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Adaptive Control of Mechanical Systems

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2014

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

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Date

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Huang Jiangshuai

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Abstract

Nonholonomic underactuated mechanical systems are special systems full of research interest and practical sense. Referring to those mechanical systems whose number of control variables is less than the degrees of freedom, the underactuated systems are abundant in real life, ranging from landing vehicles, surface ships, underwater vehicles to spacecrafts. Mobile robots and surface vessels are two typical nonholonomic underactuated systems that receive tremendous consideration in literature. For the tracking and stabilization control of underactuated mechanical systems, many methodologies have been proposed for controlling these systems by researchers. However, there are still a lot of open problems in this area. Among them the tracking control with input saturation and output-feedback control in the presence of parametric uncertainties are two important problems to be solved in this thesis.

For tracking and stabilization control of nonholonomic mobile robots with input saturation, a new adaptive scheme is proposed to ensure that the bounds of the control torques are functions of only design parameters and reference trajectory. Thus suitable design parameters can be chosen so that such bounds are within the given saturation limits. For adaptive output-feedback tracking control of nonholonomic mobile robots, a new adaptive control scheme is proposed including designing a new adaptive state feedback controller and two high-gain observers to estimate the unknown linear and angular velocities respectively. For tracking control of underactuated ships we design the yaw axis torque in such a way that its corresponding subsystem is finite time stable, which makes it be decoupled from the second subsystem after a finite time. This enables us to design the torque in the surge axis independently. For adaptive output-feedback track-

ing control of underactuated ships, by using the prescribed performance bound technique, the position error and orientation error can be guaranteed converging to arbitrarily small residual sets at a pre-specified exponential rate.

Distributed coordination for a group of dynamic agents has attracted many researchers in recent years, such as cooperative control of unmanned air vehicles (UAVs), formation control, flocking control, distributed sensor networks, attitude alignment for a clusters of satellites, congestion control in communication networks etc. An important performance indicator for the consensus problem is the convergence rate. Most of the existing consensus control schemes for multi-agent systems so far achieve asymptotical convergence. This implies that the convergence rate at best is exponential, and it needs infinite time for the tracking errors to converge to the origin. Thus finite-time consensus control of a group of mechanical systems with parametric uncertainties is an important problem and will be considered in this thesis.

New adaptive finite time continuous distributed control algorithms are proposed for multi-agent systems under two different scenarios. For the leaderless multi-agent systems, it is shown that the states of the mechanical systems can reach a consensus within finite time. For the leader-follower control of a group of mechanical systems, two control schemes are proposed. The first one is based on the hierarchical decomposition of the graph. It is demonstrated that perfect consensus can be reached or the leader-following errors converge to a region with arbitrarily small radius in finite time. The second one is based on distributed estimators, which also achieves finite-time consensus. Transient performances in terms of convergence rates and time are also analyzed. Finally simulation results illustrate and verify the effectiveness of the proposed schemes.

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Symbols and Acronyms

Algebraic Operators

A^T	Transpose of matrix A
A^{-1}	Inverse of matrix A
$\det(A)$	Determinant of matrix A
$\rho(A)$	Spectral radius of matrix A
$\text{diag}(a_1, \dots, a_n)$	Diagonal matrix with main given diagonal numbers
$\ \cdot\ $	ℓ^2 norm for vector or the induced norm for matrix
$\ \cdot\ _\infty$	ℓ^∞ norm for vector or the induced norm for matrix
$M > 0$ ($M \geq 0$)	M is positive definite(semi-definite)
$M_1 > M_2$ ($M_1 \geq M_2$)	$M_1 - M_2$ is positive definite(semi-definite)
$A \otimes B$	Kronecker product of A and B

Sets

$\mathbb{R}$	Set of real numbers
$\mathbb{Z}$	Set of integers
$\mathbb{N}$	Set of nonnegative integers

Others

$\mathbf{0}$	Zero vector with a compatible dimension
$\mathbf{1}$	Vector with a compatible dimension and all elements of one
$\lambda_{\max}(P)$	maximum eigenvalue of matrix P
$\lambda_{\min}(P)$	minimum eigenvalue of matrix P

Chapter 1

Introduction

1.1 Underactuated Mechanical Systems

In the past few decades, the control of underactuated mechanical systems has been widely investigated in the control community. Underactuated mechanical systems refer to those mechanical systems which have more degree of freedom than the number of actuators. For example an unicycle mobile robot, which has two input signals while the degree of freedom is three, i.e., the positions in X and Y coordinate, and its heading angle. Underactuated mechanical systems appear almost everywhere in real life, e.g., robotics, aerospace systems, surface marine systems, underwater marine systems, helicopters, road vehicles, etc. In general, the linear approximation approach is not applicable to this type of systems. Therefore controller design for underactuated mechanical systems is a challenge task and it has both theoretical interest and practical significance. The reasons of “underactuation” are mainly due to the following reasons: 1) kinematics or dynamics of the system is inherently underactuated, e.g. nonholonomic mobile robots, aircraft, spacecraft. 2) part of actuators are removed for reducing cost or other practical purposes, e.g. surface vessel without sway force or satellite with two thrusters. 3) actuator failure, 4) artificially creating a complex low-order nonlinear system to obtain insight of high-order underactuated systems, e.g. inverted pendulum, Beam-and-Ball system, etc.

1.2 Nonholonomic Constraints

Most of the underactuated mechanical systems are suffered from nonholonomic constraints. Let $q \in Q$ be the configuration vector of a mechanical system, then nonholonomic constraint can be defined as a set of non-integrable conditions satisfied by certain functions, for example, for the condition that $\phi(q)\dot{q} = 0$, there does not exist a $\Phi(q) = 0$ such that $d\Phi(q) = \phi(q)\dot{q}$. In classical mechanical systems, nonholonomic constraints can be divided into two classes, the first order nonholonomic constraints and the second order nonholonomic constraints. For the first order nonholonomic constraints, the constraints are defined on the coordinates and velocities of the mechanical systems, whereas for the second order nonholonomic constraints, the constraints involve accelerations of the mechanical systems. The first-order nonholonomic constraint is widely found in land vehicles like mobile robots, tractor and trailer system, while the second order nonholonomic condition is normally required by surface marine vehicles, under-water vehicles and space vehicles like ships and spacecraft.

1.2.1 First-order Nonholonomic Constraint

In order to illustrate the first order nonholonomic constraint, we show an example of nonholonomic mobile robot by considering a two-wheeled mobile robot with its kinematic model is given by

$$\dot{x} = v \cos \theta$$

$$\dot{y} = v \sin \theta$$

$$\dot{\theta} = w$$

where the forward velocity v and angular velocity w are inputs that can be controlled independently and it is assumed the front castor wheel and the rear wheels just roll without slipping, (x, y) denotes the coordinates of the center of mass and θ denotes the angle between the heading direction and x -axis. Let the center-line be the line in the middle of two actuated wheels and orthogonal to the line con-

necting two wheels. Then the rolling without slipping condition of mobile robot requires that the velocity orthogonal to the center-line is equal to zero, which means

$$\dot{x} \sin \theta - \dot{y} \cos \theta = 0 \quad (1.1)$$

Constraint (1.1) can not be integrated, i.e., it cannot be written as a time-derivative of some function of the state, i.e. $G(x, y, \theta)$. Therefore it is called a nonholonomic constraint. In fact, it can be shown that a first-order nonholonomic system cannot be stabilized by any smooth time-invariant static state-feedback based on Brockett's necessary lemma as in [1].

1.2.2 Second-order Nonholonomic Constraint

We consider the general form of an underactuated mechanical system with $q = (q_1, q_2, \dots, q_n)$ as follows:

$$M_{11}(q)\ddot{q}_i + M_{12}(q)\ddot{q}_j + C_1(q, \dot{q}) = F(q)u \quad (1.2a)$$

$$M_{21}(q)\ddot{q}_i + M_{22}(q)\ddot{q}_j + C_2(q, \dot{q}) = 0 \quad (1.2b)$$

We partition the state space configuration into two parts, q_i and q_j , where $q_i \in R^m$ denotes the subsystem actuated directly by control $u \in R^m$, and $q_j \in R^{n-m}$ denotes the subsystem that is not directly actuated by actuators leaving q_j as an un-actuated subsystem. Actually if there is no nontrivial function $\Psi(t, q, \dot{q})$ with $d\Psi/dt = 0$ being the solution to (1.2b), then (1.2b) becomes a nonholonomic constraint.

Typical examples of underactuated nonholonomic systems with second order nonholonomic constraints include robot manipulator, autonomous under-water vehicles, underactuated surface vehicles, Acrobot system, vertical take-off and land spacecraft and aircraft. In contrast to systems with first-order nonholonomic constraint, the systems with second-order nonholonomic constraints have drift-terms which make the controller design more difficult. Such systems also do not

satisfy Brockett's necessary lemma, which means there is no time-invariant state feedback controller that makes the systems asymptotically stable. As an example, consider the dynamics of an underactuated ship, which can be described as

$$\dot{\eta} = J(\eta)\nu \quad (1.3)$$

$$M\dot{\nu} + C(\eta, \nu)\nu + D(\nu)\nu = \begin{bmatrix} \tau \\ 0 \end{bmatrix} \quad (1.4)$$

where $\eta = [x, y, \phi]^T$ denotes the position and the orientation of the ship in the body-fixed frame, $\nu = [u, v, r]^T$ denotes the linear and angular velocities of the ship, M is the inertia matrix, $C(\eta, \nu)$ denotes the Coriolis and centripetal matrix, $D(\nu)$ is the damping matrix.

1.2.3 Literature Review of Underactuated Mechanical Systems

Typical examples of underactuated mechanical systems are nonholonomic mobile robots and underactuated ships, which have received lots of attention from the control community. In control theory, stabilization is usually regarded as a special case of the tracking problem. However for controlling underactuated mechanical systems with nonholonomic constraints such as nonholonomic mobile robots, the stabilization and tracking problems are totally different, and thus they are normally considered separately.

For nonholonomic mobile robots, due to the Brockett's necessary condition [1], there exists no continuous time-invariant state-feedback controller that asymptotically stabilizes the mobile robot. For the stabilization problem, the past abundant literature aims at developing suitable discontinuous time-invariant stabilizer [3] or time-varying stabilizer [2] or hybrid stabilizers [4, 5]. The seminal work of [2] introduced the first time-varying feedback stabilizer for a wheeled mobile robot. For tracking control, in [6] a time-varying state-feedback tracking controller is proposed for kinematics and simplified dynamics of mobile robots based on backstepping [8]. In [9], an adaptive tracking controller is designed for

nonholonomic mobile robots with parametric uncertainties. In [10] by using coordinate transformation an adaptive controller is designed for nonholonomic mobile robots such that the tracking and stabilization problems can be solved with only one controller. In [11] a global time-varying output feedback controller that can solve tracking and stabilization for nonholonomic mobile robots is proposed based on a coordinate transformation. In [12], transverse function approach is proposed for the stabilization of arbitrary reference trajectories for nonholonomic mobile robots based on a change of coordinates.

For the underactuated ships, In [13], the method developed for chained-form systems is applied for tracking control of underactuated ships through a coordinate transformation. Exponential stability of the closed-loop system is ensured and the position and the angle of the ship are steered to track the reference trajectory. In [14], two tracking control schemes are proposed using Lyapunov's direct and passivity approaches for underactuated ships. In [15] a tracking controller is developed for underactuated surface ships to globally asymptotically track a reference which is allowed to be a curve including a straight line. In [16] an universal controller is proposed for underactuated ships to achieve stabilization and tracking control simultaneously based on Lyapunov's direct method and backstepping technique. In [17], a continuous time-varying tracking controller is proposed to yield globally bounded tracking by transforming the ship error system into a skew-symmetric form and generating a time-varying dynamic oscillator. In [18] a global robust adaptive controller is proposed for underactuated ship to follow a reference path under both constant and time-varying disturbances and parametric uncertainties. In [19] a global smooth controller is proposed for underactuated ships that achieves the practical stabilization of arbitrary reference trajectories including fixed points and non-admissible trajectories based on the transverse function approach developed in [12]. In [20] a nonlinear model-based adaptive output feedback controller was proposed for global asymptotic tracking in the presence of parametric uncertainties associated with nonlinear ship dynamics. In [22] by nonlinear coordinate changes to transform the ship dynamics to a system affine in the ship's velocities, observers are designed to estimate unmeasured velocities,

and tracking controllers are then proposed based on Lyapunov's direct method and backstepping.

For general underactuated mechanical systems, several researchers have proposed controllers to achieve stabilization and tracking objectives. In [23] a sliding mode control scheme which robustly stabilizes a class of underactuated mechanical systems with parametric uncertainties that are not linearly controllable is proposed. In [24] a passivity based interconnection and damping assignment control scheme is proposed to address the stabilization of underactuated mechanical systems by modifying both the potential and kinetic energies. In [25] cascade normal forms including strict feedback form, feedforward form and nontriangular quadratic form are introduced for underactuated mechanical systems to facilitate control design. In [26] a hybrid switching control strategy that combines switching control, proportional control and derivative control is proposed for underactuated mechanical systems. In [27] the controllability and stabilizability results are derived for a class of underactuated mechanical systems characterized by nonintegrable dynamics relations.

1.3 Control of Multi-agent Systems

A multi-agent system is a computerized system composed of multiple interacting intelligent agents, and it can be used to solve a problem that can not be solved by an individual agent or a combination of several agents without interaction. Cooperative multi-agent problem has a long history in computer science which originated in 1960s. Distributed computing over a network of multiple interacting intelligent agents started in the field of systems and control in [32], [33] and [34].

In a network of multiple interacting intelligent agents, consensus problem means all the agents reach an agreement with certain level of interest depending on the states of all agents through the exchange of information among the agents and the respective neighbors in the network. The theoretic framework of consensus problem of a network of dynamic systems is introduced by Olfati-Saber and Murray in [35]. Graph Laplacians and the spectral properties are playing an

important role in the graph related consensus convergence analysis, which is also utilized in this thesis. The second smallest eigenvalue of the Laplacians which qualifies the convergence rate of consensus is called 'algebraic connectivity'. The research on multi-agent systems mainly can be divided into several categories, including consensus [28–31, 35, 57], collective behavior of flocks and swarms [36, 54], sensor fusion [38, 39], synchronization of coupled oscillators [40, 41], formation control of multiple vehicles [42–44, 46, 57], optimization based cooperative control [47, 48], etc.

Consensus of a group of agents means that the agents reach an agreement on a common value by exchanging information with their respective neighbors. Consensus has been studied extensively recently in the context of broad applications including cooperative control of multiple mechanical systems [49, 123], formation control [51, 52], flocking control of mobile agents [53–55] etc. The consensus of first-order multi-agent systems has been widely investigated. In [56] a Nyquist criterion that uses the eigenvalues of the graph Laplacian matrix to determine the effect of the topology on the stability of formation of multiple vehicles is established. In [35] the first-order consensus problems for networks of dynamic agents with directed and fixed/switching topology as well as undirected and fixed topology are investigated. In [58] first-order multi-agent systems with coupling delays and switching topologies is investigated and it is shown that the consensus is delay-robustness and it holds as long as the graph is uniformly quasi-strongly connected and switch with a dwell time.

For consensus algorithm of second-order multi-agent systems, the analysis tools from algebraic graph theory have been adopted to solve the consensus problem. It is shown that for an undirected graph, consensus could be reached if the graph is connected, and for a directed graph, it has been shown that consensus could be reached if the graph has a spanning tree and the control gains are carefully selected. In [59] the consensus algorithms for networked double-integrators with or without a reference are investigated under four different cases. In [60] the consensus of networked double integrator systems is considered when the inputs are suffered from input saturation and the velocities of the agents are

not available for feedback. In [61] some necessary and sufficient conditions for second-order consensus of multi-agent dynamical systems are investigated. It is shown that both the real and imaginary parts of the eigenvalues of the Laplacian matrix corresponding to the communication network play key roles in achieving consensus, and a consensus algorithm is derived for the second-order multi-agent system facing communication delays. In [62] consensus problems for second-order agents under directed and arbitrarily switching topologies without or with communication delay are investigated. In [63] the consensus problem of second order multi-agent systems with diverse input delays is studied based on the frequency-domain analysis. Decentralized consensus condition is obtained for the systems with symmetric coupling weights and the robustness of the symmetric system with asymmetric perturbation is investigated.

The consensus control of multiple mechanical systems has also received a lot of attention from the control community in recent years. In [64] the consensus of a group of uncertain mechanical systems with a trajectory known to a part of agents is investigated. In [65] the flocking control of a group of nonholonomic mobile robots with trajectory only available to a part of agents. With backstepping technique and singular perturbation theory, it can be shown that the consensus errors are of the order of ε where ε is an arbitrarily small number. In [123] a distributed coordinated tracking problem for multiple networked Euler-Lagrange systems is studied to track a leader whose generalized coordinates are time varying and available to a subset of the followers. In [52] and [67] the formation control of a group of nonholonomic mobile robots and underactuated ships are studied respectively.

1.4 Motivations and Control Objectives

In practical applications, the inputs of the underactuated mechanical systems like mobile robots and underactuated ships are provided by the motors, whose outputs can not be unlimited. Thus when we design controllers for the underactuated mechanical systems the input saturation problem must be put into consideration.

The tracking or stabilization control of nonholonomic mobile robots and underactuated ships with input saturation is seldom considered in the existing literature. Thus in this thesis the tracking and stabilization control of nonholonomic mobile robots with parametric uncertainties and external disturbance in the presence of input saturation, and the tracking control of underactuated ships with input saturation, are considered.

For state-feedback control of underactuated mechanical systems the velocities of the mechanical systems must be measured by devices like tachometers. But in practice such devices may not be used either because they are contaminated by noise or they are expensive. Thus we are concerned with the cases where the velocities of the mechanical systems are not needed for the implementation of tracking control. The output feedback tracking control of underactuated mechanical systems like mobile robots and underactuated ships has received attentions from control community. But it should be pointed out that the output feedback tracking control of nonholonomic mobile robots or underactuated ships with parametric uncertainties has never been considered before. Due to the existence of parametric uncertainties, the output feedback tracking controller for nonholonomic mobile robots and underactuated ships will become much more difficult due to the simultaneous existence of nonholonomic constraints, unknown system parameters and a quadratic term of unmeasurable states in the dynamic system as well as their couplings.

The control of multi-agent systems is receiving a lot of attention from control community recently. The consensus control of a group of mechanical systems with parametric uncertainties is an interesting topic particularly. An important performance indicator for the consensus problem is the convergence rate. Most of the existing consensus control schemes for multi-agent systems so far achieve asymptotical convergence, namely the convergence rate at best is exponential, which means it needs infinite time for the tracking errors to converge to the origin. Thus designing a controller for each agent such that consensus could be reached within finite time is part of the objective of this thesis. However, achieving adaptive finite-time control is rather challenging. Certain key techniques employed in

existing adaptive control literature cannot be applied. To solve adaptive finite-time control problems, some new techniques have to be developed.

1.5 Major Contributions of the Thesis

The main results of this thesis are summarized as follows.

1. In Chapter 2, the tracking and stabilization control of nonholonomic mobile robots with parametric uncertainties and external disturbance in the presence of input saturation is investigated. A new adaptive scheme is proposed to ensure that the bounds of the control torques are functions of only design parameters and reference trajectory and thus computable in advance. Thus suitable design parameters can be chosen so that such bounds are within the given saturation limits. To compensate for the disturbances, the unknown bounds are estimated through adaptive law and the estimates are employed in controller design. The system stability, perfect tracking of the given trajectory and stabilization to the origin are also established.

2. In Chapter 3, tracking control of underactuated ship in the presence of input saturation is addressed. By dividing the tracking error dynamic system into a cascade of two subsystems, the torques in surge and yaw axis are designed separately using the backstepping technique. More specifically, we design the yaw axis torque in such a way that its corresponding subsystem is finite time stable, which makes it be de-coupled from the second subsystem after a finite time. This enables us to design the torque in the surge axis independently. It is shown that the closed-loop system is stable and the mean-square tracking errors can be made arbitrarily small by choosing design parameters.

3. In Chapter 4, an adaptive output feedback tracking controller for nonholonomic mobile robots is proposed. The major difficulties involved are caused by simultaneous existence of nonholonomic constraints, unknown system parameters and a quadratic term of unmeasurable states in the mobile robot dynamic system as well as their couplings. To overcome these difficulties, a new adaptive control scheme is proposed including designing a new adaptive state feedback controller

and two high-gain observers to estimate the unknown linear and angular velocities respectively. It is shown that the closed loop adaptive system is stable and the tracking errors are guaranteed to be within the arbitrarily small pre-specified bounds.

4. In Chapter 5, both state feedback and output feedback robust adaptive controllers for an underactuated ship are proposed to make the ship follow a pre-defined path with guaranteed transient performance and arbitrarily small steady state position error and orientation error, despite the presences of environmental disturbances induced by wave, wind and ocean current and unknown system parameters. By using the prescribed performance bound technique, the position error and orientation error can be guaranteed converging to arbitrarily small residual sets at a pre-specified exponential rate. It is shown that the mean-square tracking errors are of order of an arbitrarily small design parameter, and the position error and the orientation error also decrease at a pre-specified exponential rate.

5. In Chapter 6, the finite-time consensus control of a group of mechanical systems with parametric uncertainties is considered. New adaptive finite time continuous distributed control algorithms are proposed for the multi-agent systems under two different scenarios. Firstly for the leaderless multi-agent systems, it is shown that the states of the mechanical systems can reach a consensus within finite time. Under this scenario, the rendezvous seeking problem is also solved when only some of the agents have access to the rendezvous location. Secondly in the leader-follower scenario of a group of mechanical systems, two methodologies are proposed, i.e. hierarchical decomposition and distributed estimator method, and it is demonstrated that perfect consensus can be reached or the leader-following errors converge to a ball with arbitrarily small radius in finite time. Transient performances in terms of convergence rates and time are also analyzed. Finally simulation results illustrate and verify the effectiveness of the proposed schemes.

1.6 Organization of the Thesis

This thesis is mainly divided into two categories: underactuated mechanical systems control (Chapters 1-6) and networked multi-agent consensus control (Chapters 7).

In Chapter 1, we review the most relevant results in the literature. In Chapter 3 the adaptive tracking and stabilization control of nonholonomic mobile robots in the presence of input saturation is addressed. In Chapter 4 the tracking control of underactuated ships with input saturation is investigated. The adaptive tracking control of nonholonomic mobile robots and underactuated ships are investigated in Chapter 5 and Chapter 6 respectively. In Chapter 7 the finite-time consensus control of a group of mechanical systems with parametric uncertainties under undirected/directed topology is considered respectively.

Chapter 2

Preliminaries

Some basic concepts and definitions are firstly introduced in this chapter.

2.1 Definitions and Useful Lemmas for Finite-time Control

Finite-time stability is defined as follows.

Definition 2.1. [127]. Consider a dynamic system

$$\dot{x} = f(x, t), \quad f(0, t) = 0, \quad x \in U_0 \subset R^n \quad (2.1)$$

where $f : U_0 \times R^+ \rightarrow R^n$ is continuous on an open neighborhood U_0 of the origin $x = 0$. The equilibrium $x = 0$ of the system is (locally) finite-time stable if it is Lyapunov stable and for any initial condition $x(t_0) = x_0 \in U$ where $U \subset U_0$, if there is a settling time $T > t_0$, such that every solution $x(t; t_0, x_0)$ of system (2.1) satisfies $x(t; t_0, x_0) \in U \setminus \{0\}$ for $t \in [t_0, T)$, and

$$\lim_{t \rightarrow T} x(t; t_0, x_0) = 0, \quad x(t; t_0, x_0) = 0 \quad \forall t > T$$

If $U = R^n$, then the origin $x = 0$ is a globally finite-time stable equilibrium.

The following lemmas are useful for establishing finite time system stability.

Lemma 2.1. [127]. Suppose there is a C^1 positive definite Lyapunov function $V(x, t)$ defined on $U \times \mathbb{R}^+$ where $U \subset U_0$ is the neighborhood of the origin, and there are positive real constants $c > 0$ and $0 < \alpha < 1$, such that $\dot{V}(x, t) + cV^\alpha(x, t)$ is negative semidefinite on U . Then $V(x, t)$ is locally in finite-time convergent with a settling time

$$T \leq \frac{V^{1-\alpha}(x_0, t)}{c(1-\alpha)}$$

for any given initial condition $x(t_0)$ in the neighborhood of the origin in U .

Lemma 2.2. [134]. If $0 < p = p_1/p_2 \leq 1$, where $p_1 > 0$ and $p_2 > 0$ are positive odd integers, then $|x^p - y^p| \leq 2^{1-p}|x - y|^p$.

Lemma 2.3. [136]. For $x_i \in \mathfrak{R}$, $i = 1, \dots, n$, $0 < p \leq 1$, then

$$\left(\sum_{i=1}^n |x_i| \right)^p \leq \sum_{i=1}^n |x_i|^p \leq n^{1-p} \left(\sum_{i=1}^n |x_i| \right)^p$$

Lemma 2.4. [134]. Let c and d be positive constants and $\gamma(x, y) > 0$ is a real value function. Then

$$|x|^c |y|^d \leq \frac{c\gamma(x, y)|x|^{c+d}}{c+d} + \frac{d\gamma^{-c/d}(x, y)|y|^{c+d}}{c+d}$$

Lemma 2.5. [35]. For a connected undirected graph G , the Laplacian matrix L has the following property: (1) L is semi-definite. (2) 0 is a simple eigenvalue of L and $\mathbf{1}$ is the associated eigenvector. (3) Assuming the eigenvalue of L is denoted as $0, \lambda_2, \dots, \lambda_n$ satisfying $0 \leq \lambda_2 \leq \dots \leq \lambda_n$, then the second smallest eigenvalue $\lambda_2 > 0$. Furthermore, if $\mathbf{1}^T x = 0$, then $x^T L x \geq \lambda_2 x^T x$.

Consider $y = x^{\frac{p}{q}}$ where p is a positive integer and q is a positive odd integer. If we ignore all the complex roots, then obviously $y = \text{sign}(x)|x|^{\frac{p}{q}}$ if p is an odd integer; otherwise $y = |x|^{\frac{p}{q}}$.

We will introduce some concepts of homogeneous functions following [80, 81].

Definition 2.2. Suppose $(r_1, \dots, r_n) \in \mathbb{R}^n$ where $r_i > 0$, $i = 1, \dots, n$, and $V(x_1, \dots, x_n) : \mathbb{R}^n \rightarrow \mathbb{R}$ being a continuous function. Then $V(x_1, \dots, x_n)$ is homogeneous of degree

$\sigma > 0$ with respect to $(r_1, \dots, r_n)$, if for any given ε ,

$$V(\varepsilon^{r_1}x_1, \dots, \varepsilon^{r_n}x_n) = \varepsilon^\sigma V(x_1, \dots, x_n) \quad (2.2)$$

Let $x = (x_1, \dots, x_n)^T$ and $f(x) = (f_1(x), \dots, f_n(x))^T$ be a continuous vector field. $f(x)$ is said to be homogeneous of degree $\kappa \in \mathbb{R}$ with respect to $(r_1, \dots, r_n)$ if for any $\varepsilon > 0$,

$$f_i(\varepsilon^{r_1}x_1, \dots, \varepsilon^{r_n}x_n) = \varepsilon^{\kappa+r_i} f_i(x), \quad i = 1, \dots, n, \quad x \in \mathbb{R}^n \quad (2.3)$$

A given system $\dot{x} = f(x)$ is said to be homogeneous if $f(x)$ is homogeneous.

For a homogeneous system, we have the following lemma.

Lemma 2.6. [127] Suppose $\dot{x} = f(x)$ is homogeneous of degree κ . Then the origin of the system is finite-time stable if the origin is asymptotically stable and $\kappa < 0$.

Lemma 2.7. [81] Consider the following system

$$\dot{x} = f(x) + \tilde{f}(x), \quad x \in \mathbb{R}^n, \quad f(0) = 0, \quad \tilde{f}(0) = 0 \quad (2.4)$$

where $f(x)$ is a continuous homogeneous vector field of degree $\kappa < 0$ with respect to $(r_1, \dots, r_n)$. Assume $x = 0$ is an asymptotically stable equilibrium of the system $\dot{x} = f(x)$. Then $x = 0$ is also a locally finite-time stable equilibrium of the system (2.4) if

$$\lim_{\varepsilon \rightarrow 0} \frac{\tilde{f}_i(\varepsilon^{r_1}x_1, \dots, \varepsilon^{r_n}x_n)}{\varepsilon^{\kappa+r_i}} = 0, \quad i = 1, \dots, n, \quad \forall x \neq 0 \quad (2.5)$$

2.2 Graph Theory

The communications among n agents are represented by a graph $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, \dots, n\}$ denotes the set of indexes (or vertices) corresponding to each agent, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges between two distinct agents. If $\mathcal{G}$ is

undirected, then the edge $(i, j) \in \mathcal{E}$ also means $(j, i) \in \mathcal{E}$ which indicates that agents i and j can obtain state and structure information from each other. In this case, agent j is called a neighbor of agent i , and vice versa. Otherwise if the graph is directed, then $(i, j) \in \mathcal{E}$ means only agent i can get information from agent j . We denote the set of neighbors for agent i as $\mathcal{N}_i \triangleq \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. Note that self edges (i, i) is not allowed, thus $(i, i) \notin \mathcal{E}$ and $i \notin \mathcal{N}_i$. If the graph is undirected, then $\mathcal{G}$ is connected means that there is an undirected sequence of edges between every pair of distinct agents [133]. Otherwise $\mathcal{G}$ is connected means that the graph contains a spanning tree. A directed spanning tree is a directed tree that contains all indexes of the graph. A graph has a spanning tree if there is a directed spanning tree as a subset of the graph, which means there exists at least one node having a directed path to all of the other nodes. The connectivity matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is defined that $a_{ij} \neq 0$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$ if $(i, j) \notin \mathcal{E}$. Clearly the diagonal elements $a_{ii} = 0$ and for an undirected graph A is symmetric. We introduce an in-degree matrix Δ such that $\Delta = \text{diag}(\Delta_i) \in \mathbb{R}^{n \times n}$ with $\Delta_i = \sum_{j \in \mathcal{N}_i} a_{ij}$ being the i th row sum of A . Then, the Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L} = \Delta - A$.

A digraph is called strongly connected if any two distinct nodes of the graph can be connected by a directed path. The connectivity matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is defined that $a_{ij} \neq 0$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$ if $(i, j) \notin \mathcal{E}$. We introduce an in-degree matrix Δ such that $\Delta = \text{diag}(\Delta_i) \in \mathbb{R}^{n \times n}$ with $\Delta_i = \sum_{j \in \mathcal{N}_i} a_{ij}$ being the i th row sum of A . Then, the Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L} = \Delta - A$.

Note that if $\mathcal{G}$ is strongly connected, then there exists a positive column vector $\nu = [\nu_1, \dots, \nu_n]^T$ such that $\nu^T \mathcal{L} = 0$ and $\nu^T \mathbf{1}_N = 1$. With this notation, the algebraic connectivity [61] of a strongly connected digraph defined below has the following qualities.

Lemma 2.8. *If $\mathcal{L}$ is the Laplacian matrix of a strongly connected graph, then the algebraic connectivity $a(\mathcal{L})$ defined below has the following quality:*

$$a_{\mathcal{L}} = \min_{x^T \nu = 0, x \neq 0} \frac{x^T \tilde{\mathcal{L}} x}{x^T \text{diag}(\nu) x} > 0 \quad (2.6)$$

where $\tilde{\mathcal{L}} = \frac{1}{2} \left(\text{diag}(\nu) \mathcal{L} + \mathcal{L}^T \text{diag}(\nu) \right)$ is a symmetric semi-positive definite matrix. $a_{\mathcal{L}}$ can be calculated as $a_{\mathcal{L}} = \max(\delta)$ where δ is a positive constant such that $Q^T(\tilde{\mathcal{L}} - \delta \text{diag}(\nu))Q$ is semi-positive definite, where $Q = \begin{pmatrix} I_{n-1} \\ -\frac{\tilde{v}^T}{v_n} \end{pmatrix} \in R^{n \times (n-1)}$ and $\tilde{v} = [v_1, \dots, v_{n-1}]^T$.

2.3 The Property of Parameter Projection

Lemma 2.9. [8] *The following projection*

$$\text{Proj}\{\tau\} = \begin{cases} \tau & \hat{\theta} \in \Pi \text{ or } \nabla_{\hat{\theta}} P^T \tau \leq 0 \\ (I - \Gamma \frac{\nabla_{\hat{\theta}} P \nabla_{\hat{\theta}} P^T}{\nabla_{\hat{\theta}} P^T \Gamma \nabla_{\hat{\theta}} P}) \tau & \hat{\theta} \in \partial \Pi \text{ or } \nabla_{\hat{\theta}} P^T \tau > 0 \end{cases} \quad (2.7)$$

has the following properties if $|\hat{b}(t_0)| \leq b_M$:

- (1) $|\hat{b}(t)| \leq b_M + \epsilon, \forall 0 \leq t_0 \leq t < \infty$.
- (2) $\text{Proj}(a, \hat{b})$ is Lipschitz continuous.
- (3) $|\text{Proj}(a, \hat{b})| \leq |a|$
- (4) $\tilde{b} \text{Proj}(a, \hat{b}) \geq \tilde{b}a$ where $\tilde{b} = b - \hat{b}$.

Chapter 3

Nonholonomic Mobile Robots with Input Saturation

In this chapter, the stabilization and tracking control of nonholonomic mobile robots with parametric uncertainties and external disturbance in the presence of input saturation is addressed. The torques are designed such that the bounds of the torques are functions of only design parameters and reference trajectory, which are computable in advance. For a given reference trajectory, design parameters are chosen such that the bounds of the torques are within the given saturation limits. To handle the disturbances, the unknown bounds are estimated through adaptive law and the estimates are employed in controller design. The system stability is established.

3.1 Introduction

The two-wheeled mobile robot, a two-input three-output system, is one of the well-known benchmark nonholonomic systems, which leads to the development of various novel controller design schemes. Much effort has been devoted to the stabilization and tracking of this dynamic system, see [9, 12, 68] and the references therein. By means of Brockett's necessary condition [1] for asymptotic stabilization, there exists no continuous time-invariant state-feedback controller that asymptotically stabilizes the mobile robot. Therefore, the past abundant

literature aims at developing suitable time-varying controllers for mobile robots. In [6, 69] the stabilization problem is dealt with for a driftless controllable system using bounded state-feedback approach. In [70] the simultaneous stabilization and tracking problem of mobile robots is studied when saturation is imposed on linear and angular velocities. In practice, a mobile robot is ultimately driven by a motor which can only provide limited amount of torque. Thus the magnitude of the control signal is always constrained. It often severely limits system performance, giving rise to undesirable inaccuracy or leading to instability as in [71, 72]. The development of an adaptive control scheme for a two-wheeled mobile robot with input saturation has been a task of major practical interest as well as theoretical significance. As far as we know, the stabilization and tracking of a nonholonomic mobile robot in the presence of input saturation and external disturbance is rarely addressed in past literature. In [73], a passivity-based, saturated, smooth feedback controller is designed based on the kinematic model of a mobile robot. However, there has been no result addressing the issue of input torque saturation at the dynamic model level for a wheeled mobile robot due to the difficulty of the problem. Also there has been no consideration on external disturbances. The main challenge is how to handle the effects of unknown parameters, the saturation and the external disturbance to achieve internal asymptotic stability and desirable performance.

In this chapter, we consider these effects in solving the stabilization and tracking problem for a wheeled mobile robot. By exploiting the special structure of the mobile robot, a new control scheme is proposed to design controllers and parameter estimators. Through normalizing certain signals appropriately, we are able to make the designed ‘saturated’ torques bounded by functions of design parameters and reference trajectories only. In this way, we can compute the bounds in advance and thus ensure them to be within the saturation limits by suitably choosing the design parameters. Therefore with these parameters, the control torques are free of saturation and the control law for the ‘saturated’ torque is the control law we need. To compensate for the disturbances, we estimate their unknown bounds and employ the estimates in controller design. This enables us

to achieve perfect tracking of reference trajectories or stabilization to the origin, in addition to the system stability. To illustrate the controller design procedure, simulation studies are conducted for the set-point stabilization and the tracking of a line and a circle in a plane. The obtained numerical results are in line with our theoretical findings.

3.2 System Model and Problem Statement

We consider a wheeled mobile robot shown in Fig.3.1 which is described by the following kinematic and dynamic models

$$\dot{\eta} = J(\eta)\omega \quad (3.1)$$

$$M\dot{\omega} + C(\dot{\eta})\omega + D\omega + \bar{\varepsilon} = \text{sat}(\tau) \quad (3.2)$$

where $\eta = (x, y, \phi)$ denotes the position and orientation of the robot, $\omega = (\omega_1, \omega_2)^T$ denotes the angular velocities of the left and right wheels, $\tau = (\tau_1, \tau_2)^T$ represents the control torques applied to the wheels, and M is a symmetric, positive definite inertia matrix, $C(\dot{\eta})$ is the centripetal and coriolis matrix, D denotes the surface friction, and $\bar{\varepsilon}$ represents external time-varying disturbances. The expressions of $J(\eta)$, M , $C(\dot{\eta})$ and D are the same as those in [10], which are also given as below for completeness

$$J(\eta) = \frac{r}{2} \begin{bmatrix} \cos \phi & \cos \phi \\ \sin \phi & \sin \phi \\ b^{-1} & -b^{-1} \end{bmatrix}, M = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{11} \end{bmatrix}$$

$$C(\dot{\eta}) = \begin{bmatrix} 0 & c\dot{\phi} \\ -c\dot{\phi} & 0 \end{bmatrix}, D = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}$$

Figure 3.1: A two-wheeled nonholonomic mobile robot

$$m_{11} = 0.25b^{-2}r^2(mb^2 + I) + I_w$$

$$m_{12} = 0.25b^{-2}r^2(mb^2 - I)$$

$$m = m_c + 2m_w$$

$$I = m_c d^2 + 2m_w b^2 + I_c + 2I_m$$

$$c = 0.5b^{-1}r^2m_c d.$$

In these expressions, b is the half width of the mobile robot and r is the radius of the wheel, d is the distance between the center of the mass, P_c , of the robot and the middle point between the left and right wheels, I_c , I_w and I_m are the moment of inertia of the body about the vertical axis through P_c , the wheel with a motor about the wheel axis, and the wheel with a motor about the diameter, respectively. The positive constants d_{ii} , $i = 1, 2$, are the damping coefficients.

$sat(\tau)$ represents the input torques subject to saturation described by $sat(\tau) = \{sat(\tau_1), sat(\tau_2)\}^T$ and

$$sat(\tau_i) = \begin{cases} sign(\tau_i)u_{Mi} & |\tau_i| \geq u_{Mi} \\ \tau_i & |\tau_i| < u_{Mi} \end{cases}$$

where u_{Mi} is a known saturation bound of τ_i . Let v and w denote the linear and angular velocities of robot, respectively. Then the relationship between ω_1 , ω_2

and v, w is described as follows

$$(v, w)^T = B^{-1}(\omega_1, \omega_2)^T, B = \frac{1}{r} \begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix}. \quad (3.3)$$

Substituting (3.3) into (3.1), we get

$$\begin{aligned} \dot{\eta} &= L(\eta)\varpi \\ \dot{\varpi} &= -B^{-1}M^{-1}C(\dot{\eta})B\varpi - B^{-1}M^{-1}DB\varpi + B^{-1}M^{-1}\bar{\varepsilon} + B^{-1}M^{-1}sat(\tau) \end{aligned} \quad (3.4)$$

where

$$L(\eta) = \begin{bmatrix} \cos \phi & 0 \\ \sin \phi & 0 \\ 0 & 1 \end{bmatrix}, \quad \varpi = [v \ w]^T \quad (3.5)$$

Let

$$\bar{B} = \frac{1}{r} \begin{bmatrix} -b & -1 \\ -b & 1 \end{bmatrix}.$$

It is obvious that $C(\dot{\eta})B = \bar{B}C(\dot{\eta})$ and thus we can replace $C(\dot{\eta})B$ in (3.4) by $\bar{B}C(\dot{\eta})$. Multiplying (3.4) by $\bar{B}^{-1}MB$, we get

$$\dot{\eta} = L(\eta)\varpi \quad (3.6)$$

$$R\dot{\varpi} = -\bar{C}(w)\varpi - D_R\varpi + \varepsilon + B_Rsat(\tau) \quad (3.7)$$

where $R = -\bar{B}^{-1}MB$, $\bar{C}(w) = -C(w)$, $D_R = -\bar{B}^{-1}DB$, $B_R = -\bar{B}^{-1}$ and R is a positive definite matrix, $\varepsilon = -\bar{B}^{-1}\bar{\varepsilon}$ is a disturbances-like signal. Explicitly, we have

$$R = \begin{bmatrix} (m_{11} + m_{12})/b & 0 \\ 0 & (m_{11} - m_{12})b \end{bmatrix}, \quad D_R = \begin{bmatrix} \frac{d_{11}+d_{22}}{2b} & \frac{d_{11}-d_{22}}{2} \\ \frac{d_{11}-d_{22}}{2} & \frac{b(d_{11}+d_{22})}{2} \end{bmatrix}, \quad B_R = \frac{r}{2b} \begin{bmatrix} 1 & 1 \\ b & -b \end{bmatrix}. \quad (3.8)$$

Now let $(x_r, y_r, \phi_r)^T$ denote the desired reference position and orientation of the

virtual robot the motion of which is described by:

$$\begin{aligned}\dot{x}_r &= v_r \cos \phi_r \\ \dot{y}_r &= v_r \sin \phi_r \\ \dot{\phi}_r &= w_r\end{aligned}\tag{3.9}$$

where v_r and w_r denote the linear and angular velocities of the the virtual robot. As often done in control of mobile robot ([9]), we define x_e , y_e and ϕ_e as

$$\begin{bmatrix} x_e \\ y_e \\ \phi_e \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - x_r \\ y - y_r \\ \phi - \phi_r \end{bmatrix}\tag{3.10}$$

which describe the tracking errors between the actual robot position, direction and those of the virtual robot. It can be directly checked that these tracking errors satisfy the following differential equations:

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \\ \dot{\phi}_e \end{bmatrix} = \begin{bmatrix} wy_e + v - v_r \cos \phi_e \\ -wx_e + v_r \sin \phi_e \\ w - w_r \end{bmatrix}.\tag{3.11}$$

The control objective is to find torque τ applied to the mobile robot, which is subject to saturation, to ensure the desired reference trajectories are tracked in the presence of external disturbances. If v_r and w_r are zero or both converge to zero as $t \rightarrow \infty$, the tracking problem becomes set-point stabilization of robots according to (3.10). To solve the control problem, we need the following assumptions:

Assumption 3.1. *The reference linear and angular velocities v_r and w_r and their first-order derivatives are bounded.*

Assumption 3.2. *The unknown parameters of the mobile robot are in known compact sets.*

Assumption 3.3. *The disturbances are bounded, namely, $\varepsilon_1 < E_1$ and $\varepsilon_2 < E_2$ where E_1 and E_2 are positive constants.*

To compensate for the effects of the disturbances, we will obtain an adaptive law to estimate the unknown bounds $E = (E_1, E_2)^T$ for the design of controllers.

3.3 Controller Design

The controller design is divided into two steps. Firstly we develop virtual control laws for v and w based on the kinematic model in such a way that the mobile robot will follow the pre-established reference trajectory. Secondly we derive a design law for $\text{sat}(\tau)$ based on the dynamic model of (3.2) so that v and w would converge to the designed virtual controls. Through our design and derivation, suitable normalization is introduced to certain signals. This enables us to make the bound of $\text{sat}(\tau)$ as a function of design parameters and reference trajectories only. In this way, we can compute the bound in advance and ensure it within the saturation limit by suitably choosing the design parameters. Therefore with these parameters, $\text{sat}(\tau)$ is the same as τ and the derived design law for $\text{sat}(\tau)$ is also the control law for τ .

3.3.1 Adaptive control of kinematic model

Introduce a new variable $\bar{\phi}_e = \phi_e + \arcsin\left(\frac{k(t)y_e}{\sqrt{1+x_e^2+y_e^2}}\right)$ where

$$k(t) = \lambda_1 v_r + \lambda_2 \sin(t), \quad (3.12)$$

λ_1 and λ_2 are positive constants chosen so that $\lambda_1 |v_r|^{\max} + \lambda_2 \leq 0.5$, where $|v_r|^{\max}$ denotes the maximum value of $|v_r|$. This will ensure that $k(t) \leq 0.5$ for all t . Then (3.11) is transformed to

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \\ \dot{\bar{\phi}}_e \end{bmatrix} = \begin{bmatrix} wy_e + v + f_1 \\ -wx_e - \frac{k(t)v_r y_e}{\Gamma_1} + f_2 \\ w\left(1 - \frac{k(t)x_e}{\Gamma_1}\right) + f_3 - v\frac{k(t)x_e y_e}{\Gamma_1^2 \Gamma_2} \end{bmatrix} \quad (3.13)$$

where

$$\begin{aligned}
\Gamma_1 &= \sqrt{1 + x_e^2 + y_e^2}, \quad \Gamma_2 = \sqrt{1 + x_e^2 + (1 - k^2(t))y_e^2} \\
f_1 &= -\frac{v_r \cos \bar{\phi}_e \Gamma_2}{\Gamma_1} - \frac{v_r k(t) y_e \sin \bar{\phi}_e}{\Gamma_1}, \\
f_2 &= -\frac{v_r (\cos \bar{\phi}_e - 1) \Gamma_2}{\Gamma_1} - \frac{v_r k(t) y_e \sin \bar{\phi}_e}{\Gamma_1} \\
f_3 &= -w_r + \frac{\dot{k}(t) y_e + k(t) f_2 - k(t) (x_e f_1 + y_e f_2) / \Gamma_1^2}{\Gamma_2}.
\end{aligned} \tag{3.14}$$

Define the virtual control error $\tilde{v}$ and $\tilde{w}$ as

$$\tilde{v} = v - v_c, \quad \tilde{w} = w - w_c \tag{3.15}$$

where v_c and w_c are the virtual controls designed for v and w . We have

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \\ \dot{\bar{\phi}}_e \end{bmatrix} = \begin{bmatrix} (w_c + \tilde{w}) y_e + v_c + \tilde{v} + f_1 \\ -(w_c + \tilde{w}) x_e - \frac{k(t) v_r y_e}{\Gamma_1} + f_2 \\ (w_c + \tilde{w}) (1 - \frac{k(t) x_e}{\Gamma_1}) + f_3 - (v_c + \tilde{v}) \frac{k(t) x_e y_e}{\Gamma_1^2 \Gamma_2} \end{bmatrix}. \tag{3.16}$$

Define a Lyapunov function

$$V_0 = \sqrt{1 + x_e^2 + y_e^2} + \sqrt{1 + \bar{\phi}_e^2} - 2.$$

The time derivative of V_0 is

$$\begin{aligned}
\dot{V}_0 &= -\frac{k(t) v_r y_e^2}{\Gamma_1^2} + \frac{x_e}{\Gamma_1} (v_c + f_1) + \frac{y_e f_2}{\Gamma_1} + \frac{\bar{\phi}_e}{\Gamma_3} [w_c (1 - \frac{k(t) x_e}{\Gamma_1}) + f_3 \\
&\quad - v_c \frac{k(t) x_e y_e}{\Gamma_1^2 \Gamma_2}] + \tilde{v} (\frac{x_e}{\Gamma_1} - \frac{k(t) x_e y_e \bar{\phi}_e}{\Gamma_1^2 \Gamma_2 \Gamma_3}) + \tilde{w} \frac{\bar{\phi}_e}{\Gamma_3} (1 - \frac{k(t) x_e}{\Gamma_1})
\end{aligned}$$

where $\Gamma_3 = \sqrt{1 + \bar{\phi}_e^2}$. The virtual controllers v_c and w_c are chosen as:

$$\begin{aligned}
v_c &= -k_1 \frac{x_e}{\Gamma_1} - f_1 \\
w_c &= \frac{1}{(1 - \frac{k(t) x_e}{\Gamma_1})} (-k_2 \frac{\bar{\phi}_e}{\Gamma_3} - f_3 + v_c \frac{k(t) x_e y_e}{\Gamma_1^2 \Gamma_2}).
\end{aligned} \tag{3.17}$$

Since we have $k(t) \leq 0.5$ from (12) and $\frac{x_e}{\Gamma_1} < 1$, so we have $1 - \frac{k(t)x_e}{\Gamma_1} > 0.5$. The derivative of $V_0(t)$ is

$$\begin{aligned} \dot{V}_0 = & -k_1 \frac{x_e^2}{\Gamma_1^2} - \frac{k(t)v_r y_e^2}{\Gamma_1^2} - k_2 \frac{\bar{\phi}_e^2}{\Gamma_3^2} + \frac{y_e f_2}{\Gamma_1} + \tilde{v} \left(\frac{x_e}{\Gamma_1} - \frac{k(t)x_e y_e \bar{\phi}_e}{\Gamma_1^2 \Gamma_2} \right) \\ & + \tilde{w} \frac{\bar{\phi}_e}{\Gamma_3} \left(1 - \frac{k(t)x_e}{\Gamma_1} \right). \end{aligned}$$

Remark 3.1. Γ_1 , Γ_2 and Γ_3 are introduced to normalize x_e , y_e and $\bar{\phi}_e$ so that the bounds of virtual control signals based on the kinematic model are only dependent on system parameters and reference signals, and can be pre-determined. This normalization technique will also be applied to the dynamic model when designing control law for $\text{sat}(\tau)$.

3.3.2 Adaptive control of dynamic model

Define $\varpi_c = [v_c \ w_c]^T$, $\tilde{\varpi} = \varpi - \varpi_c$, and a new Lyapunov function $V_1 = V_0 + \frac{1}{2} \tilde{\varpi}^T R \tilde{\varpi}$, then

$$\begin{aligned} \dot{V}_1 = & \dot{V}_0 + \tilde{\varpi}^T R \dot{\tilde{\varpi}} = \dot{V}_0 + \tilde{\varpi}^T R (\dot{\varpi} - \dot{\varpi}_c) \\ = & \dot{V}_0 + \tilde{\varpi}^T [-\bar{C}(w)\varpi - D_R \varpi + \varepsilon + B_R \text{sat}(\tau) - R \dot{\varpi}_c] \\ = & \dot{V}_0 + \tilde{\varpi}^T [-(\bar{C}(w) + D_R)\tilde{\varpi} + \varepsilon + B_R \text{sat}(\tau) - Y_c P] \end{aligned} \quad (3.18)$$

where $Y_c P = R \dot{\varpi}_c + C(w)\varpi_c + D_R \varpi_c$ and

$$Y_c = \begin{bmatrix} \dot{v}_c & 0 & w_c w & v_c & w_c & 0 \\ 0 & \dot{w}_c & -v_c w & 0 & v_c & w_c \end{bmatrix} \quad (3.19)$$

$$P = \begin{bmatrix} \frac{m_{11}+m_{22}}{b} & (m_{11} - m_{22})b & c & \frac{d_{11}+d_{22}}{2b} & \frac{d_{11}-d_{22}}{2} & \frac{b(d_{11}+d_{22})}{2} \end{bmatrix}^T. \quad (3.20)$$

Remark 3.2. Unlike the typical way of designing virtual controllers for (3.11) as in [9, 10], we use v and w as the virtual controllers. By doing this, the tracking error dynamic system in (3.13) and the virtual controllers v_c and w_c in (3.17) do

not contain any unknown system parameters. So when taking the time-derivatives of v_c and w_c in the matrix Y_c in (3.19) to get the torque τ_i , no unknown parameter will appear. This will reduce the number of unknown parameters in (3.20) from 10 in [10] to 6 and thus greatly simplify the matrix Y_c . Therefore parameter estimation is simplified compared to [10].

