Dynamic pricing for perishable assets and multiunit demand

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Dynamic Pricing for Perishable Assets and Multiunit Demand

A THESIS

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Abstract

With the widespread application of dynamic pricing strategies across a variety of industries, the traditional dynamic pricing is usually implemented by coupling with technique from other disciplines. Thus, in this dissertation, we analyze three dynamic pricing problems in the context of nonuniform pricing from economics, supply chain, and sales effort from marketing respectively.

Motivated by simultaneous multi-unit demand and customer choice behavior in the retailing industry, we first endogenize the purchase quantity and study the problem of dynamic pricing of limited inventories over a finite horizon to maximize expected revenues. We examine three types of dynamic pricing schemes: the dynamic nonuniform pricing (DNP) scheme, the dynamic uniform pricing (DUP) scheme, and the dynamic block pricing (DBP) scheme. For DNP scheme, we have identified a necessary and sufficient condition for the structural properties of optimal policy. The relationship among these three schemes is examined and the magnitude of revenue impact for these schemes is explored.

Second, we study a supply chain with one supplier and a retailer where the retailer practices dynamic pricing. Compared to the decentralized system, we find the centralized one is a Pareto improvement in terms of profit and consumer surplus. Moreover, we develop a stylized approach to evaluate various supply chain contracts, and find a necessary and sufficient condition for an independent contract to coordinate the system. Extensive numerical experiments are conducted to evaluate the values of pricing flexibility and coordination.

Chapter 4 addresses the problem for a firm that dynamically adjusts both effort and price for selling limited quantities of product before some given time. We model the retailer's problem as

a dynamic program, in which both the revenue from selling the product and the cost for exerting sales effort are embedded in each period. We characterize the optimal effort and price as functions of the inventory level and the remaining selling time. Furthermore, we demonstrate that the optimal effort level is increasing with the remaining inventory and decreasing with the remaining selling time, regardless of whether the retailer revises the price dynamically or not.

Finally, we summarize and give some future research directions.

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Chapter 1

Introduction

1.1 Overview

Recent years have witnessed the widespread application of dynamic pricing strategies across a variety of industries (Talluri and van Ryzin 2004). Several factors contribute to the rapid growth of dynamic pricing. The most important reason is always the profit. A recent McKinsey study (Marn et al. 2003) estimates that for a typical S&P 1500 company, a 1% improvement in pricing can lead to an 8% improvement in profits. Moreover, as for the fashion industry, retail managers face rapid changes in customers' preferences and hence the short selling period highlights the importance of better management of inventory through dynamic pricing. Third, advances in information technology (e.g., e-commerce) have made it possible to track sales and inventory, as well as adjust prices with negligible cost. Fourth, decision support systems allow firms to have extensive reach to customers, collect market data, learn about customer behavior and change prices dynamically.

While these industries are enjoying the benefit of dynamic pricing, managers often encounter new problems during the application of dynamic pricing technique. On the one side, these problems raise challenges to the existing decision support systems, on the other side, they also provide new research opportunities for the researcher. For example, while most of the research on revenue management focuses on single unit demand, managers from the fashion retailing industry (e.g., G2000, Gap) often face two or more units demand in practice. Some attempts have

been tried by using compound Poisson to model the underlying customer's purchasing process. Nevertheless, a further thinking puts this simplification into question; because how many units a customer purchases depends not only on the decision of purchase-or-not but also on the price itself. Intuitively, when the price is low, the customer is willing to buy more units; otherwise the customer only purchases one unit or makes no purchase. This indicates that the study of multi-unit demand in dynamic pricing must be coupled with customer choice model, which is basically a dynamic nonuniform pricing problem.

Another problem is the research for the supply chain with dynamic pricing retailer. For the centralized system, as shown in Zhao and Zheng (2000), the optimal initial inventory is well established as long as the procurement cost is linear or convex. However there is a need to study the decentralized system where the retailer acts as a dynamic pricing newsvendor. Because the decisions on production and sales of the product are often made by different entities; for instance, Sport Obermeyer sells its products through a network of over 600 retailers. Furthermore, it is well known that double marginalization leads to inefficiency of the system. Thus, it is important to study the supply chain coordination problem for such a system.

The third problem is how to coordinate the retailer's sales effort (e.g., advertisement) and pricing decision. Traditional revenue management only considers the influence of pricing to coordinate the demand and inventory; however it has long been acknowledged that retailers' sales effort is also important in influencing demand for fashion retail products. For example, retailers can boost demand by providing attractive shelf space, guiding consumer purchases with sales personnel and operating longer hours. Hence it is important to study the impact of sales effort under traditional dynamic pricing framework.

The power of pricing is noted in Operations Management since the seminal work by Whittin (1955) who studies the single period pricing and inventory management problem for a perishable product. As Weatherford and Bodily (1992), a product or a service is called a perishable asset if there is one date before or on which the product or service is available and after which it is either not available or it obsoletes. In this thesis, we focus on the case where the capacity is fixed or there is no replenishment opportunity for the product after the sales season begins. Examples include seats for the airline or a sporting event; rooms for a hotel; fashion or high-tech goods; electricity and other utilities and online advertising time slots (see Talluri and van Ryzin 2004b for a review). Kincaid and Darling (1963) and Miller (1968) are the first papers that study the dynamic pricing problem for a perishable product. Since the deregulation of the US airline industry in the 1970s, the dynamic seat allocation problem, which basically is a dynamic pricing problem, gains popularity. Belobaba (1987), Weatherford and Bodily (1992), McGill and van Ryzin (1999), and Talluri and van Ryzin (2004b) provide comprehensive reviews for this stream of literature.

Due to the application in fashion industry, similar to Kincaid and Darling (1963)'s dynamic pricing setting, Gallego and van Ryzin (1994), Bitran and Mondschein (1997), Bitran et al. (1998), and Zhao and Zheng (2000) extend the problem by focusing on the structural properties of the optimal policy and the heuristics. Bitran and Caldentey (2003) and Elmaghraby and Keshinocak (2003) survey the related literature along this line of research. Our works belong to this stream, but are further coupled with research from other fields. To characterize customer's choice behavior among different purchase units, we bring in the classic research of nonuniform pricing in economics (e.g., Spence 1977, Goldman et al. 1984, Maskin and Riley 1984). For general reviews on this subject, refer to Tirole (1988), Wilson (1993) and Stole (2008). The

decentralized dynamic pricing system and its coordination problem are theoretically motivated by the huge research on coordination problem for fixed and price-setting newsvendors. Lariviere (1999) and Cachon (2003) provide comprehensive reviews on supply chain contracting literature. The sales effort is a classic topic in marketing, but most of the research assumes that the price is exogenous or fixed during the sales season. Basu et al. (1985) and Kok et al. (2008) review related literature. Of course, it is desirable to study these three problems in a common setting. However, due to the complexity of each problem, we study them one at a time.

Motivated by simultaneous multi-unit demand and customer choice behavior in retailing industry, Chapter 2 studies a dynamic pricing model for a retailer with limited inventories over a finite time horizon where an individual's purchase quantity is endogenous. We handle this issue by analyzing the underlying utility function; a rational customer will optimize the purchase quantity by maximizing the utility. We examine three types of intrinsically related dynamic pricing schemes: the dynamic nonuniform pricing (DNP) scheme, the dynamic uniform pricing (DUP) scheme, and the dynamic block pricing (DBP) scheme. For DNP scheme, we have identified a necessary and sufficient condition for structural properties to hold for the optimal policy. A surprising finding is that the concavity of the value function is not a necessary condition for the monotonicity of optimal price. We also give an example to show that a value function without structural properties can exhibit structural properties before some truncated time. Similar phenomena are also found under DUP scheme. Moreover, the condition for the validation of classic single-unit demand is analyzed for DUP scheme. Furthermore, we develop a novel methodology to obtain the optimal solution for DUP and DBP schemes, which not only simplify the computation process but also facilitate understanding of the underlying sales process. Finally, under some mild assumptions, we show that DNP scheme dominates DBP scheme, which

outperforms DUP scheme. It is shown that the potential revenue improvement of DNP over DUP scheme ranges from 30% to 90%. Most importantly, in our numerical studies DBP always achieve more than 97% of the revenue from DNP scheme. Hence for practical purpose, all we need is DBP scheme.

Chapter 3 studies a supply chain with one supplier and a retailer where the retailer practices dynamic pricing. Meanwhile, the retailer also faces a newsvendor problem of deciding the initial stocking level. Compared to the decentralized supply chain, we find the centralized one leads to Pareto improvement in both profit and consumer surplus. Later, we develop a stylized approach to evaluate various supply chain contracts. In particular, we find a necessary and sufficient condition for an independent contract to coordinate the underlying system. Moreover, we demonstrate the structural properties for both the revenue function and optimal pricing policy for such a contract. Extensive numerical experiments are conducted to evaluate the values of pricing flexibility and coordination. It is interesting to find that the values of pricing flexibility are similar for decentralized and centralized systems; and they mainly depend on the characteristics of market demand. As the relative variability of the heterogeneity among the customer decreases and the obsolescence rate of the good increases, the value of pricing flexibility increases and is so significant that the decentralized dynamic pricing can outperform the centralized static pricing system. Moreover, the benefit of dynamic flexibility under decentralized system is symmetrically shared between the supplier and the retailer. On the other hand, the value of coordination decreases as relative variability decreases. Furthermore, we find that the dynamic pricing policy could alleviate the competition between the supplier and the retailer, and hence the coordination is not as important as it is under static pricing one.

Chapter 4 addresses the problem for a firm that dynamically adjusts both effort and price for selling limited quantities of product before some specific time. While price is the main factor in affecting the demand, the retailer's sales effort (e.g., attractive shelf space and guiding consumer purchases with sales personnel) is also an important determinant in practice. To measure the combined impact of price and effort, one must take into account the interactions among inventory, pricing and sales effort. We model the retailer's problem as a dynamic program, where both the revenue from selling the product and the cost for exerting sales effort are embedded in each period. We characterize the optimal effort and price as functions of the inventory level and the remaining selling time. Moreover, we demonstrate that the optimal effort level is increasing with the remaining inventory and decreasing with the remaining selling time, regardless of whether the retailer revises the price dynamically or not. Even though the retailer can choose the initial price (effort), our numerical study shows that the potential profit improvement is still significant from dynamically adjusting the effort (price respectively). However there is not much benefit from simultaneously adjusting both the effort and price dynamically. Finally, we find that the value of dynamic effort is decreasing with the cost rate for the effort and the coefficient of variation of the demand, and increasing with the proportion of the potential market that is unaware of the product.

1.2 Organization of the Dissertation

To pinpoint the contribution of our work, we review literature again in each chapter. Occasionally, we refer back and forth to discuss some articles that are relevant to more than one chapter. Moreover, the notation in each chapter is self-contained.

The rest of the thesis is organized as follows. Chapter 2 studies a dynamic pricing model for perishable assets where an individual's purchase quantity is endogenous. Chapter 3 studies a decentralized supply chain with one supplier and a retailer where the retailer practices dynamic pricing, and the associated coordination problem. Chapter 4 addresses the problem for a firm that dynamically adjusts both effort and price for selling limited quantities of product before some specific time. The last chapter summarizes the main contributions of the thesis and points out some future research directions.

Chapter 2

Dynamic Pricing of Limited Inventories with Multiunit Demand

2.1 Introduction

A standard assumption for traditional dynamic pricing in revenue management (RM) is that a customer purchases at most one unit. While this assumption is valid for travel industry, it is problematic in retailing setting since customers do often purchase more than one unit and more importantly, realizing this opportunity, retailers commonly adopt promotional tools that tout sales of multiple units to propel their depressed inventories (e.g., Brandweek 2002). The ubiquitous business practice of multi-unit promotion, which entails a price reduction when customers make multi-unit purchase (e.g., Buy 2 for 20% off, 2nd piece at 50%, Now 2 for \$60), requires explicit treatment of customers' purchase quantity. The promotional issue has been intensively studied in marketing (e.g. Dolan 1987, Harlam and Lodish 1995, Foubert and Gijbrecchts 2007). Dilip and Sara (2009) highlights that customers' purchase quantity, resulting from either low price and high volume or high price and low volume, is one of the key factors for managing customers' value. Under these circumstances, the customer's decision is to choose how many units to purchase given different prices. Correspondingly, the retailer's problem is to design the nonuniform (or nonlinear) pricing scheme. The origin of nonuniform pricing in static case is from economics, for example, Goldman et al. (1984), Maskin and Riley (1984) and Tirole (1988). The main purpose of this chapter is to fill an important gap in the literature by studying

nonuniform pricing problem in dynamic setting. In the context of RM, our main contribution is to make the dynamic pricing more relevant and useful to the retailing industry.

Following the tradition from the economics literature (e.g., Spence 1977, Goldman et al. 1984 and Maskin and Riley 1984), rational consumer behavior is characterized by utility maximization. That is, given the retailer's pricing scheme, a customer makes the optimal quantity choice to maximize her utility. Motivated by practices in retailing, we examine three distinct dynamic pricing schemes in this chapter. The first one is the dynamic nonuniform pricing (DNP) scheme, which allows the retailer to dynamically and simultaneously set prices for a single unit and bundles of multiple units. Customers make optimal purchase decision among these provided bundles. This scheme captures many retailers' pricing behavior in practice (e.g., Buy 2 for 20% off). Most importantly, it is the dynamic extension of static nonuniform pricing model in economics (e.g., Goldman et al. 1984). The second type is the dynamic uniform pricing (DUP) scheme, where the retailer dynamically optimizes the unit price of the product while customers make the optimal purchase-quantity decision. It is evident that DUP model extends Gallego and van Ryzin (1994)'s single-unit demand case to multi-unit demand case. The third model is the dynamic block pricing (DBP) scheme, where the retailer dynamically and simultaneously designs the purchase quantity blocks and sets prices for these blocks. Many fashion retailers (e.g., G2000, Giordano) are implementing such block pricing scheme (e.g., 20% off up to 2 or 3 units). It is also widely used for software products, drinks and beverages, fruits, among others. This model extends block pricing literature (e.g., Leland and Meyer 1976) to dynamic setting. With the ability to handle multi-unit demand, we have substantially broadened the scope of revenue management. In particular, to our knowledge, group-pricing in revenue management has not

been properly addressed in the literature. Our models make a first step toward a better understanding of this issue.

For DNP scheme, we show that the price differences for the optimal prices of adjacent bundles are only determined by the associated maximal utility differences and the marginal expected value of the additional units. We provide a full analysis of structural properties for the optimal policy, referring to concavity of the value function and monotonicity of optimal prices with respect to both inventory and time. Specifically, a necessary and sufficient condition for the concavity of the value function is that the bundle schedule is consecutive from one. Under this condition, both the optimal prices and the associated price differences exhibit both inventory monotonicity property, that is, the optimal prices decrease in the number of left inventory and time monotonicity property, namely, the optimal prices decrease over time. Without this condition, the concavity of expected revenue function breaks down in general. Nevertheless, a value function without structural properties may exhibit monotonicity properties prior to some truncated time. Furthermore, the optimal prices may still exhibit time monotonicity property.

For DUP scheme, we identify a condition for the existence of a bounded myopic price, which implies, there exists a maximum quantity that a consumer would purchase under this scheme. Moreover, we show that the optimal price can be obtained by limiting the selection from a few price candidates. When the largest purchase quantity is bounded by two, we find that the structural properties depend on customers' utility sensitivity of the second unit over the first unit. When the utility sensitivity is weak, meaning that customers are much less willing to buy the second unit, DUP scheme degenerates to the traditional dynamic pricing of single-unit demand, which possesses the standard structural properties (Gallego and van Ryzin 1994). As the utility sensitivity increases, examples show that the concavity of the value function might break down.

However, similar to DNP scheme, the value function without structural properties may also display truncated structural properties. Moreover, the optimal price possesses both time and inventory monotonicity properties.

For DBP scheme, we first establish the existence of optimal policy consisting of the optimal block scheme and the optimal prices. Following the idea of finding the optimal price candidates in DUP scheme, we develop a novel methodology to obtain the optimal solution for DBP scheme. The comparisons of expected revenues among these three schemes are examined. Under some mild assumptions, we show DNP dominates DBP scheme, which in turn outperforms DUP scheme. A similar finding for the static case was found in Leland and Meyer (1976). When the inventory is high enough, the selling processes are the same for DBP and DNP schemes. Consequently, the expected revenues from these two schemes are the same.

The magnitude of revenue impact for these three schemes is examined through numerical examples. The potential improvement of DNP over DUP scheme ranges from 30% to 90% depending on different levels of largest purchase quantity: the more units customers are willing to purchase, the higher potential for adopting DNP over DUP scheme. Most importantly, in our numerical studies DBP always achieves almost the same revenue ($> 97\%$) as DNP scheme. Consequently, from a practical point of view, it may be enough to offer a DBP scheme.

The rest of this chapter is organized as follows. In Section 2.2, we review the relevant literature. In Section 2.3, we examine DNP scheme and its structural properties. DUP scheme and the corresponding structural properties are analyzed in Section 2.4. Section 2.5 is for DBP scheme and its solution. In Section 2.6, we provide numerical comparisons among these three schemes. The heuristics for the three schemes are developed in Section 2.7. Finally, we conclude in

Section 2.8, including managerial insights and future research directions. All proofs are provided in Section 2.9.

2.2 Literature Review

Our research is closely related to several streams of literature. The first one is the dynamic allocation of perishable resource (e.g., seats in airline industry) with different customer segments in revenue management. The structural properties, including both inventory and time monotonicity, have been well established for single-unit demand case. For a general review, see McGill and van Ryzin (1999). Here we focus on these papers with explicit consideration of multi-unit demand. Lee and Hersh (1993) first study the dynamic seat allocation problem with multi-seat demand for different booking classes in airline industry, where they note the breakdown of inventory monotonicity but report the time monotonicity of the marginal value. Brumelle and Walczak (2003) extend this model to semi-Markov process by focusing on multiple seats demand. They give a counterexample to Lee and Hersh (1993)'s claim on time monotonicity property. Moreover, they show that the time monotonicity continues to break down even if requests can be partially satisfied in the event of inventory shortage. Papastavrou et al. (1996) study the dynamic and stochastic knapsack problem (DSKP) with deadline, which serve as a general case of seat inventory control in airline industry. They give necessary conditions for ensuring the structural properties for some special cases with multi-unit demand, and provide several examples showing breakdown of structural properties if these conditions do not hold. Kleywegt and Papastavrou (2001) investigate the continuous version of DSKP with multi-unit demand and holding cost for both the finite-horizon and infinite-horizon cases. Van Slyke and Young (2000) consider the DSKP with non-homogeneous arriving rates, which is important for the travel industry. They also provide an example showing non-monotonic properties.

All these DSKP-related papers have made a common assumption that customers from different segments can be separated and hence are independent. This assumption becomes potentially problematic even in single-unit demand case (Talluri and van Ryzin 2004) and is clearly not applicable in a typical multi-unit demand retailing setting. Our research in this chapter intends to rectify this problem by incorporating customer choice behavior under different pricing schemes. To the best of our knowledge, our work is the first attempt to incorporate customer choice behavior into a dynamic RM model with multi-unit demand. Another interesting phenomenon common to those above-mentioned papers is that even though the structural properties disappear at the proximity of deadline, they seem to hold before some given remaining time. This conjecture of truncated structural properties is verified in our context, which is new to the literature.

We now turn to the literature on dynamic pricing of resource with customer choice behavior. For general literature on dynamic pricing, refer to Bitran and Caldentey (2003) and Elmaghraby and Keskinocak (2003); while Shen and Su (2009) give a review on customer behavior in RM. The dynamic pricing model of single product, such as Gallego and van Ryzin (1994), Bitran and Mondschein (1997) and Zhao and Zheng (2000), can be seen as the earliest dynamic models in RM that incorporate customer choice behavior where a customer's choice is to buy or not to buy. A major common finding for these papers is that the optimal policy exhibits both inventory and time monotonicity properties. However, a common assumption in these papers is that a customer buys either one unit of the product or none, which is restrictive to many industries, especially retailing and fashion. Our DUP model contributes to the literature by filling this gap. We also discuss the condition that makes the single-unit demand assumption appropriate.

By segmenting customers into different demand streams, Maglaras and Meissner (2006) study a multiproduct dynamic pricing problem with multidimensional demand functions that map prices into demand rates associated with a common resource. Aydin and Ziya (2008) consider the dynamic pricing of promotional product with the possibility of upselling to customers who have already purchased a regular product. Along this direction, given the information at individual level, Aydin and Ziya (2009) study the personalized dynamic pricing of limited inventories. Kuo et al. (2011) study the dynamic pricing problem with negotiating customers. Under certain regularity conditions, the structural properties for the optimal policy can be established, as demonstrated in abovementioned papers. Our dynamic nonuniform pricing and dynamic block pricing models are in line with this stream of research in the sense of a single resource with multiple customer streams. However, the different customer streams in our models arise from different purchasing quantities rather than the knowledge of customers' private information.

While the customer behavior in aforementioned dynamic pricing models is implicit, Talluri and van Ryzin (2004) explicitly incorporate a general discrete choice model into the problem of optimal control policy for a single-leg model of RM. Zhang and Cooper (2005) analyze customer choice behavior among parallel flights in the same market. Liu and van Ryzin (2008) extend Talluri and van Ryzin (2004)'s single-leg setting to network. These papers focus on the question of which product to choose, rather than what quantity or which bundle to purchase in our context. Akcay et al. (2010) is closely related to our dynamic nonuniform pricing model. They study the joint dynamic pricing problem of multiple substitutable and perishable products that are either horizontally or vertically differentiated assortments. When products are vertically differentiated in term of quality, they prove that the optimal prices possess monotonicity properties with respect to quality, inventory and time. Our research focuses on customer choice in quantity,

which hence differentiates our work from all aforementioned dynamic pricing models. Moreover, we show that both the monotonicity properties and the prices depend on the underlying business model in term of different pricing practice such as DNP, DUP or DBP and the demand characteristics captured by customer preference.

The last stream of literature is related to price discrimination. For general reviews on this subject, refer to Tirole (1988), Wilson (1993) and Stole (2008). Pigou (1920) distinguishes three kinds of price discrimination. The first-degree price discrimination is perfect price discrimination that requires perfect information on each customer's reservation value, which is unlikely in practice. In second-degree price discrimination, price varies according to purchased quantity or/and product quality, which is commonly practiced in many industries such as retailing. Akcay et al. (2010)'s vertical differentiation model can be seen as dynamic second-degree price discrimination via quality. Along this direction, our DNP and DBP models contribute to the literature by studying dynamic second-degree price discrimination via quantity. The third-degree price discrimination uses the customer's specific characteristics (e.g., age, occupation, location) to segment customers. Effective third-degree price discrimination requires that the segments have different price elasticities and can be properly separated. All DSKP-type models in the revenue management literature, such as Aydin and Ziya (2008, 2009) and Kuo et al. (2011), can be classified as dynamic third-degree price discrimination.

Handling different purchase quantities is a difficult problem in the field of operations research; hence direct literature is scarce. Hence we need to reply the theoretical development of nonlinear pricing from economics literature, which is overwhelmingly large. We here highlight a few relevant papers only. Oi (1971), Feldstein (1972), and Ng and Weisser (1974) study the two-part pricing problem, which consists of a fixed fee and a constant unit price. Leland and Meyer (1976)

analyze block pricing problem, which consists of a sequence of marginal prices for different demand blocks. Our dynamic block pricing model follows this line of research, which is a dynamic extension of their model with application in revenue management. The general nonuniform pricing problem has been examined by Spence (1977), Goldman et al. (1984) and Maskin and Riley (1984). This chapter extends this to a dynamic setting. Finally the nonuniform pricing problem is also relevant to the quantity discount problem in OM and marketing literature, such as Monahan (1984), Lal and Staelin (1984), Kohli and Park (1989), among others. Refer to Dolan (1987) for a review this topic.

2.3 Dynamic Nonuniform Pricing

In this section, we first introduce the nonuniform pricing framework motivated from economics literature. We then formulate our dynamic nonuniform pricing (DNP) model, followed by the analysis of the structural properties of the value function and the optimal prices.

2.3.1 The Customer Choice Model

To characterize consumer's quantity choice behavior, we follow the standard method in economics literature, for example, Spence (1977), Goldman et al. (1984) and Maskin and Riley (1984). It is assumed that consumer's heterogeneity is captured by a single parameter θ which varies according to certain characteristic such as taste, brand loyalty, incomes, among others. A type θ consumer's preference is characterized by the utility function $u(\theta, n)$, where n is the number of units purchased. Given the pricing schedule $p(n)$ that is the total price of n units, a consumer's optimal quantity decision is derived from optimizing her consumer surplus $v(\theta, n) = u(\theta, n) - p(n)$. By imposing some regularity conditions on $u(\theta, n)$, for example, $u_{\theta n} > 0$ in Spence (1977) and similar conditions in other papers, one can obtain some

monotonicity properties of optimal nonlinear price. However, it is difficult to get an explicit expression for the optimal solution in general; even a basic question like “how the marginal price varies” on quantity discount (Spence 1977) has no answer. To make the problem tractable and to gain more insight into the problem, Spence (1977) assumes a multiplicative utility function, namely, $u(\theta, n) = \theta q(n)$. Maskin and Riley (1984) uses the same type of utility function with a further simplification by choosing $u(\theta, n) = \theta n^\gamma$.

An interesting feature of Spence’s nonlinear pricing model is that it can be used to study pricing problem of quality-differentiated products. The intrinsic reason is that nonlinear pricing problem and quality pricing problem are analytically equivalent. Maskin and Riley (1984) shows that the monopoly pricing of product quality is just a reinterpretation of the nonlinear pricing model. Tirole (1988, p.150) highlights the similarity between quantity and quality discrimination and states that at a formal level the two models are identical. When using a vertical quality model to substitute nonuniform pricing, Stole (2008, p.87) simply states that “we take q to represent quality, but it could equally well represent quantities.” A simple example may help understand this insight: it is difficult and unnecessary to distinguish different (unit) prices associated with a 250ml Apple Juice and a 1000ml Apple Juice as a result of differentiation by quality or discrimination by quantity. Recently Spence’s multiplicative specification has also been used in OM literature, for example, see Bhargava and Choudhary (2008), Akcay et al. (2010), and Liu and Zhang (2013). Following those papers, our subsequent developments are based on Spence’s multiplicative utility model, which leads to the following specification of the consumer’s surplus

$$v(\theta, n) = \theta q(n) - p(n), 0 \leq \theta \leq \bar{\theta} \quad (2.1)$$

where $q(n)$ is concave in n . Here $q(n)$ can be interpreted as the maximal total utility value for consuming n units of the product. Without loss of generality, we rescale θ so that it is uniformly distributed on the unit interval $[0, 1]$.

We now turn to the retailer, which is selling one product according to K different bundles with different quantity levels, denoted by $\mathbf{n} = (n_1, n_2, \dots, n_K)$ where $1 \leq n_1 < n_2 < \dots < n_K$, under a price schedule $\mathbf{p} = (p_1, p_2, \dots, p_K)$. Here p_k is the total price for the k th bundle with n_k units. Note that n_k 's are not necessarily consecutive, for instance, a retailer may offer a discount if the customer buys three units but there is no discount if he buys two units. For technical purpose, we rule out any arbitrage opportunity, which is valid in a typical retailing setting. It is also assumed that a consumer either buys exactly one bundle from the K offered bundles or makes no purchase. This precludes the case that a customer purchases some combination of the bundles. However, this assumption will be removed for DUP and DBP models.

Finally, we assume that the firm knows the distribution function of consumer type θ , which itself is private information to the particular consumer. Given the specification of preferences and the price schedule $(\mathbf{n}, \mathbf{p})$, the consumer's surplus becomes

$$v(\theta, n_k) = \theta q_k - p_k \text{ for } k = 0, 1, 2, \dots, K. \quad (2.2)$$

where $q_k = q(n_k)$, and $p_0 \equiv 0$ and $n_0 \equiv 0$ imply the case of zero expenditure when customer makes no-purchase. By either examining the index of the lowest consumer type who purchases the bundle k or higher $\theta(n_k)$ as in Goldman et al. (1984) or just reinterpreting the argument as in Akcay et al. (2010), we can substantially reduce the choices of price schedules as shown in the following lemma.

