Index of lattice dirac operators

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School of Physical and Mathematical Sciences

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Abstract

The general topic of this thesis is how to define and compute the index of discretized “lattice” versions of the Dirac operator coupled to a topologically non-trivial gauge field. The famous Atiyah–Singer Index Theorem, applied to the usual continuum Dirac operator, says that its index is equal to the topological charge of the gauge field. This fact plays an important role in theoretical particle physics. However, in the physics context one often solves problems by discretizing the physical theory and simulating it on a computer. This raises the question of whether the Index Theorem (and its physical implications) can be maintained in the discretized lattice setting. This is a subtle mathematical issue since the usual definition of the index of the Dirac operator breaks down in the discretized setting due to its finite-dimensional nature. This thesis reviews how the index can be extracted in another, indirect, way in the discretized setting, and then goes on to consider the index of a new discretization of the Dirac operator introduced recently by D. Adams. This new lattice Dirac operator has the virtue of being more computationally efficient for use in computer simulations of particle physics theory, but before it can be used with confidence a secure theoretical foundation needs to be established. One part of this is to show that its index correctly reproduces the continuum index in the continuum limit where the lattice spacing goes to zero. This is the main goal and new research contribution of this thesis. It is done here by adapting the methods and techniques used to establish this result previously for the standard discretization of the Wilson-Dirac operator, and involves dealing with new issues specific to the new discretization.

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Introduction

The Atiyah-Singer Index Theorem [7] states that for an elliptic differential operator on a compact manifold

$$\textbf{Analytic index} = \textbf{Topological index}.$$

The analytic index is an integer associated with the solution space of a partial differential equation (PDE), and topological index is an integer defined from topological data of the PDE, that is, from its boundary conditions. Because it is usually hard to evaluate the analytic index directly, the theorem makes it easier to find its value, as the topological index is often straightforward to calculate. Apart from giving a deep connection between analysis and topology, the Atiyah-Singer Index Theorem plays a unifying role for several other theorems as well [29, 18].

The PDEs for which the Index Theorem applies have the form

$$Df(x) = 0, \tag{1.0.1}$$

where D is a partial differential operator acting on functions f , which belong to some infinite-dimensional vector space. The domain of the functions f is a compact manifold and the functions take values in a vector bundle over that manifold. In this thesis we will be concerned with the simplest case, where the functions live on $\mathbb{R}^n$ and take values in a complex vector space and satisfies certain boundary conditions involving topological data.

The partial differential operator D maps the vector space of these functions to themselves and must satisfy the technical condition of being elliptic. The condition of being elliptic roughly means that D is non-degenerate and is in some sense as close as a partial differential operator can get to being invertible. The important consequences of D being elliptic are I. The solution space of (1.0.1) is finite dimensional, which is as we shall see later essential for the definition of Analytic Index. II. Solutions of the PDE are smooth functions which implies that without loss of generality we can work with smooth functions.

Dirac operators which originally arose in the works of theoretical physicist Paul Dirac [12], are some of the most interesting and widely studied examples of elliptic differential operators. These operators also provide important examples of the Index Theorem [29]. The Index Theorem for the Dirac operators play a central role in theoretical physics especially in quantum chromodynamics which is the quantum field theory of the strong nuclear force. For example it is needed to explain the large mass of the η' meson [39, 38, 43, 40]. In theoretical physics the mass of nuclear particles including η' mesons is calculated by discretizing the theory of quantum chromodynamics and simulating them numerically on computers. This presents us with a very interesting theoretical question. Does the

Index Theorem also remain valid in the discretized setting? If not, then the discretization approach taken for the calculation of e.g. the mass of the η' meson will not give the correct answer.

Intuitively it appears that some version of the Index Theorem should hold in the lattice setting as the solutions of the discretized Dirac operator should approximate the solutions of the continuum Dirac operator. However in the lattice setting the vector space of functions on which the naive Dirac operator acts is finite dimensional and it can be shown by a general argument that the index always vanishes in this case. This apparent paradox happens because in the discretized setting there are additional solutions which do not have any continuum counterpart.

A solution of how to reproduce the Index Theorem in a discretized setting was provided by theoretical physicists [36, 24]. The key idea is to carry out the discretization of the naive Dirac operator in such a way that a distinction can be made between solutions which have continuum approximations and those which do not. This was achieved by adding a "Wilson term" to the discretized Dirac operator.¹ The additional Wilson term has a small effect on the would-be smooth solutions, but ensures that the additional "spurious" solutions of the discretized Dirac operator are no longer solutions of the modified Dirac operator. Thus the Wilson Dirac operator no longer has exact solutions hence the usual definition of analytic index does not hold. Therefore we need to extract the index in other

¹In fact the Wilson term had been introduced earlier in the pioneering work of K.Wilson on discretization of quantum field theories [42]. The main reason for the introduction of the Wilson term was to solve the problem regarding the additional fermion species described by the discretized Dirac operator (the so-called "doubling problem") compared to the continuum case. The doubling problem is similar to the one we are discussing about additional rough solutions of the discretized PDE and it is quite natural that the Wilson term also solves the latter problem.

indirect ways [36, 24]. The concept of a topological index also survives in this setting and it is given by the topological charge of the gauge field, coupled to the Dirac operator. The gauge field however has to satisfy an approximate smoothness condition [31]. Thus one can make a conjecture for the Index Theorem in the discrete setting. Although there is no analytic proof of the theorem, numerous numerical studies have provided overwhelming evidence towards the validity of this conjecture [36, 24, 17, 13]. The only analytic proof in this direction has been to show that in the continuum limit the analytic index of the discretized Dirac operator with the Wilson term does agree with the index of the continuum Dirac operator [1]. The topological charge then also equals the continuum topological index, thus the Index Theorem holds in the continuum limit of the discretized setting and therefore must hold in the discretized setting provided the lattice spacing is sufficiently small. (Since the Index Theorem is an equality between integers, if it holds in the continuum limit then it must also hold slightly away from the continuum limit in the discretized setting.)

The main topic of this thesis is the analytic index of the Dirac operator in the discretized setting and henceforth we will refer to it just as index. We note that the topological index is simply the topological charge of the gauge field which are coupled with the Dirac operator and henceforth will be referred to as topological charge.

As we have discussed before, the usual continuum definition for the index does not work in the lattice setting and needs to be extracted in an indirect way. After reviewing this, the thesis will focus on the index for a new discretization of the Dirac operator introduced recently in [5]. The new research contribution of this thesis is to show that the index of this new discretization of the Dirac operator

correctly reproduces the index of the continuum Dirac operator in the continuum limit, thus establishing a similar analytic result that was shown earlier in [1].

Index of Dirac operators in the Discretized Setting

There are two ways of extracting the index of the Dirac operator in the discretized setting.

Spectral Flow Approach: In the usual continuum setting we can extract the index by introducing an operator $H(m)$, which is a hermitian version of the Dirac operator and depends on the parameter m . The eigenvalues of $H(m)$ are functions of m and can be shown to be non-zero except possibly when $m = 0$. As m varies from negative to positive values some of the eigenvalues can cross from positive to negative (or vice-versa) at $m = 0$. If we denote the number of eigenvalues that crosses from positive to negative by n_+ and the ones crossing from negative to positive by n_- , then the index of the Dirac operator can be expressed as $n_+ - n_-$. We can work in a similar way to extract the index in the discrete setting, when the Wilson term is added to the discretized Dirac operator, by considering a discretized version of the hermitian operator $H(m)$. However unlike in the continuum case the eigenvalues of $H(m)$ no longer need to be nonzero when $m \neq 0$ and the crossings occur at arbitrary values of m . However it has been shown [22, 34, 3] that when the lattice gauge field satisfies certain approximate smoothness conditions then the eigenvalue crossings only occur close to $m = 0, 2, 4, 6, 8$. The eigenvalue crossings near $m = 0$ in the discrete setting corresponds to the eigenvalue crossings at $m = 0$ in the continuum setting. Thus we define the index of the Dirac operator by $n_+ - n_-$ where $n_{\pm}$ now denotes the number of eigenvalue crossings near $m = 0$. This approach of extracting the index

is called the spectral flow approach.

The number of positive and negative eigenvalues of $H(m)$ are equal for $m < 0$. Furthermore the eigenvalues are nonzero in both the continuum and discretized settings. Thus if $n_{\pm}$ number of eigenvalues change crossings from positive to negative and vice versa near $m = 0$, then the difference between number of positive and negative eigenvalues of $H(m)$ will be $2(n_- - n_+)$. Using these facts we can derive a formula for the spectral flow which plays a central role in the thesis. The formula reads

$$\text{index} = n_+ - n_- = -\frac{1}{2}\text{Tr}\left(\frac{H(m)}{\sqrt{H(m)^2}}\right) \quad (1.0.2)$$

where Tr denotes trace over internal and space-time indices, the inverse square root of $H(m)^2$ is defined via the spectral theory. We will now provide a short discussion on the validity of this formula. If $f_m(x)$ is an eigenfunction of $H(m)$ with eigenvalue $\lambda(m) (\neq 0)$ then $f_m(x)$ is also an eigenfunction of $H(m)/\sqrt{H(m)^2}$ with eigenvalue $\lambda(m)/\sqrt{\lambda(m)^2} = \text{sign}(\lambda(m)) = \pm 1$. Therefore, the operator trace in (1.0.2) counts $+1$ (or -1) for each eigenfunction of $H(m)$ with positive (or negative) eigenvalue, hence it gives the difference between the number of positive and negative eigenvalues of $H(m)$. As we have mentioned before the difference is $2(n_- - n_+)$, and hence (1.0.2) follows. In order for us to calculate the index, the parameter m has to be chosen in such a way that it is greater than the values of m at which the low-lying eigenvalue crossings occur (near $m = 0$), but less than the values of m at which the higher-lying eigenvalue crossing occur (i.e. close to $m = 2, 4, 6, 8$ for the usual discretization of the Dirac operator with Wilson term).

Overlap Dirac Operator Approach: The second way of extracting the index involves a more complicated version of the discretized Dirac operator known as the overlap Dirac operator. Whereas in the case of usual discretization of the Dirac operator, which is known as the Wilson Dirac operator there were no exact solutions, the most important feature of the overlap Dirac operator is that it preserves the property of having exact solutions of the PDE given in equation (1.0.1). Hence we can extract the index just like in the continuum setting from the solution space of D . The overlap Dirac operator turns out to satisfy [33] a special relation called the Ginsparg-Wilson relation [19] which enables us to derive a formula for its index. The formula precisely turns out to be the expression (1.0.2) [21]. . Thus the second approach to the index, based on the overlap Dirac operator, turns out to give precisely the same index as the aforementioned spectral flow approach. So we have a consistent way to define the index in the discretized setting using either of these approaches.

After the general review of the index of the Dirac operator in the discrete setting, this thesis focuses on the index of a new discretization of the Dirac operator introduced recently in [5]. The construction is within the very computationally efficient formalism of "staggered" lattice fermions [27, 20]. This is an important discretization of fermionic fields, since realistic computer simulations require enormous computing resources and are currently performed on some of the world's most powerful supercomputers – see, e.g., [11] for a description. The new discretization of the Dirac operator has the potential to make a big impact on the computational efficiency of these simulations. However, before it can be used with confidence, a secure theoretical foundation has to be established for it. One of the most important things that need to be verified is that the Dirac operator

satisfies the discretized version of the Index Theorem. In fact this has already been checked in some numerical studies [4, 10]. However, one should also show that the previous analytic result, namely that the index of the discretized Dirac operator reproduces the topological charge of the gauge field in the continuum limit (which is equal to the index of the continuum Dirac operator by the continuum Index Theorem), also holds for this new discretization of the Dirac operator. This is done in this thesis and is its original research contribution.

The new discretization of the Dirac operator in [5] contains a version of the Wilson term, which plays a similar role to the Wilson term in the Wilson Dirac operator discussed above. Therefore it is called the staggered Wilson Dirac operator. Both the approaches of extracting the index of the Wilson Dirac operator discussed above remain valid for the new operator and both give the same value for the index. In particular, the index can be expressed as in (1.0.2), where $H(m)$ is now a certain hermitian version of the new staggered Wilson Dirac operator. We use this expression as the starting point for showing that the index of the discretized Dirac operators are equal to the topological charge of the gauge field in the continuum limit, both for the usual Wilson Dirac discretization (shown in [1]) and for the new staggered Wilson Dirac discretization considered here. We will now sketch the main steps in deriving this result. This covers both the steps of the previous proof for the usual Wilson Dirac case in [1] and the proof that we give in this thesis for the new staggered Wilson Dirac case.

The discretization is based on a hyper-cubic lattice on a 4-dimensional box, regarded as a domain in $\mathbb{R}^4$ (and corresponding to "space-time" in the physics quantum field theory), the lattice spacing is denoted by a , the continuum limit refers to " $a \rightarrow 0$ ". The functions live on the vertices, called the sites of the lattice,

and take values in a vector space which we will describe later. Since the 4-box has finite length, the lattice on it has a finite number of lattice sites, and hence the vector spaces of functions on the lattice sites are finite-dimensional. The values of the functions at opposite faces of the 4-box are related by boundary conditions, which encode the topology of an arbitrary smooth continuum $SU(N)$ gauge field with topological charge $Q \in \mathbb{Z}$. The lattice transcript of this gauge field is a lattice gauge field which we couple to the discretized lattice Dirac operator. Our aim is to show $\text{index} \rightarrow Q$ for $a \rightarrow 0$ for the index of the lattice Dirac operator. The $a \rightarrow 0$ limit is taken by successive subdivisions of the lattice such that for each subdivision the lattice spacing is reduced by half.

We will first rewrite the index expression (1.0.2) in terms of an "index density" $q(x)$ as follows:

$$\text{index} = \sum_x a^4 q(x) , \quad q(x) = -\frac{1}{2} \text{tr} \frac{H(m)}{\sqrt{H(m)^2}}(x, x) \quad (1.0.3)$$

where we use tr to denote trace over internal indices, the sum is over the lattice sites, generically denoted by " x " here and in the the rest of the thesis. We note that a^4 is the volume of a lattice hyper-cube, so the sum in (1.0.3) is the "lattice integral" of $q(x)$. We are using the concept of the kernel $\mathcal{O}(x, y)$ of a linear operator $\mathcal{O}$ defined on a space of functions $\{f(x)\}$ living on the lattice sites; it is given by $(\mathcal{O}f)(x) = \sum_y a^4 \mathcal{O}(x, y) f(y)$.

The rest of the proof is to show

$$q(x) = q_{\text{cont}}(x) + O(a)(x) \quad (1.0.4)$$

where the quantities on the right-hand side are as follows. The $q_{cont}(x)$ is the topological charge density of the continuum gauge field. Thus its integral over the 4-box gives the topological charge Q of the continuum gauge field. Explicitly, it is given in terms of the curvature tensor $F_{\mu\nu}(x)$ of the gauge field by

$$q_{cont}(x) = \frac{-1}{32\pi^2} \epsilon_{\mu\nu\sigma\rho} \text{tr}(F_{\mu\nu}(x)F_{\sigma\rho}(x)). \quad (1.0.5)$$

The trace is taken over a matrix algebra $\text{su}(N)$, which is the Lie algebra corresponding to the gauge group $\text{SU}(N)$. The curvature tensor F takes values in $\text{su}(N)$. μ, ν, σ, ρ are labels for the 4 coordinate axes of the 4-box, hence each of them take values in $\{1, 2, 3, 4\}$.

$$\epsilon_{\mu\nu\sigma\rho} = \begin{cases} 1 & \text{if } (\mu, \nu, \sigma, \rho) \text{ is an even permutation of } (1, 2, 3, 4) \\ -1 & \text{if } (\mu, \nu, \sigma, \rho) \text{ is a odd permutation of } (1, 2, 3, 4) \\ 0 & \text{otherwise.} \end{cases}$$

Here and throughout this thesis we follow the Einstein summation convention, i.e. there is an implicit sum over each of μ, ν, σ, ρ in (1.0.5) with each sum being over the values 1, 2, 3, 4.

The quantity $O(a)(x)$ on the right-hand side of (1.0.4), denotes an unspecified function of the lattice sites $\{x\}$ which satisfies a uniform bound of the form

$$|O(a)(x)| \leq Ka \quad (1.0.6)$$

for some positive constant K which is independent of x .

Once these facts are established, the desired result for the continuum limit of the

index is obtained after substituting (1.0.4) for $q(x)$ in (1.0.3). The contribution of $O(a)(x)$ in (1.0.3) vanishes in the continuum limit $a \rightarrow 0$ limit: its norm is

$$\begin{aligned}
 \left| \sum_x a^4 O(a)(x) \right| &\leq \sum_x a^4 |O(a)(x)| \\
 &\leq \sum_x K a^5 \quad \text{using (1.0.6)} \\
 &= aK \sum_x a^4 = aKV \quad (V \equiv \text{volume of the 4-box}) \\
 &\xrightarrow{a \rightarrow 0} 0.
 \end{aligned} \tag{1.0.7}$$

Now the contribution to (1.0.3), from the smooth function $q_{cont}(x)$ in (1.0.4), is given by

$$\sum_x a^4 q_{cont}(x) \xrightarrow{a \rightarrow 0} \int_{4\text{-box}} d^4x q_{cont}(x) = Q. \tag{1.0.8}$$

(1.0.7) and (1.0.8) imply that in the continuum limit $a \rightarrow 0$ the expression (1.0.3) for the index of the discretized Dirac operator converges to the topological charge Q of the continuum gauge field, which is the desired result.

Thus we note that the most important step is to prove (1.0.4) for the discrete index density $q(x)$. Once we prove this step then the desired result of the discretized Dirac operator holds by the above argument.

We note that the nature of the approach followed here for the index density is completely different. Normally proving results for some discretized quantity involves straightforward and tedious arguments between deriving bounds between the discretized quantity and its continuum counterpart and showing that such bounds tend to zero in the continuum limit. This approach however derives an estimate which relates the (analytic) index in the discretized setting to the topological charge (topological index) in the continuum setting. Thus the approach

is essentially a new way of proving the Index Theorem. Based on a discretization approach in the case of the Dirac operator coupled to a gauge field and none of the approaches used for proving Index Theorems in the past involve discretization. Although the possibility of a discrete approach to Index Theory was developed by physicists in the lattice quantum field theory since 1980's, the method still remains underutilized by pure mathematicians.

Returning to the problem of proving (1.0.4), this was done for the usual Wilson Dirac discretization in [1], based on earlier results in [2].² The proof involved the expansion of inverse square root in the expression for $q(x)$ in (1.0.3) as a power series and then identifying the terms of the power series that contribute to the $a \rightarrow 0$ limit. The evaluation of such terms require a number of techniques, including re-expressing them using a Fourier basis for the functions on the lattice. For the completion of the evaluation it is also necessary to consider the infinite volume setting that is $\mathbb{R}^4$ and not just the 4 -dimensional box. Thus the vector spaces are no longer finite dimensional, thus raising a new technical question. However these problems are solved in [1] using spectral theory for bounded hermitian linear operators on infinite dimensional Hilbert spaces.

The research problem addressed in this thesis aims to prove (1.0.4) for the new staggered Wilson Dirac operator. The same general approach as in [1] will be followed. But, there will be some significant differences in the details, since the derivation relies on the explicit form of the discretized Dirac operator. Also an

²The index density $q(x)$ in (1.0.3) is closely related to a quantity called the axial anomaly in quantum field theory, and previous works showed that it reproduced the continuum axial anomaly in the $a \rightarrow 0$ limit, which is closely related to the continuum topological charge density q_{cont} given by (1.0.5) [26, 15, 37, 2]. These amounted to heuristic derivations of (1.0.4), which were not able to accommodate topologically nontrivial gauge fields and did not establish (1.0.6) as a uniform bound (i.e. with K independent of x), which was essential for establishing (1.0.7) above.

extension of the methods in [1] will be required. This is because the staggered formalism used in the new discretization is more complicated than the formalism that previously used in [1] and no longer exists in the staggered formalism.

Chapter 2

A Short Discussion on Index Theorems

The Atiyah-Singer Index Theorem, one of the most celebrated theorems in mathematics, asserts that the number of solutions of an elliptic differential operator (Analytic Index) acting on a compact manifold can be obtained from the shape or topology of the manifold (Topological Index). In this sense, the theorem forms a bridge between analysis and topology, two very important branches of mathematics. In this chapter we will illustrate the analytical and topological nature of the index of a Fredholm operator through some elementary examples and theorems. We will discuss a toy model in $\mathbb{R}^2$ which will serve as an example of the Atiyah-Singer Index Theorem. We will further show how to obtain a formula for the index of a Fredholm operator in terms of a topological quantity and in the process prove the Toeplitz Index Theorem which is a very special case of the Atiyah-Singer Index Theorem.

2.1 Index of a Linear Operator

Let V and W be vector spaces and $L : V \rightarrow W$ be a linear operator.

The kernel of L is defined as

$$\ker L = \{v \in V \mid Lv = 0\}.$$

The cokernel of L is defined as

$$\operatorname{coker} L = W/L(V),$$

is the equivalence classes of vectors in W where $w_1 \sim w_2 \iff w_1 - w_2 \in L(V)$.

