Advanced simulation methods for reliability analysis and updating of stochastic dynamic systems with applications to structural dynamics, seismic risk and loss assessment

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**ADVANCED SIMULATION METHODS FOR RELIABILITY
ANALYSIS AND UPDATING OF STOCHASTIC DYNAMIC
SYSTEMS WITH APPLICATIONS TO STRUCTURAL
DYNAMICS, SEISMIC RISK AND LOSS ASSESSMENT**

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SCHOOL OF CIVIL AND ENVIRONMENTAL ENGINEERING

2014

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SYSTEMS WITH APPLICATIONS TO STRUCTURAL
DYNAMICS, SEISMIC RISK AND LOSS ASSESSMENT

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TABLE OF CONTENTS

ACKNOWLEDGEMENT.....	iii
TABLE OF CONTENTS	v
ABSTRACT.....	ix
LIST OF TABLES	xiii
LIST OF FIGURES	xv
1 Introduction.....	1
1.1 Motivation and Objectives	1
1.2 Overview of the Thesis	5
2 Stochastic Sampling Based Bayesian Model Updating	9
2.1 Introduction	9
2.2 The Proposed Approach	12
2.3 Conditional PDFs	17
2.3.1 Samples from $p(\theta_1 \theta_2, \theta_3, \theta_4, D)$	17
2.3.2 Samples from $p(\theta_2 \theta_1, \theta_3, \theta_4, D)$	18
2.3.3 Samples from $p(\theta_3 \theta_1, \theta_2, \theta_4, D)$	19
2.3.4 Samples from $p(\theta_4 \theta_1, \theta_2, \theta_3, D)$	20
2.4 Summary of the Proposed Algorithm	20
2.5 Convergence Issues	21
2.6 Numerical Issues	22
2.7 Illustrative Examples.....	23
2.7.1 Example 1	23
2.7.2 Example 2	30
3 Updating Robust Failure Probability of Dynamic System.....	39

3.1	Introduction.....	39
3.1.1	Bayesian Model Updating of a Linear Dynamic System Using Model Data	43
3.1.2	Original Subset Simulation.....	43
3.1.3	Modified Metropolis-Hastings algorithm.....	45
3.2	The Proposed Approach.....	46
3.3	Sampling from $p(\boldsymbol{\theta}_s, \mathbf{Z} F_i, D)$ for Linear Dynamic System	47
3.3.1	Sampling from $p(\mathbf{Z} \boldsymbol{\theta}_s, F_i, D)$	48
3.3.2	Sampling from $p(\boldsymbol{\theta}_s \mathbf{Z}, F_i, D)$	51
3.3.3	Summary of steps for simulating samples according to $p(\boldsymbol{\theta}_s, \mathbf{Z} F_i, D)$	53
3.4	Sampling from $p(\boldsymbol{\theta}_s, \boldsymbol{\theta}_{nl}, \mathbf{Z} F_i, D)$ for Nonlinear Dynamic System.....	54
3.4.1	Candidate state for $\mathbf{Z}$ and $\boldsymbol{\theta}_{nl}$	55
3.4.2	Candidate state for $\boldsymbol{\theta}_s$	55
3.4.3	Summary of steps for simulating samples according to $p(\boldsymbol{\theta}_s, \boldsymbol{\theta}_{nl}, \mathbf{Z} F_i, D)$	55
3.5	Statistical Properties of the Estimator.....	57
3.6	Illustrative Examples	59
3.6.1	Results for Example 1.....	61
3.6.2	Results for Example 2.....	64
	Appendix 3.A: Conditional Distribution of Input Random Vector	68
	Appendix 3.B: Proof for Stationary Distribution	68
4	A New Stochastic Simulation Based Approach for Seismic Loss Analysis	73
4.1	Introduction.....	73
4.2	Nonlinear Stochastic Dynamic Reliability Analysis for Loss Estimation .	76

4.3	The Proposed Modified Subset Simulation Based Method	79
4.3.1	The Original Subset Simulation.....	79
4.3.2	The Proposed Approach.....	80
4.3.3	Simulating samples distributed according to $p(\boldsymbol{\theta} F_i)$ and $p(L \boldsymbol{\theta}, F_i)$	82
4.3.4	Summary of Steps for Simulating Samples	83
4.4	Statistical Properties of the Estimator	84
4.5	An Illustrative Example	86
4.5.1	Stochastic Ground Motion Model.....	87
4.5.2	Structural Model	91
4.5.3	Fragility and Loss Functions.....	92
4.5.4	Simulation Parameters	93
4.5.5	Results.....	94
	Appendix 4.A: Proof for Stationary Distribution	105
	Appendix 4.B: Proof for Some Statistical Properties of the Proposed Estimators	107
5	Updating Probabilistic Seismic Loss Estimation	109
5.1	Introduction	109
5.2	The Proposed Approach	109
5.3	An Illustrative Example	113
6	Reliability Analysis with Multiple Performance Objectives	117
6.1	Introduction	117
6.2	Reliability Problem with Multiple Performance Objectives	118
6.3	The Proposed Approach	120
6.3.1	Selection of intermediate thresholds vectors $\{\mathbf{q}_i : i = 1, \dots, H\}$	125
6.4	Illustrative Examples.....	126

6.4.1	Simulation Parameters (Case 1)	127
6.4.2	Results (Case 1).....	127
6.4.3	Simulation Parameters (Case 2)	132
6.4.4	Results (Case 2).....	132
7	Conclusions.....	135
7.1	Summary and Main Conclusions	135
7.2	Future Works	139
8	References.....	141

ABSTRACT

Given the fact that often low probability events can lead to disastrous consequences, performance assessments of the civil engineering systems and critical infrastructure systems taking into account the impact of these events are very important. To assess the system performance subjected to dynamic excitations, a stochastic system analysis considering all the important uncertainties involved should be performed. This requires evaluating the dynamic response of the system subjected to any uncertainty that may arise due to, for example, the uncertainty in future excitations, environmental loadings, the imperfection or incomplete knowledge in the modeling of physical systems, or a combination of these, and evaluating damage or loss statistics and probability based on the response to obtain the system performance probabilities. In this thesis, the focus is on the development of new stochastic simulation algorithms for Bayesian model updating, robust reliability updating, extreme-event probability computation, conditional failure sample simulation, and their applications to the evaluation of reliability and risk/loss assessment of civil engineering structures.

Model updating using measured system dynamic response has a wide range of applications in system response and control, health monitoring, or reliability and risk assessment. To quantify the uncertainties and plausibility of the model parameters and model inadequacy, a Bayesian approach is developed. A new stochastic simulation method for Bayesian model updating of a linear dynamic system is developed to quantify the uncertainties associated with the model parameters based on system data. Convergence issues and numerical issues arising in the case of high-dimensionality of the problem are addressed and solutions to tackle these problems are proposed.

The condition of a structure may change from time to time during its operation and may deteriorate which may lead to a significant reduction of its reliability. Therefore, it is essential to assess the reliability of a dynamic system from time to time during its operation. A new stochastic simulation approach is presented for

computing the updated robust failure probability (or its complement robust reliability) of a dynamic system using modal data when the system is subjected to future stochastic excitation. The proposed approach is robust to the number of uncertain parameters and random variables, and the dimension of vibration data involved in the problem.

Performance-based earthquake engineering (PBEE) has emerged as an important development in the field of Earthquake Engineering. It aims at providing information to improve seismic risk decision-making for structures through assessment and design methods which employ realistic behavior models and properly account for the underlying uncertainties. A new stochastic simulation based approach is proposed for the evaluation of seismic loss exceedance probability as a function of threshold without repeated reliability analyses, and the generation of samples of input random variables and any function of them conditioned on different levels of loss exceedance for a more comprehensive seismic risk and loss analysis and investigation. It allows for a more comprehensive characterization of the probability distribution of the loss including the tail parts due to combinations of scenarios which can lead to extreme and catastrophic consequences. The approach is robust to the number of random variables involved.

Taking into account the most up-to-date condition of a structure by collecting vibration data to infer, especially structural model parameters can provide more accurate prediction of its future performance. Therefore, the newly developed computational method to update the robust reliability of the structural system based on system data is integrated with the newly developed exceedance probability based loss estimation method to evaluate the updated probability distribution (including the tail distribution) of risk/loss.

Often an engineering system has multiple performance objectives which require simultaneous consideration when evaluating its reliability. These performance objectives are often probabilistically dependent, because they can depend on a common set of uncertain variables. In this thesis, a new stochastic simulation

approach is proposed to evaluate the failure probability as a function of multiple failure thresholds with each threshold corresponding to one performance objective in a problem with multiple performance objectives.

LIST OF TABLES

Table 2–1 SOME STATISTICAL PROPERTIES OF THE POSTERIOR MARGINAL PDFs OF THE CONTRIBUTION PARAMETERS.....	27
Table 2–2 POSTERIOR MEAN ESTIMATES OF PREDICTION ERROR VARIANCES.....	27
Table 2–3 POSTERIOR MEAN ESTIMATES OF MODE SHAPE COMPONENTS	27
Table 2–4 PROPERTIES OF STRUCTURAL MEMBERS	31
Table 2–5 POSTERIOR MEAN ESTIMATES OF PREDICTION ERROR VARIANCES.....	33
Table 2–6 SOME STATISTICAL PROPERTIES OF THE POSTERIOR MARGINAL PDFs OF THE CONTRIBUTION PARAMETERS.....	34

LIST OF FIGURES

Figure 2.1. The 4-DOF mechanical system	24
Figure 2.2. Posterior mean and c.o.v estimates for mass, stiffness and damping contribution parameters as functions of the number of simulated samples, showing convergence after about 30,000 samples.....	28
Figure 2.3. Prior marginal PDFs and posterior marginal PDFs fitted by kernel sampling density estimates for α_4 , β_4 and η_4 , and posterior samples and contours for the posterior joint PDFs obtained by kernel sampling density estimates for pairs $\{\alpha_4, \beta_4\}$, $\{\alpha_4, \eta_4\}$ and $\{\beta_4, \eta_4\}$	29
Figure 2.4. Diagram of modified benchmark structure	30
Figure 2.5. Prediction error variances (shown all together).....	33
Figure 2.6. Posterior mean and c.o.v estimates for damping contribution parameters as functions of the number of simulated samples, showing convergence after about 30,000 samples	36
Figure 2.7. Posterior mean and c.o.v estimates for stiffness contribution parameters as functions of the number of simulated samples, showing convergence after about 30,000 samples	37
Figure 2.8. Prior marginal PDFs and posterior marginal PDFs fitted by kernel sampling density estimates for $\eta_{c,1}$, $\eta_{Rx,1}$ and $\eta_{Ry,1}$, and posterior samples and contours for the posterior joint PDFs obtained by kernel sampling density estimates for pairs $\{\eta_{c,1}, \eta_{Rx,1}\}$, $\{\eta_{c,1}, \eta_{Ry,1}\}$ and $\{\eta_{Rx,1}, \eta_{Ry,1}\}$	38
Figure 3.1 Failure events $F^{(j)}$, and the corresponding design points in standard Gaussian space	48
Figure 3.2. Updated robust failure probability against threshold, obtained by exploiting the linear property, 50 independent simulation runs	62
Figure 3.3. Updated robust failure probability against threshold, obtained without exploiting the linear property, 50 independent simulation runs	62
Figure 3.4. Sample mean estimate of updated robust failure probability estimator against threshold, obtained from 50 independent simulation runs	63

Figure 3.5. Sample c.o.v estimate of updated robust failure probability estimator obtained from 50 independent simulation runs	63
Figure 3.6. Unit c.o.v of the proposed updated robust failure estimator and MCS estimator	64
Figure 3.7. Updated robust failure probability against threshold, obtained by exploiting the linear property, 50 independent simulation runs	65
Figure 3.8. Updated robust failure probability against threshold, obtained without exploiting the linear property, 50 independent simulation runs	66
Figure 3.9. Sample mean estimate of updated robust failure probability estimator against threshold, obtained from 50 independent simulation runs	66
Figure 3.10. Sample c.o.v estimate of updated robust failure probability estimator obtained from 50 independent simulation runs	67
Figure 3.11. Unit c.o.v of the proposed updated robust failure estimator and MCS estimator	67
Figure 4.1 Amplitude spectrum using the stochastic ground motion model for different moment magnitudes and epicentral distances.....	89
Figure 4.2 Envelop function using the stochastic ground motion model for different moment magnitudes and epicentral distances	89
Figure 4.3. Fragility functions and loss functions for an interior column in story 1	93
Figure 4.4. Exceedance probability estimates for different threshold levels of economic loss from 50 independent simulation runs	94
Figure 4.5. Sample mean of exceedance probability estimates from 50 independent simulation runs.	95
Figure 4.6. Sample c.o.v of exceedance probability estimates from 50 independent simulation runs	95
Figure 4.7. Sample ‘unit c.o.v’ of exceedance probability estimates from 50 independent simulation runs	96
Figure 4.8. Frequency histogram with $P(F_{i+1} \theta, F_i)$ with uncertain θ on the x-axis and the number of θ samples for each value of $P(F_{i+1} \theta, F_i)$ on the y- axis.....	97
Figure 4.9 $\{p(L \theta^{(k)}, F_i) : k = 1, 2, \dots, N\}$	98

Figure 4.10. Estimates of the exceedance probability curve from single simulation run	98
Figure 4.11. Estimates of c.o.v assuming the conditional probability estimators are uncorrelated and fully correlated (upper bound)	99
Figure 4.12. Conditional <i>Magt</i> and <i>dist</i> samples at conditional levels 0,1,2,3	101
Figure 4.13. Conditional <i>IDR</i> samples for the fourth story at conditional loss threshold levels 0,1,2,3	102
Figure 4.14. Number of columns in different damage states at various conditional loss threshold levels	103
Figure 4.15. Mean total loss from each story at various conditional loss threshold levels	104
Figure 5.1 Prior robust, posterior robust and nominal exceedance probability against the threshold of total economic loss (Case 1, fixed <i>Magt</i> and <i>dist</i>)..	114
Figure 5.2 Prior robust, posterior robust and nominal exceedance probability against the threshold of total economic loss (Case 2, uncertain <i>Magt</i> and <i>dist</i>)	114
Figure 6.1 Failure probability contour lines in two-dimensional and contour surfaces in three-dimensional threshold vector space.....	120
Figure 6.2 Failure probability contour surface π_1 in the threshold vector space and failure domain F_1 in θ space corresponding to point $\mathbf{q}_1$ on π_1 (show systematically in 2-D space)	123
Figure 6.3 Threshold vectors $\mathbf{C}$ with failure probability estimates that can be obtained using samples distributed according to the PDF $p(\theta F_i)$.	124
Figure 6.4 Discontinuity surface between different conditional levels	124
Figure 6.5 Selection of intermediate thresholds vector $\mathbf{q}_i$	125
Figure 6.6. Sample mean of failure probability contour surface estimates obtained from 60 independent simulation runs and failure probability contour surface estimate using MCS	129
Figure 6.7. Sample mean of failure probability contour surface estimates and sample c.o.v of failure probability estimates corresponding to points on the curve shown obtained from 60 independent simulation runs	129

Figure 6.8. Failure probability contour surface estimate obtained from a single simulation run and failure probability contour surface estimate using MCS.....	130
Figure 6.9. <i>IDR</i> and <i>FA</i> samples at various conditional levels.....	130
Figure 6.10. $P_F([c_{IDA} \ c_{FA} = 9\text{m/s}^2]^T)$	131
Figure 6.11. $P_F([c_{IDA} = 2\% \ c_{FA}]^T)$	131
Figure 6.12. Sample mean of failure probability contour surface estimates from 60 independent simulation runs and failure probability contour surface estimates using MCS	133
Figure 6.13. Sample mean of failure probability contour surface estimates and sample c.o.v of failure probability estimates corresponding to points on the curve shown obtained from 60 independent simulation runs	133
Figure 6.14. Failure probability estimates from a single simulation runs and failure probability estimates from MCS.....	134

Chapter 1

Introduction

1.1 Motivation and Objectives

A catastrophic event is often a relatively rare event with disastrous consequences. For example, small earthquakes occur throughout the world, but large earthquakes that are catastrophic in scale occur very rarely. The larger the earthquake, the less likely is its occurrence and more severe might be the consequences. This is true for the other natural hazards that may become catastrophic such as tsunamis, hurricanes or large volcanic eruptions etc. These catastrophic events can have both economic as well as social implications. Damage to infrastructure such as roads, buildings, highways or life line systems can severely impede economic activity. Social impact can include the loss of life, injury, homelessness and disruption of communities and society.

Given the fact that these low probability events can have disastrous consequences, prior assessment of buildings, manufacturing units, dams, bridges, offshore platforms, nuclear power plants and critical infrastructure is important. Understanding where future damage/loss is likely to occur can help in reducing potential losses and assist recovery. In the case of any large impact event, for optimal decision makings on resilient structural, infrastructural systems and urban risk/loss mitigation strategies, it is essential to develop evaluation methods for risk/loss assessment of structures, which are both effective and efficient.

Performance-based engineering aims to quantify the performance of a system based on quantifiable and probabilistic performance objectives. Performance objectives are statements of acceptable performance of the system, defined by the performance quantities of interest attached to certain specified thresholds. Quantities of interest

can take the form of conventional system response parameters (e.g., stress, deflection, drift) or their derivatives (e.g., dollar losses, downtime). Probabilistic performance objectives need to take into consideration any uncertainty that may arise because of the uncertainty in the future excitations, environmental loadings, the imperfection or lack of accurate information in the modeling of physical systems, or a combination of these.

The first objective of this study is to develop a Bayesian model updating technique to quantify the uncertainties associated with the model parameters based on system data with application to structural systems. In real practice, it is impossible to measure the physical properties of a system completely in space and time. The estimates of the model parameters of the mathematical model used to represent the behavior of the real system always involve uncertainties due to limitations of the mathematical model and the presence of measurement error in the data etc. To assess the system performance subjected to dynamic excitations, a stochastic system analysis considering all the important uncertainties involved should be performed. Reliability considering model uncertainties in addition to the modeling of the uncertain excitation is termed as ‘robust reliability’. For this, the uncertainty characterization of the system modeling parameters and modeling error becomes an important task. Several researchers (Ching and Chen, 2007; Jaishi and Ren, 2005; Modak et al., 2002; Mottershead and Friswell, 1993; Papadimitriou et al., 2012) have presented works on the updating of the model parameters based on system data. However, there are relatively few papers in model updating literature in which probabilistic model updating is considered (Beck and Katafygiotis, 1998; Beck and Au, 2002; Ching and Chen, 2007; Ching et al., 2006b; Papadimitriou and Papadioti, 2013; Vanik et al., 2000a; Yuen and Beck, 2003).

The second objective of this study is to develop a new computational method to update the robust reliability of a dynamic system using the response data measured from the system. The condition of a structure may change from time to time during its operation and may deteriorate which may lead to a significant reduction of its reliability. Therefore, it is essential to assess the reliability of a dynamic system

from time to time during its operation. Over the past few years, several methods have been presented to evaluate the updated robust reliability. Papadimitriou et al. (2001) presented Laplace's asymptotic approximation for estimating updated robust reliability. However, it can be computationally challenging in a high-dimensional parameter space and can be inaccurate when the Gaussian assumption is not valid for the global identifiable case. Beck and Au (2002) proposed a level-adaptive Metropolis-Hastings algorithm with a global proposal probability density function (PDF) to obtain the samples from the posterior PDF and then use these samples to update the system reliability by evaluating the system reliability conditional on each of these samples. The approach experiences difficulty when the number of uncertain model parameters is large and is computationally inefficient because it requires multiple reliability analyses. Ching and Beck (2007) proposed a method to update the reliability based on combining a Kalman filter and smoother and modifying the algorithm ISEE (Au and Beck 2001a). Such an approach is only applicable to a linear system with no uncertainties in model parameters. Ching and Hsieh (2006) proposed a method based on the Bayes' theorem and an analytical approximation of some of the required PDFs by maximum entropy PDFs. The method is applicable regardless of the number of uncertain model parameters but can only be applied to the case with very low-dimensional system output data. In practice, system data are of very high dimension (say of the order of hundreds or thousands). Cheung and Beck (2007) proposed a stochastic simulation method which can handle general nonlinear dynamic system and the case with high-dimensional system output data but may encounter problems if the number of uncertain model parameters is huge. Jensen et al. (2013) integrated Bayesian probabilistic framework for model updating with advanced simulation tools for updating the robust reliability of a structural system using the dynamic data. However, the performance of the algorithm can deteriorate with an increase in the number of uncertain model parameters.

The third objective of the present work is to evaluate seismic exceedance probability as a function of threshold including the tail distribution, and the generation of samples of input random variables and any function of them conditioned on different levels of loss exceedance for a more comprehensive

seismic risk and loss analysis and investigation. Performance-based earthquake engineering (PBEE) has emerged as an important development in the field of Earthquake Engineering. It aims at providing information to improve seismic risk decision-making for structures through assessment and design methods which employ realistic behavior models and properly account for the underlying uncertainties. The Pacific Earthquake Engineering Center (PEER) has developed a modular approach for PBEE (Moehle and Deierlein, 2004; Porter, 2003). The outputs of such an analysis can be, for example, exceedance probability against a certain threshold. Over the last few years, several studies have demonstrated the implementation of this framework, for example, Aslani and Miranda (2005), Mitrani-Reiser (2007), Ramirez (2010), Yang et al. (2009). However, since a small number of analyses are used to establish the distribution of structural response quantities, the more complete probabilistic information of the performance of the system (especially the tail parts of the performance PDF) is not obtained. The results obtained from such a study are generally exceedance probabilities that focus on the high probability regions which may be sufficient for estimating the first and second-order statistics such as mean and standard deviation or large exceedance probability (e.g., > 0.1). The framework falls short of accurately considering small exceedance probability (e.g., $<< 0.1$) that characterizes rarer events which can lead to significant consequences. Mahsuli and Haukaas (2013) presented a FORM-based approach to evaluate the loss exceedance probability for a particular threshold. Their approach provides accurate estimates for low probability events, however, requiring repeated reliability analyses to obtain the loss exceedance probability as a function of thresholds.

The fourth objective of this study is to integrate the newly developed computational method to update the robust reliability of the structural system based on system data with newly developed fully probability-based loss estimation formulation to estimate the updated probability distribution of risk/loss. Taking into account the most up-to-date condition of a structure by collecting vibration data to infer, especially structural model parameters can provide more accurate prediction of its future performance.

The fifth objective of this study is to evaluate the failure probability as a function of various combinations of failure thresholds with each threshold corresponding to one performance objective in a problem with multiple performance objectives. Often an engineering system has multiple performance objectives, all of which are taken into consideration for estimating its reliability. These performance objectives are often probabilistically dependent, because they can depend on a common set of uncertain variables. Most reliability assessment techniques (Au and Beck, 2001a, 2001b; Katafygiotis et al., 2010; Katafygiotis et al., 2007; Pradlwarter et al., 2007) focus on obtaining failure probabilities as a function of a single threshold. They assume a single performance function, or if there are multiple performance functions for a system, they are correlated to each another in some way to obtain a single performance functions, such as in Parallel Subset Simulation by Hsu and Ching (2010) where a principal variable is introduced which is correlated with all performance functions. For a problem with multiple performance functions which involves evaluation of failure probability as a function of multiple thresholds, the aforementioned techniques will require their repetitive application that can be very inefficient (unless the combination of thresholds corresponding to which the failure probability is estimated is known in prior). To the author's best knowledge, no works have looked into this problem.

1.2 Overview of the Thesis

In this thesis, the focus is on the development of new stochastic simulation algorithms for Bayesian model updating, robust reliability updating, extreme-event probability computation, conditional failure sample simulation, and their applications to the evaluation of reliability and risk/loss assessment of civil engineering structures.

Chapter 2 presents a new stochastic simulation method for Bayesian model updating of a linear dynamic system based on incomplete modal data namely modal frequencies, damping ratios and partial mode shapes of some of the dominant modes. A new Gibbs sampling based algorithm is proposed that allows for efficient sampling from the posterior probability distribution of the uncertain parameters. In

addition to the model parameters, the probability distribution of complete mode shapes and prediction errors are also updated. Convergence issues and numerical issues arising in the case of high-dimensionality of the problem are addressed and solutions to tackle these problems are proposed. Two numerical examples are considered for illustrating the effectiveness and accuracy of the proposed methods. The first example involves a 4-degree of freedom (DOF) shear building model. The second example involves a modified version of 120-DOF four-story, two-bay by two-bay steel frame originally designed for IASC-ASCE Phase-I Simulated Structural Health Monitoring Benchmark Problem.

In Chapter 3, a new stochastic simulation approach is presented for computing the updated robust failure probability (or its complement robust reliability) of a dynamic system using modal data when the system is subjected to future stochastic excitation. The updating is based on incomplete modal data including modal frequencies, damping ratios and partial mode shapes of some of the dominant modes. Uncertainties from structural modeling and the modeling of the uncertain excitation that the structure will experience during its lifetime are taken into account. The proposed approach is robust to the number of random variables involved and efficient to compute small failure probabilities. The proposed approach integrates a newly-developed stochastic simulation method for Bayesian model updating of a linear dynamic system presented in Chapter 2 and a very efficient algorithm called Subset Simulation (Au and Beck, 2001a, 2003). Two new algorithms, Constrained Metropolis-within-Gibbs and Constrained Multi-group Metropolis-within-Gibbs, are developed for efficient sampling from the conditional distribution, one for the case involving a linear dynamic system and one for the case involving a nonlinear dynamic system. Two numerical examples are considered for illustrating the effectiveness, efficiency and accuracy of the proposed approach.

In Chapters 4 and 5, the focus is shifted to the estimation of losses in a structure due to future earthquakes. In Chapter 4, a new stochastic simulation based approach for the evaluation of seismic loss exceedance probability as a function of threshold without repeated reliability analyses, and the generation of samples of input random

variables and any function of them conditioned on different levels of loss exceedance for a more comprehensive seismic risk and loss analysis and investigation is presented. Stochastic ground motion model coupled with nonlinear stochastic dynamic model, and probabilistic fragility and loss functions are considered. The proposed approach involves the modification of the simulation algorithms in the Subset Simulation (Au and Beck, 2001a, 2003) and development of new estimators to tackle the estimation problem of seismic loss probability function. Exceedance probability as a function of threshold and the corresponding curve are obtained at thresholds using much fewer dynamic analyses than those would be required by Monte Carlo Simulation (MCS). Like MCS, the proposed approach is robust to the number of random variables involved.

An example is presented to show the application of the stochastic simulation based approach for the estimation of seismic loss exceedance probability as a function of threshold. The building used in the illustration is a hotel structure located in Van Nuys. It is a seven story reinforced concrete structure that was severely damaged during the 1994 Northridge Earthquake. The stochastic ground motion records are generated by adopting a point-source model developed for California seismicity, characterized by the moment magnitude and the epicentral distance. Exceedance probabilities, and samples for the involved random variables and some functions of these are obtained when the risk related decision variable is economic loss.

In Chapter 5, the proposed approach for the estimation of seismic loss exceedance probability is integrated with the algorithm for computing the updated robust failure probability based on the incomplete modal data, for the estimation of updated seismic loss exceedance probability as a function of threshold. Using the same example as in Chapter 4, the updated exceedance probabilities are obtained as a function of economic loss threshold.

Chapter 6 presents a new stochastic simulation based approach for the evaluation of failure probabilities of the response variables of a dynamic system where failure is defined by a union of multiple performance objectives. The proposed approach

allows for the simultaneous consideration of multiple performance objectives and the corresponding thresholds. The approach modifies the Subset Simulation algorithm which can efficiently compute small failure probabilities. A new procedure is proposed to effectively propagate samples between different conditional levels in a high-dimensional threshold space. Using the same example as in Chapter 4, the exceedance probabilities as a function of two thresholds, inter-story drift ratio and floor acceleration, and economic loss and downtime, are obtained.

Finally, Chapter 7 concludes the thesis and presents plans for further study.

Chapter 2

Stochastic Sampling Based Bayesian Model Updating

2.1 Introduction

The need of model updating is usually motivated by the desire to improve the accuracy of prediction of the system response and control (Friswell and Mottershead, 1995; Hemez and Doebling, 2001), health monitoring (Ching et al., 2006b; Vanik et al., 2000b), or reliability and risk assessment (Papadimitriou et al., 2001; Vanik et al., 2000a). There always exist modeling errors and uncertainties associated with the process of constructing a mathematical model of a system arising either because of incomplete knowledge or simplifying assumptions made during the modeling of the physical problem. These uncertainties in the modeling process can cause the predicted system response to be different from the true system response. If experimental data measured from the system are available, then these data can be used to update the uncertainties in the model parameters.

The usual approach to update a linear dynamic system model is to first identify its modal properties (especially when the data are obtained during ambient vibration) and then use these to update the modeling parameters. There are several ambient or forced vibration based modal identification techniques available (Ang et al., 2003; Brincker et al., 2000; Katafygiotis and Yuen, 2001; Parloo, 2003; Peeters and De Roeck, 2001; Yang et al., 2003; Yuen and Katafygiotis, 2001) that provide optimal estimates of the modal parameters. Probabilistic model updating techniques, particularly the Bayesian approach, provide estimates of the optimal parameters along with their probability density function (PDF) that can be used to provide a

The work presented in this chapter have been submitted for publication to *Computer Methods in Applied Mechanics and Engineering*, and parts of the work appear in the conference papers (Cheung and Bansal, 2013a, 2013d).

comprehensive quantification of the uncertainty. Several researchers (Jaishi and Ren, 2005; Modak et al., 2002; Mottershead and Friswell, 1993; Papadimitriou et al., 2012) have presented works on the updating of the Finite element models based on experimental modal data. However, there are relatively few papers in model updating literature in which probabilistic model updating is considered (Beck and Katafygiotis, 1998; Beck and Au, 2002; Ching et al., 2006b; Papadimitriou and Papadioti, 2013; Vanik et al., 2000a; Yuen and Beck, 2003). Ching et al. (2006b) proposed a new Gibbs sampling based simulation approach for the model updating of linear dynamic systems with classical damping. However, in their algorithm parameters defining the damping matrix were not considered.

In this chapter, a stochastic simulation algorithm based on Gibbs sampler is presented for Bayesian model updating of a linear dynamic system with non-classical damping based on incomplete complex modal data, namely modal frequencies, damping ratios and partial complex mode shapes of some of the dominant modes. The available data will always be incomplete because the measurements will only be taken over a limited frequency range and at a limited number of locations. In the proposed algorithm, the damping matrix in the identification model is represented as a sum of individual substructures in the case of viscous damping, in terms of mass and stiffness matrices in the case of Rayleigh damping or a combination of the former. Convergence issues and numerical issues arising in the case of high-dimensionality of the problem are addressed and solutions to tackle these problems are proposed. The proposed method is robust to the dimension of the problem. Finally, to demonstrate the effectiveness and accuracy of the proposed method, two numerical examples are presented.

A Bayesian model updating approach provides a robust and rigorous framework to characterize modeling uncertainties. Given $\boldsymbol{\theta} \in R^{n_\theta}$ the uncertain model parameter vector with the prior PDF $p(\boldsymbol{\theta} | \mathcal{M})$ specified by a model class $\mathcal{M}$ and the data D , by applying the Bayes' theorem the posterior (updated) PDF $p(\boldsymbol{\theta} | D, \mathcal{M})$ can be written as (Beck and Katafygiotis, 1998):

$$p(\boldsymbol{\theta} | D, \mathcal{M}) = \frac{p(D | \boldsymbol{\theta}, \mathcal{M})p(\boldsymbol{\theta}, \mathcal{M})}{\int p(D | \boldsymbol{\theta}, \mathcal{M})p(\boldsymbol{\theta}, \mathcal{M})d\boldsymbol{\theta}} \quad (2.1)$$

As the integral in the denominator of Equation (2.1) is often not known explicitly *a priori*, $p(\boldsymbol{\theta} | D, \mathcal{M})$ is only known up to a normalizing constant. Based on the topology $p(\boldsymbol{\theta} | D, \mathcal{M})$ in the parameter space, $\boldsymbol{\theta}$ that maximizes $p(\boldsymbol{\theta} | D, \mathcal{M})$ which implies maximization of $p(D | \boldsymbol{\theta}, \mathcal{M})p(\boldsymbol{\theta}, \mathcal{M})$ can be classified into three different categories (Beck and Katafygiotis, 1998): globally identifiable (unique optimal estimate), locally identifiable (finite number of optimal estimates) and unidentifiable (continuum of optimal estimates). Beck and Katafygiotis (1998) adopted Laplace's method of asymptotic approximation which requires a non-convex optimization (Papadimitriou et al., 1997) to obtain the posterior PDF of the model parameters. However, the accuracy of such an approximation is questionable when either the amount of data is not sufficiently large or the chosen class of models turns out to be unidentifiable based on the available data. Also, the approach is computationally challenging, especially in a high-dimensional parameter space or when the model class is not globally identifiable. To avoid these limitations, in recent years focus has shifted to stochastic simulation methods for Bayesian updating especially Markov Chain Monte Carlo (MCMC) methods, such as the Gibbs sampling (Ching et al., 2006b), Metropolis–Hasting (Beck and Au, 2002) and Hybrid Monte Carlo algorithms (Cheung and Beck, 2009). Stochastic simulation methods allow generating samples which are distributed according to the posterior PDF without the need of evaluating the normalizing constant in the Bayes' Theorem. The Gibbs Sampler (Geman and Geman, 1984) is a special a case of Metropolis-Hastings algorithm (Hastings, 1970; Metropolis et al., 1953) that allow sampling from an arbitrary multivariate PDF if sampling according to the PDF of each group of uncertain parameters conditioned on all the others groups is possible. The advantage of Gibbs Sampling methods to other MCMC algorithms is in decomposing a high-dimensional problem into lower-dimensional simpler and more manageable form problems exploiting the structure of full conditional distribution of the parameters.

Assume that $\boldsymbol{\theta}$ is partitioned into G groups of uncertain parameter vectors, i.e.,

$\boldsymbol{\theta} = [\boldsymbol{\theta}_j : j = 1, \dots, G]$. In the original version of Gibbs sampling algorithm, the full conditional PDFs $p_j(\boldsymbol{\theta}_j^* | \boldsymbol{\theta}_{1:j-1}^*, \boldsymbol{\theta}_{j+1:G}, D)$ are required for $j = 1, \dots, G$ where $\boldsymbol{\theta}_{1:j-1}^* = [\boldsymbol{\theta}_1^{*T} \ \boldsymbol{\theta}_2^{*T} \ \dots \ \boldsymbol{\theta}_{j-1}^{*T}]^T$ and $\boldsymbol{\theta}_{j+1:G} = [\boldsymbol{\theta}_{j+1}^T \ \boldsymbol{\theta}_{j+2}^T \ \dots \ \boldsymbol{\theta}_G^T]^T$. Usually some initial portion of Markov chain samples are discarded before the stationary stage is reached. After the burn-in period, the Markov chain samples obtained are distributed according to the target PDF. Statistical measures such as the mean, variance or PDFs can be estimated using the remaining samples.

Let $D \equiv \{\hat{\omega}_{m,s}, \hat{\zeta}_{m,s}, \hat{\psi}_{m,s} : m = 1 \dots M, s = 1 \dots S\}$ be the experimentally obtained modal data from any system, consisting of modal frequencies $\hat{\omega}_{m,s} \in R^+$, damping ratios $\hat{\zeta}_{m,s} \in R^+$ and mode shape components $\hat{\psi}_{m,s} \in C^{N_o}$, where N_o is the number of measured DOF, M is the number of observed modes and S is the number of modal data sets available.

