Dispersion in oscillatory flows

Song, Jie

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DISPERSION IN OSCILLATORY FLOWS

SONG JIE

SCHOOL OF CIVIL AND ENVIRONMENTAL ENGINEERING

2014

DISPERSION IN OSCILLATORY FLOWS

SONG JIE

SONG JIE

School of Civil and Environmental Engineering

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Table of Contents

Acknowledgements	i
Summary.....	vi
List of Tables	ix
List of Figures.....	x
List of Symbols	xiv
Chapter 1 Introduction.....	1
1.1 General introduction.....	1
1.2 Objectives.....	2
1.3 Outline.....	4
Chapter 2 Literature review	5
2.1 Dispersion in coastal environments.....	5
2.2 Longitudinal dispersion in oscillatory pipe/channel flows	7
2.2.1 Laminar oscillatory pipe/channel flows.....	7
2.2.2 Turbulent oscillatory pipe/channel flows.....	9
2.2.3 Relevance to isobaric pressure exchanger	10
2.3 Dispersion in electro-osmotic flows.....	13
2.4 Summary	16
Chapter 3 One-dimensional stochastic diffusion in wave motion.....	17
3.1 Introduction	17
3.2 Analytical methods.....	18
3.2.1 Monochromatic progressive waves.....	21
3.2.2 Standing waves	24

3.3	Results and discussion.....	26
3.4	Conclusions	29

Chapter 4 Longitudinal dispersion of turbulent oscillatory flows in a circular pipe ..31

4.1	Introduction	31
4.2	Theoretical study	33
4.2.1	Velocity field	33
4.2.2	Concentration field.....	40
4.2.3	Theoretical results and discussion	45
4.3	Experimental study.....	54
4.3.1	Experimental apparatus and conditions	54
4.3.2	Experimental methodology	59
4.3.3	Experimental results and analysis	69
4.4	Comparison	83
4.4.1	Comparison with laminar flows.....	83
4.4.2	Comparison with turbulent flows.....	85
4.5	Conclusions	86

Chapter 5 Longitudinal dispersion of turbulent oscillatory flows in a two-dimensional channel.....89

5.1	Introduction	89
5.2	Turbulent flows driven by synchronously oscillating walls.....	90
5.2.1	Velocity field	90
5.2.2	Concentration field.....	96
5.3	Turbulent flows driven by single wall oscillation	101
5.3.1	Velocity field	101
5.3.2	Concentration field.....	103
5.4	Results and discussion.....	105
5.5	Conclusions	113

Chapter 6 Longitudinal dispersion of two-dimensional oscillatory electro-osmotic flows with reactive walls.....114

6.1	Introduction	114
6.2	Problem formulation	115
6.3	Mass transfer	118
6.4	Results and discussion.....	126
6.5	Conclusions	135
Chapter 7	Conclusions.....	137
	References.....	140
	Appendix.....	150

Summary

Compared to the in-depth understanding of dispersion in steady flows, the knowledge of dispersion in oscillatory flows is relatively un-developed, even though the topic is relevant to many practical problems. A few topics on the dispersion in oscillatory flows have been investigated in this thesis. They include the dispersion in progressive and standing waves, the longitudinal dispersion in pressure-driven turbulent oscillatory pipe flows and in turbulent oscillatory flows generated by vibrating boundaries, and the dispersion of oscillatory electro-osmotic flows in nano- and micro-channels.

First, progress is made on the dispersion in one-dimensional progressive and standing waves. Mathematical analysis is performed by considering inert pollutants that flow with the carrier fluid while undergoing Brownian motion at the same time. The focus is on the nonlinear interactions between the stochastic diffusion and deterministic wave motions, and the scope is limited to cases whereby a small parameter exists between the advective and diffusive displacements, which then allows a perturbation analysis to be performed. With a sinusoidal progressive wave, the results show that the deterministic wave motion can either increase or decrease the embedded stochastic diffusion depending on the wave characteristics. Longer wave lengths and shorter wave periods tend to significantly promote diffusion, while shorter wave lengths and longer wave periods act in the opposite manner but with a much smaller effect. Numerical simulations are also performed and the results agree well with the analytical predictions.

Second, an analytical analysis is performed using the homogenization approach to predict the magnitude of the longitudinal dispersion induced by a turbulent oscillatory flow forced by a sinusoidal pressure gradient inside a circular pipe. An axisymmetric

co-axial eddy viscosity model is adopted to resolve the radial distribution of velocities and turbulent shear stresses. The results demonstrate that a dimensionless parameter α , which is the ratio of the velocity amplitude divided by the frequency and pipe radius, determines the flow structure and the magnitude of the induced longitudinal dispersion coefficient. Experiments are also conducted to measure the longitudinal dispersion coefficient under different oscillation frequencies and velocity magnitudes. The measurement approaches are based on the non-invasive laser imaging techniques of Particle Image Velocimetry (PIV) and Planar Laser Induced Fluorescence (PLIF). The experimental results show that the dispersion coefficient increases significantly with the Reynolds number. In the laminar regime, the measurement results generally exceed the analytical predictions from earlier literature. In the turbulent regime, which occurs when the Reynolds number exceeds ~ 500 based on the PIV vector maps, a reasonably good agreement is obtained between the present analytical predictions and the measurements results. This study provides understanding on how the mixing in the pressure exchanger can be reduced.

Third, using the same analytical method, the dispersion in turbulent oscillatory flows in a two-dimensional channel with vibrating boundaries is investigated. Two specific conditions, i.e. synchronous vibration of both boundaries and vibration of only one boundary, are examined and compared. Similarly, a control parameter α , which equals to the ratio of the amplitude of wall vibrating velocity over the vibrating frequency and channel height, characterizes the flow structure as well as the induced longitudinal dispersion coefficient. The dispersion coefficient for the one boundary vibration is smaller than that of the dual boundary vibration with small α , but reverses when α is large. This subject has the potential to be applied to flat-sheet membrane filtration

systems where vibration can be employed to increase the shear stress and mass transfer efficiency, and thus improve the filtration performance.

Finally, the dispersion in two-dimensional oscillatory electro-osmotic flows (laminar flows) in nano- and micro-scales driven by alternating currents with reactive walls is analyzed. Due to the involvement of various electrical and chemical factors, the situation is more complicated and additional operational parameters need to be considered to characterize the dispersion performance. A general expression for the dispersion coefficient under different zeta potentials as well as various sorption conditions at the walls is derived analytically. The dispersion coefficient is found to be dependent on the oscillation frequency, the Debye parameter, the species partition coefficient, the reaction kinetics and the wall potentials. The results demonstrate that the presence of wall sorption tends to enhance the dispersion when the oscillation frequency is low, but the situation reverses in high-frequency oscillatory flows. The results can help improve the understanding of pollutant removal in ground waters by using electric fields.

List of Tables

Table 4.1 Key parameters of the experimental apparatus	56
Table 4.2 Summary of experimental conditions	58
Table 4.3 Specifications of the Dantec DualPower 65-15 laser.....	61
Table 4.4 Summary of the measured dispersion coefficients	81

List of Figures

Figure 2.1 Schematic layout of SWRO process.....	11
Figure 2.2 Working principle of rotary isobaric pressure exchanger (Zhou et al., 2009)...	12
Figure 2.3 Electro-osmotic flow	14
Figure 3.1 Relationship between the modulation coefficient R/ε^2 and σ	27
Figure 3.2 Comparison between analytical predictions and numerical results. The symbols represent the numerical simulations and the lines represent the analytical predictions for different values of σ	28
Figure 3.3 Growth of variance with time under various σ for small value of $\varepsilon = 0.2$ by MATLAB simulation	29
Figure 4.1 Schematic diagram of the coordinate axis for analysis	34
Figure 4.2 Variations of (a) friction velocity $\overline{u_f}'$, (b) complex constant $a^{(2)}$, and (c) inner boundary layer thickness δ_l' with α . The solid curves are for $r_0' = 0.001$ and dashed curves for $r_0' = 0.005$	47
Figure 4.3 Profiles of eddy viscosity ν_T' at various phases within one period under different α (left: $r_0' = 0.001$, right: $r_0' = 0.005$)	49
Figure 4.4 Profiles of velocity u' at various phases within half a period under different α (left: $r_0' = 0.001$, right: $r_0' = 0.005$)	50
Figure 4.5 Distribution of shear stress τ' at various phases within half a period under different α (left: $r_0' = 0.001$; right: $r_0' = 0.005$)	51

Figure 4.6 Longitudinal dispersion coefficient D_l' as a function of α ($\phi = 0.6, r_0' = 0.001$ (solid), $\phi = 0.6, r_0' = 0.005$ (dash), $\phi = 1, r_0' = 0.001$ (dash dot), and $\phi = 1, r_0' = 0.005$ (dash dot dot))	53
Figure 4.7 Turbulent dispersivity $\langle\langle D_t' \rangle\rangle$ as a function of α ($\phi = 0.6, r_0' = 0.001$ (solid), $\phi = 0.6, r_0' = 0.005$ (dash), $\phi = 1, r_0' = 0.001$ (dash dot), and $\phi = 1, r_0' = 0.005$ (dash dot dot))	54
Figure 4.8 Photo of the experimental apparatus.....	55
Figure 4.9 Working principle of PIV technique.....	60
Figure 4.10 Configuration of the experimental setup (unit: mm)	62
Figure 4.11 Calibration picture for PIV experiments.....	63
Figure 4.12 (a) A pair of original PIV images under the condition of $Re_\delta = 567$ ($V = 25.1 \text{ cm}^3$, $\omega = 2\pi \text{ rad/s}$); (b) vector map obtained through the cross- correlation algorithm	64
Figure 4.13 Calibration images under various concentrations for $Re_\delta = 501$ ($V = 31.4$ cm^3 , $\omega = \pi \text{ rad/s}$)	67
Figure 4.14 Calibration curves for $Re_\delta = 501$ ($V = 31.4 \text{ cm}^3$, $\omega = \pi \text{ rad/s}$).....	68
Figure 4.15 Velocity vector maps at different phases for various experimental conditions (unit: mm)	71
Figure 4.16 Velocity variation within a period at the centerline for different experimental condtions.....	74
Figure 4.17 Cross-sectional velocity profiles at various phases within half period.....	75
Figure 4.18 Longitudinal dispersion process within one cycle for $Re_\delta = 425$ ($V = 18.8$ cm^3 , $\omega = 2\pi \text{ rad/s}$)	77

Figure 4.19 Concentration distribution as a function of distance from a maintained source based on complementary error function.....	79
Figure 4.20 Variation of dimensionless longitudinal dispersion coefficient with (a) dimensionless stroke volume and (b) Womersley number.....	80
Figure 4.21 Relationship between longitudinal dispersion coefficient and Reynolds number.....	82
Figure 4.22 Comparison between experimental results and theoretical predictions for laminar flows	85
Figure 4.23 Comparison between experimental results and theoretical predictions for turbulent flows.....	86
Figure 5.1 Schematic diagram of the coordinate axis for the case of synchronous dual walls vibration	91
Figure 5.2 Schematic diagram of the coordinate axis for the case of single wall vibration	101
Figure 5.3 Variations of (a) friction velocity $\overline{u_f}'$ and (b) complex constant $a^{(2)}$ with α . The solid lines are for dual walls oscillation and dashed lines for single wall oscillation	106
Figure 5.4 Velocity profiles at various phases within half an oscillation period at different α with $y_0' = 0.001$ (left: dual walls oscillation, right: single wall oscillation).....	107
Figure 5.5 Distribution of shear stress at various phases within half an oscillation period at different α with $y_0' = 0.001$ (left: dual walls oscillation, right: single wall oscillation).....	108

Figure 5.6 Longitudinal dispersion coefficient D_l' as a function of α at different ϕ_t with $y_0' = 0.001$. ((a) dual walls oscillation and (b) single wall oscillation. $\phi_t = 0.5$ (dash), $\phi_t = 1.0$ (solid) and $\phi_t = 2.0$ (dash dot)).....	110
Figure 5.7 Turbulent dispersivity $\langle\langle \overline{D_t'} \rangle\rangle$ as a function of α at different ϕ_t with $y_0' = 0.001$. ((a) dual walls oscillation and (b) single wall oscillation. $\phi_t = 0.5$ (dash), $\phi_t = 1.0$ (solid) and $\phi_t = 2.0$ (dash dot)).....	112
Figure 6.1 Schematic diagram of the problem concerned.....	115
Figure 6.2 Scaled velocity profiles at various phases within half an oscillation for $\psi_2' = 1$ and $Sc = 1000$	128
Figure 6.3 Dispersion coefficient D_{Tw}' as a function of δ' for $k' = 100$, $\psi_2' = 1$, $\xi' = 1$ and $Sc = 1000$. (a) Comparison among different Da and (b) Comparison among different sorption conditions with $Da = 1$	130
Figure 6.4 Dispersion coefficient D_{Tw}' as a function of k' for $\psi_2' = 1$, $\xi' = 1$ and $Sc = 1000$. (a) and (c) Comparison among different Da , and (b) and (d) Comparison among different sorption conditions with $Da = 1$	132
Figure 6.5 Variation of D_{Tw}' with ξ' for $\psi_2' = 1$ and $Sc = 1000$. ((a)-(d) for sorption at one wall, and (e)-(h) for sorption at both walls)	133
Figure 6.6 Relationship between the relative bottom wall potential ψ_2' and the dispersion coefficient under different sorption conditions for $\xi' = 1$, $Da = 1$ and $Sc = 1000$	135

List of Symbols

a	Radius of pipe
$a^{(2)}$	Complex constant representing the time variation of v_T
A_w	Wave amplitude
c_0	Ion concentration far from the charged walls
C	Concentration
C_{cs}	Constant-source concentration
C_s	Concentration of immobile phase
C_{rT}	Measured radial-direction and period mean concentration
d_p	Diameter of pipe
d_{pc}	Diameter of piston chamber
D	Molecular diffusion coefficient
Da	Damkohler number
D_c	Combined period-averaged diffusion coefficient
D_l	Longitudinal dispersion coefficient
D_t	Eddy diffusivity
D_{eff}	Measured effective dispersion coefficient
D_{lam}	Predicted dispersion coefficient based on laminar flow assumption
D_{tur}	Predicted dispersion coefficient based on turbulent flow assumption
D_{Tw}	Longitudinal dispersion coefficient of EOF

e	Electron charge
E	Electric field
E_0	Amplitude of electric field
h	Half of channel height
k	Reciprocal of Debye length
k_b	Equivalent Nikuradse roughness
k_w	Wave number
K	Taylor dispersion coefficient of steady flow
l_p	Length of pipe
l_{pc}	Length of piston chamber
l_{ra}	Length of rotating arm
N	Rotating speed
p	Pressure
P	Amplitude of pressure gradient
r	Radial direction
r_0	Roughness of pipe wall
R	Modulation parameter
R_a	Retardation factor
R_B	Boltzmann constant
Re_δ	Reynolds number ($\text{Re}_\delta = U\delta/\nu$)
Sc	Schmidt number ($Sc = \nu/D$)
t	Time
T	Period of oscillation

T_a	Absolute temperature
u	Axial velocity
$\overline{u_f}$	Friction velocity
U	Amplitude of cross-sectional mean velocity
U_s	Cross-sectional mean velocity of steady flow
U_{wall}	Amplitude of wall velocity
U_{HS}	Helmholtz-Smoluchowski velocity
V	Stroke volume
W_t	Particle displacement due to Brownian motion
x	Axial direction
X_t	Particle trajectory in waves
y	Vertical direction
y_0	Roughness of channel wall
z	Valence of the co- and counter- ions in the carrier liquid

Greek symbols

σ	Diffusion constant due to Brownian motion
σ_c	Critical value of diffusion constant
ε	A small parameter equals to $k_w A_w / \omega$
ω	Angular frequency
ϖ	Permittivity of liquid
κ	von Karman's constant

δ	Stokes boundary layer thickness ($\delta = \sqrt{2\nu/\omega}$)
δ_I	Thickness of the inner boundary layer
α	Control parameter for dispersion in turbulent oscillatory flows
ϕ	Reciprocal of time-averaged turbulent Schmidt number
ϕ_t	Reciprocal of turbulent Schmidt number
θ	Womersley number ($\theta = a\sqrt{\omega/\nu}$)
ξ	Partition coefficient
χ	Reaction rate constant
ρ	Density
ρ_e	Electric charge density
ψ	Electric potential
ν	Kinematic viscosity
ν_T	Eddy viscosity
μ	Dynamic viscosity
τ	Shear stress
τ_b	Wall shear stress

Note: parameters with prime are non-dimensional.

Chapter 1 Introduction

1.1 General introduction

The dispersion of soluble matter in fluids is a common phenomenon. It is a key consideration in numerous practical problems in the area of environmental and water resources engineering, such as pollutant transport in surface water, discharge of wastewater and brine, chemical species mixing and separation in the laboratory, and so on. In static liquids, matter naturally diffuses due to the Brownian motion (i.e. molecular diffusion), while in moving liquids, matter experiences the Brownian motion and the movements induced by the carrier simultaneously. For unidirectional flows in channels and pipes, the longitudinal dispersion process is more complicated as the flow velocity varies over the cross section, which is known as the Taylor dispersion (Taylor, 1953).

Comparing to steady flows, research on the dispersion in oscillatory flows are relatively rare. Some classical studies were conducted on the dispersion in laminar oscillatory flows, e.g. Aris (1960), Chatwin (1975) and Watson (1983), however, very limited research has been performed on the dispersion in turbulent oscillatory flows. The present study contributes to an in-depth understanding of longitudinal dispersion of turbulent oscillatory flows in a circular pipe, as well as some studies on the pollutant dispersion in the wave environment and inside microfluidic devices.

1.2 Objectives

A key objective of the present study is to quantify the longitudinal dispersion in turbulent oscillatory flows in a circular pipe, which can be applied to analyze the mixing in isobaric pressure exchangers in seawater reverse osmosis (SWRO) processes. For large scale SWRO plants, the energy cost could represent up to 70% of the final costs of the water product (Stover, 2004). Isobaric pressure exchanger is one type of the most updated energy recovery devices for SWRO. The core component of the device is a rotor with a certain number of symmetric ducts. During the operation, the duct fills/empties as the rotor turns and inlets to the different streams open/close. In the device, the pressure energy from the membrane reject (brine) is transferred directly to the feedwater based on the positive displacement principle, and the two streams with different salinities move forward and backward periodically inside an isobaric duct. In the isobaric pressure exchanger, the efficiency, defined by the pressure transfer ratio, can be up to 97% as there is no energy transformation occurs in it. This efficiency advantage makes it possible to dramatically reduce the energy cost of SWRO by as much as 60% compared to those without energy recovery (Stover, 2007). However, mixing, which could increase the energy cost, occurs spontaneously in pressure exchanger as there is no barrier between the two streams. The flow in the pressure exchanger is turbulent under most operating conditions, hence the knowledge on the mass dispersion in the turbulent oscillating flow provides the theoretical foundation to quantify the mixing inside the pressure exchanger. In this report, an experimental setup in which oscillatory turbulent flows can be generated in a single pipe is built. Particle image velocimetry (PIV) and planar laser induced fluorescence (PLIF) techniques are employed to quantitatively measure the dispersion coefficients under different operating conditions. In parallel, an analytical analysis based on a co-axial eddy

viscosity model and using the homogenization approach is performed to examine this problem as well. The velocity profiles of the oscillatory flows are predicted by the analysis, and parameters that affect the dispersion process are compared and discussed in details.

Using a similar approach as above, the longitudinal dispersion of turbulent oscillatory flows in a two-dimensional channel is investigated analytically. Different from the pressure-driven oscillatory flows, the flow is induced by the periodic moving walls in this case. The dispersion coefficients under both synchronously oscillating walls and single oscillating wall are discussed in details.

The scope in this study also includes the study of mass dispersion in a wave environment. The stochastic diffusion of pollutants carried by the wavy fluid is studied theoretically. The combined effects of Brownian motion and wave motion are examined by deducing a modulation parameter to represent the period-averaged diffusion coefficient in both monochromatic progressive waves and standing waves. It is shown that there is a critical value above which the diffusion is weakened, but under which the diffusion is enhanced. Numerical simulation is applied to verify the predictions, and a satisfactory agreement is obtained.

The pollutant dispersion in oscillatory flows with micro- and nano-scales, i.e. electro-osmotic flow (EOF), is also examined in this study. Different from the above macro-scale flows, EOF is driven by the electric field in micro-channels. Furthermore, chemical reactions between the pollutants and walls may exist to affect the dispersion process. The effects of various parameters on the dispersion coefficient of oscillatory EOFs are investigated comprehensively. They provide the theoretical foundation on the mixing and separation of species which may be useful in ground water pollution.

1.3 Outline

The structure of this report is as follow. Chapter 2 gives an extensive review of the literature relevant to this research study. It includes studies on the dispersion in various oscillatory flows, i.e. wave environment, pipe and channel flows, and micro- and nano-flows. Chapter 3 presents the theoretical analysis on the one-dimensional stochastic diffusion in wave motion. The derivation process is discussed in detail, and specific results are given. Chapter 4 presents the major topic of this study, i.e. the longitudinal dispersion of turbulent oscillatory flows in a circular pipe. The velocity profiles based on the co-axial eddy viscosity model are first obtained, followed by the theoretical predictions on the dispersion coefficient using the homogenization approach. Subsequently, the experimental setup and procedures are presented in details, and the experimental results are discussed and compared with the theoretical predictions. Based on the results, suggestions on how to reduce the mixing in isobaric pressure exchangers are given. Chapter 5 focuses on the analysis of the dispersion in a two-dimensional channel with moving walls, which possesses some similarities with the analytical analysis in Chapter 4. Chapter 6 discusses the dispersion in oscillatory EOFs with reversible sorption on the walls. The effects of various parameters on the dispersion coefficient are presented in details. Finally, Chapter 7 summarizes the findings in this thesis, and provides suggestions for future research.

Chapter 2 Literature review

The dispersion of soluble matter in steady laminar pipe flows was first studied by Taylor (1953). He showed that the longitudinal dispersion in the steady pipe flow is much faster than that in static fluid due to the variation of the flow velocity over the cross-section. A specific formula for the dispersion coefficient was obtained through mathematical analysis, $K = \frac{a^2 U_s^2}{192 D}$, where K is the Taylor dispersion coefficient, U_s is the cross-sectional average velocity of the steady flow, a is the pipe radius and D is the molecular diffusivity. Later, he extended his work to dispersion in steady turbulent flows through a pipe, and pointed out that $K = 10.1 a v_*$, where v_* is the ‘friction velocity’ (Taylor, 1954). Since the pioneering work of Taylor, many research studies had been conducted on the longitudinal dispersion in steady flows. A good summary can be found in Fischer et al. (1979). Compared to the extensive work on steady flows, the understanding on the longitudinal dispersion in unsteady flows is relatively undeveloped. In this chapter, previous studies on the dispersion in several different kinds of oscillatory flows, including surface waves, oscillatory pipe/channel flows, and microflows, are reviewed.

2.1 Dispersion in coastal environments

Due to the development of industries and population expansion, the concern on the pollution of natural water body has become more and more serious in the past few decades. Smith (1981) investigated the contaminant dispersion in estuaries. He showed

that there may be a time span in which the contaminant cloud is contracting, i.e. the longitudinal dispersion coefficient is negative, if the time scale for cross-sectional mixing is equal or larger than the flow period. Csanady (1983) analyzed the long-term dispersion of persistent contaminants in the ocean with tidal, wind-driven and other variable currents in a coastal case study. His results showed that the concentration plume widens with time compared with the source dimensions. Yasuda (1984) showed that longitudinal dispersion occurs for both steady and oscillatory shear currents. The dispersion processes from the initial to stationary stages were analytically analyzed. West and Mangat (1986) compared two existing semi-empirical formulae for dispersion coefficients in estuaries with field data collected from the Conwy estuary. The field data showed that both vertical and transverse shear stresses contribute to the dispersion. Farmer and Li (1994) investigated the dispersion of oil in the ocean, which can be broken up into many small droplets. They found that the joint effect of circulation and turbulence can homogenize the oil concentration. Mei et al. (1998) examined the transport and resuspension of fine particles in an oscillatory tidal boundary near a small peninsula. A general expression of shear-induced dispersion coefficient was derived in terms of the ambient flow conditions. Law (2000) demonstrated that longitudinal dispersion would also occur under surface waves due to the depth-wise variation of the induced wave-drift profile. The dispersion coefficient was found to be dependent upon a wave-dispersion parameter, which is related to the wave height and wave period. Ng and Wu (2007, 2008) examined the dispersion of suspended particles in the wave boundary layers over a mud bed. They showed that the dispersion coefficient is a non-monotonic function of the thickness of the mud layer due to the rheology of the mud. Recently, Huang and Law (2011) examined the longitudinal dispersion induced by Stokes drift under random surface waves. The results showed that the longitudinal

dispersion induced by random waves is significantly stronger than an equivalent regular sea state with a similar wave energy density.

2.2 Longitudinal dispersion in oscillatory pipe/channel flows

2.2.1 Laminar oscillatory pipe/channel flows

In a laminar oscillatory pipe flow, the passive solute particles undergo simultaneous molecular diffusion and convective motion that is driven by a periodic pressure gradient. Aris (1960) studied the dispersion in pulsating flows by tracing the development of the moments of solute distribution in an infinitely long pipe. He found that the dispersion coefficient contains terms that are proportional to the square of the harmonics' amplitude. However, the effects contribute little to the total dispersion if the amplitude of fluctuations is small compared to the mean pressure gradient. Chatwin (1975) examined this question by means of statistical arguments which are similar to those used by Taylor, and obtained a contrary conclusion. He showed that the harmonic terms in the effective longitudinal diffusion coefficient may have a noticeable effect on the dispersion, and the effect depends on both the flow frequency and Schmidt number. Similarly, Watson (1983) deduced an expression for the effective diffusivity in another form. In his paper, two special cases were studied in detail, and a table showing the coefficient under different values of frequency and Schmidt number in circular pipe flow was presented. A successful experiment was performed by Joshi et al. (1983), who determined the effective diffusion coefficient for the axial transport of a gas flow in a circular pipe with laminar oscillatory flows. The results were compared with the predictions of Watson (1983) and found to be in good agreement. Some other classical

works in this area include Harris and Goren (1967), Mukherjee and Mazumder (1988) and Farrell and Larson (1973).

In recent years, Pizzuto (1987) developed a numerical model for the dispersion of dyed tracers in an oscillatory flow based on a grid of cells. The model was equivalent to a finite-difference approximation of the diffusion equation. Pedley and Kamm (1988) found that for the axial mass transport in an oscillatory pipe flow with steady secondary flow, a maximum transport rate can be achieved when the secondary-flow time equals the oscillation period. Elad et al. (1992) derived a mathematical theory for the dispersion in a fully developed volume-cycled oscillatory pipe flow. The effects of various flow parameters, e.g. tidal volume, were examined. Gaver et al. (1992) then studied the gas dispersion in a volume-cycled pipe flow through tracer bolus experiments, which corresponded to the theoretical prediction of Elad et al. (1992) and showed good agreement. Hazra et al. (1996) analyzed the solute dispersion in a channel during a periodic oscillating flow, and pointed out that the longitudinal dispersion due to molecular diffusion and non-uniform cross-section velocity can be influenced by the frequency and amplitude of the flow. Bandyopadhyay and Mazumder (1999) investigated the streamwise dispersion by using the Aris-Barton method of moments, and obtained the coefficient for three different cases of steady, periodic and combined effect of steady and periodic flows. The results revealed that the dispersion coefficient can reach a stationary state after a certain time. Experiments in an oscillatory flow inside a continuous chemical reactor were conducted by Palma and Giudici (2003). The effects of various properties of the pulsation, such as frequency, flowrate and amplitude, on mixing were investigated. It was shown that the axial dispersion coefficient increases with the amplitude and frequency of pulsation. Jansons (2006) developed an

alternative way of calculating the Taylor dispersion coefficient which is more suited with computer algebra packages to get exact expressions.

