Mechanics of soft composites under axial, torsional, shear and dilatational loadings

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**NANYANG
TECHNOLOGICAL
UNIVERSITY**

**MECHANICS OF SOFT COMPOSITES UNDER
AXIAL, TORSIONAL, SHEAR AND
DILATATIONAL LOADINGS**

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SCHOOL OF MECHANICAL & AEROSPACE ENGINEERING

2015

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AXIAL, TORSIONAL, SHEAR AND
DILATATIONAL LOADINGS**

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Abstract

The mechanics of soft matter has stimulated much interest due to diverse applications of soft materials in medicine and engineering. A critical issue is the nonlinear mechanical behavior of composites under different loadings. This research focuses on characterizing the nonlinear phenomena of multilayered soft materials under dilatation, axial loading, torsion, shear and combined loadings. The constitutive equations are those of second-order elasticity and the governing equations are Lagrangian equilibrium equations in terms of the first Piola-Kirchhoff stresses.

First, the problem of soft bilayered spherical and multilayered cylindrical hydrogels subjected to various dilatation profiles is studied. The results show that: (1) elastic nonlinearity and inhomogeneity play a crucial role in the mechanical state, (2) a wide range of mechanical states can be designed for specific applications by manipulating the layer elasticity, interface positions and dilatation profile, and (3) the displacement and stresses can be characterized by a reduced set of geometric-elastic constants.

Second, analytical solutions are obtained for homogeneous and bilayered cylinders under torsion, axial and combined loadings. Explicit parameters are given for judging the sign of the Poynting effect, in which a cylinder elongates or contracts axially under torsion. It is found that the effect in a soft composite may be significantly amplified over that in homogeneous materials and that it is strongly

influenced by the interface position and by the material configuration in the composite. A coupled axial force-twist effect under combined loading, i.e., the twist of a torsionally loaded cylinder can be affected by the axial loading, is also found. Comparison of the predictions with the torque-tension-twist data for cardiac papillary muscles shows reasonable agreement. The solutions also provide the basis for a mechanistic method of determining third-order elastic constants.

Third, the problems of the shear of homogenous and bilayered blocks are treated in terms of a generalized shear displacement. In this generalization, the applied shear displacement is neither uniform nor is the shear displacement of the block assumed *a priori* to be a linear function of the height. The shear displacement solutions are characterized by the product of sine and hyperbolic sine functions of the height and depth variables, respectively. The height dependence of the shear displacement is a combination of linear and sinusoidal functions, and is verified against the test data of homogeneous and bilayered agar-gelatin cuboidal blocks. The calculation of stresses reveals the presence of not only negative normal stresses but also sinusoidally varying shear stresses on the lateral planes tending to distort the block about the height direction. These results can be of great importance in tissue/cell mechanics.

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List of Symbols

Symbol	Definition
a	Position vector in the undeformed states
$\mathbf{E}$	Lagrangian strain matrix
E_1, E_2, E_3	Principal strains
$\mathbf{F}$	Deformation gradient
$\mathbf{I}$	Identity matrix
k	Boltzmann constant; a marker
J_1	First strain invariant
J_2	Second strain invariant
J_3	Third strain invariant
l, m, n	Third-order elastic constants
N	Number of polymer chains per unit volume of dry polymer; number of layers
P	Axial loading
p	Hydrostatic stress
r	Radial coordinate
T	Torsional loading
$\mathbf{T}$	Stress matrix
T_{xx}	Normal stress in the x -direction

T_{yy}	Normal stress in the y -direction
T_{zz}	Normal stress in the z -direction
T_{xy}	Shear stress in the x - y plane
T_{rr}	Radial stress
$T_{\theta\theta}$	Meridional stress
$T_{\phi\phi}$	Azimuthal stress
T_{rr}^L	Linear radial stress
T_{rr}^{NL}	Nonlinear radial stress
$T_{\theta\theta}^L$	Linear meridional stress
$T_{\theta\theta}^{NL}$	Nonlinear meridional stress
$T_{\phi\phi}^L$	Linear azimuthal stress
$T_{\phi\phi}^{NL}$	Nonlinear azimuthal stress
u_r	Displacement in r -direction
u_θ	Displacement in θ -direction
u_z	Displacement in z -direction
ν	Possion ratio
W	Energy density density
ρf	Body force per unit volume
$d\mathbf{S}$	Oriented surface element
λ, μ	Lamé constants
$\mathcal{V}$	Dilatation

Chapter 1 Introduction

The present research aims to study the mechanics of soft layered composites under several fundamental loading modes, i.e., dilatational, axial, shear and torsional loadings. The influence of nonlinearity and the associated phenomena, particularly in connection with the elastic inhomogeneity and composite architecture, is emphasized. Elastic inhomogeneity refers to the difference between the elastic moduli of the layers. Composite architecture refers to the geometry (shape and size) of the layers. In this research, the layers are similar in shape but may have different thicknesses.

This chapter is organized in the following manner. The research background is introduced in Section 1.1, followed by an explanation of the research motivation in Section 1.2. Thereafter, the objectives of this research thesis are stated in Section 1.3, and the scope of the thesis work is outlined in Section 1.4.

1.1. Background

Soft materials can be characterized as a class of materials that give a "large response to small perturbations" (Hirst, 2012). This provides us with a clear and straightforward description of the behavior of soft materials under various loadings. Any material that deforms easily under an external stimulus (e.g., mechanical load,

electric fields, chemical stimulus, etc.) can be considered as "soft". Soft materials, for example, include synthetic and biological materials.

1.1.1. Synthetic soft materials

Synthetic soft materials have applications in many fields. Hydrogels are one kind of synthetic soft materials. They are essentially polymeric networks with a three-dimensional configuration capable of imbibing a large amount of water or biological fluids. They consist of three phases, namely, the three-dimensional polymeric network or solid matrix, interstitial fluid and ionic species. Hydrogels have a diverse range of applications, e.g., stem cell culture platforms (Lutolf et al., 2009), drug delivery vehicles (Brazel and Peppas, 1999), actuators (Beebe et al., 2000), tissue scaffolding (Stammen et al., 2001), miniature sensors (Han et al., 2002), bioseparations (Kim and Park, 1998), food science and green technologies (Ehrick et al., 2005).

1.1.2. Biological soft materials

Most organs and biological tissues are soft materials. Their elastic moduli generally range from the order of 10^2 Pa for the brain to 10^5 Pa for soft cartilage (Levental et al., 2006). Compared with synthetic materials, biological soft materials are more complex. For example, their structures may form from the self-assembly (directed or spontaneous) of several components, which may be proteins, nucleic

acids, lipids or polysaccharides (Hamley, 2007). Biological materials are often structurally hierarchical and may show highly desirable structure-property relations that can be used as templates for bio-inspired materials (Wu, 2011).

1.2. Motivation

In the application of soft materials, it is vital to understand, model and simulate their behavior under different loadings, taking into account their composite nature and nonlinearity. More generally, the issues of nonlinearity, finite strains, anisotropy and viscoelasticity of soft materials are also important and have been active areas of research (Limbert and Taylor, 2002).

1.2.1. Mechanics of soft materials

Over the past few decades, soft materials have gained much interest in materials and biological science. While a majority of studies have focused on the chemical and biological aspects of soft materials, it has becoming increasingly clear that the mechanical properties of soft materials also play a key part in their functions. For example, Engler et al. (2006) used polyacrylamide gel substrates with variable elastic moduli to culture human mesenchymal stem cells (hMSCs). The results showed that the same hMSCs differentiated predominantly into neurogenic, myogenic and osteogenic cells when cultured on substrates with elastic moduli in the range of 0.1-1 kPa, 8-17 kPa, and 25-40 kPa, which mimic the elastic moduli of

the brain, muscle and bone, respectively. Similarly, studies of hMSCs on methacrylated hyaluronic acid hydrogels with moduli in the range of 3-100 kPa showed that hMSCs exhibited enhanced proliferation on the stiffer gels and their behavior depended critically on the local elasticity (Marklein and Burdick, 2010). Kloxin et al. (2010) also demonstrated the influence of microenvironment elasticity on the differentiation and migration of valvular interstitial cells.

The health of tissues may often be revealed by their mechanical (elastic) properties. Emphysema, characterized by a reduction in the mechanical elasticity of the lungs, is a chronic lung disease (Levental et al, 2007) in which the alveoli are damaged. The elastic modulus of healthy lung tissue is in the range of 5 - 30 kPa when deformed at physiologically relevant rates (Cavalcante et al., 2005). However, the progression of alveolar disease is characterized by a reduction in the elastic modulus of between 33% and 47% (Yuan et al., 2000). In contrast, the elastic modulus of the lung tissue will increase around 50% in the case of lung fibrosis (Dolhnikoff et al., 1999). Hence, the elastic properties of tissues can be used for diagnostic purposes. In the context of mechanics, this presents an inverse problem in which the measured deformations of a tissue are used to infer its elastic constants.

In solving such mechanics problems, knowledge or investigation of the materials, geometries, loadings and boundary conditions is required. In the following, the issues of geometry, material nonlinearity and loadings are highlighted.

1.2.2. Composite architecture

The structure of soft biological materials is typically hierarchical, with one level of structure built upon underlying levels of structure (Dutcher and Marangoni, 2004). In addition, many soft materials are constituted of different components, i.e., they are composites. The interfaces between components play an important role in determining their mechanical properties. The functional adaption of the structure at all levels of hierarchy results in their exceptional mechanical properties (Fratzl and Weinkamer, 2007). In this thesis work, the hierarchical nature of the materials is not considered. The focus is on their composite nature, as further reviewed below.

Many natural biological materials such as skin, heart valves, articular cartilage, tendons, small-intestinal submucosa and vascular tissues are multilayered or reinforced with various components, see for example Diridollou et al. (2000), Sofer et al. (2002), Alexopoulos et al. (2003), Stella and Sacks (2007), Zhao et al. (2011) and Browning et al. (2012). Criscione et al. (1999) investigated the twist rate of cardiac capillary muscles under a constant axial stretch as a function of the twisting moment. The muscles are considered as layered composites consisting of a core of myocardium inside a thin sheath of endocardium. An example of a reinforced biological composite is the stroma of the cornea, made up of collagen fibrils embedded in a hydrated matrix of proteoglycans, glycoproteins and keratocytes. The biomechanical response plays a critical part in corrective surgeries such as LASIK and diseases such as keratoconus and corneal dystrophies (Boyce *et al.*, 2007). In general, connective tissues such as tendons and ligaments are fiber-reinforced soft

composite materials.

Furthermore, multilayered hydrogel networks can also be synthesized (Cuchiara et al., 2010) for many applications such as tissue regeneration, wound healing and drug delivery (Detzel et al., 2011; Jessel et al., 2006). Zhu *et al.* (2004) used a polyelectrolyte multilayer technique to create multilayer coating on poly-(DL-lactide) to promote chondrocyte attachment and growth. A layer-by-layer technique was also used to create multilayer coatings of chitosan and heparin on biomaterials to control MG-63 osteoblast adhesion and growth (Kirchhof et al., 2009). A multilayered biomimetic scaffold mimicking the structure of cancellous and cortical bones was designed for bone tissue engineering, which shows an enhanced mechanical strength and larger pore size in the center (Kong et al., 2007).

1.2.3. Nonlinearity

Soft materials are inherently nonlinear and exhibit many unique phenomena, e.g., nonlinear stress-strain relationship, the Poynting effect, and a negative normal stress. The Poynting effect refers to the lengthening of a cylinder under pure torsional loading. If the cylinder shortens, the Poynting effect is negative. A negative normal stress is experienced by a material whose faces tend to be drawn together under shear loading, corresponding to a negative Poynting effect.

It is noted that many biological materials, including blood vessels, mesentery tissue, lung parenchyma, cornea and blood clots, stiffen as they are strained, thereby

preventing overly large deformations that could threaten their integrity (Storm et al., 2005). Cross-linked semiflexible or rigid filaments exhibit strain stiffening and negative normal stress when samples are under simple shear (Kang et al., 2009). Experiments also show that the shear modulus of human medial collateral ligament increases significantly with increasing shear strain, reaching a maximum of 1.72 ± 0.4871 MPa (Weiss et al., 2002).

A significant finding of some of these works is the existence of negative normal stresses or the negative Poynting effect when certain soft, particularly biological, solids are subjected to shear or torsion. The shearing of biological materials is common in their physiological environments (Horgan and Murphy, 2011), and understanding of nonlinear phenomena such as the Poynting effect is therefore of great importance. The large stresses generated by the Poynting effect can result in a significant effect on the overall force balance in the cytoskeleton under shear (Janmey et al., 2007), and the Poynting effect is important in understanding the interaction between surgical instruments and tissues (Misra et al., 2010).

1.2.4. Complex loadings

Both biological and synthetic materials are often subjected to different kinds of loadings, e.g., dilatational, axial, torsional, shear and combined loadings. Dilatational loading refers to a “body force” type of loading, such as temperature and chemical concentrations. In order to fully understand their mechanical responses,

it is therefore necessary to incorporate these different, and fundamental, loading types in the mechanics study.

In a physiological environment, biological materials are frequently subjected to complex loads. The kinematic response of lumbar spinal units subjected to torsion, compression and flexion/extension were investigated experimentally by Haberl et al. (2004). The mechanical parameters of arterial walls, specifically related to human brain aneurysm (bulge along a blood vessel), were studied by Toth et al. (2005). The arterial wall was modeled as a three-layer cylindrical composite subjected to combined extension, torsion and inflation; this mimics real physiological conditions. Hydrogels have diverse applications and complex loadings may be applied on them. The loading on hydrogels may be caused by internal stimuli, such as the various concentrations due to drugs, proteins and growth factors, resulting in a dilatation field. In addition, external stimuli may also be applied on hydrogels. The mechanical behaviors of physical and chemical gels of poly(vinyl alcohol) under uniaxial, pure shear and biaxial loadings were studied by Urayama et al. (2006).

1.2.5. Application of the mechanics of soft materials

The mechanics of soft materials is essential for the understanding, modeling, and simulation of soft materials and devices under working conditions. Soft materials include natural biological materials from biogels to tissues. They may also refer to bioinspired (or biomimicking) materials, whose working principles are

likely to be strongly influenced by the mechanical properties, or to biomaterials specifically designed for optimum compatibility with biological systems, e.g., implants.

Whatever the specific applications and materials, the mechanical properties of the bio-environment have a profound influence on the processes occurring in it. In spite of intensive research, the mechanics of the soft materials under different loadings remains poorly understood, especially in connection with the multilayer architecture and material nonlinearity. Most current works focus on homogeneous soft materials and dilatational loading is rarely considered, as will be reviewed in Chapter 2

1.3. Objectives

The proposed research aims to conduct an in-depth investigation into the mechanics of soft layered composite materials under various loadings. The specific objectives are to:

- (1) develop an analytical model for multilayered spheres and cylinders under dilatational loadings and study the stress and displacement fields,
- (2) investigate the mechanics of soft homogeneous and multilayered cylinders under axial, torsional and axial-torsional loadings.
- (3) study the generalized shear of soft homogeneous and multilayered rectangular blocks,

- (4) study the Poynting effect and to explore the existence of other nonlinear effects,
and
- (5) investigate the influence of elastic and geometric inhomogeneities on the stresses,
displacements and nonlinear effects.

1.4. Outline of the Thesis

Following this introduction, the other chapters of the thesis are organized as follows.

In Chapter 2, the experimental and theoretical works on the mechanical behaviors of soft solids subjected to various loadings are reviewed. The scope of the dissertation work is discussed.

In Chapter 3, the second-order nonlinear elasticity model of Murnaghan (1951) is reviewed first. This model is used as the basic constitutive theory throughout this thesis. The problems of the bilayered spherical and multilayered cylindrical hydrogels under dilatations are next defined. Analytical solutions for the displacements and stresses are derived using the second-order nonlinear elasticity model. Numerical results are obtained. The theory of swelling of spheres and cylinders is verified against experiment data.

In Chapter 4, the analytical solutions are obtained for both homogeneous and bilayered cylinders under axial loading. Experimental data are used to validate the theory. Numerical analysis is carried out to study the dependence of second-order stresses and displacements on the elastic constants, layer inhomogeneity and

geometry.

In Chapter 5, the problems of homogeneous and bilayered cylinders under torsional loading are investigated. The effects of the material nonlinearity and geometrical complexities are presented. The existence of second-order normal stresses (Poynting effect) is studied for both homogeneous and bilayered cylinders.

In Chapter 6, the analytical solutions are obtained for both homogeneous and bilayered cylinders under combined torsional-axial loadings. The effects of the material nonlinearity and geometrical complexities are presented, highlighting the axial force-twist nonlinear effect.

In Chapter 7, the problems of homogeneous and bilayered soft rectangular blocks under generalized shear are solved. The gravity effect can also be taken into account. The shapes of experimentally deformed blocks of agar-gelatin are compared with the theoretical predictions. The dependence of negative or positive second-order normal/shear stresses on the elastic constants are investigated.

Chapter 2 Literature Review

This chapter briefly reviews experimental and theoretical works on the mechanical behavior of soft solids subjected to various loadings. The specific loading types are dilatation, axial loading, torsion and shear. The scope of the dissertation work is then discussed, based on some of the challenging issues that have not been resolved.

2.1. Dilatation

A dilatation in a material may result from various fields such as temperature, moisture, pH, ionic strength, and generally chemical concentrations such as those due to drugs, proteins and growth factors. It may be considered as a loading on the material associated with these fields. The loading profile (temperature, moisture, chemical concentration), or equivalently the dilatation profile, naturally plays a crucial role in the mechanical response (stresses and displacements). Hence, the design of the elasticity of the material, in particular composite material, should take into account the dilatation profile within the material.

Growth has been treated as the process by which a material increases in size and mass (Alberts et al., 2008; Ben Amar & Goriely, 2005). Growth and swelling have many similarities. A common approach in modeling growth is to treat changes in mass via variations in concentrations (Ambrosi et al., 2011), which is very similar to the method we used to simulate the swelling of polymer gels under dilatation.

Kirchner (1974) investigated the stress and displacement fields of a finite linear elastic body containing precipitates with a spherically symmetric concentration fluctuation. The local dilatation is a function of the local concentration, which can be treated as equivalent to spherically symmetric temperature distribution within the matrix. In the paper, the dilatation is assumed to be $3\alpha T[c(r)]$, where α is a coefficient, T the temperature, c the local concentration and r the radial coordinate. The equilibrium equation for a sphere under the dilatation is

$$\frac{d^2 w}{dr^2} + \frac{2}{r} \frac{dw}{dr} - \frac{2w}{r^2} = \alpha \left(\frac{1+\nu}{1-\nu} \right) \frac{dT}{dr}, \quad (2.1)$$

where w is the radial displacement and ν the Poisson ratio. The radial strain u_{rr} , the meridional strain $u_{\theta\theta}$ and circumferential strain $u_{\varphi\varphi}$ are then obtained via the following relations:

$$u_{rr} = dw/dr, u_{\theta\theta} = u_{\varphi\varphi} = w/r. \quad (2.2)$$

The corresponding stresses are obtained through the constitutive equations as:

$$\sigma_{rr} = \lambda(u_{rr} + 2u_{\varphi\varphi}) + 2\mu u_{rr} - \alpha T(3\lambda + 2\mu), \quad (2.3)$$

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = \lambda(u_{rr} + 2u_{\varphi\varphi}) + 2\mu u_{\varphi\varphi} - \alpha T(3\lambda + 2\mu), \quad (2.4)$$

where λ and μ are the Lamé constants. With appropriate boundary conditions, the

exact displacement and stress fields are obtained.

The mechanical behaviors of an unconstrained sphere and a spherical shell bonded to a rigid core due to a dilatation $\mathcal{G}(r)$ were studied by Wu and Kirchner (2011). The dilatation is assumed to be spherically symmetric and defined as a function of the radius r . Here the dilatation is considered from a general standpoint; the field producing the dilatation profile is not specified, but can be due to chemical concentrations, etc. as mentioned above. The materials are assumed to be second-order nonlinear elastic with the following energy density function (Murnaghan, 1951):

$$W = \frac{\lambda + 2\mu}{2} J_1^2 - 2\mu J_2 + \frac{l + 2m}{3} J_1^3 - 2m J_1 J_2 + n J_3, \quad (2.5)$$

where λ, μ are the second-order and l, m, n the third-order elastic constants, respectively. J_1, J_2 and J_3 are the strain invariants. Due to symmetry, each point in the sphere undergoes a radial displacement $u(r)$. The radial displacement u is assumed as the sum of the first- and second-order displacement u_1 and w :

$$u = u_1 + w. \quad (2.6)$$

The equilibrium equations can be written as:

$$u_1'' + 2\frac{u_1'}{r} - 2\frac{u_1}{r^2} = \frac{3\lambda + 2\mu}{\lambda + 2\mu} \frac{d\mathcal{G}}{dr} \quad (2.7)$$

and

$$\begin{aligned}
 (\lambda + 2\mu)(w'' + 2\frac{w'}{r} - 2\frac{w}{r^2}) = & -2(\lambda + 3\mu + 2m)[(u_1'' + 2\frac{u_1'}{r} - 2\frac{u_1}{r^2})u_1' - \frac{1}{r}(u_1' - \frac{u_1}{r})^2] \\
 & - (\lambda + 2l)(u_1'' + 2\frac{u_1'}{r} - 2\frac{u_1}{r^2})(u_1' + 2\frac{u_1}{r}) \\
 & - (\lambda + \frac{2}{3}\mu - 2l - \frac{2n}{9})9g\frac{d\vartheta}{dr},
 \end{aligned} \tag{2.8}$$

for first- and second-order separately. With appropriate boundary conditions on interfaces and boundaries, the analytical solutions are obtained in closed-form. Results show that the nonlinear stresses and displacements can be significantly larger than the linear part and they may have different signs. The stress and displacement distributions are studied under different dilatation profiles. It shows that the dilatation profiles can significantly change the stresses and displacements.

2.2. Axial loading

Axial loading is one of the most fundamental modes of loading. Not surprisingly, there has been much interest in the mechanical behaviors of both biomaterials and synthetic materials under axial loading, e.g., hydrogels, muscles, bones, ligaments and tendons.

The mechanical behaviors of soft materials under axial loading (and pure shear) have been studied experimentally in many works. Urayama (2006) investigated the differences in elastic properties of poly(vinyl alcohol) (PVA) gels with different types of crosslinks, physical crosslinks and chemical crosslinks. The stress-strain curves for physical PVA gels (P-GEL) and chemical PVA gels (C-GEL) under

uniaxial loading are presented in Fig. 2.1. The stress is the nominal stress, which is defined as the force per unit cross-sectional area in the undeformed state. The stress-strain curve for P-GEL-15 is slightly convex, which shows a strain-stiffening effect. However, C-GEL-15 shows a strain-softening effect with a concave curve. The effects depends on the gels' crosslinked types, i.e., physically crosslinked gels tend to

Fig. 2.1. Principal stresses (σ_x and σ_y) as a function of λ_x in equi-biaxial, pure shear and uniaxial deformations for (a) P-GEL-15 and (b) C-GEL-10 (Urayama, 2006).

Fig. 2.2. Stress–strain curve of collagen with three characteristic stages; I: Toe; II: linear elastic; III: failure (Meyers et al., 2008).

exhibit strain-stiffening effect, while chemically crosslinked gels prefer to exhibit strain softening effect.

A tendon is a tough band of fibrous connective tissue that usually connects muscle to bone and is capable of withstanding tension. Fig 2.2 shows the stress-strain curve of tendon (Meyers et al., 2008). There exists three regions. Region I is in the range of $0 < \varepsilon < 0.04$, which is the normal physiological condition under which tendon operates. When the strain increases in region I, the magnitudes of the stress increase from 5 to 8 MPa. Region II is the linear part with a constant modulus. Region III shows the fracture part. The maximum strain is around 17% and the maximum stress is approximately 70 MPa.

Fig. 2.3. Tensile and compressive stress–strain curves for cortical bone in longitudinal and transverse directions (Lucas et al., 1999).

Cortical bone is one of the two types of osseous tissue that form bones. Cortical bone facilitates bone's main function: to support the whole body, protect organs, provide levers for movement, and store and release chemical elements, mainly calcium. Fig. 2.3 shows the tensile and compressive stress-strain curves for cortical bone in longitudinal and transverse directions (Lucas et al., 1999). It can be seen the bone is anisotropic, with larger moduli in the longitudinal direction than in the transverse direction. The bone can bear larger stress in compression than in tension. The stress-strain curve in longitudinal direction is a linear curve followed by a

plateau. This plateau is produced by the local buckling of the fibrils. The maximum stress in the longitudinal direction is around 150-200MPa, with the maximum strain in the range of 2.5%-3.5%.

Many theoretical models were developed to simulate the behaviors of soft matter under axial loading. In order to model elastic properties of articular cartilage, a transversely isotropic and transversely homogeneous microstructural model was developed (Federico et al, 2005). The results show that the axial elastic modulus decreases from the deep zone to the articular surface. Finite element simulations were also carried out to explore the behavior of articular cartilage under compressive loading. The force response, fluid flow and displacement fields were obtained.

Provenzano et al (2002) used the nonlinear theory of Schapery and the modified superposition method to model the nonlinear behavior of ligament under axial loading. The results show that both theories can capture the elastic and viscous nonlinearities and the modified superposition method allows for a more direct interpretation of the relationship between model parameters and physical behaviors.

The stress-strain relationship of Schapery's theory under uniaxial loading is:

$$\sigma(\varepsilon, t) = h_e(\varepsilon)E_e\varepsilon + h_1(\varepsilon)\int_0^t \Delta E(\rho(t) - \rho'(\tau)) \frac{dh_2(\varepsilon)\varepsilon}{d\tau} d\tau, \quad (2.9)$$

where σ is the stress, ε the strain, t the time, ρ is the reduced time, E_e the equilibrium value of the elastic modulus, ΔE the transient modulus, h_e and h_1 the strain-dependent material properties.

The stress-strain relationship of modified superposition method is:

$$\sigma(\varepsilon, t) = \int_0^t E(t - \tau, \varepsilon(\tau)) \frac{d\varepsilon(\tau)}{d\tau} d\tau. \quad (2.10)$$

Piolette and Rakotomanana (2000) developed a conceptual framework to test different viscoelastic constitutive laws. The relationship between stress and strain under axial loading was presented. The method has the ability to model the immediate, short time memory and long time memory contribution. Different types of energy density functions were used to identify mechanical tests performed on human patellar tendons.

Buehler (2006) investigated the mechanical properties of tropocollagen molecules via a combined atomistic-mesoscopic model under different types of mechanical loading including tension, compression, shear and bending. The results show that under tensile loading a tropocollagen fiber undergoes strain stiffening at around 20% strain. The theory predicts that the fiber buckles under a compressive load of around 1000 pN at around 5% strain.

2.3. Torsion

Many natural biomaterials are often subjected to complex multiaxial loads in their physiological environment, such as torsional or combine torsional-axial loadings. Experiments have been conducted to observe their behaviors.

Fig. 2.4. Measured response of the intact muscle and estimated response of the myocardial core during a twist protocol. The core response was estimated by subtracting the endocardial contribution with full wrinkling resistance (open circles) and without any wrinkling resistance (Criscione et al., 1999).

Criscione et al. (1999) conducted finite axial loading and torsion experiments on rabbit cardiac papillary muscles to characterize their constitutive behavior. The forces/moment and strain relationship of the papillary muscles was measured in the experiments. Five right ventricular papillary muscles were selected. The extension ratio of the specimens was fixed at 1.04 during the test. Representative results from

one set of specimens were presented in Fig.2.4.

They modeled the material as a bi-layered composite consisting of a core of myocardium inside a thin sheath of endocardium. The thin sheath of endocardium was modeled by a Fung-type strain energy function:

$$W = c_0(e^Q - 1) + 0.5c_4E_{11}^2 + c_5E_{11}E_{22} + 0.5c_6E_{22}^2, \quad (2.11)$$

where

$$Q = 0.5c_1E_{11}^2 + c_2E_{11}E_{22} + 0.5c_3E_{22}^2. \quad (2.12)$$

E_{11} and E_{22} are the extensional Green strains; $c_0 - c_6$ are the material parameters.

The twisting moment and average axial stress are plotted as a function of twist.

Results show that the endocardium sheath has a larger effect on the axial moment than on the axial force.

Haberl et al. (2004) investigated experimentally the kinematic response of lumbar spinal units subjected to torsion, compression and flexion/extension. These units can be considered as a composite material with a soft material (nucleus pulposus) between them. 10 human cadaveric lumbar specimens were selected. The following loading

Fig. 2.5. Average total torsion of all specimens after application of an axial moment of 12.5 Nm (Haberl et al., 2004).

modes were performed:

1. Pure torsion.
2. Torsion with a compressive preload of 200 N.
3. Torsion with a combined flexion-compression preload (200 N and 3 or 6 Nm)
4. Torsion with an extension-compression preload (200 N and 3 or 6 Nm).

Fig.2.5 shows that the average range of motion tends to decrease with the application of preload and with the application of flexion-compression compared to the no-preload load case.

Many theoretical models are proposed to investigate their behaviors under torsion or combined torsional and axial loadings. Tóth et al. (2005) studied the mechanical parameters of arterial walls, specifically related to human brain

aneurysm (bulge along a blood vessel). The authors modeled the arterial wall as a three-layer cylindrical composite subjected to combined extension, torsion and inflation (which mimic real physiological conditions).

Single-walled carbon nanotubes (SWCNTs) under torsion are studied using molecular dynamics simulations (Zhao and Luo, 2011). The Poynting effect (elongation of a specimen under pure torsion) is found and it depends largely on the structures. A torsion induced bending effect is also observed and it only exists for large aspect ratios and small chiral angles. The bending effect is related to the buckling of nanotubes under large torsions.

Combined torsion, inflation and longitudinal tension were applied to the esophagus to measure its shear modulus (Yang et al., 2004). Results show that the esophageal wall is anisotropic. The shear modulus of the esophagus can range from the order of 1kPa to the order of 100 kPa.

The axial tensile forces generated when human femurs under torsion were measured by Zdero et al. (2011). Results show that human femurs exhibit positive Poynting effect. Strong linear relationship between applied torsion and resulting axial loading exists with torsion/tension ratio = 551.7 ± 283.8 mm for the external rotation tests.

Horgan and Murphy (2012), employing a logarithmic form for the strain energy densities of incompressible transversely isotropic materials, investigated the problem of torsion superimposed on extension of a circular cylinders. The analysis is in good agreement with experimental data for rabbit papillary muscles. Results

showed that the negative Poynting effect is generated in the presence of large material anisotropy and is also present under certain loading conditions. Wu and Kirchner (2010) showed that the negative Poynting effect is generally possible for biogels. Their work is based on second-order elasticity. Zubov (2001) developed an expression to determine the Poynting effect in isotropic incompressible cylindrical body under combined torsion and extension/compression loading, for which an explicit expression was given for a neo-Hookean material. Earlier works on second-order torsion such as those of Bhargava and Gupta (1979, 1976) and Blackburn and Green (1957) also study their nonlinear behaviors under combined torsion-axial loading. Green and Shield (1951) examined the problem of a small twist superposed on finite extension, and an explicit expression was given.

2.4. Shear

Shearing of soft materials such as biopolymers and hydrogels is also of great interest. A significant finding is the existence of negative normal stresses or the negative Poynting effect when certain soft solids are subjected to shear. Some experiments have been carried out to study the shear of soft materials.

Storm et al. (2005) measured the shear stress and strain relationships for a series of crosslinked biopolymer networks as shown in Fig. 2.6. Specifically, the shear modulus G or storage modulus G' is plotted against the shear strain γ . It can be seen that actin, collagen, fibrin, vimentin and neurofilaments all exhibit strain-stiffening

Fig. 2.6. Stress-strain relationships for polyacrylamide, actin, collagen, fibrin, vimentin and neurofilaments (Storm et al., 2005).

effects. The stiffness and the strains they tolerate can vary over orders of magnitude. The actin ruptures with a strain at about 20% and a shear modulus at about 100 Pa, while the neurofilament can deform up to 400% with a shear modulus at about 30 Pa.

Janmey et al. (2007) measured the shear and normal stress for actin, collagen, fibrin, neurofilaments, matrigel and polyacrylamide under shear as shown in Fig 2.7. It can be observed that F-actin, collagen, fibrin, neurofilaments and matrigel all show a negative normal stress, i.e., negative Poynting effect. However, the Polyacrylamide exhibits a positive normal stress (positive Poynting effect) during shear and does not show much evidence of strain-stiffening.

Fig. 2.7. Shear (circles) and normal (triangles) stresses as the strain is gradually increased in gels made from actin crosslinked by biotin/avidin (a), collagen (b), fibrin (c), neurofilaments (d), Matrigel, a complex extracellular matrix containing mainly collagen and laminin (e) and polyacrylamide (f) (Janmey et al., 2007).

Many theoretical models have been proposed to simulate the shear of soft materials. Briefly reviewed below are microstructural and phenomenological models. A molecular model was proposed by Storm et al (2005) to study the strain-stiffening effect under shear deformation. Universal stress-strain relations are also revealed using the theory. The Cauchy stress tensor is calculated by averaging the force on the links as:

$$\sigma_{ij} = \frac{\rho}{\det \Lambda} \left\langle f(|\Lambda r|) \frac{(\Lambda_{il} r_l)(\Lambda_{jk} r_k)}{|\Lambda r|} \right\rangle_{P(r)}, \quad (2.13)$$

where Λ is the Cauchy deformation gradient, ρ is the number density of links, $P(r)$ is the probability distribution function of separation r . $\langle . \rangle$ is the averaging operator over $P(r)$. The theory shows that the strain-stiffening effect is invariable at low strains for systems of filaments arranged in an open crosslinked mesh.

There are quite a number of phenomenological models. In the strain formulation of the simple shear problem, e.g., Rivlin (1948), the starting point is the deformation of a block as given by:

$$x = X + \kappa Y, \quad y = Y, \quad z = Z, \quad (2.14)$$

where (X, Y, Z) and (x, y, z) are the referential and current Cartesian coordinates respectively, and κ is the shear deformation. Murnaghan (1951) also started from Eq. (2.14) in studying simple shear within the framework of second-order elasticity. Calculation of the stresses then reveals that normal stresses must exist to support such a deformation. Recently, Destrade et al. (2012) pursued a stress formulation for hyperelastic materials in which the starting point is a uniform shear stress $\sigma_{12} = \sigma_{21}$.

The resulting deformation is given by:

$$x = \lambda_1 X + \lambda_2 \sqrt{1 - \lambda_1^2 \lambda_2^{-2}} Y, \quad y = \lambda_2 Y, \quad z = \lambda_3 Z, \quad (2.15)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches. This means that under a pure shear stress the deformation consists of a simple shear of the type given by Eq. (2.14) and triaxial extensions. Hence, under the strain formulation the resulting stress field consists of shear stresses and triaxial normal stresses, while under the stress formulation the resulting deformation field consists of simple shear and triaxial stretches.

Mihai and Goriely (2013) considered generalized shear in which Eq. (2.14) is modified into:

$$x = X + \kappa(Y), \quad y = Y, \quad z = Z, \quad (2.16)$$

where κ is now a function of Y . Specifically, $\kappa(Y)$ is assumed to be of the form of $Y^2/2$, i.e., a quadratic function of Y . Destrade and Saccomandi (2010) also considered the effect of gravity in the form of a body force $-\rho_0 g$ (ρ_0 is the mass density and g is the gravitational constant). They took the deformation to be:

$$x = X + u(Y), \quad y = Y + v(Y), \quad z = Z, \quad (2.17)$$

where, upon solving the governing equations and assuming the Murnaghan strain energy density, $u(Y)$ was found to be the sum of linear and quadratic parts.

The simple shear testing of human medial collateral ligament is simulated using a finite element method by Gardiner and Weiss (2001). It shows that the Poynting

effect exists, i.e., normal stresses are needed to maintain zero normal strains. The effect of sample geometry, bulk modulus and clamping prestrain are also investigated. The behaviors of the ligaments are described by a transversely isotropic hyperelastic model with the strain energy as:

$$W = F_1(\tilde{I}_1) + F_2(\tilde{\lambda}) + \frac{K}{2}[\ln(J)]^2, \quad (2.18)$$

where F_1 represents the strain energy of the matrix, which is a function of the first deviatoric invariant $\tilde{I}_1$. F_2 represents the strain energy from fiber, which is a function of the deviatoric part of the stretch ratio along the local fiber direction. The last term represents the contribution from the tissue bulk response. K is the bulk modulus and J is the determinant of the deformation gradient.

Two different strain energy densities of fiber-reinforced cylindrical biological materials were applied to study the mechanical behavior of the soft biological tissues under simple shear (Horgan and Murphy, 2011). The first model is the standard reinforcing model:

$$W = \frac{\mu}{2}(I_1 - 3) + \frac{E(I_4 - 1)^2}{4}, \quad (2.19)$$

where E is a parameter that measures the degree of anisotropy. The second model considers the strain-stiffening effect of collagen fibers as

$$W = \frac{\mu}{2}(I_1 - 3) - \frac{E}{2}J[(I_4 - 1) + J \ln(1 - \frac{(I_4 - 1)}{J})]. \quad (2.20)$$

Results show that the negative Poynting effect is generated in the presence of large material anisotropy and is also present under certain loading conditions.

Misra et al. (2009) experimentally showed that the negative normal stress in anisotropic myocardial tissue under shear is larger than in Sylgard gel, which they also described theoretically using an exponential form and an Ogden form of the strain energy densities for the tissue and the gel, respectively. The exponential form strain energy density for the myocardial tissue is

$$W = \frac{c}{2}(e^Q - 1), \quad (2.21)$$

where c is a parameter and Q is a combination of strain tensor components. This strain energy accounts for the anisotropic properties of tissues. The Ogden form of strain energy density of Sylgard gel is:

$$W = \frac{2\mu}{\alpha^2}(\lambda_1^\alpha + \lambda_2^\alpha + \lambda_3^\alpha - 3), \quad (2.22)$$

where μ and α are material parameters, λ_1 , λ_2 and λ_3 are the stretch ratios.

Moreira and Nunes (2013) compares the simple shear and pure shear for an incompressible isotropic hyperelastic material under large deformation. Results show that simple shear cannot be treated as pure shear plus a rotation as usually considered in small deformation. Experiments and theoretical models are both

presented. The simple shear is achieved by single lap joint testing, and pure shear is by planar tension testing. The deformation gradients of simple shear and pure shear are:

$$F_{ss} = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.23)$$

and

$$F_{ps} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}, \quad (2.24)$$

respectively. The strain energy density functions are chosen as:

$$W = \sum_{p=1}^N \frac{\mu_p}{\alpha_p} (\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3), \quad (2.25)$$

where μ_p and α_p are material parameters.

Kang et al (2009) developed microstructural models which consider filament bending and tension energies as well as thermal fluctuations. These are used to predict the direction and magnitude of the normal stress under shear.

2.5. Challenging/Unresolved issues

The structures of natural materials are typically hierarchical and made up of different components, which may themselves be elastically anisotropic. Soft

biological materials such as skins, heart valves and articular cartilages are multilayered, for example. Criscione et al (1999) considered rabbit cardiac papillary muscles as a bilayered composite consisting of a core of myocardium inside a thin sheath of endocardium. Furthermore, there is also an increasing trend for soft synthetic materials to be composites, in order to increase and improve their functionalities, e.g., multilayered capsules for drug delivery with specific release profile. Hence, the issues of multilayered architecture and elastic anisotropy should be considered when studying natural biological materials. Most works, however, do not consider these issues due to the complexities in the theoretical formulation and the excessive computational time in numerical studies.

Soft materials are inherently nonlinear and may exhibit unusual nonlinear phenomena. The Poynting effect is one such “unusual” phenomenon, discovered more than a century ago by Poynting (1909). The negative Poynting effect has only been recently emphasized, if not discovered, in connection with soft biological materials, e.g., Storm et al. (2005). Most works reviewed above take into account the nonlinearity, although some use only linear elasticity theory to simplify numerical (especially finite element) calculations. In addition, the theoretical modeling involves assumptions which should be carefully reweighed, for the “discovery” of nonlinear phenomena depends critically on them.

Experiments also show that soft materials exhibit time-dependent/rate dependent behaviors. The shear experiments conducted by Janmey et al (2007) show that the materials are strain-rate dependent. Thus, the rate dependent behavior of soft

materials may also need to be considered.

In a physiological environment, biological materials and synthetic biomaterials (such as artificial tissues) are often subjected to complex loading which may involve torsion, axial loading (tension or compression), shear and dilatation. Similarly, in industrial applications soft devices such as sensors and actuators may operate in a demanding environment which imposes a combination of loading types. Hence, studying the combined loading of soft materials is of great practical significance.

In summary, the issues of multilayered architecture, material anisotropy and rate dependence, nonlinearity, as well as combined loading render the investigation of soft materials highly complicated. In this dissertation work, the issues of multilayered architecture, nonlinearity and combined loading are explored theoretically. A number of relatively simple experiments are also carried out to verify the theoretical predictions.

Chapter 3 Dilatational Loading

In this chapter, the problems of multilayered spherical and cylindrical hydrogels subjected to dilatation are solved. The problem of bilayered spherical hydrogels is solved in Section 3.1. The problem of multilayered cylindrical hydrogels is solved in Section 3.2. Comparison with available experimental data is presented in Section 3.3.

A dilatation is understood to be a volumetric strain, and in the present context it may result from various stimuli such as mechanical load, temperature, moisture absorption and chemical concentration. The relationship between a stimulus S and the dilatation profile $\mathcal{G}(S)$ can be very complex, but for specific elasticity problems one can obtain elastic solutions of the stresses and displacements without knowing explicitly $\mathcal{G}(S)$. For instance, if the relevant spatial variable is the radial coordinate r , one starts directly with a loading function $\mathcal{G}(r)$, instead of considering $\mathcal{G}(S(r))$. In other words, the starting point is the dilatation profile $\mathcal{G}(r)$, which can assume various functional forms (linear, stepped, quadratic, etc.) to capture the various possibilities. This was the approach taken by the seminal work of Kirchner (1974).

A note regarding the notation for stresses is made here. By “stress” is meant the first Piola-Kirchhoff stress, unless otherwise stated. Another stress measure, the Cauchy stress, will be stated explicitly as such. To simplify notation, the first Piola-Kirchhoff stress components are written using the notations T_{rr} , $T_{\theta\theta}$, $T_{\theta r}$, without differentiating the subscript coordinates in the deformed and undeformed

configurations. If we denote (r, θ) and (ρ, ψ) as coordinates in the undeformed and deformed configurations, respectively, then T_{rr} should strictly be written as $T_{\rho r}$, which represents the stress along the ρ -direction in the deformed configuration with respect to the r -face in the undeformed configuration. Similarly, $T_{\theta r}$ should be written as $T_{\psi r}$ in the strict notation.

3.1. Bilayered Spheres

In this section, we develop closed-form analytical solutions for the stresses and displacements in a bilayered spherical hydrogel subjected to a spherically symmetric dilatation. Emphasis is placed on investigating the influence of inhomogeneity arising from dissimilar layer elastic constants for achieving a wide spectrum of mechanical states. The Section is organized in the following manner. In Section 3.1.1, the nonlinear elastic problem is defined and the solution approach is reviewed. In Section 3.1.2, the solutions are presented in detail. This is followed by numerical results in Section 3.1.3, a discussion in Section 3.1.4, and a set of conclusions in Section 3.1.5.

3.1.1. Nonlinear elastic model for dilatations

3.1.1.1. Definition of problem

Figure 3.1 shows a spherical hydrogel consisting of an outer layer of inner and outer radii r_a and r_b , respectively, and an inner core of radius r_a . It is assumed that the outermost surface of the hydrogel is traction-free and the inner core-outer

layer interface is perfectly bonded with no defects. The second-order elasticity theory of Murnaghan (1951) is used to model the elastic nonlinearity of the two layers. Two second-order elastic constants λ_i , μ_i and three third-order elastic constants l_i , m_i , n_i , where $i = 1, 2$ (denoting the inner core and outer layer respectively), are used in this theory. A dilatation of $\mathcal{G}(r)$ is applied to the hydrogel, where r is the radial coordinate. In the current study, a constant dilatation of magnitude $\mathcal{G} = \mathcal{G}_0$ is applied to the outer layer only. The objective is to derive the displacement field and the stress field. To facilitate the derivations, the nonlinear elastic model of Murnaghan (1951) is briefly reviewed next.

3.1.1.2. Outline of nonlinear elastic model

The energy density W of Murnaghan (1951) is written as a function of the strain invariants J_1 , J_2 , J_3 of the Lagrangian strain $\mathbf{E}$:

$$W = \frac{\lambda + 2\mu}{2} J_1^2 - 2\mu J_2 + \frac{l + 2m}{3} J_1^3 - 2m J_1 J_2 + n J_3, \quad (3.1)$$

where λ, μ are the second-order and l, m, n the third-order elastic constants, respectively. In terms of the principal strains E_1 , E_2 and E_3 of $\mathbf{E}$, the three strain invariants take the following form:

$$J_1 = E_1 + E_2 + E_3, \quad J_2 = E_1 E_2 + E_2 E_3 + E_3 E_1, \quad J_3 = E_1 E_2 E_3. \quad (3.2)$$

Fig. 3.1. A spherical hydrogel composite consisting of a core of radius r_a and an outer layer of outer radius r_b . A constant dilatation g_0 is applied to the outer layer.

As the governing equation (the equilibrium equation) will be formulated in terms of displacements, the above strain invariants are expressed as a function of the radial displacement u_r and its derivative u'_r with respect to r .

The stress in the undeformed coordinate system is written as the matrix product between the deformation gradient $\mathbf{F}$ and the derivative of the energy density with respect to the Lagrangian strain:

$$\mathbf{T} = \mathbf{F} \frac{\partial W}{\partial \mathbf{E}}. \quad (3.3)$$

The Lagrangian strain is given by:

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^*\mathbf{F} - \mathbf{I}), \quad (3.4)$$

where $\mathbf{F}^*$ denotes the transpose of $\mathbf{F}$ and $\mathbf{I}$ is the identity matrix.

The equilibrium equation takes the following form:

$$\nabla \cdot \mathbf{T} + \rho \mathbf{f} = 0, \quad (3.5)$$

where ρ is the mass density and $\rho \mathbf{f}$ the body force per unit volume. Note that T is diagonal and its components $T_{\theta\theta} = T_{\phi\phi}$ due to spherical symmetry of the dilatation.

Hence, it can be shown that the r -component of Eq. (3.5) is the only relevant equation and can be written as (Wu and Kirchner, 2011):

$$T'_{rr} + \frac{2}{r}T_{rr} - \frac{2}{r}T_{\theta\theta} = -\rho F_r, \quad T_{rr}(r = r_b) = 0, \quad (3.6)$$

where T_{rr} and $T_{\theta\theta}$ are the radial and meridional (or azimuthal) components of the stress in the undeformed coordinate system, ρ is the mass density, and the force per unit volume ρF_r is a function of the dilatation:

$$\rho F_r = -k\left(\lambda + \frac{2}{3}\mu\right)\frac{d(3\vartheta)}{dr} + k^2\left(\lambda + \frac{2}{3}\mu - 2l - \frac{2n}{9}\right)(3\vartheta)\frac{d(3\vartheta)}{dr}. \quad (3.7)$$

The symbols k and k^2 denote the first- and second-order terms, respectively. The derivation of the solutions to this equation is shown in Section 3.1.2.

3.1.2. Solutions

The first step is to rewrite the equilibrium equation, Eq. (3.6), in terms of the radial displacement u_r and its derivative u'_r with respect to r . This is a spherically symmetric problem and other displacement components are zero. According to the perturbation procedure of Murnaghan (1951), u_r is written as the sum of the first- and second- order parts u and w :

$$u_r = u + kw, \quad (3.8)$$

where k keeps track of the approximation order. The total radial displacement is the direct sum of the first- and second-order displacements; k being merely a marker. The replacement of u_r by $u + kw$ is to be made in all expressions for the purpose of derivation. Omitting the details, the stresses T_{rr} and $T_{\theta\theta}$ obtained through the differentiation of W with respect to $\mathbf{E}$, take the following form:

$$T_{rr} = kT_{rr}^L + k^2T_{rr}^{NL}, \quad (3.9)$$

where T_{rr}^L and T_{rr}^{NL} are the linear and nonlinear radial stresses, respectively:

$$T_{rr}^L = \lambda(u' + 2\frac{u}{r}) + 2\mu u', \quad T_{rr}^{NL} = \left[\begin{array}{l} \lambda(\frac{3}{2}u'^2 + 2\frac{uu'}{r} + \frac{u^2}{r^2}) \\ + 3\mu u'^2 + l(u' + 2\frac{u}{r})^2 + 2m(u'^2 - \frac{u^2}{r^2}) + n\frac{u^2}{r^2} \\ + \lambda(w' + 2\frac{w}{r}) + 2\mu w' \end{array} \right], \quad (3.10)$$

and

$$T_{\theta\theta} = kT_{\theta\theta}^L + k^2T_{\theta\theta}^{NL}, \quad (3.11)$$

where $T_{\theta\theta}^L$ and $T_{\theta\theta}^{NL}$ are the linear and nonlinear meridional stresses, respectively:

$$T_{\theta\theta}^L = \lambda(u' + 2\frac{u}{r}) + 2\mu\frac{u}{r}, \quad T_{\theta\theta}^{NL} = \left[\begin{array}{l} \lambda(\frac{1}{2}u'^2 + \frac{uu'}{r} + 3\frac{u^2}{r^2}) \\ + 3\mu\frac{u^2}{r^2} + l(u' + 2\frac{u}{r})^2 + 2m(\frac{u^2}{r^2} - \frac{uu'}{r}) + n\frac{uu'}{r} \\ + \lambda(w' + 2\frac{w}{r}) + 2\mu\frac{w}{r} \end{array} \right]. \quad (3.12)$$

The total stresses are again the direct sum of their linear and nonlinear components.

The nonlinear stresses contain both u and w and their first-order derivatives. These

stress expressions and the body force expressions of Eq. (3.7) are substituted into Eq.

(3.6).

Upon extracting the k term of Eq. (3.6), the first-order part can be written as:

$$(\lambda + 2\mu)(u'' + 2\frac{u'}{r} - 2\frac{u}{r^2}) = (3\lambda + 2\mu)\frac{d\vartheta}{dr}, \quad (3.13)$$

which can be rewritten as:

$$\frac{d}{dr} \left[\frac{1}{r^2} \frac{d(r^2 u)}{dr} \right] = L \frac{d\vartheta}{dr}, \quad (3.14)$$

where the dimensionless parameter L is given by:

$$L = \frac{3\lambda + 2\mu}{\lambda + 2\mu} = \frac{1 + \nu}{1 - \nu}, \quad (3.15)$$

with ν being the Poisson's ratio. The second-order equilibrium equation is associated with the k^2 term and can be written as:

$$\begin{aligned} & (\lambda + 2\mu) \left(w'' + 2 \frac{w'}{r} - 2 \frac{w}{r^2} \right) \\ &= -2(\lambda + 3\mu + 2m) \left[\left(u'' + 2 \frac{u'}{r} - 2 \frac{u}{r^2} \right) u' - \frac{1}{r} \left(u' - \frac{u}{r} \right)^2 \right] \\ & - (\lambda + 2l) \left(u'' + 2 \frac{u'}{r} - 2 \frac{u}{r^2} \right) \left(u' + 2 \frac{u}{r} \right) - \left(\lambda + \frac{2}{3}\mu - 2l - \frac{2n}{9} \right) 9\vartheta \frac{d\vartheta}{dr}. \end{aligned} \quad (3.16)$$

For a bilayered hydrogel, certain interface continuity conditions must be satisfied. In the following subsections, the solutions for the stresses and radial displacement are derived for the core (Material 1) and the outer layer (Material 2). For convenience, subscripts 1 and 2 are attached to the symbols for the stresses and displacements in these two materials, e.g., T_{1r}^L , T_{2r}^L , u_1 , u_2 , etc. Note also that the stresses in Eqs. (3.10) and (3.12) should be computed from $u_r / r - \vartheta$ and $u'_r - \vartheta$ (rather than

u_r / r and u'_r) as it is the difference between the fields that preserves the continuity of the body.

3.1.2.1. Solutions for Material 1

As stated previously, the core has the radius r_a and no dilatation is applied to it. To satisfy the interface boundary conditions, both the radial displacement and radial stress must be continuous across the interface at $r = r_a$. The linear and nonlinear parts of the solutions are derived below.

