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FREE VIBRATION ANALYSIS OF DELAMINATED BEAMS

LIU YANG

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FREE VIBRATION ANALYSIS OF DELAMINATED BEAMS

LIU YANG

School of Mechanical and Aerospace Engineering

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Abstract

In this thesis, new non-dimensional parameters are introduced, and analytical solutions are developed to study the vibration of delaminated beams in civil and structural engineering, including beams with step axial force and cross-section, beams subjected to static end moments and beams fully or partially supported by elastic foundation; as well as the vibration of delaminated beams in aerospace engineering, including beams with edge cracks, rotating Timoshenko beams and functionally graded beams.

Euler-Bernoulli beam theory is adopted for the study of beams with step axial force and cross-section, beams subjected to axial force static end moments, beams supported by elastic foundation and beams with edge cracks; while Timoshenko beam theory is adopted to study the vibration of delaminated rotating beams and Kirchhoff-Love hypothesis is adopted for the study of functionally graded beams with single delamination. Both ‘Free mode’ and ‘constrained mode’ assumptions in delamination vibration are studied in this thesis.

This thesis focuses on how the effects of delamination (its size, thickness-wise location and length-wise location) on natural frequencies, mode shapes, buckling loads and critical moments for lateral instability are affected by the stepped axial force (its amplitude and location), the stepped cross-section (the stepped cross-section ratio and location), the static end moments, the elastic foundation (its stiffness, length and location), the crack (its depth and location), the rotating speed and Timoshenko effect (shear deformation and rotary inertia), as well as the material gradient for functionally graded beams.

The use of these analytical solutions allows inexpensive simulations of slight variations in the system, such as changes in the physical parameters (delamination lengths and locations) and boundary conditions. This thesis provides analytical and exact solutions that can also serve as the benchmark for FEM and other numerical solutions.

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List of Symbols

A_i	Cross-sectional area of the i th sub-beam
a	Length of delamination
B	Nondimensional crack depth
β	Constant defining material property varying along the thickness direction for functionally graded beams
b	Depth of crack
C_i, S_i, CH_i, SH_i	Arbitrary constants
D	Nondimensional crack location
d	Distance from the center of the beam to the center of the delamination
E	Young's modulus of an isotropic material
ED	Effective delamination length
El_i	Bending stiffness i th sub-beam
Φ	Cross-section area ratio
G	Shear modulus
H_i	Thickness of the i th sub-beam
I_p	Polar moment of inertia
J	St Venant torsion constant

k	Stiffness of elastic foundation
L	Length of the beam
$\lambda_i = \left(\frac{\omega^2 \rho_i A_i}{EI_i} \right)^{\frac{1}{4}} L$	Non-dimensional frequency of the i th sub-beam
λ_d	Non-dimensional frequency of intact beam
M_{zz}	Static end moment
P_i	Axial force of the i th sub-beam
ρ_i	Mass density of the i th sub-beam
R_1	Nondimensional axial force step location
T	Centrifugal force exerted on the cross section of a beam due to rotation
θ	Torsion displacement
ψ	Slope of the deflection curve of beam under bending vibration
r	relative slenderness ratio
W_i	Displacement mode shape of the i th sub-beam
ω	Natural frequency of the delaminated beam
ω_0	Natural frequency of the undelaminated beam
η	Nondimensional rotating speed

Chapter 1

Introduction

This thesis presents studies on the vibration of delaminated beams, taking into consideration of various vibration, boundary, loading, damage and material conditions commonly seen in modern industries. The motivation and relevance of these studies are presented in this chapter, together with the objectives and outline of the thesis.

1.1 Motivation

1.1.1 Vibration of delaminated beams

Delaminations in structures may arise during manufacturing (e.g., incomplete wetting, air entrapment) or during service (e.g., low velocity impact) (Della and Shu, 2007a). They may not be visible on the surface since they are embedded within the structures. However, the presence of delamination may significantly reduce the stiffness and strength of the structures (Tay, 2003) and may affect their vibration characteristics (e.g., natural frequencies and mode shapes).

Understanding the changes in the vibration parameters of damaged structures is of particular interest to the community of vibration-based structural health monitoring (SHM). SHM, which has emerged as a well-recognized field of technology, is defined as an autonomous system for the continuous monitoring, inspection and damage detection of a structure with minimum labor requirement (Kessler *et al.*, 2002). By analysing the changes in natural frequencies and mode shapes of a damaged structure, some predictions on the characteristics of the damage (type, size and location) can be made. Therefore, considering the wide applications of beam-structures and common occurrence of delamination in various aspects of modern industries, it is of great practical importance to understand how the presence of delamination (its length and location) affects the natural frequencies and mode shapes of the structures under different types of vibration, boundary, loading, damage, and material conditions.

1.1.2 Vibration of delaminated beams in civil and structural engineering

In recent decades, due to the importance of beams in industry, many scientists and engineers have published papers in the field of static, free and forced vibration analysis of beams with different geometry, boundary, and loading conditions. In many aspects of civil construction, axial loads and static end moments are commonly exerted on beams by connections. Furthermore, beams resting on elastic foundations are very common in structural and geotechnical engineering and its dynamic analysis is an important part of structural investigation (Ayvaz and Ozgan, 2002). Therefore, to obtain a better understanding on the vibration of delaminated beams in civil and structural engineering,

the effects of axial loads, static end moments and elastic foundation need to be considered.

1.1.3 Vibration of delaminated beams in aerospace engineering

In aerospace engineering, the helicopter rotor blades, turbine and compressor blades, wind turbine blades, and propeller blades are routinely modeled as rotating beams (Yoo *et al.*, 2005). Also, cracks frequently occur in practical rotating structures due to cyclic fatigue or manufacturing defects (Cheng *et al.*, 2011). Furthermore, functionally graded materials are increasingly used in aerospace engineering (Atmane *et al.*, 2011). Therefore, to obtain a better understanding on the vibration of delaminated beams in aerospace engineering, the effects of rotating, crack and functionally graded material need to be considered.

1.2 Scope of Work

1.2.1 Vibration of delaminated beams in civil and structural engineering

1.2.1.1 Vibration and buckling of multiple delaminated beams with step axial force and cross-section

A study on the free vibration of multiple delaminated beams with step changes in axial force and cross-section is presented here. This study aims to:

1. introduce new nondimensional parameters of *normalized distance of the step change in axial force, effective delamination length, normalized axial force and cross-section ratio*;

2. develop analytical solutions for the free vibration of multiple delaminated beams with step changes in axial force and cross-section;
3. investigate how the effects of delamination (its size, thickness-wise location and length-wise location) on natural frequencies, mode shapes and buckling loads are affected by the stepped axial force (its amplitude and location) and the stepped cross-section (the cross-section ratio and the location);
4. compare the ‘free mode’ and ‘constrained mode’ frequencies of the delaminated beams under the influences of the stepped axial force (its amplitude and location) and stepped cross-section (the cross-section ratio and the location).

1.2.1.2 Coupled bending-torsion vibration of delaminated beams subjected to axial load and static end moments

A study on the coupled bending-torsion vibration of delaminated beam subjected to axial force and static end moments is presented here and aims to:

1. develop analytical solutions for the coupled bending-torsion vibration of delaminated beams subjected to axial force and static end moments;
2. verify the analytical solutions by comparing with previous analytical and experimental results and ANSYS finite element modeling;
3. investigate how the effects of delamination (its size, thickness-wise location and length-wise location) on natural frequencies, mode shapes, buckling loads and critical moment for lateral instability are affected by the axial load and static end moment.

1.2.1.3 Vibration of delaminated bimaterial beams fully or partially supported by elastic foundation

A study on the free vibration of multiple delaminated bimaterial beams fully or partially supported by elastic foundation is presented here. This study aims to:

1. introduce new nondimensional parameters of *normalized axial stiffness ratio* and *normalized bending stiffness ratio*;
2. develop analytical solutions for the free vibration of delaminated bimaterial beams fully or partially supported by elastic foundation;
3. investigate how the effects of delamination (its size, thickness-wise location and length-wise location) on natural frequencies and mode shapes are affected by the stiffness, length and location of the elastic foundation;
4. compare the ‘free mode’ and ‘constrained mode’ frequencies of the delaminated beams under the influences of elastic foundation.

1.2.2 Vibration of delaminated beams in aerospace engineering

1.2.2.1 Vibration of delaminated beams with edge cracks

A study on the free vibration of delaminated beams with edge cracks is presented here. This study aims to:

1. introduce nondimensional parameters of *normalized crack location* and *normalized crack depth*;
2. develop analytical solutions for the free vibration of delaminated beams with edge cracks;

3. investigate how the effects of delamination (its size, thickness-wise location and length-wise location) on natural frequencies are affected by the depth and location of the crack while considering the two different cases of cracks occurring outside or inside the delaminated area;
4. compare the ‘free mode’ and ‘constrained mode’ frequencies of the delaminated beams under the influences of the edge crack (its depth and location).

1.2.2.2 Vibration of rotating Timoshenko beams with multiple delaminations

There is no study in the literature on the effect of delamination on the vibration of rotating beams.

A study on the free vibration of rotating Timoshenko beams with multiple delaminations is presented here. This study aims to:

1. introduce nondimensional parameters of *relative slenderness ratio* and *normalized rotating speed*;
2. develop analytical solutions for the free vibration of rotating Timoshenko beams with multiple delaminations;
3. investigate how the effects of delamination (its size, thickness-wise location and length-wise location) on natural frequencies and mode shapes are affected by the rotating, shear deformation and rotary inertia;
4. compare the ‘free mode’ and ‘constrained mode’ frequencies of the delaminated beams under the influences of rotating speed and Timoshenko effect (shear deformation and rotary inertia).

1.2.2.3 Vibration of functionally graded beams with single delamination

A study on the free vibration of functionally graded beams with a single delamination is presented here. This study aims to:

1. develop analytical solutions for the free vibration of functionally graded beam with a single delamination;
2. investigate how the effects of delamination (its size, thickness-wise location and length-wise location) on natural frequencies are affected by the material gradient while considering different boundary conditions;
3. compare the ‘free mode’ and ‘constrained mode’ frequencies of the delaminated beams under the influences of different material gradient.

1.3 Thesis Outline

Following this introduction, Chapter 2 presents a literature review. Chapter 3 studies the vibration of multiple delaminated beams with step axial forces and cross-sections. Chapter 4 studies the coupled bending-torsion vibration of multiple delaminated beams subjected to axial force and static end moments. Chapter 5 studies the vibration of multiple delaminated bimaterial beams fully or partially supported by elastic foundation. Chapter 6 studies the vibration of delaminated beams with edge cracks. Chapter 7 studies the vibration of rotating Timoshenko beams with multiple delaminations. Chapter 8 studies the vibration of exponential functionally graded beams with a single delamination. Chapter 9 presents the conclusions and publications resulting from the thesis. Recommendations are made for future works.

Chapter 2

Literature Review

2.1 Introduction

This chapter is outlined as follows. Section 2.2 reviews the vibration of beams in civil and structural engineering, focusing on the vibration of beams subjected to axial load, static end moments and the effect of elastic foundation. Section 2.3 reviews the vibration of beams in aerospace engineering, focusing on the vibration of beams with edge cracks, the vibration of rotating beams as well as the vibration of functionally graded beams. In Section 2.4 the available mathematical models for the vibration of delaminated composite laminates, as well as the influence of the delamination on the natural frequencies and mode shapes are reviewed. In Section 2.5, the summary of the chapter, and the issues that are to be addressed in this thesis are presented.

2.2 Vibration of Beams in Civil and Structural Engineering

2.2.1 Vibration of beams subjected to axial force

The vibrations of axially loaded beams are extensively studied for its wide application. For example, since buckling occurs when the lowest frequency decreases to zero, it may be possible to non-destructively estimate buckling loads by measuring frequencies at several

load levels and extrapolating the results (Platu and Virgin, 1990; Chuang and Wang, 1991). Furthermore, the natural frequencies of axially loaded beams are often required in helicopter and turbine blades design among other applications (De Borbón and Ambrosini, 2010).

The influence of axial loading on the free vibration frequencies of homogeneous beams is well established. Generally speaking, tension tends to increase free vibration frequencies, while compression tends to reduce them (Virgin and Platu, 1993). Virgin and Platu (1993) presented the analytical solution to the axially loaded forced vibration of beams with damping. The influence of axial load on the response is investigated, with attention focused on the maximum amplitude of the central deflection over all possible forcing frequencies. Matsunaga (1996) analyzed the natural frequencies and buckling loads of a simply supported beam with small length-to-depth ratio and sufficiently thin rectangular cross-sections subjected to initial axial tensile or compressive forces. Naguleswaran (2003, 2005) studied the vibration and stability of Euler-Bernoulli homogeneous beams with step changes in axial force and in cross-section (Figure 2.1). It is shown that stepped shafts are inferior machine elements if dynamic properties are the prime consideration. Naguleswaran (2004) also provided the solution to the vibration of a uniform Euler-Bernoulli beam under linearly varying axial force.

Figure 2.1. The three-step beam, the axial force applied at the steps, the co-ordinate system and free body diagrams of elements at steps (Naguleswaran, 2003).

The composite beam members subjected to axial loading have been increasingly used over the past few decades in the fields of aerospace, civil, and mechanical engineering due to

their excellent engineering features. Banerjee (1998) employed the dynamic stiffness matrix method to study the free vibration of axially loaded composite Timoshenko beams. This is accomplished by developing an exact dynamic stiffness matrix of a composite beam with the effects of axial force, shear deformation, and rotary inertia taken into account. The theory includes the (material) coupling between the bending and torsion modes of deformations which is usually present in laminated composite beams due to ply orientation. Li *et al.* (2008) used the same method to further study the buckling behaviors of axially loaded laminated composite beam having arbitrary lay-up.

There are few studies on the vibration of axially loaded beams with delaminations. Chang and Liang (1998) studied the free and forced vibrations of post-buckled delaminated homogeneous isotropic beam-plates. They concluded that if a beam-plate is under a post-buckled state, the natural frequencies, mode shapes of the structure will be substantially affected by the changes of the size and location of delamination and by the magnitude of the axial compression. The vibration and buckling of delaminated composite orthotropic beam-plates with multiple delaminations have been studied by (Jane and Harn, 2000) using the reduced flexural stiffness method. Lee *et al.* (2002) provided the free vibration analysis for the multi-delaminated composite beam-columns subjected to axial compression load. Della and Shu (2007b, 2009) provided the analytical solution the vibration of multiple delaminated homogeneous isotropic beams. The Euler-Bernoulli beam theory and both the ‘free mode’ and the ‘constrained mode’ assumption in delamination vibration are adopted. In addition, a parametric study is performed to study the influence of the axial compressive load on the vibration characteristics of the multiple-delaminated beam. The study shows that the effects of delamination on natural frequency and mode shape are significantly influenced by the axial load.

Although many studies have been presented for the vibration of delaminated beams with axial force due to the wide application of axially loaded structures, studies on delaminated beams with step changes in axial load and cross-section have not been presented in the literature.

Figure 2.2. One dimensional model for axially loaded beam with a single delamination (Jane and Harn, 2000).

2.2.2 Coupled bending-torsion vibration of beams

Previous work related to the coupled bending-torsion vibrations of beams can be broadly classified into three groups: i) the coupling due to the cross-sectional symmetry that influences the location of shear center and mass center; ii) material coupling between bending and torsional vibrations only due to ply orientation in composite laminated beams. iii) the coupling induced by static end moments exerted on the beam.

i. Coupled bending-torsional vibration of beams due to the cross-sectional symmetry:

- 1) Double cross-sectional symmetry: for a uniform beam having cross-sectional symmetry in the two perpendicular directions, shear center and centroid are coincident, thus bending and torsional vibrations are uncoupled. The solution of free vibration frequencies is well known (Bishop and Johnson, 1979).