Definition 2.1.1. Let V and W be vector spaces and

$$L : V \rightarrow W$$

be a linear operator. If the kernel and cokernel of L are both finite dimensional we define the index of the linear operator L as

$$\operatorname{ind} L = \dim \ker L - \dim \operatorname{coker} L \tag{2.1.1}$$

Fact 2.1.2. *If V and W are both finite-dimensional then $\operatorname{ind}(L) = \dim V - \dim W$. In particular, if V and W have the same finite dimension then the index of L automatically vanishes.*

Proof. By elementary linear algebra for finite-dimensional vector spaces we know that $\dim V = \dim \ker(L) + \dim L(V)$ and $\dim W = \dim L(V) + \dim \operatorname{coker}(L)$.

Hence

$$\begin{aligned}
 \text{index } L &= \dim \ker(L) - \dim \text{coker}(L) \\
 &= \dim V - \dim L(V) - (\dim W - \dim \text{coker}(L)) \\
 &= \dim V - \dim W.
 \end{aligned}$$

□

Thus we see that the index of all linear operators acting between the finite dimensional vector spaces V and W is the same and depends only on the dimension of the vector spaces. However the index becomes much more interesting for certain kinds of linear operators called Fredholm operators acting between infinite dimensional vector spaces. This will be illustrated with examples later in the chapter. When the index does exist (even though the vector spaces V and W are infinite-dimensional), and is not independent of the linear operator L , but depends only on certain topological data in the definitions of L , V and W , then the index defined above is called the analytic index. Its dependence on the topological data is expressed by the Atiyah-Singer Index Theorem, which states that it is equal to an integer determined from the topological data, called the topological index. Simple illustrations are provided throughout the chapter.

We will now consider the examples of differential operators on $C^\infty(R)$ and $C^\infty(S^1)$ and explain the analytical and topological nature of the index for these linear operators through a couple of examples [30].

Example 2.1.3. Consider the map

$$L = \frac{d}{dx} : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}).$$

The kernel of L given by $\ker L = \{f \in C^\infty(\mathbb{R}) \mid f' = 0\}$ is of dimension 1.

Given any $f \in C^\infty(\mathbb{R})$ we can find a $g \in C^\infty(\mathbb{R})$ such that $f(x) = \int_0^x g(t)dt$ and $Lf = g$. Hence L is surjective and the cokernel of L has dimension 0.

The index of $L = 1 - 0 = 1$.

Example 2.1.4. Consider the map

$$L = \frac{d}{d\theta} : C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1).$$

The kernel of L is given by

$$\ker L = \{f \in C^\infty(\mathbb{S}^1) : f' = 0\} = \mathbb{C}$$

In order to calculate the cokernel we first show that for $f \in C^\infty(\mathbb{S}^1)$,

$$\int_0^{2\pi} f d\theta = 0 \Leftrightarrow f \in \operatorname{Im} L.$$

We will first prove the $\Leftarrow$ direction.

If $f \in \operatorname{Im} L$, then $f = F'$ for some periodic function F with period 2π . Thus we have,

$$\int_0^{2\pi} f d\theta = \int_0^{2\pi} F' d\theta = F(2\pi) - F(0) = 0.$$

We will now prove the $\Rightarrow$ direction.

Let,

$$\int_0^{2\pi} f d\theta = 0,$$

Our goal is to find a periodic function F with period 2π such that $F' = f$. Using

the fundamental theorem of calculus, we put F to be

$$F(x) = \int_0^x f d\theta.$$

Clearly $F' = f$ and F is also periodic since,

$$\begin{aligned} F(x + 2\pi) - F(x) &= \int_x^{x+2\pi} f d\theta \\ &= \int_x^0 f d\theta + \int_0^{2\pi} f d\theta + \int_{2\pi}^{x+2\pi} f d\theta \\ &= \int_0^{2\pi} f d\theta \\ &= 0. \end{aligned}$$

We will now show that the cokernel of $L = \mathbb{C}^\infty(\mathbb{S}^1)/\text{Im } L \cong \mathbb{C}$ hence 1-dimensional.

Let $[f]$ denote the equivalence class of f in $\mathbb{C}^\infty(\mathbb{S}^1)/\text{Im } L$. Thus $[f] = [g]$ means $f - g \in \text{Im } L$.

Consider the linear map $T : \mathbb{C}^\infty(\mathbb{S}^1)/\text{Im } L \rightarrow \mathbb{C}$ given by

$$[f] \mapsto \int_0^{2\pi} f d\theta.$$

The map T is well defined and is nonzero. To show that T is an isomorphism, it suffices to show that T is one-to-one. This follows because

$$\int_0^{2\pi} (f - g) d\theta = 0 \Rightarrow f - g \in \text{Im } L.$$

Thus $[f] = [g]$. Hence we have, $\text{coker } L \cong \mathbb{C}$.

Putting everything together, index of L is $1 - 1 = 0$.

2.1.1 The Index is Analytical.

Let V and W be vector spaces and $L : V \rightarrow W$ be a linear operator. Consider the linear equation

$$Lv = w, \quad v \in V \text{ and } w \in W. \quad (2.1.2)$$

Remark 2.1. We can think of $\dim \ker L$ as the number of constraints required for the uniqueness of solutions to equation (2.1.2) [30].

Let $v_1, v_2, \dots, v_k$ be a basis for $\ker L$. By linearity we now have,

$$L(v + a_1v_1 + a_2v_2 + \dots + a_kv_k) = w,$$

where $a_1, a_2, \dots, a_k$ are scalars. In order to obtain a unique solution to equation (2.1.2) we need to impose constraints on the constants $a_1, a_2, \dots, a_k$.

Remark 2.2. We can think of $\dim \operatorname{coker} L$ as the number of constraints required for the existence of solutions in equation (2.1.2) [30].

Let there be a basis of $[w_1], [w_2], \dots, [w_l]$ of $\operatorname{coker} L$. Given any $w \in W$, we can write $[w] = b_1[w_1] + b_2[w_2] + \dots + b_l[w_l]$ where $b_1, b_2, \dots, b_l$ are scalars. Thus there exists a solution $v \in V$ of equation (2.1.2) iff $b_1 = 0, b_2 = 0, \dots, b_l = 0$.

In examples 2.1.3 and 2.1.4 L is a differential operator. Hence we can think of $\operatorname{Ind} L$ as the net number of constraints needed for uniqueness and existence of solutions of L . We can therefore consider $\operatorname{Ind} L$ to be a quantity of interest in analysis, related to the existence and uniqueness of solutions of differential equations.

2.1.2 The Index is Topological

In example 2.1.3 we saw that the index of the map

$$\frac{d}{dx} : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$$

is equal to 1.

The Euler characteristic $\chi(\mathbb{R})$ of $\mathbb{R}$ is 1 and is a topological invariant. Thus index of the differential operator $\frac{d}{dx}$ in example 2.1.3 is equal to a topological invariant $\chi(\mathbb{R})$ of the underlying space $\mathbb{R}$.

Similarly, from example 2.1.4 we can see that index of $\frac{d}{d\theta} : C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$ is equal to the Euler Characteristic of $\mathbb{S}^1$ which is zero.

Stability of the Index

Example 2.1.5. Consider the space $C^\infty(\mathbb{R})$ and the map

$$L_{a,b} : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$$

where,

$$L_{a,b} = a \frac{d}{dx} + b, a, b \neq 0.$$

The kernel of L is $\{Ae^{-ax} : A \text{ is an arbitrary constant}\}$ hence has dimension 1.

For any function $g \in C^\infty(\mathbb{R})$ the function

$$f(x) = e^{(\frac{-b}{a})x} \left[\int \frac{g(x)}{a} e^{(\frac{b}{a})x} dx + c \right]$$

is in the domain of $L_{a,b}$ and $L_{a,b}(f) = g$. Thus $L_{a,b}$ is surjective. The dimension of cokernel of $L_{a,b} = 0$ and $\text{ind } L = 1$. Thus we see that the index of $\frac{d}{dx}$ remains constant under perturbation [30].

2.2 Fredholm Operator and Fredholm Index

In this section we will extend the definition of index to Fredholm operator. We will begin by introducing the concept of Fredholm operators. The index of the Fredholm operator satisfies some "stability" properties which allows us to write the index of Fredholm operators in terms of other topological quantities. One example is the Toeplitz Index Theorem, a very special case of the Atiyah-Singer Index Theorem, which we will prove in section 2.3. The materials of this section are taken from [23].

Definition 2.2.1. Let $T : H_1 \rightarrow H_2$ be a linear operator between Hilbert spaces. T is said to be *bounded* if the operator norm

$$\|T\| = \sup_{0 \neq x \in H_1} \frac{\|Tx\|}{\|x\|} \text{ is finite.}$$

Definition 2.2.2. Let H be a Hilbert space with a inner product $\langle \cdot, \cdot \rangle$ and $A \subset H$. The orthogonal compliment of A is given by $A^\perp = \{a \in H : \langle a, h \rangle = 0, \forall h \in A\}$.

Definition 2.2.3. Let H_1 and H_2 be Hilbert spaces. The adjoint operator $T^* : H_2 \rightarrow H_1$ of $T : H_1 \rightarrow H_2$ is defined as

$$\langle h_2, Th_1 \rangle = \langle T^*h_1, h_2 \rangle \quad \forall h_1 \in H_1, h_2 \in H_2$$

Fact 2.2.4. $(\text{range } T)^\perp = \ker T^*$.

Proof. Let $f \in \text{range } T, g \in (\text{range } T)^\perp$, then

$$0 = \langle g, f \rangle = \langle g, Te \rangle = \langle T^*g, e \rangle \quad \forall e \in \text{domain of } T.$$

Consequently we have $T^*g = 0$ or $g \in \ker T^*$ and vice versa. Thus, $(\text{range } T)^\perp = \ker T^*$. $\square$

Definition 2.2.5. Let H_1, H_2 be Hilbert spaces and $T : H_1 \rightarrow H_2$ be a bounded linear map. T is called Fredholm if $\ker T$ and $\text{coker } T$ are finite dimensional. The Fredholm index of T is given by

$$\text{index } T = \dim \ker T - \dim \text{coker } T. \quad (2.2.1)$$

Using fact 2.2.4 we have an alternative definition of Fredholm index for an operator T as

$$\text{index } T = \dim \ker T - \dim \ker T^*. \quad (2.2.2)$$

By equation (2.2.2) we note that whenever T is self-adjoint, we have $T = T^*$ hence $\text{index } T = 0$.

Example 2.2.6. Let $\{\phi_n\}$ be the complete orthonormal basis of the Hilbert space $L^2(\mu)$ which consists of set of all square-integrable function with respect to a measure μ .

We define a linear operator $T : L^2(\mu) \rightarrow L^2(\mu)$ by

$$T\phi_n = \begin{cases} \phi_{n-1} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ \phi_n & \text{if } n < 0. \end{cases}$$

T is called the shift operator.

We see that $\ker T = \{\phi_0 : T\phi_0 = 0\}$ and clearly T is Fredholm. Therefore $\text{Index } T = 1$. It can be easily shown that, $\text{index } T^n = n$, for every $n \in \mathbb{Z}$.

2.2.1 Stability of Fredholm Index

Definition 2.2.7. A C^* -algebra is a Banach $*$ -algebra which is isometrically $*$ -isomorphic to a norm-closed $*$ -subalgebra of the set of all bounded operators on some Hilbert space H .

Remark 2.3. We refer to [23] for the definitions on Banach $*$ -algebras, $*$ -isomorphisms and $*$ -subalgebra. We also note that our definition of C^* algebra is not the conventional one. However the equivalence of our definition and the conventional definition can be seen from the Gelfand-Naimark representation theorem [23].

Definition 2.2.8. Let H be a Hilbert space. A rank-one operator on H is an operator T of the form

$$T(x) = (Ty, x)z,$$

for some fixed vectors $y, z \in H$. The rank-one operators generate a subalgebra of the set of all bounded operators called the algebra of finite-rank operators, and the norm closure of the algebra of finite-rank operators is the algebra of compact operators on H and is denoted by $K(H)$.

Remark 2.4. We again note that our definition of compact operators is not the definition that one encounters in textbooks. However this definition of compact operators will be useful when we prove the Toeplitz Index Theorem in section 2.3. Our definition of compact operators can be easily derived from the conventional definition of compact operators using the theorems presented in [28].

The stability property of the linear operator in example 2.1.5 also holds for Fredholm operators. We state the property without proof [23] in the following few lines.

Theorem 2.2.9. *If T is Fredholm and K is compact then $T + K$ is Fredholm, and $\text{index}(T) = \text{index}(T + K)$.*

Definition 2.2.10. Let H be a Hilbert space, $B(H)$ be the set of all bounded operators and $K(H)$ be the set of all compact operators on H . The quotient C^* algebra $B(H)/K(H)$ is called the Calkin algebra, and is denoted by $C(H)$.

Theorem 2.2.9 follows [23] as a corollary of Atkinsons's theorem which states that

Theorem 2.2.11. *Let H be a Hilbert space, a bounded operator $T \in B(H)$ is Fredholm if and only if its image in the Calkin algebra is invertible.*

In section 2.3 we will present a formula for the index of a Fredholm operator in terms of topological quantities. Such formulas indicate deep connections between analysis and topology and tend to have powerful applications.

2.3 Toeplitz Index Theorem

Our aim in this section is to reproduce a proof of the Toeplitz Index Theorem which is the simplest example of the Atiyah-Singer Index Theorem and show that the index of a Fredholm operator can be obtained in terms of a topological quantity. The treatment on Hardy spaces closely follows [25] and we refer to [14, 23]

Let us consider the real line $\mathbb{R}$ and the Lebesgue measure μ on $\mathbb{R}$. Consider the mapping

$$f : \mathbb{R} \rightarrow \mathbb{S}^1, \quad f(t) = e^{it}.$$

With this mapping we have

$$\mathbb{R}/2\pi\mathbb{Z} \cong \mathbb{S}^1. \quad (2.3.1)$$

Let us restrict f to $[-\pi, \pi)$. We can now define a pushforward of the Lebesgue measure in $\mathbb{R}$ to a Lebesgue measure in $\mathbb{S}^1$. Because of the identification in equation (2.3.1) we see that there is a 1-1 correspondence between 2π periodic function on $\mathbb{R}$ and functions on $\mathbb{S}^1$. A function f is integrable over $\mathbb{S}^1$ if the corresponding 2π periodic function in $\mathbb{R}$ is integrable and

$$\int_{\mathbb{S}^1} f(t) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx.$$

Thus $L^1(\mathbb{S}^1) = \{f : \mathbb{S}^1 \rightarrow \mathbb{R} : \int_{\mathbb{S}^1} f(t) dt < \infty\}$. Similarly we can define $L^p(\mathbb{S}^1)$ for $2 \leq p < \infty$. Consider the space $L^2(\mathbb{S}^1)$. We now define an inner product on $L^2(\mathbb{S}^1)$ by

$$\langle f, \bar{g} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \cdot \overline{g(t)} dt.$$

$\{\chi_n = e^{int} : n \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{S}^1)$.

Definition 2.3.1. Let S^1 denote the unit circle. The Hardy space is a closed subspace of $L^2(\mathbb{S}^1)$ which is spanned by $\{\chi_n = e^{int} : n \in \mathbb{N} \cup \{0\}\}$ and is denoted by $H^2(\mathbb{S}^1)$.

Definition 2.3.2. A Toeplitz operator on $H^2(\mathbb{S}^1)$ is a bounded operator T_g of the form

$$T_g(f) = P(gf), \quad f \in H^2(\mathbb{S}^1) \text{ and } g \in L^\infty(\mathbb{S}^1). \quad (2.3.2)$$

P is the orthogonal projection from $L^2(\mathbb{S}^1)$ onto $H^2(\mathbb{S}^1)$. The function g is called the symbol of T_g .

Before we prove the Toeplitz Index Theorem we state the following fact.

Theorem 2.3.3. *If $f \in C^0(\mathbb{S}^1)$ and P is the orthogonal projection from $L^2(\mathbb{S}^1)$ onto $H^2(\mathbb{S}^1)$ then $P[P, f]$ is a compact operator.*

Proof. If $f = \sum_{k=-N}^N a_k e^{ikt}$ then $P[P, f]$ is of finite rank.

Otherwise using fact 6.9 proved in the Appendix, we can show that

$$||P[P, \sigma_n f] - P[P, f]|| \rightarrow 0 \text{ as } n \rightarrow \infty$$

Thus $P[P, f]$ is either a finite rank operator or can be approximated by a sequence of finite rank operators, hence is compact. $\square$

Theorem 2.3.4. *If T_g is a Toeplitz operator on $H^2(\mathbb{S}^1)$ whose symbol g is a continuous and nowhere vanishing function on $\mathbb{S}^1$ then T_g is a Fredholm operator and*

$$\text{Index}(T_g) = - \text{Winding number of } (g).$$

Proof. Let g_1 and g_2 be two non vanishing continuous functions on $\mathbb{S}^1$. Using fact 2.3.3 we have,

$$\begin{aligned} T_{g_1}T_{g_2} &= (Pg_1)(Pg_2) \\ &= P(g_1g_2) - P[P, g_1]g_2 \\ &= T_{g_1g_2} + \text{Compact Operators.} \end{aligned}$$

In particular if g is continuous and nowhere zero then $T_{g^{-1}}$ is the inverse of T_g modulo compact operators. Thus by Atkinson's theorem 2.2.11, we have T_g is Fredholm. The continuity of the Fredholm index shows that index T_g only depends on the homotopy class of the continuous function $g : \mathbb{S}^1 \rightarrow \mathbb{C}\setminus\{0\}$. So we will check the theorem on a representative of each homotopy class, say for example $f_k(\theta) = (e^{ik\theta})$.

Let us consider $f_k(\theta) = (e^{ik\theta})$. A short calculation shows that if $k > 0$ then cokernel of T_{f_k} is of dimension k and if $k < 0$ then kernel of T_{f_k} is of dimension k . Thus index of $T_{f_k} = -k$.

Let g be a function with winding number k . Then the map $e^{-ik\theta}g(e^{i\theta})$ has winding number 0 hence is homotopic to a constant map. Thus there exists a homotopy

$$F : [0, 1] \times \mathbb{S}^1 \rightarrow \mathbb{C}$$

such that

$$F_0 = e^{-ik\theta}g, \quad F_1 = 1.$$

For a fixed $t \in [0, 1]$

$$F_t : \mathbb{S}^1 \rightarrow \mathbb{C}\setminus\{0\}$$

Now $G_t = e^{ik\theta} F_t, t \in (0, 1)$ is a homotopy between g and f_k . Since G_t is non-vanishing, T_{G_t} is a continuous family of Fredholm operators. Now from the continuity of the Fredholm index we have $\text{Index } T_{f_k} = \text{Index } T_g$.

We have already shown that $\text{Index } T_{f_k} = -k$. Hence $\text{Index } T_g = -k$. Therefore we have,

$$\text{Index } T_g = - \text{winding number of } g.$$

□

2.4 Dirac Type Operators and Index

This thesis deals with Dirac operators and its various discretizations. We will now discuss the index of Dirac operators in a generalized setting.

Definition 2.4.1. A hermitian (or anti hermitian) operator $D : H \rightarrow H$ on a Hilbert space H is called Dirac type if there exists a linear operator $\Gamma : H \rightarrow H$ with the following properties

$$(i) \quad \Gamma^2 = \mathbf{1} \text{ (the identity map)}$$

$$(ii) \quad \Gamma D = -D\Gamma$$

Γ is the Dirac matrix γ_5 for the usual Dirac operator in the continuum and the standard discretization of the Dirac operator. For the new discretization in the staggered formalism Γ is given by Γ_{55} , introduced in (4.1.22).

Definition 2.4.2. We introduce operators

$$P_{\pm} = \frac{1}{2} (1 \pm \Gamma)$$

with the following properties,

$$P_{\pm}^2 = P_{\pm}, \quad P_+ P_- = 0, \quad P_+ + P_- = 1.$$

We can now decompose H as

$$H = H_+ \oplus H_- \quad \text{where } \Gamma = \pm 1 \text{ on } H_{\pm}.$$

$$H_{\pm} = P_{\pm}(H).$$

Fact 2.4.3. *A Dirac-type operator D maps $H_{\pm}$ to $H_{\mp}$.*

Proof. $h \in H_{\pm} \Leftrightarrow \Gamma h = \pm h \Rightarrow \Gamma(Dh) = -D\Gamma h = \mp Dh \Leftrightarrow Dh \in H_{\mp}$. $\square$

We will now discuss another definition of index called the Dirac index which are used for the Dirac type operators. However before doing that we will introduce some notations.

The restriction of D to $H_{\pm}$ are denoted by $D_{\pm}$ and

$$D_{\pm} : H_{\pm} \rightarrow H_{\mp}$$

If D is hermitian, then $(D_{\pm})^* = D_{\mp}$. If D is anti-hermitian then $(D_{\pm})^* = -D_{\mp}$.