Linear part

Substitute $\mathcal{G} = 0$ into Eq. (3.13). The solution to this first-order equation which fulfils a nonsingular u_1 at $r = 0$ is:

$$u_1 = A_1 r, \quad (3.17)$$

where A_1 is a dimensionless constant to be determined. Substituting u_1 into the linear stress expressions of Eqs. (3.10) and (3.12), the following explicit form for the stresses is obtained:

$$T_{1r}^L = T_{1\theta}^L = (3\lambda_1 + 2\mu_1)A_1. \quad (3.18)$$

Nonlinear part

Substitute $\mathcal{G} = 0$ into Eq. (3.16) and the solution to the second-order equation which fulfils a nonsingular w_1 at $r = 0$ is:

$$w_1 = C_1 r \quad (3.19)$$

where C_1 is a dimensionless constant to be determined. Substituting w_1 into the nonlinear stress expressions of Eqs. (3.10) and (3.12), the following explicit form is obtained:

$$T_{1rr}^{NL} = T_{1\theta\theta}^{NL} = \left(\frac{9}{2}\lambda_1 + 3\mu_1 + 9l_1 + n_1\right)A_1^2 + (3\lambda_1 + 2\mu_1)C_1. \quad (3.20)$$

From Eqs. (3.17) and (3.19), it can be observed that both the linear and nonlinear displacements of the core vary linearly with r . On the other hand, the core is subjected to a constant hydrostatic stress according to Eqs. (3.18) and (3.20).

3.1.2.2. Solutions for Material 2

The outer layer has the inner radius r_a and the outer radius r_b . A constant dilatation $\mathcal{G} = \mathcal{G}_0$ is applied to this layer. The following boundary conditions are enforced: the outermost surface $r = r_b$ is traction-free and the radial displacement and radial stress are continuous across the interface at $r = r_a$.

Linear part

Substituting $\mathcal{G} = \mathcal{G}_0$ into Eqs. (3.13), the linear parts of the solutions for the radial displacement and the radial and meridional stresses in Material 2 can be obtained as:

$$u_2 = A_2 r + \frac{r_a^3}{r^2} B_2, \quad (3.21)$$

$$T_{2r}^L = (3\lambda_2 + 2\mu_2) \left(A_2 - \frac{L_2 \mathcal{G}_0}{3} \right) - \frac{4\mu_2 L_2 \mathcal{G}_0}{3} - \frac{4\mu_2 r_a^3}{r^3} B_2, \quad (3.22)$$

$$T_{2\theta}^L = (3\lambda_2 + 2\mu_2) \left(A_2 - \frac{L_2 \mathcal{G}_0}{3} \right) - \frac{4\mu_2 L_2 \mathcal{G}_0}{3} + \frac{2\mu_2 r_a^3}{r^3} B_2, \quad (3.23)$$

where A_2 and B_2 are dimensionless constants and $L_2 = (3\lambda_2 + 2\mu_2) / (\lambda_2 + 2\mu_2) = (1 + \nu_2) / (1 - \nu_2)$. At this point, a total of four constants A_1, C_1, A_2, B_2 have appeared in the solutions. Application of the following boundary conditions

$$u_2(r_a) = u_1(r_a), \quad T_{2r}^L(r_a) = T_{1r}^L(r_a), \quad T_{2r}^L(r_b) = 0, \quad (3.24)$$

which respectively refer to the continuity of radial displacement and radial stress, and vanishing traction on the outermost surface, three of the constants A_1, A_2, B_2 can be determined:

$$A_1 = -\frac{4(r_a^3 - r_b^3) \mathcal{G}_0 \mu_2 (3\lambda_2 + 2\mu_2)}{M}, \quad (3.25)$$

$$A_2 = \frac{\mathcal{G}_0 (3\lambda_2 + 2\mu_2) (-4\mu_2 r_a^3 + (3\lambda_1 + 2\mu_1 + 4\mu_2) r_b^3)}{M}, \quad (3.26)$$

$$B_2 = -\frac{\mathcal{G}_0(3\lambda_1 + 2\mu_1)(3\lambda_2 + 2\mu_2)r_b^3}{M}, \quad (3.27)$$

where

$$M = 4\mu_2(3\lambda_1 - 3\lambda_2 + 2\mu_1 - 2\mu_2)r_a^3 + (3\lambda_2 + 2\mu_2)(3\lambda_1 + 2\mu_1 + 4\mu_2)r_b^3. \quad (3.28)$$

Nonlinear part

Substituting u_2 from Eq. (3.21) into Eq. (3.16), i.e., the second-order radial equilibrium equation, yields a differential equation in the variable w_2 :

$$(\lambda_2 + 2\mu_2)(w_2'' + 2\frac{w_2'}{r} - 2\frac{w_2}{r^2}) = 18(\lambda_2 + 3\mu_2 + 2m_2)B_2^2 \frac{r_a^6}{r^7}, \quad (3.29)$$

where a constant dilatation $\mathcal{G} = \mathcal{G}_0$ has been assumed. Equation (3.29) has the solution:

$$w_2 = C_2 r + \frac{D_2 r_a^3}{r^2} + \frac{H_2 r_a^6}{r^5}, \quad (3.30)$$

where C_2, D_2, H_2 are all dimensionless constants. The last constant takes the form

$$H_2 = \frac{\lambda_2 + 3\mu_2 + 2m_2}{\lambda_2 + 2\mu_2} B_2^2, \quad (3.31)$$

and C_1 (which appears in Eq. (3.19)), C_2 and D_2 can be determined from the boundary conditions, which are the nonlinear counterparts to Eq. (3.20):

$$w_2(r_a) = w_1(r_a), \quad T_{rr}^{(2)NL}(r_a) = T_{rr}^{(1)NL}(r_a), \quad T_{rr}^{(2)NL}(r_b) = 0. \quad (3.32)$$

Using Eq. (3.30) for w_2 and the nonlinear part of Eq. (3.10) for T_{2rr}^{NL} , the constants

C_1 , C_2 and D_2 are determined to be :

$$C_1 = \frac{1}{2r_b^3 M} (\xi_1 r_a^6 + \xi_2 r_a^3 r_b^3 + \xi_3 r_b^6), \quad (3.33)$$

$$C_2 = -\frac{1}{2(\lambda_2 + 2\mu_2)r_b^3 M} (\xi_4 r_a^6 + \xi_5 r_a^3 r_b^3 + \xi_6 r_b^6), \quad (3.34)$$

$$D_2 = \frac{1}{Mr_b^3} (\xi_7 r_a^6 + \xi_8 r_a^3 r_b^3 + \xi_9 r_b^6), \quad (3.35)$$

and ξ_i , $i = 1, 2, \dots, 9$ are functions of the two sets of elastic constants given in

Appendix A.

The second-order radial and meridional stresses can be written out explicitly with

the aid of the second parts of Eqs. (3.10) and (3.12):

$$T_{rr}^{(2)NL} = \frac{1}{2(\lambda_2 + 2\mu_2)} \left(\zeta_1 + \frac{\zeta_2 r_a^3}{r^3} + \frac{\zeta_3 r_a^6}{r^6} \right), \quad (3.36)$$

$$T_{\theta\theta}^{(2)NL} = \frac{1}{2(\lambda_2 + 2\mu_2)} \left(\zeta_1 - \frac{\zeta_2 r_a^3}{2r^3} - \frac{2\zeta_3 r_a^6}{r^6} \right), \quad (3.37)$$

where ζ_1 , ζ_2 , ζ_3 are also functions of the elastic constants and given in Appendix

A.

From these results for Material 2, it can be seen that the displacement and stresses assume the form of a power-law. Specifically, the linear part of the displacement is dependent on r and r^{-2} , while the nonlinear part is dependent on r , r^{-2} and r^{-5} . For the stresses, all the corresponding power law exponents are one degree lower. The higher negative exponents for the nonlinear displacement and stresses imply that nonlinearity gains more influence as r becomes smaller. Moreover, the displacement expressions given in Eqs. (3.17), (3.19), (3.21) and (3.30) reveal that a reduced set of four geometric-elastic constants can completely characterize the displacement field. Similarly, the stresses given in Eqs. (3.18), (3.22), (3.23), (3.36) and (3.37) can be characterized by another reduced set of four constants. The original material and geometric parameters consist of ten elastic constants and the radii r_a, r_b .

3.1.3. Numerical results

In summary, the analytical solutions for the linear parts of the radial displacement and radial and meridional stresses in Material 1 are given by Eqs. (3.17) and (3.18), respectively. The corresponding nonlinear parts are given by Eqs. (3.19) and (3.20). For Material 2, the corresponding equations are Eqs. (3.21)-(3.23) for the linear part and Eqs. (3.30), (3.36) and (3.37) for the nonlinear part. All numerical results below are obtained using these equations.

The dilatation is chosen to be 0.35. The ratio of the radii is chosen to be $r_b/r_a = 2$ in all the figures, except Fig. 3.5. The elastic constants are expressed in the units of

NkT , where N is the number of polymer chains per unit volume of dry polymer, k the Boltzmann constant and T the absolute temperature. Typical values for NkT lie in the range 10^{-5} to 10^{-2} GPa. For ease of reference, the elastic constants used in the figures are summarized in Table 3.1. The following values $\lambda = 3570 NkT$, $\mu = 1030 NkT$, $l = -3560 NkT$, $m = -2420 NkT$ and $n = -2350 NkT$, taken from Wu and Kirchner (2010), are chosen as the reference from which the variations are investigated. However, not only can the elastic constants vary over a wide range, but they can also be orders of magnitude different from each other. For instance, the elastic modulus was varied over three orders of magnitude by manipulating the degree of polymer crosslinking (Engler et al., 2006; Marklein and Burdick, 2010). Also, Catheline et al. (2003) obtained the following values in kPa for agar-gelatin based phantoms via the method of transient elastography: λ (2.25×10^6), μ (6.35 to 9.67), l (-2×10^6 to 55×10^6), m (-26×10^6 to -12×10^6) and n (-101 to -64). The constant λ is five to six orders of magnitude larger than μ , while l and m are four to five orders of magnitude larger than n .

3.1.3.1. Influence of elastic constants on swelling behavior

Figures 3.2(a) and 3.2(b) plot the dependence of $u_r(r = r_b) / r_a$ (the normalized radial displacement at the outer boundary) on the linear (λ_2, μ_2) and nonlinear (l_2, m_2, n_2) elastic constants, respectively. Table 3.1 shows the elastic constants selected for the simulations, where all elastic constants in Materials 1 take their reference values while those in Material 2 are fixed at the tabulated values except for the particular

elastic constant that is being varied for each curve.

Several observations can be made. First, the radial displacement is nonlinearly dependent on the linear elastic constants, but linearly dependent on the nonlinear constants. This can be inferred from the relevant equations. It can also be observed from Fig. 3.2(a) that larger values of the second-order constants, i.e., λ_2 and μ_2 , give rise to smaller swelling. In general, if a material has relatively large (small)

Fig. 3.2. Variation of the normalized radial displacement at the outer boundary with (a) second-order and (b) third-order elasticity parameters of Material 2 in a hydrogel composite.

Fig. 3.3. Variation of the normalized radial displacement at the outer boundary with a multiplicative factor α .

values of the second-order constants, it is said to be “linearly hard (soft)”. Furthermore, increasing the third-order elastic constant m_2 (negatively) increases substantially the swelling of the hydrogel composite, while varying l_2 and n_2 cause less change to the swelling, see Fig. 3.2(b). This shows that m_2 is the more dominating third-order parameter, which can be used to design the swelling characteristic of the composite. In general, if the magnitude of m for a material is large (small), it is said to be “nonlinearly soft (hard)”.

Figure 3.3 plots $u_r(r=r_b)/r_a$ against a multiplicative factor α , with $\lambda_1 = 35700\alpha$, $\mu_1 = 10300\alpha$, $l_1 = -3560$, $m_1 = -2420/\alpha$, $n_1 = -2350$; $\lambda_2 = 357/\alpha$, $\mu_2 = 103/\alpha$, $l_2 = -3560$, $m_2 = -242000\alpha$, $n_2 = -2350$. As α increases from 0 to 1, this implies that Material 1 becomes both linearly and

Fig. 3.4. Variation of the ratio of nonlinear to linear radial stress at the interface $r = r_a$ with λ_2 , μ_2 , l_2 , m_2 and n_2 in a hydrogel composite.

nonlinearly hard, while Material 2 becomes both linearly and nonlinearly soft. For this range of variation, the ratio $u_r(r = r_b)/r_a$ increases twenty five-fold. Hence, it may be possible to control the swelling behavior to a large degree by properly designing the linear and nonlinear elasticity of the core and the outer layer.

3.1.3.2. Ratio of nonlinear to linear radial stress

To explore the relative importance of nonlinear and linear effects, the ratio of nonlinear to linear radial stress at the interface $r = r_a$ is plotted against λ_2 , μ_2 , l_2 , m_2 and n_2 in Fig. 3.4. The elastic constants in both the core and the shell are

held fixed, except for the particular one being varied for each curve shown in the figure. Refer to Table 3.1 for the actual values.

The nonlinear part of the stress can be less than, comparable to, or larger than the linear part. For example, the ratio of the stresses exceeds 1 in the range $0 < \lambda_2 < 2000 NkT$. Also, the ratio can be negative, implying that the nonlinear and linear parts will add up to a smaller magnitude compared to the larger magnitude of the two parts. Furthermore, the dependence of the ratio on the linear constants is non-monotonic, while its dependence on the nonlinear constants is linear. These results, however, are subject to the restriction of second-order theory. Overall, Fig. 3.4 suggests that the characterization of the mechanical property of a hydrogel by a single linear elastic constant in applications such as culturing stem cells on hydrogel substrates is unlikely to be accurate and may be misleading.

3.1.3.3. Influence of interface position

Figure 3.5 plots $u_r(r = r_b) / (r_b / 2)$ against $r_a / (r_b / 2)$ with r_b fixed. Two cases are considered: homogeneous material in Fig. 3.5(a) and a composite in Fig. 3.5(b). As $r_a / (r_b / 2)$ changes between 0 and 2, the interface changes its position from a location close to the spherical center to a location near to the outer surface. This means that when $r_a / (r_b / 2)$ is close to zero, the composite is made up almost entirely of Material 2. The elastic constants chosen for the homogeneous hydrogel are their reference values, while those chosen for the composite are similar to those of Catheline et al. (2003). For the composite, the core is linearly hard while the shell is

Fig. 3.5. Variation of the normalized surface radial displacement with interface position $r_a/(r_b/2)$ while keeping r_b fixed for different sets of elastic constants shown in Table 3.1: (a) Set 1, and (b) Set 2.

linearly soft ($\lambda_1 / \lambda_2 = 10$, while all other corresponding constants assume the same values). It is also notable that the shear modulus $\mu_1 = \mu_2 = 8 \text{ } NkT$ is considerably smaller than all the other constants.

Figure 3.5(a) shows that as the interface moves from the spherical center to the outer boundary, the swelling decreases monotonically. This is expected since the region in which the dilatation is applied, i.e., the outer layer, decreases in size. It can also be seen that the swelling is zero in the limit when the interface reaches the outer boundary. In contrast, Fig. 3.5(b) shows that the swelling first increases but subsequently decreases to zero with interface position, attaining a maximum value at $r_a/(r_b/2) \sim 1.6$. Also, the swelling in Fig. 3.5(b) is much more significant compared to that in Fig. 3.5(a) for $r_a/(r_b/2)$ between 0.5 and 1.8. The composite gel may thus offer a richer and more complex displacement profile that can be exploited for applications.

Further numerical simulations (omitted for brevity) reveal that the stresses also vary significantly as the interface position changes. Hence, the swelling and internal stress characteristics of a hydrogel composite can be effectively controlled by varying the interface position, besides varying the elastic constants of the two layers.

3.1.3.4. Influence of elastic constants on stress state

In this section, smaller elastic constants of the order of 10^{-3} to $1 \text{ } NkT$ are chosen for the hydrogel layers in order to mimic the elastic constants of cells or tissues. As

stated in the Introduction, the mechanical properties of the substrates for growing cells can control their growth characteristics.

Figure 3.6 plots the radial stress T_r against the radial coordinate r/r_a for two different sets of elastic constants as indicated in Table 3.1. It can be seen that T_r is constant within the core as predicted by theory, is continuous across the interface as required by the continuity condition, and tends towards zero at the outer boundary, as required by the free-traction boundary condition. The compressive stress within the core reaches a value of $\sim -0.41 NkT$ in Fig. 3.6(a), while it only reaches a value of $\sim -0.026 NkT$ in Fig. 3.6(b). The value of $-0.41 NkT$ is large compared to the order of the magnitude of the elastic constants. It may cause buckling instability of the hydrogel. Indeed, the buckling of a ring of tumor cells into various mode shapes has been investigated, e.g., Goriely and Ben Amar (2005).

Further numerical experiments confirm that the stresses inside the composite can also be tensile, depending on the elastic constants. The control of stress magnitude and sign is of significance in many applications; for example a large tensile stress may cause nucleation and extension of cracks in the material, or may influence the adherence and growth characteristics of tissues cultured on hydrogel substrates.

An important question remains as to the multiaxial stress state in the hydrogel that will have a direct bearing on biological processes. Figure 3.7 plots the variation of T_r and T_θ with the normalized radial coordinate r/r_a . For this set of material parameters (see Table 3.1), the core experiences hydrostatic tension while the outer

layer may experience triaxial tension-compression or triaxial compression. Hence, the use of a multilayered hydrogel allows much greater control of the stress state for applications in bioengineering.

Fig. 3.6. Variation of the radial stress with the radial coordinate r/r_a for two different parameter sets: (a) large compressive radial stress (Set 1), and (b) smaller compressive radial stress (Set 2).

Fig. 3.7. Variation of the radial and meridional stresses with the radial coordinate r/r_a in a hydrogel composite.

3.1.4. Discussion

The work presented is based on second-order elasticity, characterized by five elastic constants. Analytical solutions for nonlinear composites are, generally speaking, not easily available, and hence the obtained power law expressions are a contribution that permits parametric studies and facilitates the observations of key physical trends. From the results in Section 3.1.3, we can make some assessments of the mechanical behavior of bilayered hydrogels as related to their elasticity, geometry and loading.

First, both the swelling and internal stresses can be manipulated via the judicious choice of the elasticity of the two layers. For instance, large swelling can be obtained by selecting a material characterized by a large negative value of the third-order constant m . It is also possible to make use of elastic inhomogeneity to

elicit larger swelling compared to that obtainable with a homogeneous sphere of the same size.

Second, the material nonlinearity plays an essential role in the mechanical response. A very large range of values for the elastic constants can be exhibited by hydrogels of various types (e.g., pH, enzyme, electrical, temperature sensitive), dependent on factors such as chemical composition, methods of synthesis and cross-linking. Hence, the proper characterization of the three third-order constants in relation to mechanics-dependent biological processes is of great importance in designing bioenvironements.

Third, the mechanical behavior can also be effectively controlled via the selection of the relative thicknesses of the composite layers, i.e., the interface position. Various combinations of the interface position and the elastic parameters may be chosen to yield a composite with either large or negligible swelling, or one with the desired multiaxial stress state facilitating the intended functions.

Fourth, our theoretical results show that the mechanical behavior can be described by a total of only eight geometric-elastic constants, out of a total of twelve constants. This leads to greater flexibility in parameter selection, since the same mechanical state can be achieved by multiple parameter sets and the final choice can be made on the basis of other (non-mechanical) criteria.

Finally, it is emphasized that the loading on the composite sphere is caused by various concentrations, such as those due to drugs, proteins and growth factors, resulting in a dilatation field. The dilatation profile naturally plays a crucial role in

the mechanical response. Hence, the design of the elasticity and layer thicknesses must also take into account the concentration profile within the composite.

Further possible issues and challenges are discussed in the following. An increasing number of theoretical and experimental works on determining the nonlinear elastic constants of soft materials have been attempted, including those based on third-order elasticity (Gennisson et al., 2007; Réiner et al., 2008). This lays the foundation for solving multilayered soft materials using higher-order elasticity. However, the relationship between dilatation and the underlying physical phenomena (e.g., absorption of nutrients) should be explored. Kirchner (1974) considered various possible dilatation profiles in a linear elastic medium, but the relationship of these profiles to the contributing physical phenomena may be unclear.

The current work does not consider transient or time-dependent effects, which may be important for certain investigations such as cell proliferation on compliant substrates. Also, the existence of large compressive stresses in the hydrogel may induce buckling with potential adverse effects. Instability of nonlinear materials has been studied in a number of works (Sekimoto and Kawasaki, 1989; Haughton, 2004; Goriely and Ben Amar, 2005). Furthermore, only bilayered hydrogels are considered. Multilayered hydrogels have gained increasing importance, and generalization of the analytical solutions for a periodic multilayered hydrogel with alternate layers deserves future investigation.

The importance of interdisciplinary research should also be emphasized. While

the bioengineers have characterized the mechanical property of the bioenvironment with a linear elastic modulus, much of the relationship between mechanical variables (such as stresses) and biological processes remains to be explored. For instance, it is not known how multiaxiality and sign of the stress state affect various biological processes. Some simulations have been attempted by the method of mechanics, e.g., the influence of cyclic substrate deformation on the growth orientation of endothelial cells (McGarry et al., 2005). Such computational studies were largely inspired by experimental investigations, e.g., Wang et al. (2001).

An interesting challenge is the computation of the isotropic effective moduli of a composite sphere. For a linear elastic medium containing a spherical inclusion, the effective bulk and shear moduli and their bounds have been studied previously, e.g., Hashin (1962) and Christensen and Lo (1979). In a later work, Herve and Zaoui (1993) derived the effective bulk and shear moduli of a linear elastic medium containing a multilayered spherical inclusion. An essential step in the derivation is the computation of the average strains and stresses in each phase based on the separation of hydrostatic and deviatoric components, and the use of a strain (or stress) condition which requires that the average strain (stress) in the n -layered sphere to be the same as the uniform macroscopic strain (stress) imposed at infinity. While a similar principle can be applied to nonlinear elastic spherical composites, additional works are anticipated. First, it is necessary to solve boundary value problems related to the imposition of hydrostatic pressure and simple shear; the shear problem is not solved in the current work. Second, determination of the

effective third-order elastic constants would likely require the solution of additional boundary value problems, yet to be identified, since five effective parameters are to be determined. These problems will be addressed in the future.

In spite of the many outstanding issues, the current model represents a significant step towards the investigation of multilayered hydrogels. The obtained solutions have considered several essential elements: inhomogeneity arising from dissimilar layers, nonlinear elasticity and finite deformation. Incorporation of these elements has revealed the importance of nonlinearity in the elastic modeling of the hydrogel, the possible design of composite hydrogels based on dissimilar layer elasticity and the interface position, and the diverse range of hydrogel responses to even a simple dilatation.

3.1.5. Conclusion

Based on a second-order elasticity model, power law solutions have been obtained for a spherical hydrogel composite subjected to a dilatation. Two second- and three third-order elastic constants are used in the model. The dilatation is assumed to be constant and applied to the outer layer only.

The results show that nonlinearity is a significant part of the displacement and stresses, which can be completely characterized by a reduced set of eight geometric-elastic constants. Among the three nonlinear elastic parameters, m has a more dominant influence on the stresses and displacement compared to l and n . Studies are underway to investigate the relative importance of these parameters.

Table 3.1. Elastic constants for Figs. 3.2–3.7 (unit: NkT).

Figure	λ_1	μ_1	l_1	m_1	n_1	λ_2	μ_2	l_2	m_2	n_2
3.2a	3570	1030	-3560	-2420	-2350		103	-3560	-24200	-2350
	3570	1030	-3560	-2420	-2350	357		-3560	-24200	-2350
3.2b	3570	1030	-3560	-2420	-2350	357	103		-24200	-2350
	3570	1030	-3560	-2420	-2350	357	103	-3560		-2350
	3570	1030	-3560	-2420	-2350	357	103	-3560	-24200	
3.3	35700α	10300α	-3560	$-2420/\alpha$	-2350	$357/\alpha$	$103/\alpha$	-3560	-242000α	-2350
3.4	35700	10300	-3560	-2420	-2350		103	-3560	-24200	-2350
	35700	10300	-3560	-2420	-2350	3570		-3560	-24200	-2350
	35700	10300	-3560	-2420	-2350	3570	103		-24200	-2350
	35700	10300	-3560	-2420	-2350	3570	103	-3560		-2350
	35700	10300	-3560	-2420	-2350	3570	103	-3560	-24200	
3.5a Set 1	3570	1030	-3560	-2420	-2350	3570	1030	-3560	-2420	-2350
3.5b Set 2	2.25×10^6	8	-2×10^6	-20×10^6	-100	2.25×10^5	8	-2×10^6	-20×10^6	-100
3.6a Set 1	0.00357	0.00103	-0.3560	-0.00242	-0.2350	3.5700	1.0300	-0.3560	-2.4200	-0.2350
3.6b Set 2	3.5700	1.0300	-0.3560	-2.4200	-0.2350	0.00357	0.00103	-0.3560	-0.00242	-0.2350
3.7	3.5700	1.0300	-0.3560	-0.00242	-0.2350	0.00357	0.00103	-0.3560	-2.4200	-0.2350

Furthermore, a very wide range of swelling and multiaxial stress states are realizable by manipulating the layer elasticity and the layer thicknesses. Such results, in conjunction with experimental data, will help to elucidate the influence of microenvironment mechanical property on biological processes such as cell adhesion, proliferation and apoptosis.

3.2. Multilayered Cylinders

In the present work, solutions for the displacements and stresses are developed for a multilayered cylindrical hydrogel using a second-order elasticity model. The section is organized as followed. The problem of an N -layer cylinder is defined in Section 3.2.1. General solutions are derived in Section 3.2.2. Numerical results highlighting the influence of nonlinearity, material dissimilarity, non-uniform layer thicknesses and dilatation profiles are presented and discussed in Section 3.2.3. A list of conclusions is given in Section 3.2.4.

3.2.1. Nonlinear elastic model for dilatations

3.2.1.1. Definition of problem

Figure 3.8 shows an N -layered circular hydrogel cylinder with interfaces located at the radial coordinates $r = r_j, j = 1, N$. The layers are nonlinearly elastic, isotropic, and perfectly bonded to one another, but can be of dissimilar material compositions and thicknesses. The outermost surface of the hydrogel is assumed to be traction-free. The second-order elasticity theory of Murnaghan (1951) is used to model the elastic nonlinearity of the gel layers. Two second-order elastic constants λ_j, μ_j and three third-order elastic constants l_j, m_j, n_j , where $j = 1, N$, are used in this theory.

The cylinder is subjected to an axisymmetric dilatation profile, which can however fluctuate along r . For definiteness, step dilatation profiles are assumed, in which the dilatation in each layer is constant but it may vary from layer to layer.

Physically, fluctuations such as temperature $T(r)$ and chemical concentration $c(r)$ result in a local dilatation field $\mathcal{G}(r)$. The relation between T and $\mathcal{G}$ or between c and $\mathcal{G}$ is given by the coefficient of thermal expansion or the change in lattice parameter per unit concentration, respectively. However, it is not necessary to specify these relations in the elasticity problem. The field $\mathcal{G}(r)$ is considered as the fluctuation loading, and may be controlled, as in the embedding of drugs, nucleic acids, nutrients and growth factors of various concentrations in the layers of sub-micrometer thicknesses. The objective is to derive the displacement field and the stress field under the dilatation loading. To facilitate the derivations, the second-order elastic model is briefly reviewed.

3.2.1.2. Second-order elastic model

The energy density W of Murnaghan (1951) is written as a function of the strain invariants J_1, J_2, J_3 of the Lagrangian strain $\mathbf{E}$:

$$W = \frac{\lambda + 2\mu}{2} J_1^2 - 2\mu J_2 + \frac{l + 2m}{3} J_1^3 - 2m J_1 J_2 + n J_3, \quad (3.38)$$

where λ, μ are the second-order and l, m, n the third-order elastic constants, respectively. In terms of the principal strains E_1, E_2 and E_3 of $\mathbf{E}$, the three strain invariants are:

$$J_1 = E_1 + E_2 + E_3, \quad J_2 = E_1 E_2 + E_2 E_3 + E_3 E_1, \quad J_3 = E_1 E_2 E_3. \quad (3.39)$$

Fig. 3.8. A hydrogel cylinder composite consisting of N layers, with core radius r_1 and outermost radius r_N .

As the governing equation (the equilibrium equation) will be formulated in terms of displacements, the above strain invariants are expressed as a function of the radial displacement u_r and its derivative u_r' with respect to r . For a two-dimensional problem, J_1 and J_2 of Eq. (3.39) can be written as:

$$J_1 = k(u_r' + \frac{u_r}{r}) + \frac{1}{2}k^2[u_r'^2 + \left(\frac{u_r}{r}\right)^2], \quad J_2 = k^2 \frac{u_r u_r'}{r}, \quad (3.40)$$

where k is introduced to keep track on the order of approximation of the theory. Note that J_3 does not appear, since the strain matrix $\mathbf{E}$ is two-dimensional rather than three dimensional. For this reason, the third-order elastic constant n in Eq. (3.38) will also not appear in the solutions.

For the linear and second-order approximations, only terms up to the first- and second-order of k are retained, respectively. Equation (3.40) can be derived from the form of $\mathbf{E}$ given below in Eq. (3.44). The stress in the undeformed coordinate system is written as the matrix product between the deformation gradient $\mathbf{F}$ and the derivative of the energy density with respect to the Lagrangian strain:

$$\mathbf{T} = \mathbf{F} \frac{\partial W}{\partial \mathbf{E}}, \quad (3.41)$$

where

$$\frac{\partial W}{\partial \mathbf{E}} = \lambda J_1 \mathbf{I} + 2\mu \mathbf{E} + (IJ_1^2 - 2mJ_2) \mathbf{I} + 2mJ_1 \mathbf{E}. \quad (3.42)$$

The deformation gradient $\mathbf{F}$ in Eq. (3.41) can be written in terms of u_r and u'_r as:

$$\mathbf{F} = \text{diag}[1 + ku'_r, 1 + \frac{ku_r}{r}], \quad (3.43)$$

as shown in Appendix B. The Lagrangian strain in Eq. (3.42) can then be expressed as:

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^* \mathbf{F} - \mathbf{I}) = k \operatorname{diag}[u'_r, \frac{u_r}{r}] + \frac{1}{2}k^2 \operatorname{diag}[u_r'^2, \frac{u_r^2}{r^2}]. \quad (3.44)$$

The equilibrium equation takes the following form:

$$\nabla \cdot \mathbf{T} + \rho \mathbf{f} = 0, \quad (3.45)$$

where ρ is the mass density and $\rho \mathbf{f}$ the body force per unit volume, while the boundary condition at the free surface of the hydrogel is :

$$\mathbf{T} d\mathbf{S} = 0, \quad (3.46)$$

where $d\mathbf{S}$ is the oriented surface element. Note that $\mathbf{T}$ is diagonal with the radial and polar components T_{rr} and $T_{\theta\theta}$. Only the radial components of Eqs. (3.45) and (3.46) are relevant. These can be written as:

$$T'_{rr} + \frac{1}{r}T_{rr} - \frac{1}{r}T_{\theta\theta} = -\rho F_r, \quad (3.47)$$

where F_r is the radial component of $\mathbf{F}$. An expression can be obtained for the body force as follows. According to Murnaghan (1951), a uniform dilatation $\mathcal{G}$ causes a hydrostatic stress p given by:

$$p = -(3\lambda + 2\mu)\mathcal{G} + \frac{1}{2}(\lambda + \frac{2}{3}\mu - 2l - \frac{2n}{9})(3\mathcal{G})^2. \quad (3.48)$$

If $\mathcal{G}$ varies with r , the variation of p with r is treated as a body force, i.e., ρF_r is written as the derivative of p with respect to r :

$$\rho F_r = -k\left(\lambda + \frac{2}{3}\mu\right)\frac{d(3\mathcal{G})}{dr} + k^2\left(\lambda + \frac{2}{3}\mu - 2l - \frac{2n}{9}\right)(3\mathcal{G})\frac{d(3\mathcal{G})}{dr}, \quad (3.49)$$

where k and k^2 separate out the first- and second-order terms. The expressions in Eqs. (3.47) and (3.49) complete the form of the governing equation. The derivation of the solutions is shown in Section 3.

3.2.2. Solutions

Eq. (3.47) is re-written in terms of u_r and u'_r , where u_r is written as the sum of the first- and second- order displacements u and w :

$$u_r = u + kw, \quad (3.50)$$

and k keeps track of the approximation order. The total radial displacement is the direct sum of the first- and second-order displacements; k being merely a marker. From Eqs. (3.41) and (3.42), and noting that J_1 and J_2 , $\mathbf{J}_a$ and $\mathbf{E}$ are all functions of u_r and u'_r according to Eqs. (3.40) and (3.43)-(3.44), the expressions for T_{rr} and $T_{\theta\theta}$ can be written as:

$$T_{rr} = kT_{rr}^L + k^2T_{rr}^{NL}, \quad (3.51)$$

where the linear and nonlinear radial stresses T_{rr}^L and T_{rr}^{NL} are, respectively:

$$T_{rr}^L = \lambda(u' + \frac{u}{r}) + 2\mu u', \quad T_{rr}^{NL} = \begin{bmatrix} \lambda(\frac{3}{2}u'^2 + \frac{uu'}{r} + \frac{1}{2}\frac{u^2}{r^2}) \\ +3\mu u'^2 + l(u' + \frac{u}{r})^2 + 2mu'^2 \\ +\lambda(w' + \frac{w}{r}) + 2\mu w' \end{bmatrix}, \quad (3.52)$$

and similarly

$$T_{\theta\theta} = kT_{\theta\theta}^L + k^2T_{\theta\theta}^{NL}, \quad (3.53)$$

where the linear and nonlinear circumferential stresses $T_{\theta\theta}^L$ and $T_{\theta\theta}^{NL}$ are, respectively:

$$T_{\theta\theta}^L = \lambda(u' + \frac{u}{r}) + 2\mu \frac{u}{r}, \quad T_{\theta\theta}^{NL} = \begin{bmatrix} \lambda(\frac{1}{2}u'^2 + \frac{uu'}{r} + \frac{3}{2}\frac{u^2}{r^2}) \\ +3\mu \frac{u^2}{r^2} + l(u' + \frac{u}{r})^2 + 2m \frac{u^2}{r^2} \\ +\lambda(w' + \frac{w}{r}) + 2\mu \frac{w}{r} \end{bmatrix}. \quad (3.54)$$

The total stresses are the direct sum of their linear and nonlinear components. The nonlinear stresses contain both u and w and their first-order derivatives. These stress expressions and the body force expression of Eq. (3.49) are substituted into the equilibrium equation, i.e., Eq. (3.47), to complete the displacement formulation.

Upon extracting the k term of the equilibrium equation, the first-order part can be written as:

$$(\lambda + 2\mu)(u'' + \frac{u'}{r} - \frac{u}{r^2}) = (3\lambda + 2\mu) \frac{d\mathcal{G}}{dr}, \quad (3.55)$$

while the second-order equilibrium equation is associated with the k^2 term and can be written as:

$$\begin{aligned} & (\lambda + 2\mu)(w'' + \frac{w'}{r} - \frac{w}{r^2}) \\ & = -2(\lambda + 3\mu + 2m)(u' u'' + \frac{1}{2r} u'^2 - \frac{u^2}{2r^3}) \\ & - (\lambda + 2l)(u'' + \frac{u'}{r} - \frac{u}{r^2})(u' + \frac{u}{r}) - (\lambda + \frac{2}{3}\mu - 2l - \frac{2n}{9})9\mathcal{G} \frac{d\mathcal{G}}{dr}. \end{aligned} \quad (3.56)$$

These are second-order ordinary differential equations. Appropriate boundary conditions are applied. In the following subsections, the solutions for the stresses and radial displacement are derived. For convenience, subscripts j are attached to the symbols for the stresses and displacements in the j th-layer. It should be noted that in computing the stresses using Eqs. (3.52) and (3.54), u_r/r and u'_r should be replaced by $u_r/r - \mathcal{G}$ and $u'_r - \mathcal{G}$ respectively, as it is the difference between the fields that preserves the continuity of the body.

For each j th layer, it is assumed that the dilatation is a constant $\mathcal{G}_j$, $j = 1, N$. Substituting $\mathcal{G} = \mathcal{G}_j$ into the first-order governing equation (Eq. (3.55)), the linear displacement solutions can be obtained in the analytical form of a power law:

$$u_j = A_j r + \frac{B_j r_{j-1}^2}{r}, \quad (3.57)$$

where A_j and B_j are dimensionless constants to be determined. Substituting $\vartheta = \vartheta_j$ and Eq. (3.57) into the second-order governing equation (Eq. (3.56)), the nonlinear displacement solutions can be obtained as:

$$w_j = C_j r + \frac{D_j r_{j-1}^2}{r} + \frac{B_j^2 (\lambda + 3\mu + 2m) r_{j-1}^4}{2(\lambda + 2\mu) r^3}, \quad (3.58)$$

where C_j and D_j are dimensionless constants to be determined. Note that $B_1 = D_1 = 0$ in order to fulfill the nonsingular displacement condition u and w at $r = 0$. Also, for $j = 1$ the inner radius of the core $r_{j-1} = r_0 = 0$. There remain $4N-2$ unknown dimensionless parameters A_1, C_1 and $A_j, B_j, C_j, D_j, j = 2, N$. These parameters depend on $\lambda_j, \mu_j, l_j, m_j$, the layer thicknesses $r_j - r_{j-1}$, $j = 1, N$ and the dilatations ϑ_j .

Substituting these solutions of displacement into Eq. (3.52) and Eq. (3.54), the radial and circumferential stresses in each layer j can be written as:

$$T_{rr}^{(j)L} = 2(\lambda_j + \mu_j)(A_j - \vartheta_j) - 2B_j \mu_j \frac{r_{j-1}^2}{r^2} \quad (3.59)$$

$$T_{\theta\theta}^{(j)L} = 2(\lambda_j + \mu_j)(A_j + \vartheta_j) + 2B_j \mu_j \frac{r_{j-1}^2}{r^2} \quad (3.60)$$

$$T_{rr}^{(j)NL} = \frac{1}{(\lambda_j + 2\mu_j)r^4} (H_{1j} + H_{2j}r^2 + H_{3j}r^4) \quad (3.61)$$

$$T_{\theta\theta}^{(j)NL} = \frac{1}{(\lambda_j + 2\mu_j)r^4} (-3H_{1j} - H_{2j}r^2 + H_{3j}r^4) \quad (3.62)$$

where

$$H_{1j} = -(\lambda_j + 3\mu_j + 2m_j)\mu_j B_j^2 r_{j-1}^4 \quad (3.63)$$

$$H_{2j} = 2(\lambda_j + 2\mu_j)r_{j-1}^2(-D_j\mu_j - (\lambda_j + 3\mu_j + 2m_j)B_j(A_j - \mathcal{G}_j)) \quad (3.64)$$

$$H_{3j} = (\lambda_j + 2\mu_j)(2C_j(\lambda_j + \mu_j) + (3\lambda_j + 3\mu_j + 2m_j + 4l_j)(A_j - \mathcal{G}_j)^2) \quad (3.65)$$

Eqs.(3.57)–(3.65) are all power-law expressions. To solve for the $4N-2$ constants, the continuity of u_r across the interfaces at $r_j, j = 1, N-1$ requires:

$$u_j(r_j) = u_{j+1}(r_j), \quad w_j(r_j) = w_{j+1}(r_j) \quad j = 1, N-1, \quad (3.66)$$

while the continuity of T_{rr} at the same interfaces requires:

$$T_{rr}^{(j)L}(r_j) = T_{rr}^{(j+1)L}(r_j), \quad T_{rr}^{(j)NL}(r_j) = T_{rr}^{(j+1)NL}(r_j) \quad j = 1, N-1, \quad (3.67)$$

Finally, the traction-free condition at the outmost surface requires:

$$T_{rr}^{(N)L}(r_N) = T_{rr}^{(N)NL}(r_N) = 0. \quad (3.68)$$

These constitute a set of $4N-2$ equations which permit the solution for the same number of constants. The $4N-2$ constants take very complex forms. For a bilayer cylinder, the six constants are written out explicitly in Appendix C.

3.2.3. Numerical results

The objective of the numerical study is to investigate the tunability of the stresses and displacements, particularly the influence of elastic inhomogeneity,

interface positions (or thickness inhomogeneity) and dilatation profile. Homogeneous, bilayer and four-layer cylinders are considered. Unless otherwise stated, the layers are of equal thickness $(r_j - r_{j-1})/r_1 = 1$, $j = 1, N$. The elastic constants used for the numerical results are based on the earlier works (Wu and Kirchner, 2010; 2011; Catheline et al., 2003) and are listed in Table 3.2. For the constants expressed in terms of NkT (typically 10^{-5} to 10^{-2} GPa), N is the number of polymer chains per unit volume of dry polymer, k the Boltzmann's constant and T the absolute temperature. Two different dilatation profiles, step-up and step-down, are considered.

3.2.3.1. Tunable displacement and stresses

Consider a periodic four-layered cylindrical composite with Layers 1 and 3, and Layers 2 and 4, identical respectively. All layers are of equal thickness, such that $(r_j - r_{j-1})/r_1 = 1$, $j = 1, 4$. The elastic constants are $\lambda_1 = 3570NkT$, $\mu_1 = 1030NkT$, $l_1 = l_2 - 3560NkT$, $m_1 = -2420NkT$; the remaining ones will be varied. The step-up dilation profile is denoted by $\mathcal{G}_{p1}$ with $\mathcal{G}_1 = 0.1$, $\mathcal{G}_2 = 0.2$, $\mathcal{G}_3 = 0.3$ and $\mathcal{G}_4 = 0.4$. Fig. 3.9 plots the surfaces of constant u_r at the outermost boundary, i.e., $u_r(r = r_4)/r_1 = 1.3, 1.6, 3$ in the $\lambda_2 - \mu_2 - m_2$ space. Essentially, this quantity typifies the amount of swelling when subjected to the specific dilatation. Several observations can be made as follows.

First, the bounds on λ_2, μ_2, m_2 can be estimated if u_r is to lie within or outside a certain range. For instance, if u_r is to be greater than 1.6, then λ_2, μ_2 and m_2 should

Fig. 3.9. The surfaces of equal radial displacement $u_r(r=r_4)/r_1 = 1.3, 1.6, 3$ for a four-layer hydrogel cylinder subjected to a dilatation profile $\mathcal{G}_{p1}$. The dot represents a homogeneous cylinder.

Table 3.2. Elastic constants for Figs. 3.9–3.15 (unit for Figs. 3.9-3.13 and Fig. 3.15: NkT , unit for Fig. 3.14: kPa)

Figure	λ_1	μ_1	l_1	m_1	λ_2	μ_2	l_2	m_2
3.9	3570	1030	-3560	-2420			-3560	
3.10	3570	1030	-3560	-2420			-3560	
3.11	35700	10.3	-3560	-2420	357		-100000/-500	-2420
3.12	$35700 \times 10^\alpha$	$10300 \times 10^\alpha$	-3560	$-2420/10^\alpha$	$357/10^\alpha$	$103/10^\alpha$	-3560	$-242000 \times 10^\alpha$
3.13	35700	10300	-3560	-242	357	103	-3560	-24200
3.14	22.5×10^6	80	-2×10^6	-2×10^6	-0.225×10^6	0.8	-2×10^6	-200×10^6
3.15	3570	1030	-3560	-2420	3570	1030	-3560	-2420

be selected from the space under the middle surface represented by $u_r(r = r_4)/r_1 = 1.6$. These results suggest that the elasticity of the gels can be tuned to achieve a targeted functionality. In practice, gel stiffness can be varied over several orders of magnitude via cross-linking, photo-chemical modulation and other means (Engler et al., 2006; Kloxin et al., 2010).

Second, the surface shrinks in size and moves towards the vertical m_2 axis with increasing swelling. On the smaller and steeper surfaces, λ_2, μ_2 assume small values while m_2 may have large values. High compliance is characterized by small values of λ_2 and μ_2 (“linearly soft”), while high second-order nonlinear compliance is characterized by large negative values of m_2 (“nonlinearly soft”). Thus m_2 , in addition to the second-order elastic constants, can be effectively used to modify the swelling response. It is also more difficult to achieve large swelling as the surface shrinks and steepens with increase in the radial displacement.

Third, Fig. 3.9 shows the advantage of layer inhomogeneity in a composite over homogeneity in a single-material cylinder. The dot in Fig. 3.9 locates the point where all the elastic constants of Layers 2 and 4 are identical to those of Layers 1 and 3. The corresponding value of $u_r(r = r_4)/r_1$ is 1.246 for this homogeneous cylinder. It appears that larger swelling can be achieved by increasing the layer inhomogeneity.

Using the same dilatation profile and the same fixed elastic constants as in Fig. 3.9, the dependence of the interfacial radial stress $T_r(r = r_1)/NkT$ on λ_2, μ_2 and m_2 is illustrated in Fig. 3.10. The surfaces correspond to the stress values of 300, 0 and

-300, respectively. The dot refers to the homogeneous cylinder. The figure shows theoretically how both the stress magnitude and sign can be tuned. The middle surface is a zero radial stress surface, which partitions the space into tensile and compressive regions. The elastic constants corresponding to zero or small compressive stresses may be desirable, as tensile stresses encourage interfacial delamination while large compressive stresses may cause instability of the gel.

Within the range of λ_2 , μ_2 and m_2 shown in the figure, it is not apparent that the surfaces shrink significantly as the radial stress evolves from tensile to compressive. However, the closer proximity of the surfaces representing zero and compressive stress implies a greater stress sensitivity to elastic parameter control in compression than in tension.

The radial stress at the position of r_1 in the homogeneous cylinder is tensile, with a magnitude of 356. Thus, the possibility of using layer inhomogeneity to effectively control the stress magnitude and sign is demonstrated in this figure.

By comparing Figs. 3.9 and 3.10, It is interesting to note that a larger swelling is associated with a compressive interfacial radial stress between Layers 1 and 2. However, the detailed stress characteristics may vary with the form of the dilatation profile and the elastic constants, as will be shown subsequently.

Fig. 3.10. The $\lambda_2 - \mu_2 - m_2$ surfaces of equal radial stress $T_r(r=r_1)/NkT = -300, 0, 300$ for a four-layer hydrogel cylinder subjected to a dilatation profile $\mathcal{G}_{p1}$. The dot represents a homogeneous cylinder.

Fig. 3.11. Variation of the normalized radial displacement at the outer boundary with the shear modulus μ_2 for different values of l_2 in a bilayer hydrogel cylinder.

Fig. 3.12. Variation of the normalized radial displacement at the outer boundary of a four-layer cylinder subjected to the dilatation profile $\mathcal{G}_{p1}$ with the multiplicative factor α for the elastic constants. The cylinder is homogeneous when $\alpha = -1$.

3.2.3.2. Influence of elastic constants

For Fig. 3.11, a bilayer cylinder is considered. The elastic constants are given by $\lambda_1 = 35700NkT$, $\mu_1 = 10.3NkT$, $l_1 = -3560NkT$, $m_1 = -2420NkT$, and $\lambda_2 = 357NkT$, $m_2 = -2420NkT$. The applied dilatation profile is $\mathcal{G}_1 = 0$ and $\mathcal{G}_2 = 0.5$. The figure plots surface displacement $u_r(r = r_2)/r_1$ against μ_2 for two values of l_2 , i.e., -500 and -100000 NkT . The results show that l_2 can also be used for controlling the swelling of the composite. The non-monotonous variation also suggests that a complex coupling exists between the elastic constants.

To illustrate the effect of elastic inhomogeneity, consider a periodic four-layer composite with identical Layers 1 and 3, and identical Layers 2 and 4, respectively. The layers are of the same thickness: $(r_j - r_{j-1})/r_1 = 1$, $j = 1, 4$. The composite is loaded by the step-up dilatation profile $\mathcal{G}_{p1}$. Fig. 3.12 plots the outermost displacement $u_r(r=r_4)/r_1$ against a multiplicative factor α of the elastic constants, given by: $\lambda_1 = 35700 \times 10^\alpha \text{ NkT}$, $\mu_1 = 10300 \times 10^\alpha \text{ NkT}$, $l_1 = -3560 \text{ NkT}$, $m_1 = -2420 / 10^\alpha \text{ NkT}$ for Layer 1, and $\lambda_2 = 357 / 10^\alpha \text{ NkT}$, $\mu_2 = 103 / 10^\alpha \text{ NkT}$, $l_2 = -3560 \text{ NkT}$, $m_2 = -242000 \times 10^\alpha \text{ NkT}$ for Layer 2. As α increases from -2 to 0, Layers 1 and 3 become more linearly and nonlinearly stiff, while Layers 2 and 4 become more linearly and nonlinearly soft. It can be seen that the outermost displacement varies significantly with the elastic inhomogeneity. In fact, minimum swelling occurs when $\alpha \approx -1$, i.e., the composite is almost homogeneous, while larger swelling is associated with strong layer dissimilarity. These results suggest that it may be possible to control the swelling by properly designing the linear and nonlinear elasticity of the different layers of the composite.

To illustrate the relative importance of the nonlinear and linear effect, the ratio of nonlinear to linear circumferential stress $T_{\theta\theta}^{NL} / T_{\theta\theta}^L$ at the surface $r = r_4$ of a periodic four-layer cylinder with equal layer thickness is plotted against $\lambda_2, \mu_2, l_2, m_2$ in Fig. 3.13. The dilatation profile is also $\mathcal{G}_{p1}$. The elastic constants, shown in Table 3.2, indicate that Layers 2 and 4 are more compliant (linearly and nonlinearly) than Layers 1 and 3. For each curve labeled by the constant λ_2, μ_2, l_2 or m_2 , the constant is varied independently, keeping all the remaining seven constants fixed at the

Fig. 3.13. Variation of the ratio of nonlinear to linear circumferential stress at the surface $r = r_4$ with λ_2 , μ_2 , l_2 and m_2 for a four-layer cylinder subjected to the dilatation profile $\mathcal{G}_{p1}$.

values indicated in Table 3.2. It can be observed that the stress ratio is rather large for most values of the varied constants, supporting the intuition that nonlinear components of stresses in soft materials can be dominant.

3.2.3.3. Influence of interface position

The effect of modifying the layer thickness of a periodic four-layer composite (with Layers 1 and 3, and Layers 2 and 4, identical in material respectively) is also

Fig. 3.14. Dependence of the surface displacement on r_2 and r_3 (r_1 and $r_4=4r_1$ fixed) for a four-layer hydrogel cylinder subjected to the dilatation profile $\mathcal{G}_{p1}$.

investigated. Figure 3.14 plots the variation of $u_r(r = r_4)/r_1$ versus r_2/r_1 and r_3/r_1 , holding r_1 and $r_4 = 4r_1$ fixed. The dilatation profile is $\mathcal{G}_{p1}$. The elastic constants used for the simulations are suggested by the experimental data of Catheline et al. (2003) for agar-gelatin (a model for soft-tissue), in which λ is four to five orders of magnitude larger than μ . A maximum in u_r occurs when r_2/r_1 is slightly less than

3 and r_3 / r_1 is slightly larger than 3. Hence, the composite with maximum swelling will resemble the configuration indicated in Fig. 3.14, which shows the “near-elimination” of the stiff Layer 3, resulting in a bilayer with unequal layer thicknesses. In contrast, if a small swelling is desired, Fig. 3.14 suggests a configuration with near-elimination of the compliant Layer 4 but with no specific requirement for the thickness of Layer 2. In general, larger swelling appears to be more easily obtained by changing the layer thicknesses than the elastic constants (compare Figs. 3.14 and 3.9). It should be mentioned that the dependence of such maximum/minimum swelling response on the interface positions is also evident when the values of λ and μ are comparable.

3.2.3.4. Influence of dilatation profile

In order to explore the effect of different dilatation profiles, the radial displacement, radial stress and circumferential stress are plotted against the radial coordinate in a *homogeneous* cylinder, as shown in Fig. 3.15. The elastic constants are given in Table 3.2. The two dilatation profiles are $\mathcal{G}_{p1}$ and $\mathcal{G}_{p2}$, where the latter is a step-down profile with $\mathcal{G}_1 = 0.4$, $\mathcal{G}_2 = 0.3$, $\mathcal{G}_3 = 0.2$ and $\mathcal{G}_4 = 0.1$. In the current simulation, the dilatations are constant in the regions $0 \leq r \leq r_1$, $r_1 \leq r \leq r_2$, $r_2 \leq r \leq r_3$, $r_3 \leq r \leq r_4$.