Lemma 2.1 It is sufficient to restrict the price schedule $\mathbf{p}$ to the following set of preference-aligned prices, denoted by $\wp$:

$$\wp = \left\{ \mathbf{p}: 0 \leq \frac{p_1}{q_1} \leq \frac{p_2 - p_1}{q_2 - q_1} \leq \dots \leq \frac{p_{K-1} - p_{K-2}}{q_{K-1} - q_{K-2}} \leq \frac{p_K - p_{K-1}}{q_K - q_{K-1}} \leq 1 \right\}.$$

Under the preference-aligned prices $\wp$, it is evident that $\theta(n_k) = (p_{kt} - p_{k-1,t})/(q_k - q_{k-1})$, which means that the ratio of price increment over incremental utility, $(p_{kt} - p_{k-1,t})/(q_k - q_{k-1})$ is increasing in k . Otherwise, a customer purchasing a lower bundle would be better off by upgrading to a higher bundle, which implies that there would be no demand for this lower bundle. Throughout this chapter, we use increasing/decreasing and positive/negative in the weak sense unless stated otherwise.

The preference-aligned prices $\wp$ partition the interval $\theta \in [0, 1]$ into $K + 1$ subintervals with each subinterval corresponding to customers that would purchase $0, n_1, n_2, \dots, n_K$ units respectively, from low type to high type. Given $\mathbf{n}$ and $\mathbf{p}$, let $\alpha_k(\mathbf{p})$ be the probability that an arriving consumer chooses to buy the k th bundle. By restricting the prices $\mathbf{p}$ to the set $\wp$, we have

$$\alpha_k(\mathbf{p}) = \begin{cases} \frac{p_1}{q_1}, & k = 0; \\ \frac{p_{k+1} - p_k}{q_{k+1} - q_k} - \frac{p_k - p_{k-1}}{q_k - q_{k-1}}, & k = 1, 2, \dots, K - 1; \\ 1 - \frac{p_K - p_{K-1}}{q_K - q_{K-1}}, & k = K, \end{cases} \quad (2.3)$$

where $\alpha_0(\mathbf{p})$ is the probability that the arriving customer makes no purchase. This explicit expression of $\alpha_k(\mathbf{p})$ not only facilitates the understanding of customer quantity choice behavior, but also makes the dynamic pricing problem mathematically tractable.

2.3.2 Dynamic Programming Formulation

We now examine the DNP problem for a retailer with fixed units of inventory at the beginning of the selling season. Following the approach by Bitran and Mondschein (1997) and Akcay et al. (2010), we divide the selling season into T periods, each of which is short enough that there is at most one customer arrival. The time periods are ordered in reverse: $t = T$ is the beginning and $t = 0$ is the end of selling season. Let λ_t denote the probability of one customer arrival in period t . Given the nonuniform scheme with quantity bundles $\mathbf{n} = (n_1, n_2, \dots, n_K)$, the retailer's problem is to find a price schedule $\mathbf{p} = (p_1, p_t, \dots, p_K) \in \wp$ in each period to maximize the total expected revenue during the whole selling season.

Given $(\mathbf{n}, \mathbf{p})$, as discussed above, the probability that a consumer buys the k th bundle is $\alpha_k(\mathbf{p})$. Let $V_t(x)$ be the retailer's optimal expected revenue from period t to the end of the season with x units of inventory in stock. Then the retailer's problem can be formulated as the following dynamic problem:

$$V_t(x) = \sup_{\mathbf{p} \in \wp} \left\{ \sum_{k=1}^K \lambda_t \alpha_k(\mathbf{p}) (p_k + V_{t-1}(x - n_k)) + \lambda_t \alpha_0(\mathbf{p}) V_{t-1}(x) + (1 - \lambda_t) V_{t-1}(x) \right\},$$

with boundary conditions $V_t(0) = 0$ for $t = 1, \dots, T$ and $V_0(x) = 0$ for all x . The first term of $V_t(x)$ is the revenue-to-go after an arriving customer purchases one of the provided bundles; the second term is revenue-to-go if an arriving customer makes no purchase; and the third term is the revenue-to-go when there is no customer arrival in this period. After some simple algebraic manipulation, we can rewrite $V_t(x)$ as follows

$$V_t(x) = \sup_{\mathbf{p} \in \wp} \left\{ \sum_{k=1}^K \lambda_t \alpha_k(\mathbf{p}) (p_k + V_{t-1}(x - n_k) - V_{t-1}(x)) \right\} + V_{t-1}(x). \quad (2.4)$$

For ease of presentation, we define the difference functions of $V_t(x)$ with respect to inventory x and time t by

$$\Delta_n V_t(x) = V_t(x) - V_t(x - n) \text{ for } n > 0,$$

and

$$\Delta V_t(x) = V_t(x) - V_{t-1}(x) \text{ for } t = 1, \dots, T,$$

respectively. Here the function $\Delta_n V_t(x)$ can be interpreted as the marginal expected value of n units which represents the opportunity loss for reducing the inventory level x by n units at time t . $\Delta V_t(x)$ is the marginal expected value of time representing the opportunity loss for selling nothing in period t at the inventory level x . Using these notations, we define

$$G_t(x, \mathbf{p}) = \sum_{k=1}^K \alpha_k(\mathbf{p}) \left(\mathbf{p} - \Delta_{n_k} V_{t-1}(x) \right), \quad (2.5)$$

which is the expected gain in period t by selling some bundle to a customer. Therefore, the dynamic optimization formulation (2.4) has been transformed into the following problem:

$$\Delta V_t(x) = V_t(x) - V_{t-1}(x) = \lambda_t \max_{\mathbf{p} \in \mathcal{P}} \{G_t(x, \mathbf{p})\}.$$

Note that the purchasing probability $\alpha_k(\mathbf{p})$ depends only on the adjacent prices and utility differences. Hence we define the difference between the prices of k th bundle and $(k-1)$ th bundle as Δp_{kt} , namely, $\Delta p_{kt} = p_{kt} - p_{k-1,t}$ and similarly the difference between the maximal utility of k th bundle and $(k-1)$ th bundle as Δq_k , i.e., $\Delta q_k = q_k - q_{k-1}$ for $k = 1, \dots, K$. The purchasing probabilities in (2.3) can then be expressed as

$$\alpha_k(\mathbf{p}) = \frac{\Delta p_{k+1}}{\Delta q_{k+1}} - \frac{\Delta p_k}{\Delta q_k}, \quad k = 1, \dots, K, \quad (2.6)$$

where $\frac{\Delta p_{K+1,t}}{\Delta q_{K+1}} \equiv 1$. Since Δp_{kt} is a transformation of p_{kt} , it follows that finding the optimal prices $\mathbf{p}$ to maximize $G_t(x, \mathbf{p})$ is equivalent to finding the optimal price differences $\Delta \mathbf{p}$. Substituting (2.6) into (2.5),

$$G_t(x, \mathbf{p}) \equiv \sum_{k=1}^K \left[\left(\frac{\Delta p_{k+1,t}}{\Delta q_{k+1}} - \frac{\Delta p_{kt}}{\Delta q_k} \right) \left(\sum_{i=1}^k \Delta p_i - \Delta_{n_k} V_{t-1}(x) \right) \right]. \quad (2.7)$$

Note that $\Delta_{n_k} V_{t-1}(x)$ can be rewritten as $\Delta_{n_k} V_{t-1}(x) = \sum_{i=1}^k [V_{t-1}(x - n_{i-1}) - V_{t-1}(x - n_i)]$.

Let $d_k = n_k - n_{k-1}$. Now substituting $\Delta_{n_k} V_{t-1}(x) = \sum_{i=1}^k \Delta_{d_i} V_{t-1}(x - n_{i-1})$ into (2.7), we obtain

$$\begin{aligned} G_t(x, \mathbf{p}) &= \sum_{k=1}^K \left[\left(\frac{\Delta p_{k+1,t}}{\Delta q_{k+1}} - \frac{\Delta p_{kt}}{\Delta q_k} \right) \left(\sum_{i=1}^k \left(\Delta p_{it} - \Delta_{d_i} V_{t-1}(x - n_{i-1}) \right) \right) \right] \\ &= \sum_{i=1}^K \left[\left(\Delta p_{it} - \Delta_{d_i} V_{t-1}(x - n_{i-1}) \right) \left(\sum_{k=i}^K \left(\frac{\Delta p_{k+1,t}}{\Delta q_{k+1}} - \frac{\Delta p_{kt}}{\Delta q_k} \right) \right) \right] \\ &= \sum_{i=1}^K \left[\left(1 - \frac{\Delta p_i}{\Delta q_i} \right) \left(\Delta p_i - \Delta_{d_i} V_{t-1}(x - n_{i-1}) \right) \right], \end{aligned} \quad (2.8)$$

where the last equation follows from the following identity:

$$\sum_{k=i}^K \left(\frac{\Delta p_{k+1,t}}{\Delta q_{k+1}} - \frac{\Delta p_{kt}}{\Delta q_k} \right) = \frac{\Delta p_{K+1,t}}{\Delta q_{K+1}} - \frac{\Delta p_{it}}{\Delta q_i} = 1 - \frac{\Delta p_{it}}{\Delta q_i}, i = 1, \dots, K.$$

The expression (2.8) conveys another interpretation of the expected additional gain realized in period t . Recall that $\Delta p_{it}/\Delta q_i$ is the lowest consumer type that purchases the i th or higher bundle, so $1 - \Delta p_{it}/\Delta q_i$ is the probability that an arriving customer buys at least the i th bundle; and $\Delta p_{it} - \Delta_{d_i} V_{t-1}(x - n_{i-1})$ is the additional gain that the firm could achieve by selling additional d_i units after the firm had sold n_{i-1} units. Hence Equation (2.8) states that the expected additional gain realized in period t can be also measured by adding the expected additional gains

from selling d_i ($i = 1, \dots, K$) units of inventory. All told, under DNP scheme, the retailer's problem of setting optimal prices $\mathbf{p}$ for the offered bundles to maximize the expected revenue in (2.5) is converted to finding the optimal price differences $\Delta \mathbf{p}$ to maximize the expected additional gains given by (2.8) subject to the condition $\mathbf{p} \in \wp$.

Proposition 2.1 Under DNP scheme, there exists a unique optimal solution $\mathbf{p}(t, x) \in \wp$.

Moreover, let $\mathbf{p}^*$ such that

$$\Delta p_k^* = \frac{\Delta q_k + \Delta_{d_k} V_{t-1}(x - n_{k-1}) \wedge \Delta q_k}{2} \text{ for } k = 1, \dots, K, \quad (2.9)$$

where $x \wedge y \equiv \min(x, y)$. If $\mathbf{p}^* \in \wp$, then $\mathbf{p}(t, x) = \mathbf{p}^*$.

The intuition behind the optimal price is straightforward: the retailer tries to find the best tradeoff between the expected gain in the future and the potential increase from an arriving customer. When the future incremental value of d_k units is more than the incremental utility of additional d_k units, the retailer will not sell these additional d_k units to the customer. Otherwise, the retailer will sell these d_k units at the optimal price $\Delta p_k(t, x)$ equating to the average of the incremental utility of additional d_k units and the future incremental value of d_k units. Note that when $\Delta q_k > \Delta_{d_k} V_{t-1}(x - n_{k-1})$, it implies that the purchase probability for the k th bundle or higher bundle is strictly positive; but it does not necessarily mean that someone purchases exactly the k th bundle. When $\Delta q_k \leq \Delta_{d_k} V_{t-1}(x - n_{k-1})$, there will be no demand for the k th or higher bundle.

It follows from (2.9) that the optimal price for each bundle is

$$p_k(t, x) = \sum_{i=1}^k \frac{\Delta q_i + \Delta_{d_i} V_{t-1}(x - n_{i-1}) \wedge \Delta q_i}{2} \text{ for } k = 1, \dots, K. \quad (2.10)$$

Substituting (2.10) into (2.4), we obtain the following expression for the value function

$$V_t(x) = \lambda_t \sum_{k=1}^K \frac{[\Delta q_k - \Delta d_k V_{t-1}(x - n_{k-1}) \wedge \Delta q_k]^2}{4\Delta q_k} + V_{t-1}(x). \quad (2.11)$$

Next we consider the structural properties for the optimal policy.

2.3.3 Structural Properties

The structural properties of the optimal policy, which not only shed managerial insight but also facilitate the computation procedure of the optimal solution, has been well recognized in the literature. In this subsection, we first identify a necessary and sufficient condition for the structural properties of the value function. Then we present an example showing that a value function without structural properties can exhibit structural properties before some truncated time. Moreover, while the concavity of the value function breaks down, the optimal prices still display time monotonicity during the whole time horizon.

Definition 2.1 The bundle schedule $\mathbf{n}$ is said to be consecutive if $\mathbf{n} = \{1, 2, \dots, K\}$.

Proposition 2.2 For DNP scheme, the value function $V_t(x)$ is concave if and only if the bundle schedule is consecutive.

The necessity for Proposition 2.2 is straightforward. However, the sufficiency is nontrivial and is in fact derived from the intrinsic structure of DNP scheme. Intuitively, if the retailer has full control of the pricing process through a consecutive bundle schedule, he can always adjust the selling process to smoothen the value function so that it is “well-behaved”. The following corollary is a direct result of Proposition 2.2.

Corollary 2.1 Under DNP scheme with a consecutive bundle schedule, it is always true that

- (a) The marginal value of inventory $\Delta_1 V_t(x)$ is increasing in t and decreasing in x .
- (b) The marginal value of time $\Delta V_t(x)$ is increasing in x .
- (c) If $\lambda_t \geq \lambda_{t+1}$, then the marginal value of time holds with $\Delta V_t(x) \geq \Delta V_{t+1}(x)$.

The monotonicity results in above corollary are in fact intuitive. Part (a) implies that having extra inventory is off greater value when the available selling time is longer; while having extra inventory is off smaller value when the available inventory is larger. Part (b) says that the marginal gain for having an extra selling opportunity is of greater value when the available inventory is higher. Part (c) characterizes the change for the marginal value of time. If the probability of making a sale becomes less at time $t + 1$ ($\lambda_t \geq \lambda_{t+1}$), then the marginal gain for having the selling opportunity in period $t + 1$ would not exceed the marginal gain at time t .

We now turn to the monotonicity of optimal prices. To gain more insight into the pricing process, we define the unit price $\bar{p}_k(t, x)$ as

$$\bar{p}_k(t, x) = p_k(t, x)/k \text{ for } k = 1, \dots, K. \quad (2.12)$$

In reality, the posted price schedule can be either in the form of bundle price $\mathbf{p}(t, x)$, or in term of the unit price $\bar{\mathbf{p}}(t, x)$, or even price differences $\Delta \mathbf{p}(t, x)$.

Proposition 2.3 For DNP scheme with consecutive bundle schedule, the following properties hold:

- (a) The optimal prices $\mathbf{p}(t, x)$, the unit price $\bar{\mathbf{p}}(t, x)$ and price differences $\Delta \mathbf{p}(t, x)$ are all decreasing in x for any t ;
- (b) The optimal prices $\mathbf{p}(t, x)$, the unit price $\bar{\mathbf{p}}(t, x)$ and price differences $\Delta \mathbf{p}(t, x)$ are all increasing in t for any x ;

(c) The optimal price $p_k(t, x)$ is increasing in purchase quantity k .

The key implication of Proposition 2.3 is that three representations of the optimal prices are well-behaved. The monotonicity in inventory level is due to the monotonicity of marginal value of inventory. Proposition 2.3(c) shows the fact that the more a customer buys the more she pays.

We now examine the structural properties when the bundle schedule is not consecutive. Recall that for the case of single-unit demand, since the optimal price is an increasing function of the marginal value of inventory (e.g., Gallego and van Ryzin 1994), the concavity of the value function can always assure the monotonicity of optimal price, and vice versa. However, this equivalence no longer holds for multi-unit demand case. Research on DSKP-type problem (e.g., Lee and Hersh 1993 and Brumelle and Walczak 2003) has noticed the breakdown of concavity of the value function while the optimal prices may be still both inventory and time monotonic. Even though the breakdown of structural properties of optimal policy is common for DSKP-type problems, examples from Lee and Hersh (1993) and Van Slyke and Young (2000) indicate that the breakdown happens only near the end of the selling season. Is it possible that a value function exhibit structural properties before some time? The following example confirms this conjecture in our context.

Example 2.1 Consider that a retailer with inventory $x = 3$ is implementing a DNP scheme with the following parameters: $\lambda_t = 0.8$, $(q(1), q(2), q(3)) = (10, 15, 19)$, and $\mathbf{n} = \{1, 3\}$.

Figure 2.1(a) displays the marginal expected value $\Delta_1 V_t(x)$ and $\Delta_2 V_t(2)$ during period $1 \leq t \leq 10$. As the bundle schedule is not consecutive, the structural properties for the optimal policy break down. However it is clear that the marginal expected value $\Delta_2 V_t(2)$ is greater than the quality difference $\Delta q_2 = 9$ at $t = 6$, from (2.9) we know that the retailer will set the price

difference $\Delta p_2(7,3) = 9$. It implies that no customer will purchase the three-unit bundle at time $t = 7$, namely, the bundle schedule for the effective prices is $\{1\}$. Moreover, $V_6(x)$ is concave in x . Recall that the sufficiency in Proposition 2.2 is proved by induction, with the two conditions one can analogously show that $V_t(x)$ is concave in x for any $t \geq 7$. The monotonicity of the optimal prices for $t \geq 7$ is just a direct result from the concavity of the value function. Taken together, the value function displays truncated structural properties.

Figure 2.1 Marginal expected values $\Delta_1 V_t(x)$ and price differences $\Delta p_k(t, x)$

Note that the example can be generalized to more complicated cases. The induction procedure for Proposition 2.2 ensures that as long as the following two conditions hold: (1) the bundle schedule for the effective price is consecutive at some $t' \geq 0$; (2) the value function $V_t(x)$ is concave at $t = t'$, then the optimal policy will display structural properties for $t \geq t' + 1$. It is also worth highlighting that this finding is not only mathematically insightful but also important for managerial and computational purposes. Essentially, the truncated structural properties of the

optimal policy can achieve almost the same goal of the global structural properties, which is a special case of truncated structural properties where the truncated time is zero.

While the structural properties break down during the whole time horizon in Example 2.1, however, we find the optimal prices still exhibit time monotonicity. As it has been shown that the optimal prices display time monotonicity for $t \geq 7$, it suffices to show the optimal prices are increasing in time t as $t \leq 7$. Based on (2.9), a sufficient condition is to show the associated price differences are increasing in t , which is clear from Figure 2.1(b).

Last, careful readers may have noticed a subtle issue: we must show that for Example 2.1 no customer will purchase two units. This is indeed the case because the price for one unit is at least 5, which is greater the quality difference $q(2) - q(1)$. Hence no customer has incentive to purchase two units.

2.4 Dynamic Uniform Pricing

In practice, uniform pricing is more common for many reasons. First, rules and regulations may prevent discriminating pricing practice. Second, uniform pricing have become a standard practice in the industry and any deviation from it can be costly to the company. Last, but not the least, the uniform pricing is simple to implement. Therefore, it is important to study the dynamic uniform pricing (DUP) problem with customer choice on purchase quantity, which extends the classic dynamic pricing model of single-unit demand (e.g. Gallego and van Ryzin 1994) to multi-unit demand case.

2.4.1 Dynamic Programming Formulation

Under DUP scheme, the retailer offers a common unit price for the product at any time, regardless of how many units that a customer purchases. We assume that the customer has the same preference in Section 2.3. Let $\rho(n) = q(n) - q(n - 1)$ ($n \geq 1$), representing the marginal maximal utility for consuming the n th unit of the product. Hence for a customer with type θ , $\theta\rho(n)$ represents her maximum willingness-to-pay for consuming the n th unit of the product.

Note that the retailer will always set the price such that it is less than $\rho(1)$, otherwise there is no demand. For any price $p \in [\rho(k + 1), \rho(k))$, the price vector $(p, 2p, \dots, kp)$ belongs to preference-aligned prices $\wp$ with $K = k$ as in Lemma 2.1. Moreover, since $\frac{p}{\rho(k+1)} \geq 1$, it implies that no customer would purchase $k + 1$ units or more. Hence the individual rationality assures that the underlying customers' choice process is consistent with self-selection. Therefore given that the inventory is sufficiently large ($x \geq k$) and $p \in [\rho(k + 1), \rho(k))$, the purchase probability $\alpha_i(p)$ that an arriving consumer chooses to buy i units is given by

$$\alpha_i(p) = \begin{cases} \frac{p}{\rho(1)}, & i = 0; \\ \frac{p}{\rho(i+1)} - \frac{p}{\rho(i)}, & i = 1, 2, \dots, k - 1; \\ 1 - \frac{p}{\rho(i)}, & i = k; \\ 0, & i > k. \end{cases} \quad (2.13)$$

With a slight abuse of notation, we use the same notation as in DNP scheme. With a minor modification of (2.4), the DUP problem can then be formulated as follows

$$V_t(x) = \lambda_t \sup_p \{G_t(x, p)\} + V_{t-1}(x),$$

where

$$G_t(x, p) = \sum_{i=1}^{k \wedge x} \left(1 - \frac{p}{\rho(i)}\right) (p_t - \Delta_1 V_{t-1}(x - i + 1)) \quad (2.14)$$

such that $\rho(k+1) \leq p < \rho(k)$ if $1 \leq k < x$ and $0 \leq p < \rho(x)$ if $k = x$. This expression of $G_t(x, p)$ is quite straightforward and similar to $G_t(x, \mathbf{p})$ in DNP scheme, except that here the price difference is always p . The condition of $0 \leq p < \rho(x)$ for $k = x$ is due to the fact that the largest purchased quantity cannot exceed x . Given time t , and left inventory x , it is easy to check that the function $G_t(x, p)$ is continuous at any point of $p = \rho(k)$, $1 \leq k \leq x$, which implies that $G_t(x, p)$ is continuous in the closed set $[0, \rho(1)]$. This leads to the following result on the existence of optimal price for DUP scheme.

Lemma 2.2 There always exists a price $p(t, x)$ that solves the retailer's DUP problem.

The expression of $G_t(x, p)$ in (2.14) also gives a direct method to obtain the optimal solution. We only need search the x intervals by solving the associated constrained maximization problem. However, when x is large, this method becomes very time-consuming and inefficient. In fact, if there is a positive lower bound for the optimal price, a complete search for all intervals becomes unnecessary. Next we identify a sufficient condition that guarantees the existence of a bounded price for the DUP scheme. For easy reference, we define the condition as Assumption 2.1.

Assumption 2.1 The marginal maximal utility series $\rho(k)$ is $o(1/k)$, namely, $\lim_{k \rightarrow \infty} \rho(k) \cdot k = 0$.

Assumption 2.1 implies that the marginal maximal utility series decreases quick enough, which basically indicates demand elasticity is less than 1 as inventory increases (Zhao and Zheng 2000). Then it precludes the case that the retailer lowers price to any small value to increase the revenue rate.

Proposition 2.4 Given that Assumption 2.1 holds, then for DUP scheme,

- (a) there exists the largest purchase quantity K ; and
- (b) for any inventory x and at any time t , the optimal price $p(t, x) \geq p^*$, where $p^* = K/[2 \sum_{i=1}^K 1/\rho(i)]$ is the myopic price.

It is worth highlighting that Assumption 2.1 cannot be weakened in general. This can be shown by just considering $\rho(k) = 1/k$:

$$\sup_{p \geq 0} G_1(x, p) = \sup_{p \geq 0, k \geq 1} \left\{ -\frac{k(k+1)}{2} p^2 + kp \right\} I\{\rho(k+1) \leq p < \rho(k)\}.$$

When $p \in [\rho(k+1), \rho(k))$, the supremum is attained at $p = \frac{1}{k+1}$. Hence we have

$\sup_{p \geq 0} G_1(x, p) = \sup_{k \geq 1} \left\{ \frac{k}{2(k+1)} \right\} = 1/2$. This means that as the inventory increases, the retailer will set the price approaching zero to pursue more profit; hence there is no bounded price that maximizes the revenue rate. We call a utility function satisfying Assumption 2.1 a *regular utility function*. For the rest of this section, we suppose Assumption 2.1 holds; hence the largest purchase quantity K is well defined.

Now $\sup_{p \geq 0} G_t(x, p)$ becomes

$$\max_{p \geq 0, 1 \leq k \leq K \wedge x} \sum_{i=1}^k \left(1 - \frac{p}{\rho(i)} \right) (p - \Delta_1 V_{t-1}(x - i + 1))$$

such that $\rho(k+1) \leq p < \rho(k)$ for $1 \leq k < x$ and $0 \leq p < \rho(x)$ if $k = x$. By searching the local optimal solution among these $K \wedge x$ disjoint intervals, one can find the global optimal solution.

It seems that the varying expressions for $G_t(x, p)$ in different intervals make the problem difficult and troublesome. However it actually provides a novel way to obtain the global optimal price. To facilitate the computation process, we rewrite $G_t(x, p)$ in (2.14) as follows

$$G_t(x, p) = -\left(\sum_{i=1}^k \frac{1}{\rho(i)}\right)p^2 + \left(k + \sum_{i=1}^k \frac{\Delta_1 V_{t-1}(x-i+1)}{\rho(i)}\right)p - \Delta_k V_{t-1}(x). \quad (2.15)$$

Given that $\rho(k+1) \leq p < \rho(k)$, one can use the standard Lagrangean method to solve this problem. With the existence of the optimal price, we know there must exist some k such that $\rho(k+1) \leq p < \rho(k)$. There are two possibilities: (1) $\rho(k+1) < p < \rho(k)$; and (2) $p = \rho(k+1)$. For the first case, a necessary condition for the price to be optimal is that it satisfies the first order condition (FOC) for the associated unconstrained problem of (2.15). For the second case, to ensure $\rho(k)$ is the best solution for $\rho(k+1) \leq p < \rho(k)$, a necessary condition is the solution for the associated unconstrained problem of (2.15) is no more than $\rho(k+1)$. This analysis leads to a simple way to find the optimal price, summarized as the following proposition.

Proposition 2.5 For DUP scheme, the optimal price $p(t, x)$ satisfies

$$p(t, x) \in \underset{1 \leq k \leq K \wedge x}{argmax} G_t(x, p^k),$$

where p^k 's are the optimal price candidates, which are determined as follows. When $k < K \wedge x$,

$$p^k = \begin{cases} p^k(t, x), & \text{if } \rho(k+1) < p^k(t, x) < \rho(k); \\ \rho(k+1), & \text{if } p^k(t, x) \leq \rho(k+1); \\ null, & \text{otherwise;} \end{cases}$$

otherwise (when $k = K \wedge x$),

$$p^k = \begin{cases} p^k(t, x), & \text{if } p^k(t, x) < \rho(k); \\ null, & \text{otherwise;} \end{cases}$$

where $p^k(t, x) = [k + \sum_{i=1}^k \frac{\Delta_1 V_{t-1}(x-i+1)}{\rho(i)}] / [2 \sum_{i=1}^k \frac{1}{\rho(i)}]$.

Note that DUP scheme is solved by selecting the optimal price from the set of qualified candidates, so the optimal price is not necessarily unique. Moreover, if it is not, one can specify any rule to break the tie.

2.4.2 Structural Properties for the Case of $K \leq 2$

Since there is no explicit expression for the optimal price in general, consequently, establishing structural properties for the general case is extremely difficult. Hence we limit our discussion to the case of $K \leq 2$ and hope to uncover the structural properties of DUP scheme under this simple situation. For ease of exposition, let $\beta = \rho(2)/\rho(1)$, which reflects customers' utility sensitivity of the second unit over the first unit. Specifically, we demonstrate that the structural properties for the optimal policy still hold for $\beta \leq 1/3$. Otherwise, the concavity for the value function may disappear, whereas the optimal price still exhibits both time and inventory monotonicity.

Proposition 2.6 For the DUP problem with $K \leq 2$, when $\beta \leq 1/3$, customers will purchase at most one unit under the optimal policy for any inventory x and time t .

Proposition 2.6 implies that, when customers' utility sensitivity is weak (i.e., $\beta \leq 1/3$), the DUP problem degenerates to the classic dynamic pricing problem with single-unit demand (e.g., Gallego and van Ryzin 1994 and Bitran and Mondschein 1997). Accordingly, the optimal policy displays structural properties.

Corollary 2.2 For DUP problem with $K \leq 2$, if $3\rho(2) \leq \rho(1)$, the value function $V_t(x)$ is concave in both x and t . Furthermore, the optimal price $p(t, x)$ is decreasing in x and increasing in t .

Now we consider the case that $\beta > 1/3$. Recall that concavity of value function $V_t(x)$ requires $V_t(x)$ to be concave in x for any time t . Hence the first step is to examine the concavity of $V_1(x)$. It is simple to check the concavity of $V_1(x)$ under this case. Next we consider $V_2(x)$. The following example shows the result.

Example 2.2 Consider that a retailer with inventory $x = 4$ is facing a DUP problem with parameters $\lambda_t = 0.8$ and $K \leq 2$.

