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Theoretical Modelling of Oscillating Water Column Wave Energy Converters

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Summary

For reasons of more available wave energy, and favourable wave climate and geography, some devices for extracting energy from ocean waves will likely be installed at nearshore or offshore. Based on matching eigenfunction expansions and a fast convergence technique, four specific Oscillating Water Columns (OWC), which are bottom-mounted or fully-truncated, are analytically studied in this thesis.

An analytical model of an oscillating water column supported by a coaxial tube-sector-shaped structure extending to the sea floor is developed. In order to speed up the convergence in numerical computations, an auxiliary radial velocity function is introduced, which is expanded in orthogonal polynomials and includes the singular behaviors of the flow at sharp corners. Power-takeoff (PTO) by Wells turbines, which are installed on the top of a pneumatic chamber, is modeled by including the compressibility of air and assumed to be linear (i.e., there is no lag between the mass flux through the turbines and the air pressure drop). It is shown that the coaxial tube-sector-shaped support with an opening in the range of $\pi/2$ - $5\pi/4$ can significantly increase the extraction efficiency and widen the frequency range over which the extraction efficiency is high. Optimization by controlling the turbine characteristics and resulting capture length are examined for a wide range of angular frequencies.

As an extension of the above OWC device, a bottom-mounted oscillating water column converter with a V-shaped channel is analytically developed by using the same approach. The auxiliary radial velocity functions are expanded in the form of both vertical and azimuthal basis functions, which includes the singular behaviors of the flow in vertical and azimuthal directions respectively. It is found that in this configuration, the V-shaped channel can significantly raise the extraction efficiency and widen the range of frequency over which the OWC system can operate at a high efficiency. It is also shown that in view of

the capture length, the OWC-type devices with a V-shaped channel is about 6-7 times more efficient than the fully truncated one and 3-4 times more efficient than the one without V-shaped channel.

A free floating OWC converter in heaving motion is also developed analytically. Compared with the previous two OWC devices, another radiated wave field is generated by the heaving motion of the system. The correction of the optimization of turbine characteristics presented by Mavrakos & Konispoliatis (2012) has been given. It is found that the maximum extraction efficiency, under the corrected optimization of turbine characteristics, can be the same as that where a circular OWC device with thin walls is fixed in waves. In addition, a small size of circular floating platform, involving the width and submergence depth, is beneficial for widening the bandwidth of high extraction efficiency.

Finally, a heaving OWC converter restrained by a linear system of spring-damper, which can be considered as another power-takeoff (PTO) model (for example, a hydraulic extraction device), is developed as well. The diffraction and radiation problems are similar with those for a free floating OWC converter in heaving motion. The results show that a moderate intensity of damper is beneficial for the OWC device placed at the sea condition where wavelength is relative long. In addition, a relatively strong intensity of spring is beneficial for widening the bandwidth of high extraction efficiency.

The main contributions of this thesis are: (1) four analytical models for physical insights involving the understanding of hydrodynamic performances and prediction of extraction efficiency; (2) a fast convergence technique by including the singular behaviors of flow in both vertical and azimuthal directions; (3) a power-takeoff (PTO) model incorporating the added spring-damper system.

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List of Abbreviation

OWC	Oscillating Water Column
OWCs	Oscillating Water Columns
PTO	Power Take-off
WEC	Wave Energy Converter
WECs	Wave Energy Converters
SWL	Still Water Level
RAO	Response Amplitude Operator

List of Symbols

λ	Wavelength
k	Wavenumber
k_l	$l = 1, 2, 3, \dots$, positive real roots of the dispersion relation for evanescent waves
A	Wave amplitude
t	Time
C_g	Group velocity of waves
ω	Angular frequency
α	Incidence angle
g	Acceleration of gravity
R	Reflection coefficient
T	Transmission coefficient
h	Constant water depth
$Oxyz$	Cartesian coordinate system
$Or\theta z$	Cylindrical coordinate system
Φ	Velocity potential
ϕ	Time-independent complex velocity potential
ϕ_I	Time-independent complex velocity potential of incident waves
ϕ_D	Time-independent complex diffraction potential
ϕ_S	Time-independent complex scattering potential
ϕ_R	Time-independent complex radiation potential
ϕ_P	Time-independent complex radiation potential due to the fluctuation of air pressure
ϕ_3	Time-independent complex radiation potential in heave
v_3	Instantaneous vertical velocity of the floating platform
V_3	Complex amplitude of vertical velocity of the floating platform
$\mathcal{W}$	Denoting ϕ_D or ϕ_R
$\mathcal{W}^{(s)}$	$s = 1, 2, \dots$, denoting complex velocity potential in subregion s
η	Free surface elevation

ζ	Time-independent free surface elevation
ζ_I	Time-independent surface elevation of incident waves
ρ	Density of water
ρ_a	Density of air
ρ_a^0	Air density at rest
δ_{ml}	Kronecker delta
P_a	Air pressure
p_a	Complex-valued amplitude of air pressure
P_0	Atmospheric pressure (usually, set to zero)
P	hydrodynamic pressure
$\mathbf{n}$	Inward normal vector
j	modes of the body motion
i	$\sqrt{-1}$, imaginary unit
$\mathcal{M}$	Mass of the air
V	Instantaneous volume of the air chamber
V_0	Air-chamber volume at rest
$\hat{K}$	Dimensionless damping coefficient
D	Outer diameter of the turbine rotor
$\hat{N}$	Rotational speed of the turbine blades
Q	Volume flux of the air inside the OWC chamber
q	Complex-valued amplitude of volume flux Q
q_D	Volume flux due to the wave diffraction
q_R	Volume flux due to the wave radiation
c_a	Sound speed in air
Γ	Diffraction flux factor
$\mathcal{B}$	Radiation conductance
$\mathcal{C}$	Radiation susceptance
$\tilde{\Gamma}$	Dimensionless diffraction flux factor
$\tilde{\mathcal{B}}$	Dimensionless radiation conductance
$\tilde{\mathcal{C}}$	Dimensionless radiation susceptance
L	Capture length
kL	Dimensionless capture length
kL_{opt}	Optimum dimensionless capture length
kL_{max}	Maximum dimensionless capture length

χ	Turbine characteristics
χ_{opt}	Optimum turbine characteristics
κ	Characterizing the air chamber
P_{out}	Extracted wave power flux
P_w	the wave power per unit crest length of the incident waves
J_m	Bessel function of the first kind of m order
H_m	Hankel function of the first kind of m order
I_m	Modified Bessel function of the first kind of m order
K_m	Modified Bessel function of the second kind of m order
$\mathcal{F}_3$	Vertical hydrodynamic reaction force
f_{33}	Complex amplitude of vertical hydrodynamic reaction force
f_{d3}	Complex amplitude of vertical wave force due to wave diffraction
f_{p3}	Complex amplitude of vertical wave force due to oscillating air pressure
Γ_{d3}	vertical wave force factor due to wave diffraction
S_i	plane area of the air chamber
S_w	plane area of the wetted floating platform
$\mathcal{M}$	Mass of the OWC device incorporating platform
μ_{33}	Added mass
ν_{33}	Radiation damping
$\mathcal{D}$	Linear damping constant
$\mathcal{K}$	Linear spring constant
kL_{tur}	Dimensionless capture length by Wells turbines
kL_{tur}^{opt}	optimum capture length by Wells turbines
P_{out}^D	Time-averaged rate of work done by linear damper
kL_D	Dimensionless capture length by the PTO system
kL_{tot}	Total dimensionless capture length
Θ_{dif}	Phase difference

Chapter 1

Introduction

1.1 Background

Due to the increasing energy demand and climate change, more attention is being paid to alternative renewable energy resources. Renewable energy could be collected from different natural resources such as solar, wind, geothermal, ocean, and so on (Falcão, 2010; Fadaeenejad *et al.*, 2014). The ocean is acknowledged by experts as a vast prospective energy resource, which holds a tremendous amount of untapped energy, existing in various forms such as marine currents, tidal currents, geothermal vents, and waves (Brooke, 2003). The wave power resources, as one type of renewable energy resources, is estimated 1-10 *TW* of wave power in the entire ocean, and on average, each wave crest transmit about 10-50 *kW* per meter (Vining, 2005; Saket & Etemad-Shahidi, 2012). Meanwhile, wave energy is predictable and can be captured. Thus wave energy is considered to be one of the potential alternative sources (Liberti *et al.*, 2013), although the utilization of wave energy is currently still at the early stage of industrial level (Falnes, 2007).

Ocean waves essentially result from the transfer of solar energy. Because of the uneven heating from the sunlight, the winds above the ocean surface are induced. When the winds blow over the ocean surface, part of the wind energy is converted to ocean wave energy. The utilization of ocean waves as a source of renewable energy has several significant advantages as follows:

- (i) Ocean waves can offer the highest energy intensity (Clément *et al.*, 2002). Solar energy intensity of typically 0.1-0.3 *kW/m²* of horizontal surface of the earth is converted to an average power flow intensity of 2-3 *kW/m²*

of vertical plane perpendicular to the direction of wave propagation just below the water surface (Falnes, 2007; Drew *et al.*, 2009; Hughes & Heap, 2010).

- (ii) Limited negative environmental impact in use.
- (iii) Waves can propagate thousands of miles with little energy loss.
- (iv) Wave energy converters (WECs) can work up to 90% of the time, compared to 20-30% for wind and solar power devices (Drew *et al.*, 2009).

The propagation of ocean waves carries vast energy, and this energy can be captured at any time with proper operations. The mechanism of utilizing ocean wave energy, such as electricity generation, sea water desalination and the pumping of water, can be traced back to the late eighteenth century or even earlier (Ross, 1995). However, the systematic studies only began in 1970s. Till now, there are large number of concepts for wave energy conversion, and over 1000 wave energy conversion techniques have been patented in Japan, North America, and Europe (Drew *et al.*, 2009). Despite the large variation in design, these devices are generally categorized by their installed location:

(i) Shoreline devices

Shoreline devices are very close to the electricity grid. They are easy to maintain, and less susceptible to damage in extreme sea conditions. However, the available energy to be extracted is relatively small.

(ii) Nearshore devices

Nearshore devices are installed in relatively shallow water and close to shoreline. These devices have suitable stationary bases against which oscillating bodies can work. Like shoreline devices, the available energy to be extracted is also relatively small.

(iii) Offshore devices

Offshore devices can harvest greater amount of energy. However, these devices are more difficult to construct and maintain, and more expensive due to the higher chance of exposure to extreme sea conditions. It is argued that floating devices can offer greater structural economy.

Figure 1.1 Point absorber device: AWS Point Absorber (Drew *et al.*, 2009).

In general, starting with the method used to capture wave energy, neglecting the installed sites, the proposed wave energy converters can be further classified into four types:

- Point absorber

A point absorber possesses small horizontal dimensions relative to the incident wavelength. It consists of floating structures heaving up and down on the water surface or submerged below the surface to work by pressure differential. One of the examples is AWS Point Absorber as shown in Figure 1.1. The disadvantages of point absorbers are the complexity of the drive mechanisms and the difficulty of mooring in an offshore environment (Drew *et al.*, 2009).

- Oscillating water column

The oscillating water column (OWC) wave energy converter has received most comprehensive attention due to its promising commercial potential Falcão (2010). In general, an OWC device consists basically of two key parts: a trapped air chamber, which acts as a collector taking energy from waves and storing it into the air inside the chamber, and a power take-off (PTO) system (usually, Wells turbines installed at the top of the

Figure 1.2 Illustration of an oscillating water column (Fadaeenejad *et al.*, 2014).

chamber), which further converts the pneumatic energy into some other useful form such as electricity. The concept is that under the incident waves, the interior water free-surface will raise and fall, which further induces an oscillating air pressure to derive one/more unidirectional air turbines to rotate for the fulfillment of energy conversion. The illustration of an oscillating water column is shown in Figure 1.2. The shortcoming is that this type of device is difficult to scale up to an appreciable size and suitable locations are difficult to find (Falnes, 2007).

- Oscillating wave surge converter

An oscillating wave surge converter is generally consist of a hinged deflector, which is perpendicular the wave direction. The concept is that under the incident waves, the deflector moves forth and back to derive a hydraulic system for energy conversion. An example is the Aquamarine Power Oyster, which has been constructed, as shown in Figure 1.3. Some disadvantages of this type of devices are as follows: (1) the installation and production are difficult and expensive and (2) the operation may interfere with marine mammal life and fish life (Drew *et al.*, 2009).

- Overtopping device

An overtopping device captures sea water of incident waves in a reservoir above the sea level, then releases the water back to sea through turbines.

Figure 1.3 Oscillating wave surge converter: Aquamarine Power Oyster (Drew *et al.*, 2009).

Figure 1.4 Overtopping wave energy converter: the Wave Dragon (Fadaeenejad *et al.*, 2014).

An example of such a device is the Wave Dragon, as shown in Figure 1.4. This device works well at capturing energy in deeper water environments, however it has a large footprint and produces the least amount of energy per unit area of any current wave power generation system (Drew *et al.*, 2009).

In this thesis, the absorption efficiency of wave energy devices is measured by the so-called capture length, which is the ratio of the extracted power flux to the wave power per unit crest length of the incident waves.

Compared to wind or solar energy converter technologies, the technology of wave energy converter is still immature due to its low extraction efficiency. Commercialization of wave energy converters has mostly occurred in the U.S. with several installations planned along the coasts. Europe still regards this technology to be in the research stage even though at least two designs have

been incorporated by European utilities for prototyping purposes (Falnes, 2007; Falcão, 2010). If the technology can be successfully developed, the market potential is enormous.

Though many ingenious wave energy converter devices have been proposed, the most widely studied fall in two major types, i.e., point absorber and oscillating water column. In this thesis, the later will be investigated theoretically. Specially, in order to convert more available energy from ocean waves, the nearshore and offshore OWC-type water energy converters will be analytically studied.

1.2 Objectives and scope

Let us first briefly view four circular OWC-type configurations shown in Figure 1.5. Figure 1.5(a) describes a circular OWC device supported by a semi-circular structure, which is beneficial to enhance the partial standing waves underneath the bottom opening of the OWC chamber. As an extension, two arms shown in Figure 1.5(b), which form a V-shaped channel, are attached to the OWC device with semi-circular supporting structure. This V-shaped channel is used to collect wave power and beneficial to enhance the amplitude of standing waves. A free floating OWC device in heave is shown in Figure 1.5(c). Furthermore, a linear spring-damper system, which acts as another power take-off (PTO) system (the first one is represented by the turbine system), is attached to the free floating OWC device in heave, as shown in Figure 1.5(d).

1.2.1 Objectives

The main objective of this study is to analytically predict the hydrodynamic characteristics of four OWC-type wave energy converters and model their extraction efficiencies. Of most importance is to improve these four systems performances so that high extraction efficiency is achieved over a wide range of frequencies around the spectral peak of the incident sea.

Another objective of this study is to develop auxiliary radial velocities at common boundaries between subregions, which are expanded in one or two sets of orthogonal basis functions, to speed up the convergence in numerical computations. Special emphasis is that the orthogonal basis functions include the singular behaviors of the flow at sharp corners in both vertical and azimuthal directions.

Figure 1.5 Schematic diagram of four OWC-type wave energy converters.

1.2.2 scope

The scope of this project includes:

(i) Bottom-mounted nearshore OWC devices

- An OWC supported by a coaxial tube-sector-shaped structure is modelled analytically to investigate the effects of the supporting wall on the hydrodynamic performances and extraction efficiency.
- A bottom-mounted OWC converter with a V-shaped channel is modelled analytically to examine the effects of the V-shaped channel on the hydrodynamic performances and extraction efficiency. The singular behaviour of the flow at sharp corner in azimuthal direction is included in the expansions of auxiliary radial velocities as well.

(ii) Floating offshore OWC devices

- An OWC centered at a circular floating platform with a finite width is modelled analytically to investigate the effects of the coupling response of the OWC system and air pressure inside the chamber on the hydrodynamic performances and extraction efficiency.
- A floating OWC device restrained by a linear spring-damper system, which is used to model another power take off (PTO), is modelled analytically to investigate the effects of the spring and damping constants on the hydrodynamic performances and extraction efficiency.

1.3 Outline of thesis

This thesis is organized as follows:

The first chapter provides the theoretical background as well as the objectives and scope of the present study.

Chapter 2 provides a literature review on water wave diffraction and radiation by barriers, and wave energy converters. It also provides a generalized approach used to solving the coming problems analytically.

Chapter 3 studies the wave power extraction by an OWC converter supported by a coaxial tube-sector-shaped structure. The effects of the supporting wall are discussed. The optimum turbine characteristics and resulting capture length are presented as well.

Chapter 4 investigates the wave power extraction by a bottom-mounted OWC converter with a V-shaped channel, which is based on the OWC device in Chapter 3. The effects of the V-shaped channel are examined. The optimum turbine characteristics and resulting capture length are presented as well.

Chapter 5 investigated a floating OWC device centered at a circular floating platform of finite width. The effects of the coupling of the response of structure and the air pressure inside the column on the hydrodynamic performances and extraction efficiency are examined.

Chapter 6 studies a floating OWC device restrained by a spring-damper power take-off (PTO) system, which is a extension of the OWC device in Chapter 5. The effects of the spring and damping constants on the hydrodynamic performances and extraction efficiency are examined.

In the last chapter, we summarize the major conclusions and discuss about the directions and schedule of our future research.

Finally, the analytical solutions of integrals and original Matlab codes used to numerically compute the results to the problems in Chapter 3, 4, 5, 6, respectively, can be found in the Appendix A.

Chapter 2

Literature review

In this chapter, literature review and generalized procedure solving the boundary-value problems in the following chapters are given.

2.1 Literature review

The literature on the hydrodynamic characteristics of ocean waves by obstacles, including wave energy converters is reviewed. Indeed, there are numerous experimental and numerical studies in the literature, however the following review will be only concentrated on the theoretical aspects owing to the analytical focus of the thesis.

The real sea surface is complex, nonlinear and random. The application of linearized theory to wave diffraction/radiation problems is a very useful approach to gain an important insight into hydrodynamic effects (Porter, 1995*a*). Many problems can be approached by analytical method due to two pronounced advantages, which are (i) the reduction of the computational effort to a minimum, and (ii) the ease of revealing interesting wave phenomena. In analytical studies involving practical problems, linear water wave theory is usually employed to the problems of wave interactions with objects placed in ocean. For fully theoretical exploitation of hydrodynamics, these objects are usually idealized and modelled by structures associated with simple and normal geometries such as infinitely thin plates, rectangular blocks, or circular cylinders.

2.1.1 Diffraction problems

Water wave diffraction phenomena can be found visibly when water waves are incident upon relatively large floating or submerged objects. In investigating

the diffraction problems analytically, due to the mixed surface boundary condition on the mean free surface of the fluid, there are only a few such problems which can be solved analytically in a closed form under linear water-wave theory. Therefore, before proceeding with this review, it is necessary to distinguish between small and large objects of typical dimension a in comparison with the characteristic wavelength λ ($\lambda = 2\pi/k$, where k is the wavenumber) and the wave amplitude A . For small ka and large A/a , vortex shedding and flow separation are dominant. On the other hand, separation becomes insignificant while diffraction is crucial for small A/a and large ka ($\geq O(1)$) (Mei *et al.*, 2005). The later category will be addressed in the following.

2.1.1.1 Diffraction by thin barriers

Under linear water-wave theory, water wave diffraction problems by fixed vertical thin barriers have been investigated over the last six decades (as sketched in Figure 2.1). The explicit solution of the diffraction problem by a semi-infinite flat plate, extending from sea bottom of infinite depth to a point a below the free surface, was obtained by Dean (1945) through reduction method. He showed that the reflection coefficient $|R|$ and transmission coefficient $|T|$ could be expressed in terms of modified Bessel functions of argument ka , where $k = 2\pi/\lambda$ and λ is the wavelength. Two years later, Ursell (1947) employed Havelock's expansion of water-wave potential to solve the complementary problem, namely a fixed thin vertical barrier partially immersed to a depth a beneath the surface. He provided the similar exact expressions for reflection and transmission coefficients, and also treated wave generation by a small amplitude rolling of the barrier.

Various aspects of this topic were further investigated. Haskind (1959) derived expressions in terms of Bessel and related functions for the forces and moments on a barrier oscillating in an arbitrary manner under a train of incident waves with the same frequency. Lewin (1963) and Mei (1966) showed the diffraction of an incident wave by any number of in-line vertical barriers with arbitrary gaps between them in deep water. Evans (1970) solved the problems with generation and diffraction of waves by a submerged finite vertical plate making small rolling oscillations in a train of incident waves of the same frequency as the plate oscillations. He employed the complex-variable theory

Figure 2.1 Water wave diffraction by thin vertical barriers.

leading to the solution of a Riemann-Hilbert problem and obtained the reflection and transmission coefficients in terms of some definite integrals. Porter (1972) considered water-wave transmission through a submerged gap in a thin vertical wall from two approaches. One of the approaches was based on the use of Green's integral theorem, and the other was based on complex-variable theory. Both approaches lead to the solution of a Riemann-Hilbert problem. He also obtained the reflection and transmission coefficients in terms of some computable definite integrals.

Explicit solution can also be obtained for the case of two identical vertical barriers in infinite depth, which are either partially immersed to the same depth below the mean free surface or completely submerged from the same depth and extending infinitely downwards. Levine & Rodemich (1958) studied the case and showed that this problem has an exact but complicated solution, although they did not obtain the detailed solution. Jarvis (1971) solved the complementary problem of two identical semi-infinite barriers extending downwards from a point a beneath the surface, and he also obtained the curves of reflection and transmission coefficients as a function of frequency, barrier spacing, and submergence depth of barriers.

A substantial amount of work, which involves two or more thin vertical barriers in deep as well as finite-depth water, has been published in the literature. Some approximate analytical and numerical methods have been employed to calculate the numerical estimates for the reflection and transmission coefficients. As a special case of the theory, Levine & Rodemich (1958) and Evans & Morris (1972) considered two vertical parallel barriers immersed to a given depth

beneath the surface of water of infinite depth by complementary variational approximation. They obtained an approximate but more descriptive solution. They also showed that there existed certain wavelengths for each barrier configuration at which total reflection of the incident wave occurs. However, they gave bounds only for large barrier spacing. Consequently, Newman (1974) solved the complementary problem by a matched asymptotic method. He obtained an approximate result for the reflection coefficients of two barriers with close spacing.

McIver (1985) used matched eigenfunction expansions and orthogonal expansions in forms of trigonometric functions to solve the case of two closely spaced surface-piercing barriers with equal length, and obtained an infinite system of equations with an infinite number of unknowns. Dalrymple & Martin (1990) considered a long linear array of offshore breakwaters through variational and eigenfunction expansion methods. The reflection and transmission coefficients were presented. Porter & Evans (1995) used Galerkin approximation method by introducing a trial function to solve diffraction problems in several cases involving vertical barriers in finite depth and a vertical barrier in a rectangular tank. They showed how highly accurate complementary bounds could be computed with relative ease. Recently, Soumen *et al.* (2009) reinvestigated the two-barrier problem of Jarvis (1971). They solved a system of Abel integral equations by Havelock's expansion of velocity potential. They also obtained the reflection and transmission coefficients in terms of computable integrals. Based on eigenfunction expansion method, Chang *et al.* (2012) selected a set of appropriate basis functions, capturing the singular behaviors in velocity fields near salient corners and cusped edges, to solve a modified V-shaped breakwater with thin thickness. They presented the efficiency and accuracy compared with the results of the usual eigenfunction expansion method.

2.1.1.2 Diffraction by relatively large obstacles

There are numerous investigations on the problem of water-wave diffraction by relatively large obstacles (as sketched in Figure 2.2) based on potential theory. Among all of these, the method of eigenfunction expansion is most widely used in wave-structure interaction problems. The solutions are obtained by reducing the original problem to a linear algebraic system through using a matching

Figure 2.2 Water wave diffraction by large obstacles.

technique, which fulfilled by the eigenfunctions in each artificially separated region.

For the problem of diffraction of water wave by a singular cylinder in water of infinite depth, Havelock (1940) first developed an analytical solution. McCamy & Fuchs (1954) further extended the solution to finite water depth. Detailed reviews of the early work can be found in Mei *et al.* (2005).

The geometries of ocean-engineering interest (like breakwaters, platforms and subsurface storage tanks) exist in various forms. Problems of diffraction of water waves over submerged obstacles have been studied theoretically by many investigators (Newman, 1965; Miles, 1967; Mei & Black, 1969).

The problem of diffraction of gravity waves by a truncated circular dock was formulated by Miles & Gilbert (1968). They used a variational approximation to scale their trial function, in order to calculate the near field, the peripheral disturbance and the horizontal force acting on the dock. The same problem, solved by Miles & Gilbert (1968) first, was then corrected by Garrett (1971). He expanded an auxiliary function in terms of two sets of vertical eigenfunctions, and then obtained the solutions with phase independent of depth. Based on Galerkin's method, he also calculated the wave forces exerting on the circular dock.

Following the same mathematical descriptions as Miles & Gilbert (1968), Garrett (1970) solved the wave diffraction by a hollow cylinder partially immersed in water of finite depth. He used a variational approximation to calculate the resulting wave motion inside the cylinder, and obtained that the sharpness

of a given resonance depended largely on the value $k(d - h)$ (where k was the wavenumber, $d - h$ was the immersed depth). Zhu & Mitchell (2011) revisited the work done by Garrett (1970). They used a different but more direct method to numerically solve the problem and obtained the solution for the diffracted wave potential.

The above reviews focus only on the interaction between wave motion and a single obstacle, while there are many circumstances where it is important to understand the hydrodynamic interactions that occur as a result of the wave motion between neighbors in an array of offshore structures. Various aspects of wave interactions with arrays of structures have been studied by many authors in the literature. Matsui & Tamaki (1981) investigated the simplest case of the interaction between two truncated cylinders. Simon (1982) obtained a direct matrix solution for a uniformly spaced linear array of axisymmetric bodies. This simplification is only valid for large spacing. McIver & Evans (1984) extended this approach to study wave loading on arrays of fixed vertical circular cylinders and obtained improved results at smaller cylinder separations.

The diffraction of water waves by arrays of fixed vertical circular cylinders was solved exactly by Linton & Evans (1990). They improved the above direct matrix method by developing a simplified expression for the velocity potential in the local vicinity of a cylinder. Maniar & Newman (1997) examined the case of diffraction by a stationary array of cylinders. They showed that frequency dependent resonance effects could lead to large increment in the hydrodynamic force acting on any element of the array. More recently, based on linear NewWave theory, Walker & Taylor (2005) investigated the diffraction of regular waves by linear arrays of vertical bottom-mounted circular cylinders. NewWave is a focused wave group to model behaviours in the vicinity of extreme waves. As an example, in regions such as the North Sea and the Gulf of Mexico, the design of offshore structures is principally concerned with responses to environmental loads and peak free-surface elevations generated by extreme storms. Therefore, the convenient ‘design-wave’ known as NewWave is used to be an alternative to running many hours of random time domain simulation.

2.1.2 Radiation problems

Waves can be generated by the motion of rigid bodies (floating or submerged) or the oscillating pressures applying over a segment of the free surface (see

Figure 2.3 Water waves generated by motion of a rigid body.

Figures 2.3 and 2.4). The most common phenomenon in people’s daily life is the wave generation by a small stone thrown into a pond.

However, wave interaction with rigid bodies is complex. In order to understand the hydrodynamic behaviours of wave radiation, it is conventional to assume that the radiated waves are generated by the motion of rigid bodies in the absence of ocean waves and currents (i.e., calm water). Problems of water-wave radiation (i.e., wave generation) have been investigated theoretically by many researchers in various aspects. Among these, the sources of water waves may fall into two primary types, which are rigid-body motion and oscillating pressure distribution. The hydrodynamic behaviours of both cases are examined in terms of added mass and radiation damping coefficients.

2.1.2.1 Rigid-body motion

The study of the radiation problem on ocean structures can provide important information of the hydrodynamic properties of structures, including added mass and radiation damping. The problem of linearized motion of a freely floating rigid body is characterised by six components corresponding to six degrees of freedom or modes of motion. It was explained in detail by Wehausen (1971) and Newman (1977).

Many researchers have contributed to different aspects of the problem, which includes theoretical, experimental and numerical studies. In the aspect of theoretical studies, Black & Mei (1971) examined the radiation of surface waves due to small oscillation of horizontal rectangular and vertical circular cylinders by the use of Schwinger’s variational formulation. They presented radiated wave

amplitudes and calculated wave forces by means of Haskind's theorem. Yeung (1981) considered a truncated vertical circular cylinder oscillating in sway, heavy, and roll modes in finite-depth waters. He employed the eigenfunction expansion method to obtain added mass and radiation damping for heave, sway, and roll motion, and also coupled coefficients for sway and roll. Simon (1981a) considered the problem of water-wave radiation from a vertical circular duct of radius a with a submerged mouth at depth h in an ocean of infinite depth. He formulated an integral-equation and obtained variational bounds to the amplitude of, the axisymmetric component of wave diffraction. In addition to these, he also determined the radiation damping and added mass through indirect methods. Shortly afterwards, Miles (1982) solved the complementary problem. He successfully developed an integral-equation and variational formulation for the radiation of surface waves from a submerged cylindrical duct, and also provided variational representations of the radiation impedance (including added mass and radiation damping).

Mavrakos (1988) used eigenfunction expansion method to consider a thick-walled bottomless cylindrical body, free floating in water of finite depth. By implementing Galerkin's method the numerical results to the added mass and damping coefficients for surge, heave and pitch modes are obtained, respectively. Lee (1995) presented an analytical solution to the heave radiation problem of a rectangular structure. He calculated the generated waves, added mass, damping coefficients and the hydrodynamic effect of the submergence and width of the structure. Martin & Farina (1997) solved the radiation of water waves by a heaving submerged horizontal disc. They presented a method for axisymmetric motions of a horizontal disc in deep water and obtained the heaving added mass and damping. Zheng *et al.* (2005) conducted the research on the interaction of a submerged rectangular buoy with incident linear waves on finite water depth and attained the solution of radiated potential and diffracted potential. Nevertheless, Farina (2010) extended this work by considering the effects of taking the disc very close to the free surface and relating the hydrodynamic force to resonant frequencies.

2.1.2.2 Oscillating pressure distribution

As an oscillating pressure is applied uniformly over a segment of the free surface, the free surface will be disturbed and radiated waves will be generated (as shown

Figure 2.4 Water waves generated by oscillating pressure.

in Figure. 2.4).

Lamb (1905) outlined, and Stoker (1957) worked out in detail about the two-dimensional theory of the waves generated in deep water by an oscillating pressure applied uniformly over a segment of the free surface. Wehausen & Laitone (1960) presented the more general expressions. They considered the cases of two- and three-dimensional non-uniform oscillating pressure distributions on water of finite depth. Ogilvie (1969) examined a two-dimensional oscillating surface-pressure applied uniformly on two equally submerged vertical plates.

Evans (1982) considered the general case of wave-power absorption by two- and three-dimensional systems of oscillating surface-pressure distributions. He derived reciprocity relations for the applied-pressure and diffraction properties, and presented general expressions for the time-averaged power which developed by the pressure forces. Fernandes (1983) dealt with the more complicated problem of an oscillating body with chambers that enclose portions of the free surface. He obtained numerical results for an axisymmetric buoy with pneumatic wave-power absorption by the use of a singularity distribution method. Sarmiento & Falcão (1985) extended work by Stoker (1957) and considered the two-dimensional wave-energy problem.

2.1.3 Wave energy converters

Since the 1970s, the science and technology of power extraction from sea waters have been steadily advancing, and inventors have proposed many different

Figure 2.5 Point absorber wave energy converter.

devices for utilizing wave power for human purposes (Evans, 1981; McCormick, 1981; Falnes, 2002; Cruz, 2008; Heath, 2012). Among these devices, the most widely investigated fall into two major types: oscillating floating body and oscillating water column (OWC).

2.1.3.1 Oscillating floating body

Energy can be extracted from the oscillations of floating bodies (shown in Figure 2.5) such as buoys, cams, and floats. The first patent dates back to 1799 (Lindroth & Leijon, 2011), while wave energy research was intensified during the 1970s due to the oil crisis. In particular, the turning point for systematic study for extraction of wave energy was the publication of an article in *Nature* by Salter (1974). He concluded that the attractiveness of wave energy was immediate. Shortly afterwards, Mei (1976) proposed a design which is based on the idea of a tethered-float breakwater. He obtained the theoretical design criteria for maximum power extraction from a general floating cylinder with one or two degrees of freedom. Evans (1978) developed a theory to predict the absorption of the power for the device presented by Salter (1974). He presented the general expressions for the efficiency of wave absorption when the body oscillated in one or two modes. Bundal & Falnes (1977) considered a resonance-tuned absorber of ocean-wave power. The absorber utilized a machine to control its movement in an optimum way relative to the time evolution of the incident wave. As a result, the improved absorber became theoretically as efficient in non-harmonic waves as the resonance-tuned absorber is in harmonic waves. Newman (1979)

Figure 2.6 Schematic diagram of an OWC device.

used slender-body approximations to predict the maximum rate of energy absorption by an elongated floating vessel which performed vertical motions of varying amplitude and phase along its length. He obtained the estimates for the amplitude and phase of particular mode shapes as well as the corresponding power absorption.

2.1.3.2 Oscillating water column

It is well-known that the typical frequency of the motion of ocean waves is of the order of 10^{-1} Hz, while the typical frequency of electrical energy is of the order of 50 Hz. The conversion from low to high frequency provides a challenge to engineers. Oscillating water column (OWC) device, which is one of the wave energy converter devices, presents a solution to avoid this problem. In principle, a vertical circular cylindrical tube is open at the lower end with a Wells turbine at the top, as shown in Figure 2.6. As an train of incident wave arrives at the tube, a time-varying pressure field in the air chamber inside the device is induced and power is extracted via a turbine and generator. A Wells turbine, which rotates in the same direction independent of the direction of the air flow, is usually employed.

Early theoretical models relevant to OWC devices adapted existing ship hydrodynamic theory to simulate the power take-off model. For instance, Evans (1978) considered a device consisting of two closely spaced plates in two dimensions or a narrow tube of circular cross section in three dimensions. He

assumed that the spacing between the two plates or the radius was small such that the surface of air-water interface inside the chamber could be considered as a weightless piston. He employed the method of matched asymptotic expansions and obtained an expression for the efficiency of wave-energy absorption.

Later, Falcão & Sarmento (1980) sought to improve on this model of an OWC device by allowing for an oscillating air pressure inside the column and for the possibility of a non-plane interior water surface. Subsequently, Evans (1982) generalized this work. He addressed the model of OWC-type devices by considering the effect of a uniform pressure distribution over the air-water interface inside the chamber, and showed that the hydrodynamic characteristics, analogous to the added mass and radiation damping coefficients for rigid body motion, carried over to the radiation susceptance and conductance coefficients associated with pressure distributions.

Based on the theory of uniform pressure distributions, there are rich literatures on wave energy extraction by OWC devices. For two-dimensional OWCs, Sarmento & Falcão (1985) were the pioneers to consider the compressibility of air inside the chamber. They presented the variation of the efficiency with wave length, chamber size, water depth, air compressibility, and turbine coefficient. Evans & Porter (1995) considered a two-dimensional OWC device consisting of a vertical thin plate next to a plane wall in finite depth water. They used matched eigenfunction expansions to establish integral equations for the volume flux and then employed the Galerkin method of approximation to reduce this problem to a finite system of algebraic equations. An isolated axisymmetric OWC in the open sea had been considered by Evans & Porter (1997). They modelled the system by a thin-walled circular cylinder with its open bottom elevated above the seabed, thus allowing waves to enter the column. As an extension of the OWC device studied by Evans & Porter (1995), Sentürk & Özdamar (2012) used matched eigenfunction expansions and a Galerkin approximation to investigate an OWC wave energy device with a gap on the fully submerged front wall. They presented that the improvements on the efficiency curve of the OWC is possible to obtain by choosing appropriate geometrical parameters.

Apart from the local wave climate, position of wave-energy absorbers also depends on the different engineering costs, such as construction, operation, maintenance, storage, and grid connection (Martins-Rivas, 2008; Lovas, 2010). More

recently, Martins-Rivas & Mei (2009*b*) analyzed a theoretical model of an OWC standing at the tip of a thin breakwater. They found that air compressibility could be used to optimize the power absorption efficiency and the angle of incidence strongly affected the waves outside the cylinder but not the averaged response inside or the capture length of energy absorption. A vertical OWC on a straight and cliff-like coast was studied by Martins-Rivas & Mei (2009*a*). Their works showed that reflection from the shore roughly doubles the energy absorption, and the air/turbine system of power-takeoff produced high efficiency over a broad frequency band. Lovas *et al.* (2010) extended the analytical theories by Martins-Rivas & Mei (2009*b*) to examine a large circular oscillating water column (OWC) installed at the tip of a coastal corner.

All the above studies, either shoreline or offshore OWC devices, were assumed the whole structures to be stationary in waves.

Falcão *et al.* (2012) considered an OWC spar buoy wave energy converter, which consists of a submerged vertical tail-tube-fixed to an axisymmetric floater that oscillates essentially in heave. They presented the frequency-dependent hydrodynamic coefficient and the optimization of the tube geometry and the turbine characteristic. A free floating vertical axisymmetric OWC device with thick wall was studied by Mavrakos & Konispoliatis (2012) through matched eigenfunction expansions. They presented a general formulation of the hydrodynamic problem of a floating or restrained OWC device. The horizontal and vertical exciting waves forces, the rolling moment, the hydrodynamics parameters, the volume flows, and the drift forces were presented also. However, it was found that the optimization strategy to the turbine characteristic was wrong, which will be corrected in Chapter 5.

The analytical studies of two types of wave energy converters, i.e., oscillating floating body and oscillating water column, have been reviewed at the above. In this thesis, a nearshore (before wave breaking) OWC-type wave energy converter with or without a V-shaped channel, supported by a coaxial tube-sector-shaped structure, is proposed. Comparing to the existing fully-truncated OWC device by Evans & Porter (1997), the coaxial tube-sector-shaped supporting structure is beneficial for enhancing the spatial standing waves. Furthermore, the attached V-shaped channel, acting as a “wave guide”, is useful for wave gathering and amplitudes of standing waves in the field bounded by the V-shaped channel.

For harnessing wave energy in deeper water, a floating OWC device with a circular pontoon, which is connected to sea bed by a mooring system attached by a linear power takeoff (PTO) system, is proposed as well. Compared to the free floating OWC device by Mavrakos & Konispoliatis (2012), the integrated mooring system is beneficial to control the phase difference between the wave motion inside the air chamber and the motion of the pontoon and improve the efficiency of the floating OWC system.

2.2 Eigenfunction expansions incorporating local singular behaviours

In the literature, the eigenfunction expansion method is widely used to solve the wave-structure interaction problems. The solutions (Porter, 1995*b*; Manam *et al.*, 2006) are obtained by matching technique across the interface at the discontinuities, provided by the eigenfunctions in each region which form a complete set of orthogonal basis functions. Hence, using the orthogonal relation, the boundary-value problem can be reduced to a problem requiring the solution of a well-behaved infinite system of linear equations. However, its efficiency in numerical computations is low, specially, when the structure has sharp corners where the flow field behaves singularly. Mathematically, by the separation of variables, the solutions of the problems are constructed by one or more infinite summations with several unknown expansion coefficients. By the directly matching, a large number of terms is needed in finite summations to converge the solution to a value even with two-figures precision in case of singularity. Therefore, a fast convergence technique in numerical computations is developed, by introducing auxiliary functions involving local singular behaviours to represent local radial velocities (see Porter & Evans (1995); Fernyhough & Evans (1995)), to solve this type of problems. In the rest of this section, we will take a simple example to demonstrate the procedure of solving the problem of wave-body interaction by the fast convergence technique, which facilitates the studies in the coming chapters.

Without loss of generality, let us assume a truncated circular cylinder of radius a freely floating on water surface of finite water depth, as shown in Figure 2.7. Based on the geometrical properties of structure, a Cartesian coordinate

Figure 2.7 Schematic diagram of a floating circular cylinder.

system $Oxyz$ as well as a cylindrical coordinate system $Or\theta z$ are chosen with the Oxy and $Or\theta$ planes lying on the undisturbed free surface of the fluid with the z -axis pointing vertically upwards and the water depth occupying the space $-h < z < 0$. The origin coincides with the center of the still water surface occupying by the cylinder at calm. The immersed part of the structure is $-d < z < 0$. Throughout the thesis we will use the theory of classical linear water waves; for a detailed account of the derivation of the following theory see, for example, Mei *et al.* (2005) and Wehausen & Laitone (1960). Then there exists a velocity potential $\Phi(r, \theta, z; t)$, which satisfies

$$\nabla^2 \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0, \quad \text{in fluid}, \quad (2.1)$$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial z}, \quad \text{on } z = 0, \quad (2.2)$$

where $\eta(r, \theta; t)$ denotes the elevation of the free surface measured upwards from $z = 0$,

$$\frac{\partial \Phi}{\partial t} + g\eta = \frac{-P_a}{\rho}, \quad \text{on } z = 0, \quad (2.3)$$

where ρ is the (constant) density of the fluid, P_a is the air pressure just above the free surface (without loss of generality, let us set the atmospheric pressure equal to zero) and g is the acceleration of gravity,

$$\frac{\partial \Phi}{\partial z} = 0, \quad \text{on } z = -h, \quad (2.4)$$

$$\frac{\partial \Phi}{\partial n} = n_j, \quad j = 1, 3, 5, \quad \text{on wetted solid boundaries}, \quad (2.5)$$

where $\mathbf{n}$ denotes the inward normal vector to the wetted solid body surface and n_j represents the unit inward normal in j mode ($j = 1, 3, 5$ denote the three modes of the body motion, they are, surge, heave, pitch, respectively), and

$$\text{outgoing waves at infinity,} \quad (2.6)$$

where Eqns. (2.2) and (2.3) are the free surface kinematic and dynamic boundary conditions, respectively; Eqn. (2.4) is the bottom boundary condition; Eqn. (2.5) is the no-flux condition on wetted body surface; Eqn. (2.6) implies that the scattered and radiated waves satisfy the Sommerfeld radiation condition at infinity.

For a time-harmonic motion with angular frequency ω , complex-valued time-independent quantities, ϕ , ζ , p_a can be introduced in the form of

$$\Phi(r, \theta, z; t) = \text{Re} \{ \phi(r, \theta, z) e^{-i\omega t} \}, \quad (2.7)$$

$$\eta(r, \theta; t) = \text{Re} \{ \zeta(r, \theta) e^{-i\omega t} \}, \quad (2.8)$$

and

$$P_a(t) = \text{Re} \{ p_a e^{-i\omega t} \}, \quad (2.9)$$

respectively. Where i ($i = \sqrt{-1}$) is the imaginary unit.

Substitution of Eqns. (2.7)-(2.9) into Eqns. (2.1)-(2.5) and then combining the resulting Eqns. (2.2) and (2.3), we have the following boundary-value problem,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \quad \text{in fluid,} \quad (2.10)$$

$$\frac{\partial \phi}{\partial z} - \frac{\omega^2}{g} \phi = \frac{i\omega p_a}{\rho}, \quad \text{on } z = 0, \quad (2.11)$$

$$\frac{\partial \phi}{\partial z} = 0, \quad \text{on } z = -h, \quad (2.12)$$

$$\frac{\partial \phi}{\partial n} = n_j, \quad j = 1, 3, 5, \quad \text{on wetted solid boundaries.} \quad (2.13)$$

In addition to the above boundary conditions, the scattered and radiated complex-valued velocity potentials satisfy the radiation condition at infinity as well.

It is conventional to divide the total fluid field into several regions. In this simple case, there are two regions, they are

- region (1) occupying the space: $r > a$, $0 \leq \theta < 2\pi$, $-h < z < 0$;
- region (2) occupying the space: $r < a$, $0 \leq \theta < 2\pi$, $-h < z < -d$.

And there is a common interface at $r = a$ for $-h < z < -d$. Hence, let us introduce $\phi^{(1)}$ to represent the complex potential in region (1) and $\phi^{(2)}$ to represent the complex potential in region (2).

Before we go on to construct the formal expression of the complex potential in each region by separation of variables, let us introduce a set of vertical orthogonal basis functions $Z_l(z)$ ($l = 0, 1, 2, \dots$) (Mei *et al.*, 2005):

$$N_l = \frac{1}{2} \left(1 + \frac{\sin 2k_l h}{2k_l h} \right), \quad Z_l(z) = N_l^{-1/2} \cos k_l(z + h), \quad (2.14)$$

with

$$N_0 = \frac{1}{2} \left(1 + \frac{\sinh 2kh}{2kh} \right), \quad Z_0(z) = N_0^{-1/2} \cosh k(z + h), \quad (2.15)$$

where k represents the wavenumber and is the positive real root of the following dispersion relation for progressive waves,

$$k \tanh kh - \omega^2/g = 0, \quad (2.16)$$

and k_l ($l = 1, 2, 3, \dots$) are the positive real roots of the following dispersion relation for evanescent waves,

$$k_l \tan k_l h + \omega^2/g = 0. \quad (2.17)$$

Note that it is convenient to introduce k_0 ($k_0 = -ik$). After inserting the expression of k_0 into Eqns. (2.14) and (2.17), the resulting equations can reduce to Eqns. (2.15) and (2.16), respectively.

By Mei *et al.* (2005), the functions $Z_l(z)$ ($l = 0, 1, 2, \dots$) form a complete set of orthogonal functions over the interval $[-h, 0]$, namely,

$$\frac{1}{h} \int_{-h}^0 Z_l(z) Z_p(z) dz = \delta_{lp}, \quad \text{for } l, p = 0, 1, 2, \dots, \quad (2.18)$$

where

$$\delta_{lp} = \begin{cases} 1, & l = p, \\ 0, & l \neq p. \end{cases} \quad (2.19)$$

Note that the set of base functions $\{Z_l(z)\}$ is suitable for the case in which the fluid field extends from the constant water depth to the free water surface.

Starting with the linearization, the total complex potential ϕ can be decomposed into a sum,

$$\phi = \phi_D + \phi_R, \quad (2.20)$$

where ϕ_R denotes the radiation potential in the absence of incident waves. ϕ_D denotes the diffraction potential for the body keeping at calm, which can be further written as a sum of the incident wave potential ϕ_I and the scattering potential ϕ_S ,

$$\phi_D = \phi_I + \phi_S. \quad (2.21)$$

For simplification and summarization, let us introduce $\mathcal{W}$ to uniformly denote ϕ_D or ϕ_R (i.e., the diffraction potential or radiation potential). Therefore, by separation of variables, the formal expression of $\mathcal{W}^{(1)}$ in region (1), satisfying Eqns. (2.10)-(2.12), can be written as

$$\mathcal{W}^{(1)} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{A}_{ml} \mathcal{R}_{ml}^{(1)}(r) \mathcal{Q}_m^{(1)}(\theta) Z_l(z), \quad (2.22)$$

where $\mathcal{A}_{ml}$ are unknown coefficients; $\mathcal{R}_{ml}^{(1)}(r)$ are functions of r in region (1); $\mathcal{Q}_m^{(1)}(\theta)$ are expansion basis functions in θ -direction for region (1).

The formal expression of $\mathcal{W}^{(2)}$ in region (2), satisfying Eqns. (2.10), (2.12) and (2.13), can be written as

$$\mathcal{W}^{(2)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{B}_{mn} \mathcal{R}_{mn}^{(2)}(r) \mathcal{Q}_m^{(2)}(\theta) f_n(z), \quad (2.23)$$

where $\mathcal{B}_{mn}$ are unknown coefficients; $\mathcal{R}_{mn}^{(2)}(r)$ are functions of r in region (2); $\mathcal{Q}_m^{(2)}(\theta)$ are expansion basis functions in θ -direction for region (2); $f_n(z)$ are expansion basis functions in z -direction for region (2), which are orthogonal to each other over the interval $[-h, -d]$.

Now let us introduce an auxiliary function $U(\theta, z)$ to represent the radial velocity at $r = a$, which can be expanded as

$$U(\theta, z) = \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{sp} \Theta_s(\theta) u_p(z), \quad \text{for } 0 \leq \theta < 2\pi, -h < z < -d, \quad (2.24)$$

where Λ_{sp} are unknown coefficients needed to be determined; $\Theta_s(\theta)$ denote a set of expansion basis functions in θ -direction, which are orthogonal to each other over the interval $[0, 2\pi]$; $u_p(z)$ denote another set of expansion basis functions in

z -direction, which are orthogonal to each other over the interval $[-h, -d]$ and particularly include the singular behavior of the fluid flow at $z = -d$.

Next, based on the continuity of the radial velocity across the common interface, let us determine the expression of unknown coefficients $\mathcal{A}_{ml}$ and $\mathcal{B}_{mn}$ in terms of Λ_{sp} .

At $r = a$, we have the following relation for $\mathcal{W}^{(1)}$,

$$\frac{\partial \mathcal{W}^{(1)}}{\partial r} \Big|_{r=a} = \begin{cases} 0, & 0 \leq \theta < 2\pi, -d < z < 0, \\ U(\theta, z), & 0 \leq \theta < 2\pi, -h < z < -d. \end{cases} \quad (2.25)$$

Multiplying the above equation by $\mathcal{Q}_{m'}^{(1)}(\theta) Z_{l'}(z)/h$ ($m', l' = 0, 1, 2, \dots$) and then integrating over the corresponding interval, gives

$$\mathcal{A}_{ml} = \frac{1}{\mathcal{R}_{mn}^{(1)'}(a) \int_0^{2\pi} \left(\mathcal{Q}_m^{(1)}(\theta) \right)^2 d\theta} \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{sp} X_{sm}^{(1)} V_{pl}^{(1)} \quad (2.26)$$

where

$$X_{sm}^{(1)} = \int_0^{2\pi} \Theta_s(\theta) \mathcal{Q}_m^{(1)}(\theta) d\theta, \quad (2.27)$$

and

$$V_{pl}^{(1)} = \frac{1}{h} \int_{-h}^{-d} u_p(z) Z_l(z) dz. \quad (2.28)$$

Here and hereafter, a prime represents taking differentiation of a function with respect to its argument

Similarly, at $r = a$, we have the following relation for $\mathcal{W}^{(2)}$,

$$\frac{\partial \mathcal{W}^{(2)}}{\partial r} \Big|_{r=a} = U(\theta, z), \quad 0 \leq \theta < 2\pi, -h < z < -d. \quad (2.29)$$

Multiplying the above equation by $\mathcal{Q}_{m'}^{(2)}(\theta) f_{n'}(z)/(h-d)$ and then integrating over the corresponding interval, we have

$$\mathcal{B}_{mn} = \frac{1}{\mathcal{R}_{mn}^{(2)'}(a) \int_0^{2\pi} \left(\mathcal{Q}_m^{(2)}(\theta) \right)^2 d\theta} \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{sp} X_{sm}^{(2)} V_{pn}^{(2)}, \quad (2.30)$$

where

$$X_{sm}^{(2)} = \int_0^{2\pi} \Theta_s(\theta) \mathcal{Q}_m^{(2)}(\theta) d\theta, \quad (2.31)$$

and

$$V_{pn}^{(2)} = \frac{1}{h-d} \int_{-h}^{-d} u_p(z) f_n(z) dz. \quad (2.32)$$

Observing Eqns. (2.26) and (2.30), once the unknown coefficients Λ_{sp} is obtained, the values of $\mathcal{A}_{ml}$ and $\mathcal{B}_{mn}$ can be easily obtained as well.

As the last step, let us apply the continuity of pressure across the common interface at $r = a$ to calculate the unknown coefficients Λ_{sp} .

Therefore, at $r = a$, we have the following relation due to the continuity of pressure,

$$\mathcal{W}^{(1)}|_{r=a} = \mathcal{W}^{(2)}|_{r=a}, \quad \text{for } 0 \leq \theta < 2\pi, -h < z < -d. \quad (2.33)$$

Multiplying the above equation by $\Theta_{s'}(\theta) u_{p'}(z)$ ($s', p' = 0, 1, 2, \dots$) and integrating over the corresponding interval, after using (2.22), (2.23), (2.24), (2.26)-(2.28), (2.30)-(2.32), we have

$$\begin{aligned} & \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{sp} h \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\mathcal{R}_{mn}^{(1)'}(a) \mathcal{C}_m^{(1)}} \mathcal{R}_{ml}^{(1)}(a) X_{sm}^{(1)} X_{s'm}^{(1)} V_{pl}^{(1)} V_{p'l}^{(1)} \\ &= \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{sp} (h-d) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\mathcal{R}_{mn}^{(2)'}(a) \mathcal{C}_m^{(2)}} \mathcal{R}_{mn}^{(2)}(a) X_{sm}^{(2)} X_{s'm}^{(2)} V_{pn}^{(2)} V_{p'n}^{(2)}, \end{aligned} \quad (2.34)$$

where

$$\mathcal{C}_m^{(1)} = \int_0^{2\pi} (\mathcal{Q}_m^{(1)}(\theta))^2 d\theta, \quad \mathcal{C}_m^{(2)} = \int_0^{2\pi} (\mathcal{Q}_m^{(2)}(\theta))^2 d\theta. \quad (2.35)$$

After solving the above infinite linear algebraic system, the unknown coefficients Λ_{sp} can be obtained numerically. As a result, the other quantities, such as $\mathcal{A}_{ml}$, $\mathcal{B}_{mn}$ can be calculated as well.

The generalized procedure of solving the wave-structure interaction problem by introducing an auxiliary function expanded in terms of basis functions incorporating the singular behavior of the flow at corners has been presented in the above. In the coming chapters, we will apply this procedure to solve specific problems under corresponding modifications.

Chapter 3

A bottom-mounted OWC device at nearshore

This chapter aims to study the effects of a semi-circular supporting structure on the absorption efficiency of a suspended, circular OWC system. A simple circular geometry allows the eigen-function expansion method to study wave interaction with an OWC converter in finite depth of water. By the assumption of linearization, the hydrodynamics can be coupled with the power take-off model to predict the extracted power and efficiency.

3.1 Description

We shall consider an OWC device consisting of a circular hollow cylinder of radius a and supported by a semi-circular structure of the radius same as that of the OWC device. At the supporting structure, there is an opening allowing the wave energy to come in and go out. A singular Wells turbine is installed at the top of the cylinder. Under the excitation of incoming waves, the water free surface inside the hollow cylinder force the air inside the chamber to go through the Wells turbine. The unidirectional rotation of the turbine is used to convert the wave energy to a useful energy. This is sketched in Figure 3.1. We assume the sea depth h to be constant, and the walls of the coast and structure are all vertical. The draft of the circular OWC is d , and the semi-circular supporting structure can partly reflect the incoming waves, enhancing the partial standing waves underneath the bottom opening of the OWC chamber. It is assumed that both the supporting structure and the OWC chamber wall are thin.

Figure 3.1 Schematic diagram of an OWC supported by a semi-circular structure. The opening in the support is $h - d$ in height and occupies the arc section defined by $\theta = 0$ and $\theta = \psi$.

To mathematically describe the problem, we first define a Cartesian coordinate system $Oxyz$ with the still water surface being its Oxy plane. The axis-symmetric line of the OWC chamber goes through the origin O , the z -axis points vertically upward, and the x -axis touches one edge of the opening. The direction of the incident waves is specified by the angle $\theta = \alpha$ with respect to the x -axis. We only consider small amplitude waves of a single frequency ω . To solve the problem using eigen-expansion method, a cylindrical coordinate system $Or\theta z$ is also established with its origin and z -axis coinciding with those of the Cartesian coordinate system. In the cylindrical coordinate system $Or\theta z$, the opening of the supporting pile occupies the space defined by $r = a$, $0 \leq \theta \leq \psi$, $-h \leq z \leq -d$.

3.2 Power-takeoff model

Because of the high sound speed in air and the low frequency of ocean waves, the air pressure is assumed to be spatially uniform throughout the OWC chamber (Martins-Rivas & Mei, 2009b). The mass flux through a linear power takeoff device (e.g., a Wells turbine) is proportional to the difference of the air pressure $P_a(t)$ inside the chamber and the zero atmospheric pressure outside. Following Sarmiento & Falcão (1985) and Falcão & Rodrigues (2002), after applying linear water wave theory (which implies that the relative change in the density is small and the relative change of the volume of the chamber air is also small compared to the undisturbed volume), the mass flux $\mathcal{M}$ of the air through a linear turbine

is related to the turbine characteristics by

$$\frac{d\mathcal{M}}{dt} = -\frac{d(\rho_a V)}{dt} = -\rho_a^0 \frac{dV}{dt} - V_0 \frac{d\rho_a}{dt} = \frac{\hat{K}D}{\hat{N}} P_a, \quad (3.1)$$

where $\rho_a(t)$ is the instantaneous air density, $V(t)$ the instantaneous air-chamber volume, ρ_a^0 the air density at rest, V_0 the air-chamber volume at rest, $\hat{K}$ a dimensionless damping coefficient depending only on the geometry of the linear turbine (but not the turbine size, rotational speed and fluid density), D the outer diameter of the turbine rotor, and $\hat{N}$ the rotational speed of the turbine blades (radians per unit time). Physically, dV/dt is the rate of change of the total volume of the air inside the OWC chamber, calculated by

$$Q = -\frac{dV}{dt} = \int_0^a \int_0^{2\pi} w(r, \theta, z=0, t) r dr d\theta \quad (3.2)$$

where w is the vertical velocity of the water surface inside the OWC chamber. Assuming the air to be an ideal gas and the air compression inside the chamber to be an isentropic process (Falcão & Rodrigues, 2002) and neglecting mechanical losses induced by the turbine (i.e., a perfect turbine), we have $c_a^2 (d\rho_a/dt) = dP_a/dt$, where c_a is the sound speed in air at a constant temperature. Combining Eqn. (3.1) and Eqn. (3.2) and using $Q = \text{Re} \{q e^{-i\omega t}\}$ for monochromatic waves gives

$$q = \left(\frac{\hat{K}D}{\hat{N}\rho_a^0} - \frac{i\omega V_0}{c_a^2 \rho_a^0} \right) p_a. \quad (3.3)$$

For small waves the linearized velocity potential in water can be treated as the sum of radiation and diffraction problems. Hence, the volume flux amplitude q can be decomposed into the contribution from the diffracted waves (q_D) and radiated waves (q_R), i.e.,

$$q = q_D + q_R. \quad (3.4)$$

Let us define a diffraction flux factor Γ (Martins-Rivas & Mei, 2009b) for unit wave amplitude by

$$q_D \equiv \Gamma A, \quad (3.5)$$

where A is the incident wave amplitude. Following Evans (1982), the radiation flux amplitude q_R can be written as

$$q_R \equiv -(\mathcal{B} - i\mathcal{C}) p_a, \quad (3.6)$$

where the real coefficients $\mathcal{B}$ and $\mathcal{C}$ represent the radiation conductance and radiation susceptance (Falnes, 2002), respectively. Note that, physically, $\mathcal{C}$ is in phase with the flux acceleration and amounts to the added hydrodynamic inertia. On the other hand, $\mathcal{B}$ is in phase with the flux velocity and amounts to radiation damping.