2.2 The Proposed Approach

The equation of a linear dynamic system with p inputs and n state variables can be written in the following state-space form:

$$\left. \begin{aligned} \dot{\mathbf{X}}(t) &= \mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{U}(t) \\ \mathbf{X}(0) &= \mathbf{X}_0 \end{aligned} \right\} \quad (2.2)$$

where $\mathbf{X}(t) \in R^n$ and $\mathbf{U}(t) \in R^p$ denote the state and excitation vectors at time t , respectively; $\mathbf{A}$ is the system matrix, $\mathbf{B}$ is the input matrix and $\mathbf{X}_0$ denotes the initial conditions. The proposed method is valid for any linear dynamic system whose equations of motion can be written in the form of Equation (2.2). For illustration, consider a 2^{nd} -order linear dynamical system with the system matrix of the form:

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \quad (2.3)$$

where $\mathbf{M} \in R^{N_d \times N_d}$, $\mathbf{C} \in R^{N_d \times N_d}$ and $\mathbf{K} \in R^{N_d \times N_d}$ are the mass, damping and stiffness matrices, and $n = 2N_d$ (N_d is the number of DOFs). Complex eigenvalues

$\lambda_m \in C$ and eigenvector $\Psi_m \in C^{2N_d}$ for $m=1, \dots, M$ can be obtained from the solution of the eigenvalue problem corresponding to the system matrix $\mathbf{A}$ as:

$$\left. \begin{aligned} \mathbf{A}\Psi_m &= \lambda_m \Psi_m \\ \Psi_m &= \begin{Bmatrix} \psi_m \\ \lambda_m \psi_m \end{Bmatrix} \\ \lambda_m &= -\zeta_m \omega_m + i \omega_m \sqrt{1 - \zeta_m^2} \end{aligned} \right\} \quad (2.4)$$

The eigenvalues and eigenvectors occur in complex conjugate pairs. Using Equations (2.3)-(2.4), the following relationship between modal properties and dynamic model parameters can be obtained:

$$\lambda_m^2 \mathbf{M} \psi_m + \lambda_m \mathbf{C} \psi_m + \mathbf{K} \psi_m = 0 \quad (2.5)$$

Replacing system eigenvalues λ_m with experimentally obtained eigenvalues $\hat{\lambda}_{m,s}$ gives:

$$\hat{\lambda}_{m,s}^2 \mathbf{M} \psi_m + \hat{\lambda}_{m,s} \mathbf{C} \psi_m + \mathbf{K} \psi_m = \boldsymbol{\varepsilon}_{m,s} \quad (2.6)$$

where the system mode shape ψ_m is mathematically related to the experimentally obtained mode shape $\hat{\psi}_{m,s}$ through a selection matrix $\mathbf{L} \in R^{N_o \times N_d}$ that selects only those DOFs where measurements are made:

$$\hat{\psi}_{m,s} - \mathbf{L} \psi_m = \mathbf{e}_{m,s} \quad (2.7)$$

In the above equation $\boldsymbol{\varepsilon}_{m,s}$ and $\mathbf{e}_{m,s}$ are the complex random vectors representing the model prediction errors, i.e., the errors between the response of the system under consideration and that of the assumed model. The mass and stiffness matrices in Equation (2.6) are represented as a linear sum of contribution of the corresponding mass and stiffness matrices from the individual prescribed substructures:

$$\mathbf{M}(\boldsymbol{\alpha}) = \mathbf{M}_0 + \sum_{i=1}^{N_\alpha} \alpha_i \mathbf{M}_i \quad (2.8)$$

$$\mathbf{K}(\boldsymbol{\eta}) = \mathbf{K}_0 + \sum_{i=1}^{N_\eta} \eta_i \mathbf{K}_i \quad (2.9)$$

The damping matrix, in general, can be represented in terms of mass and stiffness matrix (as in the case of Rayleigh damping), and contribution from other damping sources (as in the case of viscous damping):

$$\mathbf{C}(\boldsymbol{\beta}, \mathbf{c}, \boldsymbol{\alpha}, \boldsymbol{\eta}) = \mathbf{C}_0 + \sum_{i=1}^{N_\beta} \beta_i \mathbf{C}_i + c_0 a_0 \mathbf{M}(\boldsymbol{\alpha}) + c_1 a_1 \mathbf{K}(\boldsymbol{\eta}) \quad (2.10)$$

In Equations (2.8)-(2.10), $\boldsymbol{\alpha} = [\alpha_1 \cdots \alpha_{N_\alpha}]^T$ and $\boldsymbol{\eta} = [\eta_1 \cdots \eta_{N_\eta}]^T$ are the mass and stiffness contribution parameters, $\boldsymbol{\beta} = [\beta_1 \cdots \beta_{N_\beta}]^T$ are the damping contribution parameters for non-classical damping and $\mathbf{c} = [c_0 \ c_1]^T$ are the scaling parameters for Rayleigh damping coefficients $\mathbf{a} = [a_0 \ a_1]^T$. Rayleigh damping coefficients $\mathbf{a} = [a_0 \ a_1]^T$ are given by:

$$a_0 = \frac{2\omega_i\omega_j}{\omega_j^2 - \omega_i^2} (\omega_j\zeta_i - \omega_i\zeta_j) \quad (2.11)$$

$$a_1 = \frac{2}{\omega_j^2 - \omega_i^2} (\omega_j\zeta_j - \omega_i\zeta_i) \quad (2.12)$$

where $[\omega_i \ \omega_j]^T$ and $[\zeta_i \ \zeta_j]^T$ are modal frequencies and damping ratios of the nominal system for the i -th and the j -th mode, respectively. Equations (2.8)-(2.10) with $\boldsymbol{\alpha} = [1 \cdots 1]^T$, $\boldsymbol{\beta} = [1 \cdots 1]^T$, $\mathbf{c} = [1 \ 1]^T$ and $\boldsymbol{\eta} = [1 \cdots 1]^T$ give the nominal mass, damping and stiffness matrices, respectively. The joint PDF of parameters $[\boldsymbol{\alpha}^T \ \boldsymbol{\beta}^T \ \mathbf{c}^T \ \boldsymbol{\eta}^T]^T$ are to be updated by the data D . Other parameters which are unknown in Equations (2.6)-(2.7) and whose PDF need to be updated are the system mode shapes $\{\psi_m : m = 1, \dots, M\}$ and the parameters defining the probabilistic models of the model prediction errors. The system mode shapes are dependent on the model parameters $[\boldsymbol{\alpha}^T \ \boldsymbol{\beta}^T \ \mathbf{c}^T \ \boldsymbol{\eta}^T]^T$, but here they are introduced as additional uncertain parameters. The uncertain parameter space that originally comprises of model parameters and parameters defining the probabilistic models of the model prediction errors is extended by introducing additional parameter (i.e., mode shapes) to get conditional distributions that are easier to handle algorithmically. By treating mode shapes as uncertain parameters full conditional distributions are obtained that enables exact sampling of the full conditional distributions involved within the proposed Gibbs sampling based algorithm.

By equating the real and imaginary parts on both sides of Equations (2.6)-(2.7), respectively, the following equations are obtained:

$$\text{Re}\left(\hat{\lambda}_{m,s}^2 \mathbf{M}(\boldsymbol{\alpha})\boldsymbol{\psi}_m + \hat{\lambda}_{m,s} \mathbf{C}(\boldsymbol{\beta}, \mathbf{c}, \boldsymbol{\alpha}, \boldsymbol{\eta})\boldsymbol{\psi}_m + \mathbf{K}(\boldsymbol{\eta})\boldsymbol{\psi}_m\right) = \text{Re}(\boldsymbol{\epsilon}_{m,s}) \quad (2.13)$$

$$\text{Im}\left(\hat{\lambda}_{m,s}^2 \mathbf{M}(\boldsymbol{\alpha})\boldsymbol{\psi}_m + \hat{\lambda}_{m,s} \mathbf{C}(\boldsymbol{\beta}, \mathbf{c}, \boldsymbol{\alpha}, \boldsymbol{\eta})\boldsymbol{\psi}_m + \mathbf{K}(\boldsymbol{\eta})\boldsymbol{\psi}_m\right) = \text{Im}(\boldsymbol{\epsilon}_{m,s}) \quad (2.14)$$

$$\text{Re}(\hat{\boldsymbol{\psi}}_{m,s} - \mathbf{L}\boldsymbol{\psi}_m) = \text{Re}(\mathbf{e}_{m,s}) \quad (2.15)$$

$$\text{Im}(\hat{\boldsymbol{\psi}}_{m,s} - \mathbf{L}\boldsymbol{\psi}_m) = \text{Im}(\mathbf{e}_{m,s}) \quad (2.16)$$

Based on the Principle of Maximum Entropy (Jaynes, 1978), the PDFs for vectors $\text{Re}(\boldsymbol{\epsilon}_{m,s})$, $\text{Im}(\boldsymbol{\epsilon}_{m,s})$, $\text{Re}(\mathbf{e}_{m,s})$ and $\text{Im}(\mathbf{e}_{m,s})$ are taken to be Gaussian. Their means are assumed to be equal to zero and covariance matrices equal to scaled versions of the identity matrix $\mathbf{I}$ of appropriate order, respectively:

$$\text{Re}(\boldsymbol{\epsilon}_{m,s}) \sim N(0, \sigma_{\text{Re},m}^2 \mathbf{I}) \quad (2.17)$$

$$\text{Im}(\boldsymbol{\epsilon}_{m,s}) \sim N(0, \sigma_{\text{Im},m}^2 \mathbf{I}) \quad (2.18)$$

$$\text{Re}(\mathbf{e}_{m,s}) \sim N(0, \delta_{\text{Re},m}^2 \mathbf{I}) \quad (2.19)$$

$$\text{Im}(\mathbf{e}_{m,s}) \sim N(0, \delta_{\text{Im},m}^2 \mathbf{I}) \quad (2.20)$$

Overall, the above errors are taken to be independent but the variances of errors are not equal. Thus, the likelihood PDF $p(D | \boldsymbol{\theta})$ is given by:

$$p(D | \boldsymbol{\theta}) = \prod_{s=1}^S \prod_{m=1}^M p(D_{m,s} | \boldsymbol{\theta}) \quad (2.21)$$

where $D \equiv \{\hat{\omega}_{m,s}, \hat{\zeta}_{m,s}, \hat{\boldsymbol{\psi}}_{m,s}\}$. $p(D_{m,s} | \boldsymbol{\theta})$ is given by:

$$\begin{aligned} p(D_{m,s} | \boldsymbol{\theta}) &= \frac{1}{(2\pi)^{-N_d} |\mathbf{U}_m|^{1/2}} \exp\left(-\frac{1}{2} [\text{Re}(\boldsymbol{\epsilon}_{m,s})^T \text{Im}(\boldsymbol{\epsilon}_{m,s})^T] \mathbf{U}_m^{-1} [\text{Re}(\boldsymbol{\epsilon}_{m,s})^T \text{Im}(\boldsymbol{\epsilon}_{m,s})^T]^T\right) \\ &\quad \times \frac{1}{(2\pi)^{-N_o} |\mathbf{V}_m|^{1/2}} \exp\left(-\frac{1}{2} [\text{Re}(\mathbf{e}_{m,s})^T \text{Im}(\mathbf{e}_{m,s})^T] \mathbf{V}_m^{-1} [\text{Re}(\mathbf{e}_{m,s})^T \text{Im}(\mathbf{e}_{m,s})^T]^T\right) \end{aligned} \quad (2.22)$$

$$\text{where } \mathbf{U}_m = \begin{bmatrix} \sigma_{\text{Re},m}^2 \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sigma_{\text{Im},m}^2 \mathbf{I} \end{bmatrix} \in \mathbb{R}^{2N_d \times 2N_d} \text{ and } \mathbf{V}_m = \begin{bmatrix} \delta_{\text{Re},m}^2 \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \delta_{\text{Im},m}^2 \mathbf{I} \end{bmatrix} \in \mathbb{R}^{2N_o \times 2N_o}.$$

The variance parameters $\{\delta_{\text{Re},m}^2, \delta_{\text{Im},m}^2 : m=1, \dots, M\}$ are assumed to be known or are directly estimated from the sample variance of the experimental modal data:

$$\hat{\delta}_{\text{Re},m}^2 = \frac{1}{SN_o} \sum_{s=1}^S |\text{Re}(\hat{\psi}_{m,s} - \bar{\psi}_m)|^2 \quad (2.23)$$

$$\hat{\delta}_{\text{Im},m}^2 = \frac{1}{SN_o} \sum_{s=1}^S |\text{Im}(\hat{\psi}_{m,s} - \bar{\psi}_m)|^2 \quad (2.24)$$

where $\bar{\psi}_m = \sum_{s=1}^S \hat{\psi}_{m,s} / S$ is the averaged mode shape for the m -th mode. The joint

PDF of variance parameters $\{\sigma_{\text{Re},m}^2 : m=1, \dots, M\}$ and $\{\sigma_{\text{Im},m}^2 : m=1, \dots, M\}$ is to be updated by the data D . In total, the uncertain parameters whose joint PDF is to be updated are the contribution parameters $[\boldsymbol{\alpha}^T \boldsymbol{\beta}^T \mathbf{c}^T \boldsymbol{\eta}^T]^T$, mode shapes $[\text{Re}(\psi_1)^T \text{Im}(\psi_1)^T \cdots \text{Re}(\psi_M)^T \text{Im}(\psi_M)^T]^T$ and prediction error variance $[\sigma_{\text{Re},1}^2 \sigma_{\text{Im},1}^2 \cdots \sigma_{\text{Re},M}^2 \sigma_{\text{Im},M}^2]^T$.

Similar in spirit to how Ching et al. (2006) derived their proposed algorithm; our proposed algorithm is derived as shown as follows. In the proposed Gibbs Sampling based algorithm, four groups of parameters are considered:

$$\boldsymbol{\theta}_1 = [\boldsymbol{\alpha}^T \boldsymbol{\beta}^T \boldsymbol{\eta}^T]^T \in R^{N_1}$$

$$\boldsymbol{\theta}_2 = \mathbf{c} \in R^{N_2}$$

$$\boldsymbol{\theta}_3 = [\text{Re}(\psi_1)^T \text{Im}(\psi_1)^T \cdots \text{Re}(\psi_M)^T \text{Im}(\psi_M)^T]^T \in R^{2MN_d}$$

$$\boldsymbol{\theta}_4 = [\sigma_{\text{Re},1}^2 \sigma_{\text{Im},1}^2 \cdots \sigma_{\text{Re},M}^2 \sigma_{\text{Im},M}^2]^T \in R^{2M}$$

It can be seen that Equations (2.13)-(2.16) are linear with respect to each of the groups $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$ and $\boldsymbol{\theta}_3$ given the remaining groups with their values fixed. Therefore, given the form of the likelihood function in Equation (2.21), it is appropriate to choose Bayesian conjugate priors which will allow exact sampling from the full conditional PDFs $p(\boldsymbol{\theta}_1 | \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D)$, $p(\boldsymbol{\theta}_2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D)$, $p(\boldsymbol{\theta}_3 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_4, D)$ and $p(\boldsymbol{\theta}_4 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, D)$. The prior PDF for $\boldsymbol{\theta}_1$ is taken to be a Gaussian PDF, i.e., $\boldsymbol{\theta}_1 \sim N(\boldsymbol{\theta}_1^{(0)}, \mathbf{P}_1^{(0)})$ with $\boldsymbol{\theta}_1^{(0)}$ as the most probable values and $\mathbf{P}_1^{(0)}$ a diagonal matrix

of variances to express the initial uncertainties. Similarly, the prior PDF for $\boldsymbol{\theta}_2$ is taken to be a Gaussian PDF, i.e., $\boldsymbol{\theta}_2 \sim N(\boldsymbol{\theta}_2^{(0)}, \mathbf{P}_2^{(0)})$. The prior PDF for the system mode shapes $\boldsymbol{\theta}_3$ is taken to be the product of either independent Gaussian PDFs in the case where any prior information is available, or independent uniform PDFs in the case where no prior information is available (as for the case with unknown components of the mode shapes). The prior PDF for prediction error variances $\boldsymbol{\theta}_4$ is taken to be the product of independent inverse gamma PDFs. An inverse gamma PDF $IG(\rho_0, \kappa_0)$ of, for example σ^2 , with prespecified parameters ρ_0 and κ_0 is:

$$p(\sigma^2) = \frac{\kappa_0^{\rho_0}}{\Gamma(\rho_0)} (\sigma^2)^{-\rho_0-1} \exp\left(-\frac{\kappa_0}{\sigma^2}\right) \quad (2.25)$$

where ρ_0 and κ_0 can be expressed in terms of the mean and coefficient of variation (c.o.v) of σ^2 .

2.3 Conditional PDFs

2.3.1 Samples from $p(\boldsymbol{\theta}_1 | \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D)$

Given $\boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4$ and data D , Equations (2.13)-(2.14) are linear with respect to $\boldsymbol{\theta}_1$ and can be written in the following form:

$$\mathbf{Y}_1 - \mathbf{A}_1 \boldsymbol{\theta}_1 = \mathbf{E}_1 \quad (2.26)$$

with $\mathbf{Y}_1$ and $\mathbf{A}_1$ given as follows:

$$\begin{aligned} \mathbf{Y}_1 &= \begin{bmatrix} \text{Re}(\mathbf{b}_{1,1})^T & \text{Im}(\mathbf{b}_{1,1})^T & \cdots & \text{Re}(\mathbf{b}_{m,s})^T & \text{Im}(\mathbf{b}_{m,s})^T & \cdots & \text{Re}(\mathbf{b}_{M,S})^T & \text{Im}(\mathbf{b}_{M,S})^T \end{bmatrix}^T \\ &\in R^{2N_d MS} \\ \mathbf{A}_1 &= -\begin{bmatrix} \text{Re}(\mathbf{d}_{1,1})^T & \text{Im}(\mathbf{d}_{1,1})^T & \cdots & \text{Re}(\mathbf{d}_{m,s})^T & \text{Im}(\mathbf{d}_{m,s})^T & \cdots & \text{Re}(\mathbf{d}_{M,S})^T & \text{Im}(\mathbf{d}_{M,S})^T \end{bmatrix}^T \\ &\in R^{2N_d MS \times N_1} \end{aligned}$$

where

$$\mathbf{b}_{m,s} = [\mathbf{M}_0 \mathbf{x}_{m,s} + \mathbf{C}_0 \mathbf{y}_{m,s} + \mathbf{K}_0 \mathbf{z}_{m,s}] \in C^{N_d}$$

$$\mathbf{d}_{m,s} = \begin{bmatrix} \mathbf{M}_1 \mathbf{x}_{m,s} \cdots \mathbf{M}_{N_\alpha} \mathbf{x}_{m,s} & \mathbf{C}_1 \mathbf{y}_{m,s} \cdots \mathbf{C}_{N_\beta} \mathbf{y}_{m,s} & \mathbf{K}_1 \mathbf{z}_{m,s} \cdots \mathbf{K}_{N_\eta} \mathbf{z}_{m,s} \end{bmatrix} \in C^{N_d \times N_1}$$

and where $\mathbf{x}_{m,s} = (\hat{\lambda}_{m,s}^2 + \hat{\lambda}_{m,s} c_0 a_0) \psi_m$, $\mathbf{y}_{m,s} = \hat{\lambda}_{m,s} \psi_m$ and $\mathbf{z}_{m,s} = (\hat{\lambda}_{m,s} c_1 a_1 + 1) \psi_m$. $\mathbf{E}_1$ follows a Gaussian distribution with zero mean and covariance matrix:

$$\Sigma_1 = \begin{bmatrix} \mathbf{Q} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{Q} \end{bmatrix} \in R^{2N_d MS \times 2N_d MS} \quad (2.27)$$

$$\text{where } \mathbf{Q} = \begin{bmatrix} \mathbf{U}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{U}_M \end{bmatrix} \in R^{2N_d M \times 2N_d M}.$$

$\mathbf{Y}_1$ and $\mathbf{A}_1$ are fixed matrices given $\boldsymbol{\theta}_2$, $\boldsymbol{\theta}_3$ and data D , and Σ_1 is a fixed covariance matrix given $\boldsymbol{\theta}_4$. Then, the full conditional PDF $p(\boldsymbol{\theta}_1 | \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D)$ is Gaussian with mean and covariance matrix given by (Gelfand et al., 1990):

$$\mathbf{E}(\boldsymbol{\theta}_1 | \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D) = \boldsymbol{\theta}_1^{(0)} + \mathbf{P}_1^{(0)} \mathbf{A}_1^T (\Sigma_1 + \mathbf{A}_1 \mathbf{P}_1^{(0)} \mathbf{A}_1^T)^{-1} (\mathbf{Y}_1 - \mathbf{A}_1 \boldsymbol{\theta}_1^{(0)}) \quad (2.28)$$

$$\mathbf{V}(\boldsymbol{\theta}_1 | \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D) = \mathbf{P}_1^{(0)} - \mathbf{P}_1^{(0)} \mathbf{A}_1^T (\Sigma_1 + \mathbf{A}_1 \mathbf{P}_1^{(0)} \mathbf{A}_1^T)^{-1} \mathbf{A}_1 \mathbf{P}_1^{(0)} \quad (2.29)$$

2.3.2 Samples from $p(\boldsymbol{\theta}_2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D)$

Given $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_3$, $\boldsymbol{\theta}_4$ and data D , Equations (2.13)-(2.14) are also linear with respect to $\boldsymbol{\theta}_2$ and can be written in the following form:

$$\mathbf{Y}_2 - \mathbf{A}_2 \boldsymbol{\theta}_2 = \mathbf{E}_2 \quad (2.30)$$

with $\mathbf{Y}_2$ and $\mathbf{A}_2$ given as follows:

$$\mathbf{Y}_2 = \begin{bmatrix} \text{Re}(\mathbf{g}_{1,1})^T & \text{Im}(\mathbf{g}_{1,1})^T \cdots \text{Re}(\mathbf{g}_{m,s})^T & \text{Im}(\mathbf{g}_{m,s})^T \cdots \text{Re}(\mathbf{g}_{M,S})^T & \text{Im}(\mathbf{g}_{M,S})^T \end{bmatrix}^T \in R^{2N_d MS}$$

$$\mathbf{A}_2 = -\begin{bmatrix} \text{Re}(\mathbf{h}_{1,1})^T & \text{Im}(\mathbf{h}_{1,1})^T \cdots \text{Re}(\mathbf{h}_{m,s})^T & \text{Im}(\mathbf{h}_{m,s})^T \cdots \text{Re}(\mathbf{h}_{M,S})^T & \text{Im}(\mathbf{h}_{M,S})^T \end{bmatrix}^T \in R^{2N_d MS \times N_2}$$

where

$$\mathbf{g}_{m,s} = \left[\hat{\lambda}_{m,s}^2 \mathbf{M} \psi_m + \hat{\lambda}_{m,s} \left(\mathbf{C}_0 + \sum_{i=1}^{N_\beta} \beta_i \mathbf{C}_i \right) \psi_m + \mathbf{K} \psi_m \right] \in C^{N_d}$$

$$\mathbf{h}_{m,s} = \left[\hat{\lambda}_{m,s} a_0 \mathbf{M} \psi_m \quad \hat{\lambda}_{m,s} a_1 \mathbf{K} \psi_m \right] \in C^{N_d \times N_2}$$

$\mathbf{Y}_2$ and $\mathbf{A}_2$ are fixed matrices given $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_3$ and data D . $\mathbf{E}_2$ follows a Gaussian distribution with zero mean and covariance matrix $\boldsymbol{\Sigma}_2$. $\boldsymbol{\Sigma}_2$ is a fixed covariance matrix given $\boldsymbol{\theta}_4$ and is equal to $\boldsymbol{\Sigma}_1$. The full conditional PDF $p(\boldsymbol{\theta}_2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D)$ is Gaussian with mean $E(\boldsymbol{\theta}_2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D)$ and covariance matrix $V(\boldsymbol{\theta}_2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D)$ given by Equations (2.28) and (2.29), respectively, with subscript 1 and 2 interchanged.

2.3.3 Samples from $p(\boldsymbol{\theta}_3 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_4, D)$

For each mode, the random vectors in Equations (2.17)-(2.20) are taken to be independent. Denoting $\boldsymbol{\theta}_{3,m} = [\text{Re}(\psi_m)^T, \text{Im}(\psi_m)^T]^T$ and conditioned on $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, $\boldsymbol{\theta}_4$ and data D , Equations (2.13)-(2.16) can be written in the following form:

$$\mathbf{Y}_{3,m} - \mathbf{A}_{3,m} \boldsymbol{\theta}_{3,m} = \mathbf{E}_{3,m} \quad (2.31)$$

with $\mathbf{Y}_{3,m}$ and $\mathbf{A}_{3,m}$ given as follows:

$$\mathbf{Y}_{3,m} = [\mathbf{0}^{1 \times 2N_d} \quad \text{Re}(\hat{\psi}_{m,1})^T \quad \text{Im}(\hat{\psi}_{m,1})^T \quad \cdots \quad \mathbf{0}^{1 \times 2N_d} \quad \text{Re}(\hat{\psi}_{m,S})^T \quad \text{Im}(\hat{\psi}_{m,S})^T]^T \in R^{2(N_d+N_o)S}$$

$$\mathbf{A}_{3,m} = \begin{bmatrix} \text{Re}(\mathbf{w}_{m,1}) & -\text{Im}(\mathbf{w}_{m,1}) \\ \text{Im}(\mathbf{w}_{m,1}) & \text{Re}(\mathbf{w}_{m,1}) \\ \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \\ & \vdots \\ \text{Re}(\mathbf{w}_{m,S}) & -\text{Im}(\mathbf{w}_{m,S}) \\ \text{Im}(\mathbf{w}_{m,S}) & \text{Re}(\mathbf{w}_{m,S}) \\ \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix} \in R^{2(N_d+N_o)S \times 2N_d}$$

where

$$\mathbf{w}_{m,s} = \left[\hat{\lambda}_{m,s}^2 \mathbf{M} + \hat{\lambda}_{m,s} \mathbf{C} + \mathbf{K} \right] \in C^{N_d \times N_d}$$

$\mathbf{E}_{3,m}$ follows a Gaussian distribution with zero mean and covariance matrix:

$$\Sigma_{3,m} = \begin{bmatrix} \mathbf{R}_m & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{R}_m \end{bmatrix} \in R^{2(N_o+N_d)S \times 2(N_o+N_d)S} \quad (2.32)$$

where $\mathbf{R}_m = \begin{bmatrix} \mathbf{U}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_m \end{bmatrix} \in R^{2(N_o+N_d) \times 2(N_o+N_d)}$.

For $m=1,..,M$, PDF $p(\boldsymbol{\theta}_{3,m} | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_4, D)$ is Gaussian with mean $\mathbf{E}(\boldsymbol{\theta}_{3,m} | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_4, D)$ and covariance matrix $\mathbf{V}(\boldsymbol{\theta}_{3,m} | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_4, D)$ given by Equations (2.28) and (2.29), respectively, with subscript 1 and 3,m interchanged.

2.3.4 Samples from $p(\boldsymbol{\theta}_4 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, D)$

With the previous construction it can be seen that the posterior PDFs of prediction error variances given $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3$ and data D are independent. Thus, $p(\sigma_{\text{Re},m}^2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, D)$ and $p(\sigma_{\text{Im},m}^2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, D)$ are inverse gamma (Gelfand et al., 1990) given as follows:

$$p(\sigma_{\text{Re},m}^2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, D) = IG \left(\rho_0 + \frac{N_d S}{2}, \kappa_0 + \frac{1}{2} \sum_{s=1}^S \text{Re}(\boldsymbol{\epsilon}_{m,s})^T \text{Re}(\boldsymbol{\epsilon}_{m,s}) \right) \quad (2.33)$$

$$p(\sigma_{\text{Im},m}^2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, D) = IG \left(\rho_0 + \frac{N_d S}{2}, \kappa_0 + \frac{1}{2} \sum_{s=1}^S \text{Im}(\boldsymbol{\epsilon}_{m,s})^T \text{Im}(\boldsymbol{\epsilon}_{m,s}) \right) \quad (2.34)$$

2.4 Summary of the Proposed Algorithm

Let $\boldsymbol{\theta}_j^{(k)}$ denote the k -th sample of $\boldsymbol{\theta}_j$.

1. Draw one from the prior PDF or use nominal values as the starting point $\boldsymbol{\theta}_1^{(0)}$ and let $k = 0$.
2. Sample $\boldsymbol{\theta}_1^{(k+1)}$ from $p(\boldsymbol{\theta}_1 | \boldsymbol{\theta}_2^{(k)}, \boldsymbol{\theta}_3^{(k)}, \boldsymbol{\theta}_4^{(k)}, D)$ as described in Section 2.3.1.
3. Sample $\boldsymbol{\theta}_2^{(k+1)}$ from $p(\boldsymbol{\theta}_2 | \boldsymbol{\theta}_1^{(k+1)}, \boldsymbol{\theta}_3^{(k)}, \boldsymbol{\theta}_4^{(k)}, D)$ as described in Section 2.3.2.

4. Sample $\boldsymbol{\theta}_3^{(k+1)}$ by sampling $\boldsymbol{\theta}_{3,m}^{(k+1)}$ from $p(\boldsymbol{\theta}_{3,m} | \boldsymbol{\theta}_1^{(k+1)}, \boldsymbol{\theta}_2^{(k+1)}, \boldsymbol{\theta}_4^{(k)}, D)$ for $m = 1, \dots, M$ as described in Section 2.3.3.
5. Sample $\boldsymbol{\theta}_4^{(k+1)}$ by sampling $\sigma_{\text{Re},m}^{2(k+1)}$ and $\sigma_{\text{Im},m}^{2(k+1)}$ from $p(\sigma_{\text{Re},m}^2 | \boldsymbol{\theta}_1^{(k+1)}, \boldsymbol{\theta}_2^{(k+1)}, \boldsymbol{\theta}_3^{(k+1)}, D)$ and $p(\sigma_{\text{Im},m}^2 | \boldsymbol{\theta}_1^{(k+1)}, \boldsymbol{\theta}_2^{(k+1)}, \boldsymbol{\theta}_3^{(k+1)}, D)$ for $m = 1, \dots, M$ as described in Section 2.3.4.
6. Let $k=k+1$ and go to step 2, until N samples are obtained.

2.5 Convergence Issues

In general, the model updating problem with a high-dimensional parameter space will typically have convergence problems due to over parameterized linear identification model or data with incomplete information. The likelihood function in Equation (2.21) has singularities (i.e., nonsensical maxima), and depending on the initial values, the sampling run may converge to a local maximum. For example, a local maximum where a run may get trapped involves variance parameters approaching extreme values (zero or infinity). If this happens, the results using the Gibbs sampler may be meaningless.

To tackle the aforementioned problems, one may first consider the case where all the variance parameters are forced to be equal. For the problem with all variance parameters set equal, Equations (2.33)-(2.34) become:

$$\begin{aligned}
 & p(\sigma^2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, D) \\
 &= IG \left(\rho_0 + MN_d S, \kappa_0 + \frac{1}{2} \sum_{m=1}^M \sum_{s=1}^S [\text{Re}(\boldsymbol{\epsilon}_{m,s})^T \text{Im}(\boldsymbol{\epsilon}_{m,s})^T] [\text{Re}(\boldsymbol{\epsilon}_{m,s})^T \text{Im}(\boldsymbol{\epsilon}_{m,s})^T]^T \right)
 \end{aligned} \tag{2.35}$$

For the case with equal variance parameters, the proposed Gibbs sampling based approach as presented above can be used to explore the posterior PDF or for finding the optimal points for the posterior PDF by iteratively optimizing the conditional PDFs to find the estimates for the most probable model parameters for the case with equal variance parameters until convergence. For a Gaussian conditional PDF of $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$ and $\boldsymbol{\theta}_3$ shown in Sections 2.3.1, 2.3.2 and 2.3.3, respectively, given the other

parameters fixed at the most current values, the corresponding optimal point for θ_1 , θ_2 and θ_3 is given by the corresponding conditional mean given by Equation (2.28), with subscripts same as shown in the Equation, subscripts 1 and 2 interchanged, and subscripts 1 and 3 interchanged, respectively. The optimal point of an inverse Gamma conditional PDF of σ^2 as shown in Equation (2.35) is given by $\beta/(\alpha+1)$ where α and β are given by the first and the second numbers in the parenthesis in Equation (2.35), respectively. The most probable model parameter estimates obtained in the above way can be used as a starting point of the Markov chain in the proposed algorithm for the model with unequal variance parameters.

2.6 Numerical Issues

Computing the first two moments in Equations (2.28)-(2.29) requires matrices inversion. The size of these matrices can be huge for a high dimensional problem, so the inversion of these matrices is computationally expensive. A sequential way of computing the first two moments is adopted (Duncan and Horn, 1972; Sorenson, 1970) without the need to invert these huge matrices. Let the linear equations in Equation (2.26) be partitioned into n_G groups $\{I_1 \cdots I_{n_G}\}$ such that (the subscript 1 is dropped for brevity):

$$\begin{bmatrix} \mathbf{Y}_{I_1} \\ \vdots \\ \mathbf{Y}_{I_{n_G}} \end{bmatrix} - \begin{bmatrix} \mathbf{A}_{I_1} \\ \vdots \\ \mathbf{A}_{I_{n_G}} \end{bmatrix} \boldsymbol{\theta} = \begin{bmatrix} \mathbf{E}_{I_1} \\ \vdots \\ \mathbf{E}_{I_{n_G}} \end{bmatrix} \quad (2.36)$$

Then, the mean and covariance matrix of the full conditional PDF $p(\theta_1 | \theta_2, \theta_3, \theta_4, D)$ are given by repeating the following set of equation from $i = 0$ to $n_G - 1$:

$$\boldsymbol{\theta}^{(i+1)} = \boldsymbol{\theta}^{(i)} + \mathbf{P}^{(i)} \mathbf{A}_{I_{i+1}}^T \left(\boldsymbol{\Sigma}_{I_{i+1}} + \mathbf{A}_{I_{i+1}} \mathbf{P}^{(i)} \mathbf{A}_{I_{i+1}}^T \right)^{-1} \left(\mathbf{Y}_{I_{i+1}} - \mathbf{A}_{I_{i+1}} \boldsymbol{\theta}^{(i)} \right) \quad (2.37)$$

$$\mathbf{P}^{(i+1)} = \mathbf{P}^{(i)} - \mathbf{P}^{(i)} \mathbf{A}_{I_{i+1}}^T \left(\boldsymbol{\Sigma}_{I_{i+1}} + \mathbf{A}_{I_{i+1}} \mathbf{P}^{(i)} \mathbf{A}_{I_{i+1}}^T \right)^{-1} \mathbf{A}_{I_{i+1}} \mathbf{P}^{(i)} \quad (2.38)$$

where $\boldsymbol{\theta}^{(0)}$ and $\mathbf{P}^{(0)}$ are the prior mean and covariance matrix of uncertain parameters θ_1 , respectively, and $E(\theta_1 | \theta_2, \theta_3, \theta_4, D) = \boldsymbol{\theta}^{(n_G)}$ and

$V(\boldsymbol{\theta}_1 | \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D) = \mathbf{P}^{(n_G)}$. The above set of equations involves inversion of matrices with size equal to the size of the partitioned groups, with one for all the groups being a special case. Similar steps can be adopted for efficiently computing the mean and covariance matrix of the PDFs $p(\boldsymbol{\theta}_2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D)$ and $p(\boldsymbol{\theta}_3 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_4, D)$.

2.7 Illustrative Examples

To demonstrate the effectiveness and accuracy of the proposed method, two numerical examples are presented.

2.7.1 Example 1

The linear structure system selected for this illustrative example is modeled as a 4-DOF shear building as shown in Figure 2.1, with the following properties: mass $m_1 = 103,000 \text{ kg}$, $m_2 = 93,000 \text{ kg}$, $m_3 = 78,000 \text{ kg}$, $m_4 = 60,000 \text{ kg}$; spring stiffness $k_1 = 100,000 \text{ kN/m}$, $k_2 = 60,100 \text{ kN/m}$, $k_3 = 43,500 \text{ kN/m}$, $k_4 = 127,800 \text{ kN/m}$; damping coefficient for viscous dampers $c_1 = 900 \text{ kN-s/m}$, $c_2 = 600 \text{ kN-s/m}$, $c_3 = 400 \text{ kN-s/m}$, $c_4 = 1200 \text{ kN-s/m}$; and classical damping equal to 1% for all the modes. The modal data for the updating consist of five sets of modal data ($S = 5$) with the first two modal frequencies, modal damping ratios and partial complex mode shapes (corresponding to DOFs - one, two and four, $N_o = 3$) identified for each data set ($M = 2$). Noisy measured modal data are generated by adding to the exact frequencies, damping ratios and complex mode shapes components, random values chosen from zero-mean Gaussian distribution with standard deviation equal to 2% times the exact value.

For identification and uncertainty quantification, the same 4-DOF system is considered. The mass, stiffness and damping matrices for the model are parameterized as follows:

$$\mathbf{M}(\boldsymbol{\alpha}) = \sum_{i=1}^4 \alpha_i \mathbf{M}_i \quad (2.39)$$

$$\mathbf{K}(\boldsymbol{\eta}) = \sum_{i=1}^4 \eta_i \mathbf{K}_i \quad (2.40)$$

$$\mathbf{C}(\mathbf{c}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\eta}) = c_0 a_0 \mathbf{M}(\boldsymbol{\alpha}) + c_1 a_1 \mathbf{K}(\boldsymbol{\eta}) + \sum_{i=1}^4 \beta_i \mathbf{C}_i \quad (2.41)$$

where $\mathbf{M}_i$, $\mathbf{C}_i$ and $\mathbf{K}_i$ are the mass, damping and stiffness sub-matrices, respectively, defined for the i -th story. Rayleigh damping constants $[a_0 \ a_1]$ are determined using the modal frequencies and damping ratio of the first two modes of the nominal system. The uncertain parameters whose PDF to be updated for this model class are contribution parameters $[\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4]$, $[\beta_1 \ \beta_2 \ \beta_3 \ \beta_4]$, $[\eta_1 \ \eta_2 \ \eta_3 \ \eta_4]$, Rayleigh damping scaling parameters $[c_0 \ c_1]$, prediction error variances $[\sigma_{\text{Re},1}^2 \ \sigma_{\text{Im},1}^2 \ \sigma_{\text{Re},2}^2 \ \sigma_{\text{Im},2}^2]$ and complete complex mode shapes for the first two modes $[\psi_1 \ \psi_2]$.