- 2) Single cross-sectional symmetry: in this case shear center and centroid are separated by a distance that results in coupling between torsion and bending vibrations in the perpendicular direction of the axis of symmetry (Bishop and Johnson, 1979).
 - 3) No cross-sectional symmetry: this is the most general case in which bending vibrations in two perpendicular directions are coupled with torsion vibrations. In the eigen-frequency determination of solid cross-section turbine blades, Rao and Carnegie (1970) used a Galerkin procedure. Finite element modeling was adopted by Tanaka and Bercin (1997) to study the coupled bending-torsional vibration of uniform beams with an arbitrary cross-section. However, the warping and rotary inertia effect was neglected by removing some of the entries of stiffness and mass matrices.
- ii. The coupled vibration of composite laminated beams caused by material coupling due to ply orientation.

Banerjee (1998) presented the free vibration analysis of axially loaded composite Timoshenko beams by using the dynamic stiffness matrix method. The theory includes the (material) coupling between the bending and torsional modes of deformations which is usually present in laminated composite beams due to ply orientation. Ellakany *et al.* (2004) and Mei (2005) studied the same problem by using combined transfer matrix and analogue beam method and wave vibration approach, respectively.

- iii. The coupled vibration of beams induced by static end moments.

End moments are commonly exerted on beams by the connections, which are expected to transmit axial force, shear, and moment between connecting members (Chen and Lui, 2005). For example, welded connection behavior in structures is often semi-rigid with a resultant end moment upon the supporting members, beams, columns, plates, etc (Blodgett, 2002). The coupling between bending and torsion vibration can be induced

by the static end moments (Figure 2.3) (Voros, 2009). Analytical solution for coupled bending-torsion vibration of beams subjected to static end moments is introduced by Joshi and Suryanarayan (1984), but only for beams with simply supported boundary conditions. Subsequently, Joshi and Suryanarayan (1989) developed a unified analytical solution for various boundary conditions for the coupled bending-torsional vibration of beams subjected to axial loads and end moments. An iterative method for coupled bending-torsion vibration of initially stressed beams is also presented by Joshi and Suryanarayan (1991). Voros (2009) provided the analysis on the natural frequencies and mode shapes of straight beams where the coupling between the bending and torsion is induced by static end moments. The governing differential equations and boundary conditions for coupled vibrations of Euler–Bernoulli–Vlasov beams are performed by using the virtual work principle which includes the second order terms of finite beam rotations.

There is no work available in the literature on the study of the coupled vibration of delaminated beams subjected to static end moments. Consequently, the current work could provide valuable insight on how static end moments change the effects of delamination on natural frequencies and mode shapes.

Figure 2.3. Geometry of a beam subjected to axial force and static end moments (Joshi and Suryanarayan, 1984).

2.2.3 Vibration of beams supported by elastic foundation

Beams resting on elastic foundations can model many soil-structure interaction problems, and have wide applications in many engineering fields (Wang *et al.*, 2013). Knowledge of the natural frequencies of vibration and associated mode shapes for beams resting on elastic foundation is also useful in vibration isolation problems, and in general as indicators of anticipated dynamic response.

Many researchers use the pioneering Winkler model (1867) for soil-structure interaction in the static and dynamic analysis of homogeneous beams resting on elastic foundations, where the vertical surface displacement of the beam is assumed to be proportional at every point to the contact pressure at that point (Hetenyi, 1950). In the Winkler model, it is assumed that the foundation soil consists of linear elastic springs which are closely spaced and independent of each other, as is shown in Figure 2.4.

Figure 2.4. A beam on a Winkler elastic foundation.

Analytical solutions for the vibration of homogeneous beams supported by elastic foundation are provided by many researchers. Eisenberger *et al.* (1985) studied analytically the vibration of beams fully or partially supported by elastic foundation. Wang (1991) provided the analytical solution to the vibration of stepped beams on elastic foundations. Zhou (1993) provided a general solution to vibration of beams on variable Winkler elastic foundation. The exact solution of the dynamic response of the beam is obtained by considering the reaction force of the foundation on the beam as the external force acting on the beam.

Among various numerical methods, finite element method is mostly used by many researchers to study the vibration of beams on elastic foundation. A simple finite element method is developed and applied to treat the free vibration analysis supported on elastic foundation (Thambiratnam and Zhuge, 1996). Franciosi and Masi (1993) used the shape functions to provide a finite element free vibration analysis of beams on a two-parameter elastic foundation. Yokoyama (1996) also used finite element technique to extend Franciosi and Masi's work to include the effect of shear deformation and rotary inertia by using Timoshenko beam theory.

The studies based on free vibration of composite laminates resting on elastic foundation are increasingly used in engineering applications and occupy a prominent place for solving many practical problems related to structural mechanics such as nuclear reactors, aircraft

runways, building foundation slabs, bridges, etc (Lal *et al.*, 2008). Malekzadeh and Vosoughi (2009) presented an efficient and accurate differential quadrature large amplitude free vibration analysis of laminated composite thin beams on nonlinear elastic foundation. Baghani *et al.* (2011) provided the analytical expressions for large amplitude free vibration and post-buckling analysis of unsymmetrical laminated composite beams on elastic foundation.

There is no study available in the literature on the free vibration of delaminated bimaterial beams fully or partially supported by the elastic foundation. The current work provides the first study in literature on the effects of elastic foundation (its stiffness, location and length) on the natural frequencies and mode shapes of homogeneous and bimaterial beams with multiple delaminations.

2.3 Vibration of Beams in Aerospace Engineering

In aerospace engineering, the helicopter rotor blades, turbine and compressor blades, wind turbine blades and propeller blades are routinely modeled as rotating beams (Yoo *et al.*, 2005). Also, cracks frequently occur in practical rotating structures due to cyclic fatigue or manufacturing defects (Cheng *et al.*, 2011). Furthermore, functionally graded materials are increasingly used in aerospace engineering (Atmane *et al.*, 2011). Therefore, to obtain a better understanding on the vibration of delaminated beams in aerospace engineering, the effects of rotating, crack and functionally graded material need to be considered.

2.3.1 Vibration of beams with edge cracks

Cracks frequently occur in practical rotating structures for the sake of cyclic fatigue or manufacturing defects (Cheng *et al.*, 2011). These cracks may cause changes in the structural

parameters (e.g., the stiffness of a structural member such as beam elements), which can change the dynamic properties (such as natural frequencies and mode shapes) (Kienzler and Herrmann, 1986; Doebling *et al.*, 1995). When a structural component is subjected to a crack, the crack induces a local flexibility which is a function of the crack depth, thereby changing its dynamic behavior and its stability characteristics (Shifrin and Ruotolo, 1999). Consequently, cracks may significantly affect the vibration behavior of these structures.

Various analytical methods have been used to illustrate the dynamic behavior of cracked beams. One method models the crack by reducing the suitable section modulus (Chondros and Dimarogonas, 1989), while another method tries to evaluate a local flexibility for the cracked section (Adams *et al.*, 1978). Neglecting the shear effects in bending, some models have been proposed, in which the cracked sections are replaced by a rotational massless spring (Rizos *et al.*, 1990; Liang *et al.*, 1992). The physical model of a beam with a discrete single crack can be depicted in Figure 2.5, in which the local flexibility introduced by a crack is represented by a massless rotational spring, with a spring constant k_R . The rotational spring model was adopted by Lin *et al.* (2002) to provide a hybrid analytical/numerical method that permits the efficient computation of the eigen-solutions for an arbitrary number of cracks of a beam with various boundary conditions. Carneiro and Inman (2002) used the same model to include the effect of shear deformation and rotary inertia by adopting Timoshenko beam theory.

Another class of crack modeling approaches using numerical methods (e.g., finite element methods) was proposed (Chinchalkar, 2001; Saavedra and Cuitino, 2001). Kisa (2004) studied the effects of cracks on the dynamical characteristics of a cantilever composite beam. The finite element and the component mode synthesis methods are used to model the problem. The cantilever composite beam is divided into several components from the crack sections. In the work of Zheng and Kessissoglou (2004), an ‘overall additional flexibility

matrix', instead of the 'local additional flexibility matrix' (used by Kisa (2004)), is added to the flexibility matrix of the corresponding intact beam element to obtain the total flexibility matrix, and therefore the stiffness matrix. Compared with analytical results, the new stiffness matrix obtained using the overall additional flexibility matrix can give more accurate natural frequencies than those resulted from using the local additional flexibility matrix.

Figure 2.5. Mechanical model of beam with single discrete crack (Rizos et al., 1990).

The current work provides the first study on the combined effects of crack and delamination, the two most common failure modes, on the vibration of beams. It focuses on how the effects of crack (its location and depth) on natural frequencies and mode shapes are affected by the presence of delamination.

2.3.2 Vibration of rotating beams

Rotating beams are usually simplified to one-dimensional Euler–Bernoulli or Timoshenko beams subjected to centrifugal tension force. The governing differential equations of motion for rotating tapered homogeneous beams contain variable coefficients introduced due to the variation of the centrifugal force and the geometry along the beam length. The closed-form solutions for such equations therefore are difficult to obtain. As a result, there are only a handful of analytical studies on the vibration of rotating beams. The method of Frobenius is used by Wright and Smith (1982) to solve for the exact frequencies

and mode shapes for rotating beams in which both the flexural rigidity and the mass distribution vary linearly. Du *et al.* (1994) presented the equations of motion for a rotating beam based on the Timoshenko beam theory which includes the effects of rotary inertia and shear deformation. This study provided a convergent power series expression to solve analytically the two variable-coefficient differential equations, for which only approximate solutions have been used in previous analyses.

As a result, different approximate methods have been suggested to analyze the vibration problems of rotating homogeneous beams. Hodges and Rutkowski (1981) presented a variable-order finite element method to capture the free vibration properties of rotating beams. Naguleswaran (1994) studied lateral vibration of a rotating uniform Euler–Bernoulli beam based on the general solution of the mode shape equation as the superposition of four linearly independent functions. Rao and Gupta (2001) applied the finite element method to obtain natural frequencies and mode shapes of a rotating twisted and tapered beam. Gunda and Ganguli (2008) then assumed the transverse displacement to vary as a fourth order function and obtained new shape functions that would satisfy the static part of the homogeneous governing differential equation. Later Gunda *et al.* (2009) proposed a new finite element for free vibration analysis of high-speed rotating beams using shape functions that are a linear combination of the solution of the governing static differential equation of a stiff string and a cubic polynomial.

Composite rotor blades can be considered as rotating beams (Figure 2.6) undergoing bending, torsion and extensional motions. Therefore, the vibration of rotating composite laminated beams also receives much attention from researchers. Chandiramani *et al.* (2002) studied the free vibration behavior of rotating blades modeled as laminated composite, hollow (single celled) boxed beams. Centrifugal and Coriolis force field effects are considered in the formulation. Yoo *et al.* (2005) presented a modelling method for the flap-

wise bending vibration analysis of a rotating multi-layered composite beam. Linear and nonlinear free vibrations of rotating composite Timoshenko beams are investigated by Arvin and Bakhtiari-Nejad (2013). The Galerkin discretization approach is applied to the linearized updated equations of motion to determine the linear normal modes and the associated natural frequencies of the rotating composite Timoshenko beam.

Figure 2.6. The schematic of the rotating composite beam.

There is no study available in the literature on the study of the free vibration of rotating Timoshenko beams with multiple delaminations. Therefore, the current work could provide valuable insight on how the effects of delamination on natural frequencies and mode shapes are affected by rotating speed, rotary inertia and shear deformation.

2.3.3 Vibration of functionally graded beams

Functionally graded materials (FGMs) are microscopically inhomogeneous composites that are usually made from a mixture of metals and ceramics (Figure 2.7). FGMs are regarded as one of the most promising candidates for future advanced composites in many engineering sectors such as the aerospace, aircraft, automobile, and defense industries, and most recently

the electronics and biomedical sectors (Ichikawa, 2000). Consequently, there have been many studies in recent years on the static, vibration and buckling of functionally graded materials.

Analytical solutions for the vibration of functionally graded beams are provided by many researchers. Exact solutions are presented by Mahi *et al.* (2010) to study the free vibration of a beam made of symmetric functionally graded materials. Material properties are taken to be temperature-dependent and vary continuously through the thickness according to a power law distribution, or an exponential law distribution or a sigmoid law distribution. Atmane *et al.* (2011) studied the free vibration behavior of exponential functionally graded beams with varying cross-section. The governing equation is reduced to an ordinary differential equation in spatial coordinate for a family of cross-section geometries with exponentially varying width.

Figure 2.7. The geometry of a functionally graded beam with top layer as ceramic and bottom layer as metal.

Finite element method is another commonly adopted method for the study on the vibration of functionally graded beam. Free and forced lateral vibration analysis of beams made of functionally graded materials was conducted by Azadi (2011) using finite element method. The temperature dependency of material properties along with damping had not previously been taken into account in vibration analysis. The large-amplitude free vibration

analysis of functionally graded beams is also investigated by means of a finite element formulation (Hemmatnezhad *et al.*, 2013). The Von-Karman type nonlinear strain-displacement relationships are employed where the ends of the beam are constrained to move axially. The effects of the transverse shear deformation and rotary inertia are included based upon the Timoshenko beam theory. The material properties are assumed to be graded in the thickness direction according to the power-law distribution.

For FGMs, one major failure mode is cracking perpendicular to the material gradient direction which corresponds to delamination as real FGMs are usually multi-layered materials (Kawasaki and Watanabe, 1993) and the crack grows along the interface. Delamination also occurs as one of four distinctive stages of the damage development of functionally graded thermal barrier coatings (Bahr *et al.*, 2003). The buckling driven delamination problem under mechanical and thermal loadings was investigated (Chiu and Erdogan, 2003; Erdogan and Chiu, 2003) by using analytical and finite element methods.

Literature review shows that although there are quite a few papers presenting crack and fracture analyses of FGM structures, no work investigating the vibration behavior of delaminated FGM structures has been reported. This is the first study on the free vibration of exponential functionally graded beams with a single delamination. The current work focuses on how the effects of material gradient on natural frequencies are affected by the presence of delamination.

2.4 Vibration of Delaminated Beams

A considerable amount of analytical models and numerical analyses have been reported for the vibration analysis of composite laminates with delaminations. The available mathematical models can be divided into three categories:

1. Region approach models: the delaminated laminate is divided into sub-laminates or segments and the continuity conditions are imposed at the delamination junctions. Each of these sub-laminates is analyzed using the *equivalent single layer* (ESL) theories. The ESL theories consist of the classical laminate theory (CLT), first-order shear deformation theory (FSDT) and the third-order shear deformation theory (TSDT). The ESL theories are derived from the 3-D elasticity theory by making suitable assumptions concerning the deformation of the stress state through the thickness of the laminate (Reddy, 2004).
2. Layerwise models: the laminate is modeled using the *layerwise theories*, which assume separate displacement field expansions within each material layer, thus providing a kinematically correct representation of the strain field in discrete layer laminates, and allowing accurate determination of ply level stresses (Reddy, 2004). The layerwise theories are subdivided into two classes: *partial layerwise* and *full layerwise*. The partial layerwise theories are based on piece-wise linear variation of the inplane displacement components and a constant transverse displacement through the thickness (i.e. the transverse shear strain is piece-wise constant, while the transverse normal strain is zero through the thickness). Such layerwise theories are referred to as partial layerwise theories since the transverse normal displacement does not have a layerwise representation. The full layerwise theories use layerwise expansions for all three displacement components, and thus include both discrete layer transverse shear effects and transverse normal effects. The delamination can be modeled as an embedded layer or by introducing discontinuity functions in the displacement fields.
3. Reduced stiffness models: the stiffness of the laminate is considered to be reduced at the delamination region.