Fact 2.4.4. Γ maps $\ker(D)$ to itself.

Proof. Let $h \in \ker D$. Then $Dh = 0$. Using the second property of Dirac type matrices we obtain

$$D(\Gamma h) = -\Gamma Dh = 0.$$

Thus kernel of D is mapped to itself by Γ . $\square$

Using the above fact we get an analogous decomposition for $\ker D$, given by

$$\ker(D) = \ker(D)_{+} \oplus \ker(D)_{-} \quad \text{where } \Gamma = \pm 1 \text{ on } \ker(D)_{\pm}. \quad (2.4.1)$$

We are now ready to define the Dirac index.

Definition 2.4.5. The Dirac index for Dirac type operators is given by

$$\text{Dirac-index}(D) = \dim \ker(D)_{+} - \dim \ker(D)_{-} \quad (2.4.2)$$

Theorem 2.4.6. *Using equation 2.2.2 to D_+ and recalling that in the hermitian case $(D_+)^* = D_-$ we obtain,*

$$\begin{aligned}
 \text{index}(D_+) &= \dim \ker(D_+) - \dim \ker((D_+)^*) \\
 &= \dim \ker(D_+) - \dim \ker(D_-) \\
 &= \dim \ker(D)_+ - \dim \ker(D)_- \\
 &= \text{Dirac index } (D).
 \end{aligned}$$

In all our subsequent discussions whenever Dirac operators and its discretization are discussed, $\text{index } (D)$ will refer to the Dirac index as mentioned above.

We will end this section by giving a short discussion on the index of Dirac type operators in a finite dimensional setting, which holds for the discretization of the standard Dirac operator discussed in subsequent chapters.

The following fact follows immediately from theorem 2.4.6

Fact 2.4.7. *Let $D : H \rightarrow H$ be a Dirac-type operator as above. If H is finite-dimensional and $\dim H_+ = \dim H_-$ then $\text{Dirac index}(D) = 0$.*

If H is finite dimensional then the trace of linear operators on H are always well defined. Since $\Gamma = \pm 1$ on $H_{\pm}$ we have,

$$\text{Tr}(\Gamma) = \dim H_+ - \dim H_-.$$

Thus the Dirac index of a Dirac type operator vanishes in finite dimensional settings when $\text{Tr}(\Gamma) = 0$. This is exactly the case for the discretization of standard Dirac operator and explains why the usual definition of the index vanishes for the

discretized Dirac operator. As we have mentioned in the introduction we can get around this difficulty by extracting the index in an indirect way. We will discuss this in details later in the thesis.

2.5 Elliptic Differential Operators

In this section we will describe a particular class of differential operators called elliptic operators. We will describe these operators on $\mathbb{R}^n$, but they can also be defined on general manifolds [9]. At the end of the section we will state a theorem which connects elliptic operators to Fredholm operators. The theorem tells us that elliptic operators are also Fredholm and hence making them suitable candidates for the study of index theory.

Definition 2.5.1. Let W be an open subset of $\mathbb{R}^n$ and let D_i be the operator $\frac{\partial}{\partial x_i}$. A differential operator D of order k is given by

$$D : C^\infty(W) \rightarrow C^\infty(W)$$

$$Du = \sum_{|\alpha| \leq k} a_\alpha D^\alpha w, \quad a_\alpha \in C^\infty(W).$$

$$\text{where } D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n} \text{ and } |\alpha| = \alpha_1 + \dots + \alpha_n.$$

Definition 2.5.2. Let D be a differential operator of order k on smooth functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$. The principal symbol of D is the function P on $W \times \mathbb{R}^n$ is given by

$$P(x, \xi) = \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha.$$

Definition 2.5.3. A differential operator D is called elliptic if its principal symbol $P(x, \xi)$ not equal to 0, $\forall x \in W$ and $\xi \in \mathbb{R}^n - 0$.

Example 2.5.4. Consider the Laplace operator given by

$$D \equiv \left(\frac{\partial}{\partial x_1} \right)^2 + \left(\frac{\partial}{\partial x_2} \right)^2 + \dots + \left(\frac{\partial}{\partial x_n} \right)^2$$

D is an elliptic operator since the principal symbol of D is $\xi^2 = \xi_1^2 + \xi_2^2 + \dots + \xi_n^2$ and is non-zero for all ξ .

Example 2.5.5. Consider the d' Alembert operator given by

$$D \equiv \left(\frac{\partial}{\partial x_1} \right)^2 + \left(\frac{\partial}{\partial x_2} \right)^2 + \dots - \left(\frac{\partial}{\partial x_n} \right)^2$$

D is not an elliptic operator since the principal symbol of D given by $\xi^2 = \xi_1^2 + \xi_2^2 + \dots + \xi_{n-1}^2 - \xi_n^2$ is equal to 0 when $\xi_n^2 = \xi_1^2 + \xi_2^2 + \dots + \xi_{n-1}^2$.

We will now state the main theorem [9] of this section.

Theorem 2.5.6. *An elliptic differential operator over a compact boundaryless manifold is a Fredholm operator.*

2.6 A Toy Model in the Continuum

In this section we will give an elementary example of the Atiyah-Singer Index Theorem. The example is for a Dirac operator acting on smooth functions on $\mathbb{R}^2$ satisfying certain periodicity conditions. We will build our model in the lattice from this model in the continuum.

Let us consider the operator

$$D = \gamma_1 \frac{\partial}{\partial x_1} + \gamma_2 \frac{\partial}{\partial x_2} \quad (2.6.1)$$

Here γ_μ are 2×2 complex matrices that satisfy

$$\gamma_\mu^* = \gamma_\mu, \quad \gamma_\mu^2 = 1, \quad \mu = 1, 2 \quad \gamma_1 \gamma_2 = -\gamma_2 \gamma_1 \quad (2.6.2)$$

2.6.1 Gauge Fields and Topological Charge

Let W be the vector space of smooth functions $f : \mathbb{R}^2 \rightarrow \mathbb{C}^2$ satisfying the following condition

$$\begin{cases} f(x_1 + 1, x_2) = f(x_1, x_2) & \text{periodic in 1st variable} \\ f(x_1, x_2 + 1) = e^{i\theta(x_1)} f(x_1, x_2) & \text{twisted periodic in 2nd variable} \end{cases} \quad (2.6.3)$$

Fact 2.6.1. *In order for the functions defined in equation (2.6.3) to be well defined we must have*

$$e^{i\theta(x_1+1)} = e^{i\theta(x_1)} \quad (2.6.4)$$

From equation (2.6.4) we have

$$\theta(x_1 + 1) = \theta(x_1) + 2\pi Q \text{ for all } x_1 \in \mathbb{R} \text{ and } Q \in \mathbb{Z}$$

For the deduction we refer to remark 6.1 in the Appendix.

Q is the topological winding number of the periodic map,

$$x_1 \rightarrow e^{i\theta(x_1)} \text{ from } [0, 1] \rightarrow S^1$$

Fact 2.6.2. *In order for these functions to be differentiable they need to satisfy the periodicity conditions after taking partial derivatives. (see the discussion in remark 6.2 in the appendix) However the continuum Dirac operator defined by*

$$D \equiv \gamma_1 \frac{\partial}{\partial x_1} + \gamma_2 \frac{\partial}{\partial x_2} \quad (2.6.5)$$

does not preserve the boundary conditions and we introduce functions

$$A_\mu : \mathbb{R}^2 \rightarrow \mathbb{R}. \quad \mu = 1, 2$$

called gauge fields which satisfy the following boundary conditions ,

$$\left\{ \begin{array}{l} A_1(x_1 + 1, x_2) = A_1(x_1, x_2) \\ A_1(x_1, x_2 + 1) = A_1(x_1, x_2) - \theta'(x_1) \\ A_2(x_1 + 1, x_2) = A_2(x_1, x_2) \\ A_2(x_1, x_2 + 1) = A_2(x_1, x_2). \end{array} \right. \quad (2.6.6)$$

Details of the above derivation are provided in remark 6.2 in the appendix.

Fact 2.6.3. *The following quantity*

$$\frac{1}{2\pi} \int_{[0,1]^2} F_{12}(x_1, x_2) dx_1 dx_2,$$

where

$$F_{12} = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$$

is an integer and doesn't depend on the choice of gauge fields.

Proof. We perform a direct computation:

$$\begin{aligned} \int_{[0,1]^2} \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right) dx_1 dx_2 &= \int_{[0,1]^2} \frac{\partial A_2}{\partial x_1} dx_1 dx_2 - \int_{[0,1]^2} \frac{\partial A_1}{\partial x_2} dx_2 dx_1 \\ &= \int_0^1 (A_2(1, x_2) - A_2(0, x_2)) dx_2 \\ &\quad - \int_0^1 (A_1(x_1, 1) - A_1(x_1, 0)) dx_1 \\ &= \int_0^1 \theta'(x_1) dx_1 = \theta(1) - \theta(0) \\ &= 2\pi Q, \end{aligned}$$

where we've used the twisted periodicity conditions from equation (2.6.3) in the third equality. The integer Q obtained is precisely the winding number of $x_1 \mapsto e^{i\theta(x_1)}$ and hence doesn't depend on the gauge field. $\square$

We call the integer Q the *topological charge* of the gauge field A .

2.6.2 Dirac Operator

We now couple the naive Dirac operator defined in equation (2.6.5) with the gauge fields A_μ and obtain the following Dirac operator

$$D = \gamma_1 \nabla_1 + \gamma_2 \nabla_2, \quad \text{where } \nabla_\mu = \frac{\partial}{\partial x_\mu} + A_\mu, \quad \mu = 1, 2 \quad (2.6.7)$$

For D to be elliptic γ_1 and γ_2 have to be 2×2 matrices and need to satisfy the following conditions

$$\gamma_\mu^2 = I, \gamma_\mu^* = \gamma_\mu, \quad \mu = 1, 2 \quad \gamma_1 \gamma_2 = -\gamma_2 \gamma_1 \quad (2.6.8)$$

2.6.3 Chirality and the Index of the Dirac operator.

The main goal in this section is to define the index of the Dirac operator introduced in equation (2.6.7). Our definition will closely follow the treatment already introduced in sections 2.2.

In order to define the index we first need the following fact.

Fact 2.6.4. *Let D be the Dirac operator as defined in equation (2.6.7) acting on the vector space W of smooth functions satisfying conditions of equation (2.6.3). D is elliptic and the functions f in the vector space W can be thought of as sections of a vector bundle over a twisted torus (because of the periodic boundary conditions in equation (2.6.3)). The twisted torus being compact we use theorem 2.5.6 to conclude that $\ker D$ is finite dimensional.*

In all our subsequent discussions we will denote $\ker D$ by V .

Definition 2.6.5. Let us define the matrix γ_3 as $\gamma_3 = -i\gamma_1\gamma_2$.

The matrix γ_3 also called the chirality matrix has the following properties

Fact 2.6.6.

$$\gamma_3^* = \gamma_3$$

$$\gamma_3^2 = 1$$

$$\gamma_3 \gamma_\mu = -\gamma_\mu \gamma_3, \quad \mu = 1, 2$$

$$\text{tr}(\gamma_3) = 0$$

See remark 6.3 in the appendix

Fact 2.6.7. *The Dirac operator in our example is anti-hermitian. That is*

$$D^* = -D, \text{ (for the proof see remark in 6.4 the appendix.)}$$

Thus from the definition of Fredholm index in equation (2.2.2) we have index $D = 0$.

Thus it might seem that the Dirac operator in our example is uninteresting from the point of view of Index Theorems. However we will now introduce operators called Weyl operators which are "projections" of the Dirac operator and express the index of D in terms of the index of Weyl operators.

Fact 2.6.8. *$\ker D$ is invariant under γ_3 .*

Proof. We have proved in remark 6.5 in appendix that

$$\gamma_3 D = -D \gamma_3.$$

We also have $D(\gamma_3 f) = -\gamma_3 D(f) = 0$. Thus $\ker D$ is invariant under γ_3 . $\square$

Definition 2.6.9. Let ϕ be a solution or a zero mode of the Dirac operator D . ϕ is said to be of positive chirality if $\gamma_3 \phi = \phi$ and ϕ is of negative chirality if $\gamma_3 \phi = -\phi$.

Definition 2.6.10. Consider the projection operators

$$P_+ = \frac{1}{2}(I + \gamma_3), \quad P_- = \frac{1}{2}(I - \gamma_3)$$

We define the Weyl operator as

$$D_+ = DP_+ \text{ and } D_- = DP_-$$

Let us also define $P_\pm(\ker D) = V_\pm$. Since $\ker D$ is invariant under γ_3 , we know that $V_\pm$ are subspaces of $V = \ker D$.

Fact 2.6.11. *Let D be a Dirac operator as explained in equation (2.6.7). Then*

$$V = V_+ \oplus V_-,$$

Proof: Take any $\phi \in \ker D$, we have

$$\phi = I(\phi) = (P_+ + P_-)\phi = P_+\phi + P_-\phi$$

Now we want to show that $V_+ \cap V_- = \{0\}$.

Let $\phi \in V_+ \cap V_-$. Then $\phi = P_+(\phi_1) + P_-(\phi_2)$ for some $\phi_1, \phi_2 \in \ker D$.

Definition 2.6.12. We define the index of the Dirac Operator D to be

$$\text{index } D = \dim V_+ - \dim V_- . \quad (2.6.9)$$

2.6.4 A Simple Illustration of the Atiyah-Singer Index Theorem

We conclude our toy model by making some simple choices for gauge fields, gamma matrices and stating the Atiyah-Singer Index Theorem for this particular example.

Let

$$\begin{cases} A_1(x_1, x_2) = -2\pi Q x_2 \\ A_2(x_1, x_2) = 0 \\ \theta(x_1) = 2\pi Q(x_1) \end{cases} \quad (2.6.10)$$

where Q is the integer obtained from equation (2.6.4)

and

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Then the Dirac operator D is given by

$$D \equiv \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}, \quad D_{\pm} = \frac{\partial}{\partial x_1} \pm i \frac{\partial}{\partial x_2} - i 2\pi Q x_2 \quad (2.6.11)$$

The Atiyah-Singer Index Theorem in this case states that

$$\text{Index } D = -Q$$

.

This tells us that we can extract information about the number of solutions of the Dirac operator in equation (2.6.11) by looking at the topology of the underlying vector space of functions on which it acts. In particular, if $Q \neq 0$, then we know that the Dirac equation has non trivial solutions.

The Lattice Setup

Discretization appears naturally in Physics. For example in field theory, one has to deal with an infinite number of degrees of freedom, labeled by the coordinates x and, in general, by some additional discrete indices. The functional integrals are only formally defined. One of the motivations behind discretization is to give the path integrals a precise meaning. We will therefore have to discretize space and time; i.e., we will be forced to introduce a space-time lattice. Eventually we will have to remove this lattice structure. This is a quite non-trivial task.

From the mathematical point of view, discretization is a technique which appears frequently. For instance, to solve PDE numerically, we need to discretize the domain and the differential operators. In topology, we can decompose topological spaces into discrete “cells” and then can describe some topological invariants combinatorially (the Euler characteristic is a famous example).

3.1 A Toy Model on the Lattice

In this section we will build a toy model on the lattice from the model in the continuum discussed in section 2.6. First we discretize our space by constructing a two dimensional rectangular lattice L , a set of discrete points on $[0, 1]^2$. Lattice sites are denoted by x_1, x_2 , and lattice directions are denoted by μ, ν . The spacing between two lattice sites is given by a . Without causing confusion, we can use the same symbols μ, ν to denote the corresponding unit vectors. Thus, for example, $x_1 + \hat{\mu}$, is the same site as $x_1 + ae_\mu$ and is displaced from x_1 by one unit along direction μ . The same convention holds for x_2 . Since a unit vector is also a link between two sites, μ also denotes a directed link. We can uniquely associate two directed links with each site space-time. For a much more detailed description of this model we refer to [35, 41]

3.1.1 Gauge fields and Dirac operators on the Lattice.

Let us consider the vector space $\Omega = \{f | f : L \rightarrow \mathbb{C}\}$ consisting of functions f , that satisfy the same periodicity condition as in the continuum

$$\begin{aligned} f(x_1 + 1, x_2) &= f(x_1, x_2) \\ f(x_1, x_2 + 1) &= e^{i\theta(x_1)} f(x_1, x_2) \end{aligned} \tag{3.1.1}$$

Therefore $\Omega \times \Omega = \Omega \times \mathbb{C}^2$.

We note that the discretized functions $f(x_1, x_2)$ now live on the lattice sites. Hence the smoothness or continuity requirements on $f(x_1, x_2)$ will now be replaced by approximate smoothness conditions. We have provided a very short discussion

on approximate smoothness conditions in 6.2 in the appendix.

We can also endow $\Omega \times \Omega$ with a inner product structure given by

$$\langle f, g \rangle = a^2 \sum_{x \in \Gamma} g(x)^* f(x), \quad f, g \in \Omega \times \Omega.$$

and where $\Gamma = \{x : x = (n_1 a, n_2 a) \in L, n_1, n_2, N \in \mathbb{N}, 0 < n_1, n_2 < N\}$.

However unlike in the continuum the underlying vector space $\Omega \times \Omega$ is finite dimensional since a function f on the lattice is completely determined by the values it takes on Γ .

Keeping the anti hermicity (2.6.7) property of the Dirac operator in mind we replace the continuum derivative by the finite difference operator:

$$\nabla_\mu f(x) = \frac{1}{2a} [f(x + \hat{\mu}) - f(x - \hat{\mu})], \quad (3.1.2)$$

where $\hat{\mu}$ is the unit vector in the μ direction. Just like in the continuous case , we need to introduce a lattice version of the continuum gauge fields A_μ and in order to preserve boundary conditions.

We do this by replacing $iA_\mu(x)$ by $U(x, x + \hat{\mu})$. $U_\mu(x) = U(x, x + \hat{\mu})$ are called link variables because they live on the edges of the lattice. The link variables are also known as lattice gauge fields and take values in $U(1) = \{z \in \mathbb{C} : |z| = 1\}$.

We then replace $\frac{\partial}{\partial x_\mu} + iA_\mu(x)$ by ∇_μ where,

$$\nabla_\mu f(x) = \frac{1}{2a} [U_\mu(x) f(x + \hat{\mu}) - U_\mu^{-1}(x - \hat{\mu}) f(x - \hat{\mu})] \quad (3.1.3)$$

Proceeding just like in the continuum case (2.6.6), we can analogously show that in order to preserve the boundary condition the lattice gauge fields have to satisfy the following conditions:

$$\left\{ \begin{array}{l} U_1(x_1 + a, x_2) = U_1(x_1, x_2) \\ U_1(x_1, x_2 + a) = e^{i\theta(x_1)} U_1(x_1, x_2) e^{-i\theta(x_1+a)} \\ U_2(x_1 + a, x_2) = U_2(x_1, x_2) \\ U_2(x_1, x_2 + a) = U_2(x_1, x_2) \end{array} \right. \quad (3.1.4)$$

The lattice gauge fields U_μ are related to the continuum gauge fields A_μ by the following relation

$$U_\mu(x) = e^{i\alpha_\mu(x)}, \quad \alpha_\mu(x) = \int_0^a A_\mu(x + (a-t)e_\mu) dt, \quad \mu = 1, 2. \quad (3.1.5)$$

Fact 3.1.1. *In the continuum limit the lattice operator ∇_μ approach $\partial_\mu + iA_\mu$*

Proof. We first recall that $x + \hat{\mu}$ denotes $x + ae_\mu$ and we then expand U_μ and U_μ^{-1} as defined in equation (3.1.5) in the powers of a up to the first order term to obtain

$$U_\mu(x) = 1 + iaA_\mu(x) + \mathcal{O}(a^2)$$

$$U_\mu^{-1}(x) = 1 - iaA_\mu(x) + \mathcal{O}(a^2)$$

Substituting the above expressions in equation (3.1.3) we have

$$\begin{aligned}
\nabla_\mu f(x) &= \frac{1}{2a} [U_\mu(x)f(x + \hat{\mu}) - U_\mu^{-1}(x - \hat{\mu})f(x - \hat{\mu})] \\
&= \frac{1}{2a} \left[(1 + iaA_\mu(x) + \mathcal{O}(a^2))f(x + \hat{\mu}) \right. \\
&\quad \left. - ((1 - iaA_\mu(x) + \mathcal{O}(a^2))f(x - \hat{\mu})) \right] \\
&= \frac{f(x + \hat{\mu}) - f(x - \hat{\mu})}{2a} + \frac{i}{2a} (A_\mu(x)f(x + \hat{\mu}) + A_\mu(x - \hat{\mu})f(x - \hat{\mu})) \\
&\quad + \mathcal{O}(a)(f(x + \hat{\mu}) - f(x - \hat{\mu}))
\end{aligned}$$

Thus when $a \rightarrow 0$ we have

$$\nabla_\mu f(x) \rightarrow \partial_\mu f(x) + iA_\mu(x)f(x).$$

□

Topological Charge

We will now try to introduce the concept of topological charge for a lattice gauge field. We note that the usual definition of the topological charge based on the twisted periodicity condition does not work for lattice gauge fields since in the discrete case

$$\theta(x_1 + 1) = \theta(x_1) + 2\pi Q(x_1)$$

and in general the integer Q depends on x_1 .