The extracted wave power is defined by the time-averaged rate of work done by the chamber pressure pushing air through the turbine, i.e.,

$$P_{out} = \frac{\overline{d(\rho_a V) P_a}}{dt} = \frac{\hat{K} D}{2\hat{N}\rho_a^0} |p_a|^2 \quad (3.7)$$

We use the dimensionless capture length kL to evaluate the efficiency of power extraction. The capture length L is the ratio of the extracted power flux P_{out} to the wave power P_w ($P_w = \rho g A^2 C_g / 2$) per unit crest length of the incident waves (Evans, 1982; Martins-Rivas & Mei, 2009b). Combining Eqns. (3.3) through (3.6), we obtain the following dimensionless capture length to measure the extraction efficiency (e.g., Martins-Rivas & Mei (2009b) and Lovas *et al.* (2010))

$$kL = \frac{k P_{out}}{P_w} = \frac{kh\sqrt{g\bar{h}}}{C_g} \frac{\chi |\tilde{\Gamma}|^2}{\left(\chi + \tilde{\mathcal{B}}\right)^2 + \left(\tilde{\mathcal{C}} - \kappa\right)^2}, \quad (3.8)$$

where k is the wavenumber and C_g is the group velocity of waves, which can be calculated by

$$C_g = \frac{\omega}{2k} \left(1 + \frac{2kh}{\sinh 2kh} \right), \quad (3.9)$$

and other dimensionless parameters are defined by

$$\tilde{\mathcal{B}} = \mathcal{B} \frac{\rho \sqrt{g/h}}{h}, \quad \tilde{\mathcal{C}} = \mathcal{C} \frac{\rho \sqrt{g/h}}{h}, \quad \tilde{\Gamma} = \Gamma \frac{\sqrt{g/h}}{gh}. \quad (3.10)$$

$$\chi = \frac{\rho \hat{K} D \sqrt{g/h}}{\rho_a^0 \hat{N} h}, \quad \kappa = - \left(\frac{\omega V_0 \rho \sqrt{g/h}}{c_a^2 \rho_a^0 h} \right). \quad (3.11)$$

The first three parameters need to be obtained by solving the wave diffraction and radiation problems; the parameter χ characterizes the turbine system and κ the air chamber.

3.3 Wave diffraction problem

For the waves diffracted by the OWC structure, the pressure on the water surface inside the chamber is the constant atmospheric pressure, which is taken to

be zero, thus the diffraction potential ϕ_D satisfies the equation and boundary conditions given by Eqns. (2.10) through (2.13) with $p_a = 0$ and $n_j = 0$ (since the OWC device is fixed in waves).

3.3.1 Formal expressions for the velocity potentials inside and outside the OWC

Referring to Figure 3.1, we denote the velocity potential inside the chamber ($r < a$) by ϕ_D^c and the velocity potential outside the chamber ($r > a$) by ϕ_D^o , which can be further expressed as the sum of two parts,

$$\phi_D^o = \phi_D^D + \phi_D^{cor}, \quad (3.12)$$

with ϕ_D^D being the solution corresponding to wave diffraction by a solid circular cylinder extending the entire depth, and ϕ_D^{cor} a correction to ϕ_D^D due to the presence of the opening.

For a train of incident waves of a single frequency ω coming at an angle α to the x -axis shown in Figure 3.1, the velocity potential of the incident wave can be written as, following by Mei *et al.* (2005),

$$\begin{aligned} \phi_I &= \frac{-igA}{\omega} \frac{Z_0(z)}{Z_0(0)} e^{ikr \cos(\theta - \alpha)} \\ &= \frac{-igA}{\omega} \frac{Z_0(z)}{Z_0(0)} \sum_{m=0}^{\infty} i^m \varepsilon_m J_m(kr) \cos m(\theta - \alpha) \\ &= \frac{-igA}{\omega} Z_0(z) \sum_{m=0}^{\infty} J_m(kr) [\mathcal{E}_m \cos m\theta + \mathcal{F}_m \sin m\theta], \end{aligned} \quad (3.13)$$

where the coefficients $\mathcal{E}_m$ and $\mathcal{F}_m$ are given by

$$\mathcal{E}_m = \frac{i^m \varepsilon_m}{Z_0(0)} \cos m\alpha, \quad \mathcal{F}_m = \frac{i^m \varepsilon_m}{Z_0(0)} \sin m\alpha \quad (3.14)$$

with $\varepsilon_0 = 1$ and $\varepsilon_m = 2$ for $m \geq 1$, and J_m denotes the Bessel function of the first kind of order m .

The velocity potential ϕ_D^D can be expressed in the following form (see Mei *et al.* (2005), page 369),

$$\phi_D^D = \frac{-igA}{\omega} Z_0(z) \sum_{m=0}^{\infty} \left(J_m(kr) - \frac{J'_m(ka)}{H'_m(ka)} H_m(kr) \right) (\mathcal{E}_m \cos m\theta + \mathcal{F}_m \sin m\theta), \quad (3.15)$$

where, unless otherwise specified, H_m denotes the Hankel function of the first kind of order m . Furthermore, here and hereafter, a prime represents taking differentiation of a function with respect to its argument.

By separation of variables, the velocity potential ϕ_D^{cor} , which satisfies Laplace's equation, the free surface condition and the no-flux condition at $z = -h$ (i.e., Eqns. (2.10), (2.11) and (2.12)), can be formally written as

$$\phi_D^{cor} = \frac{-igA}{\omega} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} (A_{ml} \cos m\theta + B_{ml} \sin m\theta) \frac{K_m(k_l r)}{k_l a K'_m(k_l a)} Z_l(z), \quad (3.16)$$

where A_{ml} and B_{ml} are unknown coefficients to be determined, and

$$B_{0l} = 0, \quad l = 0, 1, 2, \dots$$

Note that K_m denotes the modified Bessel function of the second of order m . The case of $l = 0$ corresponds to an outgoing progressive wave, and

$$K_m(k_0 r) = K_m(-ikr) = \frac{\pi}{2} i^{m+1} H_m(kr). \quad (3.17)$$

Similarly, inside the chamber ($r \leq a$), the velocity potential ϕ_D^c can be formally written as

$$\phi_D^c = \frac{-igA}{\omega} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} (C_{ml} \cos m\theta + D_{ml} \sin m\theta) \frac{I_m(k_l r)}{k_l a I'_m(k_l a)} Z_l(z), \quad (3.18)$$

where C_{ml} and D_{ml} are unknown coefficients, and

$$D_{0l} = 0, \quad l = 0, 1, 2, \dots$$

Note that I_m denotes the modified Bessel function of the first kind of order m . Again, the case of $l = 0$ corresponds to progressive waves, and

$$I_m(k_0 r) = I_m(-ikr) = (-i)^m J_m(kr). \quad (3.19)$$

3.3.2 Determination of the unknowns: A_{ml} , B_{ml} , C_{ml} , and D_{ml}

At $r = a$, the radial velocity is continuous across the boundary of the opening, and must vanish on the solid walls, i.e.,

$$\frac{\partial \phi_D^o}{\partial r} \Big|_{r=a} = \begin{cases} 0, & 0 \leq \theta < 2\pi, -d < z < 0 \\ 0, & \psi < \theta < 2\pi, -h < z < -d \\ A \cdot U_D(\theta, z), & 0 < \theta < \psi, -h < z < -d, \end{cases} \quad (3.20)$$

and

$$\frac{\partial \phi_D^c}{\partial r}|_{r=a} = \begin{cases} 0, & 0 \leq \theta < 2\pi, -d < z < 0 \\ 0, & \psi < \theta < 2\pi, -h < z < -d \\ A \cdot U_D(\theta, z), & 0 < \theta < \psi, -h < z < -d, \end{cases} \quad (3.21)$$

where the unknown function $U_D(\theta, z)$ is used to denote the unknown radial velocity on the boundary of the opening.

Applying the orthogonality of cosines, sines and the vertical eigenfunctions to Eqn. (3.20) gives

$$\frac{1}{h} \int_{-h}^0 \int_0^{2\pi} \frac{\partial \phi_D^o}{\partial r}|_{r=a} \cos m' \theta Z_{l'}(z) d\theta dz = \frac{A}{h} \int_{-h}^{-d} \int_0^\psi U_D(\theta, z) \cos m' \theta Z_{l'}(z) d\theta dz, \quad (3.22)$$

and

$$\frac{1}{h} \int_{-h}^0 \int_0^{2\pi} \frac{\partial \phi_D^o}{\partial r}|_{r=a} \sin m' \theta Z_{l'}(z) d\theta dz = \frac{A}{h} \int_{-h}^{-d} \int_0^\psi U_D(\theta, z) \sin m' \theta Z_{l'}(z) d\theta dz, \quad (3.23)$$

where $m', l' = 0, 1, 2, \dots$

After making use of Eqns. (3.12), (3.15) and (3.16) in Eqns. (3.22) and (3.23) and re-arranging, we obtain the following expressions for A_{ml} and B_{ml} in term of the unknown velocity $U_D(\theta, z)$,

$$A_{ml} = \frac{ia\omega\varepsilon_m}{2\pi gh} \int_{-h}^{-d} \int_0^\psi U_D(\theta, z) \cos m\theta Z_l(z) d\theta dz, \quad (3.24)$$

and

$$B_{ml} = \frac{ia\omega\varepsilon_m}{2\pi gh(1 - \delta_{m0})} \int_{-h}^{-d} \int_0^\psi U_D(\theta, z) \sin m\theta Z_l(z) d\theta dz, \quad (3.25)$$

where δ denotes the Kronecker delta, namely

$$\delta_{ml} = \begin{cases} 0 & m \neq l, \\ 1 & m = l. \end{cases}$$

Similarly, applying the orthogonality of cosines, sines and the vertical eigenfunctions to Eqn. (3.21) gives the following two equations

$$\frac{1}{h} \int_{-h}^0 \int_0^{2\pi} \frac{\partial \phi_D^c}{\partial r}|_{r=a} \cos m' \theta Z_{l'}(z) d\theta dz = \frac{A}{h} \int_{-h}^{-d} \int_0^\psi U_D(\theta, z) \cos m' \theta Z_{l'}(z) d\theta dz, \quad (3.26)$$

and

$$\frac{1}{h} \int_{-h}^0 \int_0^{2\pi} \frac{\partial \phi_D^c}{\partial r}|_{r=a} \sin m' \theta Z_{l'}(z) d\theta dz = \frac{A}{h} \int_{-h}^{-d} \int_0^\psi U_D(\theta, z) \sin m' \theta Z_{l'}(z) d\theta dz. \quad (3.27)$$

After using Eqn. (3.18) in Eqns. (3.26) and (3.27) and re-arranging, we obtain the following expressions for C_{ml} , and D_{ml} in term of the unknown velocity $U_S(\theta, z)$,

$$C_{ml} = A_{ml} = \frac{ia\omega\varepsilon_m}{2\pi gh} \int_{-h}^{-d} \int_0^\psi U_D(\theta, z) \cos m\theta Z_l(z) d\theta dz, \quad (3.28)$$

and

$$D_{ml} = B_{ml} = \frac{ia\omega\varepsilon_m}{2\pi gh(1-\delta_{m0})} \int_{-h}^{-d} \int_0^\psi U_D(\theta, z) \sin m\theta Z_l(z) d\theta dz. \quad (3.29)$$

3.3.3 Determination of the auxiliary function $U_D(\theta, z)$

To determine the unknown function $U_D(\theta, z)$, we note that the pressure must be also continuous across the boundary at the opening, i.e.,

$$\phi_D^o = \phi_D^c, \quad \text{at } r = a, \text{ for } -h < z < -d, \ 0 < \theta < \psi. \quad (3.30)$$

After substituting Eqns. (3.12) (using Eqns. (3.15) and (3.16)) and (3.18) into Eqn. (3.30) and using Eqns. (3.28) and (3.29), we have

$$\begin{aligned} Z_0(z) \sum_{m=0}^{\infty} \mathcal{R}_m (\mathcal{E}_m \cos m\theta + \mathcal{F}_m \sin m\theta) \\ = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \left[\frac{ia\omega\varepsilon_m}{2\pi gh} \int_{-h}^{-d} \int_0^\psi U_S(\theta', z') Z_l(z') \cos m\theta' d\theta' dz' \cdot \cos m\theta \right. \\ \left. + \frac{ia\omega\varepsilon_m}{2\pi gh(1-\delta_{m0})} \int_{-h}^{-d} \int_0^\psi U_S(\theta', z') Z_l(z') \sin m\theta' d\theta' dz' \cdot \sin m\theta \right] \\ \cdot (\mathcal{I}_{ml} - \mathcal{K}_{ml}) Z_l(z), \quad -h < z < -d, \ 0 < \theta < \varphi, \end{aligned} \quad (3.31)$$

with

$$\mathcal{R}_m = J_m(ka) - \frac{J'_m(ka)}{H'_m(ka)} H_m(ka) = \frac{2i}{\pi ka H'_m(ka)}, \quad (3.32)$$

$$\mathcal{K}_{ml} = \frac{K_m(k_l a)}{k_l a K'_m(k_l a)}, \quad \mathcal{I}_{ml} = \frac{I_m(k_l a)}{k_l a I'_m(k_l a)}. \quad (3.33)$$

Let us suppose the vertical distribution of the radial velocity (at $r = a$ for $0 < \theta < \varphi$, $-h < z < -d$) to be an even function (here we have extended the interval of the radial velocity in z -direction to be $-2h + d < z < -d$) and symmetrical about $z = -h$. It is well known that the behavior of flow near the lower edge of the OWC at $z = -d$ is $(z + d)^{-1/2}$. In order to fasten the speed of convergence in the coming numerical computations and following Porter & Evans (1995) and Lovas *et al.* (2010), let us expand $U_S(\theta, z)$ as

$$U_D(\theta, z) = -\frac{ig}{\omega} \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \alpha_{sp} \mathcal{T}_s(\theta) u_p(z), \quad (3.34)$$

with

$$\mathcal{T}_s(\theta) = T_s^* \left(\frac{\theta}{\psi} \right), \quad u_p(z) = \frac{2(-1)^p}{\pi \sqrt{(h-d)^2 - (z+h)^2}} T_{2p} \left(\frac{z+h}{h-d} \right). \quad (3.35)$$

Here, T_{2p} is the Chebyshev polynomial of order $2p$, and T_s^* the shifted Chebyshev polynomial of order s . Note that the singularity at $z = -d$ has been included.

Substituting Eqn. (3.34) into Eqn. (3.31) leads to

$$\begin{aligned} \frac{h}{a} Z_0(z) & \sum_{m=0}^{\infty} \mathcal{R}_m (\mathcal{E}_m \cos m\theta + \mathcal{F}_m \sin m\theta) \\ & = \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \alpha_{sp} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\pi} \bar{C}_{sm} F_{pl} \cos m\theta (\mathcal{I}_{ml} - \mathcal{K}_{ml}) Z_l(z) \\ & \quad + \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \alpha_{sp} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\pi(1-\delta_{m0})} \bar{S}_{sm} F_{pl} \sin m\theta (\mathcal{I}_{ml} - \mathcal{K}_{ml}) Z_l(z), \end{aligned} \quad (3.36)$$

with

$$F_{pl} = \int_{-h}^{-d} u_p(z) Z_l(z) dz = N_l^{-1/2} J_{2p}(k_l(h-d)), \quad (3.37)$$

here the integral formula (see Gradshteyn & Ryzhik (2007), page 802, Eqn. 7.355) has been used, and

$$\bar{C}_{sm} = \int_0^\psi \mathcal{T}_s(\theta) \cos m\theta d\theta, \quad \bar{S}_{sm} = \int_0^\psi \mathcal{T}_s(\theta) \sin m\theta d\theta. \quad (3.38)$$

After multiplying both sides of Eqn. (3.31) by $\mathcal{T}_{s'}(\theta) u_{p'}(z)$ ($s', p' = 0, 1, 2, \dots$) and integrating the resulting equation with respect to θ and z over the ranges of $0 \leq \theta \leq \psi$ and $-h \leq z \leq -d$, respectively, we obtain a system of linear algebraic equations for α_{sp} ,

$$\begin{aligned} \frac{h}{a} F_{p'0} \sum_{m=0}^{\infty} \mathcal{R}_m (\mathcal{E}_m \bar{C}_{s'm} + \mathcal{F}_m \bar{S}_{s'm}) \\ & = \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \alpha_{sp} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\pi} (\mathcal{I}_{ml} - \mathcal{K}_{ml}) F_{pl} F_{p'l} \bar{C}_{sm} \bar{C}_{s'm} \\ & \quad + \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \alpha_{sp} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\pi(1-\delta_{m0})} \delta_{sm} F_{pl} (\mathcal{I}_{ml} - \mathcal{K}_{ml}) \bar{S}_{s'm} F_{p'l}. \end{aligned} \quad (3.39)$$

After solving this set of algebraic equations (Eqn. (3.39), $s' = 0, 1, 2, \dots, S$ and $p' = 0, 1, 2, \dots, P$) for α_{sp} , we can then calculate the unknown coefficients A_{ml} , B_{ml} , C_{ml} and D_{ml} using Eqns. (3.28) and (3.29).

3.3.4 Expression for the diffraction factor $\tilde{\Gamma}$

The expression for $\tilde{\Gamma}$ can be derived now. According to Eqn. (3.2), the diffraction volume flux is calculated by

$$q_D = \int_0^a \int_0^{2\pi} \frac{\partial \phi_D^c}{\partial z} \Big|_{z=0} r dr d\theta. \quad (3.40)$$

Applying Eqns. (3.5), (3.10), and (3.18) to Eqn. (3.40), we obtain the following expression for $\tilde{\Gamma}$

$$\tilde{\Gamma} = \frac{-2\pi i}{\omega} \frac{\sqrt{g/h}}{h} \sum_{l=0}^{\infty} C_{0l} \frac{Z_l'(0)}{k_l a I_0'(k_l a)} \int_0^a I_0(k_l r) r dr. \quad (3.41)$$

Note that the characteristics of power takeoff device and the OWC chamber are implicitly included in the parameter $\tilde{\Gamma}$ through C_{0l} .

3.4 Wave radiation problem

Now we study the wave radiation problem in the absence of incident waves. Due to the oscillation of air pressure $P_a(t)$ inside the chamber, wave energy can be radiated outward in the form of radiated waves. Note that the radiation potential ϕ_R satisfies the governing equation and boundary conditions given by Eqns. (2.10) through (2.13) with $n_j = 0$. Similar to the diffraction problem, we divide the total domain into two regions: the interior region ($r \leq a$) and the exterior region ($r > a$). We denote the radiation potential in the interior region by ϕ_R^c , the velocity potential in the exterior region by ϕ_R^o .

3.4.1 Formal expressions for the radiation potentials inside and outside the chamber

By separation of variables, in the interior region ($r \leq a$), the radiation potential ϕ_R^c can be formally written as

$$\phi_R^c = \frac{-ip_a}{\rho\omega} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} E_{ml} \frac{I_m(k_l r)}{k_l a I_m'(k_l a)} Z_l(z) \cos m(\theta - \psi/2) - \frac{-ip_a}{\rho\omega}, \quad (3.42)$$

where E_{ml} are unknown coefficients. The last term on the right-hand side is used to satisfy the inhomogeneous free surface condition, i.e. Eqn. (2.11).

In the exterior region ($r > a$), the radiation potential ϕ_R^o can be formally expressed as

$$\phi_R^o = \frac{-ip_a}{\rho\omega} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} F_{ml} \frac{K_m(k_l r)}{k_l a K'_m(k_l a)} Z_l(z) \cos m(\theta - \psi/2), \quad (3.43)$$

where F_{ml} are unknown coefficients.

Note that the term $\psi/2$ incorporated by cosine function in Eqns. (3.42)-(3.43) is required to satisfy the fact that the radiation potential in the whole field is symmetric about $\theta = \psi/2$.

3.4.2 Determination of the unknowns E_{ml} and F_{ml}

At $r = a$, the radial velocity must be continuous across the boundary at the opening, and vanish on the solid walls. Therefore, we have

$$\frac{\partial \phi_R^c}{\partial r} \Big|_{r=a} = \begin{cases} 0, & 0 \leq \theta < 2\pi, -d < z < 0 \\ 0, & \psi < \theta < 2\pi, -h < z < -d \\ p_a \cdot U_R(\theta, z), & 0 < \theta < \psi, -h < z < -d, \end{cases} \quad (3.44)$$

and

$$\frac{\partial \phi_R^o}{\partial r} \Big|_{r=a} = \begin{cases} 0, & 0 \leq \theta < 2\pi, -d < z < 0 \\ 0, & \psi < \theta < 2\pi, -h < z < -d \\ p_a \cdot U_R(\theta, z), & 0 < \theta < \psi, -h < z < -d. \end{cases} \quad (3.45)$$

Again, the function $U_R(\theta, z)$ is the unknown radial velocity on the boundary of the opening.

Applying the orthogonality of cosines and the vertical eigenfunctions to Eqn. (3.44) gives

$$\begin{aligned} \frac{1}{h} \int_{-h}^0 \int_0^{2\pi} \frac{\partial \phi_R^c}{\partial r} \Big|_{r=a} Z_{l'}(z) \cos m'(\theta - \psi/2) d\theta dz \\ = \frac{p_a}{h} \int_{-h}^{-d} \int_0^\psi U_R(\theta, z) Z_{l'}(z) \cos m'(\theta - \psi/2) d\theta dz, \end{aligned} \quad (3.46)$$

from which we obtain the expression for E_{ml} ,

$$E_{ml} = \frac{i\rho a \omega \varepsilon_m}{2\pi h} \int_{-h}^{-d} \int_0^\psi U_R(\theta, z) Z_l(z) \cos m(\theta - \psi/2) d\theta dz. \quad (3.47)$$

Similarly, from Eqn. (3.45) we obtain the following equation

$$\begin{aligned} \frac{1}{h} \int_{-h}^0 \int_0^{2\pi} \frac{\partial \phi_R^o}{\partial r} \Big|_{r=a} Z_{l'}(z) \cos m'(\theta - \psi/2) d\theta dz \\ = \frac{p_a}{h} \int_{-h}^{-d} \int_0^\psi U_R(\theta, z) Z_{l'}(z) \cos m'(\theta - \psi/2) d\theta dz, \end{aligned} \quad (3.48)$$

and thus the expression for F_{ml}

$$F_{ml} = E_{ml} = \frac{i\rho a \omega \varepsilon_m}{2\pi h} \int_{-h}^{-d} \int_0^\psi U_R(\theta, z) Z_l(z) \cos m(\theta - \psi/2) d\theta dz. \quad (3.49)$$

3.4.3 Determination of the auxiliary function $U_R(\theta, z)$

Since the pressure is continuous across the boundary of the opening, we have

$$\phi_R^o = \phi_R^c, \quad \text{at } r = a, \text{ for } -h < z < -d, 0 < \theta < \psi. \quad (3.50)$$

Substituting Eqns. (3.42) and (3.43) in Eqn. (3.50) and using the expression for F_{ml} given in (3.49) leads to

$$1 = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{i\rho a \omega}{h} \frac{\varepsilon_m}{2\pi} \int_{-h}^{-d} \int_0^{\psi} U_R(\theta', z') Z_l(z') \cos m(\theta' - \psi/2) d\theta' dz' \cdot (\mathcal{K}_{ml} - \mathcal{I}_{ml}) Z_l(z) \cos m(\theta - \psi/2). \quad (3.51)$$

Again, we expand the unknown function $U_R(\theta, z)$ (Lovas *et al.*, 2010) as

$$U_R(\theta, z) = \frac{-i}{\rho \omega} \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \alpha_{sp} \cos \left[\frac{2s}{v} (\theta - \psi/2) \right] u_p(z), \quad (3.52)$$

where $v = \psi/\pi$. Because $U_R(\theta, z)$ takes the same value at $\theta = 0$ and $\theta = \psi$, we expand it as a cosine series in θ rather than the shifted Chebyshev polynomials.

Using Eqn. (3.52) to Eqn. (3.51), we get

$$\frac{h}{a} = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\pi} \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \alpha_{sp} F_{pl} \mathcal{C}_{sm} (\mathcal{K}_{ml} - \mathcal{I}_{ml}) Z_l(z) \cos m(\theta - \psi/2), \quad (3.53)$$

where

$$\mathcal{C}_{sm} = \int_0^{\psi} \cos \left[\frac{2s}{v} (\theta - \psi/2) \right] \cos m(\theta - \psi/2) d\theta. \quad (3.54)$$

After multiplying both sides of Eqn. (3.53) by $\cos \left[\frac{2s'}{v} (\theta - \psi/2) \right] u_{p'}(z)$, using Eqn. (3.54), and integrating over the range $0 < \theta < \psi$, $-h < z < -d$, we obtain another algebraic system of equations for α_{sp} :

$$\frac{h}{a} \mathcal{C}_{s'0} \delta_{p'0} = \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \alpha_{sp} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\pi} \mathcal{C}_{sm} \mathcal{C}_{s'm} (\mathcal{K}_{ml} - \mathcal{I}_{ml}) F_{pl} F_{p'l}. \quad (3.55)$$

After solving Eqn. (3.55) for α_{sp} , the coefficients E_{ml} and F_{ml} can then be calculated.

3.4.4 The radiation conductance and susceptance: $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$

Now we can derive the expressions for the radiation conductance coefficient $\tilde{\mathcal{B}}$ and the radiation susceptance coefficient $\tilde{\mathcal{C}}$ (Falnes, 2002). According to Eqn. (3.2), the radiation volume flux is calculated by

$$q_R = \int_0^a \int_0^{2\pi} \frac{\partial \phi_R^c}{\partial z} \Big|_{z=0} r dr d\theta \quad (3.56)$$

From Eqns. (3.6), (3.10), (3.42), and Eqn. (3.56) we obtain the following expressions for $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$

$$\tilde{\mathcal{B}} = \text{Re} \left\{ \frac{2\pi i}{\sqrt{kh} \tanh kh} \frac{1}{h} \sum_{l=0}^{\infty} E_{0l} \frac{Z'_l(0)}{k_l a I'_0(k_l a)} \int_0^a I_0(k_l r) r dr \right\}, \quad (3.57)$$

$$\tilde{\mathcal{C}} = -\text{Im} \left\{ \frac{2\pi i}{\sqrt{kh} \tanh kh} \frac{1}{h} \sum_{l=0}^{\infty} E_{0l} \frac{Z'_l(0)}{k_l a I'_0(k_l a)} \int_0^a I_0(k_l r) r dr \right\}. \quad (3.58)$$

3.5 Optimum extraction efficiency

Once the dimensionless hydrodynamic coefficients $\tilde{\Gamma}$, $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ are obtained, the normalized extraction efficiency kL can be evaluated using Eqn. (3.8). For given wave conditions, the optimum extraction efficiency can be achieved by varying turbine characteristics, system dimensions, or both. Mathematically, differentiating Eqn. (3.8) with respect to both κ (the OWC-chamber characteristics) and χ (the turbine characteristics) and setting these derivatives equal to zero gives $\kappa = \tilde{\mathcal{C}}$ and $\chi = \tilde{\mathcal{B}}$ for the following maximum extraction efficiency (see, e.g., Evans (1982); Lovas *et al.* (2010))

$$kL_{\max} = \frac{kh\sqrt{gh}}{C_g} \frac{|\tilde{\Gamma}(\alpha)|^2}{4\tilde{\mathcal{B}}}. \quad (3.59)$$

For $\psi = 2\pi$, Eqn. (3.59) gives $kL_{\max} = 1$, which has been shown in Evans (1982). As did in Martins-Rivas & Mei (2009b), we will fine-tune the power-takeoff system (i.e., the parameter χ) to achieve maximum absorption efficiency for given dimensions of an OWC converter. Therefore, we differentiate Eqn. (3.8) only with respect to χ to obtain the following condition for optimum extraction efficiency of the system

$$\chi_{\text{opt}} = \sqrt{\tilde{\mathcal{B}}^2 + (\tilde{\mathcal{C}} - \kappa)^2}. \quad (3.60)$$

Using Eqn. (3.60) in Eqn. (3.8) gives the following optimum capture length

$$kL_{\text{opt}} = \frac{kh\sqrt{gh}}{C_g} \frac{\sqrt{\tilde{\mathcal{B}}^2 + (\tilde{\mathcal{C}} - \kappa)^2} |\tilde{\Gamma}|^2}{\left(\sqrt{\tilde{\mathcal{B}}^2 + (\tilde{\mathcal{C}} - \kappa)^2} + \tilde{\mathcal{B}} \right)^2 + (\tilde{\mathcal{C}} - \kappa)^2}. \quad (3.61)$$

3.6 Results and discussion

We shall examine the effects of the size and submergence of the OWC chamber, the size of the opening in the supporting structure, and the direction of incident waves on the optimum capture length given in Eqn. (3.61). For all the numerical results presented in this section, we shall use $h = 10$ m, $\rho/\rho_a^0 = 1000$, $c_a = 430$ m/s, $V_0 = \pi a^2 h$, and $g = 9.81$ m/s². The dimensional quantities are needed to calculate the parameter κ (characteristics of the air chamber) (see also, Martins-Rivas & Mei (2009b); Lovas *et al.* (2010)).

3.6.1 Convergence test

In Eqns. (3.39) and (3.55), there are four infinite summations, i.e.,

$$\sum_{m=0}^{\infty}, \quad \sum_{l=0}^{\infty}, \quad \sum_{s=0}^{\infty}, \quad \text{and} \quad \sum_{p=0}^{\infty}. \quad (3.62)$$

We need to truncate these infinite summations to finite summations for practical computations,

$$\sum_{m=0}^M, \quad \sum_{l=0}^{\mathcal{L}}, \quad \sum_{s=0}^S, \quad \text{and} \quad \sum_{p=0}^{\mathcal{P}}. \quad (3.63)$$

We shall first perform convergence tests to determine the suitable values of M , $\mathcal{L}$, S , and $\mathcal{P}$ for our problem.

For the value of $\mathcal{L}$, Garrett (1970) found that the infinite series $\sum_{l=0}^{\infty}$ converged slowly and the truncation error is roughly proportional to $1/\mathcal{L}$. We will use $\mathcal{L} = 400$ in our computations to achieve a precision of 3 decimal places. Yeung (1981) found that $\sum_{m=0}^{\infty}$ had excellent truncation characteristics, and a precision of 2 decimal places could be obtained with $M = 20$. Therefore, we shall use $M = 65$ and 100 for the diffraction and radiation problems, respectively. For the infinite series $\sum_{p=0}^{\infty}$, Evans & Porter (1997) showed that a precision of 6 decimal places could be achieved with $\mathcal{P} = 4$; therefore, we shall choose $\mathcal{P} = 5$ for both diffraction and radiation problems. Next, we choose the values of S for the diffraction and radiation problems, respectively.

Referring to Eqn. (3.39), which is for the diffraction problem, the remaining is just to truncate the infinite series $\sum_{s=0}^{\infty}$. Table 3.1 shows the convergence tests for the convergence of $\left|\tilde{\Gamma}\right|^2$ against S for $a/h = 1/2$, $d/h = 0.2$, $\psi = \pi/2$,

$\alpha = \pi/4$, $K = 0.019 \text{ rad}^2/\text{m}$. It can be seen that three-digits precision can be achieved when $S = 7$. Therefore, we shall use $\mathcal{L} = 400$, $M = 65$, $S = 6$ and $\mathcal{P} = 5$ to solve the wave diffraction problem.

Table 3.1 Convergence tests for $|\tilde{\Gamma}|^2$ against S .

S	3	4	5	6	7
$ \tilde{\Gamma} ^2$	0.185318	0.184331	0.184331	0.183947	0.183947

Eqn. (3.55), which is for the radiation problem, involves only the untruncated infinite series $\sum_{s=0}^{\infty}$. Table 3.2 shows the convergence tests for convergences of $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$ and χ_{opt} against S for $a/h = 1/2$, $d/h = 0.2$, $\psi = \pi/2$, $K = 0.019 \text{ rad}^2/\text{m}$. It can be seen from the table that a precision of at least 2 decimal places can be achieved when $S = 30$, which is sufficient for this study. Therefore, we shall use $\mathcal{L} = 400$, $N = 100$, $S = 30$ and $\mathcal{P} = 5$ to solve the wave radiation problem.

Table 3.2 Convergence tests for $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$ and χ_{opt} against S

S	$\tilde{\mathcal{B}}$	$\tilde{\mathcal{C}}$	χ_{opt}
20	0.020415	0.438325	0.729132
24	0.020407	0.438243	0.729050
28	0.020222	0.436542	0.727344
29	0.020142	0.435804	0.726604
30	0.020055	0.435008	0.725807

Note that the above convergence test includes the effects of both progressive mode and evanescent modes, since we take $\mathcal{L} = 400$. In the truncated summation $\sum_{l=0}^{\mathcal{L}}$, $l = 0$ represents the progressive mode, and $l = 1, 2, 3, \dots$ represent the evanescent modes.

3.6.2 Code verification

The code has been checked against three special cases reported in the literature. If the opening angle ψ equals to 2π , our problem reduces to the problem of a truncated hollow circular cylinder, for which Evans & Porter (1997) studied the case where the incidence angle α equals to π and the air is incompressible. Martins-Rivas & Mei (2009b) studied a single OWC installed at the tip of a thin

breakwater and considered the compressibility of the air inside the chamber; for the incidence angle of π (i.e., along the thin breakwater from $+\infty$), the effects of the thin breakwater can be neglected and the case can also be reduced to a truncated hollow circular cylinder. Lovas *et al.* (2010) considered an OWC installed at a coastal corner; they also presented the limiting case where a land mass was reduced to a thin breakwater. Figure 3.2 shows that our numerical results agree very well with those of Evans & Porter (1997), Martins-Rivas & Mei (2009b), and Lovas *et al.* (2010).

Figure 3.2 Comparison of the present results of limiting case with the previous results. (a): with the results by Evans & Porter (1997) for $a/h = 1/2$, $d/h = 1/2$; (b): with the results by Martins-Rivas & Mei (2009b) for $a/h = 1/2$, $d/h = 1/5$; (c) and (d): with the results by Lovas *et al.* (2010) results for $a/h = 1/2$, $d/h = 0.2$. Here, $\mu = \frac{gh}{\pi\omega a^2\sqrt{g/h}}\tilde{C}$ and $\tilde{C}_M = \frac{\omega h}{a\sqrt{g/h}}\tilde{C}$.

3.6.3 Effects of the OWC chamber draft

Let us first examine the effects of the draft (d/h) on the hydrodynamic coefficients and the optimum absorption efficiency for a fixed OWC size $a/h = 1/2$ and a single incident angle $\alpha = \psi/4$. The size characteristics of the OWC chamber, i.e., the parameter κ defined by Eqn. (3.11), is shown in Figure 3.3 for reference.

As an example, we take $\psi = \pi/2$. Figure 3.4 shows the effects of d/h on the variation of $|\tilde{\Gamma}|^2$; Figure 3.5 shows the effects of d/h on the variation of radiation conductance $\tilde{\mathcal{B}}$ and radiation susceptance $\tilde{\mathcal{C}}$ against $\omega^2 h/g$. There are multiple

Figure 3.3 Variation of κ against $\omega^2 h/g$ for $a/h = 1/2$.

Figure 3.4 Variation of $|\tilde{\Gamma}|^2$ against $\omega^2 h/g$ for $a/h = 1/2$, $\psi = \pi/2$, $\alpha = \pi/4$, and several values of d/h .

peaks in the curves for $\tilde{\mathcal{B}}$, radiation susceptance $\tilde{\mathcal{C}}$ and $\tilde{\Gamma}$, and this is because, except for the natural mode of the heave motion of the oscillating water column, there exists also sloshing modes in a closed cylinder (Evans & Porter, 1997). However, only the first peak, corresponding to the resonant heave motion of the oscillating water column, is usually important for designing an OWC device. For each draft, the peaks of $\tilde{\mathcal{B}}$, radiation susceptance $\tilde{\mathcal{C}}$ and $\tilde{\Gamma}$ occur at the same frequency (the resonant frequency), and the resonance frequency decreases with increasing draft d/h . For both the radiated waves and the diffracted waves, the resonant interaction between waves and the OWC occurs when the frequency of the wave motion is the same as that of the oscillating water column, which decreases with increasing the height of the water column (McCormick, 1981). As d/h increases, the resonance peaks of $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$ and $\tilde{\Gamma}$ all become higher and sharper. Note that negative radiation susceptance coefficient $\tilde{\mathcal{C}}$ occurs in certain range of

Figure 3.5 Radiation conductance coefficient $\tilde{\mathcal{B}}$ (left panel) and radiation susceptance coefficient $\tilde{\mathcal{C}}$ (right panel) as functions of $\omega^2 h / g$ for $a/h = 1/2$, $\psi = \pi/2$, $\alpha = \psi/4$, and several values of d/h .

Figure 3.6 Optimum turbine parameter χ_{opt} (left panel) and the corresponding capture length kL (right panel) as functions of $\omega^2 h / g$ for $a/h = 1/2$, $\psi = \pi/2$ and $\alpha = \pi/4$, and several values of d/h .

$\omega^2 h / g$ which is a feature present in many results in wave energy (analogous with the theory of moving bodies in a fluid) (Evans & Porter, 1997; Martins-Rivas & Mei, 2009b).

Figure 3.6 shows the effects of d/h on the optimum turbine parameter χ_{opt} and the corresponding capture length kL (calculated by Eqn. (3.61)). Note that for $d/h = 0.2$, the capture length kL can be greater than 1 within the frequency range of $0.8 < \omega^2 h / g < 3$, and for $d/h = 0.5$ only within the frequency range of $0.5 < \omega^2 h / g < 2$. A shallow draft is therefore beneficial for widening the frequency range over which an OWC device can operate efficiently. To ensure that the lower end of the OWC is immersed in water at all times, the draft should be designed in consideration of other factors such as tidal variations and extreme waves.

As another example for the effects of OWC draft, we take $\psi = 3\pi/4$ and keep other parameters the same as those in Figure 3.6. Effects of d/h on χ_{opt} and kL are shown in Figure 3.7. With this design, the capture length kL for $d/h = 0.2$ can be greater than 2 within the frequency range of $1 < \omega^2 h/g < 3.2$, which is wider than the frequency range for $\psi = \pi/2$.

Figure 3.7 Optimum turbine parameter χ_{opt} (left panel) and the corresponding capture length kL (right panel) as functions of $\omega^2 h/g$ for $a/h = 1/2$, $\psi = 3\pi/4$ and $\alpha = 3\pi/8$, and several values of d/h .

3.6.4 Effects of the OWC size

Next, we examine the effects of OWC size a/h for a constant draft $d/h = 0.2$ and a single incident angle $\alpha = \psi/2$. Again, the size characteristics of the OWC chamber, i.e., the parameter κ defined by Eqn. (3.11), is shown in Figure 3.8 for reference.

As an example, we take $\psi = \pi/2$. Figure 3.9 and 3.10 show the effects of a/h on $|\tilde{\Gamma}|^2$ as well as $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$, respectively. Again, resonant interaction between waves and the OWC occurs when the frequency of the wave motion is the same as that of the oscillating water column. For an OWC device with the same draft, the resonant frequency of the oscillating water column is changed only by its radiation susceptance (McCormick, 1981). As a/h increases, the fundamental resonance frequencies of $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$ and $\tilde{\Gamma}$ all decrease. Since diffracted waves are weaker for smaller OWC devices, the peak value of $|\tilde{\Gamma}|^2$ at the fundamental resonance frequency can be greatly reduced by reducing a/h . Figure 3.11 shows the effects of a/h on the optimum turbine parameter χ_{opt} and the corresponding capture length kL . Increasing a/h can shift the first peak in capture length to

Figure 3.8 Variation of κ against $\omega^2 h / g$ for several values of a/h .

Figure 3.9 Variation of $|\tilde{\Gamma}|^2$ against $\omega^2 h / g$ for $d/h = 0.2$, $\psi = \pi/2$, $\alpha = \pi/4$, and several values of a/h .

Figure 3.10 Variation of radiation conductance $\tilde{\mathcal{B}}$ (left panel) and radiation susceptance $\tilde{\mathcal{C}}$ (right panel) against $\omega^2 h / g$ for $d/h = 0.2$, $\psi = \pi/2$, and several values of a/h .

lower frequencies. Increasing a/h from 0.2 to 0.5 can also widen the frequency range over which the OWC can operate at high efficiency.

As another example for the effects of the OWC size, we take $\psi = 3\pi/4$ and keep other parameters the same as those in Figure 3.11. The effects of a/h on χ_{opt} and kL are shown in Figure 3.12. Again, large capture length can be achieved in a frequency range wider than that for $\psi = \pi/2$.

3.6.5 Effects of the size of the opening in the supporting structure

In the previous two sections, numerical results showed that a larger opening in the supporting structure is beneficial for wave energy conversion. However, from the structural strength point of view, the opening in the supporting structure

Figure 3.11 Optimum turbine parameter χ_{opt} (left panel) and the corresponding capture length kL (right panel) as functions of $\omega^2 h / g$ for $d/h = 0.2$, $\psi = \pi/2$, $\alpha = \pi/4$, and several values of a/h .

Figure 3.12 Optimum turbine parameter χ_{opt} (left panel) and the corresponding capture length kL (right panel) as functions of $\omega^2 h / g$ for $d/h = 0.2$, $\psi = 3\pi/4$, $\alpha = 3\pi/8$, and several values of a/h .

cannot be too large. Here we examine the effects of the size of the opening in the supporting structure (measured by the opening angle ψ) using a fixed draft $d/h = 0.2$, a fixed OWC size $a/h = 1/2$, and a single incidence angle $\alpha = \psi/2$.

Figure 3.14 shows the effects of the opening angle ψ on $|\tilde{\Gamma}|^2$ for different values of ψ . Increasing the opening angle ψ increases the resonant frequency but reduces the magnitude of the diffraction flux factor; the diffraction effect is the weakest when the supporting structure is not there (i.e., $\psi = 2\pi$).

The effects of the opening angle ψ on $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ are shown in Figure 3.15. Both the maximum radiation conductance and radiation susceptance increase with the increasing opening size; the strongest radiation effect occurs when there is no supporting structure to block the radiated waves (i.e., $\psi = 2\pi$). The effects of ψ on optimum turbine parameter χ_{opt} and the corresponding capture length kL

Figure 3.13 Schematic diagram of the variation of opening angle (ψ).

Figure 3.14 Variation of $|\tilde{\Gamma}|^2$ against $\omega^2 h/g$ for $d/h = 0.2$, $a/h = 1/2$, $\alpha = \psi/2$, and several values of ψ .

Figure 3.15 Variation of radiation conductance $\tilde{B}$ and radiation susceptance $\tilde{C}$ against $\omega^2 h/g$ for $d/h = 0.2$, $a/h = 1/2$, and several values of ψ .

are shown in Figure 3.16. Even though increasing the opening angle can widen the resonance peak of kL_{opt} for $\omega^2 h/g < 3.5$, it can reduce the peak capture length and the structural strength of the supporting structure at the same time. Our results suggested that $\pi/2 < \psi < 5\pi/4$ could yield reasonable energy conversion efficiency; however, the opening angle ψ in the supporting structure should be chosen from this range, taking the requirements on structural strength into consideration.

Let us take a typical sea environment, i.e., $\rho = 1020 \text{ kg/m}^3$, $T = 7 \text{ s}$ (T is wave period), and $A = 1 \text{ m}$, to evaluate the capacity of a given OWC device ($a = h/2 = 5 \text{ m}$, $d = 0.2 \cdot h = 2 \text{ m}$, $\psi = \pi/2$). Referring to the solid line in Figure 3.16, as $\omega^2 h/g = 0.82$ ($\omega = 2\pi/T$), the optimum capture length kL_{opt} is approximately equal to 1.5 if water waves are incident normally. In this sea

Figure 3.16 Optimum turbine parameter χ_{opt} and the corresponding capture length kL as functions of $\omega^2 h/g$ for several values of ψ . In all cases, $d/h = 0.2$, $a/h = 1/2$, $\alpha = \psi/2$.

environment, the wave power of plane incident waves per unit wave crest length is about 32.5 kW/m. The wave power extracted by the given OWC device, after using Eqn. (3.8), is about 464.7 kW.

3.6.6 Effects of wave direction

The dominant direction of coastal waves may vary seasonally. Here, we examine the effects of incident angle α (shown in Figures 3.17 and 3.19) on the capture length kL for one draft $d/h = 0.2$ and one OWC size $a/h = 1/2$ and two opening sizes: $\psi = \pi/2$ and $\psi = 3\pi/4$, and the results are shown in Figures 3.18 and 3.20. As expected, when waves arrive along the line of symmetry (i.e., $\alpha = \psi/2$, normal incidence), the system has the optimal performance in terms of wave energy conversion.

3.7 Summary

Based on potential wave theory, the wave interaction with a circular OWC converter supported by a coaxial tube-sector-shaped structure is studied analytically, with a focus on the efficiency of wave power conversion. The optimization of OWC conversion system is also discussed. The following conclusions can be drawn from this study:

- (i) The coaxial tube-sector-shaped supporting structure can increase the energy conversion efficiency and widen the frequency range over which the OWC device can operate at high efficiency.

Figure 3.17 Schematic diagram of different incident directions for $\psi = \pi/2$.

Figure 3.18 Optimum capture length kL as a function of $\omega^2 h/g$ for $\psi = \pi/2$, $d/h = 0.2$, $a/h = 1/2$, and several values of α .

Figure 3.19 Schematic diagram of different incident directions for $\psi = 3\pi/4$.

Figure 3.20 Optimum capture length kL as a function of $\omega^2 h/g$ for $\psi = 3\pi/4$, $d/h = 0.2$, $a/h = 1/2$, and several values of α .

- (ii) The opening in the supporting structure can be chosen from the range of $\pi/2 < \psi < 5\pi/4$, over which the OWC device can operate more efficiently in a wider bandwidth.
- (iii) A shallow draft is beneficial for widening the frequency range over which an OWC device can operate efficiently. To ensure that the lower end of the OWC is immersed in water at all times, the draft should be designed in consideration of other factors such as tidal variations and extreme waves.
- (iv) When incident waves arrive along the line of symmetry (i.e., $\alpha = \psi/2$, normal incidence), the system has the optimal performance in terms of wave energy conversion.

Chapter 4

A bottom-mounted OWC device with a V-shaped channel

An analytical theory is developed to model an oscillating water column (OWC) with a V-shaped channel consisting of two bottom-mounted vertical plates, supported by a coaxial tube-sector-shaped structure. The V-shaped channel is like a “wave guide”, which is beneficial to gather waves and enhance the amplitudes of standing waves in the field bounded by these two plates. To some extent, this chapter can be considered as an extension to the last chapter. A set of θ -dependent orthogonal basis functions incorporating the singular properties at sharp corners is introduced. The hydrodynamics and air pressure inside the chamber are coupled by the power-takeoff (PTO) model developed in the previous sections.

4.1 Description

A circular OWC device of radius a is installed in water of constant depth h and supported by a coaxial tube-sector-shaped structure of the same radius. Two bottom-mounted thin plates of the same length b are attached to the circular structure, forming V-shape arms of an angle of 2β . The draft of the circular OWC is d . The distance between the tip of one arm and the origin of coordinates is c . As shown in Figure 4.1, two small arcs of the same length (arc-angle γ) are factitiously added to avoid the triple singularity at the tips of the two plates. It is further assumed that a linear power-takeoff (PTO) device (e.g., a linear Well’s turbine) is installed at the top of the OWC chamber for energy extraction, and the thicknesses of the supporting structure, the OWC chamber wall and the plates are all thin.

Figure 4.1 Schematic diagram of a bottom-mounted oscillating water column converter with two arms.

In order to facilitate the application of eigenfunction expansion method, except for Cartesian coordinate system, it is necessary to establish a cylindrical coordinate system $Or\theta z$ with its origin and z -axis coinciding with those of the Cartesian coordinate system $Oxyz$. In the cylindrical coordinate system, the opening of the supporting structure occupies the space defined by $r = a$, $-\beta \leq \theta \leq \beta$, $-h \leq z \leq -d$. Given the geometrical properties of the OWC converter, it is conventional to divide the total domain into the following four sub-domains,

- (1) the region defined by $r > c$, $0 \leq \theta < 2\pi$, $-h \leq z \leq 0$;
- (2) the region defined by $a < r < c$, $-\beta < \theta < \beta$, $-h \leq z \leq 0$;
- (3) the region defined by $0 < r < a$, $0 \leq \theta < 2\pi$, $-h \leq z \leq 0$;
- (4) the region defined by $a < r < c$, $\beta < \theta < 2\pi - \beta$, $-h \leq z \leq 0$.

4.2 Wave diffraction problem

The diffraction of waves by the structure is considered in this section. The pressure inside the chamber is set to the constant atmospheric pressure, which is conventionally taken to be zero. Therefore, the complex amplitude of diffraction potential ϕ_D satisfies Laplace equation and boundary conditions given by Eqns. (2.10)-(2.12) with $p_a = 0$. Let us denote the diffraction potential in different regions by $\phi_D^{(1)}$ (in region (1)), $\phi_D^{(2)}$ (in region (2)), $\phi_D^{(3)}$ (in region (3)), $\phi_D^{(4)}$ (in region (4)), respectively, as shown in Figure 4.1(b).

4.2.1 Formal expressions for the complex amplitudes of the diffraction potentials in different sub-domains

The complex-valued velocity potential $\phi_D^{(1)}$, which is defined on $r > c$, $0 \leq \theta < 2\pi$ and $-h \leq z \leq 0$, can be further written as the sum of the following two parts,

$$\phi_D^{(1)} = \phi_D^D + \phi_D^{cor}, \quad (4.1)$$

where ϕ_D^D is the solution corresponding to the wave diffracted by an imaginary solid cylinder of radius c extending the entire depth, and ϕ_D^{cor} a correction potential to ϕ_D^D due to the opening at $r = c$.

For a train of incident waves of a single frequency ω , arriving from an angle α with respect to x -axis, as shown in 4.1(b), the complex amplitude of the incident velocity potential can be written as (Mei *et al.*, 2005)

$$\phi_I = \frac{-igA}{\omega} Z_0(z) \sum_{m=0}^{\infty} J_m(kr) [\mathcal{E}_m \cos m\theta + \mathcal{F}_m \sin m\theta], \quad (4.2)$$

where J_m denotes the Bessel function of the first kind of order m , the coefficients $\mathcal{E}_m$ and $\mathcal{F}_m$ are given by

$$\mathcal{E}_m = i^m \varepsilon_m \frac{\cos m\alpha}{Z_0(0)}, \quad \mathcal{F}_m = i^m \varepsilon_m \frac{\sin m\alpha}{Z_0(0)} \quad (4.3)$$

and $\varepsilon_0 = 1$ and $\varepsilon_m = 2$ for $m \geq 1$.

Following Mei *et al.* (2005), the complex amplitude ϕ_D^D can be expressed in the following form

$$\phi_D^D = \frac{-igA}{\omega} Z_0(z) \sum_{m=0}^{\infty} \left(J_m(kr) - \frac{J'_m(kc)}{H'_m(kc)} H_m(kr) \right) (\mathcal{E}_m \cos m\theta + \mathcal{F}_m \sin m\theta), \quad (4.4)$$

where H_m represents the Hankel function of the first kind of order m . Here and hereafter, a prime represents taking the differentiation of a function with respect to its argument.

The correction potential ϕ_D^{cor} , which satisfies the free surface condition (i.e., Eqn. (2.11)) and the bottom condition at $z = -h$ (i.e., Eqn. (2.12)), can be formally written as,

$$\phi_D^{cor} = \frac{-igA}{\omega} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{K_m(k_l r)}{k_l K'_m(k_l c)} (A_{ml} \cos m\theta + B_{ml} \sin m\theta) Z_l(z), \quad (4.5)$$

where K_m represents the modified Bessel function of the second kind of order m , A_{ml} and B_{ml} are unknown coefficients to be determined, and $B_{0l} = 0$ for $l = 0, 1, 2, \dots$

Adding up Eqns. (4.4) and (4.5) gives the expression of the complex amplitude $\phi_D^{(1)}$, namely,

$$\begin{aligned} \phi_D^{(1)} = & \frac{-igA}{\omega} Z_0(z) \sum_{m=0}^{\infty} \left[J_m(kr) - \frac{J'_m(kc)}{H'_m(kc)} H_m(kr) \right] (\mathcal{E}_m \cos m\theta + \mathcal{F}_m \sin m\theta) \\ & + \frac{-igA}{\omega} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{K_m(k_l r)}{k_l K'_m(k_l c)} (A_{ml} \cos m\theta + B_{ml} \sin m\theta) Z_l(z). \end{aligned} \quad (4.6)$$

The complex amplitude $\phi_D^{(2)}$, satisfying the free surface condition (i.e., Eqn. (2.11)), the bottom condition (i.e., Eqn. (2.12)) and the no-flux conditions at rigid boundaries (i.e., Eqn. (2.13)), can be formally written as

$$\phi_D^{(2)} = \frac{-igA}{\omega} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \left[C_{ml} \frac{I_{m\mu}(k_l r)}{k_l I'_{m\mu}(k_l c)} + D_{ml} \frac{K_{m\mu}(k_l r)}{k_l K'_{m\mu}(k_l a)} \right] \cos m\mu(\theta + \beta) Z_l(z), \quad (4.7)$$

where I_m denotes the modified Bessel function of the first kind of order m , C_{ml} and D_{ml} are unknown coefficients, and $\mu = \pi/(2\beta)$. Note here that I_m is unbounded as $r \rightarrow \infty$, while K_m is unbounded as $r = 0$ (Abramowitz & Stegun, 1972). Hence, in view of the range of radial direction r in region (2), both I_m and K_m should be included in the expression of the complex amplitude $\phi_D^{(2)}$ (see also Mavrakos & Konispoliatis (2012)). The above expression also satisfies the boundary condition at the two arms (namely, $\theta = -\beta$ and β) (see also Chang *et al.* (2012)).

Similarly, by the requirement of boundedness, the complex amplitude $\phi_D^{(3)}$ can be written in the following form,

$$\phi_D^{(3)} = \frac{-igA}{\omega} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{I_m(k_l r)}{k_l I'_m(k_l a)} (E_{ml} \cos m\theta + F_{ml} \sin m\theta) Z_l(z), \quad (4.8)$$

where E_{ml} and F_{ml} are unknown coefficients and

$$F_{0l} = 0, \quad l = 0, 1, 2, \dots$$

The complex amplitude $\phi_D^{(4)}$, satisfying the solid boundary condition at $r = a$ for $\beta < \theta < 2\pi - \beta$, can be written as

$$\phi_D^{(4)} = \frac{-igA}{\omega} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} G_{ml} \left(\frac{I_{m\nu}(k_l r)}{k_l I'_{m\nu}(k_l c)} - \frac{I'_{m\nu}(k_l a)}{I'_{m\nu}(k_l c)} \frac{K_{m\nu}(k_l r)}{k_l K'_{m\nu}(k_l a)} \right) \cos m\nu(\theta - \beta) Z_l(z), \quad (4.9)$$

where G_{ml} are unknown coefficients and $\nu = \pi/2(\pi - \beta)$. Note that the above equation satisfies the no-flux condition on the surface of $r = a$ and the two arms (namely, at $\theta = \beta$ and $2\pi - \beta$). Meanwhile, it is easy to find that the above expression is bounded in region (4), as shown in Figure 4.1.

4.2.2 Treatment of singularities at sharp corners

Referring to Figure 4.1, there exist the following three common boundaries

- The first common boundary defined by $r = c$, $-\beta < \theta < \beta$, $-h < z < 0$;
- The second common boundary defined by $r = c$, $\beta + \gamma < \theta < 2\pi - \beta - \gamma$, $-h < z < 0$;
- The third common boundary defined by $r = a$, $-\beta < \theta < \beta$, $-h < z < -d$.

On each common boundary, physically, there exist singularities at sharp corners either in z -direction or θ -direction or both: the singularity of the flow at $r = a$ or c with $\theta = -\beta$ or β behaves like $(\theta \pm \beta)^{-1/3}$; the singularity of the flow at $r = c$ with $\theta = \beta + \gamma$ or $\theta = 2\pi - \beta - \gamma$ behaves like $(\theta - \beta - \gamma)^{-1/2}$ or $(\theta - 2\pi - \beta - \gamma)^{-1/2}$; the singularity of the flow at $r = a$ with $z = -d$ behaves like $(z + d)^{-1/2}$ (see Mandal & Chakrabarti, 2000; Batchelor, 1967).