Fig. 3.15. Variation of the normalized (a) radial displacement, (b) radial stress and (c) circumferential stress with normalized radial coordinate r/r_1 in a homogeneous cylinder subjected to step-up ($\mathcal{G}_{p1}$) and step-down ($\mathcal{G}_{p2}$) dilatation profiles.

Several observations can be made. First, the radial displacement and radial stress are both continuous across the dilatation steps, while the circumferential stress is not. The radial displacement also changes linearly in each region, with a larger gradient for a larger dilatation, as shown in Fig. 3.15(a). The step-up profile also results in a larger swelling than the step-down profile. Figures 3.15(b) and (c) show that the stresses generated by the two dilatation profiles are essentially opposite of each other. For instance, the radial stress produced by a step-down profile is compressive, since the material in the outer regions expands less than that in the inner regions. In general, the results show that the stress polarity can be influenced by the dilation profile alone, keeping in mind that the cylinder is entirely homogeneous. Hence, the design of a composite cylinder should take into consideration multiple factors (dilatation profile, layer elasticities, layer thicknesses) which influence the displacement and stresses in a coupled and complex manner. The continuum solutions presented here offer relatively simple “design guidelines”, which may not be easily obtained by computation-intensive models such as those based on microstructural details.

3.2.4. Conclusion

The solutions for a multilayered cylindrical gel subjected to step-up and step-down dilatation profiles in the framework of second-order elasticity were developed. The displacement and stress solutions all assume the form of a power law with parameters dependent on the elastic constants, thickness and the dilation of

each layer. The solutions emphasize the significant role of nonlinearity, elastic dissimilarity, layer thickness inhomogeneity as well as the dilatation profile in the elastic fields. In particular, they show that a rational basis is possible for the mechanics design of complex bio-inspired materials, drug delivery vehicles and soft composites in general. For instance, the development of a multiaxial stress state conducive to successful cell culture in a bio-mimetic micro-environment may be established with the assistance of such theoretical calculations.

3.3. Validation of the theory

In this section, the theory of swelling of spheres and cylinders is verified against experimental data. The data for the swelling dependence on the pH of homogeneous polymer gels are fitted against a universal relation. The swelling is characterized by the weight ratio Q , defined as the ratio of the mass of water absorbed to the mass of the dry polymer. For spheres,

$$Q = \frac{\rho_{\text{water}}[(R + u_r)^3 - R^3]}{\rho_{\text{polymer}}R^3} = \frac{\rho_{\text{water}}[(1 + \frac{u_r}{R})^3 - 1]}{\rho_{\text{polymer}}}, \quad (3.69)$$

while for cylinders,

$$Q = \frac{\rho_{\text{water}}h[(R + u_r)^2 - R^2]}{\rho_{\text{polymer}}hR^2} = \frac{\rho_{\text{water}}[(1 + \frac{u_r}{R})^2 - 1]}{\rho_{\text{polymer}}}. \quad (3.70)$$

In the above definitions, ρ_{water} and ρ_{polymer} are the densities of water and polymer, respectively, R is the radius of the gel at dry state, $u_r \equiv u_r(R)$ is the radial

displacement at the outermost surface after swelling, and h is the thickness of the cylinder gel.

Thus, u_r/R is a quadratic function of dilatation $\mathcal{G}_j$. For a single-layer homogeneous sphere or cylinder, the following linear (and simplest) assumption is made between the dilatation and pH (which is a logarithmic measure of the concentration of hydrogen ions):

$$\mathcal{G} = a \text{ pH} + b, \quad (3.71)$$

for the region $0.1R < r < R$, and $\mathcal{G} = 0$ for a small pH-insensitive core $r < 0.1R$.

The constants a and b can be obtained through fitting the displacement model to the swelling data. Hence, u_r/R is a quadratic function of pH in the second-order elasticity model:

$$\frac{u_r}{R} = a' \text{ pH}^2 + b' \text{ pH} + c', \quad (3.72)$$

where a' , b' and c' are parameters related to the elastic constants as well as a and b of Eq. (3.71). Finally, by Eqs. (3.69) and (3.70), the swelling ratio can be written as:

$$Q = \frac{\rho_{\text{water}} [(1 + a' \text{ pH}^2 + b' \text{ pH} + c')^3 - 1]}{\rho_{\text{polymer}}} \quad (3.73)$$

for spheres and

$$Q = \frac{\rho_{water} [(1 + a' pH^2 + b' pH + c')^2 - 1]}{\rho_{polymer}} \quad (3.74)$$

for cylinders. Note that Q is either a sextic or a quartic function of pH. By fitting Eqs. (3.73) and (3.74) to the swelling data, the parameters a' , b' , c' can be determined. In a similar way as for the uniaxial tension data, universal relations for Q can be developed. Defining a modified swelling ratio $\tilde{Q} = \rho_{polymer} / \rho_{water} \bullet Q$ and a modified pH $\tilde{P}_h = a' pH^2 + b' pH + c'$, Eqs. (3.73) and (3.74) can be recast in the following universal forms:

$$\tilde{Q} = (1 + \tilde{P}_h)^3 - 1, \quad \tilde{Q} = (1 + \tilde{P}_h)^2 - 1, \quad (3.75)$$

where no material parameters appear explicitly. It should be noted that universal relations can be similarly derived for a multilayer capsule, following the steps of Eqs. (3.71) and (3.72).

Twenty sets of swelling versus pH data, labeled 1-20 in Table 3.3, are used to validate the universal relations. The polymer gels include: styrene and 4-vinylpyridine copolymers (St/4VP copolymers) (Batich et al., 1993), poly(N-isopropylacrylamide-co-sodium acrylate) gels (NIPA/SA) (Motonaga and Shibayama, 2001), poly(ethylene glycol) methyl ether methacrylates-poly(acrylic acid) double-network hydrogels (PPEGMA-PAA) (Naficy et al., 2012), poly(N-isopropylacrylamide-co-acrylic acid) gels (NIPA/AAC) (Shibayama et al., 1996), polyacrylamide/laponite clay nanocomposite hydrogels (NC) (Li et al., 2009), poly(methacrylic acid)/Poly(N-isopropylacrylamide)

Table 3.3. Parameters a' , b' and c' obtained from fitting Eqs. (3.73) and (3.74) to swelling data.

No.	system	a'	b'	c'
1	St/4VP copolymer (Batich et al., 1993)	-0.0164765	-0.0990889	1.25548
2	St/4VP copolymer (Batich et al., 1993)	-0.0848899	0.298232	0.855509
3	St/4VP copolymer (Batich et al., 1993)	-0.0145595	-0.105732	1.23548
4	NIPA/SA gel (Motonaga and Shibayama, 2001)	-0.0289723	0.481836	-1.16304
5	PPEGMA1100-PAA gel (Naficy et al., 2012)	0.107563	-0.293249	0.795212
6	PPEGMA475-PAA gel (Naficy et al., 2012)	0.0603298	0.108628	-0.107012
7	PAA gel (Naficy et al., 2012)	0.0256669	0.767233	-0.812141
8	NIPA/AAc gels (Shibayama et al., 1996)	-0.0557175	0.917362	-2.52187
9	Polyacrylamide/laponite clay nanocomposite hydrogel NC15 (Li et al., 2009)	0.0519988	-0.664552	3.68776
10	Polyacrylamide/laponite clay nanocomposite hydrogel NC20 (Li et al., 2009)	0.0792195	-0.940206	4.46783
11	Polyacrylamide/laponite clay nanocomposite hydrogel NC25 (Li et al., 2009)	0.119256	-1.30251	5.06215
12	PMAA/PNIPAAm IPN (Zhang and Peppas, 2000)	-0.0204942	0.435262	-0.936233
13	PMAA (Zhang and Peppas, 2000)	-0.0138649	2.2371	-7.39658
14	PAA gel (Philippova et al., 1997)	-0.585597	9.82948	-21.667
15	HM PAA gels C8-2.5% (Philippova et al., 1997)	-0.675248	11.6895	-31.5438
16	HM PAA gels C8-10% (Philippova et al., 1997)	-1.16268	20.9599	-76.6288
17	HM PAA gels C8-20% (Philippova et al., 1997)	0.139643	1.61668	-9.73172
18	PAA gels (Philippova et al., 1997)	-0.584954	9.84174	-21.7437
19	HM PAA gels C8-5% (Philippova et al., 1997)	-0.67579	12.0035	-34.5026

20	HM PAA gels C12-5% (Philippova Philippova et al., 1997)	-0.37782	7.82808	-22.9672
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interpenetrating polymer networks (PMAA/PNIPAAm IPN) (Zhang and Peppas, 2000), and hydrophobically modified poly(acrylic acid) gels (HM PAA gels) (Philippova et al., 1997).

Table 3.3 lists the materials tested and the corresponding values for the parameters a' , b' , c' appearing in Eq. (3.73) or Eq. (3.74). The St/4VP copolymers are spherical whereas the rest of the polymers in Table 3.3 are cylindrical. Fig. 3.16(a) shows Q versus pH for the (NIPA/AAc) gels (Shibayama et al., 1996). The weight ratio increases with pH, reaches a peak around a pH value of 8, and decreases with pH thereafter. This non-monotonic Q dependence on pH is fairly well-reproduced by Eq. (3.74), indicated as a solid curve. In contrast, Fig. 3.16(b) shows that the weight ratio of HM PAA gels C8-20% (Philippova et al., 1997) increases monotonically with pH up to the value of 9. Here the theoretical prediction of Eq. (3.74) is acceptable, although the point of inflection is not well-captured.

Fig. 3.16. Dependence of (a) the poly(N-isopropylacrylamide-co-acrylic acid) gels (NIPA/AAc) (Shibayama et al., 1996), and (b) hydrophobically modified poly(acrylic acid) gels (HM PAA C8-20%) (Philippova et al., 1997) on pH. The solid lines represent the theoretical fit of the second-order elastic model to the experimental data.

Fig. 3.17. Dependence of the modified weight ratio ($\tilde{Q}$) due to the swelling of spherical capsules on the modified pH ($\tilde{P}_h$). The materials for the data sets 1-3 are listed in Table 3.3. The solid line represents the universal line predicted by the second-order elastic model.

Taking $\rho_{\text{polymer}} / \rho_{\text{water}} \approx 1.4$ (a reasonable estimate for the polymers) and the values of a' , b' , c' in Table 3.3 for the different polymers, the modified data can be computed. Figure 3.17 plots the modified weight ratio against the modified pH for the case of spherical capsules (Batich et al., 1993). The universal curve predicted by Eq. (3.72) fits the data reasonably well. Figure 3.18 assembles the $\tilde{Q} - \tilde{P}_h$ data for cylindrical capsules in two separate plots for clarity. The upper (a) and lower (b) plots show the data labeled 4-13 and 14-20 in Table 3.3, respectively. The modified pH value in Fig. 3.18(a) is less than 5, while it is mostly between 4 and 20 in Fig. 3.18(b). Good agreement is obtained between the reduced data and the

Fig. 3.18. Dependence of the modified weight ratio ($\tilde{Q}$) due to the swelling of cylindrical capsules on the modified pH ($\tilde{P}_h$). The materials for the data sets 4-13, and those for the data sets 14-20, are listed in Table 3.3. The solid lines represent the universal line predicted by the second-order elastic model.

universal curve in Fig. 3.18. Hence, the second-order elastic model captures fairly well the swelling behavior of a diverse range of polymers and composites.

Chapter 4 Axial Loading

Murnaghan (1951) solved the problem of a homogeneous cylindrical bar under uniaxial loading within the framework of second-order elasticity. In this chapter, the problem of a bilayered cylindrical composite under axial loading is solved, starting directly from the equilibrium equations. Experimental verification is also attempted, using the data for biogels, tissues and polymers.

The chapter is organized as follows. The problem is defined in Section 4.1. The governing equations are presented in Section 4.2. The solutions for stress and displacement fields are given in Section 4.3. This is followed by numerical results in Section 4.4, validation of the theory in Section 4.5, and a set of conclusions in Section 4.7.

4.1. Problem Definition

Consider an N -layered circular cylinder of length L and radius r_N (or simply R for a homogeneous cylinder). The interfaces are located at the radial coordinates $r = r_i$, $i = 1, N-1$, with $i = 1$ denoting the innermost interface, as shown in Fig. 4.1. All the layers are nonlinear elastic, isotropic, homogeneous and perfectly bonded to each other. The composite cylinder is subjected to an axial loading P (tension or compression), with respect to the longitudinal direction. The initial coordinates of a particle of the cylinder are chosen as (r, θ, z) , which respectively represent the radial,

angular and axial coordinates. The final coordinates are denoted by (ρ, ψ, ζ) .

In the following, we consider axial loadings of both homogeneous and composite cylinders in two separate sub-sections. In this work, we solve the problems, starting from the deformations and the equilibrium equations involving the first Piola-Kirchhoff stresses. As remarked by Volokh (2006), Lagrangian equilibrium equations in cylindrical and spherical coordinates are surprisingly “rarely discussed” in the literature, but they are of major interest in the mechanics of soft biomaterials. For simplicity, we will present the solutions for cylinders up to two layers only.

Under the uniform axial stress P , the final coordinates (ρ, ψ, ζ) of the particle of the cylinder whose initial coordinates being (r, θ, z) are $(r + kF(r), \theta + kG(z), z + kH(z))$, where $kF(r)$, $kG(z)$, $kH(z)$ are respectively the radial, angular and axial displacements. As before, k is a marker indicating the order of approximation of the theory. Furthermore, the unknown radial displacement $kF(r) = k(F_1(r) + kF_2(r))$, where $F_1(r)$ and $F_2(r)$ are respectively the first- and second-order terms. These terms are assumed to be functions of r only and not of z since all the sections are axially loaded in the same manner. The unknown angular displacement $G(z) = G_1(z) + kG_2(z)$, where $G_1(z)$ and $G_2(z)$ are respectively the first- and second-order terms. They are taken to be functions of z , but will be shown to vanish identically in agreement with the physical reasoning that any twisting of the bar in one direction under an axial load will contradict with the axisymmetric nature of the axial load with respect to the circular cross-sections. A non-vanishing $kG(z)$ would

Fig. 4.1. A cylindrical composite consisting of N layers, with surfaces located at the radial coordinates, $r = r_i$, $i = 1, N$, under axial loading P .

have predicted an axial force-twist effect, i.e., twisting of the bar under axial loading.

The unknown axial displacement $H(z) = H_1(z) + k H_2(z)$, where $H_1(z)$ and $H_2(z)$ are respectively the first- and second-order terms. These are taken to be functions of z only and not of r since the bar sections elongate or contract uniformly in the

z -direction. This displacement field under tension permits displacements in all coordinate directions.

4.2. Governing Equations

The differentials of position vectors $\mathbf{a}$ and $\mathbf{x}(\mathbf{a})$ in undeformed and deformed states can be written respectively as:

$$d\mathbf{a} = \begin{pmatrix} dr \\ r d\theta \\ dz \end{pmatrix} \quad \text{and} \quad d\mathbf{x} = \begin{pmatrix} (1+kF')dr \\ (r+kF)(d\theta+kG'dz) \\ (1+kH')dz \end{pmatrix}, \quad (4.1)$$

where $F' = dF / dr = F'_1(r) + kF'_2(r)$, $G' = dG / dz = G'_1(z) + kG'_2(z)$, and $H' = dH / dz = H'_1(z) + kH'_2(z)$. The deformation gradient is thus:

$$\mathbf{F} = \frac{d\mathbf{x}}{d\mathbf{a}} = \begin{pmatrix} 1+kF' & 0 & 0 \\ 0 & 1+\frac{kF}{r} & (r+kF)kG' \\ 0 & 0 & 1+kH' \end{pmatrix}. \quad (4.2)$$

It can be shown that the equilibrium equations in cylindrical coordinates in terms of the first Piola-Kirchhoff stresses are (Volokh, 2006):

$$\frac{\partial T_{rr}}{\partial r} - T_{\theta r} \frac{\partial \psi}{\partial r} + \frac{T_{rr}}{r} + \frac{\partial T_{r\theta}}{r \partial \theta} - \frac{T_{\theta\theta}}{r} \frac{\partial \psi}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} - T_{\theta z} \frac{\partial \psi}{\partial z} = 0, \quad (4.3)$$

$$T_{rr} \frac{\partial \psi}{\partial r} + \frac{\partial T_{\theta r}}{\partial r} + \frac{T_{r\theta}}{r} \frac{\partial \psi}{\partial \theta} + \frac{T_{\theta R}}{r} + \frac{\partial T_{\theta\theta}}{r \partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + T_{rz} \frac{\partial \psi}{\partial z} = 0, \quad (4.4)$$

$$\frac{\partial T_{zr}}{\partial r} + \frac{T_{zr}}{r} + \frac{\partial T_{z\theta}}{r\partial\theta} + \frac{\partial T_{zz}}{\partial z} = 0. \quad (4.5)$$

The energy density of Murnaghan (1951) is used as shown in Eq. (3.1). The stresses can be obtained by substituting Eq. (4.2) into Eq. (3.3). Subsequently, substituting the stresses into the equilibrium equations Eqs. (4.3)–(4.5) with $\partial\psi/\partial r = 0$, $\partial T_{r\theta}/\partial\theta = 0$, $\partial\psi/\partial\theta = 1$, $\partial\psi/\partial z = kG'$, yields three first-order and three second-order equilibrium equations as distinguished by the k and k^2 multipliers:

$$k\left(\frac{F_1''(r)}{r} + \frac{F_1'(r)}{r} - \frac{F_1'(r)}{r^2}\right) = 0, \quad (4.6)$$

$$k\mu r G_1'(z) = 0, \quad (4.7)$$

$$kH_1''(z) = 0, \quad (4.8)$$

$$\begin{aligned} & k^2\{-4(2l+2m+2\lambda+3\mu)F_1(r)^2 + r^4(4m-3n+4\lambda-8\mu)G_1'(z)^2 + \\ & r[-4(\lambda+2\mu)F_2(r) - 4(2l+\lambda)F_1(r)H_1'(z)] + \\ & 4r^2[(2l+2m+2\lambda+3\mu)F_1'(r)^2 + (\lambda+2\mu)F_2'(r) + \\ & (2l+\lambda)F_1'(r)H_1'(z) + (2l+\lambda)F_1(r)F_1''(r)] + \\ & 4r^3[(2l+4m+3\lambda+6\mu)F_1'(r)F_1''(r) + (2l+\lambda)H_1'(z)F_1''(r) + (\lambda+2\mu)F_2''(r)]\} = 0, \end{aligned} \quad (4.9)$$

$$\begin{aligned} & k^2\{(m+\lambda+3\mu)F_1(r)G_1''(z) + r[\frac{1}{2}(2m-n+2\lambda)F_1'(r)G_1''(z) + \\ & (m+\lambda+2\mu)H_1'(z)G_1''(z) + \mu G_2''(z) + (m+\lambda+2\mu)G_1'(z)H_1''(z)]\} = 0, \end{aligned} \quad (4.10)$$

$$k^2 \{ r^3 (m + \lambda + 2\mu) G_1'(z) G_1''(z) + (2l + \lambda) F_1(r) H_1''(z) + r[(2l + \lambda) F_1'(r) H_1''(z) + (2l + 4m + 3\lambda + 6\mu) H_1'(z) H_1''(z) + (\lambda + 2\mu) H_2''(z)] \} = 0. \quad (4.11)$$

These equilibrium equations can be solved for the unknown functions F_1 , F_2 , G_1 , G_2 , H_1 and H_2 , for both the homogeneous and the two-layered cases.

4.3. Solutions

4.3.1. Solutions for a homogeneous cylinder under axial loading

Solving the linear equilibrium equations Eqs. (4.6)–(4.8), F_1 , G_1 and H_1 can be obtained as:

$$F_1(r) = F_{11}r + \frac{F_{12}}{r}, \quad (4.12)$$

$$G_1(z) = G_{11}z \quad (4.13)$$

$$H_1(z) = H_{11}z, \quad (4.14)$$

where F_{11} , F_{12} , G_{11} , H_{11} are constants to be determined. However, $F_{12} = 0$ since the linear radial displacement vanishes at $r = 0$. This leaves three unknowns F_{11} , G_{11} , and H_{11} to be determined from the following three boundary conditions:

- (1) The first-order radial stress T_{rr}^L (L denoting “linear”) vanishes at the outer boundary:

$$T_{rr}^L(r=R)=0, \quad (4.15)$$

- (2) The applied axial forces acting at the end faces of the cylinder are $\pi R^2 P$:

$$\int T_{zz}^L(z=0, L) dA = \pi R^2 P, \quad (4.16)$$

- (3) No torsion is applied to the cylinder:

$$\int T_{\theta z}^L(z=L) r dA = 0, \quad (4.17)$$

From Eqs. (4.15)–(4.17), the constants can then be solved as:

$$F_{11} = -\frac{P\lambda}{2\mu(3\lambda+2\mu)} = -\nu \frac{P}{E}, \quad G_{11} = 0, \quad H_{11} = \frac{P(\lambda+\mu)}{\mu(3\lambda+2\mu)} = \frac{P}{E}, \quad (4.18)$$

where $E = \mu(3\lambda+2\mu)/(\lambda+\mu)$ and $\nu = \lambda/2(\lambda+\mu)$ are the usual Young’s modulus and Poisson’s ratio, respectively. Hence, $F_1(r) = -P\lambda r/2\mu(3\lambda+2\mu)$,

$G_1(z) = 0$, $H_1(z) = P(\lambda+\mu)z/\mu(3\lambda+2\mu)$ according to Eqs. (4.12)–(4.14). Note

that there is no twisting under pure axial tension in the first-order elasticity theory.

The radial displacement is associated with the Poisson’s ratio effect.

Next, the solutions for second-order axial loading in the homogeneous case are presented. Substituting F_1 , G_1 and H_1 from Eqs. (4.12)–(4.14) into the second-order equilibrium equations given by Eqs.(4.9)–(4.11), they reduce to:

$$\frac{F_2''}{r} + \frac{F_2'}{r} - \frac{F_2}{r^2} = 0, \quad (4.19)$$

$$G_2''(z) = 0, \quad (4.20)$$

$$H_2''(z) = 0, \quad (4.21)$$

with the solutions:

$$F_2(r) = F_{21}r + \frac{F_{22}}{r}, \quad (4.22)$$

$$G_2(z) = G_{21}z, \quad (4.23)$$

$$H_2(z) = H_{21}z. \quad (4.24)$$

where F_{21} , F_{22} , G_{21} , H_{21} are constants to be determined. As in the above, $F_{22} = 0$ in order that the nonlinear radial displacement vanishes at $r = 0$. Using the following three boundary conditions:

(1) The second-order radial stress T_{rr}^{NL} (NL denoting “nonlinear”) vanishes at the outer boundary:

$$T_{rr}^{NL}(r = R) = 0, \quad (4.25)$$

(2) The second-order axial stress does not contribute to the applied axial forces acting at the ends of the cylinder:

$$\int T_{zz}^{NL}(z=0, L)dA=0, \quad (4.26)$$

(3) No torsion is applied to the cylinder:

$$\int T_{\theta z}^{NL}r(z=L)dA=0, \quad (4.27)$$

the constants can be solved as:

$$F_{21} = \frac{P^2(3n\lambda^3 + 6n\lambda^2\mu + 9\lambda^3\mu + 4n\lambda\mu^2 + 18\lambda^2\mu^2 - 8l\mu^3 + 8\lambda\mu^3)}{8\mu^3(3\lambda + 2\mu)^3}, \quad (4.28)$$

$$G_{21} = 0, \quad (4.29)$$

$$H_{21} = -\frac{P^2(3n\lambda^3 + 18m\lambda^2\mu + 3n\lambda^2\mu + 18\lambda^3\mu + 24m\lambda\mu^2 + 48\lambda^2\mu^2 + 4l\mu^3 + 8m\mu^3 + 42\lambda\mu^3 + 12\mu^4)}{4\mu^3(3\lambda + 2\mu)^3}. \quad (4.30)$$

With all the functions known, the displacement field, represented by the radial (u_r),

circumferential (u_θ) and axial (u_z) components, can be written as:

$$\begin{aligned} u_r &= kF(r) = kF_1(r) + k^2F_2(r) \\ &= -k \frac{P\lambda}{6\lambda\mu + 4\mu^2} r + k^2 \frac{P^2[n\lambda(3\lambda^2 + 6\lambda\mu + 4\mu^2) + \mu(9\lambda^3 + 18\lambda^2\mu - 8l\mu^2 + 8\lambda\mu^2)]}{8\mu^3(3\lambda + 2\mu)^3} r \\ &= -k \frac{P\lambda}{2\mu(3\lambda + 2\mu)} r + k^2 \left(\frac{P\lambda}{2\mu(3\lambda + 2\mu)} \right)^2 \gamma r, \end{aligned} \quad (4.31)$$

$$u_\theta = kG(z)r = kG_1(z)r + k^2G_2(z)r = 0, \quad (4.32)$$

$$\begin{aligned}
u_z &= kH(z) = kH_1(z) + k^2H_2(z) \\
&= k \frac{P(\lambda + \mu)}{\mu(3\lambda + 2\mu)} z - k^2 \frac{P^2(3n\lambda^2(\lambda + \mu) + 2\mu(9\lambda^3 + 24\lambda^2\mu + 21\lambda\mu^2 + m(3\lambda + 2\mu)^2 + 2\mu^2(l + 3\mu)))}{4\mu^3(3\lambda + 2\mu)^3} z \\
&= k \frac{P}{E} z + k^2 \delta \frac{P^2}{E^2} z,
\end{aligned} \tag{4.33}$$

where two new dimensionless constants are introduced as:

$$\gamma = \frac{n\lambda(3\lambda^2 + 6\lambda\mu + 4\mu^2) + \mu(9\lambda^3 + 18\lambda^2\mu - 8l\mu^2 + 8\lambda\mu^2)}{2\mu(3\lambda + 2\mu)\lambda^2}, \tag{4.34}$$

$$\delta = -\frac{3n\lambda^2(\lambda + \mu) + 2\mu[9\lambda^3 + 24\lambda^2\mu + 21\lambda\mu^2 + m(3\lambda + 2\mu)^2 + 2\mu^2(l + 3\mu)]}{4\mu(\lambda + \mu)^2(3\lambda + 2\mu)}. \tag{4.35}$$

Under pure axial loading, both the first- and second-order terms of u_θ vanish. Hence, no axial force-twist effect can occur. For u_r and u_z , the first-order components are linearly dependent on P , but the second-order ones are functions of P^2 , implying that the second-order displacement changes in the radial and axial directions are independent of the direction of the axial loading. They represent an intrinsic feature of the material. Also, the dependence of these terms on P^2 implies that the second-order effect becomes increasingly important at larger axial loadings.

The stress field is simply:

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & kP \end{pmatrix}, \tag{4.36}$$

where the only non-vanishing component is T_{zz} , which does not have a second-order component.

4.3.2. Solutions for a bilayered cylindrical composite under axial loading

Considering a bilayered cylinder as before, the solutions for the first-order radial, angular and axial displacements $F_1^{(i)}(r)$, $G_1^{(i)}(z)$ and $H_1^{(i)}(z)$, $i = 1, 2$, follow the same form for the homogeneous case, i.e., Eqs. (4.12)–(4.14):

$$F_1^{(i)}(r) = F_{11}^{(i)}r + \frac{F_{12}^{(i)}}{r}, \quad (4.37)$$

$$G_1^{(i)}(z) = G_{11}^{(i)}z, \quad (4.38)$$

$$H_1^{(i)}(z) = H_{11}^{(i)}z, \quad (4.39)$$

where (i) indicates the layer. Eight unknown constants $F_{11}^{(i)}, F_{12}^{(i)}, G_{11}^{(i)}, H_{11}^{(i)}$ are to be determined. For Layer 1, however, $F_{12}^{(1)} = 0$ since the linear radial displacement vanish at $r = 0$. The remaining seven unknowns can be determined from the following seven boundary conditions:

(1) First-order radial displacement is continuous across the interface $r = r_1$:

$$F_1^{(1)}(r = r_1) = F_1^{(2)}(r = r_1), \quad (4.40)$$

(2) First-order circumferential displacement is continuous across the interface:

$$G_{11}^{(1)} z = G_{11}^{(2)} z, \text{ or } G_{11}^{(1)} = G_{11}^{(2)}, \quad (4.41)$$

(3) First-order axial displacement is continuous across the interface:

$$H_{11}^{(1)} z = H_{11}^{(2)} z \text{ or } H_{11}^{(1)} = H_{11}^{(2)}, \quad (4.42)$$

(4) First-order radial traction T_{rr}^L is continuous across the interface:

$$T_{rr}^{L(1)}(r = r_1) = T_{rr}^{L(2)}(r = r_1), \quad (4.43)$$

(5) T_{rr}^L vanishes at the outer surface $r = r_2$:

$$T_{rr}^{L(2)}(r = r_2) = 0, \quad (4.44)$$

(6) The applied axial force acting at either end of the cylinder is $\pi r_2^2 P$:

$$\int T_{zz}^{L(i)}(z = 0, L) dA = \pi r_2^2 P, \quad (4.45)$$

(7) The contribution of the second-order shear stress $T_{\theta z}^L$ to T is 0:

$$\int T_{\theta z}^{L(i)}(z = 0, L) r dA = 0. \quad (4.46)$$

The constants can be determined as:

$$F_{11}^{(1)} = -\frac{1}{2\xi_1} P r_2^2 [r_1^2 (\lambda_1 - \lambda_2) \mu_2 + r_2^2 (\lambda_2 \mu_2 + \lambda_1 \lambda_2 + \lambda_1 \mu_2)], \quad (4.47)$$

$$H_{11}^{(1)} = H_{11}^{(2)} = \frac{1}{\xi_1} P r_2^2 [r_1^2 (\lambda_1 - \lambda_2 + \mu_1 - \mu_2) \mu_2 + r_2^2 (\lambda_2 + \mu_2) (\lambda_1 + \mu_1 + \mu_2)], \quad (4.48)$$

$$G_{11}^{(1)} = G_{11}^{(2)} = 0, \quad (4.49)$$

$$F_{11}^{(2)} = -\frac{1}{2\xi_1} Pr_2^2 [r_1^2 (\lambda_1 - \lambda_2) \mu_2 + r_2^2 \lambda_2 (\lambda_1 + \mu_1 + \mu_2)], \quad (4.50)$$

$$F_{12}^{(2)} = \frac{1}{2\xi_1} Pr_1^2 r_2^4 (\lambda_2 \mu_1 - \lambda_1 \mu_2), \quad (4.51)$$

where

$$\xi_1 = (r_1^4 (3\lambda_1 - 3\lambda_2 + 2\mu_1 - 2\mu_2)(\mu_1 - \mu_2)\mu_2 + r_2^4 \mu_2 (\lambda_1 + \mu_1 + \mu_2)(3\lambda_2 + 2\mu_2) + r_1^2 r_2^2 (\lambda_1 (3\lambda_2 (\mu_1 - \mu_2) + \mu_2 (3\mu_1 + \mu_2)) + 2(\lambda_2 (\mu_1^2 - 3\mu_2^2) + \mu_2 (\mu_1^2 + \mu_1 \mu_2 - 2\mu_2^2)))). \quad (4.52)$$

Hence, the linear displacements in the two layers can be written as:

$$u_r^{L(1)} = kF_1(r) = kF_{11}^{(1)} r, \quad (4.53)$$

$$u_r^{L(2)} = kF_2(r) = k(F_{11}^{(2)} r + \frac{F_{12}^{(2)}}{r}), \quad (4.54)$$

$$u_\theta^{L(1)} = u_\theta^{L(2)} = rkG_{11}^1(z) = rkG_{11}^2(z) = 0, \quad (4.55)$$

$$u_z^{L(1)} = u_z^{L(2)} = kH_{11}^{(1)} z = kH_{11}^{(2)} z, \quad (4.56)$$

and the linear stresses in Layer 1 can be written as:

$$\mathbf{T}^{L(1)} = \begin{pmatrix} k(H_{11}^{(1)} \lambda_1 + 2F_{11}^{(1)} (\lambda_1 + \mu_1)) & 0 & 0 \\ 0 & k(H_{11}^{(1)} \lambda_1 + 2F_{11}^{(1)} (\lambda_1 + \mu_1)) & 0 \\ 0 & 0 & k(2F_{11}^{(1)} \lambda_1 + H_{11}^{(1)} (\lambda_1 + 2\mu_1)) \end{pmatrix}. \quad (4.57)$$

$\mathbf{T}^{L(2)}$ in Layer 2 also has three diagonal terms only:

$$T_{rr}^{L(2)} = k(H_{11}^{(2)}\lambda_2 - \frac{2F_{12}^{(2)}}{r^2}\mu_2 + 2F_{11}^{(2)}(\lambda_2 + \mu_2)), \quad (4.58)$$

$$T_{\theta\theta}^{L(2)} = k(H_{11}^{(2)}\lambda_2 + \frac{2F_{12}^{(2)}}{r^2}\mu_2 + 2F_{11}^{(2)}(\lambda_2 + \mu_2)), \quad (4.59)$$

$$T_{zz}^{L(2)} = k(2F_{11}^{(2)}\lambda_2 + H_{11}^{(2)}(\lambda_2 + 2\mu_2)). \quad (4.60)$$

As for the homogeneous cylinder, the bilayered cylinder does not demonstrate an axial force-twist effect in the linear theory. Also, both Layers 1 and 2 undergo identical first-order axial displacements. Furthermore, the bilayered composite has three non-vanishing first-order normal stress components, while the homogeneous cylinder has only one. The existence of $T_{rr}^{L(i)}$ and $T_{\theta\theta}^{L(i)}$ shows the complex interaction of the two materials under pure axial loading, even in the linear theory. For instance, $T_{rr}^{L(2)}$ and $T_{\theta\theta}^{L(2)}$ in Layer 2 vary with r while $T_{zz}^{L(1)}$ and $T_{zz}^{L(2)}$ are constant.

Consider next the second-order theory. Substituting $F_1^{(i)}(r)$, $G_1^{(i)}(z)$ and $H_1^{(i)}(z)$ into the second-order governing equations, i.e., Eqs. (4.9)–(4.11), $F_2^{(i)}(r)$, $G_2^{(i)}(z)$ and $H_2^{(i)}(z)$ can be determined.

$$F_2^{(i)}(r) = \frac{(2m_i + \lambda_i + 3\mu_i)}{2r^3(\lambda_i + 2\mu_i)}(F_{12}^{(i)})^2 + F_{21}^{(i)}r + \frac{F_{22}^{(i)}}{r}, \quad (4.61)$$

$$G_2^{(i)}(z) = G_{21}^{(i)} z, \quad (4.62)$$

$$H_2^{(i)}(z) = H_{21}^{(i)} z, \quad (4.63)$$

for $i = 1, 2$. There are eight unknown constants $F_{21}^{(i)}, F_{22}^{(i)}, G_{21}^{(i)}, H_{21}^{(i)}$. Again

$F_{22}^{(1)} = 0$ for the first layer. The following seven boundary conditions allow us to solve for the values of the seven remaining constants:

(1) Second-order radial displacement is continuous across the interface:

$$F_2^{(1)}(r = r_1) = F_2^{(2)}(r = r_1), \quad (4.64)$$

(2) Second-order circumferential displacement is continuous across the interface:

$$G_{21}^{(1)} z = G_{21}^{(2)} z, \text{ or } G_{21}^{(1)} = G_{21}^{(2)}, \quad (4.65)$$

(3) Second-order axial displacement is continuous across the interface:

$$H_{21}^{(1)} z = H_{21}^{(2)} z, \text{ or } H_{21}^{(1)} = H_{21}^{(2)} \quad (4.66)$$

(4) Second-order radial traction T_{rr}^{NL} is continuous across the interface:

$$T_{rr}^{NL(1)}(r = r_1) = T_{rr}^{NL(2)}(r = r_1), \quad (4.67)$$

(5) T_{rr}^{NL} vanishes at the outer boundary:

$$T_{rr}^{NL(2)}(r = r_2) = 0, \quad (4.68)$$

(6) The contribution of the second-order axial stress to the axial force applied at either end of the cylinder is 0:

$$\int T_{zz}^{NL(i)}(z = 0, L) dA = 0, \quad (4.69)$$

(7) The contribution of the second-order shear stress $T_{\theta z}^{NL}$ to T is 0:

$$\int T_{\theta z}^{NL(i)}(z=0, L) r dA = 0. \quad (4.70)$$

Solving the constants, we can obtain $G_{21}^{(1)} = G_{21}^{(2)} = 0$. The constants $F_{21}^{(1)}, F_{21}^{(2)}, F_{22}^{(2)}, H_{21}^{(1)} = H_{21}^{(2)}$ are complex combinations of material parameters $\lambda_i, \mu_i, l_i, m_i, n_i$, geometrical parameters r_i and the applied axial loading P . Due to their lengthiness, they are not listed explicitly.

In summary, the components of the total displacement field (first- and second-order) in a composite cylinder can be written as:

$$u_r^{(i)} = k(F_{11}^{(i)} r + \frac{F_{12}^{(i)}}{r}) + k^2 \left(\frac{(2m_i + \lambda_i + 3\mu_i)}{2r^3(\lambda_i + 2\mu_i)} (F_{12}^{(i)})^2 + F_{21}^{(i)} r + \frac{F_{22}^{(i)}}{r} \right), \quad (4.71)$$

$$u_\theta^{(i)} = krG_{11}^{(i)} z + k^2 rG_{21}^{(i)} z = 0, \quad (4.72)$$

$$u_z^{(1)} = u_z^{(2)} = kH_{11}^{(1)} z + k^2 H_{21}^{(1)} z = kH_{11}^{(2)} z + k^2 H_{21}^{(2)} z. \quad (4.73)$$

As in the homogeneous case, the radial and axial displacements are not zero, while the circumferential displacement vanishes identically. There is no axial force-twist effect up to the second-order elastic theory for a bilayered composite under pure tension. This is also true for a multilayered composite.

The stresses can be written as:

$$\mathbf{T}^{(i)} = \begin{pmatrix} kT_{rr}^{L(i)} + k^2T_{rr}^{NL(i)} & 0 & 0 \\ 0 & kT_{\theta\theta}^{L(i)} + k^2T_{\theta\theta}^{NL(i)} & 0 \\ 0 & 0 & kT_{zz}^{L(i)} + k^2T_{zz}^{NL(i)} \end{pmatrix}, \quad (4.74)$$

where the second-order normal stresses assume complicated forms involving the material parameters, the radii and the torsion.

4.4. Numerical results

The influence of the interface position (denoted by $r = r_1$) and the elastic constants ($\lambda_i, \mu_i, l_i, m_i, n_i; i = 1, 2$) on the second-order stresses and displacements is discussed in this section. Several soft materials were considered whose elastic constants have been determined in comparison to the experimental data of soft materials, e.g., agar-gelatin (Catheline *et al.*, 2003). The elastic constants λ, μ, l, m, n of the agar-gelatin are $2.25 \times 10^6, 8, -2 \times 10^6, -20 \times 10^6, 80$ kPa. The large value of λ for the gelatin implies near incompressibility. For parametric studies, the elastic constants are varied from these reference values, and r_1 is also varied from 0 to the radius r_2 of the outer boundary. The applied axial loading stress $P = 1.5$ or 0.1 kPa. The cylinder has the length $L = 0.01$ m, and an outer radius of $L/10$ or $L/5$. Table 4.1 lists all the elastic constants and radii used in Figs. 4.2 – 4.4.

Fig. 4.2. Contours of the constant interfacial axial stress (a) $T_{zz}(r=r_1^+)=0$ and (b) $T_{zz}(r=r_1^-)=0$ in $\mu_2 - \mu_1$ space for a two-layer composite under axial loading $P = 1.5$ kPa.

A two-layer soft composite ($r_1 = 0.001$ m, $r_2 = 0.002$ m, $L = 0.01$ m) is subjected to pure axial loading $P = 1.5$ kPa. The material constants of the two layers are held fixed and listed in Table 4.1, except for μ_1 and μ_2 which are varied to show the dependence of the stresses T_{zz} and $T_{\theta\theta}$ at either side of the interface $r = r_1$. In Figs. 4.2(a) and (b), the contours $T_{zz}(r = r_1^+) = 0$ and $T_{zz}(r = r_1^-) = 0$ are shown respectively in the $\mu_1 - \mu_2$ space. The symbols r_1^+ and r_1^- denote radial positions in Layers 2 and 1, respectively. These interfacial stresses are not functions of θ and z . Fig. 4.2 shows that there exist Regions I and II within which T_{zz} is either positive or negative, respectively, unlike the purely homogeneous case where $T_{zz} = kP$ is constant throughout the cylinder. There are both k and k^2 normal stresses in a composite cylinder under axial loading.

Also, $T_{zz}(r = r_1^+)$ and $T_{zz}(r = r_1^-)$ are not equal, as implied by the differences in the curves shown in the two sub-figures. The dependence on the elastic constants is complex; for instance there are two values of μ_1 for a given μ_2 for which $T_{zz}(r = r_1^+) = 0$, as indicated by the loop in Fig. 4.2(a).

Similar remarks can be made for $T_{\theta\theta}(r = r_1^+) = 0$ and $T_{\theta\theta}(r = r_1^-) = 0$, whose dependence on μ_1 and μ_2 are shown in Figs. 4.3(a) and (b), respectively. Here, the shear moduli of the two layers appear to display an inverse relationship: to achieve a zero circumferential stress a larger μ_2 must be accompanied by a smaller μ_1 . Regions I and II do not occupy the same $\mu_1 - \mu_2$ domains in the sub-figures, implying that there is also a jump in the circumferential stress across the interface.

Finally, we investigate how the second-order radial displacement u_r^{NL} at the outer boundary and the second-order axial displacement u_z^{NL} at the top end, vary with the interface position of a two-layer composite and its reverse configuration. The cylinders ($r_2 = 0.001$ m, $L = 0.01$ m) are subjected to the axial loading of $P = 0.1$ kPa, and their elastic constants are given in Table 4.1. Fig. 4.4(a) plots $u_r^{NL}(r_2)/r_2$ versus r_1 , while Fig. 4.4(b) $u_z^{NL}(L)/L$ versus r_1 . Both displacements vary nonlinearly with r_1 , but $u_r^{NL}(r_2)/r_2$ in a non-monotonic and $u_z^{NL}(L)/L$ in a monotonic manner. In particular, the second-order radial displacement of the composite may be significantly larger than that of its homogeneous counterparts. This is similar to the Poynting effect of a composite and its homogeneous counterparts under pure torsion. Fig. 4.4(b) shows that the axial displacement does not display this amplification effect, however.

Fig. 4.3. Contours of the constant axial stress (a) $T_{\theta\theta}(r = r_1^-) = 0$ and (b) $T_{\theta\theta}(r = r_1^+) = 0$ in $\mu_2 - \mu_1$ space for a two-layer composite under axial loading $P = 1.5$ kPa.

Fig. 4.4. Variation of normalized second-order (a) radial displacement $u_r^{NL}(r_2)/r_2$, and (b) axial displacement $u_z^{NL}(L)/L$ with the interface position r_1 for two different two-layer composites under axial loading $P = 0.1$ kPa.

4.5. Validation of the theory

In this section, the second-order nonlinear theory of soft materials under axial loading is verified against experimental data. The equations used in this part are first presented here.

Consider an isotropic homogeneous cylindrical bar under a uniaxial tensile stress σ , defined as the applied force per unit initial cross-sectional area (engineering stress). Using a perturbation analysis, the engineering strain ε takes the form:

$$\varepsilon = \frac{\sigma}{E} + \delta \left(\frac{\sigma}{E} \right)^2 \quad (4.75)$$

where E is the Young's modulus related to λ and μ via the well-known relation

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad (4.76)$$

and δ is dimensionless and related to the five elastic constants

$$\delta = -\frac{3}{2} - \frac{\mu^2 l}{(3\lambda + 2\mu)(\lambda + \mu)^2} - \frac{(3\lambda + 2\mu)}{2(\lambda + \mu)^2} m - \frac{3\lambda^2}{4\mu(3\lambda + 2\mu)(\lambda + \mu)} n \quad (4.77)$$

The details of the derivation of Eqs. (4.75)–(4.77) can be found in the original work of Murnaghan (1951). Eq. (4.75) contains two parameters E and δ (rather than five), implying that different values of the five elastic constants can lead to the same stress-strain response provided they share the same values of E and δ according to

Eqs. (4.76) and (4.77). Moreover, δ can be positive or negative. A negative value implies a nonlinear stiffening effect as the stress produces a strain less than that predicted by linear elasticity. Conversely, a positive δ implies a softening effect.

Multiplying Eq. (4.75) by δ and defining a modified strain $\tilde{\varepsilon} = \delta\varepsilon$ and a modified (dimensionless) stress $\tilde{\sigma} = \delta\sigma / E$, the following simple universal relation is obtained:

$$\tilde{\varepsilon} = \tilde{\sigma} + \tilde{\sigma}^2, \quad (4.78)$$

where no material parameters appear explicitly. The values of E and δ can be obtained by fitting Eq. (4.75) against the stress-strain data, and can be used to calculate the modified strain and stress. The modified $\tilde{\sigma}$ versus $\tilde{\varepsilon}$ data can be compared to the universal form of Eq. (4.78). A universal representation is a convenient way to collapse different data sets into a single reduced data set, as illustrated next..

The universal relation of Eq. (4.78) for uniaxial tension is validated against fourteen data sets labeled 1-14 in Table 4.2. The materials include biopolymer networks (neurofilaments, collagen, vimentin, fibrin and actin) (Storm et al., 2005), covalently crosslinked networks based on poly(propylene glycol) bis(aceto-acetate) with either neopentyl glycol diacrylate (PPG BisAcAc-NPGDA) or with 2-hydroxyethyl acrylate derivatized bis(4-isocyanatocyclohexyl) methane (PPG BisAc-Ac-EA-HMDI-EA) (Williams et al. , 2007), extremely soft poly(acrylamide) (PAAm) hydrogels (Yohsuke et al., 2011), physical and chemical gels of

poly(vinyl alcohol) (Urayama et al., 2006), calcium-crosslinked gellan gels (Tang et al., 1997), and a longitudinal strip of human abdominal aorta (Balzani et al., 2006). In addition, molecular dynamics simulations of semiflexible networks (Dobrynin and Carrilo, 2011) are also included in the data sets.

Table 4.2 lists the materials and the corresponding parameters E and δ obtained through data fitting of Eq. (4.75). Two examples of σ – ε plot are shown in Fig. 4.5. Fig. 4.5(a) shows the curve of Eq. (4.75) and the test data for a physical gel of poly(vinyl alcohol) (labeled P-GEL-15 in the original paper of Urayama et al. (2006)), while Fig. 4.5(b) compares Eq. (4.75) to the data for a longitudinal strip of human abdominal aorta (Balzani et al., 2006). The value of δ is negative in these two cases, implying a stiffening effect associated with second-order elasticity. Table 4.2 shows that most of the materials display the stiffening effect, whereas the PPG BisAcAc-NPGDA networks, the soft PAAm gels and the chemical gels of poly(vinyl alcohol) display a softening effect with corresponding positive δ .

Fig. 4.5. Uniaxial stress-strain curves of (a) P-GEL-15 polymer networks (Urayama et al., 2006) and (b) human abdominal aorta (Balzani et al., 2006). The lines represent the best fit to the theoretical expression predicted by second-order elasticity.

Figure 4.6 plots the universal relation, i.e., Eq. (4.78), in the form of $-\tilde{\sigma}$ versus $-\tilde{\epsilon}$. It can be seen that the fourteen data sets collapse into a narrow data band, through which the universal line passes. The agreement between the nonlinear elastic model prediction and the data is reasonably good, given that the materials include both biological and synthetic gels of various types. The $\tilde{\sigma}$ versus $\tilde{\epsilon}$ behavior is significantly nonlinear and it is insufficient to describe the gel elasticity using only a second-order elastic constant such as E . A universal relation for uniaxial tension has also been derived from a microscopic model of the polymer chains as worm-like chains (Dobrynin and Carrilo, 2011). Figure 4.6 shows that a simple second-order elastic continuum model is also capable of describing such a universal relation. Indeed, as the next section shows, a universal relation can also be developed for the swelling behavior.

Fig. 4.6. Dependence of modified stress $-\tilde{\sigma}$ on modified strain $-\tilde{\epsilon}$. The materials for the data sets 1-14 are listed in Table 4.2. The solid line represents the universal line.

Table 4.1. Elastic, geometrical and loading parameters used in Figs. 4.2–4.4.

	Fig. 4.2 (a), (b) Fig. 4.3 (a), (b)	Fig. 4.4 (a) solid Fig. 4.4 (b) solid	Fig 4.4 (a) dashed Fig 4.4 (b) dashed
Material 1	Cardiomyocyte-based	Agar-gelatin	Cardiomyocyte
λ_1 (kPa)	111.812	2.25×10^6	111.812
μ_1 (kPa)		8	2.282
l_1 (kPa)	7.016×10^2	-2×10^6	7.016×10^2
m_1 (kPa)	2.65×10^3	-20×10^6	2.65×10^3
n_1 (kPa)	-20	80	-20
Material 2	Agar-gelatin-based	Cardiomyocyte	Agar-gelatin
λ_2 (kPa)	2.25×10^6	111.812	2.25×10^6
μ_2 (kPa)		2.282	8
l_2 (kPa)	-2×10^6	7.016×10^2	-2×10^6
m_2 (kPa)	-20×10^6	2.65×10^3	-20×10^6
n_2 (kPa)	80	-20	80
r_1 (m)	0.001		
r_2 (m)	0.002	0.001	0.001
L (m)	0.01	0.01	0.01
P (kPa)	1.5	0.1	0.1

Table 4.2. Elastic parameters E and δ obtained from fitting Eq. (4.75) to uniaxial tensile test data.

No.	Material	E (kPa)	δ
1	Collagen (Storm et al., 2005)	87.9282	-1.39442
2	Fibrin (steady state) (Storm et al., 2005)	53.0106	-0.6301
3	Fibrin(oscillator) (Storm et al., 2005)	124.561	-0.3108
4	Actin (Storm et al., 2005)	346.669	-1.60691
5	PPG BisAcAc NPGDA network (Williams et al., 2007)	96.4221	0.0426442
6	PPG BisAcAc-EA-HMDI-EA network (Williams et al., 2007)	643.774	-0.299947
7	$K = 2$, $N_s = 17.88 \pm 7.4$, uniaxial deformation (Dobrynin and Carrilo, 2011)	0.0222387	-0.124757
8	$K = 2$, $N_s = 18.03 \pm 6.7$, uniaxial deformation (Dobrynin and Carrilo, 2011)	0.0607035	-0.133834
9	PAAm-CG-2 (Yohsuke et al., 2011)	1.37086	0.163572
10	PAAm-CG-4 (Yohsuke et al., 2011)	2.92523	0.128837
11	P-GEL-15 (Urayama et al., 2006)	31.7298	-0.0692103
12	C-GEL-10 (Urayama et al., 2006)	17.1659	0.246195
13	Gels containing 10% gellan polymer with 6 mM Ca^{++} (Tang et al., 1997)	182.349	-1.87775
14	Longitudinal media of human abdominal aorta (Balzani et al., 2006)	52.5559	-0.715767

4.6. Conclusions

The second-order effects in homogeneous and bilayered cylinders under pure axial loading have been investigated in this chapter. The major contributions and findings can be summarized as follows.

First, analytical solutions have been derived for homogeneous and bilayered cylinders under axial loadings, starting consistently from the equilibrium equations for the first Piola-Kirchhoff stress.

Second, the second-order effects in the displacement and stress fields have been identified, and their dependence on the elastic constants and geometrical parameters have been studied.

Third, the analytical solutions obtained may be used as the theoretical basis of the experimental estimation of third-order elastic constants.

Finally, universal relations have been developed in terms of modified parameters. These relations compare well with experimental data for a variety of soft materials including biogels and synthetic polymers.