2.4.1 Region approach - one-dimensional models

There have been a considerable amount of analytical models for the vibration of composite laminates with delaminations, which can be divided into three categories: 1) region approach; 2) layerwise models; 3) reduced stiffness models. In region approach, the delaminated laminate is divided into sub-laminates or segments and the continuity conditions are imposed at the delamination junctions (Reddy, 2004). While in layerwise models, the laminate is modeled using the *layerwise theories*, and the delamination is modeled as an embedded layer or by introducing discontinuity functions in the displacement fields (Reddy, 2004). In reduced stiffness models, the stiffness of the laminate is considered to be reduced at the delamination region. Since the present study adopts the region approach, it will be illustrated with details in the following.

To illustrate the approach for this group of methods, consider a beam with a single through-width delamination as shown in Figure 2.8. The delaminated beam is divided into three span-wise regions, a delamination region and two integral regions. The delamination region is composed of two separate beam segments (delaminated layers), which are joined at their ends to the integral beam segments. The delaminated beam can now be analyzed as four interconnected beams. The equations of motion are formulated for each of the beam segments. By imposing appropriate boundary conditions at the ends of the two integral beam segments and continuity conditions at the delamination junctions, the vibration response of the beam can be obtained.

Since the delamination region is considered as two separate beam segments, there is a shift in the neutral axis of these segments with respect to the reference axis of the integral beam segment. This gives rise to the coupling between the axial and bending vibrations. Therefore, apart from the continuity conditions of transverse displacements, slopes, bending moments and shear forces at the delamination junctions, two additional conditions, the

continuity of axial forces and displacements need to be satisfied (Figure 2.9). The continuity condition of axial displacement, which produces the differential stretching of the delaminated layers, contributes substantially to bending stiffness of the delaminated beam.

Figure 2.8. Modeling of a beam with a single through-width delamination (Mujumdar and Suryanarayan (1988).

To study the influence of a through-width delamination on the free vibration of an isotropic beam, Wang *et al.* (1982) presented an analytical model using four Euler–Bernoulli beams that are joined together. They assumed that the delaminated layers deform ‘freely’ without touching each other (‘free mode’) and will have different transverse deformations. However, Mujumdar and Suryanarayan (1988) showed that in the case of off-midplane delaminations, the ‘free mode’ mode shapes are physically inadmissible. This is because the delaminated layers were assumed to deform ‘freely’ without touching each other and thus the delaminated layers will have different transverse deformation.

Figure 2.9. Deformation and stress components in the delamination region: (a) Deformation and stresses with the layers free to slide; (b) Deformation and stresses when the compatibility of axial displacement is satisfied (Mujumdar and Suryanarayan, 1988).

To avoid this kind of incompatibility, Mujumdar and Suryanarayan (1988) proposed a model based on the assumption that the delaminated layers are in touch along their whole length all the time, but are allowed to slide over each other ('constrained mode'). Thus, the delaminated layers are 'constrained' to have identical transverse deformations. In addition, the delaminated layers are assumed to be free to slide over each other in the axial direction except at their ends, which are connected to the integral segments. This model was referred to as the 'constrained mode'. To 'constrain' the delaminated layers, an equal and opposite

normal pressure acts on the lower surface of the upper layer and upper surface of the lower delaminated layer (Figure 2.10).

Figure 2.10. Modeling of the delamination region for the ‘constrained mode’: (a) Deformation in the “constrained mode”; (b) Contact pressure between the layers of the delamination region (Mujumdar and Suryanarayan, 1988).

Manoach *et al.* (2012) provided experimental support for the ‘free mode’ and ‘constrained mode’ assumption. They conducted experiments on a cantilever delaminated beam with two transducer attached to the top and bottom sub-beam of inside the delaminated region. The relative distance between the top and bottom beam are calculated from the experimental data, as shown in Figure 2.11. Results showed that the relative distance varies from zero to certain small value. This indicates that the top and bottom beam does exhibit ‘free mode’ behavior or ‘constrained mode’ behavior at certain time, but a single assumption cannot describe the total vibration of a delaminated beam.

Figure 2.11. The relative distance between the top and bottom sub-beam inside the delaminated region. (Manoach *et al.* 2012)

The ‘constrained mode’ model, however, failed to predict the opening in the mode shapes found in the experiments by Shen and Grady (1992) and Lestari and Hanagud (1999). To overcome this limitation, Luo and Hanagud (2000) used an analytical model based on the Timoshenko beam theory and piecewise-linear springs to simulate the ‘open’ and ‘closed’ behavior between the delaminated layers. As is elaborated in the work of Luo and Hanagud (2000), the effects between the delaminated surfaces depend on the relative position between the sublaminates during vibrations. They assume that after delamination, some constraints between the upper and lower delamination still exist. The spring stiffness would then be equal to zero (0) for the ‘free mode’ and infinity (∞) for the ‘constrained mode’. Wang and Tong (2002) introduced a nonlinear anti-interpenetration constraint model to prevent the overlapping of the delaminated layers during vibration. The Timoshenko beam theory was used. The contact forces between the delaminated layers were expressed as a function of the relative transverse displacements of the layers. Two types of functions were used, namely, a linear spring function and a nonlinear Hertz-type function. The contact forces cause

nonlinearity in the equations of motion, which were solved using the finite difference method.

2.4.2 Analytical Solution to the Vibration of beams with a single delamination

In this section, the analytical solution to the vibration of beams with a single delamination is presented. The method of solving the vibration of delaminated beam presented here is adopted and modified accordingly for all the different cases studied in the present work. By presenting the essential part of the analytical solution, a better understanding of the approach adopted in the present study can be achieved.

The analytical solutions reported by Shu (1995) on a ‘free mode’ and Shu and Fan (1996) on a ‘constrained mode’ and are used in the present study. Figure 2.8 shows a beam with length L and thickness H_1 . The beam is made of two distinct layers, with Young’s modulus E , and thickness H_2 and H_3 . The beam is separated along the interface by a delamination with length a and located at a distance d from the center of the beam. The beam can then be subdivided into three span-wise regions, a delamination region and two integral regions. The delamination region is comprised of two segments (delaminated layers), beam 2 and beam 3, which are joined at their ends to the integral segments, beam 1 and beam 4. Each of the four beams are treated as Euler-Bernoulli beams, thus the analytical solutions are valid provided that $L_i \gg H_i$, where $i = 1 - 4$ (Shu and Fan, 1996).

Two assumptions were made based on the literature: (a) The first assumption concerns the complicated changing contact between the delaminated layers. Wang et al. (1982) assumed that the delaminated layers deformed ‘freely’ without touching each other (‘free mode’), which was shown to be physically inadmissible. Mujumdar and Suryanarayan (1988) then proposed a ‘constrained mode’ where the delaminated layers are assumed to be in touch along their whole length all the time, but are allowed to slide over each other; (b) The second

assumption concerns the influence of the coupling between the axial and bending vibrations, which is due to the shifting of the reference axis of the delaminated layers with respect to the reference axis of the undelaminated beam.

Della (2005) conducted comprehensive studies on the vibration of delaminated composite beams by using same ‘free mode’ and ‘constrained mode’ assumptions. The detailed formulation and solution to the problem of vibration of delaminated beams are illustrated below.

2.4.2.1 ‘Free mode’

For the ‘free mode’ model, the governing equation for beam i is

$$EI_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \quad (i = 1 - 4) \quad (2.1)$$

where EI_i is the bending stiffness of beam i , $w_i(x,t)$ is the midplane deflection of the beam, ρ_i is the mass density and $A_i = H_i$ is the cross-sectional area of the beam per unit width. For plain-stress problem, E is the Young’s modulus, while for plain strain problem, E would be replaced by an equivalent Young’s modulus $E = E/(1-\nu^2)$, where ν is the Poisson’s ratio. The bending stiffness of beam i , $EI_i = E_i b H_i^3/12$.

Hereafter all the boundary and continuity conditions are written out in terms of $W_i(x)$ ($i = 1 - 4$) instead of $w_i(x,t)$. The generalized solutions of Equation 2.1 are,

$$W_i(x) = C_i \cos(\lambda_i \frac{x}{L}) + S_i \sin(\lambda_i \frac{x}{L}) + CH_i \cosh(\lambda_i \frac{x}{L}) + SH_i \sinh(\lambda_i \frac{x}{L}) \quad (2.2)$$

where

$$\lambda_i^4 = \frac{\omega^2 \rho_i A_i}{EI_i} L^4 \quad (2.3)$$

and where λ_i is the non-dimensional frequency. The 16 unknown coefficients C_i , S_i , CH_i and SH_i ($i = 1 - 4$) are determined from the 4 boundary conditions and 12 continuity conditions.

The appropriate boundary conditions that can be applied at the supports, $x = x_1$ and $x = x_4$, are $W_i = 0$ and $W_i' = 0$, if the end of the beam is clamped; $W_i = 0$ and $W_i'' = 0$, if simply supported; $W_i'' = 0$ and $W_i''' = 0$, if free, where $i = 1$ and 4 .

The continuity conditions for deflection, slope, bending moment and shear at the delamination junction $x = x_2$ are

$$W_1 = W_2 = W_3 \quad (2.4)$$

$$W_1' = W_2' = W_3' \quad (2.5)$$

$$EI_1 W_1'' + \frac{(EH_2 EH_3) H_1^2}{4a(EH_2 + EH_3)} (W_1' - W_{III}') = (EI_2 + EI_3) W_{II}'' \quad (2.6)$$

$$EI_1 W_1''' = EI_2 W_2''' + EI_3 W_3''' \quad (2.7)$$

The second term on the left-hand side in eqn. (2.6) represents the contribution to the bending moment from the differential stretching between beam 2 and beam 3. Equations (2.4) – (2.7) provide six equations. The same continuity conditions at $x = x_3$ result in another six equations.

The total number of boundary and continuity conditions is 16, which is equal to the total number of unknown coefficients C_i , S_i , CH_i and SH_i . A non-trivial solution exists only when the determinant of the coefficient matrix vanishes. The natural frequencies can be obtained as eigenvalues and the mode shapes, as eigenvectors.

2.4.2.2 ‘Constrained mode’

For the ‘constrained mode’ model, the governing equation for beam 1 and beam 4 is identical to eqn. (2.1). For beam 2 and beam 3, the governing equation is

$$(EI_2 + EI_3) W_{II}'''' + kW_{II}'' - (\rho A_2 + \rho A_3) \omega^2 W_{II} = 0 \quad (2.8)$$

The generalized solutions of the ‘constrained mode’ are identical in form with the ‘free mode’. The unknown coefficients C_i , S_i , CH_i and SH_i , however, are reduced to 12 coefficients which can be determined from the 4 boundary conditions and 8 continuity conditions.

The boundary conditions for the ‘constrained mode’ are the same with the boundary conditions for the ‘free mode’. The continuity conditions for deflection, slope, bending moment and shear at $x = x_2$ are,

$$W_1 = W_2 \quad (2.9)$$

$$W_1' = W_2' \quad (2.10)$$

$$EI_1 W_1'' + \frac{(EH_2 EH_3) H_1^2}{4a(EH_2 + EH_3)} (W_1' - W_{III}') = (EI_2 + EI_3) W_{II}'' \quad (2.11)$$

$$EI_1 W_1''' = (EI_2 + EI_3) W_2''' \quad (2.12)$$

where $(EI_2 + EI_3)$ is the bending stiffness at the delamination region. Similarly, we can derive the continuity conditions at $x = x_3$. For the ‘constrained mode’, the boundary conditions and continuity conditions provide 12 homogeneous equations for the 12 unknown coefficients C_i , S_i , CH_i and SH_i .

2.4.3 Influence of Delamination on the Vibration of Delaminated Beams

In general, delamination decreases the natural frequency and causes changes to the mode shape. This is due to the reduction of the stiffness caused by the delamination. However, results by Mujumdar and Suryanarayan (1988) showed that a short delamination (less than 25% of the beam length) do not significantly affect the fundamental and second mode frequencies of the beam for all boundary conditions (Figure 2.12). For longer delaminations (greater than 50% of the beam length), the weakening effect of the delamination is dependent on the boundary conditions. In fact, the more restrained is the structure along its edges, the greater is the effect on the natural frequency (Ju *et al.*, 1995). Other studies on composite

beams and plates showed that delamination of small size do not significantly affect the lower vibration mode frequencies (Tracy and Pardoen, 1989; Shen and Grady, 1992; Ju *et al.*, 1995; Saravanos and Hopkins, 1996). The experimental investigation by Zak *et al.* (2000) on cantilever composite beams and plates with a single edge delamination further verified the conclusions of Mujumdar and Suryanarayan (1988).

Figure 2.12. Influence of the delamination length on the fundamental and second mode frequencies of the beam (Mujumdar and Suryanarayan, 1988).

Mujumdar and Suryanarayan (1988) compared the frequency of ‘free mode’ and ‘constrained mode’ with experiment. The experiments were carried out on beams made by bonding stainless steel strips with a very thin adhesive layer. The delamination was simulated by bonding only a part of the beams surfaces to create delamination of various sizes in various locations. They showed that for midplane delaminations, the results of the two models are identical and in good agreement with the experimental data. For off-midplane

delaminations, for the fundamental frequency, the results of the two models are close with each other and were in good agreement with the experimental data. However, for the second vibration mode, the ‘constrained mode’ results were closer to the experimental values.

Studies showed that delamination locations also affect the natural frequencies. It was observed that a delamination located at midplane and near the fixed end of the beam decreases the natural frequency most (Mujumdar and Suryanarayan, 1988; Krawczuk et al., 1996a, b, 1997; Kim *et al.*, 2003).

2.4.4 Experimental Investigations on the Vibration of Delaminated Beams

Mujumdar and Suryanarayan (1988) compared the frequency results of ‘free mode’ and ‘constrained mode’ models with experimental data. The experiments were carried out on beams specimens made by bonding stainless steel strips with a very thin adhesive layer. The delamination was simulated by bonding only a part of the beams surfaces to create delamination of various sizes in various locations. They showed that for midplane delaminations, the results of the two models are identical and in good agreement with the experimental data. For off-midplane delaminations, for the fundamental frequency, the results of the two models are close with each other and were in good agreement with the experimental data. However, for the second vibration mode, the ‘constrained mode’ results were closer to the experimental values.

Experimental investigations have shown that the delamination alters the mode shape of the composites (Luo and Hanagud, 2000). Also, nonlinearities in the mode shapes have been observed. These nonlinearities can be caused by either or both of the following factors:

- opening and closing of the delamination (‘delamination breathing’); and
- contact/impact between the delaminated layers during vibration.

In the experiments by Shen and Grady (1992), the ‘delamination breathing’ was shown to cause nonlinearity in the mode shapes (Figure 2.13). However, the ‘delamination breathing’ has been shown to have little effect on the natural frequencies of low vibration modes when the delamination is small (Luo and Hanagud, 1997a, b). Further investigations by Lu *et al.* (2001) showed that for long delaminations and thin delaminated layers, the nonlinearity of the vibration mode shapes is significant. However, for short delaminations, the nonlinearity is less significant and the nonlinear mode shapes tend to linear mode shapes.

Similarly, for short delamination and lower vibration modes, the nonlinearity due to the contact/impact between the delaminated layers was shown to be less significant and becomes significant for long delamination at higher vibration modes (Luo and Hanagud, 1997a, b). The delamination opening has been shown to be affected by the boundary conditions. The more restrained the beams along its edges, the larger is the delamination opening as shown in Figure 2.15 (Shu, 1995).