Definition 3.1.2. Given a lattice gauge field, the quantity

$$U(x, \mu, \nu) = U_\mu(x)U_\nu(x + \hat{\mu})U_\mu(x + \hat{\nu})^{-1}U_\nu(x)^{-1}, \quad 1 \leq \mu \neq \nu \leq 2 \quad (3.1.6)$$

is called a plaquette variable.

The next proposition plays an important role in our definition of the discrete topological charge.

Fact 3.1.3. *The following quantity*

$$\frac{1}{2i\pi} \sum_{x \in \Gamma, \mu < \nu} \log U(x, \mu, \nu)$$

is an integer.

Proof. The key observation is the following identity

$$\prod_{x \in \Gamma, \mu < \nu} U(x, \mu, \nu) = 1.$$

Now using the twisted periodicity condition 3.1.1 we obtain

$$\begin{aligned} \prod_{x \in \Gamma, \mu < \nu} U(x, \mu, \nu) &= e^{i(\theta(x_1+a)-\theta(x_1))} e^{i(\theta(x_1+2a)-\theta(x_1+a))} \dots e^{i(\theta(x_1+1)-\theta(x_1+(N-1)a))} \\ &= e^{i(\theta(x_1+1)-\theta(x_1))} = e^{i2\pi Q(x_1)} \\ &= 1. \end{aligned}$$

It follows from the properties of the exponential function that

$$\begin{aligned} \exp \left(\sum_{x \in \Gamma, \mu < \nu} \log U(x, \mu, \nu) \right) &= \prod_{x \in \Gamma, \mu < \nu} \exp(\log U(x, \mu, \nu)) \\ &= \prod_{x \in \Gamma, \mu < \nu} U(x, \mu, \nu) \\ &= 1. \end{aligned}$$

Thus

$$\sum_{x \in \Gamma, \mu < \nu} \log U(x, \mu, \nu) = i2\pi Q$$

for some integer Q . □

Definition 3.1.4. Thus the topological charge of a lattice gauge field is given by

$$Q = \frac{1}{i2\pi} \sum_{x \in \Gamma, \mu < \nu} \log U(x, \mu, \nu).$$

We are now ready to introduce a Dirac operator on the lattice which is known as the naive lattice Dirac Operator.

Definition 3.1.5. The naive lattice Dirac operator on the lattice is defined as

$$D_{lat} = \gamma_1 \nabla_1 + \gamma_2 \nabla_2 \tag{3.1.7}$$

where γ_μ 's are as in equation (2.6.2) and ∇_μ 's are as in equation (3.1.3).

Fact 3.1.6. *Just like in the continuum case, the lattice Dirac operator is also anti hermitian.*

$$D_{lat}^* = -D_{lat}.$$

For the proof we refer to remark 6.12 in the appendix.

3.1.2 The Fermion Doubling Problem

We consider the partial differential equation

$$D_{lat} f = 0. \tag{3.1.8}$$

where D_{lat} is defined in equation (3.1.7) and $f \in \Omega \times \Omega$.

Just like in the continuum case we still have the following equality

$$\gamma_3 D_{lat} = -D_{lat} \gamma_3.$$

Let $V = (\ker D_{lat})$ and $V_{\pm} = (\ker D_{lat})_{\pm}$. Hence, as in the continuum case, we define the Dirac index of D_{lat} to be

$$\text{Index } D_{lat} = \dim (V_+) - \dim (V_-) \quad (3.1.9)$$

Fact 3.1.7. *However it can be proved algebraically (see remark 6.10 in appendix) that the index of the naive Dirac operator D_{lat} is equal to zero.*

3.1.3 A Closer Look at the Fermion Doubling Problem

Fact 3.1.8. *For simplicity let us assume that the naive Dirac operator has no gauge fields. Then the following are equivalent [35, 41].*

- I. $D_{lat}\phi = 0 \iff D_{lat}^* D_{lat}\phi = 0$.
- II. $D_{lat}^* D_{lat}\phi = 0 \iff \nabla_{\mu}^* \nabla_{\mu}\phi = 0, \mu = 1, 2$.
- III. $\nabla_{\mu}^* \nabla_{\mu}\phi = 0 \iff \nabla_{\mu}\phi = 0, \mu = 1, 2$.

Remark 3.1. Thus the solutions to $D_{lat}\phi = 0$ are equivalent to the solutions of $\nabla_{\mu}\phi = 0$. For a detailed proof we refer to remark 6.11 in appendix.

Thus any solution of $D_{lat}\phi = 0$ can be expressed as a linear combination of the following functions

$$\phi_1(x) = \begin{cases} 1, & \text{if } x = (\text{odd}, \text{odd}) \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_2(x) = \begin{cases} 1, & \text{if } x = (\text{odd}, \text{even}) \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_3(x) = \begin{cases} 1, & \text{if } x = (\text{even}, \text{odd}) \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_4(x) = \begin{cases} 1, & \text{if } x = (\text{even}, \text{even}) \\ 0 & \text{otherwise} \end{cases}$$

We note that there is only one independent solution to $D\phi = 0$ in the continuum with no gauge fields which is a constant throughout the whole space and corresponds to the case $c_1 = c_2 = c_3 = c_4$. Thus discretizing the continuum Dirac operator D given by (2.6.7) gives us additional "rough" or "spurious" solutions which makes the index of D_{lat} zero!

3.1.4 Wilson Fermions

We have seen in the previous section that when we try to discretize the Dirac operator we get some additional "rough" solutions which in turn makes the index of the Dirac operator zero. In order to rectify these problems we will we modify the lattice Dirac operator in a certain way.

We introduce the lattice version of the second derivative, Δ called the lattice

Laplace operator:

$$\Delta = \sum_{\mu} \Delta_{\mu}$$

The explicit formula for

$$\Delta_{\mu}\phi(x) = \frac{1}{a^2}(-U_{\mu}(x)\phi(x + ae_{\mu}) + 2\phi(x) - U_{\mu}^{-1}(x - ae_{\mu})\phi(x - ae_{\mu})) \quad (3.1.10)$$

We then add these Laplace operators to our lattice Dirac operator D_{lat} to obtain

Definition 3.1.9. The Dirac-Wilson operator is given by

$$D_w = D_{lat} + \frac{a}{2}\Delta \quad (3.1.11)$$

The fermions described by the Dirac-Wilson operator are called the Wilson fermions and the term $\frac{a}{2}\Delta$ is the Wilson term.

Remark 3.2. The role of Δ is to separate the zero modes that approximate the smooth continuum solutions from the spurious zero modes that only exist on the lattice.

3.1.5 Spectral Flow and Index

The spectral flow is a method that will allow us to computationally obtain the index of the Dirac operator. In the previous section we observed that the naive Dirac operator D_{lat} has rough solutions and that its index is always zero. Our aim in this section is not to give a detailed discussion on spectral flow but to give

an illustration to show how the Wilson term in equation (3.1.11) distinguishes between rough and smooth solutions.

Definition 3.1.10. Let us define

$$H(m) = \gamma_3(D - mI)$$

where D can be replaced by the continuum Dirac operator, D_{lat} or the Wilson Dirac operator D_w .

It can be easily checked that for each of the continuum Dirac operator, D_{lat} and D_w the corresponding $H(m)$ is Hermitian, hence their corresponding eigenvalues $\lambda(m)$ are all real.

Definition 3.1.11. The spectral flow of $H(m)$ is the difference of the number of eigenvalues of $H(m)$ that cross the x-axis with positive slope and the number of eigenvalues that cross the x-axis with negative slopes near $m = 0$, counting multiplicity. We define the index of D to be the negative of the spectral flow.

If we plot $\lambda(m)$ for every such m we obtain a graph which looks like:

Figure 3.1.1: Spectral flow of $H(m)$ for different Dirac operators with $Q = 1$

For the naive Dirac operator D_{lat} , the graph of the Spectral flow of $H(m)$, has equal number of eigenvalue crossings with slopes $+1$ and -1 which means observing the index using the spectral flow approach does not give us anything useful. Δ helps us in separating the rough solutions from the smooth solutions. The corresponding $H(m)$ denoted by $H_w(m)$ gives us a fair indication of the index.

We will now present some facts which are extremely important for the definition of the index of the Wilson Dirac operator D_w . For the proves we refer to 6.2 in the appendix

Fact 3.1.12. *I. For the Wilson Dirac operator D_w the eigenvalue crossings do not occur at $m = 0$.*

II. The eigenvalue crossings are excluded in a region around $m = \frac{1}{a}$ as well as $m < 0$.

Figure 3.1.2: Distinction between rough and smooth solutions

III. Δ separates the rough and the smooth solutions and the eigenvalue crossings corresponding to the smooth solutions occur when $0 < m < \frac{1}{a}$. Hence the notion of eigenvalue crossing near $m = 0$ is well defined. These eigenvalue crossings are called low lying eigenvalue crossings.

Definition 3.1.13. In the lattice setting we define the index of D as

$$\text{index } D_w = -\text{Spectral flow of } H_w(m) \text{ between } m = 0 \text{ and } m = \frac{1}{a}$$

The spectral flow of $H_w(m)$ between $m = 0$ and $m = \frac{1}{a}$ is

$$\frac{1}{2}(n_+ - n_-)$$

$n_{\pm}$ = number of +ve / - ve eigenvalues of $H(\frac{1}{a})$.

Fact 3.1.14. *I. $H_w(m)$ has the same number of +ve and -ve eigenvalues when $m \leq 0$*

II. For every positive crossing the difference of the number of eigenvalues with positive slope and the number of eigenvalues with negative slope increases by two and for every negative crossing the difference decreases by two.

III. The spectral flow of $H_w(m)$ between $m = 0$ and $m = \frac{1}{a}$ is

$$\frac{1}{2}(n_+ - n_-)$$

$n_{\pm}$ = number of +ve / - ve eigenvalues of $H_w(\frac{1}{a})$.

Based on the above facts we make the following definition:

Definition 3.1.15. Index of $D = -\frac{1}{2}(n_+ - n_-) = -\frac{1}{2}\text{Tr} \left(\frac{H_w(\frac{1}{a})}{\sqrt{H_w^2(\frac{1}{a})}} \right)$

Numerical Illustration:

Figure 3.1.3: Eigenvalue flow of $H_w(m)$ with a lattice gauge field with $Q = 1$ and -2 from computer calculations

In the first diagram we see that there is one negative low lying eigenvalue and no positive eigenvalue crossings near $m = 0$ and in the second diagram we see that there are two positive low lying eigenvalue crossings and no negative eigenvalue crossings near $m = 0$. Thus from this we can conclude that the index of the corresponding D_w are given by 1 and -2 respectively.

Index of the Discretized Dirac Operator in the Staggered Formalism

In this chapter we evaluate the continuum limit of the index of the discretized Dirac operator in the staggered Wilson formalism and show that it equals the topological charge of the gauge field. Hence we establish a similar result which was previously shown in [1] for the discretization of the usual Wilson Dirac operator.

In our calculations we follow the approach of [1], however we have to develop new methods and techniques to deal with issues which are specific to the staggered Wilson formalism.

4.1 Formulation of the Problem

The Continuum Setting

We will first describe the continuum setting, that we then discretize. More specifically we will describe the continuum Dirac operator, the gauge fields coupled to it and the space of functions on which the continuum Dirac operator acts, including their boundary conditions.

Let us consider a smooth map

$$\phi : [0, L]^3 \rightarrow SU(N) \quad (4.1.1)$$

such that ϕ is periodic at the boundaries of the three dimensional box $[0, L]^3$, hence $[0, 1]^3$ can be identified with T^3 , the 3-torus. $SU(N)$ has a smooth manifold structure and we can show that for $N > 2$,

$$SU(N) \cong S^3 \times S^5 \times \dots \times S^{2N-1},$$

where S^N denotes the unit sphere in N -dimensions.

Thus we can think of ϕ being defined on a 3 dimensional torus T^3 which wraps it around the copy of S^3 in $SU(N)$ some integer number of times. This integer known as the degree of ϕ is a generalization of the winding number discussed in earlier chapters for maps from S^1 to S^1 . The degree of a map between smooth compact oriented manifolds was originally introduced by Brouwer, and is described in [32]. We will see below that ϕ is used to give the gauge fields coupled to the Dirac operator, a periodic boundary condition such that the topological

charge of the gauge fields is precisely the degree of ϕ . Thus the degree of ϕ is the integer that gives the topological index in the Index Theorem for the Dirac operator. This is a generalization of the periodicity conditions discussed in the two dimensional toy model where the topological charge of the $U(1)$ gauge field was the winding number of the map θ .

The gauge fields $A_\mu(x)$, are elements of the Lie algebra $su(N)$ of the Lie group $SU(N)$. The Lie algebra consists of all generators of the group $SU(N)$ that is all $N \times N$ matrices A such that $\exp A \in SU(N)$. Let $x = (x_1, x_2, x_3, x_4) = (\mathbf{x}, x_4)$. Then $\phi = \phi(\mathbf{x})$ and $A_\mu(x) = A_\mu(\mathbf{x}, x_4)$.

The Dirac operator D acts on the vector space

$$W = \{f|f : \mathbb{R}^4 \rightarrow \mathbb{C}^4 \otimes \mathbb{C}^N\},$$

where the functions f satisfy the following periodicity condition

(i) $f(x)$ is unchanged under $x_k \rightarrow x_k + L$ for each $k = 1, 2, 3$.

(ii) $f(\mathbf{x}, x_4 + L) = \phi(\mathbf{x})f(\mathbf{x}, x_4)$.

We note that $\mathbb{C}^N$ carries the representation of $SU(N)$ in $\mathbb{C}^4 \otimes \mathbb{C}^N$. As $\phi(x) \in SU(N)$ and $f(x) \in \mathbb{C}^4 \otimes \mathbb{C}^N$, the multiplication $\phi(x)$ with $f(x)$ occurs in the $\mathbb{C}^N$ component of $\mathbb{C}^4 \otimes \mathbb{C}^N$. We can regard the functions $f(x)$ either as living on $\mathbb{R}^4$ with periodicity conditions or living on the 4-box $[0, L]^4 \subset \mathbb{R}^4$ with boundary conditions.

One of the most important requirements for the Dirac operator is that it should map W to itself. However just as we had seen in the 2-dimensional toy model

the "covariant derivative" given by,

$$\partial_\mu + A_\mu(x). \quad (4.1.2)$$

does not preserve the boundary conditions on $f(x)$. Hence the need to impose the boundary conditions mentioned above.

The boundary conditions on the gauge field $A_\mu(x)$ for each $\mu = 1, 2, 3, 4$ are given by

- (i) $A_\mu(x)$ is periodic when $x = x_k$ for each $k = 1, 2, 3$.
- (ii) $A_\mu(\mathbf{x}, x_4 + L) = \phi(\mathbf{x})A_\mu(\mathbf{x}, x_4)\phi(\mathbf{x})^{-1} + \phi(\mathbf{x})\partial_\mu\phi(\mathbf{x})^{-1}$.
- (iii) $A_4(\mathbf{x}, x_4 + L) = A_4(\mathbf{x}, x_4)$

The consistency of (i) and (ii) are ensured by the periodicity of ϕ discussed above. The above conditions enable us to treat the gauge fields A_μ as either defined on $\mathbb{R}^4$ with periodicity conditions or as functions defined on the 4-box with boundary conditions.

The Dirac operator D acts on the functions $f(x)$, and preserves their periodicity/boundary conditions above and can then be defined analogously to the Dirac operator in the 2-dimensional example in the toy model. It is given by

$$D = \gamma_\mu (\partial_\mu + A_\mu) \quad (\text{implicit sum over the repeated index } \mu), \quad (4.1.3)$$

where the Dirac matrices $\{\gamma_\mu\}_{\mu=1,2,3,4}$ are a collection of hermitian 4×4 matrices

satisfying the Dirac algebra:

$$\gamma_\mu^2 = \mathbf{1} \quad \text{for } \mu = 1, 2, 3, 4; \quad \gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu \quad \text{for } \mu \neq \nu. \quad (4.1.4)$$

We again note that the $\mathbb{C}^4$ factor is the representation space of the Dirac algebra. These matrices can then be multiplied onto the functions $f(x)$ by acting on the $\mathbb{C}^4$ in $\mathbb{C}^4 \otimes \mathbb{C}^N$.

The Index Theorem for D is a special case of a more general Index Theorem in [8] It states

$$\text{index}(D) = Q \quad (4.1.5)$$

where

$$Q = \frac{-1}{32\pi^2} \int_{[0,L]^4} d^4x \epsilon_{\mu\nu\sigma\rho} \text{tr}(F_{\mu\nu}(x)F_{\sigma\rho}(x)) \quad (4.1.6)$$

with $F_{\mu\nu}(x)$ being the field strength tensor of the gauge field, given by

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + A_\mu(x)A_\nu(x) - A_\nu(x)A_\mu(x), \quad (4.1.7)$$

and "tr" being the trace over $\text{su}(N)$ in 4.1.6. One of the questions that one might ask is whether Q as defined in the expression 4.1.6 is an integer. We will now sketch an argument which will tell us that Q is indeed an integer.

It can be shown that the integrand in 4.1.6 is a total derivative. We then use Stokes' theorem and reduce the integral to the 3-dimensional boundary faces of $[0, L]^4$. The integral over the opposite boundary faces across the x_1, x_2, x_3 axes can be canceled out. However the integrals over the opposite boundary faces across x_4 axis combine to give an integral over $[0, L]^3$ the integrand of which can be expressed entirely in terms of ϕ and turns out to be a formula for the calculation

of the degree of ϕ . Thus we have

$$Q = \deg(\phi) \in \mathbb{Z}. \quad (4.1.8)$$

This not only shows that Q is an integer but also shows that it is determined by topological data of the boundary conditions via ϕ .

We recall that the index of D can be expressed in terms of the spectral flow of a hermitian operator $H(m)$. In the present case the hermitian operator takes the form

$$H(m) = \gamma_5(D - m\mathbf{1}) \quad (4.1.9)$$

where

$$\gamma_5 \equiv \gamma_1\gamma_2\gamma_3\gamma_4. \quad (4.1.10)$$

From the Dirac matrix algebra [4.1.4](#) it can easily be shown that γ_5 is hermitian and has the properties

$$\gamma_5^2 = \mathbf{1} \quad , \quad \gamma_5\gamma_\mu = -\gamma_\mu\gamma_5. \quad (4.1.11)$$

Using these, the index of D can be expressed in terms of spectral flow of $H(m)$ just like in the 2-dimensional toy model discussed earlier.

4.1.1 Details of the Lattice Setup

We begin our discretization by making a hyper-cubic lattice decomposition of the 4-box $[0, L]^4$. The lattice spacing is given by $a = L/N$, where N is the number of vertices on the lattice in each direction. A vertex is also known as a lattice site

and is denoted by

$$x = an, \quad n = (n_1, n_2, n_3, n_4) \in \mathbb{Z}^4 \text{ with } 0 \leq n_\mu \leq N \text{ for each } \mu = 1, 2, 3, 4. \quad (4.1.12)$$

This can be extended to an infinite lattice on $\mathbb{R}^4$ with lattice sites $x = an, n \in \mathbb{Z}^4$. By a neighboring lattice site of x we mean all lattice sites of the form $x \pm \hat{\mu}$ for $\mu = 1, 2, 3, 4$. Throughout this and the subsequent sections $\hat{\mu}$ denotes the unit vector along the positive direction of the x_μ axis in $\mathbb{R}^4, \mu = 1, 2, 3, 4$. The edges of the lattice are called links and the faces are known as plaquettes.

In the lattice setting instead of gauge fields we have link variables $U(x, x + \hat{\mu}) \in \text{SU}(N)$, for each oriented link $[x, x + \hat{\mu}]$. The orientation is so chosen that the lattice site x is the end point (since the link variables are directed) of the link and we observe the following convention $U(x + \hat{\mu}, x) = U(x, x + \hat{\mu})^{-1}$. The link variables $U(x, x + \hat{\mu})$ are also denoted by $U_\mu(x)$. This is in contrast to the continuum setting where the gauge fields $A_\mu(x) \in \mathfrak{su}(N)$, the Lie algebra of $SU(N)$. Thus we can think of the link variables $U_\mu(x)$ as exponentiated gauge fields.