Chapter 5 Torsion Loading

The problem of elastically nonlinear cylinders subjected to torsion is one of the classical problems. Murnghan (1951) solved it using second-order elasticity as well. In this research, analytical results are obtained for multilayered cylinders. In particular, the dependence of nonlinear effects such as the Poynting effect, both positive and negative, for a bilayered cylinder is investigated. The Poynting effect is a nonlinear effect discovered more than a century ago (Poynting, 1909), generally for metal bars which extended slightly under pure torsion. This is the positive Poynting effect. The recent experimental observations that biogels under shear loading experienced negative rather than positive normal stresses highlighted the possibility of a negative Poynting effect, as the negative normal stresses correspond to contraction of the material under pure torsion (Storm et al., 2005; Janmey et al., 2007). No results, to the best of knowledge, have been reported for bilayered cylinders. Recent works, for instance, have been reported only for homogeneous materials (Mihai and Goriely, 2011, 2013). Hence, the work reported here on the Poynting effect is believed to be a new contribution. Moreover, a key finding is that the Poynting effect of a bilayered bar can be stronger than that of a purely homogeneous bar made up entirely of either of the materials that together constitute the bilayered bar. This amplification effect is dependent not only on the materials but also on the layer thicknesses.

The Chapter is organized in following manner. The problem is defined in Section 5.1. The governing equations are shown in Section 5.2. The analytical solutions are obtained for both homogeneous and bilayered cylinders under torsion in Section 5.3. In Section 5.4, the effects of material nonlinearity and geometrical complexities are presented, highlighting nonlinear phenomena such as the Poynting effect and the second-order normal stresses. Further discussions and implications of the findings are given in Section 5.5. A summary of the present work is given in Section 5.6.

5.1. Problem definition

Consider an N -layered circular cylinder of length L and radius r_N (or simply R for a homogeneous cylinder). The interfaces are located at the radial coordinates $r = r_i$, $i = 1, N-1$, with $i = 1$ denoting the innermost interface, as shown in Fig. 5.1. All the layers are nonlinear elastic, isotropic, homogeneous and perfectly bonded to each other. The composite cylinder is subjected to either a torsion T . The initial coordinates of a particle of the cylinder are chosen as (r, θ, z) , which respectively represent the radial, angular and axial coordinates. The final coordinates are denoted by (ρ, ψ, ζ) .

In the following, we consider T loadings of both homogeneous and composite cylinders in two separate sub-sections. We will focus on the Poynting effect and other second-order phenomena. For simplicity, we will present the solutions for cylinders up to two layers only.

Fig. 5.1. A cylindrical composite consisting of N layers, with surfaces located at the radial coordinates, $r = r_i$, $i = 1, N$, under torsion T .

5.2. Governing equations

Under torsion, the final coordinates (ρ, ψ, ζ) of the particle of a cylinder whose initial coordinates being (r, θ, z) are given by $(r + kF(r), \theta + k\alpha z + k^2G(z), z + kH(z))$, where $kF(r)$, $k\alpha z + k^2G(z)$, $kH(z)$ are the radial, angular and axial displacements respectively, with α denoting the angle of twist per unit length in the usual linear theory. Here k is a marker indicating the order of approximation of the theory; for second-order theory we retain terms up to k^2 . The unknown radial displacement $kF(r) = k[F_1(r) + kF_2(r)]$, where $F_1(r)$ and $F_2(r)$ are respectively the first- and second-order terms. These terms are assumed to be functions of r only and not of z since all the sections are twisted in the same manner. The unknown second-order angular displacement $k^2G(z)$ is assumed to be a function of z , but will be shown to vanish, meaning that there is no additional twist under torsion in second-order elasticity theory. The unknown axial displacement $kH(z) = k[H_1(z) + kH_2(z)]$, where $H_1(z)$ and $H_2(z)$ are respectively the first- and second-order terms. These are taken to be functions of z only and not of r since the bar sections elongate or contract uniformly. This deformation representation for pure torsion permits displacements in both the axial and radial directions. The sign of the axial displacement determines the positive or negative Poynting effect.

Let $\mathbf{a}$ and $\mathbf{x}(\mathbf{a})$ denote the position vectors in the undeformed and deformed states, respectively. Their differentials are then:

$$d\mathbf{a} = \begin{pmatrix} dr \\ r d\theta \\ dz \end{pmatrix}, \quad d\mathbf{x} = \begin{pmatrix} (1 + kF')dr \\ (r + kF)(d\theta + k\alpha dz + k^2G'dz) \\ (1 + kH')dz \end{pmatrix}, \quad (5.1)$$

where the derivatives $F' = dF / dr = F_1'(r) + kF_2'(r)$, $G' = dG(z) / dz$ and $H' = dH / dz = H_1'(z) + kH_2'(z)$. The deformation gradient $\mathbf{F}$ is thus:

$$\mathbf{F} = \frac{d\mathbf{x}}{d\mathbf{a}} = \begin{pmatrix} 1+kF' & 0 & 0 \\ 0 & 1+\frac{kF}{r} & (r+kF)(k\alpha+k^2G') \\ 0 & 0 & 1+kH' \end{pmatrix} . \quad (5.2)$$

It can be shown that the equilibrium equations in cylindrical coordinates in terms of the first Piola-Kirchhoff stresses are (Volokh, 2006):

$$\frac{\partial T_{rr}}{\partial r} - T_{\theta r} \frac{\partial \psi}{\partial r} + \frac{T_{rr}}{r} + \frac{\partial T_{r\theta}}{r \partial \theta} - \frac{T_{\theta\theta}}{r} \frac{\partial \psi}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} - T_{\theta z} \frac{\partial \psi}{\partial z} = 0 , \quad (5.3)$$

$$T_{rr} \frac{\partial \psi}{\partial r} + \frac{\partial T_{\theta r}}{\partial r} + \frac{T_{r\theta}}{r} \frac{\partial \psi}{\partial \theta} + \frac{T_{\theta R}}{r} + \frac{\partial T_{\theta\theta}}{r \partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + T_{rz} \frac{\partial \psi}{\partial z} = 0 , \quad (5.4)$$

$$\frac{\partial T_{zr}}{\partial r} + \frac{T_{zr}}{r} + \frac{\partial T_{z\theta}}{r \partial \theta} + \frac{\partial T_{zz}}{\partial z} = 0 , \quad (5.5)$$

where $\partial \psi / \partial r = 0$, $\partial T_{r\theta} / \partial \theta = 0$, $\partial \psi / \partial \theta = 1$ and $\partial \psi / \partial z = k\alpha + k^2 G'$ according to the displacement field under torsion. Strictly speaking, the component T_{rr} should be written as $T_{\rho r}$, which represents the stress along the ρ -direction in the deformed configuration with respect to the r -face in the undeformed configuration. Similarly, $T_{\theta r}$ should be written as $T_{\psi r}$ in the strict notation. The stresses can be obtained by Eq. (3.3). Substituting the stresses into Eqs. (5.3)–(5.5) yields two first-order and three second-order equilibrium equations as distinguished by the k and k^2 multipliers:

$$k\left(\frac{F_1''(r)}{r} + \frac{F_1'(r)}{r} - \frac{F_1'(r)}{r^2}\right) = 0, \quad (5.6)$$

$$kH_1''(z) = 0, \quad (5.7)$$

$$\begin{aligned} & k^2\{r^4\alpha^2(4m-3n+4\lambda-8\mu) - 4(2l+2m+2\lambda+3\mu)F_1(r)^2 - 4r(\lambda+2\mu)F_2(r) \\ & + 4r^2(2l+2m+2\lambda+3\mu)F_1'(r)^2 + 4r^2(\lambda+2\mu)F_2'(r) + 4r^3(2l+\lambda)H_1'(z)F_1''(r) \\ & + 4r(2l+\lambda)F_1(r)(-H_1'(z) + rF_1''(r)) + \\ & 4r^2F_1'(r)[(2l+\lambda)H_1'(z) + r(2l+4m+3\lambda+6\mu)F_1''(r)] + 4r^3(\lambda+2\mu)F_2''(r)\} = 0, \end{aligned} \quad (5.8)$$

$$k^2(r\alpha(m+\lambda+2\mu)H_1''(z) + \mu rG'(z)) = 0, \quad (5.9)$$

$$\begin{aligned} & k^2\{(2l+\lambda)F_1(r)H_1''(z) + r[(2l+\lambda)F_1'(r)H_1''(z) + (2l+4m+3\lambda+6\mu)H_1'(z)H_1''(z) \\ & + (\lambda+2\mu)H_2''(z)]\} = 0. \end{aligned} \quad (5.10)$$

Note that the first-order term arising from Eq. (5.4) is identically satisfied, resulting in five equilibrium equations. We next solve these five equations for the unknown functions F_1 , F_2 , G , H_1 and H_2 for both the homogeneous and the two-layered cases. Solving the second-order equations requires the solutions of the first-order ones, as can be seen from the appearance of F_1 and H_1 in Eqs. (5.8)–(5.10).

5.3. Solutions

5.3.1. Solutions for a homogeneous cylinder under torsion

Solving the first-order equilibrium equations Eqs. (5.6) and (5.7), F_1 and H_1 are obtained as:

$$F_1(r) = F_{11}r + \frac{F_{12}}{r}, \quad (5.11)$$

$$H_1(z) = H_{11}z + H_{12}, \quad (5.12)$$

where F_{11} , F_{12} , H_{11} and H_{12} are constants to be determined. However, $F_{12} = 0$ in order to fulfill the condition that the radial displacement u_r is non-singular at $r = 0$. Also, $H_{12} = 0$ if the axial displacement is to vanish at the lower end $z = 0$, as one is investigating the axial displacement with respect to that lower end. In order to determine F_{11} , H_{11} and α , appropriate boundary conditions are applied as follows:

(1) The first-order radial stress T_{rr}^L (L denoting “linear”) vanishes at the outer boundary:

$$T_{rr}^L(r = R) = 0, \quad (5.13)$$

(2) No applied axial forces act at the end faces of the cylinder:

$$\int T_{zz}^L(z = 0, L) dA = 0, \quad (5.14)$$

(3) The torsion applied to the cylinder is T :

$$\int T_{\theta z}^L(z = 0, L) r dA = T. \quad (5.15)$$

In the above equations, A is the area of either end of the cylinder. Using these boundary conditions, the constants are obtained as:

$$F_{11}=0, \quad H_{11}=0, \quad \alpha = \frac{2T}{\pi R^4 \mu}. \quad (5.16)$$

Hence, $F_1(r)=0$ and $H_1(z)=0$ according to Eqs. (5.11) and (5.12), implying that the *first-order* radial and axial displacements are all zero. Substituting F_1 and H_1 into the second order equilibrium equations given by Eqs. (5.8)–(5.10), F_2 , G and H_2 can be solved as:

$$F_2(r) = -\frac{r^3 T^2 (4\lambda - 8\mu + 4m - 3n)}{8\pi^2 R^8 \mu^2 (\lambda + 2\mu)} + F_{21}r + \frac{F_{22}}{r}, \quad (5.17)$$

$$G(z) = G_{11}z, \quad (5.18)$$

$$H_2(z) = H_{21}z. \quad (5.19)$$

In Eqs. (5.17)–(5.19), F_{21} , F_{22} , G_{11} , H_{21} , are constants. Out of the four unknown constants, $F_{22} = 0$ in order that the nonlinear radial displacement vanishes at $r = 0$.

The remaining constants F_{21} , G_{11} , and H_{21} can be determined using the following boundary conditions:

(1) The second-order radial stress T_{rr}^{NL} (NL denoting “nonlinear”) vanishes at the outer boundary:

$$T_{rr}^{NL}(r = R) = 0, \quad (5.20)$$

(2) No applied axial forces act at either end of the cylinder:

$$\int T_{zz}^{NL}(z = L) dA = 0, \quad (5.21)$$

(3) The contribution of the second-order shear stress $T_{\theta z}^{NL}$ to T is 0:

$$\int T_{\theta z}^{NL}(z = L) r dA = 0. \quad (5.22)$$

Upon solving the above three equations, F_{21} , G_{11} and H_{21} can be written as:

$$F_{21} = \frac{T^2(4\lambda^2\mu + 4\lambda\mu m - 8\mu^2 - 12\lambda\mu - 12\mu^2 + \lambda^2 n - 5\lambda\mu n - 2\mu^2 n)}{8\pi^2 R^6 \mu^3 (\lambda + 2\mu)(3\lambda + 2\mu)}, \quad (5.23)$$

$$G_{11} = 0, \quad (5.24)$$

$$H_{21} = -\frac{T^2(\lambda n + 4\mu m + 4\lambda\mu + 8\mu^2)}{4\pi^2 R^6 \mu^3 (3\lambda + 2\mu)}. \quad (5.25)$$

With all the constants known, the displacement field, represented by the radial (u_r), circumferential (u_θ) and axial (u_z) components, can be written as:

$$\begin{aligned} u_r = & \frac{k^2}{8\pi^2 R^8 \mu^3 (\lambda + 2\mu)(3\lambda + 2\mu)} (rT^2(n(3r^2\mu(3\lambda + 2\mu) + R^2(\lambda^2 - 5\lambda\mu - 2\mu^2)) - \\ & 4\mu(r^2(3\lambda^2 - 4\lambda\mu - 4\mu^2) + R^2(-\lambda^2 + 12\lambda\mu + 12\mu^2) + \\ & m(-R^2(\lambda - 2\mu) + r^2(3\lambda + 2\mu))))), \end{aligned} \quad (5.26)$$

$$u_\theta = k\alpha rz, \quad (5.27)$$

$$u_z = -\frac{k^2 T^2 (n\lambda + 4\mu(m + \lambda + 2\mu))}{4\pi^2 R^6 \mu^3 (3\lambda + 2\mu)} z. \quad (5.28)$$

Replacing T in Eq. (5.28) by α , the axial displacement can be written as $u_z = k^2 D z$,

where

$$D = -\frac{T^2 (n\lambda + 4\mu(m + \lambda + 2\mu))}{4\pi^2 R^6 \mu^3 (3\lambda + 2\mu)} \quad (5.29)$$

is defined as the Poynting effect coefficient of the homogeneous cylinder. It can be seen that under pure torsion u_θ is first-order in nature while u_r and u_z represent second-order effects. In particular, u_z or D predicts the positive or negative Poynting effect while u_r predicts the associated radial expansion or contraction. The T^2 dependence in D implies that the direction of the torque does not influence the positive or negative nature of the Poynting effect. The sign of the effect is primarily determined by the relative values of the elastic constants. Note the strong inverse dependence on R^6 . For a metal such as silver, the elastic constants are (Hirth and Lothe, 1982; Hiki and Granato, 1966): $\lambda = 91$ GPa, $\mu = 45$ GPa, $l = -265$ GPa, $m = -637$ GPa, and $n = 332$ GPa. Hence $D = 3.92 \times 10^{-4} T^2 / \pi^2 R^6$ by Eq. (5.29) and a positive Poynting effect is predicted. The Poynting effect for other materials will be discussed in a subsequent section. A further remark is that the displacement

field, as well as the stress field shown later, is independent of the third-order elastic constant l . This reduced dependence on only four constants λ , μ , m and n is a consequence of the cylindrical symmetry of the problem.

Using Eq. (3.3), the stress field, consisting of the following five non-vanishing components, can be determined explicitly:

$$T_{rr} = \frac{k^2(r^2 - R^2)T^2(n(2\lambda + \mu) + 4\mu(m + 5\lambda + 6\mu))}{4\pi^2 R^8 \mu^2 (\lambda + 2\mu)}, \quad (5.30)$$

$$T_{\theta\theta} = \frac{k^2 T^2 (n(3r^2 - R^2)(2\lambda + \mu) + 4\mu(m(3r^2 - R^2) - R^2(5\lambda + 6\mu) + r^2(11\lambda + 10\mu)))}{4\pi^2 R^8 \mu^2 (\lambda + 2\mu)}, \quad (5.31)$$

$$T_{zz} = \frac{k^2(2r^2 - R^2)T^2(3n\lambda + 8\mu(m + 3\lambda + 2\mu))}{4\pi^2 R^8 \mu^2 (\lambda + 2\mu)}, \quad (5.32)$$

$$T_{\theta z} = T_{z\theta} = \frac{2krT}{\pi R^4}. \quad (5.33)$$

The normal stresses, as represented by Eqs. (5.30)–(5.32), represent second-order effects, while the shear stress $T_{\theta z} = T_{z\theta}$ represents a first-order effect consistent with the usual linear theory. The normal stresses depend on both the second- and third-order elastic constants, while the shear stress does not depend on these constants at all.

5.3.2. Solutions of a bilayered cylinder under torsion

For an N -layered cylinder, the governing equilibrium equations are applied to each layer, and the form of the solutions is essentially identical to the homogeneous case. The essential difference lies in the requirement of interface continuity of the tractions and displacements for the N -layered problem. Considering a bilayered cylinder, the solutions for the first-order radial and axial displacements $F_1^{(i)}(r)$ and $H_1^{(i)}(r)$, $i = 1, 2$, are:

$$F_1^{(i)}(r) = F_{11}^{(i)}r + \frac{F_{12}^{(i)}}{r}, \quad i = 1, 2 \quad (5.34)$$

$$H_1^{(i)}(z) = H_{11}^{(i)}z, \quad i = 1, 2 \quad (5.35)$$

where (i) indicates the layer and six unknown constants are to be determined. In addition, there are two twist rates α_1 and α_2 to be determined, resulting in a total of eight unknowns. For Layer 1, however, $F_{12}^{(1)} = 0$ since the linear radial displacement vanishes at $r = 0$. Note that $F_{12}^{(i)} \neq 0$ for $i = 2$. Hence, there remain seven unknowns to be determined from the boundary conditions. For traction boundary conditions, the relevant tractions can be calculated from Eq. (3.3). The seven boundary conditions are:

(1) Linear radial displacement is continuous across the interface $r = r_1$:

$$F_1^{(1)}(r = r_1) = F_1^{(2)}(r = r_1), \quad (5.36)$$

(2) Linear circumferential displacement is continuous across the interface:

$$r_1 \alpha_1 z = r_1 \alpha_2 z, \quad \text{or} \quad \alpha_1 = \alpha_2, \quad (5.37)$$

(3) Linear axial displacement is continuous across the interface:

$$H_{11}^{(1)} z = H_{11}^{(2)} z \quad \text{or} \quad H_{11}^{(1)} = H_{11}^{(2)}, \quad (5.38)$$

(4) Linear radial traction T_{rr}^L is continuous across the interface:

$$T_{rr}^{L(1)}(r = r_1) = T_{rr}^{L(2)}(r = r_1), \quad (5.39)$$

(5) T_{rr}^L vanishes on the outer boundary $r = r_2$:

$$T_{rr}^{L(2)}(r = r_2) = 0, \quad (5.40)$$

(6) No applied axial forces act at either end of the cylinder:

$$\int T_{zz}^{L(i)}(z = 0, L) dA = 0, \quad (5.41)$$

(7) The torsion applied to the cylinder is T :

$$\int T_{\theta z}^{L(i)} r dA = T. \quad (5.42)$$

Using the above seven equations, the seven constants can be determined as:

$$\begin{aligned} F_{11}^{(1)} = 0, \quad H_{11}^{(1)} = 0, \quad F_{11}^{(2)} = 0, \quad F_{12}^{(2)} = 0, \quad H_{11}^{(2)} = 0, \\ \alpha_1 = \alpha_2 = \alpha = \frac{2T}{\pi(r_1^4 \mu_1 - r_1^4 \mu_2 + r_2^4 \mu_2)}. \end{aligned} \quad (5.43)$$

The twist rate α is the same for both layers. With the constants determined, the

linear displacement in the two layers can be written as:

$$\begin{aligned} u_r^{L(i)} = 0, \\ u_\theta^{L(i)} = k \alpha r z = k \frac{2T}{\pi(r_1^4 \mu_1 - r_1^4 \mu_2 + r_2^4 \mu_2)} r z, \end{aligned} \quad (5.44)$$

$$u_z^{L(i)} = 0.$$

The linear stresses can be written as:

$$\mathbf{T}^{L(i)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & k \frac{2rT\mu_i}{\pi(r_1^4(\mu_1 - \mu_2) + r_2^4\mu_2)} \\ 0 & k \frac{2rT\mu_i}{\pi(r_1^4(\mu_1 - \mu_2) + r_2^4\mu_2)} & 0 \end{pmatrix}. \quad (5.45)$$

It can be seen that there are no radial and axial displacements in the first-order theory, with the latter implying that the Poynting effect does not exist in the first-order theory. Furthermore, the two layers rotate with the same angle so that a radius remains straight all the way from the cylinder center through the interface to the outer boundary. The non-vanishing shear stresses $T_{\theta z}^{L(i)} = T_{z\theta}^{L(i)}$ in the two layers are proportional to their respective shear modulus and they experience a jump at the interface.

Moving on to the second-order theory, the solutions for the radial, angular and axial displacements $F_2^{(i)}(r)$, $G^{(i)}(z)$ and $H_2^{(i)}(z)$, $i = 1, 2$ can be derived as follows. Substituting $F_1^{(i)}(r)$, and $H_1^{(i)}(z)$ into the second-order equilibrium equations, i.e., Eqs. (5.8)–(5.10), $F_2^{(i)}(r)$, $G^{(i)}(z)$ and $H_2^{(i)}(z)$ can be determined. They assume the same form as Eqs. (5.17)–(5.19) in the homogeneous case:

$$F_2^{(i)}(r) = -\frac{r^3 T^2 (4m_i - 3n_i + 4\lambda_i - 8\mu_i)}{8\pi^2 (r_1^4 \mu_1 - r_1^4 \mu_2 + r_2^4 \mu_2)^2 (\lambda_i + 2\mu_i)} + F_{21}^{(i)} r + \frac{F_{22}^{(i)}}{r}, \quad (5.46)$$

$$G^{(i)}(z) = G_{11}^{(i)} z, \quad (5.47)$$

$$H_2^{(i)}(z) = H_{21}^{(i)} z, \quad (5.48)$$

for $i = 1, 2$. There are eight unknown constants $F_{21}^{(i)}$, $F_{22}^{(i)}$, $G_{11}^{(i)}$ and $H_{21}^{(i)}$. Again

$F_{22}^{(1)} = 0$ for the first layer. The following boundary conditions allow us to solve for the values of the seven remaining constants:

(1) Second-order radial displacement is continuous across the interface:

$$F_2^{(1)}(r = r_1) = F_2^{(2)}(r = r_1), \quad (5.49)$$

(2) Second-order circumferential displacement is continuous across the interface:

$$G_{11}^{(1)} z = G_{11}^{(2)} z, \text{ or } G_{11}^{(1)} = G_{11}^{(2)}, \quad (5.50)$$

(3) Second-order axial displacement is continuous across the interface:

$$H_{21}^{(1)} z = H_{21}^{(2)} z, \text{ or } H_{21}^{(1)} = H_{21}^{(2)}. \quad (5.51)$$

(4) The nonlinear radial traction T_{rr}^{NL} is continuous across the interface:

$$T_{rr}^{NL(1)}(r = r_1) = T_{rr}^{NL(2)}(r = r_1), \quad (5.52)$$

(5) T_{rr}^{NL} vanishes at the outer surface:

$$T_{rr}^{NL(2)}(r = r_2) = 0, \quad (5.53)$$

(6) No applied forces act at either end of the cylinder:

$$\int T_{zz}^{NL(i)}(z = 0, L) dA = 0, \quad (5.54)$$

(7) The contribution of the second-order shear stress $T_{\theta z}^{NL}$ to T is 0:

$$\int T_{\theta z}^{NL(i)}(z = 0, L) r dA = 0. \quad (5.55)$$

It can be established that $G_{11}^{(1)} = G_{11}^{(2)} = 0$, thus $G^{(i)}(z) = 0$, which proves that, as in the one-layered case, there is no second-order twist under pure torsion. The five remaining constants $F_{21}^{(1)}, F_{21}^{(2)}, F_{22}^{(2)}, H_{21}^{(1)}, H_{21}^{(2)}$ are combinations of the material constants $\lambda_i, \mu_i, m_i, n_i$, geometrical parameters r_i and the applied torsion T . Because of their complexities, they are not listed explicitly.

In summary, the components of the total displacement field (first- and second-order) can be written as:

$$\begin{aligned}
 u_r^{(i)} &= k^2 \left(-\frac{r^3 T^2 (4m_i - 3n_i + 4\lambda_i - 8\mu_i)}{8\pi^2 (r_1^4 \mu_1 - r_1^4 \mu_2 + r_2^4 \mu_2)^2 (\lambda_i + 2\mu_i)} + F_{21}^{(i)} r + \frac{F_{22}^{(i)}}{r} \right), \\
 u_\theta^{(1)} = u_\theta^{(2)} &= k \alpha r z = k \frac{2T}{\pi (r_1^4 \mu_1 - r_1^4 \mu_2 + r_2^4 \mu_2)} r z, \\
 u_z^{(1)} = u_z^{(2)} &= k^2 H_2^{(1)}(z) = k^2 H_2^{(2)}(z) = k^2 D z,
 \end{aligned} \tag{5.56}$$

where D is the Poynting effect coefficient for a bilayered cylindrical composite:

$$\begin{aligned}
D = & T^2 (-n_1 r_1^4 r_2^2 \lambda_1 \lambda_2 + n_2 r_1^4 r_2^2 \lambda_1 \lambda_2 - n_2 r_2^6 \lambda_1 \lambda_2 - 4m_1 r_1^4 r_2^2 \lambda_2 \mu_1 + 3n_2 r_1^4 r_2^2 \lambda_2 \mu_1 - \\
& 2n_2 r_1^2 r_2^4 \lambda_2 \mu_1 - n_2 r_2^6 \lambda_2 \mu_1 - 4r_1^4 r_2^2 \lambda_1 \lambda_2 \mu_1 - 8r_1^4 r_2^2 \lambda_2 \mu_1^2 - n_1 r_1^6 \lambda_1 \mu_2 + n_2 r_1^6 \lambda_1 \mu_2 + 4m_2 r_1^4 r_2^2 \lambda_1 \mu_2 - \\
& n_1 r_1^4 r_2^2 \lambda_1 \mu_2 - 2n_2 r_1^4 r_2^2 \lambda_1 \mu_2 + n_2 r_1^2 r_2^4 \lambda_1 \mu_2 - 4m_2 r_2^6 \lambda_1 \mu_2 + n_1 r_1^6 \lambda_2 \mu_2 - n_2 r_1^6 \lambda_2 \mu_2 - \\
& n_1 r_1^4 r_2^2 \lambda_2 \mu_2 + n_2 r_1^4 r_2^2 \lambda_2 \mu_2 + n_2 r_1^2 r_2^4 \lambda_2 \mu_2 - n_2 r_2^6 \lambda_2 \mu_2 + 4r_1^4 r_2^2 \lambda_1 \lambda_2 \mu_2 - \\
& 4r_2^6 \lambda_1 \lambda_2 \mu_2 - 4m_1 r_1^6 \mu_1 \mu_2 + 4m_2 r_1^6 \mu_1 \mu_2 - 4m_1 r_1^4 r_2^2 \mu_1 \mu_2 + 4m_2 r_1^4 r_2^2 \mu_1 \mu_2 - \\
& 4m_2 r_1^2 r_2^4 \mu_1 \mu_2 - 4m_2 r_2^6 \mu_1 \mu_2 - 4r_1^6 \lambda_1 \mu_1 \mu_2 - 4r_1^4 r_2^2 \lambda_1 \mu_1 \mu_2 + 4r_1^6 \lambda_2 \mu_1 \mu_2 + 20r_1^4 r_2^2 \lambda_2 \mu_1 \mu_2 - \\
& 20r_1^2 r_2^4 \lambda_2 \mu_1 \mu_2 - 4r_2^6 \lambda_2 \mu_1 \mu_2 - 8r_1^6 \mu_1^2 \mu_2 - 8r_1^4 r_2^2 \mu_1^2 \mu_2 + 4m_1 r_1^6 \mu_2^2 - 4m_2 r_1^6 \mu_2^2 - \\
& 4m_1 r_1^4 r_2^2 \mu_2^2 + 4m_2 r_1^4 r_2^2 \mu_2^2 + 4m_2 r_1^2 r_2^4 \mu_2^2 - 4m_2 r_2^6 \mu_2^2 + 4r_1^6 \lambda_1 \mu_2^2 - 12r_1^4 r_2^2 \lambda_1 \mu_2^2 + \\
& 16r_1^2 r_2^4 \lambda_1 \mu_2^2 - 8r_2^6 \lambda_1 \mu_2^2 - 4r_1^6 \lambda_2 \mu_2^2 + 4r_1^4 r_2^2 \lambda_2 \mu_2^2 + 4r_1^2 r_2^4 \lambda_2 \mu_2^2 - 4r_2^6 \lambda_2 \mu_2^2 + \\
& 16r_1^6 \mu_1 \mu_2^2 - 8r_1^2 r_2^4 \mu_1 \mu_2^2 - 8r_2^6 \mu_1 \mu_2^2 - 8r_1^6 \mu_2^3 + 8r_1^4 r_2^2 \mu_2^3 + 8r_1^2 r_2^4 \mu_2^3 - 8r_2^6 \mu_2^3) / \\
& [4\pi^2 (r_1^4 \mu_1 - r_1^4 \mu_2 + r_2^4 \mu_2)^2 (3r_1^2 r_2^2 \lambda_1 \lambda_2 \mu_1 + 2r_1^2 r_2^2 \lambda_2 \mu_1^2 - 3r_1^2 r_2^2 \lambda_1 \lambda_2 \mu_2 + 3r_2^4 \lambda_1 \lambda_2 \mu_2 + 3r_1^4 \lambda_1 \mu_1 \mu_2 + \\
& 3r_1^2 r_2^2 \lambda_1 \mu_1 \mu_2 - 3r_1^4 \lambda_2 \mu_1 \mu_2 + 3r_2^4 \lambda_2 \mu_1 \mu_2 + 2r_1^4 \mu_1^2 \mu_2 + 2r_1^2 r_2^2 \mu_1^2 \mu_2 - \\
& 3r_1^4 \lambda_1 \mu_2^2 + r_1^2 r_2^2 \lambda_1 \mu_2^2 + 2r_2^4 \lambda_1 \mu_2^2 + 3r_1^4 \lambda_2 \mu_2^2 - 6r_1^2 r_2^2 \lambda_2 \mu_2^2 + 3r_2^4 \lambda_2 \mu_2^2 - \\
& 4r_1^4 \mu_1 \mu_2^2 + 2r_1^2 r_2^2 \mu_1 \mu_2^2 + 2r_2^4 \mu_1 \mu_2^2 + 2r_1^4 \mu_2^3 - 4r_1^2 r_2^2 \mu_2^3 + 2r_2^4 \mu_2^3)].
\end{aligned}
\tag{5.57}$$

As in the one-layered case, the radial and axial displacements represent second-order effects. Unlike D in Eq. (5.29) for the homogeneous case, it is a very complex combination of the elastic constants, the radii and torsion applied of the two layers. This suggests that the overall Poynting effect of the composite may depend on the interface position in a nonlinear, and possibly unexpected manner, as shown later.

The stresses can be written as:

$$\mathbf{T}^{(i)} = \begin{pmatrix} k^2 T_{rr}^{NL(i)} & 0 & 0 \\ 0 & k^2 T_{\theta\theta}^{NL(i)} & k \frac{2rT\mu_i}{\pi(r_1^4(\mu_1 - \mu_2) + r_2^4\mu_2)} \\ 0 & k \frac{2rT\mu_i}{\pi(r_1^4(\mu_1 - \mu_2) + r_2^4\mu_2)} & k^2 T_{zz}^{NL(i)} \end{pmatrix}, \tag{5.58}$$

where the second-order normal stresses also assume complicated forms involving

the material constants, the radii and the torsion. They are not written here explicitly due to their lengthiness but they can be computed easily.

5.4. Numerical results

The section focuses on the influence of the interface position (denoted by $r = r_1$) and the elastic constants ($\lambda_i, \mu_i, l_i, m_i, n_i; i = 1, 2$) on the second-order stresses and displacements. We consider several soft materials whose elastic constants have been determined in comparison to the experimental data of soft materials, e.g., agar-gelatin (Catheline *et al.*, 2003) and poly(acrylic acid)(PAA) gels (Wang and Wu, 2013). For these two materials, the elastic constants λ, μ, l, m, n are $2.25 \times 10^6, 8, -2 \times 10^6, -20 \times 10^6, 80$ kPa, and 60, 12.21, $-42.67, -35.60, -23.50$ kPa, respectively. The large value of λ for the gelatin implies near incompressibility. In addition, we also consider several metallic materials whose elastic constants can be found in the works of Hirth and Lothe (1982), Hiki and Granato (1966), Hughes and Kelly (1953), Powell and Skove (1982), and Sindhu and Menon (2001). For parametric studies, the elastic constants are varied from these reference values, and r_1 is also varied from 0 to the radius r_2 of the outer boundary. The applied torque assumes a magnitude of $T = 6 \times 10^{-10}$ kPa $\cdot$ m³, and the applied axial loading stress $P = 1.5$ or 0.1 kPa. The cylinder has the length $L = 0.01$ m, and an outer radius of $L/10$ or $L/5$. Table 5.1 lists all the elastic constants and radii used in Figs. 5.2 – 5.5. The elastic parameters l_i are not needed for the case of torsion.

Fig. 5.2. Contours of the Poynting effect coefficient $D = 0$ (solid line) and radial displacement $S = 0$ (dashed line) in (a) $\mu - n$ space for a homogeneous cylinder, (b) $\mu - n$ space for a different homogeneous cylinder, and (c) $\mu_2 - n_2$ space for a two-layer composite cylinder under torsion.

Fig. 5.2 plots the $D = 0$ and $S = 0$ contours in the μ - n space for a cylinder of overall radius 0.002 m subjected to pure torsion. The variable S refers to the second-order radial displacement at the outer surface normalized with respect to the overall radius. In Figs. 5.2(a) and (b), the cylinders are homogeneous but with different elastic constants. In Fig. 5.2(c), the cylinder is a composite with Layers 1 and 2 respectively made of the materials in Figs. 5.2 (a) and (b).

For the homogeneous cylinder in Fig. 5.2(a), the μ - n space (holding λ and m fixed) is seen to be partitioned into three regions, indicated by I, III and IV. Regions I and II indicate regions of positive Poynting effect ($D > 0$), but with $S > 0$ and $S < 0$, respectively. Regions III and IV indicate regions of negative Poynting effect ($D < 0$), but with $S > 0$ and $S < 0$, respectively. This figure shows the absence of Region II within the μ - n space, i.e., the homogeneous cylinder cannot elongate axially and contract radially within the parameter range investigated. Roughly speaking, the negative Poynting effect is predicted for negative and relatively small positive n in this case. In Fig. 5.2(b), however, Region I is missing and the homogeneous cylinder cannot elongate axially and expand radially at the same time.

The influence of inhomogeneity via the introduction of two concentric layers in the cylinder of the same overall size ($r_1 = 0.001$ m, $r_2 = 0.002$ m) is illustrated in Fig. 5.2(c). Here the $D = 0$ and $S = 0$ contours are plotted in the μ_2 - n_2 space. All the elastic constants of Layer 1 (inner layer) are held fixed. Also, λ_2 and m_2 of Layer 2 are held fixed at the same values as those of the homogeneous material in Fig. 5.2(b). For the same range of parameter values, it can be seen that the regions in Figs. 5.2(a) and (b)

Fig. 5.3. Contours of the Poynting effect coefficient $D = 0$ (solid line) and radial displacement $S = 0$ (dashed line) in (a) $\mu_2 - \mu_1$ space and (b) $\mu_1 - m_2$ space for two different two-layer composites under torsion.

have fused and rearranged themselves so that all four regions are now present. Given the existence of the four regions, there will be either volumetric expansion or contraction of the cylinder in general. This figure shows the possibility of using elastic inhomogeneity to design for an overall positive or negative Poynting effect with radial (or volumetric) expansion or contraction in a cylinder under torsion.

As a further illustration of the complex dependence on elastic inhomogeneity, Figs. 5.3(a) and (b) plot the $D = 0$ and $S = 0$ contours over the $\mu_1-\mu_2$ and μ_1-m_2 space respectively, keeping the remaining parameters fixed as shown in Table 5.1. Fig. 5.3(a) shows that all four regions can be present through the proper selection of only the shear moduli of the two materials, although the space is dominated by Regions II and III. In contrast, Fig. 5.3(b) shows that a completely different distribution of the regions emerges when the space is spanned by a second-order parameter μ_1 and a third-order elastic parameter m_2 . Region II is missing and the space is dominated by Regions III and IV. Fig. 5.3 shows the richness of deformation modes under the influence of elastic inhomogeneity.

Figs. 5.4(a), (b) and (c) plot the variation of D with r_1 for three materially different soft composites of the same overall size ($r_2 = 0.002$ m, $L = 0.01$ m) subjected to the torque $T = 6 \times 10^{-10}$ kPa•m³. In each sub-figure, the solid and dotted lines denote the results for a composite and its reverse configuration, respectively, i.e., the materials of Layers 1 and 2 are interchanged in the two composites. At $r_1 = 0$, the composite degenerates into a homogeneous cylinder made up completely of Layer 2 material. Similarly, at $r_1 = r_2$ the homogeneous cylinder is composed of the material of

Fig. 5.4. Variation of the Poynting effect coefficient D with the interface position r_1 for three different two-layer composites (solid line) and their reverse configurations (dashed line) under pure torsion.

Fig. 5.5. Variation of the Poynting effect coefficient D with the interface position r_1 for three different two-layer composites of metals (solid line) and reverse configurations (dashed line) under pure torsion.

Table 5.1. Elastic, geometrical and loading parameters used in Figs. 5.2–5.5.

Note: poly(acrylic acid)= PAA, *N*-isopropylacrylamide = NIPA, sodium acrylate = SA

	Fig. 5.2(a)	Fig. 5.2(b)	Fig. 5.2(c)	Fig. 5.3(a)	Fig. 5.3(b)	Fig. 5.4(a) solid	Fig. 5.4(a) dashed	Fig. 5.4(b) solid
Material 1	Agar-gelatin-based	PAA gel-based	Agar-gelatin-based	Agar-gelatin-based	PAA gel-based	Agar-gelatin-based	Polymer-based	Agar-gelatin-based
λ_1 (kPa)	2.25×10^6	60	2.25×10^6	2.25×10^6	60	2.5×10^4	1000	2.25×10^6
μ_1 (kPa)			8			2000	200	800
m_1 (kPa)	-20×10^6	-42.67	-20×10^6	-20×10^6	-42.67	-20×10^6	-2450	10^9
n_1 (kPa)			80	80	-23.50	8	-2350	8
Material 2			PAA gel-based	PAA gel-based	Agar-gelatin-based	Polymer-based	Agar-gelatin-based	Polymer-based
λ_2 (kPa)			60	60	2.25×10^6	1000	2.5×10^4	1000
μ_2 (kPa)					8	200	2000	200
m_2 (kPa)			-42.67	-42.67		-2450	-20×10^6	-2450
n_2 (kPa)				-23.50	80	-2350	8	-2350
r_1 (m)		0.002	0.001	0.001	0.001			
r_2 (m)			0.002	0.002	0.002	0.001	0.001	0.001
L (m)		0.01	0.01	0.01	0.01	0.01	0.01	0.01
T (kPa $\cdot$ m ³)		6×10^{-10}	6×10^{-10}	6×10^{-10}	6×10^{-10}	6×10^{-10}	6×10^{-10}	6×10^{-10}

	Fig. 5.4(b) dashed	Fig. 5.4(c) solid	Fig. 5.4(c) dashed	Fig. 5.5(a) solid	Fig. 5.5(a) dashed	Fig. 5.5(b) solid	Fig. 5.5(b) dashed
Material 1	Polymer-based	Agar-gelatin	NIPA/SA gel-based	Ag	Cu	Al	Cu
λ_1 (kPa)	1000	2.25×10^6	40	91×10^3	121×10^3	60.41×10^3	121×10^3
μ_1 (kPa)	200	8	3.983	45×10^3	75.5×10^3	28.34×10^3	75.5×10^3
m_1 (kPa)	-2450	-20×10^6	-32.44	-637×10^3	-780×10^3	-383.9×10^3	-780×10^3
n_1 (kPa)	-2350	80	-2000	332×10^3	-380×10^3	15.4×10^3	-380×10^3
Material	Agar-gelatin-	NIPA/SA	Agar-gelat	Cu	Ag	Cu	Al

2	based	gel -based	in				
λ_2 (kPa)	2.25×10^6	40	2.25×10^6	121×10^3	91×10^3	121×10^3	60.41×10^3
μ_2 (kPa)	800	3.983	8	75.5×10^3	45×10^3	75.5×10^3	28.34×10^3
m_2 (kPa)	10^9	-32.44	-20×10^6	-780×10^3	-637×10^3	-780×10^3	-383.9×10^3
n_2 (kPa)	8	-2000	80	-380×10^3	332×10^3	-380×10^3	15.4×10^3
r_1 (m)							
r_2 (m)	0.001	0.001	0.001	0.001	0.001	0.001	0.001
L (m)	0.01	0.01	0.01	0.01	0.01	0.01	0.01
T (kPa $\cdot$ m ³)	6×10^{-10}	6×10^{-10}	6×10^{-10}	6×10^{-10}	6×10^{-10}	6×10^{-10}	6×10^{-10}

	Fig. 5.5(c) solid	Fig. 5.5(c) dashed
Material 1	TiAl	Fe
λ_1 (kPa)	105×10^3	110×10^3
μ_1 (kPa)	120×10^3	82×10^3
l_1 (kPa)		
m_1 (kPa)	-40×10^3	-1030×10^3
n_1 (kPa)	-160×10^3	1100×10^3
Material 2	Fe	TiAl
λ_2 (kPa)	110×10^3	105×10^3
μ_2 (kPa)	82×10^3	120×10^3
l_2 (kPa)		
m_2 (kPa)	-1030×10^3	-40×10^3
n_2 (kPa)	1100×10^3	-160×10^3
r_1 (m)		
r_2 (m)	0.001	0.001
L (m)	0.01	0.01
T (kPa $\cdot$ m ³)	6×10^{-10}	6×10^{-10}

Layer 1.

Several interesting and unusual phenomena can be observed. First, the curve may not vary monotonically between the two end points $r_1 = 0$ and $r_1 = r_2$, implying that the Poynting effect is *amplified* beyond that of the homogenous cylinders. This is evident in Figs. 5.4(a) and (b), but not in Fig. 5.4(c). As a case in point, consider the solid line of Fig. 5.4(b), where $D \approx +0.13 \times 10^{-5}$ and -0.83×10^{-5} for homogeneous cylinders of Layers 2 and 1, respectively. Joining these two materials into a composite of the same overall size, however, may result in $D \approx -2 \times 10^{-5}$, depending on the interface position. Second, the order in which the two materials are placed in the composite may also strongly influence the Poynting effect. In Fig. 5.4(b), reversing the makeup of the layer materials results in $D \approx -3.8 \times 10^{-5}$ (dotted line), almost twice that of the original configuration. Third, this amplification effect is not always present. Fig. 5.4(c) is an example of two materials whose composite does not have a Poynting effect stronger than the homogeneous ones. Finally, a minor point is that there may exist multiple interface positions where the Poynting effect is the same regardless of the order of the layers, as indicated by the intersection of the solid and dotted curves.

We further explore the above behaviors for several metallic composites. Fig. 5.5 plots the results for three different metal combinations. The metals in Layers 1 and 2 are silver and copper, aluminum and copper, and TiAl alloy and Armco iron for the solid lines in the three sub-figures, respectively. The dotted lines show the results for the reverse configurations. It can be observed that the curves vary monotonically

between their end points; there is no amplification of the Poynting effect. Another point worthy of note is that D is of the order of 10^{-11} to 10^{-12} , much smaller compared to the range of 10^{-2} to 10^{-5} for the soft materials investigated in Fig. 5.4. For a homogeneous cylinder of TiAl intermetallic, D is negative as shown in Fig. 5.5(c).

5.5. Discussion

The Poynting effect has mostly been investigated for homogeneous materials. The results are relevant for composite materials under complex loading. They have many applications as discussed below.

First, many natural biomaterials are composites, architecturally hierarchical, and are often subjected to complex multiaxial loads in their physiological environment. Criscione et al. (1999) conducted finite extension and torsion experiments on papillary muscles in order to characterize their constitutive behavior. They modeled the material as a bi-layered composite consisting of a core of myocardium inside a thin sheath of endocardium. We determined the third-order elastic constants l , m and n for both the core and the sheath. Haberl et al. (2004) investigated experimentally the kinematic response of lumbar spinal units subjected to torsion, compression and flexion/extension. These units can be considered as a composite material with a soft material (nucleus pulposus) between them. Tóth et al. (2005) studied the mechanical parameters of arterial walls, specifically related to human brain aneurysm (bulge

along a blood vessel). The authors modeled the arterial wall as a three-layer cylindrical composite subjected to combined extension, torsion and inflation (which mimic real physiological conditions).

Understanding of the mechanics of natural biomaterials in their operating environment is crucial to the development of artificial tissues and organs in the field of medicine. Tissue and organ damage or failure is a major health issue and there has been intensive research in the field (Persidis, 1999; Danoux et al., 2013). The results are relevant to the design of bio-mimicking composite polymers, natural and/or synthetic, for use as replacement tissues.

Soft biomaterials such as collagen, fibrin, vimentin, neurofilaments and actin demonstrate the Poynting effect, as pointed out by Janmey et al. (2007). In the movement of micrometer-sized mitochondria through the cytoskeleton of a narrow axon, the negative Poynting effect (which generates negative normal stress) may facilitate organelle motion without distending the axon diameter. The large stresses generated by the Poynting effect can cause significant effect on the overall force balance in the cytoskeleton under shear or torsion. Studies of the Poynting effect are therefore of key interest.

Second, the results may assist in the design of medical instruments. A key requirement in the development of high-fidelity surgical simulators is the realistic modeling of the interaction between surgical instruments and human organs. Human organs are generally inhomogeneous and multilayered. The Poynting effect can result in differences in force or displacement magnitude larger than the absolute

human perception threshold for force or displacement discrimination in some tissues (e.g. myocardial tissues) but not in others (e.g. brain tissues), as described in Misra et al. (2010). Using our results, the magnitude of the forces or displacements of multilayered human organs generated by the Poynting effect can be predicted.

Third, the Poynting effect can be utilized in the design and operation of actuators and sensors. A bio-inspired polymer composite actuator can be made, based on the phenomenon that an input torsion may generate a large output axial displacement, which can be used to actuate a connecting device. According to our results, materials parameters and interface positions can be used to adjust (and possibly maximize) the ratio of output displacements to input torsion. Similarly, a torsionally stressed actuator, if subjected to a further input axial force, may generate an additional output twist, according to the axial force–twist effect. The amount of twist can likewise be controlled by the material elasticity and the interface design. Conversely, a bio-inspired polymer sensor can be designed for sensing torsional moments based on the Poynting effect, i.e., measurements of the axial length change can be used to detect and indeed predict the magnitude of the torsional moment.

The celebrated Poynting effect, in which a cylindrical specimen extends under a torque, is a second-order effect that has been well-investigated for homogeneous materials. This chapter develops expressions, i.e., Eqs. (5.29) and (5.57), which permit a prediction of the Poynting effect for both homogeneous and composite cylinders, respectively. Eq. (5.29) shows that the Poynting effect can be predicted by the four elastic constants λ , μ , m and n as well as T and R . Similarly, Eq. (5.57)

requires the elastic constants λ_i , μ_i , m_i and n_i as well as T and r_i for a prediction. However, the elastic constants are lumped together as a single constant in the homogeneous case, while they are dispersed by r_i in the composite case. A careful examination of Eq. (5.57) reveals that, although it is nonlinear in λ_i , μ_i , it is strictly linear in m_i and n_i (no terms involving $m_1 m_2$, $m_1 n_2$, m_1^2 , etc.). There are at least two implications as discussed below.

The Poynting effect decreases strongly with the cylinder radius ($1/R^6$) in the homogeneous case. The decrease is expected as it is more difficult to twist a thicker cylinder and the corresponding axial displacement would be smaller. The inverse sextic relation shows that the Poynting effect is two orders of magnitude smaller than the twist produced under torsion. The strong scale importance also suggests that the Poynting effect becomes much more significant at the micro- or nano-scale. This is in general also true for a composite cylinder but the interaction between the layer materials may result in anomalous behavior, e.g., an initial increase in the Poynting effect as the overall radius increases with the radius of the inner layer fixed. This is possible because the outer layer could have a stronger Poynting effect and increasing its size relative to the inner layer could lead to an initial increase in the effect.

Eq. (5.57) may be employed usefully in a reverse manner, i.e., determination of the elastic constants of the two layers, especially the third-order ones, by measuring the coefficient D of composites having various combinations of r_i . The interspersed nature of the elastic constants and the radii results in a system of algebraic equations nonlinear in the second-order constants and linear in the third-order constants. These

equations can be solved for the unknown elastic constants. A simple illustration of this is the determination of the third-order constants m_i and n_i for the two layers, assuming that the second-order constants (the Lamé constants, or the usual Young's moduli and Poisson ratios) have been measured by other methods. By measuring the Poynting effect coefficients for four different combinations of r_i , a simple system of four linear algebraic equations where the four unknowns m_i and n_i can be solved easily. Provided that the Poynting effect is significant for the soft materials and amenable to measurements, and provided that composites can be fabricated from the constituent materials, this may offer an alternative method for the determination of third-order elastic constants, usually measured via acousto-elastic methods (Catheline *et al.*, 2003; Destrade *et al.*, 2010).

We remark that the analytical solutions for periodic and non-periodic multilayered cylinders can also be developed in the same manner as done for the bilayered composite. The solutions of the differential equations of equilibrium are not complicated, but the constants in the solutions will contain complex combinations of the elastic and geometric parameters. However, this is acceptable because the constants in the differential equations can be evaluated numerically with ease before the equations are actually solved.

5.6. Conclusions

The second-order effects in homogeneous and bilayered cylinders under pure

torsion have been investigated within the framework of second-order elasticity. The major contributions and findings can be summarized as follows.

First, analytical solutions have been derived for homogeneous and bilayered cylinders under torsion, starting consistently from the equilibrium equations for the first Piola-Kirchhoff stress.

Second, the second-order effects in the displacement and stress fields have been identified, and their dependence on the elastic constants and geometrical parameters has been studied in detail.

Third, the mechanics of the Poynting effect of a composite has been investigated in some depth. Explicit Poynting effect coefficients are given for both homogeneous and bilayered composites. The key findings include: an amplified Poynting effect is possible above that of homogeneous materials, depending on the interface position; the effect is also dependent on the material configuration within the composite; the dependence of the effect on the overall cylinder size obeys an inverse sextic law.

Fourth, the analytical solutions obtained may be used as the theoretical basis for the experimental estimation of third-order elastic constants.

Chapter 6 Combined Axial-Torsional Loadings

The problem of elastically nonlinear cylinders subjected to combined axial and torsional loadings has been studied experimentally and theoretically. Criscione *et al.* (1999) investigated the twist rate of cardiac capillary muscles under a constant axial stretch as a function of the twisting moment. Kanner and Horgan (2008) and Horgan and Murphy (2012) have recently investigated the extension-torsion of isotropic and transversely isotropic incompressible soft materials. However, a theoretical treatment of composites has received much less attention. Here, the investigation focuses on the combined torsion-axial loading of both homogeneous and bilayered cylinders, where it will be shown that the solutions for the combined case cannot be obtained from a simple superposition of the solutions for the pure loading cases. A coupled axial force-twist effect under combined loading, i.e., the twist of a torsionally loaded cylinder can be affected by the axial loading, is found. Comparison of the predictions with the torque-tension-twist data for cardiac papillary muscles shows reasonable agreement. The solutions also provide the basis for a mechanistic method of determining third-order elastic constants.

The chapter is organized as follows. The problem is defined in Section 6.1. The governing equations are developed in Section 6.2. Solutions are given in Section 6.3, while the numerical results are presented in Section 3. A further discussion is provided in Section 6.4, and the conclusions of this work are given in Section 6.5.