Figure 2.1. The 4-DOF mechanical system

Independent normal distribution is assumed for masses, stiffnesses, damping coefficients for viscous dampers and Rayleigh damping parameters. Masses are assumed to be known with sufficient accuracy, thus the prior PDFs for

$[\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4]$ are chosen with mean values equal to 1 and the c.o.v for each equal to 1%. The prior mean values for viscous damping contribution parameters and stiffness contribution parameters $[\beta_1 \ \beta_2 \ \beta_3 \ \beta_4]$ and $[\eta_1 \ \eta_2 \ \eta_3 \ \eta_4]$ are chosen to be equal to 1, with prior c.o.v for each equal to 20%. The prior PDFs for the Rayleigh damping scaling parameters $[c_0 \ c_1]$ are chosen with mean values equal to 1 and the c.o.v for each equal to 20%. Additional uncertain parameters that are considered include prediction error variances $[\sigma_{\text{Re},1}^2 \ \sigma_{\text{Im},1}^2 \ \sigma_{\text{Re},2}^2 \ \sigma_{\text{Im},2}^2]$ and complete complex mode shapes $[\psi_1 \ \psi_2]$ for the first two modes. Flat independent priors are taken for $[\psi_1 \ \psi_2]$ and non-informative independent inverse gamma prior PDFs are taken for $[\sigma_{\text{Re},1}^2 \ \sigma_{\text{Im},1}^2 \ \sigma_{\text{Re},2}^2 \ \sigma_{\text{Im},2}^2]$ (Jeffreys' non-informative prior). The total number of uncertain parameters to be updated is 34 (14 for the contribution parameters, 16 for the two mode shapes, and 4 prediction error variances).

Following the proposed Gibbs sampling based algorithm, $N = 100,000$ posterior samples of contribution parameters, mode shapes and prediction errors variances are obtained. The burn-in period is less than 1000 samples. Table 2–1 shows some statistical properties of the posterior marginal PDFs of the contribution parameters estimated using the posterior samples after discarding initial 20% samples. It shows the nominal values $\hat{\theta}$ (column 2), posterior mean $\bar{\theta}$ (column 3), the posterior standard deviation σ (column 4), the posterior c.o.v (column 5) and the normalized distance β (column 6). The normalized distance β represents the absolute value of the difference between the posterior mean $\bar{\theta}$ and the exact value $\hat{\theta}$ of structural parameters, normalized with respect to the posterior standard deviation σ . It can be seen that the posterior mean values are close to those corresponding to the exact system and their c.o.v is much smaller than that what was initially assumed. The normalized distances are less than or around 1 suggesting that the posterior samples for the structural parameters are reasonable when compared with the exact values. The posterior mean estimates of prediction error variances are shown in Table 2–2. The prediction error variances are larger, in general, for the higher modes. Relatively large variance terms for the higher modes indicate that the errors associated have larger magnitudes and the contribution of the higher mode modal data to the final parameter estimation is relatively low. Table 2–3 shows the

posterior mean of mode shape components. Again the posterior mean are quite close to the exact values.

The simulation results stabilized at around 30,000 samples, as shown in Figure 2.2, where the mean and c.o.v estimates of the mass, stiffness and damping contribution parameters are reported as functions of the number of simulated samples. Figure 2.3 shows and posterior marginal PDFs fitted by kernel sampling density estimates for α_4 , β_4 and η_4 . A kernel distribution (Wand and Jones, 1994) is a nonparametric representation of the PDF of a random variable making use of samples distributed according to the probability distribution. In this study, kernel sampling density fitting function (ksdensity) available in Matlab is used to obtain the kernel sampling density estimates. For reference, the prior marginal PDFs are also shown. The prior and the posterior marginal PDF look very similar for α_4 , as the mass contribution parameters act as scaling constants and thus no information is provided by the data to update the uncertainty in these parameters. For β_4 and η_4 , it can be seen that the spread of the posterior PDF is small when compared to the spread of the prior PDFs, indicating the update in the distribution of uncertain parameters based on the modal data. Figure 2.3 also shows posterior samples and the posterior joint PDFs obtained by kernel sampling density estimates fitted to the posterior samples projected on a 2-dimensional space for pairs of , and . It can be seen that the posterior samples exhibit skewness and tail behavior not following jointly Gaussian or any standard distributions.

Table 2–1 SOME STATISTICAL PROPERTIES OF THE POSTERIOR MARGINAL PDFs OF THE CONTRIBUTION PARAMETERS

Parameter	$\hat{\theta}$	$\bar{\theta}$	Standard Deviation σ	c.o.v (%)	$\beta = \frac{ \bar{\theta} - \hat{\theta} }{\sigma}$
α_1	1.00	1.00	0.01	0.97	0.26
α_2	1.00	1.00	0.01	0.92	0.00
α_3	1.00	1.00	0.01	0.98	0.31
α_4	1.00	1.00	0.01	0.99	0.23
β_1	1.00	1.02	0.02	1.80	1.15
β_2	1.00	1.00	0.03	2.88	0.08
β_3	1.00	1.01	0.02	1.72	0.71
β_4	1.00	1.03	0.08	8.03	0.41
η_1	1.00	1.02	0.01	1.47	1.06
η_2	1.00	1.00	0.02	2.18	0.04
η_3	1.00	1.01	0.02	1.90	0.37
η_4	1.00	1.04	0.11	10.28	0.38
c_0	1.00	0.95	0.10	10.73	0.48
c_1	1.00	1.00	0.20	19.89	0.02

Table 2–2 POSTERIOR MEAN ESTIMATES OF PREDICTION ERROR VARIANCES

	Real ($\times 10^7$)	Imag ($\times 10^7$)
Mode1	0.2211	0.0007
Mode 2	1.5492	0.0262

Table 2–3 POSTERIOR MEAN ESTIMATES OF MODE SHAPE COMPONENTS

Mode	DOF	Nominal (abs, angle)		Posterior Mean (abs, angle)	
1	1	1	0	1	0
	2	2.4694	-0.0066	2.4956	-0.0066
	3	3.9035	-0.0050	3.9384	-0.0050
	4	4.1223	-0.0050	4.1530	-0.0049
2	1	1	0	1	0
	2	1.2699	-0.0104	1.284	-0.0103
	3	0.5605	3.1366	0.5650	3.1371
	4	0.9055	3.1323	0.9045	3.1322

Figure 2.2. Posterior mean and c.o.v estimates for mass, stiffness and damping contribution parameters as functions of the number of simulated samples, showing convergence after about 30,000 samples

Figure 2.3. Prior marginal PDFs and posterior marginal PDFs fitted by kernel sampling density estimates for α_4 , β_4 and η_4 , and posterior samples and contours for the posterior joint PDFs obtained by kernel sampling density estimates for pairs $\{\alpha_4, \beta_4\}$, $\{\alpha_4, \eta_4\}$ and $\{\beta_4, \eta_4\}$

2.7.2 Example 2

Figure 2.4. Diagram of modified benchmark structure

For the second example, a 120-DOF four-story, two-bay by two-bay steel frame originally designed for IASC-ASCE Phase-I Simulated Structural Health Monitoring Benchmark Problem (Johnson et al., 2004) is considered and modified as follows. It has $2.5 \text{ m} \times 2.5 \text{ m}$ plan and is 3.6 m tall. Braces in each story located on the exterior faces are removed and replaced by viscous dampers with damping coefficient 20 kN-s/m (as shown in Figure 2.4). In addition, classical damping with 1% damping ratio for all modes is assumed. The nominal properties of other structural members are shown in Table 2–4. The x-direction is the strong direction and each floor has 4 slabs: 800 kg slabs at the first level, 600 kg slabs at the second and third level, and 400 kg slabs at the fourth level.

The simulated modal data consist of two sets of modal data ($S = 2$) with the first eight translational modes ($M = 8$, four in the x-direction and four in the y-direction), each with four observed DOF ($N_o = 4$, corresponding to translational DOF at four

levels). Noisy measured data are generated by adding random values chosen from zero-mean Gaussian distribution with standard deviation equal to 2% of the exact values.

Table 2–4 PROPERTIES OF STRUCTURAL MEMBERS
(Johnson et al., 2004)

Property	Column	Beam
Section type	B100×9	S75×11
Cross-section area A (m ²)	1.133e-3	1.43e-3
Moment of inertia (strong direction) I_y (m ⁴)	1.97e-6	1.22e-6
Moment of inertia (weak direction) I_x (m ⁴)	0.664e-6	0.249e-6
St. Venant torsion constant J (m ⁴)	8.01e-9	38.2e-9
Youngs modulus E (Pa)	2e11	2e11
Shear modulus G (Pa)	$E/2.6$	$E/2.6$
Mass per unit volume ρ (kg/m ³)	7,800	7,800

For identification and uncertainty quantification, a 36-DOF model that assumes rigid floor in the x-y plane and allows rotations along x-axis and y-axis is used. Each floor has 2 in-plane DOFs (1 in x-direction and 1 in y-direction), 1 rotational DOF (in the z-axis), and 6 rotational DOFs (3 along x-axis and 3 along y-axis). Along the x-axis and y-axis, joints having the same x-coordinates or having the same y-coordinates are assumed to have the same amount of rotation along the x-axis and y-axis, respectively. Rayleigh damping constants $[a_0 \ a_1]$ are determined from the 36-DOF identification model using the modal frequencies for the 1st mode in x-direction and 4th mode in the y-direction, and damping ratio for each mode equal to 1%.

$$\mathbf{M}(\alpha) = \sum_{i=1}^4 \alpha_i \mathbf{M}_i \quad (2.42)$$

$$\mathbf{C}(\beta, \mathbf{a}, \alpha, \eta) = \sum_{i=1}^4 \beta_i \mathbf{C}_i + c_0 a_0 \mathbf{M}(\alpha) + c_1 a_1 \mathbf{K}(\eta) \quad (2.43)$$

$$\mathbf{K}(\eta) = \sum_{i=1}^4 (\eta_{c,i} \mathbf{K}_{c,i} + \eta_{Rx,i} \mathbf{K}_{Rx,i} + \eta_{Ry,i} \mathbf{K}_{Ry,i}) \quad (2.44)$$

The mass for each story is lumped at the floor level to give four uncertain mass parameters to be updated. Masses are assumed to be known with small uncertainty,

thus the prior PDF for $[\alpha_i : i = 1, \dots, 4]$ is assumed to be independent Gaussian with mean values equal to 1 and c.o.v equal to 1%. For each floor level, the stiffness matrix is defined using three prescribed substructure stiffness matrices, one related to 2 translation and 1 rotational DOF $\{\mathbf{K}_{c,i} : i = 1, \dots, 4\}$, and two related to 6 rotational DOFs $\{\mathbf{K}_{Rx,i}, \mathbf{K}_{Ry,i} : i = 1, \dots, 4\}$. In total, there are twelve uncertain stiffness parameters to be quantified for the whole structure. The prior PDF for $[\eta_{c,i} : i = 1, \dots, 4]$ and $[\eta_{Rx,i}, \eta_{Ry,i} : i = 1, \dots, 4]$ is assumed to be independent Gaussian with mean values equal to 1 and c.o.v equal to 10%. The prior PDF $[\beta_i : i = 1, \dots, 4]$ for is taken to be a product of independent Gaussian PDFs with mean value equal to 1 and c.o.v equal to 20%. The prior PDF for the Rayleigh damping parameters $[c_0, c_1]$ is taken to be a product of independent Gaussian PDFs with mean values equal to 1 and c.o.v for each equal to 20%. Flat independent priors are taken for mode shapes and a product of independent inverse gamma non-informative prior PDFs are taken for prediction error variances. Note that the assumed model is not equivalent to the one used to generate the modal data.

The total number of uncertain parameters whose PDF is to be updated is 614 (576 from 8 mode shapes, 4 mass contribution parameters, 12 stiffness contribution parameters, 4 non-classical damping contribution parameters, 2 classical damping coefficients and 16 prediction error variances). The initial point of the Markov chain for the model in which all prediction error variance parameters are allowed to be different is obtained by optimization (presented in Section 2.5) of the model where all the prediction error variance parameters are forced to be equal (only 1 prediction error variance parameter to be updated). The starting point is simulated from the prior PDFs of the uncertain parameters.

Following the proposed Gibbs sampling based algorithm, $N = 100,000$ samples of contribution parameters, mode shapes and prediction errors variances are obtained. Figure 2.5 shows the posterior samples for prediction error variances (shown together for all modes). It is observed that their patterns stabilize very quickly. After discarding initial 20% samples, the posterior mean estimates of prediction error variances are shown in Table 2–5. The prediction error variances are larger in

general for the higher modes. Relatively large variance terms for the higher modes indicate that the errors associated have larger magnitudes and the contribution of the higher mode modal data to the final parameter estimation is relatively low.

Figure 2.5. Prediction error variances (shown all together)

Table 2–5 POSTERIOR MEAN ESTIMATES OF PREDICTION ERROR VARIANCES

Data	X-direction		Y-direction	
	Real ($\times 10^{12}$)	Imag ($\times 10^{11}$)	Real ($\times 10^{12}$)	Imag ($\times 10^{11}$)
Mode 1	0.0026	0.0009	0.0017	0.0001
Mode 2	0.0338	0.0243	0.0019	0.0147
Mode 3	1.4634	0.9089	0.0870	0.1165
Mode 4	4.0895	5.9083	0.2717	0.4627

Table 2–6 shows some statistical properties of the posterior marginal PDFs of the contribution parameters with samples obtained from the proposed Gibbs sampling based algorithm. The interpretation of each column in this table is similar to

Table 2–1. The simulation results stabilized at around 30,000 samples, as shown in

Figure 2.6 and Figure 2.7 where the mean and c.o.v estimates of the damping and stiffness contribution parameters are reported as functions of the number of simulated samples.

Table 2–6 SOME STATISTICAL PROPERTIES OF THE POSTERIOR MARGINAL PDFs OF THE CONTRIBUTION PARAMETERS

Parameter	Posterior mean	Posterior standard deviation	c.o.v (%)
α_1	0.916	0.007	0.741
α_2	0.919	0.006	0.639
α_3	0.930	0.006	0.639
α_4	0.935	0.006	0.669
$\eta_{c,1}$	0.918	0.016	1.695
$\eta_{c,2}$	0.933	0.016	1.736
$\eta_{c,3}$	0.909	0.009	0.975
$\eta_{c,4}$	0.939	0.010	1.045
$\eta_{Rx,1}$	0.900	0.055	6.114
$\eta_{Rx,2}$	0.943	0.024	2.486
$\eta_{Rx,3}$	0.911	0.014	1.575
$\eta_{Rx,4}$	0.935	0.015	1.590
$\eta_{Ry,1}$	0.929	0.033	3.534
$\eta_{Ry,2}$	0.945	0.019	2.006
$\eta_{Ry,3}$	0.918	0.011	1.141
$\eta_{Ry,4}$	0.944	0.012	1.235
β_1	0.952	0.012	1.238
β_2	0.928	0.007	0.795
β_3	0.921	0.007	0.786
β_4	0.937	0.007	0.757
c_0	1.425	0.069	4.855
c_1	0.189	0.022	11.467

Figure 2.8 shows posterior marginal PDFs fitted by kernel sampling density estimates for $\eta_{c,1}$, $\eta_{Rx,1}$ and $\eta_{Ry,1}$. For reference, the prior marginal PDFs are also shown. It can be seen that the spread of the posterior marginal PDFs is small when compared to the spread of the corresponding prior marginal PDFs, indicating the update in the distribution of uncertain parameter based on the modal data. Posterior samples for pairs $\{\eta_{c,1}, \eta_{Rx,1}\}$, $\{\eta_{c,1}, \eta_{Ry,1}\}$ and $\{\eta_{Rx,1}, \eta_{Ry,1}\}$ and their posterior joint PDFs obtained by kernel sampling density estimates fitted to the posterior samples projected on a 2-dimensional space are also shown in Figure 2.8. Clearly the

posterior PDFs does not follow Gaussian distribution very well. Although the results are shown in a low-dimensional space, the high-dimensional probability information is encapsulated in the posterior samples.

High correlation is observed between the three stiffness contribution parameters in Figure 2.8. The modal data used in this example consist of natural frequencies, damping ratios and partial mode shapes corresponding to the two translational modes. In this case, the stiffness contribution parameters related to rotational DOFs are not updated directly by the modal data but indirectly based on the two translational modes. Thus, the three stiffness contribution parameters at any level are constrained by the corresponding components of the two translational modes at that level. The scaling of the three substructure stiffness matrices at each level needs to be maintained in order for the system mode shape to match the measured mode shapes in some sense. Therefore, the three stiffness contribution parameters at each level are correlated. (The scaling of different substructure stiffness matrices within the overall stiffness matrix affects the natural frequencies and mode shapes of a system. The scaling of the overall stiffness matrix affects the natural frequencies of a system and not its mode shapes.)

Figure 2.6. Posterior mean and c.o.v estimates for damping contribution parameters as functions of the number of simulated samples, showing convergence after about 30,000 samples

Figure 2.7. Posterior mean and c.o.v estimates for stiffness contribution parameters as functions of the number of simulated samples, showing convergence after about 30,000 samples

Figure 2.8. Prior marginal PDFs and posterior marginal PDFs fitted by kernel sampling density estimates for $\eta_{c,1}$, $\eta_{Rx,1}$ and $\eta_{Ry,1}$, and posterior samples and contours for the posterior joint PDFs obtained by kernel sampling density estimates for pairs $\{\eta_{c,1}, \eta_{Rx,1}\}$, $\{\eta_{c,1}, \eta_{Ry,1}\}$ and $\{\eta_{Rx,1}, \eta_{Ry,1}\}$

Chapter 3

Updating Robust Failure Probability of Dynamic System

3.1 Introduction

In this chapter, an efficient stochastic simulation method is presented for computing the updated robust failure probability (or its complement robust reliability) of a dynamic system using system data when the system is subjected to future stochastic excitation. Most vibration data of systems under investigation are obtained under low-amplitude excitation, e.g., ambient vibration. It can be assumed that during the time when the vibration data are collected, the system behaves (even damaged system) approximately linearly. Distribution of parameters related to linearly elastic behavior of the system can be updated using such data while distribution of parameters related to nonlinear behavior is usually not updated. The updated distribution of parameters related to linear behavior, e.g. initial stiffness, can provide important information about the damage to the system. The proposed approach for computing the updated robust failure probability consider updated distribution of parameters related to linear behavior and prior distribution of parameters related to nonlinear behavior of the system. In the proposed approach, the updating is based on incomplete modal data including modal frequencies, damping ratios and partial mode shapes of some of the dominant modes. The proposed approach integrates a newly-developed stochastic simulation method for Bayesian model updating of a linear dynamic system presented in Chapter 2 and a very efficient algorithm called Subset Simulation to compute small failure probabilities (Au and Beck, 2001a). The updated system is subsequently subjected

Parts of the work presented in this chapter appear in the conference papers (Cheung and Bansal, 2013b, 2014).

to future stochastic excitation during which a system may behave nonlinearly. Two new algorithms called Constrained Metropolis-within-Gibbs and Constrained Multi-group Metropolis-within-Gibbs sampling algorithms are proposed for efficient sampling from the conditional distribution, one for the case involving a linear dynamic system and one for the case involving a nonlinear dynamic system.

Let $\boldsymbol{\theta}_s \subset R^n$ denote the uncertain model parameter vector which is related to the linear structural dynamic behavior, whose PDF is to be updated by vibration data collected from the dynamic system of interest, and $\boldsymbol{\theta}_{nl} \subset R^{n_l}$ denote the uncertain model parameter vector whose PDF is not updated by the vibration data (e.g., those related to stochastic input excitation model, those related to the nonlinear structural dynamic behavior, etc.). $\boldsymbol{\theta}_s$ is specified by a model class $\mathcal{M}$, and its prior PDF is given by $p(\boldsymbol{\theta}_s | \mathcal{M})$. $\boldsymbol{\theta}_{nl}$ is specified by a model class $\mathcal{M}$ (for parameters related to the nonlinear structural dynamic behavior) and stochastic input model $\mathcal{U}$ (for parameters related to stochastic input model), and its prior PDF is given by $p(\boldsymbol{\theta}_{nl} | \mathcal{M}, \mathcal{U})$. Let $\mathbf{Y}(\boldsymbol{\theta}_s, \boldsymbol{\theta}_{nl}, \mathbf{Z}) \in R$ denote any output (performance) quantity of interest calculated by a dynamic model, specified by $\boldsymbol{\theta}_s$, $\boldsymbol{\theta}_{nl}$ and $\mathbf{Z}$. $\mathbf{Z} \in R^{N_t}$ is a vector of a finite number of independently and identically distributed (i.i.d) standard normal random variables that define the stochastic aspects of the input model. The robust failure probability is given by the following multi-dimensional integral with respect to $\boldsymbol{\theta}_s$, $\boldsymbol{\theta}_{nl}$ and $\mathbf{Z}$:

$$P(F | \mathcal{M}, \mathcal{U}) = \int I_F(\boldsymbol{\theta}_s, \boldsymbol{\theta}_{nl}, \mathbf{Z}) p(\boldsymbol{\theta}_s | \mathcal{M}) p(\boldsymbol{\theta}_{nl} | \mathcal{M}, \mathcal{U}) p(\mathbf{Z}) d\mathbf{Z} d\boldsymbol{\theta}_{nl} d\boldsymbol{\theta}_s \quad (3.1)$$

where F denotes failure, and I_F is an indicator function that is equal to 1 if $\mathbf{Y}$ exceeds the specified threshold and otherwise is equal to 0. For the case when the dynamic system behaves linearly and subjected to Gaussian Excitation (i.e., $\boldsymbol{\theta}_{nl}$ is an empty vector), its response can be expressed as a linear combination of $\mathbf{Z} \in \mathbb{R}^{N_t}$, and the robust failure probability is given by the following multi-dimensional integral with respect to $\boldsymbol{\theta}_s$ and $\mathbf{Z}$:

$$P(F | \mathcal{M}) = \int I_F(\boldsymbol{\theta}_s, \mathbf{Z}) p(\boldsymbol{\theta}_s | \mathcal{M}) p(\boldsymbol{\theta}_{nl} | \mathcal{M}, \mathcal{U}) p(\mathbf{Z}) d\mathbf{Z} d\boldsymbol{\theta}_s \quad (3.2)$$

Given the measurement data D from the system, the posterior (updated) PDF of $\boldsymbol{\theta}_s$ is given by the Bayes' theorem:

$$p(\boldsymbol{\theta}_s | D, \mathcal{M}) = \frac{p(D | \boldsymbol{\theta}_s, \mathcal{M})p(\boldsymbol{\theta}_s | \mathcal{M})}{p(D | \mathcal{M})} \quad (3.3)$$

where $p(D | \mathcal{M})$ is the normalizing constant which makes the probability volume under the posterior PDF equal to unity and $p(D | \boldsymbol{\theta}_s, \mathcal{M})$ is the likelihood function based on the predictive PDF of the response given by model class $\mathcal{M}$. The updated robust failure probability conditioned on the data is given by replacing the prior PDF $p(\boldsymbol{\theta}_s | \mathcal{M})$ in Equation (3.1) with the posterior PDF $p(\boldsymbol{\theta}_s | D, \mathcal{M})$ in Equation (3.3):

$$P(F | D, \mathcal{M}, \mathcal{U}) = \frac{\int I_F(\boldsymbol{\theta}_s, \boldsymbol{\theta}_{nl}, \mathbf{Z}) p(D | \boldsymbol{\theta}_s, \mathcal{M}) p(\boldsymbol{\theta}_s | \mathcal{M}) p(\boldsymbol{\theta}_{nl} | \mathcal{M}, \mathcal{U}) p(\mathbf{Z}) d\mathbf{Z} d\boldsymbol{\theta}_{nl} d\boldsymbol{\theta}_s}{\int p(D | \boldsymbol{\theta}_s, \mathcal{M}) p(\boldsymbol{\theta}_s | \mathcal{M}) d\boldsymbol{\theta}_s} \quad (3.4)$$

There are several difficulties in evaluating the above integral. It can be expected that the dimension of the above integral is high due to a large number of random variables involved and the failure region in $\boldsymbol{\theta}_s$, $\boldsymbol{\theta}_{nl}$ and $\mathbf{Z}$ space has complicated geometry, and thus it will be impossible to analytically evaluate the integral. For $P(F | D, \mathcal{M}, \mathcal{U}) \ll 1$, it is infeasible to evaluate the integrals in the numerator and denominator of Equation (3.4) by simulation based methods such as MCS or importance sampling since the high-probability content region of their corresponding integrands may occupy a much smaller volume than that of the prior joint PDF for $\boldsymbol{\theta}_s$, $\boldsymbol{\theta}_{nl}$ and $\mathbf{Z}$, i.e., $p(\boldsymbol{\theta}_s | \mathcal{M})p(\boldsymbol{\theta}_{nl} | \mathcal{M}, \mathcal{U})p(\mathbf{Z})$.

Over the past few years, several methods have been presented to tackle the aforementioned difficulties in evaluating the robust failure probability. Papadimitriou et al. (2001) presented an approach based on Laplace's asymptotic approximation to compute the robust failure probability. However, this approach can be computationally challenging in a high-dimensional parameter space and can be inaccurate when the Gaussian assumption for the posterior PDF is not valid for the global identifiable case. Beck and Au (2002) proposed a level-adaptive

Metropolis-Hastings algorithm with a global proposal PDF to obtain the samples from the posterior PDF and then use these samples to update the system reliability by evaluating the system reliability conditional on each of these samples. The approach will experience difficulty when the number of uncertain model parameters is large and is computationally inefficient because it requires multiple reliability analyses. Ching and Beck (2007) proposed a method to update the reliability based on combining a Kalman filter and smoother, and modifying the algorithm ISEE (Au and Beck 2001a). Such an approach is only applicable to linear systems with no uncertainties in model parameters. Ching and Hsieh (2006) proposed a method based on the Bayes' theorem and an analytical approximation of some of the required PDFs by maximum entropy PDFs. The method is applicable regardless of the number of uncertain model parameters but can only be applied to the case with very low-dimensional system output data. In practice, system data are of a very high dimension (say of the order of hundreds or thousands). Cheung and Beck (2007) proposed a stochastic simulation method which can handle general nonlinear dynamic system and the case with high-dimensional system output data but may encounter problems if the number of uncertain model parameters is huge. Jensen et al. (2013) integrated Bayesian probabilistic framework for model updating with advanced simulation tools for updating the robust reliability of a structural system using the dynamic data. However, the performance of the algorithm can deteriorate with an increase in the number of uncertain model parameters.

Let $D \equiv \{\hat{\omega}_{m,s}, \hat{\zeta}_{m,s}, \hat{\psi}_{m,s} : m = 1 \dots M, s = 1 \dots S\}$ be the experimentally obtained modal data from a linear structural dynamic system, consisting of modal frequencies $\hat{\omega}_{m,s} \in R^+$, damping ratios $\hat{\zeta}_{m,s} \in R^+$, and complex mode shape components $\hat{\psi}_{m,s} \in C^{N_o}$, where N_o is the number of measured DOFs, M is the number of observed modes and S is the number of modal data sets available. A brief review of the Bayesian model updating of a linear dynamic system and Subset Simulation is presented before proceeding to the proposed approach for computing the updated robust failure probability. For brevity, the conditioning on $\mathcal{M}$ and $\mathcal{U}$ will be left implicit in the rest of the chapter.

3.1.1 Bayesian Model Updating of a Linear Dynamic System Using Model Data

In the proposed Gibbs Sampling based algorithm for Bayesian model updating of a linear dynamic system presented in Chapter 2, four groups of parameters are considered:

$$\begin{aligned}\boldsymbol{\theta}_1 &= [\boldsymbol{\alpha}^T \ \boldsymbol{\beta}^T \ \boldsymbol{\eta}^T]^T \\ \boldsymbol{\theta}_2 &= \mathbf{c} \\ \boldsymbol{\theta}_3 &= [\text{Re}(\psi_1)^T \ \text{Im}(\psi_1)^T \ \cdots \ \text{Re}(\psi_M)^T \ \text{Im}(\psi_M)^T]^T \\ \boldsymbol{\theta}_4 &= [\sigma_{\text{Re},1}^2 \ \sigma_{\text{Im},1}^2 \ \cdots \ \sigma_{\text{Re},M}^2 \ \sigma_{\text{Im},M}^2]^T\end{aligned}$$

Assuming Bayesian conjugate priors for $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, $\boldsymbol{\theta}_3$ and $\boldsymbol{\theta}_4$, the full conditional PDFs $p(\boldsymbol{\theta}_1 | \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D)$, $p(\boldsymbol{\theta}_2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D)$ and $p(\boldsymbol{\theta}_3 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_4, D)$ are Gaussian, and $p(\sigma_{\text{Re},m}^2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, D)$ and $p(\sigma_{\text{Im},m}^2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, D)$ for $m=1, \dots, M$ are inverse gamma.

$$\begin{aligned}p(\boldsymbol{\theta}_1 | \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D) &\sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) \\ p(\boldsymbol{\theta}_2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D) &\sim N(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2) \\ p(\boldsymbol{\theta}_3 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_4, D) &\sim N(\boldsymbol{\mu}_3, \boldsymbol{\Sigma}_3) \\ p(\sigma_{\text{Re},m}^2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, D) &\sim IG(\rho_{4,m}, \kappa_{4,m}) \quad m=1, \dots, M \\ p(\sigma_{\text{Im},m}^2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, D) &\sim IG(\rho_{4,m}, \kappa_{4,m}) \quad m=1, \dots, M\end{aligned}$$

Using the full conditional distribution, the Gibbs sampler can be used to draw samples from the target PDF $p(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4 | D)$. Parameters $[\boldsymbol{\theta}_1 \ \boldsymbol{\theta}_2]$ correspond to the mass, damping and stiffness matrices and are required for structural dynamic analysis.

3.1.2 Original Subset Simulation

System reliability is concerned with the probability that the system will not reach some specific ‘failure’ or ‘unsatisfactory’ states subjected to stochastic excitations. It involves calculating the reliability, or its complement the failure probability $P(F)$, which requires the evaluation of a multi-dimensional integral of the form:

$$P(F) = \int I_F(\boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad (3.5)$$

where $\boldsymbol{\theta} \in R^{n_\theta}$ is the parameter vector containing all the uncertain quantities of interest quantified by joint PDF $p(\boldsymbol{\theta})$, and $I_F = 1$ if $\boldsymbol{\theta} \in F$ and $I_F = 0$ otherwise. Au and Beck (Au and Beck, 2001a, 2003) proposed Subset Simulation, a very efficient algorithm to compute the integral in Equation (3.5) for small failure probabilities. The basic idea of Subset Simulation is to subdivide a failure event into a sequence of H partial failure events (subsets) $F_1 \supset F_2 \supset \dots \supset F_H = F$. The division into subsets converts a rare event simulation problem into a problem of a sequence of more frequent events that are conditioned on failing at successively increasing threshold levels. This enables the computation of the failure probability as a product of conditional probabilities $\{P(F_{i+1} | F_i) : i = 0, 1, \dots, H\}$, where $P(F_1 | F_0) = P(F_1)$ and the target failure probability is given by the last element in the sequence:

$$P(F) = P(F_H) = \left[\prod_{i=1}^{H-1} P(F_{i+1} | F_i) \right] P(F_1) \quad (3.6)$$

$P(F_1)$ is estimated by simulating samples by MCS:

$$P(F_1) \approx \frac{1}{N} \sum_{k=1}^N I_{F_1}(\boldsymbol{\theta}^{(k)}) \quad (3.7)$$

where $\{\boldsymbol{\theta}^{(k)} : k = 1, \dots, N\}$ are i.i.d samples distributed as $p(\boldsymbol{\theta})$. Similarly, $P(F_{i+1} | F_i)$ is estimated using samples distributed as $p(\boldsymbol{\theta} | F_i)$:

$$P(F_{i+1} | F_i) \approx \frac{1}{N} \sum_{k=1}^N I_{F_{i+1}}(\boldsymbol{\theta}^{(i,k)}) \quad (3.8)$$

Samples $\{\boldsymbol{\theta}^{(i,k)} : k = 1, \dots, N\}$ satisfying the conditional PDF $p(\boldsymbol{\theta} | F_i)$ are generated by a MCMC simulation technique based on the modified Metropolis–Hastings method using samples distributed according to $p(\boldsymbol{\theta} | F_i)$ obtained from the previous simulation level as seeds.

The sequence of decreasing failure events $\{F_i, i = 1, \dots, H-1\}$, is determined in an adaptive way such that the conditional probabilities $P(F_{i+1} | F_i)$ are approximately equal to some pre-specified value $p_0 \in (0, 1)$. The efficiency of Subset Simulation

directly depends on the parameter p_0 , which governs the number of intermediate failure domains F_i needed to reach the target failure domain F . A very small value of p_0 means that fewer intermediate levels are needed to reach F but at the expense of a very large N needed at each level for accurate estimation of the small conditional probabilities $P(F_{i+1} | F_i)$. A large value of p_0 will result in more number of intermediate levels needed to reach F increasing the total number of samples and the bias in the failure probability estimates. Zuev et al. (2012) showed that choosing $0.1 \leq p_0 \leq 0.3$ will practically lead to similar efficiency and it is not necessary to fine tune the value of the conditional failure probability p_0 as long as Subset Simulation is implemented properly (Ching et al., 2005; Zuev et al., 2012).

3.1.3 Modified Metropolis-Hastings algorithm

Samples distributed according to PDF $p(\boldsymbol{\theta} | F_i)$ cannot be simulated directly; therefore a MCMC simulation based technique is used to generate $\{\boldsymbol{\theta}^{(i,k)} : k=1, \dots, N\}$ samples using those that lie in F_i available from the previous simulation level F_{i-1} . At the i -th level, $p_0 N$ (rounded to the closest integer) samples out of N samples distributed according to $p(\boldsymbol{\theta} | F_{i-1})$ are distributed according to $p(\boldsymbol{\theta} | F_i)$. Each of these $p_0 N$ samples as a starting point (seed) generate a Markov Chain of $(1/p_0 - 1)$ (rounded to the closest integer) samples according to target PDF $p(\boldsymbol{\theta} | F_i)$. In total, there are $p_0 N$ Markov Chains each of length $1/p_0$. The steps for the modified Metropolis-Hastings algorithm for generating samples with limiting stationary PDF equal to $p(\boldsymbol{\theta} | F_i)$ are as follows:

1. Assuming $\boldsymbol{\theta}$ is divided into G independent groups, i.e., $\boldsymbol{\theta} = [\boldsymbol{\theta}_j : j=1, \dots, G]$.

Given a current state $\boldsymbol{\theta}^{(k)}$, a candidate state $\tilde{\boldsymbol{\theta}}^{(k+1)}$ from the PDF $p(\boldsymbol{\theta})$ is generated using the Metropolis Hastings algorithm by repeating the following steps from $j=1$ till $j=G$:

- i) For the j -th group, draw a pre-candidate component $\xi_j^{(k+1)}$ from the proposal PDF $q_j(\xi_j | \boldsymbol{\theta}_j^{(k)})$ and compute the acceptance ratio $r_j^{(k+1)}$

$$r_j^{(k+1)} = \min \left\{ 1, \frac{p(\xi_j^{(k+1)}) q_j(\boldsymbol{\theta}_j^{(k)} | \xi_j^{(k+1)})}{p(\boldsymbol{\theta}_j^{(k)}) q_j(\xi_j^{(k+1)} | \boldsymbol{\theta}_j^{(k)})} \right\}$$

$$\text{ii) Set } \tilde{\boldsymbol{\theta}}_j^{(k+1)} = \begin{cases} \xi_j^{(k+1)} & \text{with probability } r_j^{(k+1)} \\ \boldsymbol{\theta}_j^{(k)} & \text{with probability } 1 - r_j^{(k+1)} \end{cases}$$

$$2. \quad \text{Set } \boldsymbol{\theta}^{(k+1)} = \begin{cases} \tilde{\boldsymbol{\theta}}^{(k+1)} & \text{if } \tilde{\boldsymbol{\theta}}^{(k+1)} \in F_i \\ \boldsymbol{\theta}^{(k)} & \text{if } \tilde{\boldsymbol{\theta}}^{(k+1)} \notin F_i \end{cases}$$

The efficiency of Subset Simulation directly depends on the choice of the proposal distributions used in the modified Metropolis-Hastings algorithm. It is observed that the efficiency is sensitive to the spread of the proposal distributions, rather than their type (Au and Beck, 2001a). Both small and large spreads tend to increase the dependence between successive samples. When the spread of the proposal distribution is large, a candidate sample is rejected frequently leading to a high positive correlation because of many repeated samples. On the other hand when the spread is very small, most of the candidate samples are accepted and because of proximity of the samples it leads to high correlation. Zuev et al. (2012) presented observations on the optimal scaling of modified Metropolis-Hastings algorithm for efficient exploration, and developed an optimal scaling strategy for this algorithm when it is employed within Subset Simulation.