Denote $\alpha := \frac{r}{2b}$, $\beta := \frac{r}{2}$ and $m = \frac{1}{\alpha}$, $n = \frac{1}{\beta}$. Let $\hat{m}$ and $\hat{n}$ be the estimates of m and n respectively, and define $\tilde{m} = m - \hat{m}$, $\tilde{n} = n - \hat{n}$, $u_1 = \frac{\text{sat}(\tau_1) + \text{sat}(\tau_2)}{\hat{m}}$, $u_2 = \frac{\text{sat}(\tau_1) - \text{sat}(\tau_2)}{\hat{n}}$, $U = [u_1 \ u_2]^T$. Then we get

$$B_R \text{sat}(\tau) = \begin{bmatrix} \alpha & \alpha \\ \beta & -\beta \end{bmatrix} \begin{bmatrix} \text{sat}(\tau_1) \\ \text{sat}(\tau_2) \end{bmatrix} = \begin{bmatrix} \alpha \hat{m} u_1 \\ \beta \hat{n} u_2 \end{bmatrix}.$$

As $\alpha \hat{m} = \alpha(m - \tilde{m}) = 1 - \alpha \tilde{m}$ and also $\beta \hat{n} = 1 - \beta \tilde{n}$,

$$B_R \text{sat}(\tau) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} \alpha \tilde{m} u_1 \\ \beta \tilde{n} u_2 \end{bmatrix} = U - \begin{bmatrix} \alpha \tilde{m} u_1 \\ \beta \tilde{n} u_2 \end{bmatrix}. \quad (3.21)$$

Let $\Omega_1 = (\frac{x_e}{\Gamma_1} - \frac{k(t)x_e y_e \bar{\phi}_e}{\Gamma_1^2 \Gamma_2 \Gamma_3})$, $\Omega_2 = \frac{\bar{\phi}_e}{\Gamma_3} (1 - \frac{k(t)x_e}{\Gamma_1})$, and $\Omega = [\Omega_1 \ \Omega_2]^T$. We now define another new Lyapunov function

$$V_2 = V_1 + \frac{1}{2} \tilde{P}^T T \tilde{P} + \frac{1}{2} \tilde{E}^T K \tilde{E} + \frac{1}{2} \alpha \tilde{m}^2 + \frac{1}{2} \beta \tilde{n}^2$$

where $\tilde{P} = P - \hat{P}$ and $\hat{P}$ is the estimate of P , $T = aI$ with a being a positive constant; $\tilde{E} = E - \hat{E}$, $\hat{E}$ is the estimates of the bounds of unknown disturbances, K is a given positive definite matrix. Then

$$\begin{aligned} \dot{V}_2 &= \dot{V}_0 + \tilde{\omega}^T [-(\bar{C}(w) + D_R)] \tilde{\omega} + \varepsilon + B_R \text{sat}(\tau) - Y_c P \\ &\quad + \tilde{P}^T T \dot{\tilde{P}} + \tilde{E}^T K \dot{\tilde{E}} + \alpha \tilde{m} \dot{\tilde{m}} + \beta \tilde{n} \dot{\tilde{n}} \\ &= -k_1 \frac{x_e^2}{\Gamma_1^2} - \frac{k(t) v_r y_e^2}{\Gamma_1^2} - k_2 \frac{\bar{\phi}_e^2}{\Gamma_3^2} + \frac{y_e f_2}{\Gamma_1} + \alpha \tilde{m} \dot{\tilde{m}} + \beta \tilde{n} \dot{\tilde{n}} \\ &\quad + \tilde{\omega}^T [-(\bar{C}(w) + D_R)] \tilde{\omega} + \varepsilon + B_R \text{sat}(\tau) - Y_c P + \Omega] + \tilde{P}^T T \dot{\tilde{P}} - \tilde{E}^T K \dot{\tilde{E}} \end{aligned}$$

$$\begin{aligned}
&= -k_1 \frac{x_e^2}{\Gamma_1^2} - \frac{k(t)v_r y_e^2}{\Gamma_1^2} - k_2 \frac{\bar{\phi}_e^2}{\Gamma_3^2} + \frac{y_e f_2}{\Gamma_1} - \alpha \tilde{m}(\dot{\hat{m}} + u_1 \tilde{v}) - \beta \tilde{n}(\dot{\hat{n}} + u_2 \tilde{w}) \\
&\quad + \tilde{P}^T T \dot{\tilde{P}} - \tilde{E}^T K \dot{\tilde{E}} \\
&\quad + \tilde{\omega}^T [-(\bar{C}(w) + D_R))\tilde{\omega} + \varepsilon + U - Y_c P + \Omega].
\end{aligned} \tag{3.22}$$

By choosing U as

$$U = Y_c \hat{P} - \Omega - \text{sgn}(\tilde{\omega})^T * \hat{E} \tag{3.23}$$

where $\text{sgn}(\tilde{\omega})^T * \hat{E} = (\text{sgn}(\tilde{\omega}_1)\hat{E}_1, \text{sgn}(\tilde{\omega}_2)\hat{E}_2)^T$, we get

$$\begin{aligned}
\dot{V}_2 &\leq -k_1 \frac{x_e^2}{\Gamma_1^2} - \frac{k(t)v_r y_e^2}{\Gamma_1^2} - k_2 \frac{\bar{\phi}_e^2}{\Gamma_3^2} + \frac{y_e f_2}{\Gamma_1} - \tilde{\omega}^T D_R \tilde{\omega} \\
&\quad - \alpha \tilde{m}(\dot{\hat{m}} + u_1 \tilde{v}) - \beta \tilde{n}(\dot{\hat{n}} + u_2 \tilde{w}) - \tilde{P}^T T(\dot{\tilde{P}} + T^{-1}Y_c^T \tilde{\omega}) - \tilde{E}^T K(\dot{\tilde{E}} - K^{-1}|\tilde{\omega}|).
\end{aligned}$$

The torque τ is designed as

$$\text{sat}(\tau) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} \hat{m} & 0 \\ 0 & \hat{n} \end{bmatrix} U. \tag{3.24}$$

The initial values of estimates are set to be in the given compact sets. By using parameter projection, the estimates to be updated are confined in the known sets. To illustrate this, we use the update laws for $\hat{m}$ and $\hat{n}$ as examples. From Assumption 3.2, there exist $m_{\max}$, $m_{\min}$, $n_{\max}$, $n_{\min}$ and a small positive constant σ so that $m_{\min} + \sigma < m < m_{\max} - \sigma$ and $n_{\min} + \sigma < n < n_{\max} - \sigma$. The update law for $\hat{m}$ and $\hat{n}$ are designed as follows:

$$\begin{aligned}
\dot{\hat{m}} &= \begin{cases} -u_1 \tilde{v} & m_{\min} + \sigma < \hat{m} < m_{\max} - \sigma \\ -u_1 \tilde{v} + \frac{1}{2}[1 + (u_1 \tilde{v})^2] & \hat{m} \leq m_{\min} + \sigma \\ -u_1 \tilde{v} - \frac{1}{2}[1 + (u_1 \tilde{v})^2] & \hat{m} \geq m_{\max} - \sigma \end{cases} \\
\dot{\hat{n}} &= \begin{cases} -u_2 \tilde{w} & n_{\min} + \sigma < \hat{n} < n_{\max} - \sigma \\ -u_2 \tilde{w} + \frac{1}{2}[1 + (u_2 \tilde{w})^2] & \hat{n} \leq n_{\min} + \sigma \\ -u_2 \tilde{w} - \frac{1}{2}[1 + (u_2 \tilde{w})^2] & \hat{n} \geq n_{\max} - \sigma. \end{cases}
\end{aligned} \tag{3.25}$$

We choose the initial value for $\hat{m}$ and $\hat{n}$ so that $m_{\min} < \hat{m}(0) < m_{\max}$ and $n_{\min} < \hat{n}(0) < n_{\max}$. When $\hat{m} \leq m_{\min} + \sigma$, we have $\dot{\hat{m}} \geq 0$, and this law will prevent $\hat{m}$ from getting smaller than $m_{\min}$. Also we get $\tilde{m} = m - \hat{m} > 0$ and $\dot{\tilde{m}} + u_1 \tilde{v} > 0$. Therefore we have $-\alpha \tilde{m}(\dot{\tilde{m}} + u_1 \tilde{v}) < 0$. Similarly, when $\hat{m} \geq m_{\max} - \sigma$, we have $\dot{\hat{m}} \leq 0$, and this law will prevent $\hat{m}$ from getting greater than $m_{\max}$. Also we get $\tilde{m} = m - \hat{m} < 0$ and $\dot{\tilde{m}} + u_1 \tilde{v} < 0$, and thus $-\alpha \tilde{m}(\dot{\tilde{m}} + u_1 \tilde{v}) < 0$. For other parameters the same parameter projection rule is adopted. From the above analysis, we have

$$\begin{aligned} \dot{V}_2 \leq & -k_1 \frac{x_e^2}{\Gamma_1^2} - \frac{k(t)v_r y_e^2}{\Gamma_1^2} - k_2 \frac{\bar{\phi}_e^2}{\Gamma_3^2} + \frac{y_e f_2}{\Gamma_1} \\ & - \tilde{\omega}^T D_R \tilde{\omega} - \tilde{P}^T T(\dot{\tilde{P}} + T^{-1} Y_c^T \tilde{\omega}) - \tilde{E}^T K(\dot{\tilde{E}} - K^{-1} |\tilde{\omega}|). \end{aligned}$$

From Assumption 3.2, we have P in a known set which satisfies $P(i)_{\min} + \sigma < P(i) < P(i)_{\max} - \sigma$ and $E(j) < E_{\max}(j) - \sigma$ for $i = 1, 2, 3, 4, 5$ and $j = 1, 2$. The parameters update laws for $\hat{P}$ and $\hat{E}$ are then chosen as

$$\begin{aligned} \dot{\hat{P}}(i) &= \begin{cases} \Psi(i) & P(i)_{\min} + \sigma < \hat{P}(i) < P(i)_{\max} - \sigma \\ \Psi(i) + \frac{1}{2}(1 + \Psi(i)^2) & \hat{P}(i) \leq P(i)_{\min} + \sigma \\ \Psi(i) - \frac{1}{2}(1 + \Psi(i)^2) & \hat{P}(i) \geq P(i)_{\max} - \sigma \end{cases} \\ \dot{\hat{E}}(j) &= \begin{cases} \Xi(j) & \sigma \leq \hat{E}(j) < E_{\max}(j) - \sigma \\ \Xi(j) - \frac{1}{2}(1 + \Xi(j)^2) & \hat{E}(j) \geq E_{\max}(j) - \sigma \\ \Xi(j) + \frac{1}{2}(1 + \Xi(j)^2) & \hat{E}(j) < \sigma \end{cases} \end{aligned} \quad (3.26)$$

where $\Psi = -T^{-1} Y_c^T \tilde{\omega}$ and $\Xi = K^{-1} |\tilde{\omega}|$. Following the same analysis as (3.25), these parameters update law guarantee that $P(i)_{\min} < \hat{P}(i) < P(i)_{\max}$, $E(j) < E_{\max}(j)$ and $-\tilde{P}^T T(\dot{\tilde{P}} + T^{-1} Y_c^T \tilde{\omega}) - \tilde{E}^T K(\dot{\tilde{E}} - K^{-1} |\tilde{\omega}|) \leq 0$.

Thus we have

$$\dot{V}_2 \leq -k_1 \frac{x_e^2}{\Gamma_1^2} - \frac{k(t)v_r y_e^2}{\Gamma_1^2} - k_2 \frac{\bar{\phi}_e^2}{\Gamma_3^2} + \frac{y_e f_2}{\Gamma_1} - \tilde{\omega}^T D_R \tilde{\omega}. \quad (3.27)$$

Proposition 3.1. Under the controllers (3.17) and (3.24) with parameter update laws in (3.25) and (3.26), there exist appropriate design constants $k_1, k_2, \lambda_1, \lambda_2$,

δ_1 and δ_2 such that system (3.16) is uniformly stable and the trajectory of virtual robot (3.9) is asymptotically tracked or the system is stabilized to the origin..

Proof. It is proved in the Appendix that positive constants $\delta_1 \in (0, 1)$ and $\delta_2 > 0$ can be chosen such that

$$\frac{y_e f_2}{\Gamma_1} \leq \frac{\delta_1 v_r^2 y_e^2}{\Gamma_1^2} + \delta_2 \frac{\bar{\phi}_e^2}{\Gamma_3^2}. \quad (3.28)$$

Substituting (3.28) into (3.27) yields

$$\begin{aligned} \dot{V}_2 \leq & -k_1 \frac{x_e^2}{\Gamma_1^2} - \frac{(\lambda_1 - \delta_1) v_r^2 y_e^2}{\Gamma_1^2} - (k_2 - \delta_2) \frac{\bar{\phi}_e^2}{\Gamma_3^2} - \tilde{\omega}^T D_R \tilde{\omega} \\ & - \frac{\lambda_2 \sin(t) v_r y_e^2}{\Gamma_1^2}. \end{aligned} \quad (3.29)$$

On the other hand, based on the condition that $|k(t)| < 0.5$, we choose positive constants λ_1 and λ_2 to satisfy

$$\lambda_1 v_r^{\max} + \lambda_2 < 0.5 \quad (3.30)$$

where $v_r^{\max}$ denotes the maximum value of $|v_r|$. From (3.29), the choices of λ_1 , λ_2 and k_2 also meet the following condition

$$\lambda_1 > \delta_1, \quad k_2 > \delta_2, \quad v_r^2(\lambda_1 - \delta_1) - |\lambda_2 v_r| \geq 0.$$

Clearly the above choices of parameters are possible, and we choose $k_1 > 0$. Let $c_1 = k_2 - \delta_2$ and $c_2 = v_r^2(\lambda_1 - \delta_1) - |\lambda_2 v_r|$, then we get

$$\dot{V}_2 \leq -\frac{c_2 y_e^2}{\Gamma_1^2} - \frac{k_1 x_e^2}{\Gamma_1^2} - \frac{c_1 \bar{\phi}_e^2}{\Gamma_3^2} - \tilde{\omega}^T D_R \tilde{\omega}. \quad (3.31)$$

For stability analysis we consider the stabilization and tracking separately. For tracking, we have $\lim_{t \rightarrow \infty} |v_r| \neq 0$, thus we get $\lim_{t \rightarrow \infty} c_2 > 0$. Then from (3.31) we get $\lim_{t \rightarrow \infty} |x_e| + |y_e| + |\bar{\phi}_e| \rightarrow 0$.

For stabilization, we have $\lim_{t \rightarrow \infty} |v_r| + |\dot{v}_r| + |w_r| + |\dot{w}_r| = 0$. So from (3.31) we

get

$$\dot{V}_2 \leq -\frac{k_1 x_e^2}{\Gamma_1^2} - c_1 \frac{\bar{\phi}_e^2}{\Gamma_3^2} - \tilde{\omega}^T D_R \tilde{\omega} \leq 0 \quad (3.32)$$

From (3.32), $(x_e, y_e, \bar{\phi}_e)$ is bounded and $\lim_{t \rightarrow \infty} |x_e| + |\bar{\phi}_e| + |\tilde{v}| + |\tilde{w}| \rightarrow 0$. It is left to prove that $\lim_{t \rightarrow \infty} |y_e| \rightarrow 0$. From (3.16) and (3.17) we get

$$\dot{x}_e = -\frac{k_1 x_e}{\Gamma_1} + \Sigma(t) \quad (3.33)$$

where

$$\Sigma(t) = \frac{-y_e}{(1 - \frac{k(t)x_e}{\Gamma_1})} \left(k_2 \frac{\bar{\phi}_e}{\Gamma_3} + f_3 + (k_1 \frac{x_e}{\Gamma_1} + f_1) \frac{k(t)x_e y_e}{\Gamma_1^2 \Gamma_2} \right) + \tilde{w} y_e + \tilde{v}.$$

Based on (3.7) and (3.11) it is easy to check that $\ddot{x}_e$ is bounded. Since $\lim_{t \rightarrow \infty} x_e = 0$, using Barbalat's Lemma we get $\lim_{t \rightarrow \infty} \dot{x}_e = 0$, which means $\lim_{t \rightarrow \infty} \Sigma(t) - \frac{k_1 x_e}{\Gamma_1} = 0$. Since $\lim_{t \rightarrow \infty} \tilde{v} = \lim_{t \rightarrow \infty} \tilde{w} = 0$, then from (3.14) and (3.33)

$$\lim_{t \rightarrow \infty} \frac{\dot{k}(t) y_e^2}{\Gamma_2} = 0. \quad (3.34)$$

If λ_1 and λ_2 are set so that $\lim_{t \rightarrow \infty} \dot{k}(t) \neq 0$, (3.34) implies $\lim_{t \rightarrow \infty} y_e = 0$. As $k(t) = \lambda_1 v_r + \lambda_2 \sin(t)$ and λ_2 is free of choice, this condition can be satisfied if $\dot{v}_r$ is not linearly related to function $\sin(t)$. Otherwise, λ_1 and λ_2 should be carefully chosen to prevent $\lambda_1 \dot{v}_r$ and $\lambda_2 \cos(t)$ from canceling each other.

Finally from the definition of $\bar{\phi}_e$ we know that $\lim_{t \rightarrow \infty} y_e + |\bar{\phi}_e| \rightarrow 0$ guarantees $\lim_{t \rightarrow \infty} |\phi_e| \rightarrow 0$. Therefore the trajectories are asymptotically tracked or the robot is stabilized to the origin. $\square$

Note that $\frac{1}{2} r_{\min} \varpi^T \varpi \leq \frac{1}{2} \varpi^T R \varpi \leq \frac{1}{2} r_{\max} \varpi^T \varpi$ where $r_{\max} = \max(R_{11}, R_{22})$,

$r_{\min} = \min(R_{11}, R_{22})$. Taking the derivative of $V_{\varpi} = \frac{1}{2}\varpi^T R \varpi$, we get

$$\begin{aligned}
\dot{V}_{\varpi} &= \varpi^T R \dot{\varpi} = \varpi^T (-(\bar{C}(\dot{\eta}) + D_R)\varpi + \varepsilon + B_R \text{sat}(\tau)) \\
&= -\varpi^T D_R \varpi + \varpi^T \bar{s}(\tau) \\
&\leq -d_{\min} \varpi^T \varpi + 0.5\zeta \varpi^T \varpi + \frac{0.5}{\zeta} \bar{s}(\tau)^T \bar{s}(\tau) \\
&= -d_{\zeta} \varpi^T \varpi + \frac{0.5}{\zeta} \bar{s}(\tau)^T \bar{s}(\tau) \\
&\leq -\frac{d_{\zeta}}{r_{\max}} V_{\varpi} + \frac{0.5}{\zeta} \bar{s}(\tau)^T \bar{s}(\tau)
\end{aligned} \tag{3.35}$$

where $d_{\min}$ is the smallest eigenvalue of D_R , $d_{\zeta} = d_{\min} - 0.5\zeta$ and $\bar{s}(\tau) = B_R \text{sat}(\tau) + \varepsilon$ with ζ being any positive constant that satisfying $d_{\min} > 0.5\zeta$. Then we have

$$V_{\varpi} \leq e^{-\frac{d_{\zeta}}{r_{\max}} t} V_{\varpi}(0) + \frac{\bar{s}(\tau)^T \bar{s}(\tau) r_{\max}}{2\zeta d_{\zeta}}. \tag{3.36}$$

So from (3.35) and (3.36) we obtain

$$|\varpi_i| \leq \sqrt{\frac{2}{r_{\min}} \left(e^{-\frac{d_{\zeta}}{r_{\max}} t} V_{\varpi}(0) + \frac{\bar{s}(\tau)^T \bar{s}(\tau) r_{\max}}{2\zeta d_{\zeta}} \right)} \tag{3.37}$$

for $i = 1, 2$.

From (3.14) and (3.17) we know that $\dot{\varpi}_c$ and ϖ_c are bounded. From (3.27) $\hat{P}$ is bounded, and from (3.37) w is bounded. Thus the torque designed for $\text{sat}(\tau)$ is bounded and its bound is given in the following proposition.

Proposition 3.2. *For the designed torque $\text{sat}(\tau_i)$, $i = 1, 2$, there exists a bound $\tau_i^{\max}$ such that $|\text{sat}(\tau_i)| < \tau_i^{\max}$ and $\tau_i^{\max}$ is solely a function of designed parameters, reference trajectories and the bounds of the known compact sets in Assumption 3.2, namely*

$$\begin{aligned}
\tau_i^{\max} &= \mathcal{F}(v_r^{\max}, w_r^{\max}, \dot{v}_r^{\max}, \dot{w}_r^{\max}, \lambda_1, \lambda_2, k_1, k_2, \\
&\quad \hat{m}_{\max}, \hat{n}_{\max}, \hat{P}_{\max}, \hat{E}_{\max}, \varpi_{\max}), i = 1, 2
\end{aligned} \tag{3.38}$$

where

$$\begin{aligned}
\mathcal{F}(\cdot) = & (\hat{m}_{\max} \hat{P}_{3\max} w_{\max} + \hat{P}_{5\max} + \hat{n}_{\max} \hat{P}_{6\max})(k_2 + 1.15(0.2\dot{v}_r^{\max} + 0.1) \\
& + 2.7v_r^{\max} + w_r^{\max} + 0.5(k_1 + 1.11v_r^{\max})) \\
& + (\hat{m}_{\max} \hat{P}_{4\max} + \hat{P}_{3\max} \hat{n}_{\max} w_{\max})(0.5k_1 + 0.56v_r^{\max}) \\
& + (\hat{m}_{\max} \hat{P}_{1\max})((1.62k_1 + \lambda_1)v_r^{\max} + 1.52v_r^{2\max} + 12.8\dot{v}_r^{\max}) \\
& + \hat{n}_{\max} \hat{P}_{2\max}(0.85w_r^{2\max} + 1.02v_r^{2\max} + 2.5\dot{w}_r^{\max} + 4.63v_r^{\max} w_r^{\max} + \lambda_1 \dot{v}_r^{\max} + \lambda_2 \\
& + 2.1\lambda_1 v_r^{\max} \dot{v}_r^{\max} + 2.1v_r^{\max} \lambda_2) \\
& + 0.5\hat{m}_{\max}(1 + \hat{E}_{1\max} + 0.5\hat{n}_{\max}(1.5 + \hat{E}_{2\max}))
\end{aligned}$$

and $v_r^{\max}$, $w_r^{\max}$, $\dot{v}_r^{\max}$, $\dot{w}_r^{\max}$, $\hat{P}_{i\max}$, $1 \leq i \leq 6$ denote the maximum values of v_r , w_r , $\dot{v}_r$, $\dot{w}_r$ and P_i respectively.

Proof. From (3.23) and (3.24) τ_i is composed of the estimated parameters, ϖ_c and $\dot{\varpi}_c$. From (3.12), (3.13), (3.14) and (3.17), ϖ_c and $\dot{\varpi}_c$ only depend on variables $\frac{d}{dt} \frac{x_e}{\Gamma_1}$, $\frac{d}{dt} \frac{y_e}{\Gamma_1}$, $\frac{d}{dt} \frac{\bar{\phi}_e}{\Gamma_3}$, $\frac{d}{dt} \frac{x_e}{\Gamma_2}$, $\frac{d}{dt} \frac{y_e}{\Gamma_2}$, $\frac{d}{dt} \frac{1}{\Gamma_2}$, $\frac{d}{dt} \frac{\Gamma_2}{\Gamma_1}$, v_r , w_r , $\dot{v}_r$, $\dot{w}_r$, λ_1 , λ_2 , k_1 , k_2 , $\frac{x_e}{\Gamma_1}$, $\frac{y_e}{\Gamma_1}$, $\frac{\bar{\phi}_e}{\Gamma_3}$, $\frac{1}{\Gamma_2}$, $\frac{\Gamma_2}{\Gamma_1}$, $\frac{x_e}{\Gamma_2}$, $\frac{y_e}{\Gamma_2}$, $\cos(\bar{\phi}_e)$, $\sin(\bar{\phi}_e)$, $k(t)$, $\dot{k}(t)$, $\ddot{k}(t)$ and ϖ . These signals are bounded and their bounds are computable. For example, note that $\frac{d}{dt} \frac{x_e}{\Gamma_1} = \frac{\dot{x}_e}{\Gamma_1} - \frac{x_e}{\Gamma_1} \left(\frac{\dot{\Gamma}_1}{\Gamma_1} + \frac{y_e}{\Gamma_1} \frac{\dot{y}_e}{\Gamma_1} \right)$. From (3.13) we know the bounds of $\frac{\dot{x}_e}{\Gamma_1}$ and $\frac{\dot{y}_e}{\Gamma_1}$ are computable. Thus the bound of $\frac{d}{dt} \frac{x_e}{\Gamma_1}$ is also computable. It can be similarly checked that the bounds of other ones are also computable. Since $|\frac{x_e}{\Gamma_1}| < 1$, $|\frac{y_e}{\Gamma_1}| < 1$, $|\frac{\bar{\phi}_e}{\Gamma_3}| < 1$, $|\frac{1}{\Gamma_2}| < 1$, $|\frac{\Gamma_2}{\Gamma_1}| < 1$, $|\frac{x_e}{\Gamma_2}| < 1$, $|\cos(\bar{\phi}_e)| \leq 1$, $|\sin(\bar{\phi}_e)| \leq 1$ and $|\frac{y_e}{\Gamma_2}| < 1.15$, from (3.24) we know (3.38) holds. Replacing these variables with their maximum values and after a lengthy calculation, we get the expression of $\mathcal{F}(\cdot)$. $\square$

From $\mathcal{F}(\cdot)$ we can calculate $\tau_i^{\max}$ and ensure $\tau_i^{\max} \leq u_{Mi}$ by choosing suitable parameters and reference signals. If $\tau_i^{\max} \leq u_{Mi}$, we have $\text{sat}(\tau_i) = \tau_i$, $i = 1, 2$. In this case, the design law in (3.24) is our control law for τ , namely

$$\tau = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} \hat{m} & 0 \\ 0 & \hat{n} \end{bmatrix} U. \quad (3.39)$$

Remark 3.3.

- For a general class of system, it is difficult to establish a bound for the control signal as a function of design parameters. The result of Proposition 3.2 is achieved mainly due to exploiting the special structure of the mobile robot with three ideas. Firstly, $\text{sat}(\tau_i)$, $i = 1, 2$ is so designed that x_e , y_e and $\bar{\phi}_e$ appearing in the design law are normalized. Secondly, the proposed parameters update law guarantees the unknown parameters are in known compact sets and thus bounded. Thirdly, by using v and w as the virtual controllers, designing parameter update laws and control law for τ is greatly simplified, which enables us to compute the bound of τ possible.
- The actual form of $\mathcal{F}(\cdot)$ in (3.38) depends on the given system and reference trajectories. It is noted that $\tau_i^{\max}$ can be divided into two parts: one totally independent of all design parameters λ_1 , λ_2 , k_1 and k_2 mentioned in Proposition 3.1, and the other only having these design parameters. If the former one is smaller than the saturation bound, then there is a margin so that we can find such design parameters to make $\tau_i^{\max} < u_{Mi}$. How to obtain $\mathcal{F}(\cdot)$ and calculate $\tau_i^{\max}$ will be illustrated in simulation studies given in Section 2.4.

Theorem 3.1. Consider the nonholonomic system (3.1) and (3.2) with input saturation and disturbance satisfying Assumptions 3.1 - 3.3. With the application of controller (3.24) and parameters update law (3.25) and (3.26), the set-point stabilization and the asymptotical tracking for a reference are achieved simultaneously, and the following statements hold:

(1) The steady state virtual control error satisfies

$$\lim_{t \rightarrow \infty} \tilde{\omega}^T \tilde{\omega} = \lim_{t \rightarrow \infty} (\omega - \omega_c)^T (\omega - \omega_c) = 0 \quad (3.40)$$

(2) The bound of virtual control error at any time is given by

$$|\varpi_i - \varpi_{ic}| \leq \sqrt{\frac{2}{r_{\min}}} \times \sqrt{e^{-\frac{2d_{\min}}{r_{\max}}t} V_3(0) + \frac{P_m^T T P_m + E_m^T K E_m + \alpha m_m^2 + \beta n_m^2}{2} + \frac{r_{\max}(\varpi_{\max} + \varpi_c^m)^T \Omega_m}{2d_{\min}}} \quad (3.41)$$

for $i = 1, 2$, where $m_m = m_{\max} - m_{\min}$, $n_m = n_{\max} - n_{\min}$, $P_m = P_{\max} - P_{\min}$ and $E_m = E_{\max} - E_{\min}$, $\varpi_{\max}$ is defined in (3.37), ϖ_c^m and Ω_m are the maximum values of ϖ_c and Ω respectively, and V_3 is defined in (3.42).

Proof:

(1). Based on *Propositions* 3.1 and 3.2, we can ensure that the set-point stabilization and the asymptotical tracking for a reference are achieved simultaneously. From (3.29) we know that V_2 is non-increasing. Hence $\tilde{\omega}$, $\hat{P}$ and $\hat{E}$ are bounded. By applying LaSalle-Yoshizawa theorem to (3.29), we further know that $\tilde{\omega}^T \tilde{\omega} \rightarrow 0$ as $t \rightarrow \infty$, which implies (3.40) holds.

(2). Define a new Lyapunov function

$$V_3 = \frac{1}{2} \tilde{\omega}^T R \tilde{\omega} + \frac{1}{2} \tilde{P}^T T \tilde{P} + \frac{1}{2} \tilde{E}^T K \tilde{E} + \frac{1}{2} \alpha \tilde{m}^2 + \frac{1}{2} \beta \tilde{n}^2. \quad (3.42)$$

We get $\dot{V}_3 \leq -\tilde{\omega}^T D_R \tilde{\omega} - \tilde{\omega}^T \Omega$. Since $-\tilde{\omega}^T D_R \tilde{\omega} \leq -d_{\min} \varpi^T \varpi \leq -\frac{2d_{\min}}{r_{\max}} \frac{1}{2} \tilde{\omega}^T R \tilde{\omega}$, we have

$$\dot{V}_3 \leq -\frac{2d_{\min}}{r_{\max}} V_3 + \frac{d_{\min}}{r_{\max}} (\tilde{P}^T T \tilde{P} + \tilde{E}^T K \tilde{E} + \alpha \tilde{m}^2 + \beta \tilde{n}^2) + (\varpi_{\max} + \varpi_c^m)^T \Omega_m.$$

Thus we obtain

$$V_3(t) \leq e^{-\frac{2d_{\min}}{r_{\max}}t} V_3(0) + \frac{P_m^T T P_m + E_m^T K E_m + \alpha m_m^2 + \beta n_m^2}{2} + \frac{r_{\max}(\varpi_{\max} + \varpi_c^m)^T \Omega_m}{2d_{\min}}. \quad (3.43)$$

Also as we have $\frac{1}{2}\tilde{\varpi}^T R \tilde{\varpi} \geq \frac{r_{\min}}{2}\tilde{\varpi}^T \tilde{\varpi}$. This gives that

$$\begin{aligned} \tilde{\varpi}^T \tilde{\varpi} &\leq \frac{2}{r_{\min}} \left(e^{-\frac{2d_{\min}}{r_{\max}}t} V_3(0) + \frac{P_m^T T P_m + E_m^T K E_m + \alpha m_m^2 + \beta n_m^2}{2} \right. \\ &\quad \left. + \frac{r_{\max}(\varpi_{\max} + \varpi_c^m)^T \Omega_m}{2d_{\min}} \right). \end{aligned} \quad (3.44)$$

Thus it can be obtained that (3.41) holds. $\square$

3.4 Simulation Results

In this section, we illustrate the design procedure and how to compute the bound of the control torque based on design parameters and reference trajectories. Our simulation tool is Matlab.

The physical parameters of robots are set as: $b = 0.75$, $d = 0.3$, $r = 0.5$, $m_c = 10$, $m_w = 1$, $I_c = 5.6$, $I_w = 0.005$, $I_m = 0.0025$, $d_{11} = d_{22} = 1$. The unknown disturbance ε is white noise satisfying $0 \leq |\varepsilon_i| \leq 2$ for $i = 1, 2$. The initial estimates for the unknown parameters are set to be 75% of their true values.

Suppose the torque saturation bound u_{M_i} is 100, $i = 1, 2$. Based on Proposition 3.2, we need to choose suitable parameters in order to perfectly track a given reference and guarantee the designed torque strictly within the saturation bound. For simplicity, we only leave parameter k_1 as a free parameter to achieve this. The other parameters are chosen as $k_2 = 3$, $\lambda_1 = 0.2$, $\lambda_2 = 0.1$, $T = 0.06I$, $K = 0.03I$. The maximum and minimum values of parameters r , b , m_{11} , m_{12} , c , d_{11} , d_{22} are given as $\{(0.5, 1.5), (1, 2), (0.1, 0.5), (-0.1, 0.05), (0.05, 0.2), (0.2, 1.5), (0.2, 1.5)\}$ respectively. Set the initial velocities $\{\varpi_1, \varpi_2\} = \{0, 0\}$, then we have $V_w(0) = 0$. From (3.37) we know

$$|\varpi_{1,2}| \leq \sqrt{\frac{2}{r_{\min}} \frac{\bar{s}(\tau)^T \bar{s}(\tau) r_{\max}}{2\zeta d_{\zeta}}} \leq \sqrt{\frac{\bar{s}(\tau)^T \bar{s}(\tau)}{2\zeta(5 - 0.5\zeta)}}$$

Choose $\zeta = 5$, we get $|\varpi_{1,2}| \leq 28.31$, which means $|w| \leq 28.31$. Note that $f_1 < 1.11v_r$, $f_2 < 2.1v_r$, $f_3 < 1.15(0.2\dot{v}_r + 0.1) + 2.7v_r + w_r$, $v_c < k_1 + f_1$ and

$w_c < 2(3 + f_3 + 0.5v_c)$. Replacing $\hat{m}$, $\hat{n}$, m_{11} , m_{12} , c , d_{11} , d_{22} , w by their maximum values specified above, after a lengthy but simple calculation, we get polynomial $\mathcal{F}(\cdot)$ for computing $\tau_i^{\max}$

$$\begin{aligned}\tau_i^{\max} &= 4.13v_r^2 + (30.34 + 0.26k_1)v_r + 10.35 + 0.75k_1 \\ &\quad + 2.04w_r^2 + 11.12v_rw_r + 15.2\dot{v}_r + 6\dot{w}_r \\ &= \mathcal{F}_1(v_r, w_r, \dot{v}_r, \dot{w}_r) + \mathcal{F}_2(k_1) \quad i = 1, 2\end{aligned}$$

where $\mathcal{F}_1 = 4.13v_r^2 + 30.34v_r + 10.35 + 2.04w_r^2 + 11.12v_rw_r + 15.2\dot{v}_r + 6\dot{w}_r$ and $\mathcal{F}_2 = (0.75 + 0.26v_r)k_1$.

We first simulate the case of tracking a line for the first 4 seconds and a circle in the remaining 5 seconds in a plane. For the line, set $v_r = 2$, $\dot{v}_r = 0$, $w_r = 0$, $\dot{w}_r = 0$ and for the circle, set $v_r = 1.4$, $\dot{v}_r = 0$, $w_r = \pi/2$, $\dot{w}_r = 0$. It is easy to check that for the line $\mathcal{F}_1 = 87.55$ and for the circle $\mathcal{F}_1 = 90.39$. The inequality $(0.75 + 2.6v_r)k_1 < 100 - \mathcal{F}_1$ under these two paths has a solution $k_1 < 8.6$. In our simulation we choose $k_1 = 5$, and get the simulation results shown in Figure 3.2. Clearly, the magnitudes of the torques are strictly less than 100 and perfect tracking is achieved.

Remark 3.4. *Saturation constraints of input torques restrict the controllability of systems. For a given reference, there exists a minimum saturation bound that makes perfect tracking possible. Thus for a tracking problem, certain reference trajectories cannot be tracked due to saturation constraints, regardless of control scheme used. For example in our case, for a given reference and a saturation bound u_{Mi} , if the design parameters in Proposition 3.1 cannot guarantee $\tau_i^{\max} \leq u_{Mi}$, then the reference cannot be tracked under this saturation bound.*

Secondly we consider stabilization. In this case the reference velocities are: $v_r = 0$, $w_r = 0$, so we have $|k(t)| = |\lambda_1 v_r + \lambda_2 \sin(t)| \leq |\lambda_2|$. Thus we can have a larger λ_2 to reduce the converging time compared with the case of tracking. In this case we set λ_2 to be 0.5. The initial position is set to be $\{2, 2\}$, and other parameters and initial conditions are the same as the tracking case, and we have

Figure 3.2: Simulation result of tracking control of a line and a circle. Left: Robot position in (x, y) plane. Right: Control torques.

Figure 3.3: Simulation result of stabilization. Left: robot position in (x, y) plane. Middle: Tracking errors. Right: Control torques.

$\tau_i^{\max} = 24.3 + 7.32k_1$. So $24.3 + 7.32k_1 < 100$ has a solution $k_1 < 10.32$. In our simulation we still choose $k_1 = 5$. The results are shown in Figure 3.3.

Thirdly we make a comparison by considering the torque design in [9], where the torque control law $\tau = B^{-1}(-k_d \tilde{v} + Y_c \hat{p} - (\frac{\partial V_1}{\partial q} \hat{S})^T)$ is developed based on combined kinematic and dynamic model without considering torque saturation. The torque is applied to the system described above for the case of tracking a circle under input saturation. The results are shown in Fig.3.4, together with the tracking results obtained with our proposed tracking controller. Due to the presence of amplitude constraints, the torque applied to the system experiences saturation for the controller designed based on [9], and it is observed that perfect tracking cannot be achieved. However the amplitude constraint will not affect the tracking result of our controllers since the designed torque is always within the saturation bound.

Figure 3.4: Comparison of two different controllers. Left: The torque of two different controllers after suffering from input saturation. Right: Robot position in plane.

3.5 Conclusions

A global adaptive stabilization and tracking control scheme is proposed for a non-holonomic wheeled mobile robot with input saturation and external disturbances. With the proposed scheme, we can compute the bounds of control signals in advance and thus ensure them within the saturation limits to avoid the occurrence of saturation by suitably choosing design parameters. System stability and perfect tracking are established. Simulation studies are also conducted to illustrate the design procedure and verify the effectiveness of the proposed scheme. However it may be possible that the computed bound of the control signal with any choice of design parameters is larger than the saturation bound. In this case, we need to explore alternative controller design methods. Other possible future work includes extending the proposed method to the controlling of other kinds of underactuated mechanical systems subject to input saturation, such as underactuated surface vessel. Formation or flocking control of mobile robots [52, 74–78] facing obstacles and communication packets loss is also an interesting area for exploration when control input is subject to saturation constraint.

3.6 Appendix:

Proof of (3.28)

Let $f_2 = -\frac{v_r(\cos \bar{\phi}_e - 1)\Gamma_2}{\Gamma_1} - \frac{v_r k(t)y_e \sin \bar{\phi}_e}{\Gamma_1} = -v_r \Lambda$ where $\Lambda = \frac{(\cos \bar{\phi}_e - 1)\Gamma_2}{\Gamma_1} + \frac{k(t)y_e \sin \bar{\phi}_e}{\Gamma_1}$. We have $-\frac{y_e v_r \Lambda}{\Gamma_1} \leq \frac{\delta_1 v_r^2 y_e^2}{\Gamma_1^2} + \delta_1^{-1} \Lambda^2$ where $\delta_1 \in (0, 1)$ is a positive constant. It is left to prove that $\Lambda^2 \leq \frac{\kappa \bar{\phi}_e^2}{1 + \phi_e^2}$ where κ is a positive design parameter.

Denote $f_\Lambda(x) = \Lambda_x^2 = [a_1(\cos(x) - 1) + a_2 \sin(x)]^2$ and $f_\kappa(x) = \frac{\kappa x^2}{1+x^2}$, where $a_1 = \frac{\Gamma_2}{\Gamma_1}$, $a_2 = \frac{k(t)y_e}{\Gamma_1}$ and $0 < |a_i| < 1$ for $i = 1, 2$. We first prove that there exists a positive constant κ so that $|\sqrt{f_\Lambda(x)}| \leq |\sqrt{f_\kappa(x)}|$ for $x \in [0, \infty)$. Define $f_4(x) = |a_1(\cos(x) - 1) + a_2 \sin(x)|$ and $f_5(x) = \frac{\sqrt{\kappa x}}{\sqrt{1+x^2}}$. Note that $f_4(x)$ is periodic and attains its maximum value at $\varphi = \arcsin(\frac{|a_2|}{\sqrt{a_1^2 + a_2^2}})$. Also $f_4(0) = 0$, $f_5(0) = 0$ and $f_4(x)$ monotonically increases in the interval for $x \in [0, \phi]$, it is sufficient to show $\frac{\partial f_5(x)}{\partial x} > \frac{\partial f_4(x)}{\partial x}$ for $x \in [0, \phi]$. Since $\frac{\partial f_4(x)}{\partial x} = \sqrt{a_1^2 + a_2^2} \sin(\varphi - x)$ for $x \in [0, \phi)$ and $\frac{\partial f_5(x)}{\partial x} = \frac{\sqrt{\kappa}}{(1+x^2)^{1.5}}$, it is easy to choose κ so that $\frac{\partial f_5(x)}{\partial x} > \frac{\partial f_4(x)}{\partial x}$ for $x \in [0, \phi]$. For example, could be chosen as $(1 + a_2^2)^3$. Letting $\delta_2 = \delta_1^{-1} \kappa$, then the result holds. For $x \in (-\infty, 0)$, it can be proved similarly.

Chapter 4

Underactuated Ship with Input Saturation

In the previous chapter, the tracking and stabilization control of nonholonomic mobile robots is investigated by normalizing certain signals and thus calculating the bounds of the input signals. However this control scheme may not be applied to underactuated ships since the dynamic model of underactuated ships is much more complicated. In this chapter, tracking control of underactuated ship in the presence of input saturation is addressed. By dividing the tracking error dynamic system into a cascade of two subsystems, the torques in surge and yaw axis are designed separately using the backstepping technique. More specifically, we design the yaw axis torque in such a way that its corresponding subsystem is finite time stable, which makes it be de-coupled from the second subsystem after a finite time. This enables us to design the torque in the surge axis independently. It is shown that the closed-loop system is stable and the mean-square tracking errors can be made arbitrarily small by choosing design parameters.

4.1 Introduction

The control of underactuated systems with nonholonomic constraints has received vast attention over the past few decades. Typical examples of such systems include nonholonomic mobile robots, underactuated ships, underwater vehicles and

VCTOL aircrafts *et al.* There has been significant interest in tracking control of underactuated ships as evidenced by various control schemes proposed for such a problem. In [14], two effective control schemes are presented to solve the global tracking control problem for underactuated ships with Lyapunov method by exploiting the inherent cascade interconnected structure of the ship dynamics. In [79] by dividing the ship dynamics into two linear subsystems a global tracking controller is proposed. In [17] a continuous time-varying tracking controller is designed to yield global uniformly ultimately bounded tracking by using a transformation of the ship tracking system into a skew-symmetric form and designing a time-varying dynamic oscillator. In [13] the method developed for chain-form system was adopted for underactuated ships through a coordinate transformation to steer both the position variables and the course angle of the ship providing exponential stability of the reference trajectory. In [16] a single controller is designed to achieve stabilization and tracking of underactuated ships simultaneously. In [19] a controller that achieves practical stabilization of arbitrary reference trajectories for underactuated ships using the transverse function approach. Subsequent related works on the tracking control of underactuated ships include, but are not limited to, [18, 82–84] and many references therein.

In practice, an underactuated ship is ultimately driven by a motor which can only provide limited amount of torque. Thus the magnitude of the control signal is always constrained. Under such a constraint, the controllability of the system is seriously affected. Thus it often severely limits system performance, giving rise to undesirable inaccuracy or leading to instability. This brings great challenges to the design and analysis of controllers. Therefore, the development of a control scheme for an underactuated ship with input saturation has been an extremely difficult task, yet with practical interest and theoretical significance. In [85] a tracking control method is presented to address such an issue by employing the linearization and dynamic surface control, only ensuring bounded tracking errors. Other than this one, there is no other literature dealing with this problem as far as we know, due to its difficulty.

In this chapter, we solve this problem by proposing a state-feedback control

scheme. Firstly the tracking error dynamic system is divided into a cascade of two subsystems so that the torques in surge and yaw axis can be designed separately based on the two subsystems. A new finite-time control scheme is proposed to design the torque in yaw which enables the two subsystems to be de-coupled within finite time. After decoupling, a new backstepping scheme similar to that in [86] is proposed for the surge subsystem. The designed controllers for the two subsystems are analyzed, respectively. Then the overall system is studied. It is shown that the closed-loop system is stable and the mean-square tracking errors can be made arbitrarily small by adjusting design parameters. To illustrate the effectiveness of the controller, simulation studies are conducted.

4.2 Problem Formation

4.2.1 Underactuated Ship Model

Similar to [14], we consider the ship as shown in Figure 4.1. The only two propellers are the torques in surge and yaw axis. The motion of the ship dynamics is described by the following differential equations

$$\begin{aligned}\dot{x} &= u \cos \phi - v \sin \phi \\ \dot{y} &= u \sin \phi + v \cos \phi \\ \dot{\phi} &= r\end{aligned}\tag{4.1}$$

$$\begin{aligned}\dot{u} &= \frac{m_2}{m_1}vr - \frac{d_1}{m_1}u + \frac{1}{m_1}Sat_u(\tau_u) \\ \dot{v} &= -\frac{m_1}{m_2}ur - \frac{d_2}{m_2}v \\ \dot{r} &= \frac{m_1 - m_2}{m_3}uv - \frac{d_3}{m_3}r + \frac{1}{m_3}Sat_r(\tau_r)\end{aligned}\tag{4.2}$$

where (x, y) denotes the position of the ship surface in X and Y directions, respectively, ϕ denotes the heading angle of the ship, u , v and r are the surge, sway and yaw velocities, respectively. τ_u and τ_r represent the torques in surge and yaw axis, respectively, which are considered as actual control inputs. Since there is no torque introduced in the sway direction, the system modeled by (4.1)-(4.2)

Figure 4.1: The coordinates of an underactuated ship.

is underactuated. The positive constants m_j denote the ship inertia including added mass, and d_j , $1 \leq j \leq 3$ represent the hydrodynamic damping coefficients. The saturation function is defined as follows

$$Sat_{\star}(x) = \begin{cases} x & \text{if } |x| < \tau_{\star \max} \\ \tau_{\star \max} & \text{if } |x| \geq \tau_{\star \max} \end{cases} \quad (4.3)$$

where $\star$ denotes u or r respectively and $\tau_{u \max}$ and $\tau_{r \max}$ are their respective saturation limits.

Our control objective is formally stated as follows.

- *Control Objective:* Design the control inputs τ_u and τ_r to force the ship, as modeled in (4.1)-(4.2) subject to input saturation, to track a prescribed path, which is denoted by $\Omega = (x_d, y_d, \phi_d, u_d, r_d)$ generated by the dynamic equations of a virtual underactuated ship:

$$\begin{aligned} \dot{x}_d &= u_d \cos(\phi_d) - v_d \sin(\phi_d) \\ \dot{y}_d &= u_d \sin(\phi_d) + v_d \cos(\phi_d) \\ \dot{\phi}_d &= r_d \\ \dot{v}_d &= -\frac{m_{11}}{m_{22}} u_d r_d - \frac{d_2}{m_2} v_d \end{aligned} \quad (4.4)$$

4.2.2 Variable Transformation

To facilitate the control design, firstly we define the path-following errors $\{x_e, y_e, \phi_e\}$ as follows

$$\begin{bmatrix} x_e \\ y_e \\ \phi_e \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - x_d \\ y - y_d \\ \phi - \phi_d \end{bmatrix} \quad (4.5)$$

Differentiating both sides of (4.5) along (4.1) results in the kinematic error dynamics:

$$\begin{aligned} \dot{x}_e &= u_e - u_d(\cos(\phi_e) - 1) - v_d \sin(\phi_e) + r_e y_e + r_d y_e \\ \dot{y}_e &= v_e - v_d(\cos(\phi_e) - 1) + u_d \sin(\phi_e) - r_e x_e - r_d x_e \\ \dot{\phi}_e &= r_e \\ \dot{u}_e &= \frac{m_2}{m_1} v r - \frac{d_1}{m_1} u + \frac{1}{m_1} \text{Sat}_u(\tau_u) - \dot{u}_d \\ \dot{v}_e &= -\frac{m_1}{m_2} (u r - u_d r_d) - \frac{d_2}{m_2} v_e \\ \dot{r}_e &= \frac{m_1 - m_2}{m_3} u v - \frac{d_3}{m_3} r + \frac{1}{m_3} \text{Sat}_r(\tau_r) - \dot{r}_d \end{aligned} \quad (4.6)$$

where $u_e = u - u_d$, $v_e = v - v_d$ and $r_e = r - r_d$.

To achieve the control objective, the following assumption is imposed.

Assumption 1:

(a) There is a constant $\delta_r > 0$ such that, for any time period (t_0, t) ,

$$\int_{t_0}^t r_d^2(\tau) d\tau \geq \delta_r(t - t_0)$$

(b) There exists a positive constant Δ such that

$$\frac{|m_1 - m_2|}{2d_1\alpha} \tau_{u \max}^2 + \Delta \leq \tau_{r \max} \quad (4.7)$$

where $\alpha = \min\{\frac{d_1}{m_1}, \frac{2d_2}{m_2}\}$. Furthermore, r_d is chosen such that $|m_d \dot{r}_d + d_3 r_d| < \Delta$.

Remark 4.1.

- *Assumption (a) is a PE condition on the reference angular velocity r_d which is commonly required, see for example [jiang2002, lefeber2003 et al.*
- *Assumption (b) is the restrictions for the saturation limits and the reference that is needed for controller design.*

4.3 Controller Design

The control design procedure can be summarized as follows. Firstly (4.6) will be divided into a cascade of two subsystems, i.e., (ϕ_e, r_e) subsystem and (x_e, y_e, v_e, u_e) subsystem. Control τ_r is designed such that ϕ_e and r_e converge to zero within finite time T , while it guarantees that $|\tau_r| < \tau_{r \max}$. For $t > T$ the two subsystems are de-coupled since the interaction effects are only from (ϕ_e, r_e) subsystem to (x_e, y_e, v_e, u_e) subsystem, i.e. one-way interaction. Control τ_u is designed based on (x_e, y_e, v_e, u_e) subsystem with a new backstepping technique in such a way that τ_u is the output of a filter. It is then shown that the overall closed-loop system is stable and the mean-square tracking errors could be arbitrarily small by adjusting design parameters.

Now we develop the following intermediate result to reveal the bounds of u , v and r .

Lemma 4.1. *The variables u , v and r are bounded and their respective bounds satisfy*

$$\limsup_{t \rightarrow \infty} |u| \leq \sqrt{\frac{m_2}{2m_1 d_1 \alpha}} \tau_{u \max}, \quad |v| \leq \sqrt{\frac{m_1}{2m_2 d_1 \alpha}} \tau_{u \max}, \quad |r| \leq \frac{\tau_{u \max}}{\sqrt{m_3 \beta d_1}} + \frac{\tau_{r \max}}{\sqrt{m_3 \beta d_3}} \quad (4.8)$$

where $\beta = \min\{\frac{d_1}{m_1}, \frac{2d_2}{m_2}, \frac{d_3}{m_3}\}$.