Developments in this area have also been made considering more complicated situations. They include the dispersion in pipes with reactive boundaries (Mazumder and Das, 1992; Ng, 2006; Paul and Mazumder, 2011 and Mazumder and Paul, 2012a), pipes with conductive walls (Jiang and Grotberg, 1993), curved pipes (Jiang and Grotberg, 1996; Sharp et al., 1991; Eckmann, 1998 and Jayaraman et al., 1998), baffled pipes (Howes and Mackley, 1990; Ni, 1995 and Ni et al., 2002) and grooved pipes (Sobey, 1985 and Ye and Shimizu, 2001).

2.2.2 Turbulent oscillatory pipe/channel flows

Contrary to the achievement in laminar flows described in the preview section, the dispersion in turbulent oscillatory pipe/channel flows is rarely reported in the literature due to the complexity of turbulence. Lee (1984) performed experiments to qualitatively visualize the mixing of smoke inside a turbulent oscillatory pipe flow. The results suggested that the turbulence affects the radial mixing considerably as the Reynolds number increases. Ott and Mann (2000) used the particle tracking technique to investigate the turbulent diffusion of particle pairs in a three-dimensional turbulent oscillatory flow. Mondal and Mazumder (2008) examined the stream-wise dispersion of fine suspended particles in a turbulent oscillatory flow from an elevated source. It was shown that the iso-concentration lines can be affected by the settling velocity, oscillation frequency and velocity magnitude, and source height. Ye and Zhang (2002) examined the effect of turbulence on the Taylor dispersion for oscillatory pipe flows using laser techniques. They found that the longitudinal dispersion can be significantly

enhanced by the turbulence by orders of magnitude. Ng (2004) and Mazumder and Paul (2012b) investigated the longitudinal dispersion in a two-dimensional channel and the dispersion of settling particles with effect of deposition and re-entrainment, respectively, by applying a time-varying eddy viscosity model which was proposed by Trowbridge and Madsen (1984a). This model is applied in the present study as well and will be discussed in details in Chapter 4.

2.2.3 Relevance to isobaric pressure exchanger

Water is critical for human survival. Since most freshwater sources such as rivers and lakes are already being excessively exploited, water resources from alternative sources such as the seas and oceans, through desalination, have received increasing attention in recent time. Seawater reverse osmosis (SWRO) has been developed rapidly in the past few decades due to the improvement of membrane technology as well as the introduction of energy recovery devices which significantly reduce the energy cost in SWRO. Figure 2.1 displays a general overview of the SWRO processes.

In SWRO, feedwater or raw seawater is pressurised to about 50-70 bars by the high pressure pumps, and then introduced into a membrane system where the water and salts can be separated. This application of high pressure forces the flow of water through the membrane which does not allow the larger molecules of salts to pass through. The resulting products from this system are the desalted product water and highly pressurised concentrated brine stream. The role of energy recovery device is to recover the pressure energy from the brine stream, which can then be redirected to pressurise the feedwater stream.

Figure 2.1 Schematic layout of SWRO process

The isobatic pressure exchanger is one type of energy recovery devices. The basic operating principle of the pressure exchanger is that the pressure energy is transferred directly from the brine to feedwater inside an isobaric chamber based on the positive displacement principle. Figure 2.2 illustrates the working principles. The core component of the device is a rotor with a certain number of symmetric ducts. During the operation, the duct fills/empties as the rotor turns and inlets to the different streams open/close. According to the working principle, oscillatory flow is generated inside each duct during the operation. Moreover, this device works without a physical barrier. Therefore, the two streams are in direct contact. Mixing of the two streams with different concentrations happens inevitably and leads to a portion of the brine going back to the feed side.

Figure 2.2 Working principle of rotary isobaric pressure exchanger (Zhou et al., 2009)

In practice, the magnitude of mixing can be as much as 6-7% (Stover, 2004). Higher pressure has to be provided to compensate for this mixing, which means more energy demand. Thus, the minimization of mixing, which can reduce the energy consumption and lower the cost of SWRO, is significant. A few studies have been conducted to analyze the mixing in the pressure exchanger. Mei et al. (2012) analytically investigated the dispersion in the isobaric pressure exchanger by assuming that the flow is laminar. By examining the related configuration of a pressure exchanger for sea water desalination, Mei et al. (2012) showed that the mixing could be reduced by enlarging the duct length, decreasing both the velocity and the rotating frequency as well as increasing the duct radius. However, the predicted long term mixing inside the exchangers was much smaller compared to the observation in real applications. Zhou et al. (2009) proposed a two-dimensional axisymmetric CFD model with $k-\epsilon$ turbulence closure to simulate the mixing inside an isobaric pressure exchanger due to oscillatory pipe flows inside the exchanger ducts. Their results showed that steady mixing can be established in the duct flows within a short time. Recently, Liu et al. (2012) extended to a three-dimensional numerical model with RNG $k-\epsilon$ turbulence closure to analyze the

effect of flow velocity and angular frequency on the mixing inside the pressure exchanger. Quantitative results on the mixing coefficient were also obtained. Although both numerical studies were performed carefully, the analysis was performed only for the specific geometry involved, and it was also unclear whether the k - ε closure approximation had affected the quantitative mixing assessment due to turbulence.

2.3 Dispersion in electro-osmotic flows

In Section 2.2, the dispersion in pressure-driven oscillatory flows in the macro-scale has been reviewed comprehensively. In nano- and micro-scales applications, the use of electrokinetic method, i.e. electroosmosis, is becoming more common as it offers the ability to drive the fluid by external means without moving parts.

An electro-osmotic flow (EOF) refers to the movement of fluid induced by an applied electric potential. Due to the electric double layer (EDL) effect formed at the contact interface of an electrolyte and a solid surface, the ions within the EDL migrate under the external electric field and this leads to the movement of the adjacent fluid by virtue of viscous momentum transfer, as illustrated in Figure 2.3. In EOF, EDL is formed due to interaction of an ionized solution with solid surfaces which possess electrostatic charges. The counter-ions in the liquid are attracted while the co-ions are repelled by the surfaces. The counter-ions clustered near the interface are known as the Stern layer. Beyond the Stern layer, there is a diffuse layer where the counter-ions are relatively free to move. The EDL is the union of both the Stern and diffuse layers. The thickness of the EDL is denoted by the Debye length, which is the distance from the charged solid interface to a point where the electrokinetic potential energy equals the thermal energy. For an EOF with a thin EDL, weaker dispersion could be generated compared

to a pressure-driven flow due to the fact that the flow is nearly a plug flow. However, the dispersion in EOF may become significant when the EDL is not thin and a strong electric field is applied (Paul and Ng, 2012).

Figure 2.3 Electro-osmotic flow

Recently, a dramatic increase in studies focused on the analysis of electro-osmotic flow (EOF) and the associated dispersion phenomena, which is applied as a mean to control the fluid transport, mixing and separation, has been seen. For example, Griffiths and Nilson (1999) determined the dispersion coefficient of a neutral non-reacting solute in an incompressible EOF by solving the time-dependent diffusion-advection equation in transformed spatial and temporal coordinates to obtain the two-dimensional late-time concentration field for both circular pipes and plane parallel channels. The dispersion coefficient was found to be proportional to the square of the Peclet number based on the characteristic transverse dimension of the pipe or channel. Zholkovskij et al. (2003) analyzed the dispersion of a nonelectrolyte solute in EOFs with a thin double layer by

applying an aversion of the Aris-Taylor procedure. It was shown that the cross-sectional geometry and the electrolyte content of the driven solution could substantially affect the dispersion for certain surface potential and Debye length. Further study was conducted by Zholkovskij and Masliyah (2004) who examined the combined pressure-driven and electro-osmotic flows. An expression was derived to address the dispersion coefficient for an arbitrary electrokinetic potential, electrolyte type and cross-section geometry by using the same approach as Zholkovskij et al. (2003). Dutta (2007) and Dutta (2008) investigated the electro-osmotic dispersion of neutral and charged samples in a rectangular micro-channel with small zeta potentials, respectively. Datta and Ghosal (2008) considered the effect of wall interactions on the dispersion in EOFs. Studies on the dispersion in steady EOFs also included EOFs with low wall potentials (McEldoon and Datta, 1992), high wall potentials (Andreev and Lisin, 1992; Gas et al., 1995 and Griffiths and Nilson, 2000) and so forth.

Compared to steady flows, studies on the dispersion in time periodic EOFs driven by alternating currents (AC-driven EOFs), which can be applied to the separation of colloids or trapping of particles in designated regions, are relatively rare. Huang and Lai (2006) analyzed the mass transport driven by oscillatory EOFs in a two-dimensional micro-channel. The results demonstrated that species separation could be well achieved by proper choices of Debye length, oscillation frequency and tidal displacement. Wang and Wu (2010) examined the pressure-driven oscillatory flows in micro-channels with slippery walls and electro-viscous effect. Kuo et al. (2008) studied a directional EOF due to a nonlinear interaction between oscillatory axial electrical fields and oscillatory wall potential. Paul and Ng (2012) further investigated the dispersion behavior in such kind of flows. They found that the phase of the wall potential plays an important role in determining the dispersion coefficient magnitude. The minimum and

maximum dispersion coefficients can be achieved when the phase difference between the two wall potentials is zero and π , respectively.

Wall reaction also plays an important role on the dispersion in micro-scale channels. Misra and Chandra (2013) studied the oscillatory EOFs in a porous micro-channel. Ghosal (2002) analyzed the effect of sorption on EOFs, and pointed out that the sorption of charged species at the wall could influence the zeta potential in a non-uniform manner (Ghosal, 2003). Recently, Ramon et al. (2011) investigated the dispersion in oscillatory EOFs in a cylindrical micro-pipe with reactive walls by applying an approach similar to Ng (2006). It was shown that the presence of a wall reaction could enhance the dispersion process as it could temporarily ‘store’ the mass.

2.4 Summary

Based on the review of previous studies, one finds that the dispersion in oscillatory flows is a fundamental problem with a wide range of applications from environmental to industrial to biomedical. The topic has also attracted many researchers’ interests. However, a number of problems remain to be solved, especially in the area of turbulent oscillatory flows. Although a numerical method may be a feasible way for the analysis in turbulent flows, the results are often limited to specific geometries and flow conditions, and the accuracy of the numerical results is still unclear.

In this thesis, a few specific problems of dispersion in oscillatory flows with different applications are examined in details. They are described in the following chapters.

Chapter 3 One-dimensional stochastic diffusion

in wave motion*

3.1 Introduction

Passive pollutants discharged into the ocean flow with the carrier fluid while diffusing at the same time. For transportation in a steady flow field, the analysis is normally based on the advection–dispersion equation with the mean velocity transporting the pollutants downstream, while the molecular and/or turbulent diffusion act to reduce the concentration peak and to enlarge the Gaussian width. Compared to this, the studies on dispersion in an oscillating flow field are not developed as well. Generally, most of the previous studies focused on conditions whereby the advective and diffusive displacements of the pollutant can be decoupled as reviewed in Section 2.1.

Actually, an Eulerian wave-induced oscillatory flow field can influence the diffusive behavior depending on the wave characteristics. This situation is rarely addressed. One exception is Jansons (2006) who explored an alternative method to calculate the Taylor dispersivity in oscillatory flows and obtained exact expressions for some specific cases. In this chapter, the stochastic diffusion of passive substances in a progressive and standing wave field, with the focus on the nonlinear interactions between the stochastic diffusive behavior and the deterministic oscillatory advective motion, is examined. The scope is limited to the situation that a small parameter ε exists between the advective and the diffusive motion, which allows a perturbation analysis to be performed.

*This chapter is written based on “Modulation of stochastic diffusion by wave motion”, *Probabilistic Engineering Mechanics* (2011) and “Stochastic diffusion by progressive waves in turbulence”, *Journal of Hydrodynamics* (2010).

3.2 Analytical methods

The governing differential equation describing the particle involving simultaneous advective and diffusive motion can be written as (Jansons and Lythe, 1998),

$$dX_t = f(X_t, t)dt + dW_t \quad (3.1)$$

where X_t represents the particle trajectory, f is the orbital velocity in an oscillatory flow, t is time, W_t is the Brownian motion with $dW_t \sim N(0, \sigma^2 dt)$, i.e., Gaussian distribution with zero mean and variance $\sigma^2 dt$, and σ is a diffusion constant. Equation (3.1) is based on the assumption that the particles follow passively the orbital velocities of the wave motion without any lag in phase or reduction in amplitude.

To facilitate the analysis, we define the non-dimensional parameters as follows:

$$X_t \rightarrow \frac{X_t}{k_w}; t \rightarrow \frac{t}{\omega}; \sigma \rightarrow \frac{\sigma\sqrt{\omega}}{k_w}; f \rightarrow fA_w \quad (3.2)$$

where k_w is the wave number, ω is the angular frequency, and A_w is the amplitude of the wave-induced velocity.

The non-dimensional equation becomes:

$$dX_t = \varepsilon f(X_t, t)dt + dW_t \quad (3.3)$$

where $\varepsilon = k_w A_w / \omega$, and $dW_t \sim N(0, \sigma^2 dt)$.

Consider a perturbation series of X_t for the small parameter ε :

$$X_t = X_t^{(0)} + \varepsilon X_t^{(1)} + \varepsilon^2 X_t^{(2)} + O(\varepsilon^3) \quad (3.4)$$

Jansons and Lythe (1998) showed that the zero-th order motion is caused directly by the Brownian motion, and the first order motion is driven by the oscillatory flow, i.e.:

$$X_t^{(0)} = W_t \text{ and } X_t^{(1)} = \int_0^t f(W_q, q) dq \quad (3.5)$$

We now proceed to analyze the diffusion behavior of the passive substance. Writing (3.3) in an incremental form as

$$X_{t+\Delta t} = X_t + \varepsilon f(X_t, t) \Delta t + \Delta W_t \quad (3.6)$$

where $\Delta W_t = W_{t+\Delta t} - W_t$, taking variance on both sides and noting that ΔW_t is independent of the other two terms on the RHS, one obtains the diffusion behavior after rearranging and taking $\Delta t \rightarrow 0$ as:

$$\frac{d\text{Var}X_t}{dt} = 2\varepsilon \text{cov}(X_t, f(X_t, t)) + \sigma^2 \quad (3.7)$$

By definition,

$$\text{cov}(X_t, f(X_t, t)) = \langle X_t f(X_t, t) \rangle - \langle X_t \rangle \langle f(X_t, t) \rangle \quad (3.8)$$

where the angular brackets $\langle \rangle$ denote the ensemble average or mathematical expectation. Taking ensemble average of the integrated form of (3.3) and noting that $\langle dW_t \rangle = 0$, one obtains:

$$\langle X_t \rangle = \varepsilon \int_0^t \langle f(X_q, q) \rangle dq \quad (3.9)$$

The leading term of the ensemble average $\langle X_t \rangle$ is the stochastic Stokes drift, which is $O(\varepsilon^2)$. Equation (3.9) thus implies that $\langle f(X_t, t) \rangle$ must be $O(\varepsilon)$, and hence the second term in (3.8) is $O(\varepsilon^3)$, therefore to the leading order for large t , one obtains,

$$\text{cov}(X_t, f(X_t, t)) \sim \langle X_t f(X_t, t) \rangle \quad (3.10)$$

Using a Taylor series to expand $f(\cdot, t)$ as

$$f(X_t, t) = f(X_t^{(0)}, t) + \varepsilon X_t^{(1)} \hat{f}(X_t^{(0)}, t) + \varepsilon^2 \left[\frac{(X_t^{(1)})^2}{2} \hat{\hat{f}}(X_t^{(0)}, t) + X_t^{(2)} \hat{f}(X_t^{(0)}, t) \right] + O(\varepsilon^3) \quad (3.11)$$

where an angle $\wedge$ on f denotes differentiation with respect to its first (position) argument, one obtains

$$\begin{aligned} X_t f(X_t, t) &= X_t^{(0)} f(X_t^{(0)}, t) + \varepsilon \left[X_t^{(1)} f(X_t^{(0)}, t) + X_t^{(0)} X_t^{(1)} \hat{f}(X_t^{(0)}, t) \right] \\ &+ \varepsilon^2 \left[X_t^{(2)} f(X_t^{(0)}, t) + (X_t^{(1)})^2 \hat{\hat{f}}(X_t^{(0)}, t) + \frac{X_t^{(0)} (X_t^{(1)})^2}{2} \hat{\hat{f}}(X_t^{(0)}, t) + X_t^{(0)} X_t^{(2)} \hat{f}(X_t^{(0)}, t) \right] + O(\varepsilon^3) \end{aligned} \quad (3.12)$$

Thus, taking the ensemble average on both sides, we finally find

$$\langle X_t f(X_t, t) \rangle = \langle X_t^{(0)} f(X_t^{(0)}, t) \rangle + \varepsilon \left[\langle X_t^{(1)} f(X_t^{(0)}, t) \rangle + \langle X_t^{(0)} X_t^{(1)} \hat{f}(X_t^{(0)}, t) \rangle \right] + O(\varepsilon^2) \quad (3.13)$$

Two specific cases, i.e. monochromatic progressive waves and standing waves, will be examined based on the general solution we obtain here.

3.2.1 Monochromatic progressive waves

For the case in which the Eulerian flow field is a monochromatic progressive wave with the motion $f(X, t) = \cos(X - t)$, the following standard results are frequently used in deriving the ensemble average, namely, for a Gaussian random variable Y with zero mean and variance σ^2 ,

$$\langle \sin(Y - t) \rangle = -\exp(-\frac{1}{2}\sigma^2) \sin t; \langle \cos(Y - t) \rangle = \exp(-\frac{1}{2}\sigma^2) \cos t \quad (3.14)$$

$$\langle Y \sin(Y - t) \rangle = \sigma^2 \exp(-\frac{1}{2}\sigma^2) \cos t \quad (3.15)$$

$$\langle Y \cos(Y - t) \rangle = \sigma^2 \exp(-\frac{1}{2}\sigma^2) \sin t \quad (3.16)$$

With these results, we now proceed to evaluate the three items in (3.13). Firstly, using (3.5) and (3.16),

$$\langle X_t^{(0)} f(X_t^{(0)}, t) \rangle = \sigma^2 t \exp(-\frac{1}{2}\sigma^2 t) \sin t \quad (3.17)$$

which vanishes as $t \rightarrow \infty$. Note that the decay is exponentially fast at a rate of $\sigma^2/2$, which is generally true for other terms that will be discussed later. The implication is that the large time behavior may be assumed to be dominant when t is greater than some multiple of σ^2 . Next,

$$\begin{aligned} \langle X_t^{(1)} f(X_t^{(0)}, t) \rangle &= \int_0^t \langle \cos(W_q - q) \cos(W_t - t) \rangle dq \\ &= \frac{1}{2} \int_0^t \langle \cos[(W_t - W_q) - (t - q)] + \cos[(W_t + W_q) - (t + q)] \rangle dq \end{aligned} \quad (3.18)$$

where compound angle formula has been used. Note that $W_t = W_q + \Delta_{t-q}$ where Δ_{t-q} is the increment from q to t . The latter is independent of Δ_{t-q} , and is a Gaussian with zero mean and variance $\sigma^2(t-q)$. Thus $W_t + W_q = 2W_q + \Delta_{t-q}$ is a Gaussian with zero mean, and with a variance of $\sigma^2(3q+t)$ being the sum of $4\sigma^2q$ for $2W_q$ and $\sigma^2(t-q)$ for Δ_{t-q} . It can be shown using (3.14) that the integral of the second ensemble average vanishes as $t \rightarrow \infty$. The first ensemble average in the integral can be obtained using (3.14) and noting that $(W_t - W_q)$ is Gaussian with mean zero and variance $\sigma^2(t-q)$, which yields, after integration and taking limits,

$$\lim_{t \rightarrow 0} \langle X_t^{(1)} f(X_t^{(0)}, t) \rangle = \frac{\sigma^2}{\sigma^4 + 4} \quad (3.19)$$

For the last ensemble average in (3.13),

$$\begin{aligned} \langle X_t^{(0)} X_t^{(1)} \hat{f}(X_t^{(0)}, t) \rangle &= - \int_0^t \langle W_t \cos(W_q - q) \sin(W_t - t) \rangle dq \\ &= - \frac{1}{2} \int_0^t \langle W_t \sin[(W_q + W_t) - (t + q)] \rangle + \langle W_t \sin[(W_t - W_q) - (t - q)] \rangle dq \end{aligned} \quad (3.20)$$

where the compound angle formula has been used. For a similar reason as before, the first integral vanishes as $t \rightarrow \infty$. By writing $W_t = W_q + (W_t - W_q)$, the second integrand can be separated into two parts. The first part is given by $\langle W_q \sin(W_t - W_q) \rangle$, which is zero because W_q and the increment $(W_t - W_q)$ are independent. Thus, (3.20) reduces to

$$\langle X_t^{(0)} X_t^{(1)} \hat{f}(X_t^{(0)}, t) \rangle = - \frac{1}{2} \int_0^t \langle (W_t - W_q) \sin[(W_t - W_q) - (t - q)] \rangle dq \quad (3.21)$$

Using (3.15) and the fact that $(W_t - W_q)$ is a Gaussian with a zero mean and a variance of $\sigma^2(t - q)$, one obtains, after integration and taking limits,

$$\lim_{t \rightarrow 0} \left\langle X_t^{(0)} X_t^{(1)} \hat{f}(X_t^{(0)}, t) \right\rangle = 2\sigma^2 \frac{4 - \sigma^4}{(\sigma^4 + 4)^2} \quad (3.22)$$

Finally, substituting (3.17), (3.19) and (3.22) into (3.13), we have, to the leading order of ε ,

$$\lim_{t \rightarrow 0} \text{cov}(X_t, f(X_t, t)) = \varepsilon \sigma^2 \frac{12 - \sigma^4}{(\sigma^4 + 4)^2} \quad (3.23)$$

And so, using (3.7), one obtains,

$$\lim_{t \rightarrow \infty} \frac{d\text{Var}X_t}{dt} = \sigma^2(1 + R) \quad (3.24)$$

where

$$R = 2\varepsilon^2 \frac{12 - \sigma^4}{(\sigma^4 + 4)^2} \quad (3.25)$$

represents a modulation parameter. Defining a combined period-averaged diffusion

coefficient as $D_c = \frac{1}{2} \lim_{t \rightarrow \infty} \frac{d\text{Var}X_t}{dt}$, (3.24) implies that

$$D_c = D(1 + R) \quad (3.26)$$

where D is the molecular diffusion coefficient.

3.2.2 Standing waves

For the case of standing waves, which can be represented mathematically as the superposition of a left and right moving wave with equal amplitude, $f(X, t) = \cos(X - t) + \cos(X + t)$. Similar to the case of the progressive wave, we make use of the following results:

$$\langle \sin(Y - t) + \sin(Y + t) \rangle = 0 \quad (3.27)$$

$$\langle \cos(Y - t) + \cos(Y + t) \rangle = 2 \exp\left(-\frac{1}{2}\sigma^2\right) \cos t \quad (3.28)$$

$$\langle Y(\sin(Y - t) + \sin(Y + t)) \rangle = 2\sigma^2 \exp\left(-\frac{1}{2}\sigma^2\right) \cos t \quad (3.29)$$

$$\langle Y(\cos(Y - t) + \cos(Y + t)) \rangle = 0 \quad (3.30)$$

Proceeding to the three items in (3.13), we first obtain with (3.5) and (3.30) that,

$$\langle X_t^{(0)} f(X_t^{(0)}, t) \rangle = \langle W_t [\cos(W_t - t) + \cos(W_t + t)] \rangle = 0 \quad (3.31)$$

Next,

$$\begin{aligned} \langle X_t^{(1)} f(X_t^{(0)}, t) \rangle &= \int_0^t \langle [\cos(W_q - q) + \cos(W_q + q)] \times [\cos(W_t - t) + \cos(W_t + t)] \rangle dq \\ &= \int_0^t \exp\left(-\frac{1}{2}\sigma^2(t - q)\right) \cos(t - q) + \exp\left(-\frac{1}{2}\sigma^2(t - q)\right) \cos(t + q) dq \end{aligned} \quad (3.32)$$

which yields after integration and taking limits

$$\lim_{t \rightarrow \infty} \langle X_t^{(1)} f(X_t^{(0)}, t) \rangle = \frac{2\sigma^2}{\sigma^4 + 4} + \lim_{t \rightarrow \infty} \frac{2\sigma^2 \cos 2t + 4 \sin 2t}{\sigma^4 + 4} \quad (3.33)$$

For the last item in (3.13),

$$\begin{aligned} \left\langle X_t^{(0)} X_t^{(1)} \hat{f}(X_t^{(0)}, t) \right\rangle &= - \int_0^t \left\langle W_t [\cos(W_q - q) + \cos(W_q + q)] \times [\sin(W_t - t) + \sin(W_t + t)] \right\rangle dq \\ &= - \int_0^t \sigma^2(t - q) \exp\left(-\frac{1}{2}\sigma^2(t - q)\right) \cos(t - q) + \sigma^2(t - q) \exp\left(-\frac{1}{2}\sigma^2(t - q)\right) \cos(t + q) dq \end{aligned} \quad (3.34)$$

which implies

$$\lim_{t \rightarrow \infty} \left\langle X_t^{(0)} X_t^{(1)} \hat{f}(X_t^{(0)}, t) \right\rangle = 4\sigma^2 \frac{4 - \sigma^4}{(\sigma^4 + 4)^2} + 4\sigma^2 \lim_{t \rightarrow \infty} \frac{4 \cos 2t - 4\sigma^2 \sin 2t - \sigma^4 \cos 2t}{(\sigma^4 + 4)^2} \quad (3.35)$$

Finally, substituting (3.31), (3.33), and (3.35) into (3.13), and using (3.10) and (3.7), we have,

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{d\text{Var}X_t}{dt} &= 4\varepsilon^2 \frac{\sigma^2(12 - \sigma^4)}{(\sigma^4 + 4)^2} + \sigma^2 \\ &+ 2\varepsilon^2 \lim_{t \rightarrow \infty} \frac{-2\sigma^6 \cos 2t + 24\sigma^2 \cos 2t - 12\sigma^4 \sin 2t + 16 \sin 2t}{(\sigma^4 + 4)^2} \end{aligned} \quad (3.36)$$

Taking time average by omitting the latter, the combined diffusion behaviour for the standing wave for large time is

$$D_c = D(1 + 2R) \quad (3.37)$$

3.3 Results and discussion

Equation (3.25) suggests that the progressive wave will either increase or decrease the embedded molecular or turbulent diffusion depending on the non-dimensional σ , i.e.

$$D_c > D \quad \text{if} \quad \sigma < \sigma_c$$

$$D_c < D \quad \text{if} \quad \sigma > \sigma_c$$

where σ_c represents the critical value of $(12)^{1/4} \sim 1.86$ whereby the modulation effect changes from positive to negative. Figure 3.1 shows the variation of the modulation effect, $R\varepsilon^{-2}$, with σ . The demarcation point of σ_c is clearly illustrated. It is noted that the reinforcing effect when σ is small is much stronger than the diminishing effect when σ is large. Equation (3.25) also deduces that $R \rightarrow 0$ as $\sigma \rightarrow \infty$, implying that the deterministic wave effect in very strong Brownian diffusion would be insignificant.

Figure 3.1 Relationship between the modulation coefficient R/ε^2 and σ

Comparing with the monochromatic progressive wave, equation (3.37) shows that for standing waves, the modulation effects of the two opposing propagating waves to the diffusion behavior superimpose and reinforce each other.

Besides the mathematical analysis, numerical simulation based on the random walk approach was applied to verify the predictions as well. It was found that the numerical results converged when the number of realizations exceeds 100 and the time step is below 1. Hence, 10,000 realizations were used and Δt was set to be 0.01 with a total time duration of $t = 200$ in the present study. The MATLAB code is given in Appendix A. Figure 3.2 shows the relationship between ε and R with $\sigma = 0.8, 1.2, 2.0$ and 2.4 . The first two values are below the critical value of 1.86 and thus our theory deduces that the diffusion will be enhanced by the progressive wave, while the last two values would be otherwise. The comparison between analytical predictions by equation (3.25) and numerical results for different values of σ is apparent by comparing the lines and individual symbols, respectively. It is clear that the two results agreed very well.