6.1. Problem definition

Consider a multilayered cylinder with concentric layers and radii r_i , $i = 1, N$ as shown in Fig. 6.1. Under combined torsion T and axial loading P , the final coordinates (ρ, ψ, ζ) of the particle of a cylinder whose initial coordinates are (r, θ, z) are assumed to be $(r + kF(r), \theta + k\alpha z + kG(z), z + k(P/E)z + k^2(P/E)^2\delta z + kH(z))$, where $kF(r)$, $k\alpha z + kG(z)$, $k(P/E)z + k^2(P/E)^2\delta z + kH(z)$ respectively are the radial, angular and axial displacements. Here α denotes the usual angle of twist per unit length under pure torsion. As shown previously, $\alpha = 2T / \pi R^4 \mu$ for a homogeneous cylinder and $\alpha_1 = \alpha_2 = \alpha = 2T / \pi(r_1^4 \mu_1 - r_1^4 \mu_2 + r_2^4 \mu_2)$ for a bilayered composite. Also, $E = \mu(3\lambda + 2\mu) / (\lambda + \mu)$ is the usual Young's modulus related to the Lamé constants λ and μ , and δ is the parameter appearing in the expression for the second-order axial displacement under pure axial loading, (Eq. (4.35) in Chapter 4). As before, k is a marker indicating the order of approximation of the theory. Furthermore, the unknown radial displacement $F(r) = F_1(r) + kF_2(r)$, where $F_1(r)$ and $F_2(r)$ are respectively the first- and second-order terms. These terms are assumed to be functions of r only and not of z since all the sections are twisted in the same manner. Similarly, the unknown function $G(z) = G_1(z) + kG_2(z)$, where $G_1(z)$ and $G_2(z)$ are respectively the first- and second-order terms. It is taken to be a function of z . Similarly, the unknown function $H(z) = H_1(z) + kH_2(z)$, where $H_1(z)$ and $H_2(z)$ are respectively the first- and second-order terms. They are taken to be function of z only and not of r since the bar sections elongate or contract uniformly.

Fig. 6.1. A cylindrical composite consisting of N layers, with surfaces located at the radial coordinates, $r = r_i, i = 1, N$, under combined torsion T and axial loading P .

It is noted that $G(z)$ in the circumferential displacement permits the additional rotation of a cylinder when an axial loading P is imposed. A non-vanishing $G(z)$ would predict an axial force-twist effect under combined torsion-axial loading. It is

established previously via the equilibrium equations that $G(z) = 0$ for both homogeneous and bilayered composites under pure loading modes. On the other hand, $H(z)$ in the axial displacement term permits an axial displacement under torsion and hence captures the Poynting effect.

6.2. Governing equations

The differentials of position vectors $\mathbf{a}$ and $\mathbf{x}(\mathbf{a})$ in the undeformed and deformed states can be respectively written as:

$$d\mathbf{a} = \begin{pmatrix} dr \\ r d\theta \\ dz \end{pmatrix} \quad \text{and} \quad d\mathbf{x} = \begin{pmatrix} (1+kF')dr \\ (r+kF)(d\theta + k\alpha dz + kG'dz) \\ (1+k\frac{P}{E} + k^2\frac{P^2}{E^2}\delta + kH')dz \end{pmatrix}, \quad (6.1)$$

where $F' = dF / dr = F_1'(r) + kF_2'(r)$, $G' = dG / dz = G_1'(z) + kG_2'(z)$ and $H' = dH / dz = H_1'(z) + kH_2'(z)$. The deformation gradient can be written as:

$$\mathbf{F} = \frac{d\mathbf{x}}{d\mathbf{a}} = \begin{pmatrix} 1+kF' & 0 & 0 \\ 0 & 1+\frac{kF}{r} & (r+kF)(kG'+k\alpha) \\ 0 & 0 & 1+k\frac{P}{E} + k^2\frac{P^2}{E^2}\delta + kH' \end{pmatrix}. \quad (6.2)$$

The stresses related to the displacements can be obtained by substituting Eq. (6.2)

into Eq. (3.3). Substituting the stresses into the equilibrium equations Eqs.

(4.3)–(4.5) with $\frac{\partial \psi}{\partial r} = 0$, $\frac{\partial T_{r\theta}}{\partial \theta} = 0$, $\frac{\partial \psi}{\partial \theta} = 1$, $\frac{\partial \psi}{\partial z} = k\alpha + kG'$ yields three

first-order and three second-order equilibrium equations as distinguished by the k

and k^2 multipliers:

$$k\left(\frac{F_1''(r)}{r} + \frac{F_1'(r)}{r} - \frac{F_1'(r)}{r^2}\right) = 0, \quad (6.3)$$

$$kG_1''(z) = 0, \quad (6.4)$$

$$kH_1''(z) = 0, \quad (6.5)$$

$$\begin{aligned} & k^2(-4\mu(3\lambda + 2\mu)(2l + 2m + 2\lambda + 3\mu)F_1(r)^2 - 4r(2l + \lambda)F_1(r)(P(\lambda + \mu) + \\ & \mu(3\lambda + 2\mu)H_1'(z) - r\mu(3\lambda + 2\mu)F_1''(r)) + r(-4\mu(3\lambda^2 + 8\lambda\mu + 4\mu^2)F_2(r) + \\ & r(4\mu(3\lambda + 2\mu)(2l + 2m + 2\lambda + 3\mu)F_1'(r)^2 + 4\mu(3\lambda^2 + 8\lambda\mu + 4\mu^2)F_2'(r) + \\ & 4F_1'(r)(P(2l + \lambda)(\lambda + \mu) + (2l + \lambda)\mu(3\lambda + 2\mu)H_1'(z) + \\ & r\mu(3\lambda + 2\mu)(2l + 4m + 3\lambda + 6\mu)F_1''(r)) + r(2r\alpha(4m - 3n + 4\lambda - 8\mu)\mu(3\lambda + 2\mu)G_1'(z) + \\ & r(4m - 3n + 4\lambda - 8\mu)\mu(3\lambda + 2\mu)G_1'(z)^2 + 4(2l + \lambda)(P(\lambda + \mu) + \\ & \mu(3\lambda + 2\mu)H_1'(z))F_1''(r) + \mu(3\lambda + 2\mu)(r\alpha^2(4m - 3n + 4\lambda - 8\mu) + 4(\lambda + 2\mu)F_2''(r)))))) = 0, \end{aligned} \quad (6.6)$$

$$\begin{aligned} & k^2\left(\frac{1}{2\mu(3\lambda + 2\mu)}(2\mu(3\lambda + 2\mu)(m + \lambda + 3\mu)F_1(r)G_1''(z) + \right. \\ & r((2m - n + 2\lambda)\mu(3\lambda + 2\mu)F_1'(r)G_1''(z) + 2((m + \lambda + 2\mu)(P(\lambda + \mu) + \\ & \mu(3\lambda + 2\mu)H_1'(z))G_1''(z) + \alpha\mu(m + \lambda + 2\mu)(3\lambda + 2\mu)H_1''(z) + \\ & \mu(m + \lambda + 2\mu)(3\lambda + 2\mu)G_1'(z)H_1''(z) + 3\lambda\mu^2G_2''(z) + 2\mu^3G_2''(z)))))) = 0, \end{aligned} \quad (6.7)$$

$$\begin{aligned}
& k^2(r^3\mu(m+\lambda+2\mu)(3\lambda+2\mu)(\alpha+G_1'(z))G_1''(z) + \\
& ((2l+\lambda)\mu(3\lambda+2\mu)F_1(r)+r((2l+\lambda)\mu(3\lambda+2\mu)F_1'(r)+ \\
& (2l+4m+3\lambda+6\mu)(P(\lambda+\mu)+\mu(3\lambda+2\mu)H_1'(z))))H_1''(z) + \\
& r\mu(3\lambda^2+8\lambda\mu+4\mu^2)H_2''(z))=0.
\end{aligned} \tag{6.8}$$

We next solve these equilibrium equations for the unknown functions F_1, F_2, G_1, G_2, H_1 and H_2 , for both the homogeneous and the bilayered cases.

6.3. Solutions

6.3.1. Solutions of a homogeneous cylinder under combined axial-torsional loadings

Let the radius of the cylinder be denoted by R . Solving the linear equilibrium equations Eqs (6.3)–(6.5), F_1, G_1 and H_1 can be obtained:

$$F_1(r) = F_{11}r + \frac{F_{12}}{r}, \tag{6.9}$$

$$G_1(z) = G_{11}z, \tag{6.10}$$

$$H_1(z) = H_{11}z, \tag{6.11}$$

where $F_{11}, F_{12}, G_{11}, H_{11}$ are constants to be determined. However, $F_{12} = 0$ since the radial displacement vanishes at $r = 0$. For traction boundary conditions, the relevant tractions can be calculated from Eq. (3.3). In order to determine F_{11}, G_{11} and H_{11} , the following three boundary conditions are applied:

(1) The first-order radial stress T_{rr}^L vanishes at the outer boundary:

$$T_{rr}^L(r = R) = 0, \quad (6.12)$$

(2) The applied forces acting at the end faces of the cylinder are $\pi R^2 P$:

$$\int T_{zz}^L(z = 0, L) dA = \pi R^2 P, \quad (6.13)$$

(3) The torsion applied to the cylinder is T :

$$\int T_{\theta z}^L r dA = T. \quad (6.14)$$

The superscript “ L ” denotes the linear or first-order part of the component.

Application of the previous three equations determines the constants:

$$F_{11} = -\frac{P\lambda}{2\mu(3\lambda + 2\mu)}, \quad H_{11} = 0, \quad G_{11} = 0. \quad (6.15)$$

Hence, the linear radial, circumferential and axial displacements can be written as:

$$u_r^L = kF_1(r) = -k \frac{P\lambda}{2\mu(3\lambda + 2\mu)} r, \quad u_\theta^L = kr\alpha z = k \frac{2T}{\pi R^4 \mu} rz, \quad u_z^L = k \frac{P}{E} z, \quad (6.16)$$

which show that the first-order displacements in homogenous material under combined torsion-axial loading are a simple addition of the corresponding first-order displacements under pure torsion and pure axial loading. Since $G_{11} = 0$, an axial force-twist effect does not exist in a homogeneous material under combined torsion-axial loading in the first-order elastic theory. The first-order stress field can

be written as:

$$T^L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & k \frac{2Tr}{\pi R^4} \\ 0 & k \frac{2Tr}{\pi R^4} & kP \end{pmatrix}. \quad (6.17)$$

This stress field is also a simple addition of the first-order stress fields for pure torsion and pure axial loading cases.

Next, the second-order solutions for the combined torsion-axial loading of a homogeneous cylinder are presented. After substituting the linear results into the second-order equilibrium equations Eqs. (6.6)–(6.8), they reduce to:

$$k^2 \{ r^3 \alpha^2 (4m - 3n + 4\lambda - 8\mu) - 4(\lambda + 2\mu) [F_2(r) - rF_2'(r) - r^2 F_2''(r)] \} = 0, \quad (6.18)$$

$$k^2 G_2''(z) = 0, \quad (6.19)$$

$$k^2 H_2''(z) = 0. \quad (6.20)$$

Thus, F_2 , G_2 and H_2 can be obtained:

$$F_2(r) = -\frac{r^3 (4m\alpha^2 - 3n\alpha^2 + 4\alpha^2\lambda - 8\alpha^2\mu)}{32(\lambda + 2\mu)} + F_{21}r + \frac{F_{22}}{r}, \quad (6.21)$$

$$G_2(z) = G_{21}z, \quad (6.22)$$

$$H_2(z) = H_{21}z. \quad (6.23)$$

where F_{21} , F_{22} , G_{21} , H_{21} are constants to be determined. As in the above, $F_{22} = 0$.

The following boundary conditions allow us to solve for the values of the three remaining constants:

(1) The second-order radial stress T_{rr}^{NL} vanishes at the outer boundary:

$$T_{rr}^{NL}(r = R) = 0, \quad (6.24)$$

(2) The second-order axial stress does not contribute to the applied axial forces acting at the ends of the cylinder:

$$\int T_{zz}^{NL}(z = 0, L) dA = 0, \quad (6.25)$$

(3) The contribution of the second-order shear stress $T_{\theta z}^{NL}$ to T is 0:

$$\int T_{\theta z}^{NL}(z = L) r dA = 0, \quad (6.26)$$

The superscript “NL” denotes the nonlinear or second-order part of the component.

The constants F_{21} , G_{21} , H_{21} can be solved as:

$$\begin{aligned} F_{21} = & \frac{1}{32\mu^3(\lambda + 2\mu)(3\lambda + 2\mu)^3} \{ -4\mu[R^2\alpha^2\mu^2(3\lambda + 2\mu)^2(-m\lambda - \lambda^2 + 2m\mu + 12\lambda\mu + 12\mu^2) \\ & - P^2(\lambda + 2\mu)(9\lambda^3 + 18\lambda^2\mu - 8\lambda\mu^2 + 8\lambda\mu^2)] + n[R^2\alpha^2\mu^2(3\lambda + 2\mu)^2(\lambda^2 - 5\lambda\mu - 2\mu^2) \\ & + 4P^2\lambda(3\lambda^3 + 12\lambda^2\mu + 16\lambda\mu^2 + 8\mu^3)] \}, \end{aligned} \quad (6.27)$$

$$G_{21} = -\frac{PT(n\lambda + 4\mu m + 6\lambda\mu + 8\mu^2)}{2\pi R^4\mu^3(3\lambda + 2\mu)}, \quad (6.28)$$

$$\begin{aligned}
H_{21} = & -\frac{1}{16\mu^3(3\lambda+2\mu)^3} \{n\lambda(12P^2\lambda(\lambda+\mu) + R^2\alpha^2\mu^2(3\lambda+2\mu)^2) + \\
& 4\mu(R^2\alpha^2\mu^2(\lambda+2\mu)(3\lambda+2\mu)^2 + m(3\lambda+2\mu)^2(2P^2 + R^2\alpha^2\mu^2) + \\
& 2P^2((9+6\delta)\lambda^3 + 8(3+2\delta)\lambda^2\mu + 7(3+2\delta)\lambda\mu^2 + 2\mu^2(l + (3+2\delta)\mu)))\}.
\end{aligned} \tag{6.29}$$

With these constants, the displacement field can be written as:

$$\begin{aligned}
u_r = kF(r) = kF_1(r) + k^2F_2(r) = & -\frac{kPr\lambda}{6\lambda\mu + 4\mu^2} + \\
& k^2\left(\frac{1}{8\pi^2R^8\mu^3(\lambda+2\mu)(3\lambda+2\mu)}rT^2(4\mu(-(3r^2 - R^2)\lambda(m+\lambda) - 2(m(r^2 + R^2) - \right. \\
& 2(r^2 - 3R^2)\lambda)\mu + 4(r^2 - 3R^2)\mu^2) + n(3r^2\mu(3\lambda+2\mu) + R^2(\lambda^2 - 5\lambda\mu - 2\mu^2))) + \\
& \left.\frac{1}{8\mu^3(3\lambda+2\mu)^3}(P^2r(n\lambda(3\lambda^2 + 6\lambda\mu + 4\mu^2) + \mu(9\lambda^3 + 18\lambda^2\mu - 8l\mu^2 + 8\lambda\mu^2))))\right)
\end{aligned} \tag{6.30}$$

$$\begin{aligned}
u_\theta = k\alpha z + krG(z) = k\alpha z + krG_1(z) + k^2rG_2(z) \\
= kr\frac{2T}{\pi R^4\mu}z - \frac{k^2PT(n\lambda + 4\mu m + 6\lambda\mu + 8\mu^2)}{2\pi R^4\mu^3(3\lambda+2\mu)}rz,
\end{aligned} \tag{6.31}$$

$$\begin{aligned}
u_z = k\frac{P}{E}z + k^2\left(\frac{P}{E}\right)^2\delta z + kH(z) = & \frac{kP(\lambda+\mu)z}{\mu(3\lambda+2\mu)} - \\
& \frac{1}{4\pi^2R^6\mu^3(3\lambda+2\mu)^3}(k^2z(n\lambda(3P^2\pi^2R^6\lambda(\lambda+\mu) + T^2(3\lambda+2\mu)^2) + \\
& 2\mu(m(P^2\pi^2R^6 + 2T^2)(3\lambda+2\mu)^2 + 2T^2(\lambda+2\mu)(3\lambda+2\mu)^2 + \\
& P^2\pi^2R^6(9\lambda^3 + 24\lambda^2\mu + (2l+21\lambda)\mu^2 + 6\mu^3))))).
\end{aligned} \tag{6.32}$$

From the above solutions, it can be seen that u_r and u_z are simple additions of the corresponding terms under pure torsion and pure axial loading. However, u_θ is not a simple addition of the expressions for the pure loading modes. An extra non-vanishing coupled term

$-(k^2 PT(n\lambda + 4\mu m + 6\lambda\mu + 8\mu^2)rz) / (2\pi R^4 \mu^3 (3\lambda + 2\mu))$ exists, which means that the circumferential displacement will change by this value when an axial loading P is applied in the presence of a torsion T . This coupled term vanishes when $T = 0$, i.e., an axial force-twist effect is not possible under pure axial loading.

The stress field can be written as:

$$T_{rr} = \frac{k^2(r-R)(r+R)T^2(n(2\lambda + \mu) + 4\mu(m + 5\lambda + 6\mu))}{4\pi^2 R^8 \mu^2 (\lambda + 2\mu)}, \quad (6.33)$$

$$T_{r\theta} = T_{\theta r} = 0, \quad (6.34)$$

$$T_{rz} = T_{zr} = 0, \quad (6.35)$$

$$T_{\theta\theta} = \frac{k^2 T^2 (n(3r^2 - R^2)(2\lambda + \mu) + 4\mu(m(3r^2 - R^2) - R^2(5\lambda + 6\mu) + r^2(11\lambda + 10\mu)))}{4\pi^2 R^8 \mu^2 (\lambda + 2\mu)}, \quad (6.36)$$

$$T_{\theta z} = k \frac{2Tr}{\pi R^4}, \quad (6.37)$$

$$T_{z\theta} = k \frac{2Tr}{\pi R^4} - k^2 \frac{PTr}{\pi R^4 \mu}, \quad (6.38)$$

$$T_{zz} = kP + k^2 \frac{(2r^2 - R^2)T^2(3n\lambda + 8\mu m + 24\lambda\mu + 16\mu^2)}{4\pi^2 R^8 \mu^2 (\lambda + 2\mu)}. \quad (6.39)$$

The stress field in a homogeneous cylinder under combined torsion and axial loading is not symmetric because $T_{\theta z}$ is not equal to $T_{z\theta}$. There also exists a coupling term in $T_{z\theta}$, which means that the value of this stress is not a simple addition of those for pure torsion and pure axial loading. No coupling terms exist in all other stress components.

6.3.2. Solutions of a bilayered cylindrical composite under combined axial-torsional loadings

Considering a bilayered cylinder, the solutions for the first-order radial and axial displacements $F_1^{(i)}(r)$, $G_1^{(i)}(r)$ and $H_1^{(i)}(r)$, $i = 1, 2$, are:

$$F_1^{(i)}(r) = F_{11}^{(i)}r + \frac{F_{12}^{(i)}}{r}, \quad (6.40)$$

$$G_1^{(i)}(z) = G_{11}^{(i)}z, \quad (6.41)$$

$$H_1^{(i)}(z) = H_{11}^{(i)}z, \quad (6.42)$$

where i indicates the layer and eight unknown constants $F_{11}^{(i)}$, $F_{12}^{(i)}$, $G_{11}^{(i)}$ and $H_{11}^{(i)}$ are to be determined. For Layer 1, however, $F_{12}^{(1)} = 0$ since the radial displacement vanish at $r = 0$. Hence, there remain seven unknowns to be determined from the following seven boundary conditions:

(1) First-order radial displacement is continuous across the interface:

$$F_1^{(1)}(r = r_1) = F_1^{(2)}(r = r_1), \quad (6.43)$$

(2) First-order circumferential displacement is continuous across the interface:

$$G_{11}^{(1)}z = G_{11}^{(2)}z, \text{ or } G_{11}^{(1)} = G_{11}^{(2)}, \quad (6.44)$$

(3) First-order axial displacement is continuous across the interface:

$$\frac{P_1}{E_1} z + H_{11}^{(1)} z = \frac{P_2}{E_2} z + H_{11}^{(2)} z \quad \text{or} \quad \frac{P_1}{E_1} + H_{11}^{(1)} = \frac{P_2}{E_2} + H_{11}^{(2)}, \quad (6.45)$$

where P_1 and P_2 are the axial stresses acting in Layers 1 and 2, respectively,

(4) First-order radial traction T_{rr}^L is continuous across the interfaces:

$$T_{rr}^{L(1)}(r = r_1) = T_{rr}^{L(2)}(r = r_1), \quad (6.46)$$

(5) T_{rr}^L vanishes at the outer surface:

$$T_{rr}^{L(2)}(r = r_2) = 0, \quad (6.47)$$

(6) The applied force acting at either end of the cylinder is $\pi r_2^2 P$:

$$\int T_{zz}^{L(i)}(z = L) dA = \pi r_2^2 P, \quad (6.48)$$

(7) The torsion applied to the cylinder is T :

$$\int T_{\theta z}^{L(i)}(z = 0, L) r dA = T. \quad (6.49)$$

The seven unknowns can be solved as:

$$F_{11}^{(1)} = -\frac{1}{2\xi_2} (Pr_2^2 (r_1^2 (\lambda_1 - \lambda_2) \mu_2 + r_2^2 (\lambda_2 \mu_2 + \lambda_1 (\lambda_2 + \mu_2)))), \quad (6.50)$$

$$F_{11}^{(2)} = -\frac{1}{2\xi_2} (Pr_2^2 (r_1^2 (\lambda_1 - \lambda_2) \mu_2 + r_2^2 \lambda_2 (\lambda_1 + \mu_1 + \mu_2))), \quad (6.51)$$

$$F_{12}^{(2)} = \frac{1}{2\xi_2} (Pr_1^2 r_2^4 (\lambda_2 \mu_1 - \lambda_1 \mu_2)), \quad (6.52)$$

$$G_{11}^{(1)} = G_{11}^{(2)} = 0, \quad (6.53)$$

$$H_{11}^{(1)} = \frac{1}{\xi_2} (Pr_2^2 (r_1^2 (\lambda_1 - \lambda_2 + \mu_1 - \mu_2) \mu_2 + r_2^2 (\lambda_2 + \mu_2) (\lambda_1 + \mu_1 + \mu_2)) - \frac{P_1 (\lambda_1 + \mu_1)}{\mu_1 (3\lambda_1 + 2\mu_1)}), \quad (6.54)$$

$$H_{11}^{(2)} = \frac{1}{\xi_2} (Pr_2^2 (r_1^2 (\lambda_1 - \lambda_2 + \mu_1 - \mu_2) \mu_2 + r_2^2 (\lambda_2 + \mu_2) (\lambda_1 + \mu_1 + \mu_2)) - \frac{P_2 (\lambda_2 + \mu_2)}{\mu_2 (3\lambda_2 + 2\mu_2)}),$$

(6.55)

where

$$\xi_2 = (r_1^4(3\lambda_1 - 3\lambda_2 + 2\mu_1 - 2\mu_2)(\mu_1 - \mu_2)\mu_2 + r_2^4\mu_2(\lambda_1 + \mu_1 + \mu_2)(3\lambda_2 + 2\mu_2) + r_1^2r_2^2(\lambda_1(3\lambda_2(\mu_1 - \mu_2) + \mu_2(3\mu_1 + \mu_2)) + 2(\lambda_2(\mu_1^2 - 3\mu_2^2) + \mu_2(\mu_1^2 + \mu_1\mu_2 - 2\mu_2^2))))). \quad (6.56)$$

Hence, the linear displacements in a bilayered composite can be written as:

$$u_r^{L(1)} = -\frac{1}{2\xi_2} kPr_2^2(r_1^2(\lambda_1 - \lambda_2)\mu_2 + r_2^2(\lambda_2\mu_2 + \lambda_1(\lambda_2 + \mu_2))), \quad (6.57)$$

$$u_\theta^{L(1)} = \frac{2kTrz}{\pi(r_1^4(\mu_1 - \mu_2) + r_2^4\mu_2)}, \quad (6.58)$$

$$u_z^{L(1)} = k\frac{1}{\xi_2} Pzr_2^2(r_1^2(\lambda_1 - \lambda_2 + \mu_1 - \mu_2)\mu_2 + r_2^2(\lambda_2 + \mu_2)(\lambda_1 + \mu_1 + \mu_2)), \quad (6.59)$$

$$u_r^{L(2)} = -k\frac{1}{2r\xi_2} Pr_2^2(r_1^2r_2^2(-\lambda_2\mu_1 + \lambda_1\mu_2) + r^2(r_1^2(\lambda_1 - \lambda_2)\mu_2 + r_2^2\lambda_2(\lambda_1 + \mu_1 + \mu_2))), \quad (6.60)$$

$$u_\theta^{L(2)} = \frac{2kTrz}{\pi(r_1^4(\mu_1 - \mu_2) + r_2^4\mu_2)}, \quad (6.61)$$

$$u_z^{L(2)} = k\frac{1}{\xi_2} Pzr_2^2(r_1^2(\lambda_1 - \lambda_2 + \mu_1 - \mu_2)\mu_2 + r_2^2(\lambda_2 + \mu_2)(\lambda_1 + \mu_1 + \mu_2)), \quad (6.62)$$

It shows that all the first-order displacements in a bilayered composite under combined loading are a simple addition of the first-order displacements under pure torsion and pure axial loading. No axial force-twist effect exists in a bilayered composite under combined loading in the first-order elasticity theory.

The stress field can be written in matrix form as:

$$T^{L(1)} = k \begin{pmatrix} \frac{1}{\xi_2} P r_2^2 (-r_1^2 + r_2^2) & 0 & 0 \\ \times \mu_2 (-\lambda_2 \mu_1 + \lambda_1 \mu_2) & \frac{1}{\xi_2} P r_2^2 (-r_1^2 + r_2^2) & \frac{2rT\mu_1}{\pi(r_1^4(\mu_1 - \mu_2) + r_2^4\mu_2)} \\ 0 & \times \mu_2 (-\lambda_2 \mu_1 + \lambda_1 \mu_2) & \frac{1}{\xi_2} P r_2^2 (-r_1^2 \mu_2 (\lambda_1 (-3\mu_1 + \mu_2) + 2\mu_1 (\lambda_2 - \mu_1 + \mu_2)) + r_2^2 (2\mu_1 (\lambda_2 + \mu_2) (\mu_1 + \mu_2) + \lambda_1 (3\lambda_2 \mu_1 + 3\mu_1 \mu_2 + \mu_2^2))) \\ 0 & \frac{2rT\mu_1}{\pi(r_1^4(\mu_1 - \mu_2) + r_2^4\mu_2)} & \end{pmatrix}, \quad (6.63)$$

$$T^{L(2)} = k \begin{pmatrix} \frac{1}{r^2 \xi_2} P r_1^2 r_2^2 (-r^2 + r_2^2) & 0 & 0 \\ \times \mu_2 (-\lambda_2 \mu_1 + \lambda_1 \mu_2) & \frac{1}{r^2 \xi_2} P r_1^2 r_2^2 (r^2 + r_2^2) & \frac{2rT\mu_2}{\pi(r_1^4(\mu_1 - \mu_2) + r_2^4\mu_2)} \\ 0 & \times \mu_2 (\lambda_2 \mu_1 - \lambda_1 \mu_2) & \frac{1}{\xi_2} P r_2^2 \mu_2 (r_2^2 (\lambda_1 + \mu_1 + \mu_2) (3\lambda_2 + 2\mu_2) + r_1^2 (\lambda_2 (\mu_1 - 3\mu_2) + 2(\lambda_1 + \mu_1 - \mu_2) \mu_2)) \\ 0 & \frac{2rT\mu_2}{\pi(r_1^4(\mu_1 - \mu_2) + r_2^4\mu_2)} & \end{pmatrix}. \quad (6.64)$$

The stress field is also a simple addition of the first-order fields for the separate loading modes.

Moving on to the second-order theory, the solutions for radial and axial displacements $F_2^{(i)}(r)$, $G_2^{(i)}(z)$ and $H_2^{(i)}(z)$, $i = 1, 2$ can be derived as follows.

Substituting the linear solutions $F_1^{(i)}(r)$, $G_1^{(i)}(z)$ and $H_1^{(i)}(z)$ into the governing

equations, $F_2^{(i)}(r)$, $G_2^{(i)}(z)$ and $H_2^{(i)}(z)$ can be determined:

$$F_2^{(i)}(r) = \frac{1}{32r^3(\lambda_i + 2\mu_i)} (32(F_{12}^{(i)})^2 m_i - 4m_i r^6 \alpha_i^2 + 3n_i r^6 \alpha_i^2 + 16(F_{12}^{(i)})^2 \lambda_i - 4r^6 \alpha_i^2 \lambda_i + 48(F_{12}^{(i)})^2 \mu_i + 8r^6 \alpha_i^2 \mu_i) + F_{21}^{(i)} r + \frac{F_{22}^{(i)}}{r}, \quad (6.65)$$

$$G_2^{(i)}(z) = G_{21}^{(i)} z \quad (6.66)$$

$$H_2^{(i)}(z) = H_{21}^{(i)} z \quad (6.67)$$

for $i = 1, 2$. Here there are eight unknown constants $F_{21}^{(i)}$, $F_{22}^{(i)}$, $G_{21}^{(i)}$, $H_{21}^{(i)}$. Again

$F_{22}^{(1)} = 0$ for the first layer. The following seven boundary conditions allow us to

solve for the remaining seven unknowns:

(1) Second-order radial displacement is continuous across the interface:

$$F_2^{(1)}(r = r_1) = F_2^{(2)}(r = r_1), \quad (6.68)$$

(2) Second-order circumferential displacement is continuous across the interface:

$$G_{21}^{(1)} z = G_{21}^{(2)} z, \text{ or } G_{21}^{(1)} = G_{21}^{(2)}, \quad (6.69)$$

(3) Second-order axial displacement is continuous across the interface:

$$\delta_1 \left(\frac{P_1}{E_1} \right)^2 z + H_{21}^{(1)} z = \delta_2 \left(\frac{P_2}{E_2} \right)^2 z + H_{21}^{(2)} z, \text{ or } \delta_1 \left(\frac{P_1}{E_1} \right)^2 + H_{21}^{(1)} = \delta_2 \left(\frac{P_2}{E_2} \right)^2 + H_{21}^{(2)}, \quad (6.70)$$

where δ_1 and δ_2 are combinations of elastic constants whose exact forms are here omitted,

(4) Second-order radial traction T_{rr}^{NL} is continuous across the interfaces

$$T_{rr}^{NL(1)}(r = r_1) = T_{rr}^{NL(2)}(r = r_1), \quad (6.71)$$

(5) T_{rr}^{NL} vanishes at the outer boundary:

$$T_{rr}^{NL(2)}(r = r_2) = 0, \quad (6.72)$$

(6) The contribution of the second-order axial stress to the axial force applied at either end of the cylinder is 0:

$$\int T_{zz}^{NL(i)}(z = 0, L) dA = 0, \quad (6.73)$$

(7) The contribution of the second-order shear stress $T_{\theta z}^{NL}$ to T is 0:

$$\int T_{\theta z}^{NL(i)}(z = L) r dA = 0, \quad (6.74)$$

The unknowns $F_{21}^{(1)}$, $F_{21}^{(2)}$, $F_{22}^{(2)}$, $G_{21}^{(1)}$, $G_{21}^{(2)}$, $H_{21}^{(1)}$, $H_{21}^{(2)}$ can be determined. They are complex combinations of the elastic parameters λ_i , μ_i , l_i , m_i , n_i , geometrical parameters r_i , and the loadings P and T . Because of their lengthiness, they are not listed explicitly.

In summary, the components of the total displacement field (first- and second-order) can be written as:

$$u_r^{(i)} = k(F_{11}^{(i)} r + \frac{F_{12}^{(i)}}{r}) + k^2 \left(\frac{1}{32r^3(\lambda_i + 2\mu_i)} (32(F_{12}^{(i)})^2 m_i - 4m_i r^6 \alpha_i^2 + 3n_i r^6 \alpha_i^2 + \right. \quad (6.75)$$

$$\left. 16(F_{12}^{(i)})^2 \lambda_i - 4r^6 \alpha_i^2 \lambda_i + 48(F_{12}^{(i)})^2 \mu_i + 8r^6 \alpha_i^2 \mu_i \right) + F_{21}^{(i)} r + \frac{F_{22}^{(i)}}{r},$$

$$u_\theta^{(i)} = kr\alpha_i z + k^2 G_{21}^{(i)} rz = k \frac{2T}{\pi(r_1^4 \mu_1 - r_1^4 \mu_2 + r_2^4 \mu_2)} rz + k^2 G_{21}^{(i)} rz, \quad (6.76)$$

$$u_z^{(i)} = kH_{11}^{(i)} z + k \frac{P_i}{E_i} z + k^2 H_{21}^{(i)} z + k^2 \left(\frac{P_i}{E_i} \right)^2 \delta_i z. \quad (6.77)$$

It should be mentioned that after substituting the constants into $u_z^{(i)}$, P_i and δ_i will drop out of the equations. The radial, circumferential and axial displacements all

contain linear and nonlinear parts. Furthermore, $u_\theta^{(i)}$ has an additional coupled term in the $G_{21}^{(i)}$ expression, which shows the axial force-twist effect, while $u_r^{(i)}$ and $u_z^{(i)}$ do not contain such coupled terms.

The stress field can be written as:

$$\mathbf{T}^{(i)} = \begin{pmatrix} kT_{rr}^{L(i)} + k^2T_{rr}^{NL(i)} & 0 & 0 \\ 0 & kT_{\theta\theta}^{L(i)} + k^2T_{\theta\theta}^{NL(i)} & kT_{\theta z}^{L(i)} + k^2T_{\theta z}^{NL(i)} \\ 0 & kT_{z\theta}^{L(i)} + k^2T_{z\theta}^{NL(i)} & kT_{zz}^{L(i)} + k^2T_{zz}^{NL(i)} \end{pmatrix}. \quad (6.78)$$

This stress matrix is not symmetric. There are coupled terms only in $T_{\theta z}^{(i)}$ and $T_{z\theta}^{(i)}$.

To illustrate clearly the first- and second-order terms in the displacement and stress components under pure torsion (T), pure axial loading (P) and combined loading (TP), the matrices for the three modes are respectively written in the following manner.

(a) Homogeneous cylinder:

$$\begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}_T = \begin{pmatrix} k^2 \\ k \\ k^2 \end{pmatrix}, \quad \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}_P = \begin{pmatrix} k + k^2 \\ 0 \\ k + k^2 \end{pmatrix}, \quad \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}_{TP} = \begin{pmatrix} k + k^2 \\ k + k^2 \\ k + k^2 \end{pmatrix}, \quad (6.79)$$

$$\mathbf{T}_T = \begin{pmatrix} k^2 & 0 & 0 \\ 0 & k^2 & k \\ 0 & k & k^2 \end{pmatrix}, \quad \mathbf{T}_P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{pmatrix}, \quad \mathbf{T}_{TP} = \begin{pmatrix} k^2 & 0 & 0 \\ 0 & k^2 & k + k^2 \\ 0 & k + k^2 & k + k^2 \end{pmatrix}. \quad (6.80)$$

(b) Bilayered cylinder:

$$\begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}_T = \begin{pmatrix} k^2 \\ k \\ k^2 \end{pmatrix}, \quad \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}_P = \begin{pmatrix} k + k^2 \\ 0 \\ k + k^2 \end{pmatrix}, \quad \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}_{TP} = \begin{pmatrix} k + k^2 \\ k + k^2 \\ k + k^2 \end{pmatrix}, \quad (6.81)$$

$$\begin{aligned} \mathbf{T}_T &= \begin{pmatrix} k^2 & 0 & 0 \\ 0 & k^2 & k \\ 0 & k & k^2 \end{pmatrix}, & \mathbf{T}_P &= \begin{pmatrix} k + k^2 & 0 & 0 \\ 0 & k + k^2 & 0 \\ 0 & 0 & k + k^2 \end{pmatrix}, \\ \mathbf{T}_{TP} &= \begin{pmatrix} k + k^2 & 0 & 0 \\ 0 & k + k^2 & k + k^2 \\ 0 & k + k^2 & k + k^2 \end{pmatrix}. \end{aligned} \quad (6.82)$$

It can be seen that for combined loading, a direct superposition of the pure loading cases is generally not valid.

6.4. Numerical results

A bilayered cylinder under combined torsion and axial loading is considered. For the axial loading, only tension is investigated. The radial, circumferential and axial displacements are given in Eqs. (6.30)–(6.32) for a homogeneous cylinder and Eqs. (6.75)–(6.77) for a bilayered composite. The stress fields are given in Eqs. (6.33)–(6.39) for the homogeneous cylinder and Eq. (6.78) for the composite. The objective of the numerical study is to investigate the influence of elastic inhomogeneity, interface position (equivalently layer thicknesses) and loading on the displacements and stresses. Comparison with experimental data is made. Coupled terms are highlighted. The existence of a coupled term in the nonlinear

circumferential displacement is evidence of the axial force-twist effect, which implies an additional twist proportional to the torsion T and axial loading P under combined loading. The material parameters are chosen from experimental data characterized by the second-order elasticity model (Catheline *et al.*, 2003 ; Wang and Wu, 2013). The applied torque assumes a magnitude of $T = 10 \times 10^{-10}$ or 6×10^{-10} kPa•m³, and the applied axial loading $P = 1$ or 0.1 kPa. The complete list of elastic constants, geometrical parameters and the loadings used in the numerical study is given in Table 6.1.

6.4.1. Dependence of the axial force-twist effect on elastic constants and interface position

Fig. 6.2 plots the constant second-order circumferential displacement $u_{\theta}^{NL}(z=L)=0$ curves in the $\mu_2 - n_2$ space for (a) a bilayered composite of soft materials and (b) a metallic composite. The other elastic parameters are held fixed and listed in Table 6.1. This displacement component is a coupled term while the linear component is solely dependent on T , as shown in Eq. (6.76). The cylinder parameters are $r_1 = 0.001$ m, $r_2 = 0.002$ m and $L = 0.01$ m. The torque $T = 10 \times 10^{-10}$ kPa•m³ and $P = 1$ kPa.

In Fig. 6.2(a), the $\mu_2 - n_2$ space is seen to be partitioned into two regions, indicated by I and II, which means positive and negative $u_{\theta}^{NL}(z=L)$, respectively. By varying μ_2 and n_2 , $u_{\theta}^{NL}(z=L)$ can be positive or negative, implying that the

twisting angle may increase (positive axial force-twist effect) or decrease (negative axial force-twist effect) when an axial loading P is imposed on the cylinder under a torsion T . Roughly speaking, the positive effect is achieved when $\mu_2 < 350$ kPa and $n_2 < 0$

Fig. 6.2. Contours of the second-order coupled circumferential displacement $u_{\theta}^{NL}(z=L)=0$ in $\mu_2 - n_2$ space for: (a) soft composite and (b) metallic composite.

kPa or $\mu_2 > 450$ kPa and $n_2 > 50000$ kPa. Fig. 6.2 (a) illustrates the possibility of designing a biomaterial composite with positive or negative axial force-twist effect via elasticity.

In order to compare the different dependences of the axial force-twist effect for composites of soft biomaterials and metallic materials, Fig. 6.2 (b) is presented. Here the $u_\theta^{NL}(z=L)=0$ contour is also plotted in the μ_2 - n_2 space. The material in Layer 1 is chosen to be silver, and λ_2 and m_2 of Layer 2 is chosen from the values for copper with μ_2 and n_2 varied in the simulation. It can be seen that Regions I and II are very different from those shown for the soft composite. Generally speaking, the positive axial force-twist effect is obtained for negative values of n_2 , depending on m_2 , with $\mu_2 < 6 \times 10^5$ kPa as indicated by Region I. Thus for soft and metallic composites, Fig. 6.2 shows the great difference between their dependences of the axial force-twist effect on the elastic parameters.

To illustrate the dependence of the coupled, second-order angular displacement versus the interface position, the circumferential displacement $u_\theta^{NL}(z=L)$ is divided by r and plotted versus r_1 . Figs. 6.3(a) and (b) show the results for soft and metallic composites, respectively. The two composites have the same overall size ($r_2 = 0.001$ m, $L = 0.01$ m) and both are subjected to $T = 10 \times 10^{-10}$ kPa•m³ and $P = 0.1$ kPa. In each sub-figure, the solid and dotted lines denote the results for a composite and its reverse configuration, respectively. i.e., the materials in Layers 1 and 2 are interchanged in the two composites.

Fig. 6.3. Variation of the second-order angular displacement $u_{\theta}^{NL}(z=L)/r$ at the top of the cylinder with interface position r_1 for the bilayered composites of: (a) soft materials (solid line) and reverse configuration (dashed line), and (b) metals (solid line) and reverse configuration (dashed line).

The “composites” with their interfaces located at $r_1=0$ and $r_1=r_2$ degenerate into homogeneous cylinders containing solely of the material in Layers 2 and 1 of the

composite, respectively. The angular displacements of the soft homogeneous cylinders mentioned above are calculated to be respectively 0.522 rad and 0.796 rad, when they are subjected to *pure* torsion $T = 10 \times 10^{-10} \text{ kPa} \cdot \text{m}^3$. For the homogeneous metallic cylinders, the angular displacements are 1.524×10^{-4} rad and 5.308×10^{-5} rad, respectively. Pure torsion of composite and homogeneous cylinders generally produce angular displacements of comparable orders.

Several conclusions can be made with regard to Fig. 6.3. First, it is evident that the axial force-twist effect in soft composites under combined loading cannot be ignored, as the coupled angular displacement is only one order of magnitude less than that due to pure torsion. Moreover, the coupled term is proportional to P and it will become larger for a larger P . In contrast, the axial force-twist effect in metals is not obvious, with the coupled term about five orders of magnitude less than that due to pure torsion. Second, the magnitude of the coupled angular displacement in the soft composite is typically of the order of 10^{-2} , which is much larger than that in a metallic composite with a magnitude of the order of 10^{-10} . This is clear in Fig. 6.3. Third, the curves vary monotonically and nonlinearly between the two end points $r_1=0$ and $r_1=r_2$. Unlike the direct Poynting effect, it appears that the axial force-twist effect is not amplified, at least for this soft composite. Finally, there exists an interface position where the second-order angular displacement is the same regardless of the order of the layers, as indicated by the intersection of the solid and dotted curves. This may have implications on materials selection for the design of bio-composites.

Fig. 6.4. Variation of (a) the ratio $u_r^{NL}(P) / u_r^{NL}(T)$ of the second-order radial displacements at the outer boundary, and (b) the ratio $u_z^{NL}(P) / u_z^{NL}(T)$ of the second-order axial displacements at the top of the cylinder, due to tension and torsion with the interface position r_1 for a soft bilayered composite.

6.4.2. Dependence of the second-order effects in displacement on interface position

An interesting question is the relative importance of torsion versus axial loading when a composite is subjected to combined loading. This is investigated in Fig. 6.4(a), which plots the variation of $u_r^{NL}(P)/u_r^{NL}(T)$ versus r_1 , where $u_r^{NL}(P)$ and $u_r^{NL}(T)$ denote the contributions due to P and T , respectively, to the second-order radial displacement, i.e., $u_r^{NL} = u_r^{NL}(P) + u_r^{NL}(T)$. Similarly, Fig. 6.4(b) plots $u_z^{NL}(P)/u_z^{NL}(T)$ versus r_1 , where the second-order axial displacement is made up of the contributions from P and T , i.e., $u_z^{NL} = u_z^{NL}(P) + u_z^{NL}(T)$. In this figure, the soft composite is subjected to $T = 10 \times 10^{-10} \text{ kPa} \cdot \text{m}^3$ and $P = 1 \text{ kPa}$.

It can be observed that when r_1 increases, the ratio $u_r^{NL}(P)/u_r^{NL}(T)$ increases from negative to positive monotonically in Fig. 6.4 (a). Also, T dominates over P if a composite is made up of roughly equal amount of the different materials, but P dominates over T if the composite is made up of a large core with a thin outer layer or a small core with a thick outer layer. The relative contributions of P and T can also differ in sign. In Fig. 6.4 (b), there exists a singular point when $r_1 \approx 0.0002 \text{ m}$, around which the contribution of T to axial displacement nearly vanishes and the contribution of P dominates. This merely means that the Poynting effect disappears for this composite with interface located at $r_1 \approx 0.0002 \text{ m}$.

As an illustration of the importance of the nonlinear displacement, Fig. 6.5 plots the variation of the ratios of the nonlinear to linear radial displacement $u_r^{NL}/u_r^L(r=r_2)$, circumferential displacement $u_\theta^{NL}/u_\theta^L(z=L)$ and axial

Fig. 6.5. Variation of the ratio of second-order to first-order radial displacements $u_r^{NL} / u_r^L(r = r_2)$ (solid line), circumferential displacements $u_\theta^{NL} / u_\theta^L(z = L)$ (short dashed line) and axial displacements $u_z^{NL} / u_z^L(z = L)$ (long dashed line) with the interface position r_1 for a soft bilayered composite.

displacement $u_\theta^{NL} / u_\theta^L(z = L)$ with the interface position r_1 for a soft bilayered composite. It can be seen that the ratios vary between -0.4 and 0.4 generally, and the nonlinear displacement components tend to decrease with the interface position in this case. At a particular r_1 , the three ratios can differ in sign, implying that one displacement component may be overestimated while another component can be underestimated. It can be concluded that the nonlinear contributions to displacements cannot be ignored for soft materials generally.

Fig. 6.6. Variation of the second-order coupled component in the shear stress $T_{\theta z}(r_1^-)$ (solid line) and $T_{z\theta}(r_1^-)$ (dashed line) with interface position r_1 for: (a) soft composite and (b) metallic composite.

6.4.3. Dependence of stresses on interface position

A bilayered composite is subjected to the combined loading $T = 10 \times 10^{-10}$ kPa•m³ and $P = 1$ kPa. The variations of the second-order *coupled* terms in the shear stress components $T_{\theta z}(r = r_1^-)$ and $T_{z\theta}(r = r_1^-)$ with r_1 are plotted for soft and metal composites in Figs. 6.6 (a) and (b), respectively. The metallic composite consists of copper in Layer 1 and silver in Layer 2.

A number of observations can be made. The coupled terms in $T_{\theta z}(r = r_1^-)$ and $T_{z\theta}(r = r_1^-)$ are not equal. Both do not vary monotonically between the end points $r_1 = 0$ and $r_1 = r_2$ and will achieve maxima at certain values of r_1 . The end points represent homogeneous materials. Thus, although two homogeneous materials have zero or small second-order effects, their composites may have such effects significantly amplified. Moreover, these effects in the metallic composite are much smaller than those in the soft composite, with magnitudes of the order of 10^{-2} and 10^{-6} kPa, respectively, as compared to $P = 1$ kPa.

Fig. 6.7 graphs the variation of the ratios of the nonlinear to linear components with the interface position, i.e., $T_{rr}^{NL} / T_{rr}^L(r = r_1^-)$, $T_{\theta\theta}^{NL} / T_{\theta\theta}^L(r = r_1^+)$, and $T_{z\theta}^{NL} / T_{z\theta}^L(r = r_1^+)$ versus r_1 . The material, geometrical and loading parameters are given in Table 6.1. From the figure, it can be seen that the magnitude of the nonlinear normal stresses is about 3 to 4 times that of the corresponding linear components, whereas the nonlinear shear stress is an insignificant fraction of the linear component. In this case, the ratios depend weakly on r_1 .

Fig. 6.7. Variation of the ratios of the second- to first-order stress components $T_{rr}^{NL} / T_{rr}^L(r = r_1)$ (solid line), $T_{\theta\theta}^{NL} / T_{\theta\theta}^L(r = r_1^+)$ (short dashed line) and $T_{z\theta}^{NL} / T_{z\theta}^L(r = r_1^+)$ (long dashed line) with the interface position r_1 for a soft bilayered composite.

Fig. 6.8. Comparison of the theoretical predictions of (a) twisting moment T and (b) average axial stress P with the experimental data of Criscione *et al.* (1999) for cardiac papillary muscles.

Table 6.1. Elastic, geometrical and loading parameters used in Figs. 6.2–6.8. Note:
poly(acrylic acid)= PAA.

	Fig. 6.2(a)	Fig. 6.2(b)	Fig. 6.3(a) solid	Fig. 6.3(a) dashed	Fig. 6.3(b) solid	Fig. 6.3(b) dashed
Material 1	Agar-gelatin	Ag	Agar-gelatin	PAA gels	Au	TiAl
λ_1 (kPa)	2.25×10^6	91×10^3	2.25×10^6	60	163×10^3	105×10^3
μ_1 (kPa)	8	45×10^3	8	12.21	41.8×10^3	120×10^3
l_1 (kPa)	-2×10^6	-265×10^3	-2×10^6	-42.67	-461×10^3	20×10^3
m_1 (kPa)	-20×10^6	-637×10^3	-20×10^6	-35.60	-648×10^3	-40×10^3
n_1 (kPa)	80	332×10^3	80	-23.50	-48×10^3	-160×10^3
Material 2	PAA gels-based	Cu-based	PAA gels	Agar-gelatin	TiAl	Au
λ_2 (kPa)	60	121×10^3	60	2.25×10^6	105×10^3	163×10^3
μ_2 (kPa)			12.21	8	120×10^3	41.8×10^3
l_2 (kPa)	-42.67	-407×10^3	-42.67	-2×10^6	20×10^3	-461×10^3
m_2 (kPa)	-35.60	-780×10^3	-35.60	-20×10^6	-40×10^3	-648×10^3
n_2 (kPa)			-23.50	80	-160×10^3	-48×10^3
r_1 (m)	0.001	0.001				
r_2 (m)	0.002	0.002	0.001	0.001	0.001	0.001
L (m)	0.01	0.01	0.01	0.01	0.01	0.01

T (kPa $\cdot$ m ³)	10×10^{-10}	10×10^{-10}	10×10^{-10}	10×10^{-10}	10×10^{-10}	10×10^{-10}
P (kPa)	1	1	0.1	0.1	0.1	0.1

	Fig. 6.4 (a), (b)	Fig. 6.5	Fig. 6.6(a)	Fig. 6.6(b)	Fig. 6.7	Fig. 6.8
Material 1	Agar-gelat in	PAA gels	Agar-gelat in	Cu	PAA gels	Myocardial core
λ_1 (kPa)	2.25×10^6	60	2.25×10^6	121×10^3	60	32670
μ_1 (kPa)	8	12.21	8	75.5×10^3	12.21	3.3
l_1 (kPa)	-2×10^6	-42.67	-2×10^6	-407×10^3	-42.67	-2.25×10^6
m_1 (kPa)	-20×10^6	-35.60	-20×10^6	-780×10^3	-35.60	-360
n_1 (kPa)	80	-23.50	80	-380×10^3	-23.50	20
Material 2	PAA gels	Agar-gelat in	PAA gels	Ag	Agar-gelat in	Endocardial sheath
λ_2 (kPa)	60	2.25×10^6	60	91×10^3	2.25×10^6	4076
μ_2 (kPa)	12.21	8	12.21	45×10^3	8	7.0
l_2 (kPa)	-42.67	-2×10^6	-42.67	-265×10^3	-2×10^6	-2×10^4
m_2 (kPa)	-35.60	-20×10^6	-35.60	-637×10^3	-20×10^6	-4200
n_2 (kPa)	-23.50	80	-23.50	332×10^3	80	-46
r_1 (m)						0.000525

r_2 (m)	0.001	0.001	0.001	0.001	0.001	0.00055
L (m)	0.01	0.01	0.01	0.01	0.01	
T (kPa $\cdot$ m ³)	10×10^{-10}	10×10^{-10}	10×10^{-10}	10×10^{-10}	10×10^{-10}	
P (kPa)	1	1	1	1	1	

6.4.4. Comparison with experimental data

In Fig. 6.8, the prediction of the twisting moment T and the average axial stress P of the second-order model versus the twist per unit length (twist rate) is compared to the experimental data of Criscione et al. (1999) for finite extension and torsion on cardiac papillary muscles. This muscle is considered as a composite of a myocardium core and a thin endocardium sheath, with an overall radius in the range of 0.375 to 0.65 mm and a sheath thickness of less than 25 μm . Since it is transversely isotropic, the second-order isotropic model here should be regarded as an approximation. The finite extension is held fixed at a stretch ratio of 1.04. Figs. 6.8(a) and 6.8(b) plot T and P versus the twist per unit length. The elastic constants for the two materials used in the simulation are listed in Table 6.1. The shear moduli μ_i are of the order of 10 kPa, with λ_i calculated assuming near-incompressibility. The third-order elastic constants are adjusted to fit the test data. The torque-twist rate prediction appears to overestimate the data (a larger T is predicted for a given twist rate), while the axial stress-twist rate prediction fits the data reasonably well. The analytical expressions for T and P are solved from the equilibrium equations; they are not arbitrary expressions with a number of fitting constants which can be calibrated against the data. The discrepancy between theory and data may be attributed to the assumption of complete isotropy in the theory, while the test specimens are actually transversely isotropic with the property in the longitudinal direction different from that in the transverse plane. However, it is also possible to develop a transversely isotropic second- or even higher-order model, and the

associated number of elastic constants can be reduced if the further assumption of incompressibility is built directly into the theory.