For a clamped-clamped isotropic beam, the ‘constrained mode’ model shows a decreasing frequency as the delamination moves towards the midplane (Figure 2.14). This is because the bending stiffness decreases as the delamination moves towards the midplane. In contrast, the ‘free mode’ model shows an increasing frequency. Since the ‘free mode’ model is dependent on the weaker delaminated layer, as the weaker layer strengthens, the natural frequency increases (Shu, 1995). It is further observed that the ‘constrained mode’ and ‘free mode’ results are identical for midplane delamination. This is because the delaminated layers have the same natural frequency and thus vibrate in a ‘constrained’ manner.

(a)

(b)

Figure 2.13. Opening mode of the delaminated beam: (a) fundamental frequency; (b) second bending frequency (Lestari and Hanagud, 1999).

Figure 2.14. Delamination opening increases with more restrained at the beam edges (Shu, 1995).

Figure 2.15. Influence of the thickness-wise location on the fundamental frequency of a clamped-clamped beam with a central delamination (Mujumdar and Suryanarayan, 1988).

The experimental data by Shen and Grady (1992) on a composite beam with a single delamination have been widely used by many researchers to validate their models. The composite beam used in their experiment is made of T300/934 graphite/epoxy cantilever beam with a $[0^\circ/90^\circ]_{2s}$ stacking sequence. The dimensions of the 8-ply beam are $127 \times 12.7 \times 1.016 \text{ mm}^3$. The material properties for the lamina are: $E_{11} = 134 \text{ GPa}$, $E_{22} = 10.3 \text{ GPa}$, $G_{12} = 5 \text{ GPa}$, $\nu_{12} = 0.33$ and $\rho = 1.48 \times 10^3 \text{ kg/m}^3$. The delamination is at midspan, and the delamination lengths considered are 25.4 mm, 50.8 mm, 76.2 mm and 101.6 mm. Figure 2.15 shows the different locations of the delamination along the thickness-wise direction.

The comparison of the results of the analytical solutions with the experimental data is presented in Tables 2.1 – 2.4, whereas the FE results are presented in Tables 2.5 – 2.8. The results of the different models have been shown to be in good agreement with the experimental data, except at Interface 4 (Tables 2.4 and 2.8). It can be seen that the experimental results are closer to the predicted frequencies that neglect the coupling effect, which might have been an experimental error, as explained by Luo and Hanagud (2000).

Figure 2.16. Thicknesswise location of delamination in the composite beam (Shen and Grady, 1992).

Tables 2.1 – 2.8 shows no evident advantage of the shear deformation theories (FSDT and TSDT) over the CLT since the available data is from a slender beam. This indicates that the model based on the CLT provides a reasonable prediction of the natural frequency slender beams.

Table 2.1. Analytical and experimental results (Interface 1 frequencies, Hz)

Delamination length	Experimental ¹			FSDT ¹		FSDT ²		CLT ³	
	Exp 1	Exp 2	Exp 3	Cons	Free	Cons	Free	Cons	Free
Intact	79.88	79.88	79.75	82.04	82.04	81.86	81.86	81.88	81.88
25.4 mm	78.38	79.13	77.00	80.13	67.36	81.84	81.84	80.72	80.72
50.8 mm	74.38	75.00	76.75	75.29	56.48	76.81	76.81	76.36	76.36
76.2 mm	68.25	66.25	66.38	66.94	67.64	67.64	47.90	68.17	68.17
101.6 mm	57.62	57.50	57.50	57.24	40.59	56.95	56.95	58.35	58.35

¹ Shen and Grady (1992); ² Luo and Hanagud (2000); ³ Della and Shu (2005)**Table 2.2.** Analytical and experimental results (Interface 2 frequencies, Hz)

Delamination length	Experimental ¹			FSDT ¹		FSDT ²		CLT ³	
	Exp 1	Exp 2	Exp 3	Cons	Free	Cons	Free	Cons	Free
Intact	79.88	79.88	79.75	82.04	82.04	81.86	81.86	81.88	81.88
25.4 mm	78.38	78.38	76.63	81.39	68.78	80.86	80.86	81.00	81.00
50.8 mm	75.13	75.25	75.00	78.10	59.44	76.62	76.62	77.45	77.45
76.2 mm	64.00	70.00	69.88	71.16	51.18	68.80	68.80	70.43	70.42
101.6 mm	45.75	49.75	49.50	62.12	43.86	59.34	59.34	61.50	61.49

¹ Shen and Grady (1992); ² Luo and Hanagud (2000); ³ Della and Shu (2005)**Table 2.3.** Analytical and experimental results (Interface 3 frequencies, Hz)

Delamination length	Experimental ¹			FSDT ¹		FSDT ²		CLT ³	
	Exp 1	Exp 2	Exp 3	Cons	Free	Cons	Free	Cons	Free
Intact	79.88	79.88	79.75	82.04	82.04	81.86	81.86	81.88	81.88
25.4 mm	79.63	80.13	80.63	81.46	81.46	82.02	82.01	81.57	81.57
50.8 mm	79.50	81.88	77.88	79.93	79.93	80.79	80.74	80.25	80.23
76.2 mm	75.63	77.13	78.13	76.71	76.71	77.82	77.52	77.27	77.16
101.6 mm	73.38	73.63	70.38	71.66	71.66	73.15	71.73	72.66	72.20

¹ Shen and Grady (1992); ² Luo and Hanagud (2000); ³ Della and Shu (2005)**Table 2.4.** Analytical and experimental results (Interface 4 frequencies, Hz)

Delamination length	Experimental ¹			FSDT ¹		FSDT ²		CLT ³	
	Exp 1	Exp 2	Exp 3	Cons	Free	Cons	Free	Cons	Free
Intact	79.88	79.88	79.75	82.04	82.04	81.86	81.86	81.88	81.88
25.4 mm	75.38	75.25	77.25	81.60	75.83	82.04	82.03	81.72	81.72
50.8 mm	69.38	68.00	69.38	80.38	71.88	80.95	80.87	80.75	80.73
76.2 mm	65.38	59.63	-----	77.70	67.18	78.29	77.61	78.40	78.25

101.6 mm 52.75 57.88 56.25 73.15 61.70 74.05 69.44 74.59 73.81

¹ Shen and Grady (1992); ² Luo and Hanagud (2000); ³ Della and Shu (2005)

Table 2.5. FE and experimental results (Interface 1 frequencies, Hz)

Delamination length	FSDT ⁵		TSDT ⁵		TSDT ⁶	FOZZ ⁷	TOZZ ⁸
	Cons	Free	Cons	Free	Free		
Intact	82.00	82.00	81.87	81.87	82.11	82.13	81.90
25.4 mm					81.37	81.49	81.20
50.8 mm	76.64	76.64	76.52	76.52	76.73	77.19	76.60
76.2 mm					67.57	68.50	67.45
101.6 mm	77.01	77.01	56.56	56.56	56.89	56.94	57.78

⁵ Hu et al. (2002); ⁶ Thornburg and Chattopadhyay (2002); ⁷ Kim et al. (2003a);

⁸ Cho and Kim (2001)

Table 2.6. FE and experimental results (Interface 2 frequencies, Hz)

Delamination length	Cons	FSDT ⁵		TSDT ⁵		TSDT ⁶	FOZZ ⁷	TOZZ ⁸
		Free	Cons	Free	Free			
Intact		82.00	82.00	81.87	81.87	82.11	82.13	81.89
25.4 mm						81.42	81.49	81.25
50.8 mm		77.02	77.01	76.89	76.89	77.11	77.19	76.96
76.2 mm						68.42	68.50	68.30
101.6 mm		57.90	57.87	57.69	57.72	58.04	58.11	57.95

⁵ Hu et al. (2002); ⁶ Thornburg and Chattopadhyay (2002); ⁷ Kim et al. (2003a);

⁸ Cho and Kim (2001)

Table 2.7. FE and experimental results (Interface 3 frequencies, Hz)

Delamination length	Cons	FSDT ⁵		TSDT ⁵		TSDT ⁶	FOZZ ⁷
		Free	Cons	Free	Free		
Intact		82.00	82.00	81.87	81.87	82.11	82.13
25.4 mm						81.92	81.96
50.8 mm		80.56	80.61	80.50	80.45	80.65	80.74
76.2 mm						77.44	77.79
101.6 mm		72.85	71.44	72.61	71.21	71.66	73.12

⁵ Hu et al. (2002); ⁶ Thornburg and Chattopadhyay (2002); ⁷ Kim et al. (2003a);

⁸ Cho and Kim (2001)

Table 2.8. FE and experimental results (Interface 4 frequencies, Hz)

Delamination	FSDT ⁵	TSDT ⁵	TSDT ⁶	FOZZ ⁷
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length	Cons	Free	Cons	Free	Free		
Intact		82.00	82.00	81.87	81.87	82.11	82.13
25.4 mm						81.94	81.97
50.8 mm		80.75	80.74	80.64	80.62	80.83	80.88
76.2 mm						78.04	78.23
101.6 mm		73.78	72.95	73.78	72.72	73.15	73.97

⁵ Hu et al. (2002); ⁶ Thornburg and Chattopadhyay (2002); ⁷ Kim et al. (2003a);

⁸ Cho and Kim (2001)

Saravanos and Hopkins (1996) presented both analytical and experimental work on the damped free vibration of delaminated laminates and beams. An analytical solution is developed for the prediction of natural frequencies, modes and modal damping in composite beams with delaminations. Composite beam specimens with simulated delamination crack were fabricated and tested. The beams had $[0/90/45-45]_s$ and $[45/-45/90/0]_s$ laminate configurations respectively, with T300/934 graphite/epoxy plies with fiber volume ratios in the range of 0.57-0.63, and 0.127 mm nominal thickness.

Figure 2.16. The effect of delamination size on the modal damping of the $[0/90/45-45]_s$ cantilever beam. (a) Second bending mode; (b) third bending mode; (c) fourth bending mode. —, predicted; $\bigcirc$ measured. (Saravanos and Hopkins, 1996)

As shown in Figure 2.17, measured and predicted changes in modal damping values are presented. There was substantial scatter in the damping measurements, possibly because of

imperfections and friction at the clamped end. However, the results for the third and fourth mode show substantially good agreement between predictions and measurements. This may indicate that the damping change at small and medium delaminations sizes is caused by the combined reductions in the flexural rigidity. Friction effects at delamination interfaces seem to dominate the damping at high crack lengths.

2.5 Summary

The summary of the literature review is presented in this section together with the issues that will be addressed in this thesis. This review firstly presented the vibration of beams under various conditions, which are commonly seen in industrial environment, including axially loaded beams, coupled bending-torsion vibration, beams supported by elastic foundation, cracked beams, rotating beams and functionally graded beams. Secondly, the available mathematical models for the vibration of delaminated beams, as well as the influence of delaminations on natural frequencies and mode shapes have been reviewed.

Based on the review, the current literature lacks the understanding of delaminated beams when stepped axial loads and cross-section, coupled bending-torsion vibration induced by static end moments, the effect of Winkler elastic foundation, the presence of edge cracks, rotating beams, and functionally graded beams are taken into consideration. Therefore, in the present study, non-dimensional parameters are developed and analytical solutions are developed to address the following issues:

- 1) the vibration and buckling of multiple delaminated beams with step changes in axial force and cross-sections;
- 2) the coupled bending-torsion vibration of multiple delaminated beams subjected axial force and static end moments;

- 3) the vibration of multiple delaminated bimaterial beams fully or partially supported by elastic foundation;
- 4) the vibration of delaminated beams with edge cracks;
- 5) the vibration of rotating Timoshenko beams with multiple delaminations;
- 6) the vibration of exponential functionally graded beams with a single delamination.

A summary table is illustrated below to clarify the scope of the current work by addressing the major difference each study has.

Table 2.9 Summary of items studied in the present thesis.

Items Subjects	Governing Equations	Continuity Conditions	'Free' & 'Constrained' ' Mode	Generalized Solutions	Differential Stretching	Verification
Stepped Axial Load	2 nd order term on stiffening effect of tensile loading;	✓	✓	2 different dimensionless natural frequency	✓	Analytical/ FEM Experiment
Coupled Bending- Torsion	Coupled (w & θ) by 2 nd term brought by static end moment	Torsion angle;	Only 'free mode'	Solution on both deflection & torsion displacement	By decouple bending and torsion displacement	Analytical/ FEM/ Ansys/ Experiment
Beams on Elastic foundation	Foundation stiffness term	Top & Btm Layer have different material	'free' bottom layer on elastic foundation	Dimensionless frequency term include foundation stiffness	✓	Analytical FEM
Crack & Delamination	✓	Rotational spring by edge crack	✓	✓	✓	Analytical FEM Experiment
Rotating Beams	Coupled slope and deflection; Timoshenko beam model	Slope of deflection	✓	Infinite (converging) power series	Shear deformation does not change differential stretching formulation	
FGM Beams	Both axial and bending vibration	Axial force/ displacement	✓	Generalized axial and bending displacement	Neutral axis offset from mid-plane	Analytical FEM

Chapter 3

Vibration and Buckling of Multiple Delaminated Beams with Step Axial Force and Cross-Section

3.1 Introduction

Many studies have been presented for the vibration and buckling of delaminated homogeneous and composite laminated beams subjected to axial force. However, studies on the vibration of delaminated beams with step changes in axial force and cross-section have not been presented in the literature. In this chapter, new nondimensional parameters of *normalized distance of the step change in axial force*, *effective delamination length*, *normalized axial force* and *cross-section area ratio* are introduced and analytical solutions are developed to study the vibration of multiple delaminated beams with step changes in axial force and cross-section. The Euler-Bernoulli beam theory with both ‘free mode’ and ‘constrained mode’ assumptions in delamination vibration are adopted.

3.2 Formulations and Introduction of New Parameters

The analytical solutions reported by Shu (1995) on a ‘free mode’ and Shu and Fan (1996) on ‘constrained mode’ are used in the present study. Della (2005) provided detailed analytical solution to the vibration of beams subjected to axial loading, which are exerted at the ends of the delaminated beam. Figure 3.1 shows a beam with length L and Young’s modulus E . The beam has two step changes in cross section and one step change in axial force. The delamination, with length a , is located at a distance d from the center of the beam and H_4 from the top of the beam, respectively. The beam is analyzed as seven interconnected beams using the delamination and step changes as their boundaries. The continuity and boundary conditions can be satisfied between adjoining beams. The beam can then be subdivided into three span-wise regions, a delamination region and two integral regions. The delamination region is comprised of two segments (delaminated layers), beam 4 and beam 5, which are joined at their ends to the integral segments, beam 3 and beam 6. The step change of axial force occurs at the interface of beam 1 and 2. Each of the seven beams is treated as Euler-Bernoulli beam. It is assumed that the axial force is evenly distributed along the cross-section and the neutral axis remain at the same straight line within the entire span.

Figure 3.1. A delaminated beam with two step changes in cross-section and one step change in axial force.

3.2.1 Governing equations

Let $w_i(x, t)$ denote the midplane deflection of beam i , as is shown in Figure 3.1. The governing equation for the free vibration of Euler-Bernoulli beam under axial force is: ($i=1-7$)

$$EI_i \frac{\partial^4 w_i}{\partial x^4} - T_i \frac{\partial^2 w_i}{\partial x^2} + \rho A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \quad (3.1)$$

where ρ is the mass density, A_i is the cross-sectional area, I_i is the moment of area of the cross section and EI_i is the bending stiffness of beam i , respectively. T_i denotes the axial force exerted on the cross section of the beam i , positive for tensile and negative for compressive.