To handle the aforementioned singularities, we adopt the methods employed by Porter & Evans (1995), Fernyhough & Evans (1995), Chang *et al.* (2012) and Deng *et al.* (2013), and introduce the following auxiliary functions to represent the unknown radial velocities on the three common boundaries:

- $U_D^{(1)}(\theta, z)$ for the first common boundary;
- $U_D^{(2)}(\theta, z)$ for the second common boundary;
- $U_D^{(3)}(\theta, z)$ for the third common boundary.

and include the singularities in both θ and z -directions in the expressions for the auxiliary functions to speed up the convergence in numerical computations. We formally expand the auxiliary functions ($U_D^{(1)}$, $U_D^{(2)}$ and $U_D^{(3)}$) as follows,

$$U_D^{(1)}(\theta, z) = \frac{-igA}{\omega} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} f_n^{(1)}(\theta) \tilde{T}_p(z), \quad -\beta < \theta < \beta, -h < z < 0, \quad (4.10)$$

$$U_D^{(2)}(\theta, z) = \frac{-igA}{\omega} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sigma_{np} f_n^{(2)}(\theta) \tilde{T}_p(z), \quad \beta + \gamma < \theta < 2\pi - \beta - \gamma, -h < z < 0, \quad (4.11)$$

$$U_D^{(3)}(\theta, z) = \frac{-igA}{\omega} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \xi_{np} f_n^{(1)}(\theta) u_p(z), \quad -\beta < \theta < \beta, -h < z < -d. \quad (4.12)$$

Which have included singularities in both θ (azimuthal) and z (vertical) directions. The functions $f_n^{(1)}(\theta)$, $f_n^{(2)}(\theta)$, $\tilde{T}_p(z)$ and $u_p(z)$ are given by

$$f_n^{(1)}(\theta) = \frac{n! \Gamma(1/6)}{2\pi \Gamma(n + 1/3) [\beta(\beta - \theta)(\beta + \theta)]^{1/3}} C_n^{1/6} \left(\frac{\theta}{\beta} \right), \quad (4.13)$$

$$f_n^{(2)}(\theta) = \frac{1}{\pi[(\theta - \beta - \gamma)(2\pi - \beta - \gamma - \theta)]^{1/2}} T_n \left(\frac{\theta - \pi}{\pi - \beta - \gamma} \right), \quad (4.14)$$

$$\tilde{T}_p(z) = T_p^* \left(\frac{z + h}{h} \right), \quad (4.15)$$

$$u_p(z) = \frac{2(-1)^p}{\pi \sqrt{(h - d)^2 - (z + h)^2}} T_{2p} \left(\frac{z + h}{h - d} \right), \quad (4.16)$$

where C_n^v denotes the Gegenbauer polynomial of order n , T_n the Chebyshev polynomial of order n , T_p^* the shifted Chebyshev polynomial of order p , and Γ the gamma function (Abramowitz & Stegun, 1972). The unknown coefficients Λ_{np} , σ_{np} and ξ_{np} are to be determined later.

4.2.3 The coefficients in the diffraction potentials: A_{ml} through G_{ml}

The coefficients in diffraction potentials, A_{ml} through G_{ml} , are determined by requiring that the radial velocities must be continuous on all common boundaries, i.e.,

$$\frac{\partial \phi_D^{(1)}}{\partial r} \Big|_{r=c} = \begin{cases} U_D^{(1)}, & -\beta < \theta < \beta, -h < z < 0, \\ 0, & \beta < \theta < \beta + \gamma, -h < z < 0, \\ U_D^{(2)}, & \beta + \gamma < \theta < 2\pi - \beta - \gamma, -h < z < 0, \\ 0 & 2\pi - \beta - \gamma < \theta < 2\pi - \beta, -h < z < 0, \end{cases} \quad (4.17)$$

$$\frac{\partial \phi_D^{(2)}}{\partial r} \Big|_{r=c} = U_D^{(1)}, \quad -\beta < \theta < \beta, -h < z < 0, \quad (4.18)$$

$$\frac{\partial \phi_D^{(2)}}{\partial r} \Big|_{r=a} = \begin{cases} 0, & -\beta < \theta < \beta, -d < z < 0, \\ U_D^{(3)}, & -\beta < \theta < \beta, -h < z < -d, \end{cases} \quad (4.19)$$

$$\frac{\partial \phi_D^{(3)}}{\partial r} \Big|_{r=a} = \begin{cases} 0, & 0 \leq \theta < 2\pi, -d < z < 0, \\ U_D^{(3)}, & -\beta < \theta < \beta, -h < z < -d, \\ 0, & \beta < \theta < 2\pi - \beta, -h < z < -d, \end{cases} \quad (4.20)$$

and

$$\frac{\partial \phi_D^{(4)}}{\partial r} \Big|_{r=c} = \begin{cases} 0, & \beta < \theta < \beta + \gamma, -h < z < 0, \\ U_D^{(2)}, & \beta + \gamma < \theta < 2\pi - \beta - \gamma, -h < z < 0, \\ 0 & 2\pi - \beta - \gamma < \theta < 2\pi - \beta, -h < z < 0. \end{cases} \quad (4.21)$$

In the following, we shall determine the coefficients in diffraction potentials by using the fact that both $\cos m\theta$ and $Z_l(z)$ are complete sets of orthogonal functions.

Multiplying Eqn. (4.17) by $\cos m'\theta Z_{l'}(z)/h$ and integrating the resulting equation with respect to θ and z over their corresponding intervals gives

$$\begin{aligned} & \frac{1}{h} \int_{-h}^0 \int_0^{2\pi} \frac{\partial \phi_D^{(1)}}{\partial r} \Big|_{r=c} \cos m'\theta Z_{l'}(z) d\theta dz \\ &= \frac{1}{h} \int_{-h}^0 \int_{-\beta}^{\beta} U_D^{(1)} \cos m'\theta Z_{l'}(z) d\theta dz + \frac{1}{h} \int_{-h}^0 \int_{\beta+\gamma}^{2\pi-\beta-\gamma} U_D^{(2)} \cos m'\theta Z_{l'}(z) d\theta dz. \end{aligned} \quad (4.22)$$

After using the expressions for $\phi_D^{(1)}$, $U_D^{(1)}$ and $U_D^{(2)}$ (i.e., Eqns. (4.6), (4.10) and (4.11)) in Eqn. (4.22), the following expression for A_{ml} is obtained

$$A_{ml} = \frac{\varepsilon_m}{2\pi} \left(\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} X_{nm}^{(1)} V_{pl}^{(1)} + \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sigma_{np} X_{nm}^{(2)} V_{pl}^{(1)} \right), \quad (4.23)$$

where $X_{nm}^{(1)}$, $X_{nm}^{(2)}$ and $V_{pl}^{(1)}$ are given by

$$X_{nm}^{(1)} = \int_{-\beta}^{\beta} f_n^{(1)}(\theta) \cos m\theta d\theta, \quad (4.24)$$

$$X_{nm}^{(2)} = \int_{\beta+\gamma}^{2\pi-\beta-\gamma} f_n^{(2)}(\theta) \cos m\theta d\theta, \quad (4.25)$$

$$V_{pl}^{(1)} = \frac{1}{h} \int_{-h}^0 \tilde{T}_p(z) Z_l(z) dz. \quad (4.26)$$

The integrands in the above three integrals are integrable and the analytical expressions for $X_{nm}^{(1)}$, $X_{nm}^{(2)}$ and $V_{pl}^{(1)}$ are given in Appendix A.1.2.1.

Multiplying Eqn. (4.17) by $\sin m'\theta Z_{l'}(z)/h$ and integrating the resulting equation with respect to θ and z over their corresponding intervals gives

$$\begin{aligned} & \frac{1}{h} \int_{-h}^0 \int_0^{2\pi} \frac{\partial \phi_D^{(1)}}{\partial r} \Big|_{r=c} \sin m'\theta Z_{l'}(z) d\theta dz \\ &= \frac{1}{h} \int_{-h}^0 \int_{-\beta}^{\beta} U_D^{(1)} \sin m'\theta Z_{l'}(z) d\theta dz + \frac{1}{h} \int_{-h}^0 \int_{\beta+\gamma}^{2\pi-\beta-\gamma} U_D^{(2)} \sin m'\theta Z_{l'}(z) d\theta dz. \end{aligned} \quad (4.27)$$

After using the expressions for $\phi_D^{(1)}$, $U_D^{(1)}$ and $U_D^{(2)}$ (i.e., Eqns. (4.6), (4.10) and (4.11)) in Eqn. (4.27), we obtain

$$B_{ml} = \frac{1}{\pi} \left(\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} Y_{nm}^{(1)} V_{pl}^{(1)} + \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sigma_{np} Y_{nm}^{(2)} V_{pl}^{(1)} \right), \quad (4.28)$$

where $Y_{nm}^{(1)}$ and $Y_{nm}^{(2)}$ are given by

$$Y_{nm}^{(1)} = \int_{-\beta}^{\beta} f_n^{(1)}(\theta) \sin m\theta d\theta, \quad (4.29)$$

$$Y_{nm}^{(2)} = \int_{\beta+\gamma}^{2\pi-\beta-\gamma} f_n^{(2)}(\theta) \sin m\theta d\theta. \quad (4.30)$$

The integrands in the above two integrals are integrable and the analytical expressions for $Y_{nm}^{(1)}$ and $Y_{nm}^{(2)}$ are given in Appendix A.1.2.1.

Multiplying Eqns. (4.18) and (4.19) by $\cos m'\mu(\theta+\beta)Z_{l'}(z)/h$ and integrating the resulting equation with respect to θ and z over their corresponding intervals gives, respectively,

$$\begin{aligned} & \frac{1}{h} \int_{-h}^0 \int_{-\beta}^{\beta} \frac{\partial \phi_D^{(2)}}{\partial r} \Big|_{r=c} \cos m'\mu(\theta+\beta) Z_{l'}(z) d\theta dz \\ &= \frac{1}{h} \int_{-h}^0 \int_{-\beta}^{\beta} U_D^{(1)} \cos m'\mu(\theta+\beta) Z_{l'}(z) d\theta dz, \end{aligned} \quad (4.31)$$

and

$$\begin{aligned} & \frac{1}{h} \int_{-h}^0 \int_{-\beta}^{\beta} \frac{\partial \phi_D^{(2)}}{\partial r} \Big|_{r=a} \cos m'\mu(\theta+\beta) Z_{l'}(z) d\theta dz \\ &= \frac{1}{h} \int_{-h}^{-d} \int_{-\beta}^{\beta} U_D^{(3)} \cos m'\mu(\theta+\beta) Z_{l'}(z) d\theta dz. \end{aligned} \quad (4.32)$$

After using the expressions for $\phi_D^{(2)}$, $U_D^{(2)}$ and $U_D^{(3)}$ (i.e., Eqns. (4.7), (4.10) and (4.12)) in Eqns. (4.31) and (4.32), we obtain

$$C_{ml} = \frac{\varepsilon_m}{2\beta} Q_{ml}^{(1)} \left[\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} L_{nm}^{(1)} V_{pl}^{(1)} - \frac{K_{m\mu}'(k_l c)}{K_{m\mu}'(k_l a)} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \xi_{np} L_{nm}^{(1)} V_{pl}^{(2)} \right], \quad (4.33)$$

and

$$D_{ml} = \frac{\varepsilon_m}{2\beta} Q_{ml}^{(1)} \left[\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \xi_{np} L_{nm}^{(1)} V_{pl}^{(2)} - \frac{I'_{m\mu}(k_l a)}{I'_{m\mu}(k_l c)} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} L_{nm}^{(1)} V_{pl}^{(1)} \right], \quad (4.34)$$

where $L_{nm}^{(1)}$, $V_{pl}^{(2)}$ and $Q_{ml}^{(1)}$ are given by

$$L_{nm}^{(1)} = \int_{-\beta}^{\beta} f_n^{(1)}(\theta) \cos m\mu(\theta + \beta) d\theta, \quad (4.35)$$

$$V_{pl}^{(2)} = \frac{1}{h} \int_{-h}^{-d} u_p(z) Z_l(z) dz, \quad (4.36)$$

$$Q_{ml}^{(1)} = 1 / \left[1 - \frac{I'_{m\mu}(k_l a)}{I'_{m\mu}(k_l c)} \frac{K_{m\mu}'(k_l c)}{K_{m\mu}'(k_l a)} \right]. \quad (4.37)$$

The analytical expressions for $L_{nm}^{(1)}$ and $V_{pl}^{(2)}$ are given in Appendix A.1.2.1.

Multiplying Eqn. (4.20) by $\cos m'\theta Z_{l'}(z)/h$ and integrating the resulting with respect to θ and z over their corresponding intervals gives

$$\frac{1}{h} \int_{-h}^0 \int_0^{2\pi} \frac{\partial \phi_D^{(3)}}{\partial r} \Big|_{r=a} \cos m'\theta Z_{l'}(z) d\theta dz = \frac{1}{h} \int_{-h}^{-d} \int_{-\beta}^{\beta} U_D^{(3)} \cos m'\theta Z_{l'}(z) d\theta dz. \quad (4.38)$$

After using the expressions for $\phi_D^{(3)}$ and $U_D^{(3)}$ (i.e., Eqns. (4.8) and (4.12) in Eqn.(4.38)), we obtain

$$E_{ml} = \frac{\varepsilon_m}{2\pi} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \xi_{np} X_{nm}^{(1)} V_{pl}^{(2)}. \quad (4.39)$$

Similarly, multiplying Eqn. (4.20) by $\sin m'\theta Z_{l'}(z)/h$ and integrating the resulting equation with respect to θ and z over their corresponding intervals gives

$$\frac{1}{h} \int_{-h}^0 \int_0^{2\pi} \frac{\partial \phi_D^{(3)}}{\partial r} \Big|_{r=a} \sin m'\theta Z_{l'}(z) d\theta dz = \frac{1}{h} \int_{-h}^{-d} \int_{-\beta}^{\beta} U_D^{(3)} \sin m'\theta Z_{l'}(z) d\theta dz. \quad (4.40)$$

After using the expressions for $\phi_D^{(3)}$ and $U_D^{(3)}$ (i.e., Eqns. (4.8) and (4.12) in Eqn.(4.40)), we obtain

$$F_{ml} = \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \xi_{np} Y_{nm}^{(1)} V_{pl}^{(2)}. \quad (4.41)$$

Finally, using the expressions for $\phi_D^{(2)}$ and $\phi_D^{(3)}$ (i.e., Eqns. (4.7) and (4.8)) in Eqn. (4.21), multiplying the resulting equation by $\cos m'\nu(\theta - \beta) Z_{l'}(z)/h$

and integrating with respect to θ and z over their corresponding intervals gives

$$\begin{aligned} & \frac{1}{h} \int_{-h}^0 \int_{\beta}^{2\pi-\beta} \frac{\partial \phi_D^{(4)}}{\partial r} \Big|_{r=c} \cos m' \nu (\theta - \beta) Z_{\nu'}(z) d\theta dz \\ &= \frac{1}{h} \int_{-h}^0 \int_{\beta+\gamma}^{2\pi-\beta-\gamma} U_D^{(2)} \cos m' \nu (\theta - \beta) Z_{\nu'}(z) d\theta dz. \end{aligned} \quad (4.42)$$

After using the expressions for $\phi_D^{(4)}$ and $U_D^{(2)}$ (i.e., Eqns. (4.9) and (4.11)) in Eqn. (4.42), we obtain

$$G_{ml} = \frac{\varepsilon_m}{2\pi - 2\beta} Q_{ml}^{(2)} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sigma_{np} L_{nm}^{(2)} V_{pl}^{(1)}, \quad (4.43)$$

where $L_{nm}^{(2)}$ and $Q_{ml}^{(2)}$ are given by

$$L_{nm}^{(2)} = \int_{\beta+\gamma}^{2\pi-\beta-\gamma} f_n^{(2)}(\theta) \cos m \nu (\theta - \beta) d\theta, \quad (4.44)$$

$$Q_{ml}^{(2)} = 1 / \left(1 - \frac{I'_{m\nu}(k_l a)}{I'_{m\nu}(k_l c)} \frac{K'_{m\nu}(k_l c)}{K'_{m\nu}(k_l a)} \right). \quad (4.45)$$

The analytical expression for $L_{nm}^{(2)}$ is given in Appendix A.1.2.1.

All the unknown coefficients in the diffraction potentials (i.e., A_{ml} through G_{ml}) have now been expressed in terms of Λ_{np} , σ_{np} and ξ_{np} .

4.2.4 Determinations of the auxiliary functions $U_D^{(1)}$, $U_D^{(2)}$ and $U_D^{(3)}$

On all common boundaries, in addition to continuity of the radial velocities, the pressures must be continuous as well, i.e.,

$$\phi_D^{(1)}|_{r=c} = \phi_D^{(2)}|_{r=c}, \quad -\beta < \theta < \beta, -h < z < 0, \quad (4.46)$$

$$\phi_D^{(1)}|_{r=c} = \phi_D^{(4)}|_{r=c}, \quad \beta + \gamma < \theta < 2\pi - \beta - \gamma, -h < z < 0, \quad (4.47)$$

and

$$\phi_D^{(2)}|_{r=a} = \phi_D^{(3)}|_{r=a}, \quad -\beta < \theta < \beta, -h < z < -d. \quad (4.48)$$

After using the formal solutions of $\phi_D^{(1)}$ and $\phi_D^{(2)}$ (i.e., Eqns. (4.6) and (4.7)) in Eqn. (4.46), multiplying the resulting equation by $f_{n'}^{(1)}(\theta) \tilde{T}_{\nu'}(z)/h$ and integrating with respect to θ and z over their corresponding intervals and invoking

Eqns. (4.24), (4.26), (4.29) and (4.35), we obtain

$$\begin{aligned}
V_{p'0}^{(1)} \sum_{m=0}^{\infty} \mathcal{R}_m \left(\mathcal{E}_m X_{n'm}^{(1)} + \mathcal{F}_m Y_{n'm}^{(1)} \right) \\
= - \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{K_m(k_l c)}{k_l K'_m(k_l c)} \left(A_{ml} X_{n'm}^{(1)} + B_{ml} Y_{n'm}^{(1)} \right) V_{p'l}^{(1)} \\
+ \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \left[C_{ml} \frac{I_{m\mu}(k_l c)}{k_l I'_{m\mu}(k_l c)} + D_{ml} \frac{K_{m\mu}(k_l c)}{k_l K'_{m\mu}(k_l a)} \right] L_{n'm}^{(1)} V_{p'l}^{(1)},
\end{aligned} \tag{4.49}$$

where

$$\mathcal{R}_m = J_m(kc) - \frac{J'_m(kc)}{H'_m(kc)} H_m(kc) = \frac{1}{H'_m(kc)} \frac{2i}{\pi kc}. \tag{4.50}$$

Substituting the expressions of A_{ml} , B_{ml} , C_{ml} and D_{ml} (e.g. Eqns. (4.23), (4.28), (4.33) and (4.34)) into Eqn. (4.50) gives

$$\begin{aligned}
V_{p'0}^{(1)} \sum_{m=0}^{\infty} \mathcal{R}_m \left(\mathcal{E}_m X_{n'm}^{(1)} + \mathcal{F}_m Y_{n'm}^{(1)} \right) \\
= - \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{\pi} \frac{K_m(k_l c)}{k_l K'_m(k_l c)} \left(\frac{\varepsilon_m}{2} X_{n'm}^{(1)} X_{nm}^{(1)} + Y_{n'm}^{(1)} Y_{nm}^{(1)} \right) V_{p'l}^{(1)} V_{pl}^{(1)} \\
+ \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\beta} Q_{ml}^{(1)} R_{ml}^{(1)} L_{n'm}^{(1)} L_{nm}^{(1)} V_{p'l}^{(1)} V_{pl}^{(1)} \\
- \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sigma_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{\pi} \frac{K_m(k_l c)}{k_l K'_m(k_l c)} \left(\frac{\varepsilon_m}{2} X_{n'm}^{(1)} X_{nm}^{(2)} + Y_{n'm}^{(1)} Y_{nm}^{(2)} \right) V_{p'l}^{(1)} V_{pl}^{(1)} \\
+ \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \xi_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\beta} Q_{ml}^{(1)} R_{ml}^{(2)} L_{n'm}^{(1)} L_{nm}^{(1)} V_{p'l}^{(1)} V_{pl}^{(2)},
\end{aligned} \tag{4.51}$$

where

$$R_{ml}^{(1)} = \frac{I_{m\mu}(k_l c)}{k_l I'_{m\mu}(k_l c)} - \frac{I'_{m\mu}(k_l a)}{I'_{m\mu}(k_l c)} \frac{K_{m\mu}(k_l c)}{k_l K'_{m\mu}(k_l a)}, \tag{4.52}$$

and

$$R_{ml}^{(2)} = \frac{K_{m\mu}(k_l c)}{k_l K'_{m\mu}(k_l a)} - \frac{I_{m\mu}(k_l c)}{k_l I'_{m\mu}(k_l c)} \frac{K_{m\mu}'(k_l c)}{K_{m\mu}'(k_l a)}. \tag{4.53}$$

Similarly, using the formal solutions of $\phi_D^{(1)}$ and $\phi_D^{(4)}$ (i.e., Eqns. (4.6) and (4.9)) in Eqn. (4.47), multiplying the resulting equation by $f_{n'}^{(2)}(\theta) \tilde{T}_{p'}(z)/h$ and integrating with respect to θ and z over their corresponding intervals, and in-

voking Eqns. (4.25), (4.26) and (4.30) gives

$$\begin{aligned}
 & V_{p'0}^{(1)} \sum_{m=0}^{\infty} \mathcal{R}_m \left(\mathcal{E}_m X_{n'm}^{(2)} + \mathcal{F}_m Y_{n'm}^{(2)} \right) \\
 &= - \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{K_m(k_l c)}{k_l K'_m(k_l c)} \left(A_{ml} X_{n'm}^{(2)} + B_{ml} Y_{n'm}^{(2)} \right) V_{p'l}^{(1)} \\
 &+ \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} G_{ml} \left(\frac{I_{m\nu}(k_l c)}{k_l I'_{m\nu}(k_l c)} - \frac{I'_{m\nu}(k_l a)}{I'_{m\nu}(k_l c)} \frac{K_{m\nu}(k_l c)}{k_l K'_{m\nu}(k_l a)} \right) L_{n'm}^{(2)} V_{p'l}^{(1)}.
 \end{aligned} \tag{4.54}$$

Inserting the expressions for A_{ml} , B_{ml} and G_{ml} (i.e., Eqns. (4.23), (4.28) and (4.43)) into Eqn. (4.54) and re-arranging gives

$$\begin{aligned}
 & V_{p'0}^{(1)} \sum_{m=0}^{\infty} \mathcal{R}_m \left(\mathcal{E}_m X_{n'm}^{(2)} + \mathcal{F}_m Y_{n'm}^{(2)} \right) \\
 &= - \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{\pi} \frac{K_m(k_l c)}{k_l K'_m(k_l c)} \left(\frac{\varepsilon_m}{2} X_{n'm}^{(2)} X_{nm}^{(1)} + Y_{n'm}^{(2)} Y_{nm}^{(1)} \right) V_{p'l}^{(1)} V_{pl}^{(1)} \\
 &- \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sigma_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{\pi} \frac{K_m(k_l c)}{k_l K'_m(k_l c)} \left(\frac{\varepsilon_m}{2} X_{n'm}^{(2)} X_{nm}^{(2)} + Y_{n'm}^{(2)} Y_{nm}^{(2)} \right) V_{p'l}^{(1)} V_{pl}^{(1)} \\
 &+ \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sigma_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\pi - 2\beta} Q_{ml}^{(2)} R_{ml}^{(3)} L_{n'm}^{(2)} L_{nm}^{(2)} V_{p'l}^{(1)} V_{pl}^{(1)},
 \end{aligned} \tag{4.55}$$

where

$$R_{ml}^{(3)} = \frac{I_{m\nu}(k_l c)}{k_l I'_{m\nu}(k_l c)} - \frac{I'_{m\nu}(k_l a)}{I'_{m\nu}(k_l c)} \frac{K_{m\nu}(k_l c)}{k_l K'_{m\nu}(k_l a)}. \tag{4.56}$$

Finally, after using the expressions for $\phi_D^{(2)}$ and $\phi_D^{(3)}$ (i.e., Eqns. (4.7) and (4.8)) in Eqn. (4.48), multiplying the resulting equation by $f_{n'}^{(1)}(\theta) u_{p'}(z)/h$ and integrating with respect to θ and z over their corresponding intervals, and invoking Eqns. (4.24), (4.29), (4.35) and (4.36), we obtain

$$\begin{aligned}
 & \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \left[C_{ml} \frac{I_{m\mu}(k_l a)}{k_l I'_{m\mu}(k_l c)} + D_{ml} \frac{K_{m\mu}(k_l a)}{k_l K'_{m\mu}(k_l a)} \right] L_{n'm}^{(1)} V_{p'l}^{(2)} \\
 &= \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{I_m(k_l a)}{k_l I'_m(k_l a)} \left(E_{ml} X_{n'm}^{(1)} + F_{ml} Y_{n'm}^{(1)} \right) V_{p'l}^{(2)}.
 \end{aligned} \tag{4.57}$$

Inserting the expressions of C_{ml} , D_{ml} and G_{ml} (e.g. Eqns. (4.33), (4.34) and

(4.43)) into Eqn. (4.57) and re-arranging gives

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\beta} Q_{ml}^{(1)} R_{ml}^{(4)} L_{n'm}^{(1)} L_{nm}^{(1)} V_{p'l}^{(2)} V_{pl}^{(1)} \\
 & + \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \xi_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\beta} Q_{ml}^{(1)} R_{ml}^{(5)} L_{n'm}^{(1)} L_{nm}^{(1)} V_{p'l}^{(2)} V_{pl}^{(2)} \\
 & - \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \xi_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{\pi} \frac{I_m(k_l a)}{k_l I_m'(k_l a)} \left(\frac{\varepsilon_m}{2} X_{n'm}^{(1)} X_{nm}^{(1)} + Y_{n'm}^{(1)} Y_{nm}^{(1)} \right) V_{p'l}^{(2)} V_{pl}^{(2)} \\
 & = 0,
 \end{aligned} \tag{4.58}$$

where

$$R_{ml}^{(4)} = \frac{I_{m\mu}(k_l a)}{k_l I_{m\mu}'(k_l c)} - \frac{I_{m\mu}'(k_l a)}{I_{m\mu}'(k_l c)} \frac{K_{m\mu}(k_l a)}{k_l K_{m\mu}'(k_l a)}, \tag{4.59}$$

and

$$R_{ml}^{(5)} = \frac{K_{m\mu}(k_l a)}{k_l K_{m\mu}'(k_l a)} - \frac{I_{m\mu}(k_l a)}{k_l I_{m\mu}'(k_l c)} \frac{K_{m\mu}'(k_l c)}{K_{m\mu}'(k_l a)}. \tag{4.60}$$

The infinite summations $\sum_{n=0}^{\infty}$ and $\sum_{p=0}^{\infty}$ are truncated to sums of finite terms for later numerical computation, i.e., $\sum_{n=0}^N$ and $\sum_{p=0}^P$. Therefore, Eqns. (4.51), (4.55) and (4.58) have three sets of unknowns (e.g. Λ_{np} , σ_{np} and ξ_{np}), which include $3 \times (N+1) \times (P+1)$ unknowns to be solved by the equal number of algebraic equations. After solving the algebraic equations for Λ_{np} , σ_{np} and ξ_{np} , the unknown coefficients in diffraction potentials (A_{ml} through G_{ml}) can be calculated by using Eqns. (4.23), (4.28), (4.33), (4.34), (4.39), (4.41) and (4.43).

4.2.5 Expression for the diffraction factor $\tilde{\Gamma}$

Now let us derive the expression for $\tilde{\Gamma}$, which is needed in the determination of the dimensionless capture length. According to Eqn. (3.2), the diffraction volume flux q_D is calculated by

$$q_D = \int_0^a \int_0^{2\pi} \frac{\partial \phi_D^{(3)}}{\partial z} \Big|_{z=0} r dr d\theta. \tag{4.61}$$

After substituting the expression for $\phi_D^{(3)}$ (i.e., (4.8)) in Eqn. (4.2.5) and invoking the following identity

$$\int_0^a \frac{I_0(k_l r)}{k_l a I_0'(k_l a)} r dr = \frac{1}{k_l^2}, \tag{4.62}$$

we can obtain, from the definition of $\tilde{\Gamma}$ given in Eqn. (3.10), the following expression for $\tilde{\Gamma}$

$$\tilde{\Gamma} = \frac{-2\pi i}{\omega} \frac{\sqrt{g/h}}{h} \sum_{l=0}^{\infty} E_{0l} Z_l'(0) \frac{a}{k_l^2}. \quad (4.63)$$

4.3 Wave radiation problem

We now consider the wave radiation problem in the absence of incident waves. The fluctuation of the air pressure in the OWC chamber $P_a(t)$ can radiate waves outwardly. The complex amplitude of the radiated velocity potential, ϕ_R , satisfies the governing equation and boundary conditions given by Eqns. (2.10)-(2.13).

4.3.1 Formal expressions for the complex amplitudes of the radiation potentials in different sub-domains

Similar to the wave diffraction problem, it is convenient to separately study the radiation potentials in four sub-domains, as shown in Figure 4.1(b). The complex amplitudes in Regions (1), (2) and (3), respectively denoted by $\phi_R^{(1)}$, $\phi_R^{(2)}$ and $\phi_R^{(3)}$, are symmetric about $\theta = 0$ and can be formally written as

$$\phi_R^{(1)} = \frac{-ip_a}{\rho\omega} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \tilde{A}_{ml} \frac{K_m(k_l r)}{k_l K_m'(k_l c)} \cos m\theta Z_l(z), \quad (4.64)$$

$$\phi_R^{(2)} = \frac{-ip_a}{\rho\omega} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \left(\tilde{B}_{ml} \frac{I_{2m}(k_l r)}{k_l I_{2m}'(k_l c)} + \tilde{C}_{ml} \frac{K_{2m}(k_l r)}{k_l K_{2m}'(k_l a)} \right) \cos 2m\mu\theta Z_l(z), \quad (4.65)$$

$$\phi_R^{(3)} = \frac{-ip_a}{\rho\omega} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \tilde{D}_{ml} \frac{I_m(k_l r)}{k_l I_m'(k_l a)} \cos m\theta Z_l(z) + \frac{-ip_a}{\rho\omega}, \quad (4.66)$$

where $\mu = \pi/(2\beta)$. The unknown coefficients $\tilde{A}_{ml}$, $\tilde{B}_{ml}$, $\tilde{C}_{ml}$ and $\tilde{D}_{ml}$ are to be determined by matching conditions later. We remark that the above expression for $\phi_R^{(2)}$ satisfies the rigid boundary conditions at $\theta = -\beta$ and β . The last term in Eqn. (4.66) is to satisfy the inhomogeneous free surface condition (i.e., Eqn. (2.11)).

In Region (4), the complex amplitude $\phi_R^{(4)}$ is symmetric about $\theta = \pi$ and can be written as

$$\begin{aligned} \phi_R^{(4)} = & \frac{-ip_a}{\rho\omega} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \tilde{E}_{ml} \left(\frac{I_{2m}(k_l r)}{k_l I_{2m}'(k_l c)} - \frac{I_{2m}'(k_l a)}{I_{2m}'(k_l c)} \frac{K_{2m}(k_l r)}{k_l K_{2m}'(k_l a)} \right) \\ & \cdot \cos 2m\nu(\theta - \beta) Z_l(z), \end{aligned} \quad (4.67)$$

where $\tilde{E}_{ml}$ are unknown coefficients. The above expression for $\phi_R^{(4)}$ satisfies the rigid boundary conditions at $r = a$, $\theta = \beta$ and $2\pi - \beta$.

4.3.2 Treatment of singularities at sharp corners

Similar to the treatment of the singularities for the wave diffraction problem, let us introduce the following three auxiliary functions to represent the radial velocities on the three common boundaries:

- (i) $U_R^{(1)}(\theta, z)$ for the boundary defined by $r = c$ and $-\beta < \theta < \beta$,
- (ii) $U_R^{(2)}(\theta, z)$ for the boundary defined by $r = c$ and $\beta + \gamma < \theta < 2\pi - \beta - \gamma$,
and
- (iii) $U_R^{(3)}(\theta, z)$ for the boundary defined by $r = a$ and $-\beta < \theta < \beta$.

To incorporate the corresponding singularities in the expressions for these three auxiliary functions, $U_R^{(1)}(\theta, z)$, $U_R^{(2)}(\theta, z)$ and $U_R^{(3)}(\theta, z)$ are formally written as

$$U_R^{(1)}(\theta, z) = \frac{-ip_a}{\rho\omega} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} \tilde{f}_n^{(1)}(\theta) \tilde{T}_p(z), \quad -\beta < \theta < \beta, -h < z < 0, \quad (4.68)$$

$$U_R^{(2)}(\theta, z) = \frac{-ip_a}{\rho\omega} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sigma_{np} \tilde{f}_n^{(2)}(\theta) \tilde{T}_p(z), \quad \beta + \gamma < \theta < 2\pi - \beta - \gamma, -h < z < 0, \quad (4.69)$$

and

$$U_R^{(3)}(\theta, z) = \frac{-ip_a}{\rho\omega} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \xi_{np} \tilde{f}_n^{(1)}(\theta) u_p(z), \quad -\beta < z < \beta, -h < z < -d, \quad (4.70)$$

with $\tilde{f}_n^{(1)}(\theta)$ and $\tilde{f}_n^{(2)}(\theta)$ being

$$\tilde{f}_n^{(1)}(\theta) = \frac{(2n)! \Gamma(1/6)}{2\pi \Gamma(2n + 1/3) [\beta(\beta - \theta)(\beta + \theta)]^{1/3}} C_{2n}^{1/6} \left(\frac{\theta}{\beta} \right), \quad (4.71)$$

$$\tilde{f}_n^{(2)}(\theta) = \frac{1}{\pi [(\theta - \beta - \gamma)(2\pi - \beta - \gamma - \theta)]^{1/2}} T_{2n} \left(\frac{\theta - \pi}{\pi - \beta - \gamma} \right). \quad (4.72)$$

where $C_{2n}^{1/6}$ ($n = 0, 1, 2, \dots$) are Gegenbauer polynomials of order $2n$ and T_{2n} ($n = 0, 1, 2, \dots$) are Chebyshev polynomials of order $2n$. Due to the symmetric properties of the radial velocities, we use Gegenbauer and Chebyshev polynomials of even orders to expand the unknown radial functions, which are different from those used in Eqns. (4.13) and (4.14) for the wave diffraction problem. We stress here that the singularities at the tips of the structure have been included in the above expansions.

4.3.3 The coefficients in the radiation potentials: $\tilde{A}_{ml}$ through $\tilde{E}_{ml}$

Continuity of the radial velocities on the common boundaries among the four sub-domains requires:

$$\frac{\partial \phi_R^{(1)}}{\partial r} \Big|_{r=c} = \begin{cases} U_R^{(1)}, & -\beta < \theta < \beta, -h < z < 0, \\ U_R^{(2)}, & \beta + \gamma < \theta < 2\pi - \beta - \gamma, -h < z < 0, \\ 0, & \beta < \theta < \beta + \gamma, -h < z < 0, \\ 0, & 2\pi - \beta - \gamma < \theta < 2\pi - \beta, -h < z < 0, \end{cases} \quad (4.73)$$

$$\frac{\partial \phi_R^{(2)}}{\partial r} \Big|_{r=c} = U_R^{(1)}, \quad -\beta < \theta < \beta, -h < z < 0, \quad (4.74)$$

$$\frac{\partial \phi_R^{(2)}}{\partial r} \Big|_{r=a} = \begin{cases} 0, & -\beta < \theta < \beta, -d < z < 0, \\ U_R^{(3)}, & -\beta < \theta < \beta, -h < z < -d, \end{cases} \quad (4.75)$$

$$\frac{\partial \phi_R^{(3)}}{\partial r} \Big|_{r=a} = \begin{cases} 0, & 0 \leq \theta < 2\pi, -d < z < 0, \\ 0, & \beta < \theta < 2\pi - \beta, -h < z < -d, \\ U_R^{(3)}, & -\beta < \theta < \beta, -h < z < -d, \end{cases} \quad (4.76)$$

and

$$\frac{\partial \phi_R^{(4)}}{\partial r} \Big|_{r=c} = \begin{cases} 0, & \beta < \theta < \beta + \gamma, -h < z < 0, \\ 0, & 2\pi - \beta - \gamma < \theta < 2\pi - \beta, -h < z < 0, \\ U_R^{(2)}, & \beta + \gamma < \theta < 2\pi - \beta - \gamma, -h < z < 0. \end{cases} \quad (4.77)$$

In the following, we shall determine the coefficients in radiation potentials ($\tilde{A}_{ml}$ through $\tilde{E}_{ml}$) by using the fact that both $\cos m\theta$ and $Z_l(z)$ are complete sets of orthogonal functions.

Multiplying Eqn. (4.73) by $\cos m'\theta Z_{l'}(z)/h$ and integrating the resulting equation with respect to θ and z over their the corresponding intervals gives

$$\begin{aligned} & \frac{1}{h} \int_{-h}^0 \int_0^{2\pi} \frac{\partial \phi_R^{(1)}}{\partial r} \Big|_{r=c} \cos m'\theta Z_{l'}(z) d\theta dz \\ &= \frac{1}{h} \int_{-h}^0 \int_{-\beta}^{\beta} U_R^{(1)} \cos m'\theta Z_{l'}(z) d\theta dz + \frac{1}{h} \int_{-h}^0 \int_{\beta+\gamma}^{2\pi-\beta-\gamma} U_R^{(2)} \cos m'\theta Z_{l'}(z) d\theta dz. \end{aligned} \quad (4.78)$$

After using the expressions for $\phi_R^{(1)}$, $U_R^{(1)}$ and $U_R^{(2)}$ (i.e., Eqns. (4.64), (4.68) and (4.69)) in Eqn. (4.78), we obtain

$$\tilde{A}_{ml} = \frac{\varepsilon_m}{2\pi} \left(\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} \tilde{X}_{nm}^{(1)} V_{pl}^{(1)} + \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sigma_{np} \tilde{X}_{nm}^{(2)} V_{pl}^{(1)} \right), \quad (4.79)$$

where $\tilde{X}_{nm}^{(1)}$ and $\tilde{X}_{nm}^{(2)}$ are given by

$$\tilde{X}_{nm}^{(1)} = \int_{-\beta}^{\beta} \tilde{f}_n^{(1)}(\theta) \cos m\theta d\theta, \quad (4.80)$$

$$\tilde{X}_{nm}^{(2)} = \int_{\beta+\gamma}^{2\pi-\beta-\gamma} \tilde{f}_n^{(2)}(\theta) \cos m\theta d\theta, \quad (4.81)$$

and the expression of $V_{pl}^{(1)}$ has been presented in Eqn. (4.26). The analytical expressions for $\tilde{X}_{nm}^{(1)}$ and $\tilde{X}_{nm}^{(2)}$ are given in Appendix A.1.2.2.

Similarly, multiplying Eqns. (4.74) and (4.75) by $\cos(2m'\mu\theta)Z_{l'}(z)/h$ and integrating the resulting equations with respect to θ and z over their corresponding intervals gives

$$\begin{aligned} & \frac{1}{h} \int_{-h}^0 \int_{-\beta}^{\beta} \frac{\partial \phi_R^{(2)}}{\partial r} \Big|_{r=c} \cos(2m'\mu\theta) Z_{l'}(z) d\theta dz \\ &= \frac{1}{h} \int_{-h}^0 \int_{-\beta}^{\beta} U_R^{(1)} \cos(2m'\mu\theta) Z_{l'}(z) d\theta dz, \end{aligned} \quad (4.82)$$

and

$$\begin{aligned} & \frac{1}{h} \int_{-h}^0 \int_{-\beta}^{\beta} \frac{\partial \phi_R^{(2)}}{\partial r} \Big|_{r=a} \cos(2m'\mu\theta) Z_{l'}(z) d\theta dz \\ &= \frac{1}{h} \int_{-h}^{-d} \int_{-\beta}^{\beta} U_R^{(3)} \cos(2m'\mu\theta) Z_{l'}(z) d\theta dz, \end{aligned} \quad (4.83)$$

After using the expressions for $\phi_R^{(2)}$, $U_R^{(1)}$ and $U_R^{(3)}$ (i.e., Eqns. (4.65), (4.68) and (4.70)) in Eqns. (4.82) and (4.83), we obtain

$$\tilde{B}_{ml} = \frac{\varepsilon_m}{2\beta} \tilde{Q}_{ml}^{(1)} \left(\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} W_{nm}^{(1)} V_{pl}^{(1)} - \frac{K_{2m}'(k_l c)}{K_{2m}'(k_l a)} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \xi_{np} W_{nm}^{(1)} V_{pl}^{(2)} \right), \quad (4.84)$$

and

$$\tilde{C}_{ml} = \frac{\varepsilon_m}{2\beta} \tilde{Q}_{ml}^{(1)} \left(\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \xi_{np} W_{nm}^{(1)} V_{pl}^{(2)} - \frac{I_{2m}'(k_l a)}{I_{2m}'(k_l c)} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} W_{nm}^{(1)} V_{pl}^{(1)} \right), \quad (4.85)$$

where $W_{nm}^{(1)}$ and $\tilde{Q}_{ml}^{(1)}$ are given by

$$W_{nm}^{(1)} = \int_{-\beta}^{\beta} \tilde{f}_n^{(1)}(\theta) \cos(2m\mu\theta) d\theta, \quad (4.86)$$

and

$$\tilde{Q}_{ml}^{(1)} = 1 / \left(1 - \frac{I_{2m}'(k_l a)}{I_{2m}'(k_l c)} \frac{K_{2m}'(k_l c)}{K_{2m}'(k_l a)} \right). \quad (4.87)$$

The analytical expression for $W_{nm}^{(1)}$ is given in Appendix A.1.2.2.

Multiplying Eqn. (4.76) by $\cos m'\theta Z_{l'}(z)/h$ and integrating the resulting equation with respect to θ and z over their corresponding intervals gives

$$\frac{1}{h} \int_{-h}^0 \int_0^{2\pi} \frac{\partial \phi_R^{(3)}}{\partial r} \Big|_{r=a} \cos m'\theta Z_{l'}(z) d\theta dz = \frac{1}{h} \int_{-h}^{-d} \int_{-\beta}^{\beta} U_R^{(3)} \cos m'\theta Z_{l'}(z) d\theta dz. \quad (4.88)$$

After using Eqns. (4.66) and (4.70) in Eqn.(4.88), we obtain

$$\tilde{D}_{ml} = \frac{\varepsilon_m}{2\pi} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \xi_{np} \tilde{X}_{nm}^{(1)} V_{pl}^{(2)}. \quad (4.89)$$

Finally, multiplying Eqn. (4.77) by $\cos 2m'\nu(\theta - \beta) Z_{l'}(z)/h$ and integrating the resulting equation with respect to θ and z over their corresponding intervals gives

$$\begin{aligned} & \frac{1}{h} \int_{-h}^0 \int_{\beta}^{2\pi-\beta} \frac{\partial \phi_R^{(4)}}{\partial r} \Big|_{r=c} \cos 2m'\nu(\theta - \beta) Z_{l'}(z) d\theta dz \\ &= \frac{1}{h} \int_{-h}^0 \int_{\beta+\gamma}^{2\pi-\beta-\gamma} U_R^{(2)} \cos 2m'\nu(\theta - \beta) Z_{l'}(z) d\theta dz. \end{aligned} \quad (4.90)$$

After using the expressions for $\phi_R^{(4)}$ and $U_R^{(2)}$ (i.e., Eqns. (4.67) and (4.69)) in Eqn.(4.90), we obtain

$$\tilde{E}_{ml} = \frac{\varepsilon_m}{2\pi - 2\beta} \tilde{Q}_{ml}^{(1)} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sigma_{np} W_{nm}^{(2)} V_{pl}^{(1)}, \quad (4.91)$$

where $W_{nm}^{(2)}$ is given by

$$W_{nm}^{(2)} = \int_{\beta+\gamma}^{2\pi-\beta-\gamma} \tilde{f}_n^{(2)}(\theta) \cos 2m\nu(\theta - \beta) d\theta. \quad (4.92)$$

The analytical expression for $W_{nm}^{(2)}$ is given in Appendix A.1.2.2.

We have now expressed the unknown coefficients in radiation potentials ($\tilde{A}_{ml}$ through $\tilde{E}_{ml}$) in terms of Λ_{np} , σ_{np} and ξ_{np} .

4.3.4 Determinations of the auxiliary functions $U_R^{(1)}$, $U_R^{(2)}$ and $U_R^{(3)}$

The continuity of pressure across the common boundaries among the four sub-domains requires

$$\phi_R^{(1)} \Big|_{r=c} = \phi_R^{(2)} \Big|_{r=c}, \quad -\beta < \theta < \beta, -h < z < 0, \quad (4.93)$$

$$\phi_R^{(1)}|_{r=c} = \phi_R^{(4)}|_{r=c}, \quad \beta + \gamma < \theta < 2\pi - \beta - \gamma, -h < z < 0, \quad (4.94)$$

and

$$\phi_R^{(2)}|_{r=a} = \phi_R^{(3)}|_{r=a}, \quad -\beta < \theta < \beta, -h < z < -d. \quad (4.95)$$

After substituting the expressions for $\phi_R^{(1)}$ and $\phi_R^{(2)}$ (i.e., Eqns. (4.64) and (4.65)) in Eqn. (4.93), multiplying the resulting equation by $\tilde{f}_{n'}^{(1)}(\theta)\tilde{T}_{p'}(z)/h$ and integrating with respect to θ and z over their corresponding intervals, and invoking Eqns. (4.26), (4.80) and (4.86), we obtain

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \tilde{A}_{ml} \frac{K_m(k_l c)}{k_l K'_m(k_l c)} \tilde{X}_{n'm}^{(1)} V_{p'l}^{(1)} \\ &= \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \left(\tilde{B}_{ml} \frac{I_{2m}(k_l c)}{k_l I_{2m}'(k_l c)} + \tilde{C}_{ml} \frac{K_{2m}(k_l c)}{k_l K_{2m}'(k_l a)} \right) W_{n'm}^{(1)} V_{p'l}^{(1)}. \end{aligned} \quad (4.96)$$

Substituting the expressions for $\tilde{A}_{ml}$, $\tilde{B}_{ml}$ and $\tilde{C}_{ml}$ (i.e., Eqns. (4.79), (4.84) and (4.85)) in Eqn. (4.96) and re-arranging gives

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\pi} \frac{K_m(k_l c)}{k_l K'_m(k_l c)} \tilde{X}_{n'm}^{(1)} \tilde{X}_{nm}^{(1)} V_{p'l}^{(1)} V_{pl}^{(1)} \\ & - \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\beta} \tilde{Q}_{ml}^{(1)} \tilde{R}_{ml}^{(1)} W_{n'm}^{(1)} W_{nm}^{(1)} V_{p'l}^{(1)} V_{pl}^{(1)} \\ & + \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sigma_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\pi} \frac{K_m(k_l c)}{k_l K'_m(k_l c)} \tilde{X}_{n'm}^{(1)} \tilde{X}_{nm}^{(2)} V_{p'l}^{(1)} V_{pl}^{(1)} \\ & - \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \xi_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\beta} \tilde{Q}_{ml}^{(1)} \tilde{R}_{ml}^{(2)} W_{n'm}^{(1)} W_{nm}^{(1)} V_{p'l}^{(1)} V_{pl}^{(2)} \\ & = 0, \end{aligned} \quad (4.97)$$

where $\tilde{R}_{ml}^{(1)}$ and $\tilde{R}_{ml}^{(2)}$ are given by

$$\tilde{R}_{ml}^{(1)} = \frac{I_{2m}(k_l c)}{k_l I_{2m}'(k_l c)} - \frac{I_{2m}'(k_l a)}{k_l I_{2m}'(k_l c)} \frac{K_{2m}(k_l c)}{K_{2m\mu}'(k_l a)}, \quad (4.98)$$

and

$$\tilde{R}_{ml}^{(2)} = \frac{K_{2m}(k_l c)}{k_l K_{2m}'(k_l a)} - \frac{I_{2m}(k_l c)}{k_l I_{2m}'(k_l c)} \frac{K_{2m}'(k_l c)}{K_{2m}'(k_l a)}. \quad (4.99)$$

After substituting the expressions for $\phi_R^{(1)}$ and $\phi_R^{(4)}$ (i.e., Eqns. (4.64) and (4.67)), multiplying Eqn. (4.94) by $\tilde{f}_{n'}^{(2)}(\theta)\tilde{T}_{p'}(z)/h$ and integrating with respect to θ and z over their corresponding intervals, and invoking Eqns. (4.26), (4.81)

and (4.92), we obtain

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \tilde{A}_{ml} \frac{K_m(k_l c)}{k_l K'_m(k_l c)} \tilde{X}_{n'm}^{(2)} V_{p'l}^{(1)} \\ &= \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \tilde{E}_{ml} \left(\frac{I_{2m}(k_l c)}{k_l I_{2m}'(k_l c)} - \frac{I_{2m}'(k_l a)}{I_{2m}'(k_l c)} \frac{K_{2m}(k_l c)}{k_l K_{2m}'(k_l a)} \right) W_{n'm}^{(2)} V_{p'l}^{(1)}. \end{aligned} \quad (4.100)$$

Applying the expressions of $\tilde{A}_{ml}$ and $\tilde{E}_{ml}$ (e.g. Eqns. (4.79) and (4.91)) in Eq. (4.100) and re-arranging gives

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\pi} \frac{K_m(k_l c)}{k_l K'_m(k_l c)} \tilde{X}_{n'm}^{(2)} \tilde{X}_{nm}^{(1)} V_{p'l}^{(1)} V_{pl}^{(1)} \\ &+ \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sigma_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\pi} \frac{K_m(k_l c)}{k_l K'_m(k_l c)} \tilde{X}_{n'm}^{(2)} \tilde{X}_{nm}^{(2)} V_{p'l}^{(1)} V_{pl}^{(1)} \\ &- \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sigma_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\pi - 2\beta} \tilde{Q}_{ml}^{(1)} \tilde{R}_{ml}^{(1)} W_{n'm}^{(2)} W_{nm}^{(2)} V_{p'l}^{(1)} V_{pl}^{(1)} \\ &= 0. \end{aligned} \quad (4.101)$$

Similarly, after substituting the expressions for $\phi_R^{(2)}$ and $\phi_R^{(3)}$ (i.e., (4.65) and (4.66)) in Eqn. (4.95), multiplying the resulting equation by $\tilde{f}_{n'}^{(1)}(\theta) u_{p'}(z)/h$ and integrating with respect to θ and z over their corresponding intervals, and invoking Eqns. (4.36), (4.80) and (4.86), we obtain

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \left(\tilde{B}_{ml} \frac{I_{2m}(k_l a)}{k_l I_{2m}'(k_l c)} + \tilde{C}_{ml} \frac{K_{2m}(k_l a)}{k_l K_{2m}'(k_l a)} \right) W_{n'm}^{(1)} V_{p'l}^{(2)} \\ &= \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \tilde{D}_{ml} \frac{I_m(k_l a)}{k_l I_m'(k_l a)} \tilde{X}_{n'm}^{(1)} V_{p'l}^{(2)} + \tilde{X}_{n'0}^{(1)} \tilde{V}_{p'}, \end{aligned} \quad (4.102)$$

where $\tilde{V}_{p'} = \frac{1}{h} \int_{-h}^{-d} u_{p'}(z) dz$.

Inserting the expressions for $\tilde{B}_{ml}$, $\tilde{C}_{ml}$ and $\tilde{D}_{ml}$ (i.e., Eqns. (4.84), (4.85) and (4.89)) in Eqn. (4.102) and re-arranging gives

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\beta} \tilde{Q}_{ml}^{(1)} \tilde{R}_{ml}^{(3)} W_{n'm}^{(1)} W_{nm}^{(1)} V_{p'l}^{(2)} V_{pl}^{(1)} \\ &+ \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \xi_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\beta} \tilde{Q}_{ml}^{(1)} \tilde{R}_{ml}^{(4)} W_{n'm}^{(1)} W_{nl}^{(1)} V_{p'l}^{(2)} V_{pl}^{(2)} \\ &- \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \xi_{np} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\varepsilon_m}{2\pi} \frac{I_m(k_l a)}{k_l I_m'(k_l a)} \tilde{X}_{n'm}^{(1)} \tilde{X}_{nm}^{(1)} V_{p'l}^{(2)} V_{pl}^{(2)} \\ &= \tilde{X}_{n'0}^{(1)} \tilde{V}_{p'}, \end{aligned} \quad (4.103)$$

where $\tilde{R}_{ml}^{(3)}$ and $\tilde{R}_{ml}^{(4)}$ are given by

$$\tilde{R}_{ml}^{(3)} = \frac{I_{2m}(k_l a)}{k_l I_{2m}'(k_l c)} - \frac{I_{2m}'(k_l a)}{k_l I_{2m}'(k_l c)} \frac{K_{2m}(k_l a)}{K_{2m}'(k_l a)}, \quad (4.104)$$

and

$$\tilde{R}_{ml}^{(4)} = \frac{K_{2m}(k_l a)}{k_l K_{2m}'(k_l a)} - \frac{K_{2m}'(k_l c)}{k_l K_{2m}'(k_l a)} \frac{I_{2m}(k_l a)}{I_{2m}'(k_l c)}. \quad (4.105)$$

Similar to the diffraction problem, there are $3 \times (N + 1) \times (P + 1)$ unknown coefficients in Eqns. (4.97), (4.101) and (4.103): Λ_{np} , σ_{np} and ξ_{np} . After the unknown coefficients are obtained, the auxiliary functions ($U_R^{(1)}$, $U_R^{(2)}$ and $U_R^{(3)}$) and the coefficients $\tilde{A}_{ml}$ - $\tilde{E}_{ml}$ can be calculated.

4.3.5 The radiation conductance and susceptance: $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$

Now we can derive expressions for the radiation conductance coefficient $\tilde{\mathcal{B}}$ and the radiation susceptance coefficients $\tilde{\mathcal{C}}$. Referring to Eqn. (3.2), the radiation volume flux is calculated by

$$q_R = \int_0^a \int_0^{2\pi} \frac{\partial \phi_R^{(3)}}{\partial z} \Big|_{z=0} r dr d\theta. \quad (4.106)$$

After invoking the expression for $\phi_R^{(3)}$ given in Eqn. (4.66) and the expression for q_R in terms of $\mathcal{B}$ and $\mathcal{C}$ given in Eqn. (3.6), and making use of the orthogonal properties of sine and cosine functions as well as the following identity

$$\int_0^a \frac{I_0(k_l r)}{k_l a I_0'(k_l a)} r dr = \frac{1}{k_l^2}, \quad (4.107)$$

the expressions for $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ can be written as

$$\tilde{\mathcal{B}} = \text{Re} \left\{ \frac{2\pi i}{\omega} \frac{\sqrt{g/h}}{h} \sum_{l=0}^{\infty} D_{0l} \frac{a Z_l'(0)}{k_l^2} \right\}, \quad (4.108)$$

and

$$\tilde{\mathcal{C}} = \text{Im} \left\{ \frac{-2\pi i}{\omega} \frac{\sqrt{g/h}}{h} \sum_{l=0}^{\infty} D_{0l} \frac{a Z_l'(0)}{k_l^2} \right\}. \quad (4.109)$$

4.4 Optimum extraction efficiency

Referring to the expression for the dimensionless capture length kL (i.e., Eqn. (3.8)), the parameters χ and κ can be optimized to maximize the extraction efficiency. Physically, there are three strategies to optimize this system: i) change

the turbine characteristics, such as rotational speed; ii) change the system dimensions, such as chamber size; iii) change both. For the last strategy, it is required that

$$\begin{cases} \frac{\partial kL}{\partial \chi} = 0, \\ \frac{\partial kL}{\partial \kappa} = 0, \end{cases} \quad (4.110)$$

which gives the following relations for a given wave condition

$$\kappa = \tilde{\mathcal{C}}, \quad \chi = \tilde{\mathcal{B}}. \quad (4.111)$$

The maximum extraction efficiency under these conditions is

$$kL_{\max} = \frac{kh\sqrt{gh}}{C_g} \frac{|\tilde{\Gamma}(\alpha)|^2}{4\tilde{\mathcal{B}}}. \quad (4.112)$$

which has been derived by Evans (1982) and Lovas *et al.* (2010).

In practice, once an OWC device is built, it is difficult to vary its dimensions (i.e. κ). As did in Martins-Rivas & Mei (2009*b,a*), we will fine-tune the power takeoff system (i.e., the parameter χ) to maximize the extraction efficiency for given dimensions of an OWC converter; this may be achieved in practice by using part of electricity generated to dynamically fine-tune the power takeoff system. Therefore, we differentiate Eqn. (3.8) only with respect to χ to obtain the following condition for the maximum extraction efficiency

$$\chi_{opt} = \sqrt{\tilde{\mathcal{B}}^2 + (\tilde{\mathcal{C}} - \kappa)^2}. \quad (4.113)$$

After using Eqn. (4.113) in Eqn. (3.8), the following optimum capture length is obtained

$$kL_{opt} = \frac{kh\sqrt{gh}}{C_g} \frac{\sqrt{\tilde{\mathcal{B}}^2 + (\tilde{\mathcal{C}} - \kappa)^2} |\tilde{\Gamma}|^2}{\left(\sqrt{\tilde{\mathcal{B}}^2 + (\tilde{\mathcal{C}} - \kappa)^2} + \tilde{\mathcal{B}} \right)^2 + (\tilde{\mathcal{C}} - \kappa)^2}. \quad (4.114)$$

4.5 Results and discussion

Effects of the size and submergence of the OWC chamber, the angle and length of the V-shaped channel, and the direction of incident waves on optimum capture length are examined. For illustration and without loss of generality, unless otherwise specified, we shall take $h = 10$ m, $\rho/\rho_a^0 = 1000$, $c_a = 340$ m/s,

M	$ \tilde{\Gamma} ^2$	$\tilde{\mathcal{B}}$	$\tilde{\mathcal{C}}$
35	1.700386	0.709577	-1.332738
40	1.696113	0.707403	-1.330838
45	1.692307	0.705379	-1.328964
50	1.689485	0.703967	-1.327622
55	1.687845	0.703113	-1.326813

Table 4.1 Convergence tests for $|\tilde{\Gamma}|^2$, $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ against M for $a/h = 1/2$, $d/h = 0.2$, $b/h = 0.01$, $\beta = \pi/4$, $\alpha = 0$, $\mathcal{L} = 400$, $\mathcal{P} = N = 5$, $\omega^2 h/g = 1.55 \text{ rad}^2/\text{m}$, $\gamma = \pi/20 \text{ rad}$.

$g = 9.81 \text{ m/s}^2$ and $\gamma = \pi/20 \text{ rad}$ for all numerical results presented in this section. Our numerical experiments showed that the speed of convergence in numerical computations was slower for smaller values of γ . The value of γ used in this study has also been suggested by Chang *et al.* (2012)).