3.2 The Proposed Approach

First consider the following updated robust failure probability integral:

$$P(F | D) = \int I_F(\boldsymbol{\theta}_s, \boldsymbol{\theta}_{nl}, \mathbf{Z}) p(\boldsymbol{\theta}_s | D) p(\boldsymbol{\theta}_{nl}) p(\mathbf{Z}) d\mathbf{Z} d\boldsymbol{\theta}_{nl} d\boldsymbol{\theta}_s \quad (3.9)$$

where the PDF of $\mathbf{Z}$ is not updated by the data D . The above integral can be approximated by the following estimator:

$$P(F | D) \approx \frac{1}{N} \sum_{k=1}^N I_F(\boldsymbol{\theta}_s^{(k)}, \boldsymbol{\theta}_{nl}^{(k)}, \mathbf{Z}^{(k)}) \quad (3.10)$$

where samples $\{\mathbf{Z}^{(k)} : k = 1, \dots, N\}$ and $\{\boldsymbol{\theta}_{nl}^{(k)} : k = 1, \dots, N\}$ are distributed as $p(\mathbf{Z})$ and $p(\boldsymbol{\theta}_{nl})$, respectively, and samples $\{\boldsymbol{\theta}_s^{(k)} : k = 1, \dots, N\}$ are distributed as

$p(\boldsymbol{\theta}_s | D)$. Samples distributed as $p(\mathbf{Z})$ and $p(\boldsymbol{\theta}_{nl})$ can be obtained directly. Samples distributed as $p(\boldsymbol{\theta}_s | D)$ are simulated using the Gibbs sampling based algorithm presented in Chapter 2. Even though the samples obtained using the Gibbs sampler are not independent, the Monte Carlo estimator for independent samples in Equation (3.10) can still be used. However, using Equation (3.10) will be computationally expensive especially when dealing with small failure probability as the number of samples N required to achieve a given c.o.v is inversely proportional to the failure probability. The Subset Simulation algorithm is adapted to compute the updated robust failure probability as follows:

$$P(F | D) = P(F_H | D) = \left[\prod_{i=1}^{H-1} P(F_{i+1} | F_i, D) \right] P(F_1 | D) \quad (3.11)$$

where $P(F_1 | D)$ is estimated by Equation (3.10) and $P(F_{i+1} | F_i, D)$ is estimated using samples $\{\boldsymbol{\theta}_s^{(i,k)}, \boldsymbol{\theta}_{nl}^{(i,k)}, \mathbf{Z}^{(i,k)}, k=1, \dots, N\}$ distributed according to the conditional PDF $p(\boldsymbol{\theta}_s, \boldsymbol{\theta}_{nl}, \mathbf{Z} | F_i, D)$ as follows:

$$P(F_{i+1} | F_i, D) \approx \frac{1}{N} \sum_{k=1}^N I_{F_{i+1}}(\boldsymbol{\theta}_s^{(i,k)}, \boldsymbol{\theta}_{nl}^{(i,k)}, \mathbf{Z}^{(i,k)}) \quad (3.12)$$

Two new algorithms called Constrained Metropolis-within-Gibbs and Constrained Multi-group Metropolis-within-Gibbs sampling algorithms are proposed for efficient sampling from the conditional PDF $p(\boldsymbol{\theta}_s, \boldsymbol{\theta}_{nl}, \mathbf{Z} | F_i, D)$, one for the case involving a linear dynamic system and one for the case involving a nonlinear dynamic system are discussed in the following sections. For the case when the dynamic system only behaves linearly when subjected to future stochastic excitation $\boldsymbol{\theta}_{nl}$ is an empty vector, i.e., $p(\boldsymbol{\theta}_s, \boldsymbol{\theta}_{nl}, \mathbf{Z} | F_i, D) = p(\boldsymbol{\theta}_s, \mathbf{Z} | F_i, D)$.

3.3 Sampling from $p(\boldsymbol{\theta}_s, \mathbf{Z} | F_i, D)$ for Linear Dynamic System

Markov Chain samples distributed according to the conditional PDF $p(\boldsymbol{\theta}_s, \mathbf{Z} | F_i, D)$ are generated by the new proposed algorithm called Constrained Metropolis-within-Gibbs sampling algorithm. Given the most recent sample $[\boldsymbol{\theta}_s^{(k)} \mathbf{Z}^{(k)}]$ from the conditional PDF $p(\boldsymbol{\theta}_s, \mathbf{Z} | F_i, D)$, the next Markov chain sample $[\boldsymbol{\theta}_s^{(k+1)} \mathbf{Z}^{(k+1)}]$ is simulated by Gibbs sampling: first simulate $\mathbf{Z}^{(k+1)}$

according to $p(\mathbf{Z} | \boldsymbol{\theta}_s = \boldsymbol{\theta}_s^{(k)}, F_i, D) = p(\mathbf{Z} | \boldsymbol{\theta}_s = \boldsymbol{\theta}_s^{(k)}, F_i)$ and then $\boldsymbol{\theta}_s^{(k+1)}$ from $p(\boldsymbol{\theta}_s | \mathbf{Z} = \mathbf{Z}^{(k+1)}, F_i, D)$.

3.3.1 Sampling from $p(\mathbf{Z} | \boldsymbol{\theta}_s, F_i, D)$

Since the data D do not provide any information which can update the PDF of $\mathbf{Z}$, $p(\mathbf{Z} | \boldsymbol{\theta}_s = \boldsymbol{\theta}_s^{(k)}, F_i, D) = p(\mathbf{Z} | \boldsymbol{\theta}_s = \boldsymbol{\theta}_s^{(k)}, F_i)$. For a linear dynamic system subjected to future Gaussian inputs, any response $\mathbf{Y}(k) \in \mathbb{R}^{N_Y}$ can be written as a linear function of standard normal vector $\mathbf{Z}$:

$$\mathbf{Y}(k) = \sum_{r=1}^{N_t} \mathbf{Y}^{(r)}(k) Z_r \quad (3.13)$$

Figure 3.1 Failure events $F^{(j)}$, and the corresponding design points in standard Gaussian space

where $\mathbf{Y}^{(r)}(k)$ is response vector at time t_k due to a unit impulse applied at time t_r and Z_r is the r -th component of $\mathbf{Z}$. For the first-passage reliability problem, the failure of the system is defined as $\mathbf{Y}(k) \notin [-b_i, b_i]$ for some $k = 1, \dots, N_t$, where b_i is the intermediate thresholds that defines the failure domain F_i defined earlier. The failure domain F_i can be expressed as a union of $2N_Y N_t$ failure events $F_i^{(jkl)}$, $j = 1, \dots, N_Y$, $k = 1, \dots, N_t$, $l = 1, 2$, i.e.,

$$F_i = \bigcup_{\forall j,k,l} F_i^{(jkl)} \quad (3.14)$$

Here $F_i^{(jkl)}$ denotes the event that the j -th response quantity of interest exceeds its upper ($l=1$) or lower ($l=2$) threshold at the k -th time instant. For each failure event $F_i^{(jkl)}$ for a given $\boldsymbol{\theta}_s$, the corresponding linear limit state function $g^{(jkl)}(\mathbf{Z}, \boldsymbol{\theta}_s)$ that separates the safe domain $g^{(jkl)}(\mathbf{Z}, \boldsymbol{\theta}_s) \geq 0$ and the failure domain $g^{(jkl)}(\mathbf{Z}, \boldsymbol{\theta}_s) < 0$ is given by:

$$g^{(jkl)}(\mathbf{Z}, \boldsymbol{\theta}_s) = \begin{cases} -\sum_{r=1}^{N_t} Y_j^{(r)}(k) Z_r + b_i & l = 1 \\ \sum_{r=1}^{N_t} Y_j^{(r)}(k) Z_r + b_i & l = 2 \end{cases} \quad (3.15)$$

where $Y_j^{(r)}(k)$ is the j -th component of $\mathbf{Y}^{(r)}(k)$. To simplify the notation, let a single index $u = 1, \dots, 2N_y N_t$ denote different combinations of j, k, l in Equation (3.15), and let the coefficients of the corresponding linear limit state function be denoted as $\mathbf{a}^{(u)}$ and $b^{(u)}$ such that:

$$g^{(u)}(\mathbf{Z}, \boldsymbol{\theta}_s) = \mathbf{A}^{(u)}(\boldsymbol{\theta}_s)^T \mathbf{Z} + b_i \quad (3.16)$$

Here, matrix $\mathbf{a}^{(u)}$ is fixed given fixed $\boldsymbol{\theta}_s$. The design point $\mathbf{Z}^{(u)*}(\boldsymbol{\theta}_s)$ (defined as the point on the plane $g^{(u)}(\mathbf{Z}, \boldsymbol{\theta}_s) = 0$ located closest to the origin) and its distance $\lambda^{(u)}$ from the origin are given by the following expressions (shown in Figure 3.1):

$$\mathbf{Z}^{(u)*} = -\frac{b_i}{\|\mathbf{a}^{(u)}(\boldsymbol{\theta}_s)\|^2} \mathbf{a}^{(u)}(\boldsymbol{\theta}_s) \quad (3.17)$$

$$\lambda^{(u)} = \frac{b_i}{\|\mathbf{a}^{(u)}(\boldsymbol{\theta}_s)\|} \quad (3.18)$$

where $\|\mathbf{a}^{(u)}(\boldsymbol{\theta}_s)\| = \sqrt{\mathbf{a}^{(u)}(\boldsymbol{\theta}_s)^T \mathbf{a}^{(u)}(\boldsymbol{\theta}_s)}$ denote the Euclidean norm of the vector $\mathbf{a}^{(u)}(\boldsymbol{\theta}_s)$.

Given a sample $\mathbf{Z}^{(k)}$ distributed according to the conditional PDF $p(\mathbf{Z} | \boldsymbol{\theta}_s, F_i)$, a new sample $\mathbf{Z}^{(k+1)}$ also distributed according to the conditional PDF $p(\mathbf{Z} | \boldsymbol{\theta}_s, F_i)$

can be simulated using Metropolis-Hastings algorithm with the following distribution as the proposal PDF:

$$q(\mathbf{Z} | \boldsymbol{\theta}_s) = \sum_{u=1}^{2N_Y N_t} w_u p(\mathbf{Z}^{(k)} | \boldsymbol{\theta}_s, F_i^{(u)}) \quad (3.19)$$

where

$$w_u = \frac{P(F_i^{(u)})}{\sum_{u=1}^{2N_Y N_t} P(F_i^{(u)})} = \frac{\Phi(-\lambda^{(u)})}{\sum_{u=1}^{2N_Y N_t} \Phi(-\lambda^{(u)})} \quad (3.20)$$

and where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of a standard normal random variable. The candidate sample $\tilde{\mathbf{Z}}$ is accepted with acceptance probability $\min(1, a_0)$, where a_0 is given by:

$$\begin{aligned} a_0 &= \frac{p(\tilde{\mathbf{Z}} | \boldsymbol{\theta}_s, F_i)}{p(\mathbf{Z}^{(k)} | \boldsymbol{\theta}_s, F_i)} \frac{q(\mathbf{Z}^{(k)} | \boldsymbol{\theta}_s)}{q(\tilde{\mathbf{Z}} | \boldsymbol{\theta}_s)} \\ &= \frac{\frac{p(\tilde{\mathbf{Z}} | \boldsymbol{\theta}_s) I_{F_i}(\tilde{\mathbf{Z}}, \boldsymbol{\theta}_s)}{P(F_i)}}{\frac{p(\mathbf{Z}^{(k)} | \boldsymbol{\theta}_s) I_{F_i}(\mathbf{Z}^{(k)}, \boldsymbol{\theta}_s)}{P(F_i)}} \frac{q(\mathbf{Z}^{(k)} | \boldsymbol{\theta}_s)}{q(\tilde{\mathbf{Z}} | \boldsymbol{\theta}_s)} \\ &= \frac{p(\tilde{\mathbf{Z}} | \boldsymbol{\theta}_s)}{p(\mathbf{Z}^{(k)} | \boldsymbol{\theta}_s)} \frac{\sum_{u=1}^{2N_Y N_t} w_u p(\mathbf{Z}^{(k)} | \boldsymbol{\theta}_s, F_i^{(u)})}{\sum_{u=1}^{2N_Y N_t} w_u p(\tilde{\mathbf{Z}} | \boldsymbol{\theta}_s, F_i^{(u)})} \\ &= \frac{p(\tilde{\mathbf{Z}} | \boldsymbol{\theta}_s)}{p(\mathbf{Z}^{(k)} | \boldsymbol{\theta}_s)} \frac{\sum_{u=1}^{2N_Y N_t} \frac{P(F_i^{(u)})}{\sum_{u=1}^{2N_Y N_t} P(F_i^{(u)})} \frac{P(F_i^{(u)} | \mathbf{Z}^{(k)}, \boldsymbol{\theta}_s) p(\mathbf{Z}^{(k)} | \boldsymbol{\theta}_s)}{P(F_i^{(u)})}}{\sum_{u=1}^{2N_Y N_t} \frac{P(F_i^{(u)})}{\sum_{u=1}^{2N_Y N_t} P(F_i^{(u)})} \frac{P(F_i^{(u)} | \tilde{\mathbf{Z}}, \boldsymbol{\theta}_s) p(\tilde{\mathbf{Z}} | \boldsymbol{\theta}_s)}{P(F_i^{(u)})}} \\ &= \frac{\sum_{u=1}^{2N_Y N_t} P(F_i^{(u)} | \mathbf{Z}^{(k)}, \boldsymbol{\theta}_s)}{\sum_{u=1}^{2N_Y N_t} P(F_i^{(u)} | \tilde{\mathbf{Z}}, \boldsymbol{\theta}_s)} \end{aligned}$$

$$= \sum_{u=1}^{2N_Y N_t} I_{F_i^{(u)}}(\mathbf{Z}^{(k)}, \boldsymbol{\theta}_s) \bigg/ \sum_{u=1}^{2N_Y N_t} I_{F_i^{(u)}}(\tilde{\mathbf{Z}}, \boldsymbol{\theta}_s) \quad (3.21)$$

The PDF in Equation (3.19) was proposed by Au and Beck (2001b) to be used as the importance sampling density for calculating the failure probability of linear dynamic system subjected Gaussian excitations with no uncertainty in the structural models. An important sampling density such as this was proposed by Au (2004) to be used as a proposal PDF to generate conditional failure samples using Metropolis-Hastings algorithm. Candidate samples $\tilde{\mathbf{Z}}$ from the above proposal PDF can be efficiently simulated using procedures as shown in Au and Beck (2001b) and Katafygiotis and Cheung (2006). $\mathbf{Z}^{(k+1)} = \tilde{\mathbf{Z}}$ if $\tilde{\mathbf{Z}}$ is accepted, otherwise $\mathbf{Z}^{(k+1)} = \mathbf{Z}^{(k)}$. More details to simulate samples from the proposal PDF $p(\mathbf{Z} | \boldsymbol{\theta}_s, F_i)$ are included in the Appendix 3.A.

3.3.2 Sampling from $p(\boldsymbol{\theta}_s | \mathbf{Z}, F_i, D)$

If samples from PDF $p(\boldsymbol{\theta} | \mathbf{Z}, F_i, D)$ are available, samples corresponding to $\boldsymbol{\theta}_s = [\boldsymbol{\theta}_1 \ \boldsymbol{\theta}_2]$ from these samples will be distributed according to the conditional PDF $p(\boldsymbol{\theta}_s | \mathbf{Z}, F_i, D)$. The full conditional PDFs for the four groups of parameter vectors conditioned on F_i are equal to:

$$p(\boldsymbol{\theta}_1 | F_i, \mathbf{Z}, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D) \propto I_{F_i}(\mathbf{Z}, [\boldsymbol{\theta}_1, \boldsymbol{\theta}_2]) p(\boldsymbol{\theta}_1 | \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D) \quad (3.22)$$

$$p(\boldsymbol{\theta}_2 | F_i, \mathbf{Z}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D) \propto I_{F_i}(\mathbf{Z}, [\boldsymbol{\theta}_1, \boldsymbol{\theta}_2]) p(\boldsymbol{\theta}_2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D) \quad (3.23)$$

$$p(\boldsymbol{\theta}_3 | F_i, \mathbf{Z}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_4, D) \propto I_{F_i}(\mathbf{Z}, [\boldsymbol{\theta}_1, \boldsymbol{\theta}_2]) p(\boldsymbol{\theta}_3 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_4, D) \quad (3.24)$$

$$p(\boldsymbol{\theta}_4 | F_i, \mathbf{Z}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, D) \propto I_{F_i}(\mathbf{Z}, [\boldsymbol{\theta}_1, \boldsymbol{\theta}_2]) p(\boldsymbol{\theta}_4 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, D) \quad (3.25)$$

According the above, samples distributed according to the conditional PDF $p(\boldsymbol{\theta}_s | \mathbf{Z}, F_i, D)$ can be obtained using the proposed Constrained Metropolis-within-Gibbs sampling technique starting from a sample already distributed according to the conditional PDF $p(\boldsymbol{\theta}_s | \mathbf{Z}, F_i, D)$. Simulation of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ requires the Metropolis-Hastings step that involves dynamic analysis. However, instead of performing the dynamic analysis (checking for failure) at both steps, it is proposed to first simulate a candidate sample for both $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ and later check for failure

once. Simulation of θ_3 and θ_4 neither requires Metropolis-Hastings step nor any dynamic analysis and can be simulated directly from their full conditional PDFs.

Given the most recent state $\theta^{(k)} = [\theta_1^{(k)} \ \theta_2^{(k)} \ \theta_3^{(k)} \ \theta_4^{(k)}]$ already distributed according to PDF $p(\theta | \mathbf{Z}, F_i, D)$, a candidate state $\tilde{\theta}_1^{(k+1)}$ is simulated by making use of the characteristics that $p(\theta_1 | \theta_2, \theta_3, \theta_4, D)$ is a Gaussian PDF. The mean vector $\mu_1^{(k)} = \mu_1^{(k)}(\theta_2^{(k)}, \theta_3^{(k)}, \theta_4^{(k)})$ and covariance matrix $\Sigma_1^{(k)} = \Sigma_1^{(k)}(\theta_2^{(k)}, \theta_3^{(k)}, \theta_4^{(k)})$ of the PDF $p(\theta_1 | \theta_2, \theta_3, \theta_4, D)$ are given by Equations (2.28) and (2.29), respectively. To simulate a candidate state $\tilde{\theta}_1^{(k+1)} = [\tilde{\theta}_{1,j}^{(k+1)} : j=1 \dots N_1]$ (conditioned on all the other parameters and the data), instead of using an N_1 -dimensional proposal PDF to directly obtain candidate state $\tilde{\theta}_1^{(k+1)}$, we propose a sequence of one-dimensional proposals PDFs $\{q_{1,j}(\xi_{1,j} | w_{1,j}^{(k)}) : j=1, \dots, N_1\}$. $\mathbf{w}_1^{(k)} = [w_{1,j}^{(k)} : j=1, \dots, N_1]$ is a set of independent standard normal variables obtained by the following linear transformation $N_{\theta_1}(\mu_1^{(k)}, \Sigma_1^{(k)}) \rightarrow N_{\mathbf{w}_1}(\mathbf{0}, \mathbf{I})$:

$$\mathbf{w}_1^{(k)} = \mathbf{L}_1^{(k)-1} (\theta_1^{(k)} - \mu_1^{(k)}) \quad (3.26)$$

where $\mathbf{L}_1^{(k)}$ is obtained from the Cholesky decomposition of $\Sigma_1^{(k)}$, i.e., $\Sigma_1^{(k)} = \mathbf{L}_1^{(k)} \mathbf{L}_1^{(k)T}$. Each component $\xi_{1,j}^{(k+1)}$ of $\tilde{\mathbf{w}}_1^{(k+1)}$ is generated separately using a one-dimensional proposal PDF $q_{1,j}(\xi_{1,j} | w_{1,j}^{(k)})$ dependent on the j -th component $w_{1,j}^{(k)}$. Following which $\tilde{\mathbf{w}}_1^{(k+1)}$ is transformed back to θ_1 space by:

$$\tilde{\theta}_1^{(k+1)} = \mathbf{L}_1^{(k)} \tilde{\mathbf{w}}_1^{(k+1)} + \mu_1^{(k)} \quad (3.27)$$

The same procedure is adopted to generate a candidate state $\tilde{\theta}_2^{(k+1)}$ given $[\tilde{\theta}_1^{(k+1)} \ \theta_2^{(k)} \ \theta_3^{(k)} \ \theta_4^{(k)}]$. Finally, if the generated candidate state for structural parameters $\tilde{\theta}_s^{(k+1)} = [\tilde{\theta}_1^{(k+1)} \ \tilde{\theta}_2^{(k+1)}]$ belong to the failure domain (given $\mathbf{Z}^{(k+1)}$), they are accepted as the new sample. Using the most recent samples $\theta_1^{(k+1)}$ and $\theta_2^{(k+1)}$, $\theta_3^{(k+1)}$ and $\theta_4^{(k+1)}$ are simulated from $p(\theta_3 | \theta_1^{(k+1)}, \theta_2^{(k+1)}, \theta_4^{(k)}, D)$ and

$p(\boldsymbol{\theta}_4 | \boldsymbol{\theta}_1^{(k+1)}, \boldsymbol{\theta}_2^{(k+1)}, \boldsymbol{\theta}_3^{(k+1)}, D)$, respectively, using the procedures described in Sections 2.3.3 and 2.3.4.

3.3.3 Summary of steps for simulating samples according to $p(\boldsymbol{\theta}_s, \mathbf{Z} | F_i, D)$

For level zero of the proposed approach, samples $\{\mathbf{Z}^{(k)} : k = 1, \dots, N\}$ are directly simulated from the PDF $p(\mathbf{Z})$ and samples $\{\boldsymbol{\theta}_s^{(k)} : k = 1, \dots, N\}$ are simulated from the PDF $p(\boldsymbol{\theta} | D)$ using the Gibbs sampling based approach presented in Chapter 2 after discarding the samples from the burn-in period. For the i -th simulation level, $p_0 N$ samples distributed according to $p(\boldsymbol{\theta}_s, \mathbf{Z} | F_i, D)$ obtained from the $(i-1)$ -th simulation level are used as seed samples to generate Markov chains to obtain additional $N(1 - p_0)$ samples which are also distributed according to the conditional PDF $p(\boldsymbol{\theta}_s, \mathbf{Z} | F_i, D)$. The following steps are repeated for each seed sample:

1. Initialize a Markov chain, use a seed sample as the starting point and let $k = 1$.
2. Sample $\mathbf{Z}^{(k+1)}$ from $p(\mathbf{Z} | \boldsymbol{\theta}_s^{(k)}, F_i, D)$.

- i) Draw a candidate state $\tilde{\mathbf{Z}}$ from the proposal distribution $q(\mathbf{Z} | \boldsymbol{\theta}_s^{(k)})$ and compute the acceptance ratio:

$$r_{\mathbf{Z}}^{(k+1)} = \min \left\{ 1, \frac{\sum_{j=1}^{2N_y N_t} I_{F_i(j)}(\mathbf{Z}^{(k)}, \boldsymbol{\theta}_s^{(k)})}{\sum_{j=1}^{2N_y N_t} I_{F_i(j)}(\tilde{\mathbf{Z}}, \boldsymbol{\theta}_s^{(k)})} \right\}$$

- ii) Set $\mathbf{Z}^{(k+1)} = \begin{cases} \tilde{\mathbf{Z}} & \text{with probability } r_{\mathbf{Z}}^{(k+1)} \\ \mathbf{Z}^{(k)} & \text{with probability } 1 - r_{\mathbf{Z}}^{(k+1)} \end{cases}$.

Note: Impulse response function for linear dynamic system model with parameters $\boldsymbol{\theta}_s^{(k)}$ is already available.

3. Sample $\boldsymbol{\theta}^{(k+1)}$ from $p(\boldsymbol{\theta}^{(k+1)} | \mathbf{Z}^{(k+1)}, F_i, D)$ by:
 - i) Determine the mean vector $\boldsymbol{\mu}_1^{(k)}$ and covariance matrix $\boldsymbol{\Sigma}_1^{(k)}$ of the Gaussian PDF $p(\boldsymbol{\theta}_1 | \boldsymbol{\theta}_2^{(k)}, \boldsymbol{\theta}_3^{(k)}, \boldsymbol{\theta}_4^{(k)}, D)$ using Equations (2.28) and (2.29) respectively. Let $\mathbf{w}_1^{(k)} = \mathbf{L}_1^{(k)-1}(\boldsymbol{\theta}_1^{(k)} - \boldsymbol{\mu}_1^{(k)})$ where $\boldsymbol{\Sigma}_1^{(k)} = \mathbf{L}_1^{(k)} \mathbf{L}_1^{(k)T}$.
 - ii) Generate a candidate state $\tilde{\mathbf{w}}_1^{(k+1)}$ for $\mathbf{w}_1$: For each component

$j=1, \dots, N_1$, draw a pre-candidate component $\xi_{1,j}^{(k+1)}$ from proposal PDF

$q_{1,j}(\xi_{1,j} | w_{1,j}^{(k)})$ and compute the acceptance ratio:

$$r_{1,j}^{(k+1)} = \min \left\{ 1, \frac{p(\xi_{1,j}^{(k+1)}) q_{1,j}(w_{1,j}^{(k)} | \xi_{1,j}^{(k+1)})}{p(w_{1,j}^{(k)}) q_{1,j}(\xi_{1,j}^{(k+1)} | w_{1,j}^{(k)})} \right\}$$

$$\text{Set } \tilde{w}_{1,j}^{(k+1)} = \begin{cases} \xi_{1,j}^{(k+1)} & \text{with probability } r_{1,j}^{(k+1)} \\ w_{1,j}^{(k)} & \text{with probability } 1 - r_{1,j}^{(k+1)} \end{cases}$$

iii) Transform $\tilde{\mathbf{w}}_1^{(k+1)}$ back to $\boldsymbol{\theta}_1$ space: $\tilde{\boldsymbol{\theta}}_1^{(k+1)} = \boldsymbol{\mu}_1^{(k)} + \mathbf{L}_1^{(k)} \tilde{\mathbf{w}}_1^{(k+1)}$.

iv) For getting $\tilde{\boldsymbol{\theta}}_2^{(k+1)}$, repeat steps i) to iii) by reversing the subscripts 1 and 2 and replacing $\boldsymbol{\theta}_1^{(k)}$ by $\tilde{\boldsymbol{\theta}}_1^{(k+1)}$.

v) Set

$$\boldsymbol{\theta}_s^{(k+1)} = [\boldsymbol{\theta}_1^{(k+1)} \ \boldsymbol{\theta}_2^{(k+1)}] = \begin{cases} [\tilde{\boldsymbol{\theta}}_1^{(k+1)} \ \tilde{\boldsymbol{\theta}}_2^{(k+1)}] & \text{if } \mathbf{Y}([\tilde{\boldsymbol{\theta}}_1^{(k+1)} \ \tilde{\boldsymbol{\theta}}_2^{(k+1)}], \mathbf{Z}^{(k+1)}) \in F \\ [\boldsymbol{\theta}_1^{(k)} \ \boldsymbol{\theta}_2^{(k)}] & \text{if } \mathbf{Y}([\tilde{\boldsymbol{\theta}}_1^{(k+1)} \ \tilde{\boldsymbol{\theta}}_2^{(k+1)}], \mathbf{Z}^{(k+1)}) \notin F \end{cases}$$

vi) Sample $\boldsymbol{\theta}_3^{(k+1)}$ from $p(\boldsymbol{\theta}_3 | \boldsymbol{\theta}_1^{(k+1)}, \boldsymbol{\theta}_2^{(k+1)}, \boldsymbol{\theta}_4^{(k)}, D)$ as described in Section 2.3.3.

vii) Sample $\boldsymbol{\theta}_4^{(k+1)}$ from $p(\boldsymbol{\theta}_4 | \boldsymbol{\theta}_1^{(k+1)}, \boldsymbol{\theta}_2^{(k+1)}, \boldsymbol{\theta}_3^{(k+1)}, D)$ as described in Section 2.3.4.

4. Let $k=k+1$ and go to step 2, until $(1/p_0-1)$ samples are obtained.

The proof of correctness of generating samples following the above procedures is given in the Appendix 3.B.

3.4 Sampling from $p(\boldsymbol{\theta}_s, \boldsymbol{\theta}_{nl}, \mathbf{Z} | F_i, D)$ for Nonlinear Dynamic System

Markov Chain samples distributed according to the conditional PDF $p(\boldsymbol{\theta}_s, \boldsymbol{\theta}_{nl}, \mathbf{Z} | F_i, D)$ are generated using the Constrained Multi-group Metropolis-within-Gibbs sampling algorithms by first generating a candidate state $[\tilde{\boldsymbol{\theta}}_s \ \tilde{\boldsymbol{\theta}}_{nl} \ \tilde{\mathbf{Z}}]$ using the current state $[\boldsymbol{\theta}_s^{(k)} \ \boldsymbol{\theta}_{nl}^{(k)} \ \mathbf{Z}^{(k)}]$ and later accepting the candidate state if it leads to F_i (i.e., $[\boldsymbol{\theta}_s^{(k+1)} \ \boldsymbol{\theta}_{nl}^{(k+1)} \ \mathbf{Z}^{(k+1)}] = [\tilde{\boldsymbol{\theta}}_s \ \tilde{\boldsymbol{\theta}}_{nl} \ \tilde{\mathbf{Z}}]$), otherwise, the current state is taken as the next state (i.e., $[\boldsymbol{\theta}_s^{(k+1)} \ \boldsymbol{\theta}_{nl}^{(k+1)} \ \mathbf{Z}^{(k+1)}] = [\boldsymbol{\theta}_s^{(k)} \ \boldsymbol{\theta}_{nl}^{(k)} \ \mathbf{Z}^{(k)}]$).

$$p(\boldsymbol{\theta}_s, \boldsymbol{\theta}_{nl}, \mathbf{Z} | F_i, D) = \frac{I_{F_i}(\mathbf{Z}, \boldsymbol{\theta}_{nl}, \boldsymbol{\theta}_s) p(\mathbf{Z}) p(\boldsymbol{\theta}_{nl}) p(\boldsymbol{\theta}_s | D)}{P(F | D)} \quad (3.28)$$

3.4.1 Candidate state for $\mathbf{Z}$ and $\boldsymbol{\theta}_{nl}$

The candidate state $\tilde{\mathbf{Z}} = [\tilde{Z}_1 \ \tilde{Z}_2 \ \cdots \ \tilde{Z}_{N_t}]^T$ for $\mathbf{Z} = [Z_1 \ Z_2 \ \cdots \ Z_{N_t}]^T$ is simulated by individually simulating each components of the candidate state to avoid high-dimensional problems. Given a current state $Z_j^{(k)}$, a candidate state $\tilde{Z}_j$ is simulated using the Metropolis-Hastings algorithm with one-dimensional adaptive symmetric proposal PDF centered at the current state.

The candidate state for $\boldsymbol{\theta}_{nl}$ can be generated either using one-dimensional proposal for each component of $\boldsymbol{\theta}_{nl}$, or can be generated at once by using a multi-dimensional proposal. However, the later should be used only when the number of uncertain parameters is not large, for otherwise high dimensional problems may occur as the number of uncertain parameters increases. Markov chain samples from the last simulation level that lie in the failure region F_i distributed according to the conditional PDF $p(\boldsymbol{\theta}_{nl} | F_i)$ may be utilized to construct the proposal PDFs.

3.4.2 Candidate state for $\boldsymbol{\theta}_s$

The candidate state for $\boldsymbol{\theta}_s$ is generated using the procedures presented in Section 3.3.2.

3.4.3 Summary of steps for simulating samples according to $p(\boldsymbol{\theta}_s, \boldsymbol{\theta}_{nl}, \mathbf{Z} | F_i, D)$

For level zero of the proposed approach, samples $\{\mathbf{Z}^{(k)} : k = 1, \dots, N\}$ and $\{\boldsymbol{\theta}_{nl}^{(k)} : k = 1, \dots, N\}$ are directly simulated from their prior PDFs $p(\mathbf{Z})$ and $p(\boldsymbol{\theta}_{nl})$, respectively. Samples $\{\boldsymbol{\theta}_s^{(k)} : k = 1, \dots, N\}$ are simulated from the PDF $p(\boldsymbol{\theta} | D)$ using the Gibbs sampling based approach presented in Chapter 2 after discarding the samples from the burn-in period. For the i -th simulation level, $p_0 N$ samples distributed according to $p(\boldsymbol{\theta}_s, \boldsymbol{\theta}_{nl}, \mathbf{Z} | F_i, D)$ obtained from the $(i-1)$ -th simulation level are used as seed samples to generate Markov chains to obtain additional

$N(1-p_0)$ samples which are also distributed according to the conditional PDF $p(\boldsymbol{\theta}_s, \boldsymbol{\theta}_{nl}, \mathbf{Z} | F_i, D)$. The following steps are repeated for each seed sample:

1. Initialize a Markov chain, use a seed sample as the starting points and let $k=1$.

2. Given a current state $\mathbf{Z}^{(k)}$, a candidate state $\tilde{\mathbf{Z}}$ from the PDF $p(\mathbf{Z})$ is generated using the Metropolis-Hastings algorithm by repeating the following steps from $j=1$ till $j=N_t$:

- i) For the j -th group, draw a pre-candidate component $\xi_{\mathbf{Z},j}^{(k+1)}$ from the proposal PDF $q_{\mathbf{Z},j}$ and compute the following acceptance ratio:

$$r_{\mathbf{Z},j}^{(k+1)} = \min \left\{ 1, \frac{p(\xi_{\mathbf{Z},j}^{(k+1)}) q_{\mathbf{Z},j}(\mathbf{Z}_j^{(k)} | \xi_{\mathbf{Z},j}^{(k+1)})}{p(\mathbf{Z}_j^{(k)}) q_{\mathbf{Z},j}(\xi_{\mathbf{Z},j}^{(k+1)} | \mathbf{Z}_j^{(k)})} \right\}$$

- ii) Set $\tilde{\mathbf{Z}}_j^{(k+1)} = \begin{cases} \xi_{\mathbf{Z},j}^{(k+1)} & \text{with probability } r_{\mathbf{Z},j}^{(k+1)} \\ \mathbf{Z}_j^{(k)} & \text{with probability } 1 - r_{\mathbf{Z},j}^{(k+1)} \end{cases}$.

3. Given a current state $\boldsymbol{\theta}_{nl}^{(k)}$ where $\boldsymbol{\theta}_{nl} = [\boldsymbol{\theta}_{nl,j} : j=1, \dots, G]$, a candidate state $\tilde{\boldsymbol{\theta}}_{nl}$ is generated using the Metropolis-Hastings algorithm by repeating the following steps from $j=1$ till $j=G$:

- i) For the j -th group, draw a pre-candidate component $\xi_{nl,j}^{(k+1)}$ from the proposal PDF $q_{nl,j}$ and compute the following acceptance ratio

$$r_{nl,j}^{(k+1)} = \min \left\{ 1, \frac{p(\xi_{nl,j}^{(k+1)}) q_{nl,j}(\boldsymbol{\theta}_{nl,j}^{(k)} | \xi_{nl,j}^{(k+1)})}{p(\boldsymbol{\theta}_{nl,j}^{(k)}) q_{nl,j}(\xi_{nl,j}^{(k+1)} | \boldsymbol{\theta}_{nl,j}^{(k)})} \right\}$$

- ii) Set $\tilde{\boldsymbol{\theta}}_{nl,j}^{(k+1)} = \begin{cases} \xi_{nl,j}^{(k+1)} & \text{with probability } r_{nl,j}^{(k+1)} \\ \boldsymbol{\theta}_{nl,j}^{(k)} & \text{with probability } 1 - r_{nl,j}^{(k+1)} \end{cases}$

4. Generate a candidate state $\tilde{\boldsymbol{\theta}}_s$ by:

- i) Determine the mean vector $\boldsymbol{\mu}_1^{(k)}$ and covariance matrix $\boldsymbol{\Sigma}_1^{(k)}$ of the Gaussian PDF $p(\boldsymbol{\theta}_1 | \boldsymbol{\theta}_2^{(k)}, \boldsymbol{\theta}_3^{(k)}, \boldsymbol{\theta}_4^{(k)}, D)$ using Equations (2.28) and (2.29) respectively. Let $\mathbf{w}_1^{(k)} = \mathbf{L}_1^{(k)-1}(\boldsymbol{\theta}_1^{(k)} - \boldsymbol{\mu}_1^{(k)})$ where $\boldsymbol{\Sigma}_1^{(k)} = \mathbf{L}_1^{(k)} \mathbf{L}_1^{(k)T}$.

ii) Generate a candidate state $\tilde{\mathbf{w}}_1^{(k+1)}$ for $\mathbf{w}_1$:

For each component $j=1, \dots, N_1$, draw a pre-candidate component

$\xi_{1,j}^{(k+1)}$ from proposal PDF $q_{1,j}(\xi_{1,j} | w_{1,j}^{(k)})$ and compute the acceptance ratio:

$$r_{1,j}^{(k+1)} = \min \left\{ 1, \frac{p(\xi_{1,j}^{(k+1)})}{p(w_{1,j}^{(k)})} \frac{q_{1,j}(w_{1,j}^{(k)} | \xi_{1,j}^{(k+1)})}{q_{1,j}(\xi_{1,j}^{(k+1)} | w_{1,j}^{(k)})} \right\}$$

$$\text{Set } \tilde{w}_{1,j}^{(k+1)} = \begin{cases} \xi_{1,j}^{(k+1)} & \text{with probability } r_{1,j}^{(k+1)} \\ w_{1,j}^{(k)} & \text{with probability } 1 - r_{1,j}^{(k+1)} \end{cases}.$$

iii) Transform $\tilde{\mathbf{w}}_1$ back to $\boldsymbol{\theta}_1$ space: $\tilde{\boldsymbol{\theta}}_1^{(k+1)} = \boldsymbol{\mu}_1^{(k)} + \mathbf{L}_1^{(k)} \tilde{\mathbf{w}}_1^{(k+1)}$.

iv) For getting $\tilde{\boldsymbol{\theta}}_2^{(k+1)}$, repeat steps i) to iii) by reversing the subscripts 1 and 2 and replacing $\boldsymbol{\theta}_1^{(k)}$ by $\tilde{\boldsymbol{\theta}}_1^{(k+1)}$.