Proof: Consider a Lyapunov function $V_0 = \frac{m_1}{2m_2} u^2 + \frac{m_2}{2m_1} v^2$, whose derivative

is

$$\begin{aligned}
\dot{V}_0 &= -\frac{d_1}{m_2}u^2 - \frac{d_2}{m_1}v^2 + \frac{1}{m_2}u\text{Sat}_u(\tau_u) \\
&\leq -\frac{d_1}{m_2}u^2 - \frac{d_2}{m_1}v^2 + \frac{d_1}{2m_2}u^2 + \frac{1}{2d_1}\tau_{u\max}^2 \\
&\leq -\alpha V_0 + \frac{1}{2d_1}\tau_{u\max}^2
\end{aligned}$$

Without loss of generality, let $u(0) = 0$ and $v(0) = 0$. Thus $V_0 \leq \frac{\tau_{u\max}^2}{2d_1\alpha}$. So $|u| \leq \sqrt{\frac{m_2}{2m_1d_1\alpha}}\tau_{u\max}$ and $|v| \leq \sqrt{\frac{m_1}{2m_2d_1\alpha}}\tau_{u\max}$. For r , consider $V = \frac{m_1}{2}u^2 + \frac{m_2}{2}v^2 + \frac{m_3}{2}r^2$, whose derivative is

$$\begin{aligned}
\dot{V} &= -d_1u^2 - d_2v^2 - d_3r^2 + u\text{Sat}_u(\tau_u) + r\text{Sat}_r(\tau_r) \\
&\leq -\frac{d_1}{2}u^2 - d_2v^2 - \frac{d_3}{2}r^2 + \frac{\tau_{u\max}^2}{2d_1} + \frac{\tau_{r\max}^2}{2d_3} \\
&\leq -\beta V + \frac{\tau_{u\max}^2}{2d_1} + \frac{\tau_{r\max}^2}{2d_3}
\end{aligned}$$

So $V \leq \frac{\tau_{u\max}^2}{2\beta d_1} + \frac{\tau_{r\max}^2}{2\beta d_3}$ and $|r| \leq \sqrt{\frac{\tau_{u\max}^2}{m_3\beta d_1} + \frac{\tau_{r\max}^2}{m_3\beta d_3}} \leq \frac{\tau_{u\max}}{\sqrt{m_3\beta d_1}} + \frac{\tau_{r\max}}{\sqrt{m_3\beta d_3}}$. This completes the proof. $\square$

Now we design τ_r with the aim that $\sup|\tau_r| \leq \tau_{r\max}$ and ϕ_e and r_e defined in (4.6) converge to zero within finite time. Consider the following system

$$\begin{aligned}
\dot{\phi}_e &= r_e \\
\dot{r}_e &= \frac{m_1 - m_2}{m_3}uv - \frac{d_3}{m_3}r + \frac{1}{m_3}\text{Sat}_r(\tau_r) - \dot{r}_d
\end{aligned} \tag{4.9}$$

First of all, define $\text{sig}(x)^k = \text{sign}(x)|x|^k$, then τ_r is designed as

$$\begin{aligned}
\tau_r &= (m_2 - m_1)uv + m_3\dot{r}_d + d_3r_d \\
&\quad - m_3\text{Sat}_{\varepsilon_1}\left(\text{sig}(\phi_e)^{k_1}\right) - m_3\text{Sat}_{\varepsilon_2}\left(\text{sig}(r_e)^{k_2}\right)
\end{aligned} \tag{4.10}$$

where ε_1 and ε_2 are positive design parameters, k_1 and k_2 are two positive constants satisfying $0 < k_1 < 1$, $k_2 = \frac{2k_1}{1 + k_1}$.

Substituting (4.10) into (4.9),

$$\begin{aligned}\dot{\phi}_e &= r_e \\ \dot{r}_e &= -\text{Sat}_{\varepsilon_1}(\text{sig}(\phi_e)^{k_1}) - \text{Sat}_{\varepsilon_2}(\text{sig}(r_e)^{k_2}) - \frac{d_3}{m_3}r_e\end{aligned}\quad (4.11)$$

Then the following lemma is established.

Lemma 4.2. *With controller (4.10), ϕ_e and r_e in (4.11) will converge to zero within finite time. Furthermore, it is ensured that $|\tau_r| \leq \tau_{r\max} \forall t$ by choosing appropriate design parameters ε_1 and ε_2 .*

Proof: We first define V_x as

$$V_x = \begin{cases} \frac{1}{1+k_1}|x|^{1+k_1} & |x| \leq \varepsilon_1^{1/k_1} \\ \frac{1}{1+k_1}|\varepsilon_1|^{1+k_1} + \text{sgn}(x)\varepsilon_1\left(x - \text{sgn}(x)\varepsilon_1^{1/k_1}\right) & |x| > \varepsilon_1^{1/k_1} \end{cases} \quad (4.12)$$

which is plotted in Fig. 4.2. It is obvious that V_x is positive semi-definite, unbounded and is C^1 . Also $\dot{V}_{\phi_e} = \text{sat}_{\varepsilon_1}(\text{sig}(\phi_e)^{k_1})\dot{\phi}_e$. Then consider the following Lyapunov function

$$V_1 = V_{\phi_e} + \frac{1}{2}r_e^2 \quad (4.13)$$

whose derivative along (4.11) is

$$\dot{V}_1 = -\frac{d_3}{m_3}r_e^2 - r_e \text{sat}_{\varepsilon_2}(\text{sig}(r_e)^{k_2}) \quad (4.14)$$

From (4.14), r_e is globally asymptotically convergent. Similarly it can also be proved that ϕ_e is globally asymptotically convergent. Note that the saturation in (4.11) will not take effect when

$$|\phi_e| \leq \varepsilon_1^{1/k_1}, \quad |r_e| \leq \varepsilon_2^{1/k_2} \quad (4.15)$$

Figure 4.2: Shape of V_x

Thus we define a set

$$\Pi = \{(\phi_e, r_e) : \frac{1}{1+k_1}|\phi_e|^{1+k_1} + \frac{1}{2}r_e^2 \leq \theta\} \quad (4.16)$$

where $\theta = \min\{\frac{\varepsilon_1^{(1+k_1)/k_1}}{1+k_1}, \frac{1}{2}\varepsilon_2^{2/k_2}\}$. Now consider $V_2 = \frac{1}{1+k_1}|\phi_e|^{1+k_1} + \frac{1}{2}r_e^2$. When $\phi_e \in \Pi$ and $r_e \in \Pi$, the saturation does not take effect, and it can be easily checked that $\dot{V}_2 = -\frac{d_3}{m_3}r_e^2 - |r_e|^{1+k_2} \leq 0$, thus Π is an invariant set.

When $(\phi_e, r_e) \notin \Pi$, (ϕ_e, r_e) approaches Π within finite time from (4.14), and after (ϕ_e, r_e) reaches Π , we have

$$\begin{aligned} \dot{\phi}_e &= r_e \\ \dot{r}_e &= -\frac{d_3}{m_3}r_e - \left(\text{sig}(\phi_e)^{k_1} + \text{sig}(r_e)^{k_2}\right) \end{aligned} \quad (4.17)$$

Next we show that (4.17) is finite-time stable. To establish the stability of (4.17), the following fact is presented.

Fact 4.1. *The following system is globally finite-time stable.*

$$\dot{x} = y \quad (4.18a)$$

$$\dot{y} = -\text{sig}(x)^{k_1} - \text{sig}(y)^{k_2} \quad (4.18b)$$

where k_1 and k_2 are two positive constants satisfying $0 < k_1 < 1$, $k_2 = \frac{2k_1}{1+k_1}$.

Based on Fact 4.1,

$$\begin{aligned}\dot{\phi}_e &= r_e \\ \dot{r}_e &= -\text{sig}(\phi_e)^{k_1} - \text{sig}(r_e)^{k_2}\end{aligned}\tag{4.19}$$

is globally finite-time stable, and is homogeneous of degree $\kappa = k_1 - 1 < 0$ with respect to $(r_1, r_2) = (\frac{2}{1+k_1}, 1)$. Furthermore

$$\lim_{\varepsilon \rightarrow 0} \frac{d_3 \varepsilon^{r_2} r_e}{m_3 \varepsilon^{\kappa+r_2}} = \frac{d_3}{m_3} r_e \lim_{\varepsilon \rightarrow 0} \varepsilon^{-\kappa} = 0\tag{4.20}$$

which means $-\frac{d_3}{m_3} r_e$ is ‘higher degree’ with respect to $\varepsilon^{\kappa+r_2}$ in the sense of (4.20). Based on Lemma 2.7 in the Appendix, we know that (4.17) is finite-time stable if $(\phi_e, r_e) \in \Pi$.

So in summary, if $(\phi_e(0), r_e(0)) \notin \Pi$, (ϕ_e, r_e) will reach Π within finite time, and when $(\phi_e, r_e) \in \Pi$, (ϕ_e, r_e) will converge to the origin within finite time. Thus (ϕ_e, r_e) will approach to zero within a total of finite time T and therefore (4.11) is globally finite time stable.

Also from (4.10) and Lemma 4.1,

$$|\tau_r| \leq |m_2 - m_1| \frac{1}{2d_1\alpha} \tau_{u\max}^2 + m_3(\varepsilon_1 + \varepsilon_2) + |m_3 \dot{r}_d + d_3 r_d|$$

Based on Assumption 1, choose ε_1 and ε_2 to satisfy

$$|m_3 \dot{r}_d + d_3 r_d| + m_3(\varepsilon_1 + \varepsilon_2) \leq \Delta\tag{4.21}$$

Then we have

$$|\tau_r| \leq \tau_{r\max}\tag{4.22}$$

which means the saturation does not take effect for the yaw subsystem. This completes the proof. $\square$

Since ϕ_e and r_e will converge to zero within a finite time T , and due to the saturation, u , v and r are bounded, thus for $t \leq T$, x_e , y_e , ϕ_e , u_e and r_e are

bounded. As for $t > T$, $\phi_e = 0$ and $r_e = 0$, then the (x_e, y_e, v_e, u_e) subsystem becomes

$$\begin{aligned}\dot{y}_e &= v_e - r_d x_e \\ \dot{x}_e &= u_e + r_d y_e \\ \dot{v}_e &= -\frac{m_1}{m_2} u_e r_d - \frac{d_2}{m_2} v_e \\ \dot{u}_e &= \frac{m_2}{m_1} v r_d - \frac{d_1}{m_1} u + \frac{1}{m_1} Sat_u(\tau_u) - \dot{u}_d\end{aligned}\tag{4.23}$$

The model in (4.23) enables us to design τ_u for $t > T$ without considering the coupling effects from (ϕ_e, r_e) subsystem. Due to the one-way interaction, the designed control τ_u can still be applied from the initial time $t = 0$. As $Sat_u(\cdot)$ is non-smooth, thus backstepping technique, see [8], cannot be applied here. To overcome this difficulty, the saturation will be approximated by a smooth function defined as in [86]

$$s(\tau_u) = \tau_{u \max} \tanh(\tau_u / \tau_{u \max}) = \tau_{u \max} \frac{e^{\tau_u / \tau_{u \max}} - e^{-\tau_u / \tau_{u \max}}}{e^{\tau_u / \tau_{u \max}} + e^{-\tau_u / \tau_{u \max}}}\tag{4.24}$$

Then $Sat_u(\cdot)$ can be expressed as

$$Sat_u(\tau_u) = s(\tau_u) + d(t)\tag{4.25}$$

where $d(t) = Sat_u(\tau_u) - s(\tau_u)$ is a bounded function which can be deemed as a ‘disturbance’ bounded as

$$|d(t)| = |Sat_u(\tau_u) - s(\tau_u)| \leq \tau_{u \max}(1 - \tanh(1)) := D\tag{4.26}$$

Figure 4.3 shows an example of approximating the saturation function. To achieve the control objective, we augment the closed-loop system (4.23) by including the saturation approximation function and the resulting approximation error as

Figure 4.3: Approximation of the saturation function $Sat_1(\tau_u)$

follows

$$\begin{aligned}
 \dot{y}_e &= v_e - r_d x_e \\
 \dot{x}_e &= u_e + r_d y_e \\
 \dot{v}_e &= -\frac{m_1}{m_2} u_e r_d - \frac{d_2}{m_2} v_e \\
 \dot{u}_e &= \frac{m_2}{m_1} v r_d - \frac{d_1}{m_1} u + \frac{1}{m_1} s(\tau_u) - \dot{u}_d + \frac{1}{m_1} d(t) \\
 \dot{\tau}_u &= -c\tau_u + \alpha
 \end{aligned} \tag{4.27}$$

where c is a positive constant and α is an auxiliary intermediate control to be designed in the backstepping approach. Clearly all functions in (4.27) are smooth and the use of backstepping is feasible. Let α_{y_e} be the virtual controls of x_e to stabilize y_e , α_{x_e} be the virtual control of u_e and α_{τ_u} be the virtual control of $s(\tau_u)$. Then following the standard backstepping design steps as in [8], these virtual controls and the intermediate control α in (4.27) can be obtained as summarized in Table 1 where $0 < \epsilon < 1$ is a positive constant and $N(\chi)$ is a Nassbaum function. From (4.30) the choices of design parameters c_1 and c_2 are dependent on $|r_d|$ and $|\dot{r}_d|$, and are possible. Similar analysis of choosing c_1 and c_2 can also be found in [14].

The final control τ_u is generated from the output of the filter

$$\dot{\tau}_u = -c\tau_u + \alpha \tag{4.31}$$

in (4.27).

Table 1: Summary of virtual control errors, virtual controls, intermediate control and design parameter condition.

Virtual control errors:

$$\begin{aligned}\tilde{x}_e &= x_e - \alpha_{y_e} \\ \tilde{u}_e &= u_e - \alpha_{x_e} \\ \tilde{\tau}_u &= \alpha_{\tau_u} - s(\tau_u)\end{aligned}\tag{4.28}$$

Virtual Controllers and Intermediate Controller:

$$\begin{aligned}\alpha_{y_e} &= c_1 r_d y_e \\ \alpha_{x_e} &= -c_2 \tilde{x}_e \\ \alpha_{\tau_u} &= -c_4 \tilde{u}_e - m_2 v r_d + d_1 u + m_1 (\dot{u}_d + \dot{\alpha}_{x_e} \\ &\quad - \tilde{x}_e) + \frac{m_1^2 c_3}{m_2} v_e r_d - \tanh\left(\frac{1}{\varepsilon} \tilde{u}_e\right) \frac{D}{m_1} \\ \alpha &= N(\chi) \bar{\alpha} \\ \bar{\alpha} &= -c_5 \tilde{\tau}_u + c \tau_u \frac{\partial s(\tau_u)}{\partial \tau_u} + \dot{\alpha}_{\tau_u} - \frac{1}{m_1} \tilde{u}_e \\ N(\chi) &= \chi^2 \cos\left(\frac{\pi \chi}{2}\right), \quad \dot{\chi} = \bar{\alpha} \tilde{\tau}_u\end{aligned}\tag{4.29}$$

Condition for design parameters:

$$\begin{aligned}&c_2 - c_1 |r_d|^2 > 0, \\ &-\tilde{x}_e \left(c_1 \dot{r}_d y_e + c_1 r_d (v_e - c_1 r_d^2 y_e) \right) + y_e v_e + \frac{c_2 c_3 m_1}{m_2} r_d v_e \tilde{x}_e \\ &\leq (1 - \epsilon) (c_2 - c_1 |r_d|^2) \tilde{x}_e^2 - \frac{m_2}{(1 - \epsilon) d_2 c_3} y_e^2 - \frac{(1 - \epsilon) c_3 d_2}{m_2} v_e\end{aligned}\tag{4.30}$$

Remark 4.2. Although τ_u is designed based on the de-coupled surge subsystem in (4.27) for $t > T$, it is implemented for all $t \geq 0$. It has no effects on (ϕ_e, r_e) subsystem and thus does not affect the results in Lemma 4.2.

Consider the following Lyapunov function

$$V_3 = \frac{1}{2}y_e^2 + \frac{1}{2}\tilde{x}_e^2 + \frac{c_3}{2}v_e^2 + \frac{1}{2}\tilde{u}_e^2 + \frac{1}{2}\tilde{\tau}_u^2 \quad (4.32)$$

Then we have

$$\begin{aligned} \dot{V}_3 &\leq (\xi N(\chi) - 1)\tilde{\alpha}\tilde{\tau}_u - \tilde{c}V_3 + D\tilde{u}_e f(\tilde{u}_e) \\ &= (\xi N(\chi) - 1)\dot{\chi} - \tilde{c}V_3 + O(\varepsilon) \end{aligned} \quad (4.33)$$

where $O(\varepsilon) = \frac{D}{m_1}x \left(\text{sign}(x) - \tanh\left(\frac{1}{\varepsilon}x\right) \right)$ for $x \in \Re$ and $\tilde{c} = \min\{c_1r_d^2 - \frac{m_2}{(1-\varepsilon)d_2c_3}, \varepsilon(c_2 - c_1r_d^2), \frac{\varepsilon c_3 d_2}{m_2}, 2c_4, 2c_5\} > 0$, $\xi = \frac{\partial s(\tau_u)}{\partial \tau_u} = \frac{4}{(e^{\tau_u/\tau_{u\max}} + e^{-\tau_u/\tau_{u\max}})^2} \in (0, 1)$. It should be noted that $\sup_{v_x} O(\varepsilon)$ is arbitrarily small if ε is small enough.

With the proposed control, the following lemma is established.

Lemma 4.3. Consider system (4.27) under the control of (4.28)-(4.30). If Assumption 1 is satisfied and $V_3(T)$ is bounded, then all signals in (4.27) are bounded for $t > T$.

Proof:

From (4.33) we have

$$V_3(t) \leq V_3(T)e^{-\tilde{c}(t-T)} + \frac{O(\varepsilon)}{\tilde{c}}(1 - e^{-\tilde{c}(t-T)}) + e^{-\tilde{c}t} \int_T^t (\xi N(\chi)\dot{\chi} - \dot{\chi})e^{\tilde{c}\sigma} d\sigma \quad (4.34)$$

for $t > T$. The boundedness of V_3 can be established based on the Nussbaum properties via a contradiction argument. First define V_N on the time-period (t_i, t_j) as

$$V_N(t_i, t_j) = \int_{t_i}^{t_j} (\xi N(\chi) - 1)\dot{\chi} e^{-\tilde{c}(t_j - \sigma)} d\sigma \quad (4.35)$$

For notation convenience, let $V_N(t_i, t_j) = V_N(\chi_i, \chi_j)$, $T < t_i < t_j$. By noting the

fact that $0 < \xi < 1$ and $0 < e^{-\tilde{c}(t_j - \sigma)} < 1$, we have

$$\begin{aligned} |V_N(\chi_i, \chi_j)| &\leq \int_{\chi_i}^{\chi_j} (|N(\chi)| + 1) d\chi \\ &\leq (\chi_j - \chi_i) \left(\sup_{\chi \in [\chi_i, \chi_j]} |N(\chi)| + 1 \right) \end{aligned} \quad (4.36)$$

For the Nussbaum function $N(\chi) = \chi^2 \cos(\frac{\pi\chi}{2})$, we know it is positive for $\chi \in (4m-1, 4m+1)$ and negative for $\chi \in (4m+1, 4m+3)$ with an integer m . Thus we consider two time periods: $[\chi_0, \chi_1] = [\chi_0, 4m+1]$ and $[\chi_1, \chi_2] = [4m+1, 4m+3]$ for $\chi_0 > 0$ and m being a positive integer such that for $t \rightarrow t_f$, where t_f is an arbitrary positive constant, we have $\chi = \chi_2$. From (4.36) we know for $[\chi_0, \chi_1] = [\chi_0, 4m+1]$, we have

$$\begin{aligned} V_N(\chi_0, \chi_2) &= V_N(\chi_0, \chi_1) + V_N(\chi_1, \chi_2) \\ &\leq (4m+1)^2 \left(-l_2((8m+2)(1-\Psi) + (1-\Psi)^2) \right. \\ &\quad \left. + 4m+1 - \chi_0 + \frac{l_1 + l + 3}{(4m+1)^2} \right) \end{aligned} \quad (4.37)$$

where $l_2 = 2\Psi e^{-\tilde{c}^*(t_2 - t_1)} \cos(\pi\Psi/2) > 0$ and $l_3 = 2\Psi e^{-\tilde{c}^*(t_2 - t_1)} > 0$.

We now prove χ is bounded over $(T, t_f]$ by seeking a contradiction. Suppose χ is unbounded over $(T, t_f]$. Two cases should be considered: (i) χ has no upper-bound and (ii) χ has no lower-bound. If $\chi(t)$ has no upper bound on $(T, t_f]$. In this case, there must exist a time interval $[t_s, t_f]$ such that $\chi(t_s) > 0$, when $t \rightarrow t_f$, $\chi(t) = \infty$. However, from (4.37) we know $V_N(\chi_0, \chi_2) \rightarrow -\infty$ as $m \rightarrow \infty$. On the other hand, $V_3(t) > 0$ for all t . Thus we can always find a subsequence of the interval that leads to a contradiction. So $\chi(t)$ has a upper bound. On the other side, if $\chi(t)$ has no lower bound on $(T, t_f]$. Define $\chi = -\lambda$, then λ has no upper bound accordingly. Since $N(\chi)$ is an even function, thus (4.33) becomes

$$\begin{aligned} V_3(t) &\leq V_3(T) e^{-\tilde{c}(t-T)} + \frac{O(\varepsilon)}{\tilde{c}} (1 - e^{-\tilde{c}(t-T)}) \\ &\quad - \int_T^t (\xi N(\lambda) - 1) \dot{\lambda} e^{-\tilde{c}(t-\sigma)} d\sigma \\ &= V_3(T) e^{-\tilde{c}(t-T)} + \frac{O(\varepsilon)}{\tilde{c}} (1 - e^{-\tilde{c}(t-T)}) - V_N(\lambda(T), \lambda(t)) \end{aligned} \quad (4.38)$$

Therefore there exists a time interval $[t_s, t_f]$ such that $\lambda(t)$ is monotonically increasing and $\lim_{t \rightarrow t_f} \lambda(t) = \infty$ with $\lambda(t_s) > 0$. Following the same procedure, we can construct a subsequence of the interval that leads to a contradiction. Therefore χ has a lower bound on $[T, t_f]$. Then from (4.36) V_N is bounded. The above argument holds for all $t_f > T$. So χ is bounded. Thus V_3 is also bounded, which means $x_e, y_e, v_e, \tilde{u}_e$ and $\tilde{\tau}_u$ are bounded. This completes the proof. $\square$

Remark 4.3. *By examining the methodologies of stability analysis using backstepping approaches in existing literatures, it is noted that control signals are shown bounded after establishing the boundedness of virtual control signals. Based on such an idea, we can prove $s(\tau_u)$ bounded after showing the boundedness of $\tilde{\tau}_u$ and α_{τ_u} in (4.28). However, this is not helpful at in our case a $s(\tau_u)$ itself is always bounded. In other words, the boundedness of τ_u cannot be obtained from the boundedness of function $s(\tau_u)$. So such existing methodologies are not applicable here. This is the main challenge encountered to address the issue of input saturation. To overcome the challenge, the Nassbaum technique is applied here to establish the boundedness of the intermediate control α and then the boundedness of τ_u . However, due to the difficulty of applying Nassbaum to MIMO nonlinear systems, the ship dynamic system is de-coupled to two single-input subsystems with one finite-time controller designed for τ_r .*

With Lemma 4.2 and Lemma 4.3, the following theorem is established.

Theorem 4.1. *Consider the closed-loop system consisting of the underactuated ship dynamic systems (4.1)-(4.2) and the proposed controllers (4.10), (4.28)-(4.30) and (4.31) designed under Assumption 1. The orientational error ϕ_e converges to origin within finite time. The mean-square of position errors $x_e(t)$ and $y_e(t)$ are of the order of ε where ε is a design parameter..*

Proof: We consider $t \in (0, T)$. Since τ_u and τ_r are saturated, it is easy to be checked that $u, v, r, x, y, \phi, \tilde{u}_e, v_e, r_e, x_e, y_e, \phi_e$ and their derivatives are bounded for $t \in (0, T)$. From (4.29) α_{τ_u} and $\dot{\alpha}_{\tau_u}$ are bounded. Thus from (4.28) we know $\tilde{\tau}_u$ is bounded. Then based on (4.27) and (4.29), we know τ_u and χ are bounded for $t \in (0, T)$. Thus from Lemma 4.2 and Lemma 4.3 we know the closed-loop

system is stable and the orientational error ϕ_e converges to origin within finite time T .

From (4.33),

$$\begin{aligned} \int_0^{T_1} V_3(t)dt &\leq \int_0^T V_3(t)dt + \frac{V_3(T_1) - V_3(T)}{\tilde{c}} \\ &\quad + \frac{1}{\tilde{c}} \int_{\chi(T)}^{\chi(T_1)} (\xi N(\chi) - 1)d\chi + \frac{1}{\tilde{c}} \int_T^{T_1} O(\varepsilon)dt \end{aligned} \quad (4.39)$$

Since $\int_0^T V_3(t)dt$ and $\int_{\chi(T)}^{\chi(T_1)} (\xi N(\chi) - 1)d\chi$ are bounded, we have

$$\lim_{T_1 \rightarrow \infty} \frac{1}{T_1} \int_0^{T_1} V_3(t)dt \leq \frac{O(\varepsilon)}{\tilde{c}} \quad (4.40)$$

which shows that the mean-square of errors $x_e, y_e, v_e, \tilde{u}_e$ and $\tilde{\tau}_u$ are of the order of ε . So the mean-square tracking errors could be arbitrarily small in the sense of (4.40) by adjusting ε . This completes the proof. $\square$

Remark 4.4. *Although the control scheme to deal with input saturation is developed for underactuated ship, but the methodology can be applied to other mechanical systems, e.g. nonholonomic mobile robots.*

4.4 Simulations

In this section, we carry out simulations to demonstrate the effectiveness of our controllers and to validate our constructive methodology for underactuated ships. The parameters of the underactuated ship from [14] are used, namely $m_1 = 0.1$, $m_2 = 0.1$, $m_3 = 0.1$, $d_1 = 0.1$, $d_2 = 0.2$ and $d_3 = 0.1$. The saturation limits are given as $\tau_{u \max} = 0.05$ and $\tau_{r \max} = 0.12$. The reference velocities are chosen as $u_d = 0.1$ and $r_d = 0.1$, respectively. The initial position of the reference ship is given to be $(x_d(0), y_d(0), \phi_d(0), v_d(0)) = (0, 0, 0, 0)$. For the real ship, we consider the following initial position: $(x(0), y(0), \phi(0), u(0), v(0), r(0)) = (0.1, 0.2, 0.1, 0, 0, 0)$. The control parameters are selected as: $c_1 = 4$, $c_2 = 2$, $c_3 = 10$, $c_4 = 4$, $c_5 = 3$, $c = 2$, $\varepsilon_1 = 0.6$, $\varepsilon_2 = 0.6$, $\varepsilon = 0.05$. Simulation time is 40s.

Figure 4.4: The positions of the reference and real ship.

Figure 4.4 shows the reference and the trajectory of the real ship. Figure 4.5 shows the position errors and orientational error. It can be seen that the reference can be tracked with arbitrarily small error as governed by ε . Figure 4.6 shows the torque τ_u and τ_r . It is obvious that the maximum value of $|\tau_u|$ and $|\tau_r|$ are less than $\tau_{u \max}$ and $\tau_{r \max}$ respectively.

As a comparison, the control scheme from [14] without considering saturation is also applied to the above system subject to the same saturation limits and with the same reference and initial conditions. The results are shown in Fig 4.7. It can be observed that the controllers designed with our proposed scheme perform better in tracking the reference.

4.5 Conclusion

In this chapter, we consider global tracking control problem of underactuated ship subject to input saturation. First the tracking error dynamic system is divided into a cascade of two subsystems, i.e., (ϕ_e, r_e) subsystem and (x_e, y_e, v_e, u_e) subsystem. With our proposed finite-time yaw controller, the two subsystems are de-coupled within finite time T . This enables the torque in surge to be designed independently by using a new backstepping technique. It is shown that the system is stable and the mean-square tracking errors can be made arbitrarily small

Figure 4.5: The position errors x_e , y_e and the orientational error ϕ_e of the real ship

Figure 4.6: The Torque τ_u and τ_r .

Figure 4.7: The position of the reference, real ship with our control scheme and real ship with control technique from [14] subject to saturation.

by adjusting the design parameters.

4.6 Appendix

Proof of Fact 4.1

Consider a Lyapunov function $V_0 = \frac{1}{1+k_1}|x|^{1+k_1} + \frac{1}{2}y^2$. Then we get $\dot{V}_0 = -|y|^{1+k_2} \leq 0$. Note that $\dot{V}_0 \equiv 0$ implies $y \equiv 0$, which further implies $x \equiv 0$. Thus direct application of LaSalle's invariant set theorem implies that the zero solution of (4.18) is asymptotically stable. Moreover, (4.18) is homogeneous of a negative degree $\kappa = k_2 - 1 < 0$ with respect to $(\frac{2}{1+k_1}, 1)$, thus based on Lemma 2.6 we know (4.18) is globally finite-time stable.

Chapter 5

Adaptive Output-Feedback of Nonholonomic Mobile Robots

In the previous two chapters, the tracking control of nonholonomic mobile robots and underactuated ships with input saturation are investigated. In the following two chapters, adaptive output feedback control of nonholonomic mobile robots and underactuated ships will be addressed.

In this chapter, an adaptive output feedback tracking controller for nonholonomic mobile robots is proposed. The major difficulties of adaptive output feedback for nonholonomic mobile robots are caused by simultaneous existence of nonholonomic constraints, unknown system parameters and a quadratic term of unmeasurable states in the mobile robot dynamic system as well as their couplings. To overcome these difficulties, a new adaptive control scheme is proposed including designing a new adaptive state feedback controller and two high-gain observers to estimate the unknown linear and angular velocities respectively. It is shown that the closed loop adaptive system is stable and the tracking errors are guaranteed to be within the arbitrarily small pre-specified bounds.

5.1 Introduction

A two-wheeled mobile robot is one of the well-known benchmark nonholonomic systems, which leads to the development of various novel controller design

schemes. Much effort has been devoted to the stabilization and tracking control of this kind of nonholonomic systems, for example see [9–11, 87], and the references therein. In control theory, stabilization is usually regarded as a special case of the tracking problem. However for controlling underactuated mechanical systems with nonholonomic constraints such as nonholonomic mobile robots, the stabilization and tracking problems are totally different, and thus they are normally considered separately. For the stabilization problem, the past abundant literature aims at developing suitable discontinuous time-invariant stabilizer [3] or time-varying stabilizer [2] or hybrid stabilizers [4, 5]. The seminal work of [2] introduced the first time-varying feedback stabilizer for a wheeled mobile robot. For tracking control, [6] proposed a time-varying state feedback tracking controller with backstepping technique [8] for a kinematic model and a simplified dynamic model. [9] designed an adaptive controller based on both kinematic and dynamic models of nonholonomic mobile robots with unknown parameters. [10] solved both adaptive tracking and stabilization simultaneously. Subsequent related works on the stabilization and tracking control of nonholonomic mobile robots include, but are not limited to, [70, 73, 88–91] and many references therein.

It should be pointed out that most of these proposed controllers are concerned with the case that the velocities of mobile robots are measured by some devices like tachometers. But in practice such devices may not be used either because they are contaminated by noise or they are expensive. Many existing output feedback control schemes such as those proposed in [92, 93] for robot manipulators cannot be applied to mobile robots due to the nonholonomic constraint and certain quadratic cross terms. [94] proposed a global output feedback control for a class of nonholonomic dynamic system, but there is not any quadratic term of unknown states contained in the system. In [11], a time-varying output feedback controller was proposed for the dynamic model of a mobile robot, but it was assumed that all parameters must be known. In [92], an output feedback controller for single-DOF Lagrangian systems was proposed with high-gain observer, but it was difficult to be extended to systems with more DOFs. To the best of our knowledge, adaptive output feedback control for nonholonomic mobile

robots with unknown parameters still remains an open research problem. The main difficulty encountered lies on the simultaneous appearance of nonholonomic constraint, unknown system parameters and the coriolis matrix, which results in quadratic cross terms involving unmeasured states. With this in mind, the observer design and the parameter estimator design are intertwined, which makes the analysis of the resulting closed loop system difficult and challenging.

In this chapter, we propose a high-gain observer based scheme to address the output feedback tracking control of mobile robots in the presence of parametric uncertainties. To accomplish this, we first address the issue of designing an adaptive state feedback controller with suitable parameter estimators to achieve asymptotic tracking. Then we design two high-gain observers to estimate the unknown states and substitute the estimates to the state feedback controller and parameter estimators based on certainty equivalence principle. The stability of the system is analyzed and established by using Lyapunov approach. Due to the disturbance rejection property of the high-gain observers, it is shown that the state estimation errors are of order $O(\varepsilon)$, where ε is a small design parameter, and the tracking errors can be made arbitrary small. With our proposed control scheme, a solution is now presented for the outstanding problem of adaptive output-feedback control for nonholonomic mobile robots. Finally we demonstrate the effectiveness of our proposed controllers with simulation studies.

5.2 Robot Model and Problem Formulation

We consider a two-wheeled mobile robot described by the following dynamic models as in [95]

$$\dot{\eta} = J(\eta)\omega, \quad (5.1)$$

$$M\dot{\omega} + C(\dot{\eta})\omega + D\omega = \tau, \quad (5.2)$$

where $\eta = (x, y, \phi)$ denotes the position and orientation of the robot, $\omega = (\omega_1, \omega_2)^T$ denotes the angular velocities of the left and right wheels, $\tau = (\tau_1, \tau_2)^T$ represents the control torques applied to the wheels, and M is a symmetric, positive definite

inertia matrix, $C(\dot{\eta})$ is the centripetal and coriolis matrix, D denotes the surface friction. Matrices $J(\eta)$, M , $C(\dot{\eta})$ and D are the same as those in Chapter 3. In this chapter, for the sake of simplicity, we assume that the robot does not slip. Also we assume that there is no sliding between the tire and the road, i.e. there is no Coulomb-like friction.

Let v and w denote the linear and angular velocities of the robot, respectively. Then the relationship between ω_1 , ω_2 and v , w is described as follows

$$(v, w)^T = B^{-1}(\omega_1, \omega_2)^T, B = \frac{1}{r} \begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix}. \quad (5.3)$$

Substituting (5.3) into (5.1), we get

$$\frac{d}{dt} \begin{bmatrix} x \\ y \\ \phi \end{bmatrix} = \begin{bmatrix} \cos \phi & 0 \\ \sin \phi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}.$$

For the output feedback control of nonholonomic mobile robot, η is taken as the output and $\varpi = [v, w]^T$ is the state which is not directly measured in our case. Now let $(x_r, y_r, \phi_r)^T$ denote the desired reference position and orientation of the virtual robot the motion of which is described by:

$$\begin{aligned} \dot{x}_r &= v_r \cos \phi_r, \\ \dot{y}_r &= v_r \sin \phi_r, \\ \dot{\phi}_r &= w_r, \end{aligned} \quad (5.4)$$

where v_r and w_r denote the linear and angular velocities of the the virtual robot. $(x_r(0), y_r(0))$ denote the initial position and $\phi_r(0)$ is the initial orientation of the reference trajectory. The control objective is to find torque τ applied to the mobile robot to ensure the desired reference trajectories are tracked.

To achieve the objective, we need the following assumptions:

Assumption 5.1. $|x_r(t)| < a^*$, $|y_r(t)| < a^*$, where a^* is a known but arbitrary

positive constant.

Assumption 5.2. *The reference linear and angular velocities v_r and w_r and their first-order derivatives are bounded. Also there exist two positive constants $r_\star$ and T_s , such that for $T_s \leq t < \infty$, $r_\star \leq |v_r(t)| < \infty$.*

Assumption 5.3. *The unknown parameters of the mobile robot are in known compact sets.*

5.3 Adaptive State Feedback Control: An Intermediate Step

In this section, a new scheme is proposed to design an adaptive state feedback controller involving a control law and parameter estimators. However, the designed controller will not be implemented as the states are unavailable. Nevertheless, the proposed scheme is helpful for the design of output feedback controller and thus is considered as an “intermediate” step. Basically, the unknown states v and w in the control law and parameter estimators will be replaced by the estimated states $\hat{v}$ and $\hat{w}$ after appropriate modifications in Section IV.

5.3.1 Design of Virtual Control based on Kinematic Model

As we will design high-gain observers to estimate unknown v and w , so unlike the existing schemes such as those in [9, 10], we use v and w as the virtual control variables in state feedback control design. Substituting (5.3) into (5.2), we get

$$\begin{aligned}\dot{\eta} &= L(\eta)\varpi, \\ \dot{\varpi} &= -B^{-1}M^{-1}C(\dot{\eta})B\varpi - B^{-1}M^{-1}DB\varpi + B^{-1}M^{-1}\tau,\end{aligned}\tag{5.5}$$

where

$$L(\eta) = \begin{bmatrix} \cos \phi & 0 \\ \sin \phi & 0 \\ 0 & 1 \end{bmatrix}. \quad (5.6)$$

Let

$$\bar{B} = \frac{1}{r} \begin{bmatrix} -b & -1 \\ -b & 1 \end{bmatrix}.$$

It is obvious that $C(\dot{\eta})B = \bar{B}C(\dot{\eta})$ and thus we can replace $C(\dot{\eta})B$ in (5.5) by $\bar{B}C(\dot{\eta})$. Multiplying (5.5) by $\bar{B}^{-1}MB$, we get

$$\begin{aligned} \dot{\eta} &= L(\eta)\varpi, \\ R\dot{\varpi} &= -\bar{C}(w)\varpi - D_R\varpi + B_R\tau, \end{aligned} \quad (5.7)$$

where $R = -\bar{B}^{-1}MB$, $\bar{C}(w) = -C(w)$, $D_R = -RB^{-1}M^{-1}DB$, $B_R = -RB^{-1}M^{-1}$ and R is a positive definite matrix. Explicitly, we have

$$\begin{aligned} R &= \begin{bmatrix} (m_{11} + m_{22})/b & 0 \\ 0 & (m_{11} - m_{22})b \end{bmatrix}, \quad B_R = \frac{r}{2b} \begin{bmatrix} 1 & 1 \\ b & -b \end{bmatrix}, \\ D_R &= \begin{bmatrix} \frac{d_{11}+d_{22}}{2b} & \frac{d_{11}-d_{22}}{2} \\ \frac{d_{11}-d_{22}}{2} & \frac{b(d_{11}+d_{22})}{2} \end{bmatrix}. \end{aligned} \quad (5.8)$$

As often done in tracking control of mobile robots, see for examples [6, 9, 10], the tracking errors are defined as

$$\begin{bmatrix} x_e \\ y_e \\ \phi_e \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - x_r \\ y - y_r \\ \phi - \phi_r \end{bmatrix}, \quad (5.9)$$

which describe the position errors and the orientation error between the actual robot position, direction and those of the virtual robot. It can be directly checked

that these tracking errors satisfy the following differential equations:

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \\ \dot{\phi}_e \end{bmatrix} = \begin{bmatrix} wy_e + v - v_r \cos \phi_e \\ -wx_e + v_r \sin \phi_e \\ w - w_r \end{bmatrix}. \quad (5.10)$$

Introduce a new variable $\bar{\phi}_e = \phi_e + \arcsin(\frac{k(t)y_e}{\sqrt{1+x_e^2+y_e^2}})$ where

$$k(t) = \lambda v_r \quad (5.11)$$

with λ being a positive constant chosen so that $\lambda|v_r|^{\max} < 0.5$, where $|v_r|^{\max}$ denotes the maximum value of $|v_r(t)|$. Then (5.10) is transformed to

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \\ \dot{\bar{\phi}}_e \end{bmatrix} = \begin{bmatrix} wy_e + v + f_1 \\ -wx_e - \frac{k(t)v_r y_e}{\Gamma_1} + f_2 \\ w(1 - \frac{k(t)x_e}{\Gamma_1}) + f_3 - v \frac{k(t)x_e y_e}{\Gamma_1^2 \Gamma_2} \end{bmatrix}, \quad (5.12)$$

where

$$\begin{aligned} \Gamma_1 &= \sqrt{1+x_e^2+y_e^2}, \quad \Gamma_2 = \sqrt{1+x_e^2+(1-k^2(t))y_e^2}, \\ f_1 &= -\frac{v_r \cos \bar{\phi}_e \Gamma_2}{\Gamma_1} - \frac{v_r k(t) y_e \sin \bar{\phi}_e}{\Gamma_1}, \\ f_2 &= -\frac{v_r (\cos \bar{\phi}_e - 1) \Gamma_2}{\Gamma_1} - \frac{v_r k(t) y_e \sin \bar{\phi}_e}{\Gamma_1}, \\ f_3 &= -w_r + \frac{\dot{k}(t) y_e + k(t) f_2 - k(t) (x_e f_1 + y_e f_2) / \Gamma_1^2}{\Gamma_2}. \end{aligned} \quad (5.13)$$

Define virtual control errors $\tilde{v}$ and $\tilde{w}$ as

$$\tilde{v} = v - v_c, \quad \tilde{w} = w - w_c, \quad (5.14)$$

where v_c and w_c are the virtual controls to be designed for v and w . We have

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \\ \dot{\bar{\phi}}_e \end{bmatrix} = \begin{bmatrix} (w_c + \tilde{w})y_e + v_c + \tilde{v} + f_1 \\ -(w_c + \tilde{w})x_e - \frac{k(t)v_r y_e}{\Gamma_1} + f_2 \\ (w_c + \tilde{w})(1 - \frac{k(t)x_e}{\Gamma_1}) + f_3 - (v_c + \tilde{v})\frac{k(t)x_e y_e}{\Gamma_1^2 \Gamma_2} \end{bmatrix}. \quad (5.15)$$

Define a Lyapunov function

$$V_0 = \sqrt{1 + x_e^2 + y_e^2} + \sqrt{1 + \bar{\phi}_e^2} - 2.$$

The time derivative of V_0 is

$$\begin{aligned} \dot{V}_0 = & -\frac{k(t)v_r y_e^2}{\Gamma_1^2} + \frac{x_e}{\Gamma_1}(v_c + f_1) + \frac{y_e f_2}{\Gamma_1} + \frac{\bar{\phi}_e}{\Gamma_3}[w_c(1 - \\ & \frac{k(t)x_e}{\Gamma_1}) + f_3 - v_c \frac{k(t)x_e y_e}{\Gamma_1^2 \Gamma_2}] + \tilde{v}(\frac{x_e}{\Gamma_1} - \frac{k(t)x_e y_e \bar{\phi}_e}{\Gamma_1^2 \Gamma_2 \Gamma_3}) \\ & + \tilde{w} \frac{\bar{\phi}_e}{\Gamma_3}(1 - \frac{k(t)x_e}{\Gamma_1}). \end{aligned}$$

The virtual control v_c and w_c are then chosen as

$$\begin{aligned} v_c &= -k_1 \frac{x_e}{\Gamma_1} - f_1 \\ w_c &= \frac{1}{(1 - \frac{k(t)x_e}{\Gamma_1})}(-k_2 \frac{\bar{\phi}_e}{\Gamma_3} - f_3 + v_c \frac{k(t)x_e y_e}{\Gamma_1^2 \Gamma_2}) \end{aligned} \quad (5.16)$$

where $\Gamma_3 = \sqrt{1 + \bar{\phi}_e^2}$, and we get

$$\begin{aligned} \dot{V}_0 = & -k_1 \frac{x_e^2}{\Gamma_1^2} - \frac{k(t)v_r y_e^2}{\Gamma_1^2} - k_2 \frac{\bar{\phi}_e^2}{\Gamma_3^2} + \frac{y_e f_2}{\Gamma_1} \\ & + \tilde{v}(\frac{x_e}{\Gamma_1} - \frac{k(t)x_e y_e \bar{\phi}_e}{\Gamma_1^2 \Gamma_2 \Gamma_3}) + \tilde{w} \frac{\bar{\phi}_e}{\Gamma_3}(1 - \frac{k(t)x_e}{\Gamma_1}). \end{aligned}$$

Remark 5.1. Γ_1 , Γ_2 and Γ_3 are introduced to normalize x_e , y_e and $\bar{\phi}_e$ so that the virtual control signals are globally bounded. This normalization technique will also be applied to the dynamic model when designing control law for τ . The bounds of v_c and w_c are computable since $|\frac{x_e}{\Gamma_1}| < 1$, $|\frac{y_e}{\Gamma_1}| < 1$ and $|\frac{\bar{\phi}_e}{\Gamma_3}| < 1$. This will play

an important role when proving the stability of the whole system, as will be shown later.

5.3.2 Design of State-feedback Control based on Dynamic Model

Define $\varpi_c = [v_c \ w_c]^T$, $\tilde{\varpi} = \varpi - \varpi_c$, and a new Lyapunov function

$$V_1 = V_0 + \frac{1}{2} \tilde{\varpi}^T R \tilde{\varpi}. \quad (5.17)$$

Then we get

$$\begin{aligned} \dot{V}_1 &= \dot{V}_0 + \tilde{\varpi}^T R \dot{\tilde{\varpi}} = \dot{V}_0 + \tilde{\varpi}^T R (\dot{\varpi} - \dot{\varpi}_c) \\ &= \dot{V}_0 + \tilde{\varpi}^T [-\bar{C}(w)\varpi - D_R\varpi + B_R\tau - R\dot{\varpi}_c] \\ &= \dot{V}_0 + \tilde{\varpi}^T [-(\bar{C}(w) + D_R)\tilde{\varpi} + B_R\tau - Y_c H], \end{aligned}$$

where $Y_c H = R\dot{\varpi}_c + \bar{C}(w)\varpi_c + D_R\varpi_c$ and

$$Y_c = \begin{bmatrix} \dot{v}_c & 0 & -w_c w & v_c & w_c & 0 \\ 0 & \dot{w}_c & v_c w & 0 & v_c & w_c \end{bmatrix}. \quad (5.18)$$

$$H = \begin{bmatrix} \frac{m_{11}+m_{22}}{b} & (m_{11}-m_{22})b & c & \frac{d_{11}+d_{22}}{2b} & \frac{d_{11}-d_{22}}{2} & \frac{b(d_{11}+d_{22})}{2} \end{bmatrix}^T \quad (5.19)$$

Remark 5.2. By using v and w as virtual controls, the tracking error dynamic system in (5.12) and the virtual controllers v_c and w_c in (5.16) do not contain any unknown system parameters. So when taking the time-derivatives of v_c and w_c in the matrix Y_c in (5.18) to get the torque τ_i , no unknown parameter will appear. This will reduce the number of unknown parameters in (5.19) from 10 in [10] to 6 and thus greatly simplify the matrix Y_c . Therefore parameter estimation is simplified compared to [10].

Denote $\alpha := \frac{r}{2b}$, $\beta := \frac{r}{2}$ and $m = \frac{1}{\alpha}$, $n = \frac{1}{\beta}$. Let $\hat{m}$ and $\hat{n}$ be the estimates

of m and n respectively, and define $\tilde{m} = m - \hat{m}$, $\tilde{n} = n - \hat{n}$, $u_1 = \frac{\tau_1 + \tau_2}{\hat{m}}$, $u_2 = \frac{\tau_1 - \tau_2}{\hat{n}}$, $U = [u_1 \ u_2]^T$. Then we get

$$B_R \tau = \begin{bmatrix} \alpha & \alpha \\ \beta & -\beta \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} \alpha \hat{m} u_1 \\ \beta \hat{n} u_2 \end{bmatrix}. \quad (5.20)$$

As $\alpha \hat{m} = \alpha(m - \tilde{m}) = 1 - \alpha \tilde{m}$ and also $\beta \hat{n} = 1 - \beta \tilde{n}$,

$$B_R \tau = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} \alpha \tilde{m} u_1 \\ \beta \tilde{n} u_2 \end{bmatrix} = U - \begin{bmatrix} \alpha \tilde{m} u_1 \\ \beta \tilde{n} u_2 \end{bmatrix}. \quad (5.21)$$

Let $\Omega_1 = (\frac{x_e}{\Gamma_1} - \frac{k(t)x_e y_e \bar{\phi}_e}{\Gamma_1^2 \Gamma_2 \Gamma_3})$, $\Omega_2 = \frac{\bar{\phi}_e}{\Gamma_3} (1 - \frac{k(t)x_e}{\Gamma_1})$, and

$$\Omega = [\Omega_1 \ \Omega_2]^T. \quad (5.22)$$

We now define a new Lyapunov function

$$V_2 = V_1 + \frac{1}{2} \tilde{H}^T \Lambda \tilde{H} + \frac{1}{2} \alpha \tilde{m}^2 + \frac{1}{2} \beta \tilde{n}^2 \quad (5.23)$$

where $\tilde{H} = H - \hat{H}$ and $\hat{H}$ is the estimate of H , $\Lambda = \gamma I$ is a positive definite matrix and γ is a positive constant. Then

$$\begin{aligned} \dot{V}_2 &= \dot{V}_0 + \tilde{\omega}^T [-(\bar{C}(w) + D_R) \tilde{\omega} + B_R \tau - Y_c H] + \tilde{H}^T \Lambda \dot{\tilde{H}} \\ &\quad + \alpha \tilde{m} \dot{\tilde{m}} + \beta \tilde{n} \dot{\tilde{n}} \\ &= -k_1 \frac{x_e^2}{\Gamma_1^2} - \frac{k(t) v_r y_e^2}{\Gamma_1^2} - k_2 \frac{\bar{\phi}_e^2}{\Gamma_3^2} + \frac{y_e f_2}{\Gamma_1} + \alpha \tilde{m} \dot{\tilde{m}} + \beta \tilde{n} \dot{\tilde{n}} \\ &\quad + \tilde{\omega}^T [-(\bar{C}(w) + D_R) \tilde{\omega} + B_R \tau - Y_c H + \Omega] + \tilde{H}^T \Lambda \dot{\tilde{H}} \\ &= -k_1 \frac{x_e^2}{\Gamma_1^2} - \frac{k(t) v_r y_e^2}{\Gamma_1^2} - k_2 \frac{\bar{\phi}_e^2}{\Gamma_3^2} + \frac{y_e f_2}{\Gamma_1} \\ &\quad - \alpha \tilde{m} (\dot{\tilde{m}} + u_1 \tilde{v}) - \beta \tilde{n} (\dot{\tilde{n}} + u_2 \tilde{w}) + \tilde{H}^T \Lambda \dot{\tilde{H}} \\ &\quad + \tilde{\omega}^T [-(\bar{C}(w) + D_R) \tilde{\omega} + U - Y_c H + \Omega]. \end{aligned} \quad (5.24)$$

By choosing U as

$$U = -K\tilde{\omega} + Y_c\hat{H} - \Omega, \quad (5.25)$$

where $K = [\kappa_1 \ 0; 0 \ \kappa_2]$ with κ_1 and κ_2 being nonnegative constants, we get

$$\begin{aligned} \dot{V}_2 = & -k_1 \frac{x_e^2}{\Gamma_1^2} - \frac{k(t)v_r y_e^2}{\Gamma_1^2} - k_2 \frac{\bar{\phi}_e^2}{\Gamma_3^2} + \frac{y_e f_2}{\Gamma_1} - \tilde{\omega}^T (D_R + K) \tilde{\omega} \\ & - \alpha \tilde{m}(\dot{\hat{m}} + u_1 \tilde{v}) - \beta \tilde{n}(\dot{\hat{n}} + u_2 \tilde{w}) - \tilde{H}^T T(\dot{\hat{H}} + \Lambda^{-1} Y_c^T \tilde{\omega}). \end{aligned}$$

The control τ is designed as

$$\tau^{\text{state}} = \begin{bmatrix} \tau_1^{\text{state}} \\ \tau_2^{\text{state}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} \hat{m} & 0 \\ 0 & \hat{n} \end{bmatrix} U. \quad (5.26)$$

We use τ^{state} to signify it as the state feedback control in order to differentiate it from the output feedback control given later.

Now we design parameter estimators with a projection operation. Define $\theta = [m, n, H^T]^T$ and $g(v, w) = [-u_1 \tilde{v}, -u_2 \tilde{w}, -\Lambda^{-1} \tilde{\omega}^T Y_c]^T$. Let $\hat{\theta}$ be the estimate of θ and $\tilde{\theta} = \theta - \hat{\theta}$. From Assumption 5.3, there exist $\theta_{i\max}$ and $\theta_{i\min}$ so that $\theta_{i\min} < \theta_i < \theta_{i\max}$. Define $\bar{\theta}_{i\min} = \frac{2-\sqrt{2}}{2}\theta_{i\min}$ and $\bar{\theta}_{i\max} = \frac{2+\sqrt{2}}{2}\theta_{i\max}$. The update law for $\hat{\theta}_i$, $i = 1, \dots, 8$, is designed as follows:

$$\dot{\hat{\theta}}_{i\text{state}} = \begin{cases} g_i(v, w) & \text{if } \theta_{i\min} < \hat{\theta}_i < \theta_{i\max} \\ g_i(v, w) + (1 - \frac{\hat{\theta}_i}{\theta_{i\min}})^2 [1 + g_i^2(v, w)] & \text{if } \hat{\theta}_i \leq \theta_{i\min} \\ g_i(v, w) - (1 - \frac{\hat{\theta}_i}{\theta_{i\max}})^2 [1 + g_i^2(v, w)] & \text{if } \hat{\theta}_i \geq \theta_{i\max} \end{cases} \quad (5.27)$$

where $g_i(v, w) = \vartheta_i^T g(v, w)$ and ϑ_i , $1 \leq i \leq 8$, is i th unit vector defined as

$$\vartheta_i = \begin{bmatrix} \overbrace{0 \dots 0}^{i-1} & 1 & \overbrace{0 \dots 0}^{8-i} \end{bmatrix}^T.$$

Here we use $\dot{\hat{\theta}}_{i\text{state}}$ also to signify it as a “state-feedback” estimator in which state variables are used.

