, it will have very little effect on the computational efficiency and the convergence of the NDDA method, thus only the comparison between the standard NDDA method and the DDA method is carried out.

It should be noted that in both the NDDA method and the DDA, the specified time step interval does not necessarily mean the same time step interval throughout the analysis. For both methods, if the convergence of the O-C iteration is not achieved in six iterations within one time step, the program itself will reduce the time step interval to 0.3 times of the original value. Considering both the accuracy and the efficiency, the maximum number of the SOR iteration within one O-C iteration is set as 200 for both methods.

The computational speed of both methods with different time interval size is presented in Table 4-2. For the same integration time, the increase of the time interval size will reduce the number of total time step, thus reduce the number of O-C iteration and the SOR iteration. It can be found that, the DDA needs a little bit more time step and O-C iteration than the NDDA method to obtain the same integration time, but the difference between the two methods is very small. However, in case of the number of SOR iteration and the total CPU time, the difference between the DDA method and the NDDA method is

dramatic. The reason is that, when the sub-block method is introduced in the DDA block system, the number of the contact pairs increases greatly, which leads the convergence of SOR equation solver difficult, as a result, many of the SOR iteration and CPU time are needed to solve the global equation. At the same time, as the static analysis is used, the convergence of the O-C iteration will be affected only at the beginning of the simulation, and the total number of time steps and O-C iteration will not increase significantly.

While in the NDDA method, as the contact detection is only carried out at the block boundary, the increasing of the contact pairs is limited, thus few SOR iteration and CPU time are needed to solve the global equation.

Table 4-2 Comparison of the computational efficiency of DDA and NDDA

Time interval(s)	method	Integration time (s)	Total time steps	O-C iteration	SOR iteration	Total CPU time(ms)
0.05	NDDA	5	100	101	15,838	5,085
	DDA	5.015	101	114	22,914	55,693
0.005	NDDA	5	1000	1001	15,721	33,295
	DDA	5.0015	1001	1012	203,613	478,307

As reviewed previously, the DDA method is a displacement-based implicit method which is generally unconditional stable, the time-step size is not limited by minimum element size. However, when the sub-block method is introduced in the DDA block system, the convergence speed of the solution is reduced, especially when a comparable large time interval is applied.

Figure 4-8 and Figure 4-9 show the stress time-history obtained from the DDA method and the NDDA method for measurement point (1.295 1.295) when time interval (Δt) equals to 0.05s and 0.005s, respectively. As it can be seen, when $\Delta t = 0.05s$, the DDA solution fluctuates greatly at the early stage and converges to 1.164KPa gradually, while the NDDA solution gets only little fluctuation at the beginning and converges to 1.178KPa quickly. When $\Delta t = 0.005s$, the solutions by both the DDA method and the NDDA method converge quickly, however, the solution of the DDA method still gets some fluctuation at the initial stage while the solution of the NDDA method is nearly stable. Considering both the convergence speed of the solution and the CPU time consumption, for the NDDA method, $\Delta t = 0.05s$ is the suitable time interval size while for the DDA method, $\Delta t = 0.005s$ is the suitable time interval size.

Figure 4-8 Stress time-history ($\Delta t = 0.05s$) of the measurement point (1.295, 1.295)

Figure 4-9 Stress time-history ($\Delta t = 0.005s$) of the measurement point (1.295, 1.295)

Figure 4-10 shows the comparison between the numerical results obtained from different methods and analytical solution. In the sub-block DDA method, if the measurement point lies on the block corner, the obtained stress of that point will be the average stress of all the blocks converging at this corner. As it can be seen, though the difference exists, all the methods predict the general stress state of the problem. Table 4-3 presents the simulation results of σ_y corresponding with different numerical methods as well as the error compared with the analytical solution, respectively. The simulation results indicate that, with the double-minimization procedure, the NDDA program can obtain a more accurate stress state compared with the original NDDA for most of the measurement points; and for the measurement points near the block boundary, the recovery effect is more obviously.

Figure 4-10 Comparison between numerical results and analytical solution ($\Delta t = 0.005s$)

Table 4-3 σ_y obtained from different methods ($\Delta t = 0.005s$)

Measurement point	X-Coordinate	Analytical (KPa)	DDA (KPa)	Error (%)	NDDA-S (KPa)	Error (%)	NDDA-R (KPa)	Error (%)
1	0.707	0.500	0.852	70.40	0.892	78.4	0.694	38.80
2	0.884	1.025	1.015	0.98	1.036	1.07	1.005	1.95
3	1.081	1.153	1.139	1.21	1.159	0.52	1.155	0.17
4	1.295	1.165	1.163	0.17	1.178	1.16	1.165	0.00
5	1.499	1.148	1.155	0.61	1.166	1.56	1.160	1.05
6	1.718	1.126	1.135	0.80	1.146	1.78	1.139	1.15
7	1.941	1.106	1.117	0.99	1.126	1.81	1.120	1.27
8	2.181	1.089	1.101	1.10	1.108	1.74	1.107	1.65
9	2.429	1.074	1.086	1.12	1.093	1.77	1.092	1.69
10	2.711	1.061	1.074	1.23	1.080	1.79	1.080	1.79
11	3.002	1.051	1.061	0.95	1.069	1.71	1.069	1.71
12	3.309	1.043	1.058	1.44	1.060	1.63	1.059	1.53
13	3.636	1.036	1.050	1.35	1.051	1.45	1.052	1.54
14	3.974	1.030	1.044	1.36	1.044	1.36	1.044	1.36
15	4.343	1.025	1.039	1.37	1.037	1.17	1.038	1.27
16	4.725	1.022	1.033	1.08	1.031	0.88	1.032	0.98
17	5.151	1.018	1.029	1.08	1.025	0.69	1.027	0.88
18	5.581	1.016	1.025	0.89	1.020	0.39	1.021	0.49
19	6.067	1.013	1.021	0.79	1.015	0.20	1.018	0.49
20	6.552	1.011	1.017	0.59	1.010	0.10	1.012	0.10
21	7.106	1.010	1.013	0.30	1.007	0.30	1.010	0.00
22	7.661	1.008	1.008	0.00	1.004	0.40	1.005	0.30
23	8.317	1.007	1.003	0.40	1.001	0.60	1.006	0.10
24	9.087	1.006	1.001	0.50	1.000	0.60	1.001	0.50
25	10	1.005	1.000	0.50	1.00	0.50	1.000	0.50

4.3.2 Induced stress due to vertical loading on an infinite strip footing on the surface

Figure 4-11 Uniform vertical loading on an infinite strip on the surface

Figure 4-11 shows the case where a uniform vertical load of p per unit area is acting on a flexible infinite strip footing on the surface of a semi-infinite elastic mass. This is a plane-strain problem as all the planes normal to y -axis have same stress state. In this case, only the stress state on the plane xoz needs to be studied.

If the Cartesian coordinate system is used and the origin of the coordinate system is set at the center of the uniform vertical loading, as shown in Figure 4-12.

Figure 4-12 Uniform vertical loading on an infinite strip

Then the analytical solutions provided by Poulos (1973) are as follows:

$$\begin{aligned}
\sigma_x &= \frac{p}{\pi} \left[\tan^{-1} \frac{z}{x-b} - \tan^{-1} \frac{z}{x+b} + \frac{2bx(x^2 - z^2 - b^2)}{(x^2 + z^2 - b^2)^2 + 4b^2 z^2} \right] \\
\sigma_z &= \frac{p}{\pi} \left[\tan^{-1} \frac{z}{x-b} - \tan^{-1} \frac{z}{x+b} - \frac{2bx(x^2 - z^2 - b^2)}{(x^2 + z^2 - b^2)^2 + 4b^2 z^2} \right] \\
\tau_{xz} &= \frac{4bpzxz^2}{\pi[(x^2 + z^2 - b^2)^2 + 4b^2 z^2]}
\end{aligned} \tag{4-19}$$

Where p is the vertical load per unit area, (x, z) is coordinate of the calculation point and b is the half width of the strip.

DDA with sub-blocks, NDDA and NDDA with stress recovery are used to simulate this problem. Due to the symmetry, only one half of the problem needs to be modeled. The domain size is 10m×10m with the Elastic modulus $E=20$ GPa and Poisson ration $\nu = 0.25$. The problem domain is discretized into 800 triangular elements and the mesh detail is showed in Figure 4-13. A uniform vertical loading $\sigma = 1$ MPa is applied at the upper boundary with the width of 1m. The bottom boundary is fixed and the right hand side boundary is free surface. The rolling boundary is set on the symmetrical axes. 10 measurement points are set along the direction which is parallel to z -axis, as shown in Figure 4-13.

Figure 4-13 Numerical simulation model of the problem

The material properties of the infinite plane are the same with the first example (see Table 4-1). The gravity force of the material is not considered and the resultant stresses are the induced stresses only. The vertical loading is applied in a manner that it increases linearly from 0 to 1MPa at the first 0.5s and then kept as constant at the level of 1MPa. Same as the first example, two time interval size, 0.05s and 0.005s are tested, and the maximum displacement ratio is set as one fifth of the time interval.

Table 4-4 Comparison of the computational efficiency of DDA and NDDA for example 2

Time interval(s)	method	Simulation time (s)	Total time steps	O-C iteration	SOR iteration	Total CPU time(ms)
0.05	NDDA	2	40	40	8,000	2,498
	DDA	2.015	41	64	12,864	37,957
0.005	NDDA	2	400	402	80,011	23,958
	DDA	2.0015	401	429	86,430	236,865

Table 4-4 presents the computational speed of both methods with different time interval size. Same as the first example, two methods result in similar numbers of total time steps and O-C iteration, but different number of SOR iteration and total CPU time. For both methods, less CPU time is needed to obtain the similar simulation time when a comparable larger time step size is applied. And for the same time step size, the NDDA method needs much less CPU time to complete the same simulation time than the DDA method.

Figure 4-14 and Figure 4-15 show the stress time-histories obtained from the two methods at the measurement point (1, 0.5) when time interval equals to 0.05s and 0.005s, respectively. Same as the first example, for both time interval sizes, the solution of the NDDA method fluctuates less and converges quickly than that by the DDA method. And for the DDA, when a comparable small time interval is applied, the solution fluctuates less and converges quickly, while for the NDDA method, the choice of the time interval has little effect on the convergence of the solution.

Figure 4-14 Stress time-history ($\Delta t = 0.05s$) of the measurement point (1, 0.5)

Figure 4-15 Stress time-history ($\Delta t = 0.005s$) of the measurement point (1, 0.5)

Figure 4-16 shows the comparison between the numerical results and analytical solution, as it can be seen, all the methods predict the general stress state of the problem well and the maximum error appears at the boundary node.

Table 4-5 presents the simulation results of σ_z corresponding with different numerical methods as well as the error compared with the analytical solution, respectively. Similar with the previous example, the NDDA method with the double-minimization procedure obtains a more accurate stress state compared with other methods and the results obtained from the DDA and the NDDA method do not differ much.

Figure 4-16 Comparison between numerical results and analytical solution ($\Delta t = 0.005s$)

Table 4-5 Simulation results of normal stress and error for different measurement points ($\Delta t = 0.005s$)

Measurement point	Coordinate	Analytical (KPa)	DDA (KPa)	Error (%)	NDDA-S (KPa)	Error (%)	NDDA-R (KPa)	Error (%)
1	(1,0.5)	500	360	28.00	353	29.40	584	16.80
2	(1,0.5)	497	508	2.21	500	0.60	499	0.40
3	(1,1)	480	498	3.75	490	2.08	488	1.67
4	(1,1.5)	448	458	2.23	454	1.12	454	1.34
5	(1,2)	409	412	0.73	412	0.73	411	0.50
6	(1,2.5)	370	372	0.54	374	1.08	372	0.54
7	(1,3)	334	336	0.60	340	1.80	335	0.30
8	(1,3.5)	302	307	1.66	311	2.98	299	1.00
9	(1,4)	275	283	2.91	286	4.00	268	2.55
10	(1,4.5)	251	262	4.38	265	5.58	250	0.40
11	(1,5)	231	245	6.06	248	7.34	236	2.16

In the numerical simulation, as the problem domain is homogeneous and finite, the vertical displacement w_z of each point converges to a constant value. Figure 4-17 shows the vertical displacement of the measurement points obtained from the DDA and the NDDA method. Under the same input parameters, the DDA obtains bigger vertical displacement than the NDDA method. It is worth to note that in theory, as the problem domain is homogeneous and semi-infinite, the vertical displacement w_z of each point is infinite and only the evaluation of the displacement of one point relative to another is meaningful. The analytical solution of the relative vertical displacement between any point within the problem domain and the central surface is:

$$w_{z(0,0)} - w_{z(x,z)} = \frac{2pb(1-\nu^2)}{\pi E} \left\{ \frac{1}{2} \left(1 + \frac{x}{b}\right) \ln \left[\frac{z^2 + (b+x)^2}{b^2} \right] + \frac{1}{2} \left(1 - \frac{x}{b}\right) \ln \left[\frac{z^2 + (b-x)^2}{b^2} \right] \right. \\ \left. + \frac{z}{b} \tan^{-1} \left(\frac{b+x}{z} \right) + \frac{z}{b} \tan^{-1} \left(\frac{b-x}{z} \right) \right\} \quad (4-20)$$

where (x, z) is the coordinate of the point, b is the half width of the strip, p is the applied stress and E and ν are the Young's modulus and Poisson's ratio of the material, respectively.

Figure 4-17 Vertical displacement of strip on homogeneous material

4.3.3 Cantilever beam subject to point loading

Figure 4-18 shows a two dimensional cantilever beam with 10 meter in length and 1 meter in height. The left boundary of the beam is fixed and the original point is set at the lower left corner of the cantilever beam. A point load equal to 5KN is applied at (9.8735, 0.5), near the free side of the beam.

In this example, the beam is discretized by 1422 triangular elements and the mesh detail is showed in Figure 4-19. For any point within the beam, its stress state and deflection can be obtained from the beam theory:

$$\begin{aligned}\sigma_x &= \frac{Fy(l-x)}{I} \\ \tau_{xy} &= \frac{3}{2} \times \frac{F}{bh} \times \left[1 - \left(\frac{2y-1}{h} \right)^2 \right] \\ f &= \frac{Fx^2(3l-x)}{6EI}\end{aligned}\tag{4-21}$$

where F is the applied load, l is the x axis coordinate of the loading point, (x, y) is the coordinate of measure point, b is the width of the beam, h is the height of the beam, E is the Young's modulus of the material, and I is the moment of inertia of the cross-section.

DDA with sub-blocks, NDDA and NDDA with stress recovery are used to simulate this problem. The material properties of the block are the same as the one used in the previous two examples (see Table 4-1). Seven points at the top of the cantilever beam are chosen as measurements points, and the coordinates of these points can be found in Table 4-7. The point load is applied in a manner that it increases linearly from 0 to 5KPa at the first 0.5s and then kept as constant at the level of 5KPa. As the element size of this example is

smaller than the previous two examples, the time step size used in this example should also be smaller than the previous two examples. Firstly, the time interval $\Delta t = 0.05\text{s}$ is tested, the non-convergence of the contact state requires the program itself to reduce the time interval frequently, which makes the DDA program inefficient and the actual time interval much smaller than the specified one. At the same time, the calculation speed becomes slow as more O-C iterations are involved in one time step. In order to save the computational time, relatively small time interval sizes, $\Delta t = 0.005\text{s}$ and $\Delta t = 0.0005\text{s}$ are used in this example, and the maximum displacement ratio is set as one fifth of the time interval size for both methods.

Figure 4-18 Cantilever beam subject to point load

Figure 4-19 Mesh detail of the beam

Table 4-6 Comparison of the computational efficiency of DDA and NDDA

Time interval(s)	method	Integration time (s)	Total time steps	O-C iteration	SOR iteration	Total CPU time(ms)
0.005	NDDA	10	2000	2000	400,000	430,155
	DDA	10.0002	2002	2086	419,183	185,483,0
0.0005	NDDA	10	20000	20000	400,000,0	406,964,5
	DDA	10.0001	20032	20657	415,205,7	190,497,81

Table 4-6 presents the computational speed of both methods with different time interval sizes. Same as in the previous two examples, the NDDA method has better computational efficiency than the DDA as it needs less CPU time to complete similar simulation time. For both methods, less CPU time is needed to complete the same simulation time when a comparable larger time step interval is applied.

Figure 4-20 Stress time-history ($\Delta t = 0.005s$) of the measurement point (5.3163, 1)

Figure 4-21 Stress time-history ($\Delta t = 0.0005s$) of the measurement point (5.3163, 1)

Figure 4-20 and Figure 4-21 show the stress time-histories obtained from the two methods at the measurement point A (5.3163, 1) when time interval equals to 0.005s and 0.0005s, respectively. When $\Delta t = 0.005s$, the solution obtained from the DDA fluctuates greatly and converges gradually after 30 simulation seconds while the solution obtained from the NDDA method gets only small fluctuation at the first 3 seconds and converges after 5 simulation seconds. When $\Delta t = 0.0005s$ is applied, though the solution of the DDA method still gets certain fluctuation at the first 10 simulation seconds, the amplitude of the fluctuation is reduced, at the same time, the convergence speed of the solution increase. However, for the NDDA method, a too small time step interval results in the difficulty in convergence.

Consider both the consumption of the CPU time and the convergence speed of the solution, a smaller time interval size, $\Delta t = 0.0005\text{s}$ is suggested for the DDA while a bigger time interval size, $\Delta t = 0.005\text{s}$ is suggested for the NDDA method.

Table 4-7 Simulation results of normal stress and error for different measure points

Measurement point	Coordinate	Analytical] (KPa)	DDA (KPa)	Error (%)	NDDA-S (KPa)	Error (%)	NDDA-R (KPa)	Error (%)
1	(1.2656,1)	258.2	248.8	3.64	233.2	9.68	251.3	2.67
2	(2.4315,1)	220.3	212.0	3.77	198.9	9.71	214.3	2.72
3	(3.7973,1)	183.2	175.2	3.89	164.4	9.71	177.4	2.69
4	(5.3163,1)	136.4	131.0	3.96	123.5	9.46	133.1	2.42
5	(6.5821,1)	98.7	93.7	5.07	89.2	9.63	96.1	2.63
6	(7.8479,1)	60.8	56.4	5.92	54.9	9.70	59.1	2.80
7	(9.1137,1)	22.8	21.1	7.46	21.2	7.02	22.9	-0.44

Table 4-8 Simulation results of shear stress for different measure points

Measurement point	Coordinate	Analytical] (KPa)	DDA (KPa)	NDDA-S (KPa)	NDDA-R (Pa)
1	(1.2656,1)	0	-1938.9	-1728.5	-9.10
2	(2.4315,1)	0	-1926.5	-1723.2	-14.03
3	(3.7973,1)	0	-1881.8	-1710.9	-15.96
4	(5.3163,1)	0	-1826.7	-1692.5	-15.07
5	(6.5821,1)	0	-1839.4	-1675.0	-12.25
6	(7.8479,1)	0	-1700.0	-1657.4	-8.11
7	(9.1137,1)	0	-1403.2	-1437.8	-3.26

Figure 4-22 Simulation result of deflection

Table 4-7 and Table 4-8 show the simulation results of σ_x and τ_{xy} corresponding with different numerical methods as well as the error compared with the analytical solution, respectively. Also, the NDDA program with double-minimization procedure provides a

more accurate stress state compared with the original NDDA method and the sub-block DDA method.

Figure 4-22 compares the beam deflection between DDA with sub-blocks, NDDA and the analytical solution. Though the absolute error increases along the beam for both methods, NDDA has a more accurate result than DDA with sub-blocks.

4.3.4 Discussion

In this section, three classical problems with analytical solutions are presented to verify the accuracy of the proposed stress recovery method. The comparison is carried out between the standard NDDA method, the NDDA method with stress recovery, the DDA method with sub-block system and the analytical solution. The results show that: 1) the recovery method can provide more accurate stress field than the standard NDDA method and the sub-block DDA system; 2) the sub-block method is also an alternative way to solve continuum problem as the stress field obtained from the sub-block DDA is also acceptable.

Considering the results listed in Table 4-2, Table 4-4 and Table 4-6, the following conclusions regarding the computational efficiency of the NDDA method and the sub-block DDA method can be drawn: 1) for both methods, a comparable larger time interval size can reduce the numbers of the O-C iteration and SOR iteration, thus saving the CPU time; 2) as the static analysis is applied, the two methods have little difference on the number of total time step and O-C iteration; 3) in most cases, the sub-block DDA method has larger numbers of the SOR iteration than the NDDA method, and in all the three examples, the NDDA method consumes much less CPU time than the DDA when the same time interval size is applied. It should be noted that the comparison about the CPU time obtained in this section is only a preliminary results, because both the DDA method and the NDDA method are not commercialized, the computational efficiency may also be affected by the skill of the code developer.

Considering the results shown in Figure 4-8, Figure 4-9, Figure 4-14, Figure 4-15, Figure 4-20, and Figure 4-21, the following conclusions regarding the speed of convergence of both methods can be drawn: 1) for both methods, there is a range of time interval sizes which produces reasonable computational speed; 2) within the range of the time-step interval sizes, a comparable larger time interval results in large fluctuations of the solutions and slower convergence speed for both methods; however, for the sub-block DDA, this effect is very obvious while for the NDDA method, this effect is negligible (except for example 3, in which too small time interval result in the difficulty in converging); and 3) considering both the computational speed and convergence speed, a comparable larger time interval is suitable for the NDDA method while a comparable small time interval is suitable for the sub-block DDA.

Considering the results shown in Figure 4-17 and Figure 4-22, the following conclusions regarding the deformation ability of both methods can be drawn: 1) both methods are able to describe the displacement field of a continuous problem; and 2) the nodal displacement obtained from the sub-block DDA method is larger than that obtained from the NDDA method, consisted with the statement that a block refined by sub-blocks will be “softer” in a certain direction or in a certain zone, depending on the distribution of the internal artificial joints (Bao, 2010).

In summary, the NDDA method is superior to the sub-block DDA method in the following areas:

- (1) The computational efficiency of the NDDA method is better than that of the sub-block DDA method, when the same time interval size is applied, the NDDA method consumes less CPU time.
- (2) The convergence speed of the NDDA method is quicker than that of the sub-block DDA method, for the same problem, a relatively larger time interval can be applied to the NDDA method.
- (3) Though the sub-block DDA method is able to predict reasonable stress field of a continuous problem, the displacement field provided by the sub-block DDA method is not as accurate as the NDDA method because too many internal artificial joints are introduced into the system, even with a strong penalty value, a small amount of penetration of the contact springs are inevitable.

4.4 Fracture sequence

As the cracks appears in time sequence and the earlier appeared cracks will affect subsequent new cracks, a fracture sequence based on the failure time of each mesh line and the special crack inserting procedure inside each time step including node splitting and stiffness matrix updating must be considered.

In the NDDA method, the continuous regions are governed by appropriated constitutive mechanisms while the material fracturing process is controlled by appropriate failure criteria. Both tensile and shear failures are allowed to happen by incorporating the Maximum tensile strength criterion and the Mohr-coulomb criterion in the NDDA method. In the fracture scheme, only one crack is allowed to open or slide at each time step, and the new crack initiates at the mesh line that satisfies the fracture criterion first.

The Maximum tensile strength criterion is

$$\sigma = \sigma_t \quad (4-22)$$

and the Mohr-coulomb criterion is

$$\tau = c + \sigma_n \tan \phi \quad (4-23)$$

where σ_t , c and ϕ are the tensile strength, cohesion and internal friction angle of the intact rock, respectively. If $\sigma_n > \sigma_t$, a tensile failure will occur; if $\tau > c + \sigma_n \tan \phi$, a shear failure will occur.