Figure 3.3 shows the variation of $VarX_t$ with time for a small value of $\varepsilon = 0.2$, and with $\sigma = 0.8, 1.2, 2.0$ and 2.4 . The results are based on numerical simulations with the MATLAB code illustrated in Appendix B. Two curves are shown in the figure, namely in Brownian motion only and with the combination of the monochromatic progressive wave, to illustrate the modulation effect of the deterministic wave motion on the diffusive behavior. Equation (3.25) deduces that the variance will increase faster with time in the presence of the wave motion for $\sigma = 0.8$ and 1.2 while slower for $\sigma = 2.0$ and 2.4 . At the same time, from Figure 3.1, it is expected that the reinforcing effect is much more significant than the diminishing effect. The prediction is consistent with the

results shown in Figure 3.3. The enhancement can be clearly observed with $\sigma = 0.8$ and 1.2, while with $\sigma = 2.0$ and 2.4 the two lines are nearly identical indicating that the wave effect is very small.

Figure 3.2 Comparison between analytical predictions and numerical results. The symbols represent the numerical simulations and the lines represent the analytical predictions for different values of σ

(a) $\sigma = 0.8$

(c) $\sigma = 2.0$

(b) $\sigma = 1.2$

(d) $\sigma = 2.4$

Figure 3.3 Growth of variance with time under various σ for small value of $\varepsilon = 0.2$ by MATLAB simulation

3.4 Conclusions

By mathematically analyzing the motion of the solute involving one-dimensional diffusion and oscillatory motion at the same time, it is confirmed that a progressive wave field can modulate the stochastic diffusion behavior of passive scalar concentrations in the oscillatory flow field, despite the fact that the Eulerian velocities are deterministic and not random. The combined diffusion coefficient can be increased or decreased depending on the wave characteristics, and the critical criteria being σ

smaller or larger than the critical value of $(12)^{1/4}$. Larger wave lengths and shorter wave periods thus tend to promote diffusion, while smaller wave lengths and longer wave periods do otherwise. The reinforcing effect is however much more significant than the diminishing effect. In a standing wave field which can be represented by the superposition of the left and right moving progressive waves of equal amplitude, the results show that the modulation effects to the diffusion behavior due to the two opposing waves are in fact superimposed.

In this study, the wave flow is assumed to satisfy the linear wave theory. Hence, the dispersion induced by the fluid shear by currents and boundaries is not included. It should also be noted that the above analysis is applicable when the parameter ε is smaller than one. For surface gravity waves, the wave length and wave period are interacted, i.e. larger wave length accompanies with longer wave period. Hence, the physical cases of the wave flows that can significantly enhance the diffusion would likely occur in phenomenon whereby the wave length and wave period can be delinked, such as in closed conduit waves.

Chapter 4 Longitudinal dispersion of turbulent oscillatory flows in a circular pipe*

4.1 Introduction

Isobaric pressure exchanger as an energy recovery device has been widely adopted in SWRO plants due to its remarkable efficiency of pressure transfer (Stover, 2007). However, due to the direct contact of the two streams, i.e. brine and feedwater, mixing between them do occur. After a long time operation, the induced mixing increases the salt concentration of the feed side and thus is undesirable.

For a typical pressure exchanger flow, the Stokes boundary Reynolds number (simplified as Reynolds number) Re_δ is around 700 (Mei et al., 2012), where Re_δ is defined by $U\delta/\nu$, U is the amplitude of cross-sectional mean velocity, $\delta = \sqrt{2\nu/\omega}$ is the Stokes layer thickness, ω is the angular frequency of oscillation and ν is the kinematic viscosity. According to the previous research studies on the transition to turbulence of oscillatory pipe flows, the critical Reynolds number for transition can range from 250 to 760 (Hino et al., 1976; Ohmi et al., 1982; Ahn and Ibrahim, 1992 and Akhavan et al., 1991). As a result, the oscillatory flows in the pressure exchangers are likely turbulent. In this chapter, the longitudinal dispersion of turbulent oscillatory flows in a circular pipe is investigated through both analytical and experimental means. The results can provide the foundation to improve the understanding of the mixing mechanisms in the pressure exchangers.

*This chapter is written based on “Longitudinal dispersion of turbulent oscillatory pipe flows”, *Environmental Fluid Mechanics* (2014).

Comparing with laminar oscillatory flows, the longitudinal dispersion in turbulent oscillatory pipe flows is rarely reported in the literature either analytically or experimentally. Unlike axisymmetric pipe flows, the longitudinal dispersion effect is studied more extensively under surface gravity waves for both laminar and turbulent boundary layers at the bottom with a typical two-dimensional planar setting. Trowbridge and Madsen (1984a) proposed a two-layer time-varying eddy viscosity model to study the turbulent wave-induced near-bottom boundary layers, and applied the model to analyze the wave-induced mass transport (Trowbridge and Madsen, 1984b). Although their model is relatively simple compared to the turbulence closures in CFD models, it is able to yield analytical results that match the reported laboratory data in a satisfactory manner. This model has also been adopted by Ng (2004) and Mazumder and Paul (2012b) as mentioned in Section 2.2.2, where good agreements were shown by comparing with numerical simulation results.

In this chapter, we adopt an axisymmetric, co-axial eddy viscosity model that is similar to the two-layer model of Trowbridge and Madsen (1984a), to examine the turbulent oscillatory flows inside the circular pipe in Section 4.2. A perturbation analysis is then performed based on the homogenization technique with multiple scales for the dispersion coefficient. Experiments using the advanced non-invasive image techniques of Particle Image Velocimetry (PIV) and Planar Laser Induced Fluorescence (PLIF) are presented in Section 4.3. The longitudinal dispersion coefficients under different combinations of oscillation frequencies and velocity magnitudes are quantified. The experimental parameters covered a wide range of Reynolds number from 100 to 1000, and included both laminar and turbulent regimes. In Section 4.4, the experimental results are compared with the theoretical predictions and Section 4.5 gives the conclusions.

4.2 Theoretical study

4.2.1 Velocity field

We adopt a cylindrical coordinate for the pipe flow, as shown in Figure 4.1. The x -axis is the axial direction, and r is the radial direction with $r=0$ at the centreline. The axial velocity u varies with time t and radius r , i.e. $u = u(r, t)$, while the pressure p is changing with t and x (i.e. only the axial pressure gradient is considered, which should be valid for small diameters). In this manner, the Navier-Stokes equation in the cylindrical form can be reduced to

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{1}{\rho} \frac{1}{r} \frac{\partial(r\tau)}{\partial r} \quad (4.1)$$

where ρ is the fluid density and τ the turbulent Reynolds stress, which can be related to the velocity gradient by the eddy viscosity ν_T in the form,

$$\tau = -\rho \nu_T \frac{\partial u}{\partial r} \quad (4.2)$$

At the wall boundary, the velocity is assumed to be zero at $r = a - r_0$ due to the no-slip condition, i.e.

$$u = 0 \text{ at } r = a - r_0 \quad (4.3a)$$

where a is the pipe radius. For a rough wall, r_0 represents the wall roughness and is typically taken to be $r_0 = k_b / 30$, where k_b is the equivalent Nikuradse roughness. For a smooth wall, however, r_0 is not the true roughness but rather denotes the embedded viscous sub-layer thickness, and usually is in the order of 10^{-5} m.

Due to axisymmetry, the velocity gradient is zero at the centreline, i.e.

$$\frac{\partial u}{\partial r} = 0 \text{ at } r = 0 \quad (4.3b)$$

Figure 4.1 Schematic diagram of the coordinate axis for analysis

We consider a purely oscillatory flow with a single frequency and without any steady component. In this case, the pressure gradient is then independent of both x and r , and only changes with time, which can be expressed as:

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = P \cos(\omega t) = P \cdot \text{Re}[e^{i\omega t}] \quad (4.4)$$

where P is the amplitude of the pressure gradient, $\text{Re}[\]$ denotes the real part of the complex expression, and $i = \sqrt{-1}$. Equation (4.4) implies that both the velocity and the shear stress are purely sinusoidal with a period of T ($T = 2\pi/\omega$). Since forcing reverses in direction in half a period, these quantities can be expressed by the following Fourier expansions with only odd harmonics of the fundamental frequency,

$$u = \text{Re}\left[u^{(1)}e^{i\omega t} + u^{(3)}e^{3i\omega t} + \dots\right] \quad (4.5)$$

$$\tau = \text{Re}\left[\tau^{(1)}e^{i\omega t} + \tau^{(3)}e^{3i\omega t} + \dots\right] \quad (4.6)$$

where $u^{(1)}, \tau^{(1)}, \dots$ are complex functions of r .

The above forms of equations also imply that ν_T contains only even harmonics of the fundamental frequency according to (4.2), i.e.

$$\nu_T = \nu^{(0)} \cdot \text{Re}\left[1 + a^{(2)}e^{2i\omega t} + \dots\right] \quad (4.7)$$

where $\nu^{(0)}$ is a real function of r representing the time-averaged value of the eddy viscosity, and $a^{(2)}$ is a complex constant representing the time variation of ν_T .

Earlier, Trowbridge and Madsen (1984a) had shown that the time-varying part of the eddy viscosity is small in comparison with the time-averaged part, and the third-harmonic components of velocity and shear stress are smaller than the first-harmonic components, i.e.

$$\left|a^{(2)}\right|, \frac{u^{(3)}}{u^{(1)}}, \frac{\tau^{(3)}}{\tau^{(1)}} = O(\zeta) \quad (4.8)$$

where ζ is a small parameter.

Substituting Equations (4.2), (4.4), (4.5) and (4.7) into (4.1), the governing equations for the Fourier components can then be derived as:

$$u^{(1)}i\omega = P + \frac{1}{r} \frac{d}{dr} \left(r \nu^{(0)} \frac{du^{(1)}}{dr} \right) + \frac{a^{(2)}}{2} \frac{1}{r} \frac{d}{dr} \left(r \nu^{(0)} \frac{du_*^{(1)}}{dr} \right) \quad (4.9)$$

$$3u^{(3)}i\omega = \frac{1}{r} \frac{d}{dr} \left(r \nu^{(0)} \frac{du^{(3)}}{dr} \right) + \frac{a^{(2)}}{2} \frac{1}{r} \frac{d}{dr} \left(r \nu^{(0)} \frac{du^{(1)}}{dr} \right) \quad (4.10)$$

where the asterisk denotes the complex conjugate. The boundary conditions are modified accordingly as follow:

$$u^{(1)} = u^{(3)} = 0 \text{ at } r = a - r_0 \quad (4.11a)$$

$$\frac{du^{(1)}}{dr} = \frac{du^{(3)}}{dr} = 0 \text{ at } r = 0 \quad (4.11b)$$

We now adopt an axisymmetrical co-axial model to represent the time-averaged part of the eddy viscosity. The approach is similar to the two-layer planar model introduced by Trowbridge and Madsen (1984a). In this model, $\nu^{(0)}$ (which is related to the time-averaged wall shear stress) is the product of a local length scale and velocity scale in the near wall region, while becomes a constant in the outer region, i.e.

$$\nu^{(0)} = \kappa \overline{u_f} \begin{cases} (a - r) & a - \delta_I < r \leq a - r_0 \\ \delta_I & 0 \leq r \leq a - \delta_I \end{cases} \quad (4.12)$$

where $\kappa = 0.4$ is von Karman's constant, $\overline{u_f}$ the friction velocity, and δ_I the thickness of the inner boundary layer (which is typically taken as one sixth of $\overline{\kappa u_f} / \omega$). Since the maximum boundary layer thickness inside the circular pipe would be the pipe radius itself, thus the thickness can be expressed as,

$$\delta_I = \frac{1}{6} \min(a, \overline{\kappa u_f} / \omega) \quad (4.13)$$

In order to simplify the analysis, it is more convenient to switch the coordinate from r to s , where $s = a - r$, and normalize all the variables as follows (using a prime to denote non-dimensional variables),

$$(u^{(1)'}, u^{(3)'}, \overline{u_f'}) = (u^{(1)}, u^{(3)}, \overline{u_f})/U, \quad (s', r_0') = (s, r_0)/a, \quad t' = t\omega \quad (4.14)$$

where U is the amplitude of cross-sectional mean velocity defined by $U=P/\omega$. The governing equations in the non-dimensional form can be obtained by substituting (4.12) into (4.9) and (4.10):

$$\frac{\alpha \overline{u_f'}}{1-s'} \frac{d}{ds'} \left((1-s')\beta \frac{du^{(1)'}}{ds'} \right) - iu^{(1)'} = -1 - \frac{a^{(2)}}{2} \frac{\alpha \overline{u_f'}}{1-s'} \frac{d}{ds'} \left((1-s')\beta \frac{du_*^{(1)'}}{ds'} \right) \quad (4.15)$$

$$\frac{\alpha \overline{u_f'}}{1-s'} \frac{d}{ds'} \left((1-s')\beta \frac{du^{(3)'}}{ds'} \right) - 3iu^{(3)'} = -\frac{a^{(2)}}{2} \frac{\alpha \overline{u_f'}}{1-s'} \frac{d}{ds'} \left((1-s')\beta \frac{du^{(1)'}}{ds'} \right) \quad (4.16)$$

with the following boundary conditions:

$$u^{(1)'} = u^{(3)'} = 0 \quad \text{at } s' = r_0' \quad (4.17a)$$

$$\frac{du^{(1)'}}{ds'} = \frac{du^{(3)'}}{ds'} = 0 \quad \text{at } s' = 1 \quad (4.17b)$$

where α is a non-dimensional control parameter in the form of $\alpha = \kappa U / (\omega a)$,

$$\beta = \min(s', \delta_l') \quad \text{and} \quad \delta_l' = \frac{1}{6} \min(1, \alpha \overline{u_f'}).$$

To solve for the governing equations (4.15) and (4.16), we need to introduce the auxiliary solutions $F^{(n)}$, ($n=1,3$), which satisfy the corresponding homogeneous differential equation,

$$\frac{\alpha \overline{u_f'}}{1-s'} \frac{d}{ds'} \left((1-s')\beta \frac{dF^{(n)}}{ds'} \right) - i n F^{(n)} = 0 \quad (4.18)$$

Since β is not analytical at $s' = \delta_l'$, equation (4.18) needs to be solved separately within the inner boundary layer and in the outer region,

Inside the inner boundary layer, $r_0' < s' \leq \delta_I'$,

$$F^{(n)} = x_1 H\left(\left[\varphi^{(n)}, \varphi^{(n)}\right], \left[2\varphi^{(n)}\right], \frac{1}{s'}\right) s'^{-\varphi^{(n)}} + x_2 H\left(\left[\psi^{(n)}, \psi^{(n)}\right], \left[2\psi^{(n)}\right], \frac{1}{s'}\right) s'^{-\psi^{(n)}} \quad (4.19)$$

where H denotes the hypergeometric functions, $\varphi^{(n)} = \frac{1}{2} - \sqrt{\frac{-in}{\alpha u_f'}} + \frac{1}{4}$ and

$$\psi^{(n)} = \frac{1}{2} + \sqrt{\frac{-in}{\alpha u_f'}} + \frac{1}{4}.$$

Outside the inner boundary layer and in the outer region, $\delta_I' \leq s' \leq 1$,

$$F^{(n)} = x_3 J_0(\gamma^{(n)}(s'-1)) + x_4 K_0(-i\gamma^{(n)}(s'-1)) \quad (4.20)$$

where J_0 and K_0 represent the first and second kind Bessel function of order zero,

$$\text{respectively, and } \gamma^{(n)} = \sqrt{\frac{-in}{\alpha u_f' \delta_I'}}.$$

The unknown parameters can be solved by applying the matching requirements for the

velocity and shear stress solutions at $s' = \delta_I'$, i.e. $F^{(n)}(\delta_I'^-) = F^{(n)}(\delta_I'^+)$,

$$\frac{dF^{(n)}(\delta_I'^-)}{ds'} = \frac{dF^{(n)}(\delta_I'^+)}{ds'}, \text{ and the outer-edge boundary condition } \frac{dF^{(n)}(1)}{ds'} = 0. \text{ The}$$

expressions for the unknown parameters are presented in Appendix C.

With the wall boundary condition $u^{(1)'}(r_0') = 0$, the solutions of (4.15) and (4.16) can be

expressed by the auxiliary functions $F^{(n)}$ as,

$$u^{(1)'} = \left(i + \frac{ia^{(2)}}{4}\right) \frac{F^{(1)}}{F^{(1)}(r_0')} + \left(\frac{ia^{(2)}}{4} \frac{F_*^{(1)}}{F_*^{(1)}(r_0')} - i\right) + O(\varepsilon^2) \quad (4.21)$$

$$u^{(3),} = \frac{-ia^{(2)}}{4} \frac{F^{(3)}}{F^{(3)}(r_0')} + \frac{ia^{(2)}}{4} \frac{F^{(1)}}{F^{(1)}(r_0')} + O(\varepsilon^2) \quad (4.22)$$

Expressions for the Fourier components of the shear stress can then be obtained accordingly as:

$$\tau^{(1),} = i\kappa \overline{u_f'} \beta \left[\frac{dF^{(1)}/ds'}{F^{(1)}(r_0')} - \frac{a^{(2)}}{4} \left(\frac{dF_*^{(1)}/ds'}{F_*^{(1)}(r_0')} + \frac{dF^{(1)}/ds'}{F^{(1)}(r_0')} \right) + O(\varepsilon^2) \right] \quad (4.23)$$

$$\tau^{(3),} = \frac{1}{4} i\kappa \overline{u_f'} \beta a^{(2)} \left[3 \frac{dF^{(1)}/ds'}{F^{(1)}(r_0')} - \frac{dF^{(3)}/ds'}{F^{(3)}(r_0')} \right] + O(\varepsilon^2) \quad (4.24)$$

To derive the velocities and shear stresses, the unknown friction velocity $\overline{u_f'}$ and the constant $a^{(2)}$ need to be resolved. Earlier, Trowbridge and Madsen (1984a) had shown that $\overline{u_f'}$ and $\overline{u_f'} a^{(2)}$ can be linked to the instantaneous wall shear stress τ_b' in the following manner,

$$\overline{u_f'} = |\tau_b^{(1)}|^{1/2} \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} \left[1 - \frac{3}{10} \text{Re} \left(\frac{\tau_b^{(3)}, \tau_{b*}^{(1),}}{\tau_b^{(1)}, \tau_b^{(1),}} \right) + O(\varepsilon^2) \right] \quad (4.25)$$

$$\overline{u_f'} a^{(2)} = \frac{1}{\sqrt{\pi}} \frac{2}{5} |\tau_b^{(1)}|^{1/2} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} \frac{\tau_b^{(1),}}{\tau_{b*}^{(1),}} + O(\varepsilon) \quad (4.26)$$

where Γ is the Gamma function. Combining (4.25) and (4.26), we obtain the leading term as,

$$a^{(2)} = \frac{2}{5} \frac{\tau_b^{(1),}}{\tau_{b*}^{(1),}} \quad (4.27)$$

By applying the limit to the wall, $\lim_{s' \rightarrow 0} s' \frac{dF^{(n)}}{ds'}$, to (4.23) and (4.24), we can obtain $\tau_b^{(1)}$, and $\tau_b^{(3)}$, as a function of $\overline{u_f}$, for specific α . Substituting $\tau_b^{(1)}$, and $\tau_b^{(3)}$, into (4.27) and (4.25), $a^{(2)}$ can be computed and $\overline{u_f}$ can be solved iteratively. Based on the solved $\overline{u_f}$ and $a^{(2)}$, the Fourier components of the velocity and shear stress can be obtained by substituting the value of $\overline{u_f}$ and $a^{(2)}$ into (4.21-4.24). Both the velocity profiles and distribution of shear stresses in non-dimensional form can then be fully resolved. The results will be discussed in Section 4.2.3.

4.2.2 Concentration field

After the flow characteristics are determined above, we proceed to analyze the longitudinal dispersion in the turbulent oscillatory pipe flow in this section. We start by considering a finite cloud of mass dissolved homogeneously over the cross-section inside the turbulent oscillatory pipe flow. Driven by advection, the centre of the cloud would then move forward and backward periodically with a zero net displacement. During this process, however, the cloud would expand due to the turbulent dispersion and more importantly, the dispersion induced by the turbulent shear flow. In the following, we shall analyze the longitudinal dispersion based on the homogenization method described in details by Mei and Vernescu (2010).

Since the flow is only in the x direction, the scalar transport in a circular pipe is governed by the convection-diffusion equation in the cylindrical coordinates:

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = D_t \frac{\partial^2 C}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r D_t \frac{\partial C}{\partial r} \right) \quad (4.28)$$

subject to the boundary condition

$$\frac{\partial C}{\partial r} = 0 \text{ at } r = 0, a - r_0 \quad (4.29)$$

where C is the scalar concentration, and D_t is the turbulent dispersivity which is assumed to be isotropic in all directions.

The turbulent dispersivity is typically taken to be related to the eddy viscosity through the turbulent Schmidt number, which is in the order of unity (Law, 2006). Here, we examine only the case with the time-averaged turbulent dispersivity being only proportional to the time-averaged eddy viscosity, $\nu^{(0)}$, i.e.

$$D_t = \phi \nu^{(0)} \quad (4.30)$$

where $\phi = O(1)$ is the reciprocal of time-averaged turbulent Schmidt number.

We consider the situation that the pipe radius is sufficiently small such that the concentration becomes homogeneous over the cross-section after a few oscillations in the turbulent regime, i.e.

$$\frac{2\pi}{\omega} \sim \frac{a^2}{D_t} \quad (4.31)$$

Furthermore, we assume that the longitudinal length scale L is much larger than the pipe radius, which is,

$$\eta = a/L \ll 1 \quad (4.32)$$

where η denotes a parameter much smaller than one. Note that this is the condition whereby the homogenization technique can be strictly applied, although in reality the

results are similar even for larger η in many cases. Based on the assumptions of (4.31) and (4.32), the governing equation (4.28) can be rewritten using η to indicate the order of magnitude of each term,

$$\frac{\partial C}{\partial t} + \eta u \frac{\partial C}{\partial x} = \eta^2 D_t \frac{\partial^2 C}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r D_t \frac{\partial C}{\partial r} \right) \quad (4.33)$$

We now introduce the homogenization approach with the multiple-scale perturbation analysis. Adopting three sharply distinct time scales t , t_1 and t_2 to represent the three different transport processes, i.e. dispersion over the cross-section, advection and dispersion along the longitudinal direction, respectively, where

$$t, t_1 = \eta t, t_2 = \eta^2 t \quad (4.34)$$

In the same manner, the concentration and time derivative can be expanded as:

$$C = C_0 + \eta C_1 + \eta^2 C_2 + \dots \quad (4.35)$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial t_1} + \eta^2 \frac{\partial}{\partial t_2} + \dots \quad (4.36)$$

At $O(1)$, the leading order concentration C_0 has been proven to be independent of r and t (Mei and Vernescu, 2010), i.e.

$$C_0 = C_0(x, t_1, t_2) \quad (4.37)$$

At $O(\eta)$, the concentration C_1 is governed by:

$$\frac{\partial C_1}{\partial t} + \frac{\partial C_0}{\partial t_1} + u \frac{\partial C_0}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r} \left(r D_t \frac{\partial C_1}{\partial r} \right) \quad (4.38)$$

Substituting (4.5) for u and (4.30) for D_t , the above equation becomes:

$$\frac{\partial C_1}{\partial t} + \frac{\partial C_0}{\partial t_1} + \text{Re}\left[u^{(1)} e^{i\omega t}\right] \frac{\partial C_0}{\partial x} = \frac{\phi}{r} \frac{\partial}{\partial r} \left(r v^{(0)} \frac{\partial C_1}{\partial r} \right) \quad (4.39)$$

with the boundary condition:

$$\frac{\partial C_1}{\partial r} = 0 \text{ at } r = 0, a - r_0 \quad (4.40)$$

Taking the time and cross-sectional average of (4.39), C_0 can be found to be independent from t_1 as well, i.e.

$$\frac{\partial C_0}{\partial t_1} = 0 \quad (4.41)$$

Substituting (4.41) into (4.39) yields:

$$\frac{\partial C_1}{\partial t} + \text{Re}\left[u^{(1)} e^{i\omega t}\right] \frac{\partial C_0}{\partial x} = \frac{\phi}{r} \frac{\partial}{\partial r} \left(r v^{(0)} \frac{\partial C_1}{\partial r} \right) \quad (4.42)$$

In view of linearity, the first order concentration C_1 can be expressed as:

$$C_1 = \frac{\partial C_0}{\partial x} \text{Re}\left[B^{(1)} e^{i\omega t}\right] \quad (4.43)$$

where $B^{(1)}$ is a complex function of r . Substituting (4.43) into (4.42), we obtain the governing equation for $B^{(1)}$ as:

$$i\omega B^{(1)} + u^{(1)} = \frac{\phi}{r} \frac{d}{dr} \left(r v^{(0)} \frac{dB^{(1)}}{dr} \right) \quad (4.44)$$

with the boundary condition:

$$\frac{dB^{(1)}}{dr} = 0 \text{ at } r = 0, a - r_0 \quad (4.45)$$

We now proceed to consider $O(\eta^2)$. The governing equation is:

$$\frac{\partial C_0}{\partial t_2} + \frac{\partial C_1}{\partial t_1} + \frac{\partial C_2}{\partial t} + u \frac{\partial C_1}{\partial x} = D_t \frac{\partial^2 C_0}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r D_t \frac{\partial C_2}{\partial r} \right) \quad (4.46)$$

Substituting (4.41-4.43) into (4.46) yields:

$$\frac{\partial C_0}{\partial t_2} + \frac{\partial C_2}{\partial t} + \text{Re}[u^{(1)} e^{i\omega t}] \text{Re}[B^{(1)} e^{i\omega t}] \frac{\partial^2 C_0}{\partial x^2} = D_t \frac{\partial^2 C_0}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r D_t \frac{\partial C_2}{\partial r} \right) \quad (4.47)$$

Taking the time and cross-sectional average of (4.47) and after some rearrangement, we obtain the effective scalar transport equation as:

$$\frac{\partial C_0}{\partial t_2} = \left\{ \langle \langle D_t \rangle \rangle - \frac{1}{2} \text{Re} \langle \langle u^{(1)} B_*^{(1)} \rangle \rangle \right\} \frac{\partial^2 C_0}{\partial x^2} \quad (4.48)$$

where the double angular brackets $\langle \langle \rangle \rangle$ represent the cross-sectional average.

Since the second term on the RHS of (4.48) gives rise to the longitudinal dispersion (Ng, 2004), we define the longitudinal dispersion coefficient in an oscillatory flow with zero time-averaged velocity as,

$$D_l = -\frac{1}{2} \text{Re} \langle \langle u^{(1)} B_*^{(1)} \rangle \rangle \quad (4.49)$$

Solving the function $B^{(1)}$ is the key towards the determination of the magnitude of the longitudinal dispersion coefficient. Recalling the governing equation (4.44) and normalizing all the variables and parameters with the change of the coordinate from r' to s' as before, we have

$$\frac{\phi \alpha \overline{u_f}'}{1-s'} \frac{d}{ds'} \left((1-s') \beta \frac{dB^{(1)'}}{ds'} \right) - iB^{(1)'} = u^{(1)'}, \quad (4.50)$$

subject to the boundary condition:

$$\frac{dB^{(1)'}}{ds'} = 0 \text{ at } s' = 1, r_0' \quad (4.51)$$

where $B^{(1)'} = B^{(1)} \omega / U$.

Substituting the expression for $u^{(1)'}$ into (4.50), the function $B^{(1)'}$ is then solved numerically (the detailed procedure is presented in Appendix D). To be consistent dimensionally with the turbulent dispersivity, the non-dimensional longitudinal dispersion coefficient is expressed in the form following (4.49),

$$D_l' = \frac{D_l}{Ua} = -\frac{\alpha}{\kappa} \frac{1}{(1-r_0')^2} \int_0^{1-r_0'} \text{Re} \left[u^{(1)'} B_*^{(1)'} \right] r' dr' \quad (4.52)$$

where the integral is computed numerically as well.