Lemma 5.1. *The estimator (5.27) has the following property: (a) The parameter projection gives continuous $\dot{\hat{\theta}}_{i\text{state}}$. (b) Estimates $\hat{\theta}_i$ satisfies $\bar{\theta}_{i\min} \leq \hat{\theta}_i \leq \bar{\theta}_{i\max}$ for all $t \geq 0$ if $\bar{\theta}_{i\min} < \hat{\theta}_i(0) < \bar{\theta}_{i\max}$. (c) $-\alpha\tilde{m}(\dot{m} + u_1\tilde{v}) - \beta\tilde{n}(\dot{n} + u_2\tilde{w}) - \tilde{H}^T T(\dot{\hat{H}} + \Lambda^{-1}Y_c^T\tilde{\omega}) \leq 0$ for all $t \geq 0$.*

Proof: (a) It is obvious that $\dot{\hat{\theta}}_{i\text{state}}$ is continuous. (b) When $\hat{\theta}_i = \bar{\theta}_{i\min}$, we have $(1 - \frac{\hat{\theta}_i}{\bar{\theta}_{i\min}})^2 = \frac{1}{2}$, thus $\dot{\hat{\theta}}_i = \frac{1}{2}(1 + g_i(v, w))^2 \geq 0$ and this prevents $\hat{\theta}_i$ from getting smaller than $\bar{\theta}_{i\min}$. So $\hat{\theta}_i \geq \bar{\theta}_{i\min}$. Similarly, when $\hat{\theta}_i = \bar{\theta}_{i\max}$, we also have $(1 - \frac{\hat{\theta}_i}{\bar{\theta}_{i\max}})^2 = \frac{1}{2}$ and thus $\dot{\hat{\theta}}_i = -\frac{1}{2}(1 + g_i(v, w))^2 < 0$ which prevents $\hat{\theta}_i$ from getting greater than $\bar{\theta}_{i\max}$. Therefore property (b) holds. (c) If $\hat{m} \leq m_{\min}$, then $\tilde{m} = m - \hat{m} \geq 0$ and $\dot{m} + u_1\tilde{v} = (1 - \frac{\hat{m}}{m_{\min}})^2[1 + (u_1\tilde{v})^2] \geq 0$, therefore $-\tilde{m}(\dot{m} + u_1\tilde{v}) \leq 0$. Similarly, when $\hat{m} \geq m_{\max}$, $\tilde{m} = m - \hat{m} \leq 0$ and $\dot{m} + u_1\tilde{v} = -(1 - \frac{\hat{m}}{m_{\max}})^2[1 + (u_1\tilde{v})^2] \leq 0$. This ensures that $-\tilde{m}(\dot{m} + u_1\tilde{v}) \leq 0$. When $m_{\min} < \hat{m} < m_{\max}$, $(\dot{m} + u_1\tilde{v}) = 0$. We can also prove that $-\beta\tilde{n}(\dot{n} + u_2\tilde{w}) - \tilde{H}^T T(\dot{\hat{H}} + \Lambda^{-1}Y_c^T\tilde{\omega}) \leq 0$. Thus property (c) is established. $\square$

Based on Property (c) in Lemma 5.1, we have

$$\begin{aligned} \dot{V}_2 \leq & -k_1 \frac{x_e^2}{\Gamma_1^2} - \frac{k(t)v_r y_e^2}{\Gamma_1^2} - k_2 \frac{\bar{\phi}_e^2}{\Gamma_3^2} + \frac{y_e f_2}{\Gamma_1} \\ & - \tilde{\omega}^T (D_R + K) \tilde{\omega}. \end{aligned} \quad (5.28)$$

Proposition 5.1. *Under the adaptive controllers (5.22) and (5.25) - (5.27) with appropriate choices of parameters k_1 , k_2 , λ , δ_1 and δ_2 , system (5.12) is uniformly stable and the trajectories of the virtual reference robot are asymptotically tracked.*

Proof: It is proved in Chapter 3 that positive constants $\delta_1 \in (0, 1)$ and $\delta_2 > 0$ can be chosen such that

$$\frac{y_e f_2}{\Gamma_1} \leq \frac{\delta_1 v_r^2 y_e^2}{\Gamma_1^2} + \delta_2 \frac{\bar{\phi}_e^2}{\Gamma_3^2}. \quad (5.29)$$

Substituting (5.29) into (5.28) yields

$$\begin{aligned} \dot{V}_2 \leq & -k_1 \frac{x_e^2}{\Gamma_1^2} - \frac{(\lambda - \delta_1)v_r^2 y_e^2}{\Gamma_1^2} - (k_2 - \delta_2) \frac{\bar{\phi}_e^2}{\Gamma_3^2} \\ & - \tilde{\omega}^T (D_R + K) \tilde{\omega}. \end{aligned} \quad (5.30)$$

On the other hand, based on the condition that $|k(t)| < 0.5$, we choose positive constant λ to satisfy

$$\lambda v_r^{\max} < 0.5 \quad (5.31)$$

where $v_r^{\max}$ denotes the maximum value of $|v_r|$. From (5.30), the choices of λ and k_2 also need to meet the following conditions

$$\lambda > \delta_1 \quad \text{and} \quad k_2 > \delta_2. \quad (5.32)$$

Clearly the above choices of parameters are possible. Also we choose $k_1 > 0$. Let $k_3 = v_r^2(\lambda - \delta_1)$ and $k_4 = k_2 - \delta_2$, we get

$$\dot{V}_2 \leq -\frac{k_1 x_e^2}{\Gamma_1^2} - \frac{k_3 y_e^2}{\Gamma_1^2} - \frac{k_4 \bar{\phi}_e^2}{\Gamma_3^2} - \tilde{\omega}^T (D_R + K) \tilde{\omega}. \quad (5.33)$$

Since $|v_r(t)| \geq r_*$ for $t \geq T_s$, $k_3 > r_*^2(\lambda - \delta_1)$ for $t \geq T_s$. Then from (5.33) we get $\lim_{t \rightarrow \infty} |x_e| + |y_e| + |\bar{\phi}_e| + \|\tilde{\omega}\| \rightarrow 0$.

Finally from the definition of $\bar{\phi}_e$, we know that $\lim_{t \rightarrow \infty} y_e + |\bar{\phi}_e| \rightarrow 0$ guarantees $\lim_{t \rightarrow \infty} |\phi_e| \rightarrow 0$. Therefore the trajectories are asymptotically tracked. $\square$

5.4 Adaptive Output Feedback Control

The design procedure is outlined as follows. Firstly two observers are designed to estimate the unknown states v and w respectively. Then the unknown states in the “state-feedback” control and parameter estimators presented in the previous section are replaced by the their estimates respectively. To prevent the peaking phenomenon of the high-gain observers from entering the robot dynamic system,

we will invoke saturation to the controller and estimators. Finally system stability is analyzed and tracking errors are shown to remain in an arbitrarily small ball.

5.4.1 Observer Design

The main difficulty of adaptive output feedback control for nonholonomic mobile robots comes from the fact that designing state observers and parameter estimators is intertwined due to the presence of nonholonomic constraints and a quadratic term. In other words, the observers involve unknown parameters and the estimators involve unknown states as seen in (5.26) and (5.27). In this chapter we design two high-gain observers to estimate v and w respectively, which can overcome the difficulty. First, we define two new variables as follows

$$z_1 = y \cos \phi - x \sin \phi, \quad z_2 = x \cos \phi + y \sin \phi.$$

Then

$$\begin{aligned} \dot{z}_1 &= -z_2 w \\ \dot{z}_2 &= v + z_1 w \\ \dot{\phi} &= w. \end{aligned} \tag{5.34}$$

By noting the third equation of (5.34), following the standard high-gain observer design procedure in [96, 97], we design a high-gain observer for w as

$$\begin{aligned} \dot{\hat{\phi}} &= \hat{w} + \frac{1}{\varepsilon} l_1 (\phi - \hat{\phi}) \\ \dot{\hat{w}} &= \frac{1}{\varepsilon^2} l_2 (\phi - \hat{\phi}), \end{aligned} \tag{5.35}$$

where ε is a small positive parameter to be chosen, and l_i , $i = 1, 2$ are chosen so that

$$\bar{A} = \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix} \tag{5.36}$$

is Hurwitz. Noting the second equation of (5.34), the observer for v is designed as:

$$\begin{aligned}\dot{\hat{z}}_2 &= \hat{v} + \frac{l_1}{\varepsilon}(z_2 - \hat{z}_2) + z_1 \hat{w} \\ \dot{\hat{v}} &= \frac{l_2}{\varepsilon^2}(z_2 - \hat{z}_2),\end{aligned}\tag{5.37}$$

where $\hat{w}$ is from system (5.35).

5.4.2 Controller and Estimator Design

To implement the output feedback control with high-gain observers, we need to introduce saturation limits to the control and the rates of the parameter estimates mainly for two reasons. The first is to prevent the peaking phenomena of the high-gain observers from entering the robot dynamic system. For detailed description about peaking phenomena, please refer to [98]. The other is that the boundedness of the system signals is ensured by saturation. On the other hand we will show that after a short time from the operations of the observers and the estimators, the saturation will no longer take effect. This implies that the designed control signals and the rates of the parameter estimates will still remain within the saturation limits, independent of the presence of saturations.

We now define a compact set

$$\Xi = \{e : V_2 < \varrho\},\tag{5.38}$$

where V_2 is defined in (5.23), $e = \{x_e, y_e, \bar{\phi}_e, \tilde{\omega}, \tilde{H}, \tilde{m}, \tilde{n}\}$ and ρ is a constant defined as

$$\varrho = h_{\max} + \Delta_1,$$

where Δ_1 is a positive constant to be chosen such that $V_2(0) < \varrho$, and $h_{\max}$ is

defined as

$$h_{\max} = \frac{1}{2} \tilde{H}_{\max}^T \Lambda \tilde{H}_{\max} + \frac{1}{2} \alpha \tilde{m}_{\max}^2 + \frac{1}{2} \beta \tilde{n}_{\max}^2$$

From Lemma 5.1, $\tilde{H}_i$, $1 \leq i \leq 6$, $\tilde{m}$ and $\tilde{n}$ are bounded. Denote $\tilde{H}_{i\max}$, $\tilde{m}_{\max}$, $\tilde{n}_{\max}$ as the maximum values of $\tilde{H}_i$, $\tilde{m}$ and $\tilde{n}$ respectively. Since τ_i^{state} in (5.26), $i = 1, 2$, is bounded and smooth in this compact set, let

$$S_i = \max_{e \in \Xi} |\tau_i^{\text{state}}(\eta, \eta_r, \hat{\theta}, v, w)| + \Delta_2, \quad (5.39)$$

where $\eta = [x, y, \phi]^T$, $\eta_r = [x_r, y_r, \phi_r, v_r, w_r]^T$, and Δ_2 is a small positive constant to be chosen. From (5.16), (5.25), (5.26), we know S_i is computable.

The final saturated output feedback control inputs are designed as

$$\tau_i = \text{sat}_{S_i}(\tau_i^{\text{state}}(\eta, \eta_r, \hat{\theta}, \hat{v}, \hat{w})), \quad (5.40)$$

where the unknown v and w are replaced by $\hat{v}$ and $\hat{w}$ respectively and the saturation function is defined as

$$\text{sat}_{S_i}(\tau_i^{\text{state}}) = \begin{cases} \text{sign}(\tau_i^{\text{state}}) S_i & |\tau_i^{\text{state}}| \geq S_i \\ \tau_i^{\text{state}} & |\tau_i^{\text{state}}| < S_i \end{cases} \quad (5.41)$$

Now we consider the design of estimators only involving output and observer signals. Define $\tilde{v} = \hat{v} - v_c$, $\tilde{w} = \hat{w} - w_c$, $\tilde{\varpi} = \hat{\varpi} - \varpi_c$. For the same reason as to the controllers, the rates of estimators will also be constrained by saturations in a similar way. The estimators for $\hat{\theta}_i$, $i = 1, \dots, 8$, is designed as follows

$$\dot{\hat{\theta}}_i = \text{sat}_{S_{\hat{\theta}_i}}(g_{\hat{\theta}_i}(\hat{v}, \hat{w})), \quad (5.42)$$

where

$$g_{\hat{\theta}_i}(\hat{v}, \hat{w}) = \begin{cases} g_i(\hat{v}, \hat{w}) & \text{if } \theta_{i \min} < \hat{\theta}_i < \theta_{i \max} \\ g_i(\hat{v}, \hat{w}) + (1 - \frac{\hat{\theta}_i}{\theta_{i \min}})^2 [1 + g_i^2(\hat{v}, \hat{w})] & \text{if } \hat{\theta}_i \leq \theta_{i \min} \\ g_i(\hat{v}, \hat{w}) - (1 - \frac{\hat{\theta}_i}{\theta_{i \max}})^2 [1 + g_i^2(\hat{v}, \hat{w})] & \text{if } \hat{\theta}_i \geq \theta_{i \max} \end{cases}$$

$$S_{\hat{\theta}_i} = \max_{e \in \Xi} |\dot{\hat{\theta}}_{i \text{state}}(v, w)| + \Delta_2.$$

$g_i(\hat{v}, \hat{w})$ is in the same form as $g_i(v, w)$ except that the unknown v and w are replaced by $\hat{v}$ and $\hat{w}$ respectively; $\text{sat}_{S_{\hat{\theta}_i}}(\cdot)$ is defined similarly to that of (5.41);

5.4.3 Stability Analysis

In this section we will give the stability analysis of the closed-loop system and show that the tracking errors are in an arbitrarily small ball eventually. This is roughly done in three steps. In the first step we prove that after a short time the saturation for controller and estimators will not take effect. Secondly we show that Ξ is an invariant set, which in turn implies the stability of the whole system. Finally we will establish that the tracking errors are in an arbitrarily small ball eventually.

Re-write (5.5) as follows

$$\begin{aligned} \dot{x} &= v \cos \phi \\ \dot{y} &= v \sin \phi \\ \dot{\phi} &= w \\ \dot{v} &= \alpha_1 w^2 + \alpha_2 v + \alpha_3 w + \alpha_4 \tau_1(\eta, \eta_r, \hat{\theta}, \hat{v}, \hat{w}) + \alpha_5 \tau_2(\eta, \eta_r, \hat{\theta}, \hat{v}, \hat{w}) \\ \dot{w} &= \alpha_6 v w + \alpha_7 v + \alpha_8 w + \alpha_9 \tau_1(\eta, \eta_r, \hat{\theta}, \hat{v}, \hat{w}) + \alpha_{10} \tau_2(\eta, \eta_r, \hat{\theta}, \hat{v}, \hat{w}), \end{aligned} \quad (5.43)$$

where τ_1 and τ_2 are from (5.40), and a_i , $i = 1, \dots, 10$ are unknown but bounded

constants given as

$$\begin{aligned}\alpha_1 &= -\frac{H(3)}{H(1)}, \alpha_2 = -\frac{H(4)}{H(1)}, \alpha_3 = -\frac{H(5)}{H(1)}, \alpha_4 = \alpha_5 = \frac{1}{mH(1)} \\ \alpha_6 &= \frac{H(3)}{H(2)}, \alpha_7 = -\frac{H(5)}{H(2)}, \alpha_8 = -\frac{H(6)}{H(2)}, \alpha_9 = -\alpha_{10} = \frac{1}{nH(2)}.\end{aligned}$$

Define

$$\varphi_1 = \frac{1}{\varepsilon}(\phi - \hat{\phi}), \quad \varphi_2 = (w - \hat{w}) \quad (5.44)$$

and denote $\varphi = [\varphi_1, \varphi_2]^T$. Then from (5.34) and the third and fifth equations of (5.43), we get

$$\varepsilon \dot{\varphi} = \bar{A}\varphi + \varepsilon \begin{bmatrix} 0 \\ G(v, w, \hat{v}, \hat{w}, \eta, \eta_r, \hat{\theta}) \end{bmatrix}, \quad (5.45)$$

where

$$G(v, w, \hat{v}, \hat{w}, \eta, \eta_r, \hat{\theta}) = \alpha_6 vw + \alpha_7 v + \alpha_8 w + \alpha_9 \tau_1 + \alpha_{10} \tau_2.$$

Define

$$\chi = [\chi_1 \quad \chi_2]^T \quad (5.46)$$

where $\chi_1 = \frac{1}{\varepsilon}(z_2 - \hat{z}_2)$ and $\chi_2 = v - \hat{v}$. We also have

$$\varepsilon \dot{\chi} = \bar{A}\chi + \begin{bmatrix} z_1 \varphi_2 \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ H(v, w, \hat{v}, \hat{w}, \eta, \eta_r, \hat{\theta}) \end{bmatrix}, \quad (5.47)$$

where

$$H(v, w, \hat{v}, \hat{w}, \eta, \eta_r, \hat{\theta}) = \alpha_1 w^2 + \alpha_2 v + \alpha_3 w + \alpha_4 \tau_1 + \alpha_5 \tau_2. \quad (5.48)$$

Lemma 5.2. *When the system is controlled by the saturated controller (5.40) with estimator (5.42), φ defined in (5.44) is globally bounded.*

Proof: Consider a Lyapunov function $V_\varpi = \frac{1}{2}\varpi^T R \varpi$. Then from (5.2) and

(5.8) we get

$$\dot{V}_\varpi \leq -2 \frac{(\lambda_{\min}(D_R) - \frac{\varsigma}{2})}{\lambda_{\max}(R)} V_\varpi + \frac{1}{2\varsigma} \tau_b^T \tau_b$$

where $\tau_b = B_R \tau$ and ς is a constant chosen such that $\lambda_{\min}(D_R) - \frac{\varsigma}{2} > 0$. Since the torque τ_i , $i = 1, 2$ is saturated, so τ_b is also bounded. Thus v and w are globally bounded. Also from the boundedness of τ_i , there exists a positive constant g so that

$$|G(t, v, w, \hat{w}, \hat{\theta})| < g$$

Since $\bar{A}$ is a Hurwitz matrix, (5.45) is a stable system and φ is bounded. $\square$

Lemma 5.3. *For any ε , there exists a finite time $T_1(\varepsilon)$, so that for $t > T_1$, $\|\varphi\| < c_1 \varepsilon$ where c_1 is a positive constant that is computable.*

Proof: From **Lemma 5.2** we know φ is globally bounded. Define a positive Lyapunov function $\nu = \varphi^T P \varphi$ where matrix $P = [p_{11} \ p_{12}; p_{21} \ p_{22}]$ is the solution of Lyapunov equation

$$\bar{A}^T P + P \bar{A} = -I$$

Then

$$\begin{aligned} \dot{\nu} &\leq -\frac{1}{\varepsilon} \varphi^T \varphi + \varphi^T \left(\begin{bmatrix} (|p_{12}| + |p_{21}|)g & 2|p_{22}|g \end{bmatrix}^T \right) \\ &\leq -\frac{\gamma}{\varepsilon} \nu + G_3(|\varphi_1| + |\varphi_2|), \end{aligned}$$

where $\gamma = 1/\lambda_{\max}(P)$ and $G_3 = \max\{g(|p_{12}| + |p_{21}|), 2g|p_{22}|\}$. It gives that

$$\dot{\nu} \leq -\frac{\gamma}{\varepsilon} \nu + \iota \sqrt{\nu}$$

where $\iota = \frac{\sqrt{2}G_3}{\sqrt{\lambda_{\min}(P)}}$. Now we consider two cases.

Case 1: For $\nu(t) \geq \frac{4\iota^2}{\gamma^2}\varepsilon^2$, we obtain

$$\dot{\nu} \leq -\frac{\gamma}{2\varepsilon}\nu \quad (5.49)$$

which results in that $\nu(t) \leq \nu(0)e^{-\frac{\gamma}{2\varepsilon}t}$. For any bounded initial conditions for ϕ , $\hat{\phi}$, w and $\hat{w}$, and from the definition of φ in (5.44), $\nu(0)$ can be written as $\nu(0) = \frac{l_3}{\varepsilon^2}$ where $l_3 = p_{11}s^2 + (p_{21} + p_{12})\varepsilon sq + p_{22}q^2\varepsilon^2$ with $s = \phi(0) - \hat{\phi}(0)$ and $q = w(0) - \hat{w}(0)$. Thus we get $\nu(t) \leq \frac{l_3}{\varepsilon^2}e^{-\frac{\gamma}{2\varepsilon}t}$.

So when the initial condition $\frac{l_3}{\varepsilon^2}$ satisfies that $\frac{l_3}{\varepsilon^2} > \frac{4\iota^2}{\gamma^2}\varepsilon^2$, the time $T_1(\varepsilon)$ for ν to reach the ball $\nu \leq \frac{4\iota^2}{\gamma^2}\varepsilon^2$ from $\nu(0)$ is

$$T_1(\varepsilon) = \frac{2\varepsilon}{\gamma} \left(\ln \frac{l_4}{\varepsilon^4} \right) \quad (5.50)$$

where $l_4 = \frac{l_3\gamma^2}{4\iota^2}$. Since (5.50) tends to zero as $\varepsilon \rightarrow 0$, so we can choose a sufficiently small $T_1(\varepsilon)$ such that $\forall t > T_1(\varepsilon)$, we have $\nu \leq \frac{4\iota^2}{\gamma^2}\varepsilon^2$, which means $\|\varphi\| \leq c_1\varepsilon$,

where $c_1 = \sqrt{\frac{4\iota^2}{\gamma^2\lambda_{\min}(P)}}$.

Case 2: When $\nu(t) \leq \frac{4\iota^2}{\gamma^2}\varepsilon^2$, the conclusion holds automatically for any $T_1(\varepsilon) \geq 0$. $\square$

Lemma 5.4. *There is a T_3 such that for $t \in [0, T_3)$, $e(t) \in \Xi$ if $e(0) \in \Xi$. Also there exists a $T_2(\varepsilon)$, which can be arbitrarily small if ε is small enough, such that for $t \in [T_2(\varepsilon), T_3)$, $\|\chi\| \leq c_2\varepsilon$.*

Proof: From (5.24) we get

$$\begin{aligned} \dot{V}_2 &= \dot{V}_0 + \tilde{\omega}^T [-(\bar{C}(w) + D_R)\tilde{\omega} + B_R\tau - Y_c H] \\ &\quad + \tilde{H}^T \Lambda \dot{\hat{H}} + \alpha \tilde{m} \dot{\hat{m}} + \beta \tilde{n} \dot{\hat{n}} \end{aligned} \quad (5.51)$$

With saturation implemented for the torque τ and the rates of estimates $\dot{\hat{H}}$, $\dot{\hat{m}}$,

$\dot{n}$, there is a positive constant c_4 such that $\dot{V}_2 < c_4$ when $V_2 \leq \varrho$. Note that $V_2(t) = V_2(0) + \int_0^t \dot{V}_2 dt < V_2(0) + \int_0^t c_4 dt$ and $V_2(0) < \varrho$. Suppose T_3 is the first time that e exits Ξ , i.e. $V_2(T_3) = \varrho$. Then we get

$$T_3 > \frac{\varrho - V_2(0)}{c_4}$$

and the first part of the lemma is proved.

For $t \in (0, T_3)$, $|x_e| < \sqrt{\varrho^2 + 2\varrho}$ and $|y_e| < \sqrt{\varrho^2 + 2\varrho}$ from the definition of V_2 . From (5.9), $|x| < |x_r| + |x_e| + |y_e|$ and $|y| < |y_r| + |x_e| + |y_e|$. Finally from Assumption 5.1 we get

$$|z_1| = |y \cos \phi - x \sin \phi| < |x| + |y| < 2a^* + 4\sqrt{\varrho^2 + 2\varrho}$$

Following the same proof as in Lemma 5.2 and Lemma 5.3, from (5.47) we know for $t \in (0, T_3)$, χ defined in (5.46) is bounded and there exist a computable positive constant c_2 , and a $T_2(\varepsilon)$, which can be arbitrarily small if ε is small enough, such that for $t \in [T_2(\varepsilon), T_3)$, $\|\chi\| \leq c_2\varepsilon$. $\square$

Let $T = \max(T_1(\varepsilon), T_2(\varepsilon))$. From (5.50) we can find an ε^* to ensure

$$\max(T_1(\varepsilon^*), T_2(\varepsilon^*)) < T_3. \quad (5.52)$$

Then for all $\varepsilon < \varepsilon^*$, it can be guaranteed that $T < T_3$.

Now we will show that the “output-feedback” parameter estimators also have those properties stated in Lemma 5.1 for $t \in [T, T_3)$.

Lemma 5.5. *For $t \in [T, T_3)$, the estimator (5.42) has the following properties:*

(a) $\dot{\hat{\theta}}_i$, $\dot{n}$ and $\dot{H}_i$ are all continuous. (b) Estimates $\hat{\theta}_i$ satisfies $\bar{\theta}_{i\min} \leq \hat{\theta}_i \leq \bar{\theta}_{i\max}$ for all $t \geq 0$ if $\bar{\theta}_{i\min} < \hat{\theta}_i(0) < \bar{\theta}_{i\max}$. (c) $-\tilde{m}(\dot{m} + u_1\tilde{v}) - \tilde{n}(\dot{n} + \hat{u}_2\tilde{w}) - \tilde{H}^T(\dot{H} + \Lambda^{-1}Y_c^T\tilde{\omega}) \leq 0$ for all $t \geq 0$.

Proof: As $\hat{\theta}_i(T) = \hat{\theta}_i(0) + \int_0^T \dot{\hat{\theta}}_i dt$ and the rates of the estimates are saturated,

thus for $\bar{\theta}_{i\min} < \hat{\theta}_i(0) < \bar{\theta}_{i\max}$, an ε can be chosen to also guarantee

$$T \leq \min_{i=1,\dots,8} \left\{ \frac{\min(\bar{\theta}_{i\max} - \hat{\theta}_i(0), \hat{\theta}_i(0) - \bar{\theta}_{i\min})}{|S_{\hat{\theta}_i}|} \right\} \quad (5.53)$$

This will ensure

$$\bar{\theta}_{i\min} < \hat{\theta}_i(T) < \bar{\theta}_{i\max}. \quad (5.54)$$

Now from (5.25) and (5.26), there exist $\rho_{1i}(\eta, \eta_r, \hat{\theta})$ and $\rho_{2i}(\eta, \eta_r, \hat{\theta})$ such that

$$\tau_i^{\text{state}}(\eta, \eta_r, \hat{\theta}, v, w) = \bar{\tau}_i^{\text{state}}(\eta, \eta_r, \hat{\theta}) + \rho_{1i}v + \rho_{2i}w \quad (5.55)$$

where $\bar{\tau}_i^{\text{state}}(\eta, \eta_r, \hat{\theta})$ is the part that does not contain v and w . From (5.12), (5.16), (5.18), (5.25) and (5.26), $\rho_{1i}(t)$ and $\rho_{2i}(t)$ are bounded. Since x_e , y_e and $\bar{\phi}_e$ are normalized, it can also be checked that the bounds of ρ_{1i} and ρ_{2i} are computable. So $\tau_i^{\text{state}}(\eta, \eta_r, \hat{\theta}, v, w)$ and its output-feedback counterpart $\tau_i^{\text{state}}(\eta, \eta_r, \hat{\theta}, \hat{v}, \hat{w})$ have the following relationship for all $t \in [T, T_3]$

$$\begin{aligned} \tau_i^{\text{state}}(\eta, \eta_r, \hat{\theta}, \hat{v}, \hat{w}) &= \bar{\tau}_i^{\text{state}}(\eta, \eta_r, \hat{\theta}) + \rho_{1i}\hat{v} + \rho_{2i}\hat{w} \\ &< \tau_i^{\text{state}}(\eta, \eta_r, \hat{\theta}) + \rho_{1i}v + \rho_{2i}w \\ &\quad + \rho_{1i}c_2\varepsilon + \rho_{2i}c_1\varepsilon \\ &< \tau_i^{\text{state}}(\eta, \eta_r, \hat{\theta}, v, w) + c_3\varepsilon \end{aligned} \quad (5.56)$$

where c_3 is the bound of $\rho_{1i}c_2 + \rho_{2i}c_1$.

Note that parameter ε can be chosen so that

$$c_3\varepsilon < \Delta_2. \quad (5.57)$$

Thus for all $t \in [T, T_3]$, we have

$$\tau_i^{\text{state}}(\eta, \eta_r, \hat{\theta}, v, w) < \max_{e \in \Xi} |\tau_i^{\text{state}}(\eta, \eta_r, \hat{\theta}, v, w)|.$$

So from (5.39), (5.56) and (5.57) we know that during the interval $[T, T_3]$ the

saturation will not take effect because $\tau_i^{\text{state}}(\eta, \eta_r, \hat{\theta}, \hat{v}, \hat{w}) < S_i$.

With the same analysis, no saturation will take effect for the rates of all the parameter estimates during this time interval either. Then by noting (5.54) and following similar analysis in the proof of Lemma 5.1, the same properties are established. $\square$

Next we will show that T_3 in Lemma 5.4 is ∞ , namely e will never exit Ξ and thus Ξ is an invariant set.

Lemma 5.6. *The set Ξ is an invariant set.*

Proof: From Lemma 5.4, (5.52) and (5.53), $e(t) \in \Xi$ for $t \in [0, T]$ if $e(0) \in \Xi$. Now defining $\tilde{\omega} = [\tilde{v} \ \tilde{w}]^T$, we get $\dot{V}_2$ as follows

$$\begin{aligned} \dot{V}_2 = & -k_1 \frac{x_e^2}{\Gamma_1^2} - \frac{k(t)v_r y_e^2}{\Gamma_1^2} - k_2 \frac{\bar{\phi}_e^2}{\Gamma_3^2} + \frac{y_e f_2}{\Gamma_1} - \tilde{\omega}^T (D_R + K) \tilde{\omega} \\ & - \alpha \tilde{m}(\dot{\hat{m}} + \hat{u}_1 \tilde{v}) - \beta \tilde{n}(\dot{\hat{n}} + \hat{u}_2 \tilde{w}) - \tilde{H}^T \Lambda(\dot{\hat{H}} \\ & + \Lambda^{-1} \hat{Y}_c^T \tilde{\omega}) + \alpha \tilde{m}[\hat{u}_1 \tilde{v} - u_1 \tilde{v}] + \beta \tilde{n}[\hat{u}_2 \tilde{w} - u_2 \tilde{w}] \\ & + \tilde{H}^T [\hat{Y}_c^T \tilde{\omega} - Y_c \tilde{\omega}] + \tilde{\omega}^T (\hat{Y}_c - Y_c) \hat{H} \end{aligned}$$

for $t \in [T, T_3)$. According to Lemma 5.5, $\tilde{m}$, $\tilde{n}$, $\tilde{H}$, $\hat{m}$, $\hat{n}$ and $\hat{H}$ are bounded, and

$$-\alpha \tilde{m}(\dot{\hat{m}} + \hat{u}_1 \tilde{v}) - \beta \tilde{n}(\dot{\hat{n}} + \hat{u}_2 \tilde{w}) - \tilde{H}^T (\Lambda \dot{\hat{H}} + \hat{Y}_c^T \tilde{\omega}) \leq 0.$$

Note that $\chi_2 = \tilde{v} - \check{v}$ and $\varphi_2 = \tilde{w} - \check{w}$, it is easy to check that $\hat{u}_1 \tilde{v} - u_1 \tilde{v}$, $\hat{u}_2 \tilde{w} - u_2 \tilde{w}$, $(\hat{Y}_c - Y_c) \hat{H}$ and $\hat{Y}_c^T \tilde{\omega} - Y_c \tilde{\omega}$ are polynomials of χ_2 and φ_2 of degree 2 without a constant term. Since $|\varphi_2| < c_1 \varepsilon$ and $|\chi_2| < c_2 \varepsilon$ from Lemmas 5.3 and 5.4, we have

$$\dot{V}_2 \leq -\frac{k_1 x_e^2}{\Gamma_1^2} - \frac{k_3 y_e^2}{\Gamma_1^2} - \frac{k_4 \bar{\phi}_e^2}{\Gamma_3^2} - \tilde{\omega}^T (D_R + K) \tilde{\omega} + k_6 \varepsilon \quad (5.58)$$

for $t \in [T, T_3)$, where k_6 is a computable positive constant.

Figure 5.1: Ξ is divided to Ξ_1 and Ξ_2 .

Define $\mathcal{E} = \{x_e, y_e, \bar{\phi}_e, \tilde{\omega}\}$ and a new set

$$\begin{aligned} \Psi = \{ \mathcal{E} : & \frac{k_3 y_e^2}{\Gamma_1^2} + \frac{k_1 x_e^2}{\Gamma_1^2} + \frac{k_4 \bar{\phi}_e^2}{\Gamma_3^2} \\ & + \tilde{\omega}^T (D_R + K) \tilde{\omega} \leq k_6 \varepsilon \}. \end{aligned} \quad (5.59)$$

Also denote ξ as

$$\xi = \max_{\mathcal{E} \in \Psi} V_1 \quad (5.60)$$

From (5.17) and (5.59), the design parameter ε can be also chosen to satisfy

$$\xi < \frac{1}{2} \Delta_1 \quad (5.61)$$

From the definitions of V_1 and Ψ this can be easily done. We now divide Ξ into two set Ξ_1 and Ξ_2 as follows

$$\Xi_1 = \{e : \mathcal{E} \in \Psi; e \in \Xi\} \quad \Xi_2 = \{e : \mathcal{E} \notin \Psi; e \in \Xi\}.$$

Clearly $\Xi = \Xi_1 \cup \Xi_2$ and $\Xi_1 \cap \Xi_2 = \emptyset$.

Ξ is divided so that when $e \in \Xi_1$, $V_2 < V_1 + h_{\max} < \frac{1}{2} \Delta_1 + h_{\max} = \varrho - \frac{1}{2} \Delta_1$, and when $e \in \Xi_2$, $\dot{V}_2 < 0$ from (5.58) and (5.59).

When $e \in \Xi_1$, $\dot{V}_2$ may be positive. But since $\dot{V}_2 < c_4$, $V_2(t)$ is continuous. Therefore if e exits Ξ_1 , it will only enter Ξ_2 , because $V_2(t) < \varrho - \frac{1}{2} \Delta_1$ when $e \in \Xi_1$. Thus when $\varrho - \frac{1}{2} \Delta_1 \leq V_2(t) < \varrho$, e is in Ξ_2 . In another word, e will not exit Ξ from Ξ_1 directly. Ξ_1 and Ξ_2 are shown in Fig. 5.1 schematically.

Note that T_3 defined in Lemma 5.4 is the first time that e exits Ξ satisfying $V_2(T_3) = \varrho$. Then $V_2(t - \Delta t) < \varrho$, where $0 < \Delta t < T_3$ is an arbitrarily small constant. As stated, $e(t - \Delta t) \in \Xi_2$, so $\dot{V}_2(t - \Delta t) < 0$ and thus as Δt decreases $V_2(t - \Delta t)$ is monotonically decreasing. This contradicts that $V_2(T_3) = \varrho$. So e will not exit Ξ from Ξ_2 either, i.e. $T_3 = \infty$. Therefore Ξ is an invariant set. $\square$

Since Ξ is an invariant set, then from Lemma 5.3 and Lemma 5.4,

$$\tau_i^{\text{state}}(\eta, \eta_r, \hat{\theta}, \hat{v}, \hat{w}) < S_i$$

for $t \in [T, \infty)$, which means the saturation will not take effect after T , and we have

$$\dot{V}_2 \leq -\frac{k_1 x_e^2}{\Gamma_1^2} - \frac{k_3 y_e^2}{\Gamma_1^2} - \frac{k_4 \bar{\phi}_e^2}{\Gamma_3^2} - \tilde{\omega}^T (D_R + K) \tilde{\omega} + k_6 \varepsilon \quad (5.62)$$

for $t \in [T, \infty)$. So based on Lemmas 5.2-5.6, we have the following theorem.

Theorem 5.1. *Consider the system consisting of a nonholonomic mobile robot (5.1) and (5.2), observers (5.35) and (5.37), the adaptive output feedback controllers τ in (5.40) and parameter estimators (5.42). Under **Assumptions** 5.1-5.3, and if $e(0) \in \Xi$, we can calculate an ε^* based on (5.52), (5.53), (5.57) and (5.61) so that for all $\varepsilon \in (0, \varepsilon^*)$, (i) the signals in the systems are bounded; (ii) the tracking errors are of $O(\varepsilon)$; (iii) the steady state of tracking errors x_e , y_e , ϕ_e and $\tilde{\omega}$ are within an arbitrarily small ball.*

Proof: (i) Since Ξ is an invariant set, thus all signals contained in Ξ are bounded. As shown in Lemma 5.3 and Lemma 5.4, φ and χ are bounded, thus all signals in the systems are bounded.

(ii) Define

$$\epsilon_1 = \frac{x_e}{\Gamma_1}, \epsilon_2 = \frac{y_e}{\Gamma_1}, \epsilon_3 = \frac{\bar{\phi}_e}{\Gamma_3} \quad (5.63)$$

and $\epsilon = [\epsilon_1, \epsilon_2, \epsilon_3, \tilde{\omega}^T]^T$. Choose an ε^* satisfying (5.52), (5.53), (5.57) and (5.61),

from (5.62) we have

$$\dot{V}_2 \leq -\epsilon^T Q \epsilon + k_6 \varepsilon \quad (5.64)$$

where

$$\bar{K} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_3 & 0 \\ 0 & 0 & k_4 \end{bmatrix}, \quad Q = \begin{bmatrix} \bar{K} & 0 \\ 0 & D_R + K \end{bmatrix}.$$

Since V_2 is bounded and Q is positive definite, we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \epsilon^T \epsilon \leq k_7 \varepsilon \quad (5.65)$$

where $k_7 = k_6/\lambda_{\min}(Q)$, which shows that the mean-square of errors $\epsilon_1, \epsilon_2, \epsilon_3$ and $\tilde{\omega}$ are of the order of $O(\varepsilon)$ as in [97]. It can be further proved that the mean-square of tracking errors x_e, y_e, ϕ_e and $\tilde{\omega}$ are also of the order of $O(\varepsilon)$.

(iii) Since $\dot{V}_2 \leq -\epsilon^T Q \epsilon + k_6 \varepsilon \leq -\lambda_{\min}(Q)\epsilon^T \epsilon + k_6 \varepsilon$, so ϵ will finally reach the ball defined by

$$\epsilon^T \epsilon \leq k_7 \varepsilon, \quad (5.66)$$

From (5.64) we can see that $x_e, y_e, \bar{\phi}_e, \tilde{\omega}$ will finally reach the ball defined by

$$x_e^2 + y_e^2 + \bar{\phi}_e^2 + \tilde{\omega}^T \tilde{\omega} \leq \left(\frac{2k_7}{1 - k_7 \varepsilon} + k_7 \right) \varepsilon \quad (5.67)$$

whose radius can be arbitrarily small since the design parameter ε can be arbitrarily small. Finally, note that $\phi_e = \bar{\phi}_e - \arcsin(\lambda v_r \epsilon_2)$. So ϕ_e is also in an arbitrarily small ball. $\square$

Remark 5.3. For adaptive state-feedback tracking control of nonholonomic mobile robots, system stability is normally established under the persistent exciting (**PE**) condition, see for example, [1, 2, 10]. However, in order to achieve an extra tracking performance of small steady state tracking errors, in the sense that

the tracking errors will eventually enter a given arbitrarily small ball and remain inside it, we need guarantee $\lambda_{\min}(Q) > \ell$ where ℓ is a positive constant. This is ensured by Assumption 5.2.

Remark 5.4. The keys for solving the adaptive output feedback control of the nonholonomic mobile robots lie on several aspects. Firstly observers for v and w are designed separately to avoid the nonholonomic constraint. Secondly saturation is introduced to the output feedback controller and estimators. This will prevent the peaking phenomenon from entering the robot system. Thirdly a compact set Ξ of e is set up. The stability of the whole system is guaranteed by proving Ξ is an invariant set.

Finally we summarize the overall design procedures in Table 5.4.3 and Table 5.4.3.

5.5 Simulation Results

In this section, we present our simulation studies to illustrate our design scheme and verify the obtained theoretical results. The physical parameters of robots are given as: $b = 0.75$, $d = 0.3$, $r = 0.25$, $m_c = 10$, $m_w = 1$, $I_c = 5.6$, $I_w = 0.005$, $I_m = 0.0025$, $d_{11} = d_{22} = 1$. The initial estimates for the unknown parameters are set to be 60% of their true values. Based on the condition in Proposition 5.1, the following parameters are selected: $k_2 = 5$, $k_4 = 3$, $\lambda = 0.2$, $K_1 = 0$, $K_2 = 0$. From (5.52), (5.57) and (5.61), ε^* is calculated to be 0.021. We then choose Δ_1 , Δ_2 , ϱ , l_1 , l_2 , ε and Λ as 10, 1, 220, 10, 10, 0.001 and $100I$ respectively. The saturation bound for τ_i , $\hat{m}$, $\hat{n}$ and $\hat{H}_i$ are chosen as 420, 10, 10 and 20 respectively. The maximum and minimum values for $\hat{m}$, $\hat{n}$, $\hat{H}(1)$ to $\hat{H}(6)$ are $\{(5, 20), (5, 20), (-2, 1), (-2, 1), (-2, 1), (5, 10), (-5, 5), (2, 5)\}$ respectively. The initial values for v , w and $\hat{v}$, $\hat{w}$ are set as $(0, 0)$. We use a circle as the reference. The initial position is $(2.3, 2)$, and the reference velocities are set as $v_r = 2.5$, $w_r = \frac{\pi}{2}$.

Simulation results are plotted in Fig.5.2-Fig.5.5. Fig.5.2 shows the tracking errors of the robot in (x, y) plane. Fig.5.3 shows the time evolution of tracking errors x_e , y_e and ϕ_e . It can be seen from the figures that the reference tracking errors

Table 5.4.3: Summary of output feedback control for nonholonomic mobile robots.

State-feedback controller: $U = -K\tilde{\omega} + Y_c\hat{H} - \Omega$ $\tau^{\text{state}}(\eta, \eta_r, \hat{\theta}, v, w) = [\frac{1}{2}\hat{m} \quad \frac{1}{2}\hat{n}; \frac{1}{2}\hat{m} \quad -\frac{1}{2}\hat{n}]U$
Observer: $\dot{\hat{\phi}} = \hat{w} + \frac{1}{\varepsilon}l_1(\phi - \hat{\phi})$ $\dot{\hat{w}} = \frac{1}{\varepsilon^2}l_2(\phi - \hat{\phi})$ $\dot{\hat{z}}_2 = \hat{v} + \frac{l_1}{\varepsilon}(z_2 - \hat{z}_2) + z_1\hat{w}$ $\dot{\hat{v}} = \frac{l_2}{\varepsilon^2}(z_2 - \hat{z}_2)$
An invariant set: $\Xi = \{e : V_2 < \rho\}$
Output feedback controller: $S_i = \max_{e \in \Xi} \tau_i^{\text{state}}(\eta, \eta_r, \hat{\theta}, v, w) + \Delta_2$ $\tau_i = \text{sat}_{S_i}(\tau_i^{\text{state}}(\eta, \eta_r, \hat{\theta}, \hat{v}, \hat{w}))$

Table 5.4.3: Summary of parameter update laws.

Parameter estimator:

$$\begin{aligned}\dot{\hat{\theta}}_i &= \text{sat}_{S_{\hat{\theta}_i}} \left(g_{\hat{\theta}_i}(\hat{v}, \hat{w}) \right), \\ S_{\hat{\theta}_i} &= \max_{e \in \Xi} |\dot{\hat{\theta}}_{i \text{ state}}(v, w)| + \Delta_2. \\ g_{\hat{\theta}_i}(\hat{v}, \hat{w}) &= \begin{cases} g_i(\hat{v}, \hat{w}) & \text{if } \theta_{i \min} < \hat{\theta}_i < \theta_{i \max} \\ g_i(\hat{v}, \hat{w}) + \left(1 - \frac{\hat{\theta}_i}{\theta_{i \min}}\right)^2 [1 + g_i^2(\hat{v}, \hat{w})] & \text{if } \hat{\theta}_i \leq \theta_{i \min} \\ g_i(\hat{v}, \hat{w}) - \left(1 - \frac{\hat{\theta}_i}{\theta_{i \max}}\right)^2 [1 + g_i^2(\hat{v}, \hat{w})] & \text{if } \hat{\theta}_i \geq \theta_{i \max} \end{cases}\end{aligned}$$

Figure 5.2: Robot position in (x, y) plane.

are small, which is consistent with Theorem 5.1. Fig.5.4. shows the torque and it is observed that the saturation does not take effect after a short time. Fig.5.5 shows the estimates and the peaking phenomenon of the high-gain observers when estimating v and w . It can be seen that due to parameter projection the estimates are within the given bounds.

5.6 Conclusions

In this chapter, a high-gain observer based adaptive output feedback tracking control design scheme for nonholonomic mobile robots is proposed. To overcome the difficulty due to the nonholonomic constraint and quadratic velocity term, we propose a new adaptive control scheme including designing a new adaptive

Figure 5.3: Tracking errors x_e , y_e and ϕ_e . Figure 5.4: Control torque τ_i , $i = 1, 2$.

Figure 5.5: Left: The parameter estimators. Right: Velocities estimate error and peaking phenomenon.

state feedback controller and two high gain observers to estimate the unknown linear and angular velocities v and w respectively. The tracking errors can be confined to an arbitrarily small ball by choosing an appropriate design parameter ε . Simulation results validate obtained theoretical results. It should be noted that the observers can be extended to other underactuated mechanical systems, for example, underactuated ships.

Chapter 6

Adaptive Output-Feedback of Underactuated Ships

In this chapter, both state feedback and output feedback robust adaptive controllers for an underactuated ship are proposed to make the ship follow a pre-defined path with guaranteed transient performance and arbitrarily small steady state position error and orientation error, despite the presences of environmental disturbances induced by wave, wind and ocean current and unknown system parameters. By using the prescribed performance bound technique, the position error and orientation error can be guaranteed converging to arbitrarily small residual sets at a pre-specified exponential rate. It is shown that the mean-square tracking errors are of order of an arbitrarily small design parameter, and the position error and the orientation error also decrease at a pre-specified exponential rate.

6.1 Introduction

The control of underactuated systems with nonholonomic constraints has received vast attention over the past few decades. Typical examples of this kind of systems include nonholonomic mobile robots, underactuated ships, underwater vehicles and VCTOL aircrafts *et al.* Various control schemes are proposed for the path following control of underactuated ships. In [13] the method developed for chain-

form systems was adopted for underactuated ship through a coordinate transformation to steer both the position variable and the course angle of the ship, providing exponential stability for the tracking of the reference trajectory. In [99] by introducing a first-order sliding surface in terms of surge tracking errors and a second-order surface in terms of lateral motion tracking errors, a sliding-mode control law is proposed for trajectory tracking of underactuated ships. In [100] coordinated path-following controllers are designed for a group of underactuated ships to follow a reference path, where derivative of each path is left as a free input to synchronize the ships' motions. In [101] an artificial immune system (AIS)-based adaptive control of generator excitation systems for electric ships with dramatic changes of disturbances is presented to solve power quality problems caused by high-energy loads. In [17] a continuous time-varying tracking controller was designed to yield globally uniformly ultimately bounded tracking by transforming the ship tracking system into a skew-symmetric form and designing a time-varying dynamic oscillator. In [79] a simple state-feedback control law was developed to render the tracking error globally $\mathcal{K}$ -exponentially stable. In [18] a global robust adaptive controller was proposed to force an underactuated ship to follow a reference path under both constant and time-varying disturbances. In [21] a globally tracking controller was developed under the case that mass and damping matrices are not assumed to be diagonal and it was not required that the reference trajectory be generated by a ship model.

All the control schemes proposed above need the full information of the velocities. As most marine vessels only contain position sensors, the velocities in yaw, surge and sway directions are not available for feedback. This fact urges us to propose output feedback control schemes for the underactuated ships. In [22] through nonlinear coordinate changes to transform the ship dynamics to a system affine in the ship velocities, observers are designed to globally exponentially estimate unmeasured velocities. Related similar output feedback tracking control schemes are proposed in [92] for 1-DOF Lagrange systems or in [93] for multiple-DOF Lagrangian systems, where the observer design are based on a global nonlinear coordinate transformation to overcome the quadratic terms of the unmeasured

states.

It should be pointed out that the observer design methods mentioned above need the full information of the system parameters. With unknown system parameters, several output feedback tracking control schemes are proposed. In [20] a nonlinear model-based adaptive output feedback controller was proposed for global asymptotic tracking in the presence of parametric uncertainties associated with nonlinear ship dynamics. In [102] and [68] robust adaptive output feedback controller was designed for dynamic position of surface vessel. However the ship model considered in these references is a fully-actuated surface ship. The main difficulty encountered lies on the simultaneous appearance of nonholonomic constraint, unknown system parameters and the coriolis matrix, which results in quadratic cross terms involving unmeasured states. With this in mind, the observer design and the parameter estimator design are intertwined, which makes the analysis of the resulting closed loop system difficult and challenging. As far as we know, there is still no result in literature concerning the control of under-actuated ships when all system parameters are unknown and the yaw, surge and sway velocities are not available. Also the existing results guarantee that the tracking errors converge to zero only when time approaches infinity. So far there is no results considering transient performance. Note that, even though transient performance is difficult and challenge to be addressed in adaptive control, it is an important specification in controller design that cannot be ignored. For instance, sometimes it is crucial for a ship to follow a prescribed path as soon as possible, to avoid the reef for example.

In this chapter, we address such issues by designing an adaptive controllers for the underactuated ships. Firstly, if all the states are available, we design adaptive state feedback controllers and state dependent parameter estimators to achieve path following control. Secondly when the yaw, surge and sway velocities are unavailable, we design three high-gain observers to estimate these velocities. Thirdly, we replace the states in the state-feedback controllers and the parameter estimators with the estimated velocities from the observers. Due to their ability of rejecting disturbance, it is shown that the estimation errors generated by

the high-gain observers are of order $O(\epsilon)$, where ϵ is a design parameter. This in turn enables the tracking errors of underactuated ship, including the position error and the orientation error, to be made arbitrary small. Our proposed controller is shown performing effectively in tracking control of underactuated ships by simulation studies. Also by incorporating the prescribed performance bound technique [106], the designed controllers guarantee that the position error and the orientation error converge to arbitrarily small values with convergence rates faster than any pre-specified exponential speed, that is a specification for transient performance.