Assuming a harmonic motion for the free vibration of the beam then:

$$w = W_i(x) \sin(\omega t) \quad (3.2)$$

Substituting Equation (3.2) into Equation (3.1), it follows that:

$$EI_i W_i'''' - T_i W_i'' - \rho_i A_i \omega^2 W_i = 0 \quad (3.3)$$

The generalized solution for Equation (3.3) is given by

$$W_i(x) = C_i \cos\left(\alpha_i \frac{x}{L}\right) + S_i \sin\left(\alpha_i \frac{x}{L}\right) + CH_i \cosh\left(\beta_i \frac{x}{L}\right) + SH_i \sinh\left(\beta_i \frac{x}{L}\right) \quad (3.4)$$

where the 28 unknown coefficients C_i , S_i , CH_i and SH_i are determined from boundary and continuity conditions. And:

$$\alpha_i^2 = \frac{\sqrt{(T_i)^2 + 4\rho_i A_i EI_i} - T_i}{2EI_i} \quad (3.5)$$

$$\beta_i^2 = \frac{\sqrt{(T_i)^2 + 4\rho_i A_i EI_i} + T_i}{2EI_i} \quad (3.6)$$

To better understand the effect of an in-plane loading, in the present case-an axial load, on an beam (un-delaminated), one assume a simply supported boundary condition at both ends to derive the complete solution. Therefore, for simply supported boundary conditions at $x=0$ and $x=L$:

$$W(0) = W(L) = 0 \quad (3.7)$$

$$\dot{W}(0) = 0 \quad (3.8)$$

$$\dot{W}(L) = 0 \quad (3.9)$$

Substitute Equation (3.4) to Equation (3.7), (3.8) and (3.9), one can have:

$$W(x) = S \sin\left(\alpha \frac{x}{L}\right) + SH \sinh\left(\beta \frac{x}{L}\right) \quad (3.10)$$

and

$$\sinh\left(\beta \frac{x}{L}\right) \times \sin\left(\alpha \frac{x}{L}\right) = 0 \quad (3.11)$$

$\beta \frac{x}{L} > 0$ for all values of $\alpha \frac{x}{L} \neq 0$, the only roots to this equation are:

$$\alpha \frac{x}{L} = n\pi, n = 0, 1, 2, \dots \quad (3.12)$$

Therefore,

$$\omega_n = \pi^2 / l^2 \sqrt{EI / \rho A} (n^4 - n^2 P / P_{cri})^{1/2} \quad (3.13)$$

where P_{cri} is the critical buckling load.

Therefore, one can plot the normalized natural frequency of a beam when subjected to axial compression load, as the following.

Figure 3.2. The effect of axial compression load on the normalized natural frequency of a simply supported beam.

One can observe that the normalized frequency decreases with an increasing compressive load and such reduction becomes faster with a bigger loading.

3.2.2 'Free mode'

Equation (3.3) is applied to the seven interconnected sub-beams, respectively (Figure 3.1). The appropriate boundary conditions that can be applied at the supports, $x = x_1$ and $x = x_7$, are $W_i = 0$ and $W_i' = 0$ if the end of the beam is clamped; $W_i = 0$ and $W_i'' = 0$ if hinged; $W_i'' = 0$ and $W_i''' = 0$ if free, where $i=1, 7$.

The continuity conditions for deflection and slope at $x = x_4$ are:

$$W_3 = W_4 \quad (3.14)$$

$$W_3 = W_5 \quad (3.15)$$

$$W_3' = W_4' \quad (3.16)$$

$$W_3' = W_5' \quad (3.17)$$

The continuity condition for shear and bending moments at $x = x_4$ are:

$$EI_3 W_3''' = EI_4 W_4''' + EI_5 W_5''' \quad (3.18)$$

$$EI_3 W_3'' = EI_4 W_4'' + EI_5 W_5'' - P_4 \left(\frac{H_3}{2} - \frac{H_4}{2} \right) + P_5 \left(\frac{H_3}{2} - \frac{H_5}{2} \right) \quad (3.19)$$

The axial force P_i can be solved from the compatibility between the stretching/shortening of the delaminated layers and axial equilibrium (Mujumdar and Suryanarayan, 1988), as is shown in Figure 3.3, thus:

$$\frac{P_5 a}{EA_5} - \frac{P_4 a}{EA_4} = (W_3'(x_4) - W_6'(x_5)) \frac{H_3}{2} \quad (3.20)$$

$$P_3 = P_4 + P_5 = 0 \quad (3.21)$$

where EA_4 and EA_5 are the axial stiffness of beams 4 and 5, respectively. By substituting Equations (3.20) and (3.21) into Equation (3.19), the continuity condition of bending moment can be expressed as:

$$EI_3 W_3'' + \frac{(EH_4 EH_5) H_3^2}{4a(EH_4 + EH_5)} (W_3' - W_6') = EI_4 W_4'' + EI_5 W_5'' \quad (3.22)$$

$\frac{(EH_4 EH_5) H_3^2}{4a(EH_4 + EH_5)} (W_3' - W_6')$ represents the consideration of the compatibility between the stretching/shortening of the delaminated layers and axial equilibrium. And similarly, we can derive the continuity conditions at $x = x_5$.

3.2.3 ‘Constrained mode’

The ‘constrained mode’ model is simplified by the assumption that the delaminated layers are constrained to have the same transverse deformations (Mujumdar and Suryanarayan, 1988). The delaminated beam is analyzed as six beam segments I-VI.

For the ‘constrained mode’ model, the governing equations for beam 1, 2, 3, 6 and 7 is identical to Equation (3.3). For beam 2 and beam 3, i.e. section II, the governing equation is

$$(EI_4 + EI_5) W_i'''' - (T_4 + T_5) W_i'' - (\rho A_4 + \rho A_5) \omega^2 W_i = 0 \quad (3.23)$$

$$\text{where } T_4 = \frac{T_3 A_4}{A_3} \text{ and } T_5 = T_3 - T_4 \quad (3.24)$$

The boundary conditions for the ‘constrained mode’ are identical to the boundary conditions of the ‘free mode’. The continuity for deflection, slope, shear, and bending moments at $x = x_4$ are:

$$W_{III} = W_{IV} \quad (3.25)$$

$$W_{III}' = W_{IV}' \quad (3.26)$$

$$EI_3 W_{III}''' = (EI_4 + EI_5) W_{IV}''' \quad (3.27)$$

$$EI_3 W_{III}'' + \frac{(EH_4 EH_5) H_3^2}{4a(EH_4 + EH_5)} (W_{III}' - W_{VI}') = (EI_4 + EI_5) W_{IV}'' \quad (3.28)$$

And similarly, we can derive the continuity conditions at $x = x_5$.

3.2.4 Continuity conditions for step changes in axial force

For the continuity of deflection and slope and compatibility of moment and shearing force at $x = x_2$, where the step change of axial force occurs, one gets (Naguleswaran, 2005):

$$W_1 = W_2 \quad (3.29)$$

$$W_1' = W_2' \quad (3.30)$$

$$EI_1 W_1''' - \tau_1 W_1' = EI_2 W_2''' - \tau_2 W_2' \quad (3.31)$$

$$EI_1 W_1'' = EI_2 W_2'' \quad (3.32)$$

where

$$\tau_i = \frac{T_i L^2}{EI_i} \quad (3.33)$$

Figure 3.3. Deformation and stress components in the delamination region: (a) Deformation and stresses with the layers free to slide; (b) Deformation and stresses when the compatibility of axial displacement is satisfied (Mujumdar and Suryanarayan, 1988).

3.2.5 Continuity conditions for step changes in cross-section

For the continuity condition of deflection and slope and compatibility of moment and shearing force at $x = x_3$, where the step change of cross-section occurs, one gets (Naguleswaran, 2003):

$$W_1 = W_2 \quad (3.34)$$

$$W_1' = W_2' \quad (3.35)$$

$$EI_1 W_1''' = EI_2 W_2''' \quad (3.36)$$

$$EI_1 W_1'' = EI_2 W_2'' \quad (3.37)$$

3.2.6 Introduction of new nondimensional parameters

To study the effect of step change in axial force, two parameters are introduced. Firstly, nondimensional parameter $R_1 = L_1/L$ is introduced to study the effect of the location of the step change in axial force when it occurs outside the delaminated area, as is shown Figure 3.4. Secondly, when the step change in axial force occurs inside the delamination area, as is shown in Figure 3.5, nondimensional effective delamination length ED is introduced. $ED = a'/a$, where a' represents the length of the overlapping area between delaminated area and area under the effect of step change in axial force. In the following study, the effective delamination length is increased by reducing the delamination length while keeping the location of the step force still. Furthermore, axial load is normalized with respect to the critical buckling load of an undelaminated beam T_{cr} , $T_1 = T_1^0/T_{cr} = T_1^0/(4\pi^2 EI_1) L^2$.

To study the effect of step change in cross-section area, the cross-section area ratio $\Phi = A_{small} / A_{big}$, where A_{small} is the cross-section area of the relatively smaller cross-section and A_{big} is the cross-section area of the relatively big one.

Figure 3.4. Delaminated beams with one step change in axial force, the step change locates outside the delaminated area.

Figure 3.5. Delaminated beams with one step change in axial force, the step change locates inside the delaminated area.

3.3 Results and Discussions

3.3.1 Verification

To validate the present study, comparisons with published results on an Euler-Bernoulli beam with a single delamination and undelaminated beam with one step change in axial force are made. The first and second nondimensional natural frequencies of a clamped-clamped beam with a single midplane central delamination having various lengths, without step change in axial force and cross-section (by denoting $T_s=0.001$ and $\Phi = 1.0$), are compared with the analytical results of Wang *et al.* (1982) and FEM results of Lee (2000), in Table 3.1 and 3.2. Furthermore, the current work is also compared with the experimental work from

Mujumdar and Suryanarayan (1988) and the results from unconstrained-interface model from Ramtekkar (2009). Moreover, the first natural frequencies of undelaminated beams (by denoting $a/L=0.001$) with one step change in axial force is verified against the results of Naguleswaran (2005) in Table 3.4. Table 3.1-3.4 show good agreements between the previous results and present ones.

When delamination length small ($a/L < 0.2$), the change of nondimensional natural frequency is less than 0.04%. However, for practical application, when using the real natural frequency as an indicator of delamination, a 0.04% change in non-dimensional natural frequency could exhibit change in real natural frequency bigger than 4% for an aluminum beam with a span of 2m and a square cross-section of $0.5\text{m} \times 0.5\text{m}$.

Table 3.1.

Nondimensional fundamental frequency (λ^2) of a clamped-clamped homogeneous static Euler-Bernoulli beam with a midplane delamination.

Delamination length, a/L	Present Free and Constrained	Analytical Wang <i>et al.</i> (1982)	FEM Lee (2000)
0.00	22.37	22.39	22.36
0.10	22.37	22.37	22.36
0.20	22.36	22.35	22.35
0.30	22.24	22.23	22.23
0.40	21.83	21.83	21.82
0.50	20.89	20.88	20.88
0.60	19.30	19.29	19.28
0.70	17.23	17.23	17.22
0.80	15.05	15.05	15.05
0.90	13.00	13.00	12.99

Table 3.2

Nondimensional mode 2 frequency (λ^2) of a clamped-clamped homogeneous beam with a midplane delamination

Delamination length, a/L	Present Constrained and Free	Analytical Wang <i>et al.</i> (1982)	FEM Lee (2000)
0.00	61.67	61.67	61.61
0.10	60.81	60.76	60.74
0.20	56.00	55.97	55.95
0.30	49.00	49.00	48.97
0.40	43.89	43.87	43.86
0.50	41.52	41.45	41.50
0.60	41.04	40.93	41.01
0.70	40.82	40.72	40.80
0.80	39.07	39.01	39.04
0.90	35.39	35.38	35.38

Table 3.3.

First and second natural frequency ω_1, ω_2 (Hz) of cantilever beam with a single delamination, $H_2 / H_1 = 0.33$, without subjecting to axial load and static end moments ($\bar{P}_1 = 0.001$ and $\bar{M}_{z,1} = 0.001$).

Specimen span (mm)	Dimension		First mode ω_1			Second Mode ω_2			
	$L / 2$	d_1	a	Experiment (Mujumdar and Suryanarayan, 1988)	Ramtekkar (2009) Unconstrained- interface model	Present Free mode	Experiment (Mujumdar and Suryanarayan, 1988)	Ramtekkar (2009) Unconstrained- interface model	Present Free mode
240.0	153.5	96.0	31.6		31.11	32.25	172.1	156.24	159.93
250.0	104.0	148.0	31.7		31.51	31.88	190.5	197.23	198.81
175.0	133.0	88.5	56.9		56.35	57.89	339.3	244.42	248.83
200.0	122.0	106.0	46.6		46.34	47.26	291.0	258.95	260.71
155.0	122.0	61.0	69.1		68.82	69.82	363.7	249.20	281.82

Table 3.4.

Nondimensional fundamental frequency (λ^4) of an Euler-Bernoulli beam, with step change in axial force, with $R_1=0.375$

BC i/j	$\tau_1 = 10, \tau_2 = 3$		$\tau_1 = 10, \tau_2 = -3$	
	Present	Naguleswaran (2005)	Present	Naguleswaran (2005)
cl/cl	4.900	4.900	4.812	4.812
cl/sl	2.625	2.625	2.406	2.406
cl/sl	2.339	2.339	1.125	1.125
sl/cl	2.580	2.580	2.326	2.326

3.3.2 Delaminated beams with one step changes in axial force

One common scenario for beams with a step change in axial force is shown Figure 3.4. It is often seen in flagpoles. Axial forces are imposed on one end of the beam and in the middle of the beam. The location of the step change in axial force may occur outside or inside the delaminated area, which will be discussed correspondingly.

3.3.2.1 Step change in axial force occurs outside the delaminated area

As is shown in Figure 3.4, the step change in axial force occurs outside the delaminated area, with a distance L_1 from the left end of the beam. The delamination is of length a , locating in the middle of the beam. T_1 is compressive axial load. Figure 3.6 shows the normalized fundamental natural frequency λ^2/λ_d^2 versus axial compressive load, of an Euler-Bernoulli beam with one step change in axial force, considering different locations of the step change in axial force. λ_d is the normalized fundamental natural frequency of an undelaminated beam of the same configuration.

As is shown in Figure 3.6 (a), the effect of axial compressive load on reducing the natural frequencies becomes more prominent when the location of step change in axial force, R_1 , decreases from 0.3 to 0.1, i.e., moving closer to left end of the beam, which resulting in a larger portion of the beam is stiffened by the axial compressive load. The effects of

delamination length on the fundamental natural frequency and buckling load are also significantly affected by R_1 . The difference in natural frequencies between various delamination length is larger when $R_1=0.3$. The decrease of buckling load due to the increase of delamination length from $a/L = 0.1$ to $a/L = 0.2$ and $a/L = 0.2$ to $a/L = 0.3$ becomes smaller when R_1 becomes smaller. The above two observations show that the effect of delamination length on fundamental natural frequency and buckling load is more prominent with a larger R_1 .

In turn, the effects of location of the step change in axial force on fundamental natural frequency and buckling load are also influenced by the length of delamination. For delamination length $a/L = 0.1$ and $a/L = 0.2$, the difference in natural frequencies between $R_1=0.1$ and $R_1=0.3$ increases when axial compressive load becomes larger; for delamination length $a/L = 0.3$, the difference increases at first then decreases to nearly zero. The difference in buckling loads between $R_1=0.1$ and $R_1=0.3$ becomes smaller when the delamination length increases.

When one compares the results from Figure 3.6 (a) and (b), the observation of the effect of delamination's thickness-wise location with different step change location can be made. The reduction of natural frequencies, due to the increase of delamination length, decreases when the thickness-wise location of delamination moves from $H/H_1 = 0.2$ to $H/H_1 = 0.3$ and such decrease is more prominent when $R_1=0.3$.