5. Set

$$[\mathbf{Z}^{(k+1)} \boldsymbol{\theta}_{nl}^{(k+1)} \boldsymbol{\theta}_s^{(k+1)}] = \begin{cases} [\tilde{\mathbf{Z}}^{(k+1)} \tilde{\boldsymbol{\theta}}_{nl}^{(k+1)} \tilde{\boldsymbol{\theta}}_s^{(k+1)}] & \text{if } \mathbf{Y}(\tilde{\mathbf{Z}}^{(k+1)}, \tilde{\boldsymbol{\theta}}_{nl}^{(k+1)}, \tilde{\boldsymbol{\theta}}_s^{(k+1)}) \in F \\ [\mathbf{Z}^{(k)} \boldsymbol{\theta}_{nl}^{(k)} \boldsymbol{\theta}_s^{(k)}] & \text{if } \mathbf{Y}(\tilde{\mathbf{Z}}^{(k+1)}, \tilde{\boldsymbol{\theta}}_{nl}^{(k+1)}, \tilde{\boldsymbol{\theta}}_s^{(k+1)}) \notin F \end{cases}.$$

6. Sample $\boldsymbol{\theta}_3^{(k+1)}$ from $p(\boldsymbol{\theta}_3 | \boldsymbol{\theta}_1^{(k+1)}, \boldsymbol{\theta}_2^{(k+1)}, \boldsymbol{\theta}_4^{(k)}, D)$ as described in Section 2.3.3.

7. Sample $\boldsymbol{\theta}_4^{(k+1)}$ from $p(\boldsymbol{\theta}_4 | \boldsymbol{\theta}_1^{(k+1)}, \boldsymbol{\theta}_2^{(k+1)}, \boldsymbol{\theta}_3^{(k+1)}, D)$ as described in Section 2.3.4.

8. Let $k=k+1$ and go to step 2, until $(1/p_0-1)$ samples are obtained.

3.5 Statistical Properties of the Estimator

For level zero of the proposed approach: (a) samples $\{\mathbf{Z}^{(k)} : k=1, \dots, N\}$ and $\{\boldsymbol{\theta}_{nl}^{(k)} : k=1, \dots, N\}$ are directly simulated from the PDFs $p(\mathbf{Z})$ and $p(\boldsymbol{\theta}_{nl})$, respectively, therefore they are i.i.d; (b) samples $\{\boldsymbol{\theta}_s^{(k)} : k=1, \dots, N\}$ are simulated from the PDF $p(\boldsymbol{\theta} | D)$ using the Gibbs sampling based approach presented in Chapter 2, therefore they are identically distributed according to the target PDF $p(\boldsymbol{\theta} | D)$ but are dependent. Thus the c.o.v. $\delta^{(1)}$ of the estimator $\tilde{P}_1$ for $P(F_1 | D) = P_1$ is given by:

$$\delta^{(1)} = \sqrt{\frac{(1-P_1)}{NP_1}}(1+\gamma^{(1)}) \quad (3.29)$$

where $\gamma^{(1)}$ is a factor included to account for the dependence between the Markov chain samples. At the i -th simulation level, neglecting the correlation between the samples from different Markov chains, the c.o.v $\delta^{(i)}$ of the estimator $\tilde{P}_i$ for $P(F_i | F_{i-1}, D) = P_i$ is given by:

$$\delta^{(i)} = \sqrt{\frac{(1-P_i)}{NP_i}}(1+\gamma^{(i)}) \quad (3.30)$$

where $\gamma^{(i)}$ is a factor included to account for the dependence between the Markov chain samples and is given by:

$$\gamma^{(i)} = 2 \sum_{k=1}^{L_c-1} \left(1 - \frac{k}{L_c}\right) \frac{R^{(i)}(k)}{R^{(i)}(0)} \quad (3.31)$$

$$\text{where } R^{(i)}(k) \approx \left(\frac{1}{N - kN_c} \sum_{j=1}^{N_c} \sum_{l=1}^{L_c-k} I_{j,k}^{(i)} I_{j,k+l}^{(i)} \right) - \tilde{P}_i^2.$$

Like the failure probability estimator in the original Subset Simulation (Au and Beck, 2001a, 2003), due to the correlation among the conditional estimators, the updated robust failure probability estimator $\tilde{P}(F | D)$ is biased and the fraction of bias is bounded above by:

$$\left| \mathbb{E} \left[\frac{\tilde{P}(F | D) - P(F | D)}{P(F | D)} \right] \right| \leq \sum_{i>j} \delta^{(i)} \delta^{(j)} + o(1/N) = O(1/N) \quad (3.32)$$

The upper bound of the c.o.v of the $\tilde{P}(F | D)$, δ is given by:

$$\delta^2 = \mathbb{E} \left[\frac{\tilde{P}(F | D) - P(F | D)}{P(F | D)} \right]^2 \leq \sum_{i,j=1}^H \delta^{(i)} \delta^{(j)} + o(1/N) = O(1/N) \quad (3.33)$$

and the lower bound corresponds to where all the samples are i.i.d:

$$\delta^2 \geq \sum_{i=1}^H (\delta^{(i)})^2 \quad (3.34)$$

3.6 Illustrative Examples

The two numerical examples presented in Chapter 2 are considered here for illustrating the effectiveness and accuracy of the proposed methods for updating the robust reliability. In both examples, the linear structural dynamic system is subjected to future non-stationary, non-white ground acceleration with the frequency content modeled by Clough-Penzien spectrum (Clough and Penzien, 1993):

$$S(\omega) = S_0 \frac{(\omega / \omega_f)^4}{\left(1 - (\omega / \omega_f)^2\right)^2 + (2\zeta_f \omega / \omega_f)^2} \frac{1 + (2\zeta_g \omega / \omega_g)^2}{\left(1 - (\omega / \omega_g)^2\right)^2 + (2\zeta_g \omega / \omega_g)^2} \quad (3.35)$$

where S_0 is a constant determining the intensity of acceleration; ω_g and ζ_g are frequency and damping ratio associated to the site/soil condition; ω_f and ζ_f are parameters of a low-cut filter. The non-stationarity is modeled using a time envelop function $\lambda(t) = \alpha_1 t e^{-\alpha_2 t}$. In the present application the following values are used: $S_0 = 1e^{-3} \text{ m}^2/\text{s}^3$, $\omega_g = 15.7 \text{ rad/s}$, $\omega_f = 0.1\omega_g$, $\zeta_g = \zeta_f = 0.6$, $\alpha_1 = 0.45 \text{ s}$ and $\alpha_2 = 1/6 \text{ s}^{-1}$.

The updated robust failure probabilities are obtained using the two approaches presented in this chapter: one by exploiting the linear property of the system (termed as linear), and the other without exploiting the linear property of the system, to resemble the case where the dynamic system subjected to future excitation can behave nonlinearly (termed as without linear). In the later case, where the linear property of the system is not exploited for generating the $\mathbf{Z}$ sample, one-dimensional symmetric uniform distribution is adopted as proposal distribution for each additive excitation random variable, i.e., $q(\tilde{Z}_j | Z_j) = 0.5$ if $|Z_j - \tilde{Z}_j| < 1$ for $j = 1, \dots, N_t$ to generate a candidate $\tilde{\mathbf{Z}}$ sample. The proposal PDFs $\{q_{1,j}(\xi_{1,j} | w_{1,j}^{(k)}), j = 1, \dots, N_1\}$ and $\{q_{2,j}(\xi_{2,j} | w_{2,j}^{(k)}), j = 1, \dots, N_2\}$ are selected as follows:

$$\left. \begin{aligned} q_{1,j}(\xi_{1,j} | w_{1,j}^{(k)}) &\sim U(w_{1,j}^{(k)} - \sigma_{1,i,j}^{(k)}, w_{1,j}^{(k)} + \sigma_{1,i,j}^{(k)}) & j=1, \dots, N_1 \\ q_{2,j}(\xi_{2,j} | w_{2,j}^{(k)}) &\sim U(w_{2,j}^{(k)} - \sigma_{2,i,j}^{(k)}, w_{2,j}^{(k)} + \sigma_{2,i,j}^{(k)}) & j=1, \dots, N_2 \end{aligned} \right\} \quad (3.36)$$

where $U(x, y)$ denotes a uniform distribution with lower and upper limit as x and y , respectively, and where

$$\left. \begin{aligned} [\sigma_{1,i,1}^2 \dots \sigma_{1,i,N_1}^2]^T &= 2.25 \times \text{diag} \left(\mathbf{L}_1^{(k)-1} (\boldsymbol{\Sigma}_{1,i}) \mathbf{L}_1^{(k)-T} \right) \\ [\sigma_{2,i,1}^2 \dots \sigma_{2,i,N_2}^2]^T &= 2.25 \times \text{diag} \left(\mathbf{L}_2^{(k)-1} (\boldsymbol{\Sigma}_{2,i}) \mathbf{L}_2^{(k)-T} \right) \end{aligned} \right\} \quad (3.37)$$

and $\boldsymbol{\Sigma}_{1,i}$ and $\boldsymbol{\Sigma}_{2,i}$ are the covariance matrices estimated from $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ samples obtained from the most recent simulation level distributed according to the conditional PDFs $p(\boldsymbol{\theta}_1 | F_i, D)$ and $p(\boldsymbol{\theta}_2 | F_i, D)$, respectively, and $\text{diag}(\mathbf{A})$ denotes a vector with entries equal to the diagonal entries of a matrix $\mathbf{A}$.

The proposed method is used to evaluate the updated robust failure probability at different thresholds with a conditional failure probability at each level approximately equal to $p_0 = 0.1$ and with the number of samples set to $N = 500$ at each conditional level. To start with, 100,000 samples of contribution parameters, mode shapes and prediction error variances are obtained using the Gibbs sampling algorithm for Bayesian model updating of a linear dynamic system presented in Chapter 2. After discarding the burn-in samples, $N = 500$ samples are randomly drawn from the remaining samples for evaluating $P(F_1 | D)$. This is to insure that the starting samples are from the stationary distribution $p(\boldsymbol{\theta} | D)$. Following this, using the modal data, the updated robust failure probability of the linear structural dynamic system subjected to future non-stationary, non-white ground acceleration is computed using the proposed method. In both the examples, failure probability estimates ranging from $P(F | D) = 1$ to $P(F | D) = 10^{-4}$ are obtained. The total number of samples or dynamic analyses required in a single simulation run to produce the failure probability versus threshold level curve is 1850: 500 samples for MCS level and 450 samples for each subsequent level.

3.6.1 Results for Example 1

A discrete-time white noise sequence $\mathbf{Z}$ corresponding to the duration of input ground motion $T = 20$ s and sampling interval $\Delta t = 0.02$ s is considered. The system is assumed to have zero initial conditions. Failure is defined as an event where the displacement for DOF-1 exceeds a specific threshold b at any discrete time instant during the total duration of the ground acceleration.

Figure 3.2 and Figure 3.3 show the estimates of the updated robust failure probability for different threshold levels of displacements from 50 independent simulation runs obtained using the two approaches, respectively. Sample means of the updated robust failure probability estimator obtained by 50 independent simulation runs are shown in Figure 3.4. For both approaches (by exploiting and by without exploiting the linear property) the computational effort is the same as the same number of dynamic analyses are performed. Results computed using 100,000 MCS (dynamic analyses) samples are also shown for comparison. The results confirm that the proposed approaches are correct giving a practically unbiased estimator of the updated robust failure probability. Figure 3.5 compares the sample c.o.v of the updated robust failure probability estimator obtained from the two approaches, and the lower limit of the c.o.v of MCS estimator at a particular failure probability level using the same number of dynamic analyses as in the proposed approaches. The results indicate that exploiting the linear property yields better performance than without exploiting the linear property. It should be noted that further improvements can be made in both approaches by further tuning the proposal PDFs. Figure 3.6 further compares the performance of the two approaches by plotting the ‘unit c.o.v Δ ’ versus the estimated sample mean updated robust failure probability ($\Delta = \text{c.o.v} \sqrt{N}$ where N is the total number of simulation samples). It is seen that the Δ for the proposed approaches varies roughly in a logarithmic manner, i.e., $\Delta \sim \log(1/P(F|D))$ (true for any Subset Simulation based algorithm); for MCS, it grows drastically as $\Delta \sim 1/\sqrt{P(F|D)}$ for small $P(F|D)$. Both approaches provide substantial improvement in efficiency over MCS.

Figure 3.2. Updated robust failure probability against threshold, obtained by exploiting the linear property, 50 independent simulation runs

Figure 3.3. Updated robust failure probability against threshold, obtained without exploiting the linear property, 50 independent simulation runs

Figure 3.4. Sample mean estimate of updated robust failure probability estimator against threshold, obtained from 50 independent simulation runs

Figure 3.5. Sample c.o.v estimate of updated robust failure probability estimator obtained from 50 independent simulation runs

Figure 3.6. Unit c.o.v of the proposed updated robust failure estimator and MCS estimator

3.6.2 Results for Example 2

A discrete-time white noise sequence $\mathbf{Z}$ corresponding to the duration of input ground motion $T = 30$ s and sampling interval $\Delta t = 0.01$ s is considered. The system is assumed to have zero initial conditions. The failure event is defined as the event that the horizontal displacement in the y-direction at the top story exceeds a specified threshold level b at any of the discrete times during the total duration of the ground shaking.

Figure 3.7 and Figure 3.8 show the estimates of the updated robust failure probability for different threshold levels of displacements from 50 independent simulation runs for the two cases, respectively. Sample means of the updated robust failure probability estimator estimated by 50 independent simulation runs is shown in Figure 3.9. For both cases (by exploiting and by without exploiting the linear property) the computational effort is the same as the same number of dynamic analyses are performed. Results obtained in the two cases are similar as seen in Figure 3.9. Figure 3.10 compares the sample c.o.v of the updated robust failure probability estimator obtained for the two cases, and the lower limit of c.o.v of

MCS estimator at a particular failure probability level using the same number of dynamic analyses as in the proposed method. The results indicate that exploiting the linear property yields better performance than without exploiting the linear property. However, both cases provide substantial improvement in efficiency over MCS. Figure 3.11 further compares the performance in the two cases by plotting the ‘unit c.o.v Δ ’ versus the updated robust failure probability.

Figure 3.7. Updated robust failure probability against threshold, obtained by exploiting the linear property, 50 independent simulation runs

Figure 3.8. Updated robust failure probability against threshold, obtained without exploiting the linear property, 50 independent simulation runs

Figure 3.9. Sample mean estimate of updated robust failure probability estimator against threshold, obtained from 50 independent simulation runs

Figure 3.10. Sample c.o.v estimate of updated robust failure probability estimator obtained from 50 independent simulation runs

Figure 3.11. Unit c.o.v of the proposed updated robust failure estimator and MCS estimator

Appendix 3.A: Conditional Distribution of Input Random Vector

(Au and Beck, 2001b; Katafygiotis and Cheung, 2006)

To draw a candidate $\tilde{\mathbf{Z}}$ sample from the proposal distribution $q(\mathbf{Z}|\boldsymbol{\theta}_s^{(k)}, F^{(j)})$ (Equation (3.19)), first a failure event $F^{(j)}$ is selected according to weights given in Equation (3.20), followed by simulating $\tilde{\mathbf{Z}}$ from $p(\tilde{\mathbf{Z}}|F^{(j)})$. The conditional vector $\tilde{\mathbf{Z}}$ distributed according to $p(\tilde{\mathbf{Z}}|F^{(j)})$ can be represented as:

$$\tilde{\mathbf{Z}} = \alpha \mathbf{U}^{(j)*} + \mathbf{Z}'_{\perp} \quad (\text{A3.1})$$

where $\mathbf{U}^{(j)*} = \mathbf{Z}^{(j)*} / \|\mathbf{Z}^{(j)*}\|$ is the unit vector in the direction of the design point $\mathbf{Z}^{(j)*}$. $\mathbf{Z}'$ is an N_t -dimensional standard Gaussian random vector and $\mathbf{Z}'_{\perp}$ is its component orthogonal to $\mathbf{U}^{(j)*}$ (parallel to hyperplane $F^{(j)}$), given by:

$$\mathbf{Z}'_{\perp} = \mathbf{Z}' - \mathbf{Z}'_{\square} \quad (\text{A3.2})$$

where $\mathbf{Z}'_{\square} = (\mathbf{Z}'^T \mathbf{U}^{(j)*}) \mathbf{U}^{(j)*}$. Therefore $\tilde{\mathbf{Z}}$ can be represented as:

$$\tilde{\mathbf{Z}} = \mathbf{Z}' + (\alpha - \mathbf{Z}'^T \mathbf{U}^{(j)*}) \mathbf{U}^{(j)*} \quad (\text{A3.3})$$

α in the above equation is a standard Gaussian random variable conditional on $\alpha > \lambda^{(j)}$, i.e.,

$$p(\alpha) = \begin{cases} \frac{\phi(\alpha)}{\Phi(-\lambda^{(j)})} & \alpha > \lambda^{(j)} > 0 \\ 0 & \text{otherwise} \end{cases} \quad (\text{A3.4})$$

and can be obtained by first simulating a random number u that is uniformly distributed in the interval $[0,1]$ and then calculating:

$$\alpha = \Phi^{-1} \left[\Phi(\lambda^{(j)}) + u(1 - \Phi(\lambda^{(j)})) \right] \quad (\text{A3.5})$$

Appendix 3.B: Proof for Stationary Distribution

For a linear dynamic system subjected to future stochastic excitation, the Constrained Metropolis-within-Gibbs sampling technique involves the following

steps:

1. Metropolis-Hastings sampling is used to sample $\mathbf{Z}$ from

$$p(\mathbf{Z} | \boldsymbol{\theta}_s, F, D) = p(\mathbf{Z} | \boldsymbol{\theta}_s, F)$$

The transition PDF for which is given by:

$$K_{\mathbf{Z}}(\mathbf{Z}^* | \mathbf{Z}, \boldsymbol{\theta}_s) = T_{\mathbf{Z}}(\mathbf{Z}^* | \mathbf{Z}, \boldsymbol{\theta}_s) + [1 - a(\mathbf{Z}, \boldsymbol{\theta}_s)] \delta(\mathbf{Z}^* - \mathbf{Z})$$

where $a(\mathbf{Z}, \boldsymbol{\theta}_s) = \int T_{\mathbf{Z}}(\hat{\mathbf{Z}} | \mathbf{Z}, \boldsymbol{\theta}_s) d\hat{\mathbf{Z}}$.

$K_{\mathbf{Z}}(\mathbf{Z}^* | \mathbf{Z}, \boldsymbol{\theta}_s)$ satisfies a specific property known as ‘reversibility’ or ‘detailed balance’ from where the following is obtained:

$$K_{\mathbf{Z}}(\mathbf{Z}^* | \mathbf{Z}, \boldsymbol{\theta}_s) p(\mathbf{Z} | \boldsymbol{\theta}_s, F) = K_{\mathbf{Z}}(\mathbf{Z} | \mathbf{Z}^*, \boldsymbol{\theta}_s) p(\mathbf{Z}^* | \boldsymbol{\theta}_s, F)$$

which is equivalent to

$$\begin{aligned} K_{\mathbf{Z}}(\mathbf{Z}^* | \mathbf{Z}, \boldsymbol{\theta}_s, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4) p(\mathbf{Z} | \boldsymbol{\theta}_s, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D, F) \\ = K_{\mathbf{Z}}(\mathbf{Z} | \mathbf{Z}^*, \boldsymbol{\theta}_s, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4) p(\mathbf{Z}^* | \boldsymbol{\theta}_s, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D, F) \end{aligned} \quad (\text{B3.1})$$

2. Multi-group Metropolis-Hastings sampling is used to generate a candidate state $\tilde{\mathbf{w}}_1 = [\xi_1 \ \xi_2 \cdots \xi_{N_1}]^T$ for $\mathbf{w}_1$ conditioned on $\boldsymbol{\theta}_2$, $\boldsymbol{\theta}_3$ and $\boldsymbol{\theta}_4$. Each component of $\tilde{\mathbf{w}}_1$ is simulated by Metropolis-Hastings sampling. The transition PDF for which is given by:

$$\begin{aligned} K_{w_{1,j}}(\xi_{1,j} | w_{1,j}, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4) = T_{w_{1,j}}(\xi_{1,j} | w_{1,j}, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4) \\ + [1 - a(w_{1,j}, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4)] \delta(\xi_{1,j} - w_{1,j}) \end{aligned}$$

where $a(w_{1,j}, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4) = \int T_{w_{1,j}}(\hat{\xi}_{1,j} | w_{1,j}, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4) d\hat{\xi}$.

$K_{w_{1,j}}(\xi_{1,j} | w_{1,j}, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4)$ satisfies a specific property known as ‘reversibility’

from where the following is obtained:

$$\begin{aligned} K_{w_{1,j}}(\xi_{1,j} | w_{1,j}, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4) p(w_{1,j} | \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D) \\ = K_{w_{1,j}}(w_{1,j} | \xi_{1,j}, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4) p(\xi_{1,j} | \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4, D) \end{aligned}$$

Since all the components are simulated independently, the transition PDF satisfies the following property:

$$\begin{aligned}
& \prod_{j=1}^{N_1} K_{w_{1,j}}(\xi_{1,j} | w_{1,j}, \theta_2, \theta_3, \theta_4) p(w_{1,j} | \theta_2, \theta_3, \theta_4, D) \\
&= \prod_{j=1}^{N_1} K_{w_{1,j}}(w_{1,j} | \xi_{1,j}, \theta_2, \theta_3, \theta_4) p(\xi_{1,j} | \theta_2, \theta_3, \theta_4, D) \\
&\Rightarrow K_{\mathbf{w}_1}(\tilde{\mathbf{w}}_1 | \mathbf{w}_1, \theta_2, \theta_3, \theta_4) p(\mathbf{w}_1 | \theta_2, \theta_3, \theta_4, D) \\
&= K_{\mathbf{w}_1}(\mathbf{w}_1 | \tilde{\mathbf{w}}_1, \theta_2, \theta_3, \theta_4) p(\tilde{\mathbf{w}}_1 | \theta_2, \theta_3, \theta_4, D)
\end{aligned}$$

Given θ_2 , θ_3 and θ_4 , there is one to one relationship between $\tilde{\mathbf{w}}_1$ and $\tilde{\theta}_1$, therefore

$$\begin{aligned}
& K_{\theta_1}(\tilde{\theta}_1 | \theta_1, \theta_2, \theta_3, \theta_4) p(\theta_1 | \theta_2, \theta_3, \theta_4, D) \\
&= K_{\theta_1}(\theta_1 | \tilde{\theta}_1, \theta_2, \theta_3, \theta_4) p(\tilde{\theta}_1 | \theta_2, \theta_3, \theta_4, D)
\end{aligned} \tag{B3.2}$$

3. Similarly for $\mathbf{w}_2$ conditioned on $\tilde{\theta}_1$, θ_3 and θ_4 ., the transition PDF must satisfies the following property:

$$\begin{aligned}
& K_{\mathbf{w}_2}(\tilde{\mathbf{w}}_2 | \tilde{\theta}_1, \mathbf{w}_2, \theta_3, \theta_4) p(\mathbf{w}_2 | \tilde{\theta}_1, \theta_3, \theta_4, D) \\
&= K_{\mathbf{w}_2}(\mathbf{w}_2 | \tilde{\theta}_1, \tilde{\mathbf{w}}_2, \theta_3, \theta_4) p(\tilde{\mathbf{w}}_2 | \tilde{\theta}_1, \theta_3, \theta_4, D)
\end{aligned}$$

Given $\tilde{\theta}_1$, θ_3 and θ_4 , there is one to one relationship between $\tilde{\mathbf{w}}_2$ and $\tilde{\theta}_2$, therefore

$$\begin{aligned}
& K_{\theta_2}(\tilde{\theta}_2 | \tilde{\theta}_1, \theta_2, \theta_3, \theta_4) p(\theta_2 | \tilde{\theta}_1, \theta_3, \theta_4, D) \\
&= K_{\theta_2}(\theta_2 | \tilde{\theta}_1, \tilde{\theta}_2, \theta_3, \theta_4) p(\tilde{\theta}_2 | \tilde{\theta}_1, \theta_3, \theta_4, D)
\end{aligned} \tag{B3.3}$$

4. Given $\mathbf{Z}^*$, the above items 2 and 3 together are steps for multi-group Metropolis within Gibbs sampling. Given $\mathbf{Z}^*, \theta_1, \theta_2, \theta_3, \theta_4$, the transition PDF for the new sample $\theta_s^* = [\theta_1^{*T} \theta_2^{*T}]^T = [\tilde{\theta}_1^T \tilde{\theta}_2^T]^T$ for θ_s is given by:

$$\begin{aligned}
& K(\theta_s^* | \mathbf{Z}^*, \theta_s, \theta_3, \theta_4) = I_F(\mathbf{Z}^*, \theta_s^*) K_{\theta_1}(\theta_1^* | \theta_1, \theta_2, \theta_3, \theta_4) K_{\theta_2}(\theta_2^* | \theta_1^*, \theta_2, \theta_3, \theta_4) + \dots \\
& \left[1 - \int I_F(\mathbf{Z}^*, \hat{\theta}_s) K_{\theta_1}(\hat{\theta}_1 | \theta_1, \theta_2, \theta_3, \theta_4) K_{\theta_2}(\hat{\theta}_2 | \hat{\theta}_1, \theta_2, \theta_3, \theta_4) d\hat{\theta}_1 d\hat{\theta}_2 \right] \delta(\theta_s^* - \theta_s)
\end{aligned}$$

It can be shown that (proof in Appendix 4.A: Proof for Stationary Distribution)

$$p(\theta_s^* | \mathbf{Z}^*, \theta_s, \theta_3, \theta_4, D, F) = \int K(\theta_s^* | \mathbf{Z}^*, \theta_s, \theta_3, \theta_4) p(\theta_s | \mathbf{Z}^*, \theta_s, \theta_3, \theta_4, D, F) d\theta_s \tag{B3.4}$$

5. θ_3 and θ_4 are directly sampled from their full conditional PDF, thus:

$$K_{\theta_3}(\theta_3^* | \theta_1^*, \theta_2^*, \theta_3, \theta_4) = p(\theta_3^* | \theta_1^*, \theta_2^*, \theta_4, D) \equiv p(\theta_3^* | \mathbf{Z}^*, \theta_s^*, \theta_4, D, F) \tag{B3.5}$$

$$K_{\theta_4}(\theta_4^* | \theta_1^*, \theta_2^*, \theta_3^*, \theta_4) = p(\theta_4^* | \theta_1^*, \theta_2^*, \theta_3^*, D) \equiv p(\theta_4^* | \mathbf{Z}^*, \theta_s^*, \theta_3^*, D, F) \quad (\text{B3.6})$$

To prove the validity of the Constrained Metropolis-within-Gibbs sampling technique used in the proposed approach, the global transition PDF must satisfy the stationarity condition where $\theta^* = [\theta_1^{*T} \theta_2^{*T} \theta_3^{*T} \theta_4^{*T}]^T$ and $\theta = [\theta_1^T \theta_2^T \theta_3^T \theta_4^T]^T$:

$$p(\theta^*, \mathbf{Z}^* | D, F) = \int K(\theta^*, \mathbf{Z}^* | \theta, \mathbf{Z}) p(\theta, \mathbf{Z} | D, F) d\theta d\mathbf{Z} \quad (\text{B3.7})$$

Starting from the right hand side:

$$\begin{aligned} I &= \int K_{\mathbf{Z}}(\mathbf{Z}^* | \mathbf{Z}, \theta_s, \theta_3, \theta_4) \\ &\quad K_{\theta_s}(\theta_s^* | \mathbf{Z}^*, \theta_s, \theta_3, \theta_4) \\ &\quad K_{\theta_3}(\theta_3^* | \mathbf{Z}^*, \theta_s^*, \theta_3, \theta_4) \\ &\quad K_{\theta_4}(\theta_4^* | \mathbf{Z}^*, \theta_s^*, \theta_3^*, \theta_4) \\ &\quad p(\mathbf{Z}, \theta_s, \theta_3, \theta_4 | D, F) d\mathbf{Z} d\theta_s d\theta_3 d\theta_4 \end{aligned} \quad (\text{B3.8})$$

The quantity inside the integral is equal to:

$$\begin{aligned} &= K_{\mathbf{Z}}(\mathbf{Z}^* | \mathbf{Z}, \theta_s, \theta_3, \theta_4) K_{\theta_s}(\theta_s^* | \mathbf{Z}^*, \theta_s, \theta_3, \theta_4) K_{\theta_3}(\theta_3^* | \mathbf{Z}^*, \theta_s^*, \theta_3, \theta_4) \\ &\quad K_{\theta_4}(\theta_4^* | \mathbf{Z}^*, \theta_s^*, \theta_3^*, \theta_4) p(\mathbf{Z}, \theta_s, \theta_3, \theta_4 | D, F) \\ &= K_{\theta_s}(\theta_s^* | \mathbf{Z}^*, \theta_s, \theta_3, \theta_4) K_{\theta_3}(\theta_3^* | \mathbf{Z}^*, \theta_s^*, \theta_3, \theta_4) K_{\theta_4}(\theta_4^* | \mathbf{Z}^*, \theta_s^*, \theta_3^*, \theta_4) \\ &\quad \underbrace{K_{\mathbf{Z}}(\mathbf{Z}^* | \mathbf{Z}, \theta_s, \theta_3, \theta_4) p(\mathbf{Z} | \theta_s, \theta_3, \theta_4, D, F)}_{K_{\mathbf{Z}}(\mathbf{Z} | \mathbf{Z}^*, \theta_s, \theta_3, \theta_4) p(\mathbf{Z}^* | \theta_s, \theta_3, \theta_4, D, F)} p(\theta_s, \theta_3, \theta_4 | D, F) \\ &= K_{\theta_s}(\theta_s^* | \mathbf{Z}^*, \theta_s, \theta_3, \theta_4) K_{\theta_3}(\theta_3^* | \mathbf{Z}^*, \theta_s^*, \theta_3, \theta_4) K_{\theta_4}(\theta_4^* | \mathbf{Z}^*, \theta_s^*, \theta_3^*, \theta_4) \\ &\quad K_{\mathbf{Z}}(\mathbf{Z} | \mathbf{Z}^*, \theta_s, \theta_3, \theta_4) p(\mathbf{Z}^*, \theta_s, \theta_3, \theta_4 | D, F) \end{aligned}$$

Substituting the above back in the integral I in Equation (B3.8) gives:

$$\begin{aligned} I &= \int K_{\theta_s}(\theta_s^* | \mathbf{Z}^*, \theta_s, \theta_3, \theta_4) \\ &\quad K_{\theta_3}(\theta_3^* | \mathbf{Z}^*, \theta_s^*, \theta_3, \theta_4) \\ &\quad K_{\theta_4}(\theta_4^* | \mathbf{Z}^*, \theta_s^*, \theta_3^*, \theta_4) \\ &\quad K_{\mathbf{Z}}(\mathbf{Z} | \mathbf{Z}^*, \theta_s, \theta_3, \theta_4) \\ &\quad K_{\theta_s}(\theta_s^* | \mathbf{Z}^*, \theta_s, \theta_3, \theta_4) \\ &= \int K_{\theta_3}(\theta_3^* | \mathbf{Z}^*, \theta_s^*, \theta_3, \theta_4) p(\mathbf{Z}^*, \theta_s, \theta_3, \theta_4 | D, F) d\theta_s d\theta_3 d\theta_4 \\ &\quad K_{\theta_4}(\theta_4^* | \mathbf{Z}^*, \theta_s^*, \theta_3^*, \theta_4) \\ &\quad K_{\theta_s}(\theta_s^* | \mathbf{Z}^*, \theta_s, \theta_3, \theta_4) \\ &= \int K_{\theta_3}(\theta_3^* | \mathbf{Z}^*, \theta_s^*, \theta_3, \theta_4) p(\theta_s | \mathbf{Z}^*, \theta_3, \theta_4, D, F) p(\mathbf{Z}^*, \theta_3, \theta_4 | D, F) d\theta_s d\theta_3 d\theta_4 \\ &\quad K_{\theta_4}(\theta_4^* | \mathbf{Z}^*, \theta_s^*, \theta_3^*, \theta_4) \end{aligned}$$

Using Equation (B3.4) gives:

$$\begin{aligned}
I &= \int \frac{K_{\theta_3}(\theta_3^* | \mathbf{Z}^*, \theta_s^*, \theta_3, \theta_4)}{K_{\theta_4}(\theta_4^* | \mathbf{Z}^*, \theta_s^*, \theta_3^*, \theta_4)} p(\theta_s^* | \mathbf{Z}^*, \theta_3, \theta_4, D, F) p(\mathbf{Z}^*, \theta_3, \theta_4 | D, F) d\theta_3 d\theta_4 \\
&= \int \frac{K_{\theta_3}(\theta_3^* | \mathbf{Z}^*, \theta_s^*, \theta_3, \theta_4)}{K_{\theta_4}(\theta_4^* | \mathbf{Z}^*, \theta_s^*, \theta_3^*, \theta_4)} p(\mathbf{Z}^*, \theta_s^*, \theta_3, \theta_4 | D, F) d\theta_3 d\theta_4 \\
&= \int K_{\theta_4}(\theta_4^* | \mathbf{Z}^*, \theta_s^*, \theta_3^*, \theta_4) \underbrace{K_{\theta_3}(\theta_3^* | \mathbf{Z}^*, \theta_s^*, \theta_3, \theta_4)}_{p(\theta_3^* | \mathbf{Z}^*, \theta_s^*, \theta_4, D, F)} p(\theta_3 | \theta_s^*, \mathbf{Z}^*, \theta_4, D, F) p(\theta_s^*, \mathbf{Z}^*, \theta_4 | D, F) d\theta_3 d\theta_4 \\
&= \int \underbrace{K_{\theta_4}(\theta_4^* | \mathbf{Z}^*, \theta_s^*, \theta_3^*, \theta_4)}_{p(\theta_4^* | \mathbf{Z}^*, \theta_s^*, \theta_3^*, D, F)} p(\theta_3^*, \mathbf{Z}^*, \theta_s^*, \theta_4 | D, F) p(\theta_3 | \theta_s^*, \mathbf{Z}^*, \theta_4, D, F) d\theta_3 d\theta_4 \\
&= \int p(\theta_4^* | \mathbf{Z}^*, \theta_s^*, \theta_3^*, D, F) p(\theta_4 | \theta_3^*, \mathbf{Z}^*, \theta_s^*) p(\theta_3^*, \mathbf{Z}^*, \theta_s^* | D, F) d\theta_4 \\
&= p(\theta_4^* | \mathbf{Z}^*, \theta_s^*, \theta_3^*, D, F) p(\theta_3^*, \mathbf{Z}^*, \theta_s^* | D, F) \\
&= p(\theta^*, \mathbf{Z}^* | D, F)
\end{aligned}$$

The above proof can be modified to tackle the case involving the dynamic system which can behave nonlinearly when it is subjected to future stochastic excitation, the corresponding steps of proof for generating a candidate state for $\mathbf{Z}$ and θ_{nl} are similar to item 2 above. Given $\mathbf{Z}, \theta_{nl}, \theta_s, \theta_3, \theta_4$, the transition PDF for the new sample $[\mathbf{Z}^{*T} \theta_{nl}^{*T} \theta_s^{*T}]^T$ is given by:

$$\begin{aligned}
K(\mathbf{Z}^*, \theta_{nl}^*, \theta_s^* | \mathbf{Z}, \theta_{nl}, \theta_s, \theta_3, \theta_4) &= \\
&I_F(\mathbf{Z}^*, \theta_{nl}^*, \theta_s^*) K_Z(\mathbf{Z}^* | \mathbf{Z}) K_Z(\theta_{nl}^* | \theta_{nl}) K_{\theta_1}(\theta_1^* | \theta_1, \theta_2, \theta_3, \theta_4) K_{\theta_2}(\theta_2^* | \theta_1^*, \theta_2, \theta_3, \theta_4) + \dots \\
&\left[1 - \int I_F(\hat{\mathbf{Z}}, \hat{\theta}_{nl}, \hat{\theta}_s) K_Z(\hat{\mathbf{Z}} | \mathbf{Z}) K_Z(\hat{\theta}_{nl} | \theta_{nl}) K_{\theta_1}(\hat{\theta}_1 | \theta_1, \theta_2, \theta_3, \theta_4) K_{\theta_2}(\hat{\theta}_2 | \hat{\theta}_1, \theta_2, \theta_3, \theta_4) d\hat{\mathbf{Z}} d\hat{\theta}_{nl} d\hat{\theta}_1 d\hat{\theta}_2 \right] \\
&\delta(\mathbf{Z}^* - \mathbf{Z}) \delta(\theta_{nl}^* - \theta_{nl}) \delta(\theta_s^* - \theta_s)
\end{aligned}$$

Chapter 4

A New Stochastic Simulation Based Approach for Seismic Loss Analysis

4.1 Introduction

Performance based engineering aims to quantify the performance of systems based on quantifiable and probabilistic performance objectives. Performance objectives are statements of acceptable performance of the system, defined by the performance quantities of interest (QOIs) attached to certain specified thresholds. QOIs can take the form of conventional system response parameters (e.g., stress, deflection, drift) or their derivatives (e.g., dollar losses, downtime). Probabilistic performance objectives need to take into consideration any uncertainty that may arise because of the uncertainty in the future excitations, the imperfection or lack of accurate information in the modeling of physical systems, or any combination of these.