Without loss of generality, we fix $\rho(1) = 10$ and. Figure 2.2(a) shows the change of $\Delta_1 V_2(x)$ ($x = 1, 2, 3, 4$) for $1/3 < \beta \leq 1$. Obviously, $\Delta_1 V_2(x)$ is decreasing in x when $\beta \in (1/3, 0.95)$. However, when $\beta \in (0.95, 1]$, $\Delta_1 V_2(3)$ is strictly smaller than $\Delta_1 V_2(4)$, ; hence the concavity property of $V_2(x)$ breaks down. To understand the rationale for the non-concavity of $V_2(x)$ when $\beta \in (0.95, 1]$, consider the case when $\beta = 1$. Denote $V'_2(4, p)$ as the expected total revenue when the retailer sets price p at time $t = 2$ and inventory $x = 4$ and sets the optimal price for $t = 1$. As shown in Figure 2.2(b), we have $p(2, 4) < p(2, 3)$, so we have $V'_2(4, p(2, 3)) < V_2(4)$. Considering the additional gain $V'_2(4, p(2, 3)) - V_2(2)$. It exceeds zero only for the case that customers purchase at both $t = 1$ and 2. In this case, since $\beta = 1$, the purchasing customer will always buy two units at time $t = 2$. Therefore the value $V'_2(4, p(2, 3)) - V_2(2)$ only originates from time $t = 1$. Similarly, considering $V_2(3) - V_2(2)$, it also only originates from time $t = 1$ for the case that customers purchase at both $t = 1$ and 2. Recall that the customer purchases two units if it is available and $p_1(1) = p_1(2)$, hence $V'_2(4, p(2, 3)) - V_2(2) = 2[V_2(3) - V_2(2)]$. As we already know $V'_2(4, p(2, 3)) < V_2(4)$, hence $\Delta_1 V_2(4) > \Delta_1 V_2(3)$.

Figure 2.2 Marginal expected values $\Delta_1 V_2(x)$ and optimal prices $p(2, x)$ for $\beta \in (1/3, 1]$

While the structural properties break down during the whole time horizon, similar to DNP scheme, we find the value function may also display structural properties before some truncated time.

Example 2.3 Consider that a retailer with inventory $x = 4$ is implementing a DUP scheme with parameters $K \leq 2, \lambda_t = 0.8, \rho(1) = 10$ and $\rho(2) = 9.6$.

From Example 2.2, $\Delta_1 V_2(x)$ is not monotonic at $t = 2$, hence the concavity for the value function breaks down. However, note that $p^2(t, x)$ ($x = 2, 3, 4$) are greater than $\rho(2)$ at $t = 184$; moreover, $\Delta_1 V_t(x)$ is decreasing in x at $t = 183$. With these two conditions, as Example 2.1, one can analogously show that $p^2(t, x)$ ($x = 2, 3, 4$) are greater than $\rho(2)$ and $V_t(x)$ is concave in x for any $t \geq 184$. This indicates the only feasible optimal price is $p^1(t, x)$ for $t \geq 184$. Hence the optimal price exhibits both time and inventory monotonicity for $t \geq 184$. All told, the value function display structural properties for $t \geq 184$.

Furthermore, the optimal price $p(t, x)$ actually exhibits both time and inventory monotonicity during the whole time horizon. As it is shown that it holds for $t \geq 184$, we only need to show $p(t, x)$ is increasing in time x and decreasing in inventory x for $t \leq 184$, which is clear from Figure 2.3.

Figure 2.3 Optimal prices $p(t, x)$

Upon this point, we want to highlight that the truncated structural properties is not an exception but a common property in multi-unit demand case. The value function has an intrinsic force that makes the value function well-behaved as time increases. Moreover, since the inventory is fixed, as time increase, the retailer will only have incentive to satisfy one-unit demand. The combined effect of these two underlying conditions is that the value function exhibits truncated structural properties. Then if the optimal price(s) still display time or inventory monotonicity before this truncated time, the optimal price would have corresponding monotonicity during the whole time horizon.

2.5 Dynamic Block Pricing

These are many advantages for DNP scheme, for instance, the solution is simple and it generates more revenue. However DNP has its weaknesses as well. The restriction that a customer only purchases from the given bundles limits its application in many circumstances. The requirement for the bundle schedule to be consecutive can lead to too many prices ($K > 2$), which is difficult and costly to implement and may cause customers' inability to choose. On the other hand, firms using DUP scheme lose the opportunity to raise revenue by price discrimination. To rectify these issues, we study the dynamic block pricing (DBP) scheme in this section. Leland and Meyer (1976) first introduce the idea of block pricing. We bring this new idea into the dynamic pricing literature by focusing on two-block pricing, which is the most common form of block pricing in practice.

2.5.1 Dynamic Programming Formulation

Suppose that a retailer is selling one product through a two-block pricing scheme $(1, k)$ with p_1 the unit price for customers who purchase less than k units and $\bar{p}_k$ the second or “trailing block” unit price for customers who purchase k units or more. Formally, we have the following pricing scheme:

$$\bar{p}_j = \begin{cases} p_1 & j < k, \\ \bar{p}_k & j \geq k; \end{cases}$$

where $\bar{p}_j$ is the average unit price for purchasing j units. Moreover, suppose $p_1 \geq \bar{p}_k$. The condition is easy to understand; because otherwise no customer purchases at price $\bar{p}_k$ and hence the block pricing degenerates to uniform pricing. For ease of presentation, we call $(p_1, \bar{p}_k)$ a block-price scheme. Compared to DUP scheme, any retailer implementing DBP scheme exerts

some degree of price discrimination. However, DBP is not as sophisticated as the full quantity discrimination in DNP scheme with consecutive bundle schedule.

First we consider the customer's behavior. Customers have the same preference as Section 2.3. Similarly, q_n is the maximal utility for consuming n units of the product and define $\rho(n) = q_n - q_{n-1}$ for $n \geq 1$. Let $\theta(j, p_1, \bar{p}_k)$ denote the index of the lowest-type consumer who purchases at least j units when the inventory is large enough. It follows from Goldman et al. (1984) that $\theta(j, p_1, \bar{p}_k)$ is well-defined. Under block-price scheme $(p_1, \bar{p}_k)$, let θ' be the smallest type such that $\theta q_k - k\bar{p}_k \geq \theta q_i - ip_1$ for all $0 \leq i < k$. Hence $\theta' = \max_{0 \leq i < k} \frac{k\bar{p}_k - ip_1}{q_k - q_i}$. It is easy to check that $\theta' < \frac{\bar{p}_k}{\rho(k+1)}$, which is the lowest-type of customer willing to buy $k+1$ units at price $\bar{p}_k$. Thus $\theta' \wedge 1$ is exactly the lowest-type consumer who purchases at least k units. Note that $\frac{p_1}{\rho(j)}$ is the lowest-type type of customer willing to buy j units at price p_1 if the discount price $\bar{p}_k$ is not available (refer to (2.13) in DUP case), therefore $\theta(j, p_1, \bar{p}_k)$ can be explicitly expressed as

$$\theta(j, p_1, \bar{p}_k) = \begin{cases} \frac{p_1}{\rho(j)} \wedge \max_{0 \leq i < k} \frac{k\bar{p}_k - ip_1}{q_k - q_i} \wedge 1, & 1 \leq j < k; \\ \max_{0 \leq i < k} \frac{k\bar{p}_k - ip_1}{q_k - q_i} \wedge 1, & j = k; \\ \frac{\bar{p}_k}{\rho(j)} \wedge 1, & j > k. \end{cases} \quad (2.16)$$

We now study the retailer's problem. Under DBP scheme, the retailer simultaneously chooses the optimal blocks (or design blocks) and the associated optimal prices. Therefore, given inventory x and time t , the retailer needs to make two decisions: the block threshold k that enjoys a discount price and the two unit prices $(p_1, \bar{p}_k)$. For practical and technical reasons, suppose there is an upper bound $\bar{k}$ for the block threshold, hence it suffices to consider $2 \leq k \leq \bar{k}$. Moreover, as for DNP scheme, we further assume that there exists a largest purchase quantity

K such that $K \geq \bar{k}$. Essentially, this assumption implies that the optimal price $\bar{p}_k$ ($2 \leq k \leq \bar{k}$) is not lower than $\rho(K + 1)$, so it is valid when $\rho(K + 1)$ is relatively small. Given the block-price scheme $(p_1, \bar{p}_k)$ and inventory x , the probability that an arriving consumer chooses to purchase j units becomes:

$$\alpha_j(p_1, \bar{p}_k) = \begin{cases} \theta(j + 1, p_1, \bar{p}_k) - \theta(j, p_1, \bar{p}_k), & 1 \leq j < K \wedge x; \\ 1 - \theta(j, p_1, \bar{p}_k), & j = K \wedge x. \end{cases} \quad (2.17)$$

When $x = 1$, obviously, the retailer uses uniform pricing. Consider $x \geq 2$, the original problem can be decomposed into two steps: first choose the optimal blocks $(1, k)$ ($2 \leq k \leq x \wedge \bar{k}$) and then find the optimal prices $(p_1, \bar{p}_k)$ for the designed blocks. Using the same notation as DNP scheme, with minor modification of (2.4), the DUP problem becomes

$$V_t(x) = \lambda_t \max_{2 \leq k \leq x \wedge \bar{k}} \{ \max_{p_1 \geq \bar{p}_k} G_t(x, p_1, \bar{p}_k) \} + V_{t-1}(x),$$

where

$$G_t(x, p_1, \bar{p}_k) = \sum_{j=1}^{x \wedge K} \alpha_j(p_1, \bar{p}_k) [j\bar{p}_j - \Delta_j V_{t-1}(x)]. \quad (2.18)$$

The implication of $G_t(x, p_1, \bar{p}_k)$ is similar to $G_t(x, \mathbf{p},)$ in DNP model. It represents the expected additional gain realized in period t with inventory x by implementing block-price scheme $(p_1, \bar{p}_k)$. From (2.16) and (2.17), it is evident that $G_t(x, p_1, \bar{p}_k)$ is a continuous function of $(p_1, \bar{p}_k)$. Moreover, it is sufficient to restrict the domain to the compact set $\{(p_1, \bar{p}_k): \rho(1) \geq p_1 \geq \bar{p}_k \geq 0\}$. Accordingly, the existence of the optimal solution is well established.

Lemma 2.4 There always exists a block-price scheme $(p_1(t, x), \bar{p}_k(t, x))$ that solves the retailer's DBP problem.

Readers are reminded that the optimal solution is not necessarily unique. The main reason is that we cannot preclude ineffective solutions. Formally, we say that a price is *ineffective* if the sales process does not change when this price is infinite; otherwise it is *effective*. Consider a case in which only $p_1(t, x)$ is effective; it indicates that no one will buy the product at price $\bar{p}_k(t, x)$. Therefore, all $(p_1, \bar{p}_k)$ such that $p_1 = p_1(t, x)$ and $\bar{p}_k(t, x) \leq \bar{p}_k \leq p_1(t, x)$ are optimal solutions. By focusing on the effective prices, the purchase probability can be simplified and then we solve the DBP problem.

2.5.2 A Solution Algorithm for DBP Scheme

The purchase probability in (2.17) has an explicit form, however the optimization of $G_t(x, p_1, \bar{p}_k)$ is not straightforward due to the complexity of its expression. In this subsection, we develop an algorithm to find the optimal solution for DBP scheme. Recall that the solution for DUP scheme is found by searching the set of qualified price candidates. The same idea is applied here to find the solution for DBP scheme.

As Lemma 2.4 guarantees the existence of the optimal solution, now we explore the necessary condition for a solution to be optimal. Suppose $(p_1, \bar{p}_k)$ is an optimal block-price scheme for inventory x at time t . There are two cases: (A) either p_1 or $\bar{p}_k$ is ineffective (B) both p_1 and $\bar{p}_k$ are effective. Next, we analyze these cases.

Case (A). When the price $\bar{p}_k$ is ineffective, customers only purchase the product at p_1 . Hence the pricing process is the same as that in DUP scheme. Given that i ($1 \leq i \leq x \wedge K$) is the largest purchase quantity at p_1 , the purchase probability $\alpha_j(p_1, \bar{p}_k)$ has the same expression as $\alpha_j(p_1)$ in (2.13). Solving the problem as DUP scheme (Proposition 2.5), and denote the solution as $p_1^{ia}(t, x)$. If it is not null, then it becomes a qualified block-price candidate.

When the price p_1 is ineffective, customers purchase according to $\bar{p}_k$ and buy at least k units. Given that i ($k \leq i \leq x \wedge K$) is the largest purchase quantity at $\bar{p}_k$, it follows from (2.16) that the purchase probability becomes

$$\alpha_j(p_1, \bar{p}_k) = \begin{cases} \frac{\bar{p}_k}{\rho(j+1)} - \frac{k\bar{p}_k}{q_k}, & j = k; \\ \frac{\bar{p}_k}{\rho(j+1)} - \frac{\bar{p}_k}{\rho(j)}, & j = k+1, \dots, i-1; \\ 1 - \frac{\bar{p}_k}{\rho(j)}, & j = i; \\ 0, & \text{others.} \end{cases}$$

Analogy to the procedure for the price $\bar{p}_k$ is ineffective, one can find the associated block-price candidate, which is denoted as $\bar{p}_k^{ia}(t, x)$. If it is not null, then it becomes a qualified candidate.

Case (B). Since $\bar{p}_k$ is effective, some customer would purchase the product at $\bar{p}_k$. First let i ($1 \leq i < k$) denote the largest purchase quantity that a customer will purchase at p_1 . Consider the customer who is indifferent between purchasing i units at p_1 or k units at $\bar{p}_k$, namely, $\theta q_i - ip_1 = \theta q_k - k\bar{p}_k$ or $\theta = \frac{k\bar{p}_k - ip_1}{q_k - q_i}$. Note that we must have $\frac{k\bar{p}_k - ip_1}{q_k - q_i} > \frac{p_1}{\rho(i)}$; since otherwise no customer purchases i units at the price of p_1 . Moreover, $\frac{p_1}{\rho(i+1)} \geq \frac{k\bar{p}_k - ip_1}{q_k - q_i}$; because otherwise the largest purchase quantity that a customer will purchase at p_1 is more than i . Furthermore, as there are some customers that purchase the product at price $\bar{p}_k$, it must be true that $\frac{k\bar{p}_k - ip_1}{q_k - q_i} < 1$. Finally, it is evident that $\bar{p}_k < p_1$; since otherwise it becomes Case (A). Now we summarize these conditions as Incentive Condition 1 (IC1):

$$\{(p_1, \bar{p}_k): \bar{p}_k < p_1; \frac{k\bar{p}_k - ip_1}{q_k - q_i} < 1; \frac{p_1}{\rho(i+1)} \geq \frac{k\bar{p}_k - ip_1}{q_k - q_i} > \frac{p_1}{\rho(i)}\}. \quad (\text{IC1})$$

At this stage, Case (B) could be further classified into two cases: (BA) the block threshold is equal to the left inventory, namely, $k = x$; and (BB) the block threshold is strictly less than the left inventory, namely, $k < x$.

Case (BA): $k = x$. Given that the price schedule $(p_1, \bar{p}_k)$ satisfies IC1, the purchase probability $\alpha_j(p_1, \bar{p}_k)$ under this case becomes

$$\alpha_j(p_1, \bar{p}_k) = \begin{cases} \frac{p_1}{\rho(j+1)} - \frac{p_1}{\rho(j)}, & j = 1, 2, \dots, i-1; \\ \frac{k\bar{p}_k - ip_1}{q_k - q_i} - \frac{p_1}{\rho(i)}, & j = i; \\ 1 - \frac{k\bar{p}_k - ip_1}{q_k - q_i}, & j = k; \\ 0, & i < j < k \text{ or } k < j. \end{cases} \quad (2.19)$$

It is easy to check that $G_t(x, p_1, \bar{p}_k)$ is a concave function of $(p_1, \bar{p}_k)$. Hence there are two scenarios that can happen: (1) the optimal solution is an interior point or (2) the optimal solution is a boundary point, namely,

$$\begin{aligned} (1) \{ (p_{1t}, \bar{p}_{kt}) : \bar{p}_k < p_1; \frac{k\bar{p}_k - ip_1}{q_k - q_i} < 1; \frac{p_1}{\rho(i+1)} > \frac{k\bar{p}_k - ip_1}{q_k - q_i} > \frac{p_1}{\rho(i)} \}; \\ (2) \{ (p_{1t}, \bar{p}_{kt}) : \bar{p}_k < p_1; \frac{k\bar{p}_k - ip_1}{q_k - q_i} < 1; \frac{p_1}{\rho(i+1)} = \frac{k\bar{p}_k - ip_1}{q_k - q_i} > \frac{p_1}{\rho(i)} \}. \end{aligned} \quad (2.20)$$

For the first case, a necessary condition for the block-price schedule $(p_1, \bar{p}_k)$ to be optimal is that the gradients of $G_t(x, p_1, \bar{p}_k)$ are zero:

$$\frac{\partial G_t(x, p_1, \bar{p}_k)}{\partial p_1} = 0 \text{ and } \frac{\partial G_t(x, p_1, \bar{p}_k)}{\partial \bar{p}_k} = 0. \quad (2.21)$$

For the second case, a necessary condition for the block-price schedule $(p_1, \bar{p}_k)$ to be optimal is that it satisfies the first order condition:

$$\frac{dG_t(x, p_1, \bar{p}_k(p_1))}{dp_1} = 0 \text{ where } \bar{p}_k(p_1) \text{ is derived from } \frac{p_1}{\rho(i+1)} = \frac{k\bar{p}_k - ip_1}{q_k - q_i}. \quad (2.22)$$

Denote these solutions for the associated unconstrained problem (2.21) and (2.22) as $(p_1^{ib}(t, x), \bar{p}_k^{ib}(t, x))$ ($b = 1, 2$) respectively. If it further satisfies the corresponding condition (b) ($b = 1, 2$) in (2.20), then it becomes a qualified block-price candidate. Otherwise the optimal solution cannot be located in the given domain.

Case (BB): $k < x$. Let h ($k \leq h \leq x \wedge K$) represent the maximum number of units that a customer will buy at $\bar{p}_k$. Again there are two cases: (BBA) the block threshold k is equal to the largest purchase quantity at price $\bar{p}_k$: $k = h$; and (BBB) the block threshold k is strictly less than the largest purchase quantity at price $\bar{p}_k$: $k < h$.

Case (BBA): $k = h$. Note that the purchase probability $\alpha_j(p_1, \bar{p}_k)$ is as the same as (2.19), but $\bar{p}_k$ further needs to satisfy the following incentive condition (IC2):

$$\{(p_1, \bar{p}_k): \rho(h+1) \leq \bar{p}_k\}. \quad (\text{IC2})$$

Hence there are four scenarios for the optimal solution:

$$\begin{aligned} (1) \{ (p_1, \bar{p}_k): \bar{p}_k < p_1; \frac{k\bar{p}_k - ip_1}{q_k - q_i} < 1; \frac{p_1}{\rho(i+1)} > \frac{k\bar{p}_k - ip_1}{q_k - q_i} > \frac{p_1}{\rho(i)}; \rho(h+1) < \bar{p}_k \}; \\ (2) \{ (p_1, \bar{p}_k): \bar{p}_k < p_1; \frac{k\bar{p}_k - ip_1}{q_k - q_i} < 1; \frac{p_1}{\rho(i+1)} = \frac{k\bar{p}_k - ip_1}{q_k - q_i} > \frac{p_1}{\rho(i)}; \rho(h+1) < \bar{p}_k \}; \\ (3) \{ (p_1, \bar{p}_k): \bar{p}_k < p_1; \frac{k\bar{p}_k - ip_1}{q_k - q_i} < 1; \frac{p_1}{\rho(i+1)} > \frac{k\bar{p}_k - ip_1}{q_k - q_i} > \frac{p_1}{\rho(i)}; \rho(h+1) = \bar{p}_k \}; \\ (4) \{ (p_1, \bar{p}_k): \bar{p}_k < p_1; \frac{k\bar{p}_k - ip_1}{q_k - q_i} < 1; \frac{p_1}{\rho(i+1)} = \frac{k\bar{p}_k - ip_1}{q_k - q_i} > \frac{p_1}{\rho(i)}; \rho(h+1) = \bar{p}_k \}. \end{aligned} \quad (2.23)$$

For each scenario, a necessary condition for the price schedule $(p_1, \bar{p}_k)$ to be optimal is that it satisfies the corresponding first order condition with the associated boundary condition,

$$\begin{aligned} \frac{\partial G_t(x, p_1, \bar{p}_k)}{\partial p_1} = 0 \text{ and } \frac{\partial G_t(x, p_1, \bar{p}_k)}{\partial \bar{p}_k} = 0; & \quad \frac{dG_t(x, p_1, \bar{p}_k(p_1))}{dp_1} = 0 \text{ s.t. } \bar{p}_k(p_1) = \frac{p_1}{k} \left[\frac{q_k - q_i}{\rho(i+1)} + i \right]; \\ \frac{dG_t(x, p_1, \bar{p}_k)}{dp_1} = 0 \text{ where } \rho(h+1) = \bar{p}_k; & \quad \frac{p_1}{\rho(i+1)} = \frac{k\bar{p}_k - ip_1}{q_k - q_i} \text{ and } \rho(h+1) = \bar{p}_k. \end{aligned}$$

Denote these solutions for above four unconstrained problems as $(p_1^{ikb}(t, x), \bar{p}_k^{ikb}(t, x))$ ($b = 1, 2, 3, 4$) respectively. If it further satisfies the corresponding condition (b) ($b = 1, 2, 3, 4$) in (2.23), then it becomes a qualified block-price candidate.

Case (BBB): $k < h$, . The purchase probability $\alpha_j(p_1, \bar{p}_k)$ now becomes

$$\alpha_j(p_1, \bar{p}_k) = \begin{cases} \frac{p_1}{\rho(j+1)} - \frac{p_1}{\rho(j)}, & j = 1, 2, \dots, i-1; \\ \frac{k\bar{p}_k - ip_1}{q_k - q_i} - \frac{p_1}{\rho(i)}, & j = i; \\ \frac{\bar{p}_k}{\rho(k+1)} - \frac{k\bar{p}_k - ip_1}{q_k - q_i}, & j = k; \\ \frac{\bar{p}_k}{\rho(j+1)} - \frac{\bar{p}_k}{\rho(j)}, & j = k+1, \dots, h-1; \\ 1 - \frac{\bar{p}_k}{\rho(j)}, & j = h; \\ 0, & i < j < k \text{ or } x < j \leq K. \end{cases} \quad (2.24)$$

Due to the constraint condition for $\bar{p}_k$, there are still two cases: (1) $h = x$ and $x < K$ and (2) $h < x$ or $x \geq K$. When $h = x$ and $x < K$, $\bar{p}_k$ must further satisfy the following incentive condition (IC3):

$$\{(p_1, \bar{p}_k): \bar{p}_k < \rho(h)\}. \quad (\text{IC3})$$

Similar to the analysis for Case (BA), there are two scenarios. Denote the solution for the associated unconstrained problem as $(p_1^{ixb}(t, x), \bar{p}_k^{ixb}(t, x))$ respectively, if it still satisfies both

the corresponding condition (b) ($b = 1, 2$) in (2.20) and IC3, then it becomes a qualified block-price candidate. When $h < x$ or $x \geq K$, $\bar{p}_k$ needs to satisfy both IC2 and IC3. As the analysis for Case (BBA), there are four scenarios. Denote the solution for the associated unconstrained problem as $(p_1^{ihb}(t, x), \bar{p}_k^{ihb}(t, x))$ respectively, if it also satisfies both the corresponding condition (b) ($b = 1, 2, 3, 4$) in (2.23) and IC3, then it becomes a qualified block-price candidate.

We now summarize our discussions above into the following main result of this section.

Proposition 2.7 Under DBP scheme, the optimal block-price scheme $(p_1(t, x), \bar{p}_k(t, x))$ satisfies

$$(p_1(t, x), \bar{p}_k(t, x)) \in \underset{(p_1, \bar{p}_k)}{argmax} G_t(x, p_1, \bar{p}_k).$$

where $(p_1, \bar{p}_k)$ are the above obtained qualified block-price candidates, namely,

$$\left\{ \begin{array}{l} A: p_1 \text{ or } \bar{p}_k \\ B: (p_1, \bar{p}_k) \end{array} \right\} \left\{ \begin{array}{l} BA: k = x: \begin{array}{l} p_1^{ia}(t, x) \text{ or } \bar{p}_k^{ia}(t, x) \\ (p_1^{ib}(t, x), \bar{p}_k^{ib}(t, x)) \end{array} \\ BB: k < x \left\{ \begin{array}{l} BBA: h = k \quad (p_1^{ikb}(t, x), \bar{p}_k^{ikb}(t, x)) \\ BBB: h = k \left\{ \begin{array}{l} h = x \text{ and } x < K: (p_1^{ixb}(t, x), \bar{p}_k^{ixb}(t, x)) \\ h < x \text{ or } x \geq K: (p_1^{ihb}(t, x), \bar{p}_k^{ihb}(t, x)) \end{array} \end{array} \right. \end{array} \right. \end{array} \right. \begin{array}{l} (IC1) \\ (IC1 + IC2) \\ (IC1 + IC3) \\ (IC1 + IC2 + IC3) \end{array}$$

It is worthwhile highlighting that DBP scheme simultaneously optimizes the two blocks and the associated prices. There are several interesting and practical variations from DBP model. First, with minor modification, we can study the case of static blocks and dynamic pricing problem, where the retailer implements the two blocks schedule $(1, k)$ during the entire time horizon, only adjusting the prices to maximize the expected value. Moreover, this result enables us to tackle the static block design problem, namely, optimizing the static two blocks $(1, k)$. Last, but not the least, the method for DUP and DBP schemes can be generalized to multiple blocks cases. The

analytical procedure provides us a methodology to address the dynamic multi-block pricing (DMBP) problem.

2.5.3 Comparison among Different Schemes

So far, we have examined three different pricing schemes: dynamic nonuniform pricing (DNP), dynamic uniform pricing (DUP) and dynamic block pricing (DBP). Now we study the relationship of the expected revenue among these three schemes. Let $V_t^N(x)$, $V_t^U(x)$ and $V_t^B(x)$ be the expected revenue, $\mathbf{p}^N(t, x)$, $p(t, x)$ and $(p_1^B(t, x), \bar{p}_k^B(t, x))$ be the optimal price(s), and K^N , K^U and K^B be the largest purchase quantity for DNP, DUP and DBP scheme respectively. If $K^B \geq K^U$, the optimal price for DUP scheme is a feasible policy for DBP scheme. Moreover if the bundle schedule for DNP scheme is consecutive and $K^N \geq K^B$ the optimal price for DBP scheme is a feasible solution for DNP scheme. This analysis leads to the dominant relationship among the three schemes.

Proposition 2.8 (a) If $K^U \leq K^B$, $V_t^B(x) \geq V_t^U(x)$;

(b) If the bundle schedule for DNP scheme is consecutive and $K^N \geq K^B$, $V_t^N(x) \geq V_t^B(x)$.

When the inventory is large enough ($x \geq Kt$), all schemes use a myopic price policy. For DBP scheme, we further assume the highest block threshold $\bar{k}$ is equal to the largest purchase quantity K^B . If $K^N = K^B$, the effective prices under DNP and DBP schemes are the same, where only $p_K^N(t, x)$ and $\bar{p}_K^B(t, x)$ (which are equivalent) is effective respectively. Hence the selling processes for DNP and DBP schemes are the same, which implies the expected revenues for the two schemes are the same.

Proposition 2.9 When $K^N = K^B = \bar{k}$, if the inventory is large enough ($x \geq Kt$), $V_t^N(x) = V_t^B(x)$.

As indicated earlier, DBP scheme can be generalized to DMBP schemes. With minor modification, both Propositions 2.8 and 2.9 still hold when the block scheme becomes multiple blocks. Therefore, regardless of the number of blocks in a DMBP scheme, DNP scheme always outperforms DMBP scheme.

2.6 Numerical Comparison among Different Schemes

In this section, we conduct numerical analysis on the pricing behavior and performance of DNP, DUP, and DBP schemes. We first illustrate the optimal prices and the associated purchase probabilities among different schemes. Next we compare the revenue performance among the three schemes. Finally, we identify situations when DNP scheme significantly outperforms DUP scheme. Throughout this section, the bundle schedule for DNP scheme is consecutive and $K^B = \bar{k}$ for DBP scheme. For ease of exposition, we say K is the largest purchase quantity for all DNP, DUP and DBP schemes which, in effect, means $K = K^N = K^B \geq K^U$.