4.5.1 Convergence test

In addition to the two summations $\sum_{n=0}^{\infty}$ and $\sum_{p=0}^{\infty}$, the following two infinite summations also exist in Eqns. (4.51), (4.55), (4.58), (4.97), (4.101) and (4.103),

$$\sum_{m=0}^{\infty} \quad \text{and} \quad \sum_{l=0}^{\infty}. \quad (4.115)$$

For the sake of numerical computations, these two infinite summations are truncated to two sums of finite terms, say M and $\mathcal{L}$, respectively. We shall first perform convergence tests to determine the suitable values of M , $\mathcal{L}$, N , and $\mathcal{P}$ for our problem.

Following Garrett (1970) and Deng *et al.* (2013), we shall use $\mathcal{L} = 400$ in our computations for both the diffraction and radiation problems to achieve a precision of 3 decimal places. The remaining task is to determine suitable values of M , $\mathcal{P}$ and N .

Table 4.1 shows the convergence tests for $|\tilde{\Gamma}|^2$, $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ against M for $a/h = 1/2$, $d/h = 0.2$, $b/h = 0.01$, $\beta = \pi/4$, $\alpha = 0$, $\mathcal{L} = 400$, $\mathcal{P} = N = 5$, $\omega^2 h/g = 1.55 \text{ rad}^2/\text{m}$. It can be seen that at least two-digit precision can be achieved with $M = 55$. Therefore, $M = 55$ will be taken to solve both the diffraction and radiation problems.

$b/h = 0.01$				$b/h = 1.5$		
$\mathcal{P}$	$ \tilde{\Gamma} ^2$	$\tilde{\mathcal{B}}$	$\tilde{\mathcal{C}}$	$ \tilde{\Gamma} ^2$	$\tilde{\mathcal{B}}$	$\tilde{\mathcal{C}}$
6	1.751591	0.729748	-1.335318	5.233060	0.963401	-0.903220
8	1.772815	0.738779	-1.337433	5.233145	0.963416	-0.903212
10	1.803438	0.751618	-1.341121	5.233261	0.963436	-0.903200
12	1.808071	0.753617	-1.341709	5.233411	0.963462	-0.903186
14	1.819125	0.758247	-1.342867	5.233595	0.963493	-0.903168

Table 4.2 Convergence tests for $|\tilde{\Gamma}|^2$, $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ against $\mathcal{P}$ for $a/h = 1/2$, $d/h = 0.2$, $\beta = \pi/4$, $\alpha = 0$ rad, $\mathcal{L} = 400$, $M = 55$, $N = 5$, $\omega^2 h/g = 1.55$ rad²/m.

$b/h = 0.01$				$b/h = 1.5$		
N	$ \tilde{\Gamma} ^2$	$\tilde{\mathcal{B}}$	$\tilde{\mathcal{C}}$	$ \tilde{\Gamma} ^2$	$\tilde{\mathcal{B}}$	$\tilde{\mathcal{C}}$
4	1.803438	0.751398	-1.340865	5.233261	0.963438	-0.903179
6	1.806288	0.751744	-1.341268	5.034908	0.963466	-0.903236
8	1.807657	0.751831	-1.341400	5.084009	0.963522	-0.903323
10	1.809013	0.752277	-1.341831	5.095752	0.963614	-0.903437
12	1.810415	0.752523	-1.342136	5.092231	0.963759	-0.903590

Table 4.3 Convergence tests for $|\tilde{\Gamma}|^2$, $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ against N for $a/h = 1/2$, $d/h = 0.2$, $\beta = \pi/4$, $\alpha = 0$ rad, $\mathcal{L} = 400$, $M = 55$, $\mathcal{P} = 10$, $\omega^2 h/g = 1.55$ rad²/m.

Tables 4.2 and 4.3 show the convergence tests for $|\tilde{\Gamma}|^2$, $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ against $\mathcal{P}$ and N for $a/h = 1/2$, $d/h = 0.2$, $\beta = \pi/4$, $\alpha = 0$, $\mathcal{L} = 400$, $M = 55$, $\omega^2 h/g = 1.55$ rad²/m and two lengths of arms: $b/h = 0.01$ and $b/h = 1.5$, respectively. Note that the speed of the convergence in θ -direction slows down when the length of arms b/h increases, while that of the convergence in z -direction reverses. It can be seen that a precision of at least 2 decimal places can be achieved when $\mathcal{P} = 14$ and $N = 12$, which is sufficient for this study. Therefore, we shall use $\mathcal{P} = 14$ and $N = 12$ to obtain the numerical results in the diffraction and radiation problems.

4.5.2 Code verification

The numerical code was checked with two limiting cases: the first by Deng *et al.* (2013) and the other by Lovas *et al.* (2010).

By setting the length of the V-shaped channel to a very small value, say, $b/a = 0.05$, the present case for $\beta = 3\pi/8$ is reduced approximately to the limiting case studied by Deng *et al.* (2013) for an opening angle of π . Figure 4.2

Figure 4.2 Comparison of the present results of limiting case with those in Deng *et al.* (2013). In all cases, $a/h = 1/2$, $d/h = 0.2$, $b/a = 0.05$, $\beta = 3\pi/8$, $\alpha = 0$.

shows that the results of the present limiting case agree well with those reported in Deng *et al.* (2013) for $a/h = 1/2$, $d/h = 0.2$, $b/a = 0.05$, $\beta = 3\pi/8$ rad, $\alpha = 0$. A separate code was written using the method employed in this study to re-examine the problem reported in Deng *et al.* (2013) (i.e., $b = 0$ and $\gamma = 0$), and the results are shown in Figure 4.2 as the dashed lines. It is concluded that the minor differences between the results reported in Deng *et al.* (2013) and those for the limiting cases are due to the small perturbation of the non-zero values of b and γ .

By setting the length of the V-shaped channel to a very large value, say, $b/\lambda = 15$ (λ is the wavelength), the present case is reduced approximately to a limiting case where an OWC is installed at a convex corner ($v = 3/2$), which was studied by Lovas *et al.* (2010). Figure 4.3 shows that the present results also

Figure 4.3 Comparison of the present results of limiting case with those in Lovas *et al.* (2010). In all cases, $a/h = 1/2$, $d/h = 0.2$, $b/\lambda = 15$, $\beta = 3\pi/4$, $\alpha = \pi/2$.

agree well with those reported in Lovas *et al.* (2010) for $a/h = 1/2$, $d/h = 0.2$, $b/\lambda = 15$, $\beta = 3\pi/4$, $\alpha = \pi/2$ rad.

4.5.3 Effects of the length of the two arms

Physically, due to the existence of the V-shaped channel, incoming waves can be focused in the field bounded by these two arms. We shall first examine here the effects of the length of the V-shaped channel b/a on the hydrodynamic parameters and the optimum capture length kL for a fixed OWC with a size $a/h = 0.2$, a draft $d/h = 0.2$, a single incidence angle $\alpha = 0$, and two angles of the V-shaped channel: $2\beta = \pi/2, 3\pi/4$.

We take $\beta = \pi/4$ as the first example for the effects of b/h . The effects of b/h on $|\tilde{\Gamma}|^2$ is shown in Figure 4.4(a), and the effects of b/h on $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ are shown in Figure 4.5. For each b/h , $|\tilde{\Gamma}|^2$, $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ all have two resonance peaks within the range of $0 < \omega^2 h/g < 6$. The peaks of both $|\tilde{\Gamma}|^2$ and $\tilde{\mathcal{B}}$ correspond to the natural modes in a closed cylinder Lovas *et al.* (2010). As b/h changes from 0.01 to 1, all the highest peaks shift slightly toward lower frequencies and reduce in magnitude. When b/h changes from 1 to 1.5, the highest peaks of both $|\tilde{\Gamma}|^2$ and $\tilde{\mathcal{B}}$ shift toward lower frequencies but increase in magnitude, while the highest peaks of $\tilde{\mathcal{C}}$ shift toward higher frequencies but increase in magnitude. Figure 4.6 shows the effects of b/h on the optimum turbine parameter χ_{opt} and the corresponding capture length kL_{opt} . As b/h increases, the first peak of kL_{opt} becomes higher and wider, i.e., more wave energy can be captured can be extracted within a wider range of frequency.

Figure 4.4 Diffraction flux factor $|\tilde{\Gamma}|^2$ as a function of $\omega^2 h / g$ for $a/h = 1/2$, $d/h = 0.2$, $\alpha = 0$, and two opening angles: $\beta = \pi/4$ and $3\pi/8$.

Figure 4.5 Radiation conductance coefficient $\tilde{B}$ and radiation susceptance coefficient $\tilde{C}$ as functions of $\omega^2 h / g$ for $a/h = 1/2$, $d/h = 0.2$, $\beta = \pi/4$, $\alpha = 0$.

Figure 4.6 Optimum turbine parameter χ_{opt} and the corresponding capture length kL_{opt} as functions of $\omega^2 h / g$ for $a/h = 1/2$, $d/h = 0.2$, $\beta = \pi/4$, $\alpha = 0$.

Figure 4.7 Optimum turbine parameter χ_{opt} and the corresponding capture length kL_{opt} as functions of $\omega^2 h / g$ for $a/h = 1/2$, $d/h = 0.2$, $\beta = 3\pi/8$, $\alpha = 0$.

As the second example for the effects of b/h , we take $\beta = 3\pi/8$ and keep the other parameters the same as those in Figures 4.4(a) and 4.6. The effects of b/h on the diffraction flux factor $|\tilde{\Gamma}|^2$, the optimum turbine parameter χ_{opt} and the corresponding capture length kL_{opt} are shown in Figures 4.4(b) and 4.7, respectively. As b/h increases from 0.01 to 1, the highest peak of kL_{opt} becomes higher and the bandwidth of frequency over which the OWC can operate at higher efficiencies widens, i.e., more wave energy can be extracted by the wave energy converter. However, as b/h changes from 1 to 1.5, the highest peak becomes lower and the bandwidth of frequency over which the OWC can operate with high efficiency becomes narrower.

Let us take a typical sea environment, i.e., $\rho = 1020 \text{ kg/m}^3$, $T = 7 \text{ s}$ (T is wave period), and $A = 1 \text{ m}$, to evaluate the capacity of a given OWC device ($a = h/2 = 5 \text{ m}$, $d = 0.2 \cdot h = 2 \text{ m}$, $b = 1.5 \cdot h = 15 \text{ m}$, $\beta = \pi/4$). Referring to the dotted line in Figure 4.6, as $\omega^2 h / g = 0.82$ ($\omega = 2\pi/T$), the optimum capture length kL_{opt} is approximately equal to 2.6 if water waves are incident normally. In this sea environment, the wave power extracted by the given OWC device, after using Eqn. (3.8), is about 805.4 kW.

4.5.4 Effects of the opening angle of two arms

Choosing a suitable opening angle for the V-shaped channel is very important in designing this type of WECs. Referring to Figure 4.8(a), we examine the effects of the opening angle 2β on the hydrodynamic parameters and the optimum capture length under the following conditions: $a/h = 1/2$, $d/h = 0.2$, $b/h = 1.5$, and $\alpha = 0$.

Figure 4.10 Optimum turbine parameter χ_{opt} and the resulting capture length kL_{opt} as functions of $\omega^2 h / g$ for $a/h = 1/2$, $d/h = 0.2$, $b/h = 1.5$, $\alpha = 0$.

$b/h = 1.5$. It can be seen that as β increases, the highest peak of kL_{opt} becomes lower and the bandwidth of frequency over which the WEC can operate at relatively high efficiency becomes narrower as well. Theoretically, our results suggests that $\beta = \pi/4$ and $3\pi/8$ may yield a reasonable energy conversion efficiency. In practice, it is costly to construct long V-shaped channels; therefore, a compromise between conversion efficiency and construction cost need to be made when designing such WECs.

4.5.5 Effects of the OWC size

Now, let us examine effects of the OWC size (a/h) on the hydrodynamic parameters and the optimum extraction efficiency under the condition of a constant draft $d/h = 0.2$, a fixed length of the V-shaped channel $b/h = 1.5$, and a single incidence angle $\alpha = 0$. The OWC size parameter κ , calculated by using Eqn. (3.11), is shown in Figure 4.11 for reference.

Let us take $\beta = \pi/4$ as an example. As shown in the right panel of Figure 4.11, the dependence of the diffraction flux factor $|\tilde{\Gamma}|^2$ on $\omega^2 h / g$ is presented for four column radii: $a/h = 1/5, 1/4, 1/3, 1/2$. For the smaller columns of $a/h = 1/5$ and $1/4$, there is only one resonance peak in the computed range of $\omega^2 h / g$. While for the larger columns of $a/h = 1/3$ and $1/2$, multiple peaks exist. These peaks are associated with the natural modes in a hollow cylinder (Garrett, 1970). As the OWC size (a/h) increases, the highest peak shifts toward a lower frequency and increases its magnitude. Note also that the peak value of $|\tilde{\Gamma}|^2$ at the fundamental resonance frequency can be greatly reduced by using a smaller

Figure 4.11 OWC size parameter κ (left) and diffraction flux factor $|\tilde{\Gamma}|^2$ (right) as functions of $\omega^2 h / g$ for $d/h = 0.2$, $b/h = 1.5$, $\beta = \pi/4$, $\alpha = 0$.

column radius (a/h). As d/h increases, the highest peak also shifts toward a lower frequency and increases its magnitude slightly.

Figure 4.12 shows the effects of a/h on radiation conductance $\tilde{\mathcal{B}}$ and radiation susceptance $\tilde{\mathcal{C}}$ against $\omega^2 h / g$. It can be seen that the frequencies at which the peaks of radiation conductance $\tilde{\mathcal{B}}$ appear to coincide with those of diffraction flux factor $|\tilde{\Gamma}|^2$. We remark that only the first peak, corresponding to the resonant heave motion of the oscillating water column, is important for designing an OWC converter. When the frequency of the wave motion is the same as the natural frequency of the oscillating water column, a resonant interaction occurs. The natural frequency decreases with increasing the height of the water column (McCormick, 1981). We also remark that the radiation susceptance $\tilde{\mathcal{C}}$ is negative in a certain range of $\omega^2 h / g$, which is the common feature present in OWC converters (Evans, 1982; Martins-Rivas & Mei, 2009b; Deng *et al.*, 2013). As a/h increases, the highest peaks for both $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ become sharper and shift toward lower frequencies.

Figure 4.13 shows the effects of a/h on the optimum turbine parameter χ_{opt} and the resulting capture length kL_{opt} for $\beta = \pi/4$. The optimum capture length kL_{opt} can only be greater than 4.5 in a frequency range of $1.25 < \omega^2 h / g < 2.75$ for $a/h = 1/2$; however, optimum capture length kL_{opt} can be greater than 4.5 in a much wider frequency range of $2 < \omega^2 h / g < 5.2$ for $a/h = 1/5$, which is a desired feature for spectral waves. For a water depth of 9.8 m, the range of $2 < \omega^2 h / g < 5.2$ corresponding to $2 < \omega^2 < 5.2$ ($2.75 \text{ s} < \text{wave period} < 4.44 \text{ s}$); the range of $1.25 < \omega^2 h / g < 2.75$ corresponding to $1.25 < \omega^2 < 2.75$ (3.79 s

Figure 4.12 Radiation conductance $\tilde{B}$ (left) and radiation susceptance $\tilde{C}$ as functions of $\omega^2 h / g$ for $d/h = 0.2$, $b/h = 1.5$, $\beta = \pi/4$, $\alpha = 0$.

Figure 4.13 Optimum turbine parameter χ_{opt} (left) and the resulting capture length kL_{opt} (right) as functions of $\omega^2 h / g$ for $d/h = 0.2$, $b/h = 1.5$, $\beta = \pi/4$, $\alpha = 0$.

$< \text{wave period} < 5.62 \text{ s}$). Therefore, the diameter of the OWC chamber should be designed according to the dominant period of coastal waves; a larger OWC chamber is needed for longer waves.

As another example of the effects of geometric parameters, we take $\beta = 3\pi/8$ and keep other parameters the same as those in Figure 4.13. As shown in Figure 4.14, the optimum capture length kL_{opt} for this design can be greater than 4.5 in a very narrow frequency range of $2.2 < \omega^2 h / g < 2.5$ for $a/h = 1/5$, but in a narrow frequency range of $1.25 < \omega^2 h / g < 2.1$ for $a/h = 1/2$. Both frequency ranges are narrower than those for $\beta = \pi/4$, because partly the reduced wave reflection by the coaxial tube-sector-shaped structure for a larger β . Therefore, a smaller value of β is suitable only for regular waves or narrow-banded waves.

Figure 4.14 Optimum turbine parameter χ_{opt} (left) and the resulting capture length kL_{opt} (right) as functions of $\omega^2 h / g$ for $d/h = 0.2$, $b/h = 1.5$, $\beta = 3\pi/8$, $\alpha = 0$.

Figure 4.15 Diffraction flux factor $|\tilde{\Gamma}|^2$ as a function of $\omega^2 h / g$ for $\beta = \pi/4$ (left) and $3\pi/8$ (right). In all cases, $a/h = 1/2$, $b/h = 1.5$, $\alpha = 0$.

4.5.6 Effects of the OWC chamber draft

Next, we examine the effects of the draft (d/h) on the hydrodynamic parameters and the optimum extraction efficiency under the condition of a fixed OWC size $a/h = 1/2$, a fixed length of the V-shaped channel $b/h = 1.5$, and a single incident angle $\alpha = 0$. We shall examine two angles between the walls of the V-shaped channel: $2\beta = \pi/2$ and $3\pi/4$.

Figure 4.15 shows the effects of the OWC chamber draft on $|\tilde{\Gamma}|^2$ against $\omega^2 h / g$. It can be seen that resonant peaks become sharper with increasing draft (d/h). For each d/h , there are two resonance peaks and increasing d/h shifts the peak toward a lower frequency.

For $\beta = \pi/4$ (i.e., the angle between the two V-shaped arms is $\pi/2$), the effects of d/h on the radiation conductance coefficient $\tilde{\mathcal{B}}$ and radiation susceptance

Figure 4.16 Radiation conductance coefficient $\tilde{B}$ (left) and radiation susceptance coefficient $\tilde{C}$ (right) as functions of $\omega^2 h / g$ for $a/h = 1/2$, $b/h = 1.5$, $\beta = \pi/4$, $\alpha = 0$.

Figure 4.17 Optimum turbine parameter χ_{opt} (left) and resulting capture length kL_{opt} (right) as functions of $\omega^2 h / g$ for $a/h = 1/2$, $b/h = 1.5$, $\beta = \pi/4$, $\alpha = 0$.

coefficient $\tilde{C}$ are shown in Figure 4.16. Again, resonant interaction between the incident waves and the OWC occurs when the frequency of the wave motion is the same as that of the oscillating water column. For an OWC device with the same submergence, the variation of the resonance frequency of the oscillating water column only depends on its radiation susceptance (McCormick, 1981). Figure 4.17 shows the effects of d/h on the optimum turbine parameter χ_{opt} and the resulting capture length kL_{opt} . Reducing d/h can shift the highest peak to a higher frequency and also widen the frequency range over which the OWC can operate at higher efficiencies.

As another example, we take $\beta = 3\pi/8$ and keep the other parameters the same as those in Figure 4.17. The effects of d/h on the optimum χ_{opt} and the resulting capture length are described in Figure 4.18. It can be seen that the optimum capture length kL_{opt} is larger than 4 within the frequency range of

Figure 4.18 Optimum turbine parameter χ_{opt} (left) and resulting capture length kL_{opt} (right) as functions of $\omega^2 h/g$ for $a/h = 1/2$, $b/h = 1.5$, $\beta = 3\pi/8$, $\alpha = 0$.

$1.2 < \omega^2 h/g < 2.7$ for $d/h = 0.2$, but within the frequency range of $2 < \omega^2 h/g < 2.5$ for $d/h = 0.5$. Therefore, for $0.2 < d/h < 0.5$ with $\beta = 3\pi/8$, the frequency ranges over which the OWC operates at higher efficiencies are all narrower than those for $\beta = \pi/4$.

4.5.7 Effects of wave direction

Because the dominant direction of coastal waves varies seasonally, we now examine the effects of incidence angle α on the diffraction flux factor $|\tilde{\Gamma}|^2$ and the optimum capture length kL_{opt} under a typical condition of a fixed radius of the column $a/h = 1/2$, a fixed draft $d/h = 0.2$, a fixed arm length $b/h = 1.5$, and two angles between the two walls of the V-shaped channel: $2\beta = \pi/2$ and $3\pi/4$.

Figure 4.19 shows the effects of incidence angle α on the variation of optimum capture length kL_{opt} . Based on the symmetry of the structure, only half of the opening angle of two arms needs to be studied. As expected, when the incident waves propagate along the line of symmetry (i.e., $\alpha = 0$, normal incidence), the OWC converter has the optimal performance.

4.5.8 Comments on the effects of air compressibility

For the power take off model ignoring the air compressibility inside the air chamber and following Sarmiento & Falcão (1985) and Falcão & Rodrigues (2002), the mass flux of the air through a linear turbine can be written as

$$\frac{d\mathcal{M}}{dt} = \rho_a^0 Q(t) = \frac{\hat{K}D}{\hat{N}} P_a, \quad (4.116)$$

Figure 4.19 Diffraction flux factor $|\tilde{\Gamma}|^2$ (left) and optimum capture length kL_{opt} (right) as functions of $\omega^2 h / g$ for $a/h = 1/2$, $d/h = 0.2$, $b/h = 1.5$, $\beta = \pi/4$.

Figure 4.20 Diffraction flux factor $|\tilde{\Gamma}|^2$ (left) and optimum capture length kL_{opt} (right) as functions of $\omega^2 h / g$ for $a/h = 1/2$, $d/h = 0.2$, $b/h = 1.5$, $\beta = 3\pi/8$.

where $M(t)$ and $Q(t)$ denote, respectively, the instantaneous mass and volume flux. For the time harmonic motion of angular frequency ω , Eqn. (4.116) can be rewritten as

$$q = \frac{\hat{K}D}{\rho_a^0 \hat{N}} p_a. \quad (4.117)$$

Applying the expression of q in Eqn. (3.4), after using the expressions of q_S and q_R (i.e., Eqns. (3.5) and (3.6)), the complex air pressure inside the chamber can be expressed by

$$p_a = \frac{\Gamma A}{\left(\frac{\hat{K}D}{\rho_a^0 \hat{N}} + (\mathcal{B} - i\mathcal{C}) \right)}. \quad (4.118)$$

Inserting Eqn. (4.118) into Eqn. (3.8), after using the dimensionless quantities in Eqn. (3.10), the normalized capture length $\widehat{kL}$ (excluding air compress-

ibility) can be obtained by

$$\widehat{kL} = \frac{kh\sqrt{gh}}{C_g} \frac{\chi \left| \tilde{\Gamma} \right|^2}{\left(\chi + \tilde{\mathcal{B}} \right)^2 + \tilde{\mathcal{C}}^2}. \quad (4.119)$$

Note that, differing from the capture length in Eqn. (3.8), the turbine parameter χ is only involved in the above equation.

Therefore, there is only one strategy to optimize the OWC device for the optimum extraction efficiency, i.e., fine-tuning turbine parameter. Mathematically, after differentiating the capture length $\widehat{kL}$ in Eqn. (4.119) with respect to χ and setting the resulting equation to zero, the optimum turbine parameter $\hat{\chi}_{opt}$ for the case excluding air compressibility can be obtained by

$$\hat{\chi}_{opt} = \sqrt{\tilde{\mathcal{B}}^2 + \tilde{\mathcal{C}}^2}. \quad (4.120)$$

Inserting the above equation into Eqn. (4.119), the corresponding maximum capture length $\widehat{kL}_{\max}$ can be expressed as

$$\widehat{kL}_{\max} = \frac{kh\sqrt{gh}}{C_g} \frac{\sqrt{\tilde{\mathcal{B}}^2 + \tilde{\mathcal{C}}^2} \left| \tilde{\Gamma} \right|^2}{\left(\sqrt{\tilde{\mathcal{B}}^2 + \tilde{\mathcal{C}}^2} + \tilde{\mathcal{B}} \right)^2 + \tilde{\mathcal{C}}^2}. \quad (4.121)$$

Note that the above expressions of $\hat{\chi}_{opt}$ and $\widehat{kL}_{\max}$ are similar with those derived by Evans & Porter (see Evans & Porter (1997), Equations (24) and (25)). In addition, after setting $\kappa = 0$ (i.e., neglecting air compressibility), the expressions of χ_{opt} and kL_{opt} in Eqns. (3.60) and (3.61), respectively, can be reduced to those in Eqns. (4.120) and (4.121).

Referring to the expression of κ in Eqn. (3.11), the coefficient κ , which represents the air compressibility, depends on the pneumatic chamber height or volume V_0 , and which is fixed for a given OWC device. The effects of air compressibility on the optimum capture length kL_{opt} are shown in Figure 4.21 for $a/h = 1/2$, $d/h = 1/2$, $b/h = 1.5$, $\beta = \pi/4$, $\alpha = 0$ and several values of V_0 , where kL_{opt} and $kL_{\max}$ are calculated by Eqns. (3.61) and (3.59), respectively. The case ignoring the air compressibility (i.e., $\kappa = 0$), which is modeled by setting the chamber volume V_0 to zero, is described in Figure 4.21(a). Recall that the maximum capture length occurs when κ is equal to $\tilde{\mathcal{C}}$, namely, the intersections of the curves κ and $\tilde{\mathcal{C}}$ against $\omega^2 h/g$, which fact has been proved in Figure 4.21.

Figure 4.21 Radiation susceptibility $\tilde{\mathcal{C}}$ (dash dot), κ coefficient (short dashed), optimal capture length kL_{opt} (dashed) and maximum capture length $kL_{\max}$ (solid) as functions of $\omega^2 h / g$ for different chamber volumes. In all cases, $a/h = 1/2$, $d/h = 1/2$, $b/h = 1.5$, $\beta = \pi/4$, $\alpha = 0$.

In addition, as V_0 increases, more interactions can be obtained, which implies that more maximum capture lengths and wider high efficiency bandwidth can be achieved for a given device over the computed range of $\omega^2 h / g$. However, if the chamber volume equals or exceeds a critical value, the number of maximum capture lengths reduces or even vanishes, and the bandwidth of high efficiency becomes narrower. The similar feature can be found in Martins-Rivas & Mei (2009b) and Lovas *et al.* (2010). Note that the spine-shape at high frequency band is due to the occurrence of sloshing mode (Evans & Porter, 1997).

Figure 4.22 Comparisons of the maximum capture lengths kL_{max} in three cases: $b/h = 0, \beta = \pi$; $b/h = 0, \beta = \pi/4$; $b/h = 1.5, \beta = \pi/4$. In all cases, $a/h = 1/4$ (left) or $a/h = 1/2$ (right), $d/h = 1/2$, $b/h = 1.5$, $\alpha = 0$, $\beta = \pi/4$.

4.5.9 Comparison of the maximum capture lengths for three OWC devices

Here, we compare the maximum capture lengths kL_{max} of the present device with those of the other two OWC device: a bottom-mounted OWC device with an opening at the bottom (i.e. $b/h = 0$, compared with the present device), which has been studied by Deng *et al.* (2013); the other is a fully truncated and fixed offshore OWC device (i.e. $a/h = 0$ and $\beta = \pi$, compared with the present device), which has been studied by Evans & Porter (1997). Specially emphasis that all the walls of the above three OWC devices are assumed to be thin. Typical dimensions are chosen, i.e., a fixed draft $d/h = 1/2$, a fixed arms length $b/h = 1.5$ (if available), a fixed angle between the two walls of the V-channel $\beta = \pi/4$ (if available), normal incidence $\alpha = 0$, and two radii: $a/h = 1/4$ and $1/2$.

Figure 4.22 shows that under the same radius of column and draft, the device with a V-shaped channel performs much better than those without. Recall that the dimensionless maximum capture length for a fully truncated circular oscillating water column (OWC) device is unity (i.e. $kL_{max} = 1$) over the entire wave spectrum (Evans, 1982; Martins-Rivas & Mei, 2009b) (apart from the sloshing mode). In view of the capture length, the OWC-type devices with a V-shaped channel is about 6-7 times more efficient than the one studied by Evans & Porter (1997), as found in Martins-Rivas & Mei (2009a) and Lovas *et al.* (2010), and 3-4 times more efficient than the one studied by Deng *et al.* (2013).

4.6 Summary

An analytical theory of the interactions between ocean waves and an OWC converter device with two bottom-mounted plates of the same length was developed based on potential wave theory. We focused on the effects of the two plates on the energy conversion efficiency of the OWC device. The optimization of this OWC conversion system (such as air turbine character) was also discussed. We draw the following conclusions from this study:

- (i) The two vertical plates can improve the energy conversion efficiency and widen the bandwidth over which the OWC device can operate at high efficiency. In view of the capture length, the OWC-type devices with a V-shaped channel is about 6-7 times more efficient than the fully truncated one and 3-4 times more efficient than the one without V-shaped channel.
- (ii) Based on local sea conditions, the length b/h should be chosen in the range of 1-1.5, over which the OWC device can operate efficiently in a wider range of frequencies.
- (iii) The angle between two plates can be chosen in the range of $3\pi/4$ - $\pi/2$ in order to ensure that the OWC converter system performs at high efficiency.

Mathematically, the use of approximate basis functions to capture the singular behaviors both in vertical and azimuthal directions can considerably speed up the convergence of numerical computations.

Chapter 5

A free floating OWC device in heaving motion

The objective of this chapter is to exploit the possibility of deploying a floating OWC wave energy converter (shown in Figure 5.1) in water deeper than those for bottom-sitting OWC systems studied in the previous two chapters. The main concern of this chapter is again the pneumatic extraction efficiency. In particular, it is desired to know how the heave motion of the OWC will affect the extraction efficiency. The structural strength of the system is not the focus of this study.

Figure 5.1 Schematic diagram of a floating oscillating water column wave energy converter in heaving motion.

5.1 Description

A circular OWC device of inner radius a with a tube-shaped pontoon of outer radius c is placed in the water of constant depth h , as shown in Figure 5.1. To mathematically describe the problem, a Cartesian coordinate system $Oxyz$ as well as a cylindrical coordinate system $Or\theta z$ is introduced with the still water surface being Oxy or $Or\theta$ plane. The origin O coincides with the center of the undisturbed water surface inside the chamber. The submerged depth of the OWC device is d . The thickness of the pontoon is b ($b = c - a$).

5.2 Volume flux

Differing from the previous cases, in this chapter, the decomposition of the total velocity potential ϕ is written as

$$\phi = \phi_D + \phi_P + V_3\phi_3, \quad (5.1)$$

where ϕ_D represents the diffraction velocity potential for the OWC fixed in the waves with the top of the column open to the atmosphere, that is, for the pressure inside the column equal to the atmospheric one; ϕ_P is the radiation potential resulting from the oscillating air pressure $P_a(t)$ in the chamber for the OWC device fixed in the undisturbed water; ϕ_3 is the radiation potential resulting from the forced OWC device motion in heave mode with unit velocity amplitude and the top of the column open to the atmosphere; V_3 is the velocity amplitude of the forced OWC device motion in heave mode.

As a result, the volume flux through the Wells turbine (also assuming that a linear Wells turbine is installed at the top of the column) can be written as

$$\begin{aligned} q &= \int_0^a \int_0^{2\pi} \frac{\partial \phi}{\partial z} \Big|_{z=0} r d\theta dr \\ &= q_D + q_P + q_B, \end{aligned} \quad (5.2)$$

where q_D is the diffraction volume flux due to the potential ϕ_D , q_P (note that q_P is the same as q_R described in Chapters 3 and 4) the radiation volume flux due to the potential ϕ_P , and q_B the radiation volume flux due to the heaving motion of the OWC device in undisturbed water with the top of the column open to the atmosphere. q_B can be further written as

$$q_B = V_3 q_3 - V_3 S_i. \quad (5.3)$$

Here q_3 represents the volume flux due the radiation potential ϕ_3 , calculated by

$$q_3 = \int_0^a \int_0^{2\pi} \frac{\partial \phi_3}{\partial z} \Big|_{z=0} r d\theta dr = -\mathcal{B}_{3q} + i\mathcal{C}_{3q}, \quad (5.4)$$

and $S_i (= \pi a^2)$ is the plane area of the air chamber. Note that volume flux q_3 has been further decomposed as a sum of real and imaginary parts, and $\mathcal{B}_{3q}$ and $\mathcal{C}_{3q}$ are both real.

5.3 Equation of motion

Before constructing the equation of motion, let us calculate various wave forces acting on the OWC device, which can be obtained from hydrodynamic pressure. By the linearized Bernoulli's equation, for the time-harmonic motion of angular frequency ω , the hydrodynamic pressure on the device can be calculated by

$$P(r, \theta, z; t) = -\rho \frac{\partial \Phi}{\partial t} = \text{Re} \{ i\rho\omega\phi e^{-i\omega t} \}, \quad (5.5)$$

where ρ is the density of water.

The complex amplitude of vertical wave force f_3 (i.e., $\mathcal{F}_3 = \text{Re} \{ f_3 e^{-i\omega t} \}$, where $\mathcal{F}_3$ represents the vertical wave force) on the device can be calculated by

$$\begin{aligned} f_3 &= i\rho\omega \int_a^c \int_0^{2\pi} \phi|_{z=-d} r d\theta dr \\ &= f_{d3} + f_{p3} + V_3 f_{33}, \end{aligned} \quad (5.6)$$

where f_{d3} denotes the complex amplitude of vertical wave force due to the diffraction potential ϕ_D , calculated by

$$f_{d3} = i\rho\omega \int_a^c \int_0^{2\pi} \phi_D|_{z=-d} r d\theta dr = \Gamma_{d3} A, \quad (5.7)$$

where Γ_{d3} represents the vertical wave force factor due to the diffraction potential ϕ_D and A the incident wave amplitude; f_{p3} denotes the complex amplitude of vertical wave force due to the radiation potential ϕ_P , calculated by

$$f_{p3} = i\rho\omega \int_a^c \int_0^{2\pi} \phi_P r d\theta dr = (-\nu_{p3} + i\omega\mu_{p3}) p_a, \quad (5.8)$$

where both μ_{p3} and ν_{p3} are real. f_{d3} and f_{p3} can be termed as exciting forces. f_{33} denotes the vertical hydrodynamic reaction force acting on the oscillating water column device with unit amplitude of vertical velocity, calculated by

$$f_{33} = i\rho\omega \int_a^c \int_0^{2\pi} \phi_3 r d\theta dr = (-\nu_{33} + i\omega\mu_{33}), \quad (5.9)$$

where μ_{33} and ν_{33} represent added mass and radiation damping coefficients (Mei *et al.*, 2005), respectively.

Under the assumptions of linearization, the motion equation in heave mode is written as

$$\left(\nu_{33} - i\omega \left(\mathcal{M} + \mu_{33} - \frac{\rho g S_w}{\omega^2} \right) \right) V_3 + (\nu_{p3} - S_i - i\omega \mu_{p3}) p_a = \Gamma_{d3} A, \quad (5.10)$$

where $S_w (= \pi(c^2 - a^2))$ denotes the plane area of the wetted OWC device, $\mathcal{M}$ represents the mass of the OWC device, calculated by $\mathcal{M} = \rho S_w d$ (note that, the mass of the OWC device is calculated in the case that the OWC device of draft d is undisturbed on water surface).

5.4 Power take-off

Following Sarmiento & Falcão (1985) and assuming linear water waves, the mass flux of the air through the Wells turbine is related to the turbine characteristics by

$$\frac{d\mathcal{M}}{dt} = \rho_a^0 Q(t) - V_0 \frac{d\rho_a(t)}{dt} = \frac{\hat{K}D}{\hat{N}} P_a(t), \quad (5.11)$$

where $\mathcal{M}(t)$ and $\rho_a(t)$ are, respectively, the instantaneous mass and density of the air inside the chamber, $Q(t)$ the volume flux of the air inside the chamber, ρ_a^0 the air density in calm water.

Further assuming that the air is ideal and the process of air compression is isentropic, after neglecting the mechanical losses of turbine, there is a relation between the instantaneous density and pressure of the air, that is,

$$\frac{d\rho_a(t)}{dt} = \frac{1}{c_a^2} \frac{dP_a(t)}{dt}, \quad (5.12)$$

where c_a is the sound speed in air at a constant temperature.

For the time-harmonic motion with frequency ω , the volume flux can be expressed as

$$q = \left(\frac{\hat{K}D}{\rho_a^0 \hat{N}} - \frac{i\omega V_0}{\rho_a^0 c_a^2} \right) p_a. \quad (5.13)$$

Inserting Eqns. (5.2)–(5.4) into the above equation, after using Eqns. (3.5)–(3.6), gives

$$(a_{3q} - i\mathcal{C}_{3q}) V_3 + \left[\left(\frac{\hat{K}D}{\rho_a^0 \hat{N}} + \mathcal{B} \right) - i \left(\frac{\omega V_0}{\rho_a^0 c_a^2} + \mathcal{C} \right) \right] p_a = \Gamma A \quad (5.14)$$

where a_{3q} is real and given by

$$a_{3q} = \mathcal{B}_{3q} + S_i. \quad (5.15)$$

After solving the system of Eqns. (5.10) and (5.14), the pressure p_a and vertical response V_3 can be obtained, which are

$$p_a = \frac{[(\nu_{33} + ib_{33})\Gamma - (a_{3q} - i\mathcal{C}_{3q})\Gamma_{d3}]A}{(\chi\nu_{33} + \kappa b_{33} + Y_1) + i(\chi b_{33} - \kappa\nu_{33} + Y_2)}, \quad (5.16)$$

and

$$V_3 = \frac{\{[(\chi + \mathcal{B}) - i(\kappa + \mathcal{C})]\Gamma_{d3} - (a_{p3} - i\omega\mu_{p3})\Gamma\}A}{(\chi\nu_{33} + \kappa b_{33} + Y_1) + i(\chi b_{33} - \kappa\nu_{33} + Y_2)}, \quad (5.17)$$

where

$$\chi = \frac{\hat{K}D}{\rho_a^0 \hat{N}}, \quad \kappa = \frac{\omega V_0}{\rho_a^0 c_a^2}, \quad (5.18)$$

$$Y_1 = \mathcal{B}\nu_{33} + \mathcal{C}b_{33} - a_{3q}a_{p3} + \omega\mathcal{C}_{3q}\mu_{p3}, \quad (5.19)$$

and

$$Y_2 = \mathcal{B}b_{33} - \mathcal{C}\nu_{33} + \omega a_{3q}\mu_{p3} + \mathcal{C}_{3q}a_{p3}, \quad (5.20)$$

with a_{p3} and b_{33} are given by

$$a_{p3} = \nu_{p3} - S_i, \quad b_{33} = -\omega \left(\rho S_w d + \mu_{33} - \frac{\rho g S_w}{\omega^2} \right), \quad (5.21)$$

respectively. Note here that a_{p3} , b_{33} , Y_1 and Y_2 are all real, χ characterizes the turbine system and κ represents the effects of chamber size and air compressibility.

The extracted wave power by the OWC system is defined by the time-averaged rate of work done by the air pressure pushing or inhaling air through the linear turbine, namely,

$$P_{out} = \overline{Q(t)P_a(t)} = \frac{\hat{K}D}{2\rho_a^0 \hat{N}} |p_a|^2. \quad (5.22)$$

Let us use the concept of capture length to evaluate the efficiency of power absorption. The capture length L is defined as the ratio of the extracted power flux P_{out} to the wave power per unit crest length of the incident waves, which is $\rho g A^2 C_g / 2$ for progressive waves of amplitude A and group velocity C_g . After using Eqns. (5.16) and (5.22), the dimensionless capture length kL can be obtained

$$\begin{aligned} kL &= \frac{kP_{out}}{\rho g A^2 C_g / 2} \\ &= \frac{k}{\rho g C_g} \frac{\chi |[(\nu_{33} + ib_{33})\Gamma - (a_{3q} - i\mathcal{C}_{3q})\Gamma_{d3}]|^2}{(\chi\nu_{33} + \kappa b_{33} + Y_1)^2 + (\chi b_{33} - \kappa\nu_{33} + Y_2)^2}. \end{aligned} \quad (5.23)$$

Starting with the linearization, after separately solving the diffraction problem, the radiation problem due to the oscillating air pressure inside the chamber and the radiation problem due to the OWC device in heaving motion, the dimensionless capture length kL can be calculated by Eqn. (5.23).

For the limiting case where the OWC is fixed in waves, the oscillating air pressure inside the column can be calculated by Eqn. (5.14) with $V_3 = 0$, that is

$$p_a = \frac{\Gamma A}{\left(\frac{\hat{K}D}{\rho_a^0 \hat{N}} + \mathcal{B}\right) - i \left(\frac{\omega V_0}{\rho_a^0 c_a^2} + \mathcal{C}\right)}. \quad (5.24)$$

Inserting Eqn. (5.24) into Eqn. (5.23), after using Eqn. (5.22), the dimensionless capture length kL is given by

$$kL = \frac{k}{\rho g C_g} \frac{\chi |\Gamma|^2}{(\chi + \mathcal{B})^2 + (\chi + \mathcal{C})^2}. \quad (5.25)$$

Note here that the expressions in Eqn. (5.25) is the same as that presented by Martins-Rivas & Mei (2009b).

5.5 Diffraction problem

In this section, let us first consider the diffraction problem of waves by the OWC device in the case of the column top open to the atmosphere (e.g., $p_a = 0$) and the device fixed to waves. No matter which direction the incident wave train comes from, the incidence angle with respect to positive x -axis can always be set to zero by adjusting coordinate system due to the axisymmetric property of the OWC system.

Let a train of incident waves propagate along x -axis from $x \rightarrow -\infty$. Hence the incident velocity potential (see Mei *et al.* (2005)) can be written as

$$\phi_I = \frac{-igA}{\omega} \frac{Z_0(z)}{Z_0(0)} \sum_{m=0}^{\infty} i^m \varepsilon_m \varphi_m^I(r) \cos m\theta, \quad (5.26)$$

where A is the incident wave amplitude, and

$$\varepsilon_m = \begin{cases} 1, & m = 0, \\ 2, & m = 1, 2, \dots \end{cases} \quad (5.27)$$

Note here that φ_m^I is a function of r .

Referring to the expression in Eqn. (5.26), the total diffraction velocity potential ϕ_D may be expressed as

$$\phi_D = \frac{-igA}{\omega} \sum_{m=0}^{\infty} i^m \varepsilon_m \varphi_m(r, z) \cos m\theta. \quad (5.28)$$

Note that the θ -dependence has been separated in the expression due to the axisymmetric OWC system.

To apply matching eigenfunction expansions, it is conventional to divide the total domain into three regions, as sketched in Figure 5.1. Let us introduce the following notations:

- (i) $\phi_D^{(1)}$ denoting the complex-valued potential in region (1);
- (ii) $\phi_D^{(2)}$ denoting the complex-valued potential in region (2);
- (iii) $\phi_D^{(3)}$ denoting the complex-valued potential in region (3).

5.5.1 Formal expressions for the complex diffraction potentials in different regions

In this subsection, let us construct the formal expression for the complex diffraction potential in different regions.

By the separation of variables, the formal expression of diffraction complex velocity potential $\phi_D^{(1)}$ in region (1) can be written as

$$\phi_D^{(1)} = \frac{-igA}{\omega} \sum_{m=0}^{\infty} i^m \varepsilon_m \varphi_m^{(1)}(r, z) \cos m\theta, \quad (5.29)$$

where

$$\varphi_m^{(1)}(r, z) = \left(J_m(kr) - \frac{J'_m(kc)}{H'_m(kc)} H_m(kr) \right) \frac{Z_0(z)}{Z_0(0)} + \sum_{l=0}^{\infty} A_{ml} \frac{K_m(k_l r)}{k_l K'_m(k_l c)} Z_l(z). \quad (5.30)$$

Note here that the first term in Eqn. (5.30) represents the diffraction of waves by a circular solid cylinder extending to the entire water depth, while the second term denotes the correction to the first term due to the truncation of the cylinder. J_m is the m th Bessel function of the first kind, H_m the m th Hankel function of the first kind (for brevity, the superscript (1) has been omitted), and K_m the

m th modified Bessel function of the second kind. Note also that the terms with $l = 0$ correspond to outgoing waves,

$$K_m(-ikr) = \frac{\pi}{2} i^{m+1} H_m(kr). \quad (5.31)$$

The coefficients A_{ml} are yet unknown. In what follows, unless otherwise specified, the prime denotes the differentiation of the corresponding function with respect to its argument.

The formal expression of the complex potential $\phi_D^{(2)}$ in region (2) can be expressed as

$$\phi_D^{(2)} = \frac{-igA}{\omega} \sum_{m=0}^{\infty} i^m \varepsilon_m \varphi_m^{(2)}(r, z) \cos m\theta, \quad (5.32)$$

where

$$\begin{aligned} \varphi_m^{(2)}(r, z) = & B_{m0} R_{m,1}(r) + C_{m0} R_{m,2}(r) \\ & + \sum_{n=1}^{\infty} \left(B_{mn} \frac{I_m(\alpha_n r)}{\alpha_n I'_m(\alpha_n c)} + C_{mn} \frac{K_m(\alpha_n r)}{\alpha_n K'_m(\alpha_n a)} \right) f_n(z), \end{aligned} \quad (5.33)$$

with $\alpha_n = n\pi/(h-d)$,

$$f_n = \cos \alpha_n (z + h). \quad (5.34)$$

The functions $R_{m,1}(r)$ and $R_{m,2}(r)$ are given by

$$R_{m,1}(r) = \begin{cases} 1, & m = 0 \\ (r/c)^m, & m = 1, 2, 3, \dots, \end{cases} \quad (5.35)$$

and

$$R_{m,2}(r) = \begin{cases} \ln(r/a), & m = 0, \\ (a/r)^m, & m = 1, 2, 3, \dots, \end{cases} \quad (5.36)$$

respectively. B_{mn} and C_{mn} are unknown, and I_m are the m th modified Bessel function of the first kind.

Similarly, the formal expression of the complex potential $\phi_D^{(3)}$ in region (3) can be written as

$$\phi_D^{(3)} = \frac{-igA}{\omega} \sum_{m=0}^{\infty} i^m \varepsilon_m \varphi_m^{(3)}(r, z) \cos m\theta, \quad (5.37)$$

with

$$\varphi_m^{(3)}(r, z) = \sum_{l=0}^{\infty} D_{ml} \frac{I_m(k_l r)}{k_l I'_m(k_l a)} Z_l(z), \quad (5.38)$$

where D_{ml} are unknown coefficients. Note here that the requirement of boundedness at $r = 0$ has been satisfied. Again the terms with $l = 0$ correspond to progressive waves,

$$I_m(-ikr) = (-i)^m J_m(kr). \quad (5.39)$$

5.5.2 Auxiliary radial velocity functions

To speed up the speed of convergence in the following numerical computations, let us first introduce two auxiliary functions to represent the distributions of the radial velocities at the following two common interfaces:

(i) $U_D^{(1)}(\theta, z)$ for the interface defined by $r = c$ and $0 \leq \theta < 2\pi$;

(ii) $U_D^{(2)}(\theta, z)$ for the interface defined by $r = a$ and $0 \leq \theta < 2\pi$.

For the convenience on the application of matching eigenfunction expansions, following Fernyhough & Evans (1995) and Martins-Rivas & Mei (2009b), let us expand these two auxiliary functions in two sets of orthogonal functions, namely,

$$U_D^{(1)}(\theta, z) = \frac{-igA}{\omega} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \gamma_{mp} u_p(z) \cos m\theta, \quad (5.40)$$

and

$$U_D^{(2)}(\theta, z) = \frac{-igA}{\omega} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \xi_{mp} u_p(z) \cos m\theta \quad (5.41)$$

with

$$u_p(z) = \frac{(-1)^p (2p)! \Gamma(1/6)}{\pi (h-d)^{1/3} \Gamma(2p+1/3)} [(h-d)^2 - (z+h)^2]^{-1/3} C_{2p}^{1/6} \left(\frac{z+h}{h-d} \right). \quad (5.42)$$

where γ_{mp} and ξ_{mp} are unknown expansion coefficients, and $C_{2p}^{1/6}$ are Gegenbauer polynomials. It is well-known that the set $\{\cos m\theta\}$ ($m = 0, 1, 2, \dots$) is orthogonal to each other over the interval $[0, 2\pi]$ and the set $\left\{ C_{2p}^{1/6} \left(\frac{z+h}{h-d} \right) \right\}$ ($p = 0, 1, 2, \dots$), referring to Abramowitz & Stegun (1972), is orthogonal to each other over the interval $[-2h+d, -h]$ with respect to the weight function

$$\frac{(-1)^p (2p)! \Gamma(1/6)}{\pi (h-d)^{1/3} \Gamma(2p+1/3)} [(h-d)^2 - (z+h)^2]^{-1/3}. \quad (5.43)$$

Recalling the singularity of the velocity distribution at sharp corners (see Batchelor (1967) or Mandal & Chakrabarti (2000)), the vertical dependence $u_p(z)$ behaves as $(z+d)^{-1/3}$ at the sharp corner ($z = -d$) with an interior angle of 90° . Invoking the weight function in Eqn. (5.43), the singularity has been included in the expansion base function $u_p(z)$, which is beneficial to speed up the convergence in the coming numerical computations. Moreover, let us extend the auxiliary functions $U_D^{(1)}(\theta, z)$ and $U_D^{(2)}(\theta, z)$ to two even functions occupying the space defined by $0 \leq \theta < 2\pi$ for $-2h+d < z < -d$, which are symmetric about $z = -h$. Following the even or odd quality of Gegenbauer polynomials, only the even degree $2p$ is taken.

5.5.3 Determinations of the unknowns A_{ml} - D_{ml}

Due to the requirements of the continuity of radial velocities across the common interfaces, we have the following relations, at $r = c$,

$$\frac{\partial \phi_D^{(1)}}{\partial r} \Big|_{r=c} = \begin{cases} 0, & 0 \leq \theta < 2\pi, -d < z < 0, \\ U_D^{(1)}, & 0 \leq \theta < 2\pi, -h < z < -d, \end{cases} \quad (5.44)$$

and

$$\frac{\partial \phi_D^{(2)}}{\partial r} \Big|_{r=c} = U_D^{(1)}, \quad 0 \leq \theta < 2\pi, -h < z < -d. \quad (5.45)$$

At the common interface $r = a$, we have

$$\frac{\partial \phi_D^{(2)}}{\partial r} \Big|_{r=a} = U_D^{(2)}, \quad 0 \leq \theta < 2\pi, -h < z < -d, \quad (5.46)$$

and

$$\frac{\partial \phi_D^{(3)}}{\partial r} \Big|_{r=a} = \begin{cases} 0, & 0 \leq \theta < 2\pi, -d < z < 0, \\ U_D^{(2)}, & 0 \leq \theta < 2\pi, -h < z < -d. \end{cases} \quad (5.47)$$

After inserting Eqns. (5.29), (5.32), (5.37), (5.40) and (5.41) into Eqns. (5.44)-(5.47), respectively, and using the fact of orthogonality of cosine functions, we have

$$\frac{\partial \varphi_m^{(1)}}{\partial r} \Big|_{r=c} = \begin{cases} 0, & -d < z < 0, \\ \sum_{p=0}^{\infty} \gamma_{mp} u_p(z), & -h < z < -d, \end{cases} \quad (5.48)$$

$$\frac{\partial \varphi_m^{(2)}}{\partial r} \Big|_{r=c} = \sum_{p=0}^{\infty} \gamma_{mp} u_p(z), \quad -h < z < -d, \quad (5.49)$$

$$\frac{\partial \varphi_m^{(2)}}{\partial r} \Big|_{r=a} = \sum_{p=0}^{\infty} \xi_{mp} u_p(z), \quad -h < z < -d, \quad (5.50)$$

and

$$\frac{\partial \varphi_m^{(3)}}{\partial r} \Big|_{r=a} = \begin{cases} 0, & -d < z < 0, \\ \sum_{p=0}^{\infty} \xi_{mp} u_p(z), & -h < z < -d. \end{cases} \quad (5.51)$$

Multiplying both sides of Eqn. (5.48) by $Z_{l'}(z)/h$, where $l' = 0, 1, 2, \dots$, and then integrating the resulting equation with respect to z over the corresponding intervals, we have

$$\frac{1}{h} \int_{-h}^0 \frac{\partial \varphi_m^{(1)}}{\partial r} \Big|_{r=c} Z_{l'}(z) dz = \frac{1}{h} \int_{-h}^{-d} \sum_{p=0}^{\infty} \gamma_{mp} u_p(z) Z_{l'}(z) dz. \quad (5.52)$$

Inserting Eqn. (5.30) into the above equation and using the relation in Eqn. (2.18) gives

$$A_{ml} = \sum_{p=0}^{\infty} \gamma_{mp} V_{pl}^{(1)}, \quad (5.53)$$

where $V_{pl}^{(1)}$ are given by

$$V_{pl}^{(1)} = \frac{1}{h} \int_{-h}^{-d} u_p(z) Z_l(z) dz. \quad (5.54)$$

After multiplying both sides of Eqns. (5.49) and (5.50) by $f_{n'}(z)/(h-d)$, where $n' = 0, 1, 2, \dots$, and integrating the resulting equations with respect to z over the corresponding intervals, we have

$$\frac{1}{h-d} \int_{-h}^{-d} \frac{\partial \varphi_m^{(2)}}{\partial r} \Big|_{r=c} f_{n'}(z) dz = \frac{1}{h-d} \int_{-h}^{-d} \sum_{p=0}^{\infty} \gamma_{mp} u_p(z) f_{n'}(z) dz, \quad (5.55)$$

$$\frac{1}{h-d} \int_{-h}^{-d} \frac{\partial \varphi_m^{(2)}}{\partial r} \Big|_{r=a} f_{n'}(z) dz = \frac{1}{h-d} \int_{-h}^{-d} \sum_{p=0}^{\infty} \xi_{mp} u_p(z) f_{n'}(z) dz. \quad (5.56)$$

Inserting Eqn. (5.33) into Eqns. (5.55) and (5.56), and using the orthogonal relation

$$\frac{1}{h-d} \int_{-h}^{-d} f_n(z) f_{n'}(z) dz = \begin{cases} 0, & n \neq n', \\ 1, & n = n' = 0, \\ 1/2, & n = n' \neq 0, \end{cases} \quad (5.57)$$

gives

$$\begin{cases} B_{m0} R'_{m,1}(c) + C_{m0} R'_{m,2}(c) = \sum_{p=0}^{\infty} \gamma_{mp} V_{p0}^{(2)}, & n = 0, \\ \frac{1}{2} \left(B_{mn} + C_{mn} \frac{K'_m(\alpha_n c)}{K'_m(\alpha_n a)} \right) = \sum_{p=0}^{\infty} \gamma_{mp} V_{pn}^{(2)}, & n = 1, 2, \dots, \end{cases} \quad (5.58)$$

and

$$\begin{cases} B_{m0} R'_{m,1}(a) + C_{m0} R'_{m,2}(a) = \sum_{p=0}^{\infty} \xi_{mp} V_{p0}^{(2)}, & n = 0, \\ \frac{1}{2} \left(B_{mn} \frac{I'_m(\alpha_n a)}{I'_m(\alpha_n c)} + C_{mn} \right) = \sum_{p=0}^{\infty} \xi_{mp} V_{pn}^{(2)}, & n = 1, 2, \dots, \end{cases} \quad (5.59)$$

where $V_{pn}^{(2)}$ are calculated by

$$V_{pn}^{(2)} = \frac{1}{h-d} \int_{-h}^{-d} u_p(z) f_n(z) dz. \quad (5.60)$$

Referring to Eqn. (5.35), it is easy to show that as $m = 0$, $R'_{0,1}(c) = R'_{0,1}(a) = 0$. Therefore, it is convenient to derive the solutions of B_{mn} and

C_{mn} ($m, n = 0, 1, 2, \dots$) based on two cases, i.e., $m = 0$ and $m = 1, 2, 3, \dots$, respectively.