The crack inserting procedure within each time step is as follow:

Step 1. The stress traction vector on the inter-element face (mesh line) is computed using the stress recovery method presented in Section 4.2.

Step 2. Find the element edge which will fracture first according to the fracture sequence.

Suppose the current time is T and the time step interval is Δt . If there is no crack initiating at the current time, for any inter-element face k , $\sigma_T^{(k)} < \sigma$ and $\tau_T^{(k)} < \tau$ should be satisfied. If the time step interval Δt is small enough, the stress variation during this period can be considered as linear, thus the normal and shear stresses at inter-element face k at any time instant $T + \delta t$ ($0 < \delta t \leq \Delta t$) can be calculated in an interpolating form:

$$\sigma_{T+\delta t}^{(k)} = \sigma_T^{(k)} + \frac{\sigma_{T+\Delta t}^{(k)} - \sigma_T^{(k)}}{\Delta t} \delta t \quad (4-24)$$

$$\tau_{T+\delta t}^{(k)} = \tau_T^{(k)} + \frac{\tau_{T+\Delta t}^{(k)} - \tau_T^{(k)}}{\Delta t} \delta t \quad (4-25)$$

If $\sigma_{T+\Delta t}^{(k)} \geq \sigma_t > \sigma_T^{(k)}$, substituting $\sigma_{T+\delta t}^{(k)} = \sigma_t$ into Eq. (4-24), the tensile failure time $\delta t_\sigma^{(k)}$ can then be obtained

$$\delta t_\sigma^{(k)} = \frac{\sigma_t - \sigma_T^{(k)}}{\sigma_{T+\Delta t}^{(k)} - \sigma_T^{(k)}} \times \Delta t \quad (4-26)$$

If $\sigma_t > \sigma_{T+\Delta t}^{(k)}$, $\delta t_\sigma^{(k)} = 2\Delta t$ is set for numerical manipulation.

Substituting Eq. (4-24) and Eq. (4-25) into Eq. (4-23),

$$\tau_T^{(k)} + \frac{\tau_{T+\Delta t}^{(k)} - \tau_T^{(k)}}{\Delta t} \delta t_s^{(k)} = \left(\sigma_T^{(k)} + \frac{\sigma_{T+\Delta t}^{(k)} - \sigma_T^{(k)}}{\Delta t} \times \delta t_s^{(k)} \right) \tan \phi + c \quad (4-27)$$

and solve Eq. (27) to obtain the shear failure time

$$\delta t_s^{(k)} = \frac{(\tau_T^{(k)} - \sigma_T^{(k)} \tan \phi - c) \times \Delta t}{(\tau_{T+\Delta t}^{(k)} - \sigma_{T+\Delta t}^{(k)} \tan \phi - c) - (\tau_T^{(k)} - \sigma_T^{(k)} \tan \phi - c)} \quad (4-28)$$

If both $0 < \delta t_\sigma^{(k)} \leq \Delta t$ and $0 < \delta t_s^{(k)} \leq \Delta t$ are satisfied, it is necessary to compare the tensile failure time $\delta t_\sigma^{(k)}$ with the shear failure time $\delta t_s^{(k)}$ and the smaller one is the final failure time for the inter-element face k during the current time step, i.e.

$$\delta t^{(k)} = \min\{\delta t_{\sigma}^{(k)}, \delta t_s^{(k)}\} \quad (4-29)$$

and the corresponding failure mode is the final failure mode of inter-element face k .

For all the inter-element faces, find out the minimum failure time δt . If $\delta t > \Delta t$, there will be no new crack during this time step, otherwise, the crack initiates at the inter-element face which has the minimum failure time and the failure pattern depends on which kind of failure time δt belongs to. It should be noted that the failure time δt has no physical meaning, is just a numerical value obtained from the interpolating procedure, with the purpose of predicting whether the crack will occur in this time step and which type of crack it will be.

Step 3. Separate the mesh line which satisfied the fracture criterion and insert a new node if necessary. In this way, two elements sharing this line will be separated. Three cases may appear depend on the position of the mesh line: no new node, one new node and two new nodes:

(1) When two nodes that form the mesh line are internal nodes, no new node is inserted. In this case, only the mesh line is marked as a crack and the two nodes are updated as external nodes by the program (see Figure 4-23).

Figure 4-23 Schematic illustration on how a mesh line with two internal nodes becomes a crack

Figure 4-24 Schematic illustration on how a mesh line with one internal node and one boundary node becomes a crack

(2) When one of the two nodes that form the mesh line is external node of the block, while the other one is an internal node, one new node will be inserted at the location of the

external node after fracturing, and the internal node will be marked as the external node by the program (see Figure 4-24).

(3) When two nodes that form the mesh line are all external nodes, two new external nodes are inserted by the program (see Figure 4-25).

Figure 4-25 Schematic illustration on how a mesh line with two boundary nodes becomes a crack. No energy dissipation is considered in the current work, the newly inserted node is assumed to have same velocity and displacement with the old one.

A simple joint strength reduction model is applied for the new born crack. After fracturing, the tensile strength of the crack will be reduced to zero, the internal friction angle will be replaced by the joint friction angle, and the cohesion will be a small value to account for irregularity of the fracture surface in reality.

Step 4. Update the mesh and other information related with nodes and elements and go back to step 1 until the last time step.

One of the shortcomings of the proposed method is that the cracks are restricted to propagate along the mesh lines. However, with the proper size of the finite element mesh, the influence of mesh strategy on the simulation results will be insignificant. At the same time, the presented method provides a fully automatic way to describe the crack initiation and propagation, no tracing or remeshing is needed.

4.5 Modeling brittle fracture using the NDDA method

In this section, three examples are presented to verify the NDDA method for the brittle fracture problem as well as provide a better understanding of the physics of crack propagation.

4.5.1 An inclined centre crack in a square plate under uniaxial tension

Figure 4-26 shows a typical mixed mode crack problem, in which the opening-mode K_I and the sliding-mode K_{II} can be expressed as (Karihaloo, 1973)

$$\begin{aligned} K_I &= \sigma \sqrt{\pi a} \cos^2 \beta \\ K_{II} &= \sigma \sqrt{\pi a} \cos \beta \sin \beta \end{aligned} \quad (4-30)$$

And the crack extension angle θ can be calculated using the maximum circumferential stress criterion (Gdoutos, 2005)

$$K_I \sin \theta + K_{II} (3 \cos \theta - 1) = 0 \quad (4-31)$$

Substituting Eq (4-30) into Eq (4-31), we then have

$$\sin \theta + (3 \cos \theta - 1) \tan \beta = 0 \quad (4-32)$$

and the variation of the crack extension angle θ versus the crack inclination angle β is drawn in Figure 4-27.

Figure 4-26 Mixed-mode crack problem

In the numerical modeling, the square block has the dimension of $w=300\text{mm}$. The initial crack is placed at the centre of the block with half length $a=141.4\text{mm}$ and the incline angle $\beta = 45^\circ$. The bottom boundary of the block is fixed and the upper boundary of the block is fixed in y direction only. The block is loaded by boundary displacement with a speed of 0.30mm/min , as shown in Figure 4-28. There are totally 1204 triangular elements inside the block and a two-color scheme is employed to show the y direction compressive and tensile region in the block: blue for compressive σ_y , red for tensile σ_y . The material properties of the block are shown in Table 4-9.

Figure 4-27 Relationship between the crack inclination angle β and the crack extension angle θ

Figure 4-28 Applied displacement boundary condition

Table 4-9 Analytical parameter for an inclined crack under tension

Control parameters	Plane stress	Yes
	Time step size (s)	0.25
	Total time (s)	800
	SOR factor	1.1
	Max. disp. ratio	0.01
	Poisson's ratio	0.11
	Mass per unit (KG/m^3)	2300
Rock sample	Young's modulus (GPa)	50
	Friction angle	30
	Cohesion (MPa)	2.5
	Tensile strength (MPa)	1.44
Joint properties	Joint friction angle	20
	Joint cohesion (MPa)	0
	Joint tensile strength (MPa)	0

At the beginning of loading, most elements within the block enter a tensile stress state in y direction and the higher stress occurs around the crack tip area. Also, opposite to the tensile region there is a compressive region on the other side of the crack tip area. The wing cracks appear first at the tip of the crack and propagate rapidly toward the free surface of the specimen until the specimen fails into two parts. As shown in Figure 4-29.

Figure 4-29 Simulation results: (a) start loading; (b) crack appear; (c) crack propagation; (d) final failure

The wing cracks are formed by single cracks, indicate that with the double-minimization stress recovery procedure, the inter-element forces and traction can be determined more accurately, so that a more realistic fracture mode can be obtained. The simulated crack extension angle, which is restricted by the initial mesh of the block, is 43° while the theoretical solution of the extension angle is 53.13° . Regardless the discrepancy in predicting the extension angle, the final crack path obtained from the NDDA method is in generally normal to the direction of the tension, agrees with the theoretical prediction. Again, shortcoming of the presented method is that the cracks are restricted to propagate along the mesh lines; however, as it can be seen, though a comparable coarse mesh is

applied in this example, the method is able to predict the general crack path of the problem. In the next two examples, a finer element mesh will be adopted, and results show that the influence of mesh strategy is insignificant.

4.5.2 Propagation of pre-existing cracks under uniaxial compression

4.5.2.1 Back ground

The sliding wing crack model, initially proposed by Brace and Bombolakis (1963), is one of the most widely used models for tensile crack growth and brittle fracture in compression. According to this model, for a specimen containing an pre-existing crack subject to uniaxial compression, once the relative sliding of the two crack faces occurs, local tensile stresses at the crack tips produce two tensile “wing cracks” and propagate towards the loading direction (Figure 4-30).

Figure 4-30 Schematic illustration of the development of tensile wing cracks under uniaxial compression

In the past decades, the formation of both microcracks and macrocracks in rocks subjected to compressive stresses has attracted much attention. Since the macroscopic fault cannot develop from a single crack but must be the result of crack-crack interaction between multiple flaws (i.e. coalescence of crack branches) (Brace and Bombolakis, 1963), studies on specimen with multiple initial cracks becomes a more challenging issue.

Experimental studies on crack propagation and coalescence in specimens with multiple initial cracks begins with Brace and Bombolaki (1963), they performed uniaxial compression test on plexiglass containing echelon cracks. However, contrary to expectations, neither secondary cracks nor coalescence cracks of primary cracks were observed. Nemat-Nasser and Horii (1982) carried out loading tests on Columbia Resin CR

39 and the occurrence of coalescence was observed. Chen et al. (1995) carried out loading tests on marble to study coalescence patterns for multiple flaws with different arrangement patterns, and provided the sketches and result coalescence patterns. Wong and Einstein studied the crack behavior under uniaxial compression in specimens containing single flaws and flaw pairs in molded gypsum and carrara marble (2007, 2009), and classified the crack types based on the crack initiation mechanism instead of their temporal relationship.

Two numerical models on the propagation of pre-existing crack under uniaxial compression are carried out in this section, the material parameters employed in the examples are from the experiments done by Wong (2008) and the results agree well with the experimental observations.

4.5.2.2 Two inclined step flaws with the bridge angle of 60°

The specimen used in this numerical modeling is a rectangular block with 152.4 mm in length and 76.2 mm in width. Two initial flaws are designed in a position so that they can interact with each other under tension (see Figure 4-31), each crack is 12.7mm in length and has a 30° angle with respect to the horizontal direction. The bridge angle is 60° and the ligament length is 25.4 mm. The bottom boundary of the block is fixed and the upper boundary of the block is fixed in y direction only. The block is loaded by boundary displacement with a speed of 0.75mm/min. Two different meshes, a coarser one with 2352 triangular elements inside the block and a finer one with 4556 triangular element inside the block are used in the simulation. The mesh details are shown in Figure 4-32. The material properties of the block are shown in Table 4-10.

Table 4-10 Analytical parameter for two cracks under compression

Control parameters	Plane stress	Yes
	Time step size (s)	0.25
	Total time (s)	800
	SOR factor	1.1
	Max. disp. ratio	0.01
	Poisson's ratio	0.11
	Mass per unit (KG/m ³)	1560
Rock sample	Young's modulus (GPa)	5.96
	Friction angle	30
	Cohesion (MPa)	7
	Tensile strength (MPa)	3.2
Joint properties	Joint friction angle	0
	Joint cohesion (MPa)	0
	Joint tensile strength (MPa)	0

The simulation results are shown in Figure 4-33. Though difference does exist, the behaviors of the two sample blocks are quite similar with different element meshes. At the

initial loading phase, the elements near the upper and bottom of the initial cracks enter a tensile stress state while the elements at the tips of the initial cracks enter a compressive stress state. For both element meshes, tensile wing cracks appear first at the tip of the flaws, in the direction normal to the crack face, then develop as the load increased. For the coarser element mesh, the crack initiation stress σ_c is 17.11MPa, 0.12% away from the experimental result of 17.09MPa (Wang 2008); for the finer element mesh, the crack initiation stress σ_c is 15.94MPa, 6.72% away from the experimental result.

Figure 4-31 Configuration of the specimen

Figure 4-32 Mesh details of the example: (a) 2352 triangular elements; (b) 4556 triangular elements

After the crack initiation, all cracks grow with the increased load. Tensile wing cracks at the outer tip of the flaws tend to develop toward the compression zone. Tensile wing cracks initiated at the inner tip of the flaws tend to develop toward the other inner tip of the flaw. As shown in Figure 4-33, the simulation results agree with the experimental observation well. For the finer mesh, coalescence occurs as a result of increased load, though not exactly same, the coalescence behavior is similar with the experimental observation.

Figure 4-33 Simulation results: (a) wing crack propagation in NDDA simulation for coarser mesh; (b) crack development in coarser mesh; (c) wing crack propagation in NDDA simulation for finer mesh; (d) crack coalescence in finer mesh; (e) experimental observation of tensile wing crack (Wong 2008) (f) coalescence behavior observed by Wong 2008 .

4.5.2.3 Two inclined step flaws with the bridge angle of 90°

The specimen used in this numerical model has the same scale of the first model. The length and the incline angle of the crack are the same with the first model while the bridge angle is 120° and the ligament length is 25.4 mm (Figure 4-34). The boundary condition, loading condition and the material properties are also same with the first model.

Figure 4-34 Configuration of the specimen

Figure 4-35 Simulation results: (a) experimental observation of tensile wing crack; (b) wing crack propagation in the simulation; (c) crack coalescence in the simulation; (d) coalescence behavior observed by Wong (2008) in the experiment.

At the beginning of the loading, the phenomenon is similar with the previous case. Wing cracks appear first at the tips of the two existing cracks, in the direction normal to the crack face, and then develop as the load increases. The crack initiation stress σ_c

=14.64MPa is close to the average crack initiation stress 18.84MPa obtained by Wong (2008), and the error is 20.78%.

After the initiation, all cracks keep on extension as load increased. The development of the crack can be divided into two types. Wing cracks appeared at the upper face of the top flaws and lower face of the bottom flaw tend to develop toward the direction of the compression. Tensile wing cracks initiated at the lower face of the top flaw and upper face of the bottom flaw tend to develop toward the tip of the other flaw. As shown in Figure 4-35, the simulation result agreed with the experiment observation well. Finally, coalescence occurs as a result of the increasing load, though not exactly same, the coalescence behaved similar with the experimental observation.

4.6 Conclusion

In this chapter, a double-minimization procedure is introduced into the NDDA method to achieve more accurate inter-element stresses and thus to obtain more realistic brittle fracture modeling. It is also a post—processing procedure to recover the nodal stress state. Three classical problems are first simulated and the results indicate that the NDDA method with the double-minimization procedure can obtain more accurate nodal stress state than the standard NDDA method. Also, a detailed comparison between the NDDA method and the sub-block DDA method regarding the accuracy, convergence speed, and computational efficiency is carried out. Next, three brittle fracture examples are used to demonstrate that the NDDA method enhanced by the double minimization procedure is able to simulate the crack propagation problem in a more realistic way. Although the development of the crack depends on the initial mesh under the current NDDA framework, the crack propagation is simulated in a fully automatic way.

Chapter 5. The NDDA method in the analysis of toppling

5.1 Introduction of toppling

Toppling failure is one of the most commonly instability mode of rock slope, it is generally defined as the down-slope overturning of the interacting columns or rock blocks. Goodman and Bray (1976) have defined three types of toppling failure mechanisms, which are flexural toppling, block toppling and block-flexural toppling. Flexural toppling (see Figure 5-1(a)) often occurs in the rock column formed by well developed steeply dipping discontinuities, these rock columns can be assumed as an array of interacted cantilever beams fixed at a certain depth and bend forward due to their own weight and finally break in flexural. Block toppling (see Figure 5-1(b)) often occurs when single rock columns are divided by widely spaced orthogonal joints. The long columns in behind overturn and transfer thrust to the short columns in the toe zone, thus induce the sliding moment at the slope foot. Block-flexural toppling (see Figure 5-1(c)) is the pseudo-continuous flexure along columns which are divided by numerous cross joints. This type of failure is resulted from the sliding and rotating of columns on cross joints.

Figure 5-1 Toppling phenomena: (a) Flexural toppling, (b) Block toppling, (c) Block-flexural toppling. (After Goodman and Bray, 1976)

Although FEM has been the most popular numerical method in engineering sciences, its continuum-based nature limited the application of the method to toppling. In order to model a layered rock slope, special joint element is needed to be introduced into the finite element model (Kalkani and Piteau, 1976; Poisel, 1990). When the layers are sufficiently thin compared with the dimension of the slope, the flexural deformation of a layered slope can be analyzed using Cosserat theory (Adhikary, 1996).

The discontinuum based techniques, such as the distinct element code UDEC, is a more appropriate method for the modeling of toppling problem. Ishida (1987) applied the DEM on the analysis of toppling of a fissured rock slope, and the obtained result is similar with the site observation. Pritchard (1990) used the UDEC to model both the block and flexural toppling, and the results indicated that UDEC is able to model small and large scale topples of all types. Lanaro (1997) validated the UDEC modeling results against the tilt

table test and discussed the effects of friction angle, rotation velocity and sharpness of block corners during the numerical simulation. Alzoubi (2010) simulated the flexural toppling by creating randomly sized polygons in the rock column and allowing the fractures to occur along the polygonal edge, he then compared the UDEC results with the centrifuge tests and discussed the influence of the tensile strength on toppling failure.

As another member of discontinuum methods, the application and the validation of the DDA in the toppling analysis has also been carried out by few researchers. Yeung (1991) modified failure mode boundary between toppling without sliding and sliding and toppling simultaneously by incorporating the dynamic effects into the derivation and used the DDA simulation results to support his expression. McBride (2001) validated the DDA method by comparing it with the video and photographic record obtained from the tilt table test. The DDA simulation proved to be able to reproduce the failure pattern observed in the physical model and the centroidal displacement of the measured block show good agreement with the experimental data.

In fact, the discontinuum based methods are more suitable for the block toppling or block-flexural toppling in which the joints are always persistent and the relative sliding occurs along the pre-defined discontinuities. In case of flexural toppling, as there are no pre-existing discontinuities to form the basal failure plane, fracturing in the intact rock is needed. A continuum/discontinuum hybrid approach, in which the discrete fractures can be readily accommodated in the model and the new fractures are able to grow if required, maybe the most desirable one for the toppling problem.

In this chapter, the NDDA method with the double minimizing procedure introduced in Chapter 4 is applied to investigate the potential failure mechanism of toppling with persistent and non-persistent joints.

The remainder of the chapter is organized as follows. The analysis methods of the toppling are briefly review in Section 5.2. Modeling the toppling failures with NDDA method is presented in 5.3, parametric studies regarding to the persistence of the bedding plane, properties of the joint set, slope profile et al, are carried out. Section 5.4 gives the conclusion of the chapter.

5.2 Analysis methods

Though the three definitions presented by Goodman and Bray (1978) accurately describe the deformation behavior during the toppling, it should be noted that the failure mode in reality is more complicated and it is usually a combination of the above mentioned modes. In order to investigate the mechanism of the toppling failure in more depth, the analysis methods, including physical tests, analytical methods are adopted by researchers.

5.2.1 Physical models of toppling

5.2.1.1 Tile table

In order to represent the gravity in physical models, one of the most popular methods is to use tilt table. The model is first laid on a flat or an inclined plane, and then tilted to increase gravity. Barla (1995) carried out the tilt test to study mechanical characterization of the material and the discontinuities during the toppling phenomena. He also measured the displacement field and used the experimental results to validate the DEM numerical simulation.

5.2.1.2 Base friction model

Base friction model is an ideal tool for toppling mechanism study as it is convenient, inexpensive and revealing. Bray and Goodman (1981) published a detail discussion on the theory of the base friction model. To do this test, a two-dimensional model is placed on the belt of the base friction machine, then the belt is moved to drag the model against the boundary constrain. The basic principle of this model is to use the drag force acting along the underside of the model to replace the effect of the gravity.

Aydan (1992) successfully modeled the flexural toppling in underground opening which is seldom occurred compared with the widespread flexural toppling in surface slopes using the base friction apparatus.

5.2.1.3 Centrifuge model

Centrifuge model can be used to study the system such as slopes, earth retaining structures and the building or bridge foundations. The purpose of the centrifuge test is to use the centrifugal force to replace the gravity forces so that the stresses in the prototype can be modeled in the laboratory.

Adhikary (1997) did a series of centrifuge experiments to investigate the mechanism of flexural toppling and derived a set of charts for stability analysis of flexural toppling.

Zhang and Chen (2007) conducted the centrifuge model tests using gypsum columns, and used the experimental results to validate the analytical method proposed by Goodman and Bray (1976) and its extension, proposed by Chen (1995, 1996).

5.2.2 Analytical analysis of toppling

5.2.2.1 Basic relationship

Consider the simple condition that a single block with the height h and base length b is resting on an inclined plane, as shown in Figure 5-2. Hoek and Bray (1981) derived the conditions of sliding and toppling based on the static equilibrium as shown in Figure 5-3(a), where δ refers to the tangent value of the ratio between block length b and height h , ψ refers to the incline angle of the plane and ϕ refers to the friction angle of the plane

surface. The behavior of the block is divided into four modes: 1) if $\psi < \phi$ and $\delta > \psi$, the statue of the block is stable; 2) if $\psi > \phi$ and $\delta > \psi$, the statue of the block is sliding without toppling; 3) if $\psi < \phi$ and $\delta < \psi$, the statue of the block is toppling without sliding; 4) and if $\psi > \phi$ and $\delta < \psi$, the statue of the block is sliding and toppling. Goodman and Bray (1981) modified the chart proposed by Hoek and Bray by incorporating the dynamic effects into the derivation of boundary between mode 2 (sliding without toppling) and mode 4 (sliding and toppling), as illustrated in Figure 5-3(b). Yeung (1991) further modified Goodman and Bray's chart by incorporating the dynamic effects into the derivation of boundary between mode 3 (topping without sliding) and mode 4 (sliding and toppling). The boundary between mode 3 and mode 4 derived by Yeung can be expressed using the combination of ψ , ϕ , and δ as:

$$\tan \psi = \frac{3 \cos^2 \delta \tan \phi - 3 \sin \delta \cos \delta + \tan \phi}{3 \sin^2 \delta - 3 \sin \delta \cos \delta \tan \phi + 1} \quad (5-1)$$

Figure 5-2 Single block on an inclined plane

Figure 5-3 Kinematic conditions for sliding and toppling: (a) proposed by Hoek & Bray (1981) (b) modified by Goodman & Bray (1981).