4.2.3 Theoretical results and discussion

The calculation results based on the above solutions are presented in this section. The non-dimensional parameter r_0' related to r_0 and a , and thus can vary over a large range for various pipe radius. Here, we present the results based on $r_0' = 0.001$ and 0.005 as a bracketing range for discussion. Before we discuss the flow characteristics and dispersion coefficient, the values of $\overline{u_f}'$ and $a^{(2)}$ for various α are presented firstly (see Figure 4.2). Figure 4.2(a) clearly illustrates that $\overline{u_f}'$ decreases

monotonically with α . Also, with small α , $\overline{u_f'}$ with the larger roughness of $r_0'=0.005$ is higher than that of $r_0'=0.001$, however, the difference becomes negligible when α exceeds 100. In Figure 4.2(b), both the real and imaginary parts of $a^{(2)}$ oscillate with α with an amplitude less than 0.4, which generally validates the assumption made in (4.8), although the higher order terms will be relatively more significant with a larger $a^{(2)}$. Figure 4.2(c) shows the inner boundary layer thickness, which is $\delta_l' = \frac{1}{6} \min(1, \alpha \overline{u_f'})$. It increases with α first (δ_l' with $r_0'=0.001$ is smaller than with $r_0'=0.005$) and becomes a constant of 1/6 when α reaches ~ 20 . The change-over point for $r_0'=0.005$ occurs at smaller α than that for $r_0'=0.001$, which suggests that the boundary layer thickness extends to the centreline of the pipe earlier at larger r_0' .

(a)

(b)

(c)

Figure 4.2 Variations of (a) friction velocity $\overline{u_f'}$, (b) complex constant $a^{(2)}$, and (c) inner boundary layer thickness δ_l' with α . The solid curves are for $r_0' = 0.001$ and dashed curves for $r_0' = 0.005$

Based on the above results, we now proceed to present the characteristics of turbulent oscillatory flows in a circular pipe, i.e. the profiles of eddy viscosity, velocity as well as shear stress, as shown in Figures 4.3, 4.4 and 4.5, respectively. In each figure, a comparison is made among $\alpha = 0.5, 5, 50$ and 500 to illustrate the effect of α . The results for $r_0' = 0.001$ and 0.005 are also listed in the figures from (a)-(d) and (e)-(h), respectively, for comparison. Since α is inversely proportional to ω , by keeping U and a constant, we expect a smaller α when the flow oscillates faster. In Figure 4.3, for the first two cases, i.e. $\alpha = 0.5$ and 5 , when the boundary layer thickness is smaller than the pipe radius, it can be observed that the inner boundary layer is thin despite it increases with α ; while for the last two cases, the boundary layer thickness extends fully to the centreline and δ_l' reaches the constant value of $1/6$.

It is obvious from Figure 4.4 that before the boundary layer reaches the pipe centreline, i.e. $\alpha = 0.5$ and 5 , a strong phase lag exists between the flow within the boundary layer (where the velocity decreases towards the wall significantly) and in the outer region (where the flow behaves as a plug flow). This is reasonable as under high frequency oscillations, the velocity inside the boundary layer does not react to the pressure gradient immediately due to inertia. As α increases, the boundary layer extends to the pipe centreline and the flow becomes uniform over the cross-section. It can also be inferred that for all the cases, the velocity gradient under $r_0' = 0.005$ is smaller compared to $r_0' = 0.001$. The same trend can be noted in Figure 4.5. As α increases, the shear stress distribution becomes more linear when the boundary layer thickness reaches the pipe radius.

(a) $\alpha = 0.5$

(e) $\alpha = 0.5$

(b) $\alpha = 5$

(f) $\alpha = 5$

(c) $\alpha = 50$

(g) $\alpha = 50$

(d) $\alpha = 500$

(h) $\alpha = 500$

Figure 4.3 Profiles of eddy viscosity ν'_T at various phases within one period under different α (left: $r'_0 = 0.001$, right: $r'_0 = 0.005$)

(a) $\alpha = 0.5$

(e) $\alpha = 0.5$

(b) $\alpha = 5$

(f) $\alpha = 5$

(c) $\alpha = 50$

(g) $\alpha = 50$

(d) $\alpha = 500$

(h) $\alpha = 500$

Figure 4.4 Profiles of velocity u' at various phases within half a period under different α (left: $r_0' = 0.001$, right: $r_0' = 0.005$)

(a) $\alpha = 0.5$

(e) $\alpha = 0.5$

(b) $\alpha = 5$

(f) $\alpha = 5$

(c) $\alpha = 50$

(g) $\alpha = 50$

(d) $\alpha = 500$

(h) $\alpha = 500$

Figure 4.5 Distribution of shear stress τ' at various phases within half a period under different α (left: $r_0' = 0.001$; right: $r_0' = 0.005$)

The variation of the non-dimensional dispersion coefficient D_l' in turbulent oscillatory pipe flows is illustrated in Figure 4.6 as a function of the control parameter α . A comparison is made between $\phi = 0.6$ and 1, the two values being chosen to approximate the range of ϕ . As before, r_0' is set to be 0.001 and 0.005. Figure 4.6 clearly shows that D_l' increases with α first and then slightly decreases with the peak at $\alpha \sim 20$. It also suggests that: (i) keeping r_0' the same, D_l' is larger with larger ϕ when α is small, however, the situation reverses when α exceeds around 20, which can be attributed to the change-over of δ_l' . Moreover, a larger ϕ tends to have a smaller peak value that appears at a relatively smaller α ; and (ii) keeping ϕ the same, D_l' is larger with larger r_0' when α is small. As α further increases, the two reach almost the same peak value. This is reasonable because the velocity profile is more uniform with larger α , and the significance of the velocity variation near the wall to the longitudinal dispersion reduces.

The behaviour of D_l' can be explained by the cross-sectional velocity variation under different α . When α is small, the velocity variation is limited within the thin boundary layer. Hence, D_l' increases with α as the boundary layer thickness increases with α . However, after α reaching the change-over point, the boundary layer thickness becomes unchanged, further increase in α leads to more uniform velocity profile over the entire cross section and thus reduces the longitudinal dispersion coefficient.

For comparison, the cross-sectional averaged turbulent dispersivity $\langle\langle D_t' \rangle\rangle$ is plotted in Figure 4.7. From the figure, $\langle\langle D_t' \rangle\rangle$ increases with α first and then decreases, which may be attributable to the change-over of δ_l' at $\alpha \sim 20$. Similarly, $\langle\langle D_t' \rangle\rangle$ increases

with r_0' and becomes almost the same when α exceeds the change-over point. $\langle\langle D_t' \rangle\rangle$ also increases with ϕ proportionally as described in (4.30).

Figure 4.6 Longitudinal dispersion coefficient D_l' as a function of α ($\phi = 0.6, r_0' = 0.001$ (solid), $\phi = 0.6, r_0' = 0.005$ (dash), $\phi = 1, r_0' = 0.001$ (dash dot), and $\phi = 1, r_0' = 0.005$ (dash dot dot))

Comparing Figures 4.6 and 4.7, one can find that the difference between D_l' and $\langle\langle D_t' \rangle\rangle$ varies considerably depending on the value of α . For most cases, $\langle\langle D_t' \rangle\rangle$ is much smaller than D_l' . However, for $\alpha \sim O(0.1)$, the magnitude of D_l' is of the same order as $\langle\langle D_t' \rangle\rangle$, which implies that the turbulent dispersivity cannot be ignored in the analysis of dispersion in oscillatory pipe flows with small velocity but high frequency.

For a typical pressure exchanger with duct radius $a \sim O(0.01)m$, rotating frequency $\omega \sim O(100)rad/s$, and velocity amplitude $U \sim O(5)m/s$, the control parameter

$\alpha = \frac{\kappa U}{\omega a}$ is found to be $\sim O(1)$. From Figure 4.7 and by noting that $D_t = D_t' U a$, we

can conclude that smaller velocity amplitude, higher oscillating frequency and larger duct radius could reduce the mixing in pressure exchanger. It is also found that wall roughness could increase the mixing.

Figure 4.7 Turbulent dispersivity $\langle\langle D_t' \rangle\rangle$ as a function of α ($\phi = 0.6, r_0' = 0.001$ (solid), $\phi = 0.6, r_0' = 0.005$ (dash), $\phi = 1, r_0' = 0.001$ (dash dot), and $\phi = 1, r_0' = 0.005$ (dash dot dot))

4.3 Experimental study

4.3.1 Experimental apparatus and conditions

The experiments were conducted in the DHI-NTU Centre laboratory. An experimental apparatus was built to generate the oscillatory pipe flows as illustrated in Figure 4.8.

Figure 4.8 Photo of the experimental apparatus

The experimental apparatus was a straight circular acrylic pipe with inner diameter $d_p = 10\text{mm}$ and length $l_p = 1100\text{mm}$ supported by two stainless steel columns. One end of the pipe (RHS) was connected to an acrylic water tank (250 mm length $\times$ 250 mm width $\times$ 500 mm height) with a valve installed on the entrance of the pipe. The other end (LHS) was joined to a circular stainless steel chamber with the inner diameter twice of the pipe diameter, i.e. 20mm. Both connections were equipped with stainless steel bell-shaped transition with the length of 20mm to reduce the entrance effect. Two valves were installed on the top and bottom of the chamber, respectively. They helped emptying the pipe and eliminating air bubbles during the calibration process, which will be discussed in details later. All the valves were installed carefully to avoid interrupting the flows.

Flow oscillations were generated by a motor-and-crank assembly driving a low-friction stainless steel piston inside the chamber. An electric stepping motor (model: Orientalmotor ARLM66AC-PS5) was employed to provide constant rotating speed. A stainless steel disk with multiple screw holes was driven by the motor and the rotations were then converted to a sinusoidal movement in the horizontal direction to the piston

through the Scotch-yoke mechanism (the whole unit was installed in an acrylic tank for protection). The displacement of piston can be adjusted by changing the rotating arm length on the disk. The rotating speed was controlled by a control panel. All the components were carefully fabricated and polished so that the piston and shaft could move smoothly in the chamber as well as no leakage would occur. The key parameters of the apparatus are listed in Table 4.1.

Table 4.1 Key parameters of the experimental apparatus

Parameters	Value
Length of pipe l_p (mm)	1100
Diameter of pipe d_p (mm)	10
Length of piston chamber l_{pc} (mm)	200
Diameter of piston chamber d_{pc} (mm)	20
Length of rotating arm l_{ra} (mm)	10 – 50
Rotating speed N (rpm)	30 – 240

The properties of the oscillatory flows in the pipe can be obtained through the following equations:

$$\omega = \frac{\pi}{30} N \quad (4.53)$$

$$U = l_{ra} \omega \left(\frac{d_{pc}}{d_p} \right)^2 / 1000 \quad (4.54)$$

$$V = \frac{\pi d_p^2 l_{ra}}{2000} = \frac{\pi d_p^2}{2\omega} U \quad (4.55)$$

where V is the stroke volume (the flow displacement volume over a half period).

In this system, the oscillation frequency and stroke volume were controlled by the motor and rotating arm on the disk, respectively, and thus are independent. A total of 18 conditions with combination of different frequencies and stroke volumes were examined and their detailed experimental parameters are presented in Table 4.2. The experiments were conducted at the temperature of 20 °C approximately. The kinematic viscosity of water ν was therefore equal to $10^{-6} \text{ m}^2/\text{s}$. The Reynolds number Re_δ of the tested flows ranged from 100 to 1000, with the intention of covering the oscillatory flow regime from laminar to turbulent.

Table 4.2 Summary of experimental conditions

Run	V (cm ³)	ω (rad/s)	U (m/s)	$\delta = \sqrt{2\nu/\omega}$ (m)	$\text{Re}_\delta = U\delta/\nu$
1	6.3	3.14	0.126	0.000798	100
2	6.3	6.28	0.251	0.000564	142
3	6.3	12.56	0.502	0.000399	200
4	12.6	3.14	0.251	0.000798	200
5	6.3	25.12	1.005	0.000282	283
6	12.6	6.28	0.502	0.000564	283
7	18.8	3.14	0.377	0.000798	301
8	12.6	12.56	1.005	0.000399	401
9	25.1	3.14	0.502	0.000798	401
10	18.8	6.28	0.754	0.000564	425
11	31.4	3.14	0.628	0.000798	501
12	12.6	25.12	2.009	0.000282	567
13	25.1	6.28	1.005	0.000564	567
14	18.8	12.56	1.507	0.000399	601
15	31.4	6.28	1.256	0.000564	709
16	25.1	12.56	2.009	0.000399	802
17	18.8	25.12	3.014	0.000282	850
18	31.4	12.56	2.512	0.000399	1002

4.3.2 Experimental methodology

4.3.2.1 Introduction of PIV technique

There are a large number of velocity measurement tools for fluid study. Point-based velocimetry techniques such as Hot Wire Anemometry (HWA), Laser Doppler Anemometry (LDA) and Acoustic Doppler Velocimetry (ADV), have been widely employed in many laboratories. These tools can provide a time history of fluid velocity at a single point with a relatively high sampling rate. However, they cannot sample the entire flow field at one time. Flow visualisation techniques based on smoke or dye seeding have been used to obtain the entire flow field qualitatively for a long time. In recent years, significant progress in entire field measurements is made due to the development of laser, optics and computer technologies, including Nuclear-Magnetic-Resonance (NMR) imaging, planar Laser Induced Fluorescence Velocimetry (LIFV), Laser Speckle Velocimetry (LSV), Particle Tracking Velocimetry (PTV) and Particle Image Velocimetry (PIV), etc (Shao, 2010). Among these techniques, the PIV technique is relatively mature and widely used in laboratories nowadays.

The working principle of PIV is straightforward (see Figure 4.9). The flow field is seeded with tracer particles and illuminated by a laser light sheet. Two or more short-duration exposures of the particles are then recorded by a camera. The recorded pictures are subsequently analyzed through cross-correlation to obtain the instantaneous velocity field based on the kinematic definition:

Velocity = displacement / time.

Figure 4.9 Working principle of PIV technique

In other words, the particle displacements within a known short time interval are measured to obtain the velocity field of the fluid. This requires that the density of the particle must be similar to that of the fluid. Moreover, the size of the particle must be sufficiently small to make the slip velocity between the particle and fluid negligible as well as sufficiently large to ensure sufficient light scattering in order that it can be detected. In order to analyze the velocity, the camera captured image is divided into a lot of Interrogation Areas (IA), the velocities of the flow field are measured based on these IAs. Each IA generates one velocity vector. The population density of the seeding particles is dictated by the need of PIV that sufficient number of particles is present within an IA. To obtain the mean displacement in an IA, a 2D spatial correlation, either auto-correlation or cross-correlation based on Fourier transforms, is performed. All the analysing functions can be easily evaluated by computer.

4.3.2.2 PIV experiments

In this study, the characteristics of the oscillatory flows in the pipe were first investigated by the PIV technique. Tap water was used as the working fluid, and 50 μ m neutral buoyant polyamid particles (density = 1.03 g/cm³, size distribution 30~70 μ m, refractive index=1.5) were added as seeding particles. The light source employed was a Dantec DualPower 65-15 Nd: YAG pulsed laser located above the pipe with the characteristics listed in the Table 4.3.

Table 4.3 Specifications of the Dantec DualPower 65-15 laser

Parameters	Value
Maximum output (mJ)	400
Pulse duration (ns)	4
Wavelength (nm)	532
Laser medium	Nd: YAG
Model No.	NANO S65-15 PIV

The laser sheet was adjusted to coincide with the centerline of the pipe vertically so that only the particles on this plane could be observed. A Dantec FlowSense 2M charge-coupled device (CCD) digital camera was configured in front of the pipe for the image capturing with a filter lens of 532nm placed to eliminate other light sources. This type of camera is specially designed for motion analysis and thus is well suited for the cross-

correlation algorithm of PIV. The camera was adjusted to perpendicularly focus on the same plane illuminated by the laser. A length of 80mm in the middle of the pipe was observed for velocity measurement. Both the laser and camera were controlled by a computer and all the images were then transferred and stored in the computer. A configuration of the experimental setup is illustrated in Figure 4.10.

Figure 4.10 Configuration of the experimental setup (unit: mm)

Before the tests, the calibration for PIV was carried out to obtain the relationship between the scale in images and in reality. A straight scale with resolution of 1mm was located on the same plane of observation as a reference (see Figure 4.11). The scale factor in the experiments was around 6.96.

Figure 4.11 Calibration picture for PIV experiments

All the tests were conducted in a dark environment where the laser sheet was the only light source. The homogeneously mixed liquid was poured into the tank and it filled the pipe gradually. After a stable oscillatory flow was achieved in the pipe, the laser and camera were triggered synchronously in double-frame mode to capture the particle movements. The time interval between pulses was adjusted according to the flow velocity under different experimental conditions. The trigger rate (frequency of laser pulse) was set to be 12 Hz (6 Hz for the conditions of $\omega = 3.14$ rad/s). For each test, the duration of time was longer than 100 times of the oscillation period.

The captured images were analyzed using the software of DynamicStudio v3.12 installed in a desktop. Each pair of images generated one vector map based on the cross-correlation algorithm with the size of IA chosen to be 32×32 pixels and 50% (horizontal) $\times$ 50% (vertical) overlapped. Figure 4.12 (a) shows a pair of original images (the #2 picture was capture 500 μ s later than #1 picture), which the lightspots are the shining seeding particles. Figure 4.12(b) is the generated raw vector map based on Figure 4.12(a). The length of vector represents the velocity magnitude and the arrow denotes the velocity direction. The raw vector maps were refined through 1) "peak validation" - the vector was rejected if the ratio of the first highest peak to the second highest peak on the cross-correlation plane was less than 1.2, 2) "range validation" - the

vector was rejected if it was greater than 1/4 of the length of interrogation area, and 3) "moving average validation" - the vector was rejected if it deviated substantially from their surrounding vectors. All rejected vectors were substituted by the average of their surrounding vectors.

#1

#2

(a)

(b)

Figure 4.12 (a) A pair of original PIV images under the condition of $Re_\delta = 567$ ($V = 25.1 \text{ cm}^3$, $\omega = 2\pi \text{ rad/s}$); (b) vector map obtained through the cross-correlation algorithm

4.3.2.3 Introduction of PLIF technique

Planar Laser Induced Fluorescence (PLIF) is a technique to measure the solute concentration in a fluid based on using a laser. Dewey (1976) pioneered the application of PLIF for flow visualization by using photographic film as the recording medium. Subsequent research studies included Dimotakis et al. (1983), Kychakoff et al. (1982) and Dyer and Crosley (1982). With the rapid development of video and computer technologies, the use of PLIF is becoming more and more common in laboratory experiments.

The principle of PLIF is to measure the solute concentration by introducing an amount of dissolvable fluorescent dye into the fluid as tracer. When the fluorescent dye is excited by laser light, it emits fluorescent light at longer wavelengths. The intensity of the emitted light depends on both the dye concentration and the laser light intensity. Hence, the dye concentration in the flow field can be obtained by measuring the fluorescent light intensity.

In PLIF, the dyed fluid is illuminated by a laser light sheet with a thickness in the order of a millimetre. The planar area of interest in the light sheet is then captured by a camera. Thus, the solute concentration field can be obtained at one time.

To maintain the high accuracy of PLIF, high light intensity is desired which can be obtained by increasing the dye concentration. However, at higher concentrations, the laser light may be significantly attenuated due to the absorption by the Rhodamine molecules (Shao, 2010). To avoid the attenuation effect, Arcoumanis et al. (1990) showed that a linear response of fluorescence with respect to concentration could be obtained when the dye concentration was less than 80 $\mu\text{g/L}$ in tap water.

4.3.2.4 PLIF experiments

With the same experimental setup described previously, the laser imaging technique of PLIF was applied to quantify the longitudinal dispersion in the oscillatory pipe flows. The tracer employed was Rhodamine B (molecular diffusivity = $4.5 \times 10^{-10} \text{ m}^2/\text{s}$ (Kapusta, 2010)) which has a high quantum yield and a remarkable chemical stability when exposed to the laser. Rhodamine B could emit fluorescent light with a peak wavelength of 575nm on excitation by the 532nm laser light. The dye concentration was chosen in the linear range whereby the intensity of the fluorescent light was linearly proportional to the solute concentration. Hence, the concentration distribution along the pipe can be obtained by analyzing the fluorescent light intensity. A filter lens of 560nm was configured on the camera so that only the fluorescent light emitted by the excited tracer could be detected. Different from PIV experiments, a longer section of 350mm in the middle of the pipe was observed to measure the concentration variation along the pipe.

Calibration was conducted before each test to determine the relationship between the light intensity and tracer concentration. The concentrations for calibration were chosen to be roughly equal-distributed within the linear range, i.e. 0, 20, 40, 60, 80 and 100 $\mu\text{g/L}$. Figure 4.13 demonstrates the calibration images under various concentration, where the brightness of image increases with the tracer concentration. Each of these images was the average of 60 images taken sequentially to minimize the effect of the instability of laser light, and the precise concentration was measured by a fluorometer (model: Turner Designs 7000-009). Based on the images, a calibration curve can be obtained for each pixel, which provided the foundation for the subsequent experiments.

Figure 4.13 Calibration images under various concentrations for $\text{Re}_\delta = 501$ ($V = 31.4 \text{ cm}^3$, $\omega = \pi \text{ rad/s}$)

The calibration curves at some selected pixels as well as the average response (over 1600×65 pixels) are illustrated in Figure 4.14. It was found that, for each pixel, the camera response is effectively linear to the tracer concentration. The scattering among the proportionality constants may be attributed to the discrepancies in the pixel responses as well as the non-uniformity of the light intensity distribution over the observation area. With the pixel-by-pixel calibration, the concentration at different positions can be obtained precisely according to their own calibration curves.

Figure 4.14 Calibration curves for $Re_\delta = 501$ ($V = 31.4 \text{ cm}^3$, $\omega = \pi \text{ rad/s}$)

For each test, the apparatus was washed a few times to make sure the tracer dye was removed in the pipe. Tap water was then added to the tank. During the whole process, the positions of apparatus, laser and camera were kept strictly the same as that for the calibration. The dye tracer was then introduced to the water inside the tank located at the end of the pipe, and mixed homogeneously inside the tank. Note that the tank and pipe was initially separated by a valve at the pipe entrance which was closed. This was different from the method used in the previous study of Ye and Zhang (2002), which the dye was injected at the middle of the pipe through a syringe. The current method was preferred as it avoided introducing initial momentum to the tracer, as well as enabled the dye tracer to be homogeneous over the cross section before entering the pipe so that the influence of initial radial differences can be minimized. The solution concentration in the tank was set to be around $80 \mu\text{g/L}$ for all the tests (the accurate concentration was measured by the fluorometer).

Allowing sufficient time for the fluid in the tube to become motionless, the valve was open, and then the motor and laser were switched on simultaneously. The single-frame mode was applied for PLIF to capture the dispersion with the trigger rate being four times of the oscillation frequency. The duration lasted until the concentration at the middle of the pipe reached half of the source concentration, and thus the duration of each test varied case by case from $O(10^2)$ to $O(10^5)$ seconds.

The original images were analyzed pixel by pixel based on the calibration curve. The concentration distribution over the entire observation area was then obtained.

4.3.3 Experimental results and analysis

4.3.3.1 Velocity measurements

The flow velocities were obtained through the PIV experiments. Figure 4.15 shows the refined vector maps at different phases under various Reynolds numbers. It is obvious that the flow directions in the core region and near boundaries were opposite at the phases of direction-change, i.e. $\omega t = \pi$ (see Figure 4.15(a)). This clearly illustrated the existence of phase lag due to inertia, whereby the flow near the wall boundaries switched direction earlier than the core region. The reason is that the flow velocity near the wall is much smaller than that in the core region and thus can react to the change immediately. In general, for small Reynolds numbers, the PIV vector maps showed that the flows were smooth and unidirectional under most phases. However, disturbance can sometimes be observed when the flow changed direction, and these disturbance became very noticeable when Re_δ reached 283 (Figure 4.15(b)). As Re_δ continued to increase, cross-sectional vortices began to appear in the domain between the core region and wall

boundaries at the phase of direction-change when Re_δ exceeded 501 (Figure 4.15(c)). Beyond that, these vortices became prominent features in the vector maps, and the oscillatory flow entered the turbulent regime. The observation therefore implied that the oscillatory pipe flow transited from laminar to turbulent when Re_δ was around 500. As discussed before, the critical Reynolds number for transition had been reported from 250 to 760. Our observed value was within the range, and also close to the results of Akhavan et al. (1991).

Based on the refined vector maps, the ensemble-averaged vector maps for specific phases were obtained. For each particular phase, the instantaneous vector maps of 100 cycles were averaged to one map. The velocity variation at the centreline within a period is plotted in Figure 4.16, where the symbols denote the measured ensemble-averaged velocities and the solid line represents the analytical prediction based on the theoretical study presented in Section 4.2. In the figure, the velocity is normalized by the amplitude of cross-sectional mean velocity, i.e. $u(r=0)/U$. Error bars are included in Figure 4.16 to illustrate the standard deviation (SD) of the measurements,

$$\text{i.e. } SD = \sqrt{\frac{1}{100} \sum_{n=1}^{100} (x_n - \bar{x})^2}, \text{ where } x_n \text{ and } \bar{x} \text{ represent the instantaneous and ensemble-averaged value, respectively.}$$

From the figure, it can be seen that the measured velocities followed well with the anticipated predictions.

The cross-sectional velocity profile was also obtained based on the ensemble-averaged velocities. Figure 4.17 shows the velocity profiles under various phases within half an oscillation period, where the x -axis represents the measured velocity over the amplitude of cross-sectional mean velocity and y -axis denotes the normalized radial distance. For $\omega t = 0$ and π , phase lags can be clearly observed in the outer region.

(i) $\omega t = \pi/3$

(ii) $\omega t = \pi$

(iii) $\omega t = 5\pi/3$

(a) $\text{Re}_\delta = 200$ ($V = 12.6 \text{ cm}^3$, $\omega = \pi \text{ rad/s}$)

Figure 4.15 Velocity vector maps at different phases for various experimental conditions (unit: mm)

(i) $\omega t = \pi/3$

(ii) $\omega t = \pi$

(iii) $\omega t = 5\pi/3$

(b) $\text{Re}_\delta = 283$ ($V = 12.6 \text{ cm}^3$, $\omega = 2\pi \text{ rad/s}$)

Figure 4.15 Velocity vector maps at different phases for various experimental conditions (unit: mm) (continued)

(i) $\omega t = \pi/3$

(ii) $\omega t = \pi$

(iii) $\omega t = 5\pi/3$

(c) $\text{Re}_\delta = 567$ ($V = 25.1 \text{ cm}^3$, $\omega = 2\pi \text{ rad/s}$)

Figure 4.15 Velocity vector maps at different phases for various experimental conditions (unit: mm) (continued)

(a) $\text{Re}_\delta = 501$ ($V = 31.4 \text{ cm}^3$, $\omega = \pi \text{ rad/s}$)

(b) $\text{Re}_\delta = 567$ ($V = 25.1 \text{ cm}^3$, $\omega = 2\pi \text{ rad/s}$)

(c) $\text{Re}_\delta = 709$ ($V = 31.4 \text{ cm}^3$, $\omega = 2\pi \text{ rad/s}$)

Figure 4.16 Velocity variation within a period at the centerline for different experimental conditions

(a) $\text{Re}_s = 200$ ($V = 12.6 \text{ cm}^3$, $\omega = \pi \text{ rad/s}$)

(b) $\text{Re}_s = 401$ ($V = 25.1 \text{ cm}^3$, $\omega = \pi \text{ rad/s}$)

(c) $\text{Re}_s = 425$ ($V = 18.8 \text{ cm}^3$, $\omega = 2\pi \text{ rad/s}$)

Figure 4.17 Cross-sectional velocity profiles at various phases within half period

It is noted that due to the limitations of the PIV setup (the trigger rate of laser was low and the camera lens cannot capture close shot), the velocity measurements in the present study were preliminary. More detailed measurement on velocity is achievable by applying a high-frequency laser and an improved camera lens in future studies.

4.3.3.2 Concentration measurements

Figure 4.18 qualitatively shows the dispersion process within one oscillation period at $Re_\delta = 425$ ($V = 18.8 \text{ cm}^3$, $\omega = 2\pi \text{ rad/s}$). Within the oscillation period, the solute moves forward and backward from the original position while it diffuses/disperses at the same time.