For $m = 0$, Eqns. (5.58) and (5.59) can be rewritten as

$$\begin{cases} C_{00}R'_{0,2}(c) = \sum_{p=0}^{\infty} \gamma_{0p}V_{p0}^{(2)}, & n = 0, \\ \frac{1}{2} \left(B_{0n} + C_{0n} \frac{K'_0(\alpha_n c)}{K'_0(\alpha_n a)} \right) = \sum_{p=0}^{\infty} \gamma_{0p}V_{pn}^{(2)}, & n = 1, 2, \dots, \end{cases} \quad (5.61)$$

and

$$\begin{cases} C_{00}R'_{0,2}(a) = \sum_{p=0}^{\infty} \xi_{0p}V_{p0}^{(2)}, & n = 0, \\ \frac{1}{2} \left(B_{0n} \frac{I'_0(\alpha_n a)}{I'_0(\alpha_n c)} + C_{0n} \right) = \sum_{p=0}^{\infty} \xi_{0p}V_{pn}^{(2)}, & n = 1, 2, \dots, \end{cases} \quad (5.62)$$

respectively. After solving the system of Eqns. (5.61) and (5.62), we obtain

$$C_{00} = \frac{1}{R'_{0,2}(c)} \sum_{p=0}^{\infty} \gamma_{0p}V_{p0}^{(2)}, \quad \text{or} \quad C_{00} = \frac{1}{R'_{0,2}(a)} \sum_{p=0}^{\infty} \xi_{0p}V_{p0}^{(2)}, \quad (5.63)$$

$$B_{0n} = 2Q_{0n} \left(\sum_{p=0}^{\infty} \gamma_{0p}V_{pn}^{(2)} - \frac{K'_0(\alpha_n c)}{K'_0(\alpha_n a)} \sum_{p=0}^{\infty} \xi_{0p}V_{pn}^{(2)} \right), \quad (5.64)$$

and

$$C_{0n} = 2Q_{0n} \left(\sum_{p=0}^{\infty} \xi_{0p}V_{pn}^{(2)} - \frac{I'_0(\alpha_n a)}{I'_0(\alpha_n c)} \sum_{p=0}^{\infty} \gamma_{0p}V_{pn}^{(2)} \right), \quad (5.65)$$

where

$$Q_{0n} = 1 / \left(1 - \frac{I'_0(\alpha_n a)}{I'_0(\alpha_n c)} \frac{K'_0(\alpha_n c)}{K'_0(\alpha_n a)} \right), \quad n = 1, 2, 3, \dots \quad (5.66)$$

Note here that, after combining the two equations in Eqn. (5.63), we have the following relation,

$$R'_{0,2}(a) \sum_{p=0}^{\infty} \gamma_{0p}V_{p0}^{(2)} - R'_{0,2}(c) \sum_{p=0}^{\infty} \xi_{0p}V_{p0}^{(2)} = 0. \quad (5.67)$$

For $m = 1, 2, \dots$, after solving the system of Eqns. (5.58) and (5.59), we have

$$B_{m0} = Q_{m0} \left(R'_{m,2}(a) \sum_{p=0}^{\infty} \gamma_{mp}V_{p0}^{(2)} - R'_{m,2}(c) \sum_{p=0}^{\infty} \xi_{mp}V_{p0}^{(2)} \right), \quad m \neq 0, \quad (5.68)$$

$$C_{m0} = Q_{m0} \left(R'_{m,1}(c) \sum_{p=0}^{\infty} \xi_{mp}V_{p0}^{(2)} - R'_{m,1}(a) \sum_{p=0}^{\infty} \gamma_{mp}V_{p0}^{(2)} \right), \quad m \neq 0, \quad (5.69)$$

$$B_{mn} = 2Q_{mn} \left(\sum_{p=0}^{\infty} \gamma_{mp}V_{pn}^{(2)} - \frac{K'_m(\alpha_n c)}{K'_m(\alpha_n a)} \sum_{p=0}^{\infty} \xi_{mp}V_{pn}^{(2)} \right), \quad m, n \neq 0, \quad (5.70)$$

and

$$C_{mn} = 2Q_{mn} \left(\sum_{p=0}^{\infty} \xi_{mp} V_{pn}^{(2)} - \frac{I'_m(\alpha_n a)}{I'_m(\alpha_n c)} \sum_{p=0}^{\infty} \gamma_{mp} V_{pn}^{(2)} \right), \quad m, n \neq 0, \quad (5.71)$$

where Q_{m0} and Q_{mn} are given by

$$Q_{m0} = 1 / (R'_{m,2}(a) R'_{m,1}(c) - R'_{m,1}(a) R'_{m,2}(c)), \quad m \neq 0, \quad (5.72)$$

and

$$Q_{mn} = 1 / \left(1 - \frac{I'_m(\alpha_n a)}{I'_m(\alpha_n c)} \frac{K'_m(\alpha_n c)}{K'_m(\alpha_n a)} \right), \quad m, n \neq 0. \quad (5.73)$$

Similarly, multiplying both sides of Eqn. (5.51) by $Z_{l'}(z)/h$ and then integrating the resulting equation over the corresponding intervals, gives

$$\frac{1}{h} \int_{-h}^0 \frac{\partial \varphi_m^{(3)}}{\partial r} \Big|_{r=a} Z_{l'}(z) dz = \frac{1}{h} \int_{-h}^{-d} \sum_{p=0}^{\infty} \xi_{mp} u_p(z) Z_{l'}(z) dz. \quad (5.74)$$

After inserting Eqn. (5.38) into the above equation and using the orthogonality of functions $Z_l(z)$, we have

$$D_{ml} = \sum_{p=0}^{\infty} \xi_{mp} V_{pl}^{(1)}. \quad (5.75)$$

Now we have obtained the solutions of A_{ml} - D_{ml} in terms of γ_{mp} and ξ_{mp} except for B_{00} .

5.5.4 Determinations of the auxiliary functions $U_D^{(1)}$ and $U_D^{(2)}$

Starting with the continuity of pressure across the two common interfaces, we have

$$\phi_D^{(1)}|_{r=c} = \phi_D^{(2)}|_{r=c}, \quad 0 \leq \theta < 2\pi, -h < z < -d, \quad (5.76)$$

and

$$\phi_D^{(2)}|_{r=a} = \phi_D^{(3)}|_{r=a}, \quad 0 \leq \theta < 2\pi, -h < z < -d. \quad (5.77)$$

Inserting Eqns. (5.29) and (5.32) into the above two equations and using the orthogonality of cosine function, we further have

$$\varphi_m^{(1)}|_{r=c} = \varphi_m^{(2)}|_{r=c}, \quad -h < z < -d, \quad (5.78)$$

and

$$\varphi_m^{(2)}|_{r=a} = \varphi_m^{(3)}|_{r=a}, \quad -h < z < -d. \quad (5.79)$$

Multiplying both sides of Eqns. (5.78) and (5.79) by $u_{p'}(z)$ ($p' = 0, 1, 2, \dots$) and integrating the resulting equations over the interval $[-h, -d]$, gives

$$\begin{aligned} & \frac{h\mathcal{R}_m}{Z_0(0)} V_{p'0}^{(1)} + h \sum_{l=0}^{\infty} A_{ml} \frac{K_m(k_l c)}{k_l K'_m(k_l c)} V_{p'l}^{(1)} \\ &= (h-d) (B_{m0} R_{m,1}(c) + C_{m0} R_{m,2}(c)) V_{p'0}^{(2)} \\ &+ (h-d) \sum_{n=1}^{\infty} \left(B_{mn} \frac{I_m(\alpha_n c)}{\alpha_n I'_m(\alpha_n c)} + C_{mn} \frac{K_m(\alpha_n c)}{\alpha_n K'_m(\alpha_n a)} \right) V_{p'n}^{(2)}, \end{aligned} \quad (5.80)$$

and

$$\begin{aligned} & (h-d) (B_{m0} R_{m,1}(a) + C_{m0} R_{m,2}(a)) V_{p'0}^{(2)} \\ &+ (h-d) \sum_{n=1}^{\infty} \left(B_{mn} \frac{I_m(\alpha_n a)}{\alpha_n I'_m(\alpha_n c)} + C_{mn} \frac{K_m(\alpha_n a)}{\alpha_n K'_m(\alpha_n a)} \right) V_{p'n}^{(2)} \\ &= h \sum_{l=0}^{\infty} D_{ml} \frac{I_m(k_l a)}{k_l I'_m(k_l a)} V_{p'l}^{(1)}, \end{aligned} \quad (5.81)$$

where

$$\mathcal{R}_m = \left(J_m(kc) - \frac{J'_m(kc)}{H'_m(kc)} H_m(kc) \right) = \frac{2i}{\pi kc H'_m(kc)}. \quad (5.82)$$

To determine B_{00} , it is convenient to discuss Eqns. (5.80) and (5.81) based on two cases, namely, $m = 0$ and $m = 1, 2, \dots$

For $m = 0$, after inserting Eqns. (5.53), (5.63)-(5.65), and (5.75) into Eqns. (5.80)-(5.81) and re-arranging, we have

$$\begin{aligned} & (h-d) B_{00} R_{0,1}(c) V_{p'0}^{(2)} + \sum_{p=0}^{\infty} \gamma_{0p} (h-d) \frac{R_{0,2}(c)}{R'_{0,2}(c)} V_{p'0}^{(2)} V_{p'0}^{(2)} \\ & - \sum_{p=0}^{\infty} \gamma_{0p} h \sum_{l=0}^{\infty} \frac{K_0(k_l c)}{k_l K'_0(k_l c)} V_{pl}^{(1)} V_{p'l}^{(1)} + \sum_{p=0}^{\infty} \gamma_{0p} 2 (h-d) \sum_{n=1}^{\infty} Q_{0n} R_{0n}^{(1)} V_{pn}^{(2)} V_{p'n}^{(2)} \\ & + \sum_{p=0}^{\infty} \xi_{0p} 2 (h-d) \sum_{n=1}^{\infty} Q_{0n} R_{0n}^{(2)} V_{pn}^{(2)} V_{p'n}^{(2)} = \frac{h\mathcal{R}_0}{Z_0(0)} V_{p'0}^{(1)}, \end{aligned} \quad (5.83)$$

and

$$\begin{aligned} & (h-d) B_{00} R_{0,1}(a) V_{p'0}^{(2)} + \sum_{p=0}^{\infty} \gamma_{0p} 2 (h-d) \sum_{n=1}^{\infty} Q_{0n} R_{0n}^{(3)} V_{pn}^{(2)} V_{p'n}^{(2)} \\ & + \sum_{p=0}^{\infty} \xi_{0p} (h-d) \frac{R_{0,2}(a)}{R'_{0,2}(a)} V_{p'0}^{(2)} V_{p'0}^{(2)} + \sum_{p=0}^{\infty} \xi_{0p} 2 (h-d) \sum_{n=1}^{\infty} Q_{0n} R_{0n}^{(4)} V_{pn}^{(2)} V_{p'n}^{(2)} \\ & - \sum_{p=0}^{\infty} \xi_{0p} h \sum_{l=0}^{\infty} \frac{I_0(k_l a)}{k_l I'_0(k_l a)} V_{pl}^{(1)} V_{p'l}^{(1)} = 0, \end{aligned} \quad (5.84)$$

where $R_{0n}^{(1)}$, $R_{0n}^{(2)}$, $R_{0n}^{(3)}$ and $R_{0n}^{(4)}$ are given by

$$R_{0n}^{(1)} = \frac{I_0(\alpha_n c)}{\alpha_n I_0'(\alpha_n c)} - \frac{I_0'(\alpha_n a)}{I_0'(\alpha_n c)} \frac{K_0(\alpha_n c)}{\alpha_n K_0'(\alpha_n a)}, \quad (5.85)$$

$$R_{0n}^{(2)} = \frac{K_0(\alpha_n c)}{\alpha_n K_0'(\alpha_n a)} - \frac{I_0(\alpha_n c)}{\alpha_n I_0'(\alpha_n c)} \frac{K_0'(\alpha_n c)}{K_0'(\alpha_n a)}, \quad (5.86)$$

$$R_{0n}^{(3)} = \frac{I_0(\alpha_n a)}{\alpha_n I_0'(\alpha_n c)} - \frac{I_0'(\alpha_n a)}{I_0'(\alpha_n c)} \frac{K_0(\alpha_n a)}{\alpha_n K_0'(\alpha_n a)}, \quad (5.87)$$

and

$$R_{0n}^{(4)} = \frac{K_0(\alpha_n a)}{\alpha_n K_0'(\alpha_n a)} - \frac{I_0(\alpha_n a)}{\alpha_n I_0'(\alpha_n c)} \frac{K_0'(\alpha_n c)}{K_0'(\alpha_n a)}, \quad (5.88)$$

respectively.

For $m = 1, 2, \dots$, inserting Eqns. (5.53), (5.68)-(5.71) and (5.75) into Eqns. (5.80)-(5.81) and after re-arranging, gives

$$\begin{aligned} & \sum_{p=0}^{\infty} \gamma_{mp} h \sum_{l=0}^{\infty} \frac{K_m(k_l c)}{k_l K_m'(k_l c)} V_{pl}^{(1)} V_{p'l}^{(1)} + \sum_{p=0}^{\infty} \gamma_{mp} (h-d) Q_{m0} F_m^{(1)} V_{p0}^{(2)} V_{p'0}^{(2)} \\ & + \sum_{p=0}^{\infty} \xi_{mp} (h-d) Q_{m0} F_m^{(2)} V_{p0}^{(2)} V_{p'0}^{(2)} + \sum_{p=0}^{\infty} \gamma_{mp} 2(h-d) \sum_{n=1}^{\infty} Q_{mn} R_{mn}^{(1)} V_{pn}^{(2)} V_{p'n}^{(2)} \\ & + \sum_{p=0}^{\infty} \xi_{mp} 2(h-d) \sum_{n=1}^{\infty} Q_{mn} R_{mn}^{(2)} V_{pn}^{(2)} V_{p'n}^{(2)} = \frac{h \mathcal{R}_m}{Z_0(0)} V_{p'0}^{(1)}, \end{aligned} \quad (5.89)$$

and

$$\begin{aligned} & \sum_{p=0}^{\infty} \gamma_{mp} (h-d) Q_{m0} F_m^{(3)} V_{p0}^{(2)} V_{p'0}^{(2)} + \sum_{p=0}^{\infty} \gamma_{mp} 2(h-d) \sum_{n=1}^{\infty} Q_{mn} R_{mn}^{(3)} V_{pn}^{(2)} V_{p'n}^{(2)} \\ & + \sum_{p=0}^{\infty} \xi_{mp} (h-d) Q_{m0} F_m^{(4)} V_{p0}^{(2)} V_{p'0}^{(2)} + \sum_{p=0}^{\infty} \xi_{mp} 2(h-d) \sum_{n=1}^{\infty} Q_{mn} R_{mn}^{(4)} V_{pn}^{(2)} V_{p'n}^{(2)} \\ & - \sum_{p=0}^{\infty} \xi_{mp} h \sum_{l=0}^{\infty} \frac{I_m(k_l a)}{k_l I_m'(k_l a)} V_{pl}^{(1)} V_{p'l}^{(1)} = 0, \end{aligned} \quad (5.90)$$

where $F_m^{(1)}$ - $F_m^{(4)}$ are given by

$$F_m^{(1)} = R'_{m,2}(a) R_{m,1}(c) - R'_{m,1}(a) R_{m,2}(c), \quad (5.91)$$

$$F_m^{(2)} = R'_{m,1}(c) R_{m,2}(c) - R'_{m,2}(c) R_{m,1}(c), \quad (5.92)$$

$$F_m^{(3)} = R'_{m,2}(a) R_{m,1}(a) - R'_{m,1}(a) R_{m,2}(a), \quad (5.93)$$

$$F_m^{(4)} = R'_{m,1}(c) R_{m,2}(a) - R'_{m,2}(c) R_{m,1}(a), \quad (5.94)$$

and $R_{mn}^{(1)}-R_{mn}^{(4)}$ are given by

$$R_{mn}^{(1)} = \frac{I_m(\alpha_n c)}{\alpha_n I'_m(\alpha_n c)} - \frac{I'_m(\alpha_n a)}{I'_m(\alpha_n c)} \frac{K_m(\alpha_n c)}{\alpha_n K'_m(\alpha_n a)}, \quad (5.95)$$

$$R_{mn}^{(2)} = \frac{K_m(\alpha_n c)}{\alpha_n K'_m(\alpha_n a)} - \frac{I_m(\alpha_n c)}{\alpha_n I'_m(\alpha_n c)} \frac{K'_m(\alpha_n c)}{K'_m(\alpha_n a)}, \quad (5.96)$$

$$R_{mn}^{(3)} = \frac{I_m(\alpha_n a)}{\alpha_n I'_m(\alpha_n c)} - \frac{I'_m(\alpha_n a)}{I'_m(\alpha_n c)} \frac{K_m(\alpha_n a)}{\alpha_n K'_m(\alpha_n a)}, \quad (5.97)$$

$$R_{mn}^{(4)} = \frac{K_m(\alpha_n a)}{\alpha_n K'_m(\alpha_n a)} - \frac{I_m(\alpha_n a)}{\alpha_n I'_m(\alpha_n c)} \frac{K'_m(\alpha_n c)}{K'_m(\alpha_n a)}. \quad (5.98)$$

Observing Eqns. (5.83) and (5.84), except for γ_{0p} and ξ_{0p} , there is another unknown coefficient B_{00} to be determined. Therefore, let us combine Eqns. (5.67) and (5.83)-(5.84) to construct a closed infinite linear algebraic equations. Assuming $p', p = 0, 1, \dots, P$, the linear system includes $(2P + 3)$ unknowns (i.e., $B_{00}, \gamma_{00}, \dots, \gamma_{0P}, \xi_{00}, \dots, \xi_{0P}$) and $(2P + 3)$ linear equations.

Similarly, for each m ($m = 1, 2, \dots$), another closed linear algebraic system can be constructed by combining Eqns. (5.89)-(5.90), which includes $(2P + 2)$ unknowns ($\gamma_{m0}, \dots, \gamma_{mP}, \xi_{m0}, \dots, \xi_{mP}$) and $(2P + 2)$ linear equations.

Once the above linear systems have been solved numerically, the numerical results of $A_{ml}-D_{ml}$ can be calculated by Eqns. (5.53), (5.63)-(5.65), (5.68)-(5.71) and (5.75). Further, the other corresponding quantities can be obtained straightforwardly.

5.5.5 Expressions for the diffraction factors: Γ and Γ_{d3}

Now let us derive the expressions for Γ and Γ_{d3} for the determination of the dimensionless capture length kL .

Referring to Eqn. (3.2) and after using the orthogonal property of cosine functions, the diffraction volume flux q_D is calculated by

$$\begin{aligned} q_D &= \int_0^a \int_0^{2\pi} \frac{\partial \phi_D^{(3)}}{\partial z} \Big|_{z=0} r dr d\theta \\ &= \frac{-igA}{\omega} 2\pi \int_0^a \frac{\partial \varphi_0^{(3)}(r, z)}{\partial z} \Big|_{z=0} r dr. \end{aligned} \quad (5.99)$$

Inserting Eqn. (5.38) into the above equation and then using the relation in Eqn. (3.5), we obtain the following expression for Γ

$$\Gamma = \frac{-i2\pi g}{\omega} \sum_{l=0}^{\infty} D_{0l} \frac{a Z_l'(0)}{k_l^2}, \quad (5.100)$$

where the identity $\int_0^a \frac{r I_0(k_l r)}{k_l a I_0'(k_l a)} dr = \frac{1}{k_l^2}$ has been used.

According to Eqn. (5.7), the exciting force in z -direction due to the diffraction of waves by a fixed OWC device with the column open to the atmosphere can be calculated by, after using the orthogonality of cosine functions,

$$\begin{aligned} f_{d3} &= i\rho\omega \int_a^c \int_0^{2\pi} \phi_D^{(2)}|_{z=-d} r dr d\theta \\ &= 2\pi\rho g A \int_a^c \varphi_0^{(2)}(r, -d) r dr. \end{aligned} \quad (5.101)$$

Inserting Eqn. (5.33) into the just above equation and using the relation in Eqn. (5.7), we have

$$\begin{aligned} \Gamma_{d3} &= 2\pi\rho g \left[B_{00} \frac{c^2 - a^2}{2} + \frac{C_{00}}{4} (a^2 - c^2 + 2c^2 \ln(c/a)) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \left(B_{0n} \frac{c}{\alpha_n^2} \left(1 - \frac{a I_1(\alpha_n a)}{c I_1(\alpha_n c)} \right) + C_{0n} \frac{a}{\alpha_n^2} \left(\frac{c K_1(\alpha_n c)}{a K_1(\alpha_n a)} - 1 \right) \right) f_n(-d) \right]. \end{aligned} \quad (5.102)$$

Let us introduce the following dimensionless parameters,

$$\tilde{\Gamma} = \Gamma \frac{\sqrt{g/h}}{gh}, \quad \tilde{\Gamma}_{d3} = \frac{\Gamma_{d3}}{\pi\rho g c^2}. \quad (5.103)$$

Hence, $\tilde{\Gamma}$ and $\tilde{\Gamma}_{s3}$ are given by

$$\tilde{\Gamma} = \frac{-i2\pi}{\omega} \frac{\sqrt{g/h}}{h} \sum_{l=0}^{\infty} D_{0l} \frac{a Z_l'(0)}{k_l^2}, \quad (5.104)$$

and

$$\begin{aligned} \tilde{\Gamma}_{d3} &= \frac{2}{c^2} \left[B_{00} \frac{c^2 - a^2}{2} + \frac{C_{00}}{4} (a^2 - c^2 + 2c^2 \ln(c/a)) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \left(B_{0n} \frac{c}{\alpha_n^2} \left(1 - \frac{a I_1(\alpha_n a)}{c I_1(\alpha_n c)} \right) + C_{0n} \frac{a}{\alpha_n^2} \left(\frac{c K_1(\alpha_n c)}{a K_1(\alpha_n a)} - 1 \right) \right) f_n(-d) \right]. \end{aligned} \quad (5.105)$$

5.6 Radiation problem due to oscillating air pressure

In this section, let us consider the wave radiation problem due to the oscillating air pressure inside the chamber in the case where the OWC device is assumed to be fixed on water surface and the incident waves are absent. The complex potential ϕ_P satisfies Eqns. (2.10)-(2.13) with $n_j = 0$.

Similar to the diffraction problem, let us introduce $\phi_P^{(1)}$ to represent the complex-valued potential in region (1), $\phi_P^{(2)}$ the potential in region (2), $\phi_P^{(3)}$ the potential in region (3). Therefore, the linearized boundary condition on the water surface can be rewritten as

$$\left(\frac{\partial}{\partial z} - \frac{\omega^2}{g} \right) \begin{Bmatrix} \phi_P^{(1)} \\ \phi_P^{(3)} \end{Bmatrix} = \begin{cases} 0, & z = 0, r > c, \\ \frac{i\omega}{\rho g} p_a, & z = 0, r < a. \end{cases} \quad (5.106)$$

5.6.1 Formal expressions of the complex radiation potentials in different regions

As in earlier theories, we approximate the unknown air pressure inside the chamber to be uniform in space but sinusoidal in time. Based on the axisymmetric property of the OWC device, the radiation potentials in all regions are independent of θ , namely, no angular variation in the potentials. Thus, by separation of variables, only the $m = 0$ mode will be included.

The formal expressions of $\phi_P^{(1)}$, $\phi_P^{(2)}$ and $\phi_P^{(3)}$ can be written as

$$\phi_P^{(1)} = \frac{-ip_a}{\rho\omega} \sum_{l=0}^{\infty} \tilde{A}_l \frac{K_0(k_l r)}{k_l K'_0(k_l c)} Z_l(z), \quad (5.107)$$

$$\begin{aligned} \phi_P^{(2)} = & \frac{-ip_a}{\rho\omega} \left(\tilde{B}_0 + \tilde{C}_0 \ln r \right) \\ & + \frac{-ip_a}{\rho\omega} \sum_{n=1}^{\infty} \left(\tilde{B}_n \frac{I_0(\alpha_n r)}{\alpha_n I'_0(\alpha_n c)} + \tilde{C}_n \frac{K_0(\alpha_n r)}{\alpha_n K'_0(\alpha_n a)} \right) f_n(z), \end{aligned} \quad (5.108)$$

$$\phi_P^{(3)} = \frac{-ip_a}{\rho\omega} \sum_{l=0}^{\infty} \tilde{D}_l \frac{I_0(k_l r)}{k_l I'_0(k_l a)} Z_l(z) - \frac{ip_a}{\rho\omega}. \quad (5.109)$$

where $\tilde{A}_l$, $\tilde{B}_n$, $\tilde{C}_n$ and $\tilde{D}_l$ all are unknown coefficients. Note that the last term in Eqn. (5.109) is introduced to satisfy the inhomogeneous free surface condition Eqn. (5.106). In addition to the no-flux conditions on vertical solid walls at $r = c$ and a , all other conditions have been satisfied by the above expressions in Eqns. (5.107)-(5.109).

5.6.2 Auxiliary radial velocity functions

To obtain the expressions of $\tilde{A}_l$ - $\tilde{D}_l$, let us introduce two auxiliary functions to represent the radial velocity distributions on two interface (at $r = c$ and a), they are

(i) $U_P^{(1)}(z)$ the radial velocity distribution at $r = c$;

(ii) $U_P^{(2)}(z)$ the radial velocity distribution at $r = a$.

Note here that the auxiliary functions $U_P^{(1)}$ and $U_P^{(2)}$ exclude the angular variation as well due to the axisymmetric properties of OWC system.

Similar to the diffraction problem, let us expand these two auxiliary functions as

$$U_P^{(1)}(z) = \frac{-ip_a}{\rho\omega} \sum_{p=0}^{\infty} \tilde{\gamma}_p u_p(z), \quad (5.110)$$

and

$$U_P^{(2)}(z) = \frac{-ip_a}{\rho\omega} \sum_{p=0}^{\infty} \tilde{\xi}_p u_p(z). \quad (5.111)$$

The same expansion base functions $u_p(z)$ in Eqn. (5.42) have been adopted due to the same singular behavior of the flow field at sharp corners with an interior angle of 90° .

5.6.3 Determinations of unknown coefficients $\tilde{A}_l$ - $\tilde{D}_l$

The continuity of radial velocities across common boundaries at $r = c$ and a requires

$$\frac{\partial \phi_P^{(1)}}{\partial r} \Big|_{r=c} = \begin{cases} 0, & -d < z < 0, \\ U_P^{(1)}(z), & -h < z < -d, \end{cases} \quad (5.112)$$

$$\frac{\partial \phi_P^{(2)}}{\partial r} \Big|_{r=c} = U_P^{(1)}(z), \quad -h < z < -d, \quad (5.113)$$

$$\frac{\partial \phi_P^{(2)}}{\partial r} \Big|_{r=a} = U_P^{(2)}(z), \quad -h < z < -d, \quad (5.114)$$

and

$$\frac{\partial \phi_P^{(3)}}{\partial r} \Big|_{r=a} = \begin{cases} 0, & -d < z < 0, \\ U_P^{(2)}(z), & -h < z < -d. \end{cases} \quad (5.115)$$

Multiplying both sides of Eqn. (5.112) by $Z_{l'}(z)/h$ and integrating the resulting equation with respect to z over the corresponding intervals gives

$$\frac{1}{h} \int_{-h}^0 \frac{\partial \phi_P^{(1)}}{\partial r} \Big|_{r=c} Z_{l'}(z) dz = \frac{1}{h} \int_{-h}^{-d} U_P^{(1)}(z) Z_{l'}(z) dz, \quad (5.116)$$

Inserting Eqns. (5.107) and (5.110) into Eqn. (5.116), after using the orthogonal relation in Eqn. (2.18), we have

$$\tilde{A}_l = \sum_{p=0}^{\infty} \tilde{\gamma}_p V_{pl}^{(1)}. \quad (5.117)$$

Multiplying both sides of Eqns. (5.113) and (5.114) by $f_{n'}(z)/(h-d)$ and then integrating the resulting equations with respect to z over $[-h, -d]$ gives

$$\frac{1}{h-d} \int_{-h}^{-d} \frac{\partial \phi_P^{(2)}}{\partial r} \Big|_{r=c} f_{n'}(z) dz = \frac{1}{h-d} \int_{-h}^{-d} U_P^{(1)}(z) f_{n'}(z) dz, \quad (5.118)$$

and

$$\frac{1}{h-d} \int_{-h}^{-d} \frac{\partial \phi_P^{(2)}}{\partial r} \Big|_{r=a} f_{n'}(z) dz = \frac{1}{h-d} \int_{-h}^{-d} U_P^{(2)}(z) f_{n'}(z) dz. \quad (5.119)$$

After inserting Eqns. (5.108), (5.110), and (5.111) into Eqns. (5.118) and (5.119), respectively, and using the orthogonal relation in Eqn. (5.57), we have

$$\tilde{C}_0 = c \sum_{p=0}^{\infty} \tilde{\gamma}_p V_{p0}^{(2)}, \quad \text{or} \quad \tilde{C}_0 = a \sum_{p=0}^{\infty} \tilde{\xi}_p V_{p0}^{(2)}, \quad (5.120)$$

$$\left(\tilde{B}_n + \tilde{C}_n \frac{K'_0(\alpha_n c)}{K'_0(\alpha_n a)} \right) = 2 \sum_{p=0}^{\infty} \tilde{\gamma}_p V_{pn}^{(2)}, \quad n = 1, 2, \dots, \quad (5.121)$$

and

$$\left(\tilde{B}_n \frac{I'_0(\alpha_n a)}{I'_0(\alpha_n c)} + \tilde{C}_n \right) = 2 \sum_{p=0}^{\infty} \tilde{\xi}_p V_{pn}^{(2)}, \quad n = 1, 2, \dots \quad (5.122)$$

Solving the linear system of Eqns. (5.121) and (5.122) gives

$$\tilde{B}_n = 2\tilde{R}_n^{(1)} \sum_{p=0}^{\infty} \tilde{\gamma}_p V_{pn}^{(2)} - 2\tilde{R}_n^{(1)} \frac{K'_0(\alpha_n c)}{K'_0(\alpha_n a)} \sum_{p=0}^{\infty} \tilde{\xi}_p V_{pn}^{(2)}, \quad n \neq 0, \quad (5.123)$$

and

$$\tilde{C}_n = 2\tilde{R}_n^{(1)} \sum_{p=0}^{\infty} \tilde{\xi}_p V_{pn}^{(2)} - 2\tilde{R}_n^{(1)} \frac{I'_0(\alpha_n a)}{I'_0(\alpha_n c)} \sum_{p=0}^{\infty} \tilde{\gamma}_p V_{pn}^{(2)}, \quad n \neq 0, \quad (5.124)$$

where $\tilde{R}_n^{(1)}$ are given by

$$\tilde{R}_n^{(1)} = 1 / \left(1 - \frac{I'_0(\alpha_n a)}{I'_0(\alpha_n c)} \frac{K'_0(\alpha_n c)}{K'_0(\alpha_n a)} \right). \quad (5.125)$$

Note here that combining these two equations in Eqn. (5.120) gives the following relation,

$$c \sum_{p=0}^{\infty} \tilde{\gamma}_p V_{p0}^{(2)} - a \sum_{p=0}^{\infty} \tilde{\xi}_p V_{p0}^{(2)} = 0. \quad (5.126)$$

Similarly, multiplying both sides of Eqn. (5.115) by $Z_{l'}(z)/h$ and then integrating the resulting equation over the corresponding intervals gives

$$\frac{1}{h} \int_{-h}^0 \frac{\partial \phi_P^{(3)}}{\partial r} \Big|_{r=a} Z_{l'}(z) dz = \frac{1}{h} \int_{-h}^{-d} \tilde{U}_2(z) Z_{l'}(z) dz. \quad (5.127)$$

Inserting Eqns. (5.109) and (5.111) into Eqn. (5.127) and then using the orthogonal relation of function $Z_l(z)$ gives

$$\tilde{D}_l = \sum_{p=0}^{\infty} \tilde{\xi}_p V_{pl}^{(1)}. \quad (5.128)$$

Now, except $\tilde{B}_0$, the expressions of $\tilde{A}_l$ - $\tilde{D}_l$ have been derived in terms of unknown coefficients $\tilde{\gamma}_p$ and $\tilde{\xi}_p$.

5.6.4 Determinations of the auxiliary functions $U_P^{(1)}$ and $U_P^{(2)}$

In addition to the requirement of continuity of radial velocities across these two common boundaries, pressure must be continuous across these two boundaries (i.e., at $r = c$ and a) as well. Hence we have the following relations,

$$\phi_P^{(1)}|_{r=c} = \phi_P^{(2)}|_{r=c}, \quad -h < z < -d, \quad (5.129)$$

and

$$\phi_P^{(2)}|_{r=a} = \phi_P^{(3)}|_{r=a}, \quad -h < z < -d. \quad (5.130)$$

Multiplying both sides of Eqn. (5.129) by $u_{p'}(z)$ and integrating the resulting equation over $[-h, -d]$, and after using Eqns. (5.54) and (5.60), we have

$$\begin{aligned} & h \sum_{l=0}^{\infty} \tilde{A}_l \frac{K_0(k_l c)}{k_l K'_0(k_l c)} V_{p'l}^{(1)} \\ &= (h-d) \left(\tilde{B}_0 + \tilde{C}_0 \ln c \right) V_{p'0}^{(2)} \\ &+ (h-d) \sum_{n=1}^{\infty} \left(\tilde{B}_n \frac{I_0(\alpha_n c)}{\alpha_n I'_0(\alpha_n c)} + \tilde{C}_n \frac{K_0(\alpha_n c)}{\alpha_n K'_0(\alpha_n a)} \right) V_{p'n}^{(2)}. \end{aligned} \quad (5.131)$$

Inserting Eqns. (5.117), (5.120), (5.123) and (5.124) into Eqn. (5.131) and rearranging gives

$$\begin{aligned} & - (h-d) \tilde{B}_0 V_{p'0}^{(2)} + \sum_{p=0}^{\infty} \tilde{\gamma}_p h \sum_{l=0}^{\infty} \frac{K_0(k_l c)}{k_l K'_0(k_l c)} V_{pl}^{(1)} V_{p'l}^{(1)} \\ & - (h-d) c \ln c \sum_{p=0}^{\infty} \tilde{\gamma}_p V_{p0}^{(2)} V_{p'0}^{(2)} - \sum_{p=0}^{\infty} \tilde{\gamma}_p (h-d) \sum_{n=1}^{\infty} 2 \tilde{R}_n^{(1)} \tilde{Q}_n^{(1)} V_{pn}^{(2)} V_{p'n}^{(2)} \\ & - \sum_{p=0}^{\infty} \tilde{\xi}_p (h-d) \sum_{n=1}^{\infty} 2 \tilde{R}_n^{(1)} \tilde{Q}_n^{(2)} V_{pn}^{(2)} V_{p'n}^{(2)} = 0, \end{aligned} \quad (5.132)$$

where $\tilde{Q}_n^{(1)}$ and $\tilde{Q}_n^{(2)}$ are given by

$$\tilde{Q}_n^{(1)} = \frac{I_0(\alpha_n c)}{\alpha_n I_0'(\alpha_n c)} - \frac{I_0'(\alpha_n a)}{\alpha_n I_0'(\alpha_n c)} \frac{K_0(\alpha_n c)}{K_0'(\alpha_n a)}, \quad (5.133)$$

$$\tilde{Q}_n^{(2)} = \frac{K_0(\alpha_n c)}{\alpha_n K_0'(\alpha_n a)} - \frac{I_0(\alpha_n c)}{\alpha_n I_0'(\alpha_n c)} \frac{K_0'(\alpha_n c)}{K_0'(\alpha_n a)}, \quad (5.134)$$

respectively.

Multiplying Eqn. (5.130) by $u_{p'}(z)$ and integrating the resulting equation over $[-h, -d]$, and after using Eqns. (5.54) and (5.60), we obtain

$$\begin{aligned} & h \sum_{l=0}^{\infty} \tilde{D}_l \frac{I_0(k_l a)}{k_l I_0'(k_l a)} V_{p'l}^{(1)} + (h-d) V_{p'0}^{(2)} \\ &= (h-d) \left(\tilde{B}_0 + \tilde{C}_0 \ln a \right) V_{p'0}^{(2)} \\ &+ \sum_{n=1}^{\infty} \left(\tilde{B}_n \frac{I_0(\alpha_n a)}{\alpha_n I_0'(\alpha_n c)} + \tilde{C}_n \frac{K_0(\alpha_n a)}{\alpha_n K_0'(\alpha_n a)} \right) V_{p'n}^{(2)}. \end{aligned} \quad (5.135)$$

Inserting Eqns. (5.120), (5.123)-(5.124) and (5.128) into Eqn. (5.135) and rearranging gives

$$\begin{aligned} & - (h-d) \tilde{B}_0 V_{p'0}^{(2)} - \sum_{p=0}^{\infty} \tilde{\gamma}_p (h-d) \sum_{n=1}^{\infty} 2 \tilde{R}_n^{(1)} \tilde{Q}_n^{(3)} V_{pn}^{(2)} V_{p'n}^{(2)} \\ & - (h-d) a \ln a \sum_{p=0}^{\infty} \tilde{\xi}_p V_{p0}^{(2)} V_{p'0}^{(2)} + \sum_{p=0}^{\infty} \tilde{\xi}_p h \sum_{l=0}^{\infty} \frac{I_0(k_l a)}{k_l I_0'(k_l a)} V_{pl}^{(1)} V_{p'l}^{(1)} \\ & - \sum_{p=0}^{\infty} \tilde{\xi}_p (h-d) \sum_{n=1}^{\infty} 2 \tilde{R}_n^{(1)} \tilde{Q}_n^{(4)} V_{pn}^{(2)} V_{p'n}^{(2)} = - (h-d) V_{p'0}^{(2)}, \end{aligned} \quad (5.136)$$

where $\tilde{Q}_n^{(3)}$ and $\tilde{Q}_n^{(4)}$ are given by

$$\tilde{Q}_n^{(3)} = \frac{I_0(\alpha_n a)}{\alpha_n I_0'(\alpha_n c)} - \frac{I_0'(\alpha_n a)}{\alpha_n I_0'(\alpha_n c)} \frac{K_0(\alpha_n a)}{K_0'(\alpha_n a)}, \quad (5.137)$$

and

$$\tilde{Q}_n^{(4)} = \frac{K_0(\alpha_n a)}{\alpha_n K_0'(\alpha_n a)} - \frac{I_0(\alpha_n a)}{\alpha_n I_0'(\alpha_n c)} \frac{K_0'(\alpha_n c)}{K_0'(\alpha_n a)}, \quad (5.138)$$

respectively.

Observing Eqns. (5.132) and (5.136), in addition to unknown coefficients $\tilde{\gamma}_p$ and $\tilde{\xi}_p$ ($p = 0, 1, 2, \dots$), there is another unknown coefficient $\tilde{B}_0$. Hence, let us combine Eqns. (5.126), (5.132) and (5.136) to form a closed infinite linear system. Assuming the p -series truncated after the P th term, there are $(2P+3)$ unknowns (i.e., $\tilde{B}_0, \tilde{\gamma}_0, \dots, \tilde{\gamma}_P, \tilde{\xi}_0, \dots, \tilde{\xi}_P$) and $(2P+3)$ linear equations in this linear system.

Once this linear system has been solved numerically, the unknown coefficients $\tilde{A}_l, \tilde{D}_l$ can be calculated by Eqns. (5.117), (5.120), (5.123)-(5.124), and (5.128).

5.6.5 The radiation conductance and susceptance: $\mathcal{B}$ and $\mathcal{C}$

Now let us derive the expressions for the volume flux through air turbine at top and the vertical exciting force due to the oscillating air pressure inside the chamber and the corresponding quantities such as $\mathcal{B}$ (radiation conductance), $\mathcal{C}$ (radiation susceptance), μ_{p3} and ν_{p3} .

According to Eqn. (3.2), the volume flux q_P can be calculate by

$$\begin{aligned} q_P &= \int_0^a \int_0^{2\pi} \frac{\partial \phi_P^{(3)}}{\partial z} \Big|_{z=0} r dr d\theta \\ &= \frac{-ip_a}{\rho\omega} 2\pi \sum_{l=0}^{\infty} \tilde{D}_l Z'_l(0) \frac{a}{k_l^2}, \end{aligned} \quad (5.139)$$

where the identity

$$\int_0^a \frac{r I_0(k_l r)}{k_l a I'_0(k_l a)} dr = \frac{1}{k_l^2}$$

has been used.

Based on the decomposition in Eqn. (3.6), the radiation conductance $\mathcal{B}$ and radiation susceptance $\mathcal{C}$ are given by

$$\mathcal{B} = \text{Re} \left\{ \frac{2\pi i}{\rho\omega} \sum_{l=0}^{\infty} \hat{D}_l Z'_l(0) \frac{a}{k_l^2} \right\} \quad (5.140)$$

and

$$\mathcal{C} = \text{Im} \left\{ \frac{-2\pi i}{\rho\omega} \sum_{l=0}^{\infty} \hat{D}_l Z'_l(0) \frac{a}{k_l^2} \right\}, \quad (5.141)$$

respectively.

According to Eqn. (5.8), the vertical exciting force f_{p3} due to the oscillating air pressure inside the chamber is calculated by

$$\begin{aligned} f_{p3} &= i\rho\omega \int_a^c \int_0^{2\pi} \phi_P^{(2)} \Big|_{z=-d} r d\theta dr \\ &= 2\pi p_a \left[\tilde{B}_0 \frac{c^2 - a^2}{2} + \frac{\tilde{C}_0}{4} (a^2 - c^2 - 2a^2 \ln a + 2c^2 \ln c) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \left(\tilde{B}_n \frac{c}{\alpha_n^2} \left(1 - \frac{a I_1(\alpha_n a)}{c I_1(\alpha_n c)} \right) + \tilde{C}_n \frac{a}{\alpha_n^2} \left(\frac{c K_1(\alpha_n c)}{a K_1(\alpha_n a)} - 1 \right) \right) f_n(-d) \right]. \end{aligned} \quad (5.142)$$

Based on the decomposition in Eqn. (5.8), the expressions for the components μ_{p3} and ν_{p3} are given by

$$\mu_{p3} = \frac{1}{\omega} \text{Im} \{ f_{p3}/p_a \}, \quad (5.143)$$

and

$$\nu_{p3} = -\operatorname{Re}\{f_{p3}/p_a\}, \quad (5.144)$$

respectively.

Let us introduce the following dimensionless parameters,

$$\{\tilde{\mathcal{B}}, \tilde{\mathcal{C}}\} = \{\mathcal{B}, \mathcal{C}\} \frac{\rho\sqrt{g/h}}{h}, \quad \{\tilde{\mu}_{p3}, \tilde{\nu}_{p3}\} = \{\mu_{p3}, \nu_{p3}\} / (\pi c^2). \quad (5.145)$$

The dimensionless quantities $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$, $\tilde{\mu}_{p3}$ and $\tilde{\nu}_{p3}$ can be calculated easily by inserting Eqns. (5.140)-(5.141) and (5.143)-(5.144) into Eqn. (5.145).

5.7 Radiation problem for the OWC system in heaving motion

In the absence of incident waves, the radiation problem due to the OWC system oscillating in heave mode (unit amplitude of vertical velocity) with the top of the column open to the atmosphere is considered in this section.

Similar to the previous sections, the notations $\phi_3^{(1)}$, $\phi_3^{(2)}$ and $\phi_3^{(3)}$ are introduced to represent the complex radiation potential of unit velocity amplitude in region (1), (2) and (3), respectively. These three radiation potentials are also governed by Laplace's equation and the free surface condition with $p_a = 0$. On solid walls, they separately satisfy

$$\frac{\partial \phi_3^{(1)}}{\partial r} \Big|_{r=c} = 0, \quad -d < z < 0, \quad (5.146)$$

$$\frac{\partial \phi_3^{(2)}}{\partial z} \Big|_{z=-d} = n_3, \quad a < r < c, \quad (5.147)$$

$$\frac{\partial \phi_3^{(3)}}{\partial r} \Big|_{r=a} = 0, \quad -d < z < 0, \quad (5.148)$$

where n_3 ($= 1$) is the unit inward normal in z -direction. In addition, the complex potential in region (1) must behave as outgoing waves at infinity.

5.7.1 Formal expressions for the complex radiation potentials in different regions

Owing to the axisymmetric property of the OWC system and the θ -independence of the heaving motion, the three radiation potentials can be respectively expanded as follows:

$$\phi_3^{(1)} = \sum_{l=0}^{\infty} \hat{A}_l \frac{K_0(k_l r)}{k_l K'_0(k_l c)} Z_l(z), \quad (5.149)$$

$$\begin{aligned} \phi_3^{(2)} = & \hat{B}_0 + \hat{C}_0 \ln r + \sum_{n=1}^{\infty} \left(\hat{B}_n \frac{I_0(\alpha_n r)}{\alpha_n I'_0(\alpha_n c)} + \hat{C}_n \frac{K_0(\alpha_n r)}{\alpha_n K'_0(\alpha_n a)} \right) f_n(z) \\ & + \frac{(z+h)^2 - r^2/2}{2(h-d)}, \end{aligned} \quad (5.150)$$

$$\phi_3^{(3)} = \sum_{l=0}^{\infty} \hat{D}_l \frac{I_0(k_l r)}{k_l I'_0(k_l a)} Z_l(z), \quad (5.151)$$

where $\hat{A}_l$, $\hat{B}_n$, $\hat{C}_n$ and $\hat{D}_l$ are unknown expansion coefficients to be determined. Note that the last term in Eqn. (5.150) is introduced to satisfy the inhomogeneous no-flux condition Eqn. (5.147) (Yeung, 1981). Except the radial solid boundary conditions at $r = c$ and a for $-d < z < 0$, the above formal expressions satisfy all other boundary conditions.

5.7.2 Auxiliary radial velocity functions

Similar to the previous sections, let us introduce two auxiliary functions to represent the distributions of radial velocities at the following two common boundaries:

- $U_3^{(1)}(z)$ for the common boundary at $r = c$ defined by $-h < z < -d$,
- $U_3^{(2)}(z)$ for the common boundary at $r = a$ defined by $-h < z < -d$.

To facilitate the determinations of $\hat{A}_l$ - $\hat{D}_l$, the above two unknown auxiliary functions may be expanded as

$$U_3^{(1)}(z) = \sum_{p=0}^{\infty} \hat{\gamma}_p u_p(z), \quad -h < z < -d, \quad (5.152)$$

and

$$U_3^{(2)}(z) = \sum_{p=0}^{\infty} \hat{\xi}_p u_p(z), \quad -h < z < -d, \quad (5.153)$$

respectively. Where $\hat{\gamma}_p$ and $\hat{\xi}_p$ are unknown expansion coefficients. Note here that the same z -dependence expansion basis functions $u_p(z)$ as those in the previous sections are adopted due to the same singular behaviors of flow field at sharp corners.

5.7.3 Determinations of the unknowns $\hat{A}_l$ - $\hat{D}_l$

Based on the requirements of the radial velocities across these two common interfaces (at $r = c$ and a), we have

$$\frac{\partial \phi_3^{(1)}}{\partial r} \Big|_{r=c} = \begin{cases} 0, & -d < z < 0, \\ U_3^{(1)}, & -h < z < -d, \end{cases} \quad (5.154)$$

$$\frac{\partial \phi_3^{(2)}}{\partial r} \Big|_{r=c} = U_3^{(1)}, \quad -h < z < -d, \quad (5.155)$$

$$\frac{\partial \phi_3^{(2)}}{\partial r} \Big|_{r=a} = U_3^{(2)}, \quad -h < z < -d, \quad (5.156)$$

and

$$\frac{\partial \phi_3^{(3)}}{\partial r} \Big|_{r=a} = \begin{cases} 0, & -d < z < 0, \\ U_3^{(2)}, & -h < z < -d. \end{cases} \quad (5.157)$$

Multiplying both sides of Eqn. (5.154) by $Z_{l'}(z)/h$ and integrating the resulting equation over the corresponding intervals gives

$$\frac{1}{h} \int_{-h}^0 \frac{\partial \phi_3^{(1)}}{\partial r} \Big|_{r=c} Z_{l'}(z) dz = \frac{1}{h} \int_{-h}^{-d} U_3^{(1)}(z) Z_{l'}(z) dz. \quad (5.158)$$

After inserting Eqns. (5.149) and (5.152) into Eqn. (5.158) and using the orthogonal relation of $Z_l(z)$, we have

$$\hat{A}_l = \sum_{p=0}^{\infty} \hat{\gamma}_p V_{pl}^{(1)}. \quad (5.159)$$

Multiplying both sides of Eqns. (5.155) and (5.156) by $f_{n'}(z)/(h-d)$, respectively, and integrating the resulting equations over the corresponding intervals gives

$$\frac{1}{h-d} \int_{-h}^{-d} \frac{\partial \phi_3^{(2)}}{\partial r} \Big|_{r=c} f_{n'}(z) dz = \frac{1}{h-d} \int_{-h}^{-d} U_3^{(1)} f_{n'}(z) dz, \quad (5.160)$$

and

$$\frac{1}{h-d} \int_{-h}^{-d} \frac{\partial \phi_3^{(2)}}{\partial r} \Big|_{r=a} f_{n'}(z) dz = \frac{1}{h-d} \int_{-h}^{-d} U_3^{(2)} f_{n'}(z) dz. \quad (5.161)$$

After inserting Eqns. (5.150) and (5.152)-(5.153) into Eqns. (5.160)-(5.161), respectively, and using the orthogonal relation of $f_n(z)$, we obtain

$$\hat{C}_0 = c \left(\sum_{p=0}^{\infty} \hat{\gamma}_p V_{p0}^{(2)} + \frac{c}{2(h-d)} \right) \quad \text{or} \quad \hat{C}_0 = a \left(\sum_{p=0}^{\infty} \hat{\xi}_p V_{p0}^{(2)} + \frac{a}{2(h-d)} \right), \quad (5.162)$$

and

$$\begin{cases} \left(\hat{B}_n + \hat{C}_n \frac{K'_0(\alpha_n c)}{K'_0(\alpha_n a)} \right) = 2 \sum_{p=0}^{\infty} \hat{\gamma}_p V_{pn}^{(2)}, & n = 1, 2, 3, \dots, \\ \left(\hat{B}_n \frac{I'_0(\alpha_n a)}{I'_0(\alpha_n c)} + \hat{C}_n \right) = 2 \sum_{p=0}^{\infty} \hat{\xi}_p V_{pn}^{(2)}, & n = 1, 2, 3, \dots \end{cases} \quad (5.163)$$

Further, after solving the linear system in Eqn. (5.163), we have

$$\hat{B}_n = 2\hat{Q}_n^{(1)} \left(\sum_{p=0}^{\infty} \hat{\gamma}_p V_{pn}^{(2)} - \frac{K'_0(\alpha_n c)}{K'_0(\alpha_n a)} \sum_{p=0}^{\infty} \hat{\xi}_p V_{pn}^{(2)} \right), \quad n = 1, 2, 3, \dots, \quad (5.164)$$

and

$$\hat{C}_n = 2\hat{Q}_n^{(1)} \left(\sum_{p=0}^{\infty} \hat{\xi}_p V_{pn}^{(2)} - \frac{I'_0(\alpha_n a)}{I'_0(\alpha_n c)} \sum_{p=0}^{\infty} \hat{\gamma}_p V_{pn}^{(2)} \right), \quad n = 1, 2, 3, \dots, \quad (5.165)$$

with $\hat{Q}_n^{(1)}$ are given by

$$\hat{Q}_n^{(1)} = 1 / \left(1 - \frac{I'_0(\alpha_n a)}{I'_0(\alpha_n c)} \frac{K'_0(\alpha_n c)}{K'_0(\alpha_n a)} \right), \quad n = 1, 2, 3, \dots \quad (5.166)$$

Note that combining the two equations in Eqn. (5.162) gives the following relation,

$$c \sum_{p=0}^{\infty} \hat{\gamma}_p V_{p0}^{(2)} - a \sum_{p=0}^{\infty} \hat{\xi}_p V_{p0}^{(2)} + \frac{c^2 - a^2}{2(h-d)} = 0. \quad (5.167)$$

Similarly, multiplying both sides of Eqn. (5.157) by $Z_{l'}(z)/h$ and integrating over the resulting equation over the corresponding intervals gives

$$\frac{1}{h} \int_{-h}^0 \frac{\partial \phi_3^{(3)}}{\partial r} \Big|_{r=a} Z_{l'}(z) dz = \frac{1}{h} \int_{-h}^{-d} U_3^{(2)} Z_{l'}(z) dz. \quad (5.168)$$

After inserting Eqns. (5.151) and (5.153) into Eqn. (5.168) and using the orthogonal relation of $Z_l(z)$, we obtain

$$\hat{D}_l = \sum_{p=0}^{\infty} \hat{\xi}_p V_{pl}^{(1)}. \quad (5.169)$$

Except $\hat{B}_0$, the solutions to $\hat{A}_l - \hat{D}_l$ have been obtained in terms of unknown coefficients $\hat{\gamma}_p$ and $\hat{\xi}_p$.

5.7.4 Determinations of the auxiliary functions $U_3^{(1)}$ and $U_3^{(2)}$

The continuity of pressure across the common boundaries requires

$$\phi_3^{(1)}|_{r=c} = \phi_3^{(2)}|_{r=c}, \quad -h < z < -d, \quad (5.170)$$

and

$$\phi_3^{(2)}|_{r=a} = \phi_3^{(3)}|_{r=a}, \quad -h < z < -d. \quad (5.171)$$

Multiplying both sides of Eqn. (5.170) by $u_{p'}(z)$ and integrating the resulting equation with respect to z over $[-h, -d]$, after inserting Eqns. (5.149)-(5.150) and using Eqns. (5.54) and (5.60), gives

$$\begin{aligned} & h \sum_{l=0}^{\infty} \hat{A}_l \frac{K_0(k_l c)}{k_l K'_0(k_l c)} V_{p'l}^{(1)} - (h-d) \left(\hat{B}_0 + \hat{C}_0 \ln c \right) V_{p'0}^{(2)} \\ & - (h-d) \sum_{n=1}^{\infty} \left(\hat{B}_n \frac{I_0(\alpha_n c)}{\alpha_n I'_0(\alpha_n c)} + \hat{C}_n \frac{K_0(\alpha_n c)}{\alpha_n K'_0(\alpha_n a)} \right) V_{p'n}^{(2)} \\ & = \frac{\hat{V}_{p'}}{2} - \frac{c^2}{4} V_{p'0}^{(2)}, \end{aligned} \quad (5.172)$$

where $\hat{V}_p$ are given by

$$\hat{V}_p = \frac{1}{h-d} \int_{-h}^{-d} u_p(z) (z+h)^2 dz. \quad (5.173)$$

After inserting Eqns. (5.159), (5.162) and (5.164)-(5.165) into Eqn. (5.172) and re-arranging, we have

$$\begin{aligned} & - (h-d) \hat{B}_0 V_{p'0}^{(2)} + \sum_{p=0}^{\infty} \hat{\gamma}_p h \sum_{l=0}^{\infty} \frac{K_0(k_l c)}{k_l K'_0(k_l c)} V_{pl}^{(1)} V_{p'l}^{(1)} \\ & - (h-d) c \ln c \sum_{p=0}^{\infty} \hat{\gamma}_p V_{p0}^{(2)} V_{p'0}^{(2)} - \sum_{p=0}^{\infty} \hat{\gamma}_p 2(h-d) \sum_{n=1}^{\infty} \hat{Q}_n^{(1)} \hat{R}_n^{(1)} V_{pn}^{(2)} V_{p'n}^{(2)} \\ & - \sum_{p=0}^{\infty} \hat{\xi}_p 2(h-d) \sum_{n=1}^{\infty} \tilde{Q}_n^{(1)} \hat{R}_n^{(2)} V_{pn}^{(2)} V_{p'n}^{(2)} \\ & = \frac{\hat{V}_{p'}}{2} + \frac{c^2}{2} \left(\ln c - \frac{1}{2} \right) V_{p'0}^{(2)}, \end{aligned} \quad (5.174)$$

where $\hat{R}_n^{(1)}$ and $\hat{R}_n^{(2)}$ are given by

$$\hat{R}_n^{(1)} = \frac{I_0(\alpha_n c)}{\alpha_n I'_0(\alpha_n c)} - \frac{I'_0(\alpha_n a)}{I'_0(\alpha_n c)} \frac{K_0(\alpha_n c)}{\alpha_n K'_0(\alpha_n a)}, \quad (5.175)$$

and

$$\hat{R}_n^{(2)} = \frac{K_0(\alpha_n c)}{\alpha_n K'_0(\alpha_n a)} - \frac{I_0(\alpha_n c)}{\alpha_n I'_0(\alpha_n c)} \frac{K'_0(\alpha_n c)}{K'_0(\alpha_n a)}, \quad (5.176)$$

respectively.

Again, multiplying both sides of Eqn. (5.171) by $u_{p'}(z)$ and integrating the resulting equation with respect to z over $[-h, -d]$, after inserting Eqns. (5.150)-(5.151) and using Eqns. (5.54), (5.60) and (5.173), gives

$$\begin{aligned} & (h-d) \left(\hat{B}_0 + \hat{C}_0 \ln a \right) V_{p'0}^{(2)} \\ & + (h-d) \sum_{n=1}^{\infty} \left(\hat{B}_n \frac{I_0(\alpha_n a)}{\alpha_n I'_0(\alpha_n c)} + \hat{C}_n \frac{K_0(\alpha_n a)}{\alpha_n K'_0(\alpha_n a)} \right) V_{p'n}^{(2)} \\ & - h \sum_{l=0}^{\infty} \hat{D}_l \frac{I_0(k_l a)}{k_l I'_0(k_l a)} V_{p'l}^{(1)} = \frac{a^2}{4} V_{p'0}^{(2)} - \frac{\hat{V}_{p'}}{2}. \end{aligned} \quad (5.177)$$

After inserting Eqns. (5.162), (5.164)-(5.165) and (5.169) and re-arranging, we have

$$\begin{aligned} & (h-d) \hat{B}_0 V_{p'0}^{(2)} + \sum_{p=0}^{\infty} \hat{\gamma}_p 2(h-d) \sum_{n=1}^{\infty} \hat{Q}_n^{(1)} \hat{R}_n^{(3)} V_{pn}^{(2)} V_{p'n}^{(2)} \\ & + (h-d) a \ln a \sum_{p=0}^{\infty} \hat{\xi}_p V_{p0}^{(2)} V_{p'0}^{(2)} - \sum_{p=0}^{\infty} \hat{\xi}_p h \sum_{l=0}^{\infty} \frac{I_0(k_l a)}{k_l I'_0(k_l a)} V_{pl}^{(1)} V_{p'l}^{(1)} \\ & + \sum_{p=0}^{\infty} \hat{\xi}_p 2(h-d) \sum_{n=1}^{\infty} \hat{Q}_n^{(1)} \hat{R}_n^{(4)} V_{pn}^{(2)} V_{p'n}^{(2)} \\ & = \frac{a^2}{2} \left(\frac{1}{2} - \ln a \right) V_{p'0}^{(2)} - \frac{\hat{V}_{p'}}{2}, \end{aligned} \quad (5.178)$$

where $\hat{R}_n^{(3)}$ and $\hat{R}_n^{(4)}$ are separately given by

$$\hat{R}_n^{(3)} = \frac{I_0(\alpha_n a)}{\alpha_n I'_0(\alpha_n c)} - \frac{I'_0(\alpha_n a)}{I'_0(\alpha_n c)} \frac{K_0(\alpha_n a)}{\alpha_n K'_0(\alpha_n a)}, \quad (5.179)$$

and

$$\hat{R}_n^{(4)} = \frac{K_0(\alpha_n a)}{\alpha_n K'_0(\alpha_n a)} - \frac{I_0(\alpha_n a)}{\alpha_n I'_0(\alpha_n c)} \frac{K'_0(\alpha_n c)}{K'_0(\alpha_n a)}. \quad (5.180)$$

Let us combine Eqns. (5.167), (5.174) and (5.178) to form a closed linear system with infinite equations. Assuming the truncated p -series with $(\mathcal{P} + 1)$ terms (i.e., $p = 0, 1, \dots, \mathcal{P}$), the resulting finite linear system includes $(2\mathcal{P} + 3)$ unknowns (they are, $\hat{B}_0, \hat{\gamma}_0, \dots, \hat{\gamma}_{\mathcal{P}}, \hat{\xi}_0, \dots, \hat{\xi}_{\mathcal{P}}$) and $(2\mathcal{P} + 3)$ linear algebraic equations. Once this system is solved numerically, the numerical values of $\hat{A}_l - \hat{D}_l$ can be calculated by Eqns. (5.159), (5.162), (5.164)-(5.165) and (5.169), straightforwardly.