The Pacific Earthquake Engineering Center (PEER) has developed a modular approach for Performance Based Earthquake Engineering (Moehle and Deierlein, 2004; Porter, 2003) that involves four fundamental steps: hazard analysis, response analysis, damage assessment, and loss evaluation. First, the seismic hazard is characterized by adopting intensity measures that correspond to a specific annual rate of return. These intensity measures are then used to scale a suite of ground motion recordings in order to capture ground motion uncertainty. Using the generated ground motions, dynamic analyses are carried out to obtain the conditional distribution of structural response quantities. The structural response

The work presented in this chapter have been submitted for publication to journal *Structural Safety*, and parts of the work appear in the conference papers, (Cheung and Bansal, 2012, 2013c).

quantities are next linked to damage measures that describe the condition of the structure and its components. Finally, given a probabilistic description of damage, the process culminates with the calculation of exceedance probabilities of QOI (or decision variable) that can be used to make risk related decisions. The exceedance probability is the probability that a QOI is greater than a particular threshold (complementary CDF). Over the last few years, several studies have demonstrated the implementation of this framework, for example (Aslani and Miranda, 2005; Mitrani-Reiser, 2007; Ramirez, 2010; Yang et al., 2009). However, since a small number of analyses are used to establish the distribution of structural response quantities, the more complete probabilistic information of the performance of the system (especially the tail parts of the performance PDF) is not obtained. The results obtained from such a study are generally exceedance probabilities of a QOI that focus on the high probability regions which may be sufficient for estimating the first and second-order statistics such as mean and standard deviation or large exceedance probability (e.g., > 0.1). The framework falls short of accurately considering small exceedance probability (e.g., $<< 0.1$) that characterizes rarer events which can lead to significant consequences. Mahsuli and Haukaas (2013) presented a FORM-based approach to evaluate the loss exceedance probability for a particular threshold. Their approach provides accurate estimates for low probability events, however, requiring repeated reliability analyses to obtain the loss exceedance probability as a function of thresholds. Also it is not computationally feasible to use FORM to handle the case involving a stochastic nonlinear dynamic system subjected to non-stationary stochastic excitation with a large number of uncertain model parameters including those in the stochastic excitation model, physical dynamic model (say structural dynamic model), fragility function or loss function.

The loss estimation formulation presented here is similar to the four step formulation presented by most researchers in PEER; however, here the uncertainty in the ground motion is characterized using stochastic ground motion models instead of performing seismic hazard analysis. It is obvious that evaluating seismic loss probability can be viewed as a reliability analysis problem. In reliability

analysis, the ‘failure’ can be defined as unsatisfactory performance of the system. ‘The failure’ here is the seismic loss exceedance over some threshold level. Compared with conventional approaches, reliability analyses by state-of-the-art stochastic simulation based techniques have been proved to be very efficient and reliable in problems involving high stochastic dimension of interest (with thousands of random variables or more and non-stationary stochastic input ground motion considered), complex dynamic systems and rare events (Schuëller and Pradlwarter, 2007). In the literature, a few stochastic simulation based techniques that are applicable to solving such problems can be found, such as Importance Sampling (Au and Beck, 2001b), Subset Simulation (Au and Beck, 2001a, 2003), Domain Decomposition Method (Katafygiotis and Cheung, 2006), Auxiliary Domain Method (Katafygiotis et al., 2007), Line Sampling (Pradlwarter et al., 2007), Spherical Subset Simulation (Katafygiotis et al., 2010). All these techniques require the PDF of all uncertain variables and model parameters prior to analysis which allow random sampling. A stochastic ground motion model provides a probabilistic description of the ground motions expected at a building site. There are a number of comprehensive literature reviews available on stochastic modeling and simulation of ground motions, for example, Shinozuka and Deodatis (1988), Rezaeian and Kiureghian (2010). A stochastic ground motion model, for example the one due to Boore (2003), is parameterized by the earthquake characteristics (moment magnitude and epicentral distance) and parameters (corner frequencies, shear wave velocity etc.) that define the predictive relationship relating ground motion properties (frequency content, duration etc.) to these characteristics (Atkinson and Silva, 2000; Boore, 2003; Jalayer and Beck, 2008). The stochastic aspects are treated by integrating random noise with some spectrums to obtain the ground acceleration time histories and the non-stationarity aspect is treated by an envelope function.

The objective of this chapter is to: 1) evaluate seismic exceedance probability as a function of threshold including the tail distribution (graphically this is represented by exceedance probability curve with the vertical axis representing the exceedance probability in log-scale); and 2) the generation of samples of input random variables

and any function of them conditioned on different levels of loss exceedance for a more comprehensive seismic risk and loss analysis and investigation. In this chapter, a new stochastic simulation based approach is proposed, which involves the modification of the simulation algorithms in the Subset Simulation (Au and Beck, 2001a, 2003) to simulate the aforementioned conditional samples and development of new estimators to tackle the estimation problem of seismic loss probability curve without repeated reliability analyses. While being robust to the number of and type of random variables involved, MCS can be used to evaluate seismic loss probability curve and simulate conditional failure samples in one reliability analysis. However, MCS is highly inefficient when the simulation of the tail distribution is of interest. Exceedance probability as a function of threshold and the corresponding curve are obtained at thresholds using much fewer dynamic analyses than those would be required by MCS. Like MCS, the approach is robust to the number of random variables both continuous and discrete resulting from the stochastic modeling of ground motion, the structural dynamic modeling and large number of combinations of discrete damage measures and states (and corresponding loss functions) that describe the condition of the structure. Performance assessment of an existing reinforced-concrete (RC) seven-story hotel located in Van Nuys, California is considered to show the application and efficiency of the proposed approach.

In chapter 5, the approach proposed in this chapter to evaluate seismic exceedance probability will be integrated with the approach presented in chapter 3 for computing the updated robust failure/exceedance probability of a dynamic system using modal data, to compute the updated seismic exceedance probability.

4.2 Nonlinear Stochastic Dynamic Reliability Analysis for Loss Estimation

Stochastic dynamic analysis provides the mean for probabilistic assessment of seismic demand on structures subjected to uncertain excitation modeled by stochastic processes. It allows the determination of various statistics of the structural performance, such as the probability distributions of maximum responses

or the first-passage probability. Given the performance function $D(\boldsymbol{\theta})$ and the corresponding threshold b , the failure probability P_F (the complement of reliability) can be expressed by a multi-dimensional integral of the form:

$$P_F = \int I_F(D(\boldsymbol{\theta}) > b) p(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad (4.1)$$

where $\boldsymbol{\theta} \in \mathbb{R}^n$ is the vector of random variables in the problem of interest, whose joint PDF is given by $p(\boldsymbol{\theta})$, $F \subset \mathbb{R}^n$ is the failure region, and I_F is the indicator function: $I_F = 1$ if $D(\boldsymbol{\theta}) > b$ and otherwise $I_F = 0$.

To evaluate the performance of the system for any possible future event, different thresholds of performance quantity of interest need to be considered and the corresponding exceedance probabilities need to be estimated. In this study, one of the objectives is to obtain the PDF of the *Loss* including the tail parts due to the combinations of scenarios which can lead to extreme consequences. *Loss* may refer to dollar loss (repair and restoration cost), loss in functionality (downtime) or human loss (casualties) etc. Let $L \in \mathbb{R}$ denote any loss quantity. To estimate the PDF of L as a function of the threshold b , the integral in Equation (4.1) can be modified to the following:

$$P_F(b) = \int I_F(L > b) p(L | \boldsymbol{\theta}_g, \mathbf{Z}, \boldsymbol{\theta}_s, \boldsymbol{\theta}_f, \boldsymbol{\theta}_l) p(\boldsymbol{\theta}_g) p(\mathbf{Z}) p(\boldsymbol{\theta}_s) p(\boldsymbol{\theta}_f) p(\boldsymbol{\theta}_l) dL d\boldsymbol{\theta}_g d\mathbf{Z} d\boldsymbol{\theta}_s d\boldsymbol{\theta}_f d\boldsymbol{\theta}_l \quad (4.2)$$

where $\boldsymbol{\theta}_g$ is the stochastic ground motion model parameters vector, $\mathbf{Z}$ is the stochastic excitation vector (white noise), $\boldsymbol{\theta}_s$ is the structural model parameters vector, $\boldsymbol{\theta}_f$ is the fragility function model parameters vector, $\boldsymbol{\theta}_l$ is the loss function model parameters vector and I_F is an indicator function ($I_F = 1$ if $L > b$ and otherwise $I_F = 0$). It should be noted that in addition to accounting for ground motion uncertainty, uncertainties resulting from, the size and location of the earthquake, the structural parameters, and parameters defining fragility and loss functions are also directly considered. The PDF of any loss for a scenario event (which will still correspond to many possible ground motions) can be obtained by

fixing θ_g . In Equation (4.2), $p(L|\theta_g, \mathbf{Z}, \theta_s, \theta_f, \theta_l)$ is equivalent to $p(L|\mathbf{EDP}, \theta_f, \theta_l)$ where $\mathbf{EDP}$ denotes the engineering demand parameter vector (e.g., maximum inter-story drifts, peak floor accelerations). There is a one-to-one mapping from $[\mathbf{Z} \ \theta_g \ \theta_s]$ to $\mathbf{EDP}$, as for every unique set of input parameters, a unique structural model output is obtained during the structural dynamic analysis. The PDF of loss L conditioned on stochastic ground motion parameters, stochastic excitation vector, structural model parameters, fragility function model parameters, loss function model parameters is given by:

$$\begin{aligned} p(L|\theta_g, \mathbf{Z}, \theta_s, \theta_f, \theta_l) &\equiv p(L|\mathbf{EDP}, \theta_f, \theta_l) \\ &= \sum_{\forall \mathbf{dm}} p(L|\mathbf{DM} = \mathbf{dm}, \theta_l) p(\mathbf{DM} = \mathbf{dm}|\mathbf{EDP}, \theta_f) \end{aligned} \quad (4.3)$$

$$p(\mathbf{DM} = \mathbf{dm}|\mathbf{EDP}, \theta_f) = p\left(\bigcap_{j=1}^{N_c} \{DM_j = dm_j\}|\mathbf{EDP}, \theta_f\right) \quad (4.4)$$

$$p(L|\mathbf{DM} = \mathbf{dm}, \theta_l) = p\left(\sum_{j=1}^{N_c} L_j | DM_j = dm_j, \theta_l\right) \quad (4.5)$$

$$p(L|\mathbf{DM} = \mathbf{dm}, \theta_l) = p(g(L_1, \dots, L_{N_c})|\mathbf{DM} = \mathbf{dm}, \theta_l) \quad (4.6)$$

where $\mathbf{DM} = [DM_1 \ DM_2 \ \dots \ DM_{N_c}]$ is a damage state random vector and $dm_j \in \{1, 2, \dots, nds_j\}$; N_c is the total number of damageable components and nds_j is the total number of discrete damage states for the j -th component. There are $nds_1 \cdot nds_2 \cdot \dots \cdot nds_{N_c}$ possible realizations of $\mathbf{DM}$. Conditioned on $\theta_g, \mathbf{Z}, \theta_s, \theta_f, \theta_l$, the PDF of the total loss is given by the summation of the conditional distribution $p(L|\mathbf{DM} = \mathbf{dm}, \theta_l)$ weighted by $p(\mathbf{DM} = \mathbf{dm}|\mathbf{EDP}, \theta_f)$ over each possible realization $\mathbf{dm}$. $p(DM_j = dm_j|\mathbf{EDP}, \theta_f)$ can be obtained using the fragility functions for the j -th component (fragility functions are probability distributions that are used to indicate the probability that a component will be damaged to a given or more severe damage state as a function of demand parameters). $p(L_j | DM_j = dm_j)$ is the PDF for the loss of the j -th component in damage state dm_j and is defined using a loss function (that gives the probability of occurrence of a certain level of loss when a certain damage state has occurred in the component).

Depending on the type of loss quantity, $p(L|\mathbf{DM}=\mathbf{dm},\boldsymbol{\theta}_l)$ can take the form in Equation (4.5) or a general form as in Equation (4.6). It should be stressed that seismic loss is not independent of the seismic parameters, structural parameters, fragility function model parameters and loss function model parameters which can be expressed as a function of independent random variables. For brevity in presentation, let $\boldsymbol{\theta}=[\boldsymbol{\theta}_g \ \mathbf{Z} \ \boldsymbol{\theta}_s \ \boldsymbol{\theta}_f \ \boldsymbol{\theta}_l]$.

It can be expected that the dimension of random variables involved is high (of the order of thousands or more) and the failure region has complicated geometry. To evaluate the seismic loss exceedance probability as a function of the threshold c in Equation (4.2) and to simulate samples of input random variables conditioned on different levels of loss exceedance without repeated reliability analyses, a new stochastic approach which involves the modification of the simulation algorithms in the Subset Simulation (Au and Beck, 2001a, 2003) and the development of new estimators is presented in Sections 4.3.

4.3 The Proposed Modified Subset Simulation Based Method

4.3.1 The Original Subset Simulation

Au and Beck (Au and Beck, 2001a, 2003) proposed a very efficient algorithm for computing small failure probabilities of dynamic systems. The application of Subset Simulation for efficiently computing small failure probabilities encountered in seismic risk problems involving structural dynamic analysis can be found in Au and Beck (2003). The basic idea of Subset Simulation is to consider a sequence of failure events $F_1 \supset F_2 \supset \dots \supset F_H = F$ (one being the subset of another) converting a rare-event problem into a problem with a sequence of more frequent events. This enables the computation of the small failure probability as a product of conditional probabilities $\{P(F_{i+1}|F_i): i=0,1,\dots,H\}$, where $P(F_1|F_0)=P(F_1)$ and the target failure probability is given by the last element in the sequence.

$$P_F = P(F_H) = \left[\prod_{i=1}^{H-1} P(F_{i+1}|F_i) \right] P(F_1) \quad (4.7)$$

The intermediate thresholds $\{b_i : i = 1, \dots, H-1\}$, where $b_1 < b_2 < \dots < b_H$ are chosen “adaptively” so that the conditional PDFs $P(F_{i+1} | F_i)$ are approximately equal to some specified value p_0 . A brief review of the Subset Simulation is presented in Section 3.1.2. In subset simulation, it is assumed that

- (a) $P(F_{i+1} | F_i)$ and $P(F_1)$ are evaluated by considering the integrals as shown in Equations (4.8) and (4.9) where $I_{F_i} = 1$ if $\boldsymbol{\theta} \in F_i$ and otherwise $I_{F_i} = 0$. The conditional failure probability $P(F_{i+1} | F_i)$ is estimated by the sample average of $I_{F_{i+1}}(\boldsymbol{\theta})$ which is just the fraction of number of samples following the distribution $p(\boldsymbol{\theta} | F_i)$ that lead to failure F_{i+1} out of the total number of samples following the distribution $p(\boldsymbol{\theta} | F_i)$.

$$P(F_1) = \int I_{F_1}(\boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad (4.8)$$

$$P(F_{i+1} | F_i) = \int I_{F_{i+1}}(\boldsymbol{\theta}) p(F | \boldsymbol{\theta}_i) d\boldsymbol{\theta} \quad (4.9)$$

- (b) $p(\boldsymbol{\theta})$ can be written as a product of marginal PDFs of its component groups of random variables, i.e., $p(\boldsymbol{\theta}) = \prod_{j=1}^G p(\boldsymbol{\theta}_j)$ where $\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \ \boldsymbol{\theta}_2 \ \dots \ \boldsymbol{\theta}_G]$.

According to this, since the transition of individual groups are independent, so the transition PDF of the Markov chain between two states in F can be expressed as a product of the group transition PDFs as shown in Equation (4.10).

$$p(\boldsymbol{\theta}^{(k+1)} | \boldsymbol{\theta}^{(k)}) = \prod_{j=1}^G p_j(\boldsymbol{\theta}_j^{(k+1)} | \boldsymbol{\theta}_j^{(k)}) \quad (4.10)$$

4.3.2 The Proposed Approach

The proposed approach for evaluating loss exceedance probability curve selects a “sequence of failure events” based on a similar way as in the original Subset Simulation to compute the target failure probability, with several important differences as described as follows. The following integrals are first considered:

$$\begin{aligned}
P(F_1) &= \int P(F_1 | \boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta} \\
&= \int P(L > b_1 | \boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta}
\end{aligned} \tag{4.11}$$

In the above equation (or Equation (4.2)) ‘failure’ is still uncertain given $\boldsymbol{\theta}$, and is certain if L is known. The PDF of L as a function of $\boldsymbol{\theta}$ is given by Equations (4.3)-(4.6). For this problem, to estimate $P(F_1)$ in Equation (4.7), the new exceedance probability estimator is proposed as follows:

$$P_F(b_1) = P(F_1) = P(L > b_1) \approx \tilde{P}_1 = \frac{1}{N} \sum_{k=1}^N \frac{1}{M} \sum_{j=1}^M I_{F_1}(L^{(k,j)}) \tag{4.12}$$

where $\{L^{(k,j)} : j=1, \dots, M\}$ are i.i.d samples distributed as $p(L | \boldsymbol{\theta}^{(k)})$ and $\{\boldsymbol{\theta}^{(k)} : k=1, \dots, N\}$ are i.i.d samples distributed as $p(\boldsymbol{\theta})$. For any b such that $0 < b < b_1$, $P_F(b) = P(L > b)$ can be obtained using Equation (4.12) with $I_{F_1}(L^{(k,j)})$ replaced by $I_F(L^{(k,j)} > b)$. Similarly, $P(F_{i+1} | F_i)$ in Equation (4.7) can be estimated using the new conditional exceedance probability estimator proposed as follows:

$$P(F_{i+1} | F_i) \approx \tilde{P}_{i+1} = \frac{1}{N} \sum_{k=1}^N \frac{1}{M} \sum_{j=1}^M I_{F_{i+1}}(L^{(i,k,j)}) \tag{4.13}$$

where $\{L^{(i,k,j)} : j=1, \dots, M\}$ are samples distributed as $p(L | \boldsymbol{\theta}^{(i,k)}, F_i)$ and $\{\boldsymbol{\theta}^{(i,k)} : k=1, \dots, N\}$ are samples distributed as $p(\boldsymbol{\theta} | F_i)$, simulated by stochastic sampling technique proposed in the following section. For any b such that $b_i < b < b_{i+1}$ with $i > 0$, $P_F(b) = P(L > b)$ can be obtained using Equation (4.13) with $I_{F_{i+1}}(L^{(i,k,j)})$ replaced by $I_F(L^{(i,k,j)} > b)$.

Unlike the original Subset Simulation, the proposed stochastic sampling technique does not need to assume that the transition of individual groups is independent. The output of the proposed method is the exceedance probability of loss at various thresholds, i.e., the complementary CDF of loss as a function of threshold.

It should be noted that the case with $M = 1$ may look more natural to most of the readers as each $\boldsymbol{\theta}$ sample will generate one L sample. However, the advantage of having $M > 1$ is that the estimator of the exceedance probabilities and conditional exceedance probabilities has smaller variance than for the case with $M = 1$ (shown in Section 4.4). The increase in the computational cost due to $M > 1$ is negligible because no additional dynamic analyses are needed. The special case with $M = 1$ is presented by the present author in Cheung and Bansal (2012).

4.3.3 Simulating samples distributed according to $p(\boldsymbol{\theta} | F_i)$ and $p(L | \boldsymbol{\theta}, F_i)$

Samples distributed according to the PDF $p(L | F_i)$ cannot be simulated directly using MCS, therefore a MCMC simulation based technique is proposed to generate $\{L^{(i,k,j)} : k = 1, \dots, N; j = 1, \dots, M\}$ samples using those that lie in F_i available from the previous simulation level F_{i-1} . The stationary distribution of the Markov chain is equal to:

$$\begin{aligned} p(L, \mathbf{DM}, \boldsymbol{\theta} | F_i) &= \frac{P(F_i | L, \mathbf{DM}, \boldsymbol{\theta}) p(L, \mathbf{DM}, \boldsymbol{\theta})}{P(F_i)} \\ &= \frac{P(F_i | L) p(L | \mathbf{DM} = \mathbf{dm}) p(\mathbf{DM} = \mathbf{dm} | \boldsymbol{\theta}) p(\boldsymbol{\theta})}{P(F_i)} \\ &\propto I_{F_i}(L) p(L | \mathbf{DM} = \mathbf{dm}) p(\mathbf{DM} = \mathbf{dm} | \boldsymbol{\theta}) p(\boldsymbol{\theta}) \end{aligned} \quad (4.14)$$

According to the above set of equations, samples distributed according to the conditional PDF $p(L, \mathbf{DM}, \boldsymbol{\theta} | F_i)$ can be generated by starting Markov chains from the samples distributed according to the conditional PDF $p(\boldsymbol{\theta} | F_i)$. Seed samples for the Markov chains distributed according to the conditional PDF $p(\boldsymbol{\theta} | F_i)$ are obtained by propagating $p_0 N$ $\boldsymbol{\theta}$ samples from F_{i-1} to F_i according to the following weights:

$$\boldsymbol{\theta}^{(i,u)} = \boldsymbol{\theta}^{(i-1,k)} \quad \text{w.p.} \quad \frac{1}{\tilde{P}_i N M} \sum_{j=1}^M I_{F_i}(L^{(i-1,k,j)}) \quad \begin{matrix} u = 1, \dots, p_0 N \\ k = 1, \dots, N \end{matrix} \quad (4.15)$$

If based on Equation (4.15), a $\boldsymbol{\theta}$ sample is selected K times, then the Markov chain using it as the seed (starting point) will generate KN/p_0 samples (children).

Given a sample $\boldsymbol{\theta}^{(i,k)}$ distributed as $p(\boldsymbol{\theta} | F_i)$, a new sample $\boldsymbol{\theta}^{(i,k+1)}$ distributed as $p(\boldsymbol{\theta} | F_i)$ is generated by first simulating a candidate $\tilde{\boldsymbol{\theta}}^{(i,k+1)}$ sample from the PDF $p(\boldsymbol{\theta})$, using the Metropolis-Hastings algorithm (Chib and Greenberg, 1995). Using $\tilde{\boldsymbol{\theta}}^{(i,k+1)}$ and conditioned on $\tilde{\boldsymbol{\theta}}^{(i,k+1)}$, **EDP** is computed from the dynamic analysis. Conditioned on $\tilde{\boldsymbol{\theta}}^{(i,k+1)}$ and **EDP**, a candidate damage state random vector $\mathbf{DM}^{(i,k+1)}$ is simulated according to Equation (4.4). Conditioned on $\tilde{\boldsymbol{\theta}}^{(i,k+1)}$ and $\mathbf{DM}^{(i,k+1)}$, a candidate loss sample $\tilde{L}^{(i,k+1)}$ is simulated according to Equation (4.5). At last, if the simulated $\tilde{L}^{(i,k+1)}$ falls in the ‘failure’ domain F_i (i.e., if $\tilde{L}^{(i,k+1)} > b_i$), the candidate sample $\tilde{\boldsymbol{\theta}}^{(i,k+1)}$ is accepted as the next sample (i.e., $\boldsymbol{\theta}^{(i,k+1)} = \tilde{\boldsymbol{\theta}}^{(i,k+1)}$). If not, the current sample $\boldsymbol{\theta}^{(i,k)}$ is taken as the next sample (i.e., $\boldsymbol{\theta}^{(i,k+1)} = \boldsymbol{\theta}^{(i,k)}$). The reader is referred to the Appendix 4.A for the proof of correctness of generating samples following the above procedure.

Given $\{\boldsymbol{\theta}^{(i,k)} : k=1, \dots, N\}$ samples distributed according to conditional PDF $p(\boldsymbol{\theta} | F_i)$, any number of loss samples distributed according to conditional PDF $p(L | F_i)$ can be obtained by first simulating samples according to $p(L | \boldsymbol{\theta}^{(i,k)})$ (following Equations (4.4)-(4.5)) and then accepting only those that fall in F_i . This is true because the projection of $[L \ \mathbf{DM} \ \boldsymbol{\theta}]$ samples satisfying stationary PDF in Equation (4.14) to $\boldsymbol{\theta}$ space will provide the samples for the marginal PDF $p(\boldsymbol{\theta} | F_i)$ and

$$p(L, \mathbf{DM}, \boldsymbol{\theta} | F_i) = p(L | \mathbf{DM}, \boldsymbol{\theta}, F_i) p(\mathbf{DM} | \boldsymbol{\theta}, F_i) p(\boldsymbol{\theta} | F_i) \quad (4.16)$$

4.3.4 Summary of Steps for Simulating Samples

The steps for the proposed MCMC simulation procedure for generating samples with limiting stationary PDF equal to $p(L | F_i)$ are as follows:

1. Given a current state $\boldsymbol{\theta}^{(i,k)}$ with distribution PDF $p(\boldsymbol{\theta} | F_i)$, generate a new candidate sample $\tilde{\boldsymbol{\theta}}^{(i,k+1)}$ using the Metropolis Hastings algorithm (described below).
2. Given $\tilde{\boldsymbol{\theta}}^{(i,k+1)}$ generate a loss sample $\tilde{L}^{(i,k+1)}$ from the conditional PDF

$p(L|\tilde{\boldsymbol{\theta}}^{(i,k+1)})$ (equations (4.3)-(4.6)).

3. Set $\boldsymbol{\theta}^{(i,k+1)} = \begin{cases} \tilde{\boldsymbol{\theta}}^{(i,k+1)} & \text{if } \tilde{L}^{(i,k+1)} > b_i \\ \boldsymbol{\theta}^{(i,k)} & \text{otherwise} \end{cases}$
4. Given $\boldsymbol{\theta}^{(i,k+1)}$, generate $\{L^{(i,k+1,j)} : j=1,..M\}$ samples by repeating the following for $j=1,..,M$
 - i) Generate $\tilde{L}^{(i,k+1,j)}$ from PDF $p(L|\boldsymbol{\theta}^{(i,k+1)})$.
 - ii) Accept $L^{(i,k+1,j)} = \tilde{L}^{(i,k+1,j)}$ if $\tilde{L}^{(i,k+1,j)} > b_i$, otherwise return to step (i).

Assuming $\boldsymbol{\theta}$ is divided into G independent groups, i.e., $\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \boldsymbol{\theta}_2 \cdots \boldsymbol{\theta}_G]$. Given a current state $\boldsymbol{\theta}^{(i,k)}$, a candidate state $\tilde{\boldsymbol{\theta}}^{(i,k+1)}$ from the PDF $p(\boldsymbol{\theta})$ is generated using the Metropolis Hastings algorithm by repeating the following steps from $j=1$ till $j=G$:

1. For the j -th group, draw a pre-candidate component $\xi_j^{(i,k+1)}$ from the proposal PDF q_j and compute the acceptance ratio $r_j^{(i,k+1)}$

$$r_j^{(i,k+1)} = \min \left\{ 1, \frac{p(\xi_j^{(i,k+1)}) q_j(\boldsymbol{\theta}_j^{(i,k)} | \xi_j^{(i,k+1)})}{p(\boldsymbol{\theta}_j^{(i,k)}) q_j(\xi_j^{(i,k+1)} | \boldsymbol{\theta}_j^{(i,k)})} \right\}$$

2. Set $\tilde{\boldsymbol{\theta}}_j^{(i,k+1)} = \begin{cases} \xi_j^{(i,k+1)} & \text{with probability } \min(1, r_j^{(i,k+1)}) \\ \boldsymbol{\theta}_j^{(i,k)} & \text{with probability } 1 - \min(1, r_j^{(i,k+1)}) \end{cases}$

4.4 Statistical Properties of the Estimator

Assuming that the total number of samples simulated at each conditional level are N , the number of Markov chain is $N_c = p_0 N$, where the length of each chain is $L_c = N / N_c$. For simplicity in notation, let $I_{jkl}^{(i)} = I_{F_i}(L^{(i-1,jk,l)})$ where $L^{(i-1,jk,l)}$ is the l -th loss sample distributed according to PDF $p(L|\boldsymbol{\theta}^{(i-1,jk)})$ and where $\boldsymbol{\theta}^{(i-1,jk)}$ is the k -th sample in the j -th Markov chain at the $(i-1)$ -th simulation level distributed according to $p(\boldsymbol{\theta}|F_{i-1})$. Neglecting correlation between the samples from different Markov chains, it is shown in Appendix 4.B that:

$$\mathbb{E}[(\tilde{P}_i - P_i)^2] = \frac{R_0^{(i)}}{NM} [1 + \gamma_1^{(i)} + \gamma_2^{(i)}] \quad (4.17)$$

where $R_0^{(i)} = \text{var}(I_{jkl}^{(i)}) = P_i(1 - P_i)$, since $I_{jkl}^{(i)}$ is a Bernoulli random variable with parameter P_i and

$$\gamma_1^{(i)} = 2 \sum_{l_1=1}^{M-1} \left(1 - \frac{l_1}{M}\right) \frac{R_1^{(i)}(l_1)}{R_0^{(i)}} \quad (4.18)$$

$$\gamma_2^{(i)} = 2M \sum_{k_1=1}^{L_c-1} \left(1 - \frac{k_1}{L_c}\right) \frac{R_2^{(i)}(k_1)}{R_0^{(i)}} \quad (4.19)$$

and where

$$R_1^{(i)}(l) \approx \left(\frac{1}{N(M-l)} \sum_{j=1}^{N_c} \sum_{k=1}^{L_c} \sum_{l_1=1}^{M-l} I_{j,k,l_1}^{(i)} I_{j,k,l_1+l}^{(i)} \right) - \tilde{P}_i^2 \quad (4.20)$$

$$\begin{aligned} R_2^{(i)}(k) &= \frac{1}{M^2} \mathbb{E} \left[\sum_{l_1, l_2=1}^M (I_{j,k_1,l_1}^{(i)} - P_i)(I_{j,k_1+k,l_2}^{(i)} - P_i) \right] \\ &= \frac{1}{M^2} \sum_{l_1=1}^M \mathbb{E}[(I_{j,k_1,l_1}^{(i)} - P_i)(I_{j,k_1+k,l_1}^{(i)} - P_i)] \\ &\quad + \frac{2}{M^2} \sum_{l_1 > l_2}^M \mathbb{E}[(I_{j,k_1,l_1}^{(i)} - P_i)(I_{j,k_1+k,l_2}^{(i)} - P_i)] \\ &\approx \left(\frac{1}{(N - kN_c)M^2} \sum_{j=1}^{N_c} \sum_{k_1=1}^{L_c-k} \sum_{l_1=1}^M I_{j,k_1,l_1}^{(i)} I_{j,k_1+k,l_1}^{(i)} \right. \\ &\quad \left. + \frac{2}{(N - kN_c)M^2} \sum_{j=1}^{N_c} \sum_{k_1=1}^{L_c-k} \sum_{l_1=1}^M \sum_{l_2=1}^{M-l_1} I_{j,k_1,l_1}^{(i)} I_{j,k_1+k,l_2+l_1}^{(i)} \right) - \tilde{P}_i^2 \end{aligned} \quad (4.21)$$

Thus the c.o.v. of $\tilde{P}_i$, $\delta^{(i)}$ is given by:

$$\delta^{(i)} = \sqrt{\frac{(1 - P_i)}{MNP_i} (1 + \gamma_1^{(i)} + \gamma_2^{(i)})} \quad (4.22)$$

In the above expression, $\gamma_1^{(i)}$ appears because of the correlation between the loss samples obtained for a particular $\boldsymbol{\theta}^{(k)}$ and $\gamma_2^{(i)}$ is a factor included to account for the dependence between the Markov chain samples for $\boldsymbol{\theta}$. At level 1, i.e., the MCS level, all $\boldsymbol{\theta}$ samples are i.i.d thus $\delta^{(1)}$ in that case is given by:

$$\delta^{(1)} = \sqrt{\frac{(1-P_1)}{MNP_1}}(1+\gamma_1^{(1)}) < \sqrt{\frac{(1-P_1)}{NP_1}} \quad \text{if } M > 1 \quad (4.23)$$

The factor $(1+\gamma_1^{(1)})/M$ in the above equation will always be less than or equal to one, indicating reduction in the c.o.v. compared to the case where $M=1$. The factor $(1+\gamma_1^{(1)})/M$ is equal to one if and only if $R_1^{(i)}(l_1)/R_0^{(i)} = 1 \quad \forall l_1 = 1, \dots, M-1$, which is very unlikely in most cases.

Similar to the original Subset Simulation paper, it can be shown that the estimators $\{\tilde{P}_i : i=1, \dots, m\}$ are unbiased; and due to the correlation among these estimators, the failure probability estimator $\tilde{P}_F$ is biased and the fraction of bias is bounded above by:

$$\left| E \left[\frac{\tilde{P}_F - P_F}{P_F} \right] \right| \leq \sum_{i>j} \delta^{(i)} \delta^{(j)} + o(1/N) = O(1/N) \quad (4.24)$$

However, for a lot of real life applications the exceedance probability estimator is practically unbiased. And the c.o.v δ of $\tilde{P}_F$ is bounded above by:

$$\delta^2 = E \left[\frac{\tilde{P}_F - P_F}{P_F} \right]^2 \leq \sum_{i,j=1}^H \delta^{(i)} \delta^{(j)} + o(1/N) = O(1/N) \quad (4.25)$$

4.5 An Illustrative Example

The example used in the study is a hotel structure located in Van Nuys. This building is a seven story reinforced concrete (RC) structure that was severely damaged during the 1994 Northridge Earthquake. For illustration, here it is assumed that the building is in its original condition prior to the occurrence of the Northridge earthquake. The stochastic ground motion records are generated by adopting a point-source model characterized by the moment magnitude $Magt$ and the epicentral distance $dist$. Without loss of generality, uncertainty in structural parameters, fragility functions and loss functions is not considered in this example and a simplified nonlinear degradation structural model (Ching et al., 2006a) is used for structural dynamic analysis. Economic loss (building repair cost) due to earthquake

induced damage to drift sensitive structural and non-structural components is considered as a risk-related decision variable. Two cases of seismic hazard are considered, first with fixed $Magt$ and $dist$, and the second with uncertain $Magt$ and $dist$.

4.5.1 Stochastic Ground Motion Model

A brief review on the generation of synthetic ground motions using the stochastic method is presented in the following sections; for more details the reader can refer to Boore (2003). The stochastic ground motion records are generated by adopting a point-source model developed for California seismicity, characterized by the moment magnitude $Magt$ and the epicentral distance $dist$ (Atkinson and Silva, 2000; Boore, 2003; Boore and Joyner, 1997).

Amplitude spectrum

The total radiation Fourier spectrum $Y(f, Magt, dist)$ may be expressed as a product of the source $E(f, Magt)$, path $P(f, dist)$ and site $G(f)$ contributions:

$$Y(f, Magt, dist) = (2\pi f)^2 E(f, Magt) P(f, dist) G(f) \quad (4.26)$$

where the source spectrum $E(f, Magt)$ is selected based on two corner point source model developed by Atkinson and Silva (2000) for ground motion in California, which is described as:

$$E(f, Magt) = CM_0 \left\{ \frac{1 - \varepsilon}{1 + (f / f_a)^2} + \frac{\varepsilon}{1 + (f / f_b)^2} \right\} \quad (4.27)$$

In the above equation: the constant C is given by $C = R_\Theta VF_s / (4\pi R_0 \rho \beta^3)$, where R_Θ is the radiation pattern, V represents the partition of the total shear wave velocity into two horizontal components, F_s is the free surface amplification, R_0 is a reference distance, and ρ and β are the density and shear-wave velocity in the vicinity of the earthquake source, respectively; M_0 is the seismic moment; f_a and f_b , are the lower and higher corner frequencies, respectively, which are related to the size of the finite fault and subfault size; and ε is a relative weighing parameter:

$$\left. \begin{aligned} M_0 &= 10^{1.5(Magt+10.7)} \\ f_a &= 10^{2.181-0.496Magt} \\ f_b &= 10^{2.410-0.408Magt} \\ \varepsilon &= 10^{0.605-0.255Magt} \end{aligned} \right\} \quad (4.28)$$

The path effect $P(f, dist)$ is given by the multiplication of the geometrical spreading and anelastic attenuation:

$$P(f, dist) = Z(R) \exp(-\pi f R / Q(f) c_Q) \quad (4.29)$$

where $R = \sqrt{dist^2 + h^2}$ is the radial distance from the earthquake source to the site (h is the ‘equivalent point-source depth’ with $\log h = -0.05 + 0.15Magt$). The geometrical spreading function $Z(R)$ is given by a piecewise continuous series of straight lines where and c_Q is the seismic velocity used in the determination of regional attenuation function $Q(f)$.

$$Z(R) = \begin{cases} 1/R & R < 40 \text{ km} \\ (1/40)(40/R)^{0.5} & R \geq 40 \text{ km} \end{cases} \quad (4.30)$$

$$Q(f) = 180 f^{0.45} \quad (4.31)$$

The site effect $G(f)$ is given by the multiplication of an amplification factor $A(f)$ and a high-frequency diminution $D(f)$, i.e., $G(f) = A(f)D(f)$. The diminution is here expressed by combination of the κ_o filter and the f_{max} filter, expressed respectively as:

$$\left. \begin{aligned} D_1(f) &= \exp(-\pi \kappa_o f) \\ D_2(f) &= \left(1 + \left(\frac{f}{f_{max}} \right)^8 \right)^{-0.5} \end{aligned} \right\} \quad (4.32)$$

The parameters adopted in this study are: $R_\Theta = 0.55$, $\nu = 0.707$, $F_s = 2$, $R_0 = 1$, $\rho = 2.8$, $\beta = 3.5$, $c_Q = 3.5$, $\kappa_o = 0.02$ and $f_{max} = 100$. The amplification factor $A(f)$ is obtained from the empirical curves given in Boore and Joyner (1997).