Definition 4.1.1. Let us consider the plaquette in the $\mu\nu$ -plane and orient it by a boundary path starting and ending at x . The plaquette variable $U_{\mu\nu}$ is given by

$$U_{\mu\nu}(x) = U(x, x + \hat{\mu})U(x + \hat{\mu}, x + \hat{\mu} + \hat{\nu})U(x + \hat{\mu} + \hat{\nu}, x + \hat{\nu})U(x + \hat{\nu}, x) \quad (4.1.13)$$

Henceforth we take the link variables $U_\mu(x)$ to be the lattice transcripts of the continuum gauge fields. A precise definition of lattice transcripts can be found in [1] and here we state the following relations of the link variables with the

continuum gauge fields A_μ and field tensor $F_{\mu\nu}$ introduced in equation (4.1.7),

$$U_\mu(x) \equiv U(x, x + \hat{\mu}) = \mathbf{1} + aA_\mu(x) + O(a^2) \quad (4.1.14)$$

$$U_{\mu\nu}(x) = \mathbf{1} + a^2 F_{\mu\nu}(x) + O(a^3) \quad (4.1.15)$$

Just like in the continuum gauge fields $A_\mu(x)$ the link variables satisfy the following periodicity conditions,

(i) $U_\mu(x)$ is unchanged under $x_k \rightarrow x_k + L$ for each $k = 1, 2, 3$ and all $\mu = 1, 2, 3, 4$.

(ii) $U_\mu(\mathbf{x}, x_4 + L) = \phi(\mathbf{x})U_\mu(\mathbf{x}, x_4)\phi(\mathbf{x} + \hat{\mu})^{-1}$ for each $\mu = 1, 2, 3$.

(iii) $U_4(\mathbf{x}, x_4 + L) = U_4(\mathbf{x}, x_4)$.

where $x = (\mathbf{x}, x_4)$.

In the lattice setting the functions $f(x)$ live on the lattice sites x and in the standard lattice setting take values in $\mathbb{C}^4 \otimes \mathbb{C}^N$. Furthermore they satisfy the same periodicity conditions 4.1 of their continuum counterpart. The lattice covariant derivative ∇_μ is defined on the functions by

$$\nabla_\mu f(x) = \frac{1}{2a} \left(U(x, x + \hat{\mu})f(x + \hat{\mu}) - U(x, x - \hat{\mu})f(x - \hat{\mu}) \right) \quad (4.1.16)$$

Using the periodicity condition 4.1.1 of the link variables it can be shown that ∇_μ preserves the periodicity conditions of the functions f . Just like in the continuum we can regard the functions $f(x)$ either as living on $\mathbb{R}^4$ with periodicity conditions or living on the 4-box $[0, L]^4 \subset \mathbb{R}^4$ with boundary conditions. The same is true

for the link variables. However the second perspective makes it clear that the vector space of lattice functions is finite dimensional.

The naive lattice Dirac operator is defined as

$$D = \gamma_\mu \nabla_\mu \quad (4.1.17)$$

We note that the matrix $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$ plays the role of the operator Γ introduced in equation 2.4.1. Using equation 4.1.11 we have,

$$\begin{aligned} \text{tr}(\gamma_5) &= \text{tr}(\gamma_5 \gamma_\mu \gamma_\mu) = \text{tr}(\gamma_\mu \gamma_5 \gamma_\mu) = -\text{tr}(\gamma_5) \\ \Rightarrow \text{tr}(\gamma_5) &= 0. \end{aligned}$$

However as discussed in the remarks following fact 2.4.7 in chapter 2 index $D = 0$. As discussed extensively in the previous chapters we modify our Dirac operator by a Wilson term and the modified Dirac operator takes the form

$$D_W = \gamma_\mu \nabla_\mu + \Delta \quad (4.1.18)$$

where Δ is the Wilson term. The index can then be extracted from the spectral flow of the hermtian operator

$$H(m) = \gamma_5 (D_W - m) \quad (4.1.19)$$

Specifically, it is given in terms of $H(m)$ by the formula 1.0.2 in the introduction.

The continuum limit, taken by successive subdivisions of the lattice, was evaluated for the index discussed above in [1] and found to be the topological charge Q

of the gauge field, thus confirming that the continuum Index Theorem is reproduced in this case. The main research goal of this thesis is to establish the same result for a new lattice version of the Dirac operator introduced in [5]. This new version was constructed in a different formalism known as the staggered formalism. This formalism [20] has the added advantage of being computationally more efficient and hence is favored by some lattice physicists for computer simulations of quantum field theory. We will now describe the formalism.

The main difference of the staggered formalism from the standard formalism described above, concerns the lattice functions $f(x)$ and the γ matrices. The lattice, link variables and the lattice covariant operators ∇_μ are exactly same as before. Unlike in the standard formalism the γ matrices are not explicitly present in the staggered formalism and emerge in an indirect way later on. Also the lattice functions in the staggered formalism take values in $\mathbb{C}^N$ instead of $\mathbb{C}^4 \times \mathbb{C}^N$ (in physics terminology this is explained by saying that the staggered fermions do not contain Dirac structure). We note that the periodicity/boundary conditions on the lattice link variables and lattice functions $f(x)$ are still the same as before.

The lattice Dirac operator in this setting is given by [20]

$$D_{st} = \eta_\mu \nabla_\mu \quad (4.1.20)$$

where the η_μ s (which appear in place of the Dirac matrices γ_μ of the standard formalism) are defined to act on the functions by

$$\eta_\mu f(x) = (-1)^{n_1 + \dots + n_{\mu-1}} f(x) \quad (\text{using } x = an \text{ where } n \in \mathbb{Z}^4). \quad (4.1.21)$$

The η_μ s are easily seen to preserve the periodicity/boundary conditions on the functions $f(x)$ as long as the lattice has an even number of sites along each axis, which we henceforth assume to be the case.

The role of γ_5 in this formalism is played by an operator Γ_{55} defined by

$$\Gamma_{55}f(x) = (-1)^{n_1+n_2+n_3+n_4}f(x). \quad (4.1.22)$$

We can easily verify that $\Gamma_{55}^2 = \mathbf{1}$ and $\Gamma_{55}D_{st} = -D_{st}\Gamma_{55}$. Moreover D_{st} is anti-hermitian and since $\text{Tr}(\Gamma_{55}) = 0$, by the remarks following fact 2.4.7 in chapter 2 $\text{index}D_{st} = 0$. Thus just like in the standard formalism the index has to be extracted in a different way and we need a new version of Wilson term in this formalism. A suitable Wilson term for this formalism was discovered recently in [5]. It is given by

$$W_{st} = \frac{r}{a}(1 - \Gamma_{55}\Gamma_5), \quad r > 0$$

where

$$\Gamma_{55}\chi(x) = (-1)^{x_1+x_2+x_3+x_4}\chi(x),$$

$$\Gamma_5 = \eta_5 C,$$

$$\eta_5 = \eta_1\eta_2\eta_3\eta_4, \quad C = (C_1C_2C_3C_4)_{\text{sym}},$$

$$C_\mu\chi(x) = \frac{U_\mu(x)\chi(x + \hat{\mu}) + U_\mu^\dagger(x - \hat{\mu})\chi(x - \hat{\mu})}{2}.$$

The staggered Wilson-Dirac operator is then given by

$$D_{sW} = D_{st} + W_{st} \quad (4.1.23)$$

The hermitian version of this operator for extracting the index via spectral flow is

$$H_{sW}(m) = \Gamma_{55}(D_{sW} - m\mathbf{1}) \quad (4.1.24)$$

The index in this formalism is then expressed by 1.0.3 in the Introduction, with $H(m)$ being $H_{sW}(m)$. From hereon $H_{sW}(m)$ will be simply denoted by $H(m)$.

4.2 Evaluation of the Continuum Limit of the Dirac Index in the Staggered Formalism

Let us consider the index density

$$q(x) = -\frac{1}{2}\mathrm{tr}_c\left(\frac{H}{\sqrt{H^2}}\right)(x, x) \quad (4.2.1)$$

In the usual overlap case, the main step in showing that both the index and axial anomaly have the correct continuum limits is to show that the index density $q(x)$ reproduces the continuum topological charge density [2, 1]:

$$\lim_{a \rightarrow 0} q(x) = -\frac{1}{32\pi^2} \epsilon_{\mu\nu\sigma\rho} \mathrm{tr}_c(F_{\mu\nu}(x)F_{\sigma\rho}(x)) \quad (4.2.2)$$

Here x is a specific point in the space-time box, and the $a \rightarrow 0$ limit is taken by repeated subdivisions of the lattice such that x continues to be a lattice site of

all the subdivided lattices. However, in the present case, it turns out that 4.2.2 only holds after $q(x)$ is averaged over the sites of a lattice hypercube containing x . This not surprising, since the sites around a lattice hypercube can be regarded as corresponding to the same spacetime point in the staggered formalism.

Our goal in this section is to sketch the derivation of 4.2.2, i.e. with $q(x)$ averaged over the sites of a lattice hypercube containing x , and with an extra factor of 2 due to the two physical fermion flavors described by the staggered overlap fermion.

By the same arguments as in the usual overlap case [1] we have

$$\lim_{a \rightarrow 0} \frac{H}{\sqrt{H^2}}(x, x) = \lim_{a \rightarrow 0} \int_{-\pi/2a}^{3\pi/2a} d^4 q e^{-iqx} \frac{H}{\sqrt{H^2}} e^{iqx}. \quad (4.2.3)$$

Now decompose the lattice momentum $q \in [-\pi/2a, 3\pi/2a]^4$ as $q = \frac{\pi}{a}A + p$, where $p \in [-\pi/2a, \pi/2a]^4$ and $A = (A_1, A_2, A_3, A_4)$ with $A_\mu \in \{0, 1\}$. In the following, if A and B are two such vectors, we also consider $A + B$ to be a vector of this type with the components $(A + B)_\mu \in \{0, 1\} \bmod 2$. Let V be the vectorspace spanned by the 16 plane waves $\{e_A(x) \equiv e^{i\frac{\pi}{a}Ax}\}$. In [20] two commuting representations of the Dirac algebra are defined on V , namely $\{\hat{\Gamma}_\mu\}$ and $\{\hat{\Xi}_\nu\}$, given by $(\hat{\Gamma}_\mu)_{AB} = (-1)^{A_\mu} \delta_{A, B+\eta_\mu}$ and $(\hat{\Xi}_\nu)_{AB} = (-1)^{A_\mu} \delta_{A, B+\zeta_\nu}$ where η_μ and ζ_ν are vectors whose components are all zero except $(\eta_\mu)_\sigma = 1$ for $\sigma < \mu$ and $(\zeta_\nu)_\sigma = 1$ for $\sigma > \nu$.¹

The staggered Wilson Dirac operator [5] is $D_{sW} = D_{st} + W_{st}$ where D_{st} is the usual (massless) staggered Dirac operator and $W_{st} = \frac{r}{a}(1 - \Xi_5)$ is the Wilson term. Here $\Xi_5 (= \Gamma_{55}\Gamma_5)$ is an extension of $\hat{\Xi}_5$ from V to the space of staggered

¹The hats on $\hat{\Gamma}_\mu$ and $\hat{\Xi}_\nu$ are not present in the notation of [20]. We include them here, since we use the unhatted versions to denote the extensions of these operators from V to the space of one-component lattice spinor fields $\chi(x)$.

lattice fermion fields $\chi(x)$, described in [5]. For present purposes it suffices to note that the action of the staggered Wilson Dirac operator on a plane wave e^{ipx} can be expressed as follows. Let $T_{+\mu}$ denote the parallel transporter given by $T_{+\mu}\chi(x) = U(x, x+a\mu)\chi(x+a\mu)$, and let $T_{-\mu}$ denote its inverse. The symmetrized covariant derivative is $\nabla_\mu = \frac{1}{2}(T_{+\mu} - T_{-\mu})$, and we define $C_\mu = \frac{1}{2}(T_{+\mu} + T_{-\mu})$. We will also need the symmetrized product of the C_μ s, namely $C_5 \equiv \frac{1}{4!} \sum C_\mu C_\nu C_\sigma C_\rho$ where the sum is over all permutations $\{\mu, \nu, \sigma, \rho\}$ of $\{1, 2, 3, 4\}$. Then, writing the plane wave as $e^{iqx} = e_B(x)e^{ipx}$, the action of D_{sW} on it can be expressed as

$$\begin{aligned} D_{sW}(e_B(x)e^{ipx}) &= e_A(x) \left((\hat{\Gamma}_\mu)_{AB} \nabla_\mu + \frac{r}{a} \left(\delta_{AB} - (\hat{\Xi}_5)_{AB} C_5 \right) \right) e^{ipx} \\ &= e_A(x) \left(\hat{\Gamma}_\mu \nabla_\mu + \frac{r}{a} \left(\mathbf{1} - \hat{\Xi}_5 C_5 \right) \right)_{AB} e^{ipx} \\ &\equiv e_A(x) (\tilde{D}_{sW})_{AB} e^{ipx} \end{aligned} \quad (4.2.4)$$

Here and in the following there is an implicit sum over repeated indices. This includes the vector A which is regarded as an index with 16 possible values.

Since H is built from D_{sW} and Γ_{55} , and the latter acts on plane waves by $\Gamma_{55}(e_B(x)e^{ipx}) = e_A(x)(\hat{\Gamma}_5 \hat{\Xi}_5)_{AB} e^{ipx}$, it follows that the action of $H/\sqrt{H^2}$ on a plane wave has the same structure as in 4.2.4:

$$\frac{H}{\sqrt{H^2}}(e_B(x)e^{ipx}) = e_A(x) \left(\frac{\tilde{H}}{\sqrt{\tilde{H}^2}} \right)_{AB} e^{ipx} \quad (4.2.5)$$

where $\tilde{H}$ is obtained from H by replacing $D_{sW} \rightarrow \tilde{D}_{sW}$ and $\Gamma_{55} \rightarrow \hat{\Gamma}_5 \hat{\Xi}_5$.

Using the preceding to evaluate (4.2.3), we get

$$\begin{aligned} \lim_{a \rightarrow 0} \frac{H}{\sqrt{H^2}}(x, x) &= \lim_{a \rightarrow 0} \sum_B \int_{-\pi/2a}^{\pi/2a} d^4p e^{-ipx} e_B(x) \frac{H}{\sqrt{H^2}} e_B(x) e^{ipx} \\ &= \lim_{a \rightarrow 0} \sum_{A,B} e^{i\frac{\pi}{a}(A-B)x} \int_{-\pi/2a}^{\pi/2a} d^4p e^{-ipx} \left(\frac{\tilde{H}}{\sqrt{\tilde{H}^2}} \right)_{AB} e^{ipx} \end{aligned} \quad (4.2.6)$$

We will show below that (i) the contributions from the terms with $A \neq B$ vanish after averaging $q(x')$ over the sites x' in a lattice hypercube containing x , and (ii) the contribution to $q(x)$ from the terms in (4.2.6) with $A = B$ reproduces the continuum topological charge density (4.2.2). This will complete the derivation of the averaged version of (4.2.2).

We now show that the contributions to (4.2.6) from the terms in the sum over A, B with $A \neq B$ vanish when averaged over $x \in \{\text{a lattice hypercube}\}$. Writing $x = an$, $n \in \mathbb{Z}^4$, we have $e^{i\frac{\pi}{a}(A-B)x} = (-1)^{(A-B)n}$. It is easy to see that summing this over the sites x of a lattice hypercube gives zero if at least one of the components of $A - B$ is nonzero (mod 2). Consequently, the problem of showing that the hypercube-averaged terms with $A \neq B$ in (4.2.6) vanish is reduced to showing a property of the x -dependence of the integral there,

$$\int_{-\pi/2a}^{\pi/2a} d^4p e^{-ipx} \left(\frac{\tilde{H}}{\sqrt{\tilde{H}^2}} \right)_{AB} e^{ipx}, \quad (4.2.7)$$

namely, that this integral changes by $O(a)$ as x is varied among the sites of a lattice hypercube. The argument for this is as follows. Apparently, (4.2.7) diverges $\sim 1/a^4$ for $a \rightarrow 0$. However, the integrand can be expanded in powers of the continuum gauge field just like in the usual overlap case [2, 1]² and it can

²This relies on a certain bound of the spectrum of the hermitian operator H when the

be shown that only the terms of mass-dimension 4 or higher in this expansion are nonvanishing.³ For each mass-dimension of the expansion terms there is an accompanying power of a , so the nonvanishing terms contain at least a factor a^4 which balances the divergence $\sim 1/a^4$ in (4.2.7). The fact that (4.2.7) changes by $O(a)$ as x is varied among neighboring lattice sites then follows from the fact that the smooth continuum gauge field has this property.

It remains to show that the contribution in the sum with $A = B$ reproduces the correct continuum result. The sum over A, B with $A = B$ in (4.2.6) gives

$$\lim_{a \rightarrow 0} \int_{-\pi/2a}^{\pi/2a} d^4 p e^{-ipx} \text{Tr} \left(\frac{\tilde{H}}{\sqrt{\tilde{H}^2}} \right) e^{ipx} \quad (4.2.8)$$

where Tr denotes the trace of a linear operator (matrix) on the vectorspace V spanned by the $e_A(x)$'s. As shown in [20], there is a (non-unique) isomorphism $V \simeq \mathbb{C}^4 \otimes \mathbb{C}^4$ such that $\hat{\Gamma}_\mu \simeq \gamma_\mu \otimes \mathbf{1}$ and $\hat{\Xi}_\nu \simeq \mathbf{1} \otimes \gamma_\nu$, where the tensor product is interpreted as a spinor $\otimes$ flavor structure. After choosing a basis for flavor $\mathbb{C}^4$ such that γ_5 is diagonal, (4.2.4) gives

$$\tilde{D}_{sW} = (\gamma_\mu \otimes \mathbf{1}) \nabla_\mu + (\mathbf{1} \otimes \mathbf{1}) \frac{r}{a} (1 \mp C_5) \quad (4.2.9)$$

plaquette variable of the lattice gauge field satisfies an approximate smoothness condition. It was established in the usual overlap case in [22]. An analogous bound holds in the present staggered overlap case; the proof will be given elsewhere.

³This relies on two things: First, exponential locality of the integrand of (4.2.7) in the gauge field (shown in the same way as in the usual overlap case [22], which implies that the terms in the expansion become local functionals of the gauge field in the $a \rightarrow 0$ limit. Second, the gauge invariance and lattice rotation invariance of staggered overlap fermions imply that the local functionals of the continuum gauge field that arise in the expansion of (4.2.7) must have mass-dimension ≥ 4 . (There are no such terms with mass-dimension ≤ 3 , while for mass-dimension 4 there are the Yang-Mills action functional and topological charge density.)

with sign $\mp$ for the flavors on which $\gamma_5 = \pm \mathbf{1}$. Thus, on each of the two 2-dimensional flavor subspaces on which $\gamma_5 = \pm \mathbf{1}$, $\tilde{D}_{sW}$ is a hypercubic lattice Dirac operator of the form considered in [6] (note that C_5 couples opposite corners of lattice hypercubes).

In order to calculate the continuum limit we will now evaluate (4.2.8) After reconstruction and the introduction of spinor and flavor structures,

$$H = (\gamma_5 \otimes \gamma_5) [(\gamma_\mu \otimes \mathbf{1}) \nabla_\mu + r(1 - m_0 - (\mathbf{1} \otimes \gamma_5) C)]. \quad (4.2.10)$$

We now decompose H^2 as follows

$$\begin{aligned} H^2 &= \{(\gamma_5 \otimes \gamma_5) [(\gamma_\mu \otimes \mathbf{1}) \nabla_\mu + r(1 - m_0 - (\mathbf{1} \otimes \gamma_5) C)]\}^2 \\ &= L - V \end{aligned} \quad (4.2.11)$$

with

$$L = -(\mathbf{1} \otimes \mathbf{1}) \nabla^2 + r^2 (1 - m_0 - (\mathbf{1} \otimes \gamma_5) C)^2 \quad (4.2.12)$$

$$= \mathbf{1} \otimes (-\mathbf{1} \nabla^2 + \mathbf{1} r^2 (1 - m_0 - \gamma_5 C)^2),$$

$$V = r(\gamma_\mu \otimes \gamma_5) V_\mu + \frac{1}{4} ([\gamma_\mu, \gamma_\nu] \otimes \mathbf{1}) V_{\mu\nu} \quad (4.2.13)$$

and

$$V_\mu = [C, \nabla_\mu], \quad (4.2.14)$$

$$V_{\mu\nu} = [\nabla_\mu, \nabla_\nu]. \quad (4.2.15)$$

In order to deal with the term $1/\sqrt{H^2}$ in 4.2.8, we expand $H/\sqrt{H^2}$ as follows

$$\begin{aligned}
 H/\sqrt{H^2} &= H \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \frac{1}{H^2 + \sigma^2} \\
 &= H \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \left(\frac{1}{1 - (L + \sigma^2)^{-1} V} \right) \left(\frac{1}{L + \sigma^2} \right) \\
 &= H \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \sum_{k=0}^{\infty} (G_{\sigma} V)^k G_{\sigma} \\
 &= H \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} (G_{\sigma} + G_{\sigma} V G_{\sigma} + G_{\sigma} V G_{\sigma} V G_{\sigma} + \mathcal{O}(V^3)), \quad (4.2.16)
 \end{aligned}$$

where we defined $G_{\sigma} = (L + \sigma^2)^{-1}$. Hence the index density can be rewritten as

$$q(x) = -\frac{i}{a^4} \text{tr} \int_{-\pi/2}^{\pi/2} \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} e^{ipx} [H G_{\sigma} + H G_{\sigma} V G_{\sigma} + H G_{\sigma} V G_{\sigma} V G_{\sigma} + \mathcal{O}(V^3)] e^{ipx}, \quad (4.2.17)$$

where the trace is taken over spinor, flavor and color indices. We will now show that in the above expression only the third term gives a contribution in the continuum limit.