5.2.2.2 Limit equilibrium method for block toppling failure

Goodman and Bray (1976) proposed a limit equilibrium approach to predict whether toppling occurs for a system of rectangular blocks on a stepped base. For the n th block in the system, consider the limit equilibrium conditions for sliding and toppling of the block (see Figure 5-4), the lateral forces $p_{n1-l, t}$, required to prevent toppling can be calculated statically and the lateral force $P_{n1-l, s}$, required to prevent sliding can be calculated with the assumption regarding the location and direction of the inter-force. By comparing the relative magnitudes of $p_{n1-l, t}$ and $p_{n1-l, s}$, the state of each block can then be determined.

Starting from the top region, find the block n_1 , which satisfies the condition:

$$\frac{y_n}{\Delta x} > \cot \alpha \quad (5-2)$$

If the lateral forces $p_{n1-l, t}$ is larger than the lateral force $p_{n1-l, s}$, the block tends to topple, and vice versa. Using the same procedure to predict the state for the lower block (n_1-1), (n_1-2) until the block n_2 , for which $p_{n2-l, s} > p_{n2-l, t}$ is satisfied. This method separates the block systems into three groups: the blocks taller than n_1 , at the top of the slope, belong to the stable group; the blocks lower than n_2 , at the toe of the slope, belong to the sliding group; and the blocks between n_1 and n_2 belong to toppling group. Considering p_0 on the toe block 1, the criteria for slope stability are: if $p_0 > 0$, the slope is unstable; if $p_0 < 0$, the slope is stable; if p_0 is sufficiently close to zero, the slope is at the limited equilibrium state.

Figure 5-4 Limit equilibrium analysis of toppling of rectangular blocks on a stepped system
Chen (1995, 1996) extended the mechanical formulations of the Goodman-Bray method as follows:

- 1) Introduced the factor of safety using the strength reduction method:

$$\begin{aligned} c_e &= c / F \\ \tan \varphi_e &= \tan \varphi / F \\ \sigma_{te} &= \sigma_t / F \end{aligned} \quad (5-3)$$

where c and φ are the cohesion and friction angle of the discontinuity, F is the strength reduction factor.

- 2) Considered the joint persistence ratio K of the basal plane.
- 3) Introduced the strength criteria for shear and bending failures.

Also, the extended method is validated by centrifuge tests (Zhang and Chen, 2007).

Instead of treating the toppling failure as a discrete assemblage of blocks, Bobet (1999) treated it as a continuous medium and derived the analytical solutions. The assumption inherited in his method is that the thickness of the blocks is very small compared with the dimension of the slope. Verification example showed that, when the slope height over the block thickness exceeds 50, the analytical solution gives accurate results.

5.2.2.3 Limit equilibrium method for flexural toppling

In case of flexural toppling, because there are no pre-existing flatly dipping discontinuities, the method proposed by Goodman and Bray is not suitable. Aydan and Kawamoto (1992) proposed a limit equilibrium approach by considering the block columns as a series of cantilever beams and using the theory of elasticity for analysis.

Figure 5-5 Model for limit equilibrium analysis of flexural toppling: (a) slope geometry and (b) forces acting on a single column.

The assumptions involved in the theoretical model are:

- (1) As shown in Figure 5-5(a), each column is treated as a cantilever beam subjected to gravity force and side forces. The position of the acting point is assumed to be μh_j where h_j is the height of the corresponding side and μ is a parameter defining the distribution of the inter-column force.
- (2) When the tensile stress due to bending reaches the tensile strength of the rock, the fracture will be initiated and the failure is inevitable.
- (3) The fracture follows a plane inclined upward with an angle θ to the normal of the joint dip.
- (4) A limit equilibrium state exists simultaneously across all the columns along the fracture plane.

In the theory of elasticity (Timoshenko, 1951), the tensile stress σ_x at the base of the column with a unit thickness is

$$\sigma_x = \pm \frac{N}{A} + \frac{M}{I} y \quad (5-4)$$

where N is the normal force in the column, M is the bending moment, A is the cross-section area, I is the inertia modulus and y is the distance from the centroid of the section.

Substitute the forces indicated in Figure 5-5(b) and rearrange Eq. (5-4), the following expression for P_{j-1} was derived by Aydan and Kawamoto (1992):

$$P_{j-1} = \frac{p_{j+1}(\mu h_j - \mu \frac{b_j}{2}) + w_j \sin \alpha \frac{h_j + h_{j-1}}{4} - \frac{2I_j}{b_j} (\sigma_{t_j} + \frac{N_j}{b_j})}{(\mu h_{j-1} + \frac{1}{2} \mu b_j)} \quad (5-5)$$

Starting from the uppermost column which is unstable from its own weight, a step-by-step method can be used to find the resultant side force p_0 at the toe of the slope. The criteria for slope stability are: if $p_0 > 0$, the slope is unstable; if $p_0 < 0$, the slope is stable; if p_0 is sufficiently close to zero, the slope is at the limit equilibrium state.

Based on the observations of model tests, Aydan and Kawamoto (1992) suggested the value of μ should be between 0.75 and 1, and the basal failure plane to be normal to the joint dip ($\theta = 0$). Adhikary (1997) did a series of centrifuge tests and varied the parameters θ and μ to match the centrifuge collapse data, the best fitted combination of parameters found by Adhikary are $\theta = 10^\circ$ to 12° and $\mu = 0.5$ to 0.6 .

5.3 The NDDA modeling of toppling

5.3.1 One block on incline face

Figure 5-6 Kinematic conditions for sliding and toppling considering the dynamic effect when $\varphi = 30^\circ$ (after Yueng, 1991)

To seek the potential of the NDDA method in the modeling of toppling, the simplest case of one rigid block on an incline face is first tested. During this test, the friction angle φ is fixed as 30° while the tangent value δ (which is the ratio between block length b and height h), and the incline angle of the plane ψ are varied to investigate the kinematic conditions for sliding and toppling as well as compare the numerical results with the classical chart. Figure 5-6 shows the kinematic conditions provided by Yueng (1991) with the friction angle $\varphi = 30^\circ$. Comparing the Hoek and Bray's chart (Figure 5-3(a)), Goodman and Bray's chart (Figure 5-3(b)) and Yueng's Chart (Figure 5-6) it can be found: for boundary I and boundary II, all the three charts predicted the same boundary; for boundary III, Goodman and Bray's chart and Yueng's chart predicted the same one while

Hoek and Bray's chart predicted a different one; for boundary IV, all the three charts predicted a different one.

First of all, the δ and ψ values listed in Table 5-1 are chosen to test the boundary I. During the modeling, the width of the base block is fixed at 1m while the height of the base block changes with the variation of incline angle. The height of the top block is fixed at 0.5m while the width of the top block varies to obtain the different tangent value. The fixed boundary condition is applied at the base block. Two measurement points, located at upper left corner (measurement point 1) and the lower left corner (measurement point 2) are set to check the status of the top block. The input parameters for this example are listed in Table 5-2. Dynamic analysis is used to provide real time displacement and deformation of the block. A relative large penalty value is used to minimize the inaccuracy caused by the interpenetration. The high Young's modulus and Poisson's ratio are used to satisfy the rigid block assumption.

Table 5-1 Chosen δ and ψ value for boundary I

Case number	Incline angle (°)	Tangent value (°)
1-1	5	4.5
1-2	5	5.5
1-3	10	9.5
1-4	10	10.5
1-5	15	14.5
1-6	15	15.5
1-7	20	19.5
1-8	20	20.5
1-9	25	24.5
1-10	25	25.5
1-11	29.5	29
1-12	29.5	30

Table 5-2 Input parameters of case 1

Dynamic	yes
Time step size(s)	0.01
Mass per unit(Kg/m3)	2548
Poisson's ratio	0.49
Spring stiffness(GPa)	30
Young's modulus(GPa)	3
Max disp. ratio	0.001
Joint friction angle	30
Joint cohesion	0
Joint tensile strength	0

Table 5-3 presents the simulated displacements after 1 second, since only the initial behavior of the block is of interest. However, in order to show the behavior of the block

more visually, the output graphics for case 1-3 and case 1-4 after 10 seconds are presented in Figure 5-7(a) and Figure 5-7(b), respectively. The blue line appears at the measurement point indicates the displacement vector. In Figure 5-7(a), the point at the upper left corner moves while the point at the lower left corner remains stable, indicating that the status of case 1-3 is toppling without sliding. In Figure 5-7(b), both the points at the upper left corner and lower left corner remain stable, indicating the status of case 1-4 is stable.

Table 5-3 Simulation results at time=1s

case	Measurement point 1		Measurement point 2		Behavior
	X-Disp(mm)	Y-Disp(mm)	X-Disp(mm)	Y-Dips(mm)	
1-1	0.508	0.046	0.000	0.000	T
1-2	0.012	0.003	0.000	0.000	Safe
1-3	0.592	0.107	0.000	0.001	T
1-4	0.009	0.004	0.000	0.000	Safe
1-5	0.654	0.178	0.000	0.001	T
1-6	0.006	0.006	0.000	0.001	Safe
1-7	0.561	0.209	0.000	0.002	T
1-8	0.005	0.005	0.001	0.001	Safe
1-9	0.624	0.295	0.001	0.002	T
1-10	0.008	0.009	0.000	0.003	Safe
1-11	0.621	0.357	0.001	0.003	T
1-12	0.008	0.009	0.004	0.005	Safe

Figure 5-7 NDDA output graphics: (a) toppling; (b) stable.

Second, the δ and ψ values listed in Table 5-4 are chosen to test the boundary II. Same as the previous step, the width of the base block is fixed at 1m while the height of the base block changes with the variation of the incline angle. For the top block, the longer edge is set as 0.5m, and the shorter edge is changed to satisfy the required tangent value. The applied boundary condition, the location of measurement point and the input parameters for this simulation are the same with the previous case. Also, the simulated displacements after 1 second are presented, as can be seen in Table 5-5. In order to illustrate the

different behaviors of the block system, the output graphics for case 2-3 and case 2-4 after 10 seconds are presented in Figure 5-8 (a) and Figure 5-8 (b), respectively. In Figure 5-8 (a), both the measurement points at the upper left corner and the lower left corner remain stable compared with Figure 5-8 (b), in which both the measurement points at the upper left corner and lower left corner reveal considerable displacements. What's more, in Figure 5-8 (b), the displacement of the upper left corner is approximately equals to the displacement of the lower left corner, indicates that no rotation occurs. Thus, the status of case 2-3 can be regarded as stable while the status of case 2-4 can be regarded as sliding without toppling.

Table 5-4 Chosen δ and ψ value for boundary II

Case number	Incline angle	Tangent value
2-1	29.5	30.5
2-2	30.5	30.5
2-3	29.5	40
2-4	30.5	40
2-5	29.5	50
2-6	30.5	50
2-7	29.5	60
2-8	30.5	60
2-9	29.5	70
2-10	30.5	70
2-11	29.5	80
2-12	30.5	80

Table 5-5 Simulation results at time=1s

case	Measurement point 1		Measurement point 2		Behavior
	X-Disp(mm)	Y-Disp(mm)	X-Disp(mm)	Y-Dips(mm)	
2-1	0.009	0.006	0.007	0.004	Safe
2-2	0.484	0.285	0.483	0.285	Sliding
2-3	0.005	0.003	0.004	0.002	Safe
2-4	0.984	0.579	0.983	0.579	Sliding
2-5	0.004	0.003	0.004	0.002	Safe
2-6	0.999	0.587	0.999	0.588	Sliding
2-7	0.004	0.003	0.003	0.002	Safe
2-8	0.852	0.502	0.852	0.502	Sliding
2-9	0.002	0.001	0.002	0.001	Safe
2-10	0.397	0.223	0.397	0.223	Sliding
2-11	0.002	0.001	0.002	0.001	Safe
2-12	0.238	0.139	0.238	0.139	Sliding

Figure 5-8 NDDA output graphics: (a) stable (b) sliding.

Third, the δ and ψ values listed in Table 5-6 are chosen to test the boundary III. The input parameters are listed in Table 5-7. A smaller time step interval, which is equal to 0.001 second, is used to satisfy the assumption of infinitesimal displacements and the maximum displacement ratio is set as 0.0001. The simulated displacements after 0.1 second are presented in Table 5-8 to show the initial behavior of the block. Again, to illustrate the different modes of behavior, output graphics for case 3-3 and case 3-4 after 1 second are presented in Figure 5-9(a) and Figure 5-9(b), respectively. In Figure 5-9(a), the displacements of upper left corner and lower left corner differs a lot both in the direction and magnitude, while in Figure 5-9(b), the displacements of upper left corner and lower left corner are parallel and approximately equals to each other. Thus, the failure mode of case 3-3 can be classified as toppling and sliding while the failure mode of case 3-4 can be classified as sliding without toppling.

Table 5-6 Chosen δ and ψ value for boundary III

Case number	Incline angle	Tangent value
3-1	32	29.5
3-2	32	30.5
3-3	40	29.5
3-4	40	30.5
3-5	50	29.5
3-6	50	30.5
3-7	60	29.5
3-8	60	30.5
3-9	70	29.5
3-10	70	30.5
3-11	80	29.5
3-12	80	30.5

Table 5-7 Input parameters for case 3

Dynamic	yes
Time step size(s)	0.001
Mass per unit(Kg/m3)	2548
Poisson's ratio	0.49
Spring stiffness(GPa)	30
Young's modulus(GPa)	30
Max disp. ratio	0.0001
Joint friction angle	30
Joint cohesion	0
Joint tensile strength	0

Table 5-8 Simulation results at time=0.1s

case	Measurement point 1		Measurement point 2		Behavior
	X-Disp(mm)	Y-Disp(mm)	X-Disp(mm)	Y-Dips(mm)	
3-1	2.237	1.404	0.357	0.223	S+T
3-2	1.769	1.112	1.709	1.069	Sliding
3-3	9.206	7.825	5.652	4.787	S+T
3-4	7.778	6.599	7.782	6.582	Sliding
3-5	11.193	13.481	13.352	16.101	S+T
3-6	12.888	15.374	12.880	15.363	Sliding
3-7	13.955	24.210	15.071	26.156	S+T
3-8	14.648	25.126	14.652	25.423	Sliding
3-9	12.692	35.080	13.151	36.349	S+T
3-10	12.718	34.935	12.792	35.174	Sliding
3-11	6.721	38.231	6.839	38.903	S+T
3-12	6.939	39.449	6.999	39.808	Sliding

Figure 5-9 NDDA output graphics: (a) toppling and sliding; (b) sliding only.

In the last case study, the δ and ψ values listed in Table 5-9 are chosen to test the boundary IV. The input parameters are the same as that used in case 1. In the derivation of the boundary IV, the dynamic equilibrium conditions are considered, as introduced in Section 5.2.2.1. When toppling appears, the pivot point rotates $d\delta$ degree, the angle between the inclined plane of the linear acceleration changes from δ to $(\delta-d\delta)$, resulting in the increase of the critical value ψ . This means that, a block which behaves toppling and sliding at the instantaneous will have the tendency of toppling only at the next time step. As the dynamic analysis has difficulty in catching the instantaneous behavior of the block, the static analysis is used. The simulated displacements after 1 second are presented in Table 5-10 to show the initial behavior of the block. Again, to illustrate the different modes of behavior, output graphics for case 4-3 and case 4-4 after 5 seconds are presented in Figure 5-10(a) and Figure 5-10(b), respectively. In Figure 5-10(a), both points at the upper left corner and the lower left corner move, however, the displacements of the points differ in the direction and magnitude. In Figure 5-10(b), the upper left corner moves while the point at the lower left corner remains stable. Thus, the failure mode of case 4-3 can be classified as toppling and sliding while the failure mode of case 4-4 can be classified as toppling without sliding. It is worth to note that when the block behaves sliding and toppling, the displacement of the upper left corner is much larger than the displacement of the lower left corner, indicating the toppling phenomena is more dominant than the sliding phenomena at the initial state. This is caused by the fact that the toppling has the tendency to inhibit sliding. As time goes on, the sliding phenomena will become more evident, as shown in Figure 5-10.

Table 5-9 Chosen δ and ψ value for boundary IV

Case number	Incline angle	Tangent value
4-1	66.3	5
4-2	67.3	5
4-3	65	10
4-4	66	10
4-5	61.5	15
4-6	62.5	15
4-7	54.7	20
4-8	55.7	20
4-9	43.8	25
4-10	44.8	25
4-11	32.5	29
4-12	33.5	29

Table 5-10 Simulation results at time=1s

case	Measurement point 1		Measurement point 2		Behavior
	X-Disp(mm)	Y-Disp(mm)	X-Disp(mm)	Y-Dips(mm)	
4-1	16.218	42.001	0.018	0.040	Toppling
4-2	15.467	40.312	2.865	6.927	S+T
4-3	21.512	54.167	0.031	0.017	Toppling
4-4	20.628	53.878	0.898	2.096	S+T
4-5	23.237	48.955	0.028	0.015	Toppling
4-6	22.360	48.977	0.495	1.021	S+T
3-7	22.143	34.133	0.011	0.007	Toppling
4-8	22.119	35.398	0.442	0.677	S+T
4-9	16.253	16.324	0.003	0.003	Toppling
4-10	16.845	17.566	0.208	0.327	S+T
4-11	3.439	2.166	0.000	0.003	Toppling
4-12	4.330	2.902	0.112	0.077	S+T

Figure 5-10 NDDA output graphics: (a) toppling and sliding; (b) toppling only.

Figure 5-11 NDDA results compared with Yueng's chart

In the end, the NDDA results of all 48 cases are plotted on the chart with the boundary conditions provided by Yeung (1991), as illustrated in Figure 5-11. Different modes of behavior are represented by different symbols in the Figure. As can be seen, the NDDA results agree with Yueng's chart for all the boundaries, indicating that the NDDA method takes the dynamic effects into consideration automatically, thus gives correct prediction of the behavior of block system.

5.3.2 Toppling of rectangular block on a stepped base

In Bray and Hoek (1981), an idealized example is given to illustrate the practical use of the Goodman-Bray's method. As shown in Figure 5-12, the rock slope has the height of 92.5m and the slope angle of 56.6° , a joint set dipping at 60° into the hill. The block system is on a base stepped at 0.2m. The unit weight of the block is 25KN/m^3 . The same example is studied here using the DDA and the NDDA method, and the input parameters are listed in Table 5-11. For each block, two measurement points located at the lower left corner and the upper left corner are set. Table 5-12 presents the block dimension as well as the results obtained from Goodman-Bray's method and the numerical simulations.

Figure 5-12 Geometry of the block system

Table 5-11 DDA and NDDA input parameters

Dynamic	yes
Time step size(s)	0.001
Total time step	5000
Mass per unit(Kg/m3)	2548
Poisson's ratio	0.25
Max disp. ratio	0.0005
Joint friction angle(°)	38.15
Joint cohesion	0
Joint tensile strength	0

Table 5-12 Analytical result from Goodman-Bray's method and the simulation results obtained from the DDA and the NDDA method

Block number	$y_n(\text{m})$	$\Delta x(\text{m})$	Mode	Mode (DDA, $t=1.5\text{S}$)	Mode (NDDA, $t=1.5\text{s}$)
1	4	10	sliding	sliding	sliding
2	8	10			
3	12	10			
4	16	10	toppling	toppling	toppling
5	20	10			
6	24	10			
7	28	10			
8	32	10			
9	36	10			
10	40	10			
11	34	10			
12	28	10			
13	22	10			
14	16	10	stable	stable	sliding
15	10	10			
16	4	10

Figure 5-13 Block displacement when $t=1.5\text{s}$: (a) DDA result; (b) NDDA result.

The horizontal displacement of each measurement point obtained from the DDA and the NDDA method when the time $t = 1.5\text{s}$ is shown in Figure 5-13. As it can be seen, the behaviors of the block system obtained from both methods are quite similar. For block 1 and block 2, the horizontal displacements of the lower left corner and the upper left corner nearly equal to each other, indicates the initial behavior of the block 1 and block 2 is sliding. For blocks 3 to 13, the displacement of the upper left corner is much larger than that of the lower left corner, indicating though sliding occurs at the same time, and the initial behavior of blocks 3 to 13 is dominated by toppling. For blocks 14 to 16, the

displacements of the upper left corner and the lower left corner nearly equal to each other, indicating no toppling occurs. At the same time, the displacements obtained from the NDDA method are all around 0.1m, indicating the initial behavior of blocks 14 to 16 is sliding; while the displacements obtained from the DDA method are nearly to zero, indicating the initial behavior of blocks 14 to 16 is stable.

The NDDA and the DDA output are shown graphically in Figure 5-14 and Figure 5-15, respectively. The difference between the two methods becomes more obviously in case of the time $t=5s$. For the NDDA method, block 14 experiences both sliding and toppling failure while block 15 and 16 experience sliding failure. For the DDA method, blocks 14 to 16 remain stable as time goes on.

Figure 5-14 NDDA output graphic: (a) $t=1.5s$; (b) $t=5s$.

Figure 5-15 DDA output graphic: (a) $t=1.5s$; (b) $t=5s$

In this example, the modes of behavior predicted by the numerical simulations and the Goodman-Bray's analysis agree well, the discrepancy can be explained by the factors that: (1) in the Goodman-Bray's method, only the static analysis is carried out, thus the

dynamic effect is not taken into account; and (2) in the Goodman-Bray's method, the block is assumed to be either sliding or toppling, the possibility that a block slides and topples simultaneously is not considered.

And the discrepancy between the NDDA simulation and the DDA simulation can be explained by the factor that, during the simulation procedure, the failure of the upper block will cause the deformation of the base block: in the DDA method, the stress field in the base block is constant, thus the deformation in the lower part of the base block will not affect the stress distribution in the upper part of the base block; while in the NDDA method, with the help of the finite element mesh, the deformation in the lower part of the base block will cause the stress-redistribution in the upper part of the base block, thus change the status of the blocks on it.

5.3.3 Modeling of flexural toppling under seismic loading

Flexural toppling often occurs in the rock column formed by well developed steeply dipping discontinuities, such as foliation, bedding planes and so on. The study of the mechanism of flexural toppling failure has been carried out using various methods, including limiting equilibrium model, numerical analysis and physical experiments (Hittinger 1978; Aydan 1992; Adhikary and Dyskin 1996, 1997, 2007).

Aydan and Amini (2009) divided the flexural toppling into active flexural toppling and passive flexural toppling. If the vertical motion of the layers is in the same direction of gravity force, the failure mode is active flexural toppling; however, if the initial vertical motion is opposite to the direction of gravity force, the failure mode is passive flexural toppling. Active flexural toppling is very common when the body force of the rock layers is the primary cause of the flexural, however, when a surcharge load resulting from earthquake, blasting or superstructures acts on layers, passive flexural toppling may be induced. Among all the surcharge loads, the seismic load is expected to be the most prominent in inducing the flexural toppling.

In this section, three of the physical experiments done by Aydan and Amini (2009) are simulated by the NDDA method. Before the physical experiments, the column sample is created by a special mixture of BaSO₄, ZnO and Vaseline oil, and the material properties, including the friction angle, unit weight, tensile and compressive strength of the column sample are tested in the laboratory. The input parameters of the NDDA method are chosen as the average value obtained from the experiments, as listed in Table 5-13. During the physical experiments, one end of the column is firmly fixed on the shaking table (in the case of 60° inclination, a supporting base for the column is also prepared), the acceleration of the shaking increase gradually until the maximum capacity of the shaking table is reached or the failure occurs, as illustrated in Figure 5-16. While in the NDDA simulation, the input seismic loading is a periodic sinusoidal wave, and the amplitude of the

sinusoidal wave is fixed to test the stability of the column. The frequency of the acceleration wave is 5HZ. Figure 5-17 shows the input acceleration function of the NDDA simulation. If the column is stable under the current seismic loading, the amplitude of the acceleration wave will be increased, and the simulation will be run again. At each time, the increment of the amplitude is 0.01g.