The difference in buckling loads, due to the increase of delamination length, becomes smaller when the thickness-wise location of delamination moves from $H/H_1 = 0.2$ to $H/H_1 = 0.3$ and such decrease becomes more prominent when $R_1=0.3$. When the thickness-wise location of delamination moves from $H/H_1 = 0.2$ to $H/H_1 = 0.3$, the effect of R_1 on natural frequency and buckling load is also affected. The difference in natural frequencies and buckling loads between $R_1=0.1$ and $R_1=0.3$ is larger when $H/H_1 = 0.2$.

(a)

(b)

Figure 3.6. 'Free mode' natural frequencies λ^2/λ_d^2 for a homogeneous clamped-clamped beam with a single central delamination of various lengths when the step change of axial force locates outside the delaminated area, considering different locations of the step change. (a) $H/H_1=0.2$; (b) $H/H_1=0.3$.

3.3.2.2 Step change in axial force occurs inside the delaminated area

As is shown in Figure 3.5, the step change of axial force occurs inside the delaminated area. ED represents the effective delamination length. Delamination locates at the center of the beam, the thickness of the top layer $H/H_1 = 0.2$. Figure 3.7 shows the normalized natural frequencies λ^2/λ_d^2 versus axial compressive load of an Euler-Bernoulli beam with one step change in axial force, considering different delamination lengths and different values of ED .

As is shown in Figure 3.7, the fundamental natural frequency decreases when ED decreases. The difference in natural frequencies between various delamination lengths decreases when ED increases. Such decrease is larger when ED increases from 0.25 to 0.5 than from 0.5 to 0.75. It indicates that the effect of delamination length on fundamental natural frequency becomes larger when the length of the overlapping area ED becomes shorter. The effect of ED on fundamental natural frequency and buckling load is in turn influenced by the length of delamination. The difference in natural frequencies between various values of ED becomes larger with an increasing compressive load. However, such difference becomes smaller with a longer delamination length.

Figure 3.8 shows the buckling load versus ED of an Euler-Bernoulli beam with one step change in axial force, considering different delamination lengths. The increase in buckling load, due to the increase of ED , is slower with a longer delamination. Also such increase is relatively gentle $0.4 < ED < 0.6$, becomes abrupt when $ED > 0.6$. The difference in buckling loads between various delamination lengths increases when the value of ED is larger. Such effect is larger as ED increases. It indicates that the effect of delamination length on buckling load is larger when ED increases.

Figure 3.7. 'Free mode' natural frequencies λ^2/λ_d^2 for a homogeneous clamped-clamped beam with a single central midplane delamination of various lengths when the step locates inside the delaminated area.

Figure 3.8. Free mode buckling load P_{cr} for a homogeneous clamped-clamped beam with a single central delamination of various lengths, the step change locates inside the delaminated area.

3.3.2.3 'Free mode' vs. 'Constrained mode'

Figure 3.9 shows the comparison between 'free mode' and 'constrained mode'. The delaminated beam is subjected to the same condition as is shown in Figure 3.5. The difference in natural frequencies between 'free mode' and 'constrained mode' is larger with a smaller value of ED . When comparing Figure 3.9 (a) and (b), one can see that for both $ED = 0.25$ and 0.75 , the difference between 'free mode' and 'constrained mode' increases with a longer delamination length. The increase of such difference is larger when $ED = 0.25$. Furthermore, the difference in buckling loads between 'free mode' and 'constrained mode' is larger with a smaller value of ED . For both $ED = 0.25$ and 0.75 , the difference in buckling load between 'free mode' and 'constrained mode' increases with a longer delamination length. The increase of such difference is larger when $ED = 0.25$.

3.3.3 Delaminated beams with two step changes in axial force

3.3.3.1 Two step changes in axial force located symmetrically outside the delaminated area

Figure 3.10 shows a beam with two central overlapping delaminations with symmetrically located step changes in axial force, which occurs outside the delaminated area. Correspondingly, Figure 3.11 shows the fundamental natural frequency versus step change axial force T_s of beams with two step changes in axial force locating symmetrically outside the delamination area.

Figure 3.9. 'Free mode' and 'constrained mode' natural frequencies λ^2/λ_d^2 for a homogeneous clamped-clamped beam with sing central delamination. (a) $a/L = 0.2$; (b) $a/L = 0.3$.

Figure 3.10. Beam with two enveloped central delaminations and two step changes in axial force locate symmetrically outside the delaminated area.

When comparing Figure 3.11 (a) and (b), it can be observed that the difference in natural frequencies between $R_1=0.1$ and 0.3 becomes larger when T_1 increases from 0.25 to 0.5 for each length of delamination; such increase becomes larger when the delamination length becomes longer. It can also be observed that the difference in buckling loads between $R_1=0.1$ and 0.3 becomes smaller when T_1 increases from 0.25 to 0.5 for each delamination length; such decrease becomes smaller when the delamination length becomes longer.

3.3.3.2 Step changes of axial force occur in the middle of the two symmetrically separated delaminated area

Figure 3.12 shows beams with two step changes in axial force locating in the middle of two symmetrically separated delaminated areas. Correspondingly, Figure 3.13 shows the fundamental natural frequency versus step change axial force T_s , of beams with two step changes in axial force locating at the center of two symmetrically located delaminations.

(a)

(b)

Figure 3.11. 'Free mode' natural frequencies λ^2/λ_d^2 for a homogeneous clamped-clamped beam with two enveloped delamination of evenly distributed thickness-wise location, with various delamination lengths, the step locates symmetrically outside the delaminated area, considering different locations of the step change. (a) $T_1=0.25$; (b) $T_1=0.5$.

Figure 3.12. Beams with two symmetrically located midplane delamination, the two step change in axial force located at the center of the two delaminations.

As is shown in Figure 3.13, the difference in natural frequencies between $d=0.2$ and 0.3 becomes increases as T_s increases. The buckling load decreases as d increase from 0.2 to 0.3 . When T_1 increases from 0.25 to 0.5 , the difference in natural frequencies between $d = 0.2$ and 0.3 becomes larger; such increase becomes larger when T_s increases. It can also be observed that when T_1 increases from 0.25 to 0.5 , the difference in buckling loads between $d=0.2$ and 0.3 decreases.

Figure 3.13. Influence of step axial force on 'free mode' natural frequencies for beams with two symmetrically located delaminations.

3.3.4 Mode Shape of Delaminated Beams with Step Change in Axial Force

3.3.4.1 'Free mode' first mode shape

We can see from Figure 3.14 (a) that the difference in amplitude between beam 3 and beam 4 as a result of delamination becomes larger when axial compressive load increases. When comparing Figure 3.14 (a) and (b), it can be observed that the difference in amplitude between beam 3 and 4 as a result of delamination becomes larger when R_1 increases from 0.1 to 0.3; such increase is larger with a larger axial compressive load.

3.3.4.2 'Constrained mode' first mode shape

As is shown in Figure 3.15, a higher axial compressive load leads to larger amplitude of the first mode shape. When comparing Figure 3.15 (a) and (b), it can be observed that the increase of amplitude due to the increase of axial load becomes larger with a larger axial load when $R_1=0.1$, smaller when $R_1=0.3$.

Figure 3.15. Influence of step change axial force on the effect of delamination on the 'free mode' first mode shape, of beam with single central delamination of length $a/L = 0.2$, considering different step change locations. (a) $R_1 = 0.1$; (b) $R_1 = 0.3$

Figure 3.15. Influence of step change axial force on the ‘constrained mode’ first mode shape of beam with a single central delamination considering different step change locations. (a) $R_1 = 0.1$; (b) $R_1 = 0.3$

3.3.5 Single delamination with one step change in cross-section

Figure 3.16 shows a single delaminated beam with one step change in cross-section. The beam is analyzed as five sub-beams with the step change of cross-section occurring at the conjunction of beam 1 and beam 2. For the explanation purposes, Beams 2, 3, 4 and 5 that were separated by the delamination as shown in Figure 3.16 will be collectively known as the Secondary Beam 2, with Beam 1 being the Secondary Beam 1. In this study, the delamination is always at the central location of the Secondary Beam 2 in the horizontal x -direction, therefore the distance between the central point of delamination and central point of the entire composite beam, d will increase as the length of the Secondary Beam 1, L_1 increases.

Figure 3.16. A model of a beam with one delamination and one step change

Figure 3.17 shows the ‘free mode’ normalized fundamental frequency against the delamination length a/L , considering different cross-section area ratio Φ and length of secondary beam 1. As is shown in Figure 3.17, the frequency decreases as delamination length increases. The drop of frequency when delamination length a/L increases from 0.01 to 0.5 is 9.52% when $\Phi=0.5$, and 8.65% when $\Phi=1.0$, where $\Phi = A_{\text{small}} / A_{\text{big}}$. It indicates that the effect of an increasing delamination length on reducing the fundamental natural frequency becomes less prominent when secondary beam 1 has a smaller cross-section area. It can also be observed from Figure 3.17 that the fundamental natural frequency becomes smaller when the cross-section area of secondary beam 1 decreases. The difference in natural frequencies between $\Phi = 0.5$ and 0.6, 0.6 and 0.7, 0.7 and 0.8, 0.8 and 0.9, 0.9 and 1.0 becomes larger as the delamination length increases.

When comparing the results between Figure 3.17 (a), (b) and (c), it can be observed that, when $\Phi=0.5$, the fundamental natural frequency is smaller when the secondary beam 1 takes a larger portion of the entire beam. The drop of fundamental natural frequency, due to the increase of delamination length, is larger when secondary beam 1 is longer (i.e. delaminated secondary beam 2 takes a smaller portion of the entire beam).

(a)

(b)

Figure 3.17. Normalized frequencies versus the delamination length for a clamped-free beam with one delamination and one step change considering different cross –section area ratio. (a) $L_1=0.1$, (b) $L_1=0.2$ and (c) $L_1=0.3$

3.3.6 Multiple delaminated beams with two step changes in cross-section

For the analysis of multiple delaminated beams with step changes, the cross-sectional area ratio at the step changes and delamination lengths were varied. In addition, different lengths of step-changed beams (L_1) are computed to observe how the fundamental natural frequency response will be affected. Figure 3.18 shows a multiple delaminated beam with two symmetrically located step changes of cross-section. The beam is subdivided into three secondary beams, 1, 2, and 3. The three delaminations occur at the center (in the x -direction) of each secondary beam, respectively. Figure 3.18 shows the ‘free mode’ normalized frequency of a clamped-clamped beam against delamination length.

Figure 3.18. Multiple delaminated beam with two symmetrical step changes in cross-section.

It can be observed from Figure 3.19 that when the beam suffers two step changes in cross-section, the effect of step change in cross-section on natural frequencies is more prominent when compared to the results from Figure 3.17. When the delamination length a / L changes from 0.01 to 0.1, the beam suffers larger drop of fundamental natural frequency, which is partially due to the presence of three delaminations instead of one, and partially due to the two step changes in cross-section.

Figure 3.19. 'Free mode' normalized fundamental frequency versus delamination length of a multiple delaminated beam with two symmetrical step changes in cross-section, considering different cross-section area ratio.

3.4 Concluding Remarks

New nondimensional parameters are introduced and analytical solutions are developed on the vibration and stability of multiply delaminated beams subjected to step axial force and cross-section has been developed. The authors studied how the step changes in axial force and cross-section influence the effect of delamination on natural frequencies and buckling loads.

Based on the results, the following conclusions are made:

1. When the step change of axial force occurs outside the delaminated area, the effect of delamination length and thickness-wise location on fundamental natural frequency as well as buckling load is more prominent when a smaller portion of the beam is under the effect of axial compressive loading.
2. When the step change of axial force occurs inside the delaminated area, the effect of delamination length on fundamental natural frequency becomes larger when the length of effective delamination ED becomes shorter.
3. When the step change of axial force occurs inside the delaminated area, the effect of effective delamination length on the nature frequency is more prominent when the axial compressive load increases.
4. When the step change of axial force occurs inside the delaminated area, the effect of delamination length on buckling load is more significant with a longer effective delamination length ED .
5. When the step change of axial force occurs inside the delaminated area, the difference in natural frequencies and buckling loads between ‘free mode’ and ‘constrained mode’ is larger with a longer effective delamination length ED .
6. The opening of the first mode shape, under ‘free mode’ assumption, is larger when a smaller portion of the beam is under the effect of axial compressive load.

7. For single delaminated beams with one step change in cross-section area, the effect of delamination length on fundamental natural frequency become less prominent when the undelaminated segment have a smaller cross-section area compared to the delaminated segment; or as the undelaminated segment becomes longer.

Chapter 4

Coupled Bending-Torsion Vibration of Delaminated Beams

Subjected to Axial Loads and Static End Moments

4.1 Introduction

Many studies have been presented for the coupled vibration of beams subjected to axial force and static end moments. However, studies on the coupled bending-torsion vibration of delaminated beams subjected to axial force and static end moments have not been presented in the literature. In this chapter, analytical solutions are developed to study the coupled bending-torsion vibration of multiple delaminated beams subjected to axial force and static end moments. The Euler-Bernoulli beam theory with ‘free mode’ assumption in delamination vibration are adopted. The beam is analyzed as four interconnected beams using the delamination as their boundaries. The continuity and boundary conditions are satisfied between adjoining beams.

4.2 Formulations

The analytical solution to the governing equations for the natural frequency of coupled bending-torsion vibration of beams subjected to axial load and static end moments reported by Joshi and Suryanarayan (1989) is adopted here. The analytical solution reported by Della (2005) on ‘free mode’ assumption is used in the present study. Figure 4.1 (a) shows a homogeneous beam with length L and thickness H_1 . The Young’s modulus of the beam is E . The beam is separated along the interface by a delamination with length a and located at a distance d_1 from the center of the beam, the thicknesses of the top and bottom layers are H_2 and H_3 . The beam can then be subdivided into three span-wise regions, a delamination region and two integral regions. The delamination region is comprised of two segments (delaminated layers), beam 2 and beam 3, which are joined at their ends to the integral segments, beam 1 and beam 4. Each of the four beams is treated as an Euler-Bernoulli beam.

4.2.1 Governing equations

The governing equations (Joshi and Suryanarayan, 1989) for the free coupled bending-torsion vibration of beam i subjected to axial loads and static end moments are ($i=1-4$):

$$-EI_i \frac{\partial^4 w_i}{\partial x^4} + P_i \frac{\partial^2 w_i}{\partial x^2} + M_{zz,i} \frac{\partial^2 \theta_i}{\partial x^2} - \rho A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \quad (4.1)$$

$$GJ_i \frac{\partial^2 \theta_i}{\partial x^2} + \left(\frac{P_i I_{P,i}}{A_i} \right) \frac{\partial^2 \theta_i}{\partial x^2} + M_{zz,i} \frac{\partial^2 w_i}{\partial x^2} - \rho I_{p,i} \frac{\partial^2 \theta_i}{\partial t^2} = 0 \quad (4.2)$$

where $w_i(x, t)$ and $\theta_i(x, t)$ denote the midplane deflection and torsion displacement of beam i . ρ is the mass density, G is shear modulus and A_i is the cross-sectional area, J_i is the St Venant torsion constant, $I_{p,i}$ is the polar moment of inertia, EI_i is the bending stiffness of beam i ($i=1-4$).