2.6.1 Optimal Prices and Purchase Probabilities

To gain the key insight into the pricing behavior for these schemes, we consider a firm implementing dynamic pricing with parameters $\lambda_t = \lambda = 0.8$, $\rho(1) = 10$, $\rho(2) = 5$ and $K = 2$. In such a case, DNP and DBP schemes are the same, which will be evident in Section 2.6.3. Thus, we only need to compare DNP and DUP schemes. Figure 2.4(a) depicts the two optimal prices $(p_1(t, x), p_2(t, x))$ and the associated price difference $\Delta p_2(t, x)$ for DNP scheme as well as the optimal price $p(t, x)$ for DUP scheme at $t = 40$ with the inventory x varying from 1 to 60.

And Figure 2.4(b) shows the associated purchase probabilities of buying one unit and two units under DNP scheme (α_1^N and α_2^N), and under DUP scheme (α_1^U and α_2^U) respectively.

Figure 2.4 Optimal price(s) and purchase probabilities under DNP and DUP schemes

Recall that for deterministic and single-unit demand, the optimal price is the myopic price if the inventory is high and otherwise, it is the run-out price. For comparative purpose, it is more convenient to classify the inventory level here into three cases: (a) high inventory ($x \geq 44$); (b) intermediate inventory ($10 \leq x \leq 43$); (c) low inventory ($x \leq 9$). When the inventory level is high ($x \geq 44$), both DNP and DUP schemes approximately use the associated myopic price policy. While there is only one price under DUP scheme, interestingly, some customers buy one unit and some others buy two units. However, under DNP scheme, only $p_{2t}(x)$ is effective, that is, all customers purchase two units. For low inventory ($x \leq 9$), the only effective price for DNP scheme is $p_{1t}(x)$, which is approximately equal to the optimal price $p_t(x)$ for DUP scheme. Moreover, no customer buys two units under both schemes. The reason is that due to the scarcity of inventory, there is no incentive for either scheme to capture customers' second-unit demand. It

further implies that the pricing behaviors of the two schemes are similar to the pricing behavior with single-unit demand. As the left inventory falls between 10 and 43, the retailer under DNP scheme adjusts the prices for one unit and two units in a smooth way. However, under DUP scheme, the retailer has to decide between implementing the optimal price for single-unit demand ($x \leq 16$) or the optimal price for both one-unit and two-unit demand ($x \geq 17$), which results in the price jump for the optimal price.

2.6.2 Revenue Impact: DNP verses DUP and DBP

We now compare revenue performance among three schemes. Specifically, we consider five values for the largest purchase quantity with $K = (2, 3, 4, 6, 8)$. To make the considered cases eligible and representative, for each K , we fix $\rho(1) = 10$, then select $\{q(1), \dots, q(K)\}$ or $\{\rho(1), \dots, \rho(K)\}$ from four types of series. The first two types are taken from Maskin and Riley (1984): (a) $q(k) = \rho(1)k^{1/\eta}$ with $\eta = \{1.2, 1.5, 1.8, 2, 2.2, 2.5, 3\}$ and (b) $q(k) = \rho(1)(1 + a \ln k)$ with $a = \{0.2, 0.3, \dots, 1\}$. The other two are geometric series and arithmetic series respectively: (c) $\rho(k+1) = \rho(k)s$ with $s = \{0.1, 0.2, \dots, 0.9\}$ and (d) $\rho(k) = \rho(1) - (k-1)s$ with $s = \{0.2, 0.3, \dots, 1.3\}$. Hence for each K , there are 37 instances. We let $\lambda_t = \lambda = 0.8$, remaining time $t = 40$ and inventory level vary from 1 to 120.

We first examine the revenue improvement of DNP over DUP scheme. Let $R_1(x) \stackrel{\text{def}}{=} [V_t^N(x) - V_t^U(x)]/V_t^U(x)$ denote the percentage improvement of DNP over DUP scheme. Recall that the expected revenues from both schemes are almost the same when the inventory is low; hence the aim here is to identify the potential of the improvement. We use the highest percentage improvement $R_1^{\text{Max}}(x)$, which is defined as the highest $R_1(x)$ over all 37 instances, to characterize the potential of revenue improvement. Figure 2.5 depicts the highest percentage

improvement for DNP scheme over DUP scheme for different K . We obtain some managerial insights from these results. First, the potential improvement is increasing in the inventory; it indicates that the higher the inventory, the more opportunity that the retailer can improve the revenue. Moreover, the potential improvement is increasing in K . Hence if a customer is likely to purchase more units, the potential for revenue improvement from adopting DNP scheme is higher. Furthermore, even if a customer just chooses to buy two units, the revenue improvement can be as high as 30%. In other words, the potential can be huge for adopting DNP scheme and hence retailers should take advantage of this opportunity in practice. This explains the ubiquitous phenomenon of nonuniform pricing in reality.

Figure 2.5 The highest percentage improvement $R_1^{Max}(x)$ for different K

Another interesting question is why most of nonuniform pricing behaviors in practice usually have only two prices? To answer this question, we study the relative performance of DBP over DNP scheme by evaluating $R_2(x) \stackrel{\text{def}}{=} V_t^B(x)/V_t^N(x)$. For each K , we use $R_2^{Min}(x)$, the lowest $R_2(x)$ over all 37 instances, to characterize the relative performance of DBP over DNP scheme.

As shown in Figure 2.6, the key observation is that, regardless of what the inventory level or largest purchase quantity K is, DBP scheme captures more than 98% of the optimal revenue for DNP scheme.

Figure 2.6 The lowest relative performance of $R_2^{Min}(x)$ for different K

Nevertheless, this observation is derived for remaining time $t = 40$, we therefore test the robustness of this conclusion by adding the time dimension, which basically reflects the total expected arriving customer. In particular, we let $\lambda_t = \lambda = 0.8$, the remaining time now ranges from 1 to 200 and the inventory level varies from 1 to 500. Moreover, it follows from Figure 2.6 that given the remaining time, the higher the largest purchase quantity K is, generally the lower the relative performance becomes. Following the essence of the lowest relative performance, we consider the worst case of $K = 8$, as shown in Figure 2.7.

Figure 2.7 The lowest relative performance of $R_2^{Min}(x)$ for $K = 8$

Figure 2.7 indicates that, when $x \geq Kt$ (i.e., the inventory is large), or $x \leq 1$ (i.e., the inventory is small) or $T = 1$ (i.e., the time is scarce), the selling processes for DNP and DBP are the same and hence both of them generate the same revenue. Otherwise, DNP scheme outperforms DBP scheme. If we evaluate the lowest relative performance only based on remaining time t , we observe that when t is small, it drops relatively fast; but as t becomes larger, it decreases very slowly. Moreover, even for $t = 200$, the lowest relative performance is still greater than 97%. It is a surprise that the performance of DBP scheme is so close to DNP scheme. Recall that DNP and DBP are the same in Section 2.6.1. Consistent with the driving force there, as DBP scheme dynamically chooses the blocks as well as the prices, it can adjust the selling process so that the associated revenue rate approximates that for DNP scheme. Finally, note that the revenue for DNP scheme is an upper bound for any DMBP scheme, therefore DBP scheme can always capture most of the revenue generated any DMBP scheme. Thus, there is little need to adopt multiple blocks pricing, DBP scheme performs sufficiently well.

2.6.3 DUP verses DBP: when will DBP significantly outperform DUP scheme?

As shown in Section 2.6.2, DNP scheme has the potential to substantially improve the revenue over the DUP scheme. Hence the next question is when the improvement becomes significant? For the sake of practical relevance as well as ease of illustration, we study the case when $K = 2$. Consider a retailer facing customers ($\lambda_t = \lambda = 0.8$) with fixed $\rho(1) = 10$ and remaining time is 60. We evaluate $R_1(x)$, the percentage improvement of DNP over DUP scheme, according to different level of utility sensitivity (β) and initial inventory (x). Previous works (e.g., Gallego and van Ryzin 1994 and Zhao and Zheng 2000) show that dynamic pricing policy can achieve 5-10% improvement over the optimal fixed price policy. Here we use 7% as the benchmark for a significant improvement and 20% as the benchmark for enormous improvement. The results are summarized in Table 2.1.

Table 2.1 Percentage improvement $R_1(x)$ of DNP over DUP scheme (%)

β	Initial Inventory (x)												
	5	10	15	20	25	30	35	40	45	50	55	60	∞
0.1	0.00	0.02	0.08	0.28	1.01	2.98	5.40	7.29	8.57	9.35	9.76	9.93	10
0.2	0.01	0.05	0.21	0.84	3.08	7.39	11.77	15.13	17.43	18.84	19.57	19.88	20
0.3	0.01	0.10	0.46	1.98	6.34	12.66	18.70	23.30	26.46	28.40	29.41	29.83	30
0.4	0.02	0.16	0.85	3.70	9.02	14.49	18.16	20.09	21.08	21.69	22.11	22.36	22.5
0.5	0.03	0.26	1.65	5.38	9.24	11.04	11.54	11.69	11.82	12.02	12.23	12.39	12.5
0.6	0.04	0.51	2.89	5.61	6.35	6.32	6.21	6.16	6.20	6.32	6.47	6.58	6.67
0.7	0.07	1.22	3.11	3.29	3.11	2.95	2.86	2.84	2.89	2.97	3.07	3.16	3.21
0.8	0.22	1.40	1.34	1.18	1.08	1.02	1.00	1.01	1.04	1.10	1.16	1.21	1.25
0.9	0.31	0.26	0.21	0.19	0.17	0.17	0.17	0.18	0.19	0.22	0.24	0.26	0.28

Additional explanation is needed when the inventory approaches infinity, corresponding to the last column in Table 2.1. From (2.10), we know that the only effective price for DBP scheme is $p_2(60, \infty) = q(2)/2$ and the revenue rate is $\lambda_t q(2)/4$. By Proposition 2.5, it follows that the revenue rate for DUP scheme is $\lambda_t q(1)/4$ if $3\rho(2) \leq \rho(1)$ and $\lambda_t \rho(1)\rho(2)/[\rho(1) + \rho(2)]$ otherwise. Hence as the inventory goes to infinite, the percentage improvement $R_1(x) = \beta$ if $0 \leq \beta \leq \frac{1}{3}$ and $R_1(x) = \frac{(1+\beta)^2}{4\beta} - 1$ if $0 < \beta \leq 1$.

Table 2.1 tells us that the percentage improvements are negligible when the initial inventory is low or utility sensitivity is relatively high. The improvement becomes significant for high inventory and intermediate utility sensitivity and enormous for high inventory when $\beta = 0.3$ or 0.4 . These results are consistent with findings for the underlying pricing behavior in Section 2.6.1. When the inventory level is low, i.e., the resource is scarce, the selling processes for DNP and DUP schemes are almost the same. Hence the improvement is not significant. If utility sensitivity is relatively high, it incentivizes DUP scheme to capture mixed customers' demand most of the time, and hence the selling process is also similar to that of DNP scheme. Only for high inventory and intermediate β , DNP scheme significantly outperforms DUP scheme.

There is still a subtle issue to be addressed. It is expected that the DNP and DBP schemes should be the same for $K = 2$. However, Figure 2.6 shows that the expected revenue for DBP scheme can strictly less than that for DNP scheme. To understand the difference, one only needs to examine the assumptions. The bundle in DNP scheme is inseparable; hence it precludes the possibility that customers purchase some combination of the bundles. For example, when $t = 2$ and $\beta = 0.9$, the price of two-unit bundle is strictly higher than the amount for two one-unit bundles, but some customers will still purchase the two-unit bundle. This constraint is replaced

in DBP scheme by the customers' rationality condition (i.e., $p_1 \geq \bar{p}_k$). Nevertheless, it happens only for strong utility sensitivity, that is, $\beta = 0.9$ in our example, where the retailer just needs DUP scheme.

2.7 Heuristics for DNP, DBP and DUP schemes

In this section, we develop heuristics for the three schemes. In particular, for DNP and DUP schemes, we study the associated fluid models and consider the fixed-price heuristics. For DBP scheme, due to the difficulty for solving the associated fluid model, we construct a heuristic from the solution of the fluid model for DNP.

2.7.1 The heuristic for DNP scheme

Consider the following deterministic version of the problem in Section 2.3, given the left selling time t , the firm has a stock level x , a continuous quantity of product to sell. Given customer's arrival rate λ and the bundle and price schedule $(\mathbf{n}, \mathbf{p})$, the instantaneous demand rate for the k th bundle is $\lambda \alpha_k(\mathbf{p})$ which is deterministic and $\alpha_k(\mathbf{p})$ is given by (2.3). The retailer's problem is to maximize the total revenue generated during $[t, 0]$ given x , denoted by

$$V_t^{ND}(x) = \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p})$$

such that $\lambda t \sum_{k=1}^K k \alpha_k(\mathbf{p}) \leq x$ where the revenue function $R(\mathbf{p}) = \lambda t \alpha_k(\mathbf{p}) \mathbf{p}$. From (2.3), the retailer's problem is actually the quadratic program to maximize

$$R(\mathbf{p}) = \lambda_t t \alpha_k(\mathbf{p}) \mathbf{p} = \lambda_t t \sum_{k=1}^K \left[\left(1 - \frac{\Delta p_i}{\Delta q_i} \right) \Delta p_i \right]$$

such that $\lambda_t t \sum_{k=1}^K k \alpha_k(\mathbf{p}) \leq x$ and $\mathbf{p} \in \wp$. One can now solve the retailer's problem by using the standard method of quadratic programming. Note that the DNP scheme is a special case of dynamic group pricing model in Gallego and van Ryzin (1997), based on the conclusion there, one can analogously show that the fluid model heuristic is asymptotically optimal. Here we focus the performance of the fixed-price heuristic.

In particular, we study the relative performance of the fixed-price heuristic for DNP scheme by evaluating $R_3(x) \stackrel{\text{def}}{=} V_t^{ND}(x)/V_t^N(x)$. To be consistent, we keep using the numerical parameters in Section 2.6.2. For each K , we use $R_3^{Min}(x)$, the lowest $R_3(x)$ over all 37 instances, to characterize the worst relative performance of the fixed-price heuristic. As shown in Figure 2.8, when the inventory level is relatively low, the relative performance can be as low as 75%. This poor performance for the fixed-price heuristic is distinct from the result in Gallego and van Ryzin (1994). Hence the fixed-price heuristic is not a good heuristic of DNP scheme for low inventory. However when the inventory is large, the fixed-price heuristic can capture most of the revenue generated from the dynamic one.

Figure 2.8 The worst relative performance of the fixed-price heuristic for DNP scheme

2.7.2 The heuristic for DBP scheme

Analogous to the fixed-price heuristic for DNP scheme, given the block-price schedule $(p_1, \bar{p}_k)$, the revenue function for the deterministic demand is given by

$$R(p_1, \bar{p}_k) = \sum_{j=1}^K \alpha_j(p_1, \bar{p}_k) j \bar{p}_j.$$

Therefore the retailer's problem is to maximize the revenue $R(p_1, \bar{p}_k)$ subject to $\lambda_t t \sum_{j=1}^K \alpha_j(p_1, \bar{p}_k) j \leq x$. As there is no explicit expression for the purchase probability $\alpha_j(p_1, \bar{p}_k)$, it is difficult to find the solution for the associated fluid model. Hence we consider a heuristic solution that is created from the corresponding heuristic for DNP scheme. Given the solution for the fixed-price heuristic for DNP scheme is $(p_1^D, p_2^D, \dots, p_K^D)$. We first construct $K - 1$ fixed block-price candidates $(p_1, \bar{p}_2), (p_1, \bar{p}_3), \dots, (p_1, \bar{p}_K)$ by letting $p_1 = p_1^D$ and $\bar{p}_k = p_k^{FP}/k$ for $k > 1$, and then choose the best one among the $K - 1$ candidates.

Note that when the demand is deterministic, if the bundle schedule for the nonuniform pricing is consecutive, the fixed-price heuristic for DNP serves as an upper bound for the constructed heuristic for DBP. Therefore, we compare the performance between DBP heuristic and DNP heuristic by evaluating $R_4(x) \stackrel{\text{def}}{=} V_t^{BD}(x)/V_t^{ND}(x)$. As before, we keep using the numerical parameters in Section 2.6.2. For each K , we use $R_4^{Min}(x)$, the lowest $R_4(x)$ over all 37 instances, to characterize the worst relative performance of the DBP heuristic over DNP heuristic, which is shown in Figure 2.9. Analogous to the comparison between DBP and DNP in Section 2.6.2, DBP heuristic can always capture most of the revenue generated from DNP heuristic, the relative performance is always higher than 96.5% for our examples. The result highlights the finding that a little pricing flexibility or two prices are enough to generate most of the revenue from multiple

prices, regardless of whether it is in a dynamic pricing setting for stochastic demand or it is a fluid model.

Figure 2.9 The worst relative performance of DBP heuristic over DNP heuristic

Similar to the fixed-price heuristic for DNP scheme, we also consider the worst relative performance of the heuristic for DBP scheme, as shown in Figure 2.10. It is not surprising that the two heuristics for DBP and DNP show similar performance. The reason is that DBP and its heuristic are good approximations for DNP and its fixed-price heuristic respectively.

Figure 2.10 The worst relative performance of the heuristic for DBP scheme

2.7.3 The heuristic for DUP scheme

Similarly, one can consider the fluid model for DUP scheme. Given the unit price p and the largest purchase quantity K , the revenue function for the deterministic demand is given by

$$R(p) = \lambda_t t \sum_{k=1}^K \alpha_k(p) kp$$

where $\alpha_k(p)$ is given by (2.13). Hence the retailer's problem is to maximize the revenue $R(p)$ subject to $\lambda_t t \sum_{k=1}^K k \alpha_k(p) \leq x$. As the largest purchase quantity is K , so the solution for the associated unconstrained problem is the myopic price $p^* = K/[2 \sum_{k=1}^K 1/\rho(k)]$. Hence, if $\lambda_t t \sum_{k=1}^K k \alpha_k(p^*) \leq x$, then the optimal solution is p^* . Otherwise, the constraint is bound, there exists a i ($1 \leq i \leq K$) such that $\lambda_t t \sum_{k=1}^K k \alpha_k(\rho(i)) \leq x < \lambda_t t \sum_{k=1}^K k \alpha_k(\rho(i+1))$, and accordingly the optimal price is obtained by solving $\lambda_t t \sum_{k=1}^i k \alpha_k(p) = x$.

2.8 Conclusions and Future Directions

This chapter investigates the dynamic pricing problem of perishable asset with multi-unit demand under customer choice behavior. We examine three kinds of dynamic pricing schemes, namely, the dynamic nonuniform pricing, the dynamic uniform pricing and the dynamic block pricing. We present a detailed analysis of the structural properties for DNP and DUP schemes and provide a novel methodology to obtain the solutions for DUP and DBP schemes. We identify a necessary and sufficient condition for the structural properties of DNP scheme and a validation condition for classic single-unit demand dynamic pricing model in our context under DUP scheme. We further show a value function without structural properties can nevertheless exhibit truncated structural properties. Moreover, the underlying optimal price may display both time and inventory monotonicity.

Several important managerial insights arise from the extensive numerical study. When the inventory is scarce, the sales processes of all three schemes are almost the same, in which case the retailer prices the product such that customers at most purchase one unit. As the inventory increases, the benefit of implementing DNP over DUP scheme becomes significant. Our computational results reveal that the potential percentage improvement of DNP over DUP scheme ranges from 30% to 90% as the customer's largest purchase quantity increases from 2 to 8. This potential suggests that managers in the retailing industry need to identify the opportunity for nonuniform pricing. Most importantly, it further uncovers that DBP scheme achieves most of the revenue obtained by DNP scheme (more than 97%). In other words, all we need is at most two prices. This result liberates the industry from the burden of too many prices while it still enjoys the benefit of nonuniform pricing. This probably explains why most of the nonuniform pricing behavior in practice only has two prices.

Given that the potential improvement of DNP over DUP can achieve up to 30% when customers at most buy two units, we identify the circumstances for significant improvement. In particular, we find the percentage improvement becomes significant ($>7\%$) when the inventory is high and utility sensitivity ranging from 0.1 to 0.5 and enormous ($>20\%$) for high inventory with utility sensitivity between 0.3 and 0.4. This finding not only further highlights the importance of nonuniform pricing, but also pinpoints the direction for exploiting the potential of nonuniform pricing. For example, managers could improve utility sensitivity or the utility for second unit by giving customers the option to choose among different colors or styles of a fashion product, or even any two units from the similar price level storewide rather than just the identical product.

There are several directions for further research. One possible extension of our model is to study the case of general utility function. Unfortunately, work in economics (e.g., Maskin and Riley

1984) suggests this extension is not easy even without the presence of inventory consideration. Hence focusing on some other specific forms of utility function is plausible. Another possible extension is to treat the case of multiproduct quantity-dependent (Spence 1980), which is the combination of our work and Akcay et al. (2010)'s multiproduct dynamic pricing problem. The extension of DSKP with other type of customer choice is also important and demanding. Following Aviv and Pazgal (2008), the consideration of strategic customer under dynamic nonuniform pricing is both interesting and promising. To develop simple and implementable heuristics, the study on the associated fluid model is also an interesting avenue for future research.

2.9 Appendix: Proofs

Proposition 2.1 Under DNP scheme, there exists a unique optimal solution $\mathbf{p}(t, x) \in \wp$. Moreover, let $\mathbf{p}^*$ such that

$$\Delta p_k^* = \frac{\Delta q_k + \Delta_{d_k} V_{t-1}(x - n_{k-1}) \wedge \Delta q_k}{2} \text{ for } k = 1, \dots, K,$$

where $x \wedge y \equiv \min(x, y)$. If $\mathbf{p}^* \in \wp$, then $\mathbf{p}(t, x) = \mathbf{p}^*$.

Proof. First, we solve the associated problem of $G_t(x, \mathbf{p}) = \sum_{k=1}^K \left[\left(1 - \frac{\Delta p_k}{\Delta q_k}\right) (\Delta p_k - \Delta_{d_k} V_{t-1}(x - n_{k-1})) \right]$ such that $\{\mathbf{p}: 0 \leq \frac{\Delta p_k}{\Delta q_k} \leq 1 \text{ for any } 1 \leq k \leq K\}$. It is obvious that the optimal solution is $\Delta p_k^* = \frac{\Delta q_k + \Delta_{d_k} V_{t-1}(x - n_{k-1}) \wedge \Delta q_k}{2}$ for $k = 1, \dots, K$. If $\mathbf{p}^* \in \wp$, then the optimal price $\mathbf{p}(t, x) = \mathbf{p}^*$. Otherwise, suppose $\bar{k}$ is the smallest k such that $\frac{\Delta p_{\bar{k}}^*}{\Delta q_{\bar{k}}} < \frac{\Delta p_{\bar{k}-1}^*}{\Delta q_{\bar{k}-1}}$, which indicates that there will no customer purchase the $(\bar{k} - 1)$ th bundle, so we can constraint optimal solution

for (2.8) with $\frac{\Delta p_{\bar{k}-1}}{\Delta q_{\bar{k}-1}} = \frac{\Delta p_{\bar{k}}}{\Delta q_{\bar{k}}}$. Moreover, as there is no purchase for the $(\bar{k} - 1)$ th bundle, by combining $n_{\bar{k}-1}$ and $n_{\bar{k}}$, we construct a new bundle schedule $\mathbf{n}' = (n'_1, n'_2, \dots, n'_{K-1})$ such that $n'_k = n_k$ for $k < \bar{k} - 1$, $n'_{\bar{k}-1} = n_{\bar{k}-1} + n_{\bar{k}}$, and $n'_k = n_{k-1}$ for $k > \bar{k} - 1$. Accordingly, we have the corresponding $\mathbf{d}'$, $\Delta \mathbf{q}'$ and $\wp' = \left\{ \mathbf{p}' : 0 \leq \frac{\Delta p'_1}{\Delta q'_1} \leq \dots \leq \frac{\Delta p'_{K-1}}{\Delta q'_{K-1}} \leq 1 \right\}$, then Problem (2.8) becomes the degenerated problem $G_t(x, \mathbf{p}') = \sum_{k=1}^{K-1} \left[\left(1 - \frac{\Delta p'_k}{\Delta q'_k} \right) \left(\Delta p'_k - \Delta_{d'_k} V_{t-1}(x - n'_{k-1}) \right) \right]$ such that $\mathbf{p}' \in \wp'$. Based on its optimal solution, given that $\frac{\Delta p_{\bar{k}-1}}{\Delta q_{\bar{k}-1}} = \frac{\Delta p_{\bar{k}}}{\Delta q_{\bar{k}}}$, we can easily recover the optimal solution for (2.8). Note that one can repeat this procedure until there is only one bundle if necessary. Namely, the final solution is unique and hence the solution for (2.8) is unique. $\square$

Proposition 2.2 For DNP scheme, the value function $V_t(x)$ is concave if and only if the bundle schedule is consecutive.

Proof. First we show that the bundle schedule is consecutive is a necessary condition. Otherwise, if the bundle number $n_0, n_1, n_2, \dots, n_K$ is not consecutive, there exists n_k such that $n_{k-1} < n_k - 1 < n_k$. When $t = 1$, since customers only purchase n_i units, hence $V_1(n_k - 2) = V_1(n_k - 1) < V_1(n_k)$. It implies $V_1(n_k - 1) - V_1(n_k - 2) = 0 < V_1(n_k) - V_1(n_k - 1)$. Hence $V_t(x)$ is not concave.

Now we show it is also a sufficient condition. The proof is by backward induction on t . Since the bundle schedule is consecutive, namely, $d_k = 1$ and $\Delta q_k = \rho(k)$ for $1 \leq k \leq K$. When $t = 1$, from (2.11), $V_1(x) = \lambda_1 \sum_{k=1}^{K \wedge x} \frac{\rho(k)}{4}$. If $x \geq K$, $V_1(x) = V_1(K)$, hence $V_1(x + 1) - V_1(x) = 0 \leq V_1(x) - V_1(x - 1)$. When $x < K$, we have $V_1(x) = \lambda_1 \sum_{k=1}^x \frac{\rho(k)}{4}$; hence $V_1(x) - V_1(x - 1) =$

$\lambda_1 \rho(x)/4 \geq \lambda_1 \rho(x+1)/4 = V_1(x+1) - V_1(x)$. Therefore $V_1(x)$ is concave in x . Now suppose it holds for $t-1$, namely, $V_{t-1}(x) - V_{t-1}(x-1) \geq V_{t-1}(x+1) - V_{t-1}(x)$ for any $x \geq 1$, we show it holds for t . From (2.11),

$$\begin{aligned} V_t(x) &= \lambda_t \sum_{k=1}^K \frac{(\rho(k) - \Delta_1 V_{t-1}(x-k+1) \wedge \rho(k))^2}{4\rho(k)} + V_{t-1}(x) \\ &= \lambda_t \sum_{k=1}^K \frac{(\rho(k) - \Delta_1 V_{t-1}(x-k+1))^2}{4\rho(k)} I_{\{\rho(k) > \Delta_1 V_{t-1}(x-k+1)\}} + V_{t-1}(x), \end{aligned}$$

where we suppose that when $x \leq 0$, $\Delta_1 V_{t-1}(x)$ is some number large enough, for example, $\Delta_1 V_{t-1}(x) = \rho(1)$ for $x \leq 0$. Considering the function $h(k) = \rho(k) - \Delta_1 V_{t-1}(x+1-k)$, $1 \leq k \leq K+1$ where $\rho(K+1) = 0$. Since $\rho(k)$ is decreasing and $V_{t-1}(x)$ is concave in x , hence $h(k)$ is decreasing in k . Hence there must exist unique I , $1 \leq I \leq K$, such that $h(I) > 0$ and $h(I+1) \leq 0$, equivalently, $\rho(I) > \Delta_1 V_{t-1}(x+1-I)$ and $\rho(I+1) \leq \Delta_1 V_{t-1}(x+1-(I+1))$. Comparing $\rho(I)$ with $\Delta_1 V_{t-1}(x-I)$, it leads to two cases,

Case 1: $\rho(I) > \Delta_1 V_{t-1}(x+1-I)$ and $\rho(I) \leq \Delta_1 V_{t-1}(x-I)$,

Case 2: $\rho(I) > \Delta_1 V_{t-1}(x-I)$ and $\rho(I+1) \leq \Delta_1 V_{t-1}(x-I)$.

When Case 1 happens, it further corresponds to two possibilities:

Case 1.1: $\rho(I+1) > \Delta_1 V_{t-1}(x+1-I)$ and Case 1.2: $\rho(I+1) \leq \Delta_1 V_{t-1}(x+1-I)$.