Figure 4.1 shows the dependence of the amplitude spectrum on the $Magt$ and $dist$. It can be observed that increasing the $Magt$ results in increase in the spectral amplitude at all frequencies, with shift of dominant frequency content towards the

lower frequency regime. The spectral amplitude decrease at all frequencies with increase in $dist$ and there is no significant change of frequency content.

Figure 4.1 Amplitude spectrum using the stochastic ground motion model for different moment magnitudes and epicentral distances

Figure 4.2 Envelop function using the stochastic ground motion model for different moment magnitudes and epicentral distances

Time envelope function

Non-stationarity in the ground motion amplitude is modeled by using an empirical window function $e(t, Magt, dist)$ (Boore, 2003).

$$e(t, Magt, dist) = a(t / t_\eta)^b \exp(-ct / t_\eta) \quad (4.33)$$

where parameters a , b , and c are determined such that $e(t) = 1$ when $t = \varepsilon \times t_\eta$ and $e(t) = \eta$ when $t = t_\eta$. The time t_η is given by $t_\eta = 2T_w$ where T_w is the duration of the ground motion, expressed as a sum of a path dependent component and a source dependent component.

$$\left. \begin{aligned} b &= -\varepsilon \ln \eta / (1 + \varepsilon(\ln \varepsilon - 1)) \\ c &= b / \varepsilon \\ a &= (\exp(1) / \varepsilon)^b \\ T_w &= 1 / 2 f_a + 0.05R \end{aligned} \right\} \quad (4.34)$$

The parameters ε and η are taken to be 0.2 and 0.05, respectively.

Figure 4.2 shows the dependence of the envelop function on the $Magt$ and $dist$. It can be observed that increasing the $Magt$ results in increase in the duration of the envelop function and there is no significant effect of $dist$ on the envelop function.

Simulation of time series

To obtain a sample ground acceleration $u_g(t)$ for a given scenario event (i.e., a certain $Magt$ and $dist$), first a discrete-time white noise sequence $\mathbf{Z}$ with unit spectral intensity for the sampling interval Δt is generated. The noise is then windowed by multiplying it by an envelope function $e(t, Magt, dist)$. The windowed noise is transformed into frequency domain and the spectrum is normalized by the square-root of the mean square amplitude spectrum. The resulting spectrum is multiplied by the point-source spectrum $Y(f, Magt, dist)$ which is then transformed back to the time domain to yield a sample of the ground acceleration time history. The synthetic ground motion $u_g(t)$ generated from the model is thus a function of the additive excitation parameters $\mathbf{Z}$, and model parameters $Magt$ and $dist$ (Boore, 2003; Rezaeian and Der Kiureghian, 2008).

Uncertainty in Seismicity

The distribution of earthquake sizes is modeled by bounded Gutenberg-Richter

recurrence law (Kramer, 1996) and the uncertainty in the epicentral distance is described by a triangular distribution given by:

$$p(Magt) = \frac{\beta \exp(-\beta Magt)}{\exp(-\beta Magt_{\min}) - \exp(-\beta Magt_{\max})} \quad (4.35)$$

$$Magt \in [Magt_{\min}, Magt_{\max}]$$

$$p(dist) = 2dist / dist_{\max}^2 \quad (4.36)$$

$$dist \in [0, dist_{\max}]$$

where β is the regional seismicity factor. It is assumed that $\beta = 2.303$, for case 1, $Magt = 6.5$ and $dist = 20\text{km}$, and for case 2, $[Magt_{\min}, Magt_{\max}] = [5.5, 8]$ and $dist_{\max} = 50$.

4.5.2 Structural Model

Ching et al. (2006a) described in details the procedures for developing a simplified nonlinear degradation model which is able to capture the east-west responses of the Van Nuys hotel building. It is a single dimensional seven-DOF lumped-mass shear building model and is able to produce relationships for inter-story restoring force versus inter-story drift similar to those produced by the Finite-element model (Beck, 2002). With the lumped mass assumption, the stiffnesses and damping parameters are given by:

$$\left. \begin{aligned} k_{i,t} &= k_{i,0} e^{-\gamma \mu_{i,t}^\rho} \\ c_{i,t} &= \delta + \lambda \mu_{i,t} \end{aligned} \right\} \quad (4.37)$$

where $\mu_{i,t}$ is the maximum inter-story drift ratio up to time t (for story i -th with story height H_i) and is given by:

$$\left. \begin{aligned} \mu_{1,t} &= \max_{0 \leq k \leq t} (|x_{1,k}| / H_1) \\ \mu_{i,t} &= \max_{0 \leq k \leq t} (|x_{i,k} - x_{i-1,k}| / H_i) \end{aligned} \right\} \quad (4.38)$$

where $\gamma = 5.0$, $\rho = 0.4$, $\delta = 0.5 \text{ MN-s/m}$, $\lambda = 50 \text{ MN-s/m}$, $k_{1,0} \approx 100 \text{ MN/m}$, $k_{2,0} \approx 120 \text{ MN/m}$, $k_{3,0} \approx 98 \text{ MN/m}$, $k_{4,0} = k_{5,0} = k_{6,0} = k_{7,0} \approx 80 \text{ MN/m}$,

$M_1 = 120.02$ Ton, $M_2 = \dots = M_6 = 102.11$ Ton, $M_7 = 95.87$ Ton, $k_{2,0} \approx 120$ MN/m, $H_1 = 4.12$ m and $H_2 = \dots H_7 = 2.27$ m.

4.5.3 Fragility and Loss Functions

Loss due to earthquake induced damage to drift sensitive structural components is considered here as the performance function. The mapping between **EDP** and economic loss adopted here follows the Assembly-based vulnerability framework presented in Porter et al. (2001). Peak inter-story drift ratio (*IDR*) at each story is used as the **EDP**. For illustration and convenience, it is assumed that the damage experienced by different components for a given **EDP** vector is statically independent and no collapse mechanism is considered. Component specific fragility functions are defined using the lognormal distribution with their statistical parameters, logarithmic mean θ and logarithmic standard deviation β , are taken from Aslani and Miranda (2005). Given the fragility functions, the probability of the j -th component being in a damage state k (i.e., $DM_j = k$) is given by:

$$P(DM_j = k | \mathbf{EDP}) = \Phi\left(\frac{\ln(\mathbf{EDP}) - \theta_{j,k}}{\beta_{j,k}}\right) - \Phi\left(\frac{\ln(\mathbf{EDP}) - \theta_{j,k+1}}{\beta_{j,k+1}}\right) \quad (4.39)$$

where Φ is the standard normal CDF. The direct economic loss L is calculated by summing the economic loss for all components. Economic loss functions follow a lognormal distribution with logarithmic mean μ and logarithmic standard deviation σ given in Aslani and Miranda (2005). If function of loss for a particular damage state is not available, a loss function is assumed with mean equal to twice the mean of the loss function of the previous damage state and same c.o.v. The CDF when the j -th component is in damage state k is given by:

$$P(L_j \leq l | DM_j = k) = \Phi\left(\frac{\ln(l) - \mu_{j,k}}{\sigma_{j,k}}\right) \quad (4.40)$$

Figure 4.3 shows fragility functions and loss functions for an interior column in story 1 adopted from Aslani and Miranda (2005). Samples of L distributed according to $p(L | Magt, dist, \mathbf{Z})$ can be generated by MCS by first simulating a

random damage state for each component given the **EDP** using Equation (4.4), and then given the damage state, a random economic loss sample for each component is generated using Equation (4.5). Finally the sum of economic losses in individual components gives a sample of L distributed according to $p(L | Magt, dist, \mathbf{Z})$.

Figure 4.3. Fragility functions and loss functions for an interior column in story 1

4.5.4 Simulation Parameters

A discrete-time white noise sequence $\mathbf{Z}$ corresponding to a duration of input ground motion $T = 50\text{s}$ and a sampling interval $\Delta t = 0.02\text{s}$ is considered. To obtain the estimates of exceedance probability as a function of threshold level c the proposed method is applied with a conditional failure probability at each level approximately equal to $p_0 = 0.1$, $N = 500$ and $M = 1000$. One-dimensional chain adaptive symmetric uniform distribution is adopted as the proposal PDF for each additive excitation random variable, i.e., $q(\xi_j | Z_j) = 0.5$ if $|Z_j - \xi_j| < 1$. A level adaptive bivariate truncated Gaussian distribution with mean and covariance matrix (magnified by 2.25 times) estimated from samples from the most recent simulation level is adopted as the proposal for $Magt$ and $dist$. For each analysis (or set of **EDPs**), 1000 damage state vectors are simulated and for each damage state vector a single loss sample is simulated. Therefore, in total there are 500,000 loss samples at each conditional level. For comparison, results are also obtained using the proposed method with $M = 1$.

4.5.5 Results

Figure 4.4. Exceedance probability estimates for different threshold levels of economic loss from 50 independent simulation runs

Figure 4.4 shows the exceedance probability estimates for different threshold levels of economic loss from 50 independent simulation runs for the two cases, case 1 with fixed $Magt$ and $dist$, and case 2 with uncertain $Magt$ and $dist$. Results computed using 50,000 MCS samples are shown for comparison. It can be seen that the results using the proposed method and MCS agree well. Figure 4.5 shows the sample mean of exceedance probability estimates from 50 independent simulation runs. It is observed that the mean exceedance probability curve computed for $M = 1$ and $M = 1000$ coincide with MCS results, showing that the exceedance probability estimator is practically unbiased. To investigate the variability of the exceedance probability estimators, the sample c.o.v of the exceedance probability estimates over 50 independent simulation runs is computed. Figure 4.6 shows the sample c.o.v of the exceedance probability estimates over 50 independent simulation runs with $M = 1$ and $M = 1000$, and the lower limit of c.o.v of MCS estimator at a particular exceedance probability using the same number of dynamic analyses as in the proposed approach. The results indicate that the estimators with $M = 1000$ have lower c.o.v than those with $M = 1$, and both have smaller c.o.v than the MCS estimator; this shows an improvement in the quality of the estimator when M increases.

Figure 4.5. Sample mean of exceedance probability estimates from 50 independent simulation runs.

Figure 4.6. Sample c.o.v of exceedance probability estimates from 50 independent simulation runs

Figure 4.7 further compares the performance of the proposed approach by plotting the ‘unit c.o.v Δ ’ versus the estimated sample mean failure probability ($\Delta = \text{cov} \sqrt{N}$ where N is the total number of dynamic analyses performed). It is observed that a significant improvement in the computational efficiency is achieved in case 1 with $M = 1000$. On the other hand, in case 2, the improvement in the computational efficiency with increase in M is not as substantial as in case 1.

Figure 4.7. Sample 'unit c.o.v' of exceedance probability estimates from 50 independent simulation runs

To understand the difference in the performance observed in the two cases, Figure 4.8 shows the frequency histogram with $P(F_{i+1} | \boldsymbol{\theta}, F_i)$ (uncertain $\boldsymbol{\theta}$) on the x-axis and the number of $\boldsymbol{\theta}$ samples for each value of $P(F_{i+1} | \boldsymbol{\theta}, F_i)$ on the y-axis (the results shown in this figure are from a single simulation run with $N = 500$ and $M = 1000$). It can be observed that the random variable $P(F_{i+1} | \boldsymbol{\theta}, F_i)$ does not follow any particular probability distribution. Approximating the PDF $P(F_{i+1} | \boldsymbol{\theta}, F_i)$ with the Bernoulli distribution (with parameter p_0) may introduce additional uncertainties. This is what happens when M is set equal to 1 in the first case, the PDF of $P(F_{i+1} | \boldsymbol{\theta}, F_i)$ is approximated with the Bernoulli distribution, hence introducing additional uncertainties. Thus, an improvement in computational efficiency is observed with an increase in M . On the other hand, in the second case, the PDF of $P(F_{i+1} | \boldsymbol{\theta}, F_i)$ is close to Bernoulli distribution (not exactly) for the first two levels, thus the increase in the computational efficiency is not substantial with an increase in M . Figure 4.9 shows the PDFs $\{p(L | \boldsymbol{\theta}^{(k)}, F_i) : k = 1, \dots, N\}$ for the two cases. The vertical line in the figure indicates the intermediate threshold c_{i+1} . It can be observed that in the first case all the PDFs are close to each other and the vertical line cuts through almost all the PDFs in the tail region, showing that $P(F_{i+1} | \boldsymbol{\theta}, F_i)$ take values anywhere between 0 and 1. In the second case, the PDFs are distributed in a broader range and there are PDFs that lie completely on either side of the threshold line, showing that $P(F_{i+1} | \boldsymbol{\theta}, F_i)$ take values close to 0 and 1.

Figure 4.8. Frequency histogram with $P(F_{i+1} | \theta, F_i)$ with uncertain θ on the x-axis and the number of θ samples for each value of $P(F_{i+1} | \theta, F_i)$ on the y-axis

Figure 4.9 $\{p(L|\theta^{(k)}, F_i) : k = 1, 2, \dots, N\}$

Thus, the amount of improvement in computational efficiency achieved by increasing M is problem dependent; however, it cannot be denied that an increase in computational efficiency is obtained with an increase in M . For the loss estimation problem, the increase in the computational cost for increasing M is negligible because this does not involve any additional dynamic analysis. In both cases, substantial improvements are obtained using the proposed approach when compared to MCS.

Figure 4.10. Estimates of the exceedance probability curve from single simulation run

Figure 4.11. Estimates of c.o.v assuming the conditional probability estimators are uncorrelated and fully correlated (upper bound)

Figure 4.10 shows the exceedance probability estimates for different threshold levels of economic loss estimated using a single simulation run (with $M = 1000$). Figure 4.11 shows the estimates of c.o.v assuming the conditional probability estimators are uncorrelated and fully correlated (upper bound) using the procedures described in the Section 4.4. These c.o.v bounds give a good idea about the quality of exceedance probability estimator obtained with just one simulation run and without any additional simulation runs.

Another appealing feature of the proposed method is that it does not only provide the exceedance probability as a function of threshold but also conditional samples which allow in-depth ‘damage’ and ‘failure’ analysis. To illustrate the type of information which can be obtained using the proposed method, Figure 4.12 shows the scattering of $Magt$ and $dist$ samples at different conditional levels of simulation, from a single simulation run. It indicates that for higher loss exceedance level, the samples shift towards the region with large magnitude and small distance. Figure 4.13 shows the IDR for the fourth story at different conditional levels of simulation. In case 1, since the $Magt$ and $dist$ are fixed, IDR are confined to a narrow band of values, however, the samples take higher values in the band as the conditional level increases. In case 2, the IDR take higher as the conditional level increases. Figure 4.14 shows the complementary CDF of the number of columns in different damage

states. As the simulation level increases, increasing number of columns appear in higher damage states. Figure 4.15 shows the contribution of each story to the mean total loss at different conditional levels of simulation and the maximum mean loss is observed at the fourth story. This indicates that the fourth story experiences the maximum damage which further indicates that the inter-story drift ratio reaches its maximum at the fourth story.

Figure 4.12. Conditional $Magt$ and $dist$ samples at conditional levels 0,1,2,3

Figure 4.13. Conditional *IDR* samples for the fourth story at conditional loss threshold levels 0,1,2,3

Figure 4.14. Number of columns in different damage states at various conditional loss threshold levels

Figure 4.15. Mean total loss from each story at various conditional loss threshold levels

Appendix 4.A: Proof for Stationary Distribution

Assume that $\boldsymbol{\theta}$ includes all random variables, independent and dependent. The global transition PDF of sampling algorithm used in modified Subset Simulation is given by the following:

$$K(\tilde{\boldsymbol{\theta}} | \boldsymbol{\theta}) = I(\tilde{\boldsymbol{\theta}}) \prod_{j=1}^G K_j(\tilde{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) + \dots \left[1 - \int I(\hat{\boldsymbol{\theta}}) \prod_{j=1}^G K_j(\hat{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) d\hat{\boldsymbol{\theta}} \right] \delta(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}) \quad (\text{A4.1})$$

where

$$K_j(\tilde{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) = T_j(\tilde{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) + \left[1 - a_j(\tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) \right] \delta(\tilde{\boldsymbol{\theta}}_j - \boldsymbol{\theta}_j)$$

and where $a_j(\tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) = \int T_j(\hat{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) d\hat{\boldsymbol{\theta}}_j$.

$K_j(\tilde{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G})$ is the transitional PDF for ordinary Metropolis Hastings algorithm which satisfy a specific property known as ‘reversibility’:

$$\begin{aligned} T_j(\tilde{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) \pi(\boldsymbol{\theta}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_{j+1:G}) &= T_j(\boldsymbol{\theta}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \tilde{\boldsymbol{\theta}}_j, \boldsymbol{\theta}_{j+1:G}) \pi(\tilde{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_{j+1:G}) \\ K_j(\tilde{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) \pi(\boldsymbol{\theta}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_{j+1:G}) &= K_j(\boldsymbol{\theta}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \tilde{\boldsymbol{\theta}}_j, \boldsymbol{\theta}_{j+1:G}) \pi(\tilde{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_{j+1:G}) \end{aligned} \quad (\text{A4.2})$$

To prove the validity of modified Metropolis-Hastings algorithms in modified Subset Simulation directly, the global transition PDF must satisfy the stationarity condition:

$$p(\tilde{\boldsymbol{\theta}}) = \int K(\tilde{\boldsymbol{\theta}} | \boldsymbol{\theta}) p(\boldsymbol{\theta} | F) d\boldsymbol{\theta} = p(\tilde{\boldsymbol{\theta}} | F) \quad (\text{A4.3})$$

Substituting Equation (A4.1) into the left hand side of (A4.3), it can be seen that

$$\int K(\tilde{\boldsymbol{\theta}} | \boldsymbol{\theta}) p(\boldsymbol{\theta} | F) d\boldsymbol{\theta} = H + p(\tilde{\boldsymbol{\theta}} | F) - J \quad (\text{A4.4})$$

where

$$\begin{aligned}
H &= \int I(\tilde{\boldsymbol{\theta}}) \prod_{j=1}^G K_j(\tilde{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) p(\boldsymbol{\theta} | F) d\boldsymbol{\theta} \\
&= I(\tilde{\boldsymbol{\theta}}) \int \prod_{j=1}^G K_j(\tilde{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) I(\boldsymbol{\theta}) p(\boldsymbol{\theta}) / P(F) d\boldsymbol{\theta} \\
&= I(\tilde{\boldsymbol{\theta}}) / P(F) \int I(\boldsymbol{\theta}) \prod_{j=1}^G K_j(\tilde{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) p(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad (\text{A4.5})
\end{aligned}$$

$$\begin{aligned}
J &= \int \left[\int I(\hat{\boldsymbol{\theta}}) \prod_{j=1}^G K_j(\hat{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) d\hat{\boldsymbol{\theta}} \right] \delta(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}) p(\boldsymbol{\theta} | F) d\boldsymbol{\theta} \\
&= p(\tilde{\boldsymbol{\theta}} | F) \int I(\hat{\boldsymbol{\theta}}) \prod_{j=1}^G K_j(\hat{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \tilde{\boldsymbol{\theta}}_j, \tilde{\boldsymbol{\theta}}_{j+1:G}) d\hat{\boldsymbol{\theta}} \quad (\text{A4.6})
\end{aligned}$$

Taking the inner part of the integral in H and applying the reversibility condition for each group of parameters gives:

$$\begin{aligned}
&\prod_{j=1}^G K_j(\tilde{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) p(\boldsymbol{\theta}) \\
&= \prod_{j=2}^G K_j(\tilde{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) K_1(\tilde{\boldsymbol{\theta}}_1 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_{2:G}) p(\boldsymbol{\theta}_1 | \boldsymbol{\theta}_{2:G}) p(\boldsymbol{\theta}_{2:G}) \\
&= K_1(\boldsymbol{\theta}_1 | \tilde{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_{2:G}) \prod_{j=2}^G K_j(\boldsymbol{\theta}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) p(\tilde{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_{2:G}) \\
&= K_1(\boldsymbol{\theta}_1 | \tilde{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_{2:G}) \prod_{j=3}^G K_j(\boldsymbol{\theta}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) K_2(\tilde{\boldsymbol{\theta}}_2 | \tilde{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_{3:G}) p(\boldsymbol{\theta}_2 | \tilde{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_{3:G}) p(\tilde{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_{3:G}) \\
&= K_1(\boldsymbol{\theta}_1 | \tilde{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_{2:G}) K_2(\boldsymbol{\theta}_2 | \tilde{\boldsymbol{\theta}}_1, \tilde{\boldsymbol{\theta}}_2, \boldsymbol{\theta}_{3:G}) \prod_{j=3}^G K_j(\boldsymbol{\theta}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) p(\tilde{\boldsymbol{\theta}}_2 | \tilde{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_{3:G}) p(\tilde{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_{3:G}) \\
&= \prod_{j=1}^2 K_j(\boldsymbol{\theta}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \tilde{\boldsymbol{\theta}}_j, \boldsymbol{\theta}_{j+1:G}) \prod_{j=3}^G K_j(\tilde{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1:G}) p(\tilde{\boldsymbol{\theta}}_{1:2}, \boldsymbol{\theta}_{3:G}) \\
&\vdots \\
&= \prod_{j=1}^G K_j(\boldsymbol{\theta}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \tilde{\boldsymbol{\theta}}_j, \boldsymbol{\theta}_{j+1:G}) p(\tilde{\boldsymbol{\theta}}_{1:G}) \quad (\text{A4.7})
\end{aligned}$$

Substituting Equation (A4.7) in Equation (A4.5) gives:

$$\begin{aligned}
H &= I(\tilde{\boldsymbol{\theta}}) / P(F) \int I(\boldsymbol{\theta}) \prod_{j=1}^G K_j(\boldsymbol{\theta}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \tilde{\boldsymbol{\theta}}_j, \boldsymbol{\theta}_{j+1:G}) p(\tilde{\boldsymbol{\theta}}) d\boldsymbol{\theta} \\
&= I(\tilde{\boldsymbol{\theta}}) / P(F) p(\tilde{\boldsymbol{\theta}}) \int I(\boldsymbol{\theta}) \prod_{j=1}^G K_j(\boldsymbol{\theta}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \tilde{\boldsymbol{\theta}}_j, \boldsymbol{\theta}_{j+1:G}) d\boldsymbol{\theta}
\end{aligned}$$

$$= p(\tilde{\boldsymbol{\theta}} | F) \int I(\boldsymbol{\theta}) \prod_{j=1}^G K_j(\boldsymbol{\theta}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \tilde{\boldsymbol{\theta}}_j, \boldsymbol{\theta}_{1:j-1}) d\boldsymbol{\theta} \\ \neq J$$

It is seen that the global stationarity condition in Equation (A4.4) will only be satisfied if $H = J$, and that is only possible if the transition PDF for the j -th group is either only dependent on the variables in the j -th group (as in Subset Simulation) or is independent of all succeeding groups, i.e.,

$$K_j = K_j(\tilde{\boldsymbol{\theta}}_j | \tilde{\boldsymbol{\theta}}_{1:j-1}, \boldsymbol{\theta}_j) \quad (\text{A4.8})$$

The transition PDF for the loss samples in the sampling algorithm used in the proposed method is of the form $K_G(\boldsymbol{\theta}_G | \tilde{\boldsymbol{\theta}}_{1:G-1})$. Thus samples generated following the above global transition PDF satisfy the stationarity condition, though the global reversibility (detailed balanced) condition is not satisfied.

Appendix 4.B: Proof for Some Statistical Properties of the Proposed Estimators

Suppose that the total number of samples simulated at each conditional level are N , the number of Markov chain is $N_c = p_0 N$, where the length of each chain is $L_c = N / N_c$. For simplicity in notation let $I_{jkl}^{(i)} = I_{F_i}(L^{(i-1,jk,l)})$ where $L^{(i-1,jk,l)}$ is the l -th loss sample distributed according to PDF $p(L | \boldsymbol{\theta}^{(i-1,jk)})$ and where $\boldsymbol{\theta}^{(i-1,jk)}$ is the k -th sample in the j -th Markov chain at the $(i-1)$ -th simulation level. Neglecting the correlation between the samples from different Markov Chains gives:

$$\begin{aligned} \mathbb{E}[(\tilde{P}_i - P_i)^2] &= \mathbb{E}\left[\left(\frac{1}{N} \sum_{j=1}^{N_c} \sum_{k=1}^{L_c} \frac{1}{M} \sum_{l=1}^M (I_{jkl}^{(i)} - P_i)\right)^2\right] \\ &= \frac{1}{N^2} \sum_{j=1}^{N_c} \mathbb{E}\left[\left(\sum_{k=1}^{L_c} \frac{1}{M} \sum_{l=1}^M (I_{jkl}^{(i)} - P_i)\right)^2\right] \\ &= \frac{N_c}{N^2} \mathbb{E}\left[\left(\sum_{k=1}^{L_c} \frac{1}{M} \sum_{l=1}^M (I_{jkl}^{(i)} - P_i)\right)^2\right] \end{aligned} \quad (\text{B4.1})$$

For the j -th chain:

$$\begin{aligned}
& \mathbb{E} \left[\left(\sum_{k=1}^{L_c} \frac{1}{M} \sum_{l=1}^M (I_{jkl}^{(i)} - P_i) \right)^2 \right] \\
&= \sum_{k_1, k_2=1}^{L_c} \mathbb{E} \left[\frac{1}{M^2} \sum_{l_1, l_2=1}^M (I_{jk_1 l_1}^{(i)} - P_i)(I_{jk_2 l_2}^{(i)} - P_i) \right] \\
&= \sum_{k_1=1}^{L_c} \mathbb{E} \left[\frac{1}{M^2} \sum_{l_1, l_2=1}^M (I_{jk_1 l_1}^{(i)} - P_i)(I_{jk_1 l_2}^{(i)} - P_i) \right] \\
&\quad + 2 \sum_{k_1 > k_2}^{L_c} \mathbb{E} \left[\frac{1}{M^2} \sum_{l_1, l_2=1}^M (I_{jk_1 l_1}^{(i)} - P_i)(I_{jk_2 l_2}^{(i)} - P_i) \right] \\
&= \sum_{k_1=1}^{L_c} \mathbb{E} \left[\frac{1}{M^2} \sum_{l_1=1}^M (I_{jk_1 l_1}^{(i)} - P_i)(I_{jk_1 l_1}^{(i)} - P_i) + \frac{1}{M^2} \sum_{l_1 > l_2}^M (I_{jk_1 l_1}^{(i)} - P_i)(I_{jk_1 l_2}^{(i)} - P_i) \right] \\
&\quad + 2 \sum_{k_1 > k_2}^{L_c} \mathbb{E} \left[\frac{1}{M^2} \sum_{l_1, l_2=1}^M (I_{jk_1 l_1}^{(i)} - P_i)(I_{jk_2 l_2}^{(i)} - P_i) \right] \\
&= \sum_{k_1=1}^{L_c} \left[\frac{1}{M^2} \sum_{l_1=1}^M \mathbb{E} \left[(I_{jk_1 l_1}^{(i)} - P_i)(I_{jk_1 l_1}^{(i)} - P_i) \right] \right. \\
&\quad \left. + \frac{1}{M^2} \sum_{l_1 > l_2}^M \mathbb{E} \left[(I_{jk_1 l_1}^{(i)} - P_i)(I_{jk_1 l_2}^{(i)} - P_i) \right] \right] \\
&\quad + 2 \sum_{k_1 > k_2}^{L_c} \mathbb{E} \left[\frac{1}{M^2} \sum_{l_1, l_2=1}^M (I_{jk_1 l_1}^{(i)} - P_i)(I_{jk_2 l_2}^{(i)} - P_i) \right] \\
&= \sum_{k_1=1}^{L_c} \frac{R_0^{(i)}}{M} + 2 \sum_{k_1=1}^{L_c} \sum_{l_1 > l_2}^M R_1^{(i)}(l_1 - l_2) + 2 \sum_{k_1 > k_2}^{L_c} R_2^{(i)}(k_1 - k_2) \\
&= L_c \left[\frac{R_0^{(i)}}{M} + \frac{2}{M} \sum_{l=1}^{M-1} \left(1 - \frac{l}{M} \right) R_1^{(i)}(l) + 2 \sum_{k=1}^{L_c-1} \left(1 - \frac{k}{L_c} \right) R_2^{(i)}(k) \right] \tag{B4.2}
\end{aligned}$$

Substituting Equation (B4.2) in Equation (B4.1) gives:

$$\begin{aligned}
\mathbb{E} \left[(\tilde{P}_i - P_i)^2 \right] &= \frac{1}{N} \left[\frac{R_0^{(i)}}{M} + 2 \sum_{l=1}^{M-1} \left(1 - \frac{l}{M} \right) R_1^{(i)}(l) + 2 \sum_{k=1}^{L_c-1} \left(1 - \frac{k}{L_c} \right) R_2^{(i)}(k) \right] \\
&= \frac{R_0^{(i)}}{NM} \left[1 + 2 \sum_{l=1}^{M-1} \left(1 - \frac{l}{M} \right) \frac{R_1^{(i)}(l)}{R_0^{(i)}} + 2M \sum_{k=1}^{L_c-1} \left(1 - \frac{k}{L_c} \right) \frac{R_2^{(i)}(k)}{R_0^{(i)}} \right] \\
&= \frac{R_0^{(i)}}{NM} \left[1 + \gamma_1^{(i)} + \gamma_2^{(i)} \right] \tag{B4.3}
\end{aligned}$$

Chapter 5

Updating Probabilistic Seismic Loss Estimation

5.1 Introduction

Taking into account the most up-to-date condition of a structure by collecting vibration data to infer, especially structural model parameters can provide more accurate prediction of its future performance. In this chapter, the proposed approach for the estimation of seismic loss exceedance probability presented in Chapter 4 is integrated with the proposed algorithm for computing the updated robust failure probability based on the incomplete modal data presented in Chapter 3, for the estimation of updated seismic loss exceedance probability as a function of threshold. Using the same example as in Chapter 4, the updated exceedance probabilities are obtained as a function of economic loss threshold.

5.2 The Proposed Approach

In Chapter 3, an integral for the updated robust failure probability conditioned on the data was presented:

$$P(F | D) = \int I_F(\boldsymbol{\theta}) p(\boldsymbol{\theta} | D) d\boldsymbol{\theta} \quad (5.1)$$

For updating the seismic loss exceedance probability, the integral is modified to:

$$P(F | D) = \int P(L > b | \boldsymbol{\theta}, D) p(\boldsymbol{\theta} | D) d\boldsymbol{\theta} \quad (5.2)$$

where $\boldsymbol{\theta} = [\mathbf{Z} \ \boldsymbol{\theta}_g \ \boldsymbol{\theta}_s \ \boldsymbol{\theta}_f \ \boldsymbol{\theta}_l]$. The Subset Simulation algorithm is modified to compute the updated seismic loss exceedance probability as follows:

$$P(F | D) = P(F_m | D) = \left[\prod_{i=1}^{m-1} P(F_{i+1} | F_i, D) \right] P(F_1 | D) \quad (5.3)$$

where $P(F_1 | D)$ is estimated as follows:

$$P(F_1 | D) \approx \frac{1}{N} \sum_{k=1}^N \frac{1}{M} \sum_{l=1}^M I_{F_1}(L^{(k,l)}) \quad (5.4)$$

where $\{L^{(k,l)} : l=1, \dots, M\}$ are i.i.d samples distributed as $p(L | \theta^{(k)})$. Samples $\{\theta^{(k)} : k=1, \dots, N\}$ are distributed as $p(\theta | D)$. Assume data D do not provide any information which can update the PDF of $[\mathbf{Z} \ \theta_g \ \theta_f \ \theta_l]$. Samples $\{\mathbf{Z}^{(k)} : k=1, \dots, N\}$ and $\{\theta_g^{(k)} : k=1, \dots, N\}$ distributed as $p(\mathbf{Z})$ and $p(\theta_g)$, respectively, can easily be simulated since it is assumed that the input is specified by a stochastic input model. Samples $\{\theta_f^{(k)} : k=1, \dots, N\}$ and $\{\theta_l^{(k)} : k=1, \dots, N\}$ are simulated from their prior PDFs $p(\theta_f)$ and $p(\theta_l)$, respectively. Samples $\{\theta_s^{(k)} : k=1, \dots, N\}$ distributed as $p(\theta_s | D)$ can be simulated using the Gibbs sampling based algorithm presented in Chapter 2.

$P(F_{i+1} | F_i, D)$ in Equation (5.3) is estimated using samples $\{\theta^{(k)} : k=1, \dots, N\}$ distributed as $p(\theta | F_i, D)$. Simulation of samples distributed as $p(\theta | F_i, D)$ for a nonlinear problem is discussed in Chapter 3. The same procedure is adopted here with minor modifications.

Considering the same four independent groups of parameters as proposed in Gibbs Sampling based algorithm for model updating in Chapter 2:

$$\theta_1 = [\alpha^T \ \beta^T \ \eta^T]^T$$

$$\theta_2 = \mathbf{c}$$

$$\theta_3 = [\text{Re}(\psi_1)^T \ \text{Im}(\psi_1)^T \ \cdots \ \text{Re}(\psi_M)^T \ \text{Im}(\psi_M)^T]^T$$

$$\theta_4 = [\sigma_{\text{Re},1}^2 \ \sigma_{\text{Im},1}^2 \ \cdots \ \sigma_{\text{Re},M}^2 \ \sigma_{\text{Im},M}^2]^T$$

Given a current state $\theta_s^{(k)}$ a candidate state $\tilde{\theta}_s^{(k+1)}$, where $\theta_s = [\theta_1 \ \theta_2]$, is generated by:

1. Determine the mean vector $\boldsymbol{\mu}_1^{(k)}$ and covariance matrix $\boldsymbol{\Sigma}_1^{(k)}$ of the Gaussian PDF $p(\boldsymbol{\theta}_1 | \boldsymbol{\theta}_2^{(k)}, \boldsymbol{\theta}_3^{(k)}, \boldsymbol{\theta}_4^{(k)}, D)$ using Equations (2.28) and (2.29) respectively.
Let $\mathbf{w}_1^{(k)} = \mathbf{L}_1^{(k)-1}(\boldsymbol{\theta}_1^{(k)} - \boldsymbol{\mu}_1^{(k)})$ where $\boldsymbol{\Sigma}_1^{(k)} = \mathbf{L}_1^{(k)} \mathbf{L}_1^{(k)T}$.
2. Generate a candidate state $\tilde{\mathbf{w}}_1^{(k+1)} = [\tilde{\mathbf{w}}_{1,j}^{(k+1)} : j = 1, \dots, N_1]$ for $\mathbf{w}_1$:
For each component $j = 1, \dots, N_1$, draw a pre-candidate component $\xi_{1,j}^{(k+1)}$ from the proposal PDF $q_{1,j}(\xi_{1,j} | w_{1,j}^{(k)})$ and compute the acceptance ratio:

$$r_{1,j}^{(k+1)} = \min \left\{ 1, \frac{p(\xi_{1,j}^{(k+1)})}{p(w_{1,j}^{(k)})} \frac{q_{1,j}(w_{1,j}^{(k)} | \xi_{1,j}^{(k+1)})}{q_{1,j}(\xi_{1,j}^{(k+1)} | w_{1,j}^{(k)})} \right\}$$
Set $\tilde{w}_{1,j}^{(k+1)} = \begin{cases} \xi_{1,j}^{(k+1)} & \text{with probability } r_{1,j}^{(k+1)} \\ \tilde{w}_{1,j}^{(k)} & \text{with probability } 1 - r_{1,j}^{(k+1)} \end{cases}$
3. Transform $\tilde{\mathbf{w}}_1^{(k+1)}$ back to $\boldsymbol{\theta}_1$ space: $\tilde{\boldsymbol{\theta}}_1^{(k+1)} = \boldsymbol{\mu}_1^{(k)} + \mathbf{L}_1^{(k)} \tilde{\mathbf{w}}_1^{(k+1)}$.
4. For getting $\tilde{\boldsymbol{\theta}}_2^{(k+1)}$, repeat steps i) to iii) by reversing the subscripts 1 and 2 and replacing $\boldsymbol{\theta}_1^{(k)}$ by $\tilde{\boldsymbol{\theta}}_1^{(k+1)}$.
5. Sample $\tilde{\boldsymbol{\theta}}_3^{(k+1)}$ from PDF $p(\boldsymbol{\theta}_3 | \tilde{\boldsymbol{\theta}}_1^{(k+1)}, \tilde{\boldsymbol{\theta}}_2^{(k+1)}, \boldsymbol{\theta}_4^{(k)}, D)$ as described in Section 2.3.3.
6. Sample $\tilde{\boldsymbol{\theta}}_4^{(k+1)}$ from PDF $p(\boldsymbol{\theta}_4 | \tilde{\boldsymbol{\theta}}_1^{(k+1)}, \tilde{\boldsymbol{\theta}}_2^{(k+1)}, \tilde{\boldsymbol{\theta}}_3^{(k+1)}, D)$ as described in Section 2.3.4.

Assuming $\boldsymbol{\theta}_g$ is divided into G independent groups, i.e., $\boldsymbol{\theta}_g = [\theta_{g,1}, \theta_{g,2}, \dots, \theta_{g,G}]$.