5.7.5 The added mass μ_{33} and damping ν_{33}

According to Eqn. (5.9), the volume flux q_3 due to the heaving motion of the OWC system can be calculated by

$$\begin{aligned} q_3 &= \int_0^a \int_0^{2\pi} \frac{\partial \phi_3^{(3)}}{\partial z} \Big|_{z=0} r dr d\theta \\ &= 2\pi \sum_{l=0}^{\infty} \hat{D}_l \frac{a Z'_l(0)}{k_l^2}, \end{aligned} \quad (5.181)$$

where the identity $\int_0^a \frac{r I_0(k_l r)}{k_l a I'_0(k_l a)} dr = \frac{1}{k_l^2}$ has been used. Further, using the decomposition in Eqn. (5.9), the expressions of $\mathcal{B}_{3q}$ and $\mathcal{C}_{3q}$ can be written as

$$\mathcal{B}_{3q} = -\operatorname{Re} \left\{ 2\pi \sum_{l=0}^{\infty} \hat{D}_l \frac{a Z'_l(0)}{k_l^2} \right\}, \quad (5.182)$$

and

$$\mathcal{C}_{3q} = \frac{1}{\omega} \operatorname{Im} \left\{ 2\pi \sum_{l=0}^{\infty} \hat{D}_l \frac{a Z'_l(0)}{k_l^2} \right\}. \quad (5.183)$$

According to Eqn. (5.9), the vertical hydrodynamic reaction force f_{33} due to radiation potential ϕ_3 can be calculated by

$$\begin{aligned} f_{33} &= i\rho\omega \int_a^c \int_0^{2\pi} \phi_3^{(3)} r d\theta dr \\ &= i\omega\rho 2\pi \left[\left(\hat{B}_0 + \frac{h-d}{2} \right) \frac{c^2 - a^2}{2} \right. \\ &\quad + \frac{\hat{C}_0}{4} (a^2 (1 - 2 \ln a) - c^2 (1 - 2 \ln c)) - \frac{c^4 - a^4}{16 (h-d)} \\ &\quad \left. + \sum_{n=1}^{\infty} \left(\frac{c \hat{B}_n}{\alpha_n^2} \left(1 - \frac{a I_1(\alpha_n a)}{c I_1(\alpha_n c)} \right) + \frac{a \hat{C}_n}{\alpha_n^2} \left(\frac{c K_1(\alpha_n c)}{a K_1(\alpha_n a)} - 1 \right) \right) f_n(-d) \right]. \end{aligned} \quad (5.184)$$

After inserting the decomposition in Eqn. (5.9) into Eqn.(5.184), we further have

$$\mu_{33} = \frac{1}{\omega} \operatorname{Im} \{ f_{33} \}, \quad (5.185)$$

and

$$\nu_{33} = -\operatorname{Re} \{ f_{33} \}. \quad (5.186)$$

Let us introduce the following dimensionless parameters,

$$\left\{ \tilde{\mathcal{B}}_{3q}, \tilde{\mathcal{C}}_{3q} \right\} = \{ \mathcal{B}_{3q}, \mathcal{C}_{3q} \} / (\pi a^2), \quad \left\{ \tilde{\mu}_{33}, \tilde{\nu}_{33} \right\} = \{ \mu_{33}, \nu_{33} \} / (\rho c^3). \quad (5.187)$$

The dimensionless quantities $\tilde{\mathcal{B}}_{3q}$, $\tilde{\mathcal{C}}_{3q}$, $\tilde{\mu}_{33}$ and $\tilde{\nu}_{33}$ can be obtained by inserting Eqns. (5.182)-(5.183) and (5.185)-(5.186) into Eqn. (5.187) after using Eqns. (5.181) and (5.184), respectively.

5.8 Amplitudes and phases

The amplitudes and phases of the response of OWC device and the averaged surface elevation inside the chamber are dressed in this section.

Following the kinematic free surface boundary condition and after using Eqn. (5.1), the surface elevation inside the chamber can be calculated by

$$\zeta = \frac{i}{\omega} \left(\frac{\partial \phi_S^{(3)}}{\partial z} \Big|_{z=0} + \frac{\partial \phi_P^{(3)}}{\partial z} \Big|_{z=0} + \frac{\partial \phi_3^{(3)}}{\partial z} \Big|_{z=0} \right). \quad (5.188)$$

In order to study the response of the free surface inside the chamber, let us take the area-averaged surface elevation inside the chamber, that is,

$$\begin{aligned} \zeta_{av} &= \frac{1}{\pi a^2} \frac{i}{\omega} \int_0^a \int_0^{2\pi} \left(\frac{\partial \phi_D}{\partial z} \Big|_{z=0} + \frac{\partial \phi_P}{\partial z} \Big|_{z=0} + V_3 \frac{\partial \phi_3}{\partial z} \Big|_{z=0} \right) r d\theta dr \\ &= \frac{2}{a\omega} \left(\frac{gA}{\omega} \sum_{l=0}^{\infty} D_{0l} \frac{Z'_l(0)}{k_l^2} + \frac{p_a}{\rho\omega} \sum_{l=0}^{\infty} \hat{D}_l \frac{Z'_l(0)}{k_l^2} + iV_3 \sum_{l=0}^{\infty} \tilde{D}_l \frac{Z'_l(0)}{k_l^2} \right), \end{aligned} \quad (5.189)$$

where Eqns. (5.37), (5.109) and (5.151) have been used. Furthermore, the notations Λ_{av} and Θ_{av} representing the averaged amplitude and phase, respectively, are calculated by

$$\Lambda_{av} = |\zeta_{av}|, \quad (5.190)$$

and

$$\Theta_{av} = \tan^{-1} \left(\frac{\text{Im } \zeta_{av}}{\text{Re } \zeta_{av}} \right). \quad (5.191)$$

According to Eqn. (5.17), the response amplitude operator (RAO) Λ_{hv} and phase Θ_{hv} of the forced OWC device in heave mode are calculated by, respectively,

$$\Lambda_{hv} = \left| \frac{iV_3}{\omega A} \right|, \quad (5.192)$$

and

$$\Theta_{hv} = \tan^{-1} \left(\frac{\text{Im } iV_3}{\text{Re } iV_3} \right). \quad (5.193)$$

Thus, the phase difference between the heave mode of the forced OWC device and the averaged surface elevation inside the chamber is calculated by

$$\Theta_{dif} = \Theta_{hv} - \Theta_{av}. \quad (5.194)$$

5.9 Optimum extraction efficiency

Physically, for a given frequency ω , there are at least three strategies to optimize the OWC system for maximum extraction efficiency, namely, by adjusting the turbine parameter, or the chamber size, or both. Referring to Eqn. (5.23), mathematically, let us differentiate kL with respect to χ and κ , respectively, and set the resulting equations both to zeros (Martins-Rivas & Mei, 2009b; Lovas *et al.*, 2010), i.e.,

$$\begin{cases} \frac{\partial(kL)}{\partial\chi} = 0, \\ \frac{\partial(kL)}{\partial\kappa} = 0. \end{cases} \quad (5.195)$$

Inserting Eqn. (5.23) into the above equation and solving the resulting linear system, gives

$$\chi(\omega) = \sqrt{\frac{(\kappa_{opt}b_{33} + Y_1)^2 + (Y_2 - \kappa_{opt}\nu_{33})^2}{\nu_{33}^2 + b_{33}^2}}, \quad (5.196)$$

where the optimum chamber size κ_{opt} is given by

$$\kappa_{opt}(\omega) = \frac{\nu_{33}Y_2 - b_{33}Y_1}{b_{33}^2 + \nu_{33}^2}. \quad (5.197)$$

As a result, the maximum capture length kL_{max} , after inserting Eqns. (5.196)-(5.197) into Eqn. (5.23), can be calculated by

$$kL_{max} = \frac{k}{\rho g C_g} \frac{\chi(\omega) |[(\nu_{33} + ib_{33})\Gamma - (a_{3q} - i\omega C_{3q})\Gamma_{d3}]|^2}{(\chi(\omega)\nu_{33} + \kappa_{opt}(\omega)b_{33} + Y_1)^2 + (\chi(\omega)b_{33} - \kappa_{opt}(\omega)\nu_{33} + Y_2)^2}. \quad (5.198)$$

The above equation can be reduced to the maximum capture length as the OWC system is fixed on water surface.

In practice, once an OWC system is built, it is hard to vary the chamber size (i.e., κ) for different wave frequencies. Hence, as did in Martins-Rivas & Mei (2009b), we will fine-tune the turbine parameter χ to maximize the extraction efficiency for a given dimension of OWC system (i.e., κ is given). Therefore, the optimum turbine parameter χ_{opt} can be rewritten as

$$\chi_{opt} = \sqrt{\frac{(\kappa b_{33} + Y_1)^2 + (Y_2 - \kappa \nu_{33})^2}{\nu_{33}^2 + b_{33}^2}}, \quad (5.199)$$

where κ is a constant for a given OWC device.

After inserting Eqn. (5.199) into Eqn. (5.23), the dimensionless optimum capture length kL_{opt} can be expressed as

$$kL_{opt} = \frac{k}{\rho g C_g} \frac{\chi_{opt} |[(\nu_{33} + ib_{33})\Gamma - (a_{3q} - i\omega C_{3q})\Gamma_{d3}]|^2}{(\chi_{opt}\nu_{33} + \kappa b_{33} + Y_1)^2 + (\chi_{opt}b_{33} - \kappa \nu_{33} + Y_2)^2}. \quad (5.200)$$

In literature, the hydrodynamics of a free floating vertical axisymmetric oscillating water column wave energy converter in three modes (i.e., surge, heave and pitch) has been studied by Mavrakos & Konispoliatis (2012). However, in their paper, the optimization of the turbine parameter g_{Topt} ($g_T = \chi$) (the same notation in Mavrakos & Konispolistis' paper) should be corrected. As in Mavrakos & Konispolistis' paper, the capture length L , extracted by the OWC device, is calculated by (in Mavrakos & Konispoliatis (2012), equation (11.6)),

$$L = \frac{2}{\rho g A^2 C_g} \cdot \frac{g_T}{2} \frac{|q_D + q_R|^2}{(f_{pp} + g_T)^2 + (\omega V_0 / \gamma p_a + g_{PP})^2}, \quad (5.201)$$

where f_{PP} and g_{PP} in Mavrakos & Konispoliatis (2012) are equivalent to $\mathcal{B}$ and $\mathcal{C}$ in the present paper, respectively, and $\gamma = 1.4$ is the adiabatic constant.

Observing Eqn. (5.201), it is noted that the volume flux q_R , calculated by Eqn. (5.3), is a function of g_T . Therefore, when we differentiate L with respect to g_T , q_R can not be considered as a quantity of independence of g_T . However, in Mavrakos & Konospolistis' paper, the expression (in Mavrakos & Konispoliatis (2012), Equation (11.8)) of the optimum turbine parameter g_{Topt} , i.e.,

$$g_{Topt} = \sqrt{f_{PP}^2 + \left(g_{PP} + \frac{\omega V_0}{\gamma p_a} \right)^2}, \quad (5.202)$$

is wrong, because q_R has been considered to be independent of g_T .

In addition to the above theoretical verification, the numerical verification of the optimum capture lengths in terms of Eqns (5.199) and (5.201), respectively, will be performed later.

5.10 Results and discussions

Effects of the chamber size a , the submergence d of the OWC device and the width b of the tube-shaped floating platform on the optimum capture length are examined in this section, respectively. Without loss of generality, unless otherwise specified, $h = 10$ m, $\rho_a = 1$ kg/m³, $\rho/\rho_a = 1000$, $c_a = 340$ m/s, $V_0 = \pi a^2 h$, and $g = 9.81$ m/s² are taken for all numerical results presented in the following subsections.

5.10.1 Convergence test

Referring to the linear infinite systems in the previous three problems, i.e., diffraction problem, radiation problem due to the oscillatory air pressure inside and radiation problem due to the heaving motion of the OWC device, there exist four sets of infinite summations, they are,

$$\sum_{l=0}^{\infty}, \quad \sum_{m=0}^{\infty}, \quad \sum_{n=0}^{\infty}, \quad \sum_{p=0}^{\infty}. \quad (5.203)$$

In order to numerically solve the unknown coefficients (i.e., γ_p , ξ_p , $\tilde{\gamma}_p$, $\tilde{\xi}_p$, $\hat{\gamma}_p$ and $\hat{\xi}_p$), truncating these four infinite summations to approximate the numerical results is necessary. As a result, the four truncated summations are, respectively,

$$\sum_{l=0}^{\mathcal{L}}, \quad \sum_{m=0}^M, \quad \sum_{n=0}^N, \quad \sum_{p=0}^{\mathcal{P}}. \quad (5.204)$$

The remaining task is to determine the suitable values of $\mathcal{L}$, M , N and $\mathcal{P}$.

Following Garrett (1970) and Deng *et al.* (2013), we shall use $\mathcal{L} = 400$ in our numerical computations for all three problems to achieve a precision of 3 decimal places. In addition, following Yeung (1981), the m -series has a very good convergent property. Hence, in what follows, $M = 50$ will be taken.

Table 5.1 Convergence tests for $\left|\tilde{\Gamma}\right|^2$, $\left|\tilde{\Gamma}_{d3}\right|^2$, $\tilde{\mathcal{B}}$, $\tilde{\mu}_{p3}$, $\tilde{\mathcal{B}}_{3q}$ and $\tilde{\mu}_{33}$ against N .

N	$\left \tilde{\Gamma}\right ^2$	$\left \tilde{\Gamma}_{d3}\right ^2$	$\tilde{\mathcal{B}}$	$\tilde{\mu}_{p3}$	$\tilde{\mathcal{B}}_{3q}$	$\tilde{\mu}_{33}$
30	4.466475	0.009106	7.439378	-0.481325	0.693712	-0.039489
35	4.464555	0.009092	7.436181	-0.481523	0.693852	-0.039413
40	4.463202	0.009082	7.433927	-0.481661	0.693953	-0.039360
45	4.462201	0.009075	7.432259	-0.481761	0.694029	-0.039321
50	4.461434	0.009069	7.430982	-0.481836	0.694089	-0.039291

Table 5.1 shows the convergence tests for $\left|\tilde{\Gamma}\right|^2$, $\left|\tilde{\Gamma}_{d3}\right|^2$, $\tilde{\mathcal{B}}$, $\tilde{\mu}_{p3}$, $\tilde{\mathcal{B}}_{3q}$ and $\tilde{\mu}_{33}$ against N for $a/h = 1/2$, $d/h = 0.2$, $b/h = 0.1$, $\mathcal{L} = 400$, $M = 50$, $\mathcal{P} = 5$, $\omega^2 h/g = 2.3 \text{ rad}^2/\text{m}$. Where $\left|\tilde{\Gamma}\right|^2$, $\left|\tilde{\Gamma}_{d3}\right|^2$, $\tilde{\mathcal{B}}$, $\tilde{\mu}_{p3}$, $\tilde{\mathcal{B}}_{3q}$ and $\tilde{\mu}_{33}$ are calculated by Eqns. (5.104), (5.105), (5.145) and (5.187), respectively, after using Eqns. (5.140), (5.143), (5.182) and (5.184). It shows that at least three-digit precision can be achieved with $N = 50$. Thus $N = 50$ is taken to solve these three problems numerically.

Table 5.2 Convergence tests for $\left|\tilde{\Gamma}\right|^2$, $\left|\tilde{\Gamma}_{d3}\right|^2$, $\tilde{\mathcal{B}}$, $\tilde{\mu}_{p3}$, $\tilde{\mathcal{B}}_{3q}$ and $\tilde{\mu}_{33}$ against $\mathcal{P}$.

$\mathcal{P}$	$\left \tilde{\Gamma}\right ^2$	$\left \tilde{\Gamma}_{d3}\right ^2$	$\tilde{\mathcal{B}}$	$\tilde{\mu}_{p3}$	$\tilde{\mathcal{B}}_{3q}$	$\tilde{\mu}_{33}$
3	4.458693	0.009020	7.426417	-0.481368	0.693370	-0.039075
4	4.461081	0.009062	7.430395	-0.481761	0.693965	-0.039260
5	4.461434	0.009082	7.430982	-0.481836	0.694089	-0.039291
6	4.461554	0.009071	7.431182	-0.481849	0.694118	-0.039301
7	4.461700	0.009072	7.431425	-0.481845	0.694122	-0.039309

Table 5.2 shows the convergence tests for $\left|\tilde{\Gamma}\right|^2$, $\left|\tilde{\Gamma}_{d3}\right|^2$, $\tilde{\mathcal{B}}$, $\tilde{\mu}_{p3}$, $\tilde{\mathcal{B}}_{3q}$ and $\tilde{\mu}_{33}$ against $\mathcal{P}$ for $a/h = 1/2$, $d/h = 0.2$, $b/h = 0.1$, $\mathcal{L} = 400$, $M = 50$, $N = 50$, $\omega^2 h/g = 2.3 \text{ rad}^2/\text{m}$. It shows that at least three-digit precision can be achieved with $\mathcal{P} = 7$. Thus $\mathcal{P} = 7$ is taken.

5.10.2 Code verification

Let us verify the numerical codes by comparing the present results to those in two cases: one by Deng *et al.* (2013) and the other by Mavrakos (1988).

As the outer radius of the tube-shaped floating platform is small, i.e., $b/h = 0.005$, and the OWC device is fixed in waves, the present case can reduce to the case studied by Deng *et al.* (2013) where the opening angle is 2π . Figure 5.2 shows the comparisons of diffraction factor $\left|\tilde{\Gamma}\right|^2$, radiation conductance $\tilde{\mathcal{B}}$, optimum turbine parameter χ_{opt} and corresponding capture length kL_{opt} between these two limiting cases for $a/h = 1/2$, $d/h = 0.2$. The results show that the agreements are well.

When the incident waves are absent and the top of the column is open to atmosphere, the present case can reduce to the case studied by Mavrakos (1988). Figure 5.3 shows the comparisons of added mass $\tilde{\mu}_{33}$ and radiation damping $\tilde{\nu}_{33}$ between these two cases with the vertical OWC device velocity $V_3 = -i\omega$ for $a = 5 \text{ m}$, $h/a = 2$, $c/a = 1.8$, $d/a = 1/3$. The results show that our numerical results agree well with those presented by Mavrakos (1988).

5.10.3 Comparison of optimization strategies of turbine parameter χ

Theoretically, the correction to the optimization strategy of turbine parameter presented by Mavrakos & Konispoliatis (2012) has been given in Section 5.9.

Figure 5.2 Comparison of the present results of limiting case with those in Deng *et al.* (2013). In all cases, $a/h = 1/2$, $d/h = 0.2$, $b/h = 0.005$.

Figure 5.3 Comparison of the present results with those in Mavrakos (1988). In all cases, $a = 5$, $h/a = 2$, $c/a = 1.8$, $d/a = 1/3$.

In this subsection, let us take an example to numerically examine the resulting optimum capture lengths evaluated under the two optimization strategies: one

given by the present authors and the other by Mavrakos & Konispoliatis (2012).

Figure 5.4 Comparison of the results based on the present optimization strategy with those in Mavrakos (1988). In all cases, $a/h = 1/2$, $d/h = 0.2$, $b/h = 0.15$.

Figure 5.4 shows the comparisons of χ_{opt} (calculated by Eqns. (5.199) (solid line) and (5.202) (dashed line), respectively) and resulting capture length kL_{opt} for $a/h = 1/2$, $d/h = 0.2$, $b/h = 0.15$. Clearly, the higher absorption efficiency and wider bandwidth of high efficiency can be obtained under the present optimization strategy of turbine parameter. Note that the maximum extraction efficiency can be the same as the one by a fixed thin-walled OWC device (Evans & Porter, 1997), i.e., $\max \{kL_{opt}\} = 1$.

5.10.4 Effects of the dimension of tube-shaped floating platform

The effects of the dimension of circular floating platform on the hydrodynamics and absorption efficiency are examined in this subsection for a fixed radius of the column, $a/h = 1/2$, and a single submerged depth of the OWC device, $d/h = 0.2$.

Figures 5.5–5.6 show the variations of $|\tilde{\Gamma}|^2$, $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ against $\omega^2 h/g$ for several dimensions of the tube-typed platform: $b/h = 0.1, 0.3$ and 0.5 . In the computed range of $\omega^2 h/g$, both $|\tilde{\Gamma}|^2$ and $\tilde{\mathcal{B}}$ just have a single peak enhancing height and becoming sharper with the increasement of b/h . The peaks of $\tilde{\mathcal{C}}$ shift to low frequency band and become sharper with the increasement of b/h . Note that negative values of $\tilde{\mathcal{C}}$ occurs in certain range of $\omega^2 h/g$, which feature is analogous with that of moving bodies in fluid (Evans & Porter, 1995; Martins-Rivas & Mei, 2009b; Deng *et al.*, 2013).

Figure 5.5 Diffraction flux factor $|\tilde{\Gamma}|^2$ as a function of $\omega^2 h/g$ for $a/h = 1/2$, $d/h = 0.2$ and several values of b/h .

Figure 5.6 Radiation conductance $\tilde{B}$ and radiation susceptance $\tilde{C}$ for $a/h = 1/2$, $d/h = 0.2$ and several values of b/h .

Figure 5.7 shows the variations of added mass μ_{33} and radiation damping ν_{33} against $\omega^2 h/g$ for several values of b/h : 0.1, 0.3, 0.5. As b/h increases, the peaks of both μ_{33} and ν_{33} become sharper and shift to low frequency band slightly. Note that these peaks occur at the neighbourhood of $\omega^2 h/g = 2$, at which resonance situations occur (Garrett, 1970; Mavrakos, 1985, 1988).

Figure 5.8 shows the variations of optimum turbine parameter χ_{opt} and the resulting phase difference Θ_{dif} between the response amplitude operator (RAO) and the spatial averaged surface elevation against $\omega^2 h/g$ for several b/h . The peaks of both χ_{opt} and Θ_{dif} shift toward low frequency band and become flatter as b/h increases. Note that as $b/h = 0.5$, Θ_{dif} has two peaks in the computed range, at which frequencies ($\omega^2 h/g$) more energy can be extracted.

The response amplitude operator (RAO) of OWC device in waves, Λ_{hv} , and spatial average surface elevation inside the column, Λ_{av} , are shown in Figure 5.9.

Figure 5.7 Added mass μ_{33} and radiation damping ν_{33} for $a/h = 1/2$, $d/h = 0.2$ and several values of b/h .

Figure 5.8 Phase difference Θ_{dif} and optimum turbine parameter χ_{opt} for $a/h = 1/2$, $d/h = 0.2$ and several values of b/h .

There are two peaks for both Λ_{hv} and Λ_{av} in the computed range of $\omega^2 h/g$ as $b/h = 0.1$ and 0.3 . However, as $d/h = 0.5$, both Λ_{hv} and Λ_{av} have three peaks. Note also that all of those peaks shift to low frequency band and become flatter as d/h increases. Indeed, the peaks of both Λ_{hv} and Λ_{av} represent the occurrence of resonance situations at these frequencies.

Figure 5.10 shows the variations of optimum capture length kL_{opt} against $\omega^2 h/g$ for $b/h = 0.1, 0.3$ and 0.5 . Physically, referring to Figure 5.9, the maximum extraction efficiency can be obtained when the resonance situations of both the response (RAO) of the OWC device and the average surface elevation inside occur at the same frequency and, meanwhile, whose phases are inverse. Which can be achieved by adjusting the turbine parameter, referring to Figure 5.8, the phase difference equal or close to π can be obtained at certain frequencies for different b/h . Mathematically, the frequency, at which the maximum capture length occurs, is when the chamber parameter κ is equal to the optimum cham-

Figure 5.9 Response amplitude operator (RAO) Λ_{hv} and normalized interior spatial average surface elevation Λ_{av}/A for $a/h = 1/2$, $d/h = 0.2$ and several values of b/h .

Figure 5.10 Resulting capture length kL_{opt} for $a/h = 1/2$, $d/h = 0.2$ and several values of b/h .

ber parameter κ_{opt} (see Eqn. (5.197)). Note also that there is only one large peak for kL_{opt} as $b/h = 0.1$ – 0.3 , while two large peaks occur when $b/h = 0.5$. In addition, as expected, the bandwidth of high extraction efficiency becomes narrower as b/h increases.

5.10.5 Effects of chamber size

The effects of chamber size a/h on the hydrodynamics and optimum extraction efficiency are examined in this subsection for a fixed outer radius $b/h = 0.2$, a single submerged depth $d/h = 0.2$.

The variations of $|\tilde{\Gamma}|^2$, $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ against $\omega^2 h / g$ for several values of a/h are shown in Figures 5.11–5.12. It can be seen that as a/h varies from $1/4$ to $1/2$, the peaks of $|\tilde{\Gamma}|^2$, $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ all shift toward low frequency band and raise in magnitude. Figure 5.13 shows the variation of added mass μ_{33} and radiation

Figure 5.11 Diffraction flux factor $|\tilde{\Gamma}|^2$ as a function of $\omega^2 h / g$ for $d/h = 0.2$, $d/h = 0.2$ and several values of a/h .

Figure 5.12 Radiation conductance $\tilde{\mathcal{B}}$ and radiation susceptance $\tilde{\mathcal{C}}$ for $b/h = 0.2$, $d/h = 0.2$ and several values of a/h .

damping ν_{33} against $\omega^2 h / g$. The peaks of μ_{33} and ν_{33} shift to low frequency band but reduce in magnitude. It is noted that comparing with the effects of b/h (in Figures 5.5–5.7) on $|\tilde{\Gamma}|^2$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$, μ_{33} and ν_{33} , those of a/h are relatively significant.

Figure 5.14 shows the variation of the optimum turbine parameter χ_{opt} and the resulting phase difference Θ_{dif} between the response of the OWC device and the inside average surface elevation against $\omega^2 h / g$. It can be seen that as a/h increases, the peaks of both χ_{opt} and Θ_{dif} shift toward low frequency band slightly. The variations of response amplitude operator (RAO) Λ_{hv} and inside average surface elevation Λ_{av} are shown in Figure 5.15. There are two peaks for both Λ_{hv} and Λ_{av} in the computed range of $\omega^2 h / g$, and both peaks shift toward low frequency band and become flatter as a/h increases.

Figure 5.16 shows the variations of the optimum capture length kL_{opt} against

Figure 5.13 Added mass μ_{33} and radiation damping ν_{33} for $b/h = 0.2$, $d/h = 0.2$ and several values of a/h .

Figure 5.14 Phase difference Θ_{dif} and optimum turbine parameter χ_{opt} for $b/h = 0.2$, $d/h = 0.2$ and several values of a/h .

$\omega^2 h/g$ for $a/h = 1/4, 1/3$ and $1/2$. There are two peaks for kL_{opt} in the computed range of $\omega^2 h/g$. The larger one occurs at the frequency when $\kappa = \kappa_{opt}$, namely, the same resonance situations for Λ_{hv} and Λ_{av} and the phase difference close to π . The smaller one represents the local maximum extraction efficiency. Note also that the frequencies at which the large peak occurs are the same as those of Θ_{dif} (in Figure 5.14).

5.10.6 Effects of submergence

In this subsection, let us examine the effects of submerged depth d/h on the hydrodynamics and optimum extraction efficiency for a fixed chamber radius $a/h = 1/2$, a single outer radius $b/h = 0.2$.

The variations of $|\tilde{\Gamma}|^2$, $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ against $\omega^2 h/g$ are shown in Figures 5.17–5.18. There is only one peak for both $|\tilde{\Gamma}|^2$ and $\tilde{\mathcal{B}}$ in the computed range and which occurs at the same frequency. As d/h increases, the peaks of $|\tilde{\Gamma}|^2$, $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ all

Figure 5.15 Response amplitude operator (RAO) Λ_{hv} and normalized interior spatial average surface elevation Λ_{av}/A for $b/h = 0.2$, $d/h = 0.2$ and several values of a/h .

Figure 5.16 Resulting capture length kL_{opt} for $b/h = 0.2$, $d/h = 0.2$ and several values of a/h .

shift toward low frequency band. It is noted that the effects of d/h on $|\tilde{\Gamma}|^2$, $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ are relatively significant, compared with those of b/h (in Figures 5.5–5.6).

Figure 5.19 shows the effects of d/h on the variations of added mass μ_{33} and damping ν_{33} . The peaks of both μ_{33} and ν_{33} shift toward low frequency band and get in magnitude. Comparing Figure 5.19 with Figure 5.7, it is clear that b/h has no influence in the resonance location, which depends significantly on the value of d/h (Mavrakos, 1985, 1988).

Figure 5.20 shows the variations of optimum turbine parameter χ_{opt} and the resulting phase difference Θ_{dif} against $\omega^2 h / g$. In the computed range, Θ_{dif} has one single peak for $d/h = 0.2$ and two peaks for $d/h = 0.4$ and 0.6 . These peaks all shift toward low frequency band with d/h increasing.

The variations of Λ_{hv} and Λ_{av} against $\omega^2 h / g$ are shown in Figure 5.21. In the computed range, both Λ_{hv} and Λ_{av} have two peaks for $d/h = 0.2$ and three peaks

Figure 5.17 Diffraction flux factor $|\tilde{\Gamma}|^2$ as a function of $\omega^2 h / g$ for $a/h = 1/2$, $b/h = 0.2$ and several values of d/h .

Figure 5.18 Radiation conductance $\tilde{\mathcal{B}}$ and radiation susceptance $\tilde{\mathcal{C}}$ for $a/h = 1/2$, $b/h = 0.2$ and several values of d/h .

Figure 5.19 Added mass μ_{33} and radiation damping ν_{33} for $a/h = 1/2$, $b/h = 0.2$ and several values of d/h .

Figure 5.20 Phase difference Θ_{diff} and optimum turbine parameter χ_{opt} for $a/h = 1/2$, $b/h = 0.2$ and several values of d/h .

Figure 5.21 Response amplitude operator (RAO) Λ_{hv} and normalized interior spatial average surface elevation Λ_{av}/A for $a/h = 1/2$, $b/h = 0.2$ and several values of d/h .

Figure 5.22 Resulting capture length κL_{opt} for $a/h = 1/2$, $b/h = 0.2$ and several values of d/h .

for $d/h = 0.4$ and 0.6 . Which implies that as the submerged depth increases, more resonance modes occur.

Figure 5.22 shows the variations of optimum capture length kL_{opt} against $\omega^2 h/g$. In the computed range, kL_{opt} has two peaks as $d/h = 0.2$ and three peaks as $d/h = 0.4$ and 0.6 . As the submerged depth d/h increases, the peaks of kL_{opt} shift toward low frequency band and the bandwidth of high extraction efficiency becomes narrower dramatically. It is clear that the frequency at which the maximum extraction efficiency occurs is when $\kappa = \kappa_{opt}$. Note also that the other small peaks represent the local maximum extraction efficiencies.

5.11 Summary

Starting with linear water wave theory, the wave interaction with an oscillating water column (OWC) installed at the center of a circular floating platform in heaving motion is studied analytically. We focus on the effects of heaving motion on the extraction efficiency of the OWC device. The optimization and response amplitude operator (RAO) of OWC device are also discussed. The following conclusions can be drawn from this study:

- (i) By optimizing the turbine parameters, a higher extraction efficiency (about 3–4 times, compared to that by Mavrakos & Konispoliatis (2012)) and wider bandwidth (about 2 times, compared to that by Mavrakos & Konispoliatis (2012)) of high extraction efficiency can be obtained.
- (ii) The maximum extraction efficiency can be one, which is the same as that where a circular OWC device with thin walls is fixed in waves.
- (iii) A bigger outer diameter of the circular pontoon reduces the resonance frequency of the OWC device.
- (iv) A smaller outer diameter of the circular pontoon and a smaller draught both are beneficial for widening the bandwidth of high extraction efficiency. The numerical results suggest $b/h = 0.1$ and $d/h = 0.2$.

Chapter 6

A heaving thick-walled OWC device restrained by a system of spring and damper

A thick-walled OWC device in heave motion, connected to sea bottom by a linear spring-damper system acting as a power take-off (PTO), is considered in this chapter, as shown in Figure 6.1. Wave power extracted by the OWC device is consist of two parts: power extracted by the turbine system and by PTO. Let us adopt the extraction model associated with a Wells turbine, which has been developed in previous sections, to this new OWC device as well.

Figure 6.1 Schematic diagram of a floating oscillating water column wave energy converter in heaving motion.

The objective of this chapter is to exploit the possibility of optimizing the capture length of a given floating OWC wave energy converter in heaving motion by adjusting the PTO attached to the mooring structure, as sketched in Fig-

ure 6.1. In addition to extract wave energy by the PTO attached to the mooring structure, it is also possible to use the same PTO to control the phase difference between the OWC motion and the wave motion and improve the efficiency of the floating OWC. This type of system is intended to be used in water deeper than those for bottom-sitting OWC systems.

In this conceptual analysis, the PTO model is simulated by a linear spring-damper system, and mechanical losses associated with both the PTO and OWC are not considered. For a practical design, these losses need to be addressed by either physical model tests or CFD simulations.

6.1 Equation of motion and power extraction by air turbines

Due to the presence of PTO, the equation of motion in heave must be modified slightly, relative to that in Section 5.3.

In addition to the exciting forces resulting from incident waves and induced waves by the oscillating air pressure inside the column, there exists another type of external force resulting from the spring-damper system. By Newton's law, the equation of motion in heave can be written as

$$\mathcal{M}\ddot{X}_3 = -\mathcal{M}g + \rho g S_w (d - X_3) + \mathcal{F}_3 + P_a S_i - \mathcal{D}\dot{X}_3 - \mathcal{K}X_3, \quad (6.1)$$

where $X_3(t)$ denotes the vertical displacement of the OWC device, $\mathcal{D}$ represents the linear damping constant, $\mathcal{K}$ represents the linear spring constant.

For time-harmonic motion, Eqn. (6.1) can be rewritten as

$$\left(\hat{\nu}_{33} + i\hat{b}_{33}\right)V_3 + (a_{p3} - i\omega\nu_{p3})p_a = \Gamma_{s3}A, \quad (6.2)$$

where $\hat{\nu}_{33}$ and $\hat{b}_{33}$ are given by

$$\hat{\nu}_{33} = \nu_{33} + \mathcal{D}, \quad (6.3)$$

and

$$\hat{b}_{33} = -\omega \left(\rho S_w d + \mu_{33} - \frac{\mathcal{K}}{\omega^2} - \frac{\rho g S_w}{\omega^2} \right), \quad (6.4)$$

and V_3 denotes the complex amplitude of vertical velocity. Note here that the identity $\mathcal{M} = \rho S_w d$ has been used.

Combining Eqns. (6.2) and (5.14) gives

$$\begin{cases} (a_{3q} - i\mathcal{C}_{3q}) V_3 + \left[\left(\frac{KD}{\rho_a^0 \tilde{N}} + \mathcal{B} \right) - i \left(\frac{\omega V_0}{\rho_a^0 c_a^2} + \mathcal{C} \right) \right] p_a = \Gamma A, \\ \left(\hat{\nu}_{33} + i\hat{b}_{33} \right) V_3 + (a_{p3} - i\omega\mu_{p3}) p_a = \Gamma_{s3} A. \end{cases} \quad (6.5)$$

After solving the above linear system, the expressions of p_a and V_3 are separately given by

$$p_a = \frac{\left[\left(\hat{\nu}_{33} + i\hat{b}_{33} \right) \Gamma - (a_{3q} - i\mathcal{C}_{3q}) \Gamma_{s3} \right] A}{\left(\chi \hat{\nu}_{33} + \kappa \hat{b}_{33} + \hat{Y}_1 \right) + i \left(\chi \hat{b}_{33} - \kappa \hat{\nu}_{33} + \hat{Y}_2 \right)}, \quad (6.6)$$

and

$$V_3 = \frac{\{[(\chi + \mathcal{B}) - i(\kappa + \mathcal{C})] \Gamma_{s3} - (a_{p3} - i\omega\mu_{p3}) \Gamma\} A}{\left(\chi \hat{\nu}_{33} + \kappa \hat{b}_{33} + \hat{Y}_1 \right) + i \left(\chi \hat{b}_{33} - \kappa \hat{\nu}_{33} + \hat{Y}_2 \right)}, \quad (6.7)$$

where

$$\hat{Y}_1 = \mathcal{B} \hat{\nu}_{33} + \mathcal{C} \hat{b}_{33} - a_{p3} a_{3q} + \omega \mu_{p3} \mathcal{C}_{3q}, \quad (6.8)$$

and

$$\hat{Y}_2 = \mathcal{B} \hat{b}_{33} - \mathcal{C} \hat{\nu}_{33} + a_{p3} \mathcal{C}_{3q} + \omega \mu_{p3} a_{3q}. \quad (6.9)$$

Inserting Eqn. (6.6) into Eqn. (5.23), after using Eqn. (5.22), the dimensionless capture length kL_{tur} , defined by the ratio of the extracted power by the Wells turbine system to the wave power per unit crest length of the incident waves, which is $\rho g A^2 C_g / 2$, is expressed as

$$kL_{tur} = \frac{k}{\rho g C_g} \frac{\chi \left| \left[\left(\hat{\nu}_{33} + i\hat{b}_{33} \right) \Gamma - (a_{3q} - i\mathcal{C}_{3q}) \Gamma_{s3} \right] \right|^2}{\left(\chi \hat{\nu}_{33} + \kappa \hat{b}_{33} + \hat{Y}_1 \right)^2 + \left(\chi \hat{b}_{33} - \kappa \hat{\nu}_{33} + \hat{Y}_2 \right)^2}. \quad (6.10)$$

6.2 Optimum extraction efficiency by Wells turbines

The strategies of optimizing the extraction efficiency are the same as those discussed in Section 5.9. Let us derivative kL_{tur} with respect to χ and set the resulting equation to zero, i.e.,

$$\frac{\partial (kL_{tur})}{\partial \chi} = 0. \quad (6.11)$$

After inserting Eqn. (6.10) into the above equation and re-arranging, the optimum turbine parameter can be expressed as

$$\chi_{opt}(\omega) = \sqrt{\frac{\left(\kappa \hat{b}_{33} + \hat{Y}_1 \right)^2 + \left(\hat{Y}_2 - \kappa \hat{\nu}_{33} \right)^2}{\hat{\nu}_{33}^2 + \hat{b}_{33}^2}}. \quad (6.12)$$

Note here that κ is a constant for a given angular frequency ω .

Inserting Eqn. (6.12) into Eqn. (6.10), then the optimum capture length kL_{tur}^{opt} by the turbine system can be calculated by

$$kL_{tur}^{opt} = \frac{k}{\rho g C_g} \frac{\chi_{opt} \left| \left[(\hat{\nu}_{33} + i\hat{b}_{33}) \Gamma - (a_{3q} - iC_{3q}) \Gamma_{s3} \right] \right|^2}{\left(\chi_{opt} \hat{\nu}_{33} + \kappa \hat{b}_{33} + \hat{Y}_1 \right)^2 + \left(\chi_{opt} \hat{b}_{33} - \kappa \hat{\nu}_{33} + \hat{Y}_2 \right)^2}. \quad (6.13)$$

Once the problems resulting from the velocity potentials ϕ_S , ϕ_P and ϕ_3 are solved numerically, the results of kL_{tur}^{opt} can be calculated by Eqn. (6.13) straightforwardly. The solution procedures are the same as those presented in Chapter 5.

6.3 Power extracted by PTO

The wave power extracted by the spring-damper system is defined by the time-averaged rate of work done by the damper, namely,

$$P_{out}^D = \overline{\mathcal{D}v_3^2} = \frac{\mathcal{D}}{2} |V_3|^2, \quad (6.14)$$

where $v_3(t)$ represents the instantaneous vertical velocity of the OWC device, expressed as $v_3(t) = \text{Re} \{ V_3 e^{-i\omega t} \}$.

After inserting Eqn. (6.7), the dimensionless capture length kL_D , which is the ratio of the extracted power by the PTO system to the wave power per unit crest length of the incident waves, can be calculated by,

$$kL_D = \frac{k}{\rho g C_g} \frac{\mathcal{D} |[(\chi + \mathcal{B}) - i(\kappa + \mathcal{C})] \Gamma_{s3} - (a_{p3} - i\omega \mu_{p3}) \Gamma|^2}{\left(\chi \hat{\nu}_{33} + \kappa \hat{b}_{33} + \hat{Y}_1 \right)^2 + \left(\chi \hat{b}_{33} - \kappa \hat{\nu}_{33} + \hat{Y}_2 \right)^2}. \quad (6.15)$$

Further, the total dimensionless capture length kL_{tot} is the summation of the optimum capture length kL_{tur}^{opt} associated with the Wells turbine system and the dimensionless capture length kL_D associated with the PTO system, expressed as

$$kL_{tot} = kL_{tur}^{opt} + kL_D. \quad (6.16)$$

6.4 Results and discussions

The hydrodynamic parameters, resulting separately from the diffracted wave field ϕ_D , the pressure-induced radiated wave field ϕ_P and the body-induced

radiated wave field ϕ_3 , will not be given in this section. For more detailed information, one can refer to Chapter 5. In this section, the optimum turbine parameter, response amplitude operator (RAO), average surface elevation inside the column and extraction efficiency will be examined for different damping coefficients $\mathcal{D}$ and spring coefficients $\mathcal{K}$, respectively. For brevity, a fixed dimension of OWC device is chosen with $a/h = 1/2$, $b/h = 0.2$ and $d/h = 0.2$.

Before we discuss the effects of $\mathcal{D}$ and $\mathcal{K}$ on the hydrodynamics and extraction efficiency. Let us introduce $\mathcal{F}_D$ ($\mathcal{F}_D = \text{Re}\{f_D e^{-i\omega t}\}$) and $\mathcal{F}_K$ ($\mathcal{F}_K = \text{Re}\{f_K e^{-i\omega t}\}$) to denote the damping force and restoring force, respectively. Following Garrett (1971), the normalized exciting force $\tilde{f}_{d3}$ on the calm OWC device due to the incident wave potential ϕ^I can be calculated by

$$\tilde{f}_{d3} = \frac{f_{d3}}{\pi c^2 A \rho g} = \frac{\Gamma_{s3}}{\pi c^2 \rho g}. \quad (6.17)$$

Similarly, the normalized damping force resulting from the damper and restoring force resulting from the spring can be calculated by,

$$\tilde{f}_D = \frac{f_D}{\pi c^2 A \rho g} = \frac{\mathcal{D}}{\pi c^2 \rho g} \frac{V_3}{A}, \quad (6.18)$$

and

$$\tilde{f}_K = \frac{f_K}{\pi c^2 A \rho g} = \frac{\mathcal{K}}{\pi c^2 \rho g} \frac{iV_3}{\omega A}. \quad (6.19)$$

respectively.

6.4.1 Effects of damping coefficient $\mathcal{D}$

For discussing the effects of $\mathcal{D}$ on the hydrodynamics and extraction efficiency, let us first assume the spring coefficient $\mathcal{K}$ to be zero. It is needed to explain that the notation D_C in the following figures represents the damping coefficient $\mathcal{D}$.

The variations of optimum turbine parameter χ_{opt} and phase difference Θ_{dif} against $\omega^2 h/g$ are shown in Figure 6.2. As the damping coefficient $\mathcal{D}$ increases, the peak of χ_{opt} reduce in magnitude and shift toward low frequency band. Specially, when $\mathcal{D}$ are large, the difference of the curves of χ_{opt} for different large $\mathcal{D}$ is unobvious.

Figure 6.3 shows the effects of $\mathcal{D}$ on the response amplitude operator (RAO) Λ_{hv} and the interior average surface elevation Λ_{av} . As expected, when the damping coefficient $\mathcal{D}$ is enough large, the OWC device is hard to move in waves and

Figure 6.2 Phase difference Θ_{dif} and optimum turbine parameter χ_{opt} for $a/h = 1/2$, $b/h = 0.2$, $d/h = 0.2$, $\mathcal{K} = 0$ and several values of $\mathcal{D}$. Here, D_C represents the damping coefficient $\mathcal{D}$.

Figure 6.3 Response amplitude operator (RAO) Λ_{hv} and normalized interior spatial average surface elevation Λ_{av}/A for $a/h = 1/2$, $b/h = 0.2$, $d/h = 0.2$, $\mathcal{K} = 0$ and several values of $\mathcal{D}$. Here, D_C represents the damping coefficient $\mathcal{D}$.

almost stays at calm. For Λ_{av} , in the computed range of $\omega^2 h/g$, there is only one peak for moderate values of $\mathcal{D}$, i.e., $\mathcal{D} = 10^5$ and 10^6 .

Figure 6.4 shows the variations of resulting capture length kL_{opt} associated with the turbine system and capture length kL_D associated with the PTO system for $a/h = 1/2$, $b/h = 0.2$, $d/h = 0.2$ and $\mathcal{K} = 0$. For kL_{opt} , when $\mathcal{D}$ varies from 10^4 - 10^5 , the number of peaks varies from two to one, and the magnitude of peak reduces. However, when $\mathcal{D}$ increases from 10^5 - 10^8 , the number of peaks becomes two again and get in magnitude. As expected, when $\mathcal{D}$ is enough large, kL_{opt} can be described by that of a fixed OWC device. For kL_D , when $\mathcal{D}$ is enough small or large, the amount of energy extracted by the PTO system is small.

The effects of $\mathcal{D}$ on the total capture length kL and normalized damping force $\tilde{f}_D$ exerted by the PTO system are shown in Figure 6.5. It can be seen

Figure 6.4 Resulting capture length kL_{tur}^{opt} and capture length kL_D for $a/h = 1/2$, $b/h = 0.2$, $d/h = 0.2$, $\mathcal{K} = 0$ and several values of $\mathcal{D}$. Here, D_C represents the damping coefficient $\mathcal{D}$.

Figure 6.5 Total capture length kL_{tot} and normalized damping force $\tilde{f}_D$ are functions of $\omega^2 h / g$ for $a/h = 1/2$, $b/h = 0.2$, $d/h = 0.2$, $\mathcal{K} = 0$ and several values of $\mathcal{D}$. Here, D_C represents the damping coefficient $\mathcal{D}$.

that, when $\mathcal{D}$ is small, the present case can be reduced to that where the OWC device is free floating in waves. As $\mathcal{D}$ increases from 10^4 - 10^6 , the small peak of kL raise in magnitude and shift toward low frequency band, while the large peak reduces in magnitude until vanishing. When $\mathcal{D}$ continues to increase from 10^6 - 10^8 , the original large peak appears again. For $\tilde{f}_D$, as expected, when $\mathcal{D}$ is small, the damping force on the OWC device is small as well.

6.4.2 Effects of damping coefficient $\mathcal{K}$

In this subsection, let us examine the effects of damping coefficient $\mathcal{K}$ on hydrodynamics and extraction efficiency. For convenience, the damper will be taken off, i.e., $\mathcal{D} = 0$. As a result, there is no energy extracted by the PTO system based on the damper, namely, $kL_D = 0$ for all frequencies. It is specially noted

Figure 6.6 Phase difference Θ_{dif} and optimum turbine parameter χ_{opt} for $a/h = 1/2$, $b/h = 0.2$, $d/h = 0.2$, $\mathcal{D} = 0$ and several values of $\mathcal{K}$. Here, K_C represents the damping coefficient $\mathcal{K}$.

Figure 6.7 Response amplitude operator Λ_{hv} and average surface elevation Λ_{av} for $a/h = 1/2$, $b/h = 0.2$, $d/h = 0.2$, $\mathcal{D} = 0$ and several values of $\mathcal{K}$. Here, K_C represents the damping coefficient $\mathcal{K}$.

that the notation K_C in the following figures denotes the spring coefficient $\mathcal{K}$.

Figure 6.6 shows the effects of $\mathcal{K}$ on the optimum turbine parameter χ_{opt} and phase difference Θ_{dif} between the response of the OWC device and the interior average surface elevation for $a/h = 1/2$, $b/h = 0.2$ and $d/h = 0.2$. It can be seen that as $\mathcal{K}$ is relatively small (i.e., $\mathcal{K} = 10^4$ and 10^5) or relative large (i.e., $\mathcal{K} = 10^6$ and 10^8), the curves of both χ_{opt} and Θ_{dif} have a similar shape. Note also that when $\mathcal{K}$ is relative small, the phase difference Θ_{dif} close to π can be obtained at certain frequencies.

Figure 6.7 shows the variations of response amplitude operator Λ_{hv} and interior surface elevation Λ_{av} against $\omega^2 h / g$. As expected, when $\mathcal{K}$ is enough large, the OWC device stays at rest in waves. Note also that the values of $\mathcal{K}$ significantly influences the resonance situations of the OWC device.

Figure 6.8 Optimum capture length kL_{tur}^{opt} and normalized damping force $\tilde{f}_K$ are functions of $\omega^2 h / g$ for $a/h = 1/2$, $b/h = 0.2$, $d/h = 0.2$, $\mathcal{D} = 0$ and several values of $\mathcal{K}$. Here, K_C represents the damping coefficient $\mathcal{K}$.

The effects of $\mathcal{K}$ on the optimum capture length kL_{opt} and restoring force $\tilde{f}_K$ based on $\omega^2 h / g$ are shown in Figure 6.8. It can be seen that the present case can be reduced to that where the OWC device is free floating in waves when $\mathcal{K}$ is small or fixed in waves when $\mathcal{K}$ is large. Note that as $\mathcal{K} = 10^6$, a quit flat peak can be obtained, which implying that the OWC device can perform efficiently in the frequency domain of $\omega^2 h / g = 2-2.8$.

In fact, in addition to the turbine parameter χ and chamber size parameter κ , there are other strategies of optimization for the OWC wave energy converter restrained by a damper-spring system, such as floating platform size parameter, damping efficient $\mathcal{D}$ and spring coefficient $\mathcal{K}$. Specially, coefficients $\mathcal{D}$ and $\mathcal{K}$ can be variable for different wave frequencies. However, due to the large amount of work to optimize in terms of each parameter, the optimization of turbine parameter is only discussed in this thesis. The other optimization strategies my be discussed in future.

6.5 Summary

In this chapter a circular floating platform with an OWC converter, restrained by a spring-damper system acting as another power take-off model, is analytically studied based on potential theory. The effects of the damper and spring on extraction efficiency are assessed. The optimization of turbine parameter is also discussed. The following conclusions are drawn from this study:

- (i) A moderate intensity of damper is beneficial for the OWC device placed at the sea condition where wavelength is relatively long. As the intensity of

damper is small or large, the extraction efficiency can be reduced to that for a free floating OWC device in heaving motion or a fixed OWC device, respectively.

- (ii) A relatively strong intensity of spring, i.e., $\mathcal{K} = 10^5\text{--}10^6$, is beneficial for widening the bandwidth of high extraction efficiency. Specially, as shown in Figure 6.8, for $\mathcal{K} = 10^6$, a rather flat peak can be obtained for the range of $\omega^2 h/g = 2\text{--}2.8$. This is useful for the design of the mooring line system.

Chapter 7

Conclusions and Future Work

7.1 Conclusions

In this thesis, the hydrodynamic performances and extraction efficiencies for four OWC-type wave energy converters have been investigated theoretically. Based on linear water wave theory, the method of matched eigenfunction expansions is used to reduce the original boundary-value problems to linear algebraic systems by introducing auxiliary radial velocities involving the singular behaviors at sharp corners. The analytical and numerical results to optimum turbine characteristics and resulting capture length are presented. The conclusions from the studies in this thesis can be drawn as follows:

- (i) The coaxial tube-sector-shaped supporting structure in Chapter 3 can increase the energy conversion efficiency and widen the frequency range over which the OWC device can operate at high efficiency. The opening in the supporting structure is suggested to choose within the range of $\pi/2 < \psi < 5\pi/4$.
- (ii) The V-shaped channel in Chapter 4 can improve the energy conversion efficiency and widen the bandwidth over which the OWC device can operate at high efficiency. The length of the V-shaped channel can be chosen in the range of 1-1.5. In addition, the use of approximation basis functions to capture the singular behaviors in both vertical and azimuthal directions can considerably speed up the convergence of numerical computations. In view of the capture length, the OWC-type devices with a V-shaped channel is about 6-7 times more efficient than the fully truncated one and 3-4 times more efficient than the one without V-shaped channel.

- (iii) In chapter 5, our correction to the optimization of the turbine characteristics presented by Mavrakos & Konispoliatis (2012) has been presented. Under our optimization, the maximum extraction efficiency can be the same as that where a circular OWC device with thin walls is fixed in waves. In addition, in order to widen the bandwidth of high extraction efficiency, a smaller outer diameter of the circular pontoon and a smaller draught are both suggested. Our numerical results suggest $b/h = 0.1$ and $d/h = 0.2$.
- (iv) In chapter 6, a moderate intensity of damper is beneficial for the OWC device placed at the sea condition where wavelength is relative long. As the intensity of damper is small or large, the extraction efficiency can be reduced that of a free floating OWC device or a fixed OWC device, respectively. In addition, a relatively strong intensity of spring, i.e., $\mathcal{K} = 10^5$ - 10^6 , is beneficial for widening the bandwidth of high extraction efficiency. Specially, as shown in Figure 6.8, for $\mathcal{K} = 10^6$, a rather flat peak can be obtained for the range of $\omega^2 h/g = 2$ -2.8. This is useful for the design of the mooring line system.

7.2 Future work

In this thesis, the analytical model for four OWC-types wave energy converters have been conducted based on linear water wave theory, which can be considered as physical insights to practical devices and be beneficial to the design of the OWC devices. The principal areas of future work can be divided into the following main parts:

- Phase control

To widen the bandwidth over which the OWC-device can operate at high extraction efficiency. At present, for a given OWC device, it is hard to adjust the pneumatic chamber size for different angular frequency to achieve high extraction efficiency at all time. It is conventional to adjust the air mass flux through the turbine system by controlling the rotational speed of the turbine rotor. However, by phase control, the significant chamber size can be adjusted to strive for high efficiency over a broad range of frequencies around the spectral peak of the incident sea and the resonance frequency can be obtained.

- Nonlinear turbine

In practice, the characteristics of turbine is nonlinear, i.e. there is a lag between the mass flux and the air pressure drop. In this case, the optimization of turbine characteristics and resulting capture length may be considerably different from the present results or not. The nonlinear influence should be discussed and examined in future.

- Wave spectrum

The real sea surface is random and nonlinear. Therefore the assumption of monochromatic long crested waves is limited in predictions of the real performance of an OWC device. In future, the hydrodynamic performances and extraction efficiency should be examined base on wave spectrum.

- Numerical simulation

To improve the bandwidth of high efficiency, a more complex structure of OWC device will be designed, which is hard to model this structure analytically. As a result, a numerical technique should be taken to optimize this OWC device and predict its performance.

Nearshore waves can be nonlinear, and nonlinear effects on wave energy converters (WECs) need to be addressed in the future. Analytical theories for second-order wave diffraction and radiation can be found in the literature (e.g., Slavounos (1988); Newman (1996); Shao & Faltinsen (2013)), but have not been used to analyze wave energy converters. Fully nonlinear potential solvers, such as boundary element method (BEM) and finite element method (FEM), have the potential to tackle nonlinear wave-structure interactions, and can be pursued in future studies.

Appendix A

Appendix

A.1 Solutions of integrals

A.1.1 Solutions of the integrals in chapter 3

For the shifted Chebyshev polynomial, which can be written as

$$T_m^* \left(\frac{\theta}{\psi} \right) = \mathcal{T}_m(\theta) = \sum_{k=0}^m \gamma_k \theta^k. \quad (\text{A.1})$$

Therefore, for the expressions of the abbreviation $\bar{C}_{mn}$ and $\bar{S}_{mn}$ in scattering problem, after inserting Eqn. (A.1), which can be rewritten as

$$\bar{C}_{mn} = \sum_{k=0}^m \gamma_k F_{nk}^C, \quad (\text{A.2})$$

and

$$\bar{S}_{mn} = \sum_{k=0}^m \gamma_k F_{nk}^S, \quad (\text{A.3})$$

where F_{nk}^C and F_{nk}^S are given by

$$F_{nk}^C = \int_0^\psi \theta^k \cos n\theta d\theta, \quad (\text{A.4})$$

and

$$F_{nk}^S = \int_0^\psi \theta^k \sin n\theta d\theta, \quad (\text{A.5})$$

respectively.