First we show that terms beyond the third term vanish in the continuum limit.

In order to do that we will need the following equalities [2],

$$[\nabla_{\mu}^{\pm}, \nabla_{\nu}^{\pm}] e^{ipx} = e^{\pm i(p_{\mu} + p_{\nu})} [a^2 F_{\mu\nu}(x) + \mathcal{O}(a^3)] e^{ipx}, \quad (4.2.18)$$

$$[\nabla_{\mu}^{\pm}, \nabla_{\nu}^{\mp}] e^{ipx} = e^{\pm i(p_{\mu} - p_{\nu})} [a^2 F_{\mu\nu}(x) + \mathcal{O}(a^3)] e^{ipx}. \quad (4.2.19)$$

we obtain

$$\begin{aligned} [\nabla_\mu, \nabla_\nu] e^{ipx} &= \frac{1}{4} [\nabla_\mu^+ + \nabla_\mu^-, \nabla_\nu^+ + \nabla_\nu^-] e^{ipx} \\ &= \cos(p_\mu) \cos(p_\nu) [a^2 F_{\mu\nu}(x) + \mathcal{O}(a^3)] e^{ipx} \end{aligned} \quad (4.2.20)$$

We now calculate the following commutator for $\mu \neq \nu$

$$\begin{aligned} [\nabla_\mu, C_\nu] e^{ipx} &= \frac{1}{4} [\nabla_\mu^+ + \nabla_\mu^-, \nabla_\nu^+ - \nabla_\nu^- + 2] e^{ipx} \\ &= i \cos(p_\mu) \sin(p_\nu) [a^2 F_{\mu\nu}(x) + \mathcal{O}(a^3)] e^{ipx}, \end{aligned} \quad (4.2.21)$$

where we used that $C_\nu = \frac{1}{2} (\nabla_\nu^+ - \nabla_\nu^- + 2)$. This allows us to conclude that

$$\begin{aligned} [\nabla_\mu, C] e^{ipx} &= \frac{1}{4!} \left[\nabla_\mu, \sum_{\omega\nu\sigma\rho}^{\text{sym}} C_\omega C_\nu C_\sigma C_\rho \right] e^{ipx} \\ &= a^4 i \cos(p_\mu) \left(\prod_\alpha \cos p_\alpha \right) \left(\sum_\omega \frac{\sin p_\omega}{\cos p_\omega} [a^2 F_{\mu\omega}(x) + \mathcal{O}(a^3)] e^{ipx} \right), \end{aligned} \quad (4.2.22)$$

where we used that

$$C_\mu e^{ipx} = (\cos p_\mu + \mathcal{O}(a)) e^{ipx}.$$

Since

$$V = r (\gamma_\mu \otimes \gamma_5) V_\mu + \frac{1}{4} ([\gamma_\mu, \gamma_\nu] \otimes \mathbf{1}) V_{\mu\nu} \quad (4.2.23)$$

with $V_\mu = [C, \nabla_\mu]$ and $V_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$, we finally see that

$$\begin{aligned} V_\mu e^{ipx} &= -i \cos(p_\mu) \left(\prod_\alpha \cos p_\alpha \right) \left(\sum_\omega \tan p_\omega [a^2 F_{\mu\omega}(x) + \mathcal{O}(a^3)] \right) e^{ipx}, \\ V_{\mu\nu} e^{ipx} &= \cos(p_\mu) \cos(p_\nu) [a^2 F_{\mu\nu}(x) + \mathcal{O}(a^3)] e^{ipx}. \end{aligned} \quad (4.2.24)$$

Now that we have shown that terms beyond the third in (4.2.17) vanish in the continuum limit equation (4.2.17) reduces to the following expression

$$q(x) = -\frac{i}{a^4} \text{tr} \int_{-\pi/2}^{\pi/2} \frac{d^4 p}{(2\pi)^4} e^{-ipx} \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} [H G_\sigma + H G_\sigma V G_\sigma + H G_\sigma V G_\sigma V G_\sigma + \mathcal{O}(a^5)] e^{ipx}. \quad (4.2.25)$$

The first two terms vanish because in the spinor space, every term involved contains a product of gamma matrices, with one γ_5 and fewer than four γ_μ 's. Hence the trace over spinor space is 0.

Thus we are now only left with the third term in (4.2.25). Following the arguments presented in [2], we can show that $\|H - H^{(0)}\| \propto \mathcal{O}(a)$ and $\|L - L^{(0)}\| \propto \mathcal{O}(a)$. Then we can conclude that $\|G_\sigma - G_\sigma^{(0)}\| \propto \mathcal{O}(a)$. Furthermore we can show that $[G_\sigma^{(0)}, V] \propto \mathcal{O}(a^3)$. Hence we can replace H , G_σ and L by their free field versions up to an order $\mathcal{O}(a)$ term:

$$\begin{aligned} q(x) &= -\frac{i}{a^4} \text{tr} \int_{-\pi/2}^{\pi/2} \frac{d^4 p}{(2\pi)^4} e^{-ipx} \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} [H^{(0)} V^2 (G_\sigma^{(0)})^3 e^{ipx}] + \mathcal{O}(a) \\ &= -\frac{3i}{8a^4} \text{tr} \int_{-\pi/2}^{\pi/2} \frac{d^4 p}{(2\pi)^4} e^{-ipx} [H^{(0)} V^2 (L^{(0)})^{-5/2} e^{ipx}] + \mathcal{O}(a). \end{aligned} \quad (4.2.26)$$

Now the trace decomposes into a product of trace over spinor, flavor and color

spaces. For notational convenience we neglect the color space for the moment i.e. $\text{tr} = \text{tr}_s \text{tr}_f$. We first evaluate the spinor trace tr_s to find

$$\begin{aligned} & \text{tr} \left[H^{(0)} V^2 \left(L^{(0)} \right)^{-5/2} \right] \\ &= -r \varepsilon_{\mu\nu\rho\sigma} \text{tr}_f \left[2 \nabla_\mu^{(0)} (V_\nu V_{\rho\sigma} + V_{\nu\rho} V_\sigma) + \gamma_5 (1 - m_0 - \gamma_5 C^{(0)}) V_{\mu\nu} V_{\rho\sigma} \right] \left(L_f^{(0)} \right)^{-5/2} \end{aligned} \quad (4.2.27)$$

where V_μ and $V_{\mu\nu}$ are as in equation (4.2.24) and

$$L_f^{(0)} e^{ipx} = \left[1 \sum_\mu \sin^2(p_\mu) + 1r^2 \left(1 - m_0 - \gamma_5 \prod_\mu \cos(p_\mu) \right)^2 \right] e^{ipx}. \quad (4.2.28)$$

We now evaluate

$$\begin{aligned} & \left[\nabla_\mu^{(0)} V_\nu V_{\rho\sigma} \right] e^{ipx} = \sin(q_\mu) \cos(p_\nu) \cos(p_\rho) \cos(p_\sigma) \prod_\alpha \cos p_\alpha \\ & \times \sum_\omega \tan p_\omega \left[a^2 F_{\nu\omega}(x) + \mathcal{O}(a^3) \right] \left[a^2 F_{\rho\sigma}(x) + \mathcal{O}(a^3) \right] e^{ipx}, \end{aligned} \quad (4.2.29)$$

$$\begin{aligned} & \left[\nabla_\mu^{(0)} V_{\nu\rho} V_\sigma \right] e^{ipx} = \sin(p_\mu) \cos(p_\nu) \cos(p_\rho) \cos(p_\sigma) \prod_\alpha \cos p_\alpha \\ & \times \sum_\omega \tan p_\omega \left[a^2 F_{\nu\rho}(x) + \mathcal{O}(a^3) \right] \left[a^2 F_{\sigma\omega}(x) + \mathcal{O}(a^3) \right] e^{ipx}, \end{aligned} \quad (4.2.30)$$

$$\begin{aligned} \left[(1 - m_0 - \gamma_5 C^{(0)}) V_{\mu\nu} V_{\rho\sigma} \right] e^{ipx} &= \left(1 - m_0 - \gamma_5 \prod_{\omega} \cos(p_{\omega}) \right) \cos(p_{\mu}) \cos(p_{\nu}) \cos(p_{\rho}) \cos(p_{\sigma}) \\ &\times \left[a^2 F_{\mu\nu}(x) + \mathcal{O}(a^3) \right] \left[a^2 F_{\rho\sigma}(x) + \mathcal{O}(a^3) \right] e^{ipx}. \end{aligned} \quad (4.2.31)$$

Plugging in and simplifying yields

$$\begin{aligned} \text{tr} \left[H^{(0)} V^2 (L^{(0)})^{-5/2} \right] &= a^4 \sum_x \{ -r \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x) \\ &\times \text{tr}_f \frac{\prod_{\mu} \cos(p_{\mu}) \gamma_5 \left(1 - m_o - \gamma_5 \prod_{\mu} \cos p_{\mu} - \prod_{\mu} \cos p_{\mu} (\sum_{\alpha} \tan^2 p_{\alpha}) p_{\alpha} \right)}{\left(\sum_{\mu} \sin^2 p_{\mu} + r^2 \left(1 - m_0 - \gamma_5 \prod_{\mu} \cos p_{\mu} \right)^2 \right)^{5/2}} + \mathcal{O}(a^2) \}. \end{aligned} \quad (4.2.32)$$

Thus the index density

$$\begin{aligned} q(x) &= \frac{3r}{16a^4} \cdot a^4 \sum_x \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x) \\ &\times \int_{-\pi/2}^{\pi/2} \frac{d^4 p}{(2\pi)^4} \prod_{\mu} \cos(p_{\mu}) \text{tr}_f \frac{\gamma_5 (1 - m_o) - \prod_{\mu} \cos p_{\mu} - \prod_{\mu} \cos p_{\mu} (\sum_{\alpha} \tan^2 p_{\alpha})}{\left(\sum_{\mu} \sin^2 p_{\mu} + r^2 \left(1 - m_0 - \gamma_5 \prod_{\mu} \cos p_{\mu} \right)^2 \right)^{5/2}} + \mathcal{O}(a). \end{aligned} \quad (4.2.33)$$

To evaluate the flavor trace we chose the Weyl representation, where γ_5 is diagonal

and takes the form

$$\gamma_5 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \quad (4.2.34)$$

The general structure of the trace is:

$$\begin{aligned} \text{tr}_f \left[(\alpha 1 + b \gamma_5) (c 1 + d \gamma_5)^{-5/2} \right] &= \text{tr}_f \left[\begin{pmatrix} \frac{\alpha+b}{(c+d)^{5/2}} & & & \\ & \frac{\alpha+b}{(c+d)^{5/2}} & & \\ & & \frac{\alpha-b}{(c-d)^{5/2}} & \\ & & & \frac{\alpha-b}{(c-d)^{5/2}} \end{pmatrix} \right] \\ &= \frac{2(\alpha+b)}{(c+d)^{5/2}} + \frac{2(\alpha-b)}{(c-d)^{5/2}}. \end{aligned} \quad (4.2.35)$$

In our case we have the following functions:

$$\alpha = - \prod_{\mu} \cos p_{\mu} \left(1 + \sum_{\sigma} \tan^2 p_{\sigma} \right), \quad (4.2.36)$$

$$b = 1 - m_0, \quad (4.2.37)$$

$$c = \sum_{\mu} \sin^2 p_{\mu} + r^2 \left[(1 - m_0)^2 + \prod_{\mu} \cos^2 p_{\mu} \right], \quad (4.2.38)$$

$$d = -2r^2 (1 - m_0) \prod_{\mu} \cos p_{\mu}. \quad (4.2.39)$$

Combining everything we find

$$q(x) = I(r, m_0) \cdot (q_{\text{cont}}(x) + \mathcal{O}(a)) \quad (4.2.40)$$

with the continuum single-flavor topological charge density

$$q_{\text{cont}}(x) = \frac{-1}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{tr}_c [F_{\mu\nu}(x) F_{\rho\sigma}(x)] \quad (4.2.41)$$

and the prefactor in terms of the following integral:

$$I(r, m_0) = -\frac{3r}{4\pi^2} \int_{-\pi/2}^{\pi/2} d^4p \prod_{\mu} \cos(p_{\mu}) \left\{ \frac{1 - m_0 - \prod_{\mu} \cos p_{\mu} (1 + \sum_{\sigma} \tan^2 p_{\sigma})}{\left[\sum_{\mu} \sin^2 p_{\mu} + r^2 \left(1 - m_0 - \prod_{\mu} \cos p_{\mu} \right)^2 \right]^{5/2}} - \frac{1 - m_0 + \prod_{\mu} \cos p_{\mu} (1 + \sum_{\sigma} \tan^2 p_{\sigma})}{\left[\sum_{\mu} \sin^2 p_{\mu} + r^2 \left(1 - m_0 + \prod_{\mu} \cos p_{\mu} \right)^2 \right]^{5/2}} \right\}. \quad (4.2.42)$$

4.3 Evaluation of the Integral Expression

4.3.1 Analytical evaluation

We do a change of variable in the first integrand in (4.2.42) by

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} \rightarrow \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 + \pi \end{pmatrix}. \quad (4.3.1)$$

Then the integral becomes

$$\begin{aligned} I(r, m_0) &= \frac{3r}{4\pi^2} \left\{ \int_{-\pi/2}^{\pi/2} d^3\mathbf{p} \int_{\pi/2}^{3\pi/2} dp_4 \prod_{\mu} \cos(p_{\mu}) \frac{1 - m_0 + \prod_{\mu} \cos p_{\mu} (1 + \sum_{\alpha} \tan^2 p_{\alpha})}{\left[\sum_{\mu} \sin^2 p_{\mu} + r^2 \left(1 - m_0 + \prod_{\mu} \cos p_{\mu} \right)^2 \right]^{5/2}} \right. \\ &\quad \left. + \int_{-\pi/2}^{\pi/2} d^4p \prod_{\mu} \cos(p_{\mu}) \frac{1 - m_0 + \prod_{\mu} \cos p_{\mu} (1 + \sum_{\alpha} \tan^2 p_{\alpha})}{\left[\sum_{\mu} \sin^2 p_{\mu} + r^2 \left(1 - m_0 + \prod_{\mu} \cos p_{\mu} \right)^2 \right]^{5/2}} \right\} \\ &= \frac{3}{4\pi^2} \int_{-\pi/2}^{\pi/2} d^3\mathbf{p} \int_{-\pi/2}^{3\pi/2} dp_4 \prod_{\mu} \cos(p_{\mu}) \frac{r \left[1 - m_0 + \prod_{\mu} \cos p_{\mu} (1 + \sum_{\alpha} \tan^2 p_{\alpha}) \right]}{\left[\sum_{\mu} \sin^2 p_{\mu} + r^2 \left(1 - m_0 + \prod_{\mu} \cos p_{\mu} \right)^2 \right]^{5/2}}. \end{aligned} \quad (4.3.2)$$

Making use of the methods described in [16, 6], if we define a map from a hyperrectangle to a hypersphere

$$\Theta : \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]^3 \times \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right] \longrightarrow S^4, \quad \Theta(p) := \left(\frac{\theta_0}{|\theta|}, \frac{\theta_1}{|\theta|}, \dots, \frac{\theta_4}{|\theta|} \right), \quad (4.3.3)$$

where

$$\theta_0(p) = r \left(1 - m_0 + \prod_{\mu} \cos p_{\mu} \right), \quad (4.3.4)$$

$$\theta_{\mu}(p) = \sin p_{\mu}, \quad \mu = 1, \dots, 4, \quad (4.3.5)$$

$$|\theta| = \sqrt{\theta_0^2 + \sum_{\mu} \theta_{\mu}^2}, \quad (4.3.6)$$

then (4.3.2) can be written as

$$I(r, m_0) = \frac{3}{4\pi^2} \int \frac{1}{|\theta|^5} \sum_{j=0}^4 (-1)^j \theta_j d\theta_0 \wedge \dots \wedge d\theta_{j-1} \wedge d\theta_{j+1} \wedge \dots \wedge d\theta_4. \quad (4.3.7)$$

A little calculation shows that the integrand here can be rewritten as

$$I(r, m_0) = \sum_{j=0}^4 (-1)^j \frac{\theta_j}{|\theta|} d\left(\frac{\theta_0}{|\theta|}\right) \wedge \dots \wedge d\left(\frac{\theta_{j-1}}{|\theta|}\right) \wedge d\left(\frac{\theta_{j+1}}{|\theta|}\right) \wedge \dots \wedge d\left(\frac{\theta_4}{|\theta|}\right). \quad (4.3.8)$$

This is precisely the volume form Ω on the sphere S^4 embedded in $\mathbb{R}^5$, and the integral part in (4.3.2) is equal to $\deg(\Theta) \int_U \Omega$, where U is the image of Θ . Thus

$$I(r, m_0) = \frac{3}{4\pi^2} \deg(\Theta) \int_U \Omega. \quad (4.3.9)$$

If $\theta/|\theta| \in S^4 \subset \mathbb{R}^5$ is any regular point for the map Θ , then it is a well-known topological fact that $\deg(\Theta) = \sum_l s_l$, where $s_l = \pm 1$ is the sign (relative to $dp_1 \wedge dp_2 \wedge dp_3 \wedge dp_4$) of the integrand of (4.3.7) evaluated at a preimage $p^{(l)}$ of $\theta/|\theta|$ and the sum is over *all preimages* labeled by l . In our case when $m_0 \in (0, 2)$, we choose $\theta/|\theta| = (1, 0, 0, 0, 0)$. It has a unique preimage $p_1 = p_2 = p_3 = p_4 = 0$ with $s_1 = 1$. Furthermore we can show that the map is a surjection in this

parameter region, i.e. $U = S^4$. Thus

$$I(r, m_0) = \frac{3}{4\pi^2} \times 1 \times \frac{8\pi^2}{3} = 2 \quad (4.3.10)$$

as $\int_U \Omega = \text{Vol}(S^4) = 8\pi^2/3$. If $m_0 < 0$ or $m_0 > 2$, the map is no longer a surjection. This causes no problem: if we choose $\theta/|\theta| = (1, 0, 0, 0, 0)$ for the former and $\theta/|\theta| = (-1, 0, 0, 0, 0)$ for the latter, then both cases will give two preimages with $p_1 = p_2 = p_3 = 0$ and $p_4 = 0$ or $p_4 = \pi$ with signs $s_1 = 1$ and $s_2 = -1$ so that the degree is always 0. Thus

$$I(r, m_0) = \frac{3}{4\pi^2} \times 0 \times \int_U \Omega = 0. \quad (4.3.11)$$

In summary

$$I(r, m_0) = \begin{cases} 2 & m_0 \in (0, 2), \\ 0 & m_0 \notin (0, 2), \end{cases} \quad (4.3.12)$$

which shows that the operator does reproduce the correct index density for two flavors in the continuum limit if $m_0 \in (0, 2)$.

4.4 Final Remarks

The result of the previous section showed that in the continuum limit the index of the staggered Wilson Dirac operator correctly reproduces the index of the continuum Dirac operator. This is the first step towards proving a lattice version of the Index Theorem which has been conjectured as:

The index of the staggered Wilson Dirac operator equals the topological charge of

the lattice gauge field when the lattice gauge field satisfies a sufficiently stringent plaquette condition.

We have discussed the plaquette condition for the lattice gauge field in (6.2.3) in the Appendix. "Sufficiently stringent" means that the ϵ in that condition is sufficiently small. The plaquette condition can also be interpreted as an approximate smoothness condition for the lattice gauge field (see, e.g., [1]). We will now discuss why the condition is necessary for ensuring that the index and topological charge are well defined in the lattice setting.

For general lattice gauge fields the topological charge is ill defined [31]. Since lattice gauge fields can be arbitrarily rough, and the definition of topological charge for continuum gauge fields relies on the smoothness (or at least continuity) of those fields, this result does not come as a surprise. However lattice fields so have a well-defined topological charge provided they satisfy a sufficiently stringent plaquette condition [31].

The plaquette condition is also required for a well defined definition of the index of the lattice Dirac operator, since it ensures a clear separation between the "genuine" crossings and "doubler" crossings in the spectral flow definition of the index, cf. the discussion in the Appendix to Section 3.1.5. This has been shown the usual Wilson Dirac operator in [22], and has also been shown for the staggered Wilson Dirac operator in unpublished results by D. Adams (which will appear as an article in due course).

Another noteworthy consequence of the plaquette condition is, the index density $q(x)$ of the lattice Dirac operator, given by (4.2.1), is a local function of the lattice gauge field. A change in one of the link variables of the lattice gauge field only affects $q(x)$ at lattice sites x which are close to the lattice link, while $q(y)$ remains

unchanged (up to exponentially suppressed effects) at lattice sites y further away from the link. This property was shown for the usual Dirac operator in [22] and carries over to the staggered Wilson Dirac operator (as shown in the unpublished works of D. Adams).