To obtain the longitudinal dispersion coefficient, quantitative analysis is required. As the volume of Rhodamine B solution in the tank ($\approx 10\text{L}$) was much larger than the volume of tap water inside the pipe (less than 0.1L), the concentration in the tank can be treated approximately as a constant during the tests. Moreover, the period of flow oscillation was less than 2 seconds while the time scales for dispersion along the pipe ranged from $O(10^2)$ to $O(10^5)$ seconds. The large contrast implied that the time scale for longitudinal dispersion was much longer compared to that of the flow oscillation. As a result, the analysis of the measurements can be performed similar to transient constant-source diffusion/dispersion in an infinite pipe. Therefore, before the tracer reached the other end of the pipe, the concentration distribution would satisfy a complementary error function for a maintained source, which is:

Figure 4.18 Longitudinal dispersion process within one cycle for $\text{Re}_\delta = 425$ ($V = 18.8 \text{ cm}^3$, $\omega = 2\pi \text{ rad/s}$)

$$C(x,t) = C_{cs} \operatorname{erfc}\left(\frac{x}{2\sqrt{D_{eff}t}}\right) \quad (4.56)$$

with the initial condition:

$$C(x,0) = 0 \quad (4.57)$$

and the boundary conditions:

$$C(0,t) = C_{cs}; C(\infty,t) = 0 \quad (4.58)$$

where C_{cs} is the constant-source concentration, $\operatorname{erfc}()$ denotes the complementary error function and D_{eff} is the effective dispersion coefficient.

The axial concentration distribution according to the complementary error function is illustrated in Figure 4.19. By assuming the value of dispersion coefficient, one can obtain the quantitative concentration at any particular instance in time. Here, three particular positions in axial direction were selected, i.e. $x = 0.46, 0.55$ and 0.64 m ($x=0$ started from the valve on the entrance of the pipe) and the concentration variation was analyzed as a function of time at the three positions. The concentrations at the three axial positions were first averaged over the radial direction (denoted by C_r), and then among four successive images to obtain the period-averaged concentration, C_{rT} . Finally, due to the large amount of data, C_{rT} was analyzed with the time interval of 5s or 10s (according to the amount of data for different tests) for the dispersion coefficient calculations. By fitting a complementary error function to the experimental data based on the least square method, the best-fit longitudinal dispersion coefficient can be obtained. For flows with smaller Reynolds number, a longer time is required for the tracer to appear in the image window as well as disperse to the end of the pipe. Also,

the rate of increase in concentration is much lower in flows with larger Reynolds numbers. The measured concentrations for different experimental conditions are presented in Appendix E. The standard deviation, which is defined by

$$SD = \sqrt{\frac{1}{n} \sum_{i=1}^n (C_{Ei} - C_{Mi})^2} \quad (\text{where } C_E \text{ is the concentration calculated based on the}$$

complementary error function, C_M is the measured concentration, and n is the total number), is found to be small, i.e. $\sim O(10^{-2})$, for all cases.

Figure 4.19 Concentration distribution as a function of distance from a maintained source based on complementary error function

The calculated dispersion coefficients are summarized in Table 4.4, where D_{eff} is the average over the three axial positions. The influence of the two independent parameters, i.e. stroke volume and oscillating frequency, on the longitudinal dispersion coefficient is demonstrated in Figure 4.20. In the figure, D_{eff}' is the dimensionless dispersion coefficient defined by $D_{eff}' = D_{eff} / D$, with D_{eff} employed for the calculation being the average value of the measured effective dispersion coefficient over the three x

positions and $D = 4.5 \times 10^{-10} \text{ m}^2/\text{s}$ is the molecular diffusivity of Rhodamine B. Under constant oscillating frequency, D_{eff}' increased with the dimensionless stroke volume V^2/a^6 as shown in Figure 4.20(a). A similar trend can be observed in Figure 4.20(b) for a specific stroke volume. As the Womersley number $\theta = a\sqrt{\omega/\nu}$ increased, D_{eff}' also increased generally.

(a)

(b)

Figure 4.20 Variation of dimensionless longitudinal dispersion coefficient with (a) dimensionless stroke volume and (b) Womersley number

Table 4.4 Summary of the measured dispersion coefficients

Run	V (cm ³)	ω (rad/s)	$\text{Re}_\delta = U\delta/\nu$	D_{eff} (m ² /s)
1	6.3	3.14	100	1.58×10^{-6}
2	6.3	6.28	142	2.06×10^{-6}
3	6.3	12.56	200	6.01×10^{-6}
4	12.6	3.14	200	9.30×10^{-6}
5	6.3	25.12	283	1.45×10^{-5}
6	12.6	6.28	283	2.07×10^{-5}
7	18.8	3.14	301	3.75×10^{-5}
8	12.6	12.56	401	1.02×10^{-4}
9	25.1	3.14	401	1.52×10^{-4}
10	18.8	6.28	425	1.94×10^{-4}
11	31.4	3.14	501	2.21×10^{-3}
12	12.6	25.12	567	2.20×10^{-3}
13	25.1	6.28	567	2.39×10^{-3}
14	18.8	12.56	601	2.31×10^{-3}
15	31.4	6.28	709	5.65×10^{-3}
16	25.1	12.56	802	4.20×10^{-3}
17	18.8	25.12	850	4.83×10^{-3}
18	31.4	12.56	1002	1.34×10^{-2}

The relationship between D_{eff} and Re_δ is further illustrated in Figure 4.21, where errors bars are used to illustrate the standard deviation among the measurements at the three different axial positions. As Re_δ increased from 100 to 1000, the amplitude of

D_{eff} increased significantly from 10^{-6} to 10^{-2} m²/s. Comparing the dispersion coefficient with the same Reynolds number, D_{eff} was relatively larger for the conditions with larger stroke volumes and smaller angular frequencies. In other words, the oscillatory flow with smaller velocity amplitudes but longer periods introduced stronger longitudinal dispersion, even though the two Reynolds number might be identical. This is deemed reasonable as the primary cause for longitudinal dispersion is the velocity variation over the cross section. Oscillatory flows with higher velocity but shorter period have thinner boundary layers. Hence, the velocity profile is flatter and thus weaker longitudinal dispersion is induced.

Figure 4.21 Relationship between longitudinal dispersion coefficient and Reynolds number

4.4 Comparison

4.4.1 Comparison with laminar flows

A classical theoretical study on the longitudinal dispersion coefficient of laminar pipe flows under sinusoidal pressure gradient was conducted by Watson (1983). It was concluded that the dispersion coefficient can be related to the stroke volume and angular frequency as follow:

$$D_{lam} = D \left(1 + f(\theta, Sc) \frac{V^2}{a^6} \right) \quad (4.59)$$

where D_{lam} is the longitudinal dispersion coefficient for laminar flows and Sc refers to the Schmidt number. Figure 4.22 shows the ratio of the measured longitudinal dispersion coefficient D_{eff} to the theoretical prediction D_{lam} based on Equation (4.59) as a function of Reynolds number. The measured dispersion coefficients were found to deviate from the theoretical predictions, even in the laminar range with small Reynolds numbers of 100 – 200. Ye and Zhang (2002) investigated the axial dispersion of oscillatory pipe flows by injecting a bolus of tracer (diffusivity = 2.78×10^{-9} m²/s) through a syringe. They also found that the measured effective dispersion coefficients were 2~3 times larger than the predictions based on Watson (1983) when $Re_\delta = 100 - 235$. Their reported difference was smaller than ours but still noticeable. One possible reason is that Watson's theory is only suitable for matter with large molecular diffusivity (i.e. gas), which was experimentally proved by Joshi et al. (1983). Another reason might be that the oscillatory flow was not purely laminar even at this small Reynolds number range. For Re_δ between 280 and 425, the measured values of the longitudinal coefficient became an order of magnitude higher than the laminar

predictions, with the ratio D_{eff} / D_{lam} ascending to $O(10)$, which was similar to the results of Ye and Zhang (2002). Most importantly, a drastic jump occurred when Re_δ reached the full transition value of ~ 500 , as observed by the PIV velocity maps that vortices became prominent. The presence of turbulence significantly enhanced the longitudinal dispersion coefficient, and the ratio D_{eff} / D_{lam} increased to as large as $O(10^3)$. This large contrast was clearly due to the fact the analysis based on laminar flows should no longer be valid. The current ratio was larger than that reported by Ye and Zhang (2002) which was $O(10^2)$ as Re_δ increased to the range of 535 – 1000. The reason can be attributed to the fact that the molecular diffusivity of the tracer, Rhodamine B, used in the present study was almost one order of magnitude smaller than that of the tracer they used. Hence, the calculated dispersion coefficient D_{lam} based on Equation (4.59) was also much smaller in our case. At the same time, the measured D_{eff} of turbulent oscillatory flows should not be dependent on the molecular diffusivity value, thus resulting in the larger ratio. Note that the experimental results of D_{eff} were not shown in Ye and Zhang (2002), hence a more detailed comparison with our study beyond the order of magnitude assessment of the ratio was not possible.

Figure 4.22 Comparison between experimental results and theoretical predictions for laminar flows

4.4.2 Comparison with turbulent flows

With the analytical results in Section 4.2, the non-dimensional parameter α and the corresponding D_l' can be computed for each test condition. The values of α for the 18 conditions ranged from 3.2 to 16. The comparison between the experimental results and theoretical predictions for turbulent oscillatory flows as a function of Re_δ is illustrated in Figure 4.23, where $D_{tur} = D_l' Ua$ and D_l' employed is the non-dimensional dispersion coefficient under $r_0' = 0.001$ and $\phi = 1$ (Figures 4.6 and 4.7 show that the turbulent dispersivity can be ignored as it is much smaller than the longitudinal dispersion coefficient for these conditions). Clearly, for Re_δ less than 500, the predicted D_{tur} was order of magnitude larger than D_{eff} , as the perturbation analysis is not applicable in the laminar range. When Re_δ reached 500 and beyond, the measured dispersion coefficient became similar to the predicted value, and D_{eff} / D_{tur} was of the order $O(1)$, which demonstrated a satisfactory agreement in general.

Figure 4.23 Comparison between experimental results and theoretical predictions for turbulent flows

Moreover, the experiments implied that for fixed oscillation frequency, the dispersion coefficient increased with the stroke volume, which is proportional to U . This is consistent with the theoretical prediction that D_l increases with the velocity amplitude. It was also suggested from the experiments that the dispersion coefficient increased with the oscillation frequency, which seems to be contrary to the theoretical conclusion that D_l decreases with ω . The reason is that the former was under the condition of fixed stroke volume while the later was based on the assumption of fixed velocity amplitude. For fixed stroke volume, increase in frequency leads to a larger velocity amplitude. Hence, the two conclusions are in fact similar.

4.5 Conclusions

In this chapter, the longitudinal dispersion of turbulent oscillatory flows in a circular pipe was investigated both analytically and experimentally.

The analytical results showed that the non-dimensional longitudinal dispersion coefficient can be quantified by means of a control parameter α which represents the ratio of the velocity amplitude to the angular frequency and pipe radius. Generally, the magnitude of non-dimensional longitudinal dispersion coefficient is found to first increase with α and then decrease upon reaching a peak value of ~ 0.5 at $\alpha \sim 20 - 30$. The non-dimensional turbulent dispersivity follows a similar trend as D_l' but in a more moderate manner. It suggests that the longitudinal dispersion coefficient, $D_l = D_l' Ua$, can be enhanced by increasing the velocity amplitude U while keeping the other two parameters, i.e. ω and a , unchanged.

In the above analysis, the flow is assumed to be turbulent so that the eddy viscosity model is applicable. Based on previous studies as well as the PIV measurements obtained in the present study, the oscillatory pipe flow transits to turbulence when Re_δ exceeds 500~550. Hence, the theoretical prediction performed in this study is expected to be applicable when $Re_\delta > 500\sim 550$. In addition, for the homogenization approach to be applicable, it is required that the pipe radius is sufficiently small. It is also assumed that the time scale of dispersion along the axial direction to be much larger than that of the dispersion over the cross section. In other words, the result holds for long and narrow pipes, i.e. $a/L \ll 1$.

The experimental results showed that when the Reynolds number increased from 100 to 1000, the dispersion coefficient rose drastically from 10^{-6} to 10^{-2} m²/s. The dispersion coefficient was also found to increase with the stroke volume, the oscillation frequency, and the Reynolds number. By comparing the dispersion coefficient under the same Re_δ , it was observed that the oscillatory flow with lower velocity amplitude but longer oscillation period generated relatively stronger longitudinal dispersion than that with

higher velocity but shorter period. Overall, a satisfactory agreement was observed when comparing the experimental results to the theoretical predictions for the turbulent conditions when Re_δ exceeded 500, which verified the applicability of the analytical approach in this study.

Applying the above conclusions to the case of pressure exchanger, we can further conclude that in order to reduce the mixing, the design of the pressure exchanger can be improved as following:

- 1) Increase the number of ducts in the rotor as it could reduce the velocity amplitude while keeping the flow rate unchanged
- 2) Enhance the rotating speed when keeping the velocity constant or reducing the rotating speed if keeping the stroke volume fixed
- 3) Enlarge the diameter of the ducts
- 4) Use ducts with smooth walls
- 5) Increase the length of the ducts so that the concentration gradient can be reduced

Chapter 5 Longitudinal dispersion of turbulent oscillatory flows in a two-dimensional channel

5.1 Introduction

The use of vibration to enhance wall shear stresses and mass transfer has been widely applied in biotechnologies, chemical reactions and membrane filtration processes. It receives increasing attention with the emergence of new applications that require higher transfer rates and specificity in mixing rates that could not be achieved with conventional methods. Examples include the reciprocating plate column (RPC) which is a kind of gas-liquid contacting system in which surface vibration is applied to achieve uniform mixing (Gomaa and Taweel, 2005) and the vibrating-beam-based micromixer for microfluids (Sun and Wang, 2008). The vibratory shear-enhance processing (VSEP) proposed by Armando et al. (1992), in which a stacked membrane system vibrates torsionally around a vertical axis to reduce the concentration polarization and membrane fouling, has also been applied to the hollow fiber membranes to improve the filtration performance (Jaffrin, 2008 and Farhad et al., 2013).

The fundamental configuration can be conceptualized to be the longitudinal dispersion in channels/pipes with vibrating boundaries. Previous literatures mostly focused on the oscillatory flow induced by wall vibration of a flexible pipe with the form of travelling and standing waves, i.e. the radial displacement of the wall boundary (Dragon and Grotberg, 1991; Broday and Kimmel, 1999 and Carlsson et al., 2005). Gomaa and Taweel (2004) studied the mass transfer in laminar flows with vertical oscillation of the surface using both experimental and analytical approaches. Simsek et al. (2008)

investigated the dispersion of granular material on a vibrating conveyor experimentally and numerically. It was suggested that the mixing is proportional to the vertical acceleration of the conveyor but inversely proportional to the mass flow rate.

In this chapter, we perform an analytical analysis on the longitudinal dispersion of turbulent oscillatory flows inside a two-dimensional channel with axial vibrating walls but without pressure gradient by using the method similar to that in Chapter 4. Such cases occur when the vibrating velocities and the associated Reynolds number are sufficiently high, such that the oscillatory flows inside the channel are no longer laminar. Previously, it had been documented that for oscillatory pipe flows driven by periodic pressure gradient, the transition to turbulence occurs when the Reynolds number reaches 250 – 760 (Hino et al., 1976; Ohmi et al., 1982; Ahn and Ibrahim, 1992 and Akhavan et al., 1991). For wall oscillation, no such study has been reported so far, but we can expect a similar laminar-to-turbulent transition Re_δ by replacing U and ω with the velocity amplitude and angular frequency of the wall. This chapter is organized as follows. Sections 5.2 and 5.3 give the analysis as well as results on dispersion in oscillatory turbulent flows driven by synchronously oscillating walls and by single oscillating wall, respectively. A comprehensive conclusion is then made in Section 5.4.

5.2 Turbulent flows driven by synchronously oscillating walls

5.2.1 Velocity field

We first consider the case of a rectangular channel with confining wall boundaries vibrating synchronously along the channel axis. Due to the wall drag, a turbulent

oscillatory flow is generated inside the channel. We adopt a Cartesian coordinate for the flow, as shown in Figure 5.1. The x -axis is the axial direction, and y is the vertical direction with $y=0$ and $y=2h$ at the bottom and top boundaries, respectively. The axial velocity u is varying with time t and y , i.e. $u=u(y,t)$. Assuming that the planar dimensions of the channel are sufficiently large compared to the channel height such that the problem can be treated as two-dimensional. Thus, the Navier-Stokes equation can be reduced to,

$$\frac{\partial u}{\partial t} = \frac{1}{\rho} \frac{\partial \tau}{\partial y} \quad (5.1)$$

The turbulent Reynolds stress τ can be related to the velocity gradient by the eddy viscosity ν_T in the form,

$$\tau = \rho \nu_T \frac{\partial u}{\partial y} \quad (5.2)$$

Figure 5.1 Schematic diagram of the coordinate axis for the case of synchronous dual walls vibration

As the top and bottom boundaries vibrate synchronously in this case, the flow structure should be symmetric about the mid-plane $y = h$. Hence, only one half of the channel is analyzed, i.e. $0 < y \leq h$. We consider the vibration with a single frequency and without any steady drift component, i.e. $u = U_{wall} \cos(\omega t)$, where U_{wall} is the amplitude of velocity of the walls. The boundary conditions are thus,

$$\frac{\partial u}{\partial y} = 0 \text{ at } y = h \quad (5.3a)$$

$$u = U_{wall} \cos(\omega t) \text{ at } y = y_0 \quad (5.3b)$$

where y_0 is the wall roughness as r_0 defined in Chapter 4. Similarly, the velocity u , shear stress τ and eddy viscosity ν_T can be expressed by the following Fourier expansions,

$$u = \text{Re}[u^{(1)} e^{i\omega t} + u^{(3)} e^{3i\omega t} + \dots] \quad (5.4)$$

$$\tau = \text{Re}[\tau^{(1)} e^{i\omega t} + \tau^{(3)} e^{3i\omega t} + \dots] \quad (5.5)$$

$$\nu_T = \nu^{(0)} \cdot \text{Re}[1 + a^{(2)} e^{2i\omega t} + \dots] \quad (5.6)$$

Substituting (5.2), (5.4), (5.5) and (5.6) into (5.1), the governing equations for the Fourier components can be derived as,

$$u^{(1)} i\omega = \frac{d}{dy} \left(\nu^{(0)} \frac{du^{(1)}}{dy} \right) + \frac{a^{(2)}}{2} \frac{d}{dy} \left(\nu^{(0)} \frac{du^{(1)}}{dy} \right) \quad (5.7)$$

$$3u^{(3)} i\omega = \frac{d}{dy} \left(\nu^{(0)} \frac{du^{(3)}}{dy} \right) + \frac{a^{(2)}}{2} \frac{d}{dy} \left(\nu^{(0)} \frac{du^{(1)}}{dy} \right) \quad (5.8)$$

subject to the following boundary conditions:

$$\frac{du^{(1)}}{dy} = \frac{du^{(3)}}{dy} = 0 \text{ at } y = h \quad (5.9a)$$

$$u^{(1)} = U_{wall}; u^{(3)} = 0 \text{ at } y = y_0 \quad (5.9b)$$

In the following, we adopt a two-layer model also introduced by Trowbridge and Madsen (1984a) to represent the time-averaged part of the eddy viscosity, i.e.

$$\nu^{(0)} = \begin{cases} \overline{\kappa u_f} y & y_0 \leq y < \delta_I \\ \overline{\kappa u_f} \delta_I & \delta_I \leq y \leq h \end{cases} \quad (5.10)$$

where all the variables reserve the same meanings as in Chapter 4.

Substituting (5.10) into (5.7) and (5.8), and normalizing all the variables as below,

$$(u^{(1)'}, u^{(3)'}, \overline{u_f'}) = (u^{(1)}, u^{(3)}, \overline{u_f}) / U_{wall}, \quad t' = t\omega, \quad (y', y_0', \delta_I') = (y, y_0, \delta_I) / h \quad (5.11)$$

we obtain the governing equations in the non-dimensional form,

$$\alpha \overline{u_f'} \frac{d}{dy'} \left(\beta \frac{du^{(1)'}}{dy'} \right) - u^{(1)'} i = -\frac{a^{(2)}}{2} \alpha \overline{u_f'} \frac{d}{dy'} \left(\beta \frac{du_*^{(1)'}}{dy'} \right) \quad (5.12)$$

$$\alpha \overline{u_f'} \frac{d}{dy'} \left(\beta \frac{du^{(3)'}}{dy'} \right) - 3u^{(3)'} i = -\frac{a^{(2)}}{2} \alpha \overline{u_f'} \frac{d}{dy'} \left(\beta \frac{du^{(1)'}}{dy'} \right) \quad (5.13)$$

with the boundary conditions,

$$\frac{du^{(1)'}}{dy'} = \frac{du^{(3)'}}{dy'} = 0 \text{ at } y' = 1 \quad (5.14a)$$

$$u^{(1)'} = 1; u^{(3)'} = 0 \text{ at } y' = y_0' \quad (5.14b)$$

where $\alpha = \frac{\kappa U_{wall}}{\omega h}$, $\beta = \min(y', \delta_I')$ and $\delta_I' = \frac{1}{6} \min(\alpha \overline{u_f'}, 1)$.

Again, we introduce the auxiliary solutions $F^{(n)}$, ($n=1,3$), which satisfy the corresponding homogeneous differential equation,

$$\alpha \overline{u_f'}, \frac{d}{dy'} \left(\beta \frac{dF^{(n)}}{dy'} \right) - n F^{(n)} = 0 \quad (5.15)$$

Since β is not analytical at $y' = \delta_I'$, (5.15) needs to be solved separately within the inner boundary layer and in the outer region. Applying the fact that the velocity and shear stress need to be matched at $y' = \delta_I'$, and the outer-edge boundary condition,

$$\frac{dF^{(n)}(1)}{dy'} = 0, \text{ the explicit solutions for (5.15) can be obtained as illustrated below,}$$

Inside the inner boundary layer, $y_0' \leq y' < \delta_I'$,

$$F^{(n)} = x_1 I_0(\chi^{(n)} \sqrt{y'}) + x_2 K_0(\chi^{(n)} \sqrt{y'}) \quad (5.16)$$

where I_0 denotes the modified Bessel functions of the first kind of order zero, and

$$\chi^{(n)} = 2ni \sqrt{\frac{-i}{n \alpha \overline{u_f'}}}.$$

Outside the inner boundary layer and in the outer region, $\delta_I' \leq y' \leq 1$,

$$F^{(n)} = x_3 \exp(\lambda^{(n)} y') + x_4 \exp(-\lambda^{(n)} y') \quad (5.17)$$

$$\text{where } \lambda^{(n)} = \sqrt{\frac{ni}{\alpha \overline{u_f'} \delta_I'}}.$$

The expressions for the unknown parameters are presented in Appendix F for reference.

It is obvious that $u^{(1) '}$ has the form,

$$u^{(1) '}= \frac{F^{(1)}}{F^{(1)}(y_0 ')} + O(\varepsilon) \quad (5.18)$$

Substituting the above equation into the RHS of (5.12) and (5.13), after some algebra, the solutions of $u^{(1) '}$ and $u^{(3) '}$ correct to $O(\varepsilon)$ can be obtained as,

$$u^{(1) '}= \frac{F^{(1)}}{F^{(1)}(y_0 ')} - \frac{a^{(2)}}{4} \left(\frac{F_*^{(1)}}{F_*^{(1)}(y_0 ')} - \frac{F^{(1)}}{F^{(1)}(y_0 ')} \right) \quad (5.19)$$

$$u^{(3) '}= \frac{a^{(2)}}{4} \left(\frac{F^{(1)}}{F^{(1)}(y_0 ')} - \frac{F^{(3)}}{F^{(3)}(y_0 ')} \right) \quad (5.20)$$

Expressions for the Fourier components of the shear stress can then be calculated accordingly as,

$$\tau^{(1) '}= \overline{\kappa u_f} ' \beta \left[\frac{dF^{(1)} / dy'}{F^{(1)}(y_0 ')} + \frac{a^{(2)}}{4} \left(\frac{dF_*^{(1)} / dy'}{F_*^{(1)}(y_0 ')} + \frac{dF^{(1)} / dy'}{F^{(1)}(y_0 ')} \right) \right] \quad (5.21)$$

$$\tau^{(3) '}= \overline{\kappa u_f} ' \beta \left[\frac{3a^{(2)}}{4} \frac{dF^{(1)} / dy'}{F^{(1)}(y_0 ')} - \frac{a^{(2)}}{4} \frac{dF^{(3)} / dy'}{F^{(3)}(y_0 ')} \right] \quad (5.22)$$

To obtain the velocities and shear stresses, the unknown friction velocity $\overline{u_f} '$ and

constant $a^{(2)}$ need to be resolved. Applying the fact, $y' \frac{dF^{(n)}}{dy'} \rightarrow -\frac{1}{2}$ as $y' \rightarrow 0$, to (5.21)

and (5.22), the Fourier components of the wall shear stress $\tau_b '$ can be obtained as,

$$\tau_b^{(1) '}= \overline{\kappa u_f} ' \left[-\frac{1}{2F^{(1)}(y_0 ')} - \frac{a^{(2)}}{8} \left(\frac{1}{F_*^{(1)}(y_0 ')} + \frac{1}{F^{(1)}(y_0 ')} \right) \right] \quad (5.23)$$

$$\tau_b^{(3)'} = \overline{\kappa u_f'} \left[-\frac{3a^{(2)}}{8} \frac{1}{F^{(1)}(y_0')} + \frac{a^{(2)}}{8} \frac{1}{F^{(3)}(y_0')} \right] \quad (5.24)$$

Recalling (4.25) and (4.26), we have,

$$\overline{u_f'} = \frac{\kappa}{2\pi} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma^2\left(\frac{5}{4}\right)} \frac{1}{|F^{(1)}(y_0')|} \cdot \text{Re} \left\{ 1 + \frac{a^{(2)}}{4} \left[1 - \frac{4}{5} \frac{F^{(1)}(y_0')}{F_*^{(1)}(y_0')} + \frac{3}{5} \frac{F^{(1)}(y_0')F^{(1)}(y_0')}{F^{(3)}(y_0')F_*^{(1)}(y_0')} \right] + O(\varepsilon^2) \right\} \quad (5.25)$$

$$a^{(2)} = \frac{2}{5} \frac{F_*^{(1)}(y_0')}{F^{(1)}(y_0')} + O(\varepsilon) \quad (5.26)$$

For specific α and y_0' , $\overline{u_f'}$ can be solved iteratively by substituting (5.26) into (5.25), and then $a^{(2)}$ can be computed based on the solved $\overline{u_f'}$. Finally, the velocity and shear stress in non-dimensional form can be fully resolved. The results will be discussed in Section 5.4 in details.