When Case 2 happens, it also further corresponds to two possibilities:

Case 2.1: $\rho(I+1) > \Delta_1 V_{t-1}(x+1-I)$ and Case 2.2: $\rho(I+1) \leq \Delta_1 V_{t-1}(x+1-I)$.

In the following, we will examine the four cases one by one.

For Case 1.1, namely, $\rho(I) > \Delta_1 V_{t-1}(x+1-I)$, $\rho(I) \leq \Delta_1 V_{t-1}(x-I)$ and $\rho(I+1) > \Delta_1 V_{t-1}(x+1-I)$, we have

$$\begin{aligned}
& 2V_t(x) - V_t(x+1) - V_t(x-1) \\
&= \lambda_t \sum_{k=1}^{I-1} \frac{1}{4\rho(k)} \left\{ 2(\rho(k) - \Delta_1 V_{t-1}(x+1-k))^2 - (\rho(k) - \Delta_1 V_{t-1}(x+2-k))^2 \right. \\
&\quad \left. - (\rho(k) - \Delta_1 V_{t-1}(x-k))^2 \right\} \\
&\quad + \lambda_t \frac{1}{4\rho(I)} \left\{ 2(\rho(I) - \Delta_1 V_{t-1}(x+1-I))^2 - (\rho(I) - \Delta_1 V_{t-1}(x+2-I))^2 \right\} \\
&\quad - \lambda_t \frac{1}{4\rho(I+1)} (\rho(I+1) - \Delta_1 V_{t-1}(x+1-I))^2 + (\Delta_1 V_{t-1}(x) - \Delta_1 V_{t-1}(x+1))^2 \\
&\geq \lambda_t \left\{ \sum_{k=1}^{I-1} \frac{1}{4\rho(k)} \left\{ -\Delta_2 V_{t-1}(x+2-k)(\Delta_1 V_{t-1}(x+2-k) - \Delta_1 V_{t-1}(x+1-k)) \right. \right. \\
&\quad \left. \left. - \Delta_2 V_{t-1}(x+1-k)(\Delta_1 V_{t-1}(x-k) - \Delta_1 V_{t-1}(x+1-k)) \right\} \right. \\
&\quad \left. + \sum_{k=1}^{I-1} \frac{1}{2} [\Delta_1 V_{t-1}(x+2-k) + \Delta_1 V_{t-1}(x-k) - 2\Delta_1 V_{t-1}(x+1-k)] \right. \\
&\quad \left. + \frac{1}{4\rho(I)} \left\{ -\Delta_2 V_{t-1}(x+2-I)(\Delta_1 V_{t-1}(x+2-I) - \Delta_1 V_{t-1}(x+1-I)) \right\} \right. \\
&\quad \left. + \frac{1}{2} (\Delta_1 V_{t-1}(x+2-I) - \Delta_1 V_{t-1}(x+1-I)) + \frac{1}{4\rho(I)} (\rho(I) - \Delta_1 V_{t-1}(x+2-I))^2 \right. \\
&\quad \left. - \frac{1}{4\rho(I+1)} (\rho(I+1) - \Delta_1 V_{t-1}(x+1-I))^2 + (\Delta_1 V_{t-1}(x) - \Delta_1 V_{t-1}(x+1))^2 \right\}
\end{aligned}$$

The inequality holds here is just because $\lambda_t \leq 1$. Moreover, since

$$\begin{aligned}
& \frac{1}{4\rho(k)} \left\{ -\Delta_2 V_{t-1}(x+1-k)(\Delta_1 V_{t-1}(x-k) - \Delta_1 V_{t-1}(x+1-k)) \right\} \\
&\quad + \frac{1}{4\rho(k+1)} \left\{ -\Delta_2 V_{t-1}(x+2-(k+1))(\Delta_1 V_{t-1}(x+2-(k+1)) \right. \\
&\quad \left. - \Delta_1 V_{t-1}(x+1-(k+1))) \right\} \\
&= \frac{1}{4} (\Delta_1 V_{t-1}(x-k) - \Delta_1 V_{t-1}(x+1-k)) \left(\frac{1}{\rho(k+1)} - \frac{1}{\rho(k)} \right) \geq 0
\end{aligned}$$

for $1 \leq k \leq I-1$ and we have

$$\begin{aligned}
& \sum_{k=1}^{I-1} \frac{1}{2} [\Delta_1 V_{t-1}(x+2-k) + \Delta_1 V_{t-1}(x-k) - 2\Delta_1 V_{t-1}(x+1-k)] \\
&= \frac{1}{2} [(\Delta_1 V_{t-1}(x+1) - \Delta_1 V_{t-1}(x)) + (\Delta_1 V_{t-1}(x+1-I) - \Delta_1 V_{t-1}(x+2-I))]
\end{aligned}$$

Hence $2V_t(x) - V_t(x+1) - V_t(x-1)$ could be further rewritten as

$$\begin{aligned}
& 2V_t(x) - V_t(x+1) - V_t(x-1) \\
& \geq \lambda_t \left\{ \frac{1}{4\rho(1)} \{-\Delta_2 V_{t-1}(x+1)(\Delta_1 V_{t-1}(x+1) - \Delta_1 V_{t-1}(x))\} \right. \\
& \quad + \frac{1}{2}(\Delta_1 V_{t-1}(x+1) - \Delta_1 V_{t-1}(x)) + \frac{1}{4\rho(I)}(\rho(I) - \Delta_1 V_{t-1}(x+2-I))^2 \\
& \quad \left. - \frac{1}{4\rho(I+1)}(\rho(I+1) - \Delta_1 V_{t-1}(x+1-I))^2 + (\Delta_1 V_{t-1}(x) - \Delta_1 V_{t-1}(x+1)) \right\} \\
& \geq \lambda_t \left\{ \frac{1}{4\rho(1)}(\Delta_1 V_{t-1}(x) - \Delta_1 V_{t-1}(x+1))(2\rho(1) - \Delta_2 V_{t-1}(x+1)) \right. \\
& \quad + \frac{1}{4\rho(I)}(\rho(I) - \Delta_1 V_{t-1}(x+1-I))^2 - \frac{1}{4\rho(I+1)}(\rho(I+1) - \Delta_1 V_{t-1}(x+1-I))^2 \Big\} \\
& \geq \lambda_t \left\{ \frac{1}{4\rho(1)}(\Delta_1 V_{t-1}(x) - \Delta_1 V_{t-1}(x+1))(2\rho(1) - \Delta_2 V_{t-1}(x+1)) \right. \\
& \quad \left. + (\rho(I) - \rho(I+1)) \left(1 - \frac{\Delta_1 V_{t-1}(x+1-I)^2}{\rho(I)\rho(I+1)} \right) \right\} \geq 0.
\end{aligned}$$

The second inequality holds since $\rho(I) > \Delta_1 V_{t-1}(x+1-I) \geq \Delta_1 V_{t-1}(x+2-I)$ and the last holds since $\rho(1) \geq \Delta_1 V_{t-1}(x) \geq \Delta_1 V_{t-1}(x+1)$ and $\rho(I) \geq \rho(I+1) \geq \Delta_1 V_{t-1}(x+1-I)$.

For Case 1.2, namely, $\rho(I) > \Delta_1 V_{t-1}(x+1-I)$, $\rho(I) \leq \Delta_1 V_{t-1}(x-I)$ and $\rho(I+1) \leq \Delta_1 V_{t-1}(x+1-I)$, analogous to the analysis of Case 1.1, we have

$$\begin{aligned}
& 2V_t(x) - V_t(x+1) - V_t(x-1) \\
& = \lambda_t \sum_{k=1}^{I-1} \frac{1}{4\rho(k)} \left\{ 2(\rho(k) - \Delta_1 V_{t-1}(x+1-k))^2 - (\rho(k) - \Delta_1 V_{t-1}(x+2-k))^2 \right. \\
& \quad \left. - (\rho(k) - \Delta_1 V_{t-1}(x-k))^2 \right\} \\
& \quad + \lambda_t \frac{1}{4\rho(I)} \left\{ 2(\rho(I) - \Delta_1 V_{t-1}(x+1-I))^2 - (\rho(I) - \Delta_1 V_{t-1}(x+2-I))^2 \right\} \\
& \quad + (\Delta_1 V_{t-1}(x) - \Delta_1 V_{t-1}(x+1)) \\
& \geq \lambda_t \left\{ \frac{1}{4\rho(1)}(\Delta_1 V_{t-1}(x) - \Delta_1 V_{t-1}(x+1))(2\rho(1) - \Delta_2 V_{t-1}(x+1)) \right. \\
& \quad \left. + \frac{1}{4\rho(I)}(\rho(I) - \Delta_1 V_{t-1}(x+1-I))^2 \right\} \geq 0.
\end{aligned}$$

For Case 2.1, namely, $\rho(I) > \Delta_1 V_{t-1}(x - I)$, $\rho(I + 1) \leq \Delta_1 V_{t-1}(x - I)$ and $\rho(I + 1) > \Delta_1 V_{t-1}(x + 1 - I)$, we have

$$\begin{aligned}
& 2V_t(x) - V_t(x + 1) - V_t(x - 1) \\
&= \lambda_t \sum_{k=1}^I \frac{1}{4\rho(k)} \left\{ 2(\rho(k) - \Delta_1 V_{t-1}(x + 1 - k))^2 - (\rho(k) - \Delta_1 V_{t-1}(x + 2 - k))^2 \right. \\
&\quad \left. - (\rho(k) - \Delta_1 V_{t-1}(x - k))^2 \right\} - \lambda_t \frac{1}{4\rho(I + 1)} (\rho(I + 1) - \Delta_1 V_{t-1}(x + 1 - I))^2 \\
&\quad + (\Delta_1 V_{t-1}(x) - \Delta_1 V_{t-1}(x + 1)) \\
&\geq \lambda_t \left\{ \sum_{k=1}^I \frac{1}{4\rho(k)} \left\{ -\Delta_2 V_{t-1}(x + 2 - k) (\Delta_1 V_{t-1}(x + 2 - k) - \Delta_1 V_{t-1}(x + 1 - k)) \right. \right. \\
&\quad \left. \left. - \Delta_2 V_{t-1}(x + 1 - k) (\Delta_1 V_{t-1}(x - k) - \Delta_1 V_{t-1}(x + 1 - k)) \right\} \right. \\
&\quad \left. + \sum_{k=1}^I \frac{1}{2} [\Delta_1 V_{t-1}(x + 2 - k) + \Delta_1 V_{t-1}(x - k) - 2\Delta_1 V_{t-1}(x + 1 - k)] \right. \\
&\quad \left. - \lambda_t \frac{1}{4\rho(I + 1)} (\rho(I + 1) - \Delta_1 V_{t-1}(x + 1 - I))^2 + (\Delta_1 V_{t-1}(x) - \Delta_1 V_{t-1}(x + 1)) \right\} \\
&\geq \lambda_t \left\{ \frac{1}{4\rho(1)} \left\{ -\Delta_2 V_{t-1}(x + 1) (\Delta_1 V_{t-1}(x + 1) - \Delta_1 V_{t-1}(x)) \right\} \right. \\
&\quad \left. + \frac{1}{4\rho(I)} \left\{ -\Delta_2 V_{t-1}(x + 1 - I) (\Delta_1 V_{t-1}(x - I) - \Delta_1 V_{t-1}(x + 1 - I)) \right\} \right. \\
&\quad \left. + \frac{1}{2} [(\Delta_1 V_{t-1}(x + 1) - \Delta_1 V_{t-1}(x)) + (\Delta_1 V_{t-1}(x - I) - \Delta_1 V_{t-1}(x + 1 - I))] \right. \\
&\quad \left. - \lambda_t \frac{1}{4\rho(I + 1)} (\rho(I + 1) - \Delta_1 V_{t-1}(x + 1 - I))^2 + (\Delta_1 V_{t-1}(x) - \Delta_1 V_{t-1}(x + 1)) \right\}.
\end{aligned}$$

Analogous to Case 1.1, the last inequality holds here since

$$\begin{aligned}
& \frac{1}{4\rho(k)} \left\{ -\Delta_2 V_{t-1}(x + 1 - k) (\Delta_1 V_{t-1}(x - k) - \Delta_1 V_{t-1}(x + 1 - k)) \right\} \\
&\quad + \frac{1}{4\rho(k + 1)} \left\{ -\Delta_2 V_{t-1}(x + 2 - (k + 1)) (\Delta_1 V_{t-1}(x + 2 - (k + 1)) \right. \\
&\quad \left. - \Delta_1 V_{t-1}(x + 1 - (k + 1))) \right\} \geq 0,
\end{aligned}$$

for $1 \leq k \leq I - 1$ and

$$\begin{aligned}
& \sum_{k=1}^I \frac{1}{2} [\Delta_1 V_{t-1}(x+2-k) + \Delta_1 V_{t-1}(x-k) - 2\Delta_1 V_{t-1}(x+1-k)] \\
&= \frac{1}{2} [(\Delta_1 V_{t-1}(x+1) - \Delta_1 V_{t-1}(x)) + (\Delta_1 V_{t-1}(x-I) - \Delta_1 V_{t-1}(x+1-I))].
\end{aligned}$$

Hence $2V_t(x) - V_t(x+1) - V_t(x-1)$ could be further rewritten as

$$\begin{aligned}
& 2V_t(x) - V_t(x+1) - V_t(x-1) \\
& \geq \lambda_t \left\{ \frac{1}{2} (\Delta_1 V_{t-1}(x-I) - \Delta_1 V_{t-1}(x+1-I)) \left(1 - \frac{\Delta_2 V_{t-1}(x+1-I)}{2\rho(I)} \right) \right. \\
& \quad \left. - \lambda_t \frac{1}{4\rho(I+1)} (\rho(I+1) - \Delta_1 V_{t-1}(x+1-I))^2 \right\} \\
& = \frac{\lambda_t}{2} \left\{ (\rho(I+1) - \Delta_1 V_{t-1}(x+1-I)) \left(1 - \frac{\Delta_2 V_{t-1}(x+1-I)}{2\rho(I)} \right) \right. \\
& \quad \left. - \frac{\rho(I+1) - \Delta_1 V_{t-1}(x+1-I)}{2\rho(I+1)} \right\} \\
& = \frac{\lambda_t}{2} \left\{ (\rho(I+1) - \Delta_1 V_{t-1}(x+1-I)) \left(\frac{1}{2} - \frac{\Delta_2 V_{t-1}(x+1-I)}{2\rho(I)} + \frac{\Delta_1 V_{t-1}(x+1-I)}{2\rho(I+1)} \right) \right\} \\
& = \frac{\lambda_t}{4} \left\{ (\rho(I+1) - \Delta_1 V_{t-1}(x+1-I)) \left(1 - \frac{\Delta_1 V_{t-1}(x-I)}{\rho(I)} \right) \right. \\
& \quad \left. + \Delta_1 V_{t-1}(x+1-I) \left(\frac{1}{\rho(I+1)} - \frac{1}{\rho(I)} \right) \right\} \geq 0.
\end{aligned}$$

The last inequality holds since the conditions $\rho(I+1) > \Delta_1 V_{t-1}(x+1-I)$ and $\rho(I) > \Delta_1 V_{t-1}(x-I)$.

For Case 2.2, namely, $\rho(I) > \Delta_1 V_{t-1}(x-I)$, $\rho(I+1) \leq \Delta_1 V_{t-1}(x-I)$ and $\rho(I+1) \leq \Delta_1 V_{t-1}(x+1-I)$, analogous to the analysis of Case 2.1, we have

$$\begin{aligned}
& V_t(x) - V_t(x+1) - V_t(x-1) \\
&= \lambda_t \sum_{k=1}^I \frac{1}{4\rho(k)} \left\{ 2(\rho(k) - \Delta_1 V_{t-1}(x+1-k))^2 - (\rho(k) - \Delta_1 V_{t-1}(x+2-k))^2 \right. \\
&\quad \left. - (\rho(k) - \Delta_1 V_{t-1}(x-k))^2 \right\} + (\Delta_1 V_{t-1}(x) - \Delta_1 V_{t-1}(x+1)) \\
&\geq \lambda_t \left\{ \frac{1}{2} (\Delta_1 V_{t-1}(x-I) - \Delta_1 V_{t-1}(x+1-I)) \left(1 - \frac{\Delta_2 V_{t-1}(x+1-I)}{2\rho(I)} \right) \right\} \geq 0.
\end{aligned}$$

The last inequality holds since $\rho(I) > \Delta_1 V_{t-1}(x-I) \geq \Delta_1 V_{t-1}(x+1-I)$. $\square$

Corollary 2.1 Under DNP scheme with consecutive bundle schedule, it is always true that

- (a) The marginal value of inventory $\Delta_1 V_t(x)$ is increasing in t and decreasing in x .
- (b) The marginal value of time $\Delta_t V_t(x)$ is increasing in x .
- (c) If $\lambda_t \geq \lambda_{t+1}$, then the marginal value of time holds with $\Delta_t V_t(x) \geq \Delta_{t+1} V_{t+1}(x)$.

Proof. (a) To show $V_t(x) - V_t(x-1) \geq V_{t-1}(x) - V_{t-1}(x-1)$, it is equivalent to show $V_t(x) - V_{t-1}(x) \geq V_t(x-1) - V_{t-1}(x-1)$. From (2.11), it suffices to show

$$\frac{(\rho(k) - \Delta_1 V_{t-1}(x+1-k) \wedge \rho(k))^2}{4\rho(k)} \geq \frac{(\rho(k) - \Delta_1 V_{t-1}(x-k) \wedge \rho(k))^2}{4\rho(k)}$$

which is trivial since $V_t(x)$ is concave in x .

(b) To show $V_t(x) - V_{t-1}(x) \geq V_t(x-1) - V_{t-1}(x-1)$, it is equivalent to $V_t(x) - V_t(x-1) \geq V_{t-1}(x) - V_{t-1}(x-1)$, which is directly from (a).

(c) To show $V_t(x) - V_{t-1}(x) \geq V_{t+1}(x) - V_t(x)$, from (2.11), it is equivalent to show

$$\lambda_t \frac{(\rho(k) - \Delta_1 V_{t-1}(x+1-k) \wedge \rho(k))^2}{4\rho(k)} \geq \lambda_{t+1} \frac{(\Delta q_k - \Delta_1 V_t(x+1-k) \wedge \rho(k))^2}{4\rho(k)}.$$

As $\lambda_t \geq \lambda_{t+1}$, it suffices to show $\Delta_i V_{t-1}(x) \leq \Delta_i V_t(x)$, which is trivial from (a). $\square$

Corollary 2.1 Under DNP scheme with consecutive bundle schedule, it is always true that

(a) The marginal value of inventory $\Delta_1 V_t(x)$ is increasing in t and decreasing in x .

(b) The marginal value of time $\Delta_t V_t(x)$ is increasing in x .

(c) If $\lambda_t \geq \lambda_{t+1}$, then the marginal value of time holds with $\Delta_t V_t(x) \geq \Delta_{t+1} V_{t+1}(x)$.

Proof. (a) First we consider the case of $t = 1$. Let $x \rightarrow \infty$,

$$\sup_{p \geq 0} G_1(\infty, p) = \sup_{p \geq 0, k \geq 1} \sum_{i=1}^k \left(1 - \frac{p}{\rho(i)}\right) p I\{\rho(k+1) \leq p < \rho(k)\}.$$

Since

$$\sum_{i=1}^k \left(1 - \frac{p}{\rho(i)}\right) p I\{\rho(k+1) \leq p < \rho(k)\} \leq \left[-\sum_{i=1}^k \frac{1}{\rho(i)}\right] p^2 + kp,$$

hence $\sup_{p \geq 0} G_1(\infty, p) \leq \sup_{k \geq 1} \left\{ \frac{k^2}{4 \sum_{i=1}^k 1/\rho(i)} \right\} \leq \sup_{k \geq 1} \left\{ \frac{\sum_{i=1}^k i}{2 \sum_{i=1}^k 1/\rho(i)} \right\}$. When $\lim_{k \rightarrow \infty} \rho(k) \cdot k = 0$, using the discrete version of L'Hospital's rule (see Fikhtengolts 1962), we have $\lim_{k \rightarrow \infty} \sum_{i=1}^k i / (\sum_{i=1}^k 1/\rho(i)) = \lim_{k \rightarrow \infty} k\rho(k) = 0$. Therefore there exists K ($K < \infty$) such that $p \in (\rho(K+1), \rho(K)]$ to maximizes $G_1(\infty, p)$. Moreover, given this K , the optimal price for $G_1(\infty, p)$ is $p^* = K/[2 \sum_{i=1}^K 1/\rho(i)]$.

Now we show that the retailer would not set the price less than p^* for $t > 1$. We prove it by contradiction. Suppose there exist $p' < p^*$ that optimizes the DUP problem at time t and inventory level x . That is, there exists $k \geq K$ such that $\rho(k+1) \leq p' < \rho(k)$ and $\sum_{i=1}^k \left(1 - \frac{p'}{\rho(i)}\right)$

$\frac{p'}{\rho(i)} \left(p' - \Delta_1 V_{t-1}(x - i + 1) \right) > \sum_{i=1}^K \left(1 - \frac{p^*}{\rho(i)} \right) (p^* - \Delta_1 V_{t-1}(x - i + 1))$. Since p^* is the optimal price at time $t = 1$ given that inventory level $x \geq K$, we have $\sum_{i=1}^K \left(1 - \frac{p^*}{\rho(i)} \right) p^* \geq \sum_{i=1}^k \left(1 - \frac{p'}{\rho(i)} \right) p'$. Combined the two inequalities, we have $-\sum_{i=1}^k \left(1 - \frac{p'}{\rho(i)} \right) \Delta_1 V_{t-1}(x - i + 1) > -\sum_{i=1}^K \left(1 - \frac{p^*}{\rho(i)} \right) \Delta_1 V_{t-1}(x - i + 1)$, which leads to $-\sum_{i=K+1}^k \left(1 - \frac{p'}{\rho(i)} \right) \Delta_1 V_{t-1}(x - i + 1) > \sum_{i=1}^K \left(\frac{p^* - p'}{\rho(i)} \right) \Delta_1 V_{t-1}(x - i + 1)$. This is impossible since the left side is not more than zero while the right side is always not less than zero. Finally, since $p(t, x)$ is not less than p^* , the purchase quantity is no more than K . $\square$

Proposition 2.6 For the DUP problem with $K \leq 2$, when $3\rho(2) \leq \rho(1)$, customers will purchase at most one unit under the optimal policy for any inventory x and time t .

Proof. From Proposition 2.4, it is sufficient to show that $p^* = p^1(1, x) = \rho(1)/2$ for $x \geq 2$.

Based on Proposition 2.5, $p^2(1, x) = \frac{1}{\frac{1}{\rho(1)} + \frac{1}{\rho(2)}}$; hence $0 < p^2(1, x) < \rho(2)$. This implies that

$p^2(1, x)$ is an optimal price candidate, so we need further show $p^2(1, x)$ never be the optimal

solution for maximizing the expected profit, namely, $\frac{\rho(1)}{4} \geq \frac{4}{4(\frac{1}{\rho(1)} + \frac{1}{\rho(2)})}$, and which is equivalent

to $3\rho(2) \leq \rho(1)$. $\square$

Chapter 3

Supply Chain Coordination with Dynamic Pricing Newsvendor

3.1 Introduction

Recently there has been extensive research (Elmaghraby and Keskinocak 2003) on the study of replenishment and dynamic pricing problem due to its wide prevalence in practice (e.g. fashion industry). However in reality, many fashion manufacturers largely rely upon independent retailers to distribute their products. For instance, Sport Obermeyer (Hammond and Raman 1996), a leading supplier in the U.S. fashion-ski-apparel market, sells its products through a network of over 600 specialty retailers (For more details, go to: www.obermeyer.com). This widely spread network not only enables Obermeyer to have a larger market and hence enjoy the economies of scale but also to benefit from the reputation and specialized skill of those retailers. Peter Glenn, one of its retailers, is well known for using dynamic pricing policy to serve the market. This specialization raises the need to study a decentralized control of the procurement and pricing process, rather than the traditional centralized case. Furthermore, this decentralization not only occurs between an upstream supplier and a downstream retailer, but also emerges between the marketing and the production departments of the same firm.

Motivated by these applications, this chapter studies a single-period supply chain with one supplier and a retailer that uses dynamic pricing policy to serve the market. In addition to fashion industry, many other industries face a similar problem, for example, agriculture (Rajan et al.

1992), high-tech (Feng and Xiao 2000) and publishing industries. A common characteristic for previous fixed-price/price-setting newsvendor models is that the demand is either exogenous or a function of price (see Petruzzi and Dada 1999). However, in reality, the demand originates from customer's purchasing process; hence the sales process corresponds to customers' choice process. Thus, we utilize the model setting in Revenue Management (e.g., Bitran and Mondschein 1997), assuming the customers' arrival process is a Poisson process and an arriving customer chooses to purchase the product according to the reservation value. This setting enables us not only to endogenize the demand with the retailer's price, but also to examine consumer surplus and social welfare of the underlying system. We use supply chain and system interchangeably. As the first step for studying a decentralized dynamic pricing system, we focus on the firm's decisions under wholesale price. The analysis of wholesale price is not only because it is widely used due to its simplicity in terms of administration, but also serves as a benchmark in determining whether it is worth using a more sophisticated contract with higher administrative cost.

We also study the coordination problem for such a decentralized system. Cachon (2003, p31) highlights the importance of supply chain coordination with dynamic pricing retailer. Due to the complexity of the general dynamic pricing question, no answer is given there. We take the first step to shed light on this problem by focusing on our context. Traditionally (e.g., Cachon 2003), to find out which supply chain contracts coordinate a system, one needs to check the contract one by one. This method ignores the similarity among those contracts and hence is not able to identify the differences among those contracts. We develop a stylized approach to analyze various contracts, which enables us to characterize the properties of a coordinating contract and hence study the supply chain contracts by group.

Specifically, according to the dependence of a contract's parameters on themselves and other factors (i.e., the stocking decision for the retailer or the realization of the system), we say it is an independent contract if the wholesale price, shares from selling and salvaging are not affected by each other and other factors, and otherwise it is a contingent contract. For independent contracts which include wholesale, buy-back and revenue-sharing contracts, we show the retailer's revenue function is concave in the procurement quantity. Hence the stocking level of the retailer is uniquely determined. Most importantly, we identify a necessary condition for an independent contract to coordinate the supply chain is that retailer has the same share from selling and salvaging each unit of the product. We further show that a necessary and sufficient condition for an independent contract to coordinate the supply chain is sharing the same portion of gain (selling and salvaging of the product) and pain (cost). When the procurement process is independent, a prerequisite for coordinating the system is coordinating the pricing process. In such cases, the necessary condition derived for the independent contract is also useful to examine contingent contracts.

Extensive numerical experiments are conducted to explore the performance of the decentralized system. To reflect the demand process in reality, our study focuses on both demand variability among the customer and the depreciation of the product. The impact of demand variability is well recognized in literature (e.g., Lariviere and Porteus 2001). On the other hand, the depreciation of the product is intrinsic in many industries. Lazear (1980) finds some fashion goods go out of style very quickly. The obsolescence in market value has also been examined by Rajan et al. (1992) for agricultural products. Zhao and Zheng (2000) identify the main reason responsible for dynamic pricing is the decrease in customer's reservation value distribution in fashion industry. Dynamic pricing is naturally a more sophisticated policy compared to price-

setting policy; hence most of our results use the supply chain with price-setting newsvendor as a benchmark. In particular, we are most interested in following questions: (a) as the centralization is not always achievable in reality, can the decentralized dynamic pricing outperforms the centralized static pricing system in term of profit and when this result happens; (b) how do individual firms perform or how the profit is divided between the supplier and retailer; (c) what is the value of pricing flexibility and how this benefit is shared between the supplier and the retailer; and (d) compared to a centralized system, how does a decentralized system perform or what is the value of the coordination?

The remainder of the chapter is organized as follows. Section 3.2 provides a survey of relevant literature. Section 3.3 presents the model and examines the decisions of the two firms under wholesale price. An illustrative example is given to enhance the understanding and motivate the study of coordination and computational experiments. In §3.4, we evaluate various contracts for coordinating the supply chain. Intensive computational studies and numerical comparison of different systems are presented §3.5. Section 3.6 concludes with discussion and future research directions.