Hence the index density is also a local function of the lattice gauge field in this case.

Conclusions

The main topic of this thesis was discussing the subtleties of the index of Dirac operators on the lattice. To this end in chapter 2 we provided a brief introduction to index theorems, in particular we discuss the important case when dealing with a Dirac operator. We further provided a toy model in $\mathbb{R}^2$ which served as a simple illustration of the Atiyah-Singer Index Theorem. In chapter 3 we moved from the continuum to a 2-dimensional lattice and explained how the previously discussed concepts translate to the discrete setting. The main purpose of chapter 3 was to give a simple illustration – using a toy model – of the fact that the traditional definition of index breaks down in the lattice setting and how to obtain a formula for extracting the index in an indirect way using the concept of spectral flow. In chapter 4 we discuss the original research contribution of this thesis, which is calculating the continuum limit of the index of a recently proposed discretization of the Dirac operator, known as the staggered Wilson Dirac operator. This is one step towards a secure theoretical foundation of this new formalism. Our calculations reproduced the correct continuum topological charge, which is of

utmost importance for the consistency of numerical calculations of gauge theories.

Chapter 6

Appendix

6.1 Appendix to Chapter 2

Appendix to section 2.6.

Remark 6.1. In order for the functions to obey

$$\begin{cases} f(x_1 + 1, x_2) = f(x_1, x_2) & \text{periodic in 1st variable} \\ f(x_1, x_2 + 1) = e^{i\theta(x_1)} f(x_1, x_2) & \text{twisted periodic in 2nd variable} \end{cases}$$

we must have

$$e^{i\theta(x_1+1)} = e^{i\theta(x_1)}$$

Proof. From (2.6.3) we get the following equations.

$$f(x_1 + 1, x_2 + 1) = f(x_1, x_2 + 1) \quad (6.1.1)$$

$$f(x_1 + 1, x_2) = f(x_1, x_2) \quad (6.1.2)$$

$$f(x_1 + 1, x_2 + 1) = e^{i\theta(x_1+1)} f(x_1 + 1, x_2) \quad (6.1.3)$$

$$f(x_1, x_2 + 1) = e^{i\theta(x_1)} f(x_1, x_2) \quad (6.1.4)$$

Substituting (6.1.1) into (6.1.3) gives

$$f(x_1, x_2 + 1) = e^{i\theta(x_1+1)} f(x_1 + 1, x_2),$$

substituting (6.1.2) into (6.1.4) gives

$$f(x_1, x_2 + 1) = e^{i\theta(x_1)} f(x_1 + 1, x_2).$$

Hence we have,

$$e^{i\theta(x_1+1)} = e^{i\theta(x_1)}.$$

Thus

$$\theta(x_1 + 1) = \theta(x_1) + 2\pi Q, \quad Q \in \mathbb{Z}$$

□

Remark 6.2. The continuum gauge fields satisfy the following boundary conditions ,

$$\begin{cases} A_1(x_1 + 1, x_2) = A_1(x_1, x_2) \\ A_1(x_1, x_2 + 1) = A_1(x_1, x_2) - \theta'(x_1) \\ A_2(x_1 + 1, x_2) = A_2(x_1, x_2) \\ A_2(x_1, x_2 + 1) = A_2(x_1, x_2). \end{cases}$$

Proof. If the partial derivatives of the functions defined in (2.6.3) satisfy the periodicity conditions we have ,

$$\nabla_1 f(x_1 + 1, x_2) = \nabla_1 f(x_1, x_2) \quad (6.1.5)$$

$$\nabla_1 f(x_1, x_2 + 1) = e^{i\theta(x_1)} \nabla_2 f(x_1, x_2) \quad (6.1.6)$$

$$\nabla_2 f(x_1 + 1, x_2) = \nabla_2 f(x_1, x_2) \quad (6.1.7)$$

$$\nabla_2 f(x_1, x_2 + 1) = e^{i\theta(x_1)} \nabla_2 f(x_1, x_2) \quad (6.1.8)$$

We will now prove that the gauge fields $A_\mu, \mu = 1, 2$ satisfy the conditions (2.6.6).

A simple calculation using condition (2.6.3) respectively in equations (6.1.5) and (6.1.7) gives us the following conditions on the gauge fields

$$A_1(x_1, x_2) = A_1(x_1 + 1, x_2).$$

$$A_2(x_1, x_2) = A_2(x_1 + 1, x_2).$$

Let us now use condition (2.6.3) in equation (6.1.6).

The left hand side of the equation gives

$$\begin{aligned}
\nabla_1 f(x_1, x_2 + 1) &= \frac{\partial}{\partial x_1} f(x_1, x_2 + 1) + iA_1(x_1, x_2 + 1)f(x_1, x_2 + 1) \\
&= \frac{\partial}{\partial x_1} [e^{i\theta(x_1)} f(x_1, x_2)] + iA_1(x_1, x_2 + 1)e^{i\theta(x_1)} f(x_1, x_2) \\
&= i \left(\frac{\partial}{\partial x_1} \theta(x_1) \right) e^{i\theta(x_1)} f(x_1, x_2) + e^{i\theta(x_1)} \left(\frac{\partial}{\partial x_1} f(x_1, x_2) \right) \\
&\quad + iA_1(x_1, x_2 + 1)e^{i\theta(x_1)} f(x_1, x_2) \\
&= e^{i\theta(x_1)} \left\{ \left(\frac{\partial}{\partial x_1} f(x_1, x_2) \right) + i \left[\left(\frac{\partial}{\partial x_1} \theta(x_1) \right) + A_1(x_1, x_2 + 1) \right] f(x_1, x_2) \right\}.
\end{aligned}$$

The right hand side of the equation gives

$$e^{i\theta(x_1)} \nabla_1 f(x_1, x_2) = e^{i\theta(x_1)} \left\{ \left(\frac{\partial}{\partial x_1} f(x_1, x_2) \right) + iA_1(x_1, x_2)f(x_1, x_2) \right\}.$$

For these two to be equal,

$$A_1(x_1, x_2) = A_1(x_1, x_2 + 1) - \theta'(x_1)$$

Finally, applying condition (2.6.3) to the *LHS* of equation (6.1.8) gives

$$\begin{aligned}
\nabla_2 f(x_1, x_2 + 1) &= \frac{\partial}{\partial x_2} f(x_1, x_2 + 1) + iA_2(x_1, x_2 + 1)f(x_1, x_2 + 1) \\
&= \frac{\partial}{\partial x_2} e^{i\theta(x_1)} f(x_1, x_2) + iA_2(x_1, x_2 + 1)e^{i\theta(x_1)} f(x_1, x_2) \\
&= e^{i\theta(x_1)} \left[\frac{\partial}{\partial x_2} f(x_1, x_2) + iA_2(x_1, x_2 + 1)f(x_1, x_2) \right],
\end{aligned}$$

while applying condition (2.6.3) to the *RHS* of equation (6.1.8) gives

$$e^{i\theta(x_1)} \nabla_2 f(x_1, x_2) = e^{i\theta(x_1)} \left[\frac{\partial}{\partial x_2} f(x_1, x_2) + iA_2(x_1, x_2) f(x_1, x_2) \right].$$

Thus we have

$$A_2(x_1, x_2) = A_2(x_1, x_2 + 1).$$

Hence we see that the gauge fields satisfy condition (2.6.6). $\square$

Remark 6.3. γ_3 satisfies the following properties.

$$\gamma_3^* = \gamma_3$$

$$\gamma_3^2 = 1$$

$$\gamma_3 \gamma_\mu = -\gamma_\mu \gamma_3, \quad \mu = 1, 2$$

$$\text{tr}(\gamma_3) = 0$$

Proof. We will constantly use the the properties of gamma matrices introduced in (2.6.2) $\square$

I γ_3 is hermitian.

Proof. $\gamma_3^* = (-i\gamma_1\gamma_2)^* = i\gamma_2^*\gamma_1^* = i\gamma_2\gamma_1 = -i\gamma_1\gamma_2 = \gamma_3.$ $\square$

II γ_3 anti-commutes with γ_1 and γ_2

Proof. We will prove for γ_1 . The proof for γ_2 is similar.

$$\gamma_3\gamma_1 = -i\gamma_1\gamma_2\gamma_1 = i\gamma_1\gamma_1\gamma_2 = -\gamma_1\gamma_3. \quad \square$$

III $\gamma_3^2 = I$

Proof. $\gamma_3^2 = (-i\gamma_1\gamma_2)^2 = -\gamma_1\gamma_2\gamma_1\gamma_2 = \gamma_1^2\gamma_2^2 = I$ $\square$

Remark 6.4. The continuum Dirac operator D is anti hermitian.

Proof. We will first prove that $\partial_\mu^* = -\partial_\mu$, $\mu = 1, 2$.

Consider $\langle f, \partial_\mu g \rangle = \int \bar{f}(\partial_\mu g) dx$. Because $f, g \in W$ and f, g satisfy the twisted boundary conditions in (2.6.3), hence, the integral is taken over the 2-dimensional twisted torus. We will only show $\partial_1^* = -\partial_1$. The proof for ∂_2 is similar.

$$\begin{aligned}
 \langle f, \partial_1 g \rangle &= \int \bar{f}(\partial_1 g) dx \\
 &= \int \left[\int_0^1 \bar{f}(\partial_1 g) dx_1 \right] dx_2 \\
 &= \int \left[g\bar{f} \Big|_{x_1=0}^{x_1=1} - \int_0^1 (\partial_1 \bar{f}) g dx_1 \right] dx_2 \\
 &= \int g\bar{f} \Big|_{x_1=0}^{x_1=1} dx_2 - \int \overline{\partial_1 f} g dx_1 dx_2 \\
 &= \int 0 dx_2 - \langle \partial_1 f, g \rangle = \langle -\partial_1 f, g \rangle.
 \end{aligned}$$

$g\bar{f} \Big|_{x_1=0}^{x_1=1} = 0$ as g and f are periodic in the first variables. Similarly we can prove that $\partial_2^* = -\partial_2$. Thus we have,

$$\nabla_\mu^* = \left[\frac{\partial}{\partial x_\mu} + iA_\mu(x) \right]^* = \left(\frac{\partial}{\partial x_\mu} \right)^* - iA_\mu(x) = -\frac{\partial}{\partial x_\mu} - iA_\mu(x) = -\nabla_\mu, \quad \mu = 1, 2$$

Using the fact that γ_μ , $\mu = 1, 2$ are hermitian we have,

$$D^* = \left(\sum \gamma_\mu \nabla_\mu \right)^* = \sum \nabla_\mu^* \gamma_\mu^* = \sum \gamma_\mu (-\nabla_\mu) = -D$$

□

Remark 6.5. γ_3 anticommutes with the Dirac operator that is,

$$\gamma_3 D = -D \gamma_3$$

Proof.

$$\begin{aligned} \gamma_3 D &= \gamma_3 \left(\sum_{\mu=1}^2 \gamma_\mu \nabla_\mu \right) = \sum_{\mu=1}^2 \gamma_3 \gamma_\mu \nabla_\mu \\ &= \sum_{\mu=1}^2 -\gamma_\mu \gamma_3 \nabla_\mu = \left(\sum_{\mu=1}^2 -\gamma_\mu \nabla_\mu \right) \cdot \gamma_3 \\ &= -D \gamma_3 \end{aligned}$$

□

Remark 6.6. The only eigenvalues of γ_3 are ± 1 .

Proof. Let w be an eigenvector and λ an eigenvalue of γ_3 such that $\gamma_3 w = \lambda w$.

We have proved before in remark 6.3 that $\gamma_3^2 = 1$. Hence we have,

$$\begin{aligned} v = I \cdot v &= \gamma_3^2 v = \gamma_3 (\gamma_3 v) = \gamma_3 (\lambda v) = \lambda (\gamma_3 v) = \lambda^2 v \\ &\Rightarrow \lambda^2 = 1. \end{aligned}$$

Furthermore, we have proved in 6.3, γ_3 is Hermitian, and so its eigenvalues are real by the Spectral Theorem. Thus,

$$\lambda = \pm 1$$

□

Remark 6.7. Dirac gamma matrices and the chirality matrix γ_3 are traceless. In other words

$$\text{tr}(\gamma_\mu) = 0, \quad \forall \mu = 1, 2, 3.$$

Proof. We use the properties of gamma matrices introduced in (2.6.2), remark 6.3 the fact $\text{tr}(A \cdot B) = \text{tr}(B \cdot A)$:

$$\begin{aligned} \text{tr}(\gamma_\mu) &= \text{tr}(\gamma_\mu \cdot I) = \text{tr}(\gamma_\mu \gamma_\nu^2) = \text{tr}(\gamma_\mu (\gamma_\nu \gamma_\nu)) = \text{tr}((\gamma_\mu \gamma_\nu) \gamma_\nu) \\ &= \text{tr}(\gamma_\nu (\gamma_\mu \gamma_\nu)) = \text{tr}(\gamma_\nu (-\gamma_\nu \gamma_\mu)) = \text{tr}(-I \cdot \gamma_\mu) = -\text{tr}(\gamma_\mu) \\ &= 0, \end{aligned}$$

where we chose $\nu \neq \mu$, and $\mu = 1, 2, 3$. □

Fejér Kernels

Definition 6.1.1. The Fejér kernels are defined as

$$(n+1)F_n(x) = \sum_{k=0}^n D_k(x), \quad (6.1.9)$$

where the Dirchlet kernel $D_k(x)$ is given by

$$D_k(x) = \sum_{m=-k}^{m=k} e^{imx}. \quad (6.1.10)$$

Another form of the Fejér kernel:

$$\begin{aligned}
 (n+1)F_n(x) &= \sum_{k=0}^n D_k(x) \\
 &= \sum_{k=0}^n \frac{\sin[(n+\frac{1}{2})x]}{\sin \frac{x}{2}} \\
 &= \frac{1}{\sin \frac{x}{2}} \operatorname{Im} \left\{ \sum_{k=0}^n e^{i(k+1/2)x} \right\} \\
 &= \frac{1}{\sin \frac{x}{2}} \operatorname{Im} \left\{ e^{ix/2} \frac{e^{i(n+1)x} - 1}{e^{ix} - 1} \right\} \\
 &= \frac{1}{\sin \frac{x}{2}} \operatorname{Im} \left\{ \frac{e^{i(n+1)x} - 1}{e^{ix/2} - e^{-ix/2}} \right\} \\
 &= \frac{1 - \cos[(n+1)x]}{2 \sin^2 \frac{x}{2}} \\
 &= \frac{[\sin^2(n+1)\frac{x}{2}]}{2 \sin^2 \frac{x}{2}}.
 \end{aligned}$$

The second equality in the above derivation follows from the identity

$$1 + 2 \sum_{k=1}^n \cos(kx) = \frac{\sin[(n+\frac{1}{2})x]}{\sin \frac{x}{2}}$$

and De Moivre's theorem.

Remark 6.8. The Fejér kernels have the following properties

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(t) dt = 1. \quad (6.1.11)$$

$$F_n(t) \geq 0. \quad (6.1.12)$$

For each fixed $\delta > 0$,

$$\lim_{n \rightarrow \infty} \int_{\delta \leq |t| \leq \pi} F_n(t) dt = 0. \quad (6.1.13)$$

Proof. i. Let us consider (6.1.9). Substituting the definition of Dirichlet's kernel yields

$$F_n(x) = \frac{1}{n+1} \sum_{k=0}^n \sum_{m=-k}^k e^{imx}$$

Integrating $F_n(x)$ results in

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1}{n+1} \sum_{k=0}^n \sum_{m=-k}^k e^{imx} \right] dx$$

when m is non-zero, $\int_{-\pi}^{\pi} e^{imx} dx = 0$. But when $m = 0$, $\frac{1}{2\pi} \int_{-\pi}^{\pi} 1 dx = 1$. Thus

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x) dx = \frac{1}{n+1} \sum_{k=0}^n 1 = 1.$$

ii We have proved that

$$F_n(x) = \frac{1}{n+1} \frac{\sin^2[(n+1)\frac{x}{2}]}{\sin^2[\frac{x}{2}]}.$$

Thus $F_n(x) \geq 0$. iii For each fixed $\delta > 0$, $\lim_{n \rightarrow \infty} \int_{\delta \leq |x| \leq \pi} F_n(x) dx = 0$. $\square$

Definition 6.1.2. Let $f : [-\pi, \pi] \rightarrow \mathbb{R}$ be a continuous function. The n^{th} partial of the Fourier sum of f is given by

$$s_n(f) = \sum_{k=-n}^n \hat{f}(k) e^{ikt}.$$

The Cesaro sum of f is given by

$$\sigma_n(f) = \frac{1}{n+1} \sum_{k=0}^n s_k(f).$$

Fact 6.1.3. $\sigma_n(f) = F_n * f$, where F_n is the Fejér kernel.

Proof. From the above definition and from (6.1.10) we have,

$$\begin{aligned} \sigma_n(f) &= \frac{1}{n+1} \sum_{k=0}^n s_k(f) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left[\frac{1}{n+1} \sum_{k=0}^n D_n(t-x) \right] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) F_n(t-x) dx \\ &= f * F_n. \end{aligned}$$

The proof follows from the commutativity of $*$. □

Remark 6.9. If f is a continuous function on S^1 then $\sigma_n(f) \rightarrow f$ in L^∞ which is the Banach space of functions equipped with supremum norm.

Proof.

$$\begin{aligned}
\|(F_n * f) - f\|_{L^\infty} &= \left\| \frac{1}{2\pi} \int_{\mathbb{T}} F_n(\tau) f(t - \tau) d\tau - f(t) \right\|_{L^\infty} \\
&= \left\| \frac{1}{2\pi} \int_{\mathbb{T}} F_n(\tau) [f(t - \tau) - f(t)] d\tau \right\|_{L^\infty} \\
&\leq \frac{1}{2\pi} \int_{\mathbb{T}} |F_n(\tau)| \left\| f(t - \tau) - f(t) \right\|_{L^\infty} d\tau \\
&= \frac{1}{2\pi} \left(\int_{\tau \leq \delta} + \int_{\delta < |\tau| \leq \pi} \right) |F_n(\tau)| \left\| f(t - \tau) - f(\tau) \right\|_{L^\infty} d\tau \\
&\leq \left[\max_{\tau \leq \delta} \left\| f(t - \tau) - f(t) \right\|_{L^\infty} d\tau \int_{|\tau| \leq \delta} |F_n(\tau)| \right. \\
&\quad \left. + \max_{\delta \leq |\tau| \leq \pi} \left\| f(t - \tau) - f(t) \right\|_{L^\infty} d\tau \int_{\delta \leq |\tau| \leq \pi} |F_n(t)| \right] \\
&< \epsilon + \epsilon \\
&< \epsilon
\end{aligned}$$

□

In the above proof we have used (6.1.11) in the second line. In the last line we have used (6.1.12), (6.1.13) and the fact that a continuous function is bounded on a compact set.

6.2 Appendix to Chapter 3

Remark 6.10. The index of the naive Dirac operator is $D_{lat} = 0$.

We begin by proving the following statement.

$$\text{tr } (\gamma_3|_V) = \text{index } D_{lat}$$

Proof. γ_3 acts on $\Omega \times \Omega$ by $\gamma_3 : \Omega \times \Omega \rightarrow \Omega \times \Omega$

$$f \mapsto \gamma_3 f$$

We consider $\gamma_3|_V$. Since every element of $V_{\pm}$ is an eigenvector of γ_3 with eigenvalue ± 1 , γ_3 maps the subspaces $V_{\pm}$ to themselves and we have

$$\text{tr } (\gamma_3|_V) = \dim V_+ - \dim V_- = \text{index } D_{lat} \quad (6.2.1)$$

□

Our next step is to show that

$D_{lat} : V^{\perp} \rightarrow V^{\perp}$ is invertible.

Proof. Since $\gamma_3 D_{lat} = -D_{lat} \gamma_3$, we conclude that $\gamma_3 : V \rightarrow V$.

Let $v \in V^{\perp}$ and $v' \in V$. From the identity

$$\langle \gamma_3 v, v' \rangle = \langle v, \gamma_3 v' \rangle = 0, \text{ since } \gamma_3 : V \rightarrow V, \gamma_3 v' \in V.$$

we conclude that $\gamma_3 : V^{\perp} \rightarrow V^{\perp}$. Thus we see that both V and $V^{\perp}$ are invariant under γ_3 .

From elementary linear algebra we know that

$$\Omega \times \Omega = V \oplus V^\perp.$$

Where $V = \ker D_{lat}$. $V^\perp = \{\omega \in \Omega : \langle \omega, v \rangle = 0 \text{ for all } v \in V.\}$

Therefore,

$$\text{tr} (\gamma_3)_{\Omega \times \Omega} = \text{tr} (\gamma_3)_V + \text{tr} (\gamma_3)_{V^\perp}.$$

Let $v \in V^\perp$ such that $D_{lat}v = 0$. Then $v \in V$. Hence, $v \in V \cap V^\perp$. This implies $v = 0$. Thus $D_{lat} : V^\perp \rightarrow V^\perp$ is invertible. $\square$

From the identity

$$\text{tr} (\gamma_3) = \text{tr} (D_{lat}\gamma_3 D_{lat}^{-1}) = \text{tr} (\gamma_3 D_{lat} D_{lat}^{-1}) = -\text{tr} (\gamma_3)$$

we conclude that $\text{tr} (\gamma_3)_{V^\perp} = 0$.