Table 5-13 NDDA input parameters

Dynamic	yes
Time step size(s)	0.0001
Max. time displacement ratio	0.00005
Mass per unit(Kg/m ³)	1939
Poisson's ratio	0.25
Young modulus (GPa)	6
Joint friction angle(°)	38
Joint cohesion(KPa)	80
Joint tensile strength(KPa)	25

Figure 5-16 Information of the physical experiments: (a). Experimental set-up; (b). applied acceleration and displacement. (After Aydan and Amini 2009)

The simulation results obtained from the NDDA method as well as the experimental results are listed in the Table 5-14. As it can be seen, the NDDA method results in a higher maximum acceleration coefficient compared with the physical experiments, this difference can be ascribed to the factors that: (1) The input parameters used in the NDDA simulation are the average value obtained from the experiments, which may have differences with the real material properties; (2) in the NDDA simulation, as only two dimensional analysis is carried out, the effect of the thickness of the model is not considered; and (3) the seismic loading condition applied in the physical experiment and the NDDA simulation are different. In the physical experiment, both the amplitude and the frequency of the acceleration of the shaking table increased gradually, while in the NDDA simulation, they are simplified as fixed values. However, the NDDA simulation results are in the same tendency with the physical experiments. Figure 5-18 to Figure 5-20 show the output graph of the NDDA simulation.

Table 5-14 Test results of the physical experiments

Number	Inclination	Total length of the model	Width of the model	Effective length of the model	Maximum acceleration coefficient	Test results	NDDA simulation result
1-4	60	25	5	22	1.21g	Unstable	1.31g
2-1	90	25	5	20	0.32g	Unstable	0.39g
3-1	120	25	5	18	0.44g	Unstable	0.54g

Figure 5-17 Input seismic loading in the NDDA simulation

Figure 5-18 NDDA output at the 60° inclination: (a) at the beginning of the simulation; (b) Failure

Figure 5-19 NDDA output graph at the 90° inclination: (a) at the beginning of the simulation; (b) failure

Figure 5-20 NDDA output graph at the 120° inclination: (a) at the beginning of the simulation; (b) failure

5.3.4 Modeling of base friction test of flexural toppling

In this example, the NDDA method is used to reproduce a base friction table model of flexural toppling in jointed rock reported by Hittinger (1978). During the physical tests, a two-dimensional model is placed on the belt of the base friction machine, and the belt is moved to drag the model against the boundary constraint. The base friction set up and the test samples are illustrated in Figure 5-21, and thin layer in the model is made with a mixture of sand, oil and flour.

Figure 5-21 Base friction set up and test sample (After Hittinger, 1978)

In the NDDA simulation, only the gravity force is applied. To make sure the occurrence of the flexural toppling in the simulation, the NDDA model is 100 times larger than the physical model while the geometric proportions is kept unchanged. The geometries of the base friction model and the NDDA model are listed in Table 5-15.

Table 5-15 Geometry of base friction model and NDDA model

Parameter	Base friction model	NDDA model
Slope height (H)	30.48cm	30.48m
Slope face angle (θ)	78 °	78 °
Joint dip angle (α)	30 °	30 °
Column width (w)	5.08cm	5.08m

There are totally 2673 elements with 1701 nodal points in the block system and the mesh detail of the problem is shown in Figure 5-22, all the nodal points on the bottom and the right boundary are fixed. The material properties of the intact block are chosen within the range of typical sandstone and the joint friction angle is set as the material friction angle 39 ° reported by Hittinger (1978). The input parameters used in the NDDA simulation are listed in Table 5-16.

Figure 5-22 Numerical simulation for the base friction model

Table 5-16 NDDA input parameters

Control parameters	Plain stress	Yes
	Time step size (s)	0.005
	Total time (s)	100
	SOR factor	1.1
	Max. disp. ratio	0.001
	Spring stiffness(GPa)	50
Rock sample	Mass per unit(KG/ m ³)	2600
	Poisson's ratio	0.21
	Young's modulus (GPa)	10
	Friction angle(°)	39
	Cohesion strength(MPa)	2.5
	Tensile strength(MPa)	2.5
Joint properties	Joint friction angle(°)	39
	Joint cohesion (MPa)	0
	Joint tensile strength (MPa)	0

First the strength properties of the intact rock materials, such as the cohesion and tensile, are set large enough to allow the system to come to equilibrium under gravity, and then the strength parameters, as listed in Table 5-16 are applied to allow the crack propagating inside the intact block. The simulation result of the NDDA method, including the deformation and the failure zones, are presented in the Figure 5-23. The failure plane appeared first at the toe region of the slope, and then developed upper ward in a direction nearly normal to the dip of the joint. Compared with the experimental observation reported by Hittinger, as shown in Figure 5-24, it can be concluded that the NDDA method reproduced the flexural behavior of the jointed rock mass observed in the physical test as well as predicted a failure plane close to the physical test.

Figure 5-23 NDDA Simulation results: (a) failure initiates (b) final failure mode

It should be noted that the tensile strength of the rock mass adopted in the NDDA simulation is much higher than that used in the UDEC simulation (Pritchard, 1990) and uncoupled DDA-FEM framework (Khan, 2010). It probably because that without allowing the fracture propagates inside the entire rock mass, both the UDEC and

uncoupled DDA-FEM framework cannot predict the failure plane in a straight forward way, thus the tensile strength reported by these methods are not accurate.

Figure 5-24 Deformation of base friction model reported by Hittinger (1978)

5.3.5 Modeling of centrifuge test of flexural toppling failure

Adhikary et al. (1997) did a series of centrifuge experiments using three different materials and the details of their experiments are listed in Table 5-17. During the experiment, the fractures for all the tests appear first at the toe region and propagate back into the slope. The orientation of the main fracture path is 12° to 20° above the normal to the joint dip angle. For the physical model made with relatively brittle materials, such as sand-cement sheet, only fully plastic deformation with visible fractures zone and elastic deformation zone can be observed in the experiments. However, for the physical model made with relatively ductile materials, such as fiber-cement sheet, three distinct zones, which are fully plastic deformation with visible fractures zone, partially plastic deformation without visible fractures zone and elastic deformation zone can be observed in the experiments.

Table 5-17 Experimental details of flexural toppling model (Adhikary et al. 1997)

Test No.	Slope height to joint spacing ratio	Joint spacing(mm)	Slope angle(deg)	Joint dip angle(deg)	Tensile Strength (KPa)	Density (KN/m ³)	Joint friction angle (deg)
1	30	10	80	70	165-185	17	40
2	80	4.8	63	65	11000	13	17
3	81	4.8	90	70	11000	13	17
4	33	10	61	80	1070-1140	23.8	22
5	83	4.8	90	80	11000	13	17
6	33	12	61	80	1070-1140	23.8	22
7	30	12	70	69	1070-1140	23.8	22

In this section, physical test No.4 is selected to be modeled by the NDDA method. The configuration of the physical test is shown in Figure 5-25. The dimension of the numerical model is same as the physical configuration and the base and the left boundary of the numerical model is fixed during the simulation. Two measurement points A and B are placed at the corresponding positions of the numerical model.

Figure 5-25 Configuration of the physical test No.4 (after Adhikary et al., 1997)

The input parameters of the NDDA simulation are listed in Table 5-18. A small time step size is used to satisfy the small displacement assumption. The material density, tensile strength, joint friction angle are set same as the physical values. At the first 1000 steps, the gravity acceleration is fixed at g to allow the system to come to equilibrium under gravity, and then the gravity acceleration is increased gradually to model the centrifugal experiments, the increment for each time step is $0.025g$.

Table 5-18 Input parameters of the NDDA simulation

Control parameters	Plain stress	Yes
	Dynamic	Yes
	Time step size (s)	0.0001
	Max. disp. ratio	0.00005
	SOR factor	1.1
	Spring stiffness(GPa)	50
Rock sample	Mass per unit(KG/ m ³)	2380
	Poisson's ratio	0.16
	Young's modulus (GPa)	2.4
	Friction angle (°)	37
	Cohesion strength(MPa)	2.5
	Tensile strength(MPa)	0.8-1.4
Joint properties	Joint friction angle (°)	22
	Joint cohesion (KPa)	0-45
	Joint tensile strength (KPa)	0

As the flexural toppling of layered rock slopes occurs essentially due to bending, thus the tensile strength of the intact rock may play an important role in controlling the stability of the rock slope. The tensile strength for the inenite-gypsum material suggested by Adhikary (2007) ranged from 1.1 to 1.4 MPa, and in the NDDA simulation, tensile strength ranging from 0.8 to 1.4 MPa is tested to evaluate the effect of the tensile strength on the flexural toppling.

During the physical test, a small cohesion exist between the layers due to the slight roughening and bonding between sheets, however, the exact value of the cohesion is unknown. In the NDDA simulation, the joint cohesion value ranging from 0 to 4.5KPa is tested to obtain a good fit between the physical and the numerical horizontal displacement plots.

Figure 5-26 shows the simulation results of the NDDA method when the joint cohesion is set as 0 KPa and the tensile strength of the entire rock mass is set as 1.4 MPa. During the simulation, the crack appeared first at the toe region of the rock slope and then developed upward. Though the propagation of the crack path is restricted by the element edge (as mentioned in the previous chapter), the NDDA method is able to represent the failure of the rock slope in a fully automatic way. The main fracture path obtained from the NDDA simulation is about 15 ° above the normal to the joint dip, compared with the orientation of the failure surface reported by Adhikary (1997), which is 12-20 ° above the normal to the joint dip, the NDDA result agreed well with the physical tests in the rupture surface prediction.

Figure 5-27 presents the horizontal displacements of the measurement points predicted by the NDDA method when the joint cohesion strength 0KPa is applied. As it can be seen, the horizontal displacements of the measurement points increased with the increase of the gravity acceleration, and when the gravity acceleration reached a certain value, the displacements of the measurement points increased more quickly than before, indicating the collapse of the rock slope. Inspection of Figure 5-27 shows that: 1) in the physical model, there exists a critical point after which the displacements of the measurement point increased at a greater rate, i.e., about 30-40g in this case, and this critical point is probably to be caused by the loss of cohesion along the joint interfaces. However, in the numerical simulation, as 0KPa cohesion is applied, the model fails to capture the increase of the displacement rate at about 30g. 2) In the NDDA simulation, the predicted collapse load increased with the increasing of the tensile strength of the rock mass, however, even the highest value suggested by Adhikary (1997) was applied, the NDDA model collapsed at about 75g, which was still lower than the 80-85g peak loads measured by the centrifuge experiments.

Figure 5-26 NDDA simulation results (tensile strength=1.4 MPa, joint cohesion=0 KPa)

Figure 5-28 and Figure 5-29 present the horizontal displacements of the measurement points predicted by the NDDA method when the joint cohesion strength equals to 15KPa and 45KPa, respectively. As it can be seen, the NDDA results consisted with the assumption that the cohesion along the joint interfaces will affect the critical point after which the displacements will increase at a greater rate. Adhikary (2007) attempted to use the continuum model based on the Cosserat theory to match the displacements obtained from the physical tests and reported a joint cohesion of 15KPa to better fit the physical results. However, in the NDDA method, when the joint cohesion 15KPa is applied, as shown in Figure 5-28, the critical point obtained is smaller than that of the physical test (especially when the tensile strength of the rock mass is equal to 1.4MPa and 1.1MPa). If the joint cohesion increased to 45KPa, as shown in Figure 5-29, the critical point obtained from the NDDA method has a good agreement with the physical results. Inspection of Figure 5-27, Figure 5-28, and Figure 5-29 shows that the tensile strength of the rock mass

plays an important role in the toppling failure--the failure load predicted by the NDDA method increased with the increase of the tensile strength of the rock mass. However, even the tensile strength equals to 1.4 MPa is applied, compared with the physical tests, the NDDA model tend to underestimate the failure load (failure load from 65-75g).

Figure 5-27 Horizontal displacement of the measurement points predicted by the NDDA method with fixed cohesion 0KPa and varied tensile strength

Figure 5-28 Horizontal displacement of the measurement points predicted by the NDDA method with fixed cohesion 15KPa and varied tensile strength

Figure 5-29 Horizontal displacement of the measurement points predicted by the NDDA method with fixed cohesion 45KPa and varied tensile strength

Figure 5-30 shows the rupture surface predicted by the NDDA method when the tensile strength of the entire rock mass is set as 1.4MPa and the joint cohesions are set as 15KPa and 45KPa, respectively. Together with Figure 5-26, it is found that though the different joint cohesion will affect the displacement plots of the measurement points, it has little effect on the general failure mode of the slope. When the joint cohesion is set as 45KPa and the tensile strength is set 1.4MPa, there is good agreement with the physical results in both displacement plots and deformation patterns.

Figure 5-30 NDDA output graphic: (a) $\sigma_t=1.4\text{MPa}$, $c=15\text{KPa}$; (b) $\sigma_t=1.4\text{MPa}$, $c=45\text{KPa}$.

5.4 Conclusion

In this chapter, the NDDA method is applied to the analysis of toppling failure. The unique advantages of the NDDA method are that, as a continuum/discontinuum hybrid method, the discrete fractures can be readily accommodated in the model, and at the same

time, the new fractures are able to grow if necessary. In the first two examples, the joints are persistent, which are often related with block toppling or block-flexural toppling, and the simulation results show that the NDDA method is able to predict the relative movement along the pre-defined discontinuities as well as capture the right failure mode. In the last three cases, there are no pre-existing discontinuities to form a basal failure plane, which is often related with the flexural toppling, and the numerical results indicate that with the double minimizing procedure introduced in the previous chapter, the NDDA method is able to predict the rupture surface inside the rock mass and the deformation pattern of the rock slope close to the physical tests.

Chapter 6. Development and application of rockbolt element in the NDDA program

Rockbolts have a history of one hundred years and is now becoming one of the most important rock reinforcement in civil engineering works such as hard rock mining, tunneling, rock slope stability and so on. Rockbolts can be installed simply and quickly to give an immediately as well as a long time support to the rock system. It can be applied under different site conditions, working alone or together with other types of support such as concrete lining, shotcrete, and steel props. The basic concept of rock bolting is to let rock to support itself, and it is generally believed that the bolting effects are accomplished by a combination of contributing shear resistance to the rock joints/discontinuities, suspension, beam building and arch/ring building.

In order to best apply the rockbolt system, many bolt design methods having being developed by optimizing bolt length, bolt density and bolt pretension during the installation. The bolt design methods can be categorized into three groups: empirical, observational and analytical. The empirical method is based on the empirical rules or rock classification system (e. g. Q system or RMR); it does not consider the dynamic situation and specified rockbolt type. The observational method incorporates with initial empirical based design, monitoring programs and trial-and-error approaches; it gives a very site-specific ground control plan but lost time and money. The analytical method includes conceptual modeling, physical modeling, mathematical modeling and numerical modeling (Clapp, 2003). Among them, numerical modeling becomes widely used because of its low cost, high efficiency and great flexibilities.

Till now, the rockbolt element has been implemented to many numerical modeling programs, such as the continuum-based Rocscience's Phase², Itasca's FLAC; and the discontinuum-based Itasca's UDEC, Dr Shi's DDA (Shi, 1998). To more accurately capture the behavior of the rockbolts in discontinuity dominated situations such as joint opening/sliding, rock dilation and rock bursts, as well as better approach the interaction between the rockbolt and the host rock, a hybrid modeling—NDDA method seems to be an appropriate choice. The feasibilities of incorporate the rockbolt model in the NDDA method are that: 1) the discontinuities are explicitly represented in it; 2) both static and fully dynamic analysis can be performed by it; and 3) large displacement and deformation can be achieved by it. At the same time, to further extend the capacity of the NDDA method for real rock engineering applications, development of realistic rockbolt models in the NDDA method is necessary.

In this chapter, the rockbolt element will be implemented into the NDDA method to improve the ability of the method as well as assist rockbolt analysis and design. Different

types of rockbolts could be realized in the proposed numerical model and the possible failure modes, namely tensile failure of bolt shank, debonding along the bolt/rock interface and the failure of the face plate/anchor point are available in the analysis procedure.

6.1 Types of rockbolt

There are a number of prominent bolting methods used worldwide and the choice of the bolting method depends mainly on the geological conditions presented (Stillborg, 1986). In order to develop the bolt model in the NDDA method which is able to present the mechanical behavior of different bolt types, a review of the popular rockbolt systems is carried out first.

6.1.1 Mechanical bolt

Mechanical bolt belongs to the group of point anchored bolt. It usually consists of a smooth steel rod with a mechanical anchor at one end and a face plate and nut at the other end, as shown in Figure 6-1. The advantages of the mechanical bolt are that it is economical to purchase, easy to install and able to give an immediate support after the installation. It is also preferred in the geological conditions which are unsuitable for grouting.

Figure 6-1 Layout of the mechanical rockbolt (After Li, 2008)

However, the use of the mechanical bolt is limited in moderately hard to hard rock. For very hard rock, it is difficult for the anchor point to grip to rock, while for soft rock conditions, the fully grouted bolts which can provide permanent reinforcement are preferred. At the same time, the bolt is more suitable for the temporary reinforcement because it is easy to be corroded by the groundwater which flow freely along the entire bolt length.

6.1.2 Fully grouted bolt

Fully grouted bolt systems are first developed in the 1950's and have been commonly used in the civil and mining engineering during the past forty years. They consist three parts, the steel rebar or threaded bar, the bearing plate and the grout. Different from the mechanical bolt, the surface of the steel is ribbed and the anchorage is provide by installing grout along the full length of the bolt instead of using mechanical anchor device at the end of the bolt.

Figure 6-2 Layout of the fully grouted rockbolt (After Li, 2008)

The advantage of the fully grouted bolt is that it can provide permanent support as the grout prevents the groundwater corrosion. However, the support effect depends on the quality of grout and grouting, which are difficult to control during the installation.

6.1.3 Friction anchored bolt

There are two commonly used friction anchored rockbolt types, namely, the Split Set and the Swellex. Strictly speaking, the Split Set is true friction anchored rockbolt because the frictional force is the only resistant force that prevents the bolt from sliding. For the Swellex, the resistant force including both frictional force and mechanical interlock. The frictional resistance of both Split Set and Swellex comes from the radial force against the borehole wall.

Figure 6-3 Swellex (After Li, 2008)

For the friction anchored bolt, the load of the rock is transferred to the reinforcing element directly. If the load exceeds the resistant force, the bolt will start to slide, thus the bolt is able to accommodate large rock displacements. The shortcomings of the friction anchored bolt are that it is relatively expensive, have low load capacity and unless being protected against corrosion, it cannot be used as long term supporting system.

6.1.4 Dynamic rockbolt (D –bolt)

When subjected to dynamic loading, the philosophy of the ground support is no longer to balance the dead weight, but to help the rock dissipate the dynamic energy. An ideal energy-absorbing rockbolt should have the high load-bearing capacity at the same time able to accommodate large deformations. As a promising energy-absorbing rock support

device, D-bolt is a smooth steel bar with a number of anchors distributed uniformly or non-uniformly along its length, as shown in Figure 6-4 (Li, 2010). The concept of the D-bolt is to use both the strength and the deformation capacity of the bolt shank. During the installation, the bolt is fully grouted by cement or resin, the anchors will be firmly fixed in the grout while the smooth bar will have no or very weak bounding to the grout.

Figure 6-4 Layout of the D-bolt (after Li, 2010)

Laboratory test showed that, every bolt section between two adjacent anchors is equally loaded, thus the load peaks along the bolt shank due to the joint opening is avoided. At the same time, the smooth bolt sections can deform freely to absorb the energy.

6.2 Rockbolt element in the DDA program

6.2.1 Theory of the bolt element

In the original DDA program, the rockbolt element is made up of a single spring attached to the surrounding blocks by the spring's endpoints (Shi, 1993), as illustrated in Figure 6-5.

Figure 6-5 Illustration of bolt element in original DDA program

Consider a bolt contacting the i_{th} block and the j_{th} block at two end points: (x_1, y_1) (x_2, y_2) , respectively. The displacements of the two end points are:

$$\begin{aligned} d_{x1} &= u_1 = u(x_1, y_1); & d_{y1} &= v_1 = v(x_1, y_1) \\ d_{x2} &= u_2 = u(x_2, y_2); & d_{y2} &= v_2 = v(x_2, y_2) \end{aligned} \quad (6-1)$$

The length of the bolt is:

$$l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad (6-2)$$

As the blocks move, the nodes relative displacement equals to:

$$\begin{aligned} dl &= \left[(u_1 \ v_1) \begin{pmatrix} l_x \\ l_y \end{pmatrix} - (u_2 \ v_2) \begin{pmatrix} l_x \\ l_y \end{pmatrix} \right] \\ &= ([D_i]^T [T_i]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} - [D_j]^T [T_j]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix}) \end{aligned} \quad (6-3)$$

Where l_x, l_y are the direction cosines of the bolt:

$$\begin{aligned} l_x &= \frac{(x_1 - x_2)}{l} \\ l_y &= \frac{(y_1 - y_2)}{l} \end{aligned} \quad (6-4)$$

The bolt force comes from its strain while the stiffness of the bolt equals to s :

$$f = -s \frac{dl}{l} \quad (6-5)$$

The strain energy of the bolt can then be calculated as:

$$\begin{aligned} \Pi_b &= -\frac{1}{2} f dl = \frac{s}{2l} dl^2 \\ &= \frac{s}{2l} ([D_i]^T [T_i]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} - [D_j]^T [T_j]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix})^2 \\ \Pi_b &= \frac{s}{2l} [D_i]^T [T_i]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_i] [D_i] \\ &\quad - \frac{s}{l} [D_i]^T [T_i]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_j] [D_j] \\ &\quad + \frac{s}{2l} [D_j]^T [T_j]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_j] [D_j] \end{aligned} \quad (6-6)$$

To minimize Π_b , derivatives are computed with respect to the displacement of block i and j :

$$\begin{aligned} k_{rs} &= \frac{\partial^2 \Pi_b}{\partial d_{ri} \partial d_{si}} = \frac{s}{2l} \frac{\partial^2}{\partial d_{ri} \partial d_{si}} [D_i]^T [T_i]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_i] [D_i] \\ &= \frac{s}{l} [T_i]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_i] \rightarrow [K_{ii}] \end{aligned} \quad (6-7a)$$

$$\begin{aligned}
k_{rs} &= \frac{\partial^2 \Pi_b}{\partial d_{ri} \partial d_{sj}} = \frac{s}{l} \frac{\partial^2}{\partial d_{ri} \partial d_{sj}} [D_i]^T [T_i]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_j] [D_j] \\
&= \frac{s}{l} [T_i]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_j] \rightarrow [K_{ij}]
\end{aligned} \tag{6-7b}$$

$$\begin{aligned}
k_{rs} &= \frac{\partial^2 \Pi_b}{\partial d_{rj} \partial d_{si}} = \frac{s}{l} \frac{\partial^2}{\partial d_{rj} \partial d_{si}} [D_j]^T [T_j]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_i] [D_i] \\
&= \frac{s}{l} [T_j]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_i] \rightarrow [K_{ji}]
\end{aligned} \tag{6-7c}$$

$$\begin{aligned}
k_{rs} &= \frac{\partial^2 \Pi_b}{\partial d_{rj} \partial d_{sj}} = \frac{s}{2l} \frac{\partial^2}{\partial d_{rj} \partial d_{sj}} [D_j]^T [T_j]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_j] [D_j] \\
&= \frac{s}{l} [T_j]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_j] \rightarrow [K_{jj}]
\end{aligned} \tag{6-7d}$$

and the resulted four submatrices $[K_{ii}]$, $[K_{ij}]$, $[K_{ji}]$, $[K_{jj}]$ are finally incorporated into the global stiffness matrix.