3.2 Literature Review

This chapter extends pervious dynamic pricing newsvendor problem to a supply chain, hence our work is most closely related to the expansive literature on dynamic pricing and inventory problem (see Elmaghraby and Keskinocak 2003 and Chan et al. 2004 for extensive surveys). Kincaid and Darling (1963) first introduce the generic dynamic pricing model. Gallego and van Ryzin (1994) consider a continuous dynamic pricing problem where the customers arrive in Poisson process. But the demand intensity in their model is just a function of price. Bitran and

Mondschein (1997) introduce reservation value to explain the customer's purchasing process and focus on markdown in fashion industry. They also consider periodic pricing and utilize Weibull distribution to characterize customer's reservation value in their numerical analysis. Motivated by the shift of reservation value over time in fashion industry (also for airline industry), Zhao and Zheng (2000) generalize the demand to a nonhomogeneous Poisson process where the reservation value distribution is time-dependent. All of those papers find the revenue function exhibits diminishing marginal returns to inventory. Therefore the initial order quantity is well defined. Moreover, they show the optimal pricing policy exhibit monotonicity, which simplifies the implementation of pricing process. Bitran et al. (1998) conduct a real case in fashion retail chain. They validate the assumption of Poisson process and provide a method to estimate the parameters in Weibull distribution of the reservation value.

For these reasons, the retailer in our model uses this well built dynamic pricing policy. But we consider it in the framework of supply chain rather than a centralized dynamic pricing newsvendor. To the most of our knowledge, our work is the first research that considers the game behavior between the supplier and such a dynamic pricing retailer. For independent contracts, we show the concavity of the revenue function with respect to inventory and time, and the inventory monotonicity for optimal policy still hold, which is an important extension of previous case of the retailer's whole responsibility for selling and salvaging. On the other hand, note that the structural properties for independent contract are well established for decentralized fixed-price/price-setting system. For example, Lariviere and Porteus (2001) analyze the properties of wholesale contracts for fixed-price system. Song et al. (2008) characterize structural properties for buy-back contracts and Raz and Porteus (2013) examine the properties under both revenue-sharing and buy-back contract for price-setting system.

Some other pricing papers mostly related to our work are as follows. Monahan et al. (2004) study a periodic dynamic pricing newsvendor similar to Bitran and Mondschein (1997), where they show the dynamic pricing problem is just a price-setting newsvendor problem with recourse. They also develop structural properties for the optimal policy and demonstrate the monotone relationship between the optimal stocking factors. However, they utilize a specific form of demand function where the randomness in demand is price independent and multiplicative in nature, which is not able to capture customers' choice behavior and price-elastic demand over time for fashion industry. Xu and Hopp (2006) study an inventory replenishment problem with dynamic pricing where the customer arrival rate follows a geometric Brownian motion. They find closed-form optimal pricing policy, initial inventory level and expected profit. However customers in their model are identical which loses the heterogeneity of the consumer. Smith and Achabal (1997) develop clearance pricing and inventory policies for the situation which the sales rate depends on time, inventory and price. Rajan et al. (1992) examine the dynamic pricing and ordering decision with a demand function where the product exhibits both physical decay and value drop for each unit of inventory. However these models consider deterministic demand and the optimal price path is determined at the beginning of the selling season; hence they are not able to characterize the uncertainty of the demand. Another stream of literature related to our work is periodic dynamic inventory model with pricing where inventory can be replenished each period, including Zabel (1972), Federgruen and Heching (1999), Sainathan (2013) and so on. Chen and Simchi-Levi (2012) review this stream of literature.

Now we turn to the literature on decentralized supply chain in a newsvendor setting. Due to double marginalization (Spengler 1950), the supply chain with independent and self-interested firms generates less profit than the centralized system, which is manifested as stocking too little.

To induce the retailer to order system-optimal quantity, researchers have intensively studied various supply chain contracts. See Lariviere (1999) and Cachon (2003) for comprehensive reviews on supply chain contracting literature. When the retail price is exogenous, by improving retailer's value for the leftover, Pasternack (1985) shows that buy-back can coordinate the supply chain (achieving the same supply chain profits as a centralized one); by giving the retailer the option to return part of the order quantity, Tsay (1999) and Tsay and Lovejoy (1999) argue quantity-flexibility contract also coordinates the system; by designing a proper rebate on the retailer's volumes of sales, Taylor (2002) shows sales-rebate contract also achieves the system optimality.

While the aforementioned contracts induce the optimal stocking level, however for the supply chain with price-setting newsvendor, all of them distort the incentive between selling and salvaging the product and hence cannot achieve coordination. Price-discount contract (Bernstein and Federgruen 2005), in which the wholesale price and buy-back rate are adjusted linearly in the chosen retail price, coordinates the underlying system. Cachon and Lariviere (2005) illustrate by bearing the same share for cost and revenue, the revenue-sharing also coordinates the supply chain. Moreover, they compare revenue-sharing and buy-back contracts and find that while the description and implementation of them is different, they are basically the same since they generate the same flows for any realization of demand and are equivalent in administration cost. Except these two contracts, quantity discount (Jeuland and Shugan 1983) also coordinates the price-setting supply chain. As Moorthy (1987) argues, while the retailer's marginal revenue curve is untouched, the quantity discount adjusts the retailer's marginal cost curve so that the retailer's profit-maximizing quantity is the same as system's optimal quantity. All of these contracts are examined for our decentralized dynamic pricing system. Remember that the

dynamic pricing problem can be seen as a price-setting newsvendor problem with recourse; therefore a necessary condition for a contract to coordinate the supply chain is that it coordinates the price-setting system. Hence it is logical that neither buy-back nor quantity-flexibility nor sales-rebate is able to coordinate the supply chain. Finally, we show the revenue-sharing and quantity discount still coordinate the dynamic pricing supply chain. As the single posted price is not available for dynamic pricing retailer, however price-discount contract is not able to coordinate the supply chain.

3.3 Model Formulation

We consider a perishable product supply chain with one supplier and a retailer that faces one-shot inventory procurement problem and then uses dynamic pricing strategy to serve the market during the selling season. The supplier's production cost is c per-unit. For the retailer, the selling season lasts during a given time horizon $[T, 0]$. As Gallego and van Ryzin (1994) and Zhao and Zheng (2000), here the time index is reversed; i.e., $T(>0)$ indicates the starting time of the sales season, and time 0 is the end of the selling season. At any point of time, a single price p is offered, which depends on the retailer's strategy. We use a common compact set P to specify retailer's pricing strategy, which is basically determined by the market environment and the firm's long-term strategy. For example, the retailer would optimize the price from a general price set, which may be discrete prices $P = \{p_1, \dots, p_2\}$ or continuous prices $P = [p_L, p_H]$. It is easy to see that the fixed-price retailer is a special case of dynamic pricing with $P = \{p\}$. Customers arrive according to a non-homogeneous Poisson process with rate $\lambda_t, t \in [T, 0]$. Facing the posted price p , a arriving customer would purchase an item if the current price is below his or her reservation value. The retailer does not know the individual reservation value

for an arriving customer, but knows the distribution of the reservation price. Let $F_t(p)$ denotes the cumulative probability distribution of the reservation price of an arriving customer at time t . Any unmet demand for the retailer is lost.

Different from the literature in Revenue Management (e.g., Gallego and van Ryzin 1994; Bitran and Mondschein 1997; Zhao and Zheng 2000), we explicitly incorporate salvage value and let it to be s per unit for each item at the end the season. The reason is because the retailer's salvage value depends on the specific supply chain contract. For example, the salvage value for the retailer is more than s per unit under contract. It is worthwhile to indicating that our approach also applies to such system with additional fixed cost (e.g. transportation and management costs) and/or fixed transfer between the supplier and the retailer. Since when the retailer has entered the supply chain, fixed transfer and/or fixed cost would not affect the optimal decisions of two firms in the system. Moreover, with little revision of our model, a per-unit cost of retailer's inventory handling cost can be incorporated as Cachon and Lariviere (2005). Without loss of generality, we suppose such expense to be zero.

For ease of exposition, we also introduce the following notations:

x_C and x_D - optimal order quantities for centralized and decentralized system respectively with dynamic pricing newsvendor

π_C and π_D - the expected profit for centralized and decentralized system respectively with dynamic pricing newsvendor

w , π_s and π_r – the wholesale price, the expected profit for the supplier and retailer under decentralized system with dynamic pricing newsvendor

$x_C^{PS}, x_D^{PS}, \pi_C^{PS}, \pi_D^{PS}, w_s^{PS}, \pi_s^{PS}$ and π_r^{PS} are the corresponding notations for decentralized or centralized price-setting system.

3.3.1 Centralized Model

First let us quickly review the centralized system. Given pricing strategy P , let $V_t(x)$ be the supremum of expected revenue from any admissible policy over $[t, 0]$ with $x(t) = x$. A Markovian policy can be characterized by the pricing decision, which is a function from $(x, t) \in \{0, 1, \dots, x\} \times [T, 0]$ to P . When λ_t and $F_t(p)$ are continuous in t , $V_t(x)$ satisfies the following Bellman equation, which has a unique solution (see Gihman and Skorohod 1979 or Zhao and Zheng 2000)

$$\frac{\partial V_t(x)}{\partial t} = \sup_{p \in P} \lambda_t \bar{F}_t(p) [p - \Delta V_t(x)] \quad (3.1)$$

with constraints $V_0(x) = sx$ and $V_t(0) = 0$, where $\Delta V_t(x) = V_t(x) - V_t(x - 1)$ is the marginal expected value of the x th item at time t . Hence the optimal policy is the price p that maximizes the right-hand side of (3.1). Note this formulation is essentially the same as Zhao and Zheng (2000), except that here the salvage value is explicitly incorporated. Follow the argument in Gallego and van Ryzin (1994), the salvage value would not affect the structural results for expected value function. Therefore, based on Zhao and Zheng (2000), we have the following lemma.

Lemma 3.1 (a) $V_T(x)$ is concave in the order level x ; (b) $p_t(x)$ is decreasing in x for any t .

The centralizer would maximize the expected profit

$$\pi_C(x) = V_T(x) - cx.$$

From Lemma 3.1, the expected profit function for the centralizer is also concave in x , hence there exists an unique optimal order quantity level x_c to maximize the expected profit,

$$x_c = \operatorname{argmax}_{x \geq 0} \{V_T(x) - cx\}. \quad (3.2)$$

If $V_T(1) \geq c$, then $x_c = \max_{x \geq 0} \{x: \Delta V_T(x) \geq c\}$. This implies that the optimal stocking level is the largest quantity for which the marginal expected value exceeds the marginal cost.

3.3.2 Decentralized Model

For a decentralized supply chain system, the supplier (he) sets a wholesale price and then the retailer (she) optimizes the order quantity according to the price. Realizing the retailer would order different quantity level based on the proposed wholesale price, the supplier would adjust the price to maximize his profit. Hence the decision making of the two firms is a Stackelberg game (Tirole 1988): the supplier, acting as a leader, presents a wholesale price w as take-it-or-leave-it policy. The retailer, acting as a follower, chooses how many units to procurement and then sells them by setting price dynamically. The retailer accepts any contract allowing an expected profit greater than his opportunity cost, which here is set to zero. The retailer keeps using dynamic pricing to serve the market according to equation (3.1), and hence Lemma 3.1 still holds here. When the wholesale price is w , the expected profit for the retailer is a function of the order quantity x ,

$$\pi_r(x) = V_T(x) - wx$$

Obviously, the retailer's problem under a wholesale price is identical to that of the centralized system, except that here the procurement cost is w rather than c . Hence the optimal order quantity for the retailer is $x(w) = \operatorname{argmax}_{x \geq 0} \{V_T(x) - wx\}$. Suppose the supplier has the

information about the demand and knows the specific pricing strategy P that the retailer is adopting to sell the product. Hence the demand for the supplier is just the optimal order quantity for the retailer. Therefore the supplier's profit is

$$\pi_s(w) = (w - c)x(w).$$

Hence the supplier would optimize the wholesale price to pursue the maximal profit. If $x(w)$ is continuous and differential in w , we can use the standard procedure to find the optimal wholesale price. However, it is easy to find that here $x(w)$ is a step function because the demand is discrete; hence we need to find an efficient way to solve the problem.

Using the same technique in Lariviere and Porteus (2001), we study an equivalent formulation where the supplier faces the inverse demand $w(x) = \Delta V_T(x)$. Note the supplier can always lower down the wholesale price by an infinitesimal scale to induce the retailer to order x items. Hence the profit for the supplier becomes

$$\pi_s(x) = (\Delta V_T(x) - c)x.$$

The supplier would set order quantity x_D to maximize his profit, hence

$$x_D = \operatorname{argmax}_{x \geq 0} \{ (\Delta V_T(x) - c)x \} \quad (3.3)$$

where we suppose $\Delta V_T(0)$ is large enough. Obviously, from (3.3), we have $\Delta V_T(x_D) \geq c$, otherwise the profit for supplier is negative. Compared with the expression of the optimal order quantity for the centralized system x_C in (3.2), we immediately have $0 \leq x_D \leq x_C$.

Proposition 3.1 The optimal order quantity for decentralized system is always less than that for centralized system, namely, $x_D \leq x_C$.

It is well-known that double marginalization results in stocking too little in a supply chain with stochastic demand and fixed-price retailer (Lariviere and Porteus 2001). Proposition 3.1 extends this property to the supply chain with a dynamic pricing newsvendor. For fixed-price newsvendor, the higher stocking level results in a higher service level to the customer. Now we investigate the impact of stocking level on customer with dynamic pricing retailer.

In fact, using customer's utility to model the underlying sales process not only provides a mechanism to explain the purchasing process (Bitran and Mondschein 1997), but also enables us to evaluate consumers' surplus and social welfare. Following the approach in Mahajan and van Ryzin (2001), we use a sample path ω to describe the customer's arriving process. Denote $T(\omega)$ as customers' arriving times along path ω ; for any $t \in T(\omega)$, $u_t(\omega)$ is the reservation value for the customer arriving at time t . To measure the benefit from purchasing the product, as Varian (2010), we define consumer's surplus for purchasing the product as the difference between the consumer's reservation value for the product and the price the consumer actually pays, namely, $CS_t(\omega, x) = [u_t(\omega) - p_t(\omega, x)]^+$, where $p_t(\omega, x)$ is the price for the product at t along the path ω given initial inventory level x .

Lemma 3.2 Compared to decentralized system, each individual customer is better off under centralized system.

Proof. For any path ω and any arriving customer at $t \in T(\omega)$, from the definition of consumer's surplus, we only need to show $p_t(\omega, x_D) \geq p_t(\omega, x_C)$. From Proposition 3.1, we know the order quantity $x_D \leq x_C$. Let τ be the stopping time that the inventories for decentralized and centralized are the same. Hence for any $t > \tau(\omega)$, we know the inventory for decentralized

system is lower than that for centralized system, from Lemma 3.1(b) we have $p_t(\omega, x_D) \geq p_t(\omega, x_C)$. For any $t \leq \tau(\omega)$, the two systems are the same. $\square$

The implication of Lemma 3.2 is twofold. First, it indicates a high availability of the stock for centralized system. Each customer has a higher possibility to buy the product and hence it results in a higher service level. The second is that some of buyers pay less to purchase the product in centralized system, which results in the improvement of individual customer's surplus. Usually we are not terribly interested in the level of individual consumer's surplus but in the total consumer surplus. The total consumer surplus is simply the sum of all the consumer surpluses for each individual good purchased (Varian 2010). Here we consider the expected total consumer surplus,

$$CS(x) = E\left(\sum_{t \in T(\omega)} [u_t(\omega) - p_t(\omega, x)]^+\right).$$

Based on Lemma 3.2, the consumer surplus for centralized system is obviously more than that for the decentralized system. Moreover, the centralized system also generates a higher profit than the decentralized one. Combine these two points, it leads to:

Proposition 3.2 Compared to decentralized system, the centralized system is a Pareto improvement regarding to the system profit and consumer surplus.

Finally, another indicator to evaluate the performance of the system is social welfare, which is measured by the sum of system's profit and consumer surplus. Due to Proposition 3.2, it is trivial that the social welfare for the centralized system is higher than the decentralized one.

3.3 An Illustrative Example

One drawback with dynamic pricing is that there is no closed form for the revenue function; and hence we are not able to find an explicit expression for the optimal order quantity. To enhance the understanding of different systems and illustrate the magnitude of the system improvement that could be brought by dynamic pricing, now we consider an example. We also compare the results with the supply chain with price-setting newsvendor. Under price-setting policy, the retailer sets the price at the beginning of the selling season and then keeps it fixed for the duration of the season. Given price p , the arrival process of actual purchaser becomes a non-homogeneous Poisson process with rate $\lambda_t \bar{F}_t(p), t \in [T, 0]$. Therefore, the probability mass function for the number of actual purchasers $D(p)$ is Poisson distribution with mean $\int_0^T \lambda_t \bar{F}_t(p) dt$. Hence the expected revenue $V_T^{PS}(x)$ from the selling period with initial inventory x becomes $V_T^{PS}(x) = \max_{p \geq 0} E\{p[D(p) \wedge x] + s[x - D(p)]^+\}$. Using the same analysis as dynamic pricing policy, we are able to obtain the solution for the centralized and decentralized supply chain respectively.

Example 3.1 Consider a supply chain with production cost $c = 10$, customers' arrival rate $\lambda = 100$, salvage value $s = 0$, and the reservation price at time t is Weibull distribution with shape parameters $k = 5$ and scale parameter $\theta_t = 20 + 16(t - 0.5)$ ($t \in [0, 1]$).

Note that the reservation value distribution is stochastically decreasing as time elapses. Figure 3.1 displays the marginal expected revenue for both dynamic pricing and price-setting retailer. It describes the determination of order quantity for both centralized and decentralized systems. While the centralizer maximizes the whole system profit (the area of triangle AEF), the supplier in the decentralized system optimizes his own profit (the area of rectangle ABCD) which leads to

double marginalization. It is well known this double marginalization cannot be eliminated under a decentralized system and it results in under stocking of inventory and inefficiency. Compared to the benchmark model of price-setting system, the marginal expected value for dynamic pricing is higher. Therefore for centralized system, the dynamic pricing retailer will stock a higher level of inventory than the price-setting retailer.

Figure 3.1 Marginal expected revenue for dynamic pricing and price-setting newsvendor

Table 3.1 displays the outcomes of different systems. It is interesting to find that for decentralized system, both the wholesale price and the order quantity with dynamic pricing retailer is higher than the respective one with price-setting retailer; and moreover both firms improve their profit. This indicates both the retailer and the supplier have the incentive to adopt dynamic pricing to serve the market. Moreover, a surprising finding here is that the decentralized dynamic pricing outperforms the centralized price-setting system. These results explain the reasons why dynamic pricing phenomenon is so common in practice, especially in fashion industry.

Table 3.1 Performance of different systems

Systems	Order Quantity	Wholesale Price	Supplier's Profit	Retailer's Profit	System's Profit
Centralized DP	57	N.A.			444.64
Decentralized DP	28	17.24	202.84	138.89	341.73
Centralized PS	47	N.A.			340.32
Decentralized PS	22	17.09	156.09	100.71	256.80

To develop in-depth understanding of the sales process, we do 10^5 simulations for each system. Figure 3.2 depicts the dynamic evolution of the average sales price and the average transaction price (cumulative-revenue/quantity-of-sales up to time) for centralized and decentralized systems.

Figure 3.2 Simulated prices for different systems

Obviously, at the beginning of the sales season, the two prices are the same for specific system. On average, the sales price drops as the time elapses. At the end of the season, this price could only be just half or even one third of the initial price. This phenomenon mimics what we observe in practice, which is mainly due to customers' decreasing utility. It is interesting to find that the

average sales prices for centralized and decentralized systems approach each other at the end of the season. This is because both of them use myopic optimal prices (Bitran and Caldentey 2003), which are the same at the end of the season.

Table 3.2 Simulated performances for different systems

Systems	Average Sales	Sales Rate (%)	Average Transaction Price	System Profit	Consumer Surplus	Social Welfare
Centralized DP	56.69	99.45	17.88	443.65	229.58	673.23
Decentralized DP	27.99	99.96	22.20	341.49	91.57	433.06
Centralized PS	44.79	95.30	18.09	340.27	252.38	592.65
Decentralized PS	21.33	96.95	22.35	256.70	94.79	351.49

Table 3.2 is the summary for the simulated performance for different systems. With the ability to adjust the price during the whole sales season, dynamic pricing retailer clears most of the inventory; while the price-setting retailer's leftover rate is more than 3% for either decentralized or centralized system. Moreover, the average transaction price for dynamic pricing retailer is surprisingly lower than that for price-setting retailer in either centralized or decentralized system. No matter whether the retailer practices dynamic pricing or price-setting, due to higher stocking level and lower price, the centralized system significantly improves system profit, consumer surplus and consequently the social welfare. Furthermore, the centralized dynamic pricing system achieves a higher social welfare than the centralized price-setting system.

3.4 Supply Chain Contracts

Based on the example, the potential of the decentralized dynamic pricing system is far from exploited. The efficiency for the decentralized system in the example, which is measured by the profit ratio of the decentralized system to the centralized one, only achieves 76.86%. It is well known this inefficiency is due to the double marginalization of the upstream and downstream

firms. To alleviate or even eliminate the double marginalization, plenty of supply chain contracts have been examined in literature. Some intensively studied and commonly used contracts in practice are buy-back, revenue-sharing, two-part tariff, price-discount, sales-rebate, quantity discount, and so forth. Generally speaking, a supply chain contract is an option the supplier offers to the retailer which specifies the wholesale price, the gains from selling and salvaging the goods. Therefore usually a contract can be represented by one or more contract parameters (e.g., Tsay 1999 and Cachon 2003). Along this way, we define a contract as follows.

Definition 3.1 A supply chain contract corresponds to a triple variable $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ ($\beta_i \geq 0$) and transfer payment, where $\beta_1 c$ specifies the unit transfer price, β_2 is portion of sales revenue received by the retailer, and $\beta_3 s$ is salvage value for the retailer.

The ranges of β_i 's ensure that the contract we study here is broad enough to include most of the contracts in the literature (Cachon 2003). For example, revenue-sharing is such a contract where $\beta_2 = \beta_3$; sales-rebate is such one that β_2 is a function of the realized sales and $\beta_3 = 1$. Furthermore, this definition allows us to differentiate supply chain contracts by only comparing the relationship of these three parameters. Note that some of the contract parameters may depend on other parameters or factors (i.e. the decision of the player or the realization) of the system. For example, Cachon and Lariviere (2005) show that the price-discount contract is a contingent buy-back contract where both the buy-back rate and wholesale price are adjusted linearly in the retailer's selling price. Another example is quantity discount, where the wholesale price for the contract depends on the retailer's procurement quantity. Along this way, we classify the supply chain contracts into two classes. The first category is independent contract, where the parameter β_i is independent. The independence here refers to both the parameters themselves and other factors of the system. Obviously, wholesale, buy-back contract, revenue-sharing and two-part

tariff belong to this category. The other one is called contingent contract, where some of the parameter $\beta_i (i = 1, 2, 3)$ could depend on each other or be contingent on other factors of the system (e.g., sales realization and order quantity). This category includes sales-rebate, quantity discount, quantity flexibility, price-discount and so on. This classification of supply chain contracts not only facilitates us to develop a stylized method to study independent contracts, but simplify the analysis of the coordination effectiveness for contingent contracts.

Next, we first model retailer's dynamic pricing behavior under an independent contract and evaluate the properties of the retailer's ordering and pricing decisions. Then we characterize the conditions for an independent contract to coordinate the underlying supply chain. Last we utilize the derived results to examine the coordination effectiveness of contingent contracts.

3.4.1 Analysis for the Retailer's Decisions

Given an independent supply chain contract $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$, with a little abuse of notation, we still use $V_t(x)$ as the supremum of expected revenue for the retailer from any admissible policy from time t onward with inventory x . Then $V_t(x)$ satisfies the following Bellman equation,

$$\frac{\partial V_t(x)}{\partial t} = \sup_{p \in P} \lambda_t \bar{F}_t(p) [\beta_2 p - \Delta V_t(x)] \quad (3.4)$$

with constraints $V_0(x) = \beta_3 \cdot sx$, $V_t(0) = 0$ and $\Delta V_t(x) = V_t(x) - V_t(x-1)$ represents the marginal expected value of the x th item. Obviously, Problem (3.4) includes Problem (3.2) as a special case, where the retailer receives all the revenue generated from selling the product ($\beta_2 = 1$) and salvages the leftover ($\beta_3 = 1$) by herself.

Now we study the structural properties for the retailer's expected value and optimal price. The importance of structural properties is well recognized in literature, for instance, the concavity of

the revenue function is crucial for the retailer to determine the order quantity at the beginning of the selling season. Let $\hat{p} = \beta_2 p$, Problem (3.4) is converted to classic dynamic pricing problem (e.g. Zhao and Zheng 2000). Therefore the structural properties can be easily generalized to the case of independent contracts.

Lemma 3.3 Given any independent contract β ,

- (a) $V_t(x)$ is concave in x for any t , namely, $\Delta V_t(x) \geq \Delta V_t(x + 1)$ for any $x \geq 1$;
- (b) $\Delta V_t(x)$ is increasing in t for any $x \geq 1$;
- (c) The retailer's optimal price $p(x, t)$ is decreasing in x for any given t .

Lemma 3.3(a) says as long as it is an independent contract, no matter what the portion is for the retailer receiving from the sales and what the share is from the salvage value, the value function is concave in the left inventory. Lemma 3.3(b) shows the time monotonicity for the value function still holds under an independent contract. Intuitively, the more time the retailer has, the higher potential she can exploit from this marginal inventory. Lemma 3.3(c) show that the optimal price is decreasing in the inventory position. This inventory monotonicity for the optimal price is not only insightful in itself, but also useful for the implementation of the optimal policy. Due to strategic and tactical reasons, many firms restrict the price strategy P to a small discrete set. For this case, Feng and Xiao (2000) (also see Zhao and Zheng 2000) show the retailer's optimal pricing policy is in fact a threshold policy as long as the optimal price is monotonic in the left inventory. Analogously, here we can also find a set of time-dependent threshold to simplify the computation and facilitate the implementation of the optimal policy. Finally, a direct result from the concavity of the value function is that the retailer's order quantity is well defined for an independent contract.

Proposition 3.3 For any independent contract β , there exists a unique optimal order quantity x^* for the retailer such that

$$x^* = \operatorname{argmax}_{x \geq 0} \{V_T(x) - \beta_1 cx\}. \quad (5)$$

As buy-back contract is a type of independent contract, Proposition 3.3 basically extends the result of price-setting retailer (Theorem 1 in Song et al. 2004) to dynamic pricing retailer.

3.4.2 Characteristics for Coordinated Contracts

Following Cachon and Lariviere (2005), a contract is said to coordinate the supply chain if the retailer chooses the supply chain optimal actions (quantity and pricing policy) and the supply chain's profit can be arbitrarily divided between the firms. Obviously, the coordination here is specifically for full coordination, where the contract's efficiency is 100%. As it is argued before, fixed-price is a special kind of dynamic pricing policy. To preclude the cases that a contract is only effective for some special pricing strategies, we say a contract coordinating the supply chain when this contract could coordinate any setting of P . For independent contract, after the retailer makes the ordering decision, the pricing process is not relevant to the parameter β_1 . This means the coordination process can be decomposed into two stages. To coordinate the supply chain, a contract first has to coordinate the pricing process. This analysis leads to a necessary condition for achieving supply chain coordination.

Proposition 3.4 For independent contract, a necessary condition for coordinating dynamic pricing system is that it is revenue-sharing type contract, namely, $\beta_2 = \beta_3$. Moreover, when it satisfies, we have $V_t(x) = \beta_2 V_t^C(x)$ and $p(x, t) = p^C(x, t)$, where $V_t^C(x)$ and $p^C(x, t)$ are the revenue function and optimal price for the centralized system.

Proof. Let $V_t^r(x) = V_t(x) - \beta_3 sx$, hence we have

$$\frac{\partial V_t^r(x)}{\partial t} = \frac{\partial V_t(x)}{\partial t} = \sup_{p \in P} \lambda_t \bar{F}_t(p) [\beta_2 p - \beta_3 s - \Delta V_t^r(x)]$$

with constraints $V_0^r(x) = 0$ and $V_t^r(0) = 0$. If $\beta_2 \neq \beta_3$, obviously the price at time $t = 0$ does not necessarily equal to that for the centralizer. Then $p(x, t)$ does not always equal to the optimal price $p^c(x, t)$. This indicates the pricing process is distorted; and hence there must exist some systems such that $V_T(x_C) < V_T^C(x_C)$.