5.2.2 Concentration field

After the flow characteristics are determined, we proceed to analyze the longitudinal dispersion of turbulent oscillatory flows inside the channel with synchronous axial vibrating walls. Since the induced flow is only in the x direction, the scalar transport in a channel is governed by the convection-diffusion equation:

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = D_t \frac{\partial^2 C}{\partial x^2} + \frac{\partial}{\partial y} \left(D_t \frac{\partial C}{\partial y} \right) \quad (5.27)$$

subject to the following boundary conditions:

$$D_t \frac{\partial C}{\partial y} = 0 \text{ at } y = y_0, h \quad (5.28)$$

Assume the turbulent dispersivity is proportional to the eddy viscosity, i.e.

$$D_t = \phi_t \nu_T \quad (5.29)$$

where the reciprocal of turbulent Schmidt number $\phi_t = O(1)$. Keeping the same assumption as in Chapter 4, the governing equation (5.27) can be rewritten using η to indicate the order of magnitude of each term,

$$\frac{\partial C}{\partial t} + \eta u \frac{\partial C}{\partial x} = \eta^2 D_t \frac{\partial^2 C}{\partial x^2} + \frac{\partial}{\partial y} \left(D_t \frac{\partial C}{\partial y} \right) \quad (5.30)$$

Applying the multiple-scale perturbation analysis, the leading order concentration C_0 has been shown to be independent of y and t , i.e.

$$C_0 = C_0(x, t_1, t_2) \quad (5.31)$$

At $O(\eta)$, the concentration C_1 is governed by

$$\frac{\partial C_1}{\partial t} + \frac{\partial C_0}{\partial t_1} + u \frac{\partial C_0}{\partial x} = \frac{\partial}{\partial y} \left(D_t \frac{\partial C_1}{\partial y} \right) \quad (5.32)$$

Substituting the first order of (5.4) for u and (5.29) for D_t , the above equation becomes,

$$\frac{\partial C_1}{\partial t} + \frac{\partial C_0}{\partial t_1} + \text{Re} \left[u^{(1)} e^{i\omega t} \right] \frac{\partial C_0}{\partial x} = \phi_t \text{Re} \left[1 + a^{(2)} e^{2i\omega t} \right] \frac{\partial}{\partial y} \left(\nu^{(0)} \frac{\partial C_1}{\partial y} \right) \quad (5.33)$$

with the following boundary conditions:

$$\nu^{(0)} \frac{\partial C_1}{\partial y} = 0 \text{ at } y = y_0, h \quad (5.34)$$

Taking the time and vertical-direction averages of (5.33), we find C_0 to be independent from t_1 as well, i.e.

$$\frac{\partial C_0}{\partial t_1} = 0 \quad (5.35)$$

Substituting (5.35) into (5.33) yields:

$$\frac{\partial C_1}{\partial t} + \text{Re}[u^{(1)} e^{i\omega t}] \frac{\partial C_0}{\partial x} = \phi_t \text{Re}[1 + a^{(2)} e^{2i\omega t}] \frac{\partial}{\partial y} \left(\nu^{(0)} \frac{\partial C_1}{\partial y} \right) \quad (5.36)$$

In view of linearity, the first order concentration C_1 can be expressed as,

$$C_1 = \frac{\partial C_0}{\partial x} \text{Re}[B^{(1)} e^{i\omega t}] \quad (5.37)$$

Substituting (5.37) into (5.36), we obtain the governing equation for $B^{(1)}$

$$i\omega B^{(1)} + u^{(1)} = \phi_t \frac{d}{dy} \left(\nu^{(0)} \frac{dB^{(1)}}{dy} \right) + \phi_t \frac{a^{(2)}}{2} \frac{d}{dy} \left(\nu^{(0)} \frac{dB_*^{(1)}}{dy} \right) \quad (5.38)$$

with the boundary conditions

$$\nu^{(0)} \left[\frac{dB^{(1)}}{dy} + \frac{a^{(2)}}{2} \frac{dB_*^{(1)}}{dy} \right] = 0 \text{ at } y = y_0, h \quad (5.39)$$

We proceed to consider $O(\eta^2)$. The governing equation is,

$$\frac{\partial C_0}{\partial t_2} + \frac{\partial C_1}{\partial t_1} + \frac{\partial C_2}{\partial t} + u \frac{\partial C_1}{\partial x} = D_t \frac{\partial^2 C_0}{\partial x^2} + \frac{\partial}{\partial y} \left(D_t \frac{\partial C_2}{\partial y} \right) \quad (5.40)$$

Substituting (5.35-5.37) into (5.40), we find,

$$\frac{\partial C_0}{\partial t_2} + \frac{\partial C_2}{\partial t} + \text{Re}[u^{(1)} e^{i\omega t}] \text{Re}[B^{(1)} e^{i\omega t}] \frac{\partial^2 C_0}{\partial x^2} = D_t \frac{\partial^2 C_0}{\partial x^2} + \frac{\partial}{\partial y} \left(D_t \frac{\partial C_2}{\partial y} \right) \quad (5.41)$$

Taking the time and cross-sectional averages of (5.41) and after some rearrangement, we obtain the effective scalar transport equation as:

$$\frac{\partial C_0}{\partial t_2} = \left\{ \left\langle \overline{D_t} \right\rangle - \frac{1}{2} \text{Re} \left\langle \left\langle u^{(1)} B_*^{(1)} \right\rangle \right\rangle \right\} \frac{\partial^2 C_0}{\partial x^2} \quad (5.42)$$

where the overbar represents the time average. We define the longitudinal dispersion coefficient as,

$$D_t = -\frac{1}{2} \text{Re} \left\langle \left\langle u^{(1)} B_*^{(1)} \right\rangle \right\rangle \quad (5.43)$$

which is in the same form of (4.49).

Recalling the governing equation of $B^{(1)}$ and normalizing all the variables and parameters, we have

$$\phi_t \alpha \overline{u_f}, \frac{d}{dy'} \left(\beta \frac{dB^{(1),}}{dy'} \right) - iB^{(1),} = u^{(1),} - \frac{a^{(2)}}{2} \phi_t \alpha \overline{u_f}, \frac{d}{dy'} \left(\beta \frac{dB_*^{(1),}}{dy'} \right) \quad (5.44)$$

with the boundary conditions

$$\alpha \overline{u_f}, \beta \left[\frac{dB^{(1),}}{dy'} + \frac{a^{(2)}}{2} \frac{dB_*^{(1),}}{dy'} \right] = 0 \text{ at } y' = y_0', 1 \quad (5.45)$$

where $B^{(1),} = B^{(1)} \omega / U_{wall}$.

Comparing the above equation with (5.12), one finds that the two equations are similar in form. As a result, the solution for the homogeneous differential equation of (5.45) can be expressed by G , where G should be in the same form with $F^{(1)}$ but changing α to $\phi_t \alpha$, i.e.

$$G(y', \alpha) = F^{(1)}(y', \phi_t \alpha) \quad (5.46)$$

Following the same method for $u^{(1)}$, the solution for (5.46) with correction to $O(\varepsilon)$ is,

$$B^{(1)'} = \left(1 + \frac{a^{(2)}}{4}\right) \frac{F^{(1)} - G}{(\phi_t - 1)F^{(1)}(y_0')i} + \frac{a^{(2)}(3\phi_t - 1)}{4(\phi_t - 1)(\phi_t + 1)} \frac{F_*^{(1)} - G}{F_*^{(1)}(y_0')i} - \frac{a^{(2)}}{4(\phi_t - 1)} \frac{G_* - G}{F_*^{(1)}(y_0')i} \quad (5.47)$$

Since both $F^{(1)}$ and G satisfy the outer-edge boundary condition and the wall limit, (5.47) is automatically satisfied. Moreover, it is confirmed that the solution has a finite limit as $\phi_t \rightarrow 1$, hence the solution is valid for all values of $\phi_t > 0$ (Ng, 2004).

To be consistent dimensionally with the turbulent dispersivity, the non-dimensional longitudinal dispersion coefficient is expressed in the following form,

$$D_l' = -\frac{\alpha}{2\kappa(1 - y_0')} \int_{y_0'}^1 \text{Re}[u^{(1)'} B_*^{(1)'}] dy' \quad (5.48)$$

Substituting (5.19) and (5.47) into (5.48), the integral can be calculated numerically.

5.3 Turbulent flows driven by single wall oscillation

5.3.1 Velocity field

Next, we analyze the condition which the turbulent oscillatory flow is generated by the axial vibration of a single wall while the opposing wall is stationary, as shown in Figure 5.2. The governing equation remains the same as (5.1) but the boundary conditions now become

$$u = 0 \text{ at } y = 2h - y_0 \quad (5.49a)$$

$$u = U_{wall} \cos(\omega t) \text{ at } y = y_0 \quad (5.49b)$$

Figure 5.2 Schematic diagram of the coordinate axis for the case of single wall vibration

Following the same approach, the boundary conditions for the Fourier components can be obtained as,

$$u^{(1)} = u^{(3)} = 0 \text{ at } y = 2h - y_0 \quad (5.50a)$$

$$u^{(1)} = U_{wall}; u^{(3)} = 0 \text{ at } y = y_0 \quad (5.50b)$$

Given that only one wall boundary is vibrating, the velocity profile and distribution of shear stress are not symmetric about the mid-plane. Thus, the time-averaged eddy viscosity $\nu^{(0)}$ needs to be expressed using a new three-layer model as follow,

$$\nu^{(0)} = \begin{cases} \overline{\kappa u_f} y & y_0 \leq y < \delta_I \\ \overline{\kappa u_f} \delta_I & \delta_I \leq y \leq 2h - \delta_I \\ \overline{\kappa u_f} (2h - y) & 2h - \delta_I < y \leq 2h - y_0 \end{cases} \quad (5.51)$$

where $\overline{u_f}$ is induced by the vibrating wall. Since the boundary layer thickness cannot be larger than the channel height, $2h$, we have,

$$\delta_I = \frac{1}{6} \min(l, 2h) \quad (5.52)$$

Normalizing all the parameters as before, we obtain the non-dimensional governing equations that are identical to (5.12) and (5.13), but with the new boundary conditions as

$$u^{(1)'} = u^{(3)'} = 0 \text{ at } y' = 2 - y_0' \quad (5.53a)$$

$$u^{(1)'} = 1; u^{(3)'} = 0 \text{ at } y' = y_0' \quad (5.53b)$$

$$\text{where } \beta = \begin{cases} y' & y_0' \leq y' < \delta_I' \\ \delta_I' & \delta_I' \leq y' \leq 2 - \delta_I' \\ (2 - y') & 2 - \delta_I' < y' \leq 2 - y_0' \end{cases}.$$

Again, we introduce the auxiliary functions $F^{(n)}$ as in (5.15), which satisfy the homogenous differential equation. Equation (5.15) needs to be solved separately for

different ranges of y' . For $y_0' \leq y' < \delta_l'$ and $\delta_l' \leq y' \leq 2 - \delta_l'$, the solutions of $F^{(n)}$ posses the same form as (5.16) and (5.17) but with different parameters, i.e. $x_5 - x_8$.

Within $2 - \delta_l' < y' \leq 2 - y_0'$, the solution becomes,

$$F^{(n)} = x_9 I_0(\chi^{(n)} \sqrt{2 - y'}) + x_{10} K_0(\chi^{(n)} \sqrt{2 - y'}) \quad (5.54)$$

Using the continuity of the velocity and shear stress at $y' = \delta_l'$ and $2 - \delta_l'$, as well as the outer boundary condition, $F^{(n)}(2 - y_0') = 0$, the unknown parameters x_5 to x_{10} can be obtained as described in Appendix G. Following the same approach as before, we can obtain the velocity profile as well as the distribution of shear stress with various α .

5.3.2 Concentration field

After the flow characteristics are obtained, we proceed to investigate the longitudinal dispersion due to the turbulent oscillatory flow induced by the single wall vibration. As illustrated in Section 5.2.2, we have shown that the longitudinal dispersion coefficient can be obtained through Equation (5.43). In this case, the governing equation for the non-dimensional function $B^{(1) '}$ is the same as (5.44) but with the different boundary conditions,

$$\overline{\alpha u_f} \beta \left[\frac{dB^{(1) '}}{dy'} + \frac{a^{(2)}}{2} \frac{dB_*^{(1) '}}{dy'} \right] = 0 \text{ at } y' = y_0', 2 - y_0' \quad (5.55)$$

Similarly, by introducing a function G which satisfies the homogenous equation of (5.44), the analytical solution of $B^{(1) '}$ can be expressed in the following form (where the second term on the RHS of (5.44) is ignored),

$$B^{(1)'} = G + \left(1 + \frac{a^{(2)}}{4}\right) \frac{F^{(1)}}{(\phi_t - 1)F^{(1)}(y_0')i} + \frac{a^{(2)}}{4(\phi_t + 1)} \frac{F_*^{(1)}}{F_*^{(1)}(y_0')i} \quad (5.56)$$

Different from the previous case of synchronous dual walls vibration which the boundary condition (5.45) can be automatically satisfied by setting $G(y', \alpha) = F^{(1)}(y', \phi_t \alpha)$, G needs to be solved in the present case by applying the matching conditions that G and $\frac{dG}{dy'}$ should be continuous at $y' = \delta_t'$ and $2 - \delta_t'$ and the boundary condition (5.55). Recalling the governing equation for G ,

$$\phi_t \alpha u_f \overline{}', \frac{d}{dy'} \left(\beta \frac{dG}{dy'} \right) - iG = 0 \quad (5.57)$$

the solutions of G can be obtained as follows,

Within $y_0' \leq y' < \delta_t'$,

$$G = x_{11} I_0(\chi_G y') + x_{12} K_0(\chi_G y') \quad (5.58)$$

Within $\delta_t' \leq y' \leq 2 - \delta_t'$,

$$G = x_{13} \exp(\lambda_G y') + x_{14} \exp(-\lambda_G y') \quad (5.59)$$

Within $2 - \delta_t' < y' \leq 2 - y_0'$,

$$G = x_{15} I_0(\chi_G \sqrt{2 - y'}) + x_{16} K_0(\chi_G \sqrt{2 - y'}) \quad (5.60)$$

where $\chi_G = 2i \sqrt{\frac{-i}{\phi_t \alpha u_f \overline{}'}}$ and $\lambda_G = \sqrt{\frac{i}{\phi_t \alpha u_f \overline{}' \delta_t'}}$, and the unknown parameters x_{11} to

x_{16} are computed using MATLAB. Substituting the new $B^{(1)'}$ into Equation (5.43), the

non-dimensional longitudinal dispersion coefficient under one wall oscillation can be obtained.

5.4 Results and discussion

The calculation results based on the above analytical solutions are presented in this section. Firstly, the variations of $\overline{u_f}$ and $a^{(2)}$ as functions of α are illustrated in Figures 5.3(a) and 5.3(b), respectively, with a comparison between synchronous dual walls oscillation and single wall oscillation. The friction velocity decreases as α increases from 5 to 500 with negligible difference between the two cases, while $a^{(2)}$ oscillates with α with an amplitude limited within 0.4.

Based on the results shown in Figure 5.3, the profile of velocity and shear stress distribution at various phases within one vibration period are obtained as demonstrated in Figures 5.4 and 5.5, respectively. The figures at the left column are for dual walls oscillation case and the right column for single wall oscillation (note that half of the channel height, i.e. $y_0' \leq y' \leq 1$, is presented for the former while the latter presents the whole height, i.e. $y_0' \leq y' \leq 2 - y_0'$). Since $\alpha = \frac{\kappa U_{wall}}{\omega h}$, a small α would represent an oscillatory flow with low velocity and high frequency, while a large α denotes the opposite. For the dual walls oscillation, it is clear from Figures 5.4(a-c) that with small α , the velocities near the centreline ($y' = 1$) vary with an extremely small amplitude over the period. A phase lag can be observed near the wall while in the outer region, the flow behaves as a plug flow. As α increases, the velocity profiles become more uniform over the cross section, and the velocity at the centre varies considerably with

time. For single wall oscillation, notable flow oscillation only exists within the region close to the vibrating boundary at small α (Figure 5.4(d)). When α increases, the magnitude of velocity near the stationary boundary increases, and the flow oscillation can now be observed over the entire cross section (Figures 5.4(e) and (f)). Similar trend can be noted in Figure 5.5. The shear stress distribution becomes more linear as α increases.

Figure 5.3 Variations of (a) friction velocity $\overline{u_f}'$ and (b) complex constant $a^{(2)}$ with α . The solid lines are for dual walls oscillation and dashed lines for single wall oscillation

(a) $\alpha = 5$

(d) $\alpha = 5$

(b) $\alpha = 50$

(e) $\alpha = 50$

(c) $\alpha = 500$

(f) $\alpha = 500$

Figure 5.4 Velocity profiles at various phases within half an oscillation period at different α with $y_0' = 0.001$ (left: dual walls oscillation, right: single wall oscillation)

(a) $\alpha = 5$

(d) $\alpha = 5$

(b) $\alpha = 50$

(e) $\alpha = 50$

(c) $\alpha = 500$

(f) $\alpha = 500$

Figure 5.5 Distribution of shear stress at various phases within half an oscillation period at different α with $y_0' = 0.001$ (left: dual walls oscillation, right: single wall oscillation)

The relationship between D_l' and α is illustrated in Figure 5.6 for $\phi_t = 0.5, 1.0$ and 2.0 . Figure 5.6(a) shows that for the case of dual walls oscillation, D_l' increases with α generally but with a slight decrease at $\alpha = 15 \sim 20$ for the cases of $\phi_t = 1.0$ and 2.0 . After the change-over, D_l' becomes relatively constant. Similar trend can be observed for the single wall oscillation (see Figure 5.6(b)). However, the change-over of slope occurs at different α ($\alpha = 35 \sim 45$) and after that D_l' keeps increasing. The reason is that for the single wall vibration, the maximum inner boundary layer thickness is twice of that for the synchronous dual walls vibration. Moreover, D_l' for the single wall vibration is smaller than that of the dual walls vibration with small α , but reverses when α is large. This is because when α is small, the velocity profile only varies significantly near the single moving boundary, and thus induces less dispersion compared to the dual walls vibration. However, when α is large, the flow approximates to a steady shear flow, and D_l' becomes large. The effect of ϕ_t on the dispersion coefficient is the same for both cases. D_l' tends to increase with ϕ_t only moderately at small α , but decrease with ϕ_t significantly when α exceeds the change-over point. Unlike the results presented in Figure 4.6, D_l' here decreases slightly at the change-over point and then continues to increase. The reason is that for pressure driven oscillatory flows, after reaching the change-over point, further increase in α reduces the velocity variation over the cross section and thus leads to the decrease in the longitudinal dispersion. However, for wall driven oscillatory flows, where the movements are introduced from the boundaries, larger α leads to larger velocity at the core region and makes D_l' keep increasing.

(a)

(b)

Figure 5.6 Longitudinal dispersion coefficient D_l' as a function of α at different ϕ_t with $y_0' = 0.001$. ((a) dual walls oscillation and (b) single wall oscillation. $\phi_t = 0.5$ (dash), $\phi_t = 1.0$ (solid) and $\phi_t = 2.0$ (dash dot))

The time and cross-sectional average turbulent dispersivity $\langle\langle \overline{D_t'} \rangle\rangle$ is plotted in Figure 5.7. For both cases, $\langle\langle \overline{D_t'} \rangle\rangle$ first increases with α and then decreases with the change-over points around 25 and 50 for the dual walls oscillation and single wall oscillation, respectively. Comparing Figures 5.7 and 5.6, one can find that the magnitude of D_t' is typically several order larger than that of $\langle\langle \overline{D_t'} \rangle\rangle$, which suggests that the turbulent dispersivity can be mostly ignored in the axial dispersion of turbulent oscillatory flow. Note that the trend of the present results bears similarity to the results due to periodic pressure-driven turbulent oscillatory flows in Ng (2004), despite the differences in our configurations. However, it should be noted that in our case the longitudinal dispersion coefficient continues to increase when α exceeds 100, in contrast with Ng's results in which D_t' decreases slightly. The reason can be attributed to the stronger vibration of the oscillatory flow in the outer region induced by the large α with dual vibrating walls, which leads to the larger dispersion coefficient.

(a)

(b)

Figure 5.7 Turbulent dispersivity $\langle\langle \overline{D_t'} \rangle\rangle$ as a function of α at different ϕ_t with $y_0' = 0.001$. ((a) dual walls oscillation and (b) single wall oscillation. $\phi_t = 0.5$ (dash), $\phi_t = 1.0$ (solid) and $\phi_t = 2.0$ (dash dot))

5.5 Conclusions

In this chapter, an analytical analysis was conducted on the longitudinal dispersion of turbulent oscillatory flows inside a two-dimensional channel driven by vibrating wall boundaries. Two specific cases were investigated, i.e. synchronous vibration of the dual boundaries and the vibration of single wall only. Similar to the results of Chapter 4, a non-dimensional parameter, α , dominates the flow structure as well as the induced longitudinal dispersion. Generally, the magnitude of the non-dimensional longitudinal dispersion coefficient increases with α . As $\alpha \propto U_{wall}/(\omega h)$ and $D_l = D_l' U_{wall} h$, we can further conclude that the longitudinal dispersion coefficient D_l can be enhanced by reducing the channel vibrating frequency and increasing the amplitude of vibration velocity.

For the condition of small wall velocity and high oscillation frequency, the dispersion coefficient induced by synchronous dual walls vibration is larger than that by single wall vibration. However, with large velocity and low frequency, the vibration of single wall tends to generate stronger dispersion. Moreover, the effect of Schmidt number on the dispersion is not obvious at small α , although larger Schmidt number tends to induce stronger dispersion when α is large.

The assumptions made in this chapter are similar to those in chapter 4. However, it should be noted that in this chapter the Reynolds number is defined by $Re_\delta = \frac{U_{wall} \delta}{\nu}$.

Since no prior literature on the transition to turbulence in oscillatory plane channel flows can be found, the Reynolds number whereby the transition to turbulence occurs is uncertain.

Chapter 6 Longitudinal dispersion of two-dimensional oscillatory electro-osmotic flows with reactive walls*

6.1 Introduction

Electroosmosis has been widely applied in micro-devices to generate electro-osmotic flows. The associated dispersion which is related to species separation and mixing in micro- and nano-scales plays an important role in many microfluidics applications, such as pollutants removal of ground water, drug delivery, component sensing and so forth. A number of studies have been conducted on hydrodynamic dispersion in both steady and oscillatory EOFs as summarized in Section 2.3.

Reaction between the micro-channel walls and the species in fluids plays an important in EOFs. In reality, rectangular micro-channels are typically fabricated in a way that the materials of the two parallel plates are different and thus possess different electrical and chemical potentials. As a result, a general model which can take into account the effects of different zeta potentials and sorption at the two walls is desirable. In this chapter, dispersion of oscillatory EOFs in a two-dimensional micro-channel with different wall potentials as well as reversible sorption effect is theoretically investigated by using the homogenization approach. The structure of the chapter is: the problem is firstly characterized mathematically in Section 6.2, followed by applying the homogenization method to solve for the dispersion coefficient (Section 6.3), which possesses some similarities to Chapter 5. Results are presented to illustrate how the dispersion

*This chapter is written based on “Dispersion in oscillatory electro-osmotic flow through a parallel-plate channel with kinetic sorptive exchange at walls”, *Journal of Hydrodynamics* (2014).

coefficient may depend on the various controlling parameters in Section 6.4, and Section 6.5 presents the conclusions.

6.2 Problem formulation

We consider time-oscillatory electro-osmotic flow (EOF) between two parallel plates with possibly different wall potentials and sorption. A sinusoidally time-varying axial electric field is imposed on the system such that oscillatory EOFs could be generated inside the two-dimensional microchannel. As illustrated in Figure 6.1, a Cartesian coordinate system is used where the x -axis represents the axial direction and the y -axis is perpendicular to the flow direction. The distance between the top and bottom walls is $2h$. The time-varying electric field E is given by

$$E = E_0 \operatorname{Re}[e^{i\omega t}] \quad (6.1)$$

where E_0 is a constant amplitude of the electric field.

Figure 6.1 Schematic diagram of the problem concerned

The fluid is assumed to be isothermal, Newtonian and incompressible. In the absence of a pressure gradient, the governing equation for the flow field is

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + \rho_e E \quad (6.2)$$

with no-slip boundary conditions at the walls

$$u = 0 \text{ on } y = \pm h \quad (6.3)$$

where μ is the dynamic viscosity of fluid, and ρ_e is the electric charge density.

Now we proceed to consider the electric field. Assume the electric charge density ρ_e satisfies the Boltzmann distribution, i.e.

$$\rho_e = -2ezc_0 \sinh\left(\frac{ze\psi}{R_B T_a}\right) \quad (6.4)$$

where e is the electron charge, z is the valence of the co- and counter- ions in the carrier liquid, c_0 is the ion concentration far from the charged walls, R_B is the Boltzmann constant, T_a is the absolute temperature, and ψ is the electric potential, which can be expressed by the following Poisson equation (Paul and Ng, 2012),

$$\frac{\partial^2 \psi}{\partial y^2} = -\frac{\rho_e}{\varpi} \quad (6.5)$$

where ϖ is the permittivity of the liquid medium.

Substituting (6.4) into (6.5) and applying the Debye-Hückel approximation as the electric potential ψ is assumed to be sufficiently small, we obtain the following linear equation,

$$\frac{\partial^2 \psi}{\partial y^2} = k^2 \psi \quad (6.6)$$

where $k = \left(\frac{R_B T_a \varpi}{2e^2 z^2 c_0} \right)^{-1/2}$ is known as the Debye parameter, or the reciprocal of the Debye length (Paul and Ng, 2012).

The boundary conditions for Equation (6.6) are prescribed by the wall potentials. In the present study, we consider the case where the top and bottom walls may have different potentials, i.e.

$$\psi = \psi_1 \text{ on } y = h \quad (6.7a)$$

$$\psi = \psi_2 \text{ on } y = -h \quad (6.7b)$$

Equation (6.6) along with (6.7) yields the following solution for ψ :

$$\psi(y) = \psi_1 \left(\frac{1 + \psi_2' \cosh(ky)}{2 \cosh(kh)} + \frac{1 - \psi_2' \sinh(ky)}{2 \sinh(kh)} \right) \quad (6.8)$$

where ψ_2' is the ratio of the two wall potentials

$$\psi_2' = \psi_2 / \psi_1 \quad (6.9)$$

Substituting (6.8) and (6.1) into (6.2), the governing equation becomes

$$\frac{1}{\nu} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + k^2 U_{HS} \left(\frac{(1 + \psi_2') \cosh(ky)}{2 \cosh(kh)} + \frac{(1 - \psi_2') \sinh(ky)}{2 \sinh(kh)} \right) \text{Re} \left[e^{i\omega t} \right] \quad (6.10)$$

where $U_{HS} = \frac{-\varpi E_0 \psi_1}{\mu}$ is the so-called Helmholtz-Smoluchowski velocity.

Let us introduce the following form of velocity profile

$$u(y, t) = \text{Re} \left[g(y) e^{i\omega t} \right] \quad (6.11)$$

where g is a complex function of y .

Substituting (6.11) into (6.10), we obtain the governing equation for g as below:

$$\frac{i\omega}{\nu} g = \frac{d^2 g}{dy^2} + k^2 U_{HS} \left(\frac{(1 + \psi_2')}{2} \frac{\cosh(ky)}{\cosh(kh)} + \frac{(1 - \psi_2')}{2} \frac{\sinh(ky)}{\sinh(kh)} \right) \quad (6.12)$$

with the boundary conditions

$$g = 0 \text{ on } y = \pm h \quad (6.13)$$

The solution of g can be obtained readily as follows

$$g = \frac{k^2}{k^2 - \lambda^2} U_{HS} \left(\frac{(1 + \psi_2')}{2} \left[\frac{\cosh(\lambda y)}{\cosh(\lambda h)} - \frac{\cosh(ky)}{\cosh(kh)} \right] + \frac{(1 - \psi_2')}{2} \left[\frac{\sinh(\lambda y)}{\sinh(\lambda h)} - \frac{\sinh(ky)}{\sinh(kh)} \right] \right) \quad (6.14)$$

where $\lambda^2 = i\omega / \nu$ or $\lambda = (1 + i) / \delta$.

6.3 Mass transfer

For the solute dispersion, we consider that the materials of the two walls are different whereby the neutral species in the fluid undergo reversible sorption at the top wall while the bottom wall is chemically inert. The dispersion is governed by the convection-diffusion equation, i.e.

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = D \frac{\partial^2 C}{\partial y^2} + D \frac{\partial^2 C}{\partial x^2} \quad (6.15)$$

with the boundary conditions,

$$-D \frac{\partial C}{\partial y} = \frac{\partial C_s}{\partial t} = \chi(\xi C - C_s) \quad \text{on } y = h \quad (6.16a)$$

$$\frac{\partial C}{\partial y} = 0 \quad \text{on } y = -h \quad (6.16b)$$

where C is the concentration of the mobile phase (mass of species dissolved in unit volume of fluid), C_s is the concentration of the immobile phase (mass of species adsorbed on unit surface area of wall), χ is the reaction rate constant and ξ is a partition coefficient of the chemical. When at equilibrium, C_s can be related to C by,

$$\frac{C_s}{C} = \xi \quad (6.17)$$

Similarly, we introduce the homogenization approach with the multiple-scale perturbation analysis. Adopting three sharply distinct time scales t , t_1 and t_2 to represent the three different transport processes, i.e. diffusion across the channel, advection and dispersion along the axial direction, respectively, i.e.

$$t, t_1 = \eta t, t_2 = \eta^2 t \quad (6.18)$$

where η is a perturbation parameter much smaller than one.