6.2 Problem Formulation

6.2.1 System Model and Control Objective

The motion of the ship can be described by the following differential equations, see for example [18]:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ r \end{bmatrix} \quad (6.1)$$

$$\begin{aligned} \dot{u} &= \frac{m_{22}}{m_{11}}vr - \frac{d_{11}}{m_{11}}u - \sum_{i=2}^3 \frac{d_{ui}}{m_{11}}|u|^{i-1}u + \frac{1}{m_{11}}\tau_u + \frac{1}{m_{11}}\tau_{wu}(t) \\ \dot{v} &= -\frac{m_{11}}{m_{22}}ur - \frac{d_{22}}{m_{22}}v - \sum_{i=2}^3 \frac{d_{vi}}{m_{22}}|v|^{i-1}v + \frac{1}{m_{22}}\tau_{wv}(t) \\ \dot{r} &= \frac{m_{11} - m_{22}}{m_{33}}uv - \frac{d_{33}}{m_{33}}r - \sum_{i=2}^3 \frac{d_{ri}}{m_{33}}|r|^{i-1}r + \frac{1}{m_{33}}\tau_r + \frac{1}{m_{33}}\tau_{wr}(t) \end{aligned} \quad (6.2)$$

where (x, y) denotes the coordinates of surface ship, ϕ is the heading angle of the ship; u , v and r denote the velocity in surge, sway and yaw respectively. τ_u and τ_r are surge force and yaw force respectively, which are considered as inputs. Since the sway force is not available, the system (6.1) and (6.2) are underactuated. The

Figure 6.1: General framework of ship path following

positive constants m_{jj} denote the ship inertia including adding mass, and d_{jj} , d_{ui} , d_{vi} , d_{ri} , $2 \leq i \leq 3$, $1 \leq j \leq 3$ represent the hydrodynamic damping coefficients in surge, sway and yaw respectively. These constants are unknown. The bounded time-varying terms, $\tau_{wu}(t)$, $\tau_{wv}(t)$ and $\tau_{wr}(t)$, are the environmental disturbances induced by the wave, wind and ocean-current. We assume that

$$|\tau_{wu}(t)| < \tau_{wu \max}, \quad |\tau_{wv}(t)| < \tau_{wv \max}, \quad |\tau_{wr}(t)| < \tau_{wr \max} \quad (6.3)$$

where $\tau_{wu \max}$, $\tau_{wv \max}$ and $\tau_{wr \max}$ are unknown positive constants.

Control Objective: For system (6.1) and (6.2), design controls τ_u and τ_r to force the above mentioned ship to follow a prescribed path set Ω as denoted by $(x_d(s), y_d(s))$ such that the position error and the orientation error will converge to arbitrarily small values with a pre-specified exponential convergence rate, where s is a path parameter as shown in Fig.6.1. Path $(x_d(s), y_d(s))$ may be regarded as generated by a “reference ship”.

The control objective is to be achieved with both state feedback and output feedback controls subject to the following assumptions.

Assumption 6.1. $x'_d(s)^2 + y'_d(s)^2 \geq \mu$ where μ is a positive constant, $x'_d(s) = \frac{\partial x_d(s)}{\partial s}$ and $y'_d(s) = \frac{\partial y_d(s)}{\partial s}$. The minimum radius of the osculating circle of the path is larger than or equal to the minimum possible turning radius of the ships.

Assumption 6.2. *The unknown parameters are in known convex compact sets.*

Assumption 6.3. $|x_d(s)| < a^*, |y_d(s)| < a^*$, where a^* is a known but arbitrary positive constant.

Remark 6.1.

- Assumption 6.1 implies that the path Ω is regular with respect to the path parameter s . If the path is not regular, it can be divided into several different regular paths for consideration separately.
- Theoretically a^* in Assumption 6.3 could be arbitrarily large to cover any practical reference path. Normally a commercial path is composed by several point-to-point segments where each segment has a limited distance, which can be governed by a^* .

We now define the following error variables [103]:

$$\begin{aligned} x_e &= x_d - x, \quad y_e = y_d - y \\ \phi_e &= \phi - \phi_d, \quad z_e = \sqrt{x_e^2 + y_e^2} \end{aligned} \quad (6.4)$$

where $\phi_d = \arcsin \frac{y_e}{z_e}$, variables ϕ_e and z_e are shown in Figure 6.1. Since the system modeled in (6.1) and (6.2) is underactuated, this brings difficulty to controller design. To overcome this, adaptive control schemes will be proposed to ensure that z_e and ϕ_e converge to arbitrarily small values with convergence rates faster than a pre-specified exponential speed. In addition, the dynamics in the sway direction will also be stabilized. Clearly these enable our control objective is achieved.

From (6.4), we obtain that

$$\sin(\phi_e) = \frac{x_e \sin(\phi) - y_e \cos(\phi)}{z_e}, \quad \cos(\phi_e) = \frac{x_e \cos(\phi) + y_e \sin(\phi)}{z_e}. \quad (6.5)$$

From (6.4), (6.5), and (6.1), the derivatives of z_e , ϕ_e are computed as

$$\begin{aligned}\dot{z}_e &= -\cos(\phi_e)u + \sin(\phi_e)v + \left(\frac{x_e}{z_e}x'_d + \frac{y_e}{z_e}y'_d\right)\dot{s} \\ \dot{\phi}_e &= r + \left(\frac{\sin(\phi)}{z_e \cos(\phi_e)} - \frac{x_e \tan(\phi_e)}{z_e^2}\right)(x'_d \dot{s} - u \cos(\phi) + v \sin(\phi)) \\ &\quad - \left(\frac{\cos(\phi)}{z_e \cos(\phi_e)} + \frac{y_e \tan(\phi_e)}{z_e^2}\right)(y'_d \dot{s} - u \sin(\phi) - v \cos(\phi))\end{aligned}\quad (6.6)$$

where $x'_d = \frac{\partial x_d}{\partial s}$ and $y'_d = \frac{\partial y_d}{\partial s}$.

6.2.2 Performance Characterization and System Transformation

From (6.6), it can be seen that the system will be un-defined if $z_e = 0$ and $\phi_e = \pm \frac{\pi}{2}$. To avoid this, the control inputs need be designed such that $|\phi_e| < \frac{\pi}{2}$ and $z_e > \sigma_1$ holds $\forall 0 \leq t \leq \infty$ with σ_1 being a positive constant.

Based on this consideration, our proposed control scheme needs to ensure the following inequalities hold.

$$\sigma_1 < z_e(t) \leq \delta, \quad -\frac{\pi}{2} + \sigma_2 \leq \phi_e(t) \leq \frac{\pi}{2} - \sigma_2 \quad (6.7)$$

where δ and $\sigma_2 < \frac{\pi}{2}$ are positive constants. Thus an upper bound and a lower bound for the position error and orientation error are prescribed in addition to stability and steady state tracking properties. Such constraints bring new difficulties to our controller design. To overcome them, the prescribed performance bounds (PPB) approach in [106] is incorporated. We first introduce a smooth performance function $\eta(t)$ as follows

$$\eta(t) = (\eta_0 - \eta_\infty)e^{-at} + \eta_\infty, \quad (6.8)$$

where $a > 0$ and $\eta_0 \geq \eta_\infty > 0$ which are all chosen by users. Note that if $\eta_0 > \eta_\infty$, $\eta(t)$ is an exponentially decreasing function with a as the decaying rate

and $\lim_{t \rightarrow \infty} \eta(t) = \eta_\infty$. Next by satisfying the following conditions,

$$\alpha_1 \eta(t) < z_e(t) < \beta_1 \eta(t), \quad -\alpha_2 \eta(t) < \phi_e(t) < \beta_2 \eta(t) \quad (6.9)$$

with positive scalars α_1 , β_1 , α_2 and β_2 chosen as

$$\alpha_1 \geq \frac{\sigma_1}{\eta_\infty}, \quad \beta_1 \leq \frac{\delta}{\eta_0} \quad (6.10)$$

$$\alpha_2 \leq \frac{\pi - 2\sigma_2}{2\eta_0}, \quad \beta_2 \leq \frac{\pi - 2\sigma_2}{2\eta_0} \quad (6.11)$$

ensuring “constrained” errors in (6.7) is achieved. Moreover by converting (6.7) to (6.9), the transient performances of tracking errors with a as a bound of exponentially decaying rate is also guaranteed by choosing $\eta_0 > \eta_\infty$. Thus $\eta(t)$ characterizes both the transient and steady-state performances of the closed loop system. It is worthy pointing out that all the existing results on underactuated ship adaptive control do not consider transient performances, to the authors’ best knowledge.

However, it is still a challenge problem to ensure (6.9) for all $t \geq 0$. To overcome the difficulty, we transform it to solving a problem with boundedness of certain signals as the only requirements. To do this, design two smooth and strictly increasing functions $S_1(\chi)$ and $S_2(\gamma)$ with the following properties:

$$(i) \quad \alpha_1 < S_1(\chi) < \beta_1, \quad -\alpha_2 < S_2(\gamma) < \beta_2 \quad (6.12)$$

$$(ii) \quad \lim_{\chi \rightarrow +\infty} S_1(\chi) = \beta_1, \quad \lim_{\chi \rightarrow -\infty} S_1(\chi) = \alpha_1$$

$$\lim_{\gamma \rightarrow +\infty} S_2(\gamma) = \beta_2, \quad \lim_{\gamma \rightarrow -\infty} S_2(\gamma) = -\alpha_2 \quad (6.13)$$

$$(iii) \quad S_1(0) = \alpha_1 + \epsilon < \beta_1, \quad S_2(0) = 0 \quad (6.14)$$

where ϵ is a positive constant chosen sufficiently small. From properties (i) and (ii) of S_1 and S_2 , the “constrained” error conditions in (6.9) can be expressed as

$$z_e(t) = \eta(t)S_1(\chi), \quad \phi_e(t) = \eta(t)S_2(\gamma). \quad (6.15)$$

Because of the strict monotonicity property of S_1 and S_2 , and the fact that $\eta(t) \neq$

0, the inverse functions exist and are given as

$$\chi(t) = S_1^{-1} \left(\frac{z_e(t)}{\eta(t)} \right), \quad \gamma(t) = S_2^{-1} \left(\frac{\phi_e(t)}{\eta(t)} \right) \quad (6.16)$$

If $\alpha_1\eta(0) < z_e(0) < \beta_1\eta(0)$, $-\alpha_2\eta(0) < \phi_e(0) < \beta_2\eta(0)$, and the transformed errors $\chi(t)$ and $\gamma(t)$ are ensured bounded $\forall t \geq 0$ by our proposed adaptive controllers, we will have that $\alpha_1 < \frac{z_e(t)}{\eta(t)} \leq \beta_1$ and $-\alpha_2 < \frac{\phi_e(t)}{\eta(t)} \leq \beta_2$, and thus (6.9) is ensured. Furthermore, from property (iii), if $\lim_{t \rightarrow \infty} \gamma(t) = 0$ and $\lim_{t \rightarrow \infty} \chi(t) = 0$ are obtained, we get $\lim_{t \rightarrow \infty} \phi_e(t) = 0$ and $\lim_{t \rightarrow \infty} z_e(t) = \eta_\infty(\alpha_1 + \epsilon) > \eta_\infty\alpha_1 \geq \sigma_1$. One possible choice of $S_1(\chi)$ and $S_2(\gamma)$ is

$$S_1(\chi) = \frac{\beta_1 e^{(\chi+v_1)} + \alpha_1 e^{-(\chi+v_1)}}{e^{(\chi+v_1)} + e^{-(\chi+v_1)}}, \quad S_2(\gamma) = \frac{\beta_2 e^{(\gamma+v_2)} - \alpha_2 e^{-(\gamma+v_2)}}{e^{(\gamma+v_2)} + e^{-(\gamma+v_2)}} \quad (6.17)$$

where $v_1 = \frac{1}{2} \ln \frac{\epsilon}{\beta_1 - \alpha_1 - \epsilon}$, $v_2 = \frac{1}{2} \ln \frac{\alpha_2}{\beta_2}$. It can be easily shown that $S_1(\chi)$ and $S_2(\gamma)$ in (6.17) have the required properties (i)-(iii). From (6.16), the transformed errors χ and γ are then solved as

$$\begin{aligned} \chi &= S_1^{-1}(\lambda_1(t)) = \frac{1}{2} \ln [\lambda_1(t) - \alpha_1] + \frac{1}{2} \ln (\beta_1 - \alpha_1 - \epsilon) - \frac{1}{2} \ln [\epsilon\beta_1 - \epsilon\lambda_1(t)] \\ \gamma &= S_2^{-1}(\lambda_2(t)) = \frac{1}{2} \ln (\lambda_2(t)\beta_2 + \alpha_2\beta_2) - \frac{1}{2} \ln (\alpha_2\beta_2 - \lambda_2(t)\alpha_2) \end{aligned} \quad (6.18)$$

where $\lambda_1(t) = \frac{z_e(t)}{\eta(t)}$ and $\lambda_2(t) = \frac{\phi_e(t)}{\eta(t)}$. Taking the time-derivative of χ and γ , respectively, we obtain

$$\begin{aligned} \dot{\chi} &= \frac{\partial S_1^{-1}}{\partial \lambda_1} \dot{\lambda}_1 = \frac{1}{2} \left[\frac{1}{\lambda_1(t) - \alpha_1} - \frac{1}{\lambda_1(t) - \beta_1} \right] \left(\frac{\dot{z}_e}{\eta} - \frac{z_e}{\eta^2} \dot{\eta} \right) \\ \dot{\gamma} &= \frac{\partial S_2^{-1}}{\partial \lambda_2} \dot{\lambda}_2 = \frac{1}{2} \left[\frac{1}{\lambda_2(t) + \alpha_2} - \frac{1}{\lambda_2(t) - \beta_2} \right] \left(\frac{\dot{\phi}_e}{\eta} - \frac{\phi_e}{\eta^2} \dot{\eta} \right) \end{aligned} \quad (6.19)$$

Replacing $\dot{z}_e$ and $\dot{\phi}_e$ with $\dot{\chi}$ and $\dot{\gamma}$, the system dynamics is transformed to

$$\begin{aligned}
\dot{\chi} &= \xi_1 \left[-\cos(\phi_e)u + \sin(\phi_e)v + \left(\frac{x_e}{z_e}x'_d + \frac{y_e}{z_e}y'_d \right) \dot{s} - \frac{z_e\dot{\eta}}{\eta} \right] \\
\dot{\gamma} &= \xi_2 \left[r + \left(\frac{\sin(\phi)}{z_e \cos(\phi_e)} - \frac{x_e \tan(\phi_e)}{z_e^2} \right) (x'_d \dot{s} - u \cos(\phi) + v \sin(\phi)) \right. \\
&\quad \left. - \left(\frac{\cos(\phi)}{z_e \cos(\phi_e)} + \frac{y_e \tan(\phi_e)}{z_e^2} \right) (y'_d \dot{s} - u \sin(\phi) - v \cos(\phi)) - \frac{\phi_e \dot{\eta}}{\eta} \right] \\
\dot{u} &= \frac{m_{22}}{m_{11}}vr - \frac{d_{11}}{m_{11}}u - \sum_{i=2}^3 \frac{d_{ui}}{m_{11}}|u|^{i-1}u + \frac{1}{m_{11}}\tau_u + \frac{1}{m_{11}}\tau_{wu}(t) \\
\dot{v} &= -\frac{m_{11}}{m_{22}}ur - \frac{d_{22}}{m_{22}}v - \sum_{i=2}^3 \frac{d_{vi}}{m_{22}}|v|^{i-1}v + \frac{1}{m_{22}}\tau_{wv}(t) \\
\dot{r} &= \frac{m_{11} - m_{22}}{m_{33}}uv - \frac{d_{33}}{m_{33}}r - \sum_{i=2}^3 \frac{d_{ri}}{m_{33}}|r|^{i-1}r + \frac{1}{m_{33}}\tau_r + \frac{1}{m_{33}}\tau_{wr}(t) \quad (6.20)
\end{aligned}$$

where $\xi_1 = \frac{1}{2\eta} \left[\frac{1}{\lambda_1(t) - \alpha_1} - \frac{1}{\lambda_1(t) - \beta_1} \right]$ and $\xi_2 = \frac{1}{2\eta} \left[\frac{1}{\lambda_2(t) + \alpha_2} - \frac{1}{\lambda_2(t) - \beta_2} \right]$. By applying property (i) of $S_1(\chi)$ and $S_2(\gamma)$, it can also be easily checked that $\xi_1 \neq 0$ and $\xi_2 \neq 0$.

6.3 Adaptive State-feedback Control Design

Under the general framework of backstepping technique [8], we shall design the control input τ_u and τ_r to ensure stability of the transformed system (6.20) in two steps. In the first step, virtual controls u_d and r_d for u and r can be designed to guarantee the stability of χ and γ . In the second step real controls τ_u and τ_r will be designed. To address the unknown system parameters, suitable parameter estimators will be introduced.

- *Step 1.*

Define two virtual control error variables

$$\tilde{u} = u - u_d, \quad \tilde{r} = r - r_d \quad (6.21)$$

and consider the following Lyapunov function

$$V_1(t) = \frac{1}{2}\chi^2 + \frac{1}{2}\gamma^2 \quad (6.22)$$

By considering the time-derivative of (6.22), the virtual controls u_d and r_d are designed as

$$\begin{aligned} u_d &= \frac{k_1}{\xi_1 \cos(\phi_e)} \chi + \frac{1}{\cos(\phi_e)} \left[\sin(\phi_e) v + \left(\frac{x_e}{z_e} x'_d + \frac{y_e}{z_e} y'_d \right) \dot{s} - \frac{z_e \dot{\eta}}{\eta} \right] \\ r_d &= -\frac{k_2}{\xi_2} \gamma - \left(\frac{\sin(\phi)}{z_e \cos(\phi_e)} - \frac{x_e \tan(\phi_e)}{z_e^2} \right) (x'_d \dot{s} - u_d \cos(\phi) + v \sin(\phi)) \\ &\quad + \left(\frac{\cos(\phi)}{z_e \cos(\phi_e)} + \frac{y_e \tan(\phi_e)}{z_e^2} \right) (y'_d \dot{s} - u_d \sin(\phi) - v \cos(\phi)) + \frac{\phi_e \dot{\eta}}{\eta} \end{aligned} \quad (6.23)$$

where k_1 and k_2 are positive constants, and $\dot{s}$ is selected to be

$$\dot{s} = \frac{u_0(1 - \epsilon_* e^{-\epsilon^* t})}{\sqrt{x_{od}^2 + y_{od}^2}} \quad (6.24)$$

with u_0 , $\epsilon_* < 1$ and ϵ^* being positive constants. Then the derivative of V_1 becomes

$$\dot{V}_1 = -k_1 \chi^2 - k_2 \gamma^2 - \chi \xi_1 \cos(\phi_e) \tilde{u} + \xi_2 \gamma \tilde{r} \quad (6.25)$$

$$- \xi_2 \gamma \left(\cos(\phi) f_1 - \sin(\phi) f_2 \right) \tilde{u} \quad (6.26)$$

where $f_1 = \frac{\sin(\phi)}{z_e \cos(\phi_e)} - \frac{x_e \tan(\phi_e)}{z_e^2}$ and $f_2 = \frac{\cos(\phi)}{z_e \cos(\phi_e)} + \frac{y_e \tan(\phi_e)}{z_e^2}$.

Remark 6.2. Note that similar to the existing references [18], $\dot{s}$ is closely-related to the speed of the ship, which is determined to guarantee the feasibility of reference trajectory to be followed by a real ship. That is, the “reference ship” runs at a low speed in the beginning and speeds up slowly. Moreover, when a sudden turn is encountered, i.e. $\sqrt{x_{od}^2 + y_{od}^2}$ becomes larger, $\dot{s}$ turns smaller, which means the real ship will slow down. So compared to the tracking control, the existence of $\dot{s}$ makes a path more practical for a real ship to follow.

- *Step 2.* Taking the derivatives of $\tilde{u}$ and $\tilde{r}$ along (6.21) gives

$$\begin{aligned}
\dot{\tilde{u}} = & \frac{m_{22}}{m_{11}}vr - \frac{d_{11}}{m_{11}}u - \sum_{i=2}^3 \frac{d_{ui}}{m_{11}}|u|^{i-1}u + \frac{1}{m_{11}}\tau_u + \frac{1}{m_{11}}\tau_{wu}(t) \\
& - \frac{\partial u_d}{\partial \xi_1}\dot{\xi}_1 - \frac{\partial u_d}{\partial \phi_e}\dot{\phi}_e - \frac{\partial u_d}{\partial \chi}\dot{\chi} - \frac{\partial u_d}{\partial x_e}\dot{x}_e - \frac{\partial u_d}{\partial y_e}\dot{y}_e - \frac{\partial u_d}{\partial z_e}\dot{z}_e - \frac{\partial u_d}{\partial x'_d}x''_d \\
& - \frac{\partial u_d}{\partial y'_d}y''_d - \frac{\partial u_d}{\partial \dot{s}}\dot{s} - \frac{\partial u_d}{\partial \eta}\dot{\eta} - \frac{\partial u_d}{\partial \ddot{\eta}}\ddot{\eta} - \frac{\partial u_d}{\partial v}\left(-\frac{m_{11}}{m_{22}}ur\right. \\
& \left. - \frac{d_{22}}{m_{22}}v - \sum_{i=2}^3 \frac{d_{vi}}{m_{22}}|v|^{i-1}v + \frac{1}{m_{22}}\tau_{wv}(t)\right) \\
\dot{\tilde{r}} = & \frac{m_{11} - m_{22}}{m_{33}}uv - \frac{d_{33}}{m_{33}}r - \sum_{i=2}^3 \frac{d_{ri}}{m_{33}}|r|^{i-1}r + \frac{1}{m_{33}}\tau_r + \frac{1}{m_{33}}\tau_{wr}(t) \\
& - \frac{\partial r_d}{\partial \xi_2}\dot{\xi}_2 - \frac{\partial r_d}{\partial \phi_e}\dot{\phi}_e - \frac{\partial r_d}{\partial \gamma}\dot{\gamma} - \frac{\partial r_d}{\partial x_e}\dot{x}_e - \frac{\partial r_d}{\partial y_e}\dot{y}_e - \frac{\partial r_d}{\partial z_e}\dot{z}_e - \frac{\partial r_d}{\partial x'_d}x''_d \\
& - \frac{\partial r_d}{\partial y'_d}y''_d - \frac{\partial r_d}{\partial \dot{s}}\dot{s} - \frac{\partial r_d}{\partial \eta}\dot{\eta} - \frac{\partial r_d}{\partial \ddot{\eta}}\ddot{\eta} - \frac{\partial r_d}{\partial v}\left(-\frac{m_{11}}{m_{22}}ur\right. \\
& \left. - \frac{d_{22}}{m_{22}}v - \sum_{i=2}^3 \frac{d_{vi}}{m_{22}}|v|^{i-1}v + \frac{1}{m_{22}}\tau_{wv}(t)\right)
\end{aligned} \tag{6.27}$$

We define the Lyapunov function at this step as

$$V_2 = V_1 + \frac{m_{11}}{2}\tilde{u}^2 + \frac{m_{33}}{2}\tilde{r}^2 + \sum_{i=1}^3 \tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i \tag{6.28}$$

where $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$ is the parameter estimation error, $\Gamma_1 = \text{diag}\{\delta_{1j}\}$, $\Gamma_2 = \text{diag}\{\delta_{2j}\}$, $j = 1, \dots, 9$, $\Gamma_3 = \text{diag}\{\delta_{3k}\}$, $k = 1, \dots, 4$, are positive definite diagonal matrices, and the unknown parameter vector θ_i , $i = 1, 2, 3$ is defined as

$$\begin{aligned}
\theta_1 = & \left[m_{22}, d_{11}, d_{u2}, d_{u3}, m_{11}, \frac{m_{11}^2}{m_{22}}, \frac{m_{11}d_{22}}{m_{22}}, \frac{m_{11}d_{v2}}{m_{22}}, \frac{m_{11}d_{v3}}{m_{22}} \right] \\
\theta_2 = & \left[(m_{11} - m_{22}), d_{33}, d_{v2}, d_{v3}, m_{33}, \frac{m_{11}m_{33}}{m_{22}}, \frac{m_{33}d_{22}}{m_{22}}, \frac{m_{33}d_{v2}}{m_{22}}, \frac{m_{33}d_{v3}}{m_{22}} \right] \\
\theta_3 = & \left[\tau_{wu \max}, \frac{m_{11}}{m_{22}}\tau_{wv \max}, \tau_{wr \max}, \frac{m_{33}}{m_{22}}\tau_{wv \max} \right]
\end{aligned} \tag{6.29}$$

Without canceling the useful damping term, the control torques for τ_u and τ_r are

designed as

$$\begin{aligned}
\tau_{us} &= -k_3 \tilde{u} - \hat{\theta}_1^T \Omega_1 + \chi \xi_1 \cos(\phi_e) + \xi_2 \gamma \left(\cos(\phi) f_1 - \sin(\phi) f_2 \right) \\
&\quad - \hat{\theta}_{31} \tanh\left(\frac{\tilde{u} \hat{\theta}_{31}}{\epsilon_1}\right) - \hat{\theta}_{32} \frac{\partial u_d}{\partial v} \tanh\left(\frac{\partial u_d}{\partial v} \frac{\tilde{u} \hat{\theta}_{32}}{\epsilon_2}\right) \\
\tau_{rs} &= -k_4 \tilde{r} - \hat{\theta}_2^T \Omega_2 - \xi_2 \gamma - \hat{\theta}_{33} \tanh\left(\frac{\tilde{r} \hat{\theta}_{33}}{\epsilon_3}\right) - \hat{\theta}_{34} \frac{\partial r_d}{\partial v} \tanh\left(\frac{\partial r_d}{\partial v} \frac{\tilde{r} \hat{\theta}_{34}}{\epsilon_4}\right) \quad (6.30)
\end{aligned}$$

where k_3 and k_4 are positive constants, and

$$\begin{aligned}
\Omega_1 &= \left[vr, -u_d, -|u|u_d, -u^2 u_d, -\left(\frac{\partial u_d}{\partial \xi_1} \dot{\xi}_1 \right. \right. \\
&\quad \left. \left. + \frac{\partial u_d}{\partial \phi_e} \dot{\phi}_e + \frac{\partial u_d}{\partial \chi} \dot{\chi} + \frac{\partial u_d}{\partial x_e} \dot{x}_e + \frac{\partial u_d}{\partial y_e} \dot{y}_e + \frac{\partial u_d}{\partial z_e} \dot{z}_e \right. \right. \\
&\quad \left. \left. + \frac{\partial u_d}{\partial x'_d} x''_d + \frac{\partial u_d}{\partial y'_d} y''_d + \frac{\partial u_d}{\partial \dot{s}} \ddot{s} + \frac{\partial u_d}{\partial \dot{\eta}} \dot{\eta} + \frac{\partial u_d}{\partial \ddot{\eta}} \ddot{\eta} \right), \right. \\
&\quad \left. \frac{\partial u_d}{\partial v} ur, \frac{\partial u_d}{\partial v} v, \frac{\partial u_d}{\partial v} |v|v, \frac{\partial u_d}{\partial v} v^3 \right] \\
\Omega_2 &= \left[uv, -r_d, -|r|r_d, -r^2 r_d, -\left(\frac{\partial r_d}{\partial \xi_2} \dot{\xi}_2 \right. \right. \\
&\quad \left. \left. + \frac{\partial r_d}{\partial \phi_e} \dot{\phi}_e + \frac{\partial r_d}{\partial \gamma} \dot{\gamma} + \frac{\partial r_d}{\partial x_e} \dot{x}_e + \frac{\partial r_d}{\partial y_e} \dot{y}_e + \frac{\partial r_d}{\partial z_e} \dot{z}_e \right. \right. \\
&\quad \left. \left. + \frac{\partial r_d}{\partial x'_d} x''_d + \frac{\partial r_d}{\partial y'_d} y''_d + \frac{\partial r_d}{\partial \dot{s}} \ddot{s} + \frac{\partial r_d}{\partial \dot{\eta}} \dot{\eta} + \frac{\partial r_d}{\partial \ddot{\eta}} \ddot{\eta} \right), \right. \\
&\quad \left. \frac{\partial r_d}{\partial v} ur, \frac{\partial r_d}{\partial v} v, \frac{\partial r_d}{\partial v} |v|v, \frac{\partial r_d}{\partial v} v^3 \right]
\end{aligned}$$

Here τ_{us} and τ_{rs} are used to denote that they are the state-feedback controllers, differentiating them from the output-feedback controls τ_u and τ_r , which will be designed later.

The update law for $\hat{\theta}_i$ are chosen as

$$\begin{aligned}
\dot{\hat{\theta}}_{1js} &= \delta_{1j} \text{Proj}(\tilde{u} \Omega_{1j}, \hat{\theta}_{1j}), \quad 1 \leq j \leq 9 \\
\dot{\hat{\theta}}_{2js} &= \delta_{2j} \text{Proj}(\tilde{r} \Omega_{2j}, \hat{\theta}_{2j}), \quad 1 \leq j \leq 9
\end{aligned}$$

$$\begin{aligned}
\dot{\hat{\theta}}_{31s} &= \delta_{31} \text{Proj}(|\tilde{u}|, \hat{\theta}_{31}) \\
\dot{\hat{\theta}}_{32s} &= \delta_{32} \text{Proj}\left(\left|\tilde{u} \frac{\partial u_d}{\partial v}\right|, \hat{\theta}_{32}\right) \\
\dot{\hat{\theta}}_{33s} &= \delta_{33} \text{Proj}(|\tilde{r}|, \hat{\theta}_{33}) \\
\dot{\hat{\theta}}_{34s} &= \delta_{34} \text{Proj}\left(\left|\tilde{r} \frac{\partial r_d}{\partial v}\right|, \hat{\theta}_{34}\right)
\end{aligned} \tag{6.31}$$

Similarly $\dot{\hat{\theta}}_{**s}$ is used to denote it as a “state-feedback” estimator in which unknown state variables are used. The operator Proj is a Lipschitz continuous projection algorithm [105] defined as follows:

$$\text{Proj}(a, \hat{b}) = \begin{cases} a & \text{if } \mu(\hat{b}) \leq 0 \\ a & \text{if } \mu(\hat{b}) \geq 0 \text{ and } \mu'(\hat{b})a \leq 0 \\ (1 - \mu(\hat{b}))a & \text{if } \mu(\hat{b}) > 0 \text{ and } \mu'(\hat{b})a > 0 \end{cases} \tag{6.32}$$

where $\mu(\hat{b}) = \frac{\hat{b}^2 - b_M^2}{\epsilon^2 + 2\epsilon b_M}$, $\mu'(\hat{b}) = \frac{\partial \mu(\hat{b})}{\partial \hat{b}}$, ϵ is an arbitrarily small positive constant, b_M is a positive constant satisfying $|b| < b_M$. Based on [104], for any $\epsilon > 0$ and $\zeta \in \mathfrak{R}$,

$$0 \leq |\zeta| - \zeta \tanh\left(\frac{\zeta}{\epsilon}\right) \leq 0.2785\epsilon. \tag{6.33}$$

Differentiating $V_2(t)$ and using (6.30), (6.31), (6.33) and Lemma 2.9 yields that

$$\dot{V}_2 \leq -k_1 \chi^2 - k_2 \gamma^2 - (k_3 + d_{11})\tilde{u}^2 - (k_4 + d_{33})\tilde{r}^2 + 0.2785 \sum_{i=1}^4 \epsilon_i \tag{6.34}$$

By adding and subtracting $\frac{1}{2} \sum_{i=1}^3 \tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i$ on the right side of (6.34), we get

$$\dot{V}_2 \leq -k_m V_2 + \epsilon_m \tag{6.35}$$

where $k_m = \min\left(2k_1, 2k_2, \frac{2(k_3 + d_{11})}{m_{11}}, \frac{2(k_4 + d_{33})}{m_{33}}\right)$, $\epsilon_m = \frac{1}{2} \sum_{i=1}^3 \tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i +$

$0.2785 \sum_{i=1}^4 \epsilon_i$. From (6.35), it is directly shown that

$$V_2(t) \leq V_2(0)e^{-k_m t} + \frac{\epsilon_m}{k_m} \quad (6.36)$$

which means $\chi, \gamma, \tilde{u}, \tilde{r}, \tilde{\theta}_i, i = 1, 2, 3$ are bounded. Thus from (6.15) z_e and ϕ_e are bounded, and also (6.7) will be satisfied.

Let

$$u_d = a_1 v + a_2, \quad r_d = a_3 v + a_4 \quad (6.37)$$

where $a_1 = \frac{\sin(\phi_e)}{\cos(\phi_e)}$, $a_2 = \frac{k_1}{\xi_1 \cos(\phi_e)} \chi + \frac{1}{\cos(\phi_e)} \left[\left(\frac{x_e}{z_e} x'_d + \frac{y_e}{z_e} y'_d \right) \dot{s} - \frac{z_e \dot{\eta}}{\eta} \right]$, $a_3 = f_1 a_1 \cos(\phi) - f_1 \sin(\phi) - f_2 a_1 \sin(\phi) - f_2 \cos(\phi)$ and $a_4 = -\frac{k_2}{\xi_2} \gamma - f_1 (x'_d \dot{s} - a_2 \cos(\phi)) + f_2 (y'_d \dot{s} - a_2 \sin(\phi)) + \frac{\phi_e \dot{\eta}}{\eta}$. From (6.36) a_1, a_2, a_3 and a_4 are bounded. The sway velocity dynamics can be rewritten as

$$\begin{aligned} \dot{v} &= -\frac{m_{11}}{m_{22}}(a_1 v + a_2 + \tilde{u})(a_3 v + a_4 + \tilde{r}) - \frac{d_{22}}{m_{22}} v - \sum_{i=2}^3 \frac{d_{vi}}{m_{22}} |v|^{i-1} v + \frac{1}{m_{22}} \tau_{wv}(t) \\ &= a_5(t) v + a_6(t) v^2 - \frac{d_{v2}}{m_{22}} |v| v - \frac{d_{v3}}{m_{33}} v^3 + a_7(t) \end{aligned} \quad (6.38)$$

where $a_5(t) = -\frac{m_{11}}{m_{22}}(a_1(a_4 + \tilde{r}) + a_3(a_2 + \tilde{u})) - \frac{d_{22}}{m_{22}}$, $a_6(t) = -\frac{m_{11}}{m_{22}} a_1 a_3$, $a_7(t) = -\frac{m_{11}}{m_{22}}(a_2 + \tilde{u})(a_4 + \tilde{r}) + \frac{1}{m_{22}} \tau_{wv}(t)$. From the fact that $\chi, \gamma, \tilde{u}, \tilde{r}, a_1, a_2, a_3$ and a_4 are bounded, it can be easily checked that a_5, a_6 and a_7 are bounded. Also from Assumption 6.2, (6.7) and (6.36), the upper-bounds of a_5, a_6 and a_7 can also be obtained. Considering the time-derivative of Lyapunov function $V_v = \frac{1}{2} v^2$ along (6.38), we get

$$\dot{V}_v \leq -\left(\frac{d_{v3}}{m_{22}} - a_6^M \zeta\right) v^4 + \left(a_5^M + \frac{a_6^M + a_7^M}{4\zeta}\right) v^2 + a_7^M \zeta \quad (6.39)$$

where ζ is a positive constant chosen such that $\frac{d_{v3}}{m_{22}} - a_6^M \zeta > 0$. $a_i^M, i = 5, 6, 7$ are the upper-bounds of a_i respectively. Then from [112] it can be guaranteed that v is bounded with computable upper-bound. The following theorem is thus

established.

Theorem 6.1. *Consider the underactuated ship system (6.1) and (6.2), with the state-feedback controller (6.30) and parameter update laws (6.31). Under Assumptions 6.1 and 6.2, all the signals in the closed-loop system are bounded, and the reference path $x_d(s)$ and $y_d(s)$ can be followed with arbitrarily small position error and orientation error. Furthermore, the position error and orientation error will converge at a pre-specified exponential rate.*

Proof: From (6.36), the boundedness of χ , γ , $\tilde{u}$, $\tilde{r}$, $\tilde{\theta}_{1i}$, $\tilde{\theta}_{2i}$, $i = 1, \dots, 9$, $\tilde{\theta}_{3j}$, $j = 1, \dots, 4$ and V_2 are shown. From (6.38) v is bounded, thus from (6.23) u_d and r_d are bounded. Then u and r are also bounded. It is easy to check from (6.30) and (6.31) that τ_{us} , τ_{rs} , $\dot{\hat{\theta}}_{1s}$, $\dot{\hat{\theta}}_{2s}$ and $\dot{\hat{\theta}}_{3s}$ are bounded. Let

$$e = [\chi, \gamma, \tilde{u}, \tilde{r}]^T \quad (6.40)$$

and $k_{\min} = \min\{k_1, k_2, k_3 + d_{11}, k_4 + d_{33}\}$. Taking the integration of (6.34) yields

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{t=0}^T e^T e \leq \frac{0.2785 \sum_{i=1}^4 \epsilon_i}{k_{\min}}$$

which means e is of the order of $O(\epsilon_i)$, i.e., e can be adjusted arbitrarily small if ϵ_i is small enough. From (6.15), the properties of $S_1(\chi)$, $S_2(\gamma)$ and (6.11), we obtain that

$$\begin{aligned} \sigma_1 &\leq \alpha_1 \eta_\infty < z_e(t) < \beta_1 \eta_0 \leq \delta \\ -\frac{\pi}{2} + \sigma_2 &\leq -\alpha_2 \eta_0 < \phi_e(t) < \beta_2 \eta_0 \leq \frac{\pi}{2} - \sigma_2, \quad \forall t \geq 0. \end{aligned} \quad (6.41)$$

if $z_e(0) \in (\alpha_1 \eta_\infty, \beta_1 \eta_0)$ and $\phi_e(0) \in (-\pi/2 + \sigma_2, \pi/2 - \sigma_2)$. Then $\lim_{t \rightarrow \infty} \phi_e(t) = \epsilon_*$ and $\lim_{t \rightarrow \infty} z_e(t) = (\alpha_1 + \epsilon + \epsilon^*) \eta_\infty \in (\alpha_1 \eta_\infty, \beta_1 \eta_\infty)$ where ϵ_* and ϵ^* are unknown constants but are arbitrarily small due to the continuity of $S_1(\chi)$ and $S_2(\gamma)$ and the fact that χ and γ are arbitrarily small. It implies that (6.7) is always satisfied, and the underactuated ship can follow its reference trajectory with arbitrary small

position error z_e and orientation error ϕ_e .

From (6.9)

$$z_e(t) < \beta_1 \left((\eta_0 - \eta_\infty) e^{-at} + \eta_\infty \right), \quad \phi_e(t) < \beta_2 \left((\eta_0 - \eta_\infty) e^{-at} + \eta_\infty \right) \quad (6.42)$$

which means that the position error and the orientation error will converge at a rate governed by the given positive constant a . This completes the proof. $\square$

6.4 Adaptive Output-feedback Control Design

In this section, we consider the case that all the velocities are unavailable for feedback. As a starting point, three high-gain observers are designed to estimate the unknown velocities respectively. Then we replace the unknown states in the state feedback controllers and state dependent parameter estimators with the estimated velocities. Finally, the closed loop system is analyzed analysis is to show its stability and tracking properties.

6.4.1 Observer Design

The difficulty of adaptive output feedback control for underactuated ships comes from the fact that the observer design and the unknown parameters are intertwined together due to the presence of nonholonomic constraints and a quadratic term of the system states. Now we design three high-gain observers to estimate u , v and r respectively, which can avoid the difficulty mentioned above. Firstly, we make the following variable changes

$$\rho_1 = y \cos \phi - x \sin \phi, \quad \rho_2 = x \cos \phi + y \sin \phi$$

Then from (6.1)

$$\begin{aligned}\dot{\rho}_1 &= v - \rho_2 r \\ \dot{\rho}_2 &= u + \rho_1 r \\ \dot{\phi} &= r.\end{aligned}\tag{6.43}$$

Following the procedure in [97] and based on the third equation of (6.43), we design a high-gain observer for r as

$$\begin{aligned}\dot{\hat{\phi}} &= \hat{r} + \frac{1}{\varepsilon_1} l_1 (\phi - \hat{\phi}) \\ \dot{\hat{r}} &= \frac{1}{\varepsilon_1^2} l_2 (\phi - \hat{\phi})\end{aligned}\tag{6.44}$$

where ε_1 is a small positive design parameter, and l_i , $i = 1, 2$ are chosen so that

$$\bar{A} = \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix}\tag{6.45}$$

is Hurwitz. From the second equation of (6.43), the observer for u is designed as:

$$\begin{aligned}\dot{\hat{\rho}}_2 &= \hat{u} + \frac{l_1}{\varepsilon_1} (\rho_2 - \hat{\rho}_2) + \rho_1 \hat{r} \\ \dot{\hat{u}} &= \frac{l_2}{\varepsilon_1^2} (\rho_2 - \hat{\rho}_2)\end{aligned}\tag{6.46}$$

where $\hat{r}$ is from system (6.44).

From the first equation of (6.43), the observer for v is designed as:

$$\begin{aligned}\dot{\hat{\rho}}_1 &= \hat{v} + \frac{l_1}{\varepsilon_1} (\rho_1 - \hat{\rho}_1) - \rho_2 \hat{r} \\ \dot{\hat{v}} &= \frac{l_2}{\varepsilon_1^2} (\rho_1 - \hat{\rho}_1).\end{aligned}\tag{6.47}$$

6.4.2 Design of Output Feedback Controllers and Estimators

Since the output feedback controllers and the parameter estimators are implemented with the high-gain observers, we will introduce saturation to the controllers and parameter estimators based on two reasons. The first reason is to prevent the peaking phenomena of the high-gain observers from entering the ship dynamic system. The other is that the boundedness of the system signals is ensured by saturation. On the other hand we will show that after a short time from the beginning of system operation, the saturation will have no effect. This implies that the designed torques and the rates of the parameter estimates will still remain within the saturation limits, independent of the presence of saturations.

First of all, we define a compact set

$$\Xi = \{\mathcal{E} : V_2 < \varrho\} \quad (6.48)$$

where V_2 is defined in (6.28) and $\mathcal{E} = \{\tilde{u}, \tilde{r}, \chi, \gamma, \tilde{\theta}_{1i}, \tilde{\theta}_{2i}, \tilde{\theta}_{3j}\}$, $i = 1, \dots, 9$, $j = 1, \dots, 4$. Due to Assumption 6.2 and the parameter projections, $\tilde{\theta}_{1i}$, $\tilde{\theta}_{2i}$, $\tilde{\theta}_{3j}$ are bounded with computable upper-bounds. Denote $\tilde{\theta}_{1im}$, $\tilde{\theta}_{2im}$, $\tilde{\theta}_{3jm}$ as the maximum value of $\tilde{\theta}_{1i}$, $\tilde{\theta}_{2i}$, $\tilde{\theta}_{3j}$ respectively. A constant ϱ is chosen as

$$\varrho = \nu_{\max} + \Delta_1 \quad (6.49)$$

where Δ_1 is a positive constant to be selected and positive constant $\nu_{\max}$ is defined as follows

$$\nu_{\max} = \frac{1}{2}\tilde{\theta}_{1,m}^T \Gamma_1^{-1} \tilde{\theta}_{1,m} + \frac{1}{2}\tilde{\theta}_{2,m}^T \Gamma_2^{-1} \tilde{\theta}_{2,m} + \frac{1}{2}\tilde{\theta}_{3,m}^T \Gamma_3^{-1} \tilde{\theta}_{3,m} \quad (6.50)$$

where $\tilde{\theta}_{1,m} = [\tilde{\theta}_{11m}, \dots, \tilde{\theta}_{19m}]$, $\tilde{\theta}_{2,m} = [\tilde{\theta}_{21m}, \dots, \tilde{\theta}_{29m}]$ and $\tilde{\theta}_{3,m} = [\tilde{\theta}_{31m}, \dots, \tilde{\theta}_{34m}]$.

Fact 6.1. τ_{us} and τ_{rs} in (6.30) are bounded and smooth in compact set Ξ , and their bounds are computable.

Proof: See Appendix A. □

Let

$$S_{us} = \max_{\mathcal{E} \in \Xi} |\tau_{us}(u, v, r)| + \Delta_2, \quad S_{rs} = \max_{\mathcal{E} \in \Xi} |\tau_{rs}(u, v, r)| + \Delta_2 \quad (6.51)$$

where Δ_2 is a positive constant to be chosen. The final output feedback controllers are designed as

$$\tau_u = \text{Sat}_{S_{us}}(\tau_{us}(\hat{u}, \hat{v}, \hat{r})), \quad \tau_v = \text{Sat}_{S_{rs}}(\tau_{vs}(\hat{u}, \hat{v}, \hat{r})) \quad (6.52)$$

where the unknown u , v and r are replaced by $\hat{u}$, $\hat{v}$ and $\hat{r}$ respectively, and the saturation function is defined as

$$\text{Sat}_{S_\star}(\tau_\star) = \begin{cases} \text{sign}(\tau_\star)S_\star & |\tau_\star| \geq S_\star \\ \tau_\star & |\tau_\star| < S_\star \end{cases}$$

where $\tau_\star$ signifies τ_{us} or τ_{rs} , $S_\star$ denotes S_{us} or S_{rs} respectively. Now we consider the design of estimators only involving output and observer signals. For the same reasons the output feedback estimators will also be saturated. We use $\dot{\hat{\theta}}_{11}$ as an example for illustration. Let

$$S_{\hat{\theta}_{11}} = \max_{\mathcal{E} \in \Xi} |\tilde{u}\Omega_{11}| + \Delta_2 \quad (6.53)$$

Define

$$a_{\hat{\theta}_{11}} = \text{Sat}_{S_{\hat{\theta}_{11}}}(\hat{\tilde{u}}\hat{\Omega}_{11})$$

where $\hat{\tilde{u}}$ and $\hat{\Omega}_{11}$ are the values of $\tilde{u}$ and Ω_{11} obtained after the unknown u , v and r are replaced by $\hat{u}$, $\hat{v}$ and $\hat{r}$, respectively. The output feedback estimator for $\hat{\theta}_{11}$ can be designed as

$$\dot{\hat{\theta}}_{1j} = \text{Proj}(a_{\hat{\theta}_{11}}, \hat{\theta}_{11}) = \begin{cases} a_{\hat{\theta}_{11}} & \text{if } \mu(\hat{\theta}_{11}) \leq 0 \\ a_{\hat{\theta}_{11}} & \text{if } \mu(\hat{\theta}_{11}) \geq 0 \text{ and } \mu'(\hat{\theta}_{11}) \leq 0 \\ (1 - \mu(\hat{\theta}_{11}))a_{\hat{\theta}_{11}} & \text{if } \mu(\hat{\theta}_{11}) > 0 \text{ and } \mu'(\hat{\theta}_{11}) > 0. \end{cases} \quad (6.54)$$

Other estimators can be designed in a similar way.

Notation: In the remaining part of the chapter, we use $\hat{\tilde{u}}, \hat{\tilde{r}}, \hat{\Omega}_1, \hat{\Omega}_2$ to denote the value of $\tilde{u}, \tilde{r}, \Omega_1$ and Ω_2 after their unknown states u, v and r are replaced by $\hat{u}, \hat{v}$ and $\hat{r}$ respectively.

6.4.3 System Analysis

In this section we will establish the stability, the transient and steady-state tracking performances of the closed-loop system. The analysis can be divided into the following steps. Firstly we will prove that the velocities u, v and r and the state estimation errors are bounded with the saturated output-feedback controllers. Secondly we will show that there exists a time T_s , which can be adjusted as small as possible by choosing ε_1 , such that at $t = T_s$, $\mathcal{E} \in \Xi$ and after T_s , the saturations for controllers and estimators do not take effect. Thirdly, we will prove that Ξ is an invariant set, i.e., $\mathcal{E} \in \Xi \forall t$, which means that the position error and orientation error are bounded. Finally we will prove that the position error and orientation error are of $O(\varepsilon)$, and the position error and the orientation error also decrease at a pre-specified exponential rate.

Now we define the observation errors as follows

$$\begin{aligned}\varphi_1 &= \frac{1}{\varepsilon_1}(\phi - \hat{\phi}), \quad \varphi_2 = r - \hat{r} \\ \omega_1 &= \frac{1}{\varepsilon_1}(\rho_2 - \hat{\rho}_2), \quad \omega_2 = u - \hat{u} \\ \psi_1 &= \frac{1}{\varepsilon_1}(\rho_1 - \hat{\rho}_1), \quad \psi_2 = v - \hat{v}\end{aligned}\tag{6.55}$$

and denote $\varphi = [\varphi_1 \quad \varphi_2]^T$, $\omega = [\omega_1 \quad \omega_2]^T$, $\psi = [\psi_1 \quad \psi_2]^T$. From (6.43), (6.44), (6.46) and (6.47) we get

$$\varepsilon_1 \dot{\varphi} = \bar{A}\varphi + \varepsilon_1 \begin{bmatrix} 0 \\ G_r \end{bmatrix}\tag{6.56}$$

$$\varepsilon_1 \dot{\omega} = \bar{A}\omega + \varepsilon_1 \begin{bmatrix} 0 \\ G_u \end{bmatrix} + \begin{bmatrix} \rho_1 \varphi_2 \\ 0 \end{bmatrix} \quad (6.57)$$

$$\varepsilon_1 \dot{\psi} = \bar{A}\psi + \varepsilon_1 \begin{bmatrix} 0 \\ G_v \end{bmatrix} - \begin{bmatrix} \rho_2 \varphi_2 \\ 0 \end{bmatrix} \quad (6.58)$$

where $G_r = \frac{m_{11} - m_{22}}{m_{33}}uv - \frac{d_{33}}{m_{33}}r - \frac{d_{r2}}{m_{33}}|r|r - \frac{d_{r3}}{m_{33}}r^3 + \frac{1}{m_{33}}(\tau_r + \tau_{wr}(t))$, $G_u = \frac{m_{22}}{m_{11}}vr - \frac{d_{11}}{m_{11}}u - \frac{d_{u2}}{m_{11}}|u|u - \frac{d_{u3}}{m_{11}}u^3 + \frac{1}{m_{11}}(\tau_u + \tau_{wu}(t))$, $G_v = -\frac{m_{11}}{m_{22}}ur - \frac{d_{22}}{m_{22}}v - \frac{d_{v2}}{m_{22}}|v|v - \frac{d_{v3}}{m_{22}}v^3 + \frac{1}{m_{22}}\tau_{wv}(t)$.

Lemma 6.1. *When the systems (6.1) and (6.2) are controlled by the adaptive controllers in (6.52) with estimators in the form of (6.54), u , v , r and φ are globally bounded.*

Proof: By considering a Lyapunov function $V_\varpi = \frac{1}{2}\varpi^T \varpi$ where

$\varpi = [\sqrt{m_{11}}u, \sqrt{m_{22}}v, \sqrt{m_{33}}r]^T$, we get

$$\dot{V}_\varpi \leq -2\xi V_\varpi + \frac{1}{4\varsigma}(\tau_u^2 + \tau_{wu \max}^2) + \frac{1}{4\varsigma}(\tau_r^2 + \tau_{wr \max}^2) + \frac{1}{4\varsigma}\tau_{wv \max}^2$$

where $\xi = \min\{\frac{d_{11} - \varsigma}{m_{11}}, \frac{d_{22} - \varsigma}{m_{22}}, \frac{d_{33} - \varsigma}{m_{33}}\}$ and ς is a constant chosen such that $\xi > 0$. So u , v and r are globally bounded since the torques are globally saturated. Also as τ_u and τ_r are bounded, there exists positive constants g_u , g_v and g_r so that

$$|G_u| < g_u, \quad |G_v| < g_v, \quad |G_r| < g_r$$

Since $\bar{A}$ is a Hurwitz matrix, so (6.56) is a stable system and φ is bounded. $\square$

Lemma 6.2. *For any ε_1 , there exists a finite time $T_1(\varepsilon_1)$, so that for $t > T_1$, we*

have

$$\|\varphi\| < c_1 \varepsilon_1 \quad (6.59)$$

where c_1 is a positive constant that is computable.

Proof: The proof is similar to that of Lemma 5.3.

Case 1: For $\nu(t) \geq \frac{4\iota^2}{\gamma^2} \varepsilon_1^2$ we obtain

$$\dot{\nu} \leq -\frac{\gamma}{2\varepsilon_1} \nu \quad (6.60)$$

which gives that $\nu(t) \leq \nu(0)e^{-\frac{\gamma}{2\varepsilon_1}t}$. For any bounded initial conditions of ϕ , $\hat{\phi}$, r and $\hat{r}$, from the definition of φ in (6.55) we know $\nu(0)$ can be written as $\nu(0) = \frac{l_3}{\varepsilon_1^2}$ where $l_3 = p_{11}s^2 + (p_{21} + p_{12})\varepsilon_1 sq + p_{22}q^2\varepsilon_1^2$, $s = \phi(0) - \hat{\phi}(0)$, $q = r(0) - \hat{r}(0)$. Thus we get $\nu(t) \leq \frac{l_3}{\varepsilon_1^2} e^{-\frac{\gamma}{2\varepsilon_1}t}$.