Based on integration by parts, F_{nk}^C can be calculated by the following recurrence relation:

$$F_{nk}^C = \frac{\psi^k}{n} \sin n\psi + \frac{k\psi^{k-1}}{n^2} \cos n\psi - \frac{k(k-1)}{n^2} F_{n(k-2)}^C, \quad (\text{A.6})$$

with

$$F_{n0}^C = \int_0^\psi \cos n\theta d\theta = \frac{\sin n\psi}{n}, \quad (\text{A.7})$$

and

$$F_{n1}^C = \int_0^\psi \theta \cos n\theta d\theta = \frac{\cos n\psi + n\psi \sin n\psi - 1}{n^2}. \quad (\text{A.8})$$

Similarly, F_{nk}^S can be calculated by the following recurrence relation:

$$F_{nk}^S = -\frac{\psi^k \cos n\psi}{n} + \frac{k\psi^{k-1} \sin n\psi}{n^2} - \frac{k(k-1)}{n^2} F_{n(k-2)}^S, \quad (\text{A.9})$$

with

$$F_{n0}^S = \int_0^\psi \sin n\theta d\theta = \frac{1 - \cos n\psi}{n}, \quad (\text{A.10})$$

and

$$F_{n1}^S = \int_0^\psi \theta \sin n\theta d\theta = \frac{\sin n\psi - n\psi \cos n\psi}{n^2}. \quad (\text{A.11})$$

In radiation problem, the abbreviation $\mathcal{C}_{sn}$ can be calculated by

$$\begin{aligned} \mathcal{C}_{sn} &= \int_0^\psi \cos \frac{2s}{v} \left(\theta - \frac{\psi}{2} \right) \cos n \left(\theta - \frac{\psi}{2} \right) d\theta \\ &= \begin{cases} \frac{2nv^2(-1)^s \sin \frac{nv\pi}{2}}{(nv)^2 - 4s^2}, & 2s/v - n \neq 0, \\ v\pi, & 2s/v - n = 0, n = 0, \\ \frac{v\pi}{2} + \frac{\sin nv\pi}{2n}, & 2s/v - n = 0, n \neq 0. \end{cases} \end{aligned} \quad (\text{A.12})$$

A.1.2 Solutions of the integral in chapter 4

A.1.2.1 Solutions of the integrals in scattering problem

For the sake of convenience in the numerical computations, let us first introduce the following four integral formula (see Eqn. (7.324) on page 798 and Eqn. (7.355) on page 802 in Gradshteyn & Ryzhik (2007)):

$$\int_0^1 \frac{C_{2m}^{1/6}(v) \cos(uv)}{(1-v^2)^{1/3}} dv = \frac{\pi(-1)^m \Gamma(2m+1/3)}{(2m)! \Gamma(1/6) (2u)^{1/6}} J_{2m+1/6}(u), \quad (\text{A.13})$$

$$\int_0^1 \frac{C_{2m+1}^{1/6}(v) \sin(uv)}{(1-v^2)^{1/3}} dv = \frac{\pi(-1)^m \Gamma(2m+1/3)}{(2m+1)! \Gamma(1/6) (2u)^{1/6}} J_{2m+1+1/6}(u), \quad (\text{A.14})$$

$$\int_0^1 \frac{T_{2m}(v) \cos(uv)}{\sqrt{1-v^2}} dv = (-1)^m \frac{\pi}{2} J_{2m}(u), \quad (\text{A.15})$$

and

$$\int_0^1 \frac{T_{2m+1}(v) \sin(uv)}{\sqrt{1-v^2}} dv = (-1)^m \frac{\pi}{2} J_{2m+1}(u). \quad (\text{A.16})$$

After using Eqns. (4.13), (4.14), and (A.13)-(A.16) in Eqns. (4.24), (4.25), (4.29), (4.30), (4.35), (4.44) and (4.35), we have the following expressions for $X_{nm}^{(1)}$, $X_{nm}^{(2)}$, $Y_{nm}^{(1)}$, $Y_{nm}^{(2)}$, $L_{nm}^{(1)}$, and $L_{nm}^{(2)}$:

(i) The expression for $X_{nm}^{(1)}$:

$$X_{nm}^{(1)} = \int_{-\beta}^{\beta} f_n^{(1)}(\theta) \cos m\theta d\theta = \begin{cases} 0, & n \in \text{Odds}, \\ \frac{(-1)^{n/2}}{(2m\beta)^{1/6}} J_{n+1/6}(m\beta), & n \in \text{Evens}, \end{cases} \quad (\text{A.17})$$

with

$$X_{n0}^1 = \int_{-\beta}^{\beta} f_n^{(1)}(\theta) d\theta = \begin{cases} \frac{1}{2^{1/3}\Gamma(7/6)}, & n = 0, \\ 0, & n = 1, 2, 3, \dots, \end{cases} \quad (\text{A.18})$$

(ii) The expression for $X_{nm}^{(2)}$:

$$X_{nm}^{(2)} = \int_{\beta+\gamma}^{2\pi-\beta-\gamma} f_n^{(2)}(\theta) \cos m\theta d\theta = \begin{cases} 0, & n \in \text{Odds}, \\ (-1)^{m+n/2} J_n[m(\pi - \beta - \gamma)], & n \in \text{Evens}, \end{cases} \quad (\text{A.19})$$

(iii) The expression for $Y_{nm}^{(1)}$:

$$Y_{nm}^{(1)} = \int_{-\beta}^{\beta} f_n^{(1)}(\theta) \sin m\theta d\theta = \begin{cases} \frac{(-1)^{(n-1)/2}}{(2m\beta)^{1/6}} J_{n+1/6}(m\beta), & n \in \text{Odds}, \\ 0, & n \in \text{Evens}, \end{cases} \quad (\text{A.20})$$

with

$$Y_{n0}^1 = 0, \quad (\text{A.21})$$

(iv) The expression for $Y_{nm}^{(2)}$:

$$Y_{nm}^{(2)} = \int_{\beta+\gamma}^{2\pi-\beta-\gamma} f_n^{(2)}(\theta) \sin m\theta d\theta = \begin{cases} (-1)^{(n-1)/2+m} J_n[m(\pi - \beta - \gamma)], & n \in \text{Odds}, \\ 0, & b \in \text{Evens}, \end{cases} \quad (\text{A.22})$$

(v) The expression for $L_{nm}^{(1)}$:

$$\begin{aligned} L_{nm}^{(1)} &= \int_{-\beta}^{\beta} f_n^{(1)}(\theta) \cos m\mu(\theta + \beta) d\theta \\ &= \begin{cases} -\sin(m\pi/2) \frac{(-1)^{(n-1)/2}}{(m\pi)^{1/6}} J_{n+1/6}(m\pi/2), & n \in \text{Odds}, \\ \cos(m\pi/2) \frac{(-1)^{n/2}}{(m\pi)^{1/6}} J_{n+1/6}(m\pi/2), & n \in \text{Evens}, \end{cases} \end{aligned} \quad (\text{A.23})$$

with

$$L_{n0}^1 = X_{n0}^1, \quad (\text{A.24})$$

(vi) The expression for $L_{nm}^{(2)}$:

$$\begin{aligned} L_{nm}^{(2)} &= \int_{\beta+\gamma}^{2\pi-\beta-\gamma} f_n^{(2)}(\theta) \cos m\nu(\theta - \beta) d\theta \\ &= \begin{cases} -\sin(m\pi/2) (-1)^{(n-1)/2} J_n[m\nu(\pi - \beta - \gamma)], & n \in \text{Odds}, \\ \cos(m\pi/2) (-1)^{n/2} J_n[m\nu(\pi - \beta - \gamma)], & n \in \text{Evens}. \end{cases} \end{aligned} \quad (\text{A.25})$$

For the following integral

$$V_{pl}^{(1)} = \frac{1}{h} \int_{-h}^0 \tilde{T}_p(z) Z_l(z) dz, \quad (\text{A.26})$$

the function $\tilde{T}_p(z)$ can be written as

$$\tilde{T}_p(z) = T_p^* \left(\frac{z+h}{h} \right) = \sum_{s=0}^p \tau^s z^s. \quad (\text{A.27})$$

Therefore,

$$V_{pl}^{(1)} = \frac{1}{h} \sum_{s=0}^p \tau^s \int_{-h}^0 z^s Z_l(z) dz, \quad (\text{A.28})$$

where

$$\begin{aligned} &\int_{-h}^0 z^s Z_l(z) dz \\ &= N_l^{-1/2} \int_{-h}^0 z^s \cos k_l(z+h) dz \\ &= -N_l^{-1/2} \frac{s}{k_l^2} (-h)^{s-1} - \frac{s(s-1)}{k_l^2} \int_{-h}^0 z^{s-2} Z_l(z) dz. \end{aligned} \quad (\text{A.29})$$

with

$$\begin{aligned} \int_{-h}^0 Z_l(z) dz &= N_l^{-1/2} \int_{-h}^0 \cos k_l(z+h) dz \\ &= \frac{N_l^{-1/2}}{k_l} \sin k_l h, \end{aligned} \quad (\text{A.30})$$

and

$$\begin{aligned} \int_{-h}^0 z Z_l(z) dz &= N_l^{-1/2} \int_{-h}^0 z \cos k_l(z+h) dz \\ &= N_l^{-1/2} \frac{\cos k_l h - 1}{k_l^2}, \end{aligned} \quad (\text{A.31})$$

by applying Eqns. (A.30) and (A.31) in Eqn. (A.29), we can obtain

$$\int_{-h}^0 z^s Z_l(z) dz, \quad s = 2, 3, 4, \dots \quad (\text{A.32})$$

and then the values of $V_{pl}^{(1)}$ can be calculated.

To calculate the values of $V_{pl}^{(2)}$, we use Eqns. (4.16), (A.15) and (A.16) to obtain

$$\begin{aligned} V_{pl}^{(2)} &= \frac{1}{h} \int_{-h}^{-d} u_p(z) Z_l(z) dz \\ &= \frac{N_l^{-1/2}}{h} J_{2p} [k_l (h - d)]. \end{aligned} \quad (\text{A.33})$$

A.1.2.2 Solution of the integral in radiation problem

From Eqns. (4.80), (4.81), (4.86), (4.92), (A.13), and (A.15), the solutions of $\tilde{X}_{nm}^{(1)}$, $\tilde{X}_{nm}^{(2)}$, $W_{nm}^{(1)}$, $W_{nm}^{(2)}$ and $\tilde{V}_p$ can be obtained.

(i) The expression for $\tilde{X}_{nm}^{(1)}$:

$$\tilde{X}_{nm}^{(1)} = \int_{-\beta}^{\beta} \tilde{f}_n^{(1)}(\theta) \cos m\theta d\theta = \frac{(-1)^n}{(2m\beta)^{1/6}} J_{2n+1/6}(m\beta), \quad (\text{A.34})$$

with

$$\tilde{X}_{n0}^1 = \int_{-\beta}^{\beta} \tilde{f}_n^{(1)}(\theta) d\theta = \begin{cases} \frac{1}{2^{1/3}\Gamma(7/6)}, & n = 0, \\ 0, & n = 1, 2, 3, \dots \end{cases} \quad (\text{A.35})$$

(ii) The expression for $\tilde{X}_{nm}^{(2)}$:

$$\tilde{X}_{nm}^{(2)} = \int_{\beta+\gamma}^{2\pi-\beta-\gamma} \tilde{f}_n^{(2)}(\theta) \cos m\theta d\theta = (-1)^{n+m} J_{2n} [m(\pi - \beta - \gamma)]. \quad (\text{A.36})$$

(iii) The expression for $\tilde{W}_{nm}^{(1)}$:

$$W_{nm}^{(1)} = \int_{-\beta}^{\beta} \tilde{f}_n^{(1)}(\theta) \cos 2m\mu(\theta + \beta) d\theta = \frac{(-1)^{n+m}}{(2m\pi)^{1/6}} J_{2n+1/6}(m\pi), \quad (\text{A.37})$$

with

$$W_{n0}^1 = \tilde{X}_{n0}^1, \quad (\text{A.38})$$

(iv) The expression for $\tilde{W}_{nm}^{(2)}$:

$$W_{nm}^{(2)} = \int_{\beta+\gamma}^{2\pi-\beta-\gamma} \tilde{f}_n^{(2)}(\theta) \cos 2m\nu(\theta - \beta) d\theta = (-1)^{n+m} J_{2n} [2m\nu(\pi - \beta - \gamma)]. \quad (\text{A.39})$$

(v) The expression for $\tilde{V}_p$:

$$\tilde{V}_p = \frac{1}{h} \int_{-h}^{-d} u_p(z) dz = \begin{cases} \frac{1}{h}, & p = 0, \\ 0, & p = 1, 2, 3, \dots \end{cases} \quad (\text{A.40})$$

A.1.3 Solutions of the integrals in chapter 5

The abbreviation $V_{pl}^{(1)}$ in Eqn. (5.54), after using Eqns. (5.42) and (A.13), can be calculated by

$$V_{pl}^1 = \frac{N_l^{-1/2}}{h[2k_l(h-d)]^{1/6}} J_{2p+1/6}(k_l(h-d)). \quad (\text{A.41})$$

The abbreviation $V_{pn}^{(2)}$ in Eqn. (5.60), after using (5.42) and (A.13), can be calculated by

$$\begin{aligned} V_{p0}^{(2)} &= \frac{1}{h-d} \int_{-h}^{-d} u_p(z) dz \\ &= \begin{cases} 0, & p \neq 0, \\ \frac{3\Gamma(2/3)}{(h-d)^{5/6}\sqrt{\pi}\Gamma(1/3)}, & p = 0. \end{cases} \end{aligned} \quad (\text{A.42})$$

and

$$\begin{aligned} V_{pn}^2 &= \frac{1}{h-d} \int_{-h}^{-d} u_p(z) f_n(z) dz \\ &= \frac{J_{1/6+2p}(\alpha_n(h-d))}{(h-d)(2\alpha_n)^{1/6}}, \quad n = 1, 2, 3, \dots \end{aligned} \quad (\text{A.43})$$

Gegenbauer polynomials $C_{2p}^{1/6}(\sigma)$ can be rewritten as

$$C_{2p}^{1/6}(\sigma) = \sum_{s=0}^p \tau_s \sigma^{2s}. \quad (\text{A.44})$$

Therefore, the abbreviation $\hat{V}_p$ in Eqn. (5.173), after using Eqns. (5.42) and (A.44), can be calculated by

$$\begin{aligned} \hat{V}_p &= \frac{1}{h-d} \int_{-h}^{-d} u_p(z) (z+h)^2 dz \\ &= \frac{(-1)^p (2p)! \Gamma(1/6) (h-d)^{7/6}}{\pi \Gamma(2p+1/3)} \int_0^1 [1-\sigma^2]^{-1/3} C_{2p}^{1/6}(\sigma) \sigma^2 d\sigma \\ &= \frac{(-1)^p (2p)! \Gamma(1/6) (h-d)^{7/6}}{\pi \Gamma(2p+1/3)} \sum_{s=0}^p \tau_s Y_s, \end{aligned} \quad (\text{A.45})$$

where Y_s are given by

$$\begin{aligned} Y_s &= \int_0^1 [1-\sigma^2]^{-1/3} \sigma^{2(s+1)} d\sigma \\ &= \frac{\Gamma(2/3) \Gamma(3/2+s)}{2\Gamma(13/6+s)}. \end{aligned} \quad (\text{A.46})$$

A.2 Matlab codes for different chapters

A.2.1 Matlab codes for chapter 3

The Matlab codes, used to numerically compute the values of relevant quantities against corresponding arguments, are listed below.

```

1  clc
2  clear all
3  format long
4  h=10;
5  a=1/2*h;
6  d=0.2*h;
7  v=2;
8  g=9.81;
9  alpha=pi+v*pi/2;
10 rho_wa=1000;
11 ca=340;
12 V0=pi*a^2*h;
13 KCh=0.1:0.025:7;
14 num=length(KCh);
15 % Group velocity
16 k0=zeros(1,num); % wavenumber for different frequencies
17 beta=zeros(1,num);
18 C_g=zeros(1,num);
19 for ii=1:num
20     KC=KCh(ii)/h;
21     omega=sqrt(KC*g);
22     kl=wavenumber(KC,h,1);
23     k=1i*kl(1);
24     k0(ii)=k;
25     C_g(ii)=omega/(2*k)*(1+2*k*h/sinh(2*k*h));
26     beta(ii)=-(omega*V0*rho_wa*sqrt(g/h)/(ca^2*h));
27 end
28 % Radiation problem
29 L=400;
30 N=100;
31 M=50;
32 P=5;
33 B_w=zeros(1,num); % radiation damping
34 nu=zeros(1,num); % radiation damping in Evans's paper (...
    ASME 1997)
35 C_w=zeros(1,num); % added mass
36 mu=zeros(1,num); % added mass in Evans's paper (ASME 1997)
37 lmax=zeros(1,num);
38 for ii=1:num
39     % k=kh(ii)/h;
40     % KC=k*tanh(k*h);
41     KC=KCh(ii)/h; % KC=omega^2/g
42     omega=sqrt(KC*g);
43     kl=wavenumber(KC,h,L);
44     k=1i*kl(1);
45     Nl=1/2*(1+sin(2*k1*h)/(2*k1*h));

```

```

46     Zl=Nl.^(-1/2).*cos(kl*h); % as z=0
47     DZl=Nl.^(-1/2).*(-kl).*sin(kl*h); % as z=0
48 %
49     [Enl,Fnl]=radiation_problem(h,a,d,v,KC,N,L,M,P);
50     I0l=quad_I0l(kl,a);
51 %     B=g/(omega^2*pi*a^2)*real(2*pi*1i*Enl(1,:)*(DZl./(kl*...
a.*dbesseli(0,kl*a)).*I0l).');
52 %     B_bar(ii)=B;
53     B_w(ii)=real(2*pi*1i/a*Enl(1,:)*(DZl./(kl*a.*dbesseli...
(0,kl*a)).*I0l).');
54     nu(ii)=g/(pi*a*omega^2)*B_w(ii);
55     C_w(ii)=-imag(2*pi*1i/a*Enl(1,:)*(DZl./(kl*a.*dbesseli...
(0,kl*a)).*I0l).');
56     mu(ii)=g/(pi*a*omega^2)*C_w(ii);
57     lmax(ii)=nu(ii)./(k*a*((nu(ii).^2+mu(ii).^2).^(1/2)+nu(...
ii)));
58 end
59 KC=KCh/h;
60 omega=sqrt(KC*g);
61 % another dimensionless parameters
62 B_compressive=a*sqrt(g/h)*B_w./(omega*h);
63 C_compressive=a*sqrt(g/h)*C_w./(omega*h);
64 chi_opt=sqrt(B_compressive.^2+(C_compressive-beta).^2);
65 % Scattering problem
66 L=400;
67 N=65;
68 M=6;
69 P=5;
70 Gamma_w1=zeros(1,num);
71 Gamma_w2=zeros(1,num);
72 Gamma_square=zeros(1,num);
73 for ii=1:num
74     KC=KCh(ii)/h; % KC=omega^2/g
75     omega=sqrt(KC*g);
76     kl=wavenumber(KC,h,L);
77     Nl=1/2*(1+sin(2*kl*h)./(2*kl*h));
78     % N0=1/2*(1+sinh(2*k*h)/(2*k*h));
79     Zl=Nl.^(-1/2).*cos(kl*h); % as z=0
80     % Z0=N0.^(-1/2)*cosh(k*h); % as z=0
81     DZl=Nl.^(-1/2).*(-kl).*sin(kl*h);
82 %
83     [Anl,Bnl,Cnl,Dnl]=scattering(h,a,d,v,alpha,KC,N,L,M,P);
84     Gamma=-2*pi*1i*g/omega*Cnl(1,:)*(DZl./(kl*a.*dbesseli...
(0,kl*a)).*(quad_I0l(kl,a).'));
85     Gamma_w1(ii)=Gamma*omega/(a*g);
86     Gamma_w2(ii)=Gamma*sqrt(g/h)/(h*g);
87     Gamma_square(ii)=abs(Gamma_w2(ii))^2;
88 end
89 %% optimum capture length
90 kL=k0*h*g./((C_g*sqrt(g/h)).*chi_opt.*Gamma_square./((...
chi_opt+B_compressive).^2+(C_compressive-beta).^2);
91 %
92 %
93 function [Anl,Bnl,Cnl,Dnl]=scattering(h,a,d,v,alpha,KC,N,L,...
M,P)
94 kl=wavenumber(KC,h,L);

```

```

95 k=1i*k1(1);
96 N1=1/2*(1+sin(2*k1*h)/(2*k1*h));
97 % N0=1/2*(1+sinh(2*k*h)/(2*k*h));
98 Z1=N1.^(-1/2).*cos(k1*h); % as z=0
99 % Z0=N0.^(-1/2).*cosh(k*h); % as z=0
100 nn=0:N;
101 epsn=epsilon(nn);
102 En=(1i).^nn.*epsn.*cos(nn*alpha)/Z1(1);
103 Fn=1i.^nn.*epsn.*sin(nn*alpha)/Z1(1);
104 Rn=besselj(nn,k*a)-dbesselj(nn,k*a).*besselH(nn,k*a)./...
    dbesselH(nn,k*a);
105 Kn1=zeros(N+1,L+1);
106 for n=0:N
107     Kn1(n+1,:)=besselk(n,k1*a)/(k1*a.*dbesselk(n,k1*a));
108 end
109 In1=zeros(N+1,L+1);
110 for n=0:N
111     In1(n+1,:)=besseli(n,k1*a)/(k1*a.*dbesseli(n,k1*a));
112 end
113 Fp1=zeros(P+1,L+1);
114 for p=0:P
115     Fp1(p+1,:)=N1.^(-1/2).*besselj(2*p,k1*(h-d));
116 end
117 Cmn=zeros(M+1,N+1);
118 for n=0:N
119     for m=0:M
120         Cmn(m+1,n+1)=quad_TC(m,n,v);
121     end
122 end
123 Smn=zeros(M+1,N);
124 for n=1:N
125     for m=0:M
126         Smn(m+1,n)=quad_TS(m,n,v);
127     end
128 end
129 % LHS=RHS*sol
130 LHS=zeros((M+1)*(P+1),1);
131 for kp=0:(M+1)*(P+1)-1
132     mp=floor(kp/(P+1));
133     pp=kp-(P+1)*floor(kp/(P+1));
134     LHS(kp+1)=h/a*Fp1(pp+1,1)*(Rn*(En.*Cmn(mp+1,:)).'+Rn(2...
        :end)*(Fn(2:end).*Smn(mp+1,:)).');
135 end
136 RHS=zeros((M+1)*(P+1),(M+1)*(P+1));
137 for k=0:(M+1)*(P+1)-1
138     for kp=0:(M+1)*(P+1)-1
139         m=floor(k/(P+1));
140         p=k-(P+1)*floor(k/(P+1));
141         mp=floor(kp/(P+1));
142         pp=kp-(P+1)*floor(kp/(P+1));
143         RHS(kp+1,k+1)=(epsn/(2*pi).*Cmn(m+1,:).*Cmn(mp+1,:)...
            )*((In1-Kn1)*(Fp1(p+1,:).*Fp1(pp+1,:)).'+(epsn...
            (2:end)/(2*pi).*Smn(m+1,:).*Smn(mp+1,:))*((In1...
            (2:end,:)-Kn1(2:end,:))*(Fp1(p+1,:).*Fp1(pp...
            +1,:)).');
144     end

```

```

145 end
146 sol=RHS\LHS;
147 alpha_mp=zeros(M+1,P+1);
148 for p=0:P
149     for m=0:M
150         alpha_mp(m+1,p+1)=sol(m*(P+1)+p+1);
151     end
152 end
153 Anl=zeros(N+1,L+1);
154 for l=0:L
155     for n=0:N
156         Anl(n+1,l+1)=a*epsn(n+1)/(2*pi*h)*(Cmn(:,n+1).')*(...
            alpha_mp*Fpl(:,l+1));
157     end
158 end
159 Cnl=Anl;
160 Bnl=zeros(N,L+1);
161 for l=0:L
162     for n=1:N
163         Bnl(n,l+1)=a*epsn(n+1)/(2*pi*h)*(Smn(:,n).')*(...
            alpha_mp*Fpl(:,l+1));
164     end
165 end
166 Dnl=Bnl;
167 %
168 %
169 function [Enl,Fnl]=radiation_problem(h,a,d,v,KC,N,L,M,P)
170 kl=wavenumber(KC,h,L);
171 Nl=1/2*(1+sin(2*kl*h)/(2*kl*h));
172 % N0=1/2*(1+sinh(2*k*h)/(2*k*h));
173 % Zn=Nl.^(-1/2).*cos(kl*h); % as z=0
174 % Z0=N0.^(-1/2)*cosh(k*h); % as z=0
175 nn=0:N;
176 epsn=epsilon(nn);
177 Fpl=zeros(P+1,L+1);
178 for p=0:P
179     Fpl(p+1,:)=Nl.^(-1/2).*besselj(2*p,kl*(h-d));
180 end
181 Inl=zeros(N+1,L+1);
182 for n=0:N
183     Inl(n+1,:)=besseli(n,kl*a)/(kl*a.*dbesseli(n,kl*a));
184 end
185 Knl=zeros(N+1,L+1);
186 for n=0:N
187     Knl(n+1,:)=besselk(n,kl*a)/(kl*a.*dbesselk(n,kl*a));
188 end
189 Dlt=zeros(M+1,N+1);
190 for n=0:N
191     for m=0:M
192         Dlt(m+1,n+1)=Delta_mn(m,n);
193     end
194 end
195 Cmn=zeros(M+1,N+1);
196 for n=0:N
197     for m=0:M
198         Cmn(m+1,n+1)=quad_Cmn(m,n,v);

```



```

199     end
200 end
201 % LHS=RHS*sol
202 LHS=zeros((M+1)*(P+1),1);
203 for kp=0:(M+1)*(P+1)-1
204     mp=floor(kp/(P+1));
205     pp=kp-(P+1)*floor(kp/(P+1));
206     LHS(kp+1)=h/a*v*pi*Dlt(mp+1,1)*Dlt(pp+1,1);
207 end
208 RHS=zeros((M+1)*(P+1),(M+1)*(P+1));
209 for k=0:(M+1)*(P+1)-1
210     for kp=0:(M+1)*(P+1)-1
211         m=floor(k/(P+1));
212         p=k-(P+1)*floor(k/(P+1));
213         mp=floor(kp/(P+1));
214         pp=kp-(P+1)*floor(kp/(P+1));
215         RHS(kp+1,k+1)=(epsn/(2*pi).*Cmn(m+1,:).*Cmn(mp+1,:)...
                )*((Kn1-In1)*(Fpl(p+1,:).*Fpl(pp+1,:)).');
216     end
217 end
218 sol=RHS\LHS;
219 alpha_mp=zeros(M+1,P+1);
220 for p=0:P
221     for m=0:M
222         alpha_mp(m+1,p+1)=sol(m*(P+1)+p+1);
223     end
224 end
225 Enl=zeros(N+1,L+1);
226 for l=0:L
227     for n=0:N
228         Enl(n+1,l+1)=a*epsn(n+1)/(2*pi*h)*(Cmn(:,n+1).')*(...
                alpha_mp*Fpl(:,l+1));
229     end
230 end
231 Fn1=Enl;
232 %
233 %
234 function roots=wavenumber(KC,h,N)
235 % roots=wavenumber[KC,h,N]
236 fk=@(k)k.*tanh(k*h)-KC;
237 dfk=@(k)tanh(k*h)+k*h.*sech(k*h).^2;
238 fkn=@(kn)kn.*tan(kn*h)+KC;
239 dfkn=@(kn)tan(kn*h)+kn*h.*sec(kn*h).^2;
240
241 roots=zeros(1,N+1);
242 x0=0.2;
243 k=newton(fk,dfk,x0,1e-12);
244 roots(1)=-1i*k;
245 for n=1:N
246     x0=pi/(2*h)+0.005+(n-1)*pi/h;
247     roots(n+1)=newton(fkn,dfkn,x0,1e-12);
248 end
249 %
250 %
251 function x1=newton(fun,dfun,x,tol)
252 x1=x-fun(x)/dfun(x);

```

```

253 while (abs(x1-x) ≥ tol)
254     x=x1;
255     x1=x-fun(x)/dfun(x);
256 end
257 %
258 %
259 function out=besselH(m,x)
260 % out=besselH(m,x)
261 % evaluate the Hankel function of the first kind of order m
262 out=besselj(m,x)+1i*bessely(m,x);
263 %
264 %
265 function out=dbesselH(m,x)
266 out=1/2*(besselH(m-1,x)-besselH(m+1,x));
267 %
268 %
269 function out=dbesseli(nu,x)
270 out=1/2*(besseli(nu-1,x)+besseli(nu+1,x));
271 %
272 %
273 function out=dbesselj(nu,x)
274 out=1/2*(besselj(nu-1,x)-besselj(nu+1,x));
275 %
276 %
277 function out=dbesselk(nu,x)
278 % out=dbesselk(nu,x)
279 out=1/2*(-besselk(nu-1,x)-besselk(nu+1,x));
280 %
281 %
282 function out=dbessely(nu,x)
283 out=1/2*(bessely(nu-1,x)-bessely(nu+1,x));
284 %
285 %
286 function out=Δ_mn(m,n)
287 % out=Δ_mn(m,n)
288 if abs(m-n) < 1e-6
289     out=1;
290 else
291     out=0;
292 end
293 %
294 %
295 function y=epsilon(m)
296 % y=epsilon(m)
297 nn=length(m);
298 if nn==1
299     if m==0
300         y=1;
301     else
302         y=2;
303     end
304 else
305     y=1:nn;
306     for in=1:nn
307         if m(in)==0
308             y(in)=1;

```

```

309         else
310             y(in)=2;
311         end
312     end
313 end
314 %
315 %
316 function out=gamma_k(m,v)
317 % out=gamma_k(m,v)
318 syms theta
319 x=theta/(v*pi);
320 if abs(m)<1e-6
321     out=1;
322 else
323     % out=zeros(1,m+1);
324     out=sym2poly(shifted_Tm(m,x));
325 end
326 %
327 %
328 function out=shifted_Tm(m,x)
329 % out=shifted_Tm(m,x)
330 if abs(m)<1e-6
331     out=1;
332 elseif abs(m-1)<1e-6
333     out=2*x-1;
334 else
335     t0=1;
336     t1=2*x-1;
337     for ii=2:m
338         tm=2*(2*x-1)*t1-t0;
339         t0=t1;
340         t1=tm;
341     end
342     out=tm;
343 end
344 %
345 %
346 function out=quad_Cmn(m,n,v)
347 % out=quad_Cmn(m,n,v)
348 % cos(2*m/v*(theta-v*pi/2))
349 % cos(n*(theta-v*pi/2))
350 mm=2*m/v;
351 if abs(mm-n)>1e-6
352     out=(2*(n*v^2*sin((pi*n*v)/2)*cos(pi*m) - 2*m*v*sin(pi*...
353         m)*cos((pi*n*v)/2)))/(n^2*v^2 - 4*m^2);
354 else
355     if abs(n)<1e-6
356         out=pi*v;
357     else
358         out=(pi*v)/2 + sin(pi*n*v)/(2*n);
359     end
360 end
361 %
362 %
363 function out=quad_I0l(kl,a)
364 % out=quad_I0l(1)

```

```

364 out=-(a*besselj(1, a*k1*1i)*1i)./k1;
365 %
366 %
367 function out=quad_TC(m,n,v)
368 % out=quad_TC(m,n,v)
369 % Tm(x)*cos(n*theta) from 0 to v*pi
370 gk=gamma_k(m,v);
371 quad_kn=zeros(1,m+1);
372 for k=0:m
373     fun=@(theta)(theta.^k).*cos(n*theta);
374     quad_kn(m+1-k)=quadl(fun,0,v*pi);
375 end
376 out=gk*quad_kn.';
377 %
378 %
379 function out=gamma_k(m,v)
380 % out=gamma_k(m,v)
381 syms theta
382 x=theta/(v*pi);
383 if abs(m)<1e-6
384     out=1;
385 else
386     % out=zeros(1,m+1);
387     out=sym2poly(shifted_Tm(m,x));
388 end
389 %
390 %
391 function out=shifted_Tm(m,x)
392 % out=shifted_Tm(m,x)
393 if abs(m)<1e-6
394     out=1;
395 elseif abs(m-1)<1e-6
396     out=2*x-1;
397 else
398     t0=1;
399     t1=2*x-1;
400     for ii=2:m
401         tm=2*(2*x-1)*t1-t0;
402         t0=t1;
403         t1=tm;
404     end
405     out=tm;
406 end
407 %
408 %
409 function out=quad_TS(m,n,v)
410 % out=quad_TS(m,n,v)
411 % Tm(x)*sin(n*theta) theta from 0 to v*pi
412 % x=theta/(v*pi)
413 gk=gamma_k(m,v);
414 quad_kn=zeros(1,m+1);
415 for k=0:m
416     fun=@(theta)(theta.^k).*sin(n*theta);
417     quad_kn(m+1-k)=quadl(fun,0,v*pi);
418 end
419 out=gk*quad_kn.';

```

A.2.2 Matlab codes for chapter 4

A.2.2.1 Matlab codes for scattering problem

```

1 clear all
2 close all
3 clc
4 format long
5 %=====
6 h=10;
7 g=9.81;
8 d=0.2*h;
9 a=1/2*h;
10 b=0.01*h;
11 c=a+b;
12 beta=pi*1/4;
13 gam=pi*1/20;
14 alpha=pi;
15 mu=pi/(2*beta);
16 nu=pi/(2*pi-2*beta);
17 %=
18 L=400;
19 M=55;
20 P=10;
21 N=12;
22 %=====
23 Kh=0.2:0.025:6;
24 K=Kh/h;
25 omega=sqrt(K*g); % K=omega^2/g
26 num=length(Kh);
27 disp(num)
28 k0=zeros(num,1);
29 GMA=zeros(num,1);
30 %
31 epsilon=zeros(1,M+1)+2; % e0=1, em=2
32 epsilon(1,1)=1;
33 for ii=55:55
34     disp(ii)
35     kl=wavenumber(K(ii),h,L);
36     k=1i*kl(1);
37     k0(ii)=k;
38     N1=1/2*(1+sin(2*kl*h)/(2*kl*h));
39     Z1=N1.^(-1/2).*cos(kl*h); % Z(z)=N1.^(-1/2)*cos(kl*(z+h...
    ))
40     DZ1=N1.^(-1/2).*(-kl).*sin(kl*h);
41     %
42     mm=0:M;
43     Em=(1i).^mm.*epsilon.*cos(mm*alpha)/Z1(1);
44     Fm=(1i).^mm.*epsilon.*sin(mm*alpha)/Z1(1);

```

```

45     Rm=2*i./(pi*k0(ii)*c*dbesselH(mm,k0(ii)*c));
46     %
47     Q1_m1=zeros(M+1,L+1);
48     Q2_m1=zeros(M+1,L+1);
49     for m=0:M
50         Q1_m1(m+1,:)=1./(1-dIdI(m*mu,kl*a,kl*c).*dKdK(m*mu,...
51             kl*c,kl*a));
52         Q2_m1(m+1,:)=1./(1-dIdI(m*nu,kl*a,kl*c).*dKdK(m*nu,...
53             kl*c,kl*a));
54     end
55     R1_m1=zeros(M+1,L+1);
56     R2_m1=zeros(M+1,L+1);
57     R3_m1=zeros(M+1,L+1);
58     R4_m1=zeros(M+1,L+1);
59     R5_m1=zeros(M+1,L+1);
60     for m=0:M
61         R1_m1(m+1,:)=IdI(m*mu,kl*c,kl*c)./kl-dIdI(m*mu,kl*a,...
62             kl*c)...
63             .*KdK(m*mu,kl*c,kl*a)./kl;
64         R2_m1(m+1,:)=KdK(m*mu,kl*c,kl*a)./kl-IdI(m*mu,kl*c,...
65             kl*c)...
66             .*dKdK(m*mu,kl*c,kl*a)./kl;
67         R3_m1(m+1,:)=IdI(m*nu,kl*c,kl*c)./kl-dIdI(m*nu,kl*a,...
68             kl*c)...
69             .*KdK(m*nu,kl*c,kl*a)./kl;
70         R4_m1(m+1,:)=IdI(m*mu,kl*a,kl*c)./kl-dIdI(m*mu,kl*a,...
71             kl*c)...
72             .*KdK(m*mu,kl*a,kl*a)./kl;
73         R5_m1(m+1,:)=KdK(m*mu,kl*a,kl*a)./kl-IdI(m*mu,kl*a,...
74             kl*c)...
75             .*dKdK(m*mu,kl*c,kl*a)./kl;
76     end
77     Kml=zeros(M+1,L+1);
78     Iml=zeros(M+1,L+1);
79     for m=0:M
80         Kml(m+1,:)=KdK(m,kl*c,kl*c)./kl;
81         Iml(m+1,:)=IdI(m,kl*a,kl*a)./kl;
82     end
83     mm=1:M;
84     X1_nm=zeros(N+1,M+1);
85     X1_nm(1,1)=1/(2^(1/3)*gamma(7/6));
86     X2_nm=zeros(N+1,M+1);
87     X2_nm(1,1)=1;
88     for n=0:N
89         if rem(n,2)<1e-8
90             X1_nm(n+1,2:end)=(-1)^(n/2)./(2*mm*beta).^(1/6)...
91                 .*besselj(n+1/6,mm*beta);
92             X2_nm(n+1,2:end)=(-1).^(mm+n/2).*besselj(n,mm*(...
93                 pi-beta-gam));
94         end
95     end
96     Y1_nm=zeros(N+1,M+1);
97     Y2_nm=zeros(N+1,M+1);
98     for n=0:N
99         if rem(n,2)>1e-8

```

```

91         Y1_nm(n+1,2:end)=(-1)^((n-1)/2)./(2*mm*beta)...
           .^(1/6).*besselj(n+1/6,mm*beta);
92         Y2_nm(n+1,2:end)=(-1).^((n-1)/2+mm).*besselj(n,...
           mm*(pi-beta-gam));
93     end
94 end
95 L1_nm=zeros(N+1,M+1);
96 L1_nm(1,1)=1./((2.^(1/3))*gamma(7/6));
97 for n=0:N
98     if rem(n,2)<1e-8
99         L1_nm(n+1,2:end)=cos(mm*pi/2).*(-1)^(n/2)./(mm*...
           pi).^(1/6).*besselj(n+1/6,mm*pi/2);
100     else
101         L1_nm(n+1,2:end)=-sin(mm*pi/2).*(-1)^((n-1)/2)...
           ./(mm*pi).^(1/6).*besselj(n+1/6,mm*pi/2);
102     end
103 end
104 L2_nm=zeros(N+1,M+1);
105 L2_nm(1,1)=1;
106 for n=0:N
107     if rem(n,2)<1e-8
108         L2_nm(n+1,2:end)=cos(mm*pi/2).*(-1)^(n/2).*...
           besselj(n,mm*nu*(pi-beta-gam));
109     else
110         L2_nm(n+1,2:end)=-sin(mm*pi/2).*(-1)^((n-1)/2)...
           .*besselj(n,mm*nu*(pi-beta-gam));
111     end
112 end
113 V1_pl=zeros(P+1,L+1);
114 for l=0:L
115     for p=0:P
116         V1_pl(p+1,l+1)=1/h*tau(p,h)*Tp(p,kl(l+1),Nl(1...
           +1),h).';
117     end
118 end
119 V2_pl=zeros(P+1,L+1);
120 for p=0:P
121     V2_pl(p+1,:)=Nl.^(-1/2)/h.*besselj(2*p,kl*(h-d));
122 end
123 %
124 LHS=zeros(3*(N+1)*(P+1),1);
125 % equation (1):
126 for dn=0:(N+1)*(P+1)-1
127     np=floor(dn/(P+1));
128     pp=dn-(P+1)*floor(dn/(P+1));
129     LHS(dn+1,1)=V1_pl(pp+1,1)*Rm*(Em.*X1_nm(np+1,:)...
           +Fm.*Y1_nm(np+1,:)).';
130 end
131 % equation (2)
132 for dn=0:(N+1)*(P+1)-1
133     np=floor(dn/(P+1));
134     pp=dn-(P+1)*floor(dn/(P+1));
135     LHS((N+1)*(P+1)+dn+1,1)=V1_pl(pp+1,1)*Rm*(Em.*X2_nm...
           (np+1,:))...
           +Fm.*Y2_nm(np+1,:)).';
136 end
137
138 end

```

```

139     RHS=zeros(3*(N+1)*(P+1),3*(N+1)*(P+1));
140     % equation (1)
141     for rn=0:(N+1)*(P+1)-1
142         for dn=0:(N+1)*(P+1)-1
143             np=floor(dn/(P+1));
144             pp=dn-(P+1)*floor(dn/(P+1));
145             n=floor(rn/(P+1));
146             p=rn-(P+1)*floor(rn/(P+1));
147             RHS(dn+1,rn+1)=-1/pi*(epsilon/2.*X1_nm(np+1,:)...
148                 .*X1_nm(n+1,:)...
149                 +Y1_nm(np+1,:).*Y1_nm(n+1,:)).*(Kml*(V1_pl(...
150                     pp+1,:).*V1_pl(p+1,:)).')...
151                 +(epsilon/(2*beta).*L1_nm(np+1,:).*L1_nm(n...
152                     +1,:)).*...
153                 ((Q1_m1.*R1_m1)*(V1_pl(pp+1,:).*V1_pl(p...
154                     +1,:)).');
155         end
156     end
157     % equation (2)
158     for rn=0:(N+1)*(P+1)-1
159         for dn=0:(N+1)*(P+1)-1
160             np=floor(dn/(P+1));
161             pp=dn-(P+1)*floor(dn/(P+1));
162             n=floor(rn/(P+1));
163             p=rn-(P+1)*floor(rn/(P+1));
164             RHS((N+1)*(P+1)+dn+1,rn+1)=-1/pi*(epsilon/2.*...
165                 X2_nm(np+1,:).*X1_nm(n+1,:)...
166                 +Y2_nm(np+1,:).*Y1_nm(n+1,:)).*(Kml*(V1_pl(...
167                     pp+1,:).*V1_pl(p+1,:)).');
168             RHS((N+1)*(P+1)+dn+1,(N+1)*(P+1)+rn+1)=-1/pi*(...
169                 epsilon/2.*X2_nm(np+1,:)...
170                 .*X2_nm(n+1,:)+Y2_nm(np+1,:).*Y2_nm(n+1,:))...
171                 *(Kml*(V1_pl(pp+1,:).*V1_pl(p+1,:)).')...
172                 ...
173                 +(epsilon/(2*pi-2*beta).*L2_nm(np+1,:).*...
174                     L2_nm(n+1,:)).*...
175                 ((Q2_m1.*R3_m1)*(V1_pl(pp+1,:).*V1_pl(p...
176                     +1,:)).');
177         end
178     end
179     % equation (3)
180     for rn=0:(N+1)*(P+1)-1
181         for dn=0:(N+1)*(P+1)-1
182             np=floor(dn/(P+1));
183             pp=dn-(P+1)*floor(dn/(P+1));
184             n=floor(rn/(P+1));
185             p=rn-(P+1)*floor(rn/(P+1));

```



```

179         RHS(2*(N+1)*(P+1)+dn+1,rn+1)=(epsilon/(2*beta)...
180             .*L1_nm(np+1,:).*L1_nm(n+1,:))...
181             *((Q1_m1.*R4_m1)*(V2_pl(pp+1,:).*V1_pl(p...
182                 +1,:)).');
183         RHS(2*(N+1)*(P+1)+dn+1,2*(N+1)*(P+1)+rn+1)=...
184             (epsilon/(2*beta).*L1_nm(np+1,:).*L1_nm(n...
185                 +1,:))*((Q1_m1.*R5_m1)...
186                 *(V2_pl(pp+1,:).*V2_pl(p+1,:)).')...
187             -1/pi*(epsilon/2.*X1_nm(np+1,:).*X1_nm(n...
188                 +1,:)+Y1_nm(np+1,:).*Y1_nm(n+1,:))...
189             *(Iml*(V2_pl(pp+1,:).*V2_pl(p+1,:)).');
190     end
191 end
192 %
193 sol=RHS\LHS;
194 Kappa=zeros(N+1,P+1);
195 for rn=0:(N+1)*(P+1)-1
196     n=floor(rn/(P+1));
197     p=rn-(P+1)*floor(rn/(P+1));
198     Kappa(n+1,p+1)=sol(rn+1);
199 end
200 sigma=zeros(N+1,P+1);
201 for rn=0:(N+1)*(P+1)-1
202     n=floor(rn/(P+1));
203     p=rn-(P+1)*floor(rn/(P+1));
204     sigma(n+1,p+1)=sol((N+1)*(P+1)+rn+1);
205 end
206 xi=zeros(N+1,P+1);
207 for rn=0:(N+1)*(P+1)-1
208     n=floor(rn/(P+1));
209     p=rn-(P+1)*floor(rn/(P+1));
210     xi(n+1,p+1)=sol(2*(N+1)*(P+1)+rn+1);
211 end
212 %
213 Aml=zeros(M+1,L+1);
214 for m=0:M
215     Aml(m+1,:)=epsilon(m+1)/(2*pi)*(X1_nm(:,m+1).'*(...
216         Kappa*V1_pl)+...
217         X2_nm(:,m+1).'*(sigma*V1_pl));
218 end
219 Bml=zeros(M+1,L+1);
220 for m=0:M
221     Bml(m+1,:)=1/pi*(Y1_nm(:,m+1).'*(Kappa*V1_pl)+Y2_nm...
222         (:,m+1).'*(sigma*V1_pl));
223 end
224 KKml=zeros(M+1,L+1);
225 for m=0:M
226     KKml(m+1,:)=dKdK(m*mu,kl*c,kl*a);
227 end
228 Cml=zeros(M+1,L+1);
229 for m=0:M
230     Cml(m+1,:)=epsilon(m+1)/(2*beta).*Q1_m1(m+1,:).*...
231         (L1_nm(:,m+1).*(Kappa*V1_pl)-KKml(m+1,:)).*(...
232         L1_nm(:,m+1).'*...
233         *(xi*V2_pl));
234 end

```

```

228     IIm1=zeros(M+1,L+1);
229     for m=0:M
230         IIm1(m+1,:)=dIdI(m*mu,kl*a,kl*c);
231     end
232     Dm1=zeros(M+1,L+1);
233     for m=0:M
234         Dm1(m+1,:)=epsilon(m+1)/(2*beta)*Q1_m1(m+1,:).*...
235             (L1_nm(:,m+1).*(xi*V2_pl)-IIm1(m+1,:).*(L1_nm...
236                 (:,m+1).'*...
237                 *(Kappa*V1_pl)));
238     end
239     Em1=zeros(M+1,L+1);
240     for m=0:M
241         Em1(m+1,:)=epsilon(m+1)/(2*pi)*X1_nm(:,m+1).*(xi*...
242             V2_pl);
243     end
244     Fm1=zeros(M+1,L+1);
245     for m=0:M
246         Fm1(m+1,:)=1/pi*Y1_nm(:,m+1).*(xi*V2_pl);
247     end
248     Gm1=zeros(M+1,L+1);
249     for m=0:M
250         Gm1(m+1,:)=epsilon(m+1)/(2*pi-2*beta)*Q2_m1(m+1,:)...
251             .*(L2_nm(:,m+1).'*...
252             *(sigma*V1_pl));
253     end
254     GMA(ii)=abs(-2*pi*1i*sqrt(g/h)/(omega(ii)*h)*Em1(1,:).*(...
255         DZ1*a./(kl.^2)).')^2;
256 end
257 %
258 %
259 function out=besselH(m,x)
260 out=besselj(m,x)+1i*bessely(m,x);
261 %
262 %
263 function out=dbesselH(n,x)
264 % computing the derivative of the first kind Hankel ...
265 function
266 out=1./2*(besselH(n-1,x)-besselH(n+1,x));
267 %
268 %
269 function out=dIdI(m,z1,z2)
270 % out=dIdI(m,z1,z2)
271 % Im'(z1)/Im'(z2)
272 out=exp(abs(real(z1))-abs(real(z2))).*(besseli(m-1,z1,1)+...
273     besseli(m+1,z1,1))./(besseli(m-1,z2,1)+besseli(m+1,z2...
274     ,1));
275 %
276 %
277 function out=dKdK(m,z1,z2)
278 % out=dKdK(m,z1,z2)
279 % Km'(z1)/Km'(z2)
280 out=exp(z2-z1).*(besselk(m-1,z1,1)+besselk(m+1,z1,1))./(...
281     besselk(m-1,z2,1)+besselk(m+1,z2,1));
282 %
283 %

```

```

276 function out=IdI(m,z1,z2)
277 % out=IdI(m,z1,z2)
278 % Im(z1)/Im'(z2)
279 out=2*exp(abs(real(z1))-abs(real(z2))).*besseli(m,z1,1)./(...
    besseli(m-1,z2,1)+besseli(m+1,z2,1));
280 %
281 %
282 function out=KdK(m,z1,z2)
283 % out=KdK(m,z1,z2)
284 % Km(z1)/Km'(z2)
285 out=-2*exp(z2-z1).*besselk(m,z1,1)./(besselk(m-1,z2,1)+...
    besselk(m+1,z2,1));
286 function out=tau(n,h)
287 % out=tau(n)
288 % shifted_T(p,z)=sum(ts*z^s)
289 t0=1;
290 t1=[1,2/h];
291 t2=[1,8/h,8/h^2];
292 t3=[1,18/h,48/h^2,32/h^3];
293 t4=[1,32/h,160/h^2,256/h^3,128/h^4];
294 t5=[1,50/h,400/h^2,1120/h^3,1280/h^4,512/h^5];
295 t6=[1, 72/h, 840/h^2, 3584/h^3, 6912/h^4, 6144/h^5, 2048/h...
    ^6];
296 t7=[1, 98/h, 1568/h^2, 9408/h^3, 26880/h^4, 39424/h^5, ...
    28672/h^6, 8192/h^7];
297 t8=[1, 128/h, 2688/h^2, 21504/h^3, 84480/h^4, 180224/h^5, ...
    212992/h^6, ...
    131072/h^7, 32768/h^8];
298 t9=[1, 162/h, 4320/h^2, 44352/h^3, 228096/h^4, 658944/h^5, ...
    1118208/h^6, ...
    1105920/h^7, 589824/h^8, 131072/h^9];
299 t10=[1, 200/h, 6600/h^2, 84480/h^3, 549120/h^4, 2050048/h...
    ^5, 4659200/h^6, ...
    6553600/h^7, 5570560/h^8, 2621440/h^9, 524288/h^10];
300 t11=[1,242/h,9680/h^2,151008/h^3,1208064/h^4,5637632/h...
    ^5,16400384/h^6,...
    30638080/h^7, 36765696/h^8,27394048/h^9,11534336/h...
    ^10,2097152/h^11];
301 t12=[1,288/h,13728/h^2,256256/h^3,2471040/h^4,14057472/h...
    ^5,50692096/h^6,...
    120324096/h^7,190513152/h^8,199229440/h^9,132120576/h...
    ^10,50331648/h^11,8388608/h^12];
302 t13=[1, 338/h, 18928/h^2, 416416/h^3, 4759040/h^4, ...
    32361472/h^5,...
    141213696/h^6, 412778496/h^7, 825556992/h^8, ...
    1133117440/h^9,...
    1049624576/h^10, 627048448/h^11, 218103808/h^12, ...
    33554432/h^13];
303 t14=[1, 392/h, 25480/h^2, 652288/h^3, 8712704/h^4, ...
    69701632/h^5, ...
    361181184/h^6, 1270087680/h^7, 3111714816/h^8, ...
    5369233408/h^9, ...
    6499598336/h^10, 5402263552/h^11, 2936012800/h^12, ...
    939524096/h^13, ...
    134217728/h^14];
313

```

```

314 t15=[1, 450/h, 33600/h^2, 990080/h^3, 15275520/h^4, ...
      141892608/h^5, ...
315      859955200/h^6, 3572121600/h^7, 10478223360/h^8, ...
      22052208640/h^9, ...
316      33426505728/h^10, 36175872000/h^11, 27262976000/h^12,...
      ...
317      13589544960/h^13, 4026531840/h^14, 536870912/h^15];
318 t16=[1, 512/h, 43520/h^2, 1462272/h^3, 25798656/h^4, ...
      275185664/h^5, ...
319      1926299648/h^6, 9313976320/h^7, 32133218304/h^8, ...
      80648077312/h^9, ...
320      148562247680/h^10, 200655503360/h^11, 196293427200/h...
      ^12, ...
321      135291469824/h^13, 62277025792/h^14, 17179869184/h^15,...
      ...
322      2147483648/h^16];
323 t17=[1, 578/h, 55488/h^2, 2108544/h^3, 42170880/h^4, ...
      511673344/h^5, ...
324      4093386752/h^6, 22761029632/h^7, 91044118528/h^8, ...
      267776819200/h^9, ...
325      586290298880/h^10, 959384125440/h^11, 1167945891840/h...
      ^12, ...
326      1042167103488/h^13, 661693399040/h^14, 282930970624/h...
      ^15,...
327      73014444032/h^16, 8589934592/h^17];
328 t18=[1, 648/h, 69768/h^2, 2976768/h^3, 66977280/h^4, ...
      916844544/h^5, ...
329      8307167232/h^6, 52581629952/h^7, 240999137280/h^8, ...
      819082035200/h^9, ...
330      2095125626880/h^10, 4063273943040/h^11, 5977134858240/...
      h^12, ...
331      6620826304512/h^13, 5429778186240/h^14, 3195455668224/...
      h^15, ...
332      1275605286912/h^16, 309237645312/h^17, 34359738368/h...
      ^18];
333 t19=[1, 722/h, 86640/h^2, 4124064/h^3, 103690752/h^4, ...
      1589924864/h^5, ...
334      16188325888/h^6, 115630899200/h^7, 601280675840/h^8, ...
      ...
335      2334383800320/h^9, 6880289095680/h^10, 15547666268160/...
      h^11, ...
336      27039419596800/h^12, 36108024938496/h^13, ...
      36681168191488/h^14, ...
337      27827093110784/h^15, 15260018802688/h^16, ...
      5712306503680/h^17, ...
338      1305670057984/h^18, 137438953472/h^19];
339 t20=[1, 800/h, 106400/h^2, 5617920/h^3, 156900480/h^4, ...
      2677768192/h^5, ...
340      30429184000/h^6, 243433472000/h^7, 1424085811200/h^8, ...
      ...
341      6254808268800/h^9, 21002987765760/h^10, 54553214976000/...
      h^11, ...
342      110292369408000/h^12, 173752901959680/h^13, ...
      212364657950720/h^14, ...
343      199183403319296/h^15, 140552804761600/h^16, ...
      72155450572800/h^17, ...