P_1 denotes the static axial force exerted on beam 1. P_2 and P_3 can be solved from the axial force equilibrium and the axial displacement equilibrium, as are stated in Equations (4.3) and (4.4):

$$P_1 = P_2 + P_3 \quad (4.3)$$

$$\frac{P_2 a}{EA_2} = \frac{P_3 a}{EA_3} \quad (4.4)$$

hence,

$$P_2 = P_1 \frac{EA_2}{EA_2 + EA_3}, P_3 = P_1 \frac{EA_3}{EA_2 + EA_3}. \quad (4.5)$$

$M_{zz,i}$ denotes static end moments exerted on beam i , where $M_{zz,2}$ and $M_{zz,3}$ can be obtained by solving Equations (4.6) and (4.7),

$$M_{zz,1} = M_{zz,2} + M_{zz,3} \quad (4.6)$$

$$\frac{M_{zz,2}}{E \frac{b^3 H_2}{12}} = \frac{M_{zz,3}}{E \frac{b^3 H_3}{12}} = \kappa \quad (4.7)$$

where b is the width of the beam. As is shown in Figure 4.1 (b), Equation (4.6) represents the moment equilibrium and Equation (4.7) indicates that beam 2 and beam 3 have the same curvature, $\kappa = \frac{1}{\rho} = \frac{d^2 v}{dx^2}$, where v is deflection in y -direction, as a result of the static end moments. Hence,

$$M_{zz,2} = M_{zz,1} \frac{H_2}{H_2 + H_3}, M_{zz,3} = M_{zz,1} \frac{H_3}{H_2 + H_3} \quad (4.8)$$

For free harmonic vibration, the deformation mode shape can be assumed as:

$$w_i(x, t) = W_i(x) \sin \omega t \text{ and } \theta_i(x, t) = \Phi_i(x) \sin \omega t \quad (4.9)$$

Substituting Equation (4.9) in Equations (4.1) and (4.2), it follows that:

$$-EI_i \frac{\partial^4 W_i}{\partial x^4} + P_i \frac{\partial^2 W_i}{\partial x^2} + M_{zz,i} \frac{\partial^2 \Phi_i}{\partial x^2} + \rho A_i \omega^2 W_i = 0 \quad (4.10)$$

$$GJ_i \frac{\partial^2 \Phi_i}{\partial x^2} + \left(\frac{P_i l_{p,i}}{A_i} \right) \frac{\partial^2 \Phi_i}{\partial x^2} + M_{zz,i} \frac{\partial^2 W_i}{\partial x^2} + \rho I_{p,i} \omega^2 \Phi_i = 0 \quad (4.11)$$

Based on the work of Joshi and Suryanarayan (1989), a closed form solution is adopted to solve analytically for natural frequencies and corresponding mode shapes. Normalizing the above two equations, one obtains:

$$\begin{bmatrix} -D^4 + \bar{P}_i D^2 + \lambda_i^4 & \bar{M}_{z,i} \sqrt{S_i} D^2 \\ \bar{M}_{z,i} \sqrt{S_i} D^2 & (S_i + \bar{P}_i) D^2 + \lambda_i^4 \end{bmatrix} \begin{Bmatrix} W_i \\ \Phi_i \end{Bmatrix} = 0 \quad (4.12)$$

where D is the dimensional differentiation operator, $\lambda_i^4 = \frac{\rho A_i L^4}{EI_i} \omega^2$ is the dimensionless frequency parameter, $S_i = GJ_i L^2 / EI_i (I_{p,i} / A_i)$ is the uncoupled torsion value of λ_i , $\bar{P}_i = P_i L^2 / EI_i$ is the axial load parameter, $\bar{M}_{z,i} = M_{zz,i} L^2 / EI_i \sqrt{I_{p,i} / A_i}$ is the moment parameter and W_i and Φ_i are dimensionless displacements.

The characteristic root equation can be obtained from Equation (4.12) as

$$a_1 D^6 + a_2 D^4 + a_3 D^2 + a_4 = 0, \quad (4.13)$$

where

$$\begin{aligned} a_1 &= -(S_i + \bar{P}_i), \quad a_2 = \bar{P}_i (S_i + \bar{P}_i) - \lambda_i^4 - \bar{M}_{z,i} S_i \\ a_3 &= (S_i + 2\bar{P}_i) \lambda_i^4, \quad a_4 = \lambda_i^8 \end{aligned} \quad (4.14)$$

Equation (4.13) is cubic in terms of D^2 and yields three characteristic roots: $-k_1^2, k_2^2, -k_3^2$.

Then, the general solution for the displacements can be expressed as

$$\begin{aligned} W_i(\bar{x}) &= f_{1,i} \sin k_{1,i} \bar{x} + f_{2,i} \cos k_{1,i} \bar{x} + f_{3,i} \sinh k_{2,i} \bar{x} + f_{4,i} \cosh k_{2,i} \bar{x} \\ &\quad + f_{5,i} \sin k_{3,i} \bar{x} + f_{6,i} \cos k_{3,i} \bar{x} \end{aligned} \quad (4.15)$$

$$\begin{aligned} \Phi_i(\bar{x}) &= g_{1,i} (f_{1,i} \sin k_{1,i} \bar{x} + f_{2,i} \cos k_{1,i} \bar{x}) + g_{2,i} (f_{3,i} \sinh k_{2,i} \bar{x} + f_{4,i} \cosh k_{2,i} \bar{x}) \\ &\quad + g_{3,i} (f_{5,i} \sin k_{3,i} \bar{x} + f_{6,i} \cos k_{3,i} \bar{x}) \end{aligned} \quad (4.16)$$

where

$$\bar{x} = x / L, \text{ and}$$

$$g_{j,i} = -\bar{M}_{z,i} \sqrt{S_i} \left[\left(S_i + \bar{P}_i \right) + \lambda_i^4 / (-k_{j,i}^2) \right], \quad i=1-4, j=1, 2, 3. \quad (4.17)$$

Figure 4.1. Beam with a single delamination subjected to axial load and end moments. (a) Lateral view; (b) Perspective view.

4.2.2 'Free mode'

Equations (4.10) and (4.11) are applied to the four interconnected sub-beams, respectively (Figure 4.1 (a)). The appropriate boundary conditions that can be applied at the supports, $x = x_1$ and $x = x_4$, are $W_i = 0, W_i' = 0$ and $\Phi_i = 0$, if the end of the beam is clamped, $W_i = 0, W_i'' = 0$

and $\Phi_i = 0$ if the end of the beam is simply supported, where $i = 1, 4$, $W_i'' = 0$, $W_i''' = 0$ and $\Phi_i' = 0$ if the end of the beam is free.

The continuity for bending displacement, slope and torsion displacement at $x = x_2$ are:

$$W_1 = W_2 \quad (4.18)$$

$$W_1 = W_3 \quad (4.19)$$

$$W_1' = W_2' \quad (4.20)$$

$$W_1' = W_3' \quad (4.21)$$

$$\Phi_1 = \Phi_2 \quad (4.22)$$

$$\Phi_1 = \Phi_3 \quad (4.23)$$

The continuity for shear and bending moments at $x = x_2$ are:

$$EI_1 W_1''' = EI_2 W_2''' + EI_3 W_3''' \quad (4.24)$$

$$EI_1 W_1'' + \frac{(EH_2 EH_3) H_1^2}{4a(EH_2 + EH_3)} (W_1' - W_4') = EI_2 W_2'' + EI_3 W_3'' \quad (4.25)$$

$\frac{(EH_2 EH_3) H_1^2}{4a(EH_2 + EH_3)} (W_1' - W_4')$ represents the consideration of the compatibility between the stretching/shortening of the delaminated layers and axial equilibrium (see section 3.2.2). And similarly, one can derive the continuity conditions at $x = x_3$. The total number of boundary and continuity conditions is 24, which is equal to the total number of unknown coefficients. A non-trivial solution exists only when the determinant of the coefficient matrix vanishes. The natural frequencies can be obtained as eigenvalues.

4.3 Results and Discussions

4.3.1 Verification

In this section, several comparisons are conducted to validate the current study. The first and second nondimensional natural frequencies λ^2 of a clamped-clamped beam with a midplane central delamination having various lengths, without subjecting to axial load and static end moments (by denoting $\bar{P}_1 = 0.001$ and $\bar{M}_{z,1} = 0.001$), are compared with the analytical results of Wang *et al.* (1982) and FEM results of Lee (2000), as is shown in Table 4.1.

Table 4.2 shows the comparison of the current results against the experimental results of Mujumdar and Suryanarayan (1988) and the mixed-FEM results of Ramtekkar (2009). The first and second natural frequency ω_1 (Hz) of a homogeneous cantilever beam with a single delamination ($H_2 / H_1 = 0.33$.) is tabulated here.

The first natural frequencies of the coupled vibration of simply supported undelaminated beams (by denoting $a / L = 0.001$) subjected to axial loads and static end moments are compared with the analytical results of Joshi and Suryanarayan (1984), as is shown in Table 4.3. Here, the axial load is normalized with respect to the critical buckling load of an undelaminated beam P_{cr} , so that $\bar{P}_1 = P_1 / P_{cr} = P_1 L^2 / 4\pi^2 EI_1$. And the end moment is normalized with respect to the critical moment for lateral instability of an undelaminated beam M_{cr} , so that $\bar{M} = M / M_{cr} = M_1 / (\pi / L \sqrt{EI_1 GJ_1})$. Tables 4.3 shows good agreements between the present results and the results of previous work.

Table 4.4 shows the comparison between the analytical results and the FEM simulation results by conducting a pre-stressed modal analysis (using Element type of Solid 185) in ANSYS 12.1. The Young's modulus $E=193$ GPa and density $\rho=7.92 \times 10^3$ kg/m³, with $L=10$ m, $H_1=0.5$ m and $b=1$ m, where b is the width of the beam. The delamination locations at the central of the beam, with $H_2 / H_1 = 0.2$, of various delamination length, $a / L = 0.2, 0.3$ and 0.4 . The end moment is modeled as a linearly varying load across the end face of the beam, as is shown in Figure 4.2. The boundary condition studied here is clamped-clamped. One can observe that the analytical results agree well with FEM results.

(a)

(b)

Figure 4.2. The modeling of end moment as a linearly varying load across the end face of the beam.

Table 4.1.

Nondimensional fundamental frequencies (λ^2) of a clamped-clamped homogeneous beam with a central midplane delamination, without subjecting to axial load and static end moments ($\bar{P}_1 = 0.001$ and $\bar{M}_{z,1} = 0.001$).

Delamination length a / L	First Mode	Analytical Wang <i>et al.</i> (1982)	FEM Lee (2000)	Second Mode	Analytical Wang <i>et al.</i> (1982)	FEM Lee (2000)
	Present Free Mode			Present Free Mode		
0.00	22.37	22.39	22.36	61.67	61.67	61.61
0.10	22.37	22.37	22.36	60.81	60.76	60.74
0.20	22.36	22.35	22.35	56.00	55.97	55.95
0.30	22.24	22.23	22.23	49.00	49.00	48.97
0.40	21.83	21.83	21.82	43.89	43.87	43.86

Table 4.2.

First and second natural frequency ω_1, ω_2 (Hz) of cantilever beam with a single delamination, $H_2 / H_1 = 0.33$, without subjecting to axial load and static end moments ($P_1 = 0.001$ and $M_{z,1} = 0.001$).

Specimen span (mm)	Dimension		First mode ω_1			Second Mode ω_2			
	$L / 2$	d_1	a	Experiment (Mujumdar and Suryanarayan, 1988)	Ramtekkar (2009) Unconstrained-interface model	Present Free mode	Experiment (Mujumdar and Suryanarayan, 1988)	Ramtekkar (2009) Unconstrained-interface model	Present Free mode
240.0	153.5	96.0	31.6		31.11	32.25	172.1	156.24	159.93
250.0	104.0	148.0	31.7		31.51	31.88	190.5	197.23	198.81
175.0	133.0	88.5	56.9		56.35	57.89	339.3	244.42	248.83
200.0	122.0	106.0	46.6		46.34	47.26	291.0	258.95	260.71
155.0	122.0	61.0	69.1		68.82	69.82	363.7	249.20	281.82

Table 4.3.

Nondimensional mode 1 frequencies (λ^4/λ_d^4) of a simply supported homogeneous beam without delamination, slenderness ratio $b/L \cong 0$, where b is the width of the beam.

Axial force P/P_{cr}	$M/M_{cr} = 0.75$		$M/M_{cr} = 1.0$	
	Present	Joshi and Suryanarayan (1984)	Present	Joshi and Suryanarayan (1984)
0.00	0.478	0.481	0.000	0.000
0.10	0.575	0.578	0.095	0.100
0.20	0.673	0.675	0.195	0.200
0.30	0.771	0.773	0.296	0.300
0.40	0.869	0.870	0.397	0.400
0.50	0.976	0.976	0.499	0.500

Table 4.4.

First natural frequency ω_1 (Hz) of clamped-clamped beam with a single delamination, $H_2 / H_1 = 0.2$, $d_1=0$, subjected to static end moments only.

M / M_{cr}	$a / L = 0.2$		$a / L = 0.3$		$a / L = 0.4$	
	Analytical	FEM	Analytical	FEM	Analytical	FEM
0.0	159.22	158.47	157.91	157.23	152.31	151.71
0.1	158.77	157.69	157.27	156.28	151.05	150.17
0.2	157.38	155.04	155.30	153.08	146.60	144.59
0.3	155.03	151.89	151.71	148.51	134.97	132.00
0.4	151.63	146.99	145.62	141.12	88.56	85.80
0.5	147.04	139.36	131.55	124.50		
0.6	140.93	131.30				

4.3.2 Coupled vibration of beams with single delamination

4.3.2.1 Delaminated beams subjected to axial force only

Figure 4.3 shows normalized ‘free mode’ nondimensional frequency λ^2/λ_d^2 of beams subjected to axial force only, with a single central delamination of various lengths. Since there is no static end moment exerted on the beam, coupled bending-torsion vibration will not be induced in this case. The nondimensional frequency of the delaminated beam λ^2 is normalized with respect to that of undelaminated beam λ_d^2 . The axial load is normalized with respect to the critical buckling load of an undelaminated beam P_{cr} . P_1^0/P_{cr} is positive for tensile load, negative for compressive load.

$$\bar{P}_1^0 = \frac{P_1^0}{P_{cr}} = \frac{P_1^0}{4\pi^2 EI_1} L^2 \quad (4.26)$$

As is shown in Figure 4.3, the natural frequency increases as the tensile load increases or the compressive load decreases, corresponding with the results of Della and Shu (2009). The differences of natural frequencies between delamination lengths $a/L=0.2$ and 0.5 become larger when the compressive load increases, or tensile load decreases. It indicates that the effect of delamination length on natural frequency becomes less significant when compressive load decreases or tensile load increases.

Also, when the thickness-wise location of delamination H_2/H_1 decreases from 0.3 to 0.2 , the natural frequencies show a more precipitous decline, due to the increase of compressive load or decrease of tensile load, corresponding with the work of Della and Shu (2009) (Figure 2). It indicates that the farther the delamination is away from the center, the larger influence axial load has on natural frequencies. When the load $\bar{P}_1^0$ approaches the buckling load the natural frequency vanishes, indicating that the buckling load decreases when delamination length increases or when the thickness-wise location moves away from the center, which agrees well with the results of Huang and Kardomateas (1998) (Table 2).

Figure 4.3. Mode 1 ‘free mode’ frequency λ^2/λ_d^2 versus the axial load P_1^0/P_{cr} for a homogenous clamped-clamped beam with a single delamination, $d/L = 0.0$. When $\lambda = 0$, P_1^0/P_{cr} is the buckling load. (a) $H_2/H_1=0.2$; (b) $H_2/H_1=0.3$.

4.3.2.2 Delaminated beams subjected to static end moments only

Figure 4.4 shows normalized ‘free mode’ nondimensional normalized frequency λ^2/λ_d^2 of clamped-clamped beams subjected to static end moments only and no axial force, with a single central delamination of various lengths. The end moment is normalized with respect to the critical moment for lateral instability of an undelaminated beam M_{cr} .

$$\bar{M} = M / M_{cr} = \frac{M_1}{(\pi/L)\sqrt{EI_1GJ_1}} \quad (4.27)$$

The reduction of natural frequencies from delamination lengths $a / L = 0.2$ to 0.3 , 0.3 to 0.4 , 0.4 to 0.5 become larger when the static end moments increases. This may be caused by the fact that the presence of static end moments contributes to the reduction of equivalent bending stiffness (Joshi and Suryanarayan, 1984; Joshi and Suryanarayan, 1989), which enhances effect of delamination on reducing natural frequencies (Della and Shu, 2007). It indicates that the effect of delamination on natural frequencies becomes more significant when the beam is subjected to a larger static end moments.