Given a current state $\boldsymbol{\theta}_g^{(k)}$, a candidate state $\tilde{\boldsymbol{\theta}}_g^{(k+1)}$ from the PDF $p(\boldsymbol{\theta}_g)$ is generated using the multi-group Metropolis-Hastings algorithm by repeating the following steps from $j = 1$ till $j = G$:

1. For the j -th group, draw a pre-candidate component $\xi_{g,j}^{(k+1)}$ from the proposal PDF $q_{g,j}$ and compute the acceptance ratio $r_{g,j}$ as follows:

$$r_{g,j}^{(k+1)} = \min \left\{ 1, \frac{p(\xi_{g,j}^{(k+1)})}{p(\theta_{g,j}^{(k)})} \frac{q_{g,j}(\theta_{g,j}^{(k)} | \xi_{g,j}^{(k+1)})}{q_{g,j}(\xi_{g,j}^{(k+1)} | \theta_{g,j}^{(k)})} \right\}$$

2. If $u < \min(1, r_{g,j}^{(k+1)})$, where u is uniformly distributed between 0 and 1, $\xi_{g,j}^{(k+1)}$ is accepted as the candidate state, i.e., $\tilde{\theta}_{g,j}^{(k+1)} = \xi_{g,j}^{(k+1)}$, otherwise, the current state is accepted as the candidate state, i.e., $\tilde{\theta}_{g,j}^{(k+1)} = \theta_{g,j}^{(k)}$.

Similarly, given a current state $\theta_f^{(k)}$, a candidate state $\tilde{\theta}_f^{(k+1)}$ is generated using the multi-group Metropolis-Hastings algorithm with $p(\theta_f)$ as the target PDF. Given a current state $\theta_l^{(k)}$, a candidate state $\tilde{\theta}_l^{(k+1)}$ is generated using the multi-group Metropolis-Hastings algorithm with $p(\theta_l)$ as the target PDF.

Given a current state $\mathbf{Z}^{(k)}$ where $\mathbf{Z} = [Z_1, Z_2, \dots, Z_{N_t}]$, a candidate state $\tilde{\mathbf{Z}}$ from the PDF $p(\mathbf{Z})$ is generated using the multi-group Metropolis-Hastings algorithm by repeating the following steps from $j = 1$ till $j = N_t$:

1. For the j -th group, draw a pre-candidate component $\xi_{\mathbf{Z},j}^{(k+1)}$ from the proposal PDF $q_{\mathbf{Z},j}$ and compute the acceptance ratio:

$$r_{\mathbf{Z},j}^{(k+1)} = \min \left\{ 1, \frac{p(\xi_{\mathbf{Z},j}^{(k+1)}) q_{\mathbf{Z},j}(Z_j^{(k)} | \xi_{\mathbf{Z},j}^{(k+1)})}{p(Z_j^{(k)}) q_{\mathbf{Z},j}(\xi_{\mathbf{Z},j}^{(k+1)} | Z_j^{(k)})} \right\}$$

2. Set $\tilde{Z}_j^{(k+1)} = \begin{cases} \xi_{\mathbf{Z},j}^{(k+1)} & \text{with probability } r_{\mathbf{Z},j}^{(k+1)} \\ Z_j^{(k)} & \text{with probability } 1 - r_{\mathbf{Z},j}^{(k+1)} \end{cases}$

Using $\tilde{\theta}^{(k+1)} = [\tilde{\mathbf{Z}}^{(k+1)} \tilde{\theta}_g^{(k+1)} \tilde{\theta}_s^{(k+1)} \tilde{\theta}_f^{(k+1)} \tilde{\theta}_l^{(k+1)}]$, a candidate damage state random vector $\tilde{\mathbf{DM}}^{(k+1)}$ is simulated according to Equation (4.4), and conditioned on this a candidate loss sample $\tilde{L}^{(k+1)}$ is simulated according to Equation (4.5). At last, if the simulated $\tilde{L}^{(k+1)}$ falls in the failure domain F_i , i.e., if $\tilde{L}^{(k+1)} > b_i$, the candidate state $\tilde{\theta}^{(k+1)}$ is accepted as the next state. If not, the current state $\theta^{(k)}$ is taken as the next state. Given $\{\theta^{(k)} : k = 1, \dots, N\}$ samples distributed according to the PDF $p(\theta | F_i)$, any number of loss samples distributed according to PDF $p(L | F_i)$ can be obtained

by first simulating samples according to $p(L|\boldsymbol{\theta})$ (following Equations (4.4)-(4.5)) and then accepting only those that fall in F_i .

5.3 An Illustrative Example

The example presented in Chapter 4 is used in this chapter. The goal here is to calculate the posterior robust exceedance probability as a function of the threshold of total economic loss when the structure is subjected to future ground shaking from earthquakes. The future stochastic excitation model parameters, nominal structural parameters, and fragility and loss functions are given in the illustrative example in Chapter 4.

The structural parameters are assumed to be uncertain. The prior PDFs of $\{m_{i,0} : i = 1, 2, \dots, 7\}$, $\{c_{i,0} : i = 1, 2, \dots, 7\}$ and $\{k_{i,0} : i = 1, 2, \dots, 7\}$ are taken to be independent Gaussian with mean equal the optimal Finite element adjusted values (Ching et al. (2006a)) with c.o.v equal to 1% for mass, and 20% for damping and stiffness parameters. In addition, independent Gaussian PDFs are assumed for γ , ρ and λ with mean equal to optimal values and 20% c.o.v.

The damage in the structure is simulated by reducing the stiffnesses of the optimal model at the fourth and fifth story by 30%. The modal data consist of $S = 5$ sets of simulated modal data obtained from the damaged model. Each set consists of the first three modal frequencies, modal damping ratios, and partial complex mode shapes $M = 3$ (corresponding to DOFs, one, three, five and seven, $N_o = 4$). Noisy measured modal parameters are generated by adding random values chosen from zero-mean Gaussian distribution with standard deviation equal to 2% of the exact values. Using the model data, only the joint PDF of $\{m_{i,0} : i = 1, 2, \dots, 7\}$, $\{c_{i,0} : i = 1, 2, \dots, 7\}$ and $\{k_{i,0} : i = 1, 2, \dots, 7\}$ are updated, since, it is assumed that the modal data are obtained for a structure behaving approximately linearly. No information is provided by the modal data to update γ , ρ and λ , which define the nonlinear behavior of the system.

The proposal PDF for the i -th conditional level for each uncertain structural parameter is chosen as a uniform PDF centered at the current sample with width equal to twice of its standard deviation obtained using the samples distributed according to PDFs $p(\boldsymbol{\theta}_s | F_i)$.

Figure 5.1 Prior robust, posterior robust and nominal exceedance probability against the threshold of total economic loss (Case 1, fixed Mag_t and $dist$)

Figure 5.2 Prior robust, posterior robust and nominal exceedance probability against the threshold of total economic loss (Case 2, uncertain Mag_t and $dist$)

The posterior robust exceedance probability is compared with the prior robust and nominal system exceedance probability. Figure 5.1 and Figure 5.2 show the prior robust, posterior robust and nominal system exceedance probabilities for the two cases, one with fixed $Magt$ and $dist$, and the other with uncertain $Magt$ and $dist$, respectively. It can be seen very clearly in Figure 5.1 that the posterior robust exceedance probability is quite different from the other exceedance probabilities due to different levels of model uncertainties, confirming the importance of using data to update the exceedance probability. Some degree of difference is also observed in Figure 5.2 although not very significant. A possible reason for such behavior is that the damage states for each component of the building are discrete. A structure when subjected to two different excitations can have different inter-story drift ratios but may still have very similar economic loss in the case if the inter-story drift ratios correspond to the same damage states. However, the importance of using the updated robust exceedance probability for decision makings cannot be disputed based on results from one example.

Chapter 6

Reliability Analysis with Multiple Performance Objectives

6.1 Introduction

Performance-based engineering aims to quantify the performance of systems based on quantifiable and probabilistic performance functions. Performance functions are statements of acceptable performance of the system, defined by the performance quantities of interest attached to certain specified thresholds. For complex dynamic system problems involving high stochastic dimension, reliability analysis by state-of-the-art stochastic simulation based techniques has proved to be very efficient and reliable, especially for estimating small failure probabilities (Schuëller and Pradlwarter, 2007). In the literature, a few stochastic simulation based techniques that are applicable to solving the problem of interest can be found, such as Importance Sampling (Au and Beck, 2001b), Subset Simulation (Au and Beck, 2001a, 2003), Domain Decomposition Method (Katafygiotis and Cheung, 2006), Auxiliary Domain Method (Katafygiotis et al., 2007), Line Sampling (Pradlwarter et al., 2007), Spherical Subset Simulation (Katafygiotis et al., 2010). Stochastic simulation techniques for risk and loss assessment of structural systems are proposed and demonstrated in Chapters 3, 4 and 5. However, all these techniques focus on obtaining failure probabilities as a function of a single threshold. They assume a single performance function (even for the case with multiple output responses of interest), or if there are multiple performance functions for a system, they are correlated to one another in some way to obtain a single performance function, such as in Parallel Subset Simulation (Hsu and Ching, 2010) where a

Parts of the work presented in this chapter appear in the conference papers (Bansal and Cheung, 2014).

principal variable is introduced which is correlated with all performance functions. For a problem with multiple performance objectives which involves evaluation of failure probability as a function of multiple thresholds, the aforementioned techniques will require repetitive reliability analyses that can be very inefficient (unless the combination of thresholds corresponding to which the failure probability is evaluated is known a priori).

In this chapter, a stochastic simulation approach is proposed to evaluate the failure probability as a function of multiple thresholds with each threshold corresponding to one performance objective in a problem involving multiple performance objectives. The approach adopts and modifies the Subset Simulation algorithm that can efficiently compute small failure probabilities. The key idea here is to obtain a failure probability hypersurface at each conditional level in the Subset Simulation using conditional samples. Conditional samples are generated using samples belonging to a failure domain defined by a particular combination of thresholds. The proposed approach is robust with respect to the dimension of the failure probability integral, model complexity and nonlinearity.

6.2 Reliability Problem with Multiple Performance Objectives

System reliability is concerned with the probability that the system will not reach some specific ‘failure’ states subjected to stochastic excitations. It involves calculating the reliability, or its complement the failure probability P_F , which requires the evaluation of a multi-dimensional integral of the form:

$$P_F = \int_F p(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad (6.1)$$

where $\boldsymbol{\theta} \in R^{n_\theta}$ is the parameter vector containing all the uncertain quantities of interest quantified by a joint PDF $p(\boldsymbol{\theta})$ and $F = \{\boldsymbol{\theta} : g(\boldsymbol{\theta}) > 0\}$ is the failure region with $g(\boldsymbol{\theta})$ as the performance function that separates the safe domain $g(\boldsymbol{\theta}) \leq 0$ and the failure domain $g(\boldsymbol{\theta}) > 0$. One example of a performance function is:

$$g(\boldsymbol{\theta}) = u(\boldsymbol{\theta}) - c \quad (6.2)$$

where c is a threshold value and $u(\boldsymbol{\theta})$ is the quantity in terms of all output response quantities of interest of the model specified by $\boldsymbol{\theta}$. A reliability analysis with this performance function estimates the ‘failure’ probability that u exceeds the threshold c . For complex dynamic systems where the failure is defined by a union of multiple failure criteria, the failure surface can be expressed as:

$$\pi = \bigcup_{l=1}^G \{g_l(\boldsymbol{\theta}) = 0\} \quad (6.3)$$

Let vectors $\mathbf{U}(\boldsymbol{\theta}) = [u_1(\boldsymbol{\theta}), u_2(\boldsymbol{\theta}), \dots, u_G(\boldsymbol{\theta})]^T$ and $\mathbf{C} = [c_1, c_2, \dots, c_G]^T$ be the response quantity vector and the corresponding threshold vector in R^G , respectively. The failure probability as a function of threshold vector $\mathbf{C}$ can then be evaluated as:

$$P_F(\mathbf{C}) = P\left(\bigcup_{l=1}^G \{u_l(\boldsymbol{\theta}) > c_l\}\right) = \int I_F(\mathbf{U}(\boldsymbol{\theta}), \mathbf{C}) p(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad (6.4)$$

where $F = \bigcup_{l=1}^G \{u_l(\boldsymbol{\theta}) > c_l\}$ and I_F is an indicator function, which is equal to 1 in case of failure, i.e. $\boldsymbol{\theta}$ and $\mathbf{C}$ are such that $\bigcup_{l=1}^G \{u_l(\boldsymbol{\theta}) > c_l\}$ is true, and 0 otherwise. For a threshold vector of dimension G , the above expression will correspond to hypersurface in a $(G+1)$ -dimensional space. For example, in a two-dimensional threshold vector space, three-dimensional failure probability surface is obtained (represented by two-dimensional contour lines in Figure 6.1), in a three-dimensional threshold vector space, four-dimensional failure probability hypersurface is obtained (represented by three-dimensional contour surfaces in Figure 6.1), and so on. The failure probability contour surface (i.e., in G -dimensional space) corresponding to some specified failure probability p , i.e., $P_F(\mathbf{C}) = p$, is given by:

$$\pi = \{\mathbf{C} \in R^G : P_F(\mathbf{C}) = p\} \quad (6.5)$$

There are infinitely many threshold quantities for the threshold vector when $G > 1$ that correspond to the same specified failure probability, unlike in the original Subset Simulation where there is only one threshold quantity.

To efficiently evaluate the integral in Equation (6.4), the Subset Simulation based approach is modified to estimate the failure probabilities of multiple performance objectives as a function of various combinations of thresholds, each threshold corresponding to each performance objective and efficiently propagate the failure samples from one failure domain to the next failure domain. A brief review of the Subset Simulation is presented in Section 3.1.2.

Figure 6.1 Failure probability contour lines in two-dimensional and contour surfaces in three-dimensional threshold vector space

6.3 The Proposed Approach

The integral in Equation (6.4) can be approximated by the following estimator:

$$P_F(\mathbf{C}) \approx \frac{1}{N} \sum_{k=1}^N I_F(\mathbf{U}(\boldsymbol{\theta}^{(k)}), \mathbf{C}) \quad (6.6)$$

where samples $\{\boldsymbol{\theta}^{(k)} : k = 1, \dots, N\}$ are distributed according to the PDF $p(\boldsymbol{\theta})$. However, using Equation (6.6) will be computationally expensive especially when dealing with small failure probability as the number of samples N required to achieve a given c.o.v is inversely proportional to the failure probability. The proposed procedure for a reliability analysis with multiple performance functions, which involves evaluating failure probability as a function of multiple thresholds, uses the same “sequence of failure events” idea as in Subset Simulation but the way

in which the intermediate failure domains are selected and samples are propagated between successive failure domains is modified. The Subset Simulation algorithm is adapted to compute the failure probability as a function of threshold vector $\mathbf{C}$ as follows:

1. At level 0, i.e., MCS level, failure probability estimates as a function of threshold vector are obtained using Equation (6.6) using samples $\{\boldsymbol{\theta}^{(k)} : k = 1, \dots, N\}$ distributed according to the PDF $p(\boldsymbol{\theta})$. A series of failure probability contour surfaces are obtained such that all combinations of threshold vectors on each contour surface correspond to a specified failure probability.
2. For obtaining smaller failure probabilities (i.e., $< p_0$) as a function of threshold vector, conditional samples are used as in the original Subset Simulation. A failure probability contour surface with pre-specified conditional failure probability p_0 is selected from the series of failure probability contour surface estimated in step 1. This failure probability contour surface is given by:

$$\pi_1 = \{\mathbf{C} \in R^G : P_F(\mathbf{C}) = p_0\} \quad (6.7)$$

3. There are infinitely many threshold combinations when $G > 1$ that correspond to the same specified failure probability, unlike in the original Subset Simulation where there is only one threshold quantity. To tackle this problem, a point $\mathbf{q}_1 \in \pi_1$ is selected (the selection of point $\mathbf{q}_i$ is discussed later), corresponding to the failure domain given by:

$$F_1 = F_{\mathbf{q}_1} = \{\boldsymbol{\theta} : I_F(\mathbf{U}(\boldsymbol{\theta}), \mathbf{q}_1) = 1\} \quad (6.8)$$

where $P(F_1) = p_0$. Figure 6.2 shows a schematic illustration (in 2-D space) of a failure probability contour line π_1 with point $\mathbf{q}_1$ on it and failure domain F_1 in the $\boldsymbol{\theta}$ space (for illustration, shown in 2-D space) corresponding to point $\mathbf{q}_1$.

4. Let $P_1 = P_F(\mathbf{q}_1)$. There will be Np_0 $\boldsymbol{\theta}$ samples distributed according to $p(\boldsymbol{\theta} | F_1)$. Using these samples as seeds, additional $N(1-p_0)$ $\boldsymbol{\theta}$ samples which are also distributed according to $p(\boldsymbol{\theta} | F_1)$ are generated by MCMC

simulation method based on the modified Metropolis–Hastings method as in the original Subset Simulation. These samples $\{\boldsymbol{\theta}^{(1,k)} : k=1, \dots, N\}$ distributed according to the $p(\boldsymbol{\theta} | F_1)$ can be used to estimate the failure probability as a function of threshold vector as follows:

$$P_F(\mathbf{C}) \approx P_1 \times \frac{1}{N} \sum_{k=1}^N I_F(\mathbf{U}(\boldsymbol{\theta}^{(1,k)}), \mathbf{C}) \quad (6.9)$$

5. For estimating very small failure probabilities (i.e., $\ll 1$), steps 2-4 are repeated with subscript 1 replaced by an iterative index i . The target failure probability as a function of threshold vector is given by:

$$P_F(\mathbf{C}) = \prod_{i=1}^H P_i / P_{i-1} \quad (6.10)$$

where $P_0 = 1$. The failure probability contour surface π_i is given by:

$$\pi_i = \{\mathbf{C} \in R^G : P_F(\mathbf{C}) = \prod_{s=1}^i P_s / P_{s-1}\} \quad (6.11)$$

The failure domain F_i corresponding to $\mathbf{q}_i \in \pi_i$ is given by:

$$F_i = F_{\mathbf{q}_i} = \{\boldsymbol{\theta} : I_F(\mathbf{U}(\boldsymbol{\theta}), \mathbf{q}_i) = 1\} \quad (6.12)$$

If failure probability estimates are obtained for some threshold vector $\mathbf{C}$ using samples distributed according to the PDF $p(\boldsymbol{\theta} | F_i)$, the above proposed procedures hold true as long as the failure domain corresponding to threshold vector $\mathbf{C}$ is a subset of the failure domain F_i . The set of threshold vectors satisfying the above condition are shown in Figure 6.3.

The area in the grey in Figure 6.3. corresponds to threshold vectors where the estimates of failure probability can be obtained using samples distributed according to the PDF $p(\boldsymbol{\theta} | F_{i-1})$ but for these threshold vectors such that $P_F(\mathbf{C}) < P_i$, the c.o.v of the conditional probability estimator will be higher than desired. Failure probability estimates in the grey area can be improved by either increasing the conditional probability p_0 or by increasing the total number of samples N at each

conditional level, but at the expense of increasing computational effort. Still, relatively poorer estimates of failure probability are obtained in the areas shown in dark grey in Figure 6.4.

Figure 6.2 Failure probability contour surface π_1 in the threshold vector space and failure domain F_1 in θ space corresponding to point $\mathbf{q}_1$ on π_1 (show systematically in 2-D space)

The discontinuity surface between different conditional levels is shown in Figure 6.4. At any point on the discontinuity surface, two estimates of failure probability are obtained, first, from the samples distributed according to the PDF $p(\theta | F_{i-1})$, and second, from samples distributed according to the PDF $p(\theta | F_i)$. Estimates obtained from samples distributed according to the PDF $p(\theta | F_{i-1})$ will always have higher c.o.v than the estimates obtained from samples distributed according to the PDF $p(\theta | F_i)$. Assuming $i=1$ and $N=500$, two failure probability estimates are obtained at the threshold vector $\mathbf{C}_1$ (shown in Figure 6.4). For $P(\mathbf{C}_1)=0.01$, the estimator using the samples distributed according to the PDF $p(\theta)$ will have c.o.v equal to 0.44, while assuming zero correlation for Markov chain samples and full correlation between the estimators for conditional failure probability for different conditional levels, the estimates using the samples distributed according to PDF $p(\theta | F_1)$ will have c.o.v. equal to 0.23.

Figure 6.3 Threshold vectors C with failure probability estimates that can be obtained using samples distributed according to the PDF $p(\theta | F_i)$

Figure 6.4 Discontinuity surface between different conditional levels

Although the formulation has been presented for the general case of multiple performance objectives, the graphical representation of failure probabilities in more than two dimensions becomes difficult.

The proposed approach can be extended to estimate failure probabilities with multiple loss objectives as a function of multiple loss threshold vector by modifying the estimators in Equations (6.6), and using the sampling technique presented in Chapter 4 to obtain samples distributed according to the conditional PDFs.

6.3.1 Selection of intermediate thresholds vectors $\{\mathbf{q}_i : i = 1, \dots, H\}$

The intermediate threshold vectors $\{\mathbf{q}_i : i = 1, \dots, H\}$ are selected on failure probability contour surface π_i , such that the conditional PDFs P_i / P_{i-1} are approximately equal to some specified value p_0 . However, there are multiple alternatives available for selecting the intermediate threshold vectors $\{\mathbf{q}_i : i = 1, \dots, H\}$, unlike in the original Subset simulation where failure is parameterized by a single threshold parameter that allows one to select the intermediate failure events by varying the parameter. In the proposed approach, $\mathbf{q}_i$ on failure probability contour surface π_i is given by $\mathbf{q}_i = [c_{1,i} \ c_{2,i} \ \dots \ c_{G,i}]^T$ where $c_{1,i}, c_{2,i}, \dots$, and $c_{G,i}$ are thresholds that satisfy the following:

$$P_F([\infty \ c_{2,i} \ \dots \ c_{G,i}]^T) = P_F([c_{1,i} \ \infty \ \dots \ c_{G,i}]^T) = \dots = P_F([c_{1,i} \ c_{2,i} \ \dots \ \infty]^T) \quad (6.13)$$

Figure 6.5 Selection of intermediate thresholds vector $\mathbf{q}_i$

This is shown schematically in a 2-dimensional threshold space in Figure 6.5 where for $\mathbf{q}_i = [c_{1,i} \ c_{2,i}]^T$ $P_F([c_{1,i} \ \infty]^T) = P_F([\infty \ c_{2,i}]^T)$. The objective of selecting $\mathbf{q}_i$ using this procedure is to capture the curvature in the failure probability contour surface in the succeeding simulation level.

$P_F([\infty \ c_{2,i} \ \cdots \ c_{G,i}]^T)$, $P_F([c_{1,i} \ \infty \ \cdots \ c_{G,i}]^T)$, $\cdots$, $P_F([c_{1,i} \ c_{2,i} \ \cdots \ \infty]^T)$ are estimated using samples distributed according to the conditional PDF $p(\boldsymbol{\theta} | F_{i-1})$. Though these estimates are not very accurate, they provide good enough information to select $\mathbf{q}_i$.

6.4 Illustrative Examples

The example presented in Chapter 4 is considered in this chapter. Two cases are considered. In the first case, failure is defined as an event where any inter-story drift ratio (*IDR*) or floor acceleration (*FA*) exceeds specific threshold combination at any discrete time instant during the total duration.

$$P_F([c_{IDR} \ c_{FA}]^T) = \int I_F(\mathbf{U}(\boldsymbol{\theta}), [c_{IDR} \ c_{FA}]^T) p(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad (6.14)$$

In the second case, failure is defined as an event where the total economic loss (*L*) or the downtime (*D*) exceeds a specific threshold combination.

$$P_F([c_L \ c_D]^T) = \int P(\mathbf{U}(\boldsymbol{\theta}), [c_L \ c_D]^T) p(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad (6.15)$$

For both cases, *Magt* and *dist* are uncertain with $[Magt_{\min}, Magt_{\max}] = [5.5, 8]$ and $dist_{\max} = 50$.

Economic loss has already been defined in Chapter 4. Downtime (*D*) is defined here as the period of time between the occurrence of seismic event and the completion of the building repair efforts. There are various factors that affect building downtime: building inspection, damage assessment, financial planning, resource mobilization, repair duration etc. The repair time data associated with individual components are taken from Mitrani-Reiser (2007). It is assumed that they follow a lognormal distribution and their CDF is given by an equation similar to Equation (4.39). Repair time required is calculated by summing the maximum downtime for all

damaged components in each performance group. Each performance group consists of building components whose performance is similarly affected by a particular **EDP** (Yang et al., 2009).

$$D = \left(\sum_{\text{all PG}} \max \{ D_{\text{components in PG}} \} \right) / 15 + 21 \quad (6.16)$$

In the above equation, 15 is the number of working hours per day and 21 is the number of days required for mobilization of resources. The repair time data used here are for a different structure but with similar types of components and damage states, and are adopted here only for the purpose of illustration.

6.4.1 Simulation Parameters (Case 1)

A discrete-time white noise sequence $\mathbf{Z}$ corresponding to a duration of input ground motion $T = 50\text{s}$ and a sampling interval $\Delta t = 0.02\text{s}$ is considered. To obtain failure probability as a function of threshold combinations, the proposed method is applied with a conditional failure probability at each level approximately equal to $p_0 = 0.1$ and $N = 1000$. One-dimensional chain adaptive symmetric uniform distribution is adopted as a proposal for each additive excitation random variable, i.e., $q(\xi_j | Z_j) = 0.5$ if $|Z_j - \xi_j| < 1$ and the level adaptive bivariate truncated Gaussian distribution with mean and covariance matrix (magnified by three times) estimated from samples from the most recent simulation level is adopted as the proposal for $Magt$ and $dist$.

6.4.2 Results (Case 1)

The estimates of failure probability for different combinations of threshold of IDR and FA are presented using a contour diagram. Figure 6.6 shows the sample mean of failure probability contour surface estimates from 60 independent simulation runs and failure probability contour surface estimates from MCS computed using 100,000 independent samples for different combinations of threshold c_{IDR} and c_{FA} . The solid contour lines represent the sample mean estimates and the dotted contour lines represent the estimates from MCS. It can be seen that the results using the

proposed method and MCS agree well except for very small failure probabilities where the error in MCS is significant. To investigate the variability of the failure probability estimators, the sample c.o.v of the failure probability estimates over 60 independent simulation runs is computed and is shown in Figure 6.7. The solid contour lines represent the sample mean contour surface estimates and the dotted contour lines represent the sample c.o.v estimates. It can be observed, as expected, the c.o.v grows approximately in a linear fashion with the logarithm of decreasing failure probabilities, which is smaller than the c.o.v for MCS that grows exponentially with the same number of samples.

Figure 6.8 shows the failure probability contour surface estimates from a single simulation run and failure probability contour surface estimates from MCS computed using 100,000 independent samples for different combinations of threshold c_{IDR} and c_{FA} . The solid contour lines represent the sample mean estimates and the dotted contour lines represent the estimates from MCS. It can be seen that the failure probability contour surface estimates from the proposed approach and MCS are very close. The circles in the figure indicate the threshold combinations that are selected to start the next conditional level. The scatter of IDR and FA samples at each conditional level is shown in Figure 6.9. It can be observed that as the simulation level increased, the correlation between the inter-story drift ratio and floor acceleration reduces. Using the original Subset Simulation, failure probability estimates can only be obtained for a particular linear relationship between the two threshold types from a single simulation run. The proposed approach gives all combinations of thresholds c_{IDR} and c_{FA} for a given failure probability.

Figure 6.10 shows the failure probability estimates as a function of inter-story drift ratio when the threshold for floor acceleration is fixed at 9 m/s^2 . Similarly, Figure 6.11 shows the failure probability estimates as a function of floor acceleration when the threshold for inter-story drift ratio is fixed at 2%. The pointers at the bottom of the two figures indicate the thresholds where the simulation moves to next conditional level and the discontinuity in the failure probability estimates. It can be seen that at the discontinuity the difference in the failure probability estimates from two conditional levels is very small.

Figure 6.6. Sample mean of failure probability contour surface estimates obtained from 60 independent simulation runs and failure probability contour surface estimate using MCS

Figure 6.7. Sample mean of failure probability contour surface estimates and sample c.o.v of failure probability estimates corresponding to points on the curve shown obtained from 60 independent simulation runs

Figure 6.8. Failure probability contour surface estimate obtained from a single simulation run and failure probability contour surface estimate using MCS

Figure 6.9. *IDR* and *FA* samples at various conditional levels

Figure 6.10. $P_F([c_{IDA} \ c_{FA} = 9\text{m/s}^2]^T)$ Figure 6.11. $P_F([c_{IDA} = 2\% \ c_{FA}]^T)$

6.4.3 Simulation Parameters (Case 2)

In the second case, to obtain the estimates of failure probability as a function of threshold combinations, the proposed method is applied with a conditional failure probability at each level approximately equal to $p_0 = 0.1$, $N = 500$ and $M = 1000$. Rest of the parameters and conditions remain the same as the ones in the first case.

6.4.4 Results (Case 2)

The estimates of failure probability for different combinations of threshold of economic loss and downtime are presented using a contour diagram. Figure 6.12 shows the sample mean of failure probability estimates from 60 independent simulation runs and failure probability estimates from MCS computed using 100,000 independent samples for different combinations of threshold c_L and c_D . The solid contour lines represent the sample mean estimates and the dotted contour lines represent the estimates from MCS. It can be seen that the results using the proposed method and MCS agree well except for very small failure probabilities where the error in MCS is significant. To investigate the variability of the failure probability estimators, the sample c.o.v of the failure probability estimates over 60 independent simulation runs is computed and is shown in Figure 6.13. The solid contour lines represent the sample mean estimates and the dotted contour lines represent the sample c.o.v estimates. It can be observed, as expected, the c.o.v grows approximately in a linear fashion with the logarithm of decreasing failure probabilities, which is smaller than the c.o.v for MCS that grows exponentially with the same number of samples.

Figure 6.14 shows the failure probability estimates from a single simulation run and failure probability estimates from MCS computed using 100,000 independent samples for different combinations of threshold c_L and c_D . The solid contour lines represent the sample mean estimates and the dotted contour lines represent the estimates from MCS. It can be seen that the failure probability estimates from the proposed approach and MCS are very close. The circles in the figure indicate the threshold combinations that are selected to start the next conditional level.

Figure 6.12. Sample mean of failure probability contour surface estimates from 60 independent simulation runs and failure probability contour surface estimates using MCS

Figure 6.13. Sample mean of failure probability contour surface estimates and sample c.o.v of failure probability estimates corresponding to points on the curve shown obtained from 60 independent simulation runs

Figure 6.14. Failure probability estimates from a single simulation runs and failure probability estimates from MCS

Chapter 7

Conclusions

7.1 Summary and Main Conclusions

As presented in the introduction, the focus of this PhD thesis is to contribute to the development of a robust stochastic simulation approach for the evaluation of the reliability, risk and loss of large complex systems and their updating. This chapter summarizes and highlights the main contributions of all the chapters of this study.

In Chapter 2, a new Gibbs sampling based approach is proposed for Bayesian model updating of a linear structural dynamic system based on incomplete modal data including modal frequencies, damping ratios and partial mode shapes of some of the dominant modes.

- The linear system can either be assumed to be classically damped or non-classically damped.
- In addition to the model parameters, the probability distributions of complete mode shapes and prediction errors for a given model class are also updated.
- The proposed method is robust to the dimension of the problem. It allows the uncertainty of the parameters to be updated efficiently even if there are a large number of uncertain parameters.
- Convergence issues and numerical issues arising in case of high-dimensionality of the problem were addressed and solutions to tackle these problems were proposed.
- The results from the first numerical example, involving a 4-DOF shear building model, demonstrate that the posterior samples for the updated parameters are reasonable when compared with the true mean values,

indicating the effectiveness of the procedure in identifying the high probability region of the uncertain parameters.

- The results from the second numerical example, involving a 120-DOF four-story, two-bay by two-bay steel frame with 614 uncertain parameters, demonstrate that the posterior PDF of the uncertain parameters is reasonable.

In Chapter 3, an efficient stochastic simulation method is presented for computing the updated robust failure probability of a dynamic system using system data when the system is subjected to future stochastic excitation.

- The updating is based on incomplete modal data including modal frequencies, damping ratios and partial mode shapes of some of the dominant modes.
- Uncertainties from structural modeling, modeling of the uncertain stochastic excitation that the structure will experience and any other uncertainties can be taken into account.
- The proposed approach integrates a newly-developed stochastic simulation method for Bayesian model updating of a linear dynamic system presented in Chapter 2 and a very efficient algorithm called Subset Simulation to compute small failure probabilities, together with development of two new algorithms called Constrained Metropolis-within-Gibbs and Constrained Multi-group Metropolis-within-Gibbs sampling algorithms for efficient sampling from the conditional distribution, one for the case involving a linear dynamic system and one for the case involving a nonlinear dynamic system.
- The proposed approach is robust to the number of uncertain parameters and random variables, and the dimension of vibration data involved in the problem.
- Results from the illustrative examples shows that the proposed method provides substantial improvement in efficiency over MCS using samples from the posterior PDF in Bayesian model updating.

In Chapter 4, a new stochastic simulation based approach is proposed for the evaluation of seismic loss exceedance probability as a function of threshold without repeated reliability analyses.

- Stochastic ground motion model coupled with nonlinear stochastic dynamic model, and probabilistic fragility and loss functions are considered.
- The proposed approach involves the modification of the simulation algorithms in the Subset Simulation and the development of new estimators.
- A family of specially designed stochastic simulation procedures is proposed to simulate the required samples which can reduce the variance of the exceedance probability estimators without requiring additional dynamic analyses.
- In addition, samples of input random variables and any function of them conditioned on different levels of loss exceedance are generated for a more comprehensive seismic risk and loss analysis and investigation.
- The approach is robust to the number of random variables and applicable when considering both seismic uncertainty and other modeling uncertainty including structural modeling uncertainty, damage and loss modeling uncertainty.
- The quality of the prediction of course depends on (a) the stochastic ground motion model to provide realistic description of the characteristic of the ground motions expected to happen at the specific site; (b) modeling of the physical system and collapse; (c) the quality of fragility functions and loss functions for structural and nonstructural components.
- An example involving a multi-story inelastic building is presented to show the application of the proposed stochastic simulation based approach with economic loss as the risk related decision variable.

In Chapter 5, the proposed approach for the evaluation of seismic loss exceedance probability in Chapter 4 is integrated with a newly developed algorithm for Bayesian model updating of a linear dynamic system based on incomplete modal data presented in Chapter 2, for evaluation of the updated seismic loss exceedance

probability as a function of threshold.

- All types of uncertainties, including those from dynamic system modeling and modeling of the uncertain excitation, can be considered during the computation of the updated reliability of a dynamic system subjected to future uncertain excitation.
- Updated seismic loss exceedance probability as a function of threshold is estimated using the system data.
- The proposed method is illustrated by a numerical example involving a multi-story inelastic building and it is observed that the posterior robust exceedance probability as a function of loss threshold is quite different from the prior exceedance probabilities due to different levels of model uncertainties, confirming the importance of using data to update the exceedance probability.

In Chapter 6, a new stochastic simulation based approach for the evaluation of failure probabilities of the response variables of a dynamic system (e.g., structural responses, losses) where failure is defined by a union of multiple performance objectives is proposed.

- The proposed approach allows for the simultaneous consideration of multiple performance objectives and the corresponding thresholds.
- Although the formulation has been presented for the general case of multiple performance objectives, the graphical representation of failure probabilities in more than two dimensions becomes difficult.
- Exceedance probability curves for all the response variables can be obtained as a byproduct.
- An example was presented involving a multi-story inelastic building to show the application of the stochastic simulation based approach for the evaluation of multiple structural response exceedance and seismic multiple loss exceedance probability as a function of multiple thresholds. Exceedance probabilities as a function of two thresholds (graphically contour plots),

maximum inter-story drift ratio and maximum floor acceleration, and economic loss and downtime, are obtained.

7.2 Future Works

The proposed approach for Gibbs sampling based model updating is only applicable where $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}$ are linearly dependent on the contribution parameters. The cases where the assumption may not be valid, for example, Caughey damping that is nonlinearly dependent on contribution parameters, are left for future work. Another area of research is the need to consider multiple model classes for model updating and uncertainty quantification and evaluate the plausibility of each model class based on system data.

The proposed approach for computing the updated robust failure probability in chapter 3 considers the updated distribution of parameters related to the linear behavior and prior distribution of parameters related to nonlinear behavior of the system. The approach will be further improved to consider the updated distribution of parameters related to nonlinear behavior of the system.

The efficiency of the sampling procedure to explore the parameter space used in chapters 3-6 can be further improved by the tuning of the proposal distributions in the simulation algorithms in some optimal sense and this will be studied in more details in the future.

The efficiency of the approach proposed in chapter 6 with respect to number of performance functions need to be studied in future. In future the focus will also be on obtaining functional form of the relationship between threshold vectors and failure probability.

Another area of research is the need to study the influence of the error models in stochastic design process and the problems studied in this thesis. Stochastic error modeling will be advanced in the future and their integration with the topics in Chapters 2-6 together with the algorithms necessary for the computational and

simulation problems will be developed.

Methods presented in this thesis can be applicable to wide range of problems involving stochastic loading such as aerospace structures, mechanical systems, marine systems, etc. In future, applications of these methods to problems involving real and complicated dynamic systems with real data will be investigated.

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