So when the initial condition $\frac{l_3}{\varepsilon_1^2}$ satisfies that $\frac{l_3}{\varepsilon_1^2} > \frac{4\iota^2}{\gamma^2} \varepsilon_1^2$, the time $T_1(\varepsilon_1)$ for ν to reach the ball $\nu \leq \frac{4\iota^2}{\gamma^2} \varepsilon_1^2$ from $\nu(0)$ is

$$T_1(\varepsilon_1) = \frac{2\varepsilon_1}{\gamma} \left(\ln \frac{l_4}{\varepsilon_1^4} \right) \quad (6.61)$$

where $l_4 = \frac{l_3\gamma^2}{4\iota^2}$. Since (6.61) tends to zero as $\varepsilon_1 \rightarrow 0$, so we can choose a sufficiently small $T_1(\varepsilon_1)$ such that $\forall t > T_1(\varepsilon_1)$, we have $\nu \leq \frac{4\iota^2}{\gamma^2} \varepsilon_1^2$, which means

$$\|\varphi\| \leq c_1 \varepsilon_1, \text{ where } c_1 = \sqrt{\frac{4\iota^2}{\gamma^2 \lambda_{\min}(P)}}.$$

Case 2: When $\nu(t) \leq \frac{4\iota^2}{\gamma^2} \varepsilon_1^2$, the conclusion holds automatically for any $T_1(\varepsilon_1) \geq 0$. This complete the proof. $\square$

Fact 6.2. The upper-bound of $\dot{V}_2$ within Ξ along (6.52) and (6.54) is computable.

Proof: See Appendix B. $\square$

Lemma 6.3. *There exists a T_s , which can be adjusted arbitrarily small, such that for $t \in [T_s, T_e]$ the saturation for controllers (6.52) and estimators (6.54) will not take effect, where T_e is the first time instant that $\mathcal{E}$ leaves the set Ξ .*

Remark 3: In Lemma 6.4 given later we will show that $T_e = \infty$ and thus Ξ is an invariant set.

Proof: Based on Fact 6.2, we can find a constant c_4 such that $|\dot{V}_2| \leq c_4$ before $\mathcal{E}$ exits Ξ . So we have

$$V_2(T_1) \leq V_2(0) + \int_0^{T_1(\varepsilon_1)} c_4 dt = V_2(0) + c_4 T_1(\varepsilon_1).$$

From Lemma 6.2, if we choose an ε_1 such that

$$V_2(0) + c_4 T_1(\varepsilon_1) < \varrho \quad (6.62)$$

then for $t \in [0, T_1(\varepsilon_1)]$, we have $\mathcal{E} \in \Xi$.

Suppose T_e is the first time that $\mathcal{E}$ exits from Ξ . So for $t \in (0, T_e]$, we get $|\chi| < \sqrt{2\varrho}$ and $|\gamma| < \sqrt{2\varrho}$, which means $|z_e| < \eta_0 S_1(\chi)$ and $|\phi_e| < \eta_0 S_2(\gamma)$. Thus

$$\begin{aligned} |x| &< |x_d| + |x_e| < a^* + |z_e| \\ |y| &< |y_d| + |y_e| < a^* + |z_e|. \end{aligned}$$

Finally we get

$$|\rho_1| < |x| + |y| < 2a^* + 2|z_e|, \quad |\rho_2| < |x| + |y| < 2a^* + 2|z_e|. \quad (6.63)$$

From (6.46) and (6.47), ω and ψ are bounded for $t \in (0, T_e]$. Following the similar proofs of Lemma 6.1 and Lemma 6.2, there exist $T_2(\varepsilon_1)$ and $T_3(\varepsilon_1)$, which can be arbitrarily small if ε_1 is small enough, such that for $T_e > t > T_2(\varepsilon_1)$ and $T_e > t > T_3(\varepsilon_1)$, we get

$$\|\omega\| < c_2 \varepsilon_1, \quad \|\psi\| < c_3 \varepsilon_1 \quad (6.64)$$

respectively, where c_2 and c_3 are positive constants.

Fact 6.3. *There are two positive constants c_u and c_r such that*

$$\tau_{us}(\hat{u}, \hat{v}, \hat{r}) < \tau_{us}(u, v, r) + c_u \varepsilon_1, \quad \tau_{rs}(\hat{u}, \hat{v}, \hat{r}) < \tau_{rs}(u, v, r) + c_r \varepsilon_1. \quad (6.65)$$

when $t \in [T_s, T_e]$, where

$$T_s = \max\{T_1(\varepsilon_1), T_2(\varepsilon_1), T_3(\varepsilon_1)\} \quad (6.66)$$

Proof: See Appendix C. □

Since $\mathcal{E} \in \Xi$ for $t \in (0, T_e]$, we have $\tau_{us}(u, v, r) < \max_{\mathcal{E} \in \Xi} |\tau_{us}(u, v, r)|$ and $\tau_{rs}(u, v, r) < \max_{\mathcal{E} \in \Xi} |\tau_{rs}(u, v, r)|$. Now choose ε_1 such that

$$c_u \varepsilon_1 < \Delta_2, \quad c_r \varepsilon_1 < \Delta_2. \quad (6.67)$$

Then from (6.51) we have

$$\tau_{us}(\hat{u}, \hat{v}, \hat{r}) < S_{us}, \quad \tau_{rs}(\hat{u}, \hat{v}, \hat{r}) < S_{rs}$$

which means that the saturation does not take effect when $t \in [T_s, T_e]$. With the same analysis, for parameter estimators, the saturation will not take effect for $t \in [T_s, T_e]$ either. The following fact is given.

Fact 6.4. *The time derivative of V_2 in (6.28) along (6.52) and (6.54) satisfies*

$$\dot{V}_2 \leq -k_1 \chi^2 - k_2 \gamma^2 - (k_3 + d_{11}) \tilde{u}^2 - (k_4 + d_{33}) \tilde{r}^2 + 0.2785 \sum_{i=1}^4 \epsilon_i + k \varepsilon_1 \quad (6.68)$$

for $t \in [T_s, T_e]$ where k is a known positive constant depending on the initial condition and Δ_1 .

Proof: See Appendix D. □

Next we will show that T_e in Lemma 6.3 is ∞ , namely $\mathcal{E}$ will never exit Ξ and thus Ξ is an invariant set.

Lemma 6.4. Ξ is an invariant set.

Proof: Define a new set

$$\Psi = \left\{ e : k_1\chi^2 + k_2\gamma^2 + (k_3 + d_{11})\tilde{u}^2 + (k_4 + d_{33})\tilde{r}^2 \leq 0.2785 \sum_{i=1}^4 \epsilon_i + k\varepsilon_1 \right\} \quad (6.69)$$

where e is defined in (6.40). Also denote κ as

$$\kappa = \max_{e \in \Psi} V_e \quad (6.70)$$

where $V_e = V_1 + \frac{m_{11}}{2}\tilde{u}^2 + \frac{m_{33}}{2}\tilde{r}^2 = V_2 - \frac{1}{2} \sum_{i=1}^3 \tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i$. From (6.28) and (6.69), the design parameter ε_1 and ϵ_i , $i = 1, \dots, 4$ can be chosen to make κ satisfies

$$\kappa < \frac{1}{2}\Delta_1 \quad (6.71)$$

From the definitions of V_e and Ψ we know this can be easily done. We now divide Ξ into two set Ξ_1 and Ξ_2 as follows

$$\Xi_1 = \{\mathcal{E} : e \in \Psi; \mathcal{E} \in \Xi\} \quad \Xi_2 = \{\mathcal{E} : e \notin \Psi; \mathcal{E} \in \Xi\}.$$

Clearly $\Xi = \Xi_1 \cup \Xi_2$ and $\Xi_1 \cap \Xi_2 = \emptyset$.

Ξ is divided so that when $\mathcal{E} \in \Xi_1$, $V_2 < V_e + \nu_{\max} < \frac{1}{2}\Delta_1 + \nu_{\max} = \varrho - \frac{1}{2}\Delta_1$, and when $\mathcal{E} \in \Xi_2$, $\dot{V}_2 < 0$ from (6.68) and (6.69).

When $\mathcal{E} \in \Xi_1$, $\dot{V}_2$ may be positive. But since $\dot{V}_2 < c_4$, $V_2(t)$ is continuous. Therefore if $\mathcal{E}$ exits Ξ_1 , it will only enter Ξ_2 , because $V_2(t) < \varrho - \frac{1}{2}\Delta_1$ when $\mathcal{E} \in \Xi_1$. Thus when $\varrho - \frac{1}{2}\Delta_1 \leq V_2(t) < \varrho$, $\mathcal{E}$ is in Ξ_2 . In another word, $\mathcal{E}$ will not exit Ξ from Ξ_1 directly.

Note that T_e defined in Lemma 6.3 is the first time that $\mathcal{E}$ exits Ξ satisfying $V_2(T_e) = \varrho$. Then $V_2(T_e - \Delta t) < \varrho$, where $0 < \Delta t < T_e$ is an arbitrarily small constant. As stated, $\mathcal{E}(T_e - \Delta t) \in \Xi_2$, so $\dot{V}_2(T_e - \Delta t) < 0$ and thus as Δt decreases $V_2(T_e - \Delta t)$ is monotonically decreasing. This contradicts that $V_2(T_e) = \varrho$. So $\mathcal{E}$ will not exit Ξ from Ξ_2 either, i.e. $T_e = \infty$. Therefore Ξ is an invariant set. This

completes the proof. $\square$

Since Ξ is an invariant set, then from Lemma 6.2 and Lemma 6.3,

$$\tau_{us}(\hat{u}, \hat{v}, \hat{r}) < S_{us}, \quad \tau_{rs}(\hat{u}, \hat{v}, \hat{r}) < S_{rs}$$

for $t \in [T_s, \infty)$, which means the saturation will not take effect after T_s , and we have

$$\dot{V}_2 \leq -k_1\chi^2 - k_2\gamma^2 - (k_3 + d_{11})\tilde{u}^2 - (k_4 + d_{33})\tilde{r}^2 + \varepsilon \quad (6.72)$$

where $\varepsilon = 0.2785 \sum_{i=1}^4 \epsilon_i + k\varepsilon_1$ for $t \in [T_s, \infty)$. Now based on Lemmas 6.1-6.4, we have the following theorem.

Theorem 6.2. *Consider the system consisting of the underactuated ship system (6.1) and (6.2), the adaptive output feedback controllers (6.52) with observers (6.44), (6.46) and (6.47), and parameter estimators (6.54). Under **Assumptions** 6.1, 6.2 and 6.3, and assuming $\mathcal{E}(0) \in \Xi$, then there exists a computable ε_1^* satisfying (6.60), (6.64) and (6.67), such that for all $\varepsilon_1 \in (0, \varepsilon_1^*)$, (i) the signals in the closed-loop systems are bounded; (ii) the mean-square tracking errors are of $O(\varepsilon)$. (ii) the position error z_e and orientation error ϕ_e converge to arbitrarily small residual sets at a pre-specified exponential rate.*

Proof: (i) Since Ξ is an invariant set, it is obvious that all signals contained in the closed-loop system are bounded.

(ii) Due to the parameter projection, $\tilde{\theta}_{1i}$, $\tilde{\theta}_{2i}$ and $\tilde{\theta}_{3j}$, $1 \leq i \leq 9$, $1 \leq j \leq 4$ are bounded. From (6.72) we know

$$k_{\min}(\chi^2 + \gamma^2 + \tilde{u}^2 + \tilde{r}^2) + \dot{V}_2(t) \leq \varepsilon \quad (6.73)$$

where $k_{\min} = \min(k_1, k_2, k_3 + d_{11}, k_4 + d_{33})$. Integrating (6.73), and using the fact the $V_2(t)$ is bounded, we get

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^T e \leq \frac{\varepsilon}{k_{\min}}$$

which means that the mean-square position errors and virtual control errors are of the order of $O(\varepsilon)$, i.e., e can be adjusted arbitrarily small if ε is small enough. From (6.15), the properties of $S_1(\chi)$, $S_2(\gamma)$ and (6.11), we obtain that

$$\begin{aligned} \sigma &\leq \alpha_1 \eta_\infty < z_e(t) < \beta_1 \eta_0 \leq \delta \\ -\frac{\pi}{2} + \sigma_2 &\leq -\alpha_2 \eta_0 < \phi_e(t) < \beta_2 \eta_0 \leq \frac{\pi}{2} - \sigma_2, \quad \forall t \geq 0. \end{aligned} \quad (6.74)$$

if $z_e(0) \in (\alpha_1 \eta_\infty, \beta_1 \eta_0)$ and $\phi_e(0) \in (-\pi/2 + \sigma_2, \pi/2 - \sigma_2)$, which means (6.7) can always be satisfied.

We also have $\lim_{t \rightarrow \infty} \phi_e(t) = \varepsilon_\star$ and $\lim_{t \rightarrow \infty} z_e(t) = (\alpha_1 + \epsilon + \varepsilon^\star) \eta_\infty \in (\alpha_1 \eta_\infty, \beta_1 \eta_\infty)$ where $\varepsilon_\star$ and $\varepsilon^\star$ are unknown constants that can be arbitrarily small if ε is small enough, due to the continuities of $S_1(\chi)$ and $S_2(\gamma)$ and the fact that χ and γ are arbitrarily small. It implies that the underactuated ship can follow its reference trajectory with arbitrary small position error z_e and orientation error ϕ_e .

(iii) From (6.9)

$$z_e < \beta_1 \left((\eta_0 - \eta_\infty) e^{-at} + \eta_\infty \right), \phi_e < \beta_2 \left((\eta_0 - \eta_\infty) e^{-at} + \eta_\infty \right) \quad (6.75)$$

which means the position error z_e and orientation error ϕ_e will converge at a pre-specified exponential speed governed by positive constant a . This completes the proof. $\square$

Remark 6.3. *The keys for solving the adaptive output feedback control of the underactuated ships lie on several aspects. Firstly observers for v , v and r are designed separately to avoid the nonholonomic constraint. Secondly saturation is introduced to the output feedback controller and estimators. This will prevent the peaking phenomenon from entering the robot system. Thirdly a compact set Ξ of $\mathcal{E}$ is set up. The stability of the whole system is guaranteed by proving Ξ is an invariant set.*

Remark 6.4. *Our methodology is not only limited to the underactuated ships, but also applicable to the unsolved adaptive output feedback control problem of*

many other uncertain mechanical systems with nonholonomic constraints, such as underwater vehicles, VTOL aircrafts, aerospace vehicle et al.

6.5 Simulation Results

In this section, we present our simulation studies with a monohull ship having one propeller and one rudder, see Fig.6.2. The parameters of the ship are given in [18] as follows: $m_{11} = 120 \times 10^3 \text{kg}$, $m_{22} = 177.9 \times 10^3 \text{kg}$, $m_{33} = 636 \times 10^5 \text{kg}$, $d_{11} = 215 \times 10^2 \text{kg}$, $d_{22} = 147 \times 10^3 \text{kg}$, $d_{33} = 802 \times 10^4 \text{kg}$, $d_{u2} = 0.2d_{11}$, $d_{u3} = 0.1d_{11}$, $d_{v2} = 0.2d_{22}$, $d_{v3} = 0.1d_{22}$, $d_{r2} = 0.2d_{33}$, $d_{r3} = 0.1d_{33}$. These parameters are assumed to be unknown. Their maximum and minimum values are set to fluctuate 30% above and below their true values. The bounds of the external disturbances are assumed to be $\tau_{wu} = 10 \times 10^3(1 + \text{rand}(\cdot))$, $\tau_{wv} = 10 \times 10^3(1 + \text{rand}(\cdot))$, $\tau_{wr} = 10 \times 10^4(1 + \text{rand}(\cdot))$. where $\text{rand}(\cdot)$ is random noise with magnitude bounded by 1 and zero from below. This choice results in nonzero-mean disturbances.

The design parameters are chosen as follows: $k_1 = 2$, $k_2 = 3$, $k_3 = 8$, $k_4 = 8$, $\epsilon_1 = 0.2$, $\epsilon_2 = 0.2$, $\epsilon_3 = 0.1$, $\epsilon_4 = 4$, $\epsilon_* = 0.2$, $\epsilon^* = 5$, $u_0 = 10$, $\varepsilon_1 = 0.05$, $\eta_0 = 2$, $\eta_\infty = 0.1$, $a = 2$, $\alpha_1 = 1$, $\beta_1 = 40$, $\alpha_2 = 0.7$, $\beta_2 = 0.7$, $\Gamma_i^{-1} = 10I$, for $i = 1, 2, 3$. The reference trajectories are taken as $\{x_{od}, y_{od}\} = [s, 100 \sin(0.05s)]^T$. The initial conditions are set as $\{x(0), y(0), \phi(0), u(0), v(0), r(0)\} = \{-35, -10, 1.2, 0, 0, 0\}$. All initial values of parameter estimates are taken to be 60% of their assumed true values. The saturation bound calculated is $1.2 \times 10^7 \text{N}$. The results on the ship position, the tracking errors and the the torques for τ_u and τ_r evolving with time are shown in Fig.6.3, Fig.6.4, Fig.6.5 and Fig.6.6, respectively. It can be seen from the figures that the references can be followed with arbitrarily small position error and orientation error.

6.6 Conclusions

In this chapter, path following control of underactuated ships with unknown system parameters and external disturbances induced by wave, wind and ocean

Figure 6.2: A monohull ship. Courtesy <http://www.austal.com/>

Figure 6.3: Position and orientation of ship following a path $y = 100 \sin(0.05x)$

current is considered. By using the prescribed performance bound technique, we develop both state-feedback and output-feedback robust adaptive controllers. Besides showing system stability, the mean-square tracking errors are of order of an arbitrarily small design parameter, and the position error and the orientation error converge to arbitrarily small residual sets at a pre-specified exponential rate.

6.7 Appendix

Proof of Fact 6.1

From (6.30) τ_{us} and τ_{rs} are composed by $\tilde{u}$, $\tilde{r}$, ξ_1 , ξ_2 , χ , γ , ϕ_e , z_e , $\hat{\theta}_{1i}$, $\hat{\theta}_{2i}$, $\hat{\theta}_{3j}$, $i = 1, \dots, 9$, $j = 1, \dots, 4$, $\dot{s}$, η , $\dot{\eta}$, x'_d , y'_d , $\sin(\phi)$ and $\cos(\phi)$. From Assumption 1 and

Figure 6.4: Orientation error $\phi_e(t)$ Figure 6.5: Position error $z_e(t)$ Figure 6.6: Torques τ_u and τ_r with saturation limit 1.2×10^7 N

parameter projection, we know all these elements are bounded with computable upper-bounds (or within Ξ). So from (6.30) the upper-bounds of τ_{us} and τ_{rs} are computable within Ξ , i.e., there are two positive constants S'_{us} and S'_{rs} such that

$$\tau_{us}(u, v, r)|_{\mathcal{E} \in \Xi} \leq S'_{us}, \quad \tau_{rs}(u, v, r)|_{\mathcal{E} \in \Xi} \leq S'_{rs}$$

Proof of Fact 6.2

From (7.102)

$$\begin{aligned} \dot{V}_2 = & -k_1\chi^2 - k_2\gamma^2 - \chi\xi_1 \cos(\phi_e)\tilde{u} + \xi_2\gamma\tilde{r} - \xi_2\gamma(\cos(\phi)f_1 - \sin(\phi)f_2)\tilde{u} \\ & + m_{11}\tilde{u}\dot{\tilde{u}} + m_{33}\tilde{r}\dot{\tilde{r}} - \sum_{i=1}^3 \tilde{\theta}_i^T \Gamma_i^{-1} \dot{\tilde{\theta}}_i \end{aligned} \quad (6.76)$$

Since τ_u , τ_r and $\dot{\tilde{\theta}}_i$ are globally saturated. Thus from (6.27) and the proof of Fact 1 the upper-bound of $\dot{V}_2$ is computable, i.e. there exists a positive constant c_4 such that

$$\dot{V}_2 < c_4$$

Proof of Fact 6.3

From (6.55), $|u - \hat{u}| < c_2\varepsilon_1$, $|v - \hat{v}| < c_3\varepsilon_1$, $|r - \hat{r}| < c_1\varepsilon_1$ when $\mathcal{E} \in \Xi$. Thus from (6.37), $\hat{u}_d = a_1\hat{v} + a_2 < u_d + a_1c_3\varepsilon_1$, and similarly $\hat{r}_d < r_d + a_3c_3\varepsilon_1$. Taking the first item of Ω_1 as an example, $\hat{v}\hat{r} - vr = \hat{v}\hat{r} - \hat{v}r + \hat{v}r - vr < (|\hat{v}|c_1 + |r|c_3)\varepsilon_1 < (|v|c_1 + |r|c_3 + c_1c_3\varepsilon_1)\varepsilon_1 := \bar{c}\varepsilon_1$ where $\bar{c}$ is a known positive constant due to the fact that the upper-bounds of v and r are known. It can be similarly checked that this conclusion holds for each item in Ω_1 and Ω_2 when $\mathcal{E} \in \Xi$.

Suppose $\tilde{u} - \hat{\tilde{u}} < c_w\varepsilon_1$ where c_w is a positive constant, then based on the mean value theorem,

$$\tanh\left(\frac{\tilde{u}\hat{\theta}_{31}}{\epsilon_1}\right) - \tanh\left(\frac{\hat{\tilde{u}}\hat{\theta}_{31}}{\epsilon_1}\right) = \operatorname{sech}^2\left(\frac{\tilde{u}_x\hat{\theta}_{31}}{\epsilon_1}\right) \frac{\hat{\theta}_{31}}{\epsilon_1}(\tilde{u} - \hat{\tilde{u}}) \leq \frac{\hat{\theta}_{31}c_w}{\epsilon_1}\varepsilon_1$$

where $\tilde{u}_x \in (\tilde{u}, \hat{\tilde{u}})$. Similar conclusions also hold for other $\tanh(\cdot)$ in τ_{us} and τ_{rs}

when $\mathcal{E} \in \Xi$. Thus from (6.30) we can make the conclusion that when $\mathcal{E} \in \Xi$ there are two positive constants c_u and c_r such that

$$|\tau_{us}(\hat{u}, \hat{v}, \hat{r}) - \tau_{us}(u, v, r)| < c_u \varepsilon_1, \quad |\tau_{rs}(\hat{u}, \hat{v}, \hat{r}) - \tau_{rs}(u, v, r)| < c_r \varepsilon_1. \quad (6.77)$$

when $t \in [T_s, T_e]$. Thus (6.65) holds.

Proof of Fact 6.4

From (6.77), $\tilde{u}\tau_{us}(\hat{u}, \hat{v}, \hat{r}) < \tilde{u}\tau_{us}(u, v, r) + |\tilde{u}|c_u\varepsilon_1$ and $\tilde{r}\tau_{rs}(\hat{u}, \hat{v}, \hat{r}) < \tilde{r}\tau_{rs}(u, v, r) + |\tilde{r}|c_r\varepsilon_1$. Thus from (6.27) and (6.28)

$$\begin{aligned} \dot{V}_2 &= -k_1\chi^2 - k_2\gamma^2 - \chi\xi_1 \cos(\phi_e)\tilde{u} + \xi_2\gamma\tilde{r} - \xi_2\gamma(\cos(\phi)f_1 - \sin(\phi)f_2)\tilde{u} \\ &\quad + m_{11}\tilde{u}\dot{\tilde{u}} + m_{33}\tilde{r}\dot{\tilde{r}} - \sum_{i=1}^3 \tilde{\theta}_i^T \Gamma_i^{-1} \dot{\tilde{\theta}}_i \\ &< -k_1\chi^2 - k_2\gamma^2 - (k_3 + d_{11})\tilde{u}^2 - (k_4 + d_{33})\tilde{r}^2 + |\tilde{u}|c_u\varepsilon_1 + |\tilde{r}|c_r\varepsilon_1 + \tilde{u}\tilde{\theta}_1^T \Omega_1 \\ &\quad + \tilde{r}\tilde{\theta}_2^T \Omega_2 + \tilde{u}\tau_{wu}(t) + \tilde{u}\frac{m_{11}}{m_{22}}\frac{\partial u_d}{\partial v}\tau_{wv}(t) + \tilde{r}\tau_{wr}(t) + \tilde{r}\frac{m_{33}}{m_{22}}\frac{\partial r_d}{\partial v}\tau_{wv}(t) - \sum_{i=1}^3 \tilde{\theta}_i^T \Gamma_i^{-1} \dot{\tilde{\theta}}_i \\ &\quad - \tilde{u}\hat{\theta}_{31} \tanh\left(\frac{\tilde{u}\hat{\theta}_{31}}{\epsilon_1}\right) - \tilde{u}\hat{\theta}_{32} \frac{\partial u_d}{\partial v} \tanh\left(\frac{\partial u_d}{\partial v} \frac{\tilde{u}\hat{\theta}_{32}}{\epsilon_2}\right) - \tilde{r}\hat{\theta}_{33} \tanh\left(\frac{\tilde{r}\hat{\theta}_{33}}{\epsilon_3}\right) \\ &\quad - \tilde{r}\hat{\theta}_{34} \frac{\partial r_d}{\partial v} \tanh\left(\frac{\partial r_d}{\partial v} \frac{\tilde{r}\hat{\theta}_{34}}{\epsilon_4}\right) \end{aligned}$$

Obviously it has

$$\begin{aligned} \tilde{u}\tau_{wu}(t) + \tilde{u}\frac{m_{11}}{m_{22}}\frac{\partial u_d}{\partial v}\tau_{wv}(t) + \tilde{r}\tau_{wr}(t) + \tilde{r}\frac{m_{33}}{m_{22}}\frac{\partial r_d}{\partial v}\tau_{wv}(t) &\leq |\tilde{u}|\theta_{31} + |\tilde{u}|\frac{\partial u_d}{\partial v}|\theta_{32} \\ &\quad + |\tilde{r}|\theta_{33} + |\tilde{r}|\frac{\partial r_d}{\partial v}|\theta_{34} \end{aligned}$$

From (6.33) we have

$$\begin{aligned} |\tilde{u}|\hat{\theta}_{31} + |\tilde{u}|\frac{\partial u_d}{\partial v}|\hat{\theta}_{32} + |\tilde{r}|\hat{\theta}_{33} + |\tilde{r}|\frac{\partial r_d}{\partial v}|\hat{\theta}_{34} - \tilde{u}\hat{\theta}_{31} \tanh\left(\frac{\tilde{u}\hat{\theta}_{31}}{\epsilon_1}\right) - \tilde{u}\hat{\theta}_{32} \frac{\partial u_d}{\partial v} \tanh\left(\frac{\partial u_d}{\partial v} \frac{\tilde{u}\hat{\theta}_{32}}{\epsilon_2}\right) \\ - \tilde{r}\hat{\theta}_{33} \tanh\left(\frac{\tilde{r}\hat{\theta}_{33}}{\epsilon_3}\right) - \tilde{r}\hat{\theta}_{34} \frac{\partial r_d}{\partial v} \tanh\left(\frac{\partial r_d}{\partial v} \frac{\tilde{r}\hat{\theta}_{34}}{\epsilon_4}\right) \leq 0.2785 \sum_{i=1}^4 \epsilon_i \end{aligned}$$

Thus

$$\begin{aligned} \dot{V}_2 \leq & -k_1\chi^2 - k_2\gamma^2 - (k_3 + d_{11})\tilde{u}^2 - (k_4 + d_{33})\tilde{r}^2 + |\tilde{u}|c_u\varepsilon_1 + |\tilde{r}|c_r\varepsilon_1 + \hat{\tilde{u}}\tilde{\theta}_1^T\hat{\Omega}_1 \\ & + \hat{\tilde{r}}\tilde{\theta}_2^T\hat{\Omega}_2 + |\hat{\tilde{u}}|\tilde{\theta}_{31} + |\hat{\tilde{u}}|\frac{\partial u_d}{\partial v}|\tilde{\theta}_{32} + |\hat{\tilde{r}}|\tilde{\theta}_{33} + |\hat{\tilde{r}}|\frac{\partial r_d}{\partial v}|\tilde{\theta}_{34} - \sum_{i=1}^3 \tilde{\theta}_i^T \Gamma_i^{-1} \dot{\hat{\theta}}_i + \underline{k}\varepsilon_1 \end{aligned}$$

where $\underline{k}$ is a positive constant. Due to the choices of $\dot{\hat{\theta}}_i$ in (6.54), and the fact that for $t \in [T_s, T_e]$, the saturation will not take effectiveness for $\dot{\hat{\theta}}_i$, thus for $t \in [T_s, T_e]$,

$$\dot{V}_2 \leq -k_1\chi^2 - k_2\gamma^2 - (k_3 + d_{11})\tilde{u}^2 - (k_4 + d_{33})\tilde{r}^2 + 0.2785 \sum_{i=1}^4 \epsilon_i + k\varepsilon_1$$

Chapter 7

Finite-time Consensus Control of Multiple Mechanical Systems

In this chapter, the finite-time consensus control of a group of mechanical systems with parametric uncertainties under both directed and undirected topology is considered. New adaptive finite time continuous distributed control algorithms are proposed for the multi-agent systems under two different scenarios. Firstly for the leaderless multi-agent systems, it is shown that the states of the mechanical systems can reach a consensus within finite time. Under this scenario, the rendezvous seeking problem is also solved when only some of the agents have access to the rendezvous location. Secondly in the leader-follower scenario of a group of mechanical systems, it is demonstrated that perfect consensus can be reached or the leader-following errors converge to a ball with arbitrarily small radius in finite time. Transient performances in terms of convergence rates and time are also analyzed. Finally simulation results illustrate and verify the effectiveness of the proposed schemes.

7.1 Introduction

Distributed coordination for a group of dynamic agents has attracted many researchers in recent years, due to the broad applications of multi-agent systems in many areas, such as cooperative control of unmanned air vehicles (UAVs) in [140],

formation control in [52, 57, 139], flocking control in [54, 113], distributed sensor networks in [114], attitude alignment for a clusters of satellites in [115], congestion control in communication networks in [116] etc. The consensus of multi-agent systems means the states of all the agents converge to a common value by invoking some control schemes for each agent in the group. Consensus control problem is studied intensively since the state synchronization is often a basic requirement in many group behaviors, such as flocking. Related works on consensus control include, but not limited to, [35, 59, 117–120, 122].

Consensus control can be divided into constant value consensus and time-varying trajectory consensus according to the pattern of final value. In [120] the problem of static information consensus among multi-agents in the presence of limited and unreliable information exchange with dynamically changing topologies is considered, which shows that static consensus under dynamically changing interaction topologies can be achieved asymptotically if the directed interaction graphs have a spanning tree frequently enough. Asymptotical time-varying trajectory consensus is rather difficult if only parts of the agents in the group has access to the desired trajectory. In [122] multi-agent consensus problem with an active leader and variable interconnection topology is considered. To track the leader, a neighbor-based local controller together with a neighbor-based state observer is developed for each agent, while assuming leader's acceleration is partially known to all the agents in the group. Similar idea also appears in [117], where distributed observer is designed for each agent using local information while assuming leader's acceleration is globally known. In [29] the unknown time-varying references can be compensated for by introducing discontinuous *sign* functions to cope with the unknown reference trajectories in generating the control signals, however chattering phenomenon of discontinuous input may affect the performance in practical implementation. In [64] consensus problem of multiple uncertain mechanical systems with a desired trajectory is considered. The desired trajectory is available to a portion of a group of mechanical systems. By assuming that the references are composed by a set of basic vector functions, which are available to all the agents, distributed adaptive control laws are proposed such that the state of each system

asymptotically converges to the desired trajectory with information interchange between agents. In [123] adaptive distributed leader-following control for a group of uncertain Euler-Lagrange systems is considered, where only a subset of the followers have access to the leader and the followers have only local interactions.

An important performance indicator for the consensus problem is the convergence rate. Most of the existing consensus control schemes for multi-agent systems so far achieve asymptotical convergence, namely the convergence rate at best is exponential, which means it needs infinite time for the tracking errors to converge to the origin. It is shown in [35] that the second smallest eigenvalue of the interaction Laplacian matrix quantifies the rate of convergence. To get better convergence rate, several researchers try to find better interaction graph such as to get a larger ‘second smallest eigenvalue’. [124] consider the problem of finding the best vertex positional configuration so that the ‘second smallest eigenvalue’ of the associated interaction graph is maximized. In [125] by using semi-definite convex programming the weights among the agents are designed such that convergence rate is increased. However, all these efforts are only to choose proper interaction graphs, but not to find control schemes to achieve higher performance. In practice, it is often more desired that the consensus be reached in finite time. Thus several researchers invoke the finite-time control schemes to guarantee that the consensus can be reached within finite time. In addition, finite-time control schemes possess better disturbance rejection properties as shown in [128]. In [126] a finite-time formation control framework for multi-agent systems of first-order dynamics is developed. In [129] a continuous finite time control scheme is developed for the state consensus problems for multi-agent systems of first-order dynamics. [130] address the finite-time consensus problem for leaderless and leader-follower multi-agent systems of second-order double integrator dynamics. In [131] the finite-time consensus tracking problem with one leader and the finite-time containment control problem with multiple leaders are considered.

Nonetheless, uncertainties do exist in practical systems. Adaptive control is one of the most effective ways to cope with parametric uncertainties by employing on-line parameter estimators. Great deal of works has been done for adaptive con-

trol of linear and nonlinear systems with unknown parameters, see for example [8]. It should be emphasized that for adaptive control systems it is very difficult to establish exponential convergence mainly due to the use of online parameter estimators that gives arise to a highly nonlinear closed loop system, which hinders the potential applications of adaptive control. Therefore ensuring finite-time convergence will be more interesting and significant for adaptive control systems. However, achieving adaptive finite-time control is rather challenging. Certain key techniques employed in existing adaptive control literature cannot be applied. For example, the Barbalat's lemma normally adopted for analyzing asymptotic convergence cannot be applied to the analysis of finite-time convergence. On the other hand, the finite-time convergence analysis tools adopted in systems without parametric uncertainties such as those in [127] and [130] cannot be applied to adaptive finite time control directly. In [122] finite-time stabilization control for a class of single nonlinear system with parametric uncertainties is investigated with a backstepping-like recursive control scheme, but the convergence time is not expressed explicitly. Furthermore, the existing finite-time consensus control schemes mainly focus on first-order integrator as in [106] or second-order double-integrator as in [130]. Note that adaptive consensus control for uncertain multi-agent nonlinear systems with an explicitly established convergence-time is still unavailable, to the best of the authors' knowledge.

In this chapter, we address such an issue for a group of general nonlinear mechanical systems with parametric uncertainties under both undirected and directed topology. New continuous adaptive distributed finite-time controllers are proposed for each agent in the group under two different scenarios of consensus control. Our first scenario is the leaderless consensus control with an undirected graph. An adaptive distributed controller is designed such that all the positions of the agents converge to a consensus state with a configuration within finite-time. Under this scenario, by assuming that only some of the agents have accesses to the static rendezvous location, the proposed adaptive distributed controllers ensure that all the agents converge to the rendezvous location with a configuration in finite-time. The second situation is the finite-time leader-follower consensus con-

trol of multi-agent systems of a directed graph with a virtual leader. We adopt the hierarchical decomposition method proposed recently in [132] to design a distributed adaptive finite-time consensus controller for each agent. It is shown that consensus without any error can be reached if certain condition of the mechanical systems is satisfied. Otherwise an error exists. But the error can be made arbitrarily small by choosing design parameters. We also establish the transient performance for the tracking errors and the virtual control errors.

7.2 Problem Formulation

7.2.1 System Model

We consider a class of multiple mechanical nonlinear systems

$$M_i \ddot{q}_i + C_i(q_i, \dot{q}_i) \dot{q}_i + D_i(q_i) \dot{q}_i = \tau_i \quad (7.1)$$

where $q_i = (q_{i1}, \dots, q_{im})^T \in \mathbb{R}^m$, $i = 1, \dots, n$ is the state of the i th system, $\tau_i \in \mathbb{R}^m$ is the control input vector, $M_i \in \mathbb{R}^{m \times m}$ is an inertia matrix, $C_i(q_i, \dot{q}_i)$ is the centripetal and coriolis matrix and $D_i(q_i) \dot{q}_i$ denotes the friction terms. Denote $v_i = \dot{q}_i$. The following assumptions are needed for the design of finite-time adaptive consensus controllers.

Assumption 7.1. *The graph $\mathcal{G}$ is connected.*

Assumption 7.2. *$M_i = \text{diag}\{f_{i1}, \dots, f_{im}\}$ where f_{ik} , $k = 1, \dots, m$ are unknown positive constants. $(C_i(q_i, v_i)v_i + D_i(q_i)v_i) = G_i(q_i, v_i)\theta_i$ where $\theta_i \in \mathbb{R}^l$ denotes the vector of unknown parameters and $G_i(q_i, v_i) \in \mathbb{R}^{m \times l}$ does not contain unknown parameters. Furthermore, $\|G_i(q_i, v_i)\|_1 \leq \rho P(\|v_i\|)$, where ρ is a known positive constant and $P(\cdot)$ is a polynomial with no constant term, i.e. $P(0) = 0$. f_{ik} , $k = 1, \dots, m$ and θ_i are in known compact set.*

Remark 7.1.

- For a typical mechanical system, q_i appears in the centripetal force and Coriolis force. For a constant v_0 , there exists a constant p such that $\|G_i(q_i, v_0)\| <$

p. Taking the two-link robot manipulator as an example, q_i is the joint-variable vector, consisting of joint angles θ_i and a joint offset d_i . The joint angles appear in $G_i(q_i, v_i)$ in terms of $\sin(\theta_i)$ or $\cos(\theta_i)$, and a joint offset d_i is always bounded.

- *Typical examples of mechanical system satisfying Assumption 7.2 include Newton-Euler rigid body and three-link cylindrical robot manipulator, as described in [137].*

7.3 Control Design and System Analysis

System (7.1) can be written as

$$\begin{aligned}\dot{q}_i &= v_i \\ M_i \dot{v}_i + C_i(q_i, v_i)v_i + D_i(q_i)v_i &= \tau_i\end{aligned}\tag{7.2}$$

where $v_i = [v_{i1}, \dots, v_{im}]^T$, $i \in \mathcal{V}$ denotes the velocity. We will design distributed adaptive control laws by investigating three cases of the consensus problem, i.e. leaderless, rendezvous and leader-follower. In the first case the designed control law should ensure that all the agents reach consensus in finite time without additional information, or all the agents are forced to arrive at a static rendezvous location within finite time, while only some of the agents have access to this rendezvous location. In the second case, we assume that there is a virtual leader, which is available to part of the agents, and the control law is designed such that all the agents follow the virtual leader within finite time.

7.3.1 Leaderless Consensus Control with undirected topology

In this subsection, a distributed adaptive control law is designed for each agent such that the positions of all the agents converge to a consensus location based on its neighbor's information. The controller is designed using a 'backstepping-like'

procedure. In the first step virtual control v_i^* for v_i , $i = 1, \dots, n$ is designed such that the position error converge to zero in finite-time. In the second step adaptive controllers and parameter estimators are designed such that v_i converge to v_i^* in finite-time. These finite time convergence results are summarized in Theorem 7.1. The transient performance of the close-loop adaptive control systems is established and presented in Theorem 7.2, by showing that the position errors and the virtual control errors converge faster than an exponential rate.

7.3.1.1 Controller Design

• *Step 1.* Let $x_j = [q_{1j} - \delta_{1j}, \dots, q_{nj} - \delta_{nj}]^T$, $x = [x_1^T, \dots, x_m^T]^T$, $j = 1, \dots, m$, where δ_{kj} , $k = 1, \dots, n$ are constants denoting the final consensus configuration such that $q_{ij} - q_{kj} = \delta_{ij} - \delta_{kj}$, $i, k \in \mathcal{V}$. Define a Lyapunov function

$$\begin{aligned} V_1(t) &= \frac{1}{2} x^T (I_m \otimes L) x \\ &= \frac{1}{4} \sum_{j=1}^m \sum_{i=1}^n \sum_{k \in N_i} a_{ik} (q_{ij} - q_{kj} - \delta_{ij} + \delta_{kj})^2 \end{aligned} \quad (7.3)$$

Clearly, V_1 is positive definite with respect to $q_{ij} - q_{kj} - \delta_{ij} + \delta_{kj}$. Taking the derivative of V_1 yields

$$\begin{aligned} \dot{V}_1 &= \dot{x}^T (I_m \otimes L) x \\ &= \sum_{j=1}^m \sum_{i=1}^n \left[\sum_{k \in N_i} a_{ik} (q_{ij} - q_{kj} - \delta_{ij} + \delta_{kj}) \right] \dot{q}_{ij} \end{aligned}$$

Let

$$e_{ij} = \sum_{k \in N_i} a_{ik} (q_{ij} - q_{kj} - \delta_{ij} + \delta_{kj}) \quad (7.4)$$

Then

$$\dot{V}_1 = \sum_{j=1}^m \sum_{i=1}^n e_{ij} v_{ij}$$

Denoting v_{ij}^* as the virtual control of v_{ij} , we have

$$\dot{V}_1 = \sum_{j=1}^m \sum_{i=1}^n e_{ij} v_{ij}^* + \sum_{j=1}^m \sum_{i=1}^n e_{ij} (v_{ij} - v_{ij}^*)$$

By choosing the virtual control as

$$v_{ij}^* = -c_1 e_{ij}^{\frac{2\sigma-1}{2\sigma+1}} \quad (7.5)$$

where $\sigma \geq 2$ is a positive integer and c_1 is a positive constant to be designed, we get

$$\begin{aligned} \dot{V}_1 &= \sum_{j=1}^m \sum_{i=1}^n -c_1 e_{ij}^{\frac{4\sigma}{2\sigma+1}} + \sum_{j=1}^m \sum_{i=1}^n e_{ij} (v_{ij} - v_{ij}^*) \\ &\leq -c_1 \left(\sum_{j=1}^m \sum_{i=1}^n e_{ij}^2 \right)^{\frac{2\sigma}{2\sigma+1}} + \sum_{j=1}^m \sum_{i=1}^n e_{ij} (v_{ij} - v_{ij}^*) \end{aligned} \quad (7.6)$$

Let $e_j = [e_{1j}, \dots, e_{nj}]^T$, then $e_j = Lx_j$. Thus $e_j^T e_j = x_j^T L^T Lx_j = x_j^T L^2 x_j$. Since L is a diagonalizable symmetric semi-positive definite matrix, then it is easy to prove that $L^{1/2}$ is also a symmetric semi-positive definite matrix, and $L^2 = L^{1/2} L L^{1/2}$. Let $w = L^{1/2} \mathbf{1}$, we get $w^T w = (L^{1/2} \mathbf{1})^T (L^{1/2} \mathbf{1}) = \mathbf{1}^T L \mathbf{1} = 0$. Thus $w = \mathbf{0}$, which yields $w^T x_j = 0$, i.e. $\mathbf{1}^T L^{1/2} x_j = 0$. Then according to Lemma 2.5

$$e_j^T e_j = (L^{1/2} x_j)^T L (L^{1/2} x_j) \geq \lambda_2 x_j^T L x_j. \quad (7.7)$$

Let $e = [e_1^T, \dots, e_m^T]^T$, then we get

$$\sum_{j=1}^m \sum_{i=1}^n e_{ij}^2 = e^T e \geq \lambda_2 x^T (I_m \otimes L) x = 2\lambda_2 V_1 \quad (7.8)$$

• *Step 2.* Let $g = \frac{2\sigma-1}{2\sigma+1}$. Invoke the *adding a power integrator* technique as in [135] by defining another Lyapunov function

$$W_i = \frac{1}{2^{1-g}} \sum_{j=1}^m f_{ij} \int_{v_{ij}^*}^{v_{ij}} (s^{1/g} - v_{ij}^{*1/g})^{2-g} \mathbf{d}s \quad (7.9)$$

and a new variable

$$\xi_{ij} = v_{ij}^{1/g} - v_{ij}^{*1/g} \quad (7.10)$$

where f_{ij} is defined in Assumption 7.2. From [134], W_i is positive semi-definite and C^1 . Taking the time derivative of W_i , we get

$$\begin{aligned} \dot{W}_i = & \frac{1}{2^{1-g}} \sum_{j=1}^m \left[f_{ij} \xi_{ij}^{2-g} \dot{v}_{ij} \right. \\ & \left. - (2-g) f_{ij} \int_{v_{ij}^*}^{v_{ij}} (s^{1/g} - v_{ij}^{*1/g})^{1-g} \mathbf{d}s \sum_{k \in \bar{N}_i} \frac{\partial v_{ij}^{*1/g}}{\partial q_{kj}} \dot{q}_{kj} \right] \end{aligned}$$

where $\bar{N}_i = N_i \cup \{i\}$. Since

$$\begin{aligned} & \left| \int_{v_{ij}^*}^{v_{ij}} (s^{1/g} - v_{ij}^{*1/g})^{1-g} \mathbf{d}s \right| \leq |v_{ij} - v_{ij}^*| |\xi_{ij}|^{1-g} \\ & = \left| \left(v_{ij}^{1/g} \right)^g - \left(v_{ij}^{*1/g} \right)^g \right| |\xi_{ij}|^{1-g} \leq 2^{1-g} |\xi_{ij}^g| |\xi_{ij}^{1-g}| = 2^{1-g} |\xi_{ij}| \end{aligned} \quad (7.11)$$

we have

$$\begin{aligned} \dot{W}_i & \leq \sum_{j=1}^m \frac{f_{ij}}{2^{1-g}} \xi_{ij}^{2-g} \dot{v}_{ij} + (2-g) \sum_{j=1}^m f_{ij} |\xi_{ij}| \sum_{k \in \bar{N}_i} \left| \frac{\partial v_{ij}^{*1/g}}{\partial q_{kj}} \right| |v_{kj}| \\ & = \frac{1}{2^{1-g}} \xi_i^T [-(C_i(q_i, v_i) v_i + D_i v_i) + \tau_i] \\ & \quad + (2-g) \sum_{j=1}^m f_{ij} |\xi_{ij}| \sum_{k \in \bar{N}_i} \left| \frac{\partial v_{ij}^{*1/g}}{\partial q_{kj}} \right| |v_{kj}| \end{aligned}$$

where

$$\xi_i = [\xi_{i1}^{2-g}, \dots, \xi_{im}^{2-g}]^T. \quad (7.12)$$

Let $\hat{\theta}_i$, $i \in \mathcal{V}$, $j = 1, \dots, m$ denote the estimate of θ_i and define $\tilde{\theta}_i = \hat{\theta}_i - \theta_i$. Then

$$\begin{aligned} \dot{W}_i &\leq \frac{1}{2^{1-g}} \xi_i^T \left(-G_i(q_i, v_i) \hat{\theta}_i + \tau_i \right) + \frac{1}{2^{1-g}} \xi_i^T G_i(q_i, v_i) \tilde{\theta}_i \\ &\quad + (2-g)c_1^{1/g} \sum_{j=1}^m f_{ij} |\xi_{ij}| \left(\mu |v_{ij}| + \eta \sum_{k \in N_i} |v_{kj}| \right) \end{aligned}$$

where $\mu = \max_{\forall i \in \mathcal{V}} \left\{ \sum_{j \in N_i} a_{ij} \right\}$ and $\eta = \max_{\forall i, j \in \mathcal{V}} \{a_{ij}\}$. From Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned} \sum_{j=1}^m \sum_{i=1}^n e_{ij} (v_{ij} - v_{ij}^*) &\leq \sum_{j=1}^m \sum_{i=1}^n 2^{1-g} |e_{ij}| |\xi_{ij}|^g \\ &\leq \frac{2^{1-g}}{1+g} \sum_{j=1}^m \sum_{i=1}^n \left(e_{ij}^{1+g} + g \xi_{ij}^{1+g} \right) \end{aligned}$$

Define a new Lyapunov function

$$V_2 = V_1 + \sum_{i=1}^n W_i \quad (7.13)$$

Then

$$\begin{aligned} \dot{V}_2 &\leq -c_1 \sum_{j=1}^m \sum_{i=1}^n e_{ij}^{\frac{4\sigma}{2\sigma+1}} + \frac{2^{1-g}}{1+g} \sum_{j=1}^m \sum_{i=1}^n \left(e_{ij}^{1+g} + g \xi_{ij}^{1+g} \right) \\ &\quad + \frac{1}{2^{1-g}} \sum_{i=1}^n \xi_i^T \left(-G_i(q_i, v_i) \hat{\theta}_i + \tau_i \right) \\ &\quad + (2-g)c_1^{1/g} \sum_{i=1}^n \sum_{j=1}^m f_{ij} |\xi_{ij}| \left(\mu |v_{ij}| + \eta \sum_{k \in N_i} |v_{kj}| \right) \\ &\quad + \frac{1}{2^{1-g}} \sum_{i=1}^n \xi_i^T G_i(q_i, v_i) \tilde{\theta}_i \end{aligned} \quad (7.14)$$

Based on Lemmas 2.2-2.4 and (7.10),

$$\begin{aligned}
& (2-g)c_1^{1/g} \sum_{i=1}^n \sum_{j=1}^m f_{ij} |\xi_{ij}| \left(\mu |v_{ij}| + \eta \sum_{k \in N_i} |v_{kj}| \right) \\
& \leq \sum_{i=1}^n \sum_{j=1}^m \left(\frac{f_{ij} \mu g + \eta n_i g}{1+g} |e_{ij}|^{1+g} + (2-g)c_1^{1/g} \left(f_{ij} \mu + \right. \right. \\
& \quad \left. \left. \frac{f_{ij} c_1^2 (2-g)c_1^{1/g} + \eta n_i + \eta c_1^2 (2-g)c_1^{1/g} + \eta n_i g}{1+g} \right) |\xi_{ij}|^{1+g} \right)
\end{aligned}$$

where $n_i = \dim N_i$. Since $F_i^{-1} = \text{diag}(1/f_{i1}, \dots, 1/f_{im})$, the torque is designed as

$$\tau_i = G(q_i, v_i) \hat{\theta}_i - c_2 \xi_i^* \quad (7.15)$$

where $\xi_i^* = [\xi_{i1}^{2g-1}, \dots, \xi_{im}^{2g-1}]^T$ and c_2 is a positive constant to be chosen. From (7.14) and (7.15), we get

$$\begin{aligned}
\dot{V}_2 & \leq -k_1 \sum_{j=1}^m \sum_{i=1}^n e_{ij}^{\frac{4\sigma}{2\sigma+1}} - k_2 \sum_{j=1}^m \sum_{i=1}^n \xi_{ij}^{\frac{4\sigma}{2\sigma+1}} \\
& \quad + \frac{1}{2^{1-g}} \sum_{i=1}^n \xi_i^T G_i(q_i, v_i) \tilde{\theta}_i
\end{aligned} \quad (7.16)$$

where

$$\begin{aligned}
k_1 &= c_1 - \frac{2^{1-g}}{1+g} - \frac{f_{\max} \mu g + \eta n_i g}{1+g} \\
k_2 &= c_2 - \frac{g 2^{1-g}}{1+g} - (2-g)c_1^{1/g} \left(f_{\max} \mu + \right. \\
& \quad \left. \frac{f_{\max} c_1^2 (2-g)c_1^{1/g} + \eta n_i + \eta c_1^2 (2-g)c_1^{1/g} + \eta n_i g}{1+g} \right)
\end{aligned} \quad (7.17)$$

with $f_{\max} \geq \max_{j=1, \dots, m} (f_{1j}, \dots, f_{nj})$ being a positive constant known from Assumption 7.2. Thus we can find c_1 and c_2 such that $k_1 > 0$ and $k_2 > 0$.