As the end moments increase, the natural frequency declines gently at first, then it shows an abrupt drop when the end moments approach the critical load for lateral instability, where $\lambda = 0$. Identifying the location of the approximate starting point of such precipitous decline is of great practical significance, since a slight change of end moments may cause a significant change of natural frequency when the static end moments are close to the starting point of the abrupt decline.

It is also shown in Figure 4.4 that the critical load for lateral instability drops when the delamination becomes longer. It can be explained that an increasing delamination length will decrease the critical moment for lateral instability $M_{cr} = \pi / L\sqrt{EIGJ}$ (for beam 2) and since in ‘free mode’ model, the frequency is dependent upon the thinner one of the delaminated layers (Della and Shu, 2007a), the critical moment for the entire beam also decreases. However, the decrease of critical load for lateral instability from $a / L = 0.4$ to $a / L = 0.5$ is

smaller than that of $a/L = 0.2$ to $a/L = 0.3$. This result indicates that as the delamination length increases, its effect on decreasing the critical load for lateral instability becomes less prominent.

Figure 4.4. Mode 1 'free mode' frequency λ^2/λ_d^2 versus the end moment M/M_{cr} for a homogenous clamped-clamped beam with a single delamination, $d_1/L=0.0$. When $\lambda = 0$, M/M_{cr} is the static end moment corresponding to lateral instability, $H_2/H_1=0.2$.

4.3.2.3 Mode 1 and 2 'free mode' frequencies of beams with single delamination

Figure 4.5 shows the mode 1 and 2 normalized 'free mode' nondimensional natural frequencies of clamped-clamped beams with a single central delamination under the effect of axial loads and static end moments. For both mode 1 and 2, the natural frequencies decrease when static end moments increase or when the axial load increases. The differences of natural frequencies between $\bar{P}_1 = -0.5$ and 1.0 increase when the static end moments become larger. For both mode 1 and 2, the critical load for lateral instability decreases when the axial load decreases. It indicates that by increasing tensile load or decreasing compressive load one can

effectively enlarge the range of end moments beams can resist before reaching the critical point of lateral instability; yet such effect becomes less prominent with a longer delamination, when comparing the two delamination lengths $a/L = 0.1$ and $a/L = 0.4$, between (a) and (b).

Figure 4.5. Mode 1 and mode 2 ‘free mode’ frequencies λ^2/λ_d^2 versus the end moment M/M_{cr} for a homogeneous clamped-clamped beam with single delamination considering different axial load P/P_{cr} . (a) $a/L=0.1$ (b) $a/L=0.4$.

4.3.2.4 Effect of span-wise location of delamination for beams with single delamination

Figure 4.6 shows the effect of span-wise location on the natural frequencies of clamped-clamped beams with a single midplane delamination under the effect of axial load and static end moments. Critical load for lateral instability becomes smaller when the delamination span-wise location d_1 / L moves from 0.1 to 0.2. It can be explained by referring to the work of Mujumdar and Suryanarayan (1988) (Figure. 9) that natural frequency decreases as the delamination moves towards the boundary, so that less static end moments are required to bring the beam to the critical point of lateral instability, where $\lambda=0$. When $d_1 / L = 0.1$, the increase of critical load for lateral instability is 0.188 (0.442-0.254) when axial load $\bar{P}_1$ increase from 0.0 to 0.5, which is smaller compared to the case where $d_1 / L = 0.2$. Similar results can be observed for the increases of critical load for instability from $\bar{P}_1=0.5$ to 1.0. This indicates that as the span-wise location of delamination d_1 / L moves from 0.1 to 0.2, the effect of axial load on expanding the range of end moments that beams can resist before reaching lateral instability becomes larger.

(a)

(b)

Figure 4.6. Influence of span-wise location of delamination on coupled vibration mode 1 'free mode' frequencies λ^2/λ_d^2 for a homogenous clamped-clamped beam with a single delamination, $a/L=0.2$, $H_2/H_1=0.2$. (a) $d_1/L=0.1$ (b) $d_1/L=0.2$.

4.3.2.5 Effects of static end moments on the buckling load of beams with single delamination

When beams are subjected to static end moments, the natural frequencies are affected, so is the buckling load. Figure 4.7 shows the effect of static end moments on buckling load of

delaminated beam, with a clamped-clamped boundary condition. One can see from (a) and (b) that end moments will decrease the buckling load. The drops of buckling load when $\bar{M}$ increases from 0.2 to 0.4 $((0.983-0.918)/0.983=0.049)$ and 0.4 to 0.6 $((0.917-0.79)/0.917=0.055)$ are smaller when the delamination length $a/L = 0.2$, compared with the case when $a/L = 0.3$. The buckling load decreases as the delamination becomes longer and such decrease is more prominent as the static end moments increase. It can be explained that as the delamination length increases, the stiffness of the beam decreases, causing the axial compressive load necessary for buckling decreases. At the same time, as the static end moment increases, the natural frequency decreases, therefore a relatively small axial compressive force would cause the buckling.

(a)

Figure 4.7. Influence of end moment on buckling loads of a clamped-clamped beam with a single midplane delamination. (a) $a/L=0.2$ (b) $a/L=0.3$.

4.3.3 Coupled Vibration of Beams with Double Delaminations

4.3.3.1 Two equal-length delaminations

Figure 4.8 shows the mode 1 and mode 2 ‘free mode’ natural frequencies of delaminated clamped-clamped beam under the effect of axial force and static end moments. The configuration of the beam with two equal length delaminations is shown in Figure 4.9. The thickness-wise locations of the two delaminations are equally distributed. When comparing the two cases of different delamination lengths $a/L = 0.1$ and $a/L = 0.4$, it can be concluded that, for mode 1, the differences of critical load for lateral instability between $\bar{P}_1 = 0.5$ and 1.0 increase from $(1.338-0.534)/1.338 = 0.601$ when $a/L = 0.1$, to $(1.086-0.165)/1.086 = 0.848$ when $a/L = 0.4$. When compared with the results of beams with a single delamination, the critical load for lateral instability is smaller for beams with double delaminations.

Figure 4.8. Mode 1 and mode 2 'free mode' frequencies versus the end moment M/M_{cr} for homogeneous clamped-clamped beam with two enveloped delaminations considering different axial load P_1/P_{cr} . (a) $a_1/L = 0.1, a_2/L = 0.2$ (b) $a_1/L = 0.2, a_2/L = 0.4$.

Figure 4.9. Beams with two central delaminations of equal length.

4.3.3.2 Two enveloped delaminations

Figure 4.10 shows the mode 1 and mode 2 ‘free mode’ natural frequencies of beam with two enveloped central delaminations (Figure 4.11) under the effect of axial load and end moments. The thickness-wise locations of the two delaminations are equally distributed. Regarding the effect of end moments and axial load on mode 1 and 2 natural frequencies, as well as the effect of delamination on critical load for lateral instability, similar results to Figure 4.7 can be observed.

Figure 4.10. Mode 1 and mode 2 ‘free mode’ frequencies versus the end moment M/M_{cr} for homogeneous clamped-clamped beam with two enveloped delaminations considering different axial load P_1/P_{cr} . (a) $a_1/L = 0.1, a_2/L = 0.2$ (b) $a_1/L = 0.2, a_2/L = 0.4$.

Figure 4.11. Beams with two central enveloped delaminations.

4.3.4 Mode shapes

Figure 4.12 shows the influence of only static end moments $\bar{M}$ on the ‘free mode’ first mode shape of clamped-clamped beams with a single central midplane delamination. As the static end moments increase, the amplitude of first mode shape and the opening in the mode

shape increases. It can be observed that static end moments can enhance the effect of delamination on mode shapes.

Figure 4.12. First vibration bending mode shapes of beams with a single central midplane delamination considering different static end moments M/M_{cr} . (a) $M/M_{cr} = 0.1$; (b) $M/M_{cr} = 0.2$.

The 1st torsion vibration mode shape of a Cantilever beam with a single delamination under the effect of static end moments are illustrated as below. The delamination is located at the center of the beam, with a length $a/L=0.2$ and thickness-wise location $H_2/H_1 = 0.4$. One

can observe that the thinner sub-beam (beam 2), as a result of the delamination shows bigger amplitude under torsion vibration compared to the thicker sub-beam (beam 3). And such difference becomes more prominent with a bigger static end moment exerted at the ends of the delaminated beam.

(a)

(b)

Figure 4.13. First torsion vibration mode shapes of beams with a single central delamination considering different static end moments M/M_{cr} . (a) $M/M_{cr} = 0.1$; (b) $M/M_{cr} = 0.2$.

4.4 Concluding Remarks

In the present study, the analytical solution to the coupled bending-torsion vibration of delaminated beams subjected to axial load and end moments was developed. We study how axial loads and static end moments influence the effect of delamination on natural frequencies, buckling load and critical moment for lateral instability. Based on the theoretical investigations the following conclusions may be drawn.

1. The effect of delamination reduces both mode 1 and 2 natural frequencies and this effect becomes more prominent when the static end moments increase.

2. Increase in axial tensile load or decrease in axial compressive load can effectively weaken the effect of end moment on aggravating the influence of delamination on reducing natural frequency and critical moment for lateral instability.
3. The effect of delamination length and locations on buckling load and critical moments for lateral instability is aggravated by the presence of static end moments.

Chapter 5

Vibration of Delaminated Bimaterial Beams Fully or Partially Supported by Elastic Foundation

5.1 Introduction

Many studies have been presented for the vibration of beams resting on elastic foundation. However, studies on the vibration of delaminated bimaterial beam supported by elastic foundation have not been presented in the literature. In this chapter, new nondimensional parameters of *normalized axial stiffness ratio* and *normalized bending stiffness ratio* are introduced, and analytical solutions are developed to investigate the vibration of bimaterial delaminated beams supported by elastic foundation. The Winkler model, Euler-Bernoulli beam theory, and both the ‘free mode’ and ‘constrained mode’ assumptions in delamination vibration are adopted.

5.2 Formulations

Figure 5.1 (a) shows a beam with length L and thickness H_1 . The beam is made of two distinct layers, with Young's modulus E_2 and E_3 , and thickness H_2 and H_3 . The beam is separated along the interface by a delamination with length a and located at a distance d from the center of the beam. The beam can then be subdivided into three span-wise regions, a delamination region and two integral regions, as is shown in Figure 5.1 (b). The delamination region is comprised of two segments (delaminated layers), beam 2 and beam 3, which are joined at their ends to the integral segments, beam 1 and beam 4. The beam is partially supported by elastic foundation with a stiffness of k . The elastic foundation studied in this chapter provides double-sided reaction. The elastic foundation starts from the left boundary of the beam and ends at $x = x_4$. Each of the five beams are treated as Euler-Bernoulli beam. The beam is then analyzed as five interconnected beams using delamination and end of foundation as their boundaries. The continuity and boundary conditions are satisfied between adjoining beams.

5.2.1 Governing equations

The present study is carried out within the framework of small deformation theory. Let $w_i(x, t)$ denote the midplane deflection of beam i . The governing equation for the free vibration of Euler-Bernoulli beam supported by elastic foundation is: ($i=1-5$)

$$EI_i \frac{\partial^4 w_i}{\partial x^4} + k_i w_i + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \quad (5.1)$$

where ρ_i , A_i and EI_i are the mass density, the cross-sectional area and the bending stiffness of beam i . k_i is the corresponding foundation stiffness of beam i . The bending stiffness per unit width of beam 2 and beam 3 are $EI_2 = E_2 H_2^3 / 12$ and $EI_3 = E_3 H_3^3 / 12$ for rectangular cross-

sections, respectively. The bending stiffness of bimaterial beam 1, 4 and 5 is determined by $EI_1 = EI_2 + EI_3 + (E_2H_2H_3^2 + E_3H_3H_2^2) / 4$. The ρA for beam 2 and beam 3 are ρ_2H_2 and ρ_3H_3 , and $\rho_2H_2 + \rho_3H_3$ for beam 1, 4 and 5.

For free harmonic vibration, the deformation mode shape can be assumed as

$$w_i = W_i(x) \sin(\omega t) \quad (5.2)$$

Substituting Equation (5.2) in Equation (5.1), it follows that:

$$EI_i \frac{\partial^4 W_i}{\partial x^4} + k_i W_i - \rho_i A_i \omega^2 W_i = 0 \quad (5.3)$$

The generalized solution for Equation (5.3) is given by

$$W_i(x) = C_i \cos\left(\beta_i \frac{x}{L}\right) + S_i \sin\left(\beta_i \frac{x}{L}\right) + CH_i \cosh\left(\beta_i \frac{x}{L}\right) + SH_i \sinh\left(\beta_i \frac{x}{L}\right) \quad (5.4)$$

when $\lambda_i^4 - \bar{k}_i > 0$

$$W_i(x) = C_i \sin \frac{\beta_i x}{\sqrt{2} L} \sinh \frac{\beta_i x}{\sqrt{2} L} + D_i \sin \frac{\beta_i x}{\sqrt{2} L} \cosh \frac{\beta_i x}{\sqrt{2} L} \\ + CH_i \cos \frac{\beta_i x}{\sqrt{2} L} \sinh \frac{\beta_i x}{\sqrt{2} L} + SH_i \cos \frac{\beta_i x}{\sqrt{2} L} \cosh \frac{\beta_i x}{\sqrt{2} L} \quad (5.5)$$

when $\lambda_i^4 - \bar{k}_i < 0$,

$$\text{in both Equations (5.4) and (5.5), } \lambda_i^4 = \frac{\rho_i A_i \omega^2}{EI_i} L^4, \bar{k}_i = k_i L^4 / EI_i, \beta_i = \sqrt[4]{|\lambda_i^4 - \bar{k}_i|}. \quad (5.6)$$

5.2.2 'Free mode'

Equation (5.3) is applied to the five interconnected sub-beams, respectively (Figure 5.1 (b)). Under 'free mode' assumption, beam 2 and 3 vibrate freely and independent of each other, therefore, $k_2 = 0$, $k_1 = k_3 = k_4 = k$ and $k_5 = 0$. The appropriate boundary conditions that can be applied at the supports, $x = x_1$ and $x = x_5$, are $W_i = 0$, $W_i' = 0$, if the ends of the beam are clamped; $W_i = 0$ and $W_i'' = 0$ if hinged, $W_i'' = 0$ and $W_i''' = 0$ if free, where $i=1, 5$.

The continuity for deflection and slope at $x = x_2$

$$W_1 = W_2 \quad (5.7)$$

$$W_1 = W_3 \quad (5.8)$$

$$W_1' = W_2' \quad (5.9)$$

$$W_1' = W_3' \quad (5.10)$$

The continuity for shear and bending moments at $x = x_2$

$$EI_1 W_1''' = EI_2 W_2''' + EI_3 W_3''' \quad (5.11)$$

$$EI_1 W_1'' + \frac{(EH_2 EH_3) H_1^2}{4a(EH_2 + EH_3)} (W_1' - W_4') = EI_2 W_2'' + EI_3 W_3'' \quad (5.12)$$

$\frac{(EH_2 EH_3) H_1^2}{4a(EH_2 + EH_3)} (W_1' - W_4')$ represents the consideration of the compatibility between the stretching/shortening of the delaminated layers and axial equilibrium (see section 3.2.2). And similarly, one can derive the continuity conditions at $x = x_3$.

5.2.3 'Constrained mode'

The 'constrained mode' model is simplified by the assumption that the delaminated layers are constrained to have the