The relative significance of terms in (6.15) can be indicated by η as below

$$\frac{\partial C}{\partial t} + \eta u \frac{\partial C}{\partial x} = D \frac{\partial^2 C}{\partial y^2} + \eta^2 D \frac{\partial^2 C}{\partial x^2} \quad (6.19)$$

subject to the following boundary conditions:

$$-D \frac{\partial C}{\partial y} = \frac{\partial C_s}{\partial t} = \chi(\xi C - C_s) \quad \text{on } y = h \quad (6.20a)$$

$$\frac{\partial C}{\partial y} = 0 \quad \text{on } y = -h \quad (6.20b)$$

In the same manner, the concentration and time derivative can be expanded as follows:

$$C(x, y, t) = C_0 + \eta C_1 + \eta^2 C_2 + \dots \quad (6.21)$$

$$C_s(x, t) = C_{s0} + \eta C_{s1} + \eta^2 C_{s2} + \dots \quad (6.22)$$

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial t_1} + \eta^2 \frac{\partial}{\partial t_2} \quad (6.23)$$

At the leading order $O(1)$, Equation (6.19) becomes

$$0 = D \frac{\partial^2 C_0}{\partial y^2} \quad (6.24)$$

with the boundary conditions

$$-D \frac{\partial C_0}{\partial y} = \chi(\xi C_0 - C_{s0}) \quad \text{on } y = h \quad (6.25a)$$

$$\frac{\partial C_0}{\partial y} = 0 \quad \text{on } y = -h \quad (6.25b)$$

Equations (6.24) and (6.25b) imply that

$$C_0 = C_0(x, t_1, t_2) \quad (6.26)$$

Also, (6.25a) suggests

$$C_{s0} = \xi C_0 \quad (6.27)$$

At $O(\eta)$, the concentration C_1 is governed by:

$$\frac{\partial C_1}{\partial t} + \frac{\partial C_0}{\partial t_1} + u \frac{\partial C_0}{\partial x} = D \frac{\partial^2 C_1}{\partial y^2} \quad (6.28)$$

and the boundary conditions become

$$-D \frac{\partial C_1}{\partial y} = \frac{\partial C_{s0}}{\partial t_1} + \frac{\partial C_{s1}}{\partial t} = \chi(\xi C_1 - C_{s1}) \quad \text{on } y = h \quad (6.29a)$$

$$\frac{\partial C_1}{\partial y} = 0 \quad \text{on } y = -h \quad (6.29b)$$

Taking time and cross-sectional average of (6.28), we obtain

$$\frac{\partial C_0}{\partial t_1} = 0 \quad (6.30)$$

Substituting (6.30) back into (6.28), the equation is modified to

$$\frac{\partial C_1}{\partial t} + u \frac{\partial C_0}{\partial x} = D \frac{\partial^2 C_1}{\partial y^2} \quad (6.31)$$

with the boundary conditions

$$-D \frac{\partial C_1}{\partial y} = \frac{\partial C_{s1}}{\partial t} = \chi(\xi C_1 - C_{s1}) \quad \text{on } y = h \quad (6.32)$$

In view of linearity, the first order concentration C_1 can be expressed as:

$$C_1 = \text{Re} \left[B(y) e^{i\omega t} \right] \frac{\partial C_0}{\partial x} + X(x) \quad (6.33)$$

$$C_{s1} = \text{Re}[B_s e^{i\omega t}] \frac{\partial C_0}{\partial x} + X_s \quad (6.34)$$

where B is a complex function of y that will be derived later.

At $O(\eta^2)$, (6.19) and (6.20) give

$$\frac{\partial C_0}{\partial t_2} + \frac{\partial C_1}{\partial t_1} + \frac{\partial C_2}{\partial t} + u \frac{\partial C_1}{\partial x} = D \frac{\partial^2 C_0}{\partial x^2} + D \frac{\partial^2 C_2}{\partial y^2} \quad (6.35)$$

with the boundary conditions

$$-D \frac{\partial C_2}{\partial y} = \frac{\partial C_{s0}}{\partial t_2} + \frac{\partial C_{s1}}{\partial t_1} + \frac{\partial C_{s2}}{\partial t} = \chi(\xi C_2 - C_{s2}) \quad \text{on } y = h \quad (6.36a)$$

$$\frac{\partial C_2}{\partial y} = 0 \quad \text{on } y = -h \quad (6.36b)$$

Taking time and cross-sectional average of (6.35), (6.36a) and (6.36b), we obtain

$$\frac{\partial C_0}{\partial t_2} = \left(\frac{D}{R_a} - \frac{1}{2R_a} \text{Re} \langle \langle g B_* \rangle \rangle \right) \frac{\partial^2 C_0}{\partial x^2} \quad (6.37)$$

where $R_a = 1 + \xi/2h$ is called the retardation factor. Hence,

$$D_{rw} = -\frac{1}{2R_a} \text{Re} \langle \langle g B_* \rangle \rangle \quad (6.38)$$

is the dispersion coefficient due to the oscillatory flow.

Now we proceed to solve the function B , which is the key towards the determination of the dispersion coefficient. To better illustrate the problem, we introduce the normalized parameters as below,

$$g' = g / U_{HS}, B' = \frac{B}{U_{HS} h^2 / D}, y' = y / h; k' = kh, \lambda' = \lambda h, \xi' = \xi / h \quad (6.39)$$

It is easy to express g' in non-dimensional form as

$$g' = \frac{k'^2}{k'^2 - \lambda'^2} \left(\frac{(1 + \psi_2')}{2} \left[\frac{\cosh(\lambda' y')}{\cosh(\lambda')} - \frac{\cosh(k' y')}{\cosh(k')} \right] + \frac{(1 - \psi_2')}{2} \left[\frac{\sinh(\lambda' y')}{\sinh(\lambda')} - \frac{\sinh(k' y')}{\sinh(k')} \right] \right) \quad (6.40)$$

The function B' is governed by,

$$\gamma'^2 B' + g' = \frac{d^2 B'}{dy'^2} \quad (6.41)$$

with the boundary conditions

$$-\frac{dB'}{dy'} = \varphi' B' \text{ on } y' = 1 \quad (6.42a)$$

$$\frac{dB'}{dy'} = 0 \text{ on } y' = -1 \quad (6.42b)$$

where $\gamma' = h \sqrt{\frac{i\omega}{D}}$, $\varphi' = \frac{i\omega \chi \xi h}{(\chi + i\omega)D} = \frac{(\chi / \omega)i + 1}{(\chi / \omega)^2 + 1} Da \xi'$ and $Da = \frac{\chi h^2}{D}$ is the

Damkohler number.

The solution of (6.41) can be obtained as,

$$\begin{aligned} B' = & X_1 \cosh(\gamma' y') + X_2 \sinh(\gamma' y') + \frac{k'^2 (1 + \psi_2')}{2(k'^2 - \lambda'^2)(\lambda'^2 - \gamma'^2) \cosh(\lambda')} \cosh(\lambda' y') \\ & + \frac{k'^2 (1 - \psi_2')}{2(k'^2 - \lambda'^2)(\lambda'^2 - \gamma'^2) \sinh(\lambda')} \sinh(\lambda' y') \\ & - \frac{k'^2 (1 + \psi_2')}{2(k'^2 - \lambda'^2)(k'^2 - \gamma'^2) \cosh(k')} \cosh(k' y') - \frac{k'^2 (1 - \psi_2')}{2(k'^2 - \lambda'^2)(k'^2 - \gamma'^2) \sinh(k')} \sinh(k' y') \end{aligned} \quad (6.43)$$

The unknown parameters X_1 and X_2 can be determined by applying the boundary conditions. Their expressions are presented in Appendix H.

Substituting (6.40) and (6.43) into (6.38) and taking cross-sectional average, we obtain

the dimensionless dispersion coefficient $D'_{Tw} = \frac{D_{Tw}}{U_{HS}^2 h^2 / D}$ as below,

$$\begin{aligned}
D'_{Tw} = & -\frac{1}{4R_a} \text{Re} \left[X_{1*} E_1 \left(\frac{\sinh(\gamma'_* + \lambda')}{\gamma'_* + \lambda'} + \frac{\sinh(\gamma'_* - \lambda')}{\gamma'_* - \lambda'} \right) + X_{1*} F_1 \left(\frac{\sinh(\gamma'_* + k')}{\gamma'_* + k'} + \frac{\sinh(\gamma'_* - k')}{\gamma'_* - k'} \right) \right. \\
& + \frac{F_{1*} E_1}{k'^2 - \gamma'^2_*} \left(\frac{\sinh(k' + \lambda')}{k' + \lambda'} + \frac{\sinh(k' - \lambda')}{k' - \lambda'} \right) + \frac{F_{1*} F_1}{k'^2 - \gamma'^2_*} \left(\frac{\sinh(2k')}{2k'} + 1 \right) \\
& + \frac{E_{1*} E_1}{\lambda'^2_* - \gamma'^2_*} \left(\frac{\sinh(\lambda'_* + \lambda')}{\lambda'_* + \lambda'} + \frac{\sinh(\lambda'_* - \lambda')}{\lambda'_* - \lambda'} \right) + \frac{E_{1*} F_1}{\lambda'^2_* - \gamma'^2_*} \left(\frac{\sinh(\lambda'_* + k')}{\lambda'_* + k'} + \frac{\sinh(\lambda'_* - k')}{\lambda'_* - k'} \right) \\
& + X_{2*} E_2 \left(\frac{\sinh(\gamma'_* + \lambda')}{\gamma'_* + \lambda'} - \frac{\sinh(\gamma'_* - \lambda')}{\gamma'_* - \lambda'} \right) + X_{2*} F_2 \left(\frac{\sinh(\gamma'_* + k')}{\gamma'_* + k'} - \frac{\sinh(\gamma'_* - k')}{\gamma'_* - k'} \right) \\
& + \frac{F_{2*} E_2}{k'^2 - \gamma'^2_*} E_2 \left(\frac{\sinh(k' + \lambda')}{k' + \lambda'} - \frac{\sinh(k' - \lambda')}{k' - \lambda'} \right) + \frac{F_{2*} F_2}{k'^2 - \gamma'^2_*} \left(\frac{\sinh(2k')}{2k'} - 1 \right) \\
& \left. + \frac{E_{2*} E_2}{\lambda'^2_* - \gamma'^2_*} \left(\frac{\sinh(\lambda'_* + \lambda')}{\lambda'_* + \lambda'} - \frac{\sinh(\lambda'_* - \lambda')}{\lambda'_* - \lambda'} \right) + \frac{E_{2*} F_2}{\lambda'^2_* - \gamma'^2_*} \left(\frac{\sinh(\lambda'_* + k')}{\lambda'_* + k'} - \frac{\sinh(\lambda'_* - k')}{\lambda'_* - k'} \right) \right]
\end{aligned} \tag{6.44}$$

where $E_1 = \frac{k'^2}{k'^2 - \lambda'^2} \frac{1 + \psi'_2}{\cosh(\lambda')}$, $E_2 = \frac{k'^2}{k'^2 - \lambda'^2} \frac{1 - \psi'_2}{\sinh(\lambda')}$, $F_1 = -\frac{k'^2}{k'^2 - \lambda'^2} \frac{1 + \psi'_2}{\cosh(k')}$ and

$$F_2 = -\frac{k'^2}{k'^2 - \lambda'^2} \frac{1 - \psi'_2}{\sinh(k')}.$$

Particular cases I

For the case where the electric potential of the two walls are the same, i.e. $\psi'_2 = 1$,

Equation (6.44) can be simplified to

$$\begin{aligned}
 D_{Tw}' = & -\frac{1}{4R_a} \text{Re}[X_{1*}E_1 \left(\frac{\sinh(\gamma'_* + \lambda')}{\gamma'_* + \lambda'} + \frac{\sinh(\gamma'_* - \lambda')}{\gamma'_* - \lambda'} \right) + X_{1*}F_1 \left(\frac{\sinh(\gamma'_* + k')}{\gamma'_* + k'} + \frac{\sinh(\gamma'_* - k')}{\gamma'_* - k'} \right) \\
 & + \frac{F_{1*}E_1}{k'^2 - \gamma'^2} \left(\frac{\sinh(k' + \lambda')}{k' + \lambda'} + \frac{\sinh(k' - \lambda')}{k' - \lambda'} \right) + \frac{F_{1*}F_1}{k'^2 - \gamma'^2} \left(\frac{\sinh(2k')}{2k'} + 1 \right) \\
 & + \frac{E_{1*}E_1}{\lambda'^2 - \gamma'^2} \left(\frac{\sinh(\lambda'_* + \lambda')}{\lambda'_* + \lambda'} + \frac{\sinh(\lambda'_* - \lambda')}{\lambda'_* - \lambda'} \right) + \frac{E_{1*}F_1}{\lambda'^2 - \gamma'^2} \left(\frac{\sinh(\lambda'_* + k')}{\lambda'_* + k'} + \frac{\sinh(\lambda'_* - k')}{\lambda'_* - k'} \right)
 \end{aligned} \tag{6.45}$$

where $E_1 = \frac{k'^2}{k'^2 - \lambda'^2} \frac{2}{\cosh(\lambda')}$, $F_1 = -\frac{k'^2}{k'^2 - \lambda'^2} \frac{2}{\cosh(k')}$ and

$$\begin{aligned}
 X_1 = & -\left\{ \frac{(2k'\gamma' \sinh(k') \cosh(\gamma') + \varphi' \gamma' \cosh(k') \cosh(\gamma') + \varphi' k' \sinh(k') \sinh(\gamma')) F_1}{\gamma' (2\gamma' \sinh(\gamma') \cosh(\gamma') + \varphi' \cosh(\gamma')^2 + \varphi' \sinh(\gamma')^2) (k'^2 - \gamma'^2)} \right. \\
 & \left. + \frac{(2\lambda' \gamma' \sinh(\lambda') \cosh(\gamma') + \varphi' \gamma' \cosh(\lambda') \cosh(\gamma') + \varphi' \lambda' \sinh(\lambda') \sinh(\gamma')) E_1}{\gamma' (2\gamma' \sinh(\gamma') \cosh(\gamma') + \varphi' \cosh(\gamma')^2 + \varphi' \sinh(\gamma')^2) (\lambda'^2 - \gamma'^2)} \right\}.
 \end{aligned}$$

Particular case II

For the case where the two wall potentials are the same and no sorption occurs at both walls, the form of the dispersion coefficient remains the same as (6.45) with X_1 becomes

$$X_1 = -\frac{\lambda' \sinh(\lambda') E_1 / (\lambda'^2 - \gamma'^2) + k' \sinh(k') F_1 / (k'^2 - \gamma'^2)}{\gamma' \sinh(\gamma')}$$

The derived result is the same as the dispersion coefficient due to the interaction of the oscillatory electric field with the steady component of the wall potentials presented in Paul and Ng (2012).

Particular case III

For the case that the two wall potentials are identical and symmetric sorption occurs at both walls, R_a in (6.45) becomes $R_a = 1 + \xi'$, and the boundary conditions (6.42b) changes to

$$\frac{dB'}{dy'} = 0 \text{ on } y' = 0 \quad (6.46)$$

We obtain

$$X_1 = -\frac{(\lambda' \sinh(\lambda') + \varphi' \cosh(\lambda')) E_1 / (\lambda'^2 - \gamma'^2) + (k' \sinh(k') + \varphi' \cosh(k')) F_1 / (k'^2 - \gamma'^2)}{\gamma' \sinh(\gamma') + \varphi' \cosh(\gamma')}$$

6.4 Results and discussion

We evaluate the dependence of the dimensionless dispersion coefficient D'_{Tw} on various dimensionless parameters as illustrated below. We first review the physical significance of these parameters. $\delta' = \delta / h = \sqrt{2\nu / \omega} / h$ is the dimensionless Stokes boundary layer thickness. For flow with higher oscillation frequency, the Stokes

boundary layer is thinner and δ' is smaller, and vice versa. Since $\lambda' = \frac{1+i}{\delta} h$ and

$\gamma' = h \sqrt{\frac{i\omega}{D}}$, one finds that δ' is inversely proportional to λ' and γ' , i.e. $\lambda' = (1+i)/\delta'$

and $\gamma' = \sqrt{Sc}(1+i)/\delta'$, where Sc is fixed to be 1000 in the present study. EOF allows a wide range of frequencies, typically below ~ 1 MHz (Ramon et al., 2011), and thus a wide range of the Stokes boundary layer thickness will be considered in the following, i.e. $O(0.1) \sim O(10)$. k' characterizes the EDL thickness and smaller k' implies larger Debye length. Typically, the value of k' is large, i.e. $O(10) \sim O(100)$. ξ' denotes a normalized partition coefficient of the species, Da signifies the reversible sorption rate where larger Da implies faster phase exchange kinetics, and ψ'_2 represents the potential of the bottom wall relative to that of the top wall, which is kept between -1 and 1. It should be noted that the Reynolds number of EOF is typically very small.

The velocity profile is governed by the Stokes boundary layer thickness, the Debye length as well as the wall potentials. Figure 6.2 shows the velocity profile at different times within an oscillation cycle when the potentials at the two walls are the same, i.e. $\psi'_2 = 1$, where the dimensionless $u' = u/U_{HS}$. It can be seen that for the flow with a high frequency, the velocity varies significantly near the wall due to the viscous effect, and becomes almost a constant (zero) in the core region due to inertia. As ω decreases, the velocity profile becomes more uniform and finally turns to a ‘plug flow’ profile as that of a steady EOF when δ' reaches 10. The effect of Debye length is obvious; as k' increases, the plug-like core region of the flow increases in size. For the conditions that the wall potentials are different, the velocity profiles become asymmetrical about the centerline but possess similar trends.

(a) $k' = 10$, $\delta' = 0.1$

(d) $k' = 100$, $\delta' = 0.1$

(b) $k' = 10$, $\delta' = 1$

(e) $k' = 100$, $\delta' = 1$

(c) $k' = 10$, $\delta' = 10$

(f) $k' = 100$, $\delta' = 10$

Figure 6.2 Scaled velocity profiles at various phases within half an oscillation for $\psi'_2 = 1$ and $Sc = 1000$

We next proceed to evaluate the dispersion coefficient induced by the oscillatory EOFs with the presence of wall sorption effect. Ramon et al. (2011) investigated the dispersion in a circular channel with small oscillating frequency where the flow behaves as a plug flow. In the present study, we consider a much wider range of the oscillating frequency. Figure 6.3 shows the variation of D'_{Tw} with the Stokes boundary layer thickness δ' under different wall reaction conditions. It is obvious that the dimensionless D'_{Tw} increases with δ' significantly. It suggests that keeping the velocity amplitude unchanged, the dispersion coefficient is enhanced by reducing the oscillation frequency. This is reasonable as the stroke displacement increases when ω decreases and thus leads to stronger mass transfer. One noteworthy thing is that by fixing the stroke displacement, the dispersion coefficient increases with the oscillation frequency.

Figure 6.3 also compares D'_{Tw} under various wall sorption conditions, i.e. no sorption at either wall, sorption at the top wall only and symmetric sorption at both walls. The results demonstrate that the presence of wall sorption could enhance the dispersion process: the faster the reaction kinetics Da , the stronger the dispersion when δ' is large. This may be attributed to the fact that wall sorption allows for a greater temporal storage and thus enhances the mass transfer under appropriate flow conditions (Ramon et al., 2011). However, for small δ' , i.e. δ' less than around 2, stronger dispersion could be achieved when there is no sorption at the walls even though the difference is very small.

(a)

(b)

Figure 6.3 Dispersion coefficient D'_{Tw} as a function of δ' for $k'=100$, $\psi'_2=1$, $\xi'=1$ and $Sc=1000$. (a) Comparison among different Da and (b) Comparison among different sorption conditions with $Da=1$

Figure 6.4 shows the relationship between k' and D'_{Tw} under various wall sorption conditions. For the flows with the thin Stokes boundary layer, i.e. $\delta'=0.1$ (Figure 6.4(a) and 6.4(b)), the dimensionless dispersion coefficient decreases with the EDL thickness and the amplitude of D'_{Tw} is very small. As can be seen from Figure 6.2(a) and 6.2(d), the velocity profile varies only near the boundaries when the EDL thickness is thinner. Also, an inert wall, i.e. no wall sorption, tends to introduce stronger dispersion compared to wall with chemical reaction under the condition of high-frequency oscillatory EOF. For the condition of 'plug flow' (Figure 6.4(c) and 6.4(d)), k' has different effect on the dispersion coefficient depending on the wall sorption conditions: D'_{Tw} tends to increase with the EDL thickness when there is no sorption or the kinetics is slow but decrease when the kinetics is fast. Different from a high-frequency flow, as can be seen from Figures 6.2(c) and 6.2(f), the velocity profile is flatter at a thinner EDL thickness and thus leads to weaker dispersion.

The partition coefficient is a chemical constant of the species. Figure 6.5 illustrates the relationship between D'_{Tw} and ξ' under various conditions. Similar to Figure 6.3, it is found that as the normalized partition coefficient ξ' increases from 0 to 5, the dispersion coefficient declines with a decreasing rate when δ' is small, and the difference among various Da is negligible as well. For 'plug flow' with large δ' , the effect of ξ' on D'_{Tw} depends on the value of Da . It is observed that D'_{Tw} increases with ξ' and then decreases with a peak exists at $\xi' \approx 1$ when $Da=10$ for all cases. Comparing the conditions of single wall and both walls sorption, i.e. Figures 6.5(c) and 6.5(g), and Figures 6.5(d) and 6.5(h), one finds that there is no big difference on the

dispersion coefficient at small Da , however, both walls sorption could induce stronger dispersion than single wall sorption when Da is large, especially at the peak.

(a) $\delta' = 0.1$

(c) $\delta' = 10$

(b) $\delta' = 0.1$

(d) $\delta' = 10$

Figure 6.4 Dispersion coefficient D'_{Tw} as a function of k' for $\psi'_2 = 1$, $\xi' = 1$ and $Sc = 1000$. (a) and (c) Comparison among different Da , and (b) and (d) Comparison among different sorption conditions with $Da = 1$

Figure 6.5 Variation of D'_{Tw} with ξ' for $\psi'_2=1$ and $Sc=1000$. ((a)-(d) for sorption at one wall, and (e)-(h) for sorption at both walls)

The results presented above are all based on the condition of the same potential at both walls, i.e. $\psi'_2 = 1$. The effect of the relative wall potential on the dispersion coefficient under different wall sorption conditions is shown in Figure 6.6. Figures 6.6(a) and 6.6(c) demonstrate the conditions of high-frequency flows, where a minimum dispersion coefficient is achieved at $\psi'_2 = 0$ for most cases ($\psi'_2 = 0.18$ for both walls sorption in Figure 6.6(c)). The reason is that when the potential of the bottom wall is zero, the flow velocity varies near the top wall only while becoming zero at the rest area, which leads to a more limited velocity shear across the channel. Also, it is found that no sorption at walls generates stronger dispersion, which is consistent with the results shown in Figure 6.3(b). The conditions of low-frequency flows, where the flow behaves like a slowly varying steady flow, are illustrated in Figures 6.6(b) and 6.6(d). Basically, D'_{Tw} tends to increase as ψ'_2 changes from -1 to 1 for the case of sorption at both walls but decrease for the cases of no sorption and single wall sorption. The minimum D'_{Tw} , as shown in Figure 6.6(b) for the cases of no sorption, one wall sorption and both walls sorption, occurs at $\psi'_2 = 0.35, 0.24$ and -0.63 , respectively.

(a) $\delta' = 0.1, k' = 10$

(c) $\delta' = 0.1, k' = 100$

(b) $\delta' = 10, k' = 10$

(d) $\delta' = 10, k' = 100$

Figure 6.6 Relationship between the relative bottom wall potential ψ_2' and the dispersion coefficient under different sorption conditions for $\xi' = 1$, $Da = 1$ and $Sc = 1000$

6.5 Conclusions

A general expression for the dispersion coefficient of oscillatory EOFs in a two-dimensional channel with different wall potentials as well as reversible sorption effect was derived. The results show that keeping the velocity amplitude constant, the dispersion coefficient decreases significantly with the oscillation frequency. For flows with a thin EDL, the dispersion coefficient is a strong function of the sorption effects, and hence, it could be applied to chemical separation for species with different reaction

kinetics. The presence of reversible sorption at the walls could influence the dispersion coefficient in a complicated manner. Three conditions have been considered, namely, no sorption at either wall, sorption at the top wall only, and the same sorption at both walls. Generally, inert walls tend to generate slightly stronger dispersion compared to reactive walls for high-frequency flows, while for low-frequency flows, both walls with sorption of faster kinetics will lead to a higher dispersion rate.

The effect of disparate wall potentials on the dispersion coefficient was also examined. It is found that the dispersion could be minimized by making the bottom wall potential approach zero for high-frequency flows. For low-frequency flows with thin EDL, symmetrical flow, i.e. $\psi_1 = \psi_2$, could reduce the dispersion coefficient for the cases of no sorption and only top wall with sorption, while the opposite is true when both walls have sorption.

Finally, the assumptions made in this chapter are summarized as follow: 1) The electrical potential is assumed to be sufficiently small such that the Debye-Huckel approximation can be applied to the equation (6.4); 2) To apply the static Boltzmann distribution, the transience of the development of the EDLs is assumed to be of a much shorter time scale than the time variations of the applied electrical field, i.e. the oscillation frequency of the electrical field is assumed not to exceed 1MHz to avoid EDL relaxation effects; 3) The species in the fluids is assumed to be neutral so that the absorption process does not affect the wall potential and 4) The reversible reaction with the wall is assumed to be first-order kinetic.

Chapter 7 Conclusions

This thesis examines a few topics on the dispersion in oscillatory flows. Several specific problems with different applications are investigated. The main research method applied is the analytical analysis using the homogenization technique.

An analytical analysis based on the statistical approach together with numerical simulations with random walk assumption are performed to investigate the pollutant dispersion in one-dimensional wave motion. The results suggest that for pollutants discharged into the ocean, waves could strengthen or weaken the ambient dispersion process depending on the wave characteristics. The critical criteria is σ which is related to the diffusion coefficient, wave period and wave length. Larger wave lengths and shorter wave periods thus tend to promote diffusion, while smaller wave lengths and longer wave periods do otherwise. Furthermore, in a standing wave field, the modulation effects to the diffusion behavior due to two opposing waves can be superimposed.

The dispersion of turbulent oscillatory flows in an axisymmetric circular pipe is investigated with both analytical and experimental means. The analytical analysis, based on the homogenization method, suggests that the longitudinal dispersion coefficient depends on the flow characteristics as well as the pipe geometry, and the effects vary case by case. Generally, the dispersion coefficient tends to increase with the flow velocity and pipe diameter. The oscillation frequency may enhance or decline the dispersion depending on these factors. The experiments with the advanced laser techniques PIV and PLIF quantitatively measured the longitudinal dispersion coefficient under various experimental conditions. The amplitude of dispersion

coefficient was found to range from 10^{-6} to 10^{-2} m²/s when the Reynolds number increased from 100 to 1000. Overall, the experimental results agreed well with the theoretical predictions when the Reynolds number exceeds 500, which suggests the occurrence of the transisition to turbulence.

A similar analytical approach is applied to examine the longitudinal dispersion of turbulent oscillatory flows due to wall vibrations. The results show that the dispersion coefficient increases with the wall vibrating velocity and decreases with the vibration frequency. Furthermore, under the conditions of small wall velocity and high frequency, the dispersion coefficient induced by synchronous dual walls vibration is larger than that by single wall vibration. However, for flows with large velocity and low frequency, the single wall vibration tends to generate stronger dispersion.

Finally, the longitudinal dispersion of oscillatory electro-osmotic flows is investigated in a two-dimensional micro-channel with reactive boundaries. An analytical analysis based on the homogenization method is also performed but with more complicated boundary conditions. The dispersion coefficient is found to not only depend on the flow characteristics but also on the electrical field as well as the reaction on the walls. For high frequency flows, the dispersion coefficient decreases with the thickness of the electric double layer. The presence of reaction on the walls can influence the dispersion coefficient in a complicated manner. Inert walls tend to generate slightly stronger dispersion compared to reactive walls for high-frequency flows, while for low-frequency flows, reactive walls with sorption of faster kinetics lead to a higher dispersion rate.

In summary, the perturbation analysis based on the homogenization method is a powerful tool to investigate the longitudinal dispersion in oscillatory flows. In the

future, more problems on the dispersion in oscillatory flows can be examined using this approach, especially in the area of turbulent oscillatory flows.