Consider the following Lyapunov function

$$V_3 = V_2 + \frac{1}{2} \sum_{i=1}^n \tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i. \quad (7.18)$$

where Γ_i is a positive diagonal matrix. Then

$$\begin{aligned} \dot{V}_3 \leq & -k_1 \sum_{j=1}^m \sum_{i=1}^n e_{ij}^{\frac{4\sigma}{2\sigma+1}} - k_2 \sum_{j=1}^m \sum_{i=1}^n \xi_{ij}^{\frac{4\sigma}{2\sigma+1}} \\ & + \sum_{i=1}^n \tilde{\theta}_i^T \Gamma_i^{-1} \left(\dot{\theta}_i + \frac{1}{2^{1-g}} \Gamma_i G_i(q_i, v_i)^T \xi_i \right). \end{aligned}$$

The parameter update law for θ_i is

$$\dot{\hat{\theta}}_{ik} = \text{Proj}(\beta_i(k), \hat{\theta}_{ik}), \quad k = 1, \dots, l \quad (7.19)$$

where $\beta_i(k)$ is the k th element of $\beta_i = -\frac{1}{2^{1-g}} \Gamma_i G_i(q_i, v_i)^T \xi_i$. The operator $\text{Proj}(\cdot, \cdot)$ is a Lipschitz continuous projection algorithm in [105] which is defined as follows:

$$\text{Proj}(a, \hat{b}) = \begin{cases} a & \text{if } \mu(\hat{b}) \leq 0 \\ a & \text{if } \mu(\hat{b}) \geq 0 \text{ and } \mu'(\hat{b})a \leq 0 \\ (1 - \mu(\hat{b}))a & \text{if } \mu(\hat{b}) > 0 \text{ and } \mu'(\hat{b})a > 0 \end{cases} \quad (7.20)$$

where $\mu(\hat{b}) = \frac{\hat{b}^2 - b_M^2}{\epsilon^2 + 2\epsilon b_M}$, $\mu'(\hat{b}) = \frac{\partial \mu(\hat{b})}{\partial \hat{b}}$, ϵ is an arbitrarily small positive constant, b_M is a positive constant satisfying $|\hat{b}| < b_M$.

Then from (7.16), (7.18), (7.19) and based on the property of the parameter projection in [8], we get

$$\dot{V}_3 \leq -k_1 \sum_{j=1}^m \sum_{i=1}^n e_{ij}^{\frac{4\sigma}{2\sigma+1}} - k_2 \sum_{j=1}^m \sum_{i=1}^n \xi_{ij}^{\frac{4\sigma}{2\sigma+1}} \quad (7.21)$$

7.3.1.2 Stability Analysis

We are now at the position to state our first result in the following theorem.

Theorem 7.1. *Consider the undirected leaderless multi-agent uncertain systems (7.1), under the control of the distributed adaptive controllers (7.15) with parameter estimators in (7.19). If Assumptions 7.1 and 7.2 are satisfied, then the positions of the group of mechanical systems will reach consensus with specified*

configurations δ_{ij} in finite time T satisfying

$$T \leq \frac{\varrho_1^{1-\kappa}}{k_x(1-\kappa)} + \frac{V_3(t_0) - \varrho_1}{d_{v_3}} \quad (7.22)$$

where κ , k_x , ϱ_1 and d_{v_3} are computable.

Proof: From (7.21) we get $\dot{V}_3 \leq 0$. Therefore e_{ij} and $\tilde{\theta}_i$ are bounded and it can be easily checked that $\lim_{t \rightarrow \infty} e_{ij} = 0$. Since we cannot use $\dot{V}_3 + cV_3^\alpha \leq 0$ to establish the finite-time stability of e_{ij} and ξ_{ij} due to the existence of the terms $\tilde{\theta}_i^T \tilde{\theta}_i$, we will prove that there exist two positive constants k_x and $0 < \kappa < 1$ such that $\dot{V}_2 + k_x V_2^\kappa \leq 0$ which enable us to show that the systems are finite-time stable.

By using the parameter projection operation and based on Assumption 7.2, we know there exists a positive constant S_i such that $\|\tilde{\theta}_i\| \leq S_i$. Then from (7.9) and (7.11) we have

$$W_i \leq \sum_{j=1}^m f_{ij} \xi_{ij}^2 \leq f_{\max} \sum_{j=1}^m \xi_{ij}^2 \quad (7.23)$$

Thus from (7.8) we obtain

$$V_2 \leq \frac{1}{2\lambda_2} \sum_{i=1}^n \sum_{j=1}^m e_{ij}^2 + f_{\max} \sum_{i=1}^n \sum_{j=1}^m \xi_{ij}^2 \quad (7.24)$$

Suppose $P(\cdot)$ in Assumption 7.2 is

$$P(x) = p_1 x + p_2 x^2 + \dots + p_h x^h \quad (7.25)$$

where $h \geq 2$ is an integer and p_k , $k = 1, \dots, h$ are positive constants. From (7.10)

$$|v_{ij}| \leq (|\xi_{ij}| + |v_{ij}^*|^{1/g})^g \leq |\xi_{ij}|^g + c_1 |e_{ij}|^g \quad (7.26)$$

Define a compact set

$$\Xi = \{\xi_{ij}, e_{ij} : |e_{ij}| < r_{ij}, |\xi_{ij}| < d_{ij}\} \quad (7.27)$$

in the neighborhood of $e_{ij} = 0$, $\xi_{ij} = 0$ where r_{ij} and d_{ij} are positive constants chosen to be

$$r_{ij} = \left(\frac{1}{2c_1} \sqrt{\frac{1}{m}} \right)^{1/g}, \quad d_{ij} = \left(\frac{1}{2} \sqrt{\frac{1}{m}} \right)^{1/g} \quad (7.28)$$

From (7.26) it is easy to check that in this set we have $\|v_i\| \leq 1$. Thus from Assumption 7.2 we get $\|G_i(q_i, v_i)\|_1 \leq \rho h p_{\max} \|v_i\|$ where $p_{\max} = \max(p_1, \dots, p_h)$. Furthermore, $|G_i(q_i, v_i)(j, k)| < \rho h p_{\max} |v_{ij}|$, $k = 1, \dots, l$ since $|v_{ij}| \leq 1$. Then from (7.12), (7.16) and (7.26)

$$\begin{aligned} & \frac{1}{2^{1-p}} \sum_{i=1}^n \xi_i^T G_i(q_i, v_i) \tilde{\theta}_i \leq \frac{1}{2^{1-p}} \sum_{i=1}^n \left\| |\xi_i|^T |G_i(q_i, v_i)| \right\| \|\tilde{\theta}_i\| \\ & \leq \frac{S_i}{2^{1-p}} \sum_{i=1}^n \sum_{j=1}^m \rho h p_{\max} |\xi_{ij}^{2-g}| \left(|\xi_{ij}|^g + c_1 |e_{ij}|^g \right) \\ & = k_3 \sum_{i=1}^n \sum_{j=1}^m e_{ij}^2 + k_4 \sum_{i=1}^n \sum_{j=1}^m \xi_{ij}^2 \end{aligned} \quad (7.29)$$

where $|\xi_i|$ and $|G_i(q_i, v_i)|$ are taken element-wisely and k_3 and k_4 are

$$k_3 = \frac{S_{\max} \rho h p_{\max} g}{2^{2-p}}, \quad k_4 = \frac{S_{\max} \rho h p_{\max}}{2^{1-p}} \left(1 + \frac{2-g}{2c_1^2} \right) \quad (7.30)$$

in which $S_{\max}$ is defined as $S_{\max} = \max(S_1, \dots, S_n)$. Thus from (7.16) we get

$$\begin{aligned} \dot{V}_2 & \leq -\frac{k_1}{2} \sum_{j=1}^m \sum_{i=1}^n e_{ij}^{\frac{4\sigma}{2\sigma+1}} - \frac{k_2}{2} \sum_{j=1}^m \sum_{i=1}^n \xi_{ij}^{\frac{4\sigma}{2\sigma+1}} \\ & + \sum_{j=1}^m \sum_{i=1}^n \left(k_3 e_{ij}^2 - \frac{k_1}{2} e_{ij}^{\frac{4\sigma}{2\sigma+1}} \right) + \sum_{j=1}^m \sum_{i=1}^n \left(k_4 \xi_{ij}^2 - \frac{k_2}{2} \xi_{ij}^{\frac{4\sigma}{2\sigma+1}} \right) \end{aligned} \quad (7.31)$$

Now we will establish the finite-time stability in two cases. In the first case we will prove that e_{ij} and ξ_{ij} will converge to zero in finite-time if $e_{ij}(t_0) \in \Xi$ and $\xi_{ij}(t_0) \in \Xi$. In the second case we will show that e_{ij} and ξ_{ij} will converge to Ξ within finite-time and thus converge to zero in finite-time if $e_{ij}(t_0) \notin \Xi$ and/or $\xi_{ij}(t_0) \notin \Xi$.

- Case 1: The initial conditions of e_{ij} and ξ_{ij} satisfy

$$e_{ij}(t_0) \in \Xi, \quad \xi_{ij}(t_0) \in \Xi$$

From (7.17) and (7.30) we can choose the design parameters c_1 and c_2 such that $k_1 > 2k_3$ and $k_2 > 2k_4$. Since $|e_{ij}| < 1$ and $|\xi_{ij}| < 1$, it is not hard to see that $\sum_{j=1}^m \sum_{i=1}^n \left(k_3 e_{ij}^2 - \frac{k_1}{2} e_{ij}^{\frac{4\sigma}{2\sigma+1}} \right) + \sum_{j=1}^m \sum_{i=1}^n \left(k_4 \xi_{ij}^2 - \frac{k_2}{2} \xi_{ij}^{\frac{4\sigma}{2\sigma+1}} \right) < 0$ if we choose c_1 and c_2 in this way. Thus we have

$$\dot{V}_2 \leq -\frac{k_1}{2} \sum_{j=1}^m \sum_{i=1}^n e_{ij}^{\frac{4\sigma}{2\sigma+1}} - \frac{k_2}{2} \sum_{j=1}^m \sum_{i=1}^n \xi_{ij}^{\frac{4\sigma}{2\sigma+1}} \quad (7.32)$$

From (7.24) and Lemma 2.3 we get $V_2^\kappa \leq \frac{1}{2^\kappa \lambda_2^\kappa} \sum_{i=1}^n \sum_{j=1}^m e_{ij}^{\frac{4\sigma}{2\sigma+1}} + f_{\max}^\kappa \sum_{i=1}^n \sum_{j=1}^m \xi_{ij}^{\frac{4\sigma}{2\sigma+1}}$ where $\kappa = \frac{2\sigma}{2\sigma+1} \in (0, 1)$. Hence

$$\dot{V}_2 + k_x V_2^\kappa \leq 0 \quad (7.33)$$

where $k_x < \min \left\{ 2^{\kappa-1} \lambda_2^\kappa k_1, \frac{k_2}{2 f_{\max}^\kappa} \right\}$ is a positive constant. Since $0 < g < 1$, then based on Lemma 2.1, V_2 will converge to zero in a finite time T_1 which satisfies

$$T_1 \leq \frac{V_2(t_0)^{1-\kappa}}{k_x(1-\kappa)} \quad (7.34)$$

- Case 2: The initial values of e_{ij} and/or ξ_{ij} are outside the set Ξ . For the period of time when e_{ij} and/or ξ_{ij} are outside of the set Ξ , from (7.21), we know

$$\dot{V}_3 \leq -d_{v_3} \quad (7.35)$$

where $d_{v_3} \geq \min \left(k_1 \sum_{i=1}^n \sum_{j=1}^m r_{ij}^{\frac{4\sigma}{2\sigma+1}}, k_2 \sum_{i=1}^n \sum_{j=1}^m d_{ij}^{\frac{4\sigma}{2\sigma+1}} \right)$ is a positive constant.

To determine the time T_2 that e_{ij} and ξ_{ij} reach the set Ξ from the initials, we have to calculate a ϱ_1 such that $V_3 \geq \varrho_1$ when $e_{ij}(t) \notin \Xi$ and/or $\xi_{ij}(t) \notin \Xi$. Then T_2 will not be longer than $(V_3(t_0) - \varrho_1)/d_{v_3}$. From (7.13) and (7.18), such

a ϱ_1 is calculated as

$$\varrho_1 = \min \left\{ V_2|_{(e_{ij}=r_{ij}, \xi_{ij}=0)}, \left(\sum_{i=1}^n W_i \right) \Big|_{(\xi_{ij}=d_{ij}, |e_{ij}| < r_{ij})} \right\}$$

From (7.3) we know

$$\begin{aligned} V_2|_{(e_{ij}=r_{ij}, \xi_{ij}=0)} &= V_1|_{e_{ij}=r_{ij}} \\ &\geq \frac{\zeta}{4\eta^2} \sum_{j=1}^m \sum_{i=1}^n \sum_{k \in N_i} \eta^2 (q_{ij} - q_{kj} - \delta_{ij} + \delta_{kj})^2 \\ &\geq \frac{\zeta}{4\eta^2 n_{\max}} \sum_{j=1}^m \sum_{i=1}^n \left[\sum_{k \in N_i} a_{ik} (q_{ij} - q_{kj} - \delta_{ij} + \delta_{kj}) \right]^2 \\ &= \frac{\zeta}{4\eta^2 n_{\max}} \sum_{j=1}^m \sum_{i=1}^n r_{ij}^2 \end{aligned} \quad (7.36)$$

where $\zeta = \min_{\forall i,j \in \mathcal{V}} \{a_{ij}\}$, $n_{\max} = \max\{n_1, \dots, n_n\}$. $\eta = \max_{\forall i,j \in \mathcal{V}} \{a_{ij}\}$, $n_i = \dim\{N_i\}$. Based on **Fact 1** in the Appendix, it can be checked that

$$\begin{aligned} \int_{v_{ij}^*}^{v_{ij}} (s^{1/g} - v_{ij}^{*1/g})^{2-g} ds &\geq \frac{2\sigma - 1}{2^{\frac{(1-g)(2-g)}{g}} (4\sigma + 2)} \left| v_{ij} - v_{ij}^* \right|^{\frac{4\sigma+2}{2\sigma-1}} \\ &\geq \frac{2\sigma - 1}{2^{\frac{(1-g)(2-g)}{g}} (4\sigma + 2)} \left((d_{ij} + r_{ij})^g - r_{ij}^g \right)^{\frac{4\sigma+2}{2\sigma-1}} := s_{ij} \end{aligned} \quad (7.37)$$

This is because for a fixed ξ_{ij} , $v_{ij} - v_{ij}^*$ decreases monotonously as e_{ij} increases. Thus from (7.9)

$$\left(\sum_{i=1}^n W_i \right) \Big|_{\xi_{ij}=d_{ij}, |e_{ij}| < r_{ij}} = \sum_{i=1}^n \sum_{j=1}^m \frac{f_{ij}}{2^{1-g}} s_{ij} \geq \frac{f_{\min}}{2^{1-g}} \sum_{i=1}^n \sum_{j=1}^m s_{ij}$$

where $f_{\min} \leq \min_{\forall i,j \in \mathcal{V}} f_{ij}$ is a positive constant known from Assumption 7.2.

So with ϱ_1 , e_{ij} and ξ_{ij} will reach the set within a time T_2 which satisfies

$$T_2 \leq \frac{V_3(t_0) - \varrho_1}{d_{v_3}} \quad (7.38)$$

After reaching the set, e_{ij} and ξ_{ij} will stay in this set forever because $\dot{V}_2 < 0$ in this set, and it will take a time less than $T_3 = \frac{\varrho_1^{1-\kappa}}{k_x(1-\kappa)}$ to arrive at the origin. Thus in this case the system will be finite time stable with T satisfying (7.22). In summary, the closed-loop system is globally finite-time convergent. This completes the proof. $\square$

Remark 7.2. From (7.24) and (7.35) we know k_x and d_{v_3} will be larger if we choose larger controller parameters c_1 and c_2 . Since κ is a constant, thus from (7.17) and (7.21) we know (7.22) can be adjusted arbitrarily small, but it will result in larger control effort.

Remark 7.3. From the proof of Theorem 7.1 we can see that local finite-time stability and global asymptotical stability yield global finite-time stability.

7.3.1.3 Transient Performance Analysis

As mentioned in the introduction, for adaptive control of nonlinear systems it is very difficult to establish exponential convergence rate. Nevertheless we explore the transient performance of the resulting closed-loop adaptive control system here. The following theorem is established.

Theorem 7.2. Tracking errors e_{ij} and virtual control errors ξ_{ij} converge to the origin with a speed faster than an exponential rate when $e_{ij} \in \Xi$ and $\xi_{ij} \in \Xi$.

Proof: Based on (7.33) which holds when $(e_{ij}, \xi_{ij}) \in \Xi$, we get $\dot{V}_2 \leq -k_x V_2^\kappa \leq 0$. Thus $V_2 \leq V_2(t_0)$, and we can find a positive constant k_p such that

$$\dot{V}_2 + k_p V_2 \leq 0 \quad (7.39)$$

where $k_p = k_x^{1/\kappa} V_2(t_0)^{\frac{\kappa-1}{\kappa}}$. Therefore

$$V_2(t) \leq V_2(t_0) e^{-k_p t} \quad (7.40)$$

From (7.36) we get $V_2 \geq \frac{\zeta}{4\eta^2 n_{\max}} \sum_{j=1}^m \sum_{i=1}^n e_{ij}^2$. Thus the tracking errors satisfy

$$\|e(t)\|^2 = \sum_{j=1}^m \sum_{i=1}^n e_{ij}(t)^2 \leq \frac{4\eta^2 n_{\max}}{\zeta} V_2(t_0) e^{-k_p t} \quad (7.41)$$

From (7.37) we have

$$\sum_{j=1}^m \sum_{i=1}^n |v_{ij} - v_{ij}^*|^{\frac{4\sigma+2}{2\sigma-1}} \leq \vartheta_0 V_2(t_0) e^{-k_p t} \quad (7.42)$$

where $\vartheta_0 = \frac{2^{\frac{(1-g)(2-g)}{g}} (4\sigma+2)}{2\sigma-1}$. Based on (7.5), (7.41) and (7.42), we get

$$|v_{ij}| \leq |v_{ij} - v_{ij}^*| + |v_{ij}^*| \leq \vartheta_1 e^{-k_{v1} t} + \vartheta_2 e^{-k_{v2} t} \quad (7.43)$$

where $\vartheta_1 = (\vartheta_0 V_2(t_0))^{\frac{2\sigma-1}{4\sigma+2}}$, $k_{v1} = k_p \frac{2\sigma-1}{4\sigma+2}$, $k_{v2} = \frac{k_p g}{2}$, $\vartheta_2 = c_1 \left(\frac{4\eta^2 n_{\max}}{\zeta} V_2(t_0) \right)^{\frac{g}{2}}$. From (7.43) we get $|v_{ij}| \leq \vartheta_1 + \vartheta_2$, and $|v_{ij}^*| < \vartheta_2$. According to the differential mean value theorem, there exists a ψ with $|\psi| < \vartheta_1 + \vartheta_2$, such that $|v_{ij}^{1/g} - v_{ij}^{*1/g}| = \frac{1}{g} |\psi|^{\frac{1-g}{g}} |v_{ij} - v_{ij}^*|$. Since function $y = |x|^{\frac{1-g}{g}}$ is monotonously decreasing or increasing when $x < 0$ or $x \geq 0$ respectively, thus there exists a positive constant k_σ , such that

$$|v_{ij}^{1/g} - v_{ij}^{*1/g}| < k_\sigma |v_{ij} - v_{ij}^*| \quad (7.44)$$

where $k_\sigma = \frac{1}{g} (\vartheta_1 + \vartheta_2)^{\frac{1-g}{g}}$. Then from Lemma 2.3, (7.42) and (7.44) we obtain

$$\begin{aligned} \|\xi(t)\|^2 &= \sum_{j=1}^m \sum_{i=1}^n \xi_{ij}(t)^2 \leq \sum_{j=1}^m \sum_{i=1}^n k_\sigma^2 |v_{ij} - v_{ij}^*|^2 \\ &\leq k_\sigma^2 \sqrt{mn} \left(\sum_{j=1}^m \sum_{i=1}^n |v_{ij} - v_{ij}^*|^4 \right)^{\frac{1}{2}} \\ &\leq k_\sigma^2 \sqrt{mn} \left(\sum_{j=1}^m \sum_{i=1}^n |v_{ij} - v_{ij}^*|^{\frac{4\sigma+2}{2\sigma-1}} \right)^{\frac{1}{2} \frac{8\sigma-4}{4\sigma+2}} \leq k_\xi e^{-k_s t} \end{aligned} \quad (7.45)$$

where $k_\xi = k_\sigma^2 \sqrt{mn} \left(\vartheta_0 V_2(t_0) \right)^{\frac{4\sigma-2}{4\sigma+2}}$ and $k_s = k_p \frac{4\sigma-2}{4\sigma+2}$. This completes the proof. $\square$

Remark 7.4.

- From (7.41) and (7.45) we know the tracking errors and the virtual control errors converge with exponential rate at least and will converge to the origin within finite time if $(e_{ij}, \xi_{ij}) \in \Xi$.
- On the other hand, if $e_{ij}(t_0) \notin \Xi$ and/or $\xi_{ij}(t_0) \notin \Xi$, both e_{ij} and ξ_{ij} will enter Ξ within T_2 defined in (7.38). From (7.8) we know $\|e(t_0)\| \geq 2\lambda_2 V_1(t_0)$. However at $t = T_2$, $\|e\| < mnr_{ij}^2$. Thus for $t \in [t_0, T_2]$, $\|e(t)\|$ drops from $2\lambda_2 V_1(t_0)$ to mnr_{ij}^2 . For ξ_{ij} , suppose $v_{ij}(t_0) = 0$, then for $t = t_0$, $\|\xi\| \geq 2c_1^{2/g} V_1(t_0)$, and for $t = T_2$, $\|\xi(T_2)\| \leq mnd_{ij}^2$. As discussed in Remark 7.2, T_2 can be made arbitrarily small by adjusting the control parameters. Therefore, any pre-specified convergence time can be met by choosing appropriate design parameters for any initial values of tracking errors and virtual control errors.
- If the upper bound of θ is unknown, then from (7.21) we can still choose the control parameters c_1 and c_2 by incorporating $|\tilde{\theta}| \leq \sqrt{2V_3(t_0)}$ such that $\frac{1}{2^{1-g}} \sum_{i=1}^n \xi_i^T G_i(q_i, v_i) \tilde{\theta}_i \leq \frac{k_1}{2} \sum_{j=1}^m \sum_{i=1}^n e_{ij}^{\frac{4\sigma}{2\sigma+1}} + \frac{k_2}{2} \sum_{j=1}^m \sum_{i=1}^n \xi_{ij}^{\frac{4\sigma}{2\sigma+1}}$. This will ensure that e_{ij} and ξ_{ij} converge to Ξ and then converge to the origin within finite time. However this is not a global results since the initial conditions of system states must be incorporated into the controllers.

In order to consider the rendezvous problem, we now define the following set of error variables

$$z_{ij} = \sum_{k \in N_i} a_{ik} \left(q_{ij} - q_{kj} - \delta_{ij} + \delta_{kj} \right) + b_i (q_{ij} - \delta_{ij} - Y_j) \quad (7.46)$$

where $Y = \{Y_1, \dots, Y_m\}$ is a rendezvous location which is only available to part of the agents. Let $z_j = [z_{1j}, \dots, z_{nj}]^T$, Then we have

$$z_j = (L + B)(\underline{x}_j - \underline{Y}_j) \quad (7.47)$$

where $z_j = [z_{1j}, \dots, z_{nj}]^T$, $\underline{x}_j = [q_{1j} - \delta_{1j}, \dots, q_{nj} - \delta_{nj}]^T$ and $\underline{Y}_j = [Y_j, \dots, Y_j]^T \in R^n$. Define a Lyapunov function

$$V_4 = \frac{1}{2} z^T \left(I_m \otimes (L + B) \right)^{-1} z \quad (7.48)$$

where $z = [z_1^T, \dots, z_m^T]^T$. Taking the derivative of V_4 yields

$$\dot{V}_4 = z^T v = \sum_{i=1}^n \sum_{j=1}^m z_{ij} v_{ij}$$

where $v = [v_1^T, \dots, v_m^T]^T$ with $v_j = [v_{1j}, \dots, v_{nj}]$, $j = 1, \dots, m$. Choose the virtual control of v_{ij} as

$$v_{ij}^* = -c_3 z_{ij}^g \quad (7.49)$$

Then we get

$$\dot{V}_4 = -c_3 \sum_{i=1}^n \sum_{j=1}^m z_{ij}^{1+g} + \sum_{i=1}^n \sum_{j=1}^m (v_{ij} - v_{ij}^*)$$

Also from (7.48) we know

$$z^T z = \sum_{i=1}^n \sum_{j=1}^m z_{ij}^2 \geq \frac{2}{\lambda_{\max} \left(I_m \otimes (L + B) \right)^{-1}} V_4 \quad (7.50)$$

Define a new Lyapunov function

$$V_5 = V_4 + \sum_{i=1}^n \Delta_i \quad (7.51)$$

where

$$\Delta_i = \frac{1}{2^{1-g}} \sum_{j=1}^m f_{ij} \int_{v_{ij}^*}^{v_{ij}} (s^{1/g} - v_{ij}^{*1/g})^{2-g} ds \quad (7.52)$$

and design the controller τ_i as

$$\tau_i = -G_i(q_i, v_i)\hat{\theta}_i - c_2\chi_i^* \quad (7.53)$$

where $\chi_{ij} = v_{ij}^{1/g} - v_{ij}^{*1/g}$ and $\chi_i^* = [\chi_{i1}^{2g-1}, \dots, \chi_{im}^{2g-1}]$. Then

$$\begin{aligned} \dot{V}_5 \leq & -k_1 \sum_{j=1}^m \sum_{i=1}^n z_{ij}^{\frac{4\sigma}{2\sigma+1}} - k_2 \sum_{j=1}^m \sum_{i=1}^n \chi_{ij}^{\frac{4\sigma}{2\sigma+1}} \\ & + \frac{1}{2^{1-g}} \sum_{i=1}^n \chi_i^T G_i(q_i, v_i) \tilde{\theta}_i \end{aligned} \quad (7.54)$$

Thus by defining

$$V_6 = V_5 + \frac{1}{2} \sum_{i=1}^n \tilde{\theta}_i^T \Gamma^{-1} \tilde{\theta}_i, \quad (7.55)$$

and choosing the parameter update law

$$\dot{\hat{\theta}}_{ik} = \text{Proj}\left(\gamma_i(k), \hat{\theta}_{ik}\right), \quad k = 1, \dots, l \quad (7.56)$$

where $\gamma_i = -\frac{1}{2^{1-g}} \Gamma_i G_i(q_i, v_i)^T \chi_i$ and $\gamma_i(k)$ is the k th element of γ_i , we get

$$\dot{V}_6 \leq -k_1 \sum_{j=1}^m \sum_{i=1}^n z_{ij}^{\frac{4\sigma}{2\sigma+1}} - k_2 \sum_{j=1}^m \sum_{i=1}^n \chi_{ij}^{\frac{4\sigma}{2\sigma+1}} \quad (7.57)$$

Then from (7.57) the following corollary is established.

Corollary 7.1. *Consider multi-agent systems (7.1) under the control of distributed adaptive controllers (7.49) and (7.53) with parameter estimators (7.56). If Assumptions 7.1 and 7.2 are satisfied, then all the positions of the agents converge to Y with specified configurations δ_{ij} within finite time T satisfying*

$$T \leq \frac{\varrho_2^{1-\kappa}}{k_y(1-\kappa)} + \frac{V_6(t_0) - \varrho_2}{d_{v_6}}. \quad (7.58)$$

Proof: Based on (7.57) and following the similar procedure from (7.23) to (7.38), the conclusion can be established. $\square$

7.3.2 Leader-follower Consensus Control with undirected topology

In this subsection, we consider the leader-follower control for a group of identical mechanical systems given as

$$\begin{aligned}\dot{q}_i &= v_i \\ \dot{v}_i &= -M^{-1}(C(q_i, v_i)v_i + D(q_i)v_i) + M^{-1}\tau_i\end{aligned}\quad (7.59)$$

where $M = \text{diag}\{f_1, \dots, f_m\}$ and $C(q_i, v_i)v_i + D(q_i)v_i = G(q_i, v_i)\theta$ with $\theta \in R^l$ being a vector of unknown system parameters and

$$\|G(q_i, v_i)\|_1 \leq \rho P(\|v_i\|). \quad (7.60)$$

where ρ and $P(\cdot)$ are defined in Assumption 7.2. The communications among the group of agents are represented by a directed graph. The leader, which is labeled as '0', is given by

$$\begin{aligned}\dot{q}_0 &= v_0 \\ M\dot{v}_0 + C(q_0, v_0)v_0 + D(q_0)v_0 &= \tau_0\end{aligned}\quad (7.61)$$

Information on q_0 , v_0 and τ_0 is available to only a part of the agents in the group, and τ_0 is pre-defined. Furthermore, there is a positive constant ς_0 such that

$$\|v_0\| < \varsigma_0 \quad (7.62)$$

All the agents $1, \dots, N$ together with the leader form an augmented directed graph $\bar{\mathcal{G}}$. First we decompose the augmented directed graph $\bar{\mathcal{G}}$ into a hierarchical structure with r layers according to the shortest possible path corresponding to each agent originated from the leader where $2 \leq r \leq n+1$ as shown in [132] and [138]. We denote the set of the subsystems in 'Layer h ' as P_h for $1 \leq h \leq r$. The procedure of decomposition is listed as follows.

◇*Hierarchical Decomposition Algorithm:*

1. Initialization: the leader is classified as '*Layer 0*'.
2. For any agent i in $\bar{\mathcal{G}}$, if it has a direct access to the leader, then it is classified as '*Layer 1*' which is labeled as l_1 and $i \in S_1$.
3. For a subsystem $i \in \bar{\mathcal{G}} \setminus P_1$, if it can obtain the information from the agents in Layer 1 directly, it is classified as '*Layer 2* (l_2)' and $i \in P_2$.
4. ...
5. For an agent $i \in \bar{\mathcal{G}} \setminus (P_1 \cup P_2 \cdots \cup P_{h-1})$, if it can obtain information from the agents in Layer $(h-1)$ directly, it is classified as '*Layer h* (l_h)' and $i \in P_h$.
6. Once agent i is labeled with $i \in P_h$, $1 \leq h \leq r$, it only focuses on the information obtained from the agents $k \in P_{h-1}$.
7. Once all agents are labeled, the algorithm ends.

By the hierarchical decomposition algorithm, a new graph is composed based on the topology of the agents.

7.3.2.1 Controller Design

Suppose the set of agents in the k th layer are denoted by P_k , $0 \leq k \leq r-1$. So for an agent $q_k \in P_k$ to follow an agent $q_{k-1} \in P_{k-1}$, we define a new variable

$$e_k = q_k - \delta_k - q_{k-1} + \delta_{k-1} \quad (7.63)$$

where $\delta_k \in R^m$, $k = 1, \dots, r$ is a constant vector to specify the required configuration of the agent to the leader. By defining

$$\omega_k = v_k - v_{k-1} \quad (7.64)$$

where $v_k = \dot{q}_k$, we get $\dot{e}_k = \omega_k$. Choosing the virtual control for ω_{kj} , $j = 1, \dots, m$ as $\omega_{kj}^* = -c_{k1} e_{kj}^{\frac{2\sigma-1}{2\sigma+1}}$ where ω_{kj} and e_{kj} are the j th element of ω_k and e_k respectively,

then

$$\dot{e}_{kj} = -c_{k1} e_{kj}^{\frac{2\sigma-1}{2\sigma+1}} + \omega_{kj} - \omega_{kj}^* \quad (7.65)$$

Define a new Lyapunov function

$$V_7 = \frac{1}{2} \sum_{j=1}^m e_{kj}^2. \quad (7.66)$$

Then taking the derivative of V_7 yields

$$\dot{V}_7 = -c_{k1} \sum_{j=1}^m e_{kj}^{1+g} + \sum_{j=1}^m e_{kj} (\omega_{kj} - \omega_{kj}^*)$$

By defining another Lyapunov function

$$V_8 = V_7 + \sum_{j=1}^m f_j \int_{\omega_{kj}^*}^{\omega_{kj}} (s^{1/g} - \omega_{kj}^{*1/g})^{2-g} ds \quad (7.67)$$

and a new variable

$$\varphi_{kj} = \omega_{kj}^{1/g} - \omega_{kj}^{*1/g}, \quad (7.68)$$

we get

$$\begin{aligned} \dot{V}_8 &= \dot{V}_7 + \sum_{j=1}^m \left[f_j \varphi_{kj}^{2-g} \dot{\omega}_{kj} \right. \\ &\quad \left. - (2-g) f_j \int_{\omega_{kj}^*}^{\omega_{kj}} (s^{1/g} - \omega_{kj}^{*1/g})^{1-g} ds \frac{\partial \omega_{kj}^{*1/g}}{\partial e_{kj}} \dot{e}_{kj} \right] \\ &= \dot{V}_7 + \varphi_k^T (-G_k(q_k, v_k) \theta + \tau_k + G(q_{k-1}, v_{k-1}) \theta - \tau_{k-1}) \\ &\quad + (2-g) c_{k1}^{1/g} \sum_{j=1}^m \left[f_j \int_{\omega_{kj}^*}^{\omega_{kj}} (s^{1/g} - \omega_{kj}^{*1/g})^{1-g} ds \omega_{kj} \right] \end{aligned}$$

where $\varphi_k = [\varphi_{k1}^{2-g}, \dots, \varphi_{km}^{2-g}]^T$. The controller for q_k is designed as

$$\tau_k = -c_{k2}\varphi_k^* + \left(G(q_k, v_k) - G(q_{k-1}, v_{k-1})\right)\hat{\theta} + \tau_{k-1} \quad (7.69)$$

where $\varphi_k^* = [\varphi_{k1}^{2g-1}, \dots, \varphi_{km}^{2g-1}]^T$ and c_{k2} is a positive constant to be chosen. We then have

$$\begin{aligned} \dot{V}_8 = & \dot{V}_7 - c_{k2} \sum_{j=1}^m \varphi_{kj}^{1+g} + \varphi_k^T \left(G(q_{k-1}, v_{k-1}) - G(q_k, v_k)\right) \tilde{\theta} \\ & + (2-g)c_{k1}^{1/g} \sum_{j=1}^m \left[f_j \int_{\omega_{kj}^*}^{\omega_{kj}} (s^{1/g} - \omega_{kj}^{*1/g})^{1-g} ds \omega_{kj} \right] \end{aligned} \quad (7.70)$$

Note that $\sum_{j=1}^m e_{kj}(\omega_{kj} - \omega_{kj}^*) \leq \frac{2^{1-g}}{1+g} \sum_{j=1}^m (e_{kj}^{1+g} + g\varphi_{kj}^{1+g})$, and

$$\begin{aligned} & (2-g)c_{k1}^{1/g} \sum_{j=1}^m \left[f_j \int_{\omega_{kj}^*}^{\omega_{kj}} (s^{1/g} - \omega_{kj}^{*1/g})^{1-g} ds \omega_{kj} \right] \\ & \leq (2-g)c_{k1}^{1/g} 2^{1-g} \sum_{j=1}^m f_j \left(|\varphi_{kj}|^{1+g} + \frac{c_{k1}^{2+1/g} |\varphi_{kj}|^{1+g}}{1+g} \right. \\ & \quad \left. + \frac{c_{k1}^{-1/g} g |e_{kj}|^{1+g}}{1+g} \right) \end{aligned}$$

Thus from (7.70) we have

$$\begin{aligned} \dot{V}_8 \leq & -b_{k1} \sum_{j=1}^m e_{kj}^{1+g} - b_{k2} \sum_{j=1}^m \varphi_{kj}^{1+g} \\ & + \varphi_k^T \left(G(q_{k-1}, v_{k-1}) - G(q_k, v_k)\right) \tilde{\theta} \end{aligned} \quad (7.71)$$

where

$$\begin{aligned} b_{k1} = & c_{k1} - \frac{2^{1-g}}{1+g} - \frac{(2-g)2^{1-g}f'_{\max}g}{1+g} \\ b_{k2} = & c_{k2} - \frac{g2^{1-g}}{1+g} - (2-g)c_{k1}^{1/g}2^{1-g}f'_{\max}\left(1 + \frac{c_{k1}^{2+1/g}}{1+g}\right) \end{aligned} \quad (7.72)$$

where $f'_{\max} \geq \max\{f_1, \dots, f_m\}$ is a positive constant known from Assumption 7.2. Define a new Lyapunov function

$$V_9 = V_8 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \quad (7.73)$$

and choose the parameter update law

$$\dot{\hat{\theta}}_i = \text{Proj}\left(\beta(i), \hat{\theta}_i\right), i = 1, \dots, l \quad (7.74)$$

where $\beta(i)$ is the i th element of $\beta = -\Gamma\left(G(q_{k-1}, v_{k-1})^T - G(q_k, v_k)^T\right)\varphi_k$. Then

$$\dot{V}_9 \leq -b_{k1} \sum_{j=1}^m e_{kj}^{1+g} - b_{k2} \sum_{j=1}^m \varphi_{kj}^{1+g} \quad (7.75)$$

7.3.2.2 Stability Analysis

Based on (7.75), we have the following theorem.

Theorem 7.3. *Consider the leader-follower problem of a group of identical mechanical systems (7.1) with a directed graph $\bar{\mathcal{G}}$. Under the control of the adaptive controller (7.69) with parameter estimator (7.74), the following results hold.*

(a) *If $\|G(q_k, v_k) - G(q_{k-1}, v_{k-1})\|_1 \leq \rho_1 \|v_k - v_{k-1}\|$, the position errors between two agents q_k and q_{k+1} from two consecutive layers defined by (7.63) will converge to zero in finite time T given by*

$$T = \frac{V_8(t_0)^{1-\kappa}}{k_x(1-\kappa)} + \frac{V_9(t_0) - \varrho_3}{d_{v_9}} \quad (7.76)$$

(b) *Otherwise the position errors will reach an arbitrarily small region given by $\|e_k\| \leq \varepsilon_k$, $1 \leq k \leq r$ within*

$$T = \frac{V_9(t_0) - \varrho_3}{d_{v_9}} + \frac{\varrho_3 - \varrho_4}{d_{v_8}} \quad (7.77)$$

where ε_k is an arbitrarily small positive constant, and then converge to the origin when $t \rightarrow \infty$.

Proof: From (7.75), e_{kj} and φ_{kj} are asymptotically convergent. Firstly we prove that $\|\omega_k\| < 1$ is satisfied within finite-time by considering the following compact set

$$\Pi_1 = \{\varphi_{kj}, e_{kj} : |e_{kj}| < r_{kj}, |\varphi_{kj}| < d_{kj}\} \quad (7.78)$$

where r_{kj} and d_{kj} are defined similarly to that of (7.28) except that c_1 is replaced by c_{k1} . Obviously when $e_{kj} \in \Pi_1$ and $\varphi_{kj} \in \Pi_1$, we have $\|\omega_k\| < 1$. If $(e_{kj}, \varphi_{kj}) \notin \Pi_1$ at $t = t_0$, i.e., $\|\omega_k\| \geq 1$, then from (7.75) we know it will take no more than

$$T_4 = \frac{V_9(t_0) - \varrho_3}{d_{v_9}} \quad (7.79)$$

for e_{kj} and φ_{kj} to converge to Π_1 , where

$$\begin{aligned} d_{v_9} &= \min\{b_{k1} \sum_{j=1}^m r_{kj}^{1+g}, b_{k2} \sum_{j=1}^m d_{kj}^{1+g}\} \\ \varrho_3 &= \min\left\{\frac{1}{2} \sum_{j=1}^m r_{kj}^2, f_m \sum_{j=1}^m s_{kj}\right\} \end{aligned}$$

with $f_m = \min_{j=1,\dots,m}\{f_j\}$ and s_{kj} defined similarly to that of (7.37). Once e_{kj} and φ_{kj} converge to Π_1 , $\omega_{kj} < d_{kj}^g + c_{kj}r_{kj}^g$ and thus $\|\omega_k\| < 1$.

• Proof of (a):

If $\|G(q_k, v_k) - G(q_{k-1}, v_{k-1})\|_1 \leq \rho P(\|v_k - v_{k-1}\|)$, where $P(\cdot)$ is defined in (7.25), then

$$\begin{aligned} & \varphi_k^T \left(G(q_{k-1}, v_{k-1}) - G(q_k, v_k) \right) \tilde{\theta} \\ & \leq S \rho h p_{\max} \sum_{j=1}^m \left(|\varphi_{kj}|^2 + \frac{2-g}{2c_{k1}^2} |\varphi_{kj}|^2 + \frac{g}{2} |e_{kj}|^2 \right) \\ & = k_7 \sum_{j=1}^m \varphi_{kj}^2 + k_8 \sum_{j=1}^m e_{kj}^2 \end{aligned}$$

is satisfied when $\|\omega_{kj}\| \leq 1$, where $k_7 = S \rho h p_{\max} \left(1 + \frac{2-g}{2c_{k1}^2} \right)$ and $k_8 = \frac{S \rho h p_{\max} g}{2}$ and $\|\tilde{\theta}\| < S$. Thus following the similar proof from (7.21) to (7.22), it can be

shown that e_{kj} and φ_{kj} will converge to zero within the time given in (7.76) where κ and k_x are defined similar to that of (7.34).

• Proof of (b):

If $\|G(q_k, v_k) - G(q_{k-1}, v_{k-1})\|_1 \leq \rho P(\|v_k - v_{k-1}\|)$ is not satisfied, then

$$\begin{aligned} & \varphi_k^T \left(G(q_{k-1}, v_{k-1}) - G(q_k, v_k) \right) \tilde{\theta} \\ & \leq \left\| \varphi_k^T \left(G(q_{k-1}, v_{k-1}) - G(q_k, v_{k-1}) \right) \right\| \times \|\tilde{\theta}\| \\ & \quad + \left\| \varphi_k^T \left(G(q_k, v_{k-1}) - G(q_k, v_k) \right) \right\| \times \|\tilde{\theta}\| \end{aligned}$$

Obviously based on Assumption 7.2 we have $\|G(q_{k-1}, v_{k-1}) - G(q_k, v_{k-1})\|_1 \leq 2\rho P(\|v_{k-1}\|)$. To prove e_{kj} will converge to an arbitrarily small region within finite time, we use an inductive argument.

When $k = 1$, since $\|v_0\| \leq \varsigma_0$, according to (7.60), there exists a positive constant k_9 such that

$$\left\| \varphi_k^T \left(G(q_{k-1}, v_{k-1}) - G(q_k, v_{k-1}) \right) \right\| \times \|\tilde{\theta}\| \leq k_9 \sum_{j=1}^m |\varphi_{kj}|^{2-g} \quad (7.80)$$

where $k_9 = S\rho \sum_{i=1}^h p_i \varsigma_0^i$. As shown earlier, $\|\omega_k\| < 1$ will be satisfied within finite time. After $\|\omega_k\| \leq 1$, and note that $\|v_{k-1}\| \leq \varsigma_0$, $\|v_k\| \leq \varsigma_0 + 1$. From (7.25) and (7.60), there exists a positive constant $\mathfrak{p}_m$ such that

$$\|G(q_k, v_{k-1}) - G(q_k, v_k)\|_1 \leq \mathfrak{p}_m \omega_k \quad (7.81)$$

where

$$\mathfrak{p}_m = \sum_{i=1}^h p_i \left(\sum_{j=1}^i \varsigma_0^{j-1} (\varsigma_0 + 1)^{i-j} \right). \quad (7.82)$$

Then

$$\begin{aligned}
& \left\| \varphi_k^T \left(G(q_k, v_{k-1}) - G(q_k, v_k) \right) \right\| \times \|\tilde{\theta}\| \\
& \leq S\mathbf{p}_m \sum_{j=1}^m \left(\varphi_{kj}^2 + \frac{2-g}{2c_{k1}^2} \varphi_{kj}^2 + \frac{g}{2} e_{kj}^2 \right) \\
& = c_3 \sum_{j=1}^m \varphi_{kj}^2 + c_4 \sum_{j=1}^m e_{kj}^2
\end{aligned} \tag{7.83}$$

where $c_{k3} = S\mathbf{p}_m \left(1 + \frac{2-g}{2c_{k1}^2} \right)$ and $c_{k4} = \frac{S\mathbf{p}_m g}{2}$. Based on (7.71) and (7.83) we get

$$\begin{aligned}
\dot{V}_8 & \leq -b_{k1} \sum_{j=1}^m e_{kj}^{1+g} - b_{k2} \sum_{j=1}^m \varphi_{kj}^{1+g} + c_{k3} \sum_{j=1}^m \varphi_{kj}^2 \\
& + c_{k4} \sum_{j=1}^m e_{kj}^2 + k_9 \sum_{j=1}^m |\varphi_{kj}|^{2-g}
\end{aligned}$$

The control parameters c_{k1} and c_{k2} are chosen such that

$$b_{k1} > 4c_{k4}, \quad b_{k2} > 4c_{k3} \tag{7.84}$$

From (7.72) we know this choice is possible since b_{k1} and b_{k2} can be arbitrarily large. Then we have

$$\dot{V}_8 \leq -\frac{3}{4}b_{k1} \sum_{j=1}^m e_{kj}^{1+g} - \frac{3}{4}b_{k2} \sum_{j=1}^m \varphi_{kj}^{1+g} + k_9 \sum_{j=1}^m |\varphi_{kj}|^{2-g} \tag{7.85}$$

when $r_{kj} \in \Pi_1$ and $d_{kj} \in \Pi_1$.

Next define a new compact set

$$\Pi_2 = \left\{ \varphi_{kj}, e_{kj} : \sum_{j=1}^m e_{kj}^{1+g} + \sum_{j=1}^m \varphi_{kj}^{1+g} < \epsilon \right\} \tag{7.86}$$

where $\epsilon < \min_{j=1, \dots, m} \{ \min \{ r_{kj}^{1+g}, d_{kj}^{1+g} \} \}$ is an arbitrarily small positive constant.

We will prove that e_{kj} and φ_{kj} will reach Π_2 within finite time.

Fact 7.1. c_{k1} and c_{k2} can be chosen such that $b_{k1} \sum_{j=1}^m e_{kj}^{1+g} + b_{k2} \sum_{j=1}^m \varphi_{kj}^{1+g} \geq$

$4k_9 \sum_{j=1}^m |\varphi_{kj}|^{2-g}$ when $\sum_{j=1}^m e_{kj}^{1+g} + \sum_{j=1}^m \varphi_{kj}^{1+g} \geq \epsilon$.

Proof: See Appendix 7.6.

Based on **Fact 7.1**, parameters c_{k1} and c_{k2} can be chosen such that when $\{e_{kj}, \varphi_{kj}\} \notin \Pi_2$, i.e., when $\sum_{j=1}^m e_{kj}^{1+g} + \sum_{j=1}^m \varphi_{kj}^{1+g} \geq \epsilon$,

$$b_{k1} \sum_{j=1}^m e_{kj}^{1+g} + b_{k2} \sum_{j=1}^m \varphi_{kj}^{1+g} \geq 4k_9 \sum_{j=1}^m |\varphi_{kj}|^{2-g} \quad (7.87)$$

Then in this case from (7.85)

$$\dot{V}_8 \leq -\frac{1}{2}b_{k1} \sum_{j=1}^m e_{kj}^{1+g} - \frac{1}{2}b_{k2} \sum_{j=1}^m \varphi_{kj}^{1+g} \quad (7.88)$$

This yields $\dot{V}_8 \leq -d_{v_8}$, where $d_{v_8} \geq \min\{\frac{1}{2}b_{k1}\epsilon, \frac{1}{2}b_{k2}\epsilon\}$ is a positive constant. Thus it will take no more than

$$T_5 = \frac{\varrho_3 - \varrho_4}{d_{v_8}} \quad (7.89)$$

for e_{kj} and φ_{kj} to converge to Π_2 , where ϱ_4 , defined similarly to ϱ_1 , ϱ_2 or ϱ_3 , is a positive constant satisfying $V_8 \geq \varrho_4$ when $e_{kj} \notin \Pi_2$ and/or $\varphi_{kj} \notin \Pi_2$. Thus e_{kj} and φ_{kj} can reach Π_2 within the time given in (7.77).

It is obvious that $\Pi_2 \subset \Pi_1$, and thus Π_1 is an invariant set. Therefore $\|\omega_k\| < 1$ will always be satisfied thereafter. So

$$\|v_k\| < \|\omega_k\| + \|v_{k-1}\| = 1 + \varsigma_0 \quad (7.90)$$

which shows $\|v_k\|$ is also bounded. In addition,

$$\begin{aligned} \|e_k\| &= \sqrt{\sum_{j=1}^m e_{kj}^2} = \sqrt{\left(\left(\sum_{j=1}^m e_{kj}^2\right)^{\frac{4\sigma}{4\sigma+2}}\right)^{\frac{4\sigma+2}{4\sigma}}} \\ &\leq \sqrt{\left(\sum_{j=1}^m e_{kj}^g\right)^{\frac{4\sigma+2}{4\sigma}}} \leq \sqrt{\epsilon^{\frac{4\sigma+2}{4\sigma}}} := \varepsilon_k \end{aligned} \quad (7.91)$$

Since ϵ can be arbitrarily small, then $\|e_k\|$ can be arbitrarily small within finite time.

Suppose at the k th step ($k \geq 2$) the conclusion still holds, then at the $(k+1)$ th step, following the similar procedure from (7.80) to (7.91), we can prove that the conclusion also holds. This completes the proof. $\square$

Remark 7.5. ϵ can be chosen arbitrarily small, which means e_{kj} and φ_{kj} can reach an arbitrarily small invariant set Ξ_2 within finite time. But from (7.87) and **Fact 1** larger values of c_{k1} and c_{k2} must be chosen in this case, which results in larger control efforts.

7.3.3 Leaderless consensus control with directed topology

In this subsection, a distributed adaptive control law is designed for each agent such that all the agents converge to a consensus location based on its neighbor's information. The controller is designed using a 'backstepping-like' procedure, similar to [134]. In the first step virtual control v_i^* for v_i , $i = 1, \dots, n$ is designed such that the position error converge to zero in finite-time. In the second step adaptive controllers and parameter estimators are designed such that v_i converge to v_i^* in finite-time.

7.3.3.1 Controller Design

- *Step 1.* Define

$$e_{ij} = \sum_{k \in N_i} a_{ik}(q_{ij} - q_{kj} - \delta_{ij} + \delta_{kj}) \quad (7.92)$$

where δ_{kj} , $k = 1, \dots, n$ are constants denoting the final consensus configuration such that $q_{ij} - q_{kj} = \delta_{ij} - \delta_{kj}$, $i, k \in \mathcal{V}$ and let $e_j = [e_{1j}, \dots, e_{nj}]^T$, $\mathbf{e} = [e_1^T, \dots, e_m^T]^T$, $j = 1, \dots, m$. Define a Lyapunov function

$$V_1(t) = \sum_{j=1}^m \sum_{i=1}^n \frac{\nu_i}{1+g} |e_{ij}|^{1+g} \quad (7.93)$$

where $g = \frac{2\sigma-1}{2\sigma+1}$ with $\sigma > 2$ being an integer. Taking the derivative of V_1 yields

$$\dot{V}_1 = \sum_{j=1}^m \sum_{i=1}^n \nu_i \text{sig}(e_{ij})^g \dot{e}_{ij}$$

Then

$$\begin{aligned} \dot{V}_1 &= \sum_{j=1}^m \sum_{i=1}^n \nu_i \text{sig}(e_{ij})^g \left(\sum_{k \in N_i} a_{ik} [v_{ij} - v_{kj}] \right) \\ &= \text{sig}(\mathbf{e}^T)^g (\Omega \mathcal{L} \otimes I_m) v \end{aligned}$$

where $v = (v_1^T, \dots, v_m^T)^T$ with $v_j = (v_{1j}, \dots, v_{nj})^T$, $j = 1, \dots, m$ and $\Omega = \text{diag}(\nu)$.