6.5. Discussion

The celebrated Poynting effect, in which a cylindrical specimen extends under a torque, is a second-order effect that has been well-investigated for homogeneous materials. More recently, the importance of the negative Poynting effect in soft biological materials has also received much attention. A major effort of this Chapter is the investigation of the effect for composite materials. In the course of this work, it is found that the “axial force-twist effect”, in which a cylindrical specimen, homogeneous or otherwise, twists under an axial loading in the presence of a torque, is also possible within the second-order theory. This second-order effect is an intrinsically coupled phenomenon, meaning that both the torque and axial loading must be present for it to materialize. This is unlike the Poynting effect, which appears even in the absence of axial loading.

The effect appears to have been little discussed in the literature. Its importance is highlighted in this chapter, and its dependence on the elastic constants and the interface position of a composite cylinder is emphasized. Both the Poynting and axial force-twist effects, as well as other second-order effects, appear to be significant in soft materials but less so in metallic materials. Such findings could be useful for applications in regenerative medicine and for the design of bio-inspired soft

materials, e.g., through the selection of material inhomogeneity (elastic constants of layers) and geometrical inhomogeneity (layer thicknesses). The dependence of the second-order effects on these inhomogeneities may also be effectively used to determine higher-order elastic constants, as remarked in previous chapters.

Although this work considers inhomogeneity and nonlinearity, it does not consider elastic anisotropy or time-dependent phenomena such as viscoelasticity. Analytical solutions in elastically anisotropic and geometrically regular composites are likely obtainable using higher-order elasticity theories. Also, generalization of the solutions for multilayered composites can be achieved without difficulty, as shown in this chapter for bilayered and homogeneous cylinders. Although the expressions for multilayered composites will be lengthy, they can be evaluated numerically with ease. Such solutions will illustrate how the fundamental nature of the second-order effects can be affected by the additional material and geometrical complexities.

6.6. Conclusion

Nonlinear effects in soft and metallic cylindrical composites under combined torsion and axial loading are investigated within the framework of a second-order elasticity theory. Analytical expressions for homogenous and bilayered cylinders are obtained.

The major focus of this work is the identification of the second-order effects in

the displacement and stress fields, and how these effects are dependent on the second- and third-order elastic constants as well as the interface position or layer thicknesses of the composite. In particular, the axial force-twist effect, an intrinsically coupled phenomenon under combined loading, in soft composites is highlighted. The effect is absent in cylinders subjected to pure axial loading, and is present in both homogeneous and bilayered cylinders subjected to combined loading. Furthermore, the displacement and stress fields due to the combined loading are not a simple superposition of those due to the separate loading modes, as a consequence of mode coupling. These analytical results are useful for the identification of nonlinear phenomena and may have implications for technological applications involving soft materials.

Chapter 7 Generalized Shear Loading

In this chapter, the problems of homogeneous and bilayered soft rectangular blocks under generalized shear are solved. The problem of homogeneous soft block is solved in Section 7.1. The problem of bilayered soft block is solved in Section 7.2.

7.1. Homogeneous soft rectangular block under generalized shear

In nonlinear elasticity, the existence of normal stresses under simple shear has long been recognized (Rivlin, 1948) and so has the lengthening of a cylinder under pure torsion (Poynting, 1909). There has been a recent interest in the shearing and torsion of soft materials such as biopolymers and hydrogels, e.g., Storm et al. (2005), Janmey et al. (2007), Kang et al. (2009), Wu and Kirchner (2010), Destrade and Saccomandi (2010), Destrade et al. (2012), Mihai and Goriely (2011, 2013), and Horgan and Murphy (2011). A significant finding of some of these works is the existence of negative normal stresses or the negative Poynting effect when certain soft solids are subjected to shear or torsion. Negative normal stresses occur if the sheared faces of a block are drawn together under shear, and equivalently the negative Poynting effect occurs if a cylindrical rod shortens under torsion. The shearing of soft biological materials is common in their physiological environments (Horgan and Murphy, 2011), and understanding of nonlinear phenomena such as the Poynting effect is of great importance. For instance, the large stresses generated by the Poynting effect can result in a significant effect on the overall force balance in

the cytoskeleton under shear (Janmey et al., 2007), and the Poynting effect is important in understanding the interaction between surgical instruments and tissues (Misra et al., 2010).

In the strain formulation of the simple shear problem, e.g., Rivlin (1948), the starting point is the deformation of a block as given by:

$$x = X + \kappa Y, \quad y = Y, \quad z = Z, \quad (7.1)$$

where (X, Y, Z) and (x, y, z) are the referential and current Cartesian coordinates respectively, and κ is the shear deformation. Murnaghan (1951) also started from Eq. (7.1) in studying simple shear within the framework of second-order elasticity. Calculation of the stresses then reveals that normal stresses must exist to support such a deformation. Recently, Destrade et al. (2012) pursued a stress formulation for hyperelastic materials in which the starting point is a uniform shear stress $\sigma_{12} = \sigma_{21}$. The resulting deformation is given by:

$$x = \lambda_1 X + \lambda_2 \sqrt{1 - \lambda_1^2 \lambda_2^{-2}} Y, \quad y = \lambda_2 Y, \quad z = \lambda_3 Z, \quad (7.2)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches. This means that under a pure shear stress the deformation consists of a simple shear of the type given by Eq. (7.1) and triaxial extensions. Hence, under the strain formulation the resulting stress field consists of shear stresses and triaxial normal stresses, while under the stress formulation the resulting deformation field consists of simple shear and triaxial

stretches. Mihai and Goriely (2013) considered generalized shear in which Eq. (7.1) is modified into:

$$x = X + \kappa(Y), \quad y = Y, \quad z = Z, \quad (7.3)$$

where κ is now a function of Y . Specifically, $\kappa(Y)$ is assumed to be of the form of $Y^2/2$, i.e., a quadratic function of Y . Destrade and Saccomandi (2010) also considered the effect of gravity in the form of a body force $-\rho_0 g$ (ρ_0 is the mass density and g is the gravitational constant). They took the deformation to be:

$$x = X + u(Y), \quad y = Y + v(Y), \quad z = Z, \quad (7.4)$$

where, upon solving the governing equations and assuming the Murnaghan strain energy density, $u(Y)$ was found to be the sum of linear and quadratic parts.

In this section, we consider the generalized shear of a block with the X -axis as the shearing direction, the Y -axis the vertical direction and the Z -axis the depth direction. The deformation is taken to be:

$$x = X + u(Y, Z), \quad y = Y + v(Y), \quad z = Z, \quad (7.5)$$

where the specific form for $u(Y, Z)$, under various prescribed displacement profiles at the top of the block, is solved from the governing equilibrium equations. Two new features of the formulation are noted: $u(Y, Z)$ is not assumed *a priori* and it could be a function of Z . The prescribed displacements considered are constant, quadratic and

sinusoidal functions of Z . The dependence of u on Y , i.e., the deformed block shows a curved rather than straight lateral profile, has been noted by Gardiner and Weiss (2001) during their finite element simulation of the simple shear of soft tissues. The form of this dependence has been assumed to be quadratic in Eq. (7.3), and predicted to be the sum of linear and quadratic terms in the gravity formulation of Eq. (7.4). In our work, we will show that, in the absence of gravity, u is the sum of a linear function of y and a function of the product $\sin y \times \sinh z$, even if the prescribed displacement at the top of the block is uniform, i.e., $u(Y=L_y, Z) = \text{constant}$, where L_y is the height of the block. Hence, a block of soft material subjected to a constant shear displacement at the top surface is curved in both the X - Y and X - Z planes (but not in the Y - Z plane).

We also use the Murnaghan strain energy density as the constitutive law. The gravity effect can also be taken into account as was done previously by Destrade and Saccomandi (2010). We compare the shape of deformed blocks of agar-gelatin with the theoretical predictions. Furthermore, we explore the dependence of negative or positive normal stresses on the elastic constants.

This section is organized as follows. The problem is defined in Section 7.1.1. Governing equations are given in Section 7.1.2. General solutions without gravity are presented in Section 7.1.3, while Solutions to particular problems are shown in Section 7.1.4. The solutions with gravity are given in Section 7.1.5. Numerical results and experiments are presented in Section 7.1.6. Discussion is made in Section 7.1.7, and a set of concluding remarks is given in Section 7.1.8

7.1.1. Definition of problem

We consider a homogeneous isotropic rectangular block of dimensions L_X , L_Y and L_Z under generalized shear and gravity as shown in Fig. 7.1(a). In reference Cartesian coordinates (X, Y, Z) , the shear displacement is assumed to be $(U, V, 0)$, so that the final coordinates (x, y, z) are given by $(X + U, Y + V, Z)$. Here $U = u_1(Y, Z) + ku_2(Y, Z)$ denotes the shear displacement in the X -direction, with u_1 and u_2 representing the linear and nonlinear components, respectively. U is taken to be a function of both Y and Z , allowing for the possibility of a non-uniform (generalized) shear which might exist in a complex setting such as a physiological environment. Similarly, $V = v_1(Y) + kv_2(Y)$ denotes the displacement in the Y -direction resulting from gravity, with v_1 and v_2 denoting the linear and nonlinear components. However, V is solely a function of Y as this is the direction of gravity. If gravity is neglected, $V = 0$. Note also that k is a marker indicating the order of approximation of the theory. In the subsequent formulation, we retain terms up to k^2 for second-order theory.

Fig. 7.1(b) shows the possible deformed configuration of the block, which is fixed to the bottom plane $Y = 0$, and subjected to prescribed displacement at the top face $Y = L_Y$. The linear part of the prescribed displacement is proportional to Y , and the nonlinear part is described by any $g(Z)$ which is symmetric with respect to $Z = L_Z/2$. Furthermore, we prescribe a moment M_{xz} about the Z -axis, which is associated with an induced out-of-plane shear.

Fig. 7.1. (a) A homogeneous isotropic rectangular block of dimensions L_X , L_Y and L_Z at the undeformed state. (b) The block deformed under generalized shear. The prescribed displacement at the top face $Y = L_Y$ consists of the linear part $u_1(L_Y)$ and the nonlinear part $u_2(Y = L_Y, Z) = g(Z)$. The lateral profile of the block, or the deformed shaped of the four vertical edges of the box, can be written as $u_1(Y) + f(Y)$.

7.1.2. Governing equations

The governing equations are the equilibrium equations in the first Piola-Kirchoff stress. The stress components are expressed in terms of displacements via the second-order constitutive relations. The solutions to the governing equations are therefore displacements, the second-order parts of which are not polynomials, as will be shown in the subsections to follow. The second-order elasticity model is first briefly reviewed in Section 7.1.2.1, and the equilibrium equations in Section 7.1.2.2. The solutions in the absence of gravity are derived in Section 7.1.3, with particular solutions for various prescribed displacements given in Section 7.1.4. Finally, the solutions in the presence of gravity are listed in Section 7.1.5.

7.1.2.1. Second-order elasticity model

The differentials of position vectors in the undeformed and deformed states are respectively:

$$\begin{pmatrix} dX \\ dY \\ dZ \end{pmatrix}, \begin{pmatrix} dX + k\left(\frac{\partial u_1}{\partial Y} dY + \frac{\partial u_1}{\partial Z} dZ\right) + k^2\left(\frac{\partial u_2}{\partial Y} dY + \frac{\partial u_2}{\partial Z} dZ\right) \\ dY + kdv_1 + k^2 dv_2 \\ dZ \end{pmatrix}. \quad (7.6)$$

The deformation gradient $\mathbf{F}$ can then be written as:

$$\mathbf{F} = \begin{pmatrix} 1 & k\frac{\partial u_1}{\partial Y} + k^2\frac{\partial u_2}{\partial Y} & k\frac{\partial u_1}{\partial Z} + k^2\frac{\partial u_2}{\partial Z} \\ 0 & 1 + k\frac{dv_1}{dY} + k^2\frac{dv_2}{dY} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (7.7)$$

where partial derivatives are used for u_1 and u_2 . In comparison, the deformation gradient $\mathbf{F}$ in previous works (Horgan and Murphy, 2011; Gardiner and Weiss, 2001; Mihai and Goriely, 2013) take the form:

$$\mathbf{F} = \begin{pmatrix} a & \kappa & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad (7.8)$$

where a, b, c are constants, and κ is a constant or a function of y which represents the shear. Destrade and Saccomandi (2010) considers gravity and $\mathbf{F}$ takes the form:

$$\mathbf{F} = \begin{pmatrix} 1 & \frac{du(Y)}{dY} & 0 \\ 0 & 1 + \frac{dw(Y)}{dY} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (7.9)$$

where u and w represent the shear displacement and displacement caused by gravity, respectively. In our formulation, $\mathbf{F}$ in Eq. (7.6) contains an additional term $k(\partial u_1 / \partial Z) + k^2(\partial u_2 / \partial Z)$, which exists because the shear displacement is a function of both Y and Z .

The energy density W of Murnaghan is used as in Eq. (3.1). The first Piola-Kirchhoff stress can be obtained as in Eq. (3.3).

7.1.2.2. Equilibrium equations

The Lagrangian equilibrium equations in Cartesian coordinates can be written as:

$$\begin{aligned}\frac{\partial T_{xX}}{\partial X} + \frac{\partial T_{xY}}{\partial Y} + \frac{\partial T_{xZ}}{\partial Z} &= 0, \\ \frac{\partial T_{yX}}{\partial X} + \frac{\partial T_{yY}}{\partial Y} + \frac{\partial T_{yZ}}{\partial Z} - \rho_0 g &= 0, \\ \frac{\partial T_{zX}}{\partial X} + \frac{\partial T_{zY}}{\partial Y} + \frac{\partial T_{zZ}}{\partial Z} &= 0,\end{aligned}\tag{7.10}$$

where ρ_0 is the mass density in the reference configuration. The notation for the stress components is exemplified by T_{xX} , which means the stress acting in the x -direction on an element surface with X as the direction of the normal vector. Tensorially, the first Piola-Kirchhoff stress $\mathbf{T} = T_{aB} \mathbf{e}_a \otimes \mathbf{E}_B$, where T_{aB} are the components and $\mathbf{e}_a, \mathbf{E}_B$ are the basis vectors in the deformed and reference configurations. However, for convenience, this detailed notation is dropped subsequently and the stress components will simply be written as T_{xx}, T_{xy} , etc. Substituting the stresses from Eq. (3.3) into Eq. (7.10) yields the following two first-order and three second-order equations:

$$k\mu\left(\frac{\partial^2 u_1}{\partial Y^2} + \frac{\partial^2 u_1}{\partial Z^2}\right) = 0,\tag{7.11}$$

$$k^2 \left(\frac{1}{2}(2\lambda + 2m - n)v_1' \frac{\partial^2 u_1}{\partial Z^2} + (\lambda + 2\mu + m)v_1'' \frac{\partial u_1}{\partial Y} + (\lambda + 2\mu + m)v_1' \frac{\partial^2 u_1}{\partial Y^2} + \mu\left(\frac{\partial^2 u_2}{\partial Y^2} + \frac{\partial^2 u_2}{\partial Z^2}\right) \right) = 0,\tag{7.12}$$

$$k\left((\lambda + 2\mu)v_1'' - g\rho_0\right) = 0,\tag{7.13}$$

$$\begin{aligned} & \frac{1}{4}k^2((4m-n)\frac{\partial u_1}{\partial Z}\frac{\partial u_1^2}{\partial Y\partial Z} + 4(\lambda+2\mu+m)\frac{\partial u_1}{\partial Y}\frac{\partial u_1^2}{\partial Y^2} + 4(\lambda+\mu)\frac{\partial u_1}{\partial Z}\frac{\partial u_1^2}{\partial Y\partial Z} + \\ & (4\mu+n)\frac{\partial u_1}{\partial Y}\frac{\partial u_1^2}{\partial Z^2} + 4(\lambda+2\mu)v_2'' + 4v_1'v_1''(3\lambda+2l+6\mu+4m)) = 0, \end{aligned} \quad (7.14)$$

$$\frac{1}{4}k^2\left(\frac{\partial u_1}{\partial Y}\frac{\partial u_1^2}{\partial Y\partial Z}(4\lambda+4\mu+4m-n) + \frac{\partial u_1}{\partial Z}\left(4(\lambda+2\mu+m)\frac{\partial u_1^2}{\partial Z^2} + (4\mu+n)\frac{\partial u_1^2}{\partial Y^2}\right)\right) = 0. \quad (7.15)$$

The third first-order equation, which arises from the third of Eq. (7.10), is identically zero. Also, v_1'' in Eq. (7.13) denotes the second-order derivative of v_1 with respect to Y . Further note that Eq. (7.11) is the Laplace equation, Eq. (7.13) is a second-order ordinary differential equation, while the remaining appear to be rather complicated partial differential equations. The displacement u_2 appears only in Eq. (7.12).

7.1.3. General solutions without gravity

In this section, the general solutions of a rectangular block under generalized shear without gravity are obtained. Thus, $v_1(Y) = v_2(Y) = 0$ and Eqs. (7.11)–(7.15) reduce to the following four equations:

$$k\mu\left(\frac{\partial^2 u_1}{\partial Y^2} + \frac{\partial^2 u_1}{\partial Z^2}\right) = 0, \quad (7.16)$$

$$k^2\mu\left(\frac{\partial^2 u_2}{\partial Y^2} + \frac{\partial^2 u_2}{\partial Z^2}\right) = 0, \quad (7.17)$$

$$\frac{1}{4}k^2\left(\frac{\partial u_1}{\partial Z}\frac{\partial^2 u_1}{\partial Y\partial Z}(4\lambda+4\mu+4m-n) + 4(\lambda+2\mu+m)\frac{\partial u_1}{\partial Y}\frac{\partial^2 u_1}{\partial Y^2} + (4\mu+n)\frac{\partial u_1}{\partial Y}\frac{\partial^2 u_1}{\partial Z^2}\right) = 0, \quad (7.18)$$

$$\frac{1}{4}k^2((4\lambda+4\mu+4m-n)\frac{\partial u_1}{\partial Y}\frac{\partial^2 u_1}{\partial Y\partial Z} + 4(\lambda+2\mu+m)\frac{\partial u_1}{\partial Z}\frac{\partial^2 u_1}{\partial Z^2} + (4\mu+n)\frac{\partial^2 u_1}{\partial Y^2}\frac{\partial u_1}{\partial Z}) = 0. \quad (7.19)$$

We apply displacement and force boundary conditions in solving these equations, as shown below.

7.1.3.1. Application of displacement boundary conditions

The first- and second-order displacements are solved from the governing equations with displacement boundary conditions. The linear displacement u_1 can be solved from Eqs. (7.16), (7.18) and (7.19) as:

$$u_1 = \alpha_1 Y + \alpha_2 Z, \quad (7.20)$$

where α_1 and α_2 are constants. This solution represents simple shear in the first-order theory. Assuming that u_1 is a symmetric function of Z , we retain only the Y dependence so that $u_1 = \alpha Y$ with the linear shear strain α a constant. α can be determined from prescribed linear displacements at the top and bottom faces, i.e., $u_1(L_Y, Z) = \alpha L_Y$ and $u_1(0, Z) = 0$ so that $\alpha = u_1(L_Y, Z) / L_Y$. For the nonlinear displacement u_2 , it can be solved from the Laplace Equation Eq. (7.17) with the following boundary conditions on the bottom and top faces ($Y = 0, L_Y$):

$$u_2(0, Z) = 0, \quad (7.21)$$

$$u_2(L_Y, Z) = g(Z), \quad (7.22)$$

i.e., the bottom face is fixed and $g(Z)$ is the nonlinear shear deformation prescribed at the top $y = L_Y$. Furthermore, we define

$$u_2(Y, 0) = u_2(Y, L_Z) = f(Y), \quad (7.23)$$

where $f(Y)$ is the nonlinear deformation of the block at $Z = 0$ and L_Z , respectively; The shape $f(Y)$, which illustrates how the block deflects in the X – Y plane, will be predicted. The second-order displacement u_2 can then be obtained as:

$$\begin{aligned} u_2(Y, Z) = & \sum_{j=1}^{\infty} A_j \sin\left(\frac{j\pi Y}{L_Y}\right) \sinh\left(\frac{j\pi(L_Z - Z)}{L_Y}\right) + \sum_{j=1}^{\infty} B_j \sin\left(\frac{j\pi Y}{L_Y}\right) \sinh\left(\frac{j\pi Z}{L_Y}\right) \\ & + \sum_{j=1}^{\infty} D_j \sinh\left(\frac{j\pi Y}{L_Z}\right) \sin\left(\frac{j\pi Z}{L_Z}\right), \end{aligned} \quad (7.24)$$

where

$$A_j = B_j = \frac{2}{L_Y \sinh\left(\frac{j\pi L_Z}{L_Y}\right)} \int_0^{L_Y} f(Y) \sin\left(\frac{j\pi Y}{L_Y}\right) dY, \quad (7.25)$$

and

$$D_j = \frac{2}{L_Z \sinh\left(\frac{j\pi L_Y}{L_Z}\right)} \int_0^{L_Z} g(Z) \sin\left(\frac{j\pi Z}{L_Z}\right) dZ. \quad (7.26)$$

This solution is symmetric about $Z = L_Z/2$. The aim is to obtain $f(Y)$ for various prescribed top face displacements $g(Z)$. For a given $g(Z)$, the Fourier coefficients D_j can be calculated directly via Eq. (7.26). The other coefficients A_j can be solved via

force and moment boundary conditions (see below). With A_j determined, Eq. (7.25) is essentially an integral equation in the unknown $f(Y)$.

7.1.3.2. Application of force and moment boundary conditions

The particular force and moment boundary conditions to be considered depend on the problem to be solved. From the components of the first Piola–Kirchoff stress (symmetric for this case), which can be deduced from Eq. (3.3):

$$\mathbf{T} = \begin{pmatrix} \frac{1}{2}k^2\alpha^2(m+\lambda+2\mu) & k\alpha\mu + k^2\mu\frac{\partial u_2}{\partial Y} & k^2\mu\frac{\partial u_2}{\partial Z} \\ k\alpha\mu + k^2\mu\frac{\partial u_2}{\partial Y} & \frac{1}{2}k^2\alpha^2(m+\lambda+2\mu) & 0 \\ k^2\mu\frac{\partial u_2}{\partial Z} & 0 & \frac{1}{4}k^2\alpha^2(2m-n+2\lambda) \end{pmatrix}, \quad (7.27)$$

it can be seen that all the normal stresses and the shear stress T_{xz} are second-order whereas the shear stress T_{xy} consists of both first- and second-order parts. The normal components can be positive or negative; for many biogels $m + \lambda + 2\mu$ is positive (Wang and Wu, 2014), which corresponds to negative normal stresses (Janmey et al., 2007). These normal stresses are necessary for maintaining the deformation of the generalized shear. The stress T_{xy} on the Y -faces (top and bottom) gives rise to resultant forces and moments, and so do T_{xz} on the Z -faces (front and back). The nonlinear displacement u_2 , which appears as derivatives in T_{xy} and T_{xz} , can be derived for particular force and moment boundary conditions. To illustrate this, we consider the following set of boundary conditions:

(BC 1). The shear forces on the Y -faces (top and bottom) are F_{xy} , which corresponds to the prescribed displacements $u_1(0, Z)$, $u_1(L_Y, Z)$ and $u_2(0, Z)$, $u_2(L_Y, Z)$.

(BC 2). No shear forces are applied on the Z -faces (front and back).

(BC 3). The moments prescribed on the Z -faces are M_{xz} .

Moments may exist because of the existence of second-order shear stresses.

Mathematically, the above boundary conditions can be written as:

$$\int T_{xy} dXdZ = F_{xy}, \quad \text{on } Y = 0, L_Y; \quad (7.28)$$

$$\int T_{xz} dXdY = F_{xz}, \quad \text{on } Z = 0, L_Z; \quad (7.29)$$

$$\int T_{xz}(Z=0)YdXdY - \int T_{xz}(Z=L_Z)YdXdY = M_{xz}, \text{ on } Z=0, L_Z. \quad (7.30)$$

The existence of M_{xz} (which is not identically zero) implies that a constraint against rotation about the Z -axis exists. In the experiments we carried out, rotation about the Z -axis was prevented, thus inducing a moment about that axis. This is analogous to the Poynting effect, in which the block expands or contracts under simple shear (equivalently a cylinder extends or shortens under torsion); otherwise if the block is constrained it will experience a normal stress. We note that the effect associated with M_{xz} does not exist under simple shear, for which T_{xz} does not exist.

Using the displacement expressions of Eq. (7.24) to evaluate the stresses of Eq. (7.27), and substituting the resulting stresses into Eqs. (7.28)–(7.30) yields:

$$\int T_{xy} dXdZ = \left\{ \begin{aligned} k\alpha\mu L_X L_Z + k^2\mu \sum_{j=1}^{\infty} L_X [2A_j(-1 + \cosh(\frac{j\pi L_Z}{L_Y})) + D_j(1 - (-1)^j)] &= F_{xy}, & (Y=0) \\ k\alpha\mu L_X L_Z + k^2\mu \sum_{j=1}^{\infty} L_X [2A_j(-1)^j(-1 + \cosh(\frac{j\pi L_Z}{L_Y})) + D_j(1 - (-1)^j) \cosh(\frac{j\pi L_Y}{L_Z})] &= F_{xy}, & (Y=L_Y) \end{aligned} \right\}, \quad (7.31)$$

$$\int T_{xz} dXdY = \left\{ \begin{aligned} k^2\mu \sum_{j=1}^{\infty} L_X [(-1 + (-1)^j)A_j(\cosh(\frac{j\pi L_Z}{L_Y}) - 1) + D_j(-1 + \cosh(\frac{j\pi L_Y}{L_Z}))] &= 0, & (Z=0) \\ k^2\mu \sum_{j=1}^{\infty} L_X [(-1 + (-1)^j)A_j(1 - \cosh(\frac{j\pi L_Z}{L_Y})) + D_j(-1)^j(-1 + \cosh(\frac{j\pi L_Y}{L_Z}))] &= 0, & (Z=L_Z) \end{aligned} \right\}, \quad (7.32)$$

$$\int T_{xz}(Z=0)YdXdY - \int T_{xz}(Z=L_Z)YdXdY = k^2\mu(2\sum_{j=1}^{\infty} A_j L_X L_Y (-1)^j (\cosh \frac{j\pi L_Z}{L_Y} - 1) - \sum_{j=1}^{\infty} \frac{L_X}{j\pi} D_j (-j\pi L_Y \cos \frac{j\pi L_Y}{L_Z} + L_Z \sin \frac{j\pi L_Y}{L_Z}) (\cosh(j\pi) - 1)) = M_{xz}, \quad (7.33)$$

where A_j and D_j are given in Eqs. (7.25) and (7.26). It can be seen that F_{xy} in Eq.

(7.31) contains both linear (k) and nonlinear terms (k^2). Hence, we take

$F_{xy} = kF_1 + k^2F_2$. Eqs. (7.31) and (7.32) can be written out, after some elementary

rearrangement, as the following four equations:

$$k\alpha\mu L_X L_Z + k^2\mu L_X \sum_{j=1,3,5}^{\infty} [2A_j(-1 + \cosh(\frac{j\pi L_Z}{L_Y})) + 2D_j] + k^2\mu L_X \sum_{j=2,4,6}^{\infty} [2A_j(-1 + \cosh(\frac{j\pi L_Z}{L_Y}))] = kF_1 + k^2F_2, \quad (7.34)$$

$$k\alpha\mu L_X L_Z + k^2\mu L_X \sum_{j=1,3,5}^{\infty} [-2A_j(-1 + \cosh(\frac{j\pi L_Z}{L_Y})) + 2D_j \cosh(\frac{j\pi L_Y}{L_Z})] + \quad (7.35)$$

$$k^2\mu L_X \sum_{j=2,4,6}^{\infty} [2A_j(-1 + \cosh(\frac{j\pi L_Z}{L_Y}))] = kF_1 + k^2F_2,$$

$$k^2\mu L_X \sum_{j=1,3,5}^{\infty} 2A_j(\cosh(\frac{j\pi L_Z}{L_Y}) - 1) = k^2\mu L_X \sum_{j=1,3,5}^{\infty} D_j(\cosh(\frac{j\pi L_Y}{L_Z}) - 1), \quad (7.36)$$

$$k^2\mu \sum_{j=2,4,6}^{\infty} L_X [D_j(-1 + \cosh(\frac{j\pi L_Y}{L_Z}))] = 0. \quad (7.37)$$

From Eq. (7.37), which involves summation over even values of n , it can be seen that $D_j = 0$. Eq. (7.26) then shows that $g(Z)$ must be symmetric to $Z = L_Z/2$. As we restrict our study to symmetric prescribed displacement, we restrict to the case of $D_j = 0$ for n even. Substituting the above Eq. (7.36) into Eqs. (7.34) and (7.35) leads to the same equation with its linear and nonlinear parts shown below:

$$kF_1 = k\alpha\mu L_X L_Z = k\mu u_1(L_Y, Z)L_X L_Z / L_Y, \quad (7.38)$$

$$k^2(F_2 - \mu L_X \sum_{j=1,3,5}^{\infty} D_j (\cosh \frac{j\pi L_Y}{L_Z} + 1)) = k^2 \mu L_X \sum_{j=2,4,6}^{\infty} [2A_j (-1 + \cosh(\frac{j\pi L_Z}{L_Y}))]. \quad (7.39)$$

It can be seen that Eq. (7.38) is the solution for simple linear shear, in which the strain α is proportional to the stress $F_1 / L_X L_Z$ with the coefficient of shear modulus μ . From Eqs. (7.33), (7.36) and (7.39), F_2 can be solved in terms of the prescribed M_{xz} and D_j (known from prescribed displacement, see Eq. (7.26)):

$$F_2 = \frac{M_{xz}}{L_Y} + \mu L_X \sum_{j=1,3,5}^{\infty} (D_j \cosh \frac{j\pi L_Y}{L_Z}) + \sum_{j=1}^{\infty} \frac{\mu L_X}{j\pi L_Y} D_j (-j\pi L_Y \cos \frac{j\pi L_Y}{L_Z} + L_Z \sin \frac{j\pi L_Y}{L_Z}) (\cosh(j\pi) - 1). \quad (7.40)$$

A_j for j odd and even can be determined separately from Eqs. (7.36) and (7.39).

When j is odd, if Eq. (7.36) is satisfied for every j , then A_j can be derived as:

$$A_j = \frac{D_j (\cosh(\frac{j\pi L_Y}{L_Z}) - 1)}{2(\cosh \frac{j\pi L_Z}{L_Y} - 1)}. \quad (7.41)$$

When j is even, Eq. (7.39) can be used to determine A_j . A_j will be determined in next section.

Using the orthogonality of sine functions, Eq. (7.25) can be inverted to yield:

$$f(Y) = \sum_{j=1}^{\infty} A_j \sinh \frac{j\pi L_Z}{L_Y} \sin \frac{j\pi Y}{L_Y} = \sum_{j=1,3,5}^{\infty} A_j \sinh \frac{j\pi L_Z}{L_Y} \sin \frac{j\pi Y}{L_Y} + \sum_{j=2,4,6}^{\infty} A_j \sinh \frac{j\pi L_Z}{L_Y} \sin \frac{j\pi Y}{L_Y}. \quad (7.42)$$

It can be seen that $f(Y)$, which gives the shape of the block when viewed along the Z -direction, is a superposition of sine curves. Substituting Eq. (7.41) into Eq. (7.24), u_2 can be obtained. Thus the shear displacement U can be written as:

$$\begin{aligned} U &= u_1(Y) + u_2(Y, Z) \\ &= \alpha Y + \sum_{j=1,3,5}^{\infty} \left(\frac{D_j (\cosh(\frac{j\pi L_Y}{L_Z}) - 1)}{2(\cosh \frac{j\pi L_Z}{L_Y} - 1)} \left(\sin(\frac{j\pi Y}{L_Y}) \sinh(\frac{j\pi(L_Z - Z)}{L_Y}) + \sin(\frac{j\pi Y}{L_Y}) \sinh(\frac{j\pi Z}{L_Y}) \right) \right. \\ &\quad \left. + \sum_{j=1,3,5}^{\infty} (D_j \sinh(\frac{j\pi Y}{L_Z}) \sin(\frac{j\pi Z}{L_Z})) \right. \\ &\quad \left. + \sum_{j=2,4,6}^{\infty} A_j \left(\sin(\frac{j\pi Y}{L_Y}) \sinh(\frac{j\pi(L_Z - Z)}{L_Y}) + \sin(\frac{j\pi Y}{L_Y}) \sinh(\frac{j\pi Z}{L_Y}) \right) \right). \end{aligned} \quad (7.43)$$

In order to calculate A_j , $g(Z)$ should be given. Different shear displacements $g(Z)$ are considered next.

7.1.4. Solutions to particular problems

In this section, various shear displacements in the X -direction at the top face of the block are imposed. The prescribed displacement $U(L_Y, Z)$ is the sum of a linear part and a nonlinear part, i.e., $U(L_Y, Z) = u_1(L_Y) + u_2(L_Y, Z) = u_1(L_Y) + g(Z)$. Different constant values of $u_1(L_Y)$ and different functions $g(Z)$ can be prescribed. Three different forms of $g(Z)$ are chosen, as represented by constant, quadratic and sine functions. For these three functions, the lateral profile $f(Y)$ and the total displacement $U(Y, Z)$ everywhere in the block are derived in the three subsections to follow.

7.1.4.1. Constant shear displacement

In the first problem, $g(Z)$ is chosen as $g(Z) = \beta$, where β is a constant.

Substituting $g(Z) = \beta$ into Eq. (7.26), D_j can be obtained as:

$$D_j = \frac{2\beta}{j\pi \sinh(\frac{j\pi L_Y}{L_Z})} (1 - (-1)^j) \quad (7.44)$$

Substituting Eq. (7.44) into Eq. (7.39) yields:

$$\mu L_X \sum_{j=2,4,6}^{\infty} [2A_j (-1 + \cosh(\frac{j\pi L_Z}{L_Y}))] = F_2 - \mu L_X \sum_{j=1,3,5,\dots}^{\infty} \frac{4\beta}{j\pi} \frac{(1 + \cosh(\frac{j\pi L_Y}{L_Z}))}{\sinh(\frac{j\pi L_Y}{L_Z})} \quad (7.45)$$

Since $\lim_{j \rightarrow \infty} (1 + \cosh(j\pi L_Y / L_Z)) / \sinh(j\pi L_Y / L_Z) = 1$ and $\sum_{j=1,3,5,\dots}^{\infty} 1/j$ does not converge, Eq. (7.45) does not converge, unless $\beta = 0$. Thus, $\beta = 0$, which implies that in the second-order model we cannot prescribe a constant nonlinear displacement on the top face. Also, Eq. (7.44) shows that $D_j = 0$ for j both odd and even.

Next, consider j odd. Eq. (7.41) then becomes:

$$A_j = \frac{D_j (\cosh(\frac{j\pi L_Y}{L_Z}) - 1)}{2(\cosh \frac{j\pi L_Z}{L_Y} - 1)} = 0. \quad (7.46)$$

Since $A_j = 0$ for j odd, $f(Y)$ consists of only even terms, and from Eq. (7.42):

$$f(Y) = \sum_{j=1}^{\infty} A_j \sinh \frac{j\pi L_Z}{L_Y} \sin \frac{j\pi Y}{L_Y} = \sum_{j=2,4,6}^{\infty} A_j \sinh \frac{j\pi L_Z}{L_Y} \sin \frac{j\pi Y}{L_Y}. \quad (7.47)$$

In principle, A_j can be determined from minimization of the strain energy of the block subject to the constraint Eq. (7.39). A_j decreases rapidly with j because of the presence of the hyperbolic sine term in Eq. (7.47). On the other hand, our experimental results show that the second-order displacement of the side of the block is approximated well by a complete one-period sine curve. Hence, we choose $A_j = 0$, $j \neq 2$, and A_2 can then be solved from Eq. (7.39) as:

$$A_2 = \frac{F_2 - \mu L_X \sum_{j=1,3,5}^{\infty} D_j (\cosh \frac{j\pi L_Y}{L_Z} + 1)}{2\mu L_X (\cosh \frac{2\pi L_Z}{L_Y} - 1)}, \quad (7.48)$$

which, since $D_j = 0$, reduces to:

$$A_2 = \frac{F_2}{2\mu L_X (\cosh \frac{2\pi L_Z}{L_Y} - 1)}. \quad (7.49)$$

The force F_2 is determined from Eq. (7.40) as:

$$F_2 = \frac{M_{xz}}{L_Y}. \quad (7.50)$$

The final form of $f(Y)$ is thus:

$$f(Y) = \frac{M_{xz}}{2\mu L_X L_Y (\cosh \frac{2\pi L_Z}{L_Y} - 1)} \sinh \frac{2\pi L_Z}{L_Y} \sin \frac{2\pi Y}{L_Y}, \quad (7.51)$$

and the total shear displacement in Eq. (2.42) becomes:

$$U = u_1(Y) + u_2(Y, Z) = \frac{u_1(L_Y)}{L_Y} Y + \frac{M_{xz}}{2\mu L_X L_Y (\cosh \frac{2\pi L_Z}{L_Y} - 1)} \left(\sinh \frac{2\pi(L_Z - Z)}{L_Y} + \sinh \frac{2\pi Z}{L_Y} \right) \sin \frac{2\pi Y}{L_Y}. \quad (7.52)$$

The above result is a second-order solution for a block subjected to a prescribed constant top face displacement $u_1(L_Y) + \beta = u_1(L_Y)$ and a prescribed constant moment M_{xz} about the Z -axis. For these boundary conditions, the lateral profile is a sine curve (nonlinear shear displacement) superimposed on a straight line (linear shear displacement). There is also a hyperbolic sine dependence on Z even though the prescribed displacement is uniform at the top face. This is a consequence of the requirement of the equilibrium equation which reduces to the Laplace equation.

7.1.4.2. Parabolic shear displacement

In the second problem, $g(Z)$ is chosen as $g(Z) = \gamma Z(L_Z - Z)/L_Z$, where γ is an arbitrary constant. This is a parabolic profile, symmetric about $Z = L_Z/2$. D_j can be obtained using Eq. (7.26) as:

$$D_j = \frac{4\gamma L_Z}{j^3 \pi^3 \sinh(\frac{j\pi L_Y}{L_Z})} (1 - (-1)^j) \quad . \quad (7.53)$$

Using Eqs. (7.53), Eq. (7.41) becomes, for j odd:

$$A_j = \frac{D_j (\cosh(\frac{j\pi L_Y}{L_Z}) - 1)}{2(\cosh(\frac{j\pi L_Z}{L_Y}) - 1)} = \frac{4\gamma L_Z}{j^3 \pi^3 \sinh(\frac{j\pi L_Y}{L_Z})} \frac{(\cosh(\frac{j\pi L_Y}{L_Z}) - 1)}{(\cosh(\frac{j\pi L_Z}{L_Y}) - 1)} \quad . \quad (7.54)$$

For j even, A_j should satisfy Eq. (7.39). The form of $f(Y)$, from Eq. (7.42), is thus:

$$f(Y) = \sum_{j=1,3,5}^{\infty} \left(\frac{4\gamma L_z}{j^3 \pi^3 \sinh(\frac{j\pi L_y}{L_z})} \frac{(\cosh \frac{j\pi L_y}{L_z} - 1)}{(\cosh \frac{j\pi L_z}{L_y} - 1)} \sinh \frac{j\pi L_z}{L_y} \sin \frac{j\pi Y}{L_y} \right) + \sum_{j=2,4,6}^{\infty} A_j \sinh \frac{j\pi L_z}{L_y} \sin \frac{j\pi Y}{L_y}. \quad (7.55)$$

The total shear displacement U can be obtained by substituting the corresponding D_j and A_j into Eq. (7.43). The form of Eq. (7.55), which satisfies the equilibrium equation, shows that $f(Y)$ is in general a superposition of harmonics. The first part on the right-hand side of Eq. (7.55) can be summed numerically over odd values of j until convergence is reached. For the second part, it may only be necessary to sum over a finite number of even terms, and the values of A_j can be determined by comparing with the experimental profile, subject to the constraint of Eq. (7.39).

7.1.4.3. Sinusoidal shear displacement

In the final problem, $g(Z)$ is chosen as $g(Z) = \gamma \sin \pi Z / L_z$, where γ is an arbitrary constant. D_j can be obtained using Eq. (7.26) as:

$$D_1 = \frac{\gamma}{\sinh(\frac{\pi L_y}{L_z})}, \quad D_j = 0 \quad (j \neq 1). \quad (7.56)$$

Substituting the above D_j into Eq. (7.41), A_j for $j = 1$ can be obtained as:

$$A_1 = \frac{D_1(\cosh \frac{\pi L_Y}{L_Z} - 1)}{2(\cosh \frac{\pi L_Z}{L_Y} - 1)} = \frac{\gamma(\cosh \frac{\pi L_Y}{L_Z} - 1)}{2(\cosh \frac{\pi L_Z}{L_Y} - 1) \sinh(\frac{\pi L_Y}{L_Z})}, \quad (7.57)$$

with A_3, A_5, A_7 , etc. all equal to zero. For j even, A_j should satisfy Eq. (7.39). The form of $f(Y)$ can be obtained from Eq. (7.42) as:

$$f(Y) = \frac{\gamma(\cosh \frac{\pi L_Y}{L_Z} - 1)}{2(\cosh \frac{\pi L_Z}{L_Y} - 1) \sinh(\frac{\pi L_Y}{L_Z})} \sinh \frac{\pi L_Z}{L_Y} \sin \frac{\pi Y}{L_Y} + \sum_{j=2,4,6}^{\infty} A_j \sinh \frac{j\pi L_Z}{L_Y} \sin \frac{j\pi Y}{L_Y}. \quad (7.58)$$

As for the other cases, the total shear displacement U can be obtained by substituting the corresponding D_j and A_j into Eq. (7.43).

7.1.5. Solutions for generalized shear with gravity

In this section, the solutions of a rectangular block subjected to a constant shear displacement δ on the top face in the presence of gravity g are presented. The equilibrium equations are given by Eqs. (7.11)–(7.15). Specifically, Eq. (7.12) for u_2 reduces to a Poisson equation, instead of a Laplace equation. The solutions can be obtained by solving Eqs. (7.11)–(7.15) with the following boundary conditions:

(BC 1). The prescribed displacements on the Y -faces (top and bottom) are δ and zero respectively.

(BC 2). No shear forces are applied on the Z -faces (front and back).

(BC 3). The moments prescribed on the Z -faces are M_{xz} .

Using similar procedures, it can be shown that the lateral profile $f(Y)$ can be written as:

$$f(Y) = \frac{M_{xz}}{2\mu L_x L_y (\cosh \frac{2\pi L_z}{L_y} - 1)} \sinh \frac{2\pi L_z}{L_y} \sin \frac{2\pi Y}{L_y} - \frac{\alpha g \rho_0 (\lambda + 2\mu + m)}{2(\lambda + 2\mu)\mu} Y^2, \quad (7.59)$$

where the linear shear strain α is:

$$\alpha = \frac{\delta}{L_y - \frac{\rho_0 g (\lambda + 2\mu + m)}{2(\lambda + 2\mu)\mu} L_y^2}. \quad (7.60)$$

The total shear displacement U is:

$$U = \alpha Y + \frac{M_{xz}}{2\mu L_x L_y (\cosh \frac{2\pi L_z}{L_y} - 1)} (\sinh \frac{2\pi(L_z - Z)}{L_y} + \sinh \frac{2\pi Z}{L_y}) \sin \frac{2\pi Y}{L_y} - \frac{\alpha g \rho_0 (\lambda + 2\mu + m)}{2(\lambda + 2\mu)\mu} Y^2, \quad (7.61)$$

By comparing Eqs. (2.58) and (2.60) with Eqs. (2.50) and (2.51), it can be seen that gravity has an additional effect, represented by $-\alpha g \rho_0 (\lambda + 2\mu + m) Y^2 / 2(\lambda + 2\mu)\mu$, on the lateral profile and the shear displacement. The lateral profile takes the form of a straight line superimposed by a sine curve and a parabolic curve. In contrast,

Destrade and Saccomandi (2010) showed that under gravity the lateral profile is a straight line superimposed by a parabolic curve only. This difference arises because we have taken U to be a function of both Y and Z whereas they have taken it to be a function of Y only.

7.1.6. Numerical examples and experiments

In this section, we illustrate the key findings through numerical examples. Specifically, we show that if gravity is neglected: (a) the nonlinear displacement within the block can be described by the superposition of the products of hyperbolic sine functions of the depth Z and sine functions of the height Y , (b) the lateral profile of the deformed block under shear can be described by the superposition of a straight line and a complete (one-period) sine curve, (c) an out-of-plane shear stress T_{xz} is induced, which further distorts the block, and (d) normal stresses are also induced, which demonstrates the Poynting effect or the dilation/shrinking of the block. Simple experiments have also been conducted to show that the lateral profile is well-approximated by theory.

7.1.6.1. Theoretical predictions of the shape of the deformed block

Figs. 7.2(a)-(c) show the predicted shapes of a block when subjected to three different prescribed shear displacements on $Y = L_Y$: constant, parabolic and sinusoidal, respectively. For all three cases, the block dimensions are $L_X = 0.024$ m,

$L_Y = 0.072 \text{ m}$ and $L_Z = 0.020 \text{ m}$, and the prescribed moment

$$M_{xz} / \mu = -7.29 \times 10^{-6} \text{ m}^3.$$

Fig. 7.2. Deformed and undeformed states for a rectangular block under generalized shear. The prescribed nonlinear displacement $g(Z)$ at the top face is assumed to be: (a) constant, (b) quadratic, and (c) sinusoidal. The maximum shear displacements at the top and the applied moments M_{xz} on the $Z = 0, L_Z$ faces are the same in all three cases.

The prescribed displacements for Figs. 7.2(a)-(c) are given by $u_1(L_Y) = 0.04$ m and $g(Z) = 0$, $u_1(L_Y) = 0.03$ m and $g(Z) = 2Z (L_Z - Z)/L_Z$, and $u_1(L_Y) = 0.03$ m and $g(Z) = 0.01\sin(\pi Z / L_Z)$, respectively. The maximum shear displacements in all three cases equal 0.04 m. All the results are generated with $A_4 = A_6 = \dots = 0$, with A_2 calculated from M_{xz} using Eqs. (2.38) and (2.39).

It can be seen that the deformed shapes are curved with respect to both the X – Y and X – Z planes, including that under a constant prescribed shear displacement, although the curvature is not that obvious in the X – Z plane. The curvature in the X – Z plane has its origin in the hyperbolic sine term, while that in the X – Y plane reflects the sine term. The $\sin 2\pi Y/L_Y$ term results in a point of inflection in the lateral profile, which differs from what a Y^2 dependence would predict.

7.1.6.2. Experimental and theoretical studies of the lateral profile of the deformed block

To verify the lateral profile $f(Y)$, we carried some simple experiments using an edible agar-gelatin consumed as a dessert. This was easily made by dissolving different amounts of the powder in water, resulting in soft blocks of various densities. Figs. 7.3 and 7.4 show a gel made by dissolving 300 grams of powder into 1.25 and 1 liters of water, respectively. The experiments were carried out for a prescribed uniform shear: the bottom face was encased in a holder and moved rigidly to the right while holding the upper face in an immovable holder.

Fig. 7.3. Rectangular blocks of agar-gelatin gel, with $L_X = 0.024$ m, $L_Y = 0.072$ m, $L_Z = 0.014$ m, subjected to the constant shear displacement (a) 0.016 m and (b) 0.029 m at the top faces. Each block was gripped at the top and the bottom was sheared rightward. The theoretical lateral profiles are shown as dashed lines. Agreement between theory and experiments is good, with some noticeable discrepancy in the case of the larger prescribed shear.

Fig. 7.3 compares the shapes of the block subjected to two different uniform shear displacements. The block dimensions are $L_X = 0.024$ m, $L_Y = 0.072$ m, $L_Z = 0.014$ m. The prescribed total displacements are 0.016 m and 0.029 m, for Figs. 7.3(a) and (b), respectively. The lateral profiles are as shown in the photos contained in Fig. 7.3. They look similar to the predicted profile shown in Fig. 7.2(a), i.e., a superposition of a straight line and a complete sine curve with a point of inflection. The experimental lateral profiles were then fitted using our model without considering gravity, with the linear shear strain $\alpha = 0.222$ for Fig. 7.3(a) and $\alpha =$

0.403 for Fig. 7.3(b), calculated in each case by dividing the total shear displacement by the height of the block. The amplitudes of the sine curve, given by $A_2 \sinh(2\pi L_Z/L_Y)$, were estimated from the experimental profiles to be $A_2 = 0.65$ mm and 1.94 mm, respectively. The dotted lines show the theoretical predictions. It can be seen that the theoretical and experimental lateral profiles agree quite well for these rather large prescribed displacements. There is, however, some discrepancy in the case of the larger prescribed displacement. This may be attributed to three effects: (a) the inaccuracy of a second-order nonlinear elasticity model, (b) the additional displacements which arise because the block is not constrained to support the second-order normal stresses (Poynting effect), and (c) the gravity effect.

To demonstrate the influence of gravity, we carried out experiments with very long gels, with some blocks standing vertically as before and with others lying flat on a surface. For the latter, gravity should have little effect on their lateral profiles. Fig. 7.4 shows two typical examples for an agar-gelatin with $L_X = 0.055$ m, $L_Y = 0.163$ m, $L_Z = 0.025$ m. The prescribed shear displacements are 0.088 and 0.082 m, respectively for Figs. 7.4(a) and (b). Also, $\alpha = 0.54$ and 0.50, and $A_2 = 7.14$ mm for the two figures. The dotted lines show the theoretical predictions of the lateral profiles. There is greater discrepancy between theory and experiment when the block is standing up vertically than lying flat on the surface. This implies that gravity does have an effect on the deformed shape of the soft gel. However, the elastic nonlinearity and the Poynting effect may have an effect as well, as a discrepancy also exists when the

Fig. 7.4. Shear displacement of (a) vertically standing and (b) horizontally lying gels. The dashed lines are the theoretical predictions. The effect of gravity is highlighted in (a). However, significant nonlinearity and the Poynting effect may also have resulted in a discrepancy, as shown in (b), where gravity plays a minor role.

block lies flat on its side; for this case gravity should play a much smaller role in the profile.

Fig. 7.5 plots the difference between the lateral profiles at $Y = L_Y/2$ predicted with and without gravity. This difference is given by $\Delta U = U_{\text{no gravity}}(L_Y/2, L_Z) - U_{\text{with gravity}}(L_Y/2, L_Z)$. The parameters used are $L_X = 0.024$ m, $L_Y = 0.072$ m, $L_Z = 0.02$ m, $\alpha = 0.08$, and $A_2 = -1.44 \times 10^{-4}$ m. Figs. 7.5(a)

Fig. 7.5. Dependence of the difference ΔU between the lateral profiles at mid-height of a block considering and without considering gravity on the elastic constants. The block is subjected to a constant shear displacement of 5.76 mm at the top. (a) Dependence on μ and ν , (b) dependence on μ and m . ΔU is larger at small μ , small ν and large negative values of m .

and (b) plot ΔU versus μ for different values of ν and m , with $m = -50000$ Pa and $\nu = 0.45$, respectively. In order to show the effect of the Poisson ratio ν , λ is replaced by $2\mu\nu/(1-2\nu)$. The figure shows that the difference is maximum when μ approaches zero and vanishes at large values of μ (> 400 Pa, say), regardless of m and ν . On the other hand, Fig. 7.5 also shows that the difference is less important if the material is nearly incompressible or if m has a small negative value such as -1000 Pa. Roughly speaking, ΔU lies in the range of 10^{-4} to 10^{-3} m, which is one order of magnitude smaller than L_X . This implies that the gravity may play some role in the deformation of the block. In conclusion, the gravity effect becomes less significant for large μ , large ν and small negative m .

7.1.6.3. Investigation of the Poynting effect

Janmey et al. (2007) have studied the relations between the applied shear strain and the Cauchy stress components T_{Cxy} and T_{Cyy} for various soft materials. Fig. 7.6(a) shows these relations for matrigel and polyacrylamide for shear strains of up to 100%. It can be seen that matrigel experiences a negative normal stress, whereas polyacrylamide a positive normal stress, implying that the two materials exhibit negative and positive Poynting effect, respectively.