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Appendix

Appendix A. MATLAB code for Figure 3.2

```
clear all; close all;
randn('state',1);
N = 10000; % no. of particles
Td = 200; % duration of study
dt = 0.01; % time step
nt = round(Td/dt-1);
o=0.8; %  $\sigma=0.8, 1.2, 2.0$  and  $2.4$ 
e=[0.1:0.05:0.6]; %  $\varepsilon$ 
R=zeros(1,11);
Rc=zeros(1,11);
t = [0:nt-1].'*dt;
a=randn(nt-1,N);
dBt = [zeros(1,N);a]*sqrt(dt)*o;
Bt = cumsum(dBt,1);
vtfun = inline('e*cos(x-t)','e','x','t'); % velocity field
Xt = zeros(nt,N);
for j=1:11
    for i=2:nt
        Xt(i,:) = Xt(i-1,:) + vtfun(e(j),Xt(i-1,:),t(i-1))*dt + dBt(i,:);
    end
    varXt = std(Xt,[],2).^2; % (nt,1)
    R(j)=mean(varXt)/100/o^2-1; % numerical simulation
    Rc(j)= 2*e(j)^2*(12-o^4)/(4+o^4)^2; % theoretical prediction
end
R
Rc
```

Appendix B. MATLAB code for Figure 3.3

```
clear all; close all;
randn('state',1);
D = 0.00001; % diffusion coef. of Brownian motion
N = 10000; % no. of particles
Td = 200; % s, duration of study
dt = 0.01; % time step
nt = round(Td/dt-1); % including origin
o=0.8; %  $\sigma=0.8, 1.2, 2.0$  and  $2.4$ 
e=0.2; %  $\varepsilon$ 
t = [0:nt].'*dt;
a=randn(nt,N);
dBt = [zeros(1,N);a]*sqrt(dt)*o; % increment of Bt
Bt = cumsum(dBt,1); % Brownian motion
varBt = std(Bt,[],2).^2; % (nt,1)
vvfun = inline('e*cos(x-t)','e','x','t'); % velocity field
% compute particle trajectory
Xt = zeros(nt+1,N);
for i=2:nt+1
    Xt(i,:) = Xt(i-1,:) + vvfun(e,Xt(i-1,:),t(i-1))*dt + dBt(i,:);
end
varXt = std(Xt,[],2).^2;
plot(t,varBt,'k-',t,varXt,'k',t);
ylabel('Variance');
xlabel('t');
legend('Brownian motion','Monochromatic wave')
```

Appendix C. Expressions of the unknown parameters in $F^{(n)}$

The expressions for the unknown parameters in Equations (4.19) and (4.20) are presented here. Note that x_1 is set to be one due to the fact that the equation (4.18) is homogeneous.

$$x_2 = \frac{-(\delta_I' - 1)^{-\varphi^{(n)}} (A_1^{(n)} B_2^{(n)} \varphi^{(n)} + 2\varphi^{(n)} (1 - \delta_I') A_1^{(n)} B_1^{(n)} + 2\gamma^{(n)} (1 - \delta_I')^2 A_2^{(n)} B_1^{(n)})}{(\delta_I' - 1)^{-\psi^{(n)}} (A_1^{(n)} C_2^{(n)} \psi^{(n)} + 2\psi^{(n)} (1 - \delta_I') A_1^{(n)} C_1^{(n)} + 2\gamma^{(n)} (1 - \delta_I')^2 A_2^{(n)} C_1^{(n)})} \quad (C.1)$$

$$x_3 = \frac{-(\delta_I' - 1)^{-\varphi^{(n)}} (-B_1^{(n)} C_2^{(n)} \psi^{(n)} + 2\psi^{(n)} (\delta_I' - 1) B_1^{(n)} C_1^{(n)} + B_2^{(n)} C_1^{(n)} \varphi^{(n)} + 2\varphi^{(n)} (1 - \delta_I') C_1^{(n)} B_1^{(n)})}{A_1^{(n)} C_2^{(n)} \psi^{(n)} + 2\psi^{(n)} (1 - \delta_I') A_1^{(n)} C_1^{(n)} + 2\gamma^{(n)} (1 - \delta_I')^2 A_2^{(n)} C_1^{(n)}} \quad (C.2)$$

$$x_4 = 0 \quad (C.3)$$

where $A_1^{(n)} = J_0(\gamma^{(n)}(\delta_I' - 1))$; $A_2^{(n)} = J_1(\gamma^{(n)}(\delta_I' - 1))$

$$B_1^{(n)} = H\left(\left[\varphi^{(n)}, \varphi^{(n)}\right], \left[2\varphi^{(n)}\right], \frac{1}{1 - \delta_I'}\right); B_2^{(n)} = H\left(\left[\varphi^{(n)} + 1, \varphi^{(n)} + 1\right], \left[2\varphi^{(n)} + 1\right], \frac{1}{1 - \delta_I'}\right)$$

$$C_1^{(n)} = H\left(\left[\psi^{(n)}, \psi^{(n)}\right], \left[2\psi^{(n)}\right], \frac{1}{1 - \delta_I'}\right); C_2^{(n)} = H\left(\left[\psi^{(n)} + 1, \psi^{(n)} + 1\right], \left[2\psi^{(n)} + 1\right], \frac{1}{1 - \delta_I'}\right)$$

Appendix D. Solution of function $B^{(1) '}$

The solution procedures of function $B^{(1) '}$ are presented in this appendix.

Recall that the governing equation is given in (4.50) and the boundary conditions in (4.51).

When $r_0' < s' < \delta_l'$,

$$\phi \overline{\alpha u_f}' \frac{d}{ds'} \left((1-s') s' \frac{dB^{(1) '}}{ds'} \right) - i B^{(1) '} = i \left(\frac{F^{(1)}}{F^{(1)}(r_0')} - 1 \right) + \frac{ia^{(2)}}{4} \left(\frac{F_*^{(1)}}{F_*^{(1)}(r_0')} - \frac{F^{(1)}}{F^{(1)}(r_0')} \right) \quad (D.1)$$

When $\delta_l' < s' < 1$,

$$\frac{\phi \overline{\alpha u_f}' \delta_l'}{1-s'} \frac{d}{ds'} \left((1-s') \frac{dB^{(1) '}}{ds'} \right) - i B^{(1) '} = i \left(\frac{F^{(1)}}{F^{(1)}(r_0')} - 1 \right) + \frac{ia^{(2)}}{4} \left(\frac{F_*^{(1)}}{F_*^{(1)}(r_0')} - \frac{F^{(1)}}{F^{(1)}(r_0')} \right) \quad (D.2)$$

Discretizing with the second order central difference scheme, we have

$$y_x' = \frac{y_{x+1} - y_{x-1}}{2\Delta x}, y_x'' = \frac{y_{x+1} - 2y_x + y_{x-1}}{\Delta x^2} \quad (D.3)$$

where y_x' and y_x'' represent the first and second order derivative of function y at x , respectively.

Let $y = B^{(1) '}$, $y' = \frac{dB^{(1) '}}{ds'}$ and $y'' = \frac{d^2 B^{(1) '}}{ds'^2}$, equations (D.1) and (D.2) can be written in the numerical form respectively as,

$$\begin{aligned} & (-\Delta x + 2x\Delta x + 2x - 2x^2)y_{x-1} + \left(-4x + 4x^2 - \frac{2\Delta x^2}{\phi \overline{\alpha u_f}'} i \right) y_x + (\Delta x - 2x\Delta x + 2x - 2x^2)y_{x+1} \\ & = \frac{2i\Delta x^2}{\phi \overline{\alpha u_f}'} \left(\left(\frac{F^{(1)}}{F^{(1)}(r_0')} - 1 \right) + \frac{a^{(2)}}{4} \left(\frac{F_*^{(1)}}{F_*^{(1)}(r_0')} - \frac{F^{(1)}}{F^{(1)}(r_0')} \right) \right) \end{aligned} \quad (D.4)$$

$$(2 - 2x + \Delta x)y_{x-1} + \left(-4 + 4x - \frac{2\Delta x^2(1-x)}{\phi \overline{\alpha u_f}' \delta_l'} i \right) y_x + (2 - 2x - \Delta x)y_{x+1}$$

$$= \frac{2\Delta x^2(1-x)i}{\phi \alpha u_f' \delta_I'} \left(\left(\frac{F^{(1)}}{F^{(1)}(r_0')} - 1 \right) + \frac{a^{(2)}}{4} \left(\frac{F_*^{(1)}}{F_*^{(1)}(r_0')} - \frac{F^{(1)}}{F^{(1)}(r_0')} \right) \right) \quad (D.5)$$

Denoting the coefficients of y_{x-1} , y_x , and y_{x+1} as Z , X and Y , respectively, and the RHS as f , a matrix equation can be obtained as,

$$\begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ Z_1 & X_1 & Y_1 & \dots & 0 \\ & & & & \\ 0 & \dots & Z_{m-1} & X_{m-1} & Y_{m-1} \\ 0 & \dots & 0 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{m-1} \\ y_m \end{bmatrix} = \begin{bmatrix} 0 \\ f_1 \\ \vdots \\ f_{m-1} \\ 0 \end{bmatrix} \quad (D.6)$$

The first and last rows represent the boundary conditions.

For the presentation of result, $B^{(1) '}$ and D_l' are numerically computed using MATLAB with $m = 1000$.

Appendix E.

The measured concentrations at $x = 0.46, 0.55$ and 0.64m under different experimental conditions are presented in the following figures, where the symbols represent the measured data and solid lines denote the best-fitted curves based on the complementary error functions.

Figure E.1 $\text{Re}_\delta = 200$ ($V = 6.3 \text{ cm}^3$, $\omega = 4\pi \text{ rad/s}$, time interval between data = 10s)

Figure E.2 $Re_{\delta} = 200$ ($V = 12.6 \text{ cm}^3$, $\omega = \pi \text{ rad/s}$, time interval between data = 10s)

Figure E.3 $Re_{\delta} = 283$ ($V = 6.3 \text{ cm}^3$, $\omega = 8\pi \text{ rad/s}$, time interval between data = 10s)

Figure E.4 $Re_{\delta} = 283$ ($V = 12.6 \text{ cm}^3$, $\omega = 2\pi \text{ rad/s}$, time interval between data = 10s)

Figure E.5 $Re_{\delta} = 301$ ($V = 18.8 \text{ cm}^3$, $\omega = \pi \text{ rad/s}$, time interval between data = 10s)

Figure E.6 $Re_\delta = 401$ ($V = 12.6 \text{ cm}^3$, $\omega = 4\pi \text{ rad/s}$, time interval between data = 10s)

Figure E.7 $Re_\delta = 401$ ($V = 25.1 \text{ cm}^3$, $\omega = \pi \text{ rad/s}$, time interval between data = 5s)

Figure E.8 $Re_\delta = 425$ ($V = 18.8 \text{ cm}^3$, $\omega = 2\pi \text{ rad/s}$, time interval between data = 5s)

Figure E.9 $Re_\delta = 501$ ($V = 31.4 \text{ cm}^3$, $\omega = \pi \text{ rad/s}$, time interval between data = 5s)

Figure E.10 $Re_{\delta} = 567$ ($V = 25.1 \text{ cm}^3$, $\omega = 2\pi \text{ rad/s}$, time interval between data = 5s)

Figure E.11 $Re_{\delta} = 567$ ($V = 12.6 \text{ cm}^3$, $\omega = 8\pi \text{ rad/s}$, time interval between data = 1s)

Figure E.12 $Re_\delta = 601$ ($V = 18.8 \text{ cm}^3$, $\omega = 4\pi \text{ rad/s}$, time interval between data = 5s)

Figure E.13 $Re_\delta = 708$ ($V = 31.4 \text{ cm}^3$, $\omega = 2\pi \text{ rad/s}$, time interval between data = 5s)

Figure E.14 $Re_\delta = 801$ ($V = 25.1 \text{ cm}^3$, $\omega = 4\pi \text{ rad/s}$, time interval between data = 5s)

Figure E.15 $Re_\delta = 850$ ($V = 18.8 \text{ cm}^3$, $\omega = 8\pi \text{ rad/s}$, time interval between data = 1s)

Figure E.16 $\text{Re}_\delta = 1002$ ($V = 31.4\text{ cm}^3$, $\omega = 4\pi\text{ rad/s}$, time interval between data = 5s)

Appendix F. Expressions of the unknown parameters in $F^{(n)}$ for the case of synchronously dual walls oscillation

The expressions for the unknown parameters in Equations (5.16) and (5.17) are presented as follows (note that x_2 is set to be one due to the fact that the equation (5.15) is homogeneous):

$$\begin{aligned}
 x_1 = & 2\lambda^{(n)} \exp((\delta_l' - 1)\lambda^{(n)}) K_0(\sqrt{\delta_l'} \chi^{(n)}) - 2\lambda^{(n)} \exp((1 - \delta_l')\lambda^{(n)}) K_0(\sqrt{\delta_l'} \chi^{(n)}) + \frac{\chi^{(n)} \exp((\delta_l' - 1)\lambda^{(n)}) K_1(\sqrt{\delta_l'} \chi^{(n)})}{\sqrt{\delta_l'}} \\
 & + \frac{\chi^{(n)} \exp((\delta_l' - 1)\lambda^{(n)}) K_1(\sqrt{\delta_l'} \chi^{(n)})}{\sqrt{\delta_l'} (2\lambda^{(n)} \exp((1 - \delta_l')\lambda^{(n)}) I_0(\sqrt{\delta_l'} \chi^{(n)}) - 2\lambda^{(n)} \exp((\delta_l' - 1)\lambda^{(n)}) I_0(\sqrt{\delta_l'} \chi^{(n)}) + \chi^{(n)} \exp((1 - \delta_l')\lambda^{(n)}) I_1(\sqrt{\delta_l'} \chi^{(n)})} \\
 & + \frac{\chi^{(n)} \exp((\delta_l' - 1)\lambda^{(n)}) I_1(\sqrt{\delta_l'} \chi^{(n)})}{\sqrt{\delta_l'}} \quad (F.1)
 \end{aligned}$$

$$x_3 = \frac{\chi^{(n)} I_0(\sqrt{\delta_l'} \chi^{(n)}) K_1(\sqrt{\delta_l'} \chi^{(n)}) + I_1(\sqrt{\delta_l'} \chi^{(n)}) K_0(\sqrt{\delta_l'} \chi^{(n)})}{\sqrt{\delta_l'} \exp(\lambda^{(n)}) (2\lambda^{(n)} \exp((1 - \delta_l')\lambda^{(n)}) I_0(\sqrt{\delta_l'} \chi^{(n)}) - 2\lambda^{(n)} \exp((\delta_l' - 1)\lambda^{(n)}) I_0(\sqrt{\delta_l'} \chi^{(n)}) + \chi^{(n)} \exp((1 - \delta_l')\lambda^{(n)}) I_1(\sqrt{\delta_l'} \chi^{(n)}) / \sqrt{\delta_l'})} \quad (F.2)$$

$$x_4 = \frac{\chi^{(n)} \exp(\lambda^{(n)}) (I_0(\sqrt{\delta_l'} \chi^{(n)}) K_1(\sqrt{\delta_l'} \chi^{(n)}) + I_1(\sqrt{\delta_l'} \chi^{(n)}) K_0(\sqrt{\delta_l'} \chi^{(n)}))}{\sqrt{\delta_l'} (2\lambda^{(n)} \exp((1 - \delta_l')\lambda^{(n)}) I_0(\sqrt{\delta_l'} \chi^{(n)}) - 2\lambda^{(n)} \exp((\delta_l' - 1)\lambda^{(n)}) I_0(\sqrt{\delta_l'} \chi^{(n)}) + \chi^{(n)} \exp((1 - \delta_l')\lambda^{(n)}) I_1(\sqrt{\delta_l'} \chi^{(n)}) / \sqrt{\delta_l'})} \quad (F.3)$$

Appendix G. Expressions of the unknown parameters in $F^{(n)}$ for the case of single wall oscillation

The expressions for the unknown parameters in $F^{(n)}$ for the case of single wall oscillation are presented as follows (similar to Appendix F, x_6 is set to be one):

$$\begin{aligned}
 x_5 = & -\left(4\lambda^{(n)2} \exp((2\delta_l - 2)\lambda^{(n)}) I_0(\sqrt{y_0}'\chi^{(n)}) K_0(\sqrt{\delta_l}'\chi^{(n)})^2 - 4\lambda^{(n)2} I_0(\sqrt{y_0}'\chi^{(n)}) K_0(\sqrt{\delta_l}'\chi^{(n)}) / \exp((2\delta_l - 2)\lambda^{(n)})\right) \\
 & - 4\lambda^{(n)2} \exp((2\delta_l - 2)\lambda^{(n)}) K_0(\sqrt{y_0}'\chi^{(n)}) I_0(\sqrt{\delta_l}'\chi^{(n)}) K_0(\sqrt{\delta_l}'\chi^{(n)}) + \left(4\lambda^{(n)2} K_0(\sqrt{y_0}'\chi^{(n)}) I_0(\sqrt{\delta_l}'\chi^{(n)}) K_0(\sqrt{\delta_l}'\chi^{(n)}) / \exp((2\delta_l - 2)\lambda^{(n)})\right) \\
 & + \left(\chi^{(n)2} \exp((2\delta_l - 2)\lambda^{(n)}) I_0(\sqrt{y_0}'\chi^{(n)}) K_1(\sqrt{\delta_l}'\chi^{(n)})^2 / \delta_l'\right) - \left(\chi^{(n)2} I_0(\sqrt{y_0}'\chi^{(n)}) K_1(\sqrt{\delta_l}'\chi^{(n)})^2 / (\delta_l' \exp((2\delta_l - 2)\lambda^{(n)}))\right) \\
 & + \left(\chi^{(n)2} \exp((2\delta_l - 2)\lambda^{(n)}) K_0(\sqrt{y_0}'\chi^{(n)}) I_1(\sqrt{\delta_l}'\chi^{(n)}) K_1(\sqrt{\delta_l}'\chi^{(n)}) / \delta_l'\right) - \left(\chi^{(n)2} K_0(\sqrt{y_0}'\chi^{(n)}) I_1(\sqrt{\delta_l}'\chi^{(n)}) K_1(\sqrt{\delta_l}'\chi^{(n)}) / (\delta_l' \exp((2\delta_l - 2)\lambda^{(n)}))\right)
 \end{aligned} \tag{G.1}$$

$$\begin{aligned}
 x_7 = & -\left(\chi^{(n)2} \exp((\delta_l - 2)\lambda^{(n)}) I_0(\sqrt{y_0}'\chi^{(n)}) I_0(\sqrt{\delta_l}'\chi^{(n)}) K_0(\sqrt{\delta_l}'\chi^{(n)})^2 / \delta_l' + \left(\chi^{(n)2} \exp((\delta_l - 2)\lambda^{(n)}) K_0(\sqrt{y_0}'\chi^{(n)}) I_1(\sqrt{\delta_l}'\chi^{(n)})^2 K_0(\sqrt{\delta_l}'\chi^{(n)}) / \delta_l'\right)\right) \\
 & + \left(\chi^{(n)2} \exp((\delta_l - 2)\lambda^{(n)}) I_0(\sqrt{y_0}'\chi^{(n)}) I_1(\sqrt{\delta_l}'\chi^{(n)}) K_0(\sqrt{\delta_l}'\chi^{(n)}) K_1(\sqrt{\delta_l}'\chi^{(n)}) / \delta_l'\right) \\
 & + \left(\chi^{(n)2} \exp((\delta_l - 2)\lambda^{(n)}) K_0(\sqrt{y_0}'\chi^{(n)}) I_0(\sqrt{\delta_l}'\chi^{(n)}) I_1(\sqrt{\delta_l}'\chi^{(n)}) K_1(\sqrt{\delta_l}'\chi^{(n)}) / \delta_l'\right)
 \end{aligned} \tag{G.2}$$

$$\begin{aligned}
 x_8 = & -\left(\chi^{(n)2} \exp((\delta_l - 2)\lambda^{(n)}) I_0(\sqrt{y_0}'\chi^{(n)}) K_0(\sqrt{\delta_l}'\chi^{(n)}) I_0(\sqrt{\delta_l}'\chi^{(n)})^2 / \delta_l' + \left(\chi^{(n)2} \exp((\delta_l - 2)\lambda^{(n)}) K_0(\sqrt{y_0}'\chi^{(n)}) K_1(\sqrt{\delta_l}'\chi^{(n)})^2 I_0(\sqrt{\delta_l}'\chi^{(n)}) / \delta_l'\right)\right) \\
 & + \left(\chi^{(n)2} \exp((\delta_l - 2)\lambda^{(n)}) I_0(\sqrt{y_0}'\chi^{(n)}) K_1(\sqrt{\delta_l}'\chi^{(n)}) I_0(\sqrt{\delta_l}'\chi^{(n)}) I_1(\sqrt{\delta_l}'\chi^{(n)}) / \delta_l'\right) \\
 & + \left(\chi^{(n)2} \exp((\delta_l - 2)\lambda^{(n)}) K_0(\sqrt{y_0}'\chi^{(n)}) I_0(\sqrt{\delta_l}'\chi^{(n)}) K_1(\sqrt{\delta_l}'\chi^{(n)}) I_1(\sqrt{\delta_l}'\chi^{(n)}) / \delta_l'\right)
 \end{aligned} \tag{G.3}$$

$$\begin{aligned}
 x_9 = & \left(4\chi^{(n)}\lambda^{(n)} K_0(\sqrt{y_0}'\chi^{(n)}) I_0(\sqrt{\delta_l}'\chi^{(n)}) K_1(\sqrt{\delta_l}'\chi^{(n)}) + I_1(\sqrt{\delta_l}'\chi^{(n)}) K_0(\sqrt{\delta_l}'\chi^{(n)})\right) / \sqrt{\delta_l}' \\
 & + \left(4\lambda^{(n)2} K_0(\sqrt{y_0}'\chi^{(n)}) I_0(\sqrt{\delta_l}'\chi^{(n)})^2 / \exp((\delta_l - 2)\lambda^{(n)}) - 4\lambda^{(n)2} \exp((\delta_l - 2)\lambda^{(n)}) K_0(\sqrt{y_0}'\chi^{(n)}) I_0(\sqrt{\delta_l}'\chi^{(n)})^2\right) \\
 & + \left(4\chi^{(n)2} \exp((\delta_l - 2)\lambda^{(n)}) I_0(\sqrt{y_0}'\chi^{(n)}) I_0(\sqrt{\delta_l}'\chi^{(n)}) K_0(\sqrt{\delta_l}'\chi^{(n)}) - 4\chi^{(n)2} I_0(\sqrt{y_0}'\chi^{(n)}) I_0(\sqrt{\delta_l}'\chi^{(n)}) K_0(\sqrt{\delta_l}'\chi^{(n)}) / \exp((\delta_l - 2)\lambda^{(n)})\right)
 \end{aligned} \tag{G.4}$$

$$\begin{aligned}
 x_{10} = & -\left(4\chi^{(n)}\lambda^{(n)} K_0(\sqrt{y_0}'\chi^{(n)}) K_0(\sqrt{\delta_l}'\chi^{(n)}) I_1(\sqrt{\delta_l}'\chi^{(n)}) + K_1(\sqrt{\delta_l}'\chi^{(n)}) I_0(\sqrt{\delta_l}'\chi^{(n)})\right) / \sqrt{\delta_l}' \\
 & + \left(4\lambda^{(n)2} I_0(\sqrt{y_0}'\chi^{(n)}) K_0(\sqrt{\delta_l}'\chi^{(n)})^2 / \exp((\delta_l - 2)\lambda^{(n)}) - 4\lambda^{(n)2} \exp((\delta_l - 2)\lambda^{(n)}) I_0(\sqrt{y_0}'\chi^{(n)}) K_0(\sqrt{\delta_l}'\chi^{(n)})^2\right) \\
 & + \left(4\chi^{(n)2} \exp((\delta_l - 2)\lambda^{(n)}) I_0(\sqrt{y_0}'\chi^{(n)}) K_0(\sqrt{\delta_l}'\chi^{(n)}) I_0(\sqrt{\delta_l}'\chi^{(n)}) - 4\chi^{(n)2} I_0(\sqrt{y_0}'\chi^{(n)}) K_0(\sqrt{\delta_l}'\chi^{(n)}) I_0(\sqrt{\delta_l}'\chi^{(n)}) / \exp((\delta_l - 2)\lambda^{(n)})\right)
 \end{aligned} \tag{G.5}$$

Appendix H. Expressions of the unknown parameters in B'

Expressions of X_1 and X_2 in Equation (6.43) are presented as follows,

$$\begin{aligned}
 X_1 = & -\left\{ \frac{(2\lambda' \gamma' \sinh(\lambda') \cosh(\gamma') + \varphi' \gamma' \cosh(\lambda') \cosh(\gamma') + \varphi' \lambda' \sinh(\lambda') \sinh(\gamma')) E_1}{\gamma' (2\gamma' \sinh(\gamma') \cosh(\gamma') + \varphi' \cosh(\gamma')^2 + \varphi' \sinh(\gamma')^2) (\lambda'^2 - \gamma'^2)} \right. \\
 & + \frac{(\varphi' \gamma' \sinh(\lambda') \cosh(\gamma') - \varphi' \lambda' \cosh(\lambda') \sinh(\gamma')) E_2}{\gamma' (2\gamma' \sinh(\gamma') \cosh(\gamma') + \varphi' \cosh(\gamma')^2 + \varphi' \sinh(\gamma')^2) (\lambda'^2 - \gamma'^2)} \\
 & + \frac{(2k' \gamma' \sinh(k') \cosh(\gamma') + \varphi' \gamma' \cosh(k') \cosh(\gamma') + \varphi' k' \sinh(k') \sinh(\gamma')) F_1}{\gamma' (2\gamma' \sinh(\gamma') \cosh(\gamma') + \varphi' \cosh(\gamma')^2 + \varphi' \sinh(\gamma')^2) (k'^2 - \gamma'^2)} \\
 & \left. + \frac{(\varphi' \gamma' \sinh(\lambda') \cosh(\gamma') - \varphi' \lambda' \cosh(\lambda') \sinh(\gamma')) F_2}{\gamma' (2\gamma' \sinh(\gamma') \cosh(\gamma') + \varphi' \cosh(\gamma')^2 + \varphi' \sinh(\gamma')^2) (k'^2 - \gamma'^2)} \right\} \quad (H.1)
 \end{aligned}$$

$$\begin{aligned}
 X_2 = & -\left\{ \frac{(2\lambda' \gamma' \cosh(\lambda') \sinh(\gamma') + \varphi' \gamma' \sinh(\lambda') \sinh(\gamma') + \varphi' \lambda' \cosh(\lambda') \cosh(\gamma')) E_2}{\gamma' (2\gamma' \sinh(\gamma') \cosh(\gamma') + \varphi' \cosh(\gamma')^2 + \varphi' \sinh(\gamma')^2) (\lambda'^2 - \gamma'^2)} \right. \\
 & + \frac{(\varphi' \gamma' \cosh(\lambda') \sinh(\gamma') - \varphi' \lambda' \sinh(\lambda') \cosh(\gamma')) E_1}{\gamma' (2\gamma' \sinh(\gamma') \cosh(\gamma') + \varphi' \cosh(\gamma')^2 + \varphi' \sinh(\gamma')^2) (\lambda'^2 - \gamma'^2)} \\
 & + \frac{(2k' \gamma' \cosh(k') \sinh(\gamma') + \varphi' \gamma' \sinh(k') \sinh(\gamma') + \varphi' k' \cosh(k') \cosh(\gamma')) F_2}{\gamma' (2\gamma' \sinh(\gamma') \cosh(\gamma') + \varphi' \cosh(\gamma')^2 + \varphi' \sinh(\gamma')^2) (k'^2 - \gamma'^2)} \\
 & \left. + \frac{(\varphi' \gamma' \cosh(\lambda') \sinh(\gamma') - \varphi' \lambda' \sinh(\lambda') \cosh(\gamma')) F_1}{\gamma' (2\gamma' \sinh(\gamma') \cosh(\gamma') + \varphi' \cosh(\gamma')^2 + \varphi' \sinh(\gamma')^2) (k'^2 - \gamma'^2)} \right\} \quad (H.2)
 \end{aligned}$$