Denoting v_{ij}^* as the virtual control of v_{ij} , and choosing the virtual control as

$$v_{ij}^* = -c_1 \text{sig}(e_{ij})^g = -c_1 e_{ij}^g \quad (7.94)$$

where c_1 is a positive design parameter, we have

$$\begin{aligned} \dot{V}_1 &= -c_1 \text{sig}(\mathbf{e}^T)^g (\Omega \mathcal{L} \otimes I_m) \text{sig}(\mathbf{e})^g + \text{sig}(\mathbf{e}^T)^g (\Omega \mathcal{L} \otimes I_m) (v - v^*) \\ &= -\frac{c_1}{2} \text{sig}(\mathbf{e}^T)^g (\Omega \mathcal{L} \otimes I_m + \mathcal{L}^T \Omega \otimes I_m) \text{sig}(\mathbf{e})^g \\ &\quad + \sum_{j=1}^m \left[\sum_{i=1}^n (v_{ij} - v_{ij}^*) \sum_{k=1}^n l_{ki} \text{sig}(e_{kj})^g \right] \end{aligned}$$

where l_{ki} is the (k, i) th element of $\Omega \mathcal{L}$. Let $P = \frac{1}{2}(\Omega \mathcal{L} \otimes I_m + \mathcal{L}^T \Omega \otimes I_m)$, we get

$$\begin{aligned} \dot{V}_1 &= -c_1 \text{sig}(\mathbf{e}^T)^g P \text{sig}(\mathbf{e})^g + \sum_{j=1}^m \left[\sum_{i=1}^n (v_{ij} - v_{ij}^*) \sum_{k=1}^n l_{ki} \text{sig}(e_{kj})^g \right] \\ &= -c_1 \frac{\text{sig}(\mathbf{e}^T)^g P \text{sig}(\mathbf{e})^g}{\text{sig}(\mathbf{e}^T)^g \text{sig}(\mathbf{e})^g} \text{sig}(\mathbf{e}^T)^g \text{sig}(\mathbf{e})^g \\ &\quad + \sum_{j=1}^m \left[\sum_{i=1}^n (v_{ij} - v_{ij}^*) \sum_{k=1}^n l_{ki} \text{sig}(e_{kj})^g \right] \end{aligned} \quad (7.95)$$

Note that there is a positive constant $\tilde{c}$ such that $\text{sig}(\mathbf{e}^T)^g P \text{sig}(\mathbf{e})^g / \text{sig}(\mathbf{e}^T)^g \text{sig}(\mathbf{e})^g > \tilde{c}$ based on the following fact.

Fact 7.2. σ can be chosen such that

$$\frac{\text{sig}(\mathbf{e}^T)^g P \text{sig}(\mathbf{e})^g}{\text{sig}(\mathbf{e}^T)^g \text{sig}(\mathbf{e})^g} \geq \tilde{c} \quad (7.96)$$

where $\tilde{c}$ is a positive constant.

Proof: See the Appendix 7.6.

Then from (7.95)

$$\dot{V}_1 \leq -\tilde{c}_1 \sum_{j=1}^m \sum_{i=1}^n |e_{ij}|^{2g} + \sum_{j=1}^m \left[\sum_{i=1}^n (v_{ij} - v_{ij}^*) \sum_{k=1}^n l_{ki} \text{sig}(e_{kj})^g \right] \quad (7.97)$$

where $\tilde{c}_1 = c_1 \tilde{c}$ is a design parameter.

• *Step 2.* Invoke the *adding a power integrator* technique as in [135] by defining another Lyapunov function

$$W_i = \frac{1}{2^{1-g}} \sum_{j=1}^m f_{ij} \int_{v_{ij}^*}^{v_{ij}} (s^{1/g} - v_{ij}^{*1/g}) \mathrm{d}s \quad (7.98)$$

and a new variable

$$\xi_{ij} = v_{ij}^{1/g} - v_{ij}^{*1/g} \quad (7.99)$$

where f_{ij} is defined in Assumption 7.2. It is easy to prove that W_i is positive semi-definite and C^1 . Taking the time derivative of W_i , we get

$$\dot{W}_i = \frac{1}{2^{1-g}} \sum_{j=1}^m \left[f_{ij} \xi_{ij} \dot{v}_{ij} - f_{ij} (v_{ij} - v_{ij}^*) \sum_{k \in \bar{N}_i} \frac{\partial v_{ij}^{*1/g}}{\partial q_{kj}} \dot{q}_{kj} \right]$$

where $\bar{N}_i = N_i \cup \{i\}$. Since

$$|v_{ij} - v_{ij}^*| = \left| \left(v_{ij}^{1/g} \right)^g - \left(v_{ij}^{*1/g} \right)^g \right| \leq 2^{1-g} |\xi_{ij}^g| \quad (7.100)$$

we have

$$\begin{aligned}\dot{W}_i &\leq \frac{1}{2^{1-g}} \sum_{j=1}^m f_{ij} \xi_{ij} \dot{v}_{ij} + \sum_{j=1}^m f_{ij} |\xi_{ij}|^g \sum_{k \in \bar{N}_i} \left| \frac{\partial v_{ij}^{*1/g}}{\partial q_{kj}} \right| |v_{kj}| \\ &= \frac{1}{2^{1-g}} \xi_i^T [-(C_i(q_i, v_i) v_i + D_i v_i) + \tau_i] + \sum_{j=1}^m f_{ij} |\xi_{ij}|^g \sum_{k \in \bar{N}_i} \left| \frac{\partial v_{ij}^{*1/g}}{\partial q_{kj}} \right| |v_{kj}|\end{aligned}$$

where

$$\xi_i = [\xi_{i1}, \dots, \xi_{im}]^T. \quad (7.101)$$

Let $\hat{\theta}_i$, $i \in \mathcal{V}$, $j = 1, \dots, m$ denote the estimate of θ_i and define $\tilde{\theta}_i = \hat{\theta}_i - \theta_i$. Then

$$\begin{aligned}\dot{W}_i &\leq \frac{1}{2^{1-g}} \xi_i^T \left(-G_i(q_i, v_i) \hat{\theta}_i + \tau_i \right) \\ &\quad + \tilde{c}_1^{1/g} \sum_{j=1}^m f_{ij} |\xi_{ij}|^g \left(\mu |v_{ij}| + \eta \sum_{k \in N_i} |v_{kj}| \right) + \frac{1}{2^{1-g}} \xi_i^T G_i(q_i, v_i) \tilde{\theta}_i\end{aligned}$$

where $\mu = \max_{\forall i \in \mathcal{V}} \left\{ \sum_{j \in N_i} a_{ij} \right\}$ and $\eta = \max_{\forall i, j \in \mathcal{V}} \{a_{ij}\}$. From Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}&\sum_{j=1}^m \left[\sum_{i=1}^n (v_{ij} - v_{ij}^*) \sum_{k=1}^n l_{ki} \text{sig}(e_{kj})^g \right] \\ &\leq n l_{\max} 2^{-g} g^{-1} \sum_{j=1}^m \sum_{i=1}^n \left(e_{ij}^{2g} + \xi_{ij}^{2g} \right)\end{aligned}$$

where $l_{\max} = \max_{i,k=1,\dots,n} |l_{ik}|$. Define a new Lyapunov function

$$V_2 = V_1 + \sum_{i=1}^n W_i \quad (7.102)$$

Then

$$\begin{aligned}
\dot{V}_2 \leq & -\tilde{c}_1 \sum_{j=1}^m \sum_{i=1}^n e_{ij}^{2g} + n l_{\max} 2^{-g} g^{-1} \sum_{j=1}^m \sum_{i=1}^n \left(e_{ij}^{2g} + \xi_{ij}^{2g} \right) \\
& + \frac{1}{2^{1-g}} \sum_{i=1}^n \xi_i^T \left(-G_i(q_i, v_i) \hat{\theta}_i + \tau_i \right) \\
& + \tilde{c}_1^{1/g} \sum_{i=1}^n \sum_{j=1}^m f_{ij} |\xi_{ij}|^g \left(\mu |v_{ij}| + \eta \sum_{k \in N_i} |v_{kj}| \right) + \frac{1}{2^{1-g}} \sum_{i=1}^n \xi_i^T G_i(q_i, v_i) \tilde{\theta}_i
\end{aligned} \tag{7.103}$$

Based on Lemma 2.2, Lemma 2.3, Lemma 2.4 and (7.99),

$$\begin{aligned}
& \tilde{c}_1^{1/g} \sum_{i=1}^n \sum_{j=1}^m f_{ij} |\xi_{ij}|^g \left(\mu |v_{ij}| + \eta \sum_{k \in N_i} |v_{kj}| \right) \\
\leq & \sum_{i=1}^n \sum_{j=1}^m \left(\tilde{c}_1^{1/g} \left(f_{ij} \mu + \frac{f_{ij} \tilde{c}_1^{2/g} + \eta n_i + \eta \tilde{c}_1^{2/g} + \eta n_i g}{2g} \right) |\xi_{ij}|^{2g} \right. \\
& \left. + \frac{f_{ij} \mu g + \eta n_i g}{2g} |e_{ij}|^{2g} \right)
\end{aligned}$$

where $n_i = \dim N_i$. Since $F_i^{-1} = \text{diag}(1/f_{i1}, \dots, 1/f_{im})$, the torque is designed as

$$\tau_i = G(q_i, v_i) \hat{\theta}_i - c_2 \xi_i^* \tag{7.104}$$

where $\xi_i^* = [\xi_{i1}^{2g-1}, \dots, \xi_{im}^{2g-1}]^T$ and c_2 is a positive constant to be chosen. From (7.103) and (7.104), we get

$$\begin{aligned}
\dot{V}_2 \leq & -k_1 \sum_{j=1}^m \sum_{i=1}^n e_{ij}^{2g} - k_2 \sum_{j=1}^m \sum_{i=1}^n \xi_{ij}^{2g} \\
& + \frac{1}{2^{1-g}} \sum_{i=1}^n \xi_i^T G_i(q_i, v_i) \tilde{\theta}_i
\end{aligned} \tag{7.105}$$

where

$$\begin{aligned} k_1 &= \tilde{c}_1 - nl_{\max}2^{-g}g^{-1} - \frac{f_{\max}\mu g + \eta n_i g}{2g} \\ k_2 &= c_2 - nl_{\max}2^{-g}g^{-1} - \tilde{c}_1^{1/g} \left(f_{\max}\mu + \frac{f_{\max}c_1^2\tilde{c}_1^{1/g} + \eta n_i + \eta c_1^2\tilde{c}_1^{1/g} + \eta n_i g}{2g} \right) \end{aligned} \quad (7.106)$$

with $f_{\max} \geq \max_{j=1,\dots,m}(f_{1j}, \dots, f_{nj})$ being a positive constant known from Assumption 7.2. Thus we can find c_1 and c_2 such that $k_1 > 0$ and $k_2 > 0$.

Consider the following Lyapunov function

$$V_3 = V_2 + \frac{1}{2} \sum_{i=1}^n \tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i. \quad (7.107)$$

where Γ_i is a positive diagonal matrix. Then

$$\begin{aligned} \dot{V}_3 &\leq -k_1 \sum_{j=1}^m \sum_{i=1}^n e_{ij}^{2g} - k_2 \sum_{j=1}^m \sum_{i=1}^n \xi_{ij}^{2g} \\ &\quad + \sum_{i=1}^n \tilde{\theta}_i^T \Gamma_i^{-1} \left(\dot{\tilde{\theta}}_i + \frac{1}{2^{1-g}} \Gamma_i G_i(q_i, v_i)^T \xi_i \right). \end{aligned}$$

The parameter update law for θ_i is

$$\dot{\hat{\theta}}_{ik} = \text{Proj}(\beta_i(k), \hat{\theta}_{ik}), \quad k = 1, \dots, l \quad (7.108)$$

where $\beta_i(k)$ is the k th element of $\beta_i = -\frac{1}{2^{1-g}} \Gamma_i G_i(q_i, v_i)^T \xi_i$.

Then from (7.105), (7.107) and (7.108) we get

$$\dot{V}_3 \leq -k_1 \sum_{j=1}^m \sum_{i=1}^n e_{ij}^{2g} - k_2 \sum_{j=1}^m \sum_{i=1}^n \xi_{ij}^{2g} \quad (7.109)$$

7.3.3.2 Stability Analysis

We are now at the position to state our first result in the following theorem.

Theorem 7.4. *Consider the strongly connected leaderless multi-agent uncertain*

systems (7.1), under the control of the distributed adaptive controllers (7.104) with parameter estimators in (7.108). If Assumption 7.1 and Assumption 7.2 are satisfied and the graph is strongly connected, then the positions of the group of mechanical systems will reach consensus with specified configurations δ_{ij} in finite time T satisfying

$$T \leq \frac{\varrho_1^{1-\kappa}}{k_x(1-\kappa)} + \frac{V_3(t_0) - \varrho_1}{d_{v_3}} \quad (7.110)$$

where κ , k_x , ϱ_1 and d_{v_3} are computable.

Proof: The proof is similar to that of Theorem 7.1 and thus is omitted here.

Remark 7.6. From the proof of Theorem 7.4 we can see that local finite-time stability and global asymptotical stability yield global finite-time stability.

The leader-following consensus control of a group of mechanical systems with a directed graph is investigated based on hierarchical decomposition for the graph. However, the hierarchical decomposition requires that all the agents know the full information of the entire graph thus it can find a leader to follow. In this subsection, a new leader-follower consensus control scheme will be developed to relax this requirement. Based on distributed estimators, it is demonstrated that consensus can be reached or the leader-following errors converge to a region with arbitrarily small radius in finite time.

7.3.4 Leader-follower Consensus with Directed Graph

In this subsection, we consider the leader-follower control for a group of identical mechanical systems given as

$$\begin{aligned} \dot{q}_i &= v_i \\ \dot{v}_i &= -M^{-1}(C(q_i, v_i)v_i + D(q_i)v_i) + M^{-1}\tau_i \end{aligned} \quad (7.111)$$

where $M = \text{diag}\{f_1, \dots, f_m\}$ and $C(q_i, v_i)v_i + D(q_i)v_i = G(q_i, v_i)\theta$ with $\theta \in \mathbb{R}^l$ being a vector of unknown system parameters and

$$\|G(q_i, v_i)\|_1 \leq \rho P(\|v_i\|). \quad (7.112)$$

where ρ and $P(\cdot)$ are defined in Assumption 7.2. The communications among the group of agents are represented by a directed graph containing a spanning tree. The leader, which is labeled as '0', is given by

$$\begin{aligned} \dot{q}_0 &= v_0 \\ M\dot{v}_0 + C(q_0, v_0)v_0 + D(q_0)v_0 &= \tau_0 \end{aligned} \quad (7.113)$$

Information on q_0 , v_0 and τ_0 is available to only a part of the agents in the group, and τ_0 is pre-defined. Furthermore, there is a positive constant ς_0 such that

$$\|v_0\| < \varsigma_0 \quad (7.114)$$

All the agents $1, \dots, N$ together with the leader form an augmented directed graph $\bar{\mathcal{G}}$. The following assumption is needed for the design of finite-time adaptive consensus controllers.

Assumption 7.3. q_0 , v_0 and τ_0 can be expressed as

$$\begin{aligned} q_0 &= \sum_{i=1}^l f_{ri}(t)w_{ri} + c_q = f_r(t)^T w_r + c_q \\ v_0 &= \sum_{i=1}^l g_{ri}(t)y_{ri} + c_v = g_r(t)^T y_r + c_v \\ \tau_0 &= \sum_{i=1}^l h_{ri}(t)z_{ri} + c_\tau = h_r(t)^T z_r + c_\tau \end{aligned} \quad (7.115)$$

where $f_r(t) = [f_{r1}(t), \dots, f_{rl}(t)]^T \in \mathcal{R}^l$, $g_r(t) = [g_{r1}(t), \dots, g_{rl}(t)]^T \in \mathcal{R}^l$, $h_r(t) = [h_{r1}(t), \dots, h_{rl}(t)]^T \in \mathcal{R}^l$ are the vectors of basis functions which are known to all the agents. However, $w_r = [w_{r1}, \dots, w_{rl}]^T \in \mathcal{R}^l$, $y_r = [y_{r1}, \dots, y_{rl}]^T \in \mathcal{R}^l$, $z_r = [z_{r1}, \dots, z_{rl}]^T \in \mathcal{R}^l$, c_q , c_v and c_τ are constants which are only known to part

of the agents.

Assumption 7.4. Let $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ be the augmented directed graph consisting of $\mathcal{G}$ and the leader considered as a virtual node 0, i.e. $\bar{\mathcal{V}} = \{0, 1, \dots, n\}$. The leader has directed paths to all the agents in $\mathcal{G}$.

Let $\mathcal{B} = \text{diag}\{b_1, \dots, b_n\}$ where $b_i = 1$ if the i th agent has direct access to leader and otherwise $b_i = 0$. Based on Assumption 7.4, the following lemma is given.

Lemma 7.1. [78] The matrix $(\mathcal{L} + \mathcal{B})$ is nonsingular. Define

$$\begin{aligned} [\bar{p}_1, \dots, \bar{p}_n]^T &= (\mathcal{L} + \mathcal{B})^{-1}[1, \dots, 1]^T \\ \bar{P} &= \text{diag}\{p_1, \dots, p_n\} = \text{diag}\{1/\bar{p}_1, \dots, 1/\bar{p}_n\} \\ Q &= \bar{P}(\mathcal{L} + \mathcal{B}) + (\mathcal{L} + \mathcal{B})^T \bar{P} \end{aligned} \quad (7.116)$$

then $p_i > 0$ for $i = 1, \dots, n$ and Q is positive definite.

7.3.4.1 Controller Design

Since w_r , y_r , z_r , c_q , c_v and c_r are only available to part of the agents, thus we design distributed finite-time estimator for each agent these parameters. First of all, let $\hat{w}_{ri}$, $\hat{y}_{ri}$, $\hat{z}_{ri}$, $\hat{c}_{qi}$, $\hat{c}_{vi}$ and $\hat{c}_{ri}$ being the estimates of w_r , y_r , z_r , c_q , c_v and c_r respectively for the i th agent, and let $\mathcal{P} = [w_r^T, y_r^T, z_r^T, c_q, c_v, c_r]^T \in \mathcal{R}^\beta$ with $\beta = 3l + 3$ and $\hat{\mathcal{P}}_i = [\hat{w}_{ri}^T, \hat{y}_{ri}^T, \hat{z}_{ri}^T, \hat{c}_{qi}, \hat{c}_{vi}, \hat{c}_{ri}]^T$, then the distributed estimator is designed as

$$\dot{\hat{\mathcal{P}}}_i = -c_3 \text{sig} \left(\sum_{j \in \mathcal{N}_i} a_{ij}(\hat{\mathcal{P}}_i - \hat{\mathcal{P}}_j) + b_i(\hat{\mathcal{P}}_i - \mathcal{P}) \right)^\alpha \quad (7.117)$$

where $0 < \alpha < 1$ and c_3 are positive constants. With (7.117), the following lemma is given.

Lemma 7.2. With (7.117), $\hat{\mathcal{P}}_i$ will converge to $\mathcal{P}$ within finite time.

Proof: Let $e_{ik} = \sum_{j \in \mathcal{N}_i} a_{ij}(\hat{\mathcal{P}}_{ik} - \hat{\mathcal{P}}_{jk}) + b_i(\hat{\mathcal{P}}_{ik} - \mathcal{P}_k)$ where $\mathcal{P}_k$ denotes the k th element of $\mathcal{P}$, and let $\mathbf{e}_k = [e_{1k}, \dots, e_{nk}]^T$, $\mathbf{e} = [\mathbf{e}_1^T, \dots, \mathbf{e}_n^T]^T$, $\mathcal{P}_{ek} = [\mathcal{P}_{1k}, \dots, \mathcal{P}_{nk}]^T$

and $\mathcal{P}_e = [\mathcal{P}_{e1}^T, \dots, \mathcal{P}_{en}^T]^T$. Define the following Lyapunov function

$$V_e(t) = \sum_{k=1}^{\beta} \sum_{i=1}^n \frac{p_i}{1+\alpha} |e_{ik}|^{1+\alpha} \quad (7.118)$$

and take the derivative of (7.118) to get

$$\begin{aligned} \dot{V}_e(t) &= \sum_{k=1}^{\beta} \sum_{i=1}^n p_i \text{sig}(e_{ik})^{\alpha} \dot{e}_{ik} \\ &= \text{sig}(\mathbf{e})^{\alpha} (\bar{P}(\mathcal{L} + \mathcal{B}) \otimes I_{\beta}) \dot{\mathcal{P}}_e \\ &= -\frac{c_3}{2} \text{sig}(\mathbf{e}^T)^{\alpha} \left((\bar{P}(\mathcal{L} + \mathcal{B}) + (\mathcal{L} + \mathcal{B})^T \bar{P}) \otimes I_{\beta} \right) \text{sig}(\mathbf{e})^{\alpha} \\ &\leq -c_4 \sum_{k=1}^{\beta} \sum_{i=1}^n e_{ik}^{2\alpha} \end{aligned} \quad (7.119)$$

where $c_4 = \frac{c_3}{2} \lambda_{\min}(Q)$. Let $p_{\max} = \max_{i=1, \dots, n} \left\{ \left(\frac{p_i}{1+\alpha} \right)^{\frac{2\alpha}{1+\alpha}} \right\}$. If c_3 is chosen such that $c_4 > c_5 p_{\max}$ where c_5 is a positive constant, then from (7.118) and (7.119),

$$\dot{V}_e(t) + c_5 V_e(t)^{\frac{2\alpha}{1+\alpha}} \leq 0 \quad (7.120)$$

which means $\hat{\mathcal{P}}_i$ converges to $\mathcal{P}$ within

$$T_e = \frac{V_e^{\frac{1-\alpha}{1+\alpha}}(0, t)(1+\alpha)}{c_5(1-\alpha)} \quad (7.121)$$

This completes the proof. $\square$

Then we design τ_i for each agent such that q_i converge to $\hat{q}_0 = f_r(t)^T \hat{w}_{ri} + \hat{c}_q$ within finite time. We define a new variable

$$e_i = q_i - \delta_i - \hat{q}_0 + \delta_0 \quad (7.122)$$

where $\delta_i \in R^m$, $i = 1, \dots, n$ is a constant vector to specify the required configuration of the agent to the leader. By defining

$$\omega_i = v_i - v_0 \quad (7.123)$$

where $v_i = \dot{q}_i$, we get $\dot{e}_i = \omega_i + \tilde{\omega}_i$, where $\tilde{\omega}_i = v_0 - \dot{q}_0$. From Lemma 7.2 we know $\tilde{\omega}_i$ is bounded and will converge to zero within T_e . Choosing the virtual control for ω_{ij} , $j = 1, \dots, m$ as $\omega_{ij}^* = -c_{i1} e_{ij}^{\frac{2\sigma-1}{2\sigma+1}}$ where ω_{ij} and e_{ij} are the j th element of ω_i and e_i respectively, then

$$\dot{e}_{ij} = -c_{i1} e_{ij}^{\frac{2\sigma-1}{2\sigma+1}} + \omega_{ij} - \omega_{ij}^* + \tilde{\omega}_{ij} \quad (7.124)$$

Define a new Lyapunov function

$$V_7 = \frac{1}{2} \sum_{j=1}^m e_{ij}^2. \quad (7.125)$$

Then taking the derivative of V_7 yields

$$\dot{V}_7 = -c_{i1} \sum_{j=1}^m e_{ij}^{\frac{4\sigma}{2\sigma+1}} + \sum_{j=1}^m e_{ij}(\omega_{ij} - \omega_{ij}^*) + \sum_{j=1}^m e_{ij} \tilde{\omega}_{ij}$$

By defining another Lyapunov function

$$V_8 = V_7 + \sum_{j=1}^m f_j \int_{\omega_{ij}^*}^{\omega_{ij}} (s^{1/g} - \omega_{ij}^{*1/g})^{2-g} ds \quad (7.126)$$

and a new variable

$$\varphi_{ij} = \omega_{ij}^{1/g} - \omega_{ij}^{*1/g}, \quad (7.127)$$

we get

$$\begin{aligned}
\dot{V}_8 &= \dot{V}_7 + \sum_{j=1}^m \left[f_j \varphi_{ij}^{2-g} \dot{\omega}_{ij} - (2-g) f_j \int_{\omega_{ij}^*}^{\omega_{ij}} (s^{1/g} - \omega_{ij}^{*1/g})^{1-g} ds \frac{\partial \omega_{ij}^{*1/g}}{\partial e_{ij}} \dot{e}_{ij} \right] \\
&= \dot{V}_7 + \varphi_i^T M \dot{\omega}_i + (2-g) c_{i1}^{1/g} \sum_{j=1}^m \left[f_j \int_{\omega_{ij}^*}^{\omega_{ij}} (s^{1/g} - \omega_{ij}^{*1/g})^{1-g} ds \omega_{ij} \right] \\
&= \dot{V}_7 + \varphi_i^T (-G_i(q_i, v_i) \theta + \tau_i + G(q_0, v_0) \theta - \tau_0) \\
&\quad + (2-g) c_{i1}^{1/g} \sum_{j=1}^m \left[f_j \int_{\omega_{ij}^*}^{\omega_{ij}} (s^{1/g} - \omega_{ij}^{*1/g})^{1-g} ds \omega_{ij} \right]
\end{aligned}$$

where $\varphi_i = [\varphi_{i1}^{2-g}, \dots, \varphi_{im}^{2-g}]^T$. The controller for q_i is designed as

$$\tau_i = -c_{i2} \varphi_i^* + \left(G(q_i, v_i) - G(\hat{q}_0, \hat{v}_0) \right) \hat{\theta} + \hat{\tau}_0 \quad (7.128)$$

where $\varphi_i^* = [\varphi_{i1}^{2g-1}, \dots, \varphi_{im}^{2g-1}]^T$ and c_{i2} is a positive constant to be chosen. We then have

$$\begin{aligned}
\dot{V}_8 &= \dot{V}_7 - c_{i2} \sum_{j=1}^m \varphi_{ij}^{1+g} + \varphi_i^T \left(G(\hat{q}_0, \hat{v}_0) - G(q_i, v_i) \right) \tilde{\theta} \\
&\quad + (2-g) c_{i1}^{1/g} \sum_{j=1}^m \left[f_j \int_{\omega_{ij}^*}^{\omega_{ij}} (s^{1/g} - \omega_{ij}^{*1/g})^{1-g} ds \omega_{ij} \right] \\
&\quad + \varphi_i^T \left(G(q_0, v_0) - G(\hat{q}_0, \hat{v}_0) \right) \theta
\end{aligned} \quad (7.129)$$

Note that $\sum_{j=1}^m e_{ij} (\omega_{ij} - \omega_{ij}^*) \leq \frac{2^{1-g}}{1+g} \sum_{j=1}^m \left(e_{ij}^{1+g} + g \varphi_{ij}^{1+g} \right)$, and

$$\begin{aligned}
&(2-g) c_{i1}^{1/g} \sum_{j=1}^m \left[f_j \int_{\omega_{ij}^*}^{\omega_{ij}} (s^{1/g} - \omega_{ij}^{*1/g})^{1-g} ds \omega_{ij} \right] \\
&\leq (2-g) c_{i1}^{1/g} 2^{1-g} \sum_{j=1}^m f_j \left(|\varphi_{ij}|^{1+g} + \frac{c_{i1}^{2+1/g} |\varphi_{ij}|^{1+g}}{1+g} + \frac{c_{i1}^{-1/g} g |e_{ij}|^{1+g}}{1+g} \right)
\end{aligned}$$

Thus from (7.129) we have

$$\begin{aligned} \dot{V}_8 \leq & -b_{i1} \sum_{j=1}^m e_{ij}^{1+g} - b_{i2} \sum_{j=1}^m \varphi_{ij}^{1+g} + \varphi_i^T \left(G(q_0, v_0) - G(q_i, v_i) \right) \tilde{\theta} \\ & + \varphi_i^T \left(G(q_0, v_0) - G(\hat{q}_0, \hat{v}_0) \right) \theta + \sum_{j=1}^m e_{ij} \tilde{\omega}_{ij} \end{aligned} \quad (7.130)$$

where

$$\begin{aligned} b_{i1} &= c_{i1} - \frac{2^{1-g}}{1+g} - \frac{(2-g)2^{1-g}f'_{\max}g}{1+g} \\ b_{i2} &= c_{i2} - \frac{g2^{1-g}}{1+g} - (2-g)c_{i1}^{1/g}2^{1-g}f'_{\max}\left(1 + \frac{c_{i1}^{2+1/g}}{1+g}\right) \end{aligned} \quad (7.131)$$

where $f'_{\max} \geq \max\{f_1, \dots, f_m\}$ is a positive constant known from Assumption 7.2.

Define a new Lyapunov function

$$V_9 = V_8 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \quad (7.132)$$

and choose the parameter update law

$$\dot{\hat{\theta}}_i = \text{Proj}\left(\beta(i), \hat{\theta}_i\right), i = 1, \dots, l \quad (7.133)$$

where $\beta(i)$ is the i th element of $\beta = -\Gamma \left(G(q_0, v_0)^T - G(q_i, v_i)^T \right) \varphi_i$. Then

$$\dot{V}_9 \leq -b_{i1} \sum_{j=1}^m e_{ij}^{1+g} - b_{i2} \sum_{j=1}^m \varphi_{ij}^{1+g} + \varphi_i^T \left(G(q_0, v_0) - G(\hat{q}_0, \hat{v}_0) \right) \theta + \sum_{j=1}^m e_{ij} \tilde{\omega}_{ij} \quad (7.134)$$

7.3.4.2 Stability Analysis

Based on (7.134), we have the following theorem.

Theorem 7.5. *Consider the leader-follower problem of a group of identical mechanical systems (7.1) with a directed graph $\bar{\mathcal{G}}$ containing a spanning tree. Under the control of the adaptive controller (7.128) with parameter estimator (7.133), the following results hold.*

(a) If $\|G(q_i, v_i) - G(q_0, v_0)\|_1 \leq \rho_1 \|v_i - v_0\|$, the position errors between q_0 and q_i defined by (7.122) will converge to zero in finite time T given by

$$T = \frac{V_8(t_0)^{1-\kappa}}{k_x(1-\kappa)} + \frac{V_9(t_0) - \varrho_3}{d_{v_9}} + \frac{V_e^{\frac{1-\alpha}{1+\alpha}}(0, t)(1+\alpha)}{c_5(1-\alpha)} \quad (7.135)$$

(b) Otherwise the position errors will reach an arbitrarily small region given by $\|e_i\| \leq \varepsilon_i$, $1 \leq i \leq n$ within

$$T = \frac{V_9(t_0) - \varrho_3}{d_{v_9}} + \frac{\varrho_3 - \varrho_4}{d_{v_8}} + \frac{V_e^{\frac{1-\alpha}{1+\alpha}}(0, t)(1+\alpha)}{c_5(1-\alpha)} \quad (7.136)$$

where ε_i is an arbitrarily small positive constant, and then converge to the origin when $t \rightarrow \infty$.

Proof: The proof is similar to that of Theorem 7.3 thus is omitted here. $\square$

7.3.4.3 Transient Performance Analysis

Similar to Theorem 7.2, for the transient performance of e_{kj} and φ_{kj} , the following theorem is established.

Theorem 7.6. Tracking errors e_{kj} and virtual control errors φ_{kj} converge to Π_2 with a speed faster than an exponential rate when $e_{kj} \in \Pi_1$ and $\varphi_{kj} \in \Pi_1$.

Proof: As $\left| \int_{\omega_{kj}^*}^{\omega_{kj}} \left(s^{1/g} - \omega_{kj}^{*1/g} \right)^{2-g} ds \right| \leq \left| \omega_{kj} - \omega_{kj}^* \right| |\varphi_{kj}|^{2-g} = \left| (\omega_{kj}^{1/g})^g - (\omega_{kj}^{*1/g})^g \right| |\varphi_{kj}|^{1-g} \leq 2^{1-g} \left| \varphi_{kj}^g \right| |\varphi_{kj}^{2-g}| = 2^{1-g} |\varphi_{kj}|^2$, thus

$$V_8 \leq f_{max} \sum_{j=1}^m e_{kj}^2 + f_{max} \sum_{j=1}^m \varphi_{kj}^2 \quad (7.137)$$

where $f_{max} = \max \left\{ \frac{1}{2}, f_1 2^{1-g}, \dots, f_m 2^{1-g} \right\}$. Then based on Lemma 2.3 we get

$$V_8^\kappa \leq f_{max}^\kappa \sum_{j=1}^m e_{kj}^{\frac{4\sigma}{2\sigma+1}} + f_{max}^\kappa \sum_{j=1}^m \varphi_{kj}^{\frac{4\sigma}{2\sigma+1}} \quad (7.138)$$

From (7.88) we know when $(e_{kj}, \varphi_{kj}) \in \Pi_1/\Pi_2$,

$$\dot{V}_8 + \frac{b_{\min}}{f_{\max}^\kappa} V_8^\kappa \leq 0 \quad (7.139)$$

where $b_{\min} = \min\{\frac{1}{2}b_{k1}, \frac{1}{2}b_{k2}\}$. Since $V_8 \leq f_{\max} \sum_{j=1}^m r_{kj}^2 + f_{\max} \sum_{j=1}^m d_{kj}^2$ when $(e_{kj}, \varphi_{kj}) \in \Pi_1$, then there exists a positive constant $k_f = \left(\frac{b_{\min}}{f_{\max}^\kappa}\right)^{1/\kappa} \left(f_{\max} \sum_{j=1}^m r_{kj}^2 + f_{\max} \sum_{j=1}^m d_{kj}^2\right)^{\frac{\kappa-1}{\kappa}}$ such that

$$\dot{V}_8 + k_f V_8 \leq 0 \quad (7.140)$$

which means

$$V_8 \leq V_8(t_0)e^{-k_f t} \quad (7.141)$$

Therefore $\|e_k(t)\|^2 = \sum_{j=1}^m e_{kj}^2 \leq 2V_8(t_0)e^{-k_f t}$. For the convergence rate of $\|\varphi_k(t)\|$, similar analysis to the proof of *Theorem 7.2* can be made and thus we omit it. This completes the proof. $\square$

7.4 Simulation Illustrations

In this section we use four cylindrical robot arms chosen from section 3.2 in [137] to demonstrate the effectiveness of our proposed finite-time control schemes. The robot parameters are: $M = [m_1 \ 0; 0 \ m_2]$, $C = [\cos(q_1) \ \mathbf{c}_1 \dot{q}_2; \mathbf{c}_2 \dot{q}_2 \ \sin(q_2)]$, $D = [d_1 \ 0; 0 \ d_2]$ with $q = [q_1 \ q_2]^T$, $m_1 = m_2 = 5$, $\mathbf{c}_1 = \mathbf{c}_2 = 2$, $d_1 = d_2 = 3$. The control parameters are chosen as $c_1 = 5$, $c_2 = 13$, $c_{11} = c_{21} = c_{31} = 3$, $c_{12} = c_{22} = c_{32} = 10$, $\Gamma_i = \Gamma = I_4$. The initial values of the estimates are selected as 60% of their true values, respectively.

For the leaderless case with undirected topology, the connection graph is shown in Fig. 7.1. The initial positions of the robots are at $(20, 54)$, $(4, 20)$, $(44, 1)$ and $(-2, -29)$ respectively. Fig. 7.3 shows the positions of the robots. For rendezvous seeking, Fig. 7.1 is also the connection graph and only robot 1 has

access to the rendezvous point, which is given to be (43, 65). The initial positions are at (0, 4), (4, 20), (14, 1) and (12, -29). The results on positions of the robots are shown in Fig. 7.6.

The connection graph of the leader-follower case with undirected topology is shown in Fig. 7.2. The input and initial position of the virtual leader are: $q_1(0) = q_2(0) = 6$, $\dot{q}_1(0) = \dot{q}_2(0) = 0$, $\tau_r = [\sin(0.07t)^2 \cos(0.06t)]^T$. The result is shown in Fig. 7.9. It can also be seen that the leader can be tracked by all the followers within finite time.

To see the effects of control parameters on the convergence time, we consider two group of parameters: $c_1 = 20$, $c_2 = 5$ and $c_1 = 5$, $c_2 = 5$, respectively. In Fig. 7.12 and Fig. 7.13, $\|e(t)\|$ and the torque of Robot 1 are presented. These results confirm our discussions in Remark 7.2.

We make a comparison on convergence time between our proposed scheme and a typical non-finite-time based adaptive scheme that was proposed in [123], where the controller is given as

$$\begin{aligned} v_i^* &= -c_1 e_i, \tau_i = -c_2(v_i - v_i^*) + Y_i(q_i, v_i)\hat{\Theta}_i, \\ \dot{\hat{\Theta}}_i &= -\Lambda_i Y_i(q_i, v_i)^T e_i \end{aligned} \quad (7.142)$$

where v_i^* is the virtual control for the velocity v_i , $\hat{\Theta}_i$ is the estimate of unknown system parameter Θ_i , e_i is the position error defined similarly to (7.4). Both controllers are applied to the same group of robots with the same unknown parameters mentioned above for leaderless consensus. Fig. 7.14 shows the comparison results with the same controller parameters $c_1 = 5$ and $c_2 = 5$, the same initial positions and the same initial parameter estimates. It can be observed that $\|e(t)\|$ converge to 0.0001 in 0.06s with our proposed controllers whereas it takes 0.25s to converge to 0.0024 when the controller in [123] is used.

For the leaderless case with directed topology, the connection graph is shown in Fig. 7.15. The initial positions of the robots are at (20, 54), (4, 20), (44, 1) and (-2, -29) respectively. Fig. 7.17 and Fig. 7.18 show the positions and the torques of the robots respective.

Figure 7.1: Undirected graph topology for leaderless consensus and rendezvous seeking.

Figure 7.2: Directed graph topology for leader-following.

Figure 7.3: Positions of four mechanical robots in leaderless consensus with undirected topology

Figure 7.4: Errors e_{ij} of four mechanical robots in leaderless consensus with undirected topology

Figure 7.5: Torques of four mechanical robots in leaderless consensus with undirected topology

Figure 7.6: Positions of four mechanical robots in location seeking with undirected topology

Figure 7.7: Errors z_{ij} of four mechanical robots in location seeking with undirected topology

Figure 7.8: Torques of four mechanical robots in location seeking with undirected topology

Figure 7.9: Positions of four mechanical robots in leader-follower case with undirected topology

Figure 7.10: Errors e_{kj} of four mechanical robots in leader-follower case with undirected topology

Figure 7.11: Torques of four mechanical robots in leader-follower case with undirected topology

Figure 7.12: Convergence rates with different control parameters for the case of leaderless consensus with undirected topology. Black solid line: $c_1 = 20$, $c_2 = 5$. Green dash line: $c_1 = 5$, $c_2 = 5$.

Figure 7.13: Robot 1's torque with different control parameters for the leaderless consensus with undirected topology. Black lines: $c_1 = 20, c_2 = 5$. Green lines: $c_1 = 5, c_2 = 5$.

Figure 7.14: Convergence time and rates with different controllers for leaderless consensus with undirected topology. Green dash line: non-finite-time controllers in [123]; Black solid line: finite-time controllers

Figure 7.15: Directed graph topology for leaderless consensus with directed topology.

Figure 7.16: Directed graph topology for leader-following with directed topology.

Figure 7.17: Positions of four mechanical robots in leaderless consensus with directed topology

Figure 7.18: Torques of four mechanical robots in leaderless consensus with directed topology

7.5 Conclusion

In this chapter we investigate the finite-time consensus control for a group of nonlinear mechanical systems with parametric uncertainties. New continuous distributed controllers are proposed for the multi-agent systems. Firstly for the leaderless multi-agent systems, it is shown that the states of the mechanical systems can reach a consensus within finite time. We also prove that with our control schemes all the systems can reach a static rendezvous location in finite time under an directed graph when only part of the agents have accesses to the rendezvous location. Secondly in leader-follower scenario of a group of mechanical systems with a directed graph, based on hierarchical decomposition for the graph, it is demonstrated that the consensus can be reached or the leader-following errors converge to a region with arbitrarily small radius under different situations in finite time. Transient performances in terms of convergence rates and time are also analyzed and established.

7.6 Appendix

Proof of Facts 7.1

From Lemma 2 we have

$$\sum_{j=1}^m |\varphi_{kj}|^{2-g} = \sum_{j=1}^m (\varphi_{kj}^{1+g})^{\frac{2-g}{1+g}} \leq m^{\frac{2g-1}{1+g}} \left(\sum_{j=1}^m \varphi_{kj}^{1+g} \right)^{\frac{2-g}{1+g}}$$

Then we consider two cases.

Case 1: If $\sum_{j=1}^m \varphi_{kj}^{1+g} \geq \frac{\epsilon}{2}$, we choose c_{k2} such that $b_{k2} > 4k_9 m^{\frac{2g-1}{1+g}} \left(\frac{\epsilon}{2} \right)^{\frac{1-2g}{1+g}}$, then we have $b_{k2} \sum_{j=1}^m \varphi_{kj}^{1+g} \geq 4k_9 \sum_{j=1}^m |\varphi_{kj}|^{2-g}$.

Case 2: If $\sum_{j=1}^m \varphi_{kj}^{1+g} \leq \frac{\epsilon}{2}$, then $\sum_{j=1}^m e_{kj}^{1+g} \geq \frac{\epsilon}{2}$. In this case if we choose c_{k1} such that $b_{k1} > 4k_9 m^{\frac{2g-1}{1+g}} \left(\frac{\epsilon}{2} \right)^{\frac{1-2g}{1+g}}$, we have $b_{k1} \sum_{j=1}^m e_{kj}^{1+g} \geq 4k_9 \sum_{j=1}^m |\varphi_{kj}|^{2-g}$. This completes the proof. $\square$

Proof of Facts 7.2

Note that $\mathbf{e} = (\mathcal{L} \otimes I_m) \mathbf{q}$ where $\mathbf{q} = (q_i, \dots, q_m)$ with $q_j = (q_{1j} - \delta_{1j}, \dots, q_{nj} - \delta_{nj})$, $j = 1, \dots, m$. Then $(\nu^T \otimes I_m) \mathbf{e} = (\nu^T \otimes I_m) (\mathcal{L} \otimes I_m) \mathbf{q} = (\nu^T \mathcal{L} \otimes I_m) \mathbf{q} = 0$, which means $e_j \perp \nu$, thus according to Lemma 2.8

$$\frac{\mathbf{e}^T P \mathbf{e}}{\mathbf{e}^T \mathbf{e}} \geq \min(\nu_i) \frac{\mathbf{e}^T P \mathbf{e}}{\mathbf{e}^T \Omega \mathbf{e}} = \min(\nu_i) a_{\mathcal{L}}$$

It is easy to see that P has a simple eigenvalue zero with $v_0 = [1 \dots 1]^T$ being the corresponding eigenvector and $v_0^T \nu = 1$. Let θ_ν being the angle of v_0 and ν , then

$$\cos \theta_\nu = \frac{v_0^T \nu}{\|v_0\| \cdot \|\nu\|} = \frac{1}{\sqrt{n(\nu_1^2 + \dots + \nu_n^2)}} \quad (7.143)$$

Since $e_j \perp \nu$, $j = 1, \dots, m$, so from (7.143) $e_j \notin \text{span}\{v_0\}$. Let θ_j being the angle

of v_0 and e_j , then

$$\begin{aligned}
\cos \theta_j &= \frac{e_{1j} + \dots + e_{nj}}{\sqrt{n(e_{1j}^2 + \dots + e_{nj}^2)}} \\
&= \frac{e_{1j} + \dots + e_{nj} - e_{1j}\nu_1 - \dots - e_{nj}\nu_n}{\sqrt{n(e_{1j}^2 + \dots + e_{nj}^2)}} \\
&\leq \nu_{\max} \frac{|e_{1j}| + \dots + |e_{nj}|}{\sqrt{n(e_{1j}^2 + \dots + e_{nj}^2)}} \\
&\leq \nu_{\max}
\end{aligned} \tag{7.144}$$

where $\nu_{\max} = \max\{1 - \nu_1, \dots, 1 - \nu_n\} < 1$. On the other hand, let θ_{e_j} being the angle between e_j and $\text{sig}(e_j)^g$, then

$$\begin{aligned}
\cos \theta_{e_j} &= \frac{|e_{1j}|^{1+g} + \dots + |e_{nj}|^{1+g}}{\sqrt{e_{1j}^2 + \dots + e_{nj}^2} \sqrt{e_{1j}^{2g} + \dots + e_{nj}^{2g}}} \\
&\geq \frac{\left((|e_{1j}|^{1+g} + \dots + |e_{nj}|^{1+g})^{\frac{1}{1+g}}\right)^{1+g}}{(|e_{1j}| + \dots + |e_{nj}|)(|e_{1j}|^g + \dots + |e_{nj}|^g)} \\
&\geq \frac{\left(n^{g-1}(|e_{1j}| + \dots + |e_{nj}|)\right)^{1+g}}{(|e_{1j}| + \dots + |e_{nj}|)(|e_{1j}|^g + \dots + |e_{nj}|^g)} \\
&= \frac{n^{(g-1)(1+g)}(|e_{1j}| + \dots + |e_{nj}|)^g}{|e_{1j}|^g + \dots + |e_{nj}|^g} \\
&\geq n^{(g-1)(2+g)}
\end{aligned} \tag{7.145}$$

which means the angle between e_j and $\text{sig}(e_j)^g$ can be small enough by adjusting σ . So if σ is chosen to guarantee $n^{(g-1)(2+g)} > \nu_{\max}$, from (7.144) and (7.145) we know $\text{sig}(e_j)^g \notin \text{span}\{v_0\}$, so

$$\frac{\text{sig}(\mathbf{e}^T)^g P \text{sig}(\mathbf{e})^g}{\text{sig}(\mathbf{e}^T)^g \text{sig}(\mathbf{e})^g} > 0$$

Thus there exists a $\tilde{c}$ such that the conclusion holds. This completes the proof.

Chapter 8

Conclusion and Future Work

8.1 Conclusion

In this thesis, we have investigated adaptive control of underactuated mechanical systems and adaptive finite-time control of multi-agent nonlinear mechanical systems. The obtained results are summarized as follows:

1. The stabilization and tracking control of nonholonomic mobile robots with parametric uncertainties and external disturbance in the presence of input saturation is addressed in Chapter 3. The torques are designed such that the bounds of the torques are functions of only design parameters and reference trajectory and which are computable in advance. For a given reference trajectory, design parameters are chosen such that the bounds of the torques are within the given saturation limits. To handle the disturbances, the unknown bounds are estimated through adaptive law and the estimates are employed in controller design. The system stability is established.
2. Tracking control of underactuated ship in the presence of input saturation is addressed in Chapter 4. By dividing the tracking error dynamic system into a cascade of two subsystems, the torques in surge and yaw axis are designed separately using the backstepping technique. More specifically, we design the yaw axis torque in such a way that its corresponding subsystem is finite time stable, which makes it be de-coupled from the second subsystem

after a finite time. This enables us to design the torque in the surge axis independently.

3. An adaptive output feedback tracking controller for nonholonomic mobile robots is proposed in Chapter 5. The major difficulties of adaptive output feedback for nonholonomic mobile robots are caused by simultaneous existence of nonholonomic constraints, unknown system parameters and a quadratic term of unmeasurable states in the mobile robot dynamic system as well as their couplings. To overcome these difficulties, a new adaptive control scheme is proposed including designing a new adaptive state feedback controller and two high-gain observers to estimate the unknown linear and angular velocities respectively.
4. Both state feedback and output feedback robust adaptive controllers for an underactuated ship are proposed to make the ship follow a predefined path with guaranteed transient performance and arbitrarily small steady state position error and orientation error in Chapter 6, despite the presences of environmental disturbances induced by wave, wind and ocean current and unknown system parameters. By using the prescribed performance bound technique, the position error and orientation error can be guaranteed to converge to arbitrarily small residual sets at a pre-specified exponential rate.
5. The finite-time consensus control of a group of mechanical systems with parametric uncertainties is considered in Chapter 7. New adaptive finite time continuous distributed control algorithms are proposed for the multi-agent systems under two different scenarios. Firstly for the leaderless multi-agent systems, it is shown that the states of the mechanical systems can reach a consensus within finite time. Secondly in the leader-follower scenario of a group of mechanical systems, based on hierarchical decomposition and finite-time parameter estimation method for the graph, it is demonstrated that perfect consensus can be reached or the leader-following errors converge to a ball with arbitrarily small radius in finite time. Transient performances

in terms of convergence rates and time are also analyzed.

8.2 Recommendation for Future Research

Besides the work achieved in literature and this thesis, there are still several interesting research topics that are worthy to be explored in the areas of adaptive control of underactuated mechanical systems and adaptive finite-time multi-agent systems control, as outlined below.

8.2.1 Adaptive Output-feedback Stabilization of Nonholonomic Mobile Robots

Adaptive output-feedback tracking control of nonholonomic mobile robots is investigated in Chapter 5. However, it is still assumed that the linear velocity v_r satisfies the PE condition. In the case of stabilization, this assumption may not be satisfied. Thus, adaptive output-feedback stabilization of nonholonomic mobile robots is an important open problem. Furthermore, the design of a universal controller that can achieve the adaptive output feedback stabilization and tracking control simultaneously is also a potential research issue.

8.2.2 Adaptive Control of General Underactuated Mechanical Systems

There is still no result on the tracking or stabilization control of general underactuated mechanical systems,

$$\begin{aligned} M_{11}(q)\ddot{q}_1 + M_{12}(q)\ddot{q}_2 + G_1(q, \dot{q}) &= B(q)u \\ M_{21}(q)\ddot{q}_1 + M_{22}(q)\ddot{q}_2 + G_2(q, \dot{q}) &= 0 \end{aligned}$$

where q is the coordinates and u is the control. $B(q)$ is assumed to be a full-rank matrix. The second equation refers to the non-integrable second-order nonholonomic constraints. New adaptive control schemes are needed to solve the adaptive

control of the general underactuated mechanical systems.

8.2.3 Discrete-time Finite-time Consensus of Multi-agent Systems

A system is finite-time stable if there exists a time-interval such that its states do not exceed some bounds during this time-interval. Normally asymptotical stability is enough for most of the practical applications, but there are still some cases where large values of the states are not acceptable. Finite-time consensus control of multi-agent systems has received attentions from the control community. However, finite-time consensus control of discrete-time multi-agent systems has never been considered.

Author's Publications

1. Jiangshuai Huang, Changyun Wen, Wei Wang, Zhong-Ping Jiang: Adaptive stabilization and tracking control of a nonholonomic mobile robot with input saturation and disturbance. *Systems & Control Letters* 62(3): 234-241 (2013)
2. Jiangshuai Huang, Changyun Wen, Wei Wang: Adaptive Output Feedback tracking control of a nonholonomic mobile. Regular paper, *Automatica* 50(3): 821-831 (2014)
3. Jiangshuai Huang, Changyun Wen, Wei Wang, Yong-Duan Song: Adaptive Finite-time Consensus Control of a Group of Uncertain Nonlinear Mechanical Systems. *Automatica*, 51, 292-301, (2015)
4. Jiangshuai Huang, Changyun Wen, Wei Wang, Yong-Duan Song: Design of Adaptive Finite-time Controllers for Nonlinear Uncertain Systems Based on Given Transient Specifications. *Automatica* (under second review)
5. Jiangshuai Huang, Changyun Wen, Yong-Duan Song, Wei Wang: Robust Adaptive Path Following Control of Underactuated Ships with Guaranteed Transient Performance. *IEEE Trans. Industrial Electronics* (under review)
6. Jiangshuai Huang, Changyun Wen, Wei Wang, Yong-Duan Song: Adaptive Finite-time Consensus of a Group of Uncertain Nonlinear Mechanical Systems in Directed Graphs, *Systems & Control Letters*, (under review)
7. Jiangshuai Huang, Changyun Wen, Wei Wang, Yong-Duan Song: Global Tracking Control of Underactuated Ships with Input Saturation, *Systems & Control Letters*, (under second review)
8. Jiangshuai Huang, Changyun Wen, Wei Wang, Yong-Duan Song: Adaptive finite-time leader-following consensus control of a group of uncertain mechanical systems, *ICIEA 2014*, Hangzhou, 1055-1060
9. Jiangshuai Huang, Changyun Wen, Wei Wang, Yong-Duan Song: Global Stabilization Control of Underactuated Ships with Input Saturation, *CC-C2015*, Hangzhou

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