According to Eq. (7.27), second-order normal stresses are also induced in the generalized shear. The first-Piola Kirchhoff stress $\mathbf{T}$ can be converted to the Cauchy stress $\mathbf{T}_C$ (with components denoted by T_{Cxy} , T_{Cyy} , etc.) using the formula $\mathbf{T}_C = (1/\det \mathbf{F})\mathbf{T}\mathbf{F}^*$, with $\det \mathbf{F} = 1$. The Cauchy stress components T_{Cxy} and T_{Cyy} are

plotted in Fig. 7.6(b), with $\lambda = 1350$, $\mu = 150$, $m = -1000$ Pa for matrigel and $\lambda = 585$, $\mu = 65$, $m = -780$ Pa for polyacrylamide. Here we have also taken $A_2 = A_4 = \dots = 0$. Figs. 7.6(a) and (b) compare well qualitatively.

The experimental and theoretical results above show the dependence of the Poynting effect on the elastic parameters. Figs. 7.7(a) and (b) plot the variation of the stress ratio T_{Cyy}/T_{Cxy} with the shear modulus μ for different values of the Poisson's ratio ν and the third-order elastic constant m , respectively. In Fig. 7.7(a), where $m = -2000$ Pa, several results are noteworthy. First, the closer ν is to 0.5, the more likely the Poynting effect will be negative. For soft materials which are nearly incompressible, the negative Poynting effect may thus be prevalent. Second, μ also appears to have a significant influence. If $\mu > \sim 600$ Pa, a material with high or near incompressibility will experience a negative Poynting effect. If μ is very small, less than 100 Pa, say, then the Poynting effect is positive if ν is less than ~ 0.47 . In Fig. 7.7(b), where $\nu = 0.45$, the importance of m is highlighted. It can be observed that for $m > 0$, the negative Poynting effect is predicted regardless of the value of μ , while for $m < 0$, the effect depends on the value of μ . For example, the positive Poynting effect is predicted for small μ in the case of $m < 0$. The dependence of the Poynting effect on the second- and third-order elastic constants has been studied by the authors for the torsion of homogeneous and composite cylindrical bars (Wang and Wu, 2014).

Fig. 7.6. Comparison of (a) experimental shear (T_{Cxy}) and normal (T_{Cyy}) stresses versus shear strain with (b) theoretical predictions. The gels were made from matrigel and polyacrylamide (Janmey et al., 2007). Negative and positive normal stresses were found to be present, and were also predicted by theory.

Fig. 7.7. (a) Dependence of the stress ratio T_{Cyy}/T_{Cxy} with μ for different values of (a) ν between 0.3 and 0.495, and (b) m between -2000 and 2000 Pa. The induced normal stress becomes important at small μ , large ν , and large positive or negative m .

7.1.6.4. Out-of-plane shear stress

A second-order out-of-plane shear stress T_{xz} arises in generalized shear when the nonlinear displacement u_2 is a function of both Y and Z . This shear component distorts the lateral profile into a curve with an inflection point. In the second-order elasticity model, the profile is a superposition of a straight line with a complete sine curve, which well approximates our experimental results. Fig. 7.8(a) plots the ratio T_{xz} / T_{xy}^L versus the Y - and Z -coordinates, where the parameters are $L_X = 0.024$ m , $L_Y = 0.072$ m, $L_Z = 0.02$ m, $\alpha = 0.05$, and $A_2 = -1.44 \times 10^{-4}$ m. T_{xz} is of the same order of magnitude as T_{xy}^L , which is the in-plane shear stress corresponding to the linear shear strain α . Hence, T_{xz} is not negligible. Similarly, the nonlinear component T_{xy}^{NL} is also not negligible compared to T_{xy}^L . Fig. 7.8 shows that all the second-order shear stresses are generally of significant magnitude and play a fundamental role in the distortion of a soft block. Furthermore, Eq. (7.27) shows that the first- and second-order shear stresses are all proportional to μ and not to the other elastic constants.

Fig. 7.8. Dependence of the ratio of the nonlinear to linear shear stress (a) T_{xz} / T_{xy}^L and (b) T_{xy}^{NL} / T_{xy}^L on the Y and Z coordinates of a rectangular block under generalized constant shear. It can be observed that the nonlinear stresses are generally significant. Hence the simple shear model, which predicts $T_{xz} / T_{xy}^L = 0$, is inappropriate for soft gels, generally speaking.

7.1.7. Discussion

The essential difference between the current and previous works lies in the different deformation assumptions. The current work, which assumes that the second-order displacement u_2 is a function of both Y and Z , is represented by Eq. (7.5). The other works assume that it is a function of Y only, as shown in Eqs. (7.1)-(7.4). In the context of second-order elasticity, the equilibrium equation $k^2 \mu (\partial^2 u_2 / \partial Y^2 + \partial^2 u_2 / \partial Z^2) = 0$, i.e., Eq. (7.17), predicts that u_2 can at most be a linear function of Y if u_2 is a function of only Y . With a dependence on both variables, Eq. (7.17) is the Laplace equation and the solution takes the form of the product of sine and hyperbolic sine functions. Experimentally, u_2 is not seen to be a linear function of Y and hence the assumption of a dual dependence on Y and Z is justified. Furthermore, we can deduce from Eq. (7.17) that the assumption of $u_2 = Y^2$ does not satisfy the equilibrium equation. Hence, the validity of the assumption of a quadratic profile should be checked for other constitutive models. The general equation for the lateral profile, Eq. (7.42), shows that soft blocks which deform without a point of inflection over the height of the block can also be modeled, e.g., by the superposition of a straight line and a half sine curve. This profile would then resemble a quadratic one, and would also satisfy the equilibrium equation. In a more general sense, Eq. (7.42) is like a Fourier series, which can mathematically model any smooth curve.

The assumption of Eq. (7.5) also results in the existence of a second-order in-plane shear stress T_{xy}^{NL} as well as a second-order out-of-plane shear stress T_{xz} .

These shear stresses are not predicted if the other deformation assumptions are made as in Eqs. (7.1)-(7.4). They may also be significant in magnitude, as shown in Fig. 7.8. T_{xz} is analogous to, but distinct from, the normal stresses which correspond to the Poynting effect. In other words, the Poynting effect exists under simple shear, and both T_{xz} and the Poynting effect exist under generalized shear. This implies that under generalized shear, a soft block will undergo (i) an in-plane shear, (ii) a stretching or contraction in the normal directions (positive or negative Poynting effect), and (iii) an out-of-plane shear tending to distort the block about the Y -axis. If the deformations in (ii) and (iii) are constrained, the corresponding normal and out-of-plane shear stresses will be induced.

In real physiological and engineering environments, biomaterials may experience generalized shear rather than simple shear because of the presence of multiaxial loads and constraints. For instance, T_{xz} distorting a cell under generalized shear may have a significant effect on the overall force balance in the cytoskeleton. It may also alter the structure of the cell, which is sensitive to the stresses it experiences. A case in point is the uni-directional shearing of blood cells, which however may experience shearing deformation in all three planes. The red blood cell may thus be deformed, damaged or undergo structural changes (Leverett et al., 1972; Meram et al., 2013).

Second, our results may have interesting implications in the design of actuators and sensors. A multi-plane actuator can be designed, based on the phenomenon that a soft material can undergo an out-of-plane distortion under generalized shear in one

direction. The distortion of the plane may generate a moment, which can be used to actuate a connecting device. Similarly, a sensor can be designed to detect and measure the deformation in a plane which is orthogonal to the signal occurring in another plane.

The sign of the normal stresses shown in Eq. (7.27) can be explicitly used to judge the direction of the Poynting effect. Eq. (7.27) may also be used in a reverse manner to determinate the third-order elastic constants, assuming that the second-order constants λ and μ have been determined by other means. A simple illustration is as follows. By measuring the T_{yy} necessary to maintain a generalized shear deformation of a rectangular block, the elastic constant m of the material can be estimated, as T_{yy} is dependent on λ , μ and m . Next, the value of n can be estimated by measuring T_{zz} , which is dependent on λ , m and n . As pointed out by Janmey et al (2007) and predicted by our numerical results, the normal stresses can be as large as the shear stress. Thus, measuring the normal stresses may be a simple way to estimate the third-order elastic constants and provides an alternative to acousto-elastic methods (Catheline et al., 2003; Destrade et al., 2010).

Although this work considers second-order nonlinearity and generalized shear, it does not consider inhomogeneity, anisotropy, time-independent viscoelasticity, and higher-order nonlinearity. Many natural biomaterials are composites, architecturally hierarchical and exhibit time-dependent behavior. Nonlinearity can be investigated with higher-order elasticity models. The study of composite materials poses a significant challenge, which possibly can be met by adapting the method used in the

present work. Anisotropy can also be incorporated, through the use of additional invariants in the strain energy density. Viscoelasticity adds considerable difficulty to the problem, which most likely requires numerical methods of solutions. A natural extension of our current work is to the problem of composite materials under generalized shear.

Finally, the existence and uniqueness of solutions in nonlinear elastic problems deserve a special mention here, as interpretations of nonlinear phenomena such as the Poynting effect would depend on these solutions. The theorems of existence, uniqueness and analyticity of solutions for boundary value problems of nonlinear bodies have been studied by various researchers, e.g., Ball (1977), Ciarlet (1988), Valent (1988). Valent (1988) proved a number of theorems which permitted the use of Signorini's method for solving pure traction-value problems of spheres and cylinders subjected to uniform pressure. In the work reported here, the nonlinear effects, as predicted within the second-order elasticity framework of Murnaghan (1951), depend on the elastic constants. The values of these constants were derived from experimental data of real materials. Generally speaking, however, an investigation of the dependence of the nonlinear effects on the elastic constants should be treated with caution, especially if the constants have not been measured directly, or if they have not been determined via some experimental data of the mechanical response.

7.1.8. Conclusion

The analytical solutions for a homogeneous rectangular block under generalized shear in the framework of second-order elasticity were developed. The major contributions and findings can be summarized as follows.

First, the analytical solutions show that the second-order shear displacements are characterized by the product of sine and hyperbolic sine functions of the height and depth variables, respectively. The solutions also reveal that the lateral profile is a straight line superimposed on sine curves under a prescribed constant shear displacement. This profile is verified by experiments.

Second, an out-of-plane nonlinear shear stress is generated under generalized shear, which may twist/distort the block about the height direction. This phenomenon may have implications in the design of biomaterials and bio-inspired materials. .

Third, nonlinearity is a significant part of the displacements and stresses for soft materials, e.g., the out-of-plane shear stress.

Fourth, nonlinear normal stresses, which are related to the Poynting effect, are needed to maintain the deformation. The Poynting effect is significantly dependent on the elastic constants, e.g., ν and m . By measuring the normal stresses, third-order elastic constants can be estimated.

Finally, the effect of gravity has been investigated. Our results show that an extra quadratic term will arise in the dependence of the shear displacement on the

height in the presence of gravity. The gravity effect becomes less significant for large μ , large ν and small negative m .

7.2. Bilayered soft rectangular block under generalized shear

Much of the recent efforts have focused on homogeneous solids. However, bilayer systems under shear constitute an important class of problems. Many natural and synthetic material systems consist of two or more layers with different mechanical, chemical and other properties. Examples, to name a few, include skins, muscles, artificial tissue constructs, lipid bilayers, graphene bilayers, multilayer substrates for stem cell cultivation, and bio-inspired devices. The study of lipid bilayers under mechanical forces remains of considerable interest. Butler et al. (2001) studied the influence of shear on the lateral diffusion of endothelial cell membrane lipids. Jönsson et al., (2009) investigated the mechanical behavior of lipid bilayers in microfluidic channels under shear forces. Peng et al. (2013) developed computational models for lipid bilayers attached to the spectrin cytoskeletal network of red blood cell membranes. Furthermore, much interest has recently developed in graphene, which demonstrate nonlinear elasticity (Lu and Huang, 2009; Cadelano et al., 2009). Apparently the mechanical behavior of graphene bilayers is also rapidly gaining attention (Scarpa et al., 2010).

In this work, we consider the basic problem of a bilayer solid subjected to generalized shear in the framework of second-order elasticity. A uniform shear displacement is imposed at the top of the bilayer, whose bottom is held fixed. The shear is generalized in the sense that *a priori* assumption has not been made regarding the dependence of the shear displacement versus the vertical coordinate (along the height direction) of the block. The deformation takes the form:

$x = X + U(Y, Z)$, $y = Y + V(Y)$, $z = Z$, where x , y , z and X , Y , Z denote coordinates in the deformed and reference configurations, respectively, and U , V the displacements. The current study is based on the earlier work for a homogeneous soft block. The new contribution consists of the study of the dependence of (i) the induced normal stresses and shear stresses, and (ii) the interfacial stresses and displacements, on the dissimilar sets of elastic constants, as well as experimental verification of the predicted variation of the shear displacement with the vertical coordinate.

Following this introduction, we present the formulation, numerical/experimental results, and conclusions in four sections.

7.2.1. Definition of problem

We consider a bilayer rectangular block under generalized shear as shown in Fig 7.9. Each layer is homogeneous and isotropic. Denote X , Y , Z as the reference coordinates, with X along the direction of the imposed shear, Y along the height and Z along the depth. The lengths of the block in the X - and Z -directions are denoted by L_X and L_Z , respectively, while the thicknesses of the bottom (first) and top (second) layers are denoted by $L_Y^{(1)}$ and $L_Y^{(2)}$, respectively. Let also $L_Y = L_Y^{(1)} + L_Y^{(2)}$. The deformed coordinates are assumed to be $(X + U^{(i)}, Y + V^{(i)}, Z)$, with $i = 1, 2$ for the layers. Here, $U^{(i)} = u_1^{(i)}(Y, Z) + ku_2^{(i)}(Y, Z)$ denotes the shear displacement in the X -direction for the i -th layer, with $u_1^{(i)}$ and $u_2^{(i)}$ representing the linear and nonlinear components, respectively. $U^{(i)}$ is taken to be a function of both Y and Z , allowing for the possibility of a non-uniform (generalized) shear. No assumption is made

Fig. 7.9. (a) A bilayered rectangular block of dimensions L_X , $L_Y^{(1)} + L_Y^{(2)}$ and L_Z in the undeformed state. (b) The block deformed under generalized shear. The prescribed displacement at the top face $Y = L_Y$ consists of the linear part $u_1(L_Y)$ and the nonlinear part $u_2(Y = L_Y, Z) = h(Z)$. The problem of $h(Z) = 0$ is considered for simplicity. The lateral profile of the block, or the deformed shaped of the four vertical edges of the box, is written as $u_1(Y) + f(Y)$.

regarding the dependence of $U^{(i)}$ on Y or Z , as it will be solved from the equilibrium equations; this differs from the simple shear mode in which $U^{(i)}$ is assumed to be linearly dependent on Y only. Similarly, $V^{(i)} = v_1^{(i)}(Y) + kv_2^{(i)}(Y)$ denotes the displacement in the Y -direction, with $v_1^{(i)}$ and $v_2^{(i)}$ denoting the linear and nonlinear components. Note that $V^{(i)}$ is taken to be solely a function of Y . Gravity effect is neglected. Also, k is a marker indicating the order of approximation of the theory. In the subsequent formulation, we retain terms up to k^2 for the second-order theory.

7.2.2. Governing Equations

7.2.2.1. Second-order elasticity model

The differentials of position vectors in the undeformed state are:

$$\begin{pmatrix} dX \\ dY \\ dZ \end{pmatrix}, \quad (7.62)$$

while in the deformed state they are:

$$\begin{pmatrix} dX + k\left(\frac{\partial u_1^{(i)}}{\partial Y} dY + \frac{\partial u_1^{(i)}}{\partial Z} dZ\right) + k^2\left(\frac{\partial u_2^{(i)}}{\partial Y} dY + \frac{\partial u_2^{(i)}}{\partial Z} dZ\right) \\ dY + kdv_1^{(i)} + k^2 dv_2^{(i)} \\ dZ \end{pmatrix}, \quad (7.63)$$

for $i = 1, 2$ respectively. Based on Eq. (7.63), the deformation gradient $\mathbf{F}$ can be written as:

$$\mathbf{F} = \begin{pmatrix} 1 & k\frac{\partial u_1^{(i)}}{\partial Y} + k^2\frac{\partial u_2^{(i)}}{\partial Y} & k\frac{\partial u_1^{(i)}}{\partial Z} + k^2\frac{\partial u_2^{(i)}}{\partial Z} \\ 0 & 1 + k\frac{dv_1^{(i)}}{dY} + k^2\frac{dv_2^{(i)}}{dY} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (7.64)$$

In this formulation, $\mathbf{F}$ contains an additional term $k(\partial u_1^{(i)} / \partial Z) + k^2(\partial u_2^{(i)} / \partial Z)$,

which exists because $U^{(i)}$ is a function of both Y and Z .

The energy density W of Murnaghan (1951) is:

$$W = \frac{\lambda + 2\mu}{2} J_1^2 - 2\mu J_2 + \frac{l + 2m}{3} J_1^3 - 2m J_1 J_2 + n J_3, \quad (7.65)$$

where λ and μ are the second-order and l , m , n the third-order elastic constants, respectively, and J_1 , J_2 , and J_3 are the strain invariants of the Lagrangian strain $\mathbf{E}$:

$$J_1 = E_1 + E_2 + E_3, J_2 = E_1 E_2 + E_2 E_3 + E_3 E_1, \quad J_3 = E_1 E_2 E_3, \quad (7.66)$$

with E_1 , E_2 and E_3 denoting the principal values. The Lagrangian strain is expressed in terms of the deformation gradient as:

$$\mathbf{E} = \frac{1}{2} (\mathbf{F}^* \mathbf{F} - \mathbf{I}), \quad (7.67)$$

where $\mathbf{I}$ is the identity and the asterisk denotes the transpose. The first Piola-Kirchhoff stress (relating forces in the current configuration to area vectors in the undeformed configuration, and generally non-symmetric) is defined as:

$$\mathbf{T} = \mathbf{F} \frac{\partial W}{\partial \mathbf{E}} = \mathbf{F} (\lambda J_1 \mathbf{I} + 2\mu \mathbf{E} + (l J_1^2 - 2m J_2) \mathbf{I} + 2m J_1 \mathbf{E} + n \text{co } \mathbf{E}), \quad (7.68)$$

where $\text{co } \mathbf{E}$ is the cofactor matrix.

7.2.2.2. Equilibrium Equations

The Lagrangian equilibrium equations in Cartesian coordinates can be written as:

$$\begin{aligned}\frac{\partial T_{xX}}{\partial X} + \frac{\partial T_{xY}}{\partial Y} + \frac{\partial T_{xZ}}{\partial Z} &= 0, \\ \frac{\partial T_{yX}}{\partial X} + \frac{\partial T_{yY}}{\partial Y} + \frac{\partial T_{yZ}}{\partial Z} &= 0, \\ \frac{\partial T_{zX}}{\partial X} + \frac{\partial T_{zY}}{\partial Y} + \frac{\partial T_{zZ}}{\partial Z} &= 0,\end{aligned}\tag{7.69}$$

Again the notations T_{xX} , T_{xY} , etc. will be replaced by T_{xx} , T_{xy} , etc. Substituting the stresses from Eq. (7.68) into Eq. (7.69) yields the following two first-order (k) and four second-order (k^2) equations for each layer:

$$k\mu\left(\frac{\partial^2 u_1^{(i)}}{\partial Y^2} + \frac{\partial^2 u_1^{(i)}}{\partial Z^2}\right) = 0,\tag{7.70}$$

$$k^2\mu\left(\frac{\partial^2 u_2^{(i)}}{\partial Y^2} + \frac{\partial^2 u_2^{(i)}}{\partial Z^2}\right) = 0,\tag{7.71}$$

$$\frac{1}{4}k^2((4\lambda + 4\mu + 4m - n)\frac{\partial u_1^{(i)}}{\partial Z}\frac{\partial^2 u_1^{(i)}}{\partial Y\partial Z} + 4(\lambda + 2\mu + m)\frac{\partial u_1^{(i)}}{\partial Y}\frac{\partial^2 u_1^{(i)}}{\partial Y^2} + (4\mu + n)\frac{\partial u_1^{(i)}}{\partial Y}\frac{\partial^2 u_1^{(i)}}{\partial Z^2}) = 0,\tag{7.72}$$

$$\frac{1}{4}k^2((4\lambda + 4\mu + 4m - n)\frac{\partial u_1^{(i)}}{\partial Y}\frac{\partial^2 u_1^{(i)}}{\partial Y\partial Z} + 4(\lambda + 2\mu + m)\frac{\partial u_1^{(i)}}{\partial Z}\frac{\partial^2 u_1^{(i)}}{\partial Z^2} + (4\mu + n)\frac{\partial^2 u_1^{(i)}}{\partial Y^2}\frac{\partial u_1^{(i)}}{\partial Z}) = 0,\tag{7.73}$$

$$kv_1^{(i)''}(Y) = 0,\tag{7.74}$$

$$k^2v_2^{(i)''}(Y) = 0.\tag{7.75}$$

The first two are Laplace equations, and the last two are second-order linear

ordinary differential equations. The middle two are partial differential equations with mixed derivatives. These equations can be solved by applying appropriate boundary conditions.

7.2.3. General solutions

The basic boundary conditions applied to the block are: (1) the bottom of the block is fixed, (2) an applied linear displacement δ_2 is applied to the top face of the block, (3) the total height of the block is maintained, and (4) a restraining moment preventing the block from rotating about the Z -axis. This restraining moment is generally non-vanishing and arises from a second-order effect due to the existence of T_{xz} (shown later). It is denoted by M_{xz} in association with T_{xz} . Other conditions are the continuity of displacements and tractions across the interface, and no net shear forces are applied on the Z -faces. The objective is to obtain solutions for the displacements $U^{(i)} = u_1^{(i)}(Y, Z) + ku_2^{(i)}(Y, Z)$ and $V^{(i)} = v_1^{(i)}(Y) + kv_2^{(i)}(Y)$ in the two layers. The details are given in the subsections 7.2.3.1–7.2.3.3 below, in which $v_1^{(i)}(Y)$ and $v_2^{(i)}(Y)$ are obtained in Section 7.2.3.1, $u_1^{(i)}(Y, Z)$ in Section 7.2.3.2, and finally $u_2^{(i)}(Y, Z)$ in Section 7.2.3.3.

7.2.3.1. Vertical linear and nonlinear displacements

The linear and nonlinear Y -direction displacements v_1 and v_2 can be solved from Eqs. (7.74) and (7.75) as:

$$v_1^{(1)} = C_1^{(1)}Y + C_2^{(1)}, \quad v_1^{(2)} = C_1^{(2)}Y + C_2^{(2)},$$

$$v_2^{(1)} = C_3^{(1)}Y + C_4^{(1)}, \quad v_2^{(2)} = C_3^{(2)}Y + C_4^{(2)}. \quad (7.76)$$

Both $v_1^{(i)}$ and $v_2^{(i)}$ are linear functions of the Y -coordinate, with eight unknowns

$C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, C_4^{(1)}$ and $C_1^{(2)}, C_2^{(2)}, C_3^{(2)}, C_4^{(2)}$. The following boundary

conditions are applied to determine the eight unknowns:

(BC1) The bottom is fixed, i.e., there is no vertical displacement at $Y = 0$,

(BC2) The total height is maintained,

(BC3) Both the linear and nonlinear displacements are continuous at the interface

$$Y = L_Y^{(1)},$$

(BC4) Both the linear and nonlinear stresses T_{yy}^L and T_{yy}^{NL} are continuous across

the interface $Y = L_Y^{(1)}$.

The expressions for the stresses required in (BC4) can be obtained via Eq.

(7.68). Mathematically, the above boundary conditions can be written as:

$$(BC1) \quad v_1^{(1)}(Y=0) = 0 \quad \text{and} \quad v_2^{(1)}(Y=0) = 0, \quad (7.77)$$

$$(BC2) \quad v_1^{(2)}(Y = L_Y^{(1)} + L_Y^{(2)}) = 0 \quad \text{and} \quad v_2^{(2)}(Y = L_Y^{(1)} + L_Y^{(2)}) = 0, \quad (7.78)$$

$$(BC3) \quad v_1^{(1)}(Y = L_Y^{(1)}) = v_1^{(2)}(Y = L_Y^{(1)}) \quad \text{and} \quad v_2^{(1)}(Y = L_Y^{(1)}) = v_2^{(2)}(Y = L_Y^{(1)}), \quad (7.79)$$

$$(BC4) \quad T_{yy}^{(1)L}(Y = L_Y^{(1)}) = T_{yy}^{(2)L}(Y = L_Y^{(1)}) \quad \text{and} \quad T_{yy}^{(1)NL}(Y = L_Y^{(1)}) = T_{yy}^{(2)NL}(Y = L_Y^{(1)}). \quad (7.80)$$

Use the above 8 equations, the eight unknowns in Eq. (7.76) can be solved as:

$$\begin{aligned}
C_1^{(1)} &= 0, C_2^{(1)} = 0, C_1^{(2)} = 0, C_2^{(2)} = 0, \\
C_3^{(1)} &= \frac{-m_1 \alpha_1^2 + m_2 \alpha_2^2 - \alpha_1^2 \lambda_1 + \alpha_2^2 \lambda_2 - 2\alpha_1^2 \mu_1 + 2\alpha_2^2 \mu_2}{2(L_Y^{(2)}(\lambda_1 + 2\mu_1) + L_Y^{(1)}(\lambda_2 + 2\mu_2))} L_Y^{(2)}, \\
C_4^{(1)} &= 0, \\
C_3^{(2)} &= \frac{m_1 \alpha_1^2 - m_2 \alpha_2^2 + \alpha_1^2 \lambda_1 - \alpha_2^2 \lambda_2 + 2\alpha_1^2 \mu_1 - 2\alpha_2^2 \mu_2}{2(L_Y^{(2)}(\lambda_1 + 2\mu_1) + L_Y^{(1)}(\lambda_2 + 2\mu_2))} L_Y^{(1)}, \\
C_4^{(2)} &= \frac{-m_1 \alpha_1^2 + m_2 \alpha_2^2 - \alpha_1^2 \lambda_1 + \alpha_2^2 \lambda_2 - 2\alpha_1^2 \mu_1 + 2\alpha_2^2 \mu_2}{2(L_Y^{(2)}(\lambda_1 + 2\mu_1) + L_Y^{(1)}(\lambda_2 + 2\mu_2))} L_Y^{(1)}(L_Y^{(1)} + L_Y^{(2)}),
\end{aligned} \tag{7.81}$$

where α_1 and α_2 are the linear shear strains carried by the lower and upper blocks, respectively. These linear shear strains enter Eq. (2.20) through the dependence of $T_{yy}^{(i)NL}$ on α_1 and α_2 , whose explicit forms will be obtained in the next sub-section. Note that only three constants associated with the second-order effects are non-zero, and all eight constants are zero if the bilayer block reduces to a homogeneous one. Hence, second-order vertical displacements in the block arise because of the elastic dissimilarity.

7.2.3.2. Linear shear displacements

The linear shear displacement $u_1^{(i)}$ can be solved from Eqs. (7.70), (7.72) and (7.73), yielding:

$$u_1^{(1)} = \alpha_1 Y + \beta_1, \quad u_1^{(2)} = \alpha_2 Y + \beta_2. \tag{7.82}$$

There are totally four unknowns: α_1 , α_2 , β_1 and β_2 . These solutions represent the simple shear for a bilayered block in the first-order theory. The following force and displacement boundary conditions in the X -direction (resulting in four equations) are applied:

(BC1) The imposed shear displacement at the top face is δ_2 , or equivalently linear shear forces F_1 are applied at the top and bottom faces,

(BC2) The bottom face is fixed,

(BC3) The linear shear displacement is continuous at the interface.

Mathematically, they can be written as:

$$(BC1) \quad \alpha_1 \mu_1 L_x L_z = F_1 \quad \text{and} \quad \alpha_2 \mu_2 L_x L_z = F_1, \quad (7.83)$$

$$(BC2) \quad u_1^{(1)}(Y = 0) = 0, \quad (7.84)$$

$$(BC3) \quad u_1^{(1)}(Y = L_Y^{(1)}) = u_1^{(2)}(Y = L_Y^{(1)}). \quad (7.85)$$

In Eq.(7.83), $\alpha_1 = \delta_1 / L_Y^{(1)}$, $\alpha_2 = (\delta_2 - \delta_1) / L_Y^{(2)}$, where δ_1 is the shear displacement (in the X -direction) of the interface. Eq. (7.83) essentially means that the linear shear stress $T_{xy}^L = k\alpha_i \mu_i$ is constant in both layers, i.e., $\alpha_1 \mu_1 = \alpha_2 \mu_2$, with $\alpha_1 \neq \alpha_2$ generally. Using these four equations, α_1 , α_2 , β_1 , β_2 are solved in terms of F_1 or δ_2 as:

$$\alpha_1 = \frac{F_1}{\mu_1 L_X L_Z} = \frac{\delta_2}{L_Y^{(1)} + (\mu_1 / \mu_2) L_Y^{(2)}}, \quad \alpha_2 = \frac{F_1}{\mu_2 L_X L_Z} = \frac{\delta_2}{(\mu_2 / \mu_1) L_Y^{(1)} + L_Y^{(2)}},$$

$$\beta_1 = 0, \quad \beta_2 = \frac{F_1(\mu_2 - \mu_1) L_Y^{(1)}}{\mu_1 \mu_2 L_X L_Z}. \quad (7.86)$$

The coefficients $C_3^{(1)}$, $C_3^{(2)}$ and $C_4^{(2)}$ in Eq. (7.81) are thus explicitly obtained. As expected, $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2 = 0$ for a homogeneous block.

7.2.3.3. Nonlinear shear displacements

The nonlinear shear displacements $u_2^{(i)}$ are solved next. To do this, first write down the matrix of the first Piola-Kirchhoff stress using Eq. (7.68), i.e.:

$$\begin{bmatrix} \frac{1}{2} k^2 (2C_3^{(i)} \lambda + \alpha_i^2 (m_i + \lambda_i + 2\mu_i)) & k\alpha_i \mu_i + k^2 \mu_i \frac{\partial u_2^{(i)}}{\partial Y} & k^2 \mu_i \frac{\partial u_2^{(i)}}{\partial Y} \\ k\alpha_i \mu_i + k^2 \mu_i \frac{\partial u_2^{(i)}}{\partial Y} & \frac{1}{2} k^2 (m_i \alpha_i^2 + (2C_3^{(i)} + \alpha_i^2)(\lambda_i + 2\mu_i)) & 0 \\ k^2 \mu_i \frac{\partial u_2^{(i)}}{\partial Y} & 0 & \frac{1}{4} k^2 (2m_i \alpha_i^2 - n_i \alpha_i^2 + 4\lambda_i C_3^{(i)} + 2\lambda_i \alpha_i^2) \end{bmatrix}, \quad (7.87)$$

where $C_3^{(i)}$ and α_i are given in Eqs. (7.81) and (7.86), respectively. This stress matrix is symmetric. All the normal stresses T_{xx} , T_{yy} , T_{zz} and the shear stress T_{xz} are second-order whereas the shear stress T_{xy} consists of both first- and second-order parts. Two important points are highlighted here. First, T_{xz} being related to the derivative of the nonlinear shear displacement $u_2(Y, Z)$ with respect to Y , is generally non-zero. Summing the moments of the elementary shear forces $T_{xz} dXdZ$ about the X - and Y -axis leads to zero automatically, but to a non-vanishing moment M_{xz} about

the Z -axis. Hence, to maintain the block according to the imposed deformation, a restraining moment M_{xz} is necessary. In the experiments, this is imposed by providing constraints to the top and bottom faces to prevent it from undergoing any net rotation. Second, the second-order stress $T_{xz} = T_{zx}$ is not present in simple shear, in contrast to the second-order normal stresses (which correspond to the Poynting effect). Thus, for the generalized shear considered here, out-of-plane shear stresses in addition to normal stresses are induced.

The governing equation for $u_2^{(i)}$ is the Laplace equation, Eq. (7.71). The displacement boundary conditions on the top and bottom ($Y = 0, L_Y^{(1)} + L_Y^{(2)}$) faces are applied first:

$$(BC1) \quad u_2(Y = 0, Z) = 0, \quad (7.88)$$

$$(BC2) \quad h(Z) = u_2(Y = L_Y^{(1)} + L_Y^{(2)}, Z), \quad (7.89)$$

i.e., the bottom face is fixed and $h(Z)$ is the nonlinear shear displacement prescribed at the top $Y = L_Y^{(1)} + L_Y^{(2)}$. We note that the ratio of the imposed linear to nonlinear displacements is not known *a priori*. In the particular problem to be considered, $h(Z)$ will be set to the simplest choice of zero and the predicted shape of the block is compared to experimental results. Furthermore, we define:

$$u_2(Y, Z = L_Z) = u_2(Y, Z = 0) = f(Y), \quad (7.90)$$

$$u_2(Y = L_Y^{(1)}, Z) = g(Z), \quad (7.91)$$

where $f(Y)$ is the nonlinear deformation of the block at $Z = 0$ and $Z = L_Z$, respectively.

The shape $f(Y)$, which we will call the lateral profile, illustrates how the block deflects laterally in the X – Y plane. Also, $g(Z)$ is the nonlinear shear deformation of the block at the interface $Y = L_Y^{(1)}$. The second-order displacement u_2 can then be obtained for both the first and second layers, as explained next.

For the first or bottom layer, solving Eq. (7.71) leads to:

$$u_2^{(1)}(Y, Z) = \sum_{n=1}^{\infty} A_n^{(1)} \sin\left(\frac{n\pi Y}{L_Y^{(1)}}\right) \sinh\left(\frac{n\pi(L_Z - Z)}{L_Y^{(1)}}\right) + \sum_{n=1}^{\infty} B_n^{(1)} \sin\left(\frac{n\pi Y}{L_Y^{(1)}}\right) \sinh\left(\frac{n\pi Z}{L_Y^{(1)}}\right) \\ + \sum_{n=1}^{\infty} D_n^{(1)} \sinh\left(\frac{n\pi Y}{L_Z}\right) \sin\left(\frac{n\pi Z}{L_Z}\right), \quad (7.92)$$

where the coefficients

$$A_n^{(1)} = B_n^{(1)} = \frac{2}{L_Y^{(1)} \sinh\left(\frac{n\pi L_Z}{L_Y^{(1)}}\right)} \int_0^{L_Y^{(1)}} f(Y) \sin\left(\frac{n\pi Y}{L_Y^{(1)}}\right) dY, \quad (7.93)$$

and

$$D_n^{(1)} = \frac{2}{L_Z \sinh\left(\frac{n\pi L_Y^{(1)}}{L_Z}\right)} \int_0^{L_Z} g(Z) \sin\left(\frac{n\pi Z}{L_Z}\right) dZ. \quad (7.94)$$

For the second or top layer:

$$u_2^{(2)}(Y, Z) = \sum_{n=1}^{\infty} A_n^{(2)} \sin\left(\frac{n\pi(Y - L_Y^{(1)})}{L_Y^{(2)}}\right) \sinh\left(\frac{n\pi(L_Z - Z)}{L_Y^{(2)}}\right) + \sum_{n=1}^{\infty} B_n^{(2)} \sin\left(\frac{n\pi(Y - L_Y^{(1)})}{L_Y^{(2)}}\right) \sinh\left(\frac{n\pi Z}{L_Y^{(2)}}\right) \\ + \sum_{n=1}^{\infty} C_n^{(2)} \sinh\left(\frac{n\pi(L_Y^{(1)} + L_Y^{(2)} - Y)}{L_Z}\right) \sin\left(\frac{n\pi Z}{L_Z}\right) + \sum_{n=1}^{\infty} D_n^{(2)} \sinh\left(\frac{n\pi(Y - L_Y^{(1)})}{L_Z}\right) \sin\left(\frac{n\pi Z}{L_Z}\right), \quad (7.95)$$

where

$$A_n^{(2)} = B_n^{(2)} = \frac{2}{L_Y^{(2)} \sinh\left(\frac{n\pi L_Z^{(2)}}{L_Y^{(2)}}\right)} \int_{L_Y^{(1)}}^{L_Y^{(1)}+L_Y^{(2)}} f(Y) \sin\left(\frac{n\pi(Y-L_Y^{(1)})}{L_Y^{(2)}}\right) dY, \quad (7.96)$$

and

$$C_n^{(2)} = \frac{2}{L_Z \sinh\left(\frac{n\pi L_Y^{(2)}}{L_Z}\right)} \int_0^{L_Z} g(Z) \sin\left(\frac{n\pi Z}{L_Z}\right) dZ, \quad (7.97)$$

$$D_n^{(2)} = \frac{2}{L_Z \sinh\left(\frac{n\pi L_Y^{(2)}}{L_Z}\right)} \int_0^{L_Z} h(Z) \sin\left(\frac{n\pi Z}{L_Z}\right) dZ. \quad (7.98)$$

The solutions are symmetric about $Z = L_Z/2$. The aim is to obtain $f(Y)$ and $g(Z)$ for various prescribed top face displacements $h(Z)$. For a given $h(Z)$, $D_n^{(2)}$ can be calculated directly from Eq. (7.98). The other coefficients $A_n^{(1)}$, $B_n^{(1)}$, $D_n^{(1)}$ and $A_n^{(2)}$, $B_n^{(2)}$, $C_n^{(2)}$ can be solved via force, moment and displacement boundary conditions, at least numerically. The boundary conditions are the following:

(BC1) The nonlinear displacement u_2 is continuous across the interface,

(BC2) The nonlinear shear stress T_{xy}^{NL} is continuous across the interface,

(BC3) The nonlinear shear force associated with T_{xy} on the top and bottom faces is

F_2 ,

(BC4) No net shear forces associated with T_{xz} are applied on the Z-faces,

(BC5) The total moment about the Z-axis associated with T_{xz} on the Z-faces is M_{xz} .

The above five boundary conditions can be written mathematically as:

$$(BC1) \quad u_2^{(1)}(Y = L_Y^{(1)}) = u_2^{(2)}(Y = L_Y^{(1)}). \quad (7.99)$$

$$(BC2) \quad \mu_1 \frac{\partial u_2^{(1)}(Y, Z)}{\partial Y} \Big|_{Y=L_Y^{(1)}} = \mu_2 \frac{\partial u_2^{(2)}(Y, Z)}{\partial Y} \Big|_{Y=L_Y^{(1)}} \quad (7.100)$$

$$(BC3) \quad \int T_{xy}^{(1)NL} dXdZ = F_2, \text{ on } Y = 0 \text{ and } \int T_{xy}^{(2)NL} dXdZ = F_2, \text{ on } Y = L_Y^{(1)} + L_Y^{(2)}. \quad (7.101)$$

$$(BC4) \quad \int_{Y=0}^{L_Y^{(1)}} \int_{X=0}^{L_X} T_{xz}^{(1)} dXdY + \int_{Y=L_Y^{(1)}}^{L_Y^{(1)}+L_Y^{(2)}} \int_{X=0}^{L_X} T_{xz}^{(2)} dXdY = 0, \text{ on } Z = 0, L_Z. \quad (7.102)$$

$$(BC5) \quad \begin{aligned} & \int_{Y=0}^{L_Y^{(1)}} \int_{X=0}^{L_X} T_{xz}^{(1)}(Z=0) Y dXdY + \int_{Y=L_Y^{(1)}}^{L_Y^{(1)}+L_Y^{(2)}} \int_{X=0}^{L_X} T_{xz}^{(2)}(Z=0) Y dXdY \\ & - \int_{Y=0}^{L_Y^{(1)}} \int_{X=0}^{L_X} T_{xz}^{(1)}(Z=L_Z) Y dXdY - \int_{Y=L_Y^{(1)}}^{L_Y^{(1)}+L_Y^{(2)}} \int_{X=0}^{L_X} T_{xz}^{(2)}(Z=L_Z) Y dXdY = M_{xz}. \end{aligned} \quad (7.103)$$

From the first two boundary conditions, the following can be obtained:

$$\sum_{n=1}^{\infty} D_n^{(1)} \sinh \frac{n\pi L_Y^{(1)}}{L_Z} \sin \frac{n\pi Z}{L_Z} = \sum_{n=1}^{\infty} C_n^{(2)} \sinh \frac{n\pi L_Y^{(2)}}{L_Z} \sin \frac{n\pi Z}{L_Z} \quad (7.104)$$

and

$$\begin{aligned} & \sum_{n=1}^{\infty} n\pi (\mu_1 L_Y^{(1)} L_Y^{(2)} D_n^{(1)} \cosh \frac{n\pi L_Y^{(1)}}{L_Z} + \mu_2 L_Y^{(1)} L_Y^{(2)} C_n^{(2)} \cosh \frac{n\pi L_Y^{(2)}}{L_Z} - \mu_2 L_Y^{(1)} L_Y^{(2)} D_n^{(2)}) \sin \frac{n\pi Z}{L_Z} \\ & = \sum_{n=1}^{\infty} n\pi L_Z (\mu_2 L_Y^{(1)} A_n^{(2)} (\sinh \frac{n\pi(L_Z - Z)}{L_Y^{(2)}} + \sinh \frac{n\pi Z}{L_Y^{(2)}}) - (-1)^n \mu_1 A_n^{(1)} L_Y^{(2)} (\sinh \frac{n\pi(L_Z - Z)}{L_Y^{(1)}} + \sinh \frac{n\pi Z}{L_Y^{(1)}})). \end{aligned} \quad (7.105)$$

(BC3) can be written as:

$$\sum_{n=1}^{\infty} \mu_1 L_X (2A_n^{(1)} (\cosh \frac{n\pi L_Z}{L_Y^{(1)}} - 1) + D_n^{(1)} (1 - (-1)^n)) = F_2, \quad Y = 0 \quad (7.106)$$

and

$$\sum_{n=1}^{\infty} L_X \mu_2 (2A_n^{(2)} (-1)^n (\cosh \frac{n\pi L_Z}{L_Y^{(2)}} - 1) + D_n^{(2)} (1 - (-1)^n) \cosh \frac{n\pi L_Y^{(2)}}{L_Z} + C_n^{(2)} (-1 + (-1)^n)) = F_2, \quad (7.107)$$

(BC4) can be written as:

$$\begin{aligned} & \sum_{n=1}^{\infty} \mu_1 A_n^{(1)} (-1 + (-1)^n) (\cosh \frac{n\pi L_Z}{L_Y^{(1)}} - 1) + \mu_1 D_n^{(1)} (\cosh \frac{n\pi L_Y^{(1)}}{L_Z} - 1) \\ & + \mu_2 A_n^{(2)} (-1 + (-1)^n) (\cosh \frac{n\pi L_Z}{L_Y^{(2)}} - 1) + \mu_2 (C_n^{(2)} + D_n^{(2)}) (\cosh \frac{n\pi L_Y^{(2)}}{L_Z} - 1) = 0 \end{aligned} \quad (7.108)$$

and

$$\begin{aligned} & \sum_{n=1}^{\infty} -\mu_1 A_n^{(1)} (-1 + (-1)^n) (\cosh \frac{n\pi L_Z}{L_Y^{(1)}} - 1) + \mu_1 D_n^{(1)} (-1)^n (\cosh \frac{n\pi L_Y^{(1)}}{L_Z} - 1) \\ & - \mu_2 A_n^{(2)} (-1 + (-1)^n) (\cosh \frac{n\pi L_Z}{L_Y^{(2)}} - 1) + \mu_2 (C_n^{(2)} + D_n^{(2)}) (-1)^n (\cosh \frac{n\pi L_Y^{(2)}}{L_Z} - 1) = 0 \end{aligned} \quad (7.109)$$

The coefficients $A_n^{(1)}$, $B_n^{(1)}$, $D_n^{(1)}$ and $A_n^{(2)}$, $B_n^{(2)}$, $C_n^{(2)}$ should be determined using Eqs. (7.104)–(7.109). Finally, we note that the linear and nonlinear shear forces F_1 and F_2 cannot be measured separately. F_1 has been determined in terms of the imposed linear shear displacement δ_2 , as shown in Eq. (7.86). It will be shown in the next section that F_2 can be determined in terms of M_{xz} by (BC5), and thus F_1 and F_2 are both obtained in terms of the imposed quantities.

7.2.4. Particular problem

In this section, the solutions of the problem with $L_Y^{(1)} = L_Y^{(2)}$ are given

analytically. This corresponds to the problem of a bilayer block with equal layer thicknesses. In addition, we consider the simple case of $h(Z) = 0$. Based on the condition $L_Y^{(1)} = L_Y^{(2)}$, Eqs. (7.94) and (7.97) lead to:

$$D_n^{(1)} = C_n^{(2)}, \quad (7.110)$$

while Eq. (7.105) can be written as:

$$\begin{aligned} & L_Y \sum_{n=1}^{\infty} ((\mu_1 D_n^{(1)} + \mu_2 C_n^{(2)}) \cosh \frac{n\pi L_Y}{L_Z} - \mu_2 D_n^{(2)}) \sin \frac{n\pi Z}{L_Z} \\ &= L_Z \sum_{n=1}^{\infty} (\mu_2 A_n^{(2)} - (-1)^n \mu_1 A_n^{(1)}) (\sinh \frac{n\pi(L_Z - Z)}{L_Y} + \sinh \frac{n\pi Z}{L_Y}). \end{aligned} \quad (7.111)$$

Since in Eq. (7.111),

$$\sin \frac{n\pi Z}{L_Z} \neq \sinh \frac{n\pi(L_Z - Z)}{L_Y} + \sinh \frac{n\pi Z}{L_Y}, \quad (7.112)$$

one possible set of solutions is such that each of the n th coefficients multiplying the sine and sinh terms is zero:

$$(\mu_1 D_n^{(1)} + \mu_2 C_n^{(2)}) \cosh \frac{n\pi L_Y}{L_Z} - \mu_2 D_n^{(2)} = 0 \quad (7.113)$$

and

$$\mu_2 A_n^{(2)} - (-1)^n \mu_1 A_n^{(1)} = 0. \quad (7.114)$$

Eqs. (7.108) and (7.109) can be written as:

$$\begin{aligned} \sum_{n=1}^{\infty} \mu_1 A_n^{(1)} (-1 + (-1)^n) (\cosh \frac{n\pi L_Z}{L_Y} - 1) + \mu_1 D_n^{(1)} (\cosh \frac{n\pi L_Y}{L_Z} - 1) \\ + \mu_2 A_n^{(2)} (-1 + (-1)^n) (\cosh \frac{n\pi L_Z}{L_Y} - 1) + \mu_2 (C_n^{(2)} + D_n^{(2)}) (\cosh \frac{n\pi L_Y}{L_Z} - 1) = 0, \end{aligned} \quad (7.115)$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} -\mu_1 A_n^{(1)} (-1 + (-1)^n) (\cosh \frac{n\pi L_Z}{L_Y} - 1) + \mu_1 D_n^{(1)} (-1)^n (\cosh \frac{n\pi L_Y}{L_Z} - 1) \\ - \mu_2 A_n^{(2)} (-1 + (-1)^n) (\cosh \frac{n\pi L_Z}{L_Y} - 1) + \mu_2 (C_n^{(2)} + D_n^{(2)}) (-1)^n (\cosh \frac{n\pi L_Y}{L_Z} - 1) = 0. \end{aligned} \quad (7.116)$$

By adding and subtracting the above two equations, it can be shown that for each n :

$$(\mu_1 D_n^{(1)} + \mu_2 C_n^{(2)} + \mu_2 D_n^{(2)}) (\cosh \frac{n\pi L_Y}{L_Z} - 1) = 0. \quad (7.117)$$

Further, combining Eqs. (7.110), (7.113) and (7.117) leads to:

$$D_n^{(1)} = C_n^{(2)} = D_n^{(2)} = 0. \quad (7.118)$$

By using Eq. (7.118), Eqs. (7.106) and (7.107) can be written as:

$$\sum_{n=1}^{\infty} 2\mu_1 L_X A_n^{(1)} (\cosh \frac{n\pi L_Z}{L_Y} - 1) = F_2, \quad (7.119)$$

$$\sum_{n=1}^{\infty} 2\mu_2 L_X A_n^{(2)} (-1)^n (\cosh \frac{n\pi L_Z}{L_Y} - 1) = F_2. \quad (7.120)$$

There are infinitely many coefficients $A_n^{(1)}$ and $A_n^{(2)}$ in the above two equations. However, experimental results show that the lateral profile $f(Y)$ of the block is approximated well by two half-period sine curves of different amplitudes. Hence, we choose $A_n^{(i)} = 0$ for $n \geq 2$, and $A_1^{(1)}$ and $A_1^{(2)}$ can be solved from Eqs. (7.119) and (7.120) as:

$$A_1^{(1)} = \frac{F_2}{2\mu_1 L_X (\cosh \frac{\pi L_Z}{L_Y} - 1)} , \quad (7.121)$$

$$A_1^{(2)} = -\frac{F_2}{2\mu_2 L_X (\cosh \frac{\pi L_Z}{L_Y} - 1)} . \quad (7.122)$$

Finally, after integration (BC5) in Eq. (7.103) can be written as:

$$2 \sum_{n=1}^{\infty} L_X L_Y (\cosh \frac{n\pi L_Z}{L_Y} - 1) (-1)^n (\mu_1 A_n^{(1)} + \mu_2 A_n^{(2)}) = M_{xz} . \quad (7.123)$$

Substituting the solutions of $A_1^{(1)}$ and $A_1^{(2)}$ into the above equation, it can be obtained that

$$F_2 = \frac{M_{xz}}{2L_Y} . \quad (7.124)$$

With the coefficients $A_1^{(1)}$ and $A_1^{(2)}$ determined, the final form of $f(Y)$ according to

Eq. (7.90), Eq. (7.92) and Eq. (7.95) is thus:

$$f(Y) = A_1^{(1)} \sin \frac{\pi Y}{L_Y} \sinh \frac{\pi L_Z}{L_Y}, \quad (7.125)$$

for $0 < Y < L_Y$, and

$$f(Y) = A_1^{(2)} \sin \frac{\pi(Y - L_Y)}{L_Y} \sinh \frac{\pi L_Z}{L_Y}. \quad (7.126)$$

for $L_Y < Y < 2L_Y$. For later use, we further define the constants

$a_1^{(1)} = A_1^{(1)} \sinh(\pi L_Z / L_Y)$ and $a_1^{(2)} = A_1^{(2)} \sinh(\pi L_Z / L_Y)$. By using Eq. (7.124), the

total shear displacement in the composite block can be written as:

$$U^{(1)}(Y, Z) = \frac{\delta_2}{(1 + \mu_1 / \mu_2)L_Y} Y + \frac{M_{xz}}{4\mu_1 L_X L_Y (\cosh \frac{\pi L_Z}{L_Y} - 1)} \sin \frac{\pi Y}{L_Y} \left(\sinh \frac{\pi(L_Z - Z)}{L_Y} + \sinh \frac{\pi Z}{L_Y} \right), \quad (7.127)$$

for $0 < Y < L_Y$,

and

$$U^{(2)}(Y, Z) = \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2} \delta_2 + \frac{\delta_2}{(1 + \mu_2 / \mu_1)L_Y} Y + \frac{M_{xz}}{4\mu_2 L_X L_Y (\cosh \frac{\pi L_Z}{L_Y} - 1)} \sin \frac{\pi(Y - L_Y)}{L_Y} \left(\sinh \frac{\pi(L_Z - Z)}{L_Y} + \sinh \frac{\pi Z}{L_Y} \right), \quad (7.128)$$

for $L_Y < Y < 2L_Y$. In the above solutions, δ_2 is the prescribed linear displacement at the top face, and M_{xz} is the prescribed moment about the Z -axis.

Eqs. (7.125)–(7.128) are explicit expressions for the lateral profile and the shear displacements in the composite block. The coefficients $A_1^{(1)}$ and $A_1^{(2)}$ may be estimated from the experimental data for the lateral profile of the block. Knowing these two values, the predicted and experimental profiles can be compared. Similarly,

coefficients such as M_{xz}/μ_1 and M_{xz}/μ_2 can be determined to be $4A_1^{(1)}L_XL_Y(\cosh \pi L_Z / L_Y - 1)$ and $-4A_1^{(2)}L_XL_Y(\cosh \pi L_Z / L_Y - 1)$, respectively. Hence, the shear displacements can also be computed based on the experimental data.