6.2.2 Limitation of the original bolt element

From the above mentioned theory it can be seen, the original bolt element in the DDA program is a simplified model which omits many aspects of the true rockbolt. The unrealistic representation of the true rock/bolt interaction will result in many problems when doing the numerical simulation (Clap, 2003):

- (1) The first problem is that the bolt element is unable to accumulate bolt force from the current time step to the next time step. In order to reach the equilibrium, the bolt stretch linearly regardless the displacement occurred in the bolt in the previous time step.
- (2) The second problem is that the bolt-host block interaction is not considered. When the bolt crosses a discontinuity which is experiencing a shear displacement, the bolt moves freely inside the blocks instead of remaining fixed in the place, as illustrated in Figure 6-6.

Figure 6-6 Illustration of the freely movement of the rockbolt in the host rock

- (3) The third problem is that no failure criterion is incorporated in the bolt model. If the applied load keeps increase with time, the bolt will keep deforming linearly with time.
- (4) The forth problem is that the load developed in the bolt is applied to the rock at the end point only. This maybe suitable for the mechanically anchored bolts, however, for the fully grouted bolt in which the applied load dissipation gradationally away from the load point, this loading method is unrealistic.
- (5) The fifth problem is that, as introduced in the previous section, there exist two interfaces, namely, the rock-grout and the grout-bolt interface for the fully grouted rockbolt. However, in the DDA bolt element, the grout component is not considered.
- (6) The sixth problem is that the displacement of the anchor points cannot be directly monitored. In the DDA bolt model, the displacement of the anchor point is obtained by measuring the point belongs to the block which coinciding with the anchor point. In this case, the slip behavior of the bolt and the failure of the anchor points will be difficult to simulate.

6.2.3 Modification of the bolt element in the DDA

First, in order to accumulate bolt deformation over time, the bolt force obtained from the current time step will be applied on the anchor points at the beginning of the next time step as the preloading force.

The bolt force comes from its strain times its stiffness, as illustrated in Eq. (6-5). The potential energy with the force term becomes (Yeung, 1991)

$$\Pi_b = \frac{s}{2l} ([D_i]^T [T_i]^T \begin{pmatrix} lx \\ ly \end{pmatrix} - [D_i]^T [T_i]^T \begin{pmatrix} lx \\ ly \end{pmatrix} + \frac{f}{t})^2 \quad (6-8)$$

To minimize Π_b , derivatives are computed with respect to the displacement:

$$\begin{aligned}
k_{rs} &= \frac{\partial^2 \Pi_b}{\partial d_{ri} \partial d_{si}} = \frac{s}{2l} \frac{\partial^2}{\partial d_{ri} \partial d_{si}} [D_i]^T [T_i]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_i] [D_i] \\
&= \frac{s}{l} [T_i]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_i] \rightarrow [K_{ii}]
\end{aligned} \tag{6-9a}$$

$$\begin{aligned}
k_{rs} &= \frac{\partial^2 \Pi_b}{\partial d_{rj} \partial d_{si}} = \frac{s}{l} \frac{\partial^2}{\partial d_{rj} \partial d_{si}} [D_j]^T [T_j]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_i] [D_i] \\
&= \frac{s}{l} [T_j]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_i] \rightarrow [K_{ji}]
\end{aligned} \tag{6-9b}$$

$$\begin{aligned}
k_{rs} &= \frac{\partial^2 \Pi_b}{\partial d_{ri} \partial d_{sj}} = \frac{s}{l} \frac{\partial^2}{\partial d_{ri} \partial d_{sj}} [D_i]^T [T_i]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_j] [D_j] \\
&= \frac{s}{l} [T_i]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_j] \rightarrow [K_{ij}]
\end{aligned} \tag{6-9c}$$

$$\begin{aligned}
k_{rs} &= \frac{\partial^2 \Pi_b}{\partial d_{rj} \partial d_{sj}} = \frac{s}{2l} \frac{\partial^2}{\partial d_{rj} \partial d_{sj}} [D_j]^T [T_j]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_j] [D_j] \\
&= \frac{s}{l} [T_j]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} (l_x \ l_y) [T_j] \rightarrow [K_{jj}]
\end{aligned} \tag{6-9d}$$

$$\begin{aligned}
f_r &= \frac{\partial \Pi_b(0)}{\partial d_{ri}} = \frac{s}{l} \frac{\partial}{\partial d_{ri}} [D_i]^T [T_i]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} \frac{f}{t} \\
&= \frac{sf}{lt} [T_i]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} \rightarrow [F_i]
\end{aligned} \tag{6-9e}$$

$$\begin{aligned}
f_s &= \frac{\partial \Pi_b(0)}{\partial d_{sj}} = \frac{s}{l} \frac{\partial}{\partial d_{sj}} [D_j]^T [T_j]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} \frac{f}{t} \\
&= \frac{sf}{lt} [T_j]^T \begin{pmatrix} l_x \\ l_y \end{pmatrix} \rightarrow [F_j]
\end{aligned} \tag{6-9f}$$

Comparing the results obtained from the original DDA bolt model and the modified bolt model, it is found the displacement of the bolt head is markedly reduced as the stretching stops when enough resisting force is accumulated, and the first modification is proved to be effective (Clapp, 2003).

Second, in order to better investigate the interaction between the bolt and the host block, Clapp (2003) provided two types of the interface segment into the original bolt model.

The reason of using the interface segment is to offer a way to “anchor” the bolt into the host rock, so that the bolt cannot pass through it when shear movement relative to the endpoints occurs. The interface segments developed by Clapp are illustrated in Figure 6-7(a) and Figure 6-7(b), respectively.

Figure 6-7 Two different types of interface segment: (a) overlap bolt; (b) link bolt.

The first type of the segmented bolt can be defined as “overlap bolt” in which the bolt is broken into two new bolts at the joint intersection (see Figure 6-7 (a)), and the nodes of the new bolts overlapping across the joint plane. Once created, each new bolt behaves as a mechanically anchored bolt, and the displacement of the bolt head reduced because the applied load is resisted by two bolts. However, simulation result shows that the bolts are still permitted to freely pass through the host blocks, as illustrated in Figure 6-8(a).

Figure 6-8 Bolt deformation after shear displacement: (a) overlap bolt; (b) link bolt.

The second type of the segmented bolt can be referred as “link bolt” in which the bolt is broken into three bolt segments when intersecting a joint plane--with one bolt segment on each side of the joint plane and one bolt segment crossing the joint plane (see Figure 6-7(b)). The length of the link can be defined by the user, and Clapp (2003) suggested a value of 10% of the original length of the bolt. As most of the displacement occurs in the

center segment, the load developed in the bolt segment crossing the joint plane takes up almost the load afforded by the entire bolt. Simulation result shows that only the bolt segment crossing the joint plane moves freely through the host blocks (see Figure 6-8(b)), and this type of interface segment is more effective than the first one.

Also, the failure criterion is implemented into the original DDA bolt model. The bolt would break in tension if the applied load exceed the tensile resisting force, and more complete one, including the trigger of the shear yield and the tensile yield, can be found in Berger (2004).

6.3 Rockbolt model in the NDDA program

As discussed previously, though many modifications have been done, the bolt model in the DDA method is still oversimplified, especially for the case of fully grouted rockbolt and frictional rockbolt. In order to better approach the behavior of a support system, a numerical treatment, which is first introduced by Hyett (1996), is incorporated in the NDDA program.

6.3.1 Analysis of bolt-grout interactions using Finite Different approach

A reinforcement system comprises four principal components, namely, rock mass, reinforcing element, internal fixture and external fixture (Windsor, 1997). The reinforcing element refers to the bolt while the internal fixture and the external fixture refer to the interface medium (i. e., cement or resin grout) and the faceplate, respectively. When a rockbolt is subjected to the external load ΔP_i , the bonding stress F_s of internal fixture will be induced due to the relative slip between the rock and the grout. This induced bonding stress will on the other hand, act on the rock mass to restrain the further deformation of the rock. Once the external fixture is attached, it will transfer parts of the axial force, P_n , by generating pressure, F_p , on the external fixture. Pick up a small section of the bolt shank, as shown in Figure 6-9. The equilibrium equation can be written (Farmer, 1975):

$$Ad\sigma_b = -F_s dx \quad (6-10)$$

and

$$F_s = \pi d_b \tau_b \quad (6-11)$$

where F_s is the shear force developed due to the bond per unit length, d_b is the diameter of the bolt, dx is the length of this small section and A is the cross-section area of the bolt.

If the bonding stress is related to the frictional rather than the adhesive resistance between the bolt shank and the grout, as stated by Hyett (1996), it can then be expressed by assuming a linear relationship between the relative slip and the bonding stress (Hyett et al., 1996)

$$F_s = k_s (U_r - U_x) \quad (6-12)$$

where k_s is the tangential stiffness of the grout, U_r is the axial displacement of the rock mass and U_x is the axial displacement of the rockbolt. It is not an easy way to determine the tangential stiffness k_s as it depends not only on the properties of the internal fixture but also on the properties of the reinforcement and the surrounding rock mass. However, the determination of the tangential stiffness k_s is beyond the scope of this chapter, and here, only a user defined value is used.

Figure 6-9 Stress component in the small section of the bolt

The bolt is assumed to be elastically deformable, and the axial stress of the bolt can be written as

$$\sigma_x = E_b \frac{dU_x}{dx} \quad (6-13)$$

in which E_b is the Young's modulus of the rockbolt. Substitute Eq. (6-10) into Eq. (6-13)

$$\frac{d^2 U_x}{dx^2} = \frac{-F_s}{AE_b} \quad (6-14)$$

Substitute Eq. (6-14) to Eq. (6-12) and rearrange it, we have:

$$\frac{d^2 U_x}{dx^2} - \frac{k_s}{AE_b} U_x = \frac{-k_s}{AE_b} U_r \quad (6-15)$$

Consider a bolt shank model ended at point $p_1 (x_1, y_1)$ and $p_2 (x_2, y_2)$ being divided into n one dimensional elements continuously by $(n-1)$ internal nodal points. The serial number of the nodal point begins at p_1 , which may be a free end or a fixed end with a faceplate installed and ends at p_2 . In the position of each nodal point, there are actually two coinciding points, one belonging to the rock mass and the other one belonging to the bolt shank.

After implementing the finite difference scheme as shown in Figure 6-10 along the bolt length, Eq. (6-15) can then be written as:

$$\frac{\frac{U_x^{i+1} - U_x^i}{x^{i+1} - x^i} - \frac{U_x^i - U_x^{i-1}}{x^i - x^{i-1}}}{2} - \frac{k_s}{AE_b} U_x^i = \frac{-k_s}{AE_b} U_r^i \quad (6-16)$$

Rearrange Eq. (6-16), we have

$$-c^{i-1} U_x^{i-1} + (c^{i-1} + c^i + k_s (\frac{x^{i+1} - x^{i-1}}{2})) U_x^i - c^i U_x^{i+1} = k_s (\frac{x^{i+1} - x^{i-1}}{2}) U_r^i \quad (6-17)$$

where

$$c^i = AE_b / (x^{i+1} - x^i) \quad (6-18)$$

the superscript i refers to the i th nodal point.

Figure 6-10 Finite difference scheme

Figure 6-11 Illustration of the bolt model: (a) without an external fixture; (b) with an external fixture.

As the bolt can be installed with or without an external fixture, as shown in Figure 6-11, the boundary conditions of the rockbolt need to be discussed independently. For the free end condition, as illustrated in Figure 6-11(a), the differential equation at point 1 is

$$(c^1 + k_s \frac{x^2 - x^1}{2})U_x^1 - c^1 U_x^2 = k_s U_{r1} \frac{x^2 - x^1}{2} \quad (6-19a)$$

For the fixed end with a faceplate installed, as illustrated in Figure 6-11(b), the axial displacement at nodal point 1 is

$$U_x^1 = U_{r1} \quad (6-19b)$$

and the reaction force afforded by the faceplate is

$$F_x^1 = c^1 U_x^2 + c^1 U_{r1} \quad (6-19c)$$

The differential equation at the other end of the bolt (n th nodal point) is

$$-c^{n-1} U_x^{n-1} + (c^{n-1} + k_s \frac{x^n - x^{n-1}}{2})U_x^n = k_s U_r^n \frac{x^n - x^{n-1}}{2} \quad (6-20)$$

For a bolt with n nodal points, equations (6-17), (6-19) and (6-20) comprise a matrix of n simultaneous equations. In this matrix, x^i is the axial coordinate of each nodal point, U_r^i is the axial displacement of the rock mass at the position of each nodal point, and U_x^i is the axial displacement of bolt at the position of each nodal point.

The axial load along the bolt can be calculated using the equation:

$$F = AE_b \frac{du_x}{dx} \quad (6-21)$$

which can be written in the discretized form as

$$F = AE_b \frac{u_x^{i+1} - u_x^i}{x^{i+1} - x^i} \quad (6-22)$$

6.3.2 Implementation of the bolt element in the NDDA method

In the NDDA method, the axial displacement U_r^i of the rock mass can be obtained from the implicit solution, and the coefficient c^i can be obtained from problem geometry and the input parameters (i. e., bolt diameter and Young's modulus). The axial displacement U_x^i of the bolt at each nodal point can then be solved using the matrix mentioned in the previous section. After solving the axial displacement of each node along the bolt, the bonding force at each nodal point can be calculated using Eq. (6-12), and the balanced load (F) along the bolt can then be obtained using Eq. (6-10), with dx equals to the inter-nodal spacing. The calculated mobilized load is then applied back to the rock mass as preloading in the next time step to prohibit further displacement. This procedure is

continued until an equilibrium state is reached and thus a complete interaction between the reinforcing system and the rock mass will be simulated (Moosavi and Grayeli, 2006).

The direction cosines of the bolt can be calculated from the coordinate of two bolt ends,

$$l_x = \frac{1}{l}(x_2 - x_1) \quad l_y = \frac{1}{l}(y_2 - y_1) \quad (6-23)$$

and the components F_x and F_y of the preloading force equals to:

$$\begin{aligned} F_x &= -F \times l_x \\ F_y &= -F \times l_y \end{aligned} \quad (6-24)$$

The preloading force at each node is then added to the element which this node belongs to as point load. The potential energy corresponded with the point load equals to

$$\Pi_b = -(F_x u + F_y v) = -(u \ v) \begin{pmatrix} F_x \\ F_y \end{pmatrix} = -D_i^T N_i^T \begin{pmatrix} F_x \\ F_y \end{pmatrix} \quad (6-25)$$

which is derived into:

$$\begin{aligned} f_i &= \begin{Bmatrix} f_a \\ f_b \\ f_c \end{Bmatrix} = -\frac{\partial \Pi_b(0)}{\partial d_{ri}} = \frac{\partial}{\partial a_i} D_i^T N_i^T \begin{pmatrix} F_x \\ F_y \end{pmatrix} \\ &= F_x n_{1r} + F_y n_{2r}, \quad r = 1, 2 \end{aligned} \quad (6-26)$$

The component f_i ($i=a, b, c$) is a 2×1 submatrix which is added to the corresponding rows of the global equation system at the right hand side.

6.3.3 Implementation of different kinds of rockbolt

Different kinds of rockbolt model can be approached in the NDDA method by specifying proper constitutive models to each anchored point. The anchored points on the bolt can be generally divided into “real anchor point” and “virtual anchor point” by assigning different tangential stiffness k_s . Details are provided as follows.

6.3.3.1 Mechanical rockbolt

In the mechanical anchored rockbolt, the deformation which is a combination of faceplate deformation, anchor slippage and the bolt shank deformation will distribute over the entire length of the bolt shank, thus makes the mechanical rockbolt a flexible rock reinforcement system and the failure of the faceplate/anchor occurs before the ultimate tensile strength is reached. The mechanical anchor at the rear of the bolt is presented by a real anchor point while other nodal points within the mechanical anchor and the faceplate are presented by virtual anchor points, see Figure 6-12(a).

6.3.3.2 Dynamic rockbolt (D-bolt)

As introduced before, D-bolt is a smooth steel bar with a number of anchors distributed uniformly or non-uniformly along its length. It inherits the advantages of mechanical rockbolt and fully grouted rockbolt because both the strength and the deformation capacity of the bolt material are fully used. In the NDDA model, the D-bolt is realized by assigning several real anchor points distributed uniformly/non-uniformly along the bolt shank while keeping the nodal points between two real anchor points as virtual anchor points, as shown in Figure 6-12(b).

6.3.3.3 Fully grouted rockbolt

In the fully grouted rockbolt, anchorage is provide by installing grout along the full length of the bolt instead of using mechanical anchor device at the end of the bolt, thus the axial stress on the bar dissipation gradationally away from the load point. In order to describe the distribution of displacement and load along an untensioned fully grouted rockbolt, all the nodal points are assigned as real anchor points, as shown in Figure 6-12(c).

6.3.3.4 Frictional bolt

For the friction anchored rockbolt, the load of the rock is transferred to the reinforcing element directly without grouting agent or mechanical anchorage devices. Different from the fully grouted bolt which get the anchorage from grouting, the frictional bolt get the anchorage from frictional resistance (or frictional force combined with interlocking for Swellex). However, for both fully grouted bolt and frictional bolt, the anchorage exists along the full length of the bolt. In this case, the frictional bolt shares the same bolt model with the fully grouted bolt. By specifying the friction constrains instead of the tangential stiffness of grout at each real anchor point, the interaction between bolt and rock can be simulated (see Figure 6-12(d)).

(a) Mechanically anchored rockbolt

(b) D-bolt

Figure 6-12 Realization of different kinds of rockbolts

6.3.4 Bolt—joint interaction

When the rockbolt intersects a joint, the strength of the joint will be reinforced due to the combination of two effects (Ferrero, 1995):

- (1) The increment of the axial force with the bolt deformation,
- (2) The dowel effect.

The increment of the axial force along the bolt reinforces the joint strength by: 1) increasing the force normal to the joint, thus increasing the frictional forces along the joint face; and 2) increasing the force parallel to the joint, thus increasing the joint resistance. The dowel effect reinforces the joint strength by contributing the bolt's own shear strength to the strength of the joint.

The mechanical behavior of the reinforced system depends on the orientation of the reinforcement, the properties of the reinforcement and the surrounding rock mass. If the surrounding rock mass is extremely rigid, the failure occurs when the shear strength of the bolt is reached, and if the surrounding rock mass is weak, the rockbolt will probably failed due to a combination of tension and bending. In the NDDA simulation, the surrounding is assumed to be deformable and the bolt is allowed to deform to mobilize the shear resistance.

In order to model the interaction between joint and the rockbolt, an interface bolt segmentation, referred as link bolt proposed by Clapp (2003, see Figure 6-7(b)), is implemented in the NDDA method when the bolt across discontinuous. The assumed bolt deformation after shear displacement is shown in Figure 6-13. The length of the bolt segment which crosses the discontinuity and changes orientation under shear displacement

is defined as active length. According to Spang and Egger (1990), the active length ranges from 2 to 5 times of the bolt diameter is adopted in the NDDA method.

Figure 6-13 Assumed bolt deformation after shear displacement

Figure 6-14 Orientation of the shear springs: (a) before shear displacement; (b) after shear displacement.

A set of shear springs, which oriented perpendicular to the rockbolt before and after the shear displacement, will be applied at two ends of the active length, as shown in Figure 6-14. The shear force-displacement relationship is written as:

$$\Delta F_s = -K_s \times du \quad (6-27)$$

where K_s is the shear stiffness of the rockbolt and d_u is the incremental of shear displacement.

6.3.5 Implementation of different failure modes

When the surrounding rock of the expansion shell/bail crushes, the failure of the anchor point may occur. When the rock immediately contacted with the faceplate crushes or the connection between the bolt and the faceplate fails, the faceplate may lose its containment capacity. The decoupling between the rock mass and bolt shank may occur due to the decoupling of the bolt-grout interface, the decoupling of grout-rock interface, the grout failure and the broken the rock mass. And also, the failure of the bolt shank may occur under both shear and tensile displacements.

In the NDDA program, due to the difficulty in simulating the small volume of elastic-plastic grout, the grout media is omitted and only the debonding along the bolt/rock interface is considered. Other failure modes, including the failure of the face plate/anchor point and the failure of the bolt shank are also available in the NDDA bolt model.

- *Failure at the face plate/anchor point.*

When the normal or shear stress applied at the faceplate or the expansion shell/bail anchor exceeds a critical value, failure of the faceplate/anchor point occurs. In this case, the face plate/anchor point will be removed from the bolt model and corresponded end will be treated as free end (see Figure 6-15).

Figure 6-15 Failure of the face plate/anchor point

Figure 6-16 Debonding along bolt/rock interface

- *Debonding along rock/bolt interface.*

Once the shear force at the nodal point exceeds the peak shear strength, the debonding will occur (see Figure 6-16). For the fully grouted rockbolt, the shear stress drops from peak shear strength to a residual value referred as the residual shear strength. For the frictional rockbolt, the residual shear strength of the interface remains the same as the peak shear strength (refers to section 6.4.1 for the reason).

- *Failure of bolt shank.*

Axial behavior of the bolt shank is governed by an elastic-perfectly plastic constitutive model in the NDDA method, as illustrated in Figure 6-17. If the axial force of the bolt exceeds the ultimate tensile strength of the bolt (i.e. σ_y in Figure 6-17), a tensile yield will be triggered and the bolt will subjected to plastic deformation until the tensile strain of the bolt exceeds the ultimate strain (i.e. ϵ_u in Figure 6-17) and in this case, the tensile failure of the bolt will be triggered (see Figure 6-18(a)). If a fully dynamic analysis is performed, both loading and unloading need to be considered and the Young's modulus in the unloading process is assumed to be the same as that in the loading process.

Figure 6-17 Constitutive law of the NDDA bolt model

In case of the bolt experiencing shear displacement, the failure mechanism of the bolt shank is more complex. In the NDDA method, the surrounding rock is assumed to be deformable as for the fully grouted bolt, the mortar collar which is about half a diameter thickness (i.e. 22mm bolts in 45mm boreholes) is deformable, and for the mechanically anchored bolt, the bore hole is not completely filled by the bolt. In this case, the bolt does not fail in pure shear, but rather in a combination of bending and tension. Currently, the shear strength of the bolt shank is assumed as 75% of the input axial strength. If the shear force exceeds the shear strength, the shear yield will be triggered and the shear resistance will drop to 50% of the input axial strength. After the occurrence of the shear yield, tensile force will be accumulated in the bolt to balance the block system. If the tensile force exceeds the ultimate tensile strength, tensile yield will be triggered; the failure of the bolt

shank occurs when the ultimate tensile strain is reached (see Figure 6-18(b)). The deformation ability of the bolt under shear displacement in the NDDA model depends largely on the choice of the active length, which is based on the material properties of the surrounding rock mass and the type of the rockbolt applied. Currently, for the fully grouted rockbolt, the active length is defined as 2.5 times of the bolt diameter while for the mechanical anchored bolt, the active length is defined as 5 times of the bolt diameter in the NDDA bolt model.

Figure 6-18 Tensile failure of bolt shank: (a) rock bolt experiencing axial deformation; (b) rock bolt experiencing shear & axial deformation

6.4 Application of the developed rock bolt model

6.4.1 Pullout test

Figure 6-19 Sketch illustrates the pull out test in the laboratory

In order to validate the rockbolt model incorporated in the NDDA method, the conventional pull-out test is simulated by the four different types of rockbolt models. Figure 6-19 illustrates the sketch of the laboratory test. In the NDDA simulation, the laboratory model is simplified as a two-block system, the dimension of each block is $1\text{m} \times 1\text{m}$, and the length of the bolt shank is 1.5m . The left block is fixed while the right block is pulled apart. The pull-out load is applied in a sense that it increases linearly from 0 to 72KN at the first 0.1s and keeps constantly from then on. The bolt shank is installed from

the centre of the left block to the right boundary of the right block, as showed in Figure 6-20.