When $\beta_2 = \beta_3$, we first show $V_t(x) = \beta_2 V_t^C(x)$. It is trivial at $t = 0$. Suppose it holds for t , now we prove for $t + \Delta t$, with little algebraic calculation,

$$\frac{\partial V_t^r(x)}{\partial t} = \max_{p \in P} \lambda_t \beta_2 \bar{F}_t(p) (p - s - \Delta V_t^C(x)) = \beta_2 \frac{\partial V_t^C(x)}{\partial t}.$$

Hence $V_{t+\Delta t}(x) = V_t(x) + \Delta t \partial V_t^r(x) / \partial V_t^r(x) \rightarrow \beta_2 V_{t+\Delta t}^C(x)$ as $\Delta t \rightarrow 0$. From the continuity of $V_t(x)$, we have $V_t(x) = \beta_2 V_t^C(x)$. Now $p(x, t) = p^c(x, t)$ is trivial. $\square$

Proposition 3.4 shows at the stage of pricing, the retailer should have the same share from selling and salvaging an item of the product. Otherwise, the pricing process would be distorted and hence the supply chain cannot be coordinated. Now we combine the two stages together.

Proposition 3.5 Under independent contract, a necessary and sufficient condition for the dynamic pricing supply chain to achieve coordination is that it satisfies $\beta_1 = \beta_2 = \beta_3$.

Proof. First we show the necessity. From Proposition 3.4, we know a necessary condition for coordinating dynamic pricing system is $\beta_2 = \beta_3$. Given that this condition holds, then the expected profit of the retailer becomes $\pi_r(\beta) = \max_{x \geq 0} [\beta_2 V_T^C(x) - \beta_1 cx]$. Based on concavity of

$V_t^c(x)$, the optimal procurement quantity for the retailer is $\bar{x} = \operatorname{argmax}_{x \geq 0} \{\beta_2 \Delta V_T^c(x) \geq \beta_1 c\}$.

If $\beta_1 \neq \beta_2$, it is easy to find some counterexample that $\bar{x} \neq x_c$. Hence we must have $\beta_1 = \beta_2$.

Next we show the sufficiency. When $\beta_1 = \beta_2 = \beta_3$, from Proposition 3.3, the retailer's order quantity is the same as the order quantity for the centralized system. Moreover, Proposition 3.4 ensures that the pricing process is the same as the centralized system. Finally, the share of the supply chain's profit can be arbitrarily divided between the firms through adjusting β_1 . $\square$

Proposition 3.5 characterizes the equivalent condition for an independent contract to coordinates the supply chain as sharing the same portion of gain and pain. Based on the idea of recourse in Monahan et al. (2004), it is not surprising that the conditions for coordinating the dynamic pricing and price-setting system (see Cachon and Lariviere 2005) are almost the same. Moreover we show that if the coordination refers to coordinating supply chain with any setting of P , then this condition is also necessary.

Now we can directly use it to identify whether an independent contract coordinate the underlying supply chain system instead of putting the contract form to retailer's profit function to check it. Obviously, the wholesale contract ($\beta = \{\beta_1, 1, 1\}$) induces system's optimal order quantity only when the wholesale price equals to production cost, when the supplier makes no profit. Hence it does not coordinate the underlying supply chain. Pasernack (1985) consider the buy-back contract and show that by appropriately choosing the wholesale price and buy-back rate, it can fully coordinate the supply chain with fixed-price retailer. For this contract, $\beta = \{\beta_1, 1, \beta_3\}$ with both β_1 and β_3 are more than 1. From Proposition 3.5, we know it cannot coordinate our supply chain. Cachon and Lariviere (2005) demonstrate that revenue-sharing ($\beta_1 = \beta_2 = \beta_3$) coordinates a supply chain with a price-setting retailer. Obviously, revenue-sharing could also

coordinate the supply chain here and arbitrarily allocate the system's profit. For two-part tariff contract, the supplier charges a per-unit wholesale price and a fixed transfer. Obviously, coordination is achieved when and only when the wholesale price is equal to the production cost, while the fixed transfer facilitates the arbitrary allocation of the profit between the two firms. These results and the results for contingent contract are summarized in Table 3.3.

Table 3.3 Coordination result of different contracts

Contract type	Contract	Fixed-price	Price-setting	Dynamic pricing
Independent contract	wholesale	N	N	N
	Buy-back	Y	N	N
	Revenue-sharing	Y	Y	Y
	two-part tariffs	Y	Y	Y
Contingent contract	Sales-rebate	Y	N	N
	Quantity discount	Y	Y	Y
	Quantity-flexibility	Y	N	N
	Price-discount	Y	Y	N.A.

Note: all of the results for fixed-price and price-setting newsvendor can be found in Cachon (2003) or Cachon and Lariviere (2005). Here Y, N and N. A. stand for Yes, No and Not Applicable respectively.

3.4.3 The Contingent Contract

Proposition 3.5 characterizes the necessary condition for an independent contract to coordinate the supply chain. However, it is also effective to evaluate contingent contracts as long as the ordering decision is independent of pricing process. Such contingent contracts include sales-rebate and quantity flexibility. Krishnan et al. (2001) and Taylor (2002) show the sales-rebate contract coordinates the supply chain with fixed-price newsvendor. Here the supplier charges the retailer a per-unit wholesale price but gives the retailer a rebate per unit above some threshold, and the retailer continues to salvage leftover. Later on, Cachon and Lariviere (2005) show sales-rebate does not coordinates the supply chain with price-setting newsvendor. For our dynamic

pricing newsvendor, the contract here becomes $\beta = \{\beta_1, \beta_2, 1\}$ where β_2 is contingent on the sales process and can be more than 1. Based on Proposition 3.5, we know the pricing process is also distorted. Hence the sales-rebate contract does not coordinate the supply chain.

Tsay and Lovejoy (1999) and Tsay (1999) study quantity-flexibility contract for a fixed-price newsvendor. Under this contract, the retailer purchases x units at the start of the season and may return up to αx ($\alpha \in [0, 1]$) units at the end of the season for a full refund. Hence quantity-flexibility can be represented by $\beta = \{\beta_1, 1, \beta_3\}$, where the contract parameter β_3 depends on the final leftover. Proposition 3.5 indicates that this contract also distorts the retailer's pricing process and hence cannot coordinate the supply chain.

Two other contracts that need to be examined individually are quantity discount and price-discount. For quantity discount, the wholesale price is based on the quantity that the retailer purchases from the manufacturer. Hence it can be expressed as $\beta = \{\beta_1, 1, 1\}$, where β_1 depends on the quantity that the retailer orders. As Moorthy (1987) argues, while the retailer's marginal revenue curve is untouched, the quantity discount adjusts the retailer's marginal cost curve so that the retailer's profit-maximizing quantity is the same as system's optimal quantity. Due to $\beta_2 = \beta_3 = 1$, the pricing process is the same as the centralizer. Moreover, the arbitrary division of supply chain profit is achieved by adjusting the function of β_1 . Hence quantity discount also coordinates the supply chain with dynamic pricing retailer.

The final contract we consider is price-discount. Bernstein and Federgruen (2005) (also see Cachon and Lariviere 2005) find that by adjusting with the selling price p , the price-discount contract with buy-back rate $b(p) = (1 - \beta)(p - s)$ and wholesale price $w(p) = (1 - \beta)p + \beta c$ coordinates the supply chain with price-setting retailer. However, there is no such a unique price

to quote for dynamic pricing retailer since the price changes over time; hence the price-discount is not applicable here.

3.5 Computational Study

To enhance the findings in the numerical example and develop additional insights to complement those findings, we conduct extensive numerical experiments in this section. Our concern focuses on the following key aspects. First, we test the robustness of the surprising result that the performance of decentralized dynamic pricing system outperforms centralized price-setting system (π_D/π_C^{PS}) and identify when this result happens. We are also interested in the decentralized dynamic pricing system itself, the division of realized profit between the supplier and retailer. The third is the value of pricing flexibility (π_D/π_D^{PS} and π_C/π_C^{PS}); and how is the value shared between the supplier and the retailer for decentralized system. The final is the value of coordination. We investigate the performance improvement for coordinated system ($\pi_C - \pi_D$)/ π_D and reasons for the improvement; Moreover, we compare the improvement result with static pricing system to find out which system desires more for coordination.

We follow the works of Bitran and Mondschein (1997) and Bitran et al. (1998), and use Weibull distribution with parameters (k, θ) to model customer's reservation value. To cover the cases of both time-varying and time-unvarying demand, we test a wide range of supply chain parameter combinations. Without loss of generality, we fix the production cost $c = 10$ and average arriving customers $T * \lambda = 100$, then vary other parameters. The shape parameter k , which is equivalent to coefficient of variation ($CV = \sqrt{\Gamma(1 + 2/k)/\Gamma^2(1 + 1/k) - 1}$, where $\Gamma(\cdot)$ is Gamma function.) and hence captures the relative variability or the uncertainty for the demand, takes values from $\{1, 3, 5, 7, 9\}$. Hence as the parameter k increases, the relative variability decreases

($CV = \{1, 0.36, 0.23, 0.17, 0.13\}$ respectively). Moreover, note that $k = 1$ corresponds to the case of exponential distribution. The scale parameter is a function of center scale and obsolescence rate, namely, $\theta_t = \theta[1 + b(t - 0.5)]$ ($t \in [1, 0]$). The center scale, which characterizes customers' average value for the product, is chosen from values $\theta \in \{16, 18, 20, 22, 24\}$; the obsolescence rate, which captures the relative decreasing speed of scale parameter, is chosen from values $b \in \{0, 0.2, 0.4, 0.6, 0.8\}$. Hence $b = 0$ indicates the good does not obsolete with time, or equivalently, the customer's reservation value distribution is time-unvarying. Finally, the salvage value is $s = \{0, 1, 2, 3, 4\}$. Given different values of the supply chain parameters, we test 625 different combinations. In order to isolate the effect caused by the choice of pricing strategy, we suppose the retailer chooses from the price set $P \in [0, \infty)$. Finally, to eliminate the influence brought by integer order quantity, we take the average over all the experiments related to a specific level of the parameter on which we are focused. For example, if we want to see the relative performance when the shape parameter is 5 and the slope is 0.4, we take an average of the relative performance over all the 25 experiments.

3.5.1 Decentralized Dynamic Pricing vs. Centralized Static Pricing

Motivated by previous example, first we are interested in the relative performance of the decentralized dynamic pricing system compared to the centralized static pricing system, which is measured by the ratio π_D/π_C^{PS} . Figure 3.3 shows the frequency chart of the 625 experiments. Obviously, there are 91 cases satisfying that $\pi_D/\pi_C^{PS} > 1$. In other words, 15% of the decentralized dynamic pricing system outperforms the corresponding centralized static pricing system. Furthermore, it shows half of the cases that the decentralized dynamic pricing system can achieve more than 85% of the profit for corresponding centralized static pricing system.

Next we identify when the performance of decentralized dynamic pricing system can match up with centralized static pricing system. Figure 3.4 shows when the shape k increases, or when coefficient of variation CV decreases, the ratio π_D/π_C^{PS} increases. Except for exponential distribution ($k = 1$), this ratio increases with obsolescence rate b . This indicates the decentralized dynamic pricing system outperforms the centralized static pricing system for low CV and customer's preference on the good decreases gradually, which fits well with the fashion industry examined in Bitran et al. (1998).

Figure 3.3 Frequency for π_D/π_C^{PS}

Figure 3.4 π_D/π_C^{PS} versus shape and obsolescence rate

3.5.2 The Division of Profit for Decentralized System

Now we examine how the profit is divided between the supplier and retailer under decentralized system. Figure 3.5 shows the ratio of supplier's profit to the system's profit (π_s/π_D) and how it changes. On the whole, the supplier captures the majority of the system's profit, which ranges from 55% to 69%. This finding is consistent with Lariviere and Porteus (2001)'s result for fixed-price newsvendor system. Lariviere and Porteus illustrate that the supplier's share (π_s/π_D) increases as the CV falls, which is also the case for our dynamic pricing retailer. The reason is

the same, the less uncertain of the demand, and the more power for the supplier to control the supply chain. Moreover, we complement these findings and show the supplier's share also increases as obsolescence rate decreases for $k > 1$. As will show later, when obsolescence rate slows, the value for pricing flexibility decreases, hence retailer's contribution in the system drops.

Figure 3.5 π_s/π_D versus shape and obsolescence rate

3.5.3 The Value of Pricing Flexibility

The value of pricing flexibility has been well established for the centralized system. Monahan et al. (2004) first compare the effect of dynamic pricing (recourse) with price-setting model and find the value of pricing flexibility increases in the number of price changes. While they only allow several price changes, our model is the same as Xu and Hopp (2006) which study infinite number of price adjustments. Following these papers, we use π_D/π_D^{PS} and π_C/π_C^{PS} to measure the value of pricing flexibility under centralized and decentralized system respectively, which are displayed in Figure 3.6 and 3.7. It is surprising to find that these two figures are extremely

similar to each other, which indicates the value of pricing flexicibility under centralized and decentralized system are almost the same. This value mainly depends on the underlying demand characteristics, rather than on centralization or decentralization of the system. So we only need to refer to centralized system for further explanation.

Figure 3.6 π_C/π_C^{PS} versus shape and

obsolescence rate

Figure 3.7 π_D/π_D^{PS} versus shape and

obsolescence rate

For time unvarying demand ($b = 0$), the value of pricing flexicibility first increases and then decreases as CV decreases, which is consistant with Monahan et al. (2004, Figure 2a). Xu and Hopp (2006, Figure 2a) find when the arriving rate of the customer's follows a geometric Brownian motion, the value of pricing flexicibility is increasing with CV. Contrary to their result, we find the value of pricing flexicibility is decreasing in CV when obsolescence rate is large enough. The reason is that their geometric Brownian motion assumption leads to a deterministic inventory depletion process, and then dynamic pricing is effective to cope with the fluctuation of demand rate. In our case, when obsolescence rate is positive, there is a natural need for dynamic

pricing. However, the uncertainty of the demand would alleviate the need for dynamic pricing. A thought experiment provides further intuition. Suppose the case that the CV was zero ($k \rightarrow \infty$) and the arriving customer's value was known with certainty, the value of pricing flexibility is obvious when obsolescence rate is positive. Finally, it shows that value of pricing flexibility increases with obsolescence rate, which is in line with the observations of Lazear (1986) for the impact of reservation-price variability for two-period deterministic demand. The more obsolete the good becomes, the more necessary and valuable for dynamic pricing policy.

Figure 3.8 π_s/π_s^{PS} versus π_r/π_r^{PS}

After the value of pricing flexibility is exploited, now we can identify who benefits (or benefits more) this value when the retailer shifts the pricing policy from static to dynamic under decentralized system. Given the system is decentralized, denote π_s/π_s^{PS} and π_r/π_r^{PS} as the improvement of dynamic pricing over price-setting for the supplier and the retailer respectively. Figure 3.8 shows the dependence of the two ratios in two groups: $\pi_D/\pi_C^{PS} > 1$ and otherwise. It reports almost symmetric improvement of supplier and retailer. Both supplier and retailer benefit

from dynamic pricing. The improvement for the retailer becomes larger compared to that for the supplier as the ratio π_D/π_C^{PS} increases. For $\pi_D/\pi_C^{PS} > 1$, most of the improvement for the retailer is higher than that for the supplier.

3.5.4 The Value of Coordination

Proposition 3.2 shows the centralized/coordinated system can improve the system performance. Now we examine the magnitude of the improvement for coordination relative to decentralized system. Given the dynamic pricing policy, let $(\pi_C - \pi_D)/\pi_D$ denote as the percentage improvement of centralized compared to decentralized system. Moreover, to indentify the direct driver for the improvement, we compare the profit improvement with the order quantity increment of centralized to decentralized system $(x_C - x_D)/x_D$.

Figure 3.9 Percentage improvement versus
Order quantity increment

Figure 3.10 Percentage improvement versus
shape and obsolescence rate

Figure 3.9 displays the relationship between the percentage improvement and corresponding order quantity increment. It shows the profit improvement of achieve among 23%-44%, which is

significant for the system. On the other hand, the centralized system orders 60%-200% more quantity than the corresponding decentralized system. As Proposition 3.2 demonstrates the higher level of inventory will not only increase customer's availability of the product, but also decrease the sales price. Moreover, the linear relationship of the two ratios explains clearly that the profit improvement is due to the increment of procurement quantity.

Next we exploit the underlying reasons for the value of coordination. Figure 3.10 shows the profit percentage improvement with respect to the shape parameter and obsolescence rate. Similar to Lariviere and Porteus (2001)'s result for fixed-price retailer, we also find the profit improvement increases as CV increases. To our surprise, the impact of obsolescence rate on the improvement is not clear. Hence basically, the value for coordination mainly depends on the uncertainty degree of the demand. The higher the risk is, the higher the value for coordination becomes.

The coordination improves the performance for both dynamic pricing and price-setting systems. However, in reality, the coordination is not easily achievable for the supply chain. Therefore an interesting question is which system desires more for coordination. Figure 3.11 compares the cumulative distribution function (CDF) of percentage improvement for dynamic pricing system with that for price-setting system. Clearly, the improvement rate for dynamic pricing newsvendor is almost dominated by that for price-setting newsvendor, especially when this improvement is not very large (less than 35%). It implies that compared to price-setting policy, the decentralized dynamic pricing system is not so demanding for coordination.

Figure 3.11 CDFs of percentage improvement for dynamic pricing and price-setting systems

3.6 Concluding Remarks

In this chapter, we have examined a supply chain where the retailer makes the order decision at the beginning and then practices dynamic pricing during the selling period. We find the decentralized system stocks fewer inventories, which hurts both the firms and the customer. To coordinate the supply chain, we consider various contracts. For independent contract, we show the revenue function for the retailer exhibits both inventory and time concavity. This allows retailer to determine a unique stocking level. Moreover, we show the optimal price shows monotonicity in inventory, which simplifies the implementation of the dynamic pricing policy in practice. Most importantly, we find a necessary condition for a contract to achieve supply chain coordination is that it is the revenue-sharing type contract. Otherwise, the retailer's pricing process would be distorted. Moreover, we identify a necessary and sufficient condition for achieving supply chain coordination is that the retailer shares the same portion of cost and revenue in selling and salvaging. Hence neither wholesale nor buy-back contract coordinates the

supply chain, but both revenue-sharing and two-part tariffs coordinate it. According the necessary condition, the pricing processes under Sales-rebate and quantity-flexibility are also distorted; hence they cannot coordinate the supply chain. While price-discount coordinates supply chain with price-setting newsvendor, it no longer coordinates the dynamic pricing system.

Our computational study sheds light on a number of perspectives of the impact of decentralized system and pricing policy. First, as the relative variability (CV) decreases and obsolescence rate increases, even decentralized dynamic pricing system could outperform centralized price-setting system in profit. It explains why dynamic pricing is so popular in practice. We also consider the division of the profit. As the case for fixed-price newsvendor (Lariviere and Porteus 2001), we find the supplier captures the majority of the system profit, which ranges from 55% to 69%. The supplier's share increases as relative variability decreases and obsolescence rate decreases. Third, the values of pricing flexibility are similar for decentralized and centralized systems. It indicates that whether the retailer should use dynamic pricing is not depending on whether the underlying system is decentralized or centralized, but depending on the characteristics of market demand. Moreover, this benefit of dynamic flexibility under decentralized system is symmetrically shared between the supplier and the retailer. Fourth, we show the value for coordination is significant, which results in profit improvement of more than 23%. We also identify a direct reason for this improvement is due to the increase in stocking level. Furthermore, we find dynamic pricing policy could alleviate the competition between the supplier and the retailer, and hence the corresponding supply chain is not as demanding for coordination as static pricing one.

Several natural extensions of our results could be pursued. First, the coordination in this chapter is specifically for full coordination, it is meaningful to study the efficiency of different contracts in our setting. Second, the customer in our model is myopic. As we show, in average, the price is

decreasing over time; hence it is interesting to study the case where the customer is strategic (Aviv and Pazgal 2008). Moreover, an important extension of our monopoly supplier and retailer is to study the case of competitive suppliers and/or retailers. Finally, for some firms, there may exist opportunities of replenishment during the selling season, hence consider the impact of quick response (Cachon and Swinney 2009) in the supply chain would also be also useful.

Chapter 4

Dynamic Pricing for Perishable Assets with Sales Effort

4.1 Introduction

Dynamic pricing is gaining popularity in the retail industry and a great deal of research has been done on this topic in recent years. Through monitoring the availability of stock and the future's demand uncertainty, profit-maximizing firms adjust the price dynamically to control the sales of the product. While pricing is the main tool for firms to coordinate the demand and inventory, it has long been acknowledged that retailers' sales effort is also important in influencing demand for fashion retail products. Hence the aim of this work is to incorporate sales effort into traditional dynamic pricing problem.

The retailer can temporarily affect the sales by increasing the service intensity and product exposure through a variety of sales efforts. For example, the retailer can boost the sales by simply operating longer hours. Another way of boosting the sales is to provide attractive and more shelf space. Wolfe (1968) presents empirical evidence showing that the sales of women's dresses and sports clothes are proportional to the amount of displayed shelf space. The retailer can also stimulate demand by merchandising, doing point-of-sale or other advertising and guiding consumer purchases with sales personnel. Empirical evidence can be found in Lodish (1971) and Rao et al. (1988). Under these circumstances, the retailer needs not only to set the

price, but also to choose the level of sales effort, for example, the amount of shelf space and sales-force assigned to the product.

We study the selling of a single perishable product where the retailer adjusts both the effort and the price dynamically. The demand for the product is a Markovian process where the intensity of the demand is jointly determined by the selling effort and the price, whereas the effort is costly. We are interested in how the retailer will adjust the effort and price over time. In particular, we find as the inventory increases and/or the remaining selling time decreases, in order to accelerate the sales of the product, the retailer will exert more effort to attract more customers and set a lower price to motivate the arriving customer to make a purchase.

We also examine the cases when the firm has less flexibility in adjusting the sales effort and/or the price, and conduct a numerical study to explore the value of dynamically adjusting the effort and/or price. Even though the retailer is able to choose an optimal initial price (or effort), the potential profit improvement is still significant from dynamically adjusting the effort (price respectively). However we find if the retailer has the option to choose dynamically adjusting the effort or the price, there is no need to simultaneously adjust both effort and price dynamically. We also discover that the key factor that helps deciding whether to dynamically adjust the effort or the price is the relative market size of the customer segment that is unaware of the product. Finally, we find that the amount of dynamic effort is decreasing with the cost for the effort and the coefficient of variation (CV) of the demand, and increasing with the proportion of the potential market unaware of the product.

Our work is built upon the large volume of literature on dynamic pricing for perishable assets. For recent comprehensive reviews, refer to Bitran and Caldentey (2003) and Elmaghraby and

Keshinocak (2003). Gallego and van Ryzin (1994) initially study this problem where the purchasing intensity for the product is dependent on the price. To provide a mechanism for the customer's purchasing process, Bitran and Mondschein (1997) and Zhao and Zheng (2000) formulate the problem where the customer arrival rate is given and the arriving customers make the purchase decisions according to their reservation values. Later on, Bitran et al. (1998) conduct a real case in fashion retail chain under these assumptions. However, none of these papers have considered the impact of sales effort. The only exception seems to be Kuo et al. (2011), who study the dynamic pricing problem where the customer can negotiate. The negotiation process can be treated a form of retailer's sales effort.

There is a large literature on sales effort. Here, we focus on shelf space allocation and sale-force management. For shelf space allocation, Corstjens and Doyle (1981) develop a shelf-space allocation model in which the demand rate is a function of shelf space allocated to the product. Urban (1998) generalizes their result to the case that the demand rate of the product is a function of the displayed inventory level. Martínez-de-Albéniz and Roels (2011) study the competition of shelf space. See Kok et al. (2008) for a comprehensive overview along this direction. The literature on sales-force management can be roughly divided into two groups: deterministic and stochastic sales-response functions. The deterministic group studies the issue of designing optimal commission scheme to allocate sales-force for multiple products. The price here can be fixed (e.g., Farley 1964) or delegated to the sales force (e.g., Weinberg 1975 and Srinivasan 1981). More close to our work is the model by Tapiero and Farley (1975), who study sales-force commission problem where the sales effort is exerted dynamically. The stochastic group incorporates the agency theory into the salesmen's compensation problem. Similarly, the price here can be fixed (e.g. Basu et al. 1985 and Rao 1990) or delegated to the salesmen (e.g. Lal

1986 and Joseph 2001). Subsequent researches focus on extending to more complex settings, such as the joint problem of marketing incentives and manufacturing incentives among the firm and the supply chain problem. For more details, see Porteus and Whang (1991), Chen (2000, 2005), Taylor (2002), Krishnan et al. (2004) and Cachon and Lariviere (2005).

The rest of this chapter is organized as follows. In Section 4.2, we describe the model where the retailer dynamically adjusts both the effort and price. Some analytical results for this case are presented in Section 4.3. In Section 4.4, we consider the static effort and/or static price cases. Section 4.5 presents the results of our numerical study. In Section 4.6, we provide some concluding remarks.

4.2 Model Description

Consider a retailer that has a certain initial inventory at the starting of the selling season and no opportunity to replenish it during the season. Following the approach in Bitran and Mondschein (1997), we assume that the time horizon is divided into T periods, where each period is short enough such that there is at most one customer arrival in a period. We will denote the first period as period T and the last period as period 1. To model the impact of sales effort on the demand, we assume that the sales effort attracts more customers, and hence increases the customer's arrival probability. Specifically, we assume the arriving probability $\lambda(s)$ in each period is a concave and increasing function of sales effort s , where $0 < \lambda(s) < 1$ for any effort $s \geq 0$. Furthermore, the cost function $c(s)$ in each period is a convex and increasing function of sales effort s . Note that our cost function makes the linear cost models in Gerchak and Parlar (1987) and Urban (1998) special cases. Thus, the marginal effectiveness of effort is decreasing and the marginal cost of effort is increasing. This type of effort-demand model is consistent with the

literature where the sales response models have a multiplicative error term (cf. Gerchak and Parlar 1987, Rao 1990, Taylor 2002, Krishnan et al. 2004).

Each arriving customer will buy one unit of the product if the prevailing price does not exceed his or her reservation value. This reservation value is private information for the customer but its distribution is known to the retailer. Denote the cumulative distribution function as $F(\cdot)$ and the probability density function as $f(\cdot)$. Thus the probability that arriving customer purchases the product at the price p is given by $\bar{F}(p) = 1 - F(p)$. For simplicity, we assume that the salvage value of the product is zero. In each period t , the firm's objective is to maximize the total expected profit onwards by choosing an optimal combination of sales effort and price for the product. Let $V_t(x)$ denote the firm's optimal expected profit from selling the product when starting at period t with x units of inventory. The optimality equation is given by

$$V_t(x) = \max_{s,p} \{ \lambda(s) \bar{F}(p) [p + V_{t-1}(x-1)] + \lambda(s) F(p) V_{t-1}(x) + [1 - \lambda(s)] V_{t-1}(x) - c(s) \}$$

with boundary conditions $V_t(0) = 0$ for $t = 1, \dots, T$ and $V_0(x) = 0$ for all x . Given the sales effort s and the price p , the first term of $V_t(x)$ corresponds to the event that an arriving customer purchases the product, the second term corresponds to no-purchase, the third one corresponds to no arrival in the period, and the forth one is the cost for exerting effort s . We can rewrite the optimality equation as

$$V_t(x) = \max_{s,p} \{ \lambda(s) \bar{F}(p) [p - \Delta V_{t-1}(x)] - c(s) \} + V_{t-1}(x) \quad (4.1)$$

where $\Delta V_t(x) = V_t(x) - V_t(x-1)$ is the marginal value of inventory, which represents the maximum expected gain if the firm had one more unit of inventory to sell with inventory level x in period t .

Note that the domain for both the sales effort s and the price p are $[0, \infty)$, which immediately leads to the existence of the optimal (p, s) . Even for pricing-only optimization problem, Bitran and Mondschein (1997) show that the optimal price may not be unique. To impose a structure so that it is amenable to analysis, we make the following assumption on the reservation value distribution, which is standard in the revenue management literature.

Assumption 4.1 The function $F(\cdot)$ has an increasing generalized failure rate (IGFR), namely, $g(p) = pf(p)/\bar{F}(p)$ is weakly increasing in p .

The assumption provides some regularity for the value function. A variety of probability distributions satisfy this assumption, for example, the Weibull distribution, the uniform distribution and the positive part of the normal distribution. For ease of expression, let $R(p, \Delta) = \bar{F}(p)(p - \Delta)$ be the additional revenue in the initial stage of a period when the price is set at p and the marginal value of inventory is Δ , given that a customer arrives at the current period. Let $R(\Delta) = \max_p \bar{F}(p)(p - \Delta)$, given Assumption 4.1, the optimal p is unique. Moreover, $R(\Delta)$ is a decreasing convex function of Δ , and the effort maximization problem is of the form $\lambda(s)R(\Delta) - c(s)$. The retailer knows that the intensity of customer's arrival rate will be affected by the effort level, whereas the effort is costly. As $\lambda(s)$ is increasing concave and $c(s)$ is increasing convex, $\lambda''(s)R(\Delta) - c''(s) < 0$, hence the solution is unique.