7.2.5. Experimental and numerical results

In this section, experimental results on bilayer blocks of an agar-gelatin are reported and compared to the theoretical predictions. Following this, the dependence of the second-order nonlinear effects on the elastic dissimilarity, i.e., the normal stresses and the out-of-plane shear stress, as well as the interfacial displacement in the vertical direction, is studied.

7.2.5.1. Theoretical and experimental profiles of sheared bilayer blocks

Bilayer blocks with the dimensions $L_X = 30$ mm, $L_Y^{(1)} = L_Y^{(2)} = 70$ mm and $L_Z = 25$ mm were made from an agar-gelatin. In Figs. 7.10(a), (b) and (c), Layer 1 (lower layer) has a concentration of 17g of the gelatin powder in 110 mL of water, while Layer 2 (upper layer) has the concentrations of 17g/110 mL, 17g/100 mL and 17g/90 mL, respectively. Thus, the block in Fig. 7.10(a) is homogeneous, and the upper layers are stiffer than the lower layers in the other two blocks. The bottom faces of all three blocks were sheared towards the right by a distance of 45 mm.

The experimental lateral profiles $f(Y)$ are curved; they are not a linear function of Y . The homogeneous block displays a smooth curved profile. In the deformed

Fig. 7.10. Rectangular blocks of agar-gelatin gels, gripped at the top and subjected to a constant rightward shear displacement of 45 mm at the bottom. Layer 1 has the same stiffness, while Layer 2 is increasingly stiffer from (a) to (c). The theoretical lateral profiles are shown as solid lines.

Table 7.1. Parameters used in Fig. 7.10.

	Fig. 7.10(a)	Fig. 7.10(b)	Fig. 7.10(c)
Samples			
Layer 1 concentration	17 g/110 mL water	17g/110 mL water	17 g/110 mL water
Layer 2 concentration	17 g/110 mL water	17 g/100 mL water	17 g/90 mL water
L_X	30 mm	30 mm	30 mm
$L_Y^{(1)}$	70 mm	70 mm	70 mm
$L_Y^{(2)}$	70 mm	70 mm	70 mm
L_Z	25 mm	25 mm	25 mm
Applied shear displacement			
δ_2	45 mm	45 mm	45 mm
Parameters estimated from experiments			
$a_1^{(1)}$	3 mm	3 mm	2.5 mm
$a_1^{(2)}$	3 mm	1.48 mm	0.925 mm

bilayers, each block has a smooth profile but they also have a discernible kink at the interface. Comparing the slopes $f'(Y)$ of the upper blocks just above the interface, it can be seen that the stiffest block in Fig. 7.10(c) has the smallest slope. Just below the interface, the lower block in Fig. 7.10(c) appears to have a larger slope. These observations agree with intuition, since the stiffest upper block will deform the least and the corresponding lower block will deform more at the connecting interface to compensate for the smaller deformation of the upper block. The theoretical profiles, which consist of a sine curve superimposed on a straight line, are highlighted in bold. Table 7.1 shows the values of the parameters α_1 , α_2 , $a_1^{(1)}$ and $a_1^{(2)}$ used to generate these profiles. The theoretical profiles capture the essential features described above

Fig. 7.11. Deformed and undeformed states of a bilayer rectangular block under generalized shear. The prescribed shear displacement $\delta_2 = 0.045$ m at the top and the prescribed moment $M_{xz} = -0.42$ N·m.

(kinks and the slopes at the interface). Discrepancy between the theoretical and observed profiles is also evident, notably the greater sagging of the experimental samples. This could be attributed to gravity effects and the limitations of a second-order nonlinear elasticity model.

Table 7.2. Elastic parameters used in Figs. 7.11–7.16. The unit of elastic constants is kPa. The dimensions of the block are $L_X = 0.030$ m, $L_Y^{(1)} = L_Y^{(2)} = 0.070$ m, $L_Z = 0.025$ m. Also, $\delta_2 = 0.045$ m in all figures, while $M_{xz} = -0.42$ N·m in Fig. 7.11 and -0.14 N·m in Figs. 7.12–7.16.

	λ_1	μ_1	m_1	n_1
	λ_2	μ_2	m_2	n_2
Fig. 7.11	2.25×10^6	8	-20×10^6	-80
	60	12.21	-35.60	-23.50
Fig. 7.12	1024		-2400	-2350
	1024		-2400	-2350
Fig. 7.13	357		-1000	-235
		103	-1000	-235
Fig. 7.14 (a)	3570		-8465	-2350
	3570	103000		-2350
Fig. 7.14 (b)	3570		-2394	-2350
	3570	103000		-2350
Fig. 7.15	2.25×10^6	8	-20×10^6	-80
	60	21.62	-35.60	-23.50
Fig. 7.16 (a)	1×10^8	100	-2×10^8	-800
	1×10^8			-800
Fig. 7.16 (b)	1×10^8		-2×10^8	-800
	1×10^8		-2×10^8	-800

In Fig. 7.11, the three-dimensional profile of an entire deformed bilayer block is simulated for a specific set of elastic constants, prescribed displacement of 0.045 m and prescribed moment of -0.42 Nm, as listed in Table 7.2. This shows more clearly how the $X = 0$ and $X = L_X$ surfaces deform from their original positions resulting in unequally curved faces meeting at a kink at the interface, as well as the maintenance of the boundary conditions (fixed base, constant total height).

7.2.5.2. Dependence of second-order stresses on elastic inhomogeneity

In this section, the dependence of the second-order normal stresses on the layer elastic inhomogeneity (dissimilarity in the elastic constants) is studied. Three types of inhomogeneity are considered: dissimilarity in one particular constant, e.g., μ_1 / μ_2 , dissimilarity between constants of the same order, e.g., λ_2 / μ_1 , and dissimilarity between constants of different orders, e.g., m_2 / μ_1 . In all the figures to follow, $\delta_2 = 0.045$ m and $M_{xz} = -0.14$ Nm.

7.2.5.2.1. Inhomogeneity of shear moduli

Figs. 7.12 (a), (b) and (c) plot $T_{xx}^{(1)}$, $T_{yy}^{(1)}$ and $T_{zz}^{(1)}$ in Layer 1, respectively, versus the shear modulus μ_2 of Layer 2 for various constant values of $\mu_1 / \mu_2 = 0.1, 0.4, 1, 2.5$ and 10 . All other elastic constants are held constant and are the same between the layers, as shown in Table 7.2. The applied boundary conditions are also indicated in this table. The ratio $\mu_1 / \mu_2 = 1$ implies a homogeneous block. Although the normal stresses are constant in each layer of the block, they, except for $T_{yy}^{(1)}$, will generally have a jump across the interface, i.e., $T_{xx}^{(1)} \neq T_{xx}^{(2)}$, $T_{zz}^{(1)} \neq T_{zz}^{(2)}$ but $T_{yy}^{(1)} = T_{yy}^{(2)}$.

This figure shows a rich behavior of the stresses. First, the dependence on inhomogeneity is clearly seen, and is generally significant. Changing μ_1 / μ_2 from 1 to 0.1 at $\mu_2 = 200$ KPa increases $T_{xx}^{(1)}$ significantly from ~ -60 to ~ -135 kPa, for instance.

Second, within the range of μ_2 studied and depending on μ_1 / μ_2 , the stresses can

Fig. 7.12. Dependence of the stresses (a) $T_{xx}^{(1)}$, (b) $T_{yy}^{(1)}$ and (c) $T_{zz}^{(1)}$ in Layer 1 with μ_2 for inhomogeneity ratios $\mu_1 / \mu_2 = 0.1, 0.4, 1, 2.5, 10$. The other elastic constants are listed in Table 7.2. The influence of the ratios on the stresses is extrinsic, i.e., it persists even as μ_2 tends to zero.

either increase or decrease monotonically, or they might have extrema. This highlights the complexity and hence difficulty of materials selection if the aim is to minimize or maximize the normal stresses.

Third, the stresses can be either zero, positive or negative. $T_{xx}^{(1)}$ and $T_{yy}^{(1)}$ appear to be mostly compressive except at large μ_2 , whereas $T_{zz}^{(1)}$ is largely tensile if the lower layer is more compliant than the upper layer ($\mu_1 / \mu_2 < 1$). These stresses, in particular $T_{zz}^{(1)}$, demonstrate the Poynting effect under shear. This corresponds to the extension or contraction of a twisted rod in the direction perpendicular to the plane of shearing. The second-order stresses are necessary to maintain the deformation of the generalized shear. In the absence of a positive (negative) $T_{zz}^{(1)}$, the block will contract (expand) along the Z -direction, corresponding to a negative (positive) Poynting effect. Because $T_{zz}^{(1)} \neq T_{zz}^{(2)}$, there is a tendency for the bilayer block to contract or expand by different amounts.

Fourth, it is difficult to generalize whether the magnitude of the stresses will increase or decrease if the lower layer becomes more compliant compared to the upper layer. Nevertheless, some general trend can still be observed. Consider Fig. 7.12(b) for μ_2 held fixed at 1000 kPa. If the block is homogeneous, $T_{yy}^{(1)}$ is slightly tensile (~ 25 kPa). If Layer 1 becomes more compliant relative to Layer 2, $T_{yy}^{(1)}$ becomes more compressive, with $T_{yy}^{(1)} \sim -140$ kPa when $\mu_1 / \mu_2 = 0.1$. If Layer 1 becomes stiffer, $T_{yy}^{(1)}$ becomes more tensile, with $T_{yy}^{(1)} \sim 100$ kPa when $\mu_1 / \mu_2 = 10$.

Fifth, it is interesting that $T_{yy}^{(1)}$ remains the same if the materials are

interchanged across the layers, as can be verified using Fig. 7.12(b). For instance, if $\mu_2 = 500$ kPa and $\mu_1 = 200$ kPa, and if $\mu_2 = 200$ kPa and $\mu_1 = 500$ kPa, $T_{yy}^{(1)} \sim -63$ kPa in both cases. This is related to the requirement of the continuity of T_{yy} across the interface. The same does not hold for the other normal stress components, as they experience a jump in value across the interface.

Sixth, each normal stress generally tends to different values as μ_2 tends to zero, depending on the ratio μ_1 / μ_2 . Although both μ_1 and μ_2 approach zero, the induced normal stresses can be very different if μ_1 approaches zero faster or slower than μ_2 . Indeed, Fig. 7.12(c) shows that when μ_2 tends to zero, $T_{zz}^{(1)}$ is ~ 80 kPa if μ_1 is ten times smaller than μ_2 , while $T_{zz}^{(1)}$ is ~ -110 kPa if μ_1 is ten times larger. This suggests the intrinsic importance of elastic inhomogeneity: it may significantly influence the magnitude and sign of the second-order stresses even though the individual constants are themselves vanishingly small. However, exceptions do occur in the case of $T_{xx}^{(1)}$ and $T_{yy}^{(1)}$: if two ratios of μ_1 / μ_2 are reciprocals of each other, these two stresses do tend to the same values when μ_1 and μ_2 approach zero. This can be seen in Figs. 7.12(a) and (b) where the curves for $\mu_1 / \mu_2 = 10$ and 0.1 (or 2.5 and 0.4) meet each other at $\mu_2 = 0$.

7.2.5.2.2. Inhomogeneity of Lamé constants

Next, the dependence of $T_{xx}^{(1)}$, $T_{yy}^{(1)}$ and $T_{zz}^{(1)}$ on μ_1 for $\lambda_2 / \mu_1 = 0.1, 0.4, 1, 2.5$ and 10 is studied. All other elastic constants and the prescribed boundary conditions are given in Table 7.2. Note that $\lambda_2 / \mu_1 = 1$ does not imply a

Fig. 7.13. Dependence of the stresses (a) $T_{xx}^{(1)}$, (b) $T_{yy}^{(1)}$ and (c) $T_{zz}^{(1)}$ in Layer 1 with μ_1 for inhomogeneity ratios $\lambda_2 / \mu_1 = 0.1, 0.4, 1, 2.5, 10$. The other elastic constants are listed in Table 7.2. The influence of the ratios on the stresses is intrinsic, i.e., it does not persist as μ_1 tends to zero.

homogeneous block. The results are shown in Fig. 7.13. The *Lamé* first constant λ_2 is related to the Poisson ratio ν_2 and shear modulus μ_2 (all of Layer 2) according to $\nu_2 = 1/2(1 + \mu_2 / \lambda_2)$. Hence for a fixed μ_2 , as is the case here, increasing λ_2 implies that ν_2 approaches 0.5, i.e., Layer 2 tends towards incompressibility.

The dependence of all three normal stresses on μ_1 appears to be similar. Many of the remarks made with regard to Fig. 7.12 also hold here: the dependence on the inhomogeneity is significant, the stresses can vary monotonically with μ_1 or possess extrema, depending on λ_2 / μ_1 , and they can either demonstrate the positive or negative Poynting effect. An important result of Fig. 7.13 is that the Poynting effect becomes more significant (the normal stresses increase in magnitude) as Layer 2 approaches incompressibility. For the elastic constants considered here, these normal stresses are tensile at large values of λ_2 / μ_1 , i.e., the bilayer block tends to demonstrate a negative Poynting effect. A further interesting feature of the curves in Fig. 7.13 is that there appears to be a specific combination of the elastic constants that yields identical values of the stresses. This congruence of stress values occurs at $\mu_1 \sim 61$ kPa, regardless of the value of λ_2 . This can be understood from Eq. (7.81), where $C_3^{(1)}$ (which essentially determines the normal stresses) can be expressed in the form of $\gamma_1(\gamma_2 + \gamma_3\lambda_2) / (\gamma_2 + \gamma_3\lambda_2)$, where γ_1 , γ_2 and γ_3 are some combinations of elastic constants, which leads to the cancellation of the common factor in the numerator and denominator and hence elimination of λ_2 from the stress dependence.

Note also that as μ_1 tends to zero, each normal stress converges to a single negative value irrespective of λ_2 / μ_1 . This is unlike the results in Fig. 7.12. This

Fig. 7.14. Dependence of the stress $T_{xx}^{(2)}$ in Layer 2 with μ_1 for $m_2 / \mu_1 = -1000, -100, 1, 100$ and 1000 . In (a) $m_1 = -8465$ kPa; in (b) $m_1 = -2394$ kPa. The other elastic constants are listed in Table 7.2. For the most part, changing the value of m_1 results in a change in the sign of the stress.

means that as long as both μ_1 and λ_2 approach zero, each induced normal stress will be independent of how fast λ_2 approaches zero relative to μ_1 .

7.2.5.2.3. Inhomogeneity of second- and third-order elastic constants

Fig. 7.14 plots the variation of $T_{xx}^{(2)}$ in Layer 2 with μ_1 of Layer 1 for $m_2 / \mu_1 = -1000, -100, 100, 1$, and 1000 . The value of $m_1 = -8465$ kPa and -2394 kPa in Figs. 7.14(a) and 7.14(b), respectively. Other relevant parameters are given in Table 7.2.

Overall, it can be observed that changing the value of m_1 results in a change in the sign of the stress. Similar to Fig. 7.13, a notable result of Fig. 7.14 is the convergence of $T_{xx}^{(2)}$ as the Layer 1 shear modulus tends to zero. When the shear modulus of Layer 1 becomes vanishingly smaller, it appears that changing the other elastic constants of Layer 2 (λ_2 and m_2) will not alter the induced normal stresses. However, the values of these stresses would depend on the particular elastic constants. In Fig. 7.14, $T_{xx}^{(2)}$ tends to ~ -17 kPa and ~ 4 kPa for $m_1 = -8465$ kPa and -2394 kPa, respectively.

Taking Figs. (7.12)–(7.14) together, we observe an important effect of layer elastic inhomogeneity. If the inhomogeneity is between the shear moduli μ_1/μ_2 , then its influence on the normal stresses is intrinsic, meaning that the stresses may differ in magnitude and sign if these ratios are different, even if the actual values of the individual constants are vanishingly small. Exception does occur for $T_{xx}^{(1)}$ and $T_{yy}^{(1)}$ when the values of μ_1/μ_2 are reciprocals of each other. If the inhomogeneity is

between other types of elastic constants, such as λ_2/μ_1 and m_2/μ_1 , each normal stress will be influenced by this inhomogeneity ratio only when the elastic constants themselves are not vanishingly small. This may be termed an extrinsic influence, in the sense that the inhomogeneity ratio has an effect only when the individual constants are not vanishingly small. Further simulations show that all other inhomogeneity ratios such as λ_1/λ_2 , m_1/m_2 , n_1/n_2 , λ_1/μ_2 , m_1/μ_2 , m_2/λ_1 and m_1/λ_2 exert an extrinsic influence on the normal stresses.

7.2.5.2.4. Variation of the second-order shear stresses in the vertical plane

To illustrate how the second-order shear stresses vary in the block, T_{xy}^{NL} and T_{xz} are plotted versus Y and Z for the set of elastic constants and boundary conditions given in Table 7.2. These stresses are not a function of X .

Both shear stresses are of the order of 1 kPa and are not small compared to the shear moduli $\mu_1 = 8$ kPa and $\mu_2 = 21.62$ kPa. T_{xy}^{NL} reaches its maximum at mid-height and does not vary too much with Z . The distribution of T_{xz} in Fig. 7.15(b) is more complex than that of T_{xy}^{NL} in Fig. 7.15(a). First, T_{xz} is a strong function of both Y and Z . Second, the distribution is odd with respect to Y and Z . Thus, with respect to the mid-height position, $T_{xz}(Y = 0.05 \text{ m}, Z)$ and $T_{xz}(Y = 0.1 \text{ m}, Z)$ have different signs. The signs of these stresses are also reversed with respect to the mid-position of Z .

We further note that T_{xz} is not present in simple shear when the shear displacement U is taken to be a function of Y only. In generalized shear when U is a function of both Y and Z , T_{xz} exists and tends to distort the block about the Y -axis.

Fig. 7.15. Dependence of the second-order shear stresses (a) T_{xy}^{NL} and (b) T_{xz} on the Y and Z coordinates of a bilayer rectangular block under generalized shear. The sinusoidally varying T_{xz} gives rise to a generalized Poynting effect.

Thus, a block when sheared only in the X -direction (shearing of the block in the X - Y plane about the Z -direction) may also tend to shear in the out-of-plane Z -direction (shearing in the X - Z plane about the Y -direction). This effect may be called a generalized Poynting effect, in analogy with the existence of the normal stresses under simple shear. A similar terminology has been given by dell'Isola et al. (1998), in which a bar when bent stretches in the absence of second-order resultant normal force.

7.2.5.2.5. Dependence of the interface vertical displacement on elastic constants

According to Eq. (7.76), vertical displacements exist in the bilayer block. In Fig. 7.16, $V = v_1(Y = L_Y^{(1)}) + v_2(Y = L_Y^{(1)})$ at the interface versus the change in elastic constants is investigated. Fig. 7.16(a) plots V versus μ_2 for $m_2 = 0, \pm 1.5 \times 10^8, \pm 3 \times 10^7$ kPa, while Fig. 7.16(b) plots V versus μ_1 for $\mu_2 = 10, 30, 50, 100, 300$ kPa. Other parameters are given in Table 7.2.

These interfacial displacements are of the order of 0.01m, a significant fraction of the height (0.14 m) of the block in this simulation. The interface can either migrate upwards or downwards depending on the combination of elastic constants. For a homogeneous block, there will be no interfacial displacement. Hence, the interfaces in bilayer devices may be designed to move in one direction or another through a judicious selection of the layer materials.

Fig. 7.16. Dependence of the vertical displacement at the interface $V(Y = L_Y^{(1)})$ on (a) μ_2 for different values of m_2 between -1.5×10^8 and 1.5×10^8 kPa, and (b) μ_1 for different values of μ_2 between 10 and 300 kPa. Strong influence of elastic inhomogeneity on the displacement is evident.

The limiting behavior of the interfacial displacement when μ_2 and μ_1 tend to zero in Figs. 7.16(a) and (b), respectively, is interesting. Fig. 7.16(a) shows that varying m_2 leads to different values of V in the limit $\mu_2 \rightarrow 0$, while Fig. 7.16(b) shows that V is independent of μ_2 in the limit $\mu_1 \rightarrow 0$. The results in this figure show that it is possible to control the interfacial displacement using either a second- or a third-order elastic constant.

7.2.6. Conclusions

In this section, the generalized shear of a soft bilayer block is investigated using second-order elasticity. The key contributions are (i) the study of a bilayer block, in contrast to homogeneous blocks studied in most previous works, (ii) a more general assumption of the shear displacement being a function of Y and Z rather than Y alone, and (iii) verification of theoretical predictions with experiments.

In the experiments, agar-gelatin bilayer blocks deformed under shear display lateral profiles which are in good agreement with the theoretical predictions. The presence of smooth profiles of different curvatures meeting at a kink at the interface is predicted.

The numerical simulations, focusing on the dependence of the second-order effects on the elastic constants, lead to the following conclusions: (i) the layer elasticity inhomogeneity can influence significantly the sign and the magnitude of the second-order normal and shear stresses as well as the interfacial displacement, (ii) the inhomogeneity as represented by the ratio of the shear moduli has an intrinsic

influence on the stresses, while inhomogeneity between all other constants has an extrinsic influence, (iii) the Poynting effect is predicted for both homogeneous and bilayer blocks, and (iv) a generalized Poynting effect is also evident, in which an out-of-plane shear stress is induced when the block is sheared in-plane.

Chapter 8. Conclusions and Recommendations

Several major contributions can be identified in this research. They are described in the following sections. On the basis of the accomplished work, recommendations for future research are made.

8.1. Multilayered spherical and cylindrical hydrogels subjected to dilatation

The theory for multilayered spherical and cylindrical hydrogels subjected to a dilatation profile is developed in the framework of second-order elasticity. Specifically, two layers are considered for the spherical gels and two to four layers for the cylindrical hydrogels. The dilatation profile is assumed to be constant, step-up or step-down. The displacement and stress solutions all assume the form of a power law with parameters dependent on the elastic constants, thickness and the dilatation of each layer.

Several conclusions can be summarized as follows:

- (1) The results show that nonlinearity significantly influences the displacements and stresses.
- (2) The elastic fields may be completely characterized by a reduced set of geometric-elastic constants. In the case of two layers, a total of eight out of ten constants are sufficient to describe the fields.

- (2) Among the three nonlinear elastic parameters, m has a more dominant influence on the stresses and displacements compared to l and n .
- (3) Besides nonlinearity, the elastic and layer thickness inhomogeneities as well as the dilatation profile also significantly influence the elastic fields.
- (4) A very wide range of swelling and multiaxial stress states is realizable by manipulating the layer elasticity and the layer thicknesses.
- (5) Such results, in conjunction with experimental data, will help to elucidate the influence of microenvironment mechanical property on biological processes, e.g., cell adhesion, proliferation and apoptosis in stem cell culture applications.

8.2. Homogeneous and bilayered cylinders under axial loading

Analytical solutions have been derived for homogeneous and bilayered cylinders under axial loading, starting consistently from the equilibrium equations for the first Piola-Kirchhoff stress. The second-order effects in the displacement and stress fields have been identified, and their dependence on the elastic constants and geometrical parameters have been studied.

Based on the results, some important conclusions can be summarized.

- (1) The first- and second-order terms (k and k^2) in the displacement and stress components under pure axial loading can be expressed respectively for the homogeneous and bilayered cylinders as:

$$\begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}_P = \begin{pmatrix} k + k^2 \\ 0 \\ k + k^2 \end{pmatrix}, \mathbf{T}_P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{pmatrix};$$

$$\begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}_P = \begin{pmatrix} k + k^2 \\ 0 \\ k + k^2 \end{pmatrix}, \mathbf{T}_P = \begin{pmatrix} k + k^2 & 0 & 0 \\ 0 & k + k^2 & 0 \\ 0 & 0 & k + k^2 \end{pmatrix}$$

Thus,

- (a) For both homogeneous and bilayered cylinders under axial loading, the displacements u_r and u_z contain both first- and second-order components.
- (b) There is no twist under pure axial loading in either cylinder.
- (c) The stress field in the bilayered cylinder contains both first- and second-order components, while there is only a first-order stress component in the homogeneous cylinder. This highlights the influence of elastic inhomogeneity on the stress field.

Furthermore,

- (2) The analytical solutions obtained can be used for the experimental estimation of third-order elastic constants.
- (3) Universal relations can be derived using modified parameters. The experimental data are described reasonably well by the universal curve.

8.3. Homogeneous and bilayered cylinders under torsion

Analytical solutions have been derived for homogeneous and bilayered cylinders under torsion. The second-order effects in the displacement and stress fields have

been identified, and their dependence on the elastic constants and geometrical parameters has been studied in detail. The Poynting effect of a composite has been investigated in some depth. Explicit Poynting effect coefficients are given for both homogeneous and bilayered composites.

The following conclusions can be made.

- (1) The first- and second-order terms in the displacement and stress components under pure torsion can be written as:

$$\begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}_T = \begin{pmatrix} k^2 \\ k \\ k^2 \end{pmatrix}, \quad \mathbf{T}_T = \begin{pmatrix} k^2 & 0 & 0 \\ 0 & k^2 & k \\ 0 & k & k^2 \end{pmatrix}; \quad \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}_T = \begin{pmatrix} k^2 \\ k \\ k^2 \end{pmatrix},$$

$$\mathbf{T}_T = \begin{pmatrix} k^2 & 0 & 0 \\ 0 & k^2 & k \\ 0 & k & k^2 \end{pmatrix}$$

Thus,

- (a) The radial and axial displacements and all the normal stresses contain second-order components.
- (b) The Poynting effect is represented by the second-order displacement components, with the associated second-order normal stress components.

Furthermore,

- (2) An amplified Poynting effect is possible above that of homogeneous materials, depending on the interface position.
- (3) The Poynting effect is also dependent on the material configuration within the composite.
- (4) The dependence of the effect on the overall cylinder size obeys an inverse sextic

law.

- (5) The analytical solutions may be used as the theoretical basis for the experimental estimation of third-order elastic constants.

8.4. Cylinders under combined torsional-axial loadings

Nonlinear effects in soft and metallic cylindrical composites under combined torsion and axial loading are investigated within the framework of a second-order elasticity theory. Analytical expressions for homogenous and bilayered cylinders are obtained. The major focus is on the dependence of the nonlinear effects on the elastic constants and the interface position of the composite. The following conclusions can be made.

- (1) The first- and second-order terms in the displacement and stress components under combined axial-torsional loading can be written respectively for the homogeneous and bilayered cylinders as:

$$\begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}_{TP} = \begin{pmatrix} k + k^2 \\ k + k^2 \\ k + k^2 \end{pmatrix}, \quad \mathbf{T}_{TP} = \begin{pmatrix} k^2 & 0 & 0 \\ 0 & k^2 & k + k^2 \\ 0 & k + k^2 & k + k^2 \end{pmatrix};$$

$$\begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}_{TP} = \begin{pmatrix} k + k^2 \\ k + k^2 \\ k + k^2 \end{pmatrix}, \quad \mathbf{T}_{TP} = \begin{pmatrix} k + k^2 & 0 & 0 \\ 0 & k + k^2 & k + k^2 \\ 0 & k + k^2 & k + k^2 \end{pmatrix}.$$

Thus:

The axial force-twist effect, represented by the k^2 component in u_θ , is an

intrinsically coupled and unique phenomenon under combined loading. The effect is absent in cylinders subjected to pure axial loading or pure torsional loading. It is present in both homogeneous and bilayered cylinders.

- (2) The axial force-twist effect can be positive or negative, depending on the elastic parameters, interface positions and axial loading directions.
- (3) The displacement and stress fields due to the combined loading are not a simple superposition of those due to the separate loading modes.
- (4) These unique nonlinear effects may be used in technological applications, e.g., exploiting the change in twist of a torsionally loaded biosensor when it senses axial deformations.

8.5. Homogeneous and bilayered rectangular blocks under shear

The analytical solutions for homogeneous and bilayered rectangular blocks under generalized shear in the framework of second-order elasticity are obtained. The major contributions and findings can be summarized as follows.

- (1) The analytical solutions show that the second-order shear displacements are characterized by the product of sine and hyperbolic sine functions of the height and depth variables, respectively.
- (2) For a homogeneous block under a prescribed constant shear displacement at the top, the solutions reveal that the lateral profile is a straight line superimposed on sine curves. For a bilayered cylinder, the predicted lateral profile consists of two

smooth profiles of different curvatures meeting at a kink at the interface.

Experimental results agree well with the predictions.

- (3) An out-of-plane nonlinear shear stress is present under generalized shear, which may twist/distort the block about the height direction. This phenomenon may have implications in the design of biomaterials and bio-inspired materials. .
- (4) Nonlinearity significantly influences the second-order displacements and stresses e.g., the out-of-plane shear stress.
- (5). Nonlinear normal stresses, which are related to the Poynting effect, are needed to maintain the deformation. The Poynting effect is significantly dependent on the elastic constants, e.g., ν and m . By measuring the normal stresses, third-order elastic constants can be estimated.
- (6) The effect of gravity has been investigated. An extra quadratic term will arise in the dependence of the shear displacement on the height in the presence of gravity. The gravity effect becomes less significant for large μ , large ν and small negative m .
- (7) In a bilayered rectangular block, the layer elasticity inhomogeneity can influence significantly the sign and the magnitude of the second-order normal and shear stresses as well as the interfacial displacement.
- (8) In a mbilayered rectangular block, the inhomogeneity as represented by the ratio of the shear moduli has an intrinsic influence on the stresses, while inhomogeneity between all other constants has an extrinsic influence.

8.6. Recommendations for future research

The current research focuses on the displacement and stress fields of soft multilayered materials under several fundamental loadings. Other topics worthy of research include: (1) fracture and fatigue, (2) buckling, (3) bending, (4) viscoelasticity, and (5) material anisotropy.

8.6.1. Fracture and fatigue of soft composites

The study of fracture and fatigue of soft materials, although important, has received little attention thus far. Martin and Sun (2013) developed a phenomenological constitutive model to study the long-term fatigue damage of heart valve leaflets. Reyssat et al. (2012) studied the fracture of a soft solid experimentally by slicing and dicing a soft gel with a wire. They also carried out theoretical analysis and numerical simulations. Rahbar et al. (2008) investigated the interfacial fracture toughness between hard and soft materials by using a combination of adhesion/fracture mechanics theories and experiments. The interfacial fracture between parylene C and 316L stainless steel were studied which are relevant to drug-eluting stents.

Cartilage, muscle, skin, etc. suffer from cracks, tears, and other kinds of mechanical damage, which may result in pain and impaired function. Polymeric gels, as materials for tissue scaffolding and replacement, may suffer from brittleness

which accompanies their natural softness. Bioprosthetic heart valves, fabricated from bovine pericardium, suffer from fatigue-induced degradation of the tissue leaflets. Many soft materials are multilayered, thus adhesion and interfacial fracture issues are also very important. (Sacks, 2001)

8.6.2. Buckling of soft materials

Experiment results show that soft materials such as hydrogels, elastomers and biological tissues will create various surface patterns through wrinkling and morphological evolution under constrained swelling or shrinkage. The surface wrinkling and folding of core-shell soft sphere and cylinder have been investigated by Li et al. (2011) and Cao et al. (2012). Both the core and shell were simulated by the isotropic and incompressible neo-Hookean model. Li et al. (2012) have also reviewed the buckling and surface wrinkling in soft materials. However, relatively less theoretical research has been done on the buckling pattern of soft composite materials. Further in-depth investigations are therefore of interest.

8.6.3. Bending of soft materials

Bending is a common and fundamental loading mode, but the nonlinear phenomena of soft materials subjected to bending are not clear yet. Dell'isola et al (1998) studied a bent bar in the framework of second-order elasticity and found that it also underwent axial elongation. Moreover, they found that a second-order

bending deformation also existed when an infinitesimally twisted bar was deformed. These nonlinear phenomena were described as generalized Poynting effects. However, the nonlinear behaviors of bent soft materials, especially multilayered composites, remain to be explored.

8.6.4. Viscoelasticity

Experiments show that many soft materials exhibit viscoelastic behaviors (Weiss et al., 2002; Chen et al., 2010). El Sayed et al. (2008) developed a variational constitutive model of soft biological tissues, considering their viscosity. Zhang et al (2007) developed a rate-insensitive linear viscoelastic model for soft tissues. Future works may take into account the influence of viscoelasticity.

8.6.5. Anisotropy

The structures of natural materials are typically hierarchical and made up of different components, which may themselves be elastically anisotropic. Some research on the anisotropy of soft materials has been carried out (Courtney et al., 2006; Ehret and Itskov, 2009). The second-order nonlinear theory employed in this thesis may be generalized to account for elastic anisotropy through the use of certain strain invariants. Destrade et al. (2010) attempted this for an incompressible, transversely isotropic material using third-order elasticity.

8.6.6. Medical and industrial applications of the research

The mechanical behavior of soft materials may have many applications in industry. For example, the Poynting effect and the axial force-twist effect can be utilized in the design and operation of actuators and sensors. A bio-inspired polymer actuator based on the axial force-twist effect can be made, i.e., a torsionally loaded cylinder may generate an additional output twist, if subjected to an input axial force. By carefully selecting the elastic parameters of the materials and the structural dimensions, the amount of twist can be increased significantly and the desired output can be achieved.

The Poynting effect may assist in the design of medical instruments. A key requirement in the development of high-fidelity surgical simulators is the realistic modeling of the interaction between surgical instruments and human organs. Human organs are generally inhomogeneous and multilayered. The Poynting effect can result in differences in force or displacement magnitude larger than the absolute human perception threshold for force or displacement discrimination in some tissues (e.g. myocardial tissues) but not in others (e.g. brain tissues), as described in Misra et al. (2010). Using our results, the magnitude of the forces or displacements of multilayered human organs generated by the Poynting effect can be predicted.

List of Publications

- Wang, D., Wu, M.S., 2015. Generalized shear of a bilayer. *Mechanics of Materials* 83, 122-135.
- Wu, M.S., Wang, D., 2015. Nonlinear effects in composite cylinders: relations and dependence on inhomogeneities. *International Journal of Engineering Science* 90, 27-43.
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- Wang, D., Wu, M.S., 2013. Analytical solutions for bilayered spherical hydrogel subjected to constant dilatation. *Mechanics of Materials* 58, 12-22.
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Appendix

Appendix A

The parameters appearing in Eq. (3.33), Eq. (3.34) and Eq. (3.35) are given explicitly as follows:

$$\xi_1 = 48B_2^2 m_2 \mu_2 - 6B_2^2 n_2 (\lambda_2 + 2\mu_2) + 6B_2^2 \mu_2 (\lambda_2 + 6\mu_2), \quad (\text{A.1})$$

$$\begin{aligned} \xi_2 = & -48B_2^2 m_2 \mu_2 - 4A_1^3 (18l_1 + 2n_1 + 9\lambda_1 + 6\mu_1) \mu_2 + 6B_2^2 n_2 (\lambda_2 + 2\mu_2) \\ & - 2B_2^2 n_2 (3\lambda_2 + 2\mu_2) - 6B_2^2 \mu_2 (3\lambda_2 + 2\mu_2) - 6B_2^2 \mu_2 (\lambda_2 + 6\mu_2) \\ & + 4B_2 (A_2 - \mathcal{G}_0) (3\lambda_2 + 2\mu_2) (6m_2 - n_2 + 3\lambda_2 + 6\mu_2) \\ & + 4(A_2 - \mathcal{G}_0)^2 \mu_2 (18l_2 + 2n_2 + 9\lambda_2 + 6\mu_2), \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} \xi_3 = & 2B_2^2 n_2 (3\lambda_2 + 2\mu_2) - A_1^2 (18l_1 + 2n_1 + 9\lambda_1 + 6\mu_1) (3\lambda_2 + 2\mu_2) + \\ & 6B_2^2 \mu_2 (3\lambda_2 + 2\mu_2) - 4B_2 (A_2 - \mathcal{G}_0) (3\lambda_2 + 2\mu_2) (6m_2 - n_2 + 3\lambda_2 + 6\mu_2) \\ & - 4(A_2 - \mathcal{G}_0)^2 \mu_2 (18l_2 + 2n_2 + 9\lambda_2 + 6\mu_2), \end{aligned} \quad (\text{A.3})$$

$$\xi_4 = 2B_2^2 (3\lambda_1 + 2\mu_1 + 4\mu_2) (n_2 (\lambda_2 + 2\mu_2) - \mu_2 (8m_2 + \lambda_2 + 6\mu_2)), \quad (\text{A.4})$$

$$\begin{aligned} \xi_5 = & 4(B_2 \mathcal{G}_0 (3\lambda_1 + 2\mu_1) (\lambda_2 + 2\mu_2) (6m_2 - n_2 + 3\lambda_2 + 6\mu_2) \\ & - A_2^2 (\lambda_2 + 2\mu_2) \mu_2 (18l_2 + 2n_2 + 9\lambda_2 + 6\mu_2) \\ & + 2B_2^2 \mu_2 (-n_2 \lambda_2 + (2m_2 + \lambda_2) (3\lambda_1 + 2\mu_1) + (8m_2 - 2n_2 + 9\lambda_1 + \lambda_2 + 6\mu_1) \mu_2 + 6\mu_2^2) \\ & + \mu_2 (\lambda_2 + 2\mu_2) (A_1^2 (18l_1 + 2n_1 + 9\lambda_1 + 6\mu_1) - \mathcal{G}_0^2 (18l_2 + 2n_2 + 9\lambda_2 + 6\mu_2)) \\ & + A_2 (\lambda_2 + 2\mu_2) (-B_2 (3\lambda_1 + 2\mu_1) (6m_2 - n_2 + 3\lambda_2 + 6\mu_2) + 2\mathcal{G}_0 \mu_2 (18l_2 + 2n_2 + 9\lambda_2 + 6\mu_2))), \end{aligned} \quad (\text{A.5})$$

$$\xi_6 = (A_2 - \mathcal{G}_0)^2 (\lambda_2 + 2\mu_2) (3\lambda_1 + 2\mu_1 + 4\mu_2) (18l_2 + 2n_2 + 9\lambda_2 + 6\mu_2), \quad (\text{A.6})$$

$$\xi_7 = -\frac{1}{(\lambda_2 + 2\mu_2)}(B_2^2(-3\lambda_1 + 3\lambda_2 - 2\mu_1 + 2\mu_2)(n_2(\lambda_2 + 2\mu_2) - \mu_2(8m_2 + \lambda_2 + 6\mu_2))), \quad (\text{A.7})$$

$$\xi_8 = 2B_2(A_2 - \mathcal{G}_0)(-3\lambda_1 + 3\lambda_2 - 2\mu_1 + 2\mu_2)(6m_2 - n_2 + 3\lambda_2 + 6\mu_2), \quad (\text{A.8})$$

$$\begin{aligned} \xi_9 = & \left(\frac{9}{2}\lambda_2 + 3\mu_2 + 9l_2 + n_2\right)(A_2 - \mathcal{G}_0)^2(-3\lambda_1 + 3\lambda_2 - 2\mu_1 + 2\mu_2) \\ & + (3\lambda_2 + 2\mu_2)(A_1^2\left(\frac{9}{2}\lambda_1 + 3\mu_1 + 9l_1 + n_1\right) + \\ & \frac{1}{2(\lambda_2 + 2\mu_2)}(4B_2(A_2 - \mathcal{G}_0)(\lambda_2 + 2\mu_2)(6m_2 - n_2 + 3\lambda_2 + 6\mu_2) \\ & - (A_2 - \mathcal{G}_0)^2(\lambda_2 + 2\mu_2)(18l_2 + 2n_2 + 9\lambda_2 + 6\mu_2) + \\ & 2B_2^2(-n_2\lambda_2 + (2m_2 + \lambda_2)(3\lambda_1 + 2\mu_1) + (8m_2 - 2n_2 + 9\lambda_1 + \lambda_2 + 6\mu_1)\mu_2 + 6\mu_2^2))). \end{aligned} \quad (\text{A.9})$$

Finally, the parameters appearing in Eq. (3.36)-(3.37) are given by:

$$\begin{aligned} \zeta_1 = & (6C_2\lambda_2 + \mathcal{G}_0^2(18l_2 + 2n_2 + 9\lambda_2))(\lambda_2 + 2\mu_2) \\ & + 2(2C_2 + 3\mathcal{G}_0^2)\mu_2(\lambda_2 + 2\mu_2) \\ & + A_2^2(\lambda_2 + 2\mu_2)(18l_2 + 2n_2 + 9\lambda_2 + 6\mu_2) \\ & - 2A_2\mathcal{G}_0(\lambda_2 + 2\mu_2)(18l_2 + 2n_2 + 9\lambda_2 + 6\mu_2), \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} \zeta_2 = & -8D_2\mu_2(\lambda_2 + 2\mu_2) - 4A_2B_2(\lambda_2 + 2\mu_2)(6m_2 - n_2 + 3\lambda_2 + 6\mu_2) \\ & + 4B_2\mathcal{G}_0(\lambda_2 + 2\mu_2)(6m_2 - n_2 + 3\lambda_2 + 6\mu_2), \end{aligned} \quad (\text{A.11})$$

$$\zeta_3 = 2B_2^2(n_2(\lambda_2 + 2\mu_2) - \mu_2(8m_2 + \lambda_2 + 6\mu_2)). \quad (\text{A.12})$$

Appendix B

The derivation of the deformation gradient stated in Eq. (3.43) is outlined here. Consider a circular tube deforming under the restriction of circular symmetry, i.e., each point in the tube undergoes a radial displacement $u_r(r)$, where r is the radial coordinate in the plane polar coordinate system (r, θ) . According to Murnaghan (1951), $u_r(r)$ is replaced by $ku_r(r)$, where k keeps track of the order of approximation of the physical quantities. In the linear theory, terms up to the first power of k are retained, while in the second-order theory, terms up to the second powers of k are retained.

If $\mathbf{a}$ and $\mathbf{x}(\mathbf{a})$ denote the position vectors in the undeformed and deformed states of the sphere, respectively, then their differentials are :

$$d\mathbf{a} = \begin{pmatrix} dr \\ r d\theta \end{pmatrix}, \quad d\mathbf{x} = \begin{pmatrix} dr + k du_r \\ (r + k u_r) d\theta \end{pmatrix}. \quad (\text{B.1})$$

It can be easily seen that the differentials are related by the deformation gradient $\mathbf{F}$:

$$d\mathbf{x} = \mathbf{F} d\mathbf{a}, \quad \mathbf{F} = \text{diag}[1 + k u'_r, 1 + \frac{k u_r}{r}], \quad (\text{B.2})$$

where “diag” denotes a 2 by 2 diagonal matrix and the prime denotes a derivative with respect to r . The second part of Eq. (B.2) is written as Eq. (3.43). The Lagrangian strain $\mathbf{E}$ then follows from the definition given in Eq. (3.44).

Appendix C

The six constants appearing in Eqs. (3.58)–(3.65) for a bilayer cylindrical composite gel are listed in the following:

$$A_1 = \frac{r_2^2(\lambda_2 + \mu_2)(\mathcal{G}_1\lambda_1 + \mathcal{G}_1\mu_1 + \mathcal{G}_2\mu_2) + r_1^2\mu_2(\mathcal{G}_1\lambda_1 + \mathcal{G}_1\mu_1 - \mathcal{G}_2\lambda_2 - \mathcal{G}_2\mu_2)}{r_1^2(\lambda_1 - \lambda_2 + \mu_1 - \mu_2)\mu_2 + r_2^2(\lambda_2 + \mu_2)(\lambda_1 + \mu_1 + \mu_2)}, \quad (\text{C.1})$$

$$A_2 = \frac{-r_2^2\mathcal{G}_2(\lambda_2 + \mu_2)(\lambda_1 + \mu_1 + \mu_2) - r_1^2\mu_2(\mathcal{G}_1\lambda_1 + \mathcal{G}_1\mu_1 - \mathcal{G}_2\lambda_2 - \mathcal{G}_2\mu_2)}{r_1^2(\lambda_1 - \lambda_2 + \mu_1 - \mu_2)\mu_2 + r_2^2(\lambda_2 + \mu_2)(\lambda_1 + \mu_1 + \mu_2)}, \quad (\text{C.2})$$

$$B_2 = \frac{r_2^2(\mathcal{G}_1 - \mathcal{G}_2)(\lambda_1 + \mu_1)(\lambda_2 + \mu_2)}{r_1^2(\lambda_1 - \lambda_2 + \mu_1 - \mu_2)\mu_2 + r_2^2(\lambda_2 + \mu_2)(\lambda_1 + \mu_1 + \mu_2)}, \quad (\text{C.3})$$

$$C_1 = \frac{C_{11} + C_{12} + C_{13} + C_{14}}{C_{1D}}, \quad (\text{C.4})$$

$$C_2 = \frac{C_{21} + C_{22}}{C_{2D}}, \quad (\text{C.5})$$

$$D_2 = \frac{D_{21} + D_{22}}{D_{2D}}, \quad (\text{C.6})$$

where:

$$C_{11} = -r_2^2\lambda_2(4l_1r_2^2\mathcal{G}_1^2 + 2m_1r_2^2\mathcal{G}_1^2 + 4B_2m_2r_1^2\mathcal{G}_2 - 4B_2m_2r_2^2\mathcal{G}_2 + 3r_2^2\mathcal{G}_1^2\lambda_1 + 2B_2r_1^2\mathcal{G}_2\lambda_2 - 2B_2r_2^2\mathcal{G}_2\lambda_2 + 3r_2^2\mathcal{G}_1^2\mu_1) \quad (\text{C.7})$$

$$C_{12} = B_2^2r_1^2(r_1 - r_2)(r_1 + r_2)\mu_2(2m_2 + \lambda_2) + 4B_2r_2^2(-r_1^2 + r_2^2)\mathcal{G}_2\mu_2(m_2 + 2\lambda_2) \quad (\text{C.8})$$

$$C_{13} = -r_2^2\mu_2(4l_1(r_1^2 + r_2^2)\mathcal{G}_1^2 + 2m_1(r_1^2 + r_2^2)\mathcal{G}_1^2 - 4l_2r_1^2\mathcal{G}_2^2 - 2m_2r_1^2\mathcal{G}_2^2 + 4l_2r_2^2\mathcal{G}_2^2 + 2m_2r_2^2\mathcal{G}_2^2 + 3r_1^2\mathcal{G}_1^2\lambda_1 + 3r_2^2\mathcal{G}_1^2\lambda_1 - 3r_1^2\mathcal{G}_2^2\lambda_2 + 3r_2^2\mathcal{G}_2^2\lambda_2 + 3(r_1^2 + r_2^2)\mathcal{G}_1^2\mu_1) \quad (\text{C.9})$$

$$\begin{aligned}
 C_{14} = & 3(r_1 - r_2)(r_1 + r_2)(B_2^2 r_1^2 - 2B_2 r_2^2 \mathcal{G}_2 + r_2^2 \mathcal{G}_2^2) \mu_2^2 \\
 & + A_2^2 (r_1 - r_2) r_2^2 (r_1 + r_2) \mu_2 (4l_2 + 2m_2 + 3\lambda_2 + 3\mu_2) \\
 & - A_1^2 r_2^2 (4l_1 + 2m_1 + 3\lambda_1 + 3\mu_1) (r_1^2 \mu_2 + r_2^2 \lambda_2 + r_2^2 \mu_2) \\
 & + 2A_1 r_2^2 \mathcal{G}_1 (4l_1 + 2m_1 + 3\lambda_1 + 3\mu_1) (r_1^2 \mu_2 + r_2^2 \lambda_2 + r_2^2 \mu_2) \\
 & + 2A_2 (r_1 - r_2) r_2^2 (r_1 + r_2) (B_2 (\lambda_2 + \mu_2) (2m_2 + \lambda_2 + 3\mu_2) \\
 & - \mathcal{G}_2 \mu_2 (4l_2 + 2m_2 + 3\lambda_2 + 3\mu_2))
 \end{aligned} \tag{C.10}$$

$$C_{1D} = 2r_1^2 r_2^2 (\lambda_1 - \lambda_2 + \mu_1 - \mu_2) \mu_2 + 2r_2^4 (\lambda_2 + \mu_2) (\lambda_1 + \mu_1 + \mu_2) \tag{C.11}$$

$$\begin{aligned}
 C_{21} = & -B_2^2 r_1^2 (r_1 - r_2) (r_1 + r_2) \mu_2 (\lambda_1 + \mu_1 + \mu_2) (2m_2 + \lambda_2 + 3\mu_2) \\
 & + 2B_2 r_1^2 r_2^2 \mathcal{G}_2 (\lambda_1 + \mu_1) (\lambda_2 + 2\mu_2) (2m_2 + \lambda_2 + 3\mu_2) \\
 & + A_2^2 r_2^2 (-r_1^2 \mu_2 + r_2^2 \lambda_1 + r_2^2 \mu_1 + r_2^2 \mu_2) (\lambda_2 + 2\mu_2) (4l_2 + 2m_2 + 3\lambda_2 + \mu_2) \\
 & - 2A_2 B_2 r_1^2 r_2^2 (\lambda_2 + 2\mu_2) (\lambda_1 + \mu_1) (2m_2 + \lambda_2 + 3\mu_2) \\
 & - 2A_2 \mathcal{G}_2 r_2^2 (-r_1^2 \mu_2 + r_2^2 \lambda_1 + r_2^2 \mu_1 + r_2^2 \mu_2) (\lambda_2 + 2\mu_2) (4l_2 + 2m_2 + 3\lambda_2 + 3\mu_2)
 \end{aligned} \tag{C.12}$$

$$\begin{aligned}
 C_{22} = & r_2^2 (\lambda_2 + 2\mu_2) ((A_1 - \mathcal{G}_1)^2 r_1^2 (4l_1 + 2m_1) \mu_2 \\
 & + \mathcal{G}_2^2 (-r_1^2 \mu_2 + r_2^2 \lambda_1 + r_2^2 \mu_1 + r_2^2 \mu_2) (4l_2 + 2m_2) \\
 & + 3\mathcal{G}_2^2 r_2^2 (\lambda_2 + \mu_2) (\lambda_1 + \mu_1 + \mu_2) \\
 & + 3(A_1 - \mathcal{G}_1)^2 r_1^2 \mu_2 (\lambda_1 + \mu_1) - 3r_1^2 \mu_2 \mathcal{G}_2^2 (\lambda_2 + \mu_2))
 \end{aligned} \tag{C.13}$$

$$C_{2D} = -2r_1^2 r_2^2 (\lambda_2 + 2\mu_2) (\lambda_1 - \lambda_2 + \mu_1 - \mu_2) \mu_2 - 2r_2^4 (\lambda_2 + 2\mu_2) (\lambda_2 + \mu_2) (\lambda_1 + \mu_1 + \mu_2) \tag{C.14}$$

$$\begin{aligned}
 D_{21} = & \frac{(\lambda_2 + \mu_2)}{\lambda_2 + 2\mu_2} 2r_1 r_2^2 ((B_2^2 (\lambda_1 + \mu_1 + \mu_2) (2m_2 + \lambda_2 + 3\mu_2) \\
 & + 2B_2 (A_2 - \mathcal{G}_2) (\lambda_2 + 2\mu_2) (2m_2 + \lambda_2 + 3\mu_2) \\
 & + (\lambda_2 + 2\mu_2) (A_1 - \mathcal{G}_1)^2 (4l_1 + 2m_1 + 3\lambda_1 + 3\mu_1) \\
 & - (\lambda_2 + 2\mu_2) (A_2 - \mathcal{G}_2)^2 (4l_2 + 2m_2 + 3\lambda_2 + 3\mu_2))
 \end{aligned} \tag{C.15}$$

$$\begin{aligned}
 D_{22} = & 2r_1 r_2^2 (\lambda_1 - \lambda_2 + \mu_1 - \mu_2) \left(\frac{2B_2 r_1^2 \mathcal{G}_2 (2m_2 + \lambda_2 + 3\mu_2)}{r_2^2} - \frac{B_2^2 r_1^4 \mu_2 (2m_2 + \lambda_2 + 3\mu_2)}{r_2^4 (\lambda_2 + 2\mu_2)} \right. \\
 & \left. + (A_2 - \mathcal{G}_2)^2 (4l_2 + 2m_2 + 3\lambda_2 + 3\mu_2) - \frac{2A_2 B_2 r_1^2 (2m_2 + \lambda_2 + 3\mu_2)}{r_2^2} \right)
 \end{aligned} \tag{C.16}$$

$$D_{2D} = -4r_1^3 (\lambda_1 - \lambda_2 + \mu_1 - \mu_2) \mu_2 - 4r_1 r_2^2 (\lambda_2 + \mu_2) (\lambda_1 + \mu_1 + \mu_2) \tag{C.17}$$