During the NDDA simulation, both the blocks and the faceplate are assumed to be linear elastic. The bolt shank is elastic-perfectly plastic and the controlling parameters can be found in Table 6-1. The yield load of the bolt shank, which can be calculated using the bolt diameter and the bolt ultimate strength, is 141.4 KN.

Figure 6-20 NDDA model

Table 6-1 Input parameters of the NDDA simulation

Control parameters	Plain stress	Yes
	Time step size (s)	0.0005
	Total time (s)	10
	SOR factor	1.2
	Max. disp. ratio	0.0001
Bolt parameters	Young's modulus (GPa)	210
	Bolt diameter (m)	0.02
	Ultimate axial strength (MPa)	450
	Ultimate strain	0.2
Rock sample	Young's modulus (GPa)	30
	Mass per unit (Kg/m^3)	2458
	Poisson's ratio	0.25

For the mechanical rockbolt, the bolt is anchored only at the two ends of the bolt shank. Figure 6-21 shows the axial force as well as the shear force at each nodal point along the bolt shank. As it can be seen, the axial force distributes uniformly along the bolt length while the shear force only exists on the two end anchors. The axial force obtained from the numerical simulation is 72KN, exactly same with the analytical solution. If the stretch length is defined as the distance between the anchors, the strain of the bolt shank can then be obtained using the elongation divided the stretch length. The strain is constant along the free length of the bolt.

Figure 6-21 Pullout test for the mechanical anchored rockbolt: (a) NDDA model of the mechanical rock bolt; (b) axial load distribution along the bolt; (c) shear load distribution along the bolt.

For the Energy absorbing rockbolt (D-bolt), the bolt shank is divided into 5 bolt links by 4 anchor points which are distributed evenly along the bolt length. The distribution of the axial force and the shear force at each nodal point is depicted in Figure 6-22. Every bolt link between two adjacent nodal points works independently from each other. The maximum value of the tensile load appears at the bolt link which crossing over the joint, and comparable small tensile load appear at other smooth bolt links. The maximum tensile load is 72kN, same as the applied pull-out force. As the axial force is smaller than the yield load of 141.4 kN, the bolt shank is still within the elastic range.

When the applied pull-out force is increased to 3 times of the original value, i.e. 216 kN, the bolt shank yields. Figure 6-23 shows the axial force versus elongation of the bolt link which crossing over the joint. At the beginning, the axial force increases linearly until the tensile yield load, which is 142.15 kN, is reached. And the corresponding elongation is 0.65 mm. The axial force then keeps at a constant of 142.15 kN until the elongation reaches 6.01cm, and the tensile failure of the bolt shank occurs. The tensile yield load obtained from the NDDA simulation is 0.55% away from the theoretical prediction 141.4 kN. And the yield strain obtained from NDDA simulation is 20.03%, 0.15% away from the ultimate strain of the bolt shank.

Figure 6-22 Pullout test for the D-bolt: (a) NDDA model of the D-bolt; (b) axial force distribution along the bolt; (c) shear force distribution along the bolt.

Figure 6-23 Axial force vs. elongation for D-bolt

When the fully grouted bolt is applied in the pull-out test, the bolt shank is discretized into 30 segments by 31 nodal points uniformly distributed along the bolt length. These nodal points are functioned as anchored points. The left end of the bolt is a free end while the right end of the bolt can be fixed with a faceplate attached or free.

Firstly, two free ends are applied at the bolt boundary, and the tangential stiffness k_s of each anchor point is set as 100GPa. The anchored points are assumed to be reliable and no

debonding is considered. The maximum axial force obtained from the numerical simulation is 71.97KN, 0.04% away from the applied 72KN.

Li and Stillborg (1999) expressed the analytical prediction of the relationship between the joint opening displacement δ_j and the tensile stress of the bolt at the joint σ_{b0} as:

$$\delta_j \approx \frac{\sigma_{b0} d_b}{\alpha E_b}, \text{ when } \sigma_{b0} \leq \frac{2s_p}{\alpha} \quad (6-28)$$

where (Li and Stillborg, 1999)

$$\alpha^2 = \frac{2G_r G_g}{E_b [G_r \ln(\frac{d_g}{d_b}) + G_g \ln(\frac{d_0}{d_g})]} \quad (6-29)$$

$$G_r = \frac{E_r}{2(1+\nu_r)} \text{ and } G_g = \frac{E_g}{2(1+\nu_g)}$$

Otherwise (Li and Stillborg, 1999),

$$\delta_j = \frac{2}{E_b} [\sigma_{b0} x_2 - 2\omega s_p \frac{(x_2 - \Delta)^2}{d_b} - (2\omega + 1) s_p \frac{2\Delta^2}{3d_b} + \frac{s_p}{\alpha^2} d_b] \quad (6-30)$$

In equations (6-28), (6-29) and (6-30), d_b is the diameter of the rockbolt, d_g is the diameter of the borehole, d_0 is the diameter of the circle outside which the influence of the rockbolt disappears, E_b is the Young's modulus of the bolt steel, E_g is the Young's modulus of the grout material, E_r is the Young's modulus of the rock mass, ν_g is the Poisson's ratio of the grout material, ν_r is the Poisson's ratio of the rock mass, s_p is the peak shear strength of the bolt/grout interface, w is the ratio between the residual shear strength to the peak shear strength, Δ is the length of the bolt shank and x_2 position of the debonding front.

To compare the analytical prediction with the numerical simulation, several assumptions are needed because the borehole diameter and grout material is not considered in the numerical modeling. Thus, the diameter of the borehole is assumed to be the same as the diameter of the rockbolt, the Poisson's ratio of the grout material is assumed to be the same as the Poisson's ratio of the rock mass and the Young's modulus of the grout is assumed to be the tangential stiffness in the numerical modeling. As no debonding is considered in the numerical simulation, Eqn. (6-28) will be used to calculate the joint opening displacement.

The joint opening displacement obtained from the numerical simulation is 0.079 mm, and the obtained influence diameter is 0.217m. Set $d_0=0.217m$, the analytical prediction of the joint opening displacement is 0.101mm. Compare the numerical result with the analytical prediction, the difference is 22%.

The analytical prediction of the distribution of the axial force along the bolt length is (Li and Stillborg, 1999)

$$\sigma_b(x) = \pi \left(\frac{d_b}{2} \right)^2 \sigma_{b0} e^{-\frac{2\alpha}{d_b} |x-x_0|} \quad (6-31)$$

where x is the distance along and bolt and x_0 is the position of the joint opening.

Figure 6-24 Distribution of the axial force along the bolt

The comparison between the distributions of the axial force along the bolt obtained from the NDDA modeling and the analytical prediction is shown in Figure 6-24. As it can be seen, the results obtained from the two methods have a good agreement. The peak axial force appeared at the position of the joint opening. Due to the loss of the grout material, the NDDA modeling results in smaller joint opening displacement compared with the analytical prediction, thus the axial force distributed more concentrate at the joint position in the NDDA simulation.

(a)

Figure 6-25 NDDA simulation results of under three different k value (in case of two free ends): (a) distribution of bolt and rock displacement; (b) axial load; (c) shear force.

In order to obtain an indepth understanding on the effects of the tangential stiffness k_s , three different tangential stiffness $k_s = 1, 10, 100$ GPa are applied in the simulation. Figure 6-25 shows the distribution of the bolt displacement, corresponding rock displacement, axial force along the bolt and shear force along the bolt for three different tangential stiffness values. It is worth to note that, there is a displacement jump at the position of the joint in Figure 6-25(a), indicating that the initially closed joint becomes open under the pullout load. However, a smaller joint opening is corresponded with a higher k_s value while the lower k_s value results in a larger joint opening. At the same time, when a higher k_s value is applied, the relative slip between the bolt and the rock occurs only at the position closer with the joint. With the decrease of k_s , the range within which the relative slip between the bolt and the rock occurred increases. As mentioned before, the shear force developed due to the relative slip between the bolt and the rock, thus a higher k_s value results in the shear force concentrated near the joint position while a lower k_s value results in the shear force distributed nearly along the whole length of the bolt (see Figure 6-25(c)). Also, from the equilibrium equation provided by Farmer (1975), the balance between the axial force and the shear force exists at every infinitesimal element of the rockbolt. If the shear force between the rock and the bolt exists mainly at the position of the joint (large k_s value), the axial force developed in the rockbolt concentrated narrowly at the position of the joint; if the shear force between the rock and the bolt exist nearly along the whole length of the bolt (small k_s value), the axial force of the bolt develops gradually from the bolt end, and reaches its maximum at the joint position (see Figure 6-25(b)). The maximum axial force for $k_s = 1, 10, 100$ GPa, is 71.43 kN, 71.94 kN and 71.97 kN, respectively. The error is 0.79%, 0.08%, and 0.04%, as compared with the applied force of 72 kN.

Figure 6-26 NDDA simulation results of under three different k value (in case of faceplate being installed at the right end of the bolt): (a) axial load; (b) shear force.

In the second case, the faceplate is applied at the right end of the bolt. Similar with the first case, the anchor points are assumed to be reliable and no debonding is considered. Figure 6-26 shows the distribution of the axial force and shear force along the bolt under three different k_s value. From Figure 6-25, it is found that only with a lower k_s value (e.g. 1 GPa), the effects of the faceplate becomes visible (the reaction force supplied by the faceplate equals to 2.86kN in this case); otherwise, the effects of the faceplate are negligible.

From previous cases, it seems that the faceplate does not work effectively if the relative slip between the rock and the bolt is small. In this case, the effect of the faceplate is further investigated by changing the location of the joint. The distance between joint and the fixed end is defined as x and three joint locations, $x=1, 0.5, 0.25$ m, are then tested. The tangential stiffness applied is $k_s=1$ GPa and also, debonding is not considered. As shown in Figure 6-27, both the location of the peak axial load and the peak nodal shear force vary with the variation of the joint location, however, the peak load always appears at the position of the joint. The effects of the faceplate becomes more obviously when the joint moves closer to the fixed end, and the reaction forces afforded by the faceplate equal to 2.86, 18.57, 44.46kN when the joint located at 1, 0.5, 0.25m away from the fixed end, respectively.

(a)

Figure 6-27 Effects of the faceplate with different joint location: (a) illustration of joint position; (b) distribution of axial load; (c) shear force at each anchor point.

In reality, the interface has its maximum shear strength, and the assumption of reliable anchor points is unreasonable. According to Windsor (1997), the reinforcement system can be divided into continuously mechanically coupled (CMC) system, continuously frictionally coupled (CFC) system and discretely mechanically or frictionally coupled (DMFC) system. Among them, the cement or resin grouted bolts belong to the CMC system. For CMC system, the shear strength of the interface includes three parts: the adhesion/cohesion, the mechanical inter lock and the friction in the axial direction. When the peak shear strength is reached, parts of the interface shear strength will be lost due to the loss of the deformation compatibility, thus results in the appearance of the debonding front and it attenuates at an increasing distance from the point of peak shear strength. The shear strength after the loss of some strength component is called the residual shear strength in this thesis and the ratio between the residual shear strength to the peak shear strength is defined as w here.

To consider the debonding phenomenon, the shear strength of the interface is assumed to be 26 kN and the ratio w is assumed to be 0.2. The distance between the joint and the free face is $x=1\text{m}$ and the applied tangential stiffness is $k_s=10\text{GPa}$. The bolt is installed with two free ends. The distribution of the axial force and shear force along the bolt are depicted in Figure 6-28. As it can be seen, the debonding section is around 0.35 m at both sides of the joint. The shear force increases gradually from the left end, after reaching a peak value, it drops suddenly and keeps a constant (5.2 kN or -5.2 kN) within the debonding range, and again, the shear force suddenly increases to a peak value, then gradually decreases to zero within a distance about 0.3 m. Correspondingly, the axial force along the bolt increases nonlinearly within the bonded range and linearly within the debonded range from the left end of the bolt, and reaches its maximum value at the

position of the joint; then the axial load decreases from the joint position and finally drops to zero.

Figure 6-28 Bolt response in pull-out tests with debonding activation: (a) axial load along bolt length; (b) shear force at each anchor point.

Figure 6-29 Bolt responses in pull-out tests with and without debonding activation when the faceplate is installed: (a) axial load along bolt length; (b) shear force at each anchor point.

If the distance between the joint and the fixed end x is changed to 0.5m, both the bolt responses in pull-out tests with and without debonding activation are depicted in Figure 6-29. The applied tangential stiffness is $k_s = 10\text{GPa}$ and in case of debonding, the shear strength of the interface and the ratio w are same with the previous case. If the location of the peak load is not considered, the bolt response is actually similar with the previous case. What worth to note is that, as demonstrated by Figure 6-29(a), the faceplate works more effective if debonding is activated, i. e. the reaction force afforded by the faceplate is

0.58KN if all the anchors are assumed to be reliable, while if the debonding at the bolt/rock interface is accounted, the reaction force is increased to 3.70KN.

Figure 6-30 The response of frictional rock bolt under pull-out load of 72KN: (a) axial load along the bolt length; (b) shear force at each nodal point.

Figure 6-31 The response of frictional rock bolt under pull-out load of 72KN with the faceplate being installed: (a) axial load along the bolt length; (b) shear force at each nodal point.

The frictional rockbolt, such as the Split set and Swellex bolts, belongs to the family of CFC system, there are only friction component (for Split set) or the mechanical interlock and friction component (for Swellex) at the bolt interface. In order to reproduce the response of the frictional rockbolt under pull-out test, the tangential stiffness $k_s = 0.1 \text{ GPa}$ is applied, the shear strength of the interface is assumed to be 5.2KN and the magnitude of the residual shear strength after debonding is assumed to be the same as the peak value. The axial load along the bolt shank and the shear force in the anchored points are depicted in Figure 6-30(a) and Figure 6-30(b), respectively. The bolt fails through complete

debonding along the bolt/rock interface in the left block. The maximum tensile load is 71.93KN, 0.97% away from the analytical solution. As the maximum tensile load is smaller than the tensile yield strength, the bolt shank is far from tensile yield. In this case, the load capacity for the frictional rockbolt highly depends on the bolt length, the peak and residual strength of the bolt/rock interface.

In case of the faceplate installed at the right end of the bolt, the axial load in the bolt shank and the shear force in the anchored points are shown in Figure 6-31. The reaction force afforded by the faceplate is 37.74KN, indicating that the faceplate is very effective. Together with Figure 6-26(a), Figure 6-27(a), and Figure 6-29(a), it is concluded that the faceplate does not work effectively if the relative slip between the rock and the bolt is negligible, and the faceplate works effectively in case of debonding or it is located closer to the joint.

6.4.2 Shear test of fully grouted rock bolt

The block geometry of this test is shown in Figure 6-32. During the simulation, the load applied on the right boundary of the upper block increases linearly with the rate of 24KN/s. The bottom boundary is fixed while the upper boundary is fixed in y direction only to ensure that the upper block moves horizontally. The input parameters of the NDDA simulation are listed in Table 6-2.

Figure 6-32 Geometry of shear test model in NDDA

Table 6-2 NDDA input parameters

Control parameters	Plain stress	Yes
	Time step size (s)	0.0005
	Total time (s)	10s
	SOR factor	1.2
	Max. disp. ratio	0.0001
Bolt parameters	Young's modulus (GPa)	210
	Bolt diameter (m)	0.02
	Ultimate axial strength (MPa)	450
	Ultimate strain	0.17
Rock sample	Young's modulus (GPa)	30
	Mass per unit (KG/m ³)	2458
	Poisson's ratio	0.25
Joint properties	Joint friction angle(°)	0
	Joint cohesion (MPa)	0
	Joint tensile strength (MPa)	0

As mentioned in Section 6.3.5, the shear strength is set as 75% of the input axial strength, the shear resistant after shear yield is set as 50% of the input axial strength, and the active length is set as 2.5 times of the bolt diameter for fully grouted rockbolt. The simulation results are shown in Figure 6-33. As it can be seen, for both applied load vs. block displacement curve and resistant force of the bolt vs. block displacement curve, there exist two yield points, the first yield point is due to the shear yield while the second yield point is due to the tensile yield. At the initial stage, the shear strength of the bolt is the only source to balance the upper block from the applied load. When the ultimate shear strength is reached, the shear yield is triggered. After that, the shear resistance drops to 50% of the axial strength, and the tensile force is accumulated gradually to balance the upper block. Before enough tensile force can be built up, the rate of block displacement increases. The tensile yield is triggered once the tensile strength is reached and the bolt will experience plastic deformation if the applied load continues. As an elastic-perfect plastic constitutive model is used, the tensile force will keep constant in the plastic range, and the rate of block displacement increases as the resistant force of the bolt is smaller than the applied force. Finally, the tensile failure of the bolt shank appears when the ultimate tensile strain is exceeded and the resistant force of the bolt drops to 0 after failure.

A sensitive analysis is then carried out by varying one parameters while keeping other parameters unchanged, the parameters being studied including the shear strength before shear yield, the resistant shear strength after shear yield, the assumed active length and the axial strength after shear yield.

Figure 6-33 NDDA simulation result of shear test

- *Shear strength before shear yield.*

Currently, the shear strength of the rockbolt is set as 75% of the input axial strength, based on the NIOHS laboratory test, reported by McHugh and Signer (1999). The effect of the shear strength is tested by applying two different ratios, 65% and 80% of the input axial strength in the numerical simulation, and the results are depicted in Figure 6-34(a) and Figure 6-34(b), respectively. Comparing the load—displacement curves obtained from varying shear strength ratio, it is found that increasing the shear strength ratio will increase the shear yield point while decreasing the shear strength ratio will decrease the shear yield point (see Figure 6-35(a)). Also, as the shear resistance drops once the shear yield is triggered, the tensile force will be accumulated to balance the upper block against the applied force. Because the tensile force is smaller than the applied load at the moment of shear yield, the block displacement increases quickly at this stage. Once enough tensile force is built up, the speed of the block displacement slows down until the tensile yield is triggered. The higher the shear strength ratio is applied, the more the amount of block displacement before enough tensile force is built up. Based on Figure 6-35(b) it is found that once the shear yield is triggered, the resistant force of the bolt drops suddenly as a result of the drop of shear strength. Varying the shear strength ratio has the effect on the amount of the drop of the resistant force, but it does not have the effect on the points of tensile yield and tensile failure.

Figure 6-34 NDDA simulation result of shear test: (a) shear strength equals to 65% of the input axial strength; (b) shear strength equals to 80% of the input axial strength.

Figure 6-35 Effect of varying bolt shear strength before shear yield: (a) applied shear force V.S. block displacement; (b) resistant force of the bolt V.S. block displacement.

- *Resistant shear strength after shear yield.*

The resistant shear strength after shear yield is assumed as 50% of the input axial strength in the current NDDA program, and the effect of the resistant shear strength is tested by further reducing the ratio to 25% and 0% of the input axial strength in the numerical simulation. The NDDA results are depicted in Figure 6-36(a) and Figure 6-36(b), respectively. Comparing the load-displacement curves obtained from varying resistant shear strength ratio, it is found that decreasing the resistant shear strength ratio results in the increasing in the amount of the block displacement before enough tensile force can be built up (see Figure 6-37(a)). Also, as shown in Figure 6-37(b), increasing the resistant shear strength ratio has the effect of decreasing the distance between the shear yield point

and the tensile yield point, but it does not have the effect on the points of shear yield and tensile failure.

Figure 6-36 NDDA simulation result of shear test: (a) shear resistant equals to 25% of the input axial strength; (b) shear resistant equals to 0% of the input axial strength.

Figure 6-37 Effect of varying bolt shear resistance after shear yield: (a) applied shear force V.S. block displacement; (b) resistant force of the bolt V.S. block displacement

- *The active length*

The active length is assumed to be 2.5 times of bolt diameter for the type of fully grouted rockbolt, and the effect of the active length is tested by increasing the active length to 5 times of the bolt diameter. The comparison between the simulation results obtained from different active length is shown in Figure 6-38. Figure 6-38(a) indicating that increase the active length will cause the increase of the amount of the block displacement before enough axial force can be built up. Figure 6-38(b) indicating that increase the active length results in the increase of distance between the shear yield point and the tensile yield point, as well as the distance between the tensile yield point and the tensile failure point;

however, it does not have effect on the point of shear yield. This phenomenon can be explained by the fact that the axial strain and axial force of the bolt equal to

$$\varepsilon = \frac{dl}{l}$$

$$F = AE_b \varepsilon = AE_b \frac{dl}{l} \quad (6-27)$$

where E_b and A are the Young's modulus and the cross section area of the bolt, respectively. If the actively length (l) is increased, the stretch of the bolt (dl) needs to be increased to build the same amount of the axial force. At the same time, as the ultimate strain is the same for both cases, the allowable bolt stretch is increased with the increase of the bolt length.

Figure 6-38 Effect of varying active length: (a) applied shear force V.S. block displacement; (b) resistant force of the bolt V.S. block displacement

- *Bolt axial strength after shear yield*

It is somewhat unrealistic that the axial strength is kept unchanged after the occurrence of shear yield. In this case, a reduction of the axial strength after shear yield is involved. When the shear yield is triggered, the ultimate axial strength drops to 75% of the input value, the same load level as the defined shear strength. The comparison between the simulation results with/without considering the axial strength reduction is depicted in Figure 6-39. Referring to the applied force-block displacement curves (Figure 6-39(a)), two yield points can be found for both cases, however, if the axial strength is dropped to 75% of its original value after shear yield, the distance between the shear yield point and the tensile yield point is greatly reduced. At the same time, Figure 6-39(b) indicates that the axial strength reduction affects the tensile yield point, but it does not affect the shear yield point and the tensile failure point.

Figure 6-39 Effect of the reduction of axial strength after shear yield: (a) applied shear force V.S. block displacement; (b) resistant force of the bolt V.S. block displacement

6.4.3 Rock slope reinforced by rock bolts

The main purpose of this example is to demonstrate the potential of the developed rock bolt. The geometry of the slope is same as that studied in section 3.3.3, but the joint strengths are different. The friction angles for different joints are reduced from 30° , 18° , 15° to 16.102° , 9.228° and 7.631° , respectively. Under the gravity force, the upper active block deformed first, the small deformation appeared during the simulation changes the initial configuration of the slope as well as transfers the unbalance force from the upper active block to the lower slope, thus cause the instability of the slope (see Figure 6-40).

Figure 6-40 Deformation and the failure pattern of the rock slope

In order to stabilize the slope system, the fully grouted rockbolts are applied at the toe of the slope. Figure 6-41 shows the design of bolting of this problem. There are totally seven fully grouted bolts installed in the rock slope and the direction angle of the bolt is 120° . The left end of each bolt is fixed with a faceplate attached while the right end of each bolt is free. The main function of the rockbolts is, as very stiff bar element, it can afford strong

tensile force to stop the block from sliding, at the same time, enhance the joint strength by the increment of the axial force with the bolt deformation and the dowel effect.

Figure 6-41 Rock slope reinforced by the fully grouted rock bolt

The input parameter used in the numerical analysis is listed in Table 6-3. During the simulation, two measurement points, which are located at the toe of the slope (measurement point A) and the top of the slope (measurement point B) are used to judge the status of the slope. The development of the block displacement and the tensile force supplied by the bolt during the simulation are shown in Figure 6-42.

Table 6-3 Input parameter used in example

Control parameters	Static	Yes
	Time step size (s)	0.05
	Spring stiffness(GPa)	60
	SOR factor	1.4
	Max. disp. ratio	0.01
Bolt parameters	Young's modulus (GPa)	210
	Tangential stiffness(GPa)	10
	Bolt diameter (m)	0.03
	Ultimate axial strength (MPa)	450
	Ultimate strain	0.17
Rock/bolt interface	Shear strength(MPa)	128
Rock sample	Young's modulus (GPa)	6
	Mass per unit (Kg/m^3)	2458
	Poisson's ratio	0.25

As it can be seen, the bolt system increases the degree of the stability of the slope effectively. The displacement of the block only increases at the beginning of the simulation due to the effect of gravity turn on, and then stays stable. The development of the tensile force of the rockbolt has the same trend of that of the block displacement—it increases quickly at the very beginning and keeps stable as the slope is stabilized.