```

```

344     25426206392320/h^18, 5497558138880/h^19, 549755813888/h...
      ^20];
345 t21=[1, 882/h, 129360/h^2, 7537376/h^3, 232581888/h^4, ...
      4393213440/h^5,...
346     55381114880/h^6, 492952780800/h^7, 3220624834560/h^8, ...
      ...
347     15871575982080/h^9, 60144919511040/h^10, ...
      177570714746880/h^11, ...
348     411758179123200/h^12, 752567256612864/h^13, ...
      1083059755548672/h^14, ...
349     1219998345330688/h^15, 1062579203997696/h^16, ...
      700809813688320/h^17, ...
350     338168545017856/h^18, 112562502893568/h^19, ...
      23089744183296/h^20, ...
351     2199023255552/h^21];
352 t22=[1, 968/h, 155848/h^2, 9974272/h^3, 338412800/h^4, ...
      7038986240/h^5, ...
353     97905899520/h^6, 963996549120/h^7, 6988974981120/h^8, ...
      ...
354     38370843033600/h^9, 162773155184640/h^10, ...
      541167892561920/h^11, ...
355     1423506847825920/h^12, 2978414327758848/h^13, ...
      4964023879598080/h^14, ...
356     6573052309536768/h^15, 6864598984556544/h^16, ...
      5579780992794624/h^17, ...
357     3454150138396672/h^18, 1572301627719680/h^19, ...
      495879744126976/h^20, ...
358     96757023244288/h^21, 8796093022208/h^22];
359 t23=[1, 1058/h, 186208/h^2, 13034560/h^3, 484140800/h^4, ...
      11038410240/h^5, ...
360     168586629120/h^6, 1826663915520/h^7, 14613311324160/h...
      ^8, ...
361     88826010009600/h^9, 418884762992640/h^10, ...
      1555857691115520/h^11, ...
362     4599927086776320/h^12, 10898288790208512/h^13, ...
363     20758645314682880/h^14, 31782201792135168/h^15, ...
364     38958828003262464/h^16, 37917148110127104/h^17, ...
365     28889255702953984/h^18, 16848641306132480/h^19, ...
366     7257876254949376/h^20, 2174833999740928/h^21, ...
      404620279021568/h^22, ...
367     35184372088832/h^23];
368 t24=[1, 1152/h, 220800/h^2, 16839680/h^3, 682007040/h^4, ...
      16974397440/h^5, ...
369     283420999680/h^6, 3363677798400/h^7, 29544303329280/h...
      ^8, ...
370     197734422282240/h^9, 1030300410839040/h^10, ...
      4246086541639680/h^11, ...
371     13999778090188800/h^12, 37217871599763456/h^13, ...
372     80146421910601728/h^14, 140025932533465088/h^15, ...
373     198181864190509056/h^16, 226089827240509440/h^17, ...
374     205992953708019712/h^18, 147682003796361216/h^19, ...
375     81414437990301696/h^20, 33284415996035072/h^21, ...
376     9499780463984640/h^22, 1688849860263936/h^23, ...
      140737488355328/h^24];
377 if abs(n-0)<1e-8
378     out=t0;

```

```

379 elseif abs(n-1)<1e-8
380     out=t1;
381 elseif abs(n-2)<1e-8
382     out=t2;
383 elseif abs(n-3)<1e-8
384     out=t3;
385 elseif abs(n-4)<1e-8
386     out=t4;
387 elseif abs(n-5)<1e-8
388     out=t5;
389 elseif abs(n-6)<1e-8
390     out=t6;
391 elseif abs(n-7)<1e-8
392     out=t7;
393 elseif abs(n-8)<1e-8
394     out=t8;
395 elseif abs(n-9)<1e-8
396     out=t9;
397 elseif abs(n-10)<1e-8
398     out=t10;
399 elseif abs(n-11)<1e-8
400     out=t11;
401 elseif abs(n-12)<1e-8
402     out=t12;
403 elseif abs(n-13)<1e-8
404     out=t13;
405 elseif abs(n-14)<1e-8
406     out=t14;
407 elseif abs(n-15)<1e-8
408     out=t15;
409 elseif abs(n-16)<1e-8
410     out=t16;
411 elseif abs(n-17)<1e-8
412     out=t17;
413 elseif abs(n-18)<1e-8
414     out=t18;
415 elseif abs(n-19)<1e-8
416     out=t19;
417 elseif abs(n-20)<1e-8
418     out=t20;
419 elseif abs(n-21)<1e-8
420     out=t21;
421 elseif abs(n-22)<1e-8
422     out=t22;
423 elseif abs(n-23)<1e-8
424     out=t23;
425 elseif abs(n-24)<1e-8
426     out=t24;
427 end
428 %
429 %
430 function out=Tp(p,kl,Nl,h)
431 % out=Tp(p,kl,Nl,h)
432 out=zeros(1,p+1);
433 T0=sin(kl*h)/kl;
434 T1=(cos(kl*h)-1)/kl^2;

```

```

435 if abs(p)<1e-8
436     out=T0;
437 elseif abs(p-1)<1e-8
438     out(1)=T0;
439     out(2)=T1;
440 else
441     out(1)=T0;
442     out(2)=T1;
443     for s=2:p
444         out(s+1)=-s/kl^2*(-h)^(s-1)-s*(s-1)/kl^2*out(s-1);
445     end
446 end
447 out=Nl^(-1/2)*out;
448 %
449 %
450 function roots=wavenumber(KC,h,N)
451 % roots=wavenumber[KC,h,N]
452 fk=@(k)k.*tanh(k*h)-KC;
453 dfk=@(k)tanh(k*h)+k*h.*sech(k*h).^2;
454 fkn=@(kn)kn.*tan(kn*h)+KC;
455 dfkn=@(kn)tan(kn*h)+kn*h.*sec(kn*h).^2;
456 roots=zeros(1,N+1);
457 x0=0.2;
458 k=newton(fk,dfk,x0,1e-12);
459 roots(1)=-1i*k;
460 for n=1:N
461     x0=pi/(2*h)+0.005+(n-1)*pi/h;
462     roots(n+1)=newton(fkn,dfkn,x0,1e-12);
463 end
464 %
465 %
466 function x1=newton(fun,dfun,x,tol)
467 x1=x-fun(x)/dfun(x);
468 while (abs(x1-x)>=tol)
469     x=x1;
470     x1=x-fun(x)/dfun(x);
471 end

```

A.2.2.2 Matlab codes for radiation problem

```

1 clear all
2 close all
3 clc
4 format long
5 %=====
6 h=10;
7 g=9.81;
8 d=0.2*h;
9 a=1/2*h;
10 b=0.01*h;
11 c=a+b;

```

```

12 beta=pi*1/4;
13 gam=pi*1/20;
14 mu=pi/(2*beta);
15 nu=pi/(2*pi-2*beta);
16 %==
17 L=400;
18 M=55;
19 P=10;
20 N=12;
21 %=====
22 Kh=0.2:0.025:6;
23 K=Kh/h;
24 omega=sqrt(K*g); % K=omega^2/g
25 num=length(Kh);
26 disp(num)
27 k0=zeros(num,1);
28 B_wave=zeros(num,1);
29 C_wave=zeros(num,1);
30 % nu_Evans=zeros(1,num);
31 % mu_Evans=zeros(1,num);
32 epsilon=zeros(1,M+1)+2; % e0=1, em=2
33 epsilon(1,1)=1;
34 %
35 for ii=55:55
36     disp(ii)
37     kl=wavenumber(K(ii),h,L);
38     k=1i*kl(1);
39 %     waveL=2*pi/k;
40 %     b=20*waveL;
41 %     c=a+b;
42     k0(ii)=k;
43     N1=1/2*(1+sin(2*kl*h)/(2*kl*h));
44     Z1=N1.^(-1/2).*cos(kl*h); % Z(z)=N1.^(-1/2).*cos(kl*(z+h...
45     DZ1=N1.^(-1/2).*(-kl).*sin(kl*h);
46 %
47     Q1_m1=zeros(M+1,L+1);
48     R1_m1=zeros(M+1,L+1);
49     R2_m1=zeros(M+1,L+1);
50     R4_m1=zeros(M+1,L+1);
51     R5_m1=zeros(M+1,L+1);
52     for m=0:M
53         Q1_m1(m+1,:)=1./(1-dIdI(2*m,kl*a,kl*c).*dKdK(2*m,kl...
54             *c,kl*a));
55         R1_m1(m+1,:)=IdI(2*m,kl*c,kl*c)./kl-dIdI(2*m,kl*a,...
56             kl*c).*...
57             KdK(2*m,kl*c,kl*a)./kl;
58         R2_m1(m+1,:)=KdK(2*m,kl*c,kl*a)./kl-IdI(2*m,kl*c,kl...
59             *c).*...
60             dKdK(2*m,kl*c,kl*a)./kl;
61         R4_m1(m+1,:)=IdI(2*m,kl*a,kl*c)./kl-dIdI(2*m,kl*a,...
62             kl*c).*...
63             KdK(2*m,kl*a,kl*a)./kl;
64         R5_m1(m+1,:)=KdK(2*m,kl*a,kl*a)./kl-IdI(2*m,kl*a,kl...
65             *c).*...
66             dKdK(2*m,kl*c,kl*a)./kl;

```



```

62     end
63     Q2_m1=Q1_m1; %%
64     R3_m1=R1_m1;
65     %
66     mm=1:M;
67     X1_nm=zeros(N+1,M+1);
68     X1_nm(1,1)=1/(2^(1/3)*gamma(7/6));
69     for n=0:N
70         X1_nm(n+1,2:end)=(-1)^n./(2*mm*beta).^(1/6).*...
            besselj(2*n+1/6,mm*beta);
71     end
72     X2_nm=zeros(N+1,M+1);
73     X2_nm(1,1)=1;
74     for n=0:N
75         X2_nm(n+1,2:end)=(-1).^(mm+n).*besselj(2*n,mm*(pi-...
            beta-gam));
76     end
77     W1_nm=zeros(N+1,M+1);
78     W1_nm(1,1)=1/(2^(1/3)*gamma(7/6));
79     for n=0:N
80         W1_nm(n+1,2:end)=(-1)^n./(2*mm*pi).^(1/6).*besselj...
            (2*n+1/6,mm*pi);
81     end
82     W2_nm=zeros(N+1,M+1);
83     W2_nm(1,1)=1;
84     for n=0:N
85         W2_nm(n+1,2:end)=(-1).^(mm+n).*besselj(2*n,2*mm*nu...
            *(pi-beta-gam));
86     end
87     V1_pl=zeros(P+1,L+1);
88     for l=0:L
89         for p=0:P
90             V1_pl(p+1,l+1)=1/h*tau(p,h)*Tp(p,kl(l+1),Nl(l...
                +1),h).';
91         end
92     end
93     V2_pl=zeros(P+1,L+1);
94     for p=0:P
95         V2_pl(p+1,:)=Nl.^(-1/2)/h.*besselj(2*p,kl*(h-d));
96     end
97     Vp=zeros(1,P+1);
98     Vp(1)=1/h;
99     I_m1=zeros(M+1,L+1);
100    K_m1=zeros(M+1,L+1);
101    for m=0:M
102        I_m1(m+1,:)=IdI(m,kl*a,kl*a)./kl;
103        K_m1(m+1,:)=KdK(m,kl*c,kl*c)./kl;
104    end
105    % LHS*X=RHS
106    RHS=zeros(3*(N+1)*(P+1),1);
107    for dn=0:(N+1)*(P+1)-1
108        np=floor(dn/(P+1));
109        pp=dn-(P+1)*np;
110        RHS(2*(N+1)*(P+1)+dn+1,1)=X1_nm(np+1,1)*Vp(pp+1);
111    end
112    LHS=zeros(3*(N+1)*(P+1),3*(N+1)*(P+1));

```

```

113 % equation (1)
114 for rn=0:(N+1)*(P+1)-1
115     for dn=0:(N+1)*(P+1)-1
116         np=floor(dn/(P+1));
117         pp=dn-(P+1)*np;
118         n=floor(rn/(P+1));
119         p=rn-(P+1)*n;
120         LHS(dn+1,rn+1)=epsilon/(2*pi).*X1_nm(n+1,:).*...
            X1_nm(np+1,:)...
            *(K_ml*(V1_pl(p+1,:).*V1_pl(pp+1,:)).')-...
            epsilon/(2*beta)...
            .*W1_nm(n+1,:).*W1_nm(np+1,:).*((Q1_ml.*...
            R1_ml)*(V1_pl(p+1,:)...
            .*V1_pl(pp+1,:)).');
121         LHS(dn+1,(N+1)*(P+1)+rn+1)=epsilon/(2*pi).*...
            X2_nm(n+1,:).*X1_nm(np+1,:)...
            *(K_ml*(V1_pl(p+1,:).*V1_pl(pp+1,:)).');
122         LHS(dn+1,2*(N+1)*(P+1)+rn+1)=-epsilon/(2*beta)...
            .*W1_nm(n+1,:).*W1_nm(np+1,:)...
            .*((Q1_ml.*R2_ml)*(V2_pl(p+1,:).*V1_pl(pp...
            +1,:)).');
123     end
124 end
125 % equation (2)
126 for rn=0:(N+1)*(P+1)-1
127     for dn=0:(N+1)*(P+1)-1
128         np=floor(dn/(P+1));
129         pp=dn-(P+1)*floor(dn/(P+1));
130         n=floor(rn/(P+1));
131         p=rn-(P+1)*floor(rn/(P+1));
132         LHS((N+1)*(P+1)+dn+1,rn+1)=epsilon/(2*pi).*...
            X2_nm(np+1,:).*X1_nm(n+1,:)...
            *(K_ml*(V1_pl(pp+1,:).*V1_pl(p+1,:)).');
133         LHS((N+1)*(P+1)+dn+1,(N+1)*(P+1)+rn+1)=epsilon...
            /(2*pi).*X2_nm(np+1,:)...
            .*X2_nm(n+1,:)*(K_ml*(V1_pl(pp+1,:).*V1_pl(...
            p+1,:)).')...
            -epsilon/(2*pi-2*beta).*W2_nm(np+1,:).*...
            W2_nm(n+1,:)...
            .*((Q2_ml.*R3_ml)*(V1_pl(pp+1,:).*V1_pl(p...
            +1,:)).');
134     end
135 end
136 % equation (3)
137 for rn=0:(N+1)*(P+1)-1
138     for dn=0:(N+1)*(P+1)-1
139         np=floor(dn/(P+1));
140         pp=dn-(P+1)*floor(dn/(P+1));
141         n=floor(rn/(P+1));
142         p=rn-(P+1)*floor(rn/(P+1));
143         LHS(2*(N+1)*(P+1)+dn+1,rn+1)=epsilon/(2*beta).*...
            W1_nm(np+1,:)...
            .*W1_nm(n+1,:).*((Q1_ml.*R4_ml)*(V2_pl(pp...
            +1,:).*V1_pl(p+1,:)).');
144         LHS(2*(N+1)*(P+1)+dn+1,2*(N+1)*(P+1)+rn+1)=...
            epsilon/(2*beta).*W1_nm(np+1,:)...

```

```

155         .*W1_nm(n+1,:) * ((Q1_ml.*R5_ml)*(V2_pl(pp...
156             +1,:).*V2_pl(p+1,:)).') ...
157     -epsilon/(2*pi).*X1_nm(np+1,:).*X1_nm(n...
158         +1,:)*(I_ml*(V2_pl(pp+1,:))...
159         .*V2_pl(p+1,:)).');
160     end
161     sol=LHS\RHS;
162     Kappa=zeros(N+1,P+1);
163     for rn=0:(N+1)*(P+1)-1
164         n=floor(rn/(P+1));
165         p=rn-(P+1)*floor(rn/(P+1));
166         Kappa(n+1,p+1)=sol(rn+1);
167     end
168     sigma=zeros(N+1,P+1);
169     for rn=0:(N+1)*(P+1)-1
170         n=floor(rn/(P+1));
171         p=rn-(P+1)*floor(rn/(P+1));
172         sigma(n+1,p+1)=sol((N+1)*(P+1)+rn+1);
173     end
174     xi=zeros(N+1,P+1);
175     for rn=0:(N+1)*(P+1)-1
176         n=floor(rn/(P+1));
177         p=rn-(P+1)*floor(rn/(P+1));
178         xi(n+1,p+1)=sol(2*(N+1)*(P+1)+rn+1);
179     end
180     %=====
181     Aml=zeros(M+1,L+1);
182     for m=0:M
183         Aml(m+1,:)=epsilon(m+1)/(2*pi)*(X1_nm(:,m+1).'*(...
184             Kappa*V1_pl)+...
185             X2_nm(:,m+1).*(sigma*V1_pl));
186     end
187     Bml=zeros(M+1,L+1);
188     KKml=zeros(M+1,L+1);
189     for m=0:M
190         KKml(m+1,:)=dKdK(2*m,kl*c,kl*a);
191     end
192     Bml(m+1,:)=epsilon(m+1)/(2*beta)*Q1_ml(m+1,:).*(...
193         W1_nm(:,m+1).*(Kappa*V1_pl)...
194         -KKml(m+1,:).*(W1_nm(:,m+1).*(xi*V2_pl)));
195     end
196     Cml=zeros(M+1,L+1);
197     IIml=zeros(M+1,L+1);
198     for m=0:M
199         IIml(m+1,:)=dIdI(2*m,kl*a,kl*c);
200     end
201     Cml(m+1,:)=epsilon(m+1)/(2*beta)*Q1_ml(m+1,:).*(...
202         W1_nm(:,m+1).*(xi*V2_pl)...
203         -IIml(m+1,:).*(W1_nm(:,m+1).*(Kappa*V1_pl)));
204     end
205     Dml=zeros(M+1,L+1);
206     for m=0:M

```

```

205         Dm1(m+1,:)=epsilon(m+1)/(2*pi)*X1_nm(:,m+1).*(xi*...
            V2_p1);
206     end
207     Eml=zeros(M+1,L+1);
208     for m=0:M
209         Eml(m+1,:)=epsilon(m+1)/(2*pi-2*beta)*Q2_m1(m+1,:)...
            *(W2_nm(:,m+1).'*...
            *(sigma*V1_p1));
210     end
211     %=====
212     B_wave(ii)=real(2*pi*1i/omega(ii)*sqrt(g/h)/h*Dm1(1,:)...
        *(a./(kl.^2).*DZ1. ');
213     C_wave(ii)=imag(-2*pi*1i/omega(ii)*sqrt(g/h)/h*Dm1(1,:)...
        *(a./(kl.^2).*DZ1. ');
214
215 end

```

A.2.2.3 Matlab code for calculating capture length

```

1  clear all
2  clc
3  format long
4  %=====
5  h=10;
6  g=9.81;
7  d=0.2*h;
8  a=1/2*h;    %%
9  % b=0.05*a;
10 % c=a+b;
11 beta=3/8*pi;  %=%
12 gam=1/20*pi;
13 alpha=pi;
14 mu=pi/(2*beta);
15 nu=pi/(2*pi-2*beta);
16 %
17 V0=pi*a^2*h;
18 rho=1000;
19 ca=340;
20 %
21 L=400;
22 M=50;
23 P=10;
24 N=5;
25 %=====
26 Kh=0.2:0.025:6;
27 num=length(Kh);
28 Cg=zeros(1,num);
29 WaveNumber=zeros(1,num);
30 Bta=zeros(1,num);
31 for ii=1:num
32     K=Kh(ii)/h;
33     omega=sqrt(K*g);

```

```

34     kl=wavenumber(K,h,L);
35     k=1i*kl(1);
36     WaveNumber(ii)=k;
37     Cg(ii)=omega/(2*k)*(1+2*k*h/(sinh(2*k*h)));
38     Bta(ii)=-omega*V0*rho*sqrt(g/h)/(ca^2*h);
39 end
40 %-----
41 GMA=xlsread('comparisons.xlsx','Sheet1','E1:E233').';
42 B=xlsread('comparisons.xlsx','Sheet1','G1:G233').';
43 C=xlsread('comparisons.xlsx','Sheet1','I1:I233').';
44 %-----
45 % chi_opt=sqrt((B.').^2+((C.').-Bta).^2);
46 % kL_opt=WaveNumber*h*sqrt(g/h)./Cg.*(GMA.').*chi_opt./((...
    chi_opt+(B.')).^2+...
47 %      ((C.').-Bta).^2);
48 chi_opt=sqrt(B.^2+(C-Bta).^2);
49 kL_opt=WaveNumber*h*sqrt(g*h)./Cg.*GMA.*chi_opt./((chi_opt+...
    B).^2+...
50     (C-Bta).^2);
51 kL_max=WaveNumber*h*sqrt(g*h)./Cg.*GMA./(4*B);
52 xlswrite('comparisons.xlsx',chi_opt.','Sheet1','K1:K233')
53 xlswrite('comparisons.xlsx',kL_opt.','Sheet1','M1:M233')

```

A.2.3 Matlab codes for chapters 5 and 6

```

1     clc
2     clear all
3     close all
4     format long
5     global h d a c L M N P
6     %=====
7     g=9.81;
8     h=10;
9     a=1/2*h;
10    d=0.2*h;
11    b=0.2*h;
12    c=a+b;
13    %
14    L=400;
15    M=50;
16    N=50;
17    P=7;
18    %=====
19    %-----
20    rho_a=1;
21    rho_w=1000;
22    A=1; % incident wave amplitude
23    V0=pi*a^2*h;
24    ca=340;
25    Si=pi*a^2;

```

```

26 Sw=pi*(c^2-a^2);
27 %-----
28 nn=1:N;
29 mm=0:M;
30 alpha=nn*pi/(h-d);
31 fn=cos(alpha*(h-d));
32 epsn=zeros(1,M+1)+2;
33 espn(1,1)=1;
34 %
35 Kh=0.2:0.025:7;
36 K=Kh/h;
37 omega=sqrt(K*g);
38 num=length(Kh);
39 k0=zeros(1,num);
40 Cg=zeros(1,num);
41 %-----
42 % scattering problem
43 GMA_sq=zeros(1,num);
44 GMA_s3=zeros(1,num);
45 GMA_sq_nd=zeros(1,num);
46 GMA_s3_nd=zeros(1,num);
47 % radiation due to pressure
48 B_pq=zeros(1,num);
49 C_pq=zeros(1,num);
50 B_pq_nd=zeros(1,num);
51 C_pq_nd=zeros(1,num);
52 %
53 mu_p3=zeros(1,num);
54 nu_p3=zeros(1,num);
55 mu_p3_nd=zeros(1,num);
56 nu_p3_nd=zeros(1,num);
57 % radiation due to heaving motion of body
58 B_3q=zeros(1,num);
59 C_3q=zeros(1,num);
60 B_3q_nd=zeros(1,num);
61 C_3q_nd=zeros(1,num);
62 %
63 mu_33=zeros(1,num);
64 nu_33=zeros(1,num);
65 mu_33_nd=zeros(1,num);
66 nu_33_nd=zeros(1,num);
67 % averaged surface elevation inside chamber
68 zetaS_A=zeros(1,num);
69 zetaP_pa=zeros(1,num);
70 zetaB_V3=zeros(1,num);
71 %
72 mu_Mav=zeros(1,num);
73 nu_Mav=zeros(1,num);
74 %-----
75 for ii=1:num
76     disp(ii)
77     kl=wavenumber(K(ii),h,L);
78     k=1i*kl(1);
79     k0(ii)=k;
80     Cg(ii)=1/2*omega(ii)/k*(1+2*k*h/sinh(2*k*h));
81     %

```

```

82     N1=1/2*(1+sin(2*k1*h)./(2*k1*h));
83     Z1=N1.^(-1/2).*cos(k1*h);
84     DZ1=(N1.^(-1/2)).*(-k1).*sin(k1*h);
85     %
86     %% Scattering problem
87     [S_Aml,S_Bmn,S_Cmn,S_Dml]=scattering(k1);
88     % vertical force
89     for_S=2*pi*g*rho_w*A*(...
90         S_Bmn(1,1)*(c^2-a^2)/2 ...
91         +S_Cmn(1,1)/4*(a^2-c^2+2*c^2*log(c/a)) ...
92         +S_Bmn(1,2:end).*fn*(c./alpha.^2.*(1-a*besseli(1,...
93             alpha*a)./(c*besseli(1,alpha*c))))).' ...
94         +S_Cmn(1,2:end).*fn*(a./alpha.^2.*(c*besselk(1,...
95             alpha*c)./(a*besselk(1,alpha*a))-1)).');
96     GMA_s3(ii)=for_S/A;
97     GMA_s3_nd(ii)=GMA_s3(ii)/(pi*rho_w*g*c^2); % non-...
98     dimensional factor
99     %
100    % volume flux
101    vol_S=-1i*2*pi*g*A/omega(ii)*S_Dml(1,:)*(a*DZ1./k1.^2)...
102    .';
103    GMA_sq(ii)=vol_S/A;
104    GMA_sq_nd(ii)=GMA_sq(ii)*sqrt(g/h)/(g*h); % non-...
105    dimensional flux factor
106    %
107    % averaged surface elevation inside chamber nomornized ...
108    by A
109    zetaS_A(ii)=2*g/(a*(omega(ii)^2))*(S_Dml(1,:)*(DZ1./k1...
110    .^2)).');
111    %% Radiation due to pressure
112    [P_Al,P_Bn,P_Cn,P_Dl]=radiationP(k1);
113    % vertical force
114    for_P=2*pi*(P_Bn(1)*(c^2-a^2)/2 ...
115        +P_Cn(1)/4*(a^2-c^2-2*a^2*log(a)+2*c^2*log(c)) ...
116        +P_Bn(2:end).*fn*(c./alpha.^2.*(1-a*besseli(1,alpha...
117            *a)./(c*besseli(1,alpha*c))))).' ...
118        +P_Cn(2:end).*fn*(a./alpha.^2.*(c*besselk(1,alpha*c...
119            )./(a*besselk(1,alpha*a))-1)).');
120    mu_p3(ii)=1/omega(ii)*imag(for_P);
121    nu_p3(ii)=-real(for_P);
122    mu_p3_nd(ii)=mu_p3(ii)/(pi*c^2); % non-dimensional
123    nu_p3_nd(ii)=nu_p3(ii)/(pi*c^2); % non-dimensional
124    % volume flux
125    vol_P=2*pi*1i/(rho_w*omega(ii))*P_Dl*(a*DZ1./k1.^2).';
126    B_pq(ii)=real(vol_P);
127    C_pq(ii)=imag(-vol_P);
128    B_pq_nd(ii)=B_pq(ii)*rho_w*sqrt(g/h)/h; % non-...
129    dimensional
130    C_pq_nd(ii)=C_pq(ii)*rho_w*sqrt(g/h)/h; % non-...
131    dimensional
132    %
133    % averaged surface elevation inside chamber divided by ...
134    pa
135    zetaP_pa(ii)=2/(rho_w*a*omega(ii)^2)*(P_Dl*(DZ1./k1.^2)...
136    .');
137    %% Radiation due to heaving

```

```

125     [B_A1,B_Bn,B_Cn,B_D1]=radiationB(k1);
126     % added mass and damping coefficients
127     for_B=1i*2*pi*rho_w*omega(ii)*(...
128         (B_Bn(1)+(h-d)/2)*(c^2-a^2)/2 ...
129         +B_Cn(1)/4*(a^2-c^2-2*a^2*log(a)+2*c^2*log(c))...
130         -(c^4-a^4)/(16*(h-d))...
131         +B_Bn(2:end).*fn*(c./(alpha.^2).*(1-a*besseli(1,...
132             alpha*a)./(c*besseli(1,alpha*c))))).'...
133         +B_Cn(2:end).*fn*(a./(alpha.^2).*(c*besselk(1,alpha...
134             *c)./(a*besselk(1,alpha*a))-1)).');
135     mu_33(ii)=1/omega(ii)*imag(for_B);
136     nu_33(ii)=-real(for_B);
137     mu_33_nd(ii)=mu_33(ii)/(rho_w*c^3); % non-dimensional
138     nu_33_nd(ii)=nu_33(ii)/(rho_w*c^3); % non-dimensional
139     % comparising with Mavrakos 1988 Ocean Engng.
140     mu_Mav(ii)=real(-1i*omega(ii)*for_B/(omega(ii)^2*rho_w*...
141         c^3));
142     nu_Mav(ii)=imag(-1i*omega(ii)*for_B/(omega(ii)^2*rho_w*...
143         c^3));
144     %
145     % volume flux
146     vol_B=2*pi*B_D1*(a*DZ1./k1.^2).';
147     B_3q(ii)=-real(vol_B);
148     C_3q(ii)=imag(vol_B);
149     B_3q_nd(ii)=B_3q(ii)/(pi*a^2); % non-dimensional
150     C_3q_nd(ii)=C_3q(ii)/(pi*a^2); % non-dimensional
151     %
152     % averaged surface elevation inside chamber per unit ...
153     % valocity amplitude
154     zetaB_V3(ii)=2*1i/(a*omega(ii))*(B_D1*(DZ1./k1.^2).');
155 end
156 %% comparing with Martin-rivas and Mei (JFM, 2009)
157 B_Mei=B_pq./(a./(omega*rho_w));
158 C_Mei=C_pq./(a./(omega*rho_w));
159 %% comparing with Evans and Porter 1997
160 mu_Evans=rho_w*g./(omega*pi*a^2).*C_pq;
161 nu_Evans=rho_w*g./(omega*pi*a^2).*B_pq;
162 %% Power take-off
163 a_3q=B_3q+Si;
164 b_33=-omega.*(rho_w*Sw*d+mu_33-rho_w*g*Sw./(omega.^2));
165 a_p3=nu_p3-Si;
166 %-----
167 Y1=B_pq.*nu_33+C_pq.*b_33-a_3q.*a_p3+omega.*C_3q.*mu_p3;
168 Y2=B_pq.*b_33-C_pq.*nu_33+omega.*a_3q.*mu_p3+C_3q.*a_p3;
169 %
170 kappa=omega*V0/(rho_a*ca^2);
171 kappa_opt_D=(nu_33.*Y2-b_33.*Y1)./(b_33.^2+nu_33.^2);
172 %
173 chi_opt_D=sqrt(((kappa.*b_33+Y1).^2+(Y2-kappa.*nu_33).^2)...
174     ./(nu_33.^2+b_33.^2));
175 % present theory
176 chi=chi_opt_D; % choosng optimization strategy
177 GMA_power=(nu_33+1i*b_33).*GMA_sq-(a_3q-1i*C_3q).*GMA_s3;
178 kL_opt_D=k0./(rho_w*g*Cg).*chi.*(abs(GMA_power).^2)./((chi...
179     .*nu_33+kappa.*b_33+Y1).^2 ...
180     +(chi.*b_33-kappa.*nu_33+Y2).^2);

```



```

174 % Mav's theory
175 chi_opt_M=sqrt(B_pq.^2+(C_pq+kappa).^2);
176 chi=chi_opt_M;
177 kL_opt_M=k0./(rho_w*g*Cg).*chi.*(abs(GMA_power).^2)/((chi...
    .*nu_33+kappa.*b_33+Y1).^2 ...
178     +(chi.*b_33-kappa.*nu_33+Y2).^2);
179 %% Power take-off of fixed OWC device
180 chi_opt_fixed=sqrt(B_pq.^2+(C_pq+kappa).^2);
181 chi=chi_opt_fixed;
182 kL_opt_fixed=k0./(rho_w*g*Cg).*chi.*(abs(GMA_sq).^2)/((chi...
    +B_pq).^2 ...
183     +(C_pq+kappa).^2);
184 % average surface elevation in Martin-Rivas & Mei (2009, ...
    JFM)
185 pa_A_fixed=GMA_sq./((chi_opt_fixed+B_pq)-1i*(kappa+C_pq));
186 zeta_av_fixed=zetaS_A+pa_A_fixed.*zetaP_pa;
187 zeta_amp_fixed=abs(zeta_av_fixed);
188 %% pa and V3
189 % in the present case
190 chi=chi_opt_D;
191 paA=((nu_33+1i*b_33).*GMA_sq-(a_3q-1i*C_3q).*GMA_s3)./ ...
192     ((chi.*nu_33+kappa.*b_33+Y1)+1i*(chi.*b_33-kappa.*nu_33...
    +Y2));
193 V3A=((chi+B_pq)-1i*(kappa+C_pq)).*GMA_s3-(a_p3-1i*omega.*...
    mu_p3).*GMA_sq)./ ...
194     ((chi.*nu_33+kappa.*b_33+Y1)+1i*(chi.*b_33-kappa.*nu_33...
    +Y2));
195 %% RAO and spatial average of free surface elevation inside...
    column
196 % V3=-i*omega*x3
197 x3=1i./omega.*V3A;
198 RAO=abs(x3);
199 pha_3=angle(x3);
200 %
201 zeta_av=zetaS_A+paA.*zetaP_pa+V3A.*zetaB_V3;
202 zeta_amp=abs(zeta_av);
203 pha_s=angle(zeta_av);
204 dif_pha=unwrap(pha_s-pha_3);
205 %% An OWC device with a spring-damper system
206 DC=0; % damping constant
207 KC=10^8; % spring constant
208 %
209 nuw_33=nu_33+DC;
210 bw_33=-omega.*(rho_w*Sw*d+mu_33-KC./(omega.^2)-rho_w*g*Sw...
    ./(omega.^2));
211 Yw1=B_pq.*nuw_33+C_pq.*bw_33-a_p3.*a_3q+omega.*mu_p3.*C_3q;
212 Yw2=B_pq.*bw_33-C_pq.*nuw_33+a_p3.*C_3q+omega.*mu_p3.*a_3q;
213 %
214 chi_opt_SD=sqrt(((kappa.*bw_33+Yw1).^2+(Yw2-kappa.*nuw_33)...
    .^2)./...
215     (nuw_33.^2+bw_33.^2));
216 kappa_opt_SD=(nuw_33.*Yw2-bw_33.*Yw1)./(bw_33.^2+nuw_33.^2)...
    ;
217 %
218 chi=chi_opt_SD;
219 % pa_SD=pa/A

```

```

220 paA_SD=((nuw_33+1i*bw_33).*GMA_sq-(a_3q-1i*C_3q).*GMA_s3)./...
    ...
221 ((chi.*nuw_33+kappa.*bw_33+Yw1)+1i*(chi.*bw_33-kappa.*...
    nuw_33+Yw2));
222 V3A_SD=((chi+B_pq)-1i*(kappa+C_pq)).*GMA_s3-(a_p3-1i*omega...
    .*mu_p3).*GMA_sq)./ ...
223 ((chi.*nuw_33+kappa.*bw_33+Yw1)+1i*(chi.*bw_33-kappa.*...
    nuw_33+Yw2));
224 %
225 GMA_SD=(nuw_33+1i*bw_33).*GMA_sq-(a_3q-1i*C_3q).*GMA_s3;
226 kL_opt_SD=k0./(rho_w*g*Cg).*chi.*(abs(GMA_SD).^2)./ ...
227 ((chi.*nuw_33+kappa.*bw_33+Yw1).^2+(chi.*bw_33-kappa.*...
    nuw_33+Yw2).^2);
228 % RAO
229 x3_SD=1i./omega.*V3A_SD;
230 RAO_SD=abs(x3_SD);
231 pha3_SD=angle(x3_SD);
232 % average surface elevation
233 zeta_av_SD=zetaS_A+paA_SD.*zetaP_pa+V3A_SD.*zetaB_V3;
234 zeta_SD=abs(zeta_av_SD);
235 pha_zeta_SD=angle(zeta_av_SD);
236 delt_pha_SD=unwrap(pha_zeta_SD-pha3_SD);
237 % Power by damper
238 chi=chi_opt_SD;
239 GMA_D=((chi+B_pq)-1i*(kappa+C_pq)).*GMA_s3-(a_p3-1i*omega.*...
    mu_p3).*GMA_sq;
240 kL_D=k0./(rho_w*g*Cg).*DC.*(abs(GMA_D).^2)./ ...
241 ((chi.*nuw_33+kappa.*bw_33+Yw1).^2+(chi.*bw_33-kappa.*...
    nuw_33+Yw2).^2);
242 kL=kL_opt_SD+kL_D;
243 % damping force
244 fs3_nd=GMA_s3/(pi*c^2*rho_w*g);
245 fD_nd=DC/(pi*c^2*rho_w*g)*V3A_SD;
246 fK_nd=KC/(pi*c^2*rho_w*g)*1i*V3A_SD./omega;
247 %
248 fs3=abs(fs3_nd);
249 fD=abs(fD_nd);
250 fK=abs(fK_nd);
251 %
252 %
253 function [Aml,Bmn,Cmn,Dml]=scattering(kl)
254 %=====
255 global h d a c L M N P
256 %=====
257 nn=1:N;
258 mm=0:M;
259 %
260 F1m=zeros(1,M);
261 F2m=zeros(1,M);
262 F3m=zeros(1,M);
263 F4m=zeros(1,M);
264 for m=1:M
265     F1m(m)=DR2m(m,a,a)*R1m(m,c,c)-DR1m(m,a,c)*R2m(m,c,a);
266     F2m(m)=DR1m(m,c,c)*R2m(m,c,a)-DR2m(m,c,a)*R1m(m,c,c);
267     F3m(m)=DR2m(m,a,a)*R1m(m,a,c)-DR1m(m,a,c)*R2m(m,a,a);
268     F4m(m)=DR1m(m,c,c)*R2m(m,a,a)-DR2m(m,c,a)*R1m(m,a,c);

```

```

269 end
270 %
271 alpha=nn*pi/(h-d);
272 %
273 k=1i*k1(1);
274 N1=1/2*(1+sin(2*k1*h)/(2*k1*h));
275 Z1=N1.^(-1/2).*cos(k1*h);
276 %
277 Rm=2*1i./(pi*k*c*dbesselH(mm,k*c));
278 %
279 Q0n=1./(1-dIdI(0,alpha*a,alpha*c).*dKdK(0,alpha*c,alpha...
    *a));
280 Qm0=zeros(M,1);
281 for m=1:M
282     Qm0(m,1)=1/(DR2m(m,a,a)*DR1m(m,c,c)-DR1m(m,a,c)*...
        DR2m(m,c,a));
283 end
284 Qmn=zeros(M,N);
285 for m=1:M
286     Qmn(m,:)=1./(1-dIdI(m,alpha*a,alpha*c).*dKdK(m,...
        alpha*c,alpha*a));
287 end
288 %
289 R1mn=zeros(M+1,N);
290 R2mn=zeros(M+1,N);
291 R3mn=zeros(M+1,N);
292 R4mn=zeros(M+1,N);
293 for m=0:M
294     R1mn(m+1,:)=IdI(m,alpha*c,alpha*c)./alpha...
        -dIdI(m,alpha*a,alpha*c).*KdK(m,alpha*c,alpha*a...
        )./alpha;
295     R2mn(m+1,:)=KdK(m,alpha*c,alpha*a)./alpha...
        -IdI(m,alpha*c,alpha*c).*dKdK(m,alpha*c,alpha*a...
        )./alpha;
296     R3mn(m+1,:)=IdI(m,alpha*a,alpha*c)./alpha...
        -dIdI(m,alpha*a,alpha*c).*KdK(m,alpha*a,alpha*a...
        )./alpha;
297     R4mn(m+1,:)=KdK(m,alpha*a,alpha*a)./alpha...
        -IdI(m,alpha*a,alpha*c).*dKdK(m,alpha*c,alpha*a...
        )./alpha;
298 end
299 %
300 V1pl=zeros(P+1,L+1);
301 for p=0:P
302     V1pl(p+1,:)=N1.^(-1/2).*besselj(2*p+1/6,k1*(h-d))...
        ./(h*(2*k1*(h-d)).^(1/6));
303 end
304 %
305 V2pn=zeros(P+1,N+1);
306 V2pn(1,1)=1/((h-d)*2^(1/3)*gamma(7/6));
307 for p=0:P
308     V2pn(p+1,2:end)=besselj(2*p+1/6,alpha*(h-d))./(h-d...
        )^(7/6)*(2*alpha).^(1/6));
309 end
310 %
311 Kml=zeros(M+1,L+1);

```

```

316     Iml=zeros(M+1,L+1);
317     for m=0:M
318         Km1(m+1,:)=KdK(m,kl*c,kl*c)./kl;
319         Iml(m+1,:)=IdI(m,kl*a,kl*a)./kl;
320     end
321     %-----
322     gma=zeros(M+1,P+1);
323     xi=zeros(M+1,P+1);
324     %-----
325     % as m=0
326     RHS=zeros(2*P+3,1);
327     for pp=0:P
328         RHS(1+pp+1,1)=h*Rm(1)*V1pl(pp+1,1)/Zl(1);
329     end
330     LHS=zeros(2*P+3,2*P+3);
331     for p=0:P
332         LHS(1,1+p+1)=DR2m(0,a,a)*V2pn(p+1,1);
333         LHS(1,1+P+1+p+1)=-DR2m(0,c,a)*V2pn(p+1,1);
334     end
335     for p=0:P
336         for pp=0:P
337             LHS(1+pp+1,1)=(h-d)*R1m(0,c,c)*V2pn(pp+1,1);
338             LHS(1+pp+1,1+p+1)=(h-d)*R2m(0,c,a)/DR2m(0,c,a)*...
339                 V2pn(p+1,1)*V2pn(pp+1,1)...
340                 -h*Km1(1,:)*(V1pl(p+1,:).*V1pl(pp+1,:)).'...
341                 +2*(h-d)*Q0n.*R1mn(1,:)*(V2pn(p+1,2:end).*...
342                 V2pn(pp+1,2:end)).';
343             LHS(1+pp+1,P+2+p+1)=2*(h-d)*Q0n.*R2mn(1,:)*(V2pn...
344                 (p+1,2:end).*V2pn(pp+1,2:end)).';
345             LHS(P+2+pp+1,1)=(h-d)*R1m(0,a,c)*V2pn(pp+1,1);
346             LHS(P+2+pp+1,1+p+1)=2*(h-d)*Q0n.*R3mn(1,:)*(V2pn...
347                 (p+1,2:end).*V2pn(pp+1,2:end)).';
348             LHS(P+2+pp+1,P+2+p+1)=(h-d)*R2m(0,a,a)/DR2m(0,a,...
349                 a)*V2pn(p+1,1)*V2pn(pp+1,1)...
350                 +2*(h-d)*Q0n.*R4mn(1,:)*(V2pn(p+1,2:end).*...
351                 V2pn(pp+1,2:end)).'...
352             -h*Iml(1,:)*(V1pl(p+1,:).*V1pl(pp+1,:)).';
353         end
354     end
355     sol0=LHS\RHS;
356     gma(1,:)=sol0(2:P+2);
357     xi(1,:)=sol0(P+3:2*P+3);
358     for m=1:M
359         RHS=zeros(2*P+2,1);
360         for pp=0:P
361             RHS(pp+1)=h*Rm(m+1)*V1pl(pp+1,1)/Zl(1);
362         end
363         LHS=zeros(2*P+2,2*P+2);
364         for p=0:P
365             for pp=0:P
366                 LHS(pp+1,p+1)=-h*Km1(m+1,:)*(V1pl(p+1,:)...
367                     .*V1pl(pp+1,:)).'...
368                     + (h-d)*Qm0(m)*F1m(m)*V2pn(p+1,1)*...
369                     V2pn(pp+1,1)...

```

```

363         +2*(h-d)*Qmn(m,:).*R1mn(m+1,:)*(V2pn...
          (p+1,2:end).*V2pn(pp+1,2:end))...
          .';
364     LHS(pp+1,P+1+p+1)=(h-d)*Qm0(m)*F2m(m)*...
          V2pn(p+1,1)*V2pn(pp+1,1)...
365     +2*(h-d)*Qmn(m,:).*R2mn(m+1,:)*(V2pn...
          (p+1,2:end).*V2pn(pp+1,2:end))...
          .';
366     %
367     LHS(P+1+pp+1,p+1)=(h-d)*Qm0(m)*F3m(m)*...
          V2pn(p+1,1)*V2pn(pp+1,1)...
368     +2*(h-d)*Qmn(m,:).*R3mn(m+1,:)*(V2pn...
          (p+1,2:end).*V2pn(pp+1,2:end))...
          .';
369     LHS(P+1+pp+1,P+1+p+1)=(h-d)*Qm0(m)*F4m(m...
          )*V2pn(p+1,1)*V2pn(pp+1,1)...
370     +2*(h-d)*Qmn(m,:).*R4mn(m+1,:)*(V2pn...
          (p+1,2:end).*V2pn(pp+1,2:end))...
          .';
371     -h*Im1(m+1,:)*(V1pl(p+1,:).*V1pl(pp...
          +1,:)).';
372     end
373     end
374     solm=LHS\RHS;
375     gma(m+1,:)=solm(1:P+1);
376     xi(m+1,:)=solm(P+2:2*P+2);
377     end
378     %
379     Aml=gma*V1pl;
380     %
381     Kmn=zeros(M+1,N);
382     Imn=zeros(M+1,N);
383     for m=0:M
384         Kmn(m+1,:)=dKdK(m,alpha*c,alpha*a);
385         Imn(m+1,:)=dIdI(m,alpha*a,alpha*c);
386     end
387     %
388     Bmn=zeros(M+1,N+1);
389     Bmn(1,1)=sol0(1);
390     Bmn(1,2:end)=2*Q0n.*(gma(1,:)*V2pn(:,2:end)-Kmn(1,:).*(...
          xi(1,:)*V2pn(:,2:end)));
391     for m=1:M
392         Bmn(m+1,1)=Qm0(m)*(DR2m(m,a,a)*gma(m+1,:)*V2pn(:,1)-...
          DR2m(m,c,a)*xi(m+1,:)*V2pn(:,1));
393     end
394     for m=1:M
395         Bmn(m+1,2:end)=2*Qmn(m,:).*(gma(m+1,:)*V2pn(:,2:end)...
          -Kmn(m+1,:).*(xi(m+1,:)*V2pn(:,2:end)));
396     end
397     %
398     Cmn=zeros(M+1,N+1);
399     Cmn(1,1)=gma(1,:)*V2pn(:,1)/DR2m(0,c,a);
400     % C00=xi(1,:)*V2pn(:,1)/DR2m(0,a,a);
401     for m=1:M
402         Cmn(m+1,1)=Qm0(m)*(DR1m(m,c,c)*xi(m+1,:)*V2pn(:,1)...
          ...

```

```

403         -DR1m(m,a,c)*gma(m+1,:)*V2pn(:,1));
404     end
405     Cmn(1,2:end)=2*Q0n.*(xi(1,:)*V2pn(:,2:end)-Imn(1,:).*(...
        gma(1,:)*V2pn(:,2:end)));
406     for m=1:M
407         Cmn(m+1,2:end)=2*Qmn(m,:).*(xi(m+1,:)*V2pn(:,2:end)-...
            Imn(m+1,:).*(gma(m+1,:)*V2pn(:,2:end)));
408     end
409     %
410     Dml=xi*V1pl;
411     %
412     %
413     function [A1,Bn,Cn,D1]=radiationP(kl)
414     % help [A1,Bn,Cn,D1]=radiationP(kl)
415     %=====
416     global h d a c L N P
417     %=====
418     nn=1:N;
419     alpha=nn*pi/(h-d);
420     %
421     N1=1/2*(1+sin(2*kl*h)./(2*kl*h));
422     %
423     Qn=1./(1-dIdI(0,alpha*a,alpha*c).*dKdK(0,alpha*c,alpha*...
        a));
424     R1n=IdI(0,alpha*c,alpha*c)./alpha-dIdI(0,alpha*a,alpha*...
        c).*...
425         KdK(0,alpha*c,alpha*a)./alpha;
426     R2n=KdK(0,alpha*c,alpha*a)./alpha-IdI(0,alpha*c,alpha*c...
        ).*...
427         dKdK(0,alpha*c,alpha*a)./alpha;
428     R3n=IdI(0,alpha*a,alpha*c)./alpha-dIdI(0,alpha*a,alpha*...
        c).*...
429         KdK(0,alpha*a,alpha*a)./alpha;
430     R4n=KdK(0,alpha*a,alpha*a)./alpha-IdI(0,alpha*a,alpha*c...
        ).*...
431         dKdK(0,alpha*c,alpha*a)./alpha;
432     %
433     V1pl=zeros(P+1,L+1);
434     for p=0:P
435         V1pl(p+1,:)=N1.^(-1/2).*besselj(2*p+1/6,kl*(h-d)).*...
            ./(h*((2*kl*(h-d)).^(1/6)));
436     end
437     V2pn=zeros(P+1,N+1);
438     V2pn(1,1)=1/((h-d)*2^(1/3)*gamma(7/6));
439     for p=0:P
440         V2pn(p+1,2:end)=besselj(2*p+1/6,alpha*(h-d))./((h-d...
            )^(7/6)*(2*alpha).^(1/6));
441     end
442     K01=KdK(0,kl*c,kl*c)./kl;
443     I01=IdI(0,kl*a,kl*a)./kl;
444     %
445     RHS=zeros(1+P+1+P+1,1);
446     for pp=0:P
447         RHS(P+2+pp+1,1)=(h-d)*V2pn(pp+1,1);
448     end
449     LHS=zeros(2*P+3,2*P+3);

```

```

450     for p=0:P
451         LHS(1,1+p+1)=c*V2pn(p+1,1);
452         LHS(1,1+P+1+p+1)=-a*V2pn(p+1,1);
453     end
454     for p=0:P
455         for pp=0:P
456             LHS(1+pp+1,1)=-(h-d)*V2pn(pp+1,1);
457             LHS(1+pp+1,1+p+1)=h*K01*(V1pl(p+1,:).*V1pl(pp...
458                 +1,:)).'...
459                 -(h-d)*c*log(c)*V2pn(p+1,1)*V2pn(pp+1,1)...
460                 -2*(h-d)*(Qn.*R1n)*(V2pn(p+1,2:end).*V2pn(...
461                     pp+1,2:end)).';
462             LHS(1+pp+1,P+2+p+1)=-2*(h-d)*(Qn.*R2n)*(V2pn(p...
463                 +1,2:end).*V2pn(pp+1,2:end)).';
464         end
465     end
466     for p=0:P
467         for pp=0:P
468             LHS(P+2+pp+1,1)=(h-d)*V2pn(pp+1,1);
469             LHS(P+2+pp+1,1+p+1)=2*(h-d)*(Qn.*R3n)*(V2pn(p...
470                 +1,2:end).*V2pn(pp+1,2:end)).';
471             LHS(P+2+pp+1,P+2+p+1)=(h-d)*a*log(a)*V2pn(p...
472                 +1,1)*V2pn(pp+1,1)...
473                 +2*(h-d)*(Qn.*R4n)*(V2pn(p+1,2:end).*V2pn(...
474                     pp+1,2:end)).'...
475                 -h*I01*(V1pl(p+1,:).*V1pl(pp+1,:)).';
476         end
477     end
478     sol=LHS\RHS;
479     gma=sol(2:P+2).';
480     xi=sol(P+3:2*P+3).';
481     %
482     A1=gma*V1pl;
483     %
484     K0n=dKdK(0,alpha*c,alpha*a);
485     Bn=zeros(1,N+1);
486     Bn(1,1)=sol(1);
487     Bn(1,2:end)=2*Qn.*(gma*V2pn(:,2:end)-K0n.*(xi*V2pn(:,2...
488         :end)));
489     %
490     I0n=dIdI(0,alpha*a,alpha*c);
491     Cn=zeros(1,N+1);
492     Cn(1,1)=c*gma*V2pn(:,1);
493     % C0=a*xi*V2pn(:,1);
494     Cn(2:end)=2*Qn.*(xi*V2pn(:,2:end)-I0n.*(gma*V2pn(:,2...
495         :end)));
496     %
497     D1=xi*V1pl;
498     %
499     mu_Evans(ii)=imag(-2*1i*g/(omega(ii)^2*a^2)*D1*(a*DZ1...
500         ./(kl.^2)).');
501     nu_Evans(ii)=real(2*1i*g/(omega(ii)^2*a^2)*D1*(a*DZ1...
502         ./(kl.^2)).');
503 end
504 %
505 %

```

```

496 function [A1,Bn,Cn,D1]=radiationB(kl)
497 % [A1,Bn,Cn,D1]=radiationB(kl)
498 %=====
499 global h d a c L N P
500 %=====
501
502 nn=0:N;
503 alpha=nn(2:end)*pi/(h-d);
504 %
505 N1=1/2*(1+sin(2*kl*h)/(2*kl*h));
506 %
507 Q1n=1./(1-dIdI(0,alpha*a,alpha*c).*dKdK(0,alpha*c,alpha...
    *a));
508 Vp=zeros(1,P+1);
509 for p=0:P
510     Vp(p+1)=(h-d)*(-1)^p*factorial(2*p)*gamma(1/6)/(pi*...
        gamma(2*p+1/3))...
        *coefGenC(p)*Fs(p)';
511 end
512 R1n=IdI(0,alpha*c,alpha*c)./alpha-dIdI(0,alpha*a,alpha*...
    c).*...
513     KdK(0,alpha*c,alpha*a)./alpha;
514 R2n=KdK(0,alpha*c,alpha*a)./alpha-IdI(0,alpha*c,alpha*c...
    ).*...
515     dKdK(0,alpha*c,alpha*a)./alpha;
516 R3n=IdI(0,alpha*a,alpha*c)./alpha-dIdI(0,alpha*a,alpha*...
    c).*...
517     KdK(0,alpha*a,alpha*a)./alpha;
518 R4n=KdK(0,alpha*a,alpha*a)./alpha-IdI(0,alpha*a,alpha*c...
    ).*...
519     dKdK(0,alpha*c,alpha*a)./alpha;
520 I01=IdI(0,kl*a,kl*a)./kl;
521 K01=KdK(0,kl*c,kl*c)./kl;
522 %
523 V1p1=zeros(P+1,L+1);
524 for p=0:P
525     V1p1(p+1,:)=N1.^(-1/2).*besselj(2*p+1/6,kl*(h-d))...
        ./(h*(2*kl*(h-d)).^(1/6));
526 end
527 V2pn=zeros(P+1,N+1);
528 V2pn(1,1)=1/((h-d)*2^(1/3)*gamma(7/6));
529 for p=0:P
530     V2pn(p+1,2:end)=besselj(2*p+1/6,alpha*(h-d))./((2*...
        alpha).^(1/6)*(h-d)^(7/6));
531 end
532 %
533 % number of unknowns is 1+(P+1)+(P+1)=2*p+3
534 RHS=zeros(2*P+3,1);
535 RHS(1)=(a^2-c^2)/(2*(h-d));
536 for pp=0:P
537     RHS(1+pp+1)=Vp(pp+1)/2-c^2*V2pn(pp+1,1)/4+c^2*log(c...
        )*V2pn(pp+1,1)/2;
538     RHS(P+2+pp+1)=-Vp(pp+1)/2+a^2*V2pn(pp+1,1)/4-a^2*...
        log(a)*V2pn(pp+1,1)/2;
539 end
540 LHS=zeros(2*P+3,2*P+3);

```



```

542     for p=0:P
543         LHS(1,1+p+1)=c*V2pn(p+1,1);
544         LHS(1,1+P+1+p+1)=-a*V2pn(p+1,1);
545     end
546     for p=0:P
547         for pp=0:P
548             %
549             LHS(1+pp+1,1)=-(h-d)*V2pn(pp+1,1);
550             LHS(1+pp+1,1+p+1)=h*K01*(V1pl(p+1,:).*V1pl(pp...
551                 +1,:)).'...
552                 -(h-d)*c*log(c)*V2pn(p+1,1)*V2pn(pp+1,1)...
553                 -2*(h-d)*Q1n.*R1n*(V2pn(p+1,2:end).*V2pn(pp...
554                 +1,2:end)).';
555             LHS(1+pp+1,1+P+1+p+1)=-2*(h-d)*Q1n.*R2n*(V2pn(p...
556                 +1,2:end).*V2pn(pp+1,2:end)).';
557             %
558             LHS(1+P+1+pp+1,1)=(h-d)*V2pn(pp+1,1);
559             LHS(1+P+1+pp+1,1+p+1)=2*(h-d)*Q1n.*R3n*(V2pn(p...
560                 +1,2:end).*V2pn(pp+1,2:end)).';
561             LHS(1+P+1+pp+1,1+P+1+p+1)=(h-d)*a*log(a)*V2pn(p...
562                 +1,1)*V2pn(pp+1,1)...
563                 -h*I01*(V1pl(p+1,:).*V1pl(pp+1,:)).'...
564                 +2*(h-d)*Q1n.*R4n*(V2pn(p+1,2:end).*V2pn(pp...
565                 +1,2:end)).';
566         end
567     end
568     sol=LHS\RHS;
569     gma=sol(2:P+2).';
570     xi=sol(P+3:2*P+3).';
571     %
572     A1=gma*V1pl;
573     %
574     Bn=zeros(1,N+1);
575     K0n=dKdK(0,alpha*c,alpha*a);
576     Bn(1)=sol(1);
577     Bn(2:end)=2*Q1n.*(gma*V2pn(:,2:end)-K0n.*(xi*V2pn(:,2...
578         :end)));
579     %
580     Cn=zeros(1,N+1);
581     I0n=dIdI(0,alpha*a,alpha*c);
582     Cn(1)=c*(gma*V2pn(:,1)+c/(2*(h-d)));
583     % C0=a*(xi*(V2pn(:,1))+a/(2*(h-d)));
584     Cn(2:end)=2*Q1n.*(xi*V2pn(:,2:end)-I0n.*(gma*V2pn(:,2...
585         :end)));
586     %
587     D1=xi*V1pl;
588 end
589 %
590 %
591 function roots=wavenumber(KC,h,N)
592 % roots=wavenumber[KC,h,N]
593 fk=@(k)k.*tanh(k*h)-KC;
594 dfk=@(k)tanh(k*h)+k*h.*sech(k*h).^2;
595 fkn=@(kn)kn.*tan(kn*h)+KC;
596 dfkn=@(kn)tan(kn*h)+kn*h.*sec(kn*h).^2;
597

```

```

590 roots=zeros(1,N+1);
591 x0=0.2;
592 k=newton(fk,dfk,x0,1e-12);
593 roots(1)=-1i*k;
594
595 for n=1:N
596     x0=pi/(2*h)+0.005+(n-1)*pi/h;
597     roots(n+1)=newton(fkn,dfkn,x0,1e-12);
598 end
599 function x1=newton(fun,dfun,x,tol)
600 x1=x-fun(x)/dfun(x);
601 while (abs(x1-x)≥tol)
602     x=x1;
603     x1=x-fun(x)/dfun(x);
604 end
605 %
606 %
607 function out=coefGenC(p)
608 % out=coefGenC(p)
609 % order=2*p
610 if abs(p)<1e-8
611     out=1;
612 elseif abs(p-1)<1e-8
613     out=[-(1/6),7/18];
614 elseif abs(p-2)<1e-8
615     out=[7/72,-(91/108),1729/1944];
616 elseif abs(p-3)<1e-8
617     out=[-(91/1296),1729/1296,-(43225/11664)...
        ,267995/104976];
618 elseif abs(p-4)<1e-8
619     out=[1729/31104,-(43225/23328)...
        ,1339975/139968,-(9915815/629856)...
        ,60911435/7558272];
620 elseif abs(p-5)<1e-8
621     out=[-(8645/186624),1339975/559872,-(49579075/2519424)...
        ,426380045/7558272,...
        -(2984660315/45349632),32831263465/1224440064];
622 elseif abs(p-6)<1e-8
623     out=[267995/6718464,-(9915815/3359232)...
        ,2131900225/60466176,-(20892622205/136048896),...
        164156317325/544195584,-(2002707071365/7346640384)...
        ,12198306707405/132239526912];
624 elseif abs(p-7)<1e-8
625     out=[-(1416545/40310784)...
        ,426380045/120932352,-(20892622205/362797056),...
        1149094221275/3265173504,-(10013535356825/9795520512)...
        ,...
        134181373781455/88159684608,...
        -(890476389640565/793437161472),...
        773050931665985/2380311484416];
626 elseif abs(p-8)<1e-8
627     out=[60911435/1934917632,-(2984660315/725594112)...
        ,1149094221275/13060694016,...
        -(14018949499555/19591041024)...
        ,670906868907275/235092492288,...

```

```

635     -(9795240286046215/1586874322944)...
        ,70347634781604635/9521245937664,...
636     -(65709329191608725/14281868906496),...
637     1195909791287278795/1028294561267712];
638 end
639 %
640 %
641 function out=R1m(m,r,c)
642 % out=R1m(m,r,c)
643 if abs(m)<1e-8
644     out=1;
645 else
646     out=(r/c)^m;
647 end
648 %
649 %
650 function out=DR1m(m,r,c)
651 % out=DR1m(m,r,c)
652 if abs(m)<1e-8
653     out=0;
654 else
655     out=m/c*(r/c)^(m-1);
656 end
657 %
658 %
659 function out=R2m(m,r,a)
660 % out=R2m(m,r,a)
661 if abs(m)<1e-8
662     out=log(r/a);
663 %     out=log(r);
664 %     out=log(r)/log(a);
665 else
666     out=(r/a)^(-m);
667 end
668 %
669 %
670 function out=DR2m(m,r,a)
671 % out=DR2m(m,r,a)
672 if abs(m)<1e-8
673     out=1/r;
674 %     out=1/(r*log(a));
675 else
676     out=-m/a*(r/a)^(-m-1);
677 end
678 %
679 %
680 function out=Fs(p)
681 % out=Fs(p)
682 out=zeros(1,p+1);
683 for ii=0:p
684     out(ii+1)=gamma(2/3)*gamma(3/2+ii)/(2*gamma(13/6+ii));
685 end
686 %
687 %
688 function out=gebC(n,v,x)
689 % define Gegenbauer polynomials C_n^v(x)

```

```

690 % out=gebC(n,v,x)
691 C0=1;
692 C1=2*v.*x;
693 if abs(n)<1e-8
694     out=C0;
695 elseif abs(n-1)<1e-8
696     out=C1;
697 else
698     for ii=2:n
699         C2=1/ii*2*(v+ii-1)*x*C1-1/ii*(2*v+ii-2)*C0;
700         C0=C1;
701         C1=C2;
702     end
703     out=C2;
704 end
705 %
706 %
707 function out=IdI(m,z1,z2)
708 % out=IdI(m,z1,z2)
709 % Im(z1)/Im'(z2)
710 out=2*exp(abs(real(z1))-abs(real(z2)))*besseli(m,z1,1)...
711     ./...
712     (besseli(m-1,z2,1)+besseli(m+1,z2,1));
713 %
714 function out=KdK(m,z1,z2)
715 % out=KdK(m,z1,z2)
716 % Km(z1)/Km'(z2)
717 out=-2*exp(z2-z1)*besselk(m,z1,1)./(besselk(m-1,z2,1)+...
718     besselk(m+1,z2,1));
719 %
720 function out=dIdI(m,z1,z2)
721 % out=dIdI(m,z1,z2)
722 % Im'(z1)/Im'(z2)
723 out=exp(abs(real(z1))-abs(real(z2)))*(besseli(m-1,z1,1)+...
724     besseli(m+1,z1,1))...
725     ./ (besseli(m-1,z2,1)+besseli(m+1,z2,1));
726 %
727 function out=dKdK(m,z1,z2)
728 % out=dKdK(m,z1,z2)
729 % Km'(z1)/Km'(z2)
730 out=exp(z2-z1)*(besselk(m-1,z1,1)+besselk(m+1,z1,1))...
731     ./ (besselk(m-1,z2,1)+besselk(m+1,z2,1));

```

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