Proposition 4.1 Under Assumption 4.1, the optimal solution for Problem (4.1) is unique.

Moreover, the optimal price $p_t^*(x)$ is the solution for

$$p = \bar{F}(p)/f(p) + \Delta V_{t-1}(x); \quad (4.2)$$

and the optimal sales effort $s_t^*(x)$ is determined by

$$s_t^*(x) = \operatorname{argmax}_{s \geq 0} \{\lambda(s)R(\Delta V_{t-1}(x)) - c(s)\}. \quad (4.3)$$

Note that the optimal price only depends on the marginal value of inventory in next period, rather than the arrival rate in current period. This is because the price becomes effective only when a customer does arrive in this period. Otherwise when no customer arrives at the current period, any price would not affect the expected profit. After the optimal price is determined, then $\lambda(s)R(\Delta V_{t-1}(x))$ becomes the expected additional revenue from the current period and $c(s)$ is the cost. Hence the retailer chooses the optimal sales effort $s_t^*(x)$ to maximize the expected additional profit.

4.3 Analytical Results

The importance of structural properties is well recognized in literature (e.g. Gallego and van Ryzin 1994). In this section we first show the properties for the marginal value of inventory. Then using it we show the monotone properties for the optimal price and effort.

Proposition 4.2

(a) $V_t(x)$ is decreasing in x for any fixed t .

(b) $\Delta V_t(x)$ is increasing in t for any fixed x .

Proof. (a) The proof is by induction on t . First, it is trivial for $t = 0$. Assume it is true for period $t - 1$. Note that

$$\begin{aligned} \Delta V_t(x) - \Delta V_t(x-1) &= [\Delta V_{t-1}(x) - \Delta V_{t-1}(x-1)] + \{\lambda(s_t^*(x))R(\Delta V_{t-1}(x)) - c(s_t^*(x))\} \\ &\quad - 2\{\lambda(s_t^*(x-1))R(\Delta V_{t-1}(x-1)) - c(s_t^*(x-1))\} \\ &\quad + \{\lambda(s_t^*(x-2))R(\Delta V_{t-1}(x-2)) - c(s_t^*(x-2))\} \end{aligned}$$

From the definition of $(s_t^*(x-1), p_t^*(x-1))$, we have

$$\lambda(s_t^*(x-1))R(\Delta V_{t-1}(x-1)) - as_t^*(x-1) \geq \lambda(s_t^*(x))R(p_t^*(x), \Delta V_{t-1}(x-1)) - c(s_t^*(x))$$

and

$$\begin{aligned} & \lambda(s_t^*(x-1))R(\Delta V_{t-1}(x-1)) - as_t^*(x-1) \\ & \geq \lambda(s_t^*(x-2))R(p_t^*(x-2), \Delta V_{t-1}(x-1)) - C(s_t^*(x-2)) \end{aligned}$$

Hence

$$\begin{aligned} & \Delta V_t(x) - \Delta V_t(x-1) \\ & \leq [\Delta V_{t-1}(x) - \Delta V_{t-1}(x-1)] + \lambda(s_t^*(x))\bar{F}(p_t^*(x))[\Delta V_{t-1}(x-1) - \Delta V_{t-1}(x)] \\ & \quad + \lambda(s_t^*(x-2))\bar{F}(p_t^*(x-2))[\Delta V_{t-1}(x-1) - \Delta V_{t-1}(x-2)] \\ & = [1 - \lambda(s_t^*(x))\bar{F}(p_t^*(x))][\Delta V_{t-1}(x) - \Delta V_{t-1}(x-1)] \\ & \quad + \lambda(s_t^*(x-2))\bar{F}(p_t^*(x-2))[\Delta V_{t-1}(x-1) - \Delta V_{t-1}(x-2)] \leq 0 \end{aligned}$$

(b) For any $t \geq 1$, we have

$$\begin{aligned} \Delta V_t(x) - \Delta V_{t-1}(x) &= [V_t(x) - V_{t-1}(x)] - [V_t(x-1) - V_{t-1}(x-1)] \\ &= \max_{s,p} \{ \lambda(s)\bar{F}(p)(p - \Delta V_{t-1}(x)) - c(s) \} \\ & \quad - \{ \lambda(s_t^*(x-1))\bar{F}(p_t^*(x-1))[p_t^*(x-1) - \Delta V_{t-1}(x-1)] - c(s_t^*(x-1)) \} \\ & \geq \{ \lambda(s_t^*(x-1))\bar{F}(p_t^*(x-1))[p_t^*(x-1) - \Delta V_{t-1}(x)] - c(s_t^*(x-1)) \} \\ & \quad - \{ \lambda(s_t^*(x-1))\bar{F}(p_t^*(x-1))[p_t^*(x-1) - \Delta V_{t-1}(x-1)] - c(s_t^*(x-1)) \} \\ & = \lambda(s_t^*(x-1))\bar{F}(p_t^*(x-1))(\Delta V_{t-1}(x-1) - \Delta V_{t-1}(x)) \geq 0. \end{aligned}$$

This proves the result. $\square$

The first part shows the optimal value function exhibits diminishing marginal returns to inventory. The second part indicates the marginal expected value of inventory decreases over time. Both of them are consistent with the conclusions for traditional dynamic pricing problem (e.g., Gallego and van Ryzin 1994). The monotone property of the value function is not only crucial for determination of the property for optimal policy, but also is of interest in itself. For

example, the optimal initial inventory can be determined if it is a part of the decisions. Now we consider the monotonicity of the optimal policy.

Proposition 4.3 Consider a retailer that has x units of inventory with t periods to go till the end of season. Then:

- (a) For any fixed time t , as the inventory level x increases, the optimal price $p_t^*(x)$ decreases and the optimal sales effort $s_t^*(x)$ increases.
- (b) For any fixed inventory x , as the left selling time t increases, the optimal price $p_t^*(x)$ increases and the optimal sales effort $s_t^*(x)$ decreases.

Proof. Based on (4.2), the optimal price does not depend on the current sales effort, so the monotone properties for the optimal price are the same as Bitran and Mondschein (1997). Now we show the properties for the optimal sales effort. As $s_t^*(x) = \operatorname{argmax}_{s \geq 0} \{\lambda(s)R(\Delta V_{t-1}(x)) - c(s)\}$. For part (a), as the inventory level x increases, $\Delta V_{t-1}(x)$ decreases and hence $R(\Delta V_{t-1}(x))$ decreases. Recall that $\lambda(s)$ a concave and increasing function and $c(s)$ is a convex and increasing function; therefore the optimal effort $s_t^*(x)$ increases in x for any t . Similarly, the optimal effort $s_t^*(x)$ decreases in t for any x . $\square$

Proposition 4.3 implies that at a given time, when the inventory becomes larger, the retailer should not only increase the sales effort to attract more customers but also reduce the price to entice the customer to purchase. The same logic holds as the time becomes shorter with fixed inventory. Figure 4.1 shows the optimal price and effort at time $t = 20$ for an example with $\lambda(s) = 1 - 0.5 \exp(-s)$, $c(s) = 3s$, and $F(\cdot)$ is a Weibull distribution with shape and scale parameters (30, 5). When the inventory is less than or equal to 6, the retailer exerts no effort, but as the inventory increases, the retailer will increase the effort to attract more customers.

Figure 4.1 Optimal price and effort for $t = 20$

4.4 Optimal Decision Problems with Static Effort or Static Price

So far we have studied the scenario that the firm can adjust both the effort level and the price dynamically, which is called DEDP model hereafter. In this section, we consider the cases when the firm has less flexibility in adjusting the sales effort and/or the price.

4.4.1 Static Effort and Static Price (SESP)

The first case is that both the sales effort and price are fixed during the entire selling season. Here we consider the continuous approximation. Hence given the sales effort s and the price p , the arrival process of making a purchase becomes a homogeneous Poisson process with the intensity rate $\lambda(s)\bar{F}(p)$. Therefore, the probability mass function of the total demand $D_t(s, p)$ from time t onwards follows a Poisson distribution with mean $\lambda(s)\bar{F}(p)t$. Note that the total cost of sales effort will depend on the time of stockout. Given inventory x , the stockout time Z

has an Erlang distribution with shape and scale parameters x and $\lambda(s)\bar{F}(p)$ respectively.

Therefore, conditioning on Z , the total effort cost becomes

$$\begin{aligned} \int_0^t c(s)zf_1(z) dz + c(s)t \cdot P(z \geq t) &= c(s) \left[\int_0^t z dF_1(z) + t(1 - F_1(t)) \right] \\ &= c(s) \left[t - \int_0^t F_1(z) dz \right], \end{aligned}$$

where $F_1(z) = 1 - \sum_{n=0}^{x-1} e^{-\lambda(s)\bar{F}(p)z} [\lambda(s)\bar{F}(p)z]^n / n!$ and $f_1(\cdot)$ is the pdf. So the expected profit $V_t(x, p, s)$ by selling inventory x given t and p becomes

$$V_t(x, s, p) = E\{p[D_t(s, p) \wedge x]\} - c(s) \left[t - \int_0^t F_1(z) dz \right]. \quad (4.4)$$

For the associated deterministic demand problem, given the sales effort s and the price p , the revenue rate is $\lambda(s)\bar{F}(p)p$ and the cost rate is $c(s)$. To ensure the retailer has a positive profit, the profit rate $\lambda(s)\bar{F}(p)p - c(s)$ should be greater than zero. For this reason, for our stochastic demand problem, we assume $\lambda(s)\bar{F}(p)p - c(s) > 0$.

Proposition 4.4 Given the sales effort s and the price p , $\Delta V_t(x, s, p)$ is decreasing in x for any fixed t .

Proof. From (4.4), we know

$$\begin{aligned} V_t(x, s, p) - V_t(x-1, s, p) &= p \sum_{n=x}^{\infty} \frac{(\lambda(s)\bar{F}(p)t)^n e^{-\lambda(s)\bar{F}(p)t}}{n!} - c(s) \int_0^t \frac{[\lambda(s)\bar{F}(p)z]^{x-1} e^{-\lambda(s)\bar{F}(p)z}}{(x-1)!} dz \\ &= p \sum_{n=x}^{\infty} \frac{(\lambda(s)\bar{F}(p)t)^n e^{-\lambda(s)\bar{F}(p)t}}{n!} \\ &\quad - c(s) \left[\frac{e^{-\lambda(s)\bar{F}(p)z} [\lambda(s)\bar{F}(p)z]^x}{\lambda(s)\bar{F}(p)x!} + \int_0^t \frac{[\lambda(s)\bar{F}(p)z]^x e^{-\lambda(s)\bar{F}(p)z}}{x!} dz \right] \end{aligned}$$

Hence we have

$$\Delta V_t(x, s, p) - \Delta V_t(x + 1, s, p) = p \frac{(\lambda(s)\bar{F}(p)t)^x e^{-\lambda(s)\bar{F}(p)t}}{x!} - c(s) \frac{e^{-\lambda(s)\bar{F}(p)t} [\lambda(s)\bar{F}(p)t]^{x-1} t}{x!} =$$

$$[\lambda(s)\bar{F}(p)p - c(s)] \frac{e^{-\lambda(s)\bar{F}(p)t} [\lambda(s)\bar{F}(p)t]^{x-1} t}{x!} > 0.$$

The last inequality is due to $\lambda(s)\bar{F}(p)p - c(s) > 0$. $\square$

Note that when the retailer has the option to set the optimal price, our setting is similar to that in Weinberg (1975), Lal (1986) and Joseph (2001), where the retailer chooses both the optimal effort and the price. Otherwise when the price is fixed, the demand is the same as that in Basu et al (1985), which is also similar to the demand with multiplicative form in Taylor (2002) and Krishnan et al (2004).

4.4.2 Dynamic Effort and Static Price (DESP)

Consider another case that the firm charges a fixed price p over the entire horizon, but is able to choose the sales effort to adjust customers' arrival rate. Given the price p , denote $V_t(x, p)$ as the firm's optimal expected profit from selling the product when starting at time t with x units of inventory. Now the firm's dynamic problem becomes

$$V_t(x, p) = \max_s \{ \lambda(s)\bar{F}(p)[p - \Delta V_{t-1}(x, p)] - c(s) \} + V_{t-1}(x, p) \quad (4.5)$$

with boundary conditions $V_t(0, p) = 0$ for $t = 1, \dots, T$, $V_0(x, p) = 0$ for all x , and where $\Delta V_t(x, p) = V_t(x, p) - V_t(x - 1, p)$ is the marginal value of inventory for the DESP problem.

This case is similar to the problem of Tapiero and Farley (1975) in the sense of dynamically exerting sales effort. While they study the problem of how to allocate given effort among several products, our work only considers one product, whereas the effort is costly. Hence here the retailer determines the optimal effort rather than how to allocate the given effort.

For DESP problem, one can show a result analogous to Propositions 4.2: the marginal value of inventory decreases in the inventory level (x) and increases in the remaining time (t). Given such

a result on the marginal value of inventory, one can also establish the monotone property for the optimal sales effort $s_t^*(x, p)$, summarized in the following proposition.

Proposition 4.5 Given the price is fixed at p .

- (a) The marginal value of inventory $\Delta V_t(x, p)$ is decreasing in x for any given t and increasing in t for any given x .
- (b) The optimal effort level $s_t^*(x, p)$ is increasing in x for any fixed t , decreasing in t for any fixed x .

Proof. The proof is similar to the proofs of Propositions 2 and 3, hence omitted. $\square$

4.4.3 Static Effort and Dynamic Price (SEDP)

The last case we consider is that the retailer uses a fixed effort level during all the selling season, only adjust the price dynamically to maximize the expected profit. Note that the problem is similar to the traditional dynamic pricing problem (e.g. Bitran and Mondschein 1997), except that here we have a costly effort. Given the sales effort s , denote $V_t(x, s)$ as the firm's optimal expected value from selling the product when starting at period t with x units of inventory. Then the firm's problem becomes

$$V_t(x, s) = \max_p \{ \lambda(s) \bar{F}(p) [p - \Delta V_{t-1}(x, s)] - c(s) \} + V_{t-1}(x, s) \quad (4.6)$$

with boundary conditions $V_t(0, s) = 0$ for $t = 1, \dots, T$, $V_0(x, p) = 0$ for all x , and where $\Delta V_t(x, s) = V_t(x, s) - V_t(x - 1, s)$ is the marginal value of inventory for the SEDP problem. Similarly, using the same technique as DEDP model, one can also show the concavity of the expected value function and the monotone property for the optimal price $p_t^*(x, s)$.

Proposition 4.6 Given the sales effort is fixed at s .

- (a) The marginal value of inventory $\Delta V_t(x, s)$ is decreasing in x for all t and increasing in t for all x .
- (b) The optimal price $p_t^*(x, s)$ is an decreasing function of x for all t and a increasing function of t for all x .

4.5 Numerical Study

So far, we have considered four different types of effort and price policies: dynamic effort and dynamic pricing policy (DEDP), dynamic effort and static price policy (DESP), static effort and dynamic price policy (SEDP) and static effort and static price policy (SESP). In a DEDP policy, both the effort and the price are adjusted dynamically depending on the inventory level and the remaining selling time. For a DESP policy, the effort is revised dynamically, whereas the price is chosen optimally at the beginning of the season and kept the same throughout the time horizon. Likewise, under the SEDP policy, the effort is chosen optimally and kept during the whole selling season and the price is adjusted dynamically. Finally, in an SESP policy, both the effort and price are chosen optimally and then kept throughout the season.

To gain further managerial insights into the effect of retailer's sales effort along with dynamic pricing, we conduct extensive numerical experiments in this section. Following the setting of Rao (1990), we use the arrival rate $\lambda(s) = 1 - b \exp(-s)$, where b can be interpreted as the proportion of consumers who are unaware of the product. As Gerchak and Parlar (1987), the cost is linear with $c(s) = c \cdot s$, where c is the cost rate for per-unit sales effort. For the customer, we follow Bitran et al. (1998) and use a Weibull distribution with parameters (θ, k) to model customer's reservation value. We test a wide range of model parameter combinations. Without loss of generality, we fix the scale parameter $\theta = 30$ and then vary other parameters. The cost

rate for per-unit sales effort c is chosen from $\{1, 2, 3, 4, 5\}$. The parameter b has values from $\{0.1, 0.3, 0.5, 0.7, 0.9\}$. The shape parameter k , which is equivalent to the coefficient of variation (CV) and hence captures the relative variability or the degree of heterogeneity among customers, takes values from $\{1, 3, 5, 7, 9\}$. As the parameter k increases, the relative variability decreases ($CV = \{1, 0.36, 0.23, 0.17, 0.13\}$ respectively). Moreover, when $k = 1$, it is the exponential distribution. Finally, we consider a 20-period selling season and change the starting inventory level between 1 and 20.

Given different values of the model parameters, there are 2500 different instances. For all these instances, we compute the profit under each policy. Note that the policies can be ordered in a sequence of DEDP, DESP or SEDP, and SESP from more sophisticated to less sophisticated. Moreover, given that the retailer can choose to adjust either effort or price dynamically, we consider a partial dynamic policy called PD policy where the retailer uses the better policy of DESP and SEDP policies. We then compute the profit percentage improvements that would be obtained by switching from less sophisticated policies to more sophisticated ones and determine the average, maximum, and minimum improvements over the 2500 instances. The results are summarized in Table 4.1.

Table 4.1 clearly demonstrates the benefits of using dynamic policies. Numbers suggest that overall the profit improvements that would be obtained by switching from fully static policy SESP to any dynamic policy are significant. In particular, the fully dynamic policy DEDP over the fully static one SESP has an average improvement closes to 4.7%, with some cases achieving higher than 10%. In more than 72% of instances, the percentage profit improvement is greater than 3%. In about 20% of instances, the profit improvement is greater than 7%. These improvements are more significant than previous findings in Gallego and van Ryzin (1994) and

Zhao and Zheng (2000). The reason is because here the retailer is able to adjust both the price and effort level. Even for the partial dynamic policies, on average, SEDP and DESP achieve 3.9% and 3.3% respectively higher profit than SESP policy.

Table 4.1 Improvement in Profits Obtained by Switching from Less Sophisticated Policies to More Sophisticated Policies

	[0,1)	[1,2)	[2,3)	[3,4)	[4,5)	[5,6)	[6,7)	[7,8)	[8,9)	[9,10)	≥ 10	Max	Min	Average
DDvsSS	425	153	133	148	242	322	580	407	60	20	10	0.1340	0	0.0466
SDvsSS	443	146	190	258	441	537	409	76	0	0	0	0.0754	0	0.0394
DSvsSS	455	344	319	324	364	388	253	41	4	4	4	0.1110	0	0.0334
DDvsSD	1899	327	156	62	37	3	8	0	4	4	0	0.0997	0	0.0070
DDvsDS	1373	439	340	219	105	24	0	0	0	0	0	0.0579	0	0.0128
DDvsPD	2078	342	70	8	2	0	0	0	0	0	0	0.0433	0	0.0044

Notes. Numbers represent the number of instances, SS: static effort and static price, SD: static effort and dynamic price, DS: dynamic effort and static price, DS: dynamic effort and dynamic price.

A natural practical question is whether the retailer needs to implement the fully dynamic policy in practice, or partial dynamic policies are good enough? The profit improvements by switching from partial dynamic policies to DEDP give the answer. First note that DEDP over SEDP and DESP policies have an average improvement of 0.7% and 1.3% respectively, which seems to imply that they cannot be ignored. However, if the retailer uses PD policy, the average improvement of switching it to DEDP policy is only 0.44%. This indicates that it is enough to implement some partial dynamic policy. We next explore which one to use, SEDP or DESP policy.

To identify when to use dynamic policy and which partial dynamic policy to implement, we study the value of adjusting price or/and effort, which is measured by the profit improvement by

switching from SESP to SEDP, DESP and DEDP respectively, with respect to different parameters, including inventory level, effort cost, proportion of potential market unaware of the product and CV of the customers' reservation value. In reporting the results, we take the average over all the instances related to a specific level of the parameter on which we are focused. For example, if we want to see the percentage improvement when the effort cost is 2, we take an average of the percentage improvement over all the 500 instances.

Figure 4.2 illustrates the three values according to different inventory level. As long as the inventory level is not very high, the improvement percentage is significant. Moreover, the value of adjusting price is always higher than the value of adjusting effort. It is interesting to find that with the ability to adjusting the initial sales effort level, the value of pricing keeps a relatively high and stable level when the inventory is low and moderate, which is different from the stable decreasing improvement of inventory in Gallego and van Ryzin (1994).

Figure 4.2 Profit improvement percentages when switching to dynamically adjust effort and/or price with respect to inventory level

Figure 4.3 displays the three values according to the cost for sales effort. All of them decrease as the effort cost increases. As the cost increases, the burden for the retailer to adjust the effort increases; hence the impact by adjusting the values decreases. A thought experiment provides further intuition. As the effort cost goes to infinite, the value of dynamic effort becomes zero. Furthermore, the value of adjusting price is still higher than the value of adjusting effort. Similar phenomenon is also found in Figure 4.4 for the three values with respect to CV. Furthermore, as CV decreases (or k increases), all of the three values increase.

Figure 4.3 Profit improvement percentages when switching to dynamically adjust effort and/or price with respect to the cost for sales effort

Figure 4.4 Profit improvement percentages when switching to dynamically adjust effort and/or price with respect to CV

However, these values with respect to the proportion of potential market unaware of the product show different trends (Figure 4.5). As the proportion of potential market increases, the value of adjusting both effort and price dynamically is almost the same, however, the value of dynamic pricing decreases and the value of only adjusting effort increases. Moreover, when the potential market is large enough, the benefit of switching SESP to DESP policy is greater than to SEDP policy.

Figure 4.5 Profit improvement percentages when switching to dynamically adjust effort and/or price with respect to the proportion of potential market unaware of the product

In summary, when the retailer's inventory is not high, there is likely an opportunity for partial dynamic policies. Whether the retailer should dynamically adjust the effort or the price depends on the proportion of potential market unaware of the product. In general, when the proportion of potential market is high, the retailer should use dynamic effort; otherwise dynamic pricing is better.

4.6 Conclusion and Future Directions

In this chapter, we investigate the interactions among the sales effort, the price and the available inventory. We find that as the left inventory level increases or the remaining selling time decreases, to accelerate the sales of the product, the retailer will exert more effort to attract more customers no matter whether the retailer revises the price dynamically or not, and set a lower price to motivate the arriving customer to make a purchase regardless whether the retailer adjusts the effort level dynamically or not. Our numerical study indicates the profit impact of dynamic effort and price is more significant than traditional dynamic pricing. However one of the partial

dynamic policies, namely, dynamically adjusting effort or price, is enough to capture most of the improvement. A critical factor for choosing dynamically adjusting the effort or the price is the potential market unaware of the product. When the potential market is large, dynamic effort would be better; otherwise the retailer should use dynamic pricing.

Some possible extensions of this research include: (i) considering the sales effort affect both the arrival rate and customer's reservation value; (ii) allowing batch demand instead of unit purchase; (iii) studying the sales-force commission problem in the dynamic pricing environment; and (iv) incorporating the strategic behavior of customers.

Chapter 5

Summary and Future Directions

5.1 Summary of Main Contributions

Dynamic pricing has been adopted effectively to manage stochastic demand to improve revenue for retailing industry. During its application, managers must also take into account other factors from economics, supply chain, marketing and so on. Hence this thesis studies several dynamic pricing models that attempt to incorporate methodologies from such disciplines into traditional dynamic pricing.

In Chapter 2, we study a dynamic pricing model for a retailer with limited inventories over a finite time horizon in which an individual's purchase quantity is endogenous. Traditionally, a standard assumption for dynamic pricing in revenue management is that a customer purchases at most one unit. While this assumption is valid for travel industry, it is problematic in a retail setting because the buyer usually does purchase multiple units. The problem of multiunit demand has been recognized in literature even since the work of Gallego and van Ryzin (1994), but has never been properly addressed so far. We handle this issue by analyzing the underlying utility function; hence a rational customer will optimize the purchase quantity by maximizing the utility. Three types of pricing schemes are examined: the dynamic nonuniform pricing (DNP) scheme, the dynamic uniform pricing (DUP) scheme and the dynamic block pricing (DBP) scheme. We find that the potential revenue improvement of DNP over DUP ranges from 30% to 90%. Most importantly, our numerical studies reveal that DBP scheme always achieves more than 97% of

the revenue from DNP scheme. Hence for practical purpose, all we need is DBP scheme. Our results provide a theoretical explanation as to why many retailers use just two-block pricing scheme in reality.

In Chapter 3, we consider a decentralized supply chain with one supplier and one retailer in which the retailer practices dynamic pricing. The main contribution of this chapter is the analysis of the decentralized dynamic pricing system and providing mechanisms for coordinating the supply chain. As for the practitioner, we find that the benefit for the retailer switching from price-setting to dynamic pricing policy is significant when the product obsolesces fast. Moreover, this benefit of pricing flexibility is symmetrically shared between the supplier and the retailer. Therefore both the supplier and retailer would have an incentive to implement dynamic pricing policy. Furthermore, the value of coordinating this system is still significant. Based on the contracts examined in Cachon and Lariviere (2005) for price-setting system, we find that revenue-sharing, two-part tariffs and quantity discount coordinate the underlying system.

In Chapter 4, we address the problem for a firm that dynamically adjusts effort and/or price for selling limited quantities of product before some specific time. This work brings sales effort into the literature on revenue management, and hence, it will enhance the application of dynamic pricing in the retailing industry. We show the structural properties for the optimal policies under different flexibility of pricing and exerting effort. Even though the retailer is able to choose an optimal initial price (effort), a numerical study shows that the potential profit improvement is still significant from dynamically adjusting the effort (price respectively). However there is no need to simultaneously adjust both the price and the effort dynamically because the additional benefit is not so significant.

5.2 Future Directions

In this thesis, we have studied three dynamic pricing models for perishable assets regarding to multi-unit demand, decentralized supply chain and the impact of sales effort. We have addressed each of them individually due to the complexity. In the future, of course, it is necessary to combine them together. For example, there is a need to study the decentralized supply chain problem in which customers also purchase multiple units. Moreover, some other important topics in Revenue Management deserve further exploration.

5.2.1 Demand Learning

The basic framework for RM assumes full knowledge of the underlying statistical characteristics of the demand, and firms dynamically adjust price to balance the inventory level and future selling opportunity. However, this full knowledge of the demand uncertainty or the state of the market is not always available. For example, Sport Obermeyer (Hammond and Raman 1996) found that they face a “fashion gamble” because of inaccurate forecasts of demand. The ticketing sales for sports or theaters also face the same uncertain nature; the firms do not know whether the game/concert will be popular or not in advance.

In such circumstances, the seller has only a vague idea on the state of the market at the beginning of the sales season. As the sales process proceeds, the firms not only accrue revenue but also update their knowledge of the state of market through sales observations. Therefore the optimal pricing strategy needs to take into account the interactions among the inventory, the future selling opportunity and the information value from the selling process.

5.2.2 Strategic Customer Behavior

Strategic customer is an important aspect when doing dynamic pricing in practice. If the customer anticipates that the retailer is going to reduce the price of a product in the future, some of them would be willing to delay their purchases. This consideration has been examined by Su (2007, 2010), Aviv and Pazgal (2008) and so forth. However, none of them explicitly consider the multi-unit demand under customer choice. A common pricing policy for milk/juice industry is that in the first period the price is high and in the second period, a discount price for bundle purchase (more than one unit) is provided. So far, this kind of business practice is ignored in the research literature on strategic customers. Our multi-unit demand model in conjunction with strategic customer behavior is expected to make a further step to explore the rationale of such pricing behavior and provide some guideline for practitioners.

5.2.3 Competition

We have limited our study to one supplier and one retailer model in Chapter 3. But it would be interesting to study the case of multiple retailers. Lippman and McCardle (1997) and Zhao and Atkins (2008) have examined the competition for fixed-price and price-setting newsvendor respectively. For dynamic pricing newsvendor, how will the retailers make the ordering decision under competition and how can such a system with competing firms be coordinated? Martínez-Albéniz and Talluri (2011) and Liu and Zhang (2013) may have paved the way for this research. Furthermore, the study of competing suppliers will be also interesting for both academicians and practitioners.

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