Figure 6-42 Development of (a) block displacement and (b) maximum axial force of the rock bolt with time

Figure 6-43 Distribution of axial force along the bolt

The distributions of the tensile force along the bolt shanks are depicted in Figure 6-43. Generally speaking, for all the rock bolts, the tensile forces concentrate at the position of the joint opening, consistent with the observation of the pull-out test. When the bolt shank intersects several joints, i.e. the interfaces between adjacent blocks and the segments of sliding surface, the tensile force at the sliding surface is much larger than that at the block interface due to the fact that the relative slip appears more on the sliding surface, the interfaces between the blocks can be open or closed. For example, bolt No.3—No.7 intersect with the interface between the first block and the second block; however, the tensile force appears at the block interface is negligible, indicating that during the simulation, the first and the second moved together, and the joint between them is closed. Bolt No.5-No.7 intersect with the interface between the second and the third block, and the tensile force appears at this interface is a small value, indicating the slight opening of the joint. Also, though there is a faceplate attached at the left end of each rock bolt, the reaction forces supplied by the faceplates are close to zero for blot No.2-No.7, indicating

the faceplate may not work effective if no debonding occurs. For bolt No.1, as the faceplate stays close to the joint, the effect of the faceplate is obvious.

6.5 Conclusion

One of the important contributions of this thesis is the development of the rockbolt element in the NDDA framework. In this chapter, a finite different based numerical treatment, which is first introduced by Hyett (1996), is incorporated in the NDDA program to model the behavior of the rockbolt system. Four types of rockblots, namely mechanical rockbolt, friction rockbolt, fully grouted rockbolt and D-bolt are introduced. Other types of rockbolts, though not explicitly mentioned, are possible by specifying proper constitutive models to each anchored points. Three possible failure modes, including the failure of the faceplate/anchor point, debonding along the bolt/rock interface, and the tensile failure of the bolt shank are also incorporated in the developed rockbolt model. Finally, three numerical examples are carried out, the pull-out test and the shear test show the ability of the developed model while the stability analysis of the discontinuities controlled rock slopes demonstrates the potential of the developed model in engineering applications.

Chapter 7. Enhancement of the contact constrain in the DDA

The penalty method adopted in both the DDA and the NDDA method is a popular method widely used in computer codes for the treatment of constraint conditions (Vignjevic, 1999; Munjiza et al., 2000; Gu et al., 2002). It is a simple method with a clear physical meaning, and the implementation is easy and straightforward. The number of governing equations is not increased when using contact springs and the solution is easily obtained by simply adding contact components to the stiffness matrix.

However, the method has several disadvantages. The accuracy of the solution is highly dependent on the choice of the penalty value and the contact constraints are satisfied only approximately. In static problems, the penalty parameter should be an arbitrarily large number in principle. However, for a computer with a finite number of digits, it should be large enough to enforce the constraint condition, but not so large that the governing equations become ill-conditioned. On the other hand, too small a penalty parameter results in an unacceptable penetration of one body into the other and the overall response is distorted. Cheng (1998) observed that the choice of spring stiffness has a significant effect on convergence. Saad (1996) found that the penalty approach can be used with confidence for matrices of moderate size only (i.e. less than 10,000 unknowns) as it tends to decrease the conditioning of the equations.

In this chapter, the Augmented Lagrangian method (ALM)— an iterative method to obtain the exact solutions for contact forces with important advantages over the traditional Lagrangian multiplier and penalty methods is introduced into the DDA program, and the comparison between the ALM and the standard DDA method is carried out regarding the accuracy and efficiency.

7.1 The Lagrange Multiplier Method

In the classical Lagrange Multiplier Method (Papadopoulos and solberg, 1998; heintz and Hansbo, 2006), contact forces are unknowns and can be explicitly calculated. Consider two blocks i and j in contact, as show in Figure 7-1, point P_1 is the vertex of block i and has an original position in P_0 , while P_2P_3 is the boundary of block j , and P_1 penetrates into the edge P_2P_3 . Let (x_i, y_i) and (u_i, v_i) be, respectively, the coordinates and the displacement increments at points P_i ($i=0, 1, 2, 3$). A new variable λ defined as Lagrange multiplier is introduced and let λ be the unknown contact force caused by penetration depth, d . The strain energy of that contact force is:

$$\Pi_k = \lambda d_n \quad (7-1)$$

$$\mathbf{G}_n^T = \frac{1}{l} \begin{Bmatrix} y_3 - y_1 & x_1 - x_3 \end{Bmatrix} \mathbf{T}_j(x_2, y_2) + \frac{1}{l} \begin{Bmatrix} y_1 - y_2 & x_2 - x_1 \end{Bmatrix} \mathbf{T}_j(x_3, y_3) \quad (7-8)$$

and substituting into Eq. (7-6), then

$$\Delta = A_0 + l \mathbf{E}_n^T \mathbf{D}_i + l \mathbf{G}_n^T \mathbf{D}_j \quad (7-10)$$

Substituting Eq. (7-10) into Eq. (7-2), we have

$$d_n = \frac{\Delta}{l} = \frac{A_0}{l} + \mathbf{E}_n^T \mathbf{D}_i + \mathbf{G}_n^T \mathbf{D}_j \quad (7-11)$$

The normal contact sub-matrices of the Lagrange multiplier method can be deduced as follows:

$$\Pi = \lambda d_n = \lambda \left(\frac{A_0}{l} + \mathbf{E}_n^T \mathbf{D}_i + \mathbf{G}_n^T \mathbf{D}_j \right) \quad (7-12)$$

$$\mathbf{f}_{i(6 \times 1)} = - \left. \frac{\partial \Pi}{\partial \mathbf{D}_i} \right|_0 = -\lambda \mathbf{E}_n \quad (7-13)$$

$$\mathbf{f}_{j(6 \times 1)} = - \left. \frac{\partial \Pi}{\partial \mathbf{D}_j} \right|_0 = -\lambda \mathbf{G}_n \quad (7-14)$$

Similarly, the shear penetration distance can be obtained by

$$d_s = \frac{1}{l} \overrightarrow{\mathbf{P}_0 \mathbf{P}_1} \cdot \overrightarrow{\mathbf{P}_2 \mathbf{P}_3} \quad (7-15)$$

$$d_s = \frac{1}{l} \begin{Bmatrix} x_3 + u_3 - x_2 - u_2 & y_3 + v_3 - y_2 - v_2 \end{Bmatrix} \begin{Bmatrix} x_1 + u_1 - x_0 - u_0 \\ y_1 + v_1 - y_0 - v_0 \end{Bmatrix} \quad (7-16)$$

Denote

$$S_0 = \begin{Bmatrix} x_3 - x_2 & y_3 - y_2 \end{Bmatrix} \begin{Bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{Bmatrix} \quad (7-17)$$

$$\mathbf{E}_s^T = \frac{1}{l} \begin{Bmatrix} x_3 - x_2 & y_3 - y_2 \end{Bmatrix} \mathbf{T}_i(x_1, y_1) \quad (7-18)$$

$$\mathbf{G}_s^T = \frac{1}{l} \begin{Bmatrix} x_2 - x_3 & y_2 - y_3 \end{Bmatrix} \mathbf{T}_j(x_0, y_0) \quad (7-19)$$

and substitute them into Eq.(7-16), then

$$d_s = \frac{S_0}{l} + \mathbf{E}_s^T \mathbf{D}_i + \mathbf{G}_s^T \mathbf{D}_j \quad (7-20)$$

The shear contact sub-matrices of the Lagrange multiplier method can be deduced as follows:

$$\Pi'_L = \lambda' d_s = \lambda' \left(\frac{S_0}{l} + \mathbf{E}_s^T \mathbf{D}_i + \mathbf{G}_s^T \mathbf{D}_j \right) \quad (7-21)$$

$$\mathbf{f}'_{i(6 \times 1)} = - \frac{\partial \Pi'}{\partial \mathbf{D}_i} \bigg|_0 = -\lambda' \mathbf{E}_s \quad (7-22)$$

$$\mathbf{f}'_{j(6 \times 1)} = - \frac{\partial \Pi'}{\partial \mathbf{D}_j} \bigg|_0 = -\lambda' \mathbf{G}_n \quad (7-23)$$

Add Eq. (7-13), Eq. (7-14), Eq. (7-22) and Eq. (7-23) in the global Eq. (2-11), the simultaneous equilibrium equations are modified as:

$$\begin{bmatrix} [k] & [k_\lambda]^T \\ [k_\lambda] & [0] \end{bmatrix} \begin{bmatrix} [D] \\ \lambda \end{bmatrix} = \begin{bmatrix} [F] \\ -\Delta / l \end{bmatrix} \quad (7-24)$$

where

$$[K_\lambda] = \{0 \ 0 \ 0 \dots\dots E \dots\dots G \dots\dots 0 \ 0 \ 0\} \quad (7-25)$$

E and G are the 6×1 sub-matrices with components e_r and g_r ($r=1, \dots, 6$) as defined in Eq. (7-8) and Eq. (7-9).

Although the Lagrange multiplier method allows the contact constraints to be satisfied exactly from the physical point of view, it adds a gap parameter for each contact thus creating a larger system of equations and it introduces a zero onto the leading diagonal of the global equation which will cause the matrix to be non-positive definite.

7.2 The Augmented Lagrangian Method

The ALM was initially proposed by Hestenes (1969) for solving non-linear programming problems with equality constraints. It was extended to treat convex differentiable optimization problems with inequality constraints (such as the frictionless contact problem) by Rockafellar (1974). Later, it was widely used in the contact problems of numerical methods (Simo and Laursen, 1992; Carpinteri et al., 2006; Stupkiewicz et al., 2010). The ALM can be regarded as a combination of Lagrange Multiplier Method and Penalty method, both a penalty number P (representing the stiffness of the contact spring) and a Lagrange multiplier λ^* (representing the contact force λ) are used in this method. It is an iterative method in which the Lagrangian multiplier λ^* can be iteratively calculated by adding the penalty force onto the Lagrangian multiplier obtained in last iteration. The precision of solutions depends on residual forces, i.e. the penalty forces produced during

the current iteration. The residual force should be as small as possible so that a precise contact force can be obtained. It is an attractive method because from a physical point of view, the solution of the ALM is independent of the penalty value P , and it retains the simplicity of the penalty method because the size of the system matrix does not increase.

Consider again, two blocks i and j in contact, point P_1 is the vertex of block i while P_2P_3 is the boundary of block j , and P_1 penetrates into the edge P_2P_3 . A Lagrange multiplier λ^* is introduced to approximate the contact force at the contact point and the penetration depth is represented by d . The ALM does not increase the size of the system by iteratively calculating the Lagrange multiplier λ^* .

$$\lambda_{k+1}^* = \lambda_k^* + pd_k \quad (7-26)$$

λ_k^* is the Lagrange multiplier at the K^{th} iteration and λ_{k+1}^* is the updated Lagrange multiplier, and this formulation is suitable for both normal contact force and shear contact force. The penalty number, p , here can be variable and does not have to be a very large number as it is in the penalty method.

The strain energy of that contact force is

$$\Pi_k = \lambda_k^* d_k + \frac{1}{2} pd_k^2 \quad (7-27)$$

Eq. (7-27) can be divided into two components. The first component is the strain energy resulting from the iterative Lagrange Multiplier λ_k^* and the derivation of the contribution of this part is the same as the classic Lagrange multiplier method, the difference is that λ_k^* values are known and calculated from the previous iteration. The second component is the strain energy associated with the stiff spring and its contribution to the global equations can be found in Shi (1988). The difference part as compared with the original DDA method is that in the ALM, the normal contact force is not expressed by the penalty force pd_n but the Lagrangian multiplier λ^* . Hence, the friction sub-matrices need to be modified.

The friction force can be expressed as follows:

$$f = \lambda_m^* \tan \varphi \quad (7-28)$$

here, $\tan \varphi$ is the sliding friction coefficient. Refer to Figure 7-1 and let (x_i, y_i) and (u_i, v_i) be the coordinates and the displacement increments at points P_i ($i=0, 1, 2, 3$), respectively. The friction force is along edge P_2P_3 , and its directional vector is:

$$\frac{1}{l} \{ x_3 - x_2 \quad y_3 - y_2 \} \quad (7-29)$$

The friction work is done by a pair of friction forces with opposite direction on the two different contacting blocks. For block i , the potential energy of the frictional force is:

$$\Pi_f^i = \frac{f}{l} \begin{Bmatrix} x_3 - x_2 & y_3 - y_2 \end{Bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \frac{f}{l} \begin{Bmatrix} x_3 - x_2 & y_3 - y_2 \end{Bmatrix} \mathbf{T}_i(x_1, y_1) \mathbf{D}_i \quad (7-30)$$

let

$$\mathbf{H}_i^T = \frac{1}{l} \begin{Bmatrix} x_3 - x_2 & y_3 - y_2 \end{Bmatrix} \mathbf{T}_i(x_1, y_1) \quad (7-31)$$

then

$$\Pi_f^i = f \mathbf{H}_i^T \mathbf{D}_i \quad (7-32)$$

$$\mathbf{f}_{i(6 \times 1)}^f = - \left. \frac{\partial \Pi_f^i}{\partial \mathbf{D}_i} \right|_0 = -f \mathbf{H}_i = -\lambda_m^* \tan \varphi \mathbf{H}_i \quad (7-33)$$

for block j , the friction potential energy is:

$$\Pi_f^j = -\frac{f}{l} \begin{Bmatrix} x_3 - x_2 & y_3 - y_2 \end{Bmatrix} \begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix} = -\frac{f}{l} \begin{Bmatrix} x_3 - x_2 & y_3 - y_2 \end{Bmatrix} \mathbf{T}_j(x_0, y_0) \mathbf{D}_j \quad (7-34)$$

Let

$$\mathbf{H}_j^T = \frac{1}{l} \begin{Bmatrix} x_3 - x_2 & y_3 - y_2 \end{Bmatrix} \mathbf{T}_j(x_0, y_0) \quad (7-35)$$

then

$$\Pi_f^j = -f \mathbf{H}_j^T \mathbf{D}_j \quad (7-36)$$

$$\mathbf{f}_{j(6 \times 1)}^f = - \left. \frac{\partial \Pi_f^j}{\partial \mathbf{D}_j} \right|_0 = f \mathbf{H}_j = \lambda_m^* \tan \varphi \mathbf{H}_j \quad (7-37)$$

$\mathbf{f}_{i(6 \times 1)}^f$ and $\mathbf{f}_{j(6 \times 1)}^f$ are 6×1 sub-matrices and will be added to the $[F_i]$, $[F_j]$ in the global matrix, respectively.

7.3 Implementation and validation of ALM

Although the employment of ALM in the DDA method can be found in many previous research works, few detail study on its implementation and computational efficiency have been provided. Lin (1995, 1996) first implemented the ALM into the standard DDA

method with a variable penalty number function. In each time step and all iterations within that time step, the optimal penalty number was found for all contacts automatically. However, no detail information of how to find the optimal penalty number is given. Ning (2010) also applied the ALM into the DDA method and verified it with two static problems, however, these examples cannot show whether the modified program can still work properly in dynamic problems where contacts are created or removed frequently.

Actually, in the authors' opinion, the most difficult part in the ALM implementation is on how to couple the O-C iteration and the ALM procedure. This section will provide some detailed information on the implementation of the ALM into the DDA program with considering the modification of the O-C iteration. Five examples are used to validate the ALM, as well as comparing it with the standard DDA on the computational efficiency and contact precision.

7.3.1 Implementation of ALM

To enhance the DDA contact analysis with ALM, an ALM iterative procedure needs to be inserted into the time step iteration. One key question is whether the ALM iteration should be placed outside the O-C iteration or inside the O-C iteration? Both schemes are feasible but have their own advantages and disadvantages.

Putting the ALM iteration outside the O-C iteration will not increase the O-C iteration computational burden. This scheme ignores the penetration error of contacts and assumes that the error will not affect the final results of the O-C iteration. But running the ALM iteration after the O-C iteration may change the real contact status unexpectedly, which can be viewed as a contact error. If the contact spring stiffness is large enough and the penetration error is small, this kind of contact error may be not significant.

In this report, the putting-inside scheme is adopted (Figure 7-2). Since the ALM iteration is executed in the O-C iteration, the computational burden of this scheme is heavier than the putting-outside scheme. But the penetration error can be largely avoided so that a precise contact status can be achieved, and the stiffness of contact spring can be relatively softer than that in the putting-outside scheme.

Actually, the ALM iteration shares a lot of codes with the O-C iteration in the program. The major difference between an ALM iteration and an O-C iteration is that the former checks the residual force of every contacts instead of the contact status while the latter does not check the residual force but checks and updates the contact status and compares it with the last O-C iteration. Hence, the ALM iteration deals only on how to reduce penetration error in the current O-C iteration and the O-C iteration deals only on contact convergence. Once the ALM iteration reaches its converging condition, it will transfer the control to O-C iteration and update the contact status.

A single penetration control scheme is employed in O-C iterations, the advantage of using this converging criterion is that every penetration can be minimized even for the case of a few significant large contact forces among a lot of small contact forces. In this scheme, it is necessary to record the initial penetration distance at the first ALM iteration to be used in the following equation to control the ALM convergence.

$$\frac{|d_{(k)} - d_{(k-1)}|}{|d_{(1)}|} \leq \varepsilon \quad (7-38)$$

Here, $d_{(k-1)}$ and $d_{(k)}$ are the penetration distances for the last ALM iteration and the current ALM iteration, respectively; d_1 is the initial penetration distance at the first ALM iteration; ε is the tolerance which is a positive number specified by the user. In general, a smaller ε will provide a higher precision in the analysis. However, if the tolerance is too small, the ALM iteration may be increased significantly, and lead to large amount of unnecessary computation work. A value of $\varepsilon = 0.05$ is used in the following examples, which show a good balance between precision and computational efficiency. In the standard DDA method, the contact status will be checked one by one in O-C iteration, and the contact status will switch among three cases: locked, sliding and opened. If all contact status have no change over the last two iterations, the O-C iteration is considered to be converged. To check the contact status, the penetration distance is used. If a positive penetration distance is obtained, the contact is under tension. Once the tension is larger than the joint tensile strength, the contact is to be marked by ‘opened’. If the penetration distance is negative, i.e. invasion exists; the contact is marked by ‘closed’. However, in the ALM enhanced DDA, the meaning of penetration distance is different from the standard DDA method. A positive penetration does not necessarily mean a tensile contact force since the real contact force is taken by the updated Lagrangian multiplier and the penalty item indicates the imbalance force in the current iteration. Hence, corresponding modification is necessary for the program to catch the right contact status. The real contact force is employed in the ALM enhanced DDA as the criteria instead of the penetration distance which is used by the standard DDA. If the contact force between two blocks is tension, the contact will be marked by ‘opened’. If the contact force is compression, the contact will be marked by ‘closed’.

Figure 7-2 Flowchart of the ALM enhanced DDA

7.3.2 Validation of ALM: Penalty method vs. Augmented Lagrangian method

In the work of Shi (1988), the penalty parameter is recommended to have a value from $10E$ to $1000E$, where E is Young's modulus of the block material. This requirement can guarantee that the penetration distance is 10^{-3} to 10^{-4} times of the block displacement in a

time step and normally, a value of $40E$ is used. The higher the penalty number, the smaller the penetration is. However, a too high penalty value will cause ill-conditioning of the coefficient matrix of the governing equation. So far, what is an optimal selection of the penalty value in the standard DDA method is still an open question. The newly developed ALM enhanced DDA can avoid the trial error process for choosing a proper penalty value without losing too much computational efficiency.

Five numerical examples are employed in this section to compare the computational efficiency of the standard DDA (STD) and ALM enhanced DDA (ALM). The first three examples are quasi-static problems and the last two are dynamic problems.

Example 1

Figure 7-3(a) shows a system loaded by displacement boundary: the upper rigid loading plate is given a downward displacement δ toward the lower plate. The time history of the upper plate displacement is shown in Figure 7-4. All blocks, except the loading plates, have the same Young's modulus. The left and the right blocks have the same configuration while the middle block is divided into 8 small blocks. If there is no penetration between blocks, the division of the middle column will not affect the stress distribution, and the y-direction normal stress should be the same inside each block. There are three independent force transferring paths as shown in Figure 7-3(b). In the penalty method, the more contact interfaces exist on the force transferring path, the larger accumulated penetration is. The penetration between blocks is equivalent to the reduction of the compressive distance of the combinational column, which will reduce the final normal strain along the compressive direction in the column.

Figure 7-3 Three-column model with 8 layers in middle column: (a) configuration; (b) force transferring paths.

Figure 7-4 Time-displacement history of the loading boundary

The simulation results agree well with the above analysis. The error of the y-direction normal stress produced in the middle column by the standard DDA is much larger than that in the single block columns (Table 7-3). The results from the ALM enhanced DDA show a good precision for all columns.

The ALM enhanced DDA shows a higher speed than the standard DDA when the penalty value is larger than 40E and the direct equation solver is used. However, when SOR iteration solver is employed, the ALM shows a slower computational speed than the standard DDA in all three cases.

Table 7-1 DDA input parameters of example 1 and example 2

Quasi-static	Yes
Plain stress	Yes
Time step size (s)	0.003
Total time step	40
Mass per unit(Kg/m ³)	2300
Young's modulus(GPa)	10
Poisson ratio	0.25
Max disp. ratio	0.01
Joint friction angle	0
Joint cohesion	0
Joint ensile strength	0
SOR factor	1.4

Table 7-2 Y-displacements of measure points of example 1

contact stiffness	measure point	y-displacement(m)				
		analytical	STD	Error (%)	ALM	Error (%)
10E	MP1	-0.002	-1.9749E-03	1.26	-1.9993E-03	0.04
	MP2	-0.002	-1.9768E-03	1.16	-1.9993E-03	0.04
	MP3	-0.002	-1.9749E-03	1.26	-1.9993E-03	0.04
40E	MP1	-0.002	-1.9936E-03	0.32	-1.9998E-03	0.01
	MP2	-0.002	-1.9937E-03	0.32	-1.9998E-03	0.01
	MP3	-0.002	-1.9936E-03	0.32	-1.9998E-03	0.01
100E	MP1	-0.002	-1.9974E-03	0.13	-1.9999E-03	0.01
	MP2	-0.002	-1.9975E-03	0.13	-1.9999E-03	0.01
	MP3	-0.002	-1.9974E-03	0.13	-1.9999E-03	0.01

Table 7-3 Y-stress of measure points of example 1

contact stiffness	measure point	y-displacement(m)				
		analytical	STD	Error (%)	ALM	Error (%)
10E	MP1	-2.50E+7	-2.4401E+7	2.40	-2.5012E+7	-0.05
	MP2	-2.50E+7	-2.2478E+7	10.09	-2.5012E+7	-0.05
	MP3	-2.50E+7	-2.4401E+7	2.40	-2.5012E+7	-0.05
40E	MP1	-2.50E+7	-2.4862E+7	0.55	-2.5018E+7	-0.07
	MP2	-2.50E+7	-2.4332E+7	2.67	-2.5018E+7	-0.07
	MP3	-2.50E+7	-2.4862E+7	0.55	-2.5018E+7	-0.07
100E	MP1	-2.50E+7	-2.4958E+7	0.17	-2.5021E+7	-0.08
	MP2	-2.50E+7	-2.4742E+7	1.03	-2.5021E+7	-0.08
	MP3	-2.50E+7	-2.4958E+7	0.17	-2.5021E+7	-0.08

Table 7-4 Comparison of the computational efficiency of standard DDA and ALM-DDA for example1

Contact stiffness	method	Direct solver		SOR solver			
		O-C iteration	Solve times(ms)	O-C iteration	Solve times(ms)	SOR iteration	Time per SOR(ms)
10E	STD	59	109	60	452	9,742	0.046
	ALM	166	156	186	1482	37,079	0.040
40E	STD	57	171	66	562	13,114	0.043
	ALM	109	140	292	2230	58,416	0.038
100E	STD	57	156	80	655	15,929	0.041
	ALM	106	141	405	3026	81,132	0.037

Example 2

The numerical model is similar with that employed by example 1 but with the middle block being divided by 32 layers. The input parameters and loading are the same with example 1 as shown in Table 7-1.