Using 6.2.1 and the fact that $\Omega \times \Omega = V \oplus V^\perp$, we have $\text{tr} (\gamma_3)_{\Omega \times \Omega} = \text{index} (D_{lat})$.

Now

$$(\gamma_3)_{\Omega \times \Omega} = \text{Id}_\Omega \otimes \gamma_3$$

Therefore,

$$\text{tr} (\gamma_3)_{\Omega \times \Omega} = \dim \Omega \times 0,$$

since Ω is finite dimensional. Hence we have

$$\text{Index } D_{lat} = \text{tr} (\gamma_3)_{\Omega \times \Omega} = 0$$

Remark 6.11. The following are equivalent.

$$\text{I. } D_{lat}\phi = 0 \iff D_{lat}^* D_{lat}\phi = 0.$$

$$\text{II. } D_{lat}^* D_{lat}\phi = 0 \iff \nabla_\mu^* \nabla_\mu \phi = 0, \mu = 1, 2.$$

$$\text{III. } \nabla_\mu^* \nabla_\mu \phi = 0 \iff \nabla_\mu \phi = 0, \mu = 1, 2.$$

Proof.

$$\begin{aligned} ||D\psi||^2 &= \langle D\psi, D\psi \rangle = \langle \psi, D^* D\psi \rangle = \langle \psi, 0 \rangle = 0 \\ &\implies D\psi = 0. \end{aligned}$$

Next, we show $D^* D\psi = 0 \Leftrightarrow \nabla_\mu^* \nabla_\mu \psi = 0, \mu = 1, 2$:

$$\begin{aligned} D^* D &= -D^2 = -(\gamma_1 \nabla_1 + \gamma_2 \nabla_2)^2 \\ &= -(\gamma_1^2 \nabla_1^2 + \gamma_1 \gamma_2 \nabla_1 \nabla_2 + \gamma_2 \gamma_1 \nabla_1 \nabla_2 + \gamma_2^2 \nabla_2^2) \\ &= -(\nabla_1^2 + \gamma_1 \gamma_2 \nabla_1 \nabla_2 + (-\gamma_1 \gamma_2) \nabla_1 \nabla_2 + \nabla_2^2) \\ &= -(\nabla_1^2 + \nabla_2^2) = \nabla_1^* \nabla_1 + \nabla_2^* \nabla_2. \end{aligned}$$

It is clear now that $\nabla_\mu^* \nabla_\mu \psi = 0, \mu = 1, 2$ implies $D^* D\psi = 0$.

To prove that $D^*D\psi = 0$ implies $\nabla_\mu^*\nabla_\mu\psi = 0$, $\mu = 1, 2$, consider

$$\begin{aligned}
D^*D\psi = 0 &\implies \langle D^*D\psi, \psi \rangle = 0 \\
&\implies \langle (\nabla_1^*\nabla_1 + \nabla_2^*\nabla_2)\psi, \psi \rangle = 0 \\
&\implies \langle \nabla_1^*\nabla_1\psi, \psi \rangle + \langle \nabla_2^*\nabla_2\psi, \psi \rangle = 0 \\
&\implies \langle \nabla_1\psi, \nabla_1\psi \rangle + \langle \nabla_2\psi, \nabla_2\psi \rangle = 0 \\
&\implies ||\nabla_1\psi||^2 + ||\nabla_2\psi||^2 = 0 \\
&\implies \nabla_\mu\psi = 0, \mu = 1, 2 \\
&\implies \nabla_\mu^*\nabla_\mu\psi = 0, \mu = 1, 2.
\end{aligned}$$

Lastly, we will show that $\nabla_\mu^*\nabla_\mu\psi = 0 \Leftrightarrow \nabla_\mu\psi = 0$, $\mu = 1, 2$.

It is obvious that $\nabla_\mu\psi = 0 \implies \nabla_\mu^*\nabla_\mu\psi = 0$, and for the forward direction,

$$\begin{aligned}
||\nabla_\mu\psi||^2 &= \langle \nabla_\mu\psi, \nabla_\mu\psi \rangle = \langle \psi, \nabla_\mu^*\nabla_\mu\psi \rangle = \langle \psi, 0 \rangle = 0 \\
&\implies \nabla_\mu\psi = 0.
\end{aligned}$$

Therefore the solutions to $D\psi = 0$ are equivalent to the solutions of $\nabla_\mu\psi = 0$. $\square$

Appendix to section 3.1.5.

Remark 6.12. H_w is Hermitian.

Proof. We will first prove that just like ∇_μ in the continuum the difference operators ∇_μ on the lattice are also Hermitian.

$$\begin{aligned}\nabla_\mu^{(+)}\psi(x) &= \frac{1}{a}(U_\mu(x)\psi(x+ae_\mu) - \psi(x)) \\ \nabla_\mu^{(-)}\psi(x) &= \frac{1}{a}(\psi(x) - U_\mu^{-1}(x - ae_\mu)\psi(x - ae_\mu))\end{aligned}$$

We therefore have,

$$\begin{aligned}\nabla_\mu\psi(x) &= \frac{1}{2}(\nabla_\mu^{(+)} + \nabla_\mu^{(-)}) \\ &= \frac{1}{2a}(U_\mu(x)\psi(x+ae_\mu) - U_\mu^{-1}(x - ae_\mu)\psi(x - ae_\mu))\end{aligned}$$

Now we will prove that $(\nabla_\mu^\pm)^* = -(\nabla_\mu)^\mp$, $\mu = 1, 2$. $(\nabla_\mu^{(\pm)})^* = -\nabla_\mu^{(\mp)}$ (in $d = 2$).

$$\begin{aligned}\langle \nabla_\mu^{(+)}\psi, \phi \rangle &= a^2 \sum_x \overline{\left(\frac{U_\mu(x)\psi(x+ae_\mu) - \psi(x)}{a} \right)} \phi(x) \\ &= a \sum_x \left(\overline{U_\mu(x)\psi(x+ae_\mu)} - \overline{\psi(x)} \right) \phi(x) \\ &= a \sum_x \left[U_\mu^{-1}(x) \overline{\psi(x+ae_\mu)} \phi(x) - \overline{\psi(x)} \phi(x) \right]\end{aligned}$$

$$\begin{aligned}
\langle \psi, -\nabla_\mu^{(-)} \phi \rangle &= a^2 \sum_x \overline{\psi(x)} \left(\frac{-\phi(x) + U_\mu^{-1}(x - ae_\mu) \phi(x - ae_\mu)}{a} \right) \\
&= a \sum_x \overline{\psi(x)} [-\phi(x) + U_\mu^{-1}(x - ae_\mu) \phi(x - ae_\mu)] \\
&= a \sum_x \left[-\overline{\psi(x)} \phi(x) + U_\mu^{-1}(x - ae_\mu) \overline{\psi(x)} \phi(x - ae_\mu) \right] \\
&= a \sum_x -\overline{\psi(x)} \phi(x) + a \sum_x U_\mu^{-1}(x - ae_\mu) \overline{\psi(x)} \phi(x - ae_\mu) \\
&= a \sum_x -\overline{\psi(x)} \phi(x) + a \sum_{x'} U_\mu^{-1}(x') \overline{\psi(x' + ae_\mu)} \phi(x') \\
&= a \sum_x -\overline{\psi(x)} \phi(x) + a \sum_x U_\mu^{-1}(x) \overline{\psi(x + ae_\mu)} \phi(x) \quad (*) \\
\Rightarrow \langle \nabla_\mu^{(+)} \psi, \phi \rangle &= \langle \psi, -\nabla_\mu^{(-)} \phi \rangle \Rightarrow (\nabla_\mu^{(+)})^* = -\nabla_\mu^{(-)}
\end{aligned}$$

It remains to show ((*)), namely that

$$\sum_x U_\mu^{-1}(x) \overline{\psi(x + ae_\mu)} \phi(x) = \sum_{x'} U_\mu^{-1}(x') \overline{\psi(x' + ae_\mu)} \phi(x')$$

where $x' = x - ae_\mu$. For $\mu = 1$, it is clear that the sum is the same since each of the terms in the sum is periodic in that direction. For $\mu = 2$, we observe that $\overline{e^{i\theta}} e^{i\theta} = |e^{i\theta}|^2 = 1$ and thus the sums are still equal.

Also, since $(\nabla_\mu^{(+)})^* = -\nabla_\mu^{(-)}$, we have $((\nabla_\mu^{(+)})^*)^* = (-\nabla_\mu^{(-)})^*$, and this implies

$$(\nabla_\mu^{(-)})^* = -\nabla_\mu^{(+)}. \quad \square$$

The lattice derivative ∇_μ is anti-Hermitian:

$$\nabla_\mu^* = -\nabla_\mu.$$

$$\begin{aligned}
\nabla_\mu^* &= \left[\frac{1}{2} (\nabla_\mu^{(+)} + \nabla_\mu^{(-)}) \right]^* \\
&= \frac{1}{2} \left[(\nabla_\mu^{(+)})^* + (\nabla_\mu^{(-)})^* \right] \\
&= \frac{1}{2} \left[-\nabla_\mu^{(-)} - \nabla_\mu^{(+)} \right] = -\nabla_\mu
\end{aligned}$$

Hence the naive Dirac operator D_{lat} is anti hermitian.

$$\begin{aligned}
D_{lat}^* &= \left(\sum \gamma_\mu \nabla_\mu \right)^* = \sum (\gamma_\mu \nabla_\mu)^* \\
&= \sum \nabla_\mu^* \gamma_\mu^* = \sum -\nabla_\mu \gamma_\mu \\
&= -\sum \gamma_\mu \nabla_\mu = -D_{lat}
\end{aligned}$$

We note that the Wilson term (3.1.10) can be written as

$$\Delta = \sum_\mu \Delta_\mu, \quad \Delta_\mu = -\nabla_\mu^{(+)} \nabla_\mu^{(-)}.$$

Clearly Δ_μ is Hermitian. Hence Δ is hermitian.

Therefore we have

$$D_w^* = (D_{lat} + \frac{a}{2} \Delta)^* = -D_{lat} + \frac{a}{2} \Delta.$$

Moreover,

$$\gamma_3 D_w = \gamma_3 \left(D_{lat} + \frac{a}{2} \Delta \right) = -D_{lat} \gamma_3 + \frac{a}{2} \Delta \gamma_3 = D_w^* \gamma_5,$$

Thus we have,

$$\begin{aligned} H_w^*(m) &= [\gamma_3(D_w - mI)]^* = (D_w - mI)^* \gamma_3^* = (D_w^* - mI) \gamma_3 = \gamma_3(D_w - mI) \\ &= H_w(m), \end{aligned}$$

where we have used the relation

$$\gamma_3 D_w = -D_w \gamma_3.$$

□

Fact 6.2.1. *The eigenvalues of D_w are symmetric about the real axis.*

Proof. We have the relation

$$\gamma_3 D_w = -D_w \gamma_3.$$

Now suppose λ is an eigenvalue of D , i.e., there exists some nonzero vector $v \in V$ such that

$$D_w v = \lambda v.$$

Hence

$$D_w(\gamma_3 v) = -\gamma_3(D_w v) = -\gamma_3(\lambda v) = -\lambda(\gamma_3 v).$$

It follows that $\gamma_3 v$ is an eigenvector of D_w corresponding to $-\lambda$. Since γ_3 is invertible, the eigenspaces for λ and $-\lambda$ are isomorphic and thus λ and $-\lambda$ have the same multiplicities. □

Fact 6.2.2. *Let $\lambda(m)$ be an eigenvalue of $H_w(m)$, then*

$$|\lambda'(m)| \leq 1.$$

Proof. We have

$$H(m)\psi(m) = \lambda(m)\psi(m),$$

where $\psi(m)$ also depends smoothly on m and $\|\psi(m)\| = 1$ (think of $\psi(m)$ as a vector whose entries are smooth functions of m). Thus

$$\langle H(m)\psi(m), \psi(m) \rangle = \lambda(m)\langle \psi(m), \psi(m) \rangle = \lambda(m).$$

Now we can differentiate $\lambda(m)$:

$$\begin{aligned} \lambda'(m) &= \lim_{h \rightarrow 0} \frac{1}{h} (\langle H(m+h)\psi(m+h), \psi(m+h) \rangle - \langle H(m)\psi(m), \psi(m) \rangle) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} (\langle H(m+h)\psi(m+h), \psi(m+h) \rangle - \langle H(m+h)\psi(m+h), \psi(m) \rangle \\ &\quad + \langle H(m+h)\psi(m+h), \psi(m) \rangle - \langle H(m)\psi(m+h), \psi(m) \rangle \\ &\quad + \langle H(m)\psi(m+h), \psi(m) \rangle - \langle H(m)\psi(m), \psi(m) \rangle) \\ &= \langle H(m)\psi(m), \psi'(m) \rangle + \langle H'(m)\psi(m), \psi(m) \rangle + \langle H(m)\psi'(m), \psi(m) \rangle. \end{aligned}$$

Now note that

$$\langle H(m)\psi'(m), \psi(m) \rangle = \langle \psi'(m), H(m)\psi(m) \rangle = \lambda(m)\langle \psi'(m), \psi(m) \rangle.$$

Thus $\lambda'(m)$ becomes

$$\lambda'(m) = \lambda(m) (\langle \psi(m), \psi'(m) \rangle + \langle \psi'(m), \psi(m) \rangle) + \langle H'(m)\psi(m), \psi(m) \rangle.$$

Now we have

$$\frac{d}{dm} \|\psi(m)\|^2 = \frac{d}{dm} \langle \psi(m), \psi(m) \rangle = \langle \psi'(m), \psi(m) \rangle + \langle \psi(m), \psi'(m) \rangle.$$

Since $\|\psi(m)\|^2 = 1$, its derivative is 0. Thus

$$\lambda'(m) = \langle H'(m)\psi(m), \psi(m) \rangle.$$

Moreover,

$$H'(m) = \frac{d}{dm} \gamma_3 (D_w - m) = -\gamma_3.$$

Consequently,

$$\lambda'(m) = -\langle \gamma_3 \psi(m), \psi(m) \rangle.$$

Now by Cauchy–Schwarz inequality,

$$|\lambda'(m)| \leq \|\gamma_3 \psi(m)\| \|\psi(m)\| = \|\gamma_3 \psi(m)\| \leq 1$$

since

$$\|\gamma_3 \psi(m)\| \leq \|\gamma_3\| \|\psi(m)\| = 1$$

(recall that $\|\gamma_3\| = 1$).

□

Theorem 6.2.3. *There exists a positive real number ϵ such that for all lattice*

gauge fields satisfying

$$\max_{x \in L} |1 - U(x, \mu, \nu)| < \epsilon, \quad (6.2.2)$$

the operator $H_w = H_w\left(\frac{1}{a}\right)$ has no zero eigenvalue.

Unfortunately the proof of the above theorem is extremely long and we cite [35, 41] for the details.

One important consequence of Theorem 6.2.3 is

Fact 6.2.4. *There are no eigenvalue crossings of the x -axis in the interval*

$$\left(\frac{1}{a} - \frac{1}{a}\sqrt{1-5\epsilon}, \frac{1}{a} + \frac{1}{a}\sqrt{1-5\epsilon}\right)$$

for $0 < \epsilon < 1/5$.

Proof. Put $b = 1/a\sqrt{1-5\epsilon}$. Suppose $\lambda(m)$ is an eigenvalue of $H_w(m)$. From Theorem 6.2.3 we have for some unit vector $\psi \in V$,

$$\left|\lambda\left(\frac{1}{a}\right)\right| = \|H_w\psi\| > \frac{1}{a}\sqrt{1-5\epsilon} = b.$$

Now suppose there's an eigenvalue crossing of the x -axis in the interval $(1/a - b, 1/a + b)$, i.e.,

$$\lambda(x_0) = 0$$

for some $x_0 \in (1/a - b, 1/a + b)$. By the Mean Value Theorem, there exists ξ between x_0 and $1/a$ such that

$$|\lambda'(\xi)| = \left|\frac{\lambda(1/a) - \lambda(x_0)}{1/a - x_0}\right| = \left|\frac{\lambda(1/a)}{1/a - x_0}\right|.$$

Since

$$|\lambda(1/a)| > b > |1/a - x_0|,$$

we obtain $|\lambda'(\xi)| > 1$, contradicting the fact that $|\lambda'(x)| \leq 1$ (fact 6.2.2). $\square$

Thus we see that it makes sense to talk about eigenvalue crossings near $m = 0$, specifically eigenvalue crossings in the interval $(0, 1/a)$. It has been proved in [3] that the eigenvalue crossings of $H_w(m)$ only occur close to 0, 2 and 4 in unit of $1/a$. This gives us a firm ground for defining 3.1.13 the index of D to be minus the spectral flow of $H_w(m)$ in the interval $(0, 1/a)$.

We now note that since $H_w(m)$ is hermitian, the operator $H_w(m)^2$ is positive square root $\sqrt{H_w(m)^2}$ and its eigenvalues are given by $|\lambda_i|$, where λ_i are eigenvalues of $H_w(m)$. If $H_w(m)$ has only non-zero eigenvalues then the eigenvalues of $H_w(m)/\sqrt{H_w(m)^2}$ are given by

$$\frac{\lambda_i}{|\lambda_i|} = \begin{cases} +1 & \text{if } \lambda_i > 0 \\ -1 & \text{if } \lambda_i < 0 \end{cases}.$$

Therefore the following quantity

$$\text{tr} \left(\frac{H_w(m)}{\sqrt{H_w(m)^2}} \right)$$

equals the difference between positive eigenvalues and negative eigenvalues of $H_w(m)$. Recall that eigenvalues of $H_w(m)$ only cross the x -axis when $m \geq 0$. Therefore when $m < 0$ $H_w(m)$ has no zero eigenvalues and the difference between positive eigenvalues and negative eigenvalues of $H_w(m)$ is constant. Even more is true:

Proposition 6.2.5. $H_w(m)$ has the same number of positive and negative eigenvalues when $m < 0$.

Proof. From the above discussion we have that the quantity

$$\operatorname{tr} \left(\frac{H_w(m)}{\sqrt{H_w(m)^2}} \right)$$

is constant when $m < 0$. We have

$$\begin{aligned} \lim_{m \rightarrow -\infty} \operatorname{tr} \left(\frac{H_w(m)}{\sqrt{H_w(m)^2}} \right) &= \lim_{m \rightarrow -\infty} \operatorname{tr} \left(\frac{\frac{1}{|m|} H_w(m)}{\sqrt{\frac{1}{m^2} H_w(m)^2}} \right) \\ &= \lim_{m \rightarrow -\infty} \operatorname{tr} \left(\frac{\frac{1}{|m|} \gamma_3 D_w - \frac{m}{|m|} \gamma_3}{\sqrt{\frac{1}{m^2} D_w^* D_w - \frac{1}{m} (D_w^* + D_w) + 1}} \right) \\ &= \operatorname{tr}(\gamma_3: V \rightarrow V) \\ &= 0. \end{aligned}$$

Thus $H_w(m)$ has the same number of positive and negative eigenvalues when $m < 0$. □

We now come to our important result:

Theorem 6.2.6. *The index of D is given by*

$$\operatorname{Index}(D) = -\frac{1}{2} \operatorname{tr} \left(\frac{H_w}{\sqrt{H_w^2}} \right),$$

where $H_w = H_w(1/a)$.

Proof. Note that the right hand side is well-defined since we proved that H_w has no zero eigenvalue. Recall that $\operatorname{Index}(D)$ is minus the spectral flow of $H_w(m)$ in

the interval $(0, 1/a)$. Put

$$\eta(m) = \# \text{+ve eigenvalues of } H_w(m) - \# \text{-ve eigenvalues of } H_w(m)$$

when m is far away from $0, 2/a, 4/a$. We know that $\eta(m) = 0$ when $m < 0$. When $m \geq 0$, every time there's an eigenvalue crossing with positive slope, $\eta(m)$ will increase by 2 and it will decrease by 2 every time there's an eigenvalue crossing with negative slope. Thus we conclude that $\eta(1/a)$ is twice the difference between eigenvalue crossings with positive slope and eigenvalue crossings with negative slope in $(0, 1/a)$. In other words, the spectral flow of $H_w(m)$ in $(0, 1/a)$ is precisely

$$\frac{1}{2}\eta(1/a) = \frac{1}{2} \text{tr} \left(\frac{H_w}{\sqrt{H_w^2}} \right)$$

and that completes the proof. $\square$

By defining the (discrete) index of D in term of the spectral flow of $H_w(m)$, we hope to obtain some meaningful information about the index of D in the continuum. Specifically, we expect that when the lattice gauge field is the transcript of some smooth gauge field, the (discrete) index of D will reproduce the correct index when we take $a \rightarrow 0$. Proving this rigorously is not so trivial and a detailed proof can be found in [1].

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