Computational results are listed in Table 7-5, Table 7-6 and Table 7-7. The stress error of the middle block produced by the penalty method is unacceptable in all three cases, while the error of the ALM enhanced DDA is still very small. In both equation solvers, the

ALM enhanced DDA uses less computational time than the standard DDA when the penalty value is low (10E).

Based on the results in both example 1 and example 2, it can be observed that the ALM enhanced DDA has a higher computational speed than the standard DDA under a low penalty value when SOR equation solver is used.

Figure 7-5 Three-column model with 32 layers in middle column

Table 7-5 Y-displacements of measure points of example 2

contact stiffness	measure point	y-displacement(m)				
		analytical	STD	Error (%)	ALM	Error (%)
10E	MP1	-0.002	-1.9749E-03	1.26	-1.9993E-03	0.04
	MP2	-0.002	-1.9816E-03	0.92	-1.9993E-03	0.04
	MP3	-0.002	-1.9749E-03	1.26	-1.9993E-03	0.04
40E	MP1	-0.002	-1.9936E-03	0.32	-1.9998E-03	0.01
	MP2	-0.002	-1.9941E-03	0.30	-1.9998E-03	0.01
	MP3	-0.002	-1.9936E-03	0.32	-1.9998E-03	0.01
100E	MP1	-0.002	-1.9974E-03	0.13	-1.9999E-03	0.00
	MP2	-0.002	-1.9975E-03	0.13	-1.9999E-03	0.00
	MP3	-0.002	-1.9974E-03	0.13	-1.9999E-03	0.00

Table 7-6 Y-stress of measure points of example 2

contact stiffness	measure point	y-displacement(m)				
		analytical	STD	Error (%)	ALM	Error (%)
10E	MP1	-2.50E+7	-2.4402E+7	2.39	-2.5012E+7	-0.05
	MP2	-2.50E+7	-1.7698E+7	29.21	-2.5012E+7	-0.05
	MP3	-2.50E+7	-2.4402E+7	2.39	-2.5012E+7	-0.05
40E	MP1	-2.50E+7	-2.4870E+7	0.52	-2.5026E+7	-0.10
	MP2	-2.50E+7	-2.2681E+7	9.28	-2.5026E+7	-0.10
	MP3	-2.50E+7	-2.4870E+7	0.52	-2.5026E+7	-0.10
100E	MP1	-2.50E+7	-2.4966E+7	0.14	-2.5028E+7	-0.11
	MP2	-2.50E+7	-2.4035E+7	3.86	-2.5028E+7	-0.11
	MP3	-2.50E+7	-2.4966E+7	0.14	-2.5028E+7	-0.11

Table 7-7 Comparison of the computational efficiency of standard DDA and ALM-DDA for example 2

Contact stiffness	method	Direct solver		SOR solver			
		O-C iteration	Solve times(ms)	O-C iteration	Solve times(ms)	SOR iteration	Time per SOR(ms)
10E	STD	132	687	568	14,383	114,133	0.126
	ALM	128	250	465	11,903	93,363	0.127
40E	STD	79	234	1217	31,076	242,928	0.128
	ALM	118	265	744	18,564	149,491	0.124
100E	STD	58	203	1492	36,956	291,447	0.127
	ALM	118	265	1556	38,672	301,228	0.128

Example 3

The example presented here is a two dimensional cantilever beam subjected to point load. This example is the same as the one studied in section 4.3.3, and the configuration and the mesh detail of the problem can be found in Figure 4-18 and Figure 4-19, respectively. The input parameters are listed in Table 7-8.

Table 7-8 DDA input parameters of example 3

Quasi-static	Yes
Plain stress	Yes
Time step size (s)	0.0005
Max disp. ratio	0.0001
Mass per unit(Kg/m ³)	2500
Young's modulus(GPa)	20
Poisson ratio	0.25
Spring stiffness (GPa)	200
Joint friction angle	70
Joint cohesion	30e6
Joint ensile strength	15e6
SOR factor	1.4

Figure 7-6 shows the beam deflection obtained from standard DDA (DDA-S), ALM enhanced DDA and the analytical solution. As it can be seen, the standard DDA results in a bigger beam deflection compared with the ALM enhanced DDA. This is because that when the entire domain is discretized into sub-blocks, too many internal artificial joints will be introduced into the system, even with a suitable penalty value, small amount of penetration of the contact springs are inevitable. The accumulated penetration leads the block to behave “softer” than expected. While in the ALM enhanced DDA, as the contact force is taken by the Lagrangian multiplier and the penalty force, the penetration is greatly reduced, thus the block will behave “stiffer” than the standard DDA. In general, though the computational error increases along the beam length for both methods, the ALM enhanced DDA obtains a more accurate deflection value than the standard DDA.

Figure 7-6 Deflection along the beam

Table 7-9 and Table 7-10 show the simulation results of σ_x and τ_{xy} corresponding with different numerical methods as well as the error compared with the analytical solution, respectively. The σ_x obtained from ALM enhanced DDA is similar with that obtained from standard DDA while the τ_{xy} obtained from ALM-enhanced DDA is more accurate than that obtained from standard DDA in most cases.

Table 7-9 Simulation results of normal stress and error for different measure points

Measurement point	Coordinate	Analytical] (KPa)	DDA (KPa)	Error (%)	ALM-DDA (KPa)	Error (%)
1	(1.2656,1)	258.2	248.8	3.64	247.5	4.14
2	(2.4315,1)	220.3	212.0	3.77	211.0	4.22
3	(3.7973,1)	183.2	175.2	3.89	174.4	4.86
4	(5.3163,1)	136.4	131.0	3.96	130.5	4.33
5	(6.5821,1)	98.7	93.7	5.07	93.8	4.96
6	(7.8479,1)	60.8	56.4	5.92	57.1	5.71
7	(9.1137,1)	22.8	21.1	7.46	21.3	6.58

Table 7-10 Simulation results of shear stress for different measure points

Measurement point	Coordinate	Analytical (KPa)	DDA (KPa)	ALM-DDA (KPa)
1	(1.2656,1)	0	-1938.9	-2385.4
2	(2.4315,1)	0	-1926.5	-2009.9
3	(3.7973,1)	0	-1881.8	-1667.2
4	(5.3163,1)	0	-1826.7	-1243.2
5	(6.5821,1)	0	-1839.4	-892.5
6	(7.8479,1)	0	-1700.0	-543.4
7	(9.1137,1)	0	-1403.2	-119.7

Figure 7-7 shows the stress time-history obtained from the two methods at the measurement point (5.3163, 1). The solution obtained from the DDA method fluctuates greatly and converges gradually after 10 seconds while the solution obtained from the NDDA method has almost no fluctuation and get convergent slowly after 20 seconds.

Figure 7-7 Time-stress history of measurement point (5.3163, 1)

Example 4

Previous three examples are quasi-static problems, and the ALM enhanced DDA shows a good precision. At low penalty cases, the ALM enhanced DDA even has a higher computational speed than the standard DDA. In this example, the computational efficiency of the ALM enhanced DDA when analyzing dynamic problems is tested.

The model is 70 meters wide and 70 meters high with 421 blocks and four fix points, as shown in Figure 7-8. The small blocks on the slope have dimension of 1×1 m. Two kinds of contact spring stiffness, 10E and 40E, are employed to study the effect of penalty value on the computational speed of both methods. The analysis input parameters are listed in Table 7-11.

The simulation results from both standard DDA and ALM enhanced DDA are shown in Figure 7-9 and Figure 7-10, respectively. There is a tiny difference between the results but all results capture the right phenomenon of the problem. There is no analytical result to be used to judge which one is more precise, it does not matter since the purpose of this example is to test the computational efficiency of both methods when dealing with dynamic problems under different contact stiffness. The computational speed is compared in Table 7-12.

Figure 7-8 Schematic illustration of the slope model

Figure 7-9 Simulation results from the standard DDA

Figure 7-10 Simulation results from the ALM enhanced DDA

Table 7-11 DDA input parameters of example 4

Plain stress	Yes
Time step size (s)	0.00005
Total time step	200000
Max disp. ratio	0.01
Mass per unit(Kg/m^3)	2300
Young's modulus(GPa)	10
Poisson ratio	0.25
Joint friction angle	0
Joint cohesion	0
Joint ensile strength	0
SOR factor	1.2

Table 7-12 Comparison of the computational efficiency of standard DDA and ALM-DDA for example 4

Contact stiffness	method	O-C iteration	SOR iteration	Solve time (ms)	Time per SOR(ms)
10E	STD	504,040	4,573,558	7,402,357	1.619
	ALM	503,265	5,005,778	7,154,674	1.429
40E	STD	562,674	6,165,654	8,457,906	1.372
	ALM	558,951	6,974,790	8,856,224	1.270

Example 5

During the underground tunnel excavation, jointed rock mass are often encountered and played an important role on the tunnel stability. The research done by Shi and Goodman (1985) shown that joints can make blocks near the tunnel become loose and cause local failure or significant deformation. The effects of joint attributes such as joint orientations,

spacing and friction angle on the tunnel stability were also investigated by Yeung (1997) using DDA and by Shen (1997) using UDEC.

The example presented in this section is the stability analysis of a shallow tunnel. The zones disturbed by tunnel excavation can be divided into three parts (Shen, 1997): the failure zone, the open zone and the shear zone. And here, the failure zone is the main concern.

The model is shown in Figure 7-11. The rock mass has the dimension of 60 meter in length and 45 meter in height. The tunnel is horseshoe-shaped with 10 meter wide in flat invert, 5 meter height in vertical side walls and 10 meter diameter in semi-circular roof. The tunnel is excavated at the depth of 15 meter, in the middle of the rock mass model. Two joints set are used, joint set 1 has a dip angle 0°, joint direction left and joint spacing 5 meter, joint set 2 has a dip angle 30°, joint direction right and joint spacing 5 meter. The rock mass model is bounded by a fixed U-shaped block, and subjected to gravity load only.

Figure 7-11 Horseshoe-shaped tunnel with two sets of joints

Table 7-13 DDA input parameters of example 4

Type of analysis	Dynamic
Plain strain	Yes
Time step size (s)	0.005
Total time step	1600
Mass per unit(kg/m^3)	2650
Young's modulus(GPa)	20
Poisson ratio	0.25
Max disp. ratio	0.005
Joint friction angle	25
Joint cohesion	0
Joint tensile strength	0

Both standard DDA and ALM enhanced DDA are used in the simulation, two kinds of contact spring stiffness, $10E$ and $0.1E$ are employed to study the effect of the penalty value on the simulation results. The analysis input parameters are listed in Table 7-13.

The simulation results obtained from the standard DDA and the ALM enhanced DDA with different penalty values are depicted from Figure 7-12 to Figure 7-15. As it can be seen, local failure occurs for all the four cases--this result can be expected as the joint dipping angle is steeper than its friction angle. The behaviors of the tunnel are different with different penalty values when the standard DDA is involved: when the penalty value equals to $10E$, it is clear that significant deformation appears, and no interpenetration occurs during the deformation (see Figure 7-12); however, when the penalty value equals to $0.1E$, the small penalty value fails to satisfy the contact constraint and interpenetration occurs during the deformation (as highlighted in Figure 7-13). In case of ALM enhanced DDA, though slightly different exists, the behaviors of the tunnel are similar under different penalty values. Even a small penalty value is applied, the contact constraint is satisfied and no interpenetration occurs during the block contact.

Figure 7-12 Behavior of the tunnel system after 400, 800, 1200, 1600 time steps with standard DDA (penalty value= $10E$)

Figure 7-13 Behavior of the tunnel system after 400, 800, 1200, 1600 time steps of 0.005sec with standard DDA (penalty value=0.1E)

Figure 7-14 Behavior of the tunnel system after 400, 800, 1200, 1600 time steps of 0.005sec with ALM enhanced DDA (penalty value=10E)

Figure 7-15 Behavior of the tunnel system after 400, 800, 1200, 1600 time steps of 0.005sec with ALM enhanced DDA (penalty value=0.1E)

The computational speed is also compared in Table 7-14. Due to the great increase of O-C iteration times, the ALM enhanced DDA shows a lower computational speed for both director solver and SOR iteration solver.

Table 7-14 Comparison of the computational efficiency of standard DDA and ALM-DDA for example 4

Contact stiffness	method	Direct solver		SOR solver			
		O-C iteration	Solve times(ms)	O-C iteration	Solve times(ms)	SOR iteration	Time per SOR(ms)
0.1E	STD	1766	12,955	1,785	35,480	82,467	0.430
	ALM	19,385	131,486	20,186	388,487	944,932	0.411
10E	STD	5005	37,303	5,389	286,916	690,706	0.415
	ALM	19,293	138,366	17,967	1,148,503	2,886,069	0.398

Consider the results listed in Table 7-4, Table 7-7, Table 7-12, and Table 7-14, the following conclusions on the computational efficiency of the ALM enhanced DDA can be drawn: in most cases, the increasing of contact spring stiffness, i.e. the penalty value, will increase the O-C iteration number and the SOR iteration number. Consequently, this will increase the total computational time. This conclusion is true for both quasi-static and dynamic analysis when SOR equation solver is involved. The only exception is the fifth example where the contact stiffness is 10E. This conclusion can also be found in the standard DDA, with one exception for the first example when the contact stiffness is 100E.

The reason why increasing penalty value will also increase the SOR iteration is that a too large penalty value will cause ill-conditioning of the global stiffness matrix, which makes the convergence of the SOR more difficult than a small penalty value. Hence, the choice of penalty should consider two aspects. On one hand, the penalty value should be as large as possible so that the inter-block penetration can be controlled under an acceptable level. On the other hand, the penalty should be as small as possible to reduce the ill-conditioning of the coefficient matrix of the global equation.

7.4 Conclusion

In this chapter, the ALM is implemented in the standard DDA method, and four validation examples are carried out. In the standard DDA method, the penalty parameter affects not only the penetration distance but also the contact detection and the O-C iteration. While in the ALM enhanced DDA, the requirement of a super large penalty number is not needed anymore, it can often give a good precision and high computational efficiency when employ a low penalty number. As the NDDA method and the standard DDA method shares the same contact mechanism, the new implemented algorithm is also applicable to the NDDA method.

At the same time, the newly developed ALM enhanced DDA has a high precision and computational efficiency for large engineering problems where the SOR equation solver are involved. Since the penetration between blocks is no longer controlled by the penalty value in the ALM enhanced DDA, a low penalty value is recommended. As it can be seen

from the examples, a low penalty value will have less O-C iterations and SOR iterations. Hence, the total computational efficiency is higher under a low penalty value than under a high penalty value.

Chapter 8. Conclusions and recommendations

8.1 Conclusions

This research aims to develop and extend the NDDA method for rock engineering analyses. As a coupled method of the DDA and the FEM, the NDDA method chooses the nodal displacements as variables, and establishes the equilibrium equations by minimizing the total potential energy, similar to the FEM. The advantages of the NDDA method are: a) the discontinuities, such as fractures and joints in the rock mass, can be easily incorporated into the numerical model; b) with the help of the finite element mesh, the displacement and stress fields within each block can be evaluated more accurately; c) with the introduction of fracturing criterion along the FEM mesh lines, new cracks can be generated.

In this thesis, the basic theory of the NDDA method is reviewed, its differences and similarities with other numerical methods are summarized in Chapter 2. The slope stability analysis is then carried out in Chapter 3 to validate the newly developed method. Chapter 4 introduces a double-minimizing procedure into the NDDA method to better approach the crack analysis, and Chapter 5 demonstrates the practical use of the NDDA with double minimization procedure through toppling analysis. The implementation of the rockbolt element in Chapter 6 helps the NDDA method to be a powerful and versatile method for analyzing block system and finally, an ALM contact mechanism is implemented into the standard DDA method to deal with the contact constrain. The main development works are summarized as follows:

1. The validation and application studies in Chapter 3 show the capability of the NDDA method in slope stability analysis. Compared with the traditional Limit equilibrium method (LEM), a failure in the NDDA model is just one of the simulation results and no assumptions regarding the interslice forces are needed. The solution by the NDDA method satisfies all the equilibrium conditions.
2. To make the NDDA method a more powerful tool in the rock engineering analysis, a crack propagation procedure is implemented into the NDDA program. This is a fully automatic process to describe the crack initiation and propagation, so no tracing or remeshing is needed. In this crack propagation procedure:
 - The inter-element forces are computed first based on the double minimization method, and the obtained inter-element forces are then checked against the failure criterion to see whether the crack will occur in this time step.
 - Both the maximum tensile strength criterion and the Mohr-Coulomb failure criterion are incorporated in the NDDA program to allow the tensile and shear failure.

- A fracture sequence based on the failure time of each mesh line is considered and only one crack is allowed to open/slide within each time step.
 - A simple joint strength reduction model is applied for the new born crack: after fracturing, the tensile strength of the crack will be reduced to zero, the internal friction angle will be replaced by the joint friction angle, and the cohesion will be a small value to account for irregularity of the fracture surface in reality.
3. By evaluating the stress-tensor at the finite element nodes, a more accurate nodal stress state can be obtained at the same time. Three classical problems with analytical solutions are chosen to demonstrate the effectiveness of the stress recovery procedure, and the simulation results show the better accuracy of the NDDA method with stress recovery procedure.
4. A detailed comparison on the computational efficiency, convergence speed and the deformation ability between the sub-block DDA and the NDDA method is carried out. The comparison study shows that the NDDA method is superior than the sub-block DDA method in many aspects:
- The computational efficiency of the NDDA method is higher than that of the sub-block DDA method.
 - The convergence speed of the NDDA method is faster than that of the sub-block DDA.
 - The displacement field predicted by the NDDA method is more accurate than that by the sub-block DDA method.
5. The NDDA method with double minimization procedure is applied to the analysis of the toppling failure, and the NDDA results are compared with the physical tests:
- In the case of the block toppling or block-flexural toppling, the simulation results provided by the NDDA method are able to give the relative movement along the pre-defined discontinuities as well as to capture the right failure mode.
 - In the case of flexural toppling, the NDDA method is able to predict the rupture surface inside the rock mass accurately and to provide the deformation pattern of the rock slope close to the physical tests.
6. Another important contribution of this thesis is the development of the rockbolt element in the NDDA framework.
- A finite different approach is implemented in the NDDA program to analyze the rock/rockbolt interaction. The rockbolt is simulated as the bar element which does not possess independent degrees of freedom.
 - The axial behavior of the rockbolt is governed by an elastic-perfectly plastic constitutive model, with both loading and unloading capability.

- When the rockbolt crosses a discontinuous joint, an interface bolt segmentation is incorporated.
- Four types of rockbolt, namely, mechanical anchored rockbolt, energy-absorbing D-bolt, fully grouted rockbolt and frictional rockbolt, are realized by assigning different tangential stiffness k_s for the corresponding anchor points.
- Three types of failure modes, namely tensile failure of the rockbolt, debonding along the rockbolt/rock interface and loss of the faceplate/anchor points are incorporated.

The developed rockbolt models are used to investigate the rockbolt and rock mass behaviors through a few numerical tests, including the pull-out test, shear test, and the slope stabilization analysis.

8. In the last part of the thesis, an alternative contact constraint mechanism, Augmented Lagrangian method (ALM) is investigated under the DDA framework. The coupling of the open-close iteration and the ALM iteration are discussed. Validation examples show that the ALM enhanced DDA is less restricted by the choice of the penalty value. Considering both the efficiency and the accuracy, a relative low penalty value is recommended for the ALM based DDA method.

8.2 Recommendations for future work

In determining the applicability of the NDDA method to the rock engineering application, the limitations inherent in it should not be overlooked. The performance of the NDDA method is sensitive to the choice of the input parameters, even an experienced user needs to spend a lot of time in trial and error to get the most suitable parameters. The advantages associated with the implicit time integration scheme are not well demonstrated in the NDDA method, the select of the time step interval is still restricted by the element size and other relevant parameters. At the same time, as the solution of the implicit method requires extra computation, the NDDA does not exhibit better computational efficiency compared to DEM in many cases. What's more, the NDDA is currently restricted to two-dimensional representations, and the employment of only triangular mesh limits the stress distribution in each element to constant.

More extensions and improvements in the NDDA methods are still needed. Some of the potential development works includes:

1. The “crack propagation procedure” implemented in Chapter 4 shows the NDDA's capability in predicting the development of the cracks. To further improve the capability of the proposed method, the following work can be considered:

- Considering a non-linear constitutive model for material damage to realistically represent the failure behavior of the rock mass.

- Simulating the energy dissipation during the crack propagation by correlating it with material fracture energy and the fracture dimension.
2. The application of the NDDA method in the slope stability analysis and toppling analysis demonstrate the potential of the NDDA method in rock engineering. However, more work needs to be done before applying the method in general practice:
- Incorporating alternative joint constitutive model, such as Barton-Bandis model to describe the effect of loading process and joint damage.
 - Incorporating the hydro-mechanical interaction in the numerical model
3. The rockbolt model developed in Chapter 6 has been successfully used in the analysis of the behavior of different types of rockbolts under static situation. The work can be extended into the following areas:
- Calibrating the proposed model further by comparing its results with laboratory observations and other available numerical simulations, as well as the analytical predictions.
 - Testing the performance of proposed model in the dynamic scenarios, such as rock dilation and rock burst events as well as identifying the advantages and limitations of various types of rockbolts in case of dynamic situation.
 - Working together with the empirical rules and rock classification system to provide the site-specific and problem-specific rock bolting design.
4. The response of rock mass under dynamic loadings is by far, an important concern in geotechnical engineering worldwide. To extend the NDDA method as an efficient and suitable tool for dynamic problems, several modifications of the current method are necessary:
- Optimizing the implementation of the blast loads and incorporating a blast load transfer model to the NDDA method.
 - Implementing a non-reflection boundary condition which allows wave propagation through the NDDA boundary without reflection to solve the problems in infinite and semi-infinite mediums.

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List of Publications

- Journal paper:

Bao H. R., Zhao Z. Y., Tian Q., (2013). On the Implementation of Augmented Lagrangian Method in the 2D Discontinuous Deformation Analysis. International Journal for Numerical and Analytical Methods in Geomechanics. (Accepted)

Tian Q., Zhao Z.Y., Bao H. R., (2013). Block fracturing analysis using Nodal-Based Discontinuous Deformation Analysis with the double minimization procedure. International Journal for Numerical and Analytical Methods in Geomechanics. (Accepted)

Tian Q., Zhao Z. Y., He L., Sun J. P., (2013). Rock slope stability and stabilization analysis using the NDDA method. (Under preparation)

- Conference paper

Zhao Z.Y., Bao H.R., Tian Q., (2012). On the implementation of augmented lagrangian method on the 2D discontinuous deformation analysis. Advances in Discontinuous Numerical Methods and Application in Geomechanics and Geoengineering. Pp181-188.

Tian Q., Zhao Z.Y., Bao H.R., (2012). Block Fracturing Analysis Using Nodal-Based Discontinuous Deformation Analysis with Stress Recovery Procedure. Conference of Advances in Underground Space Development, Singapore, Pp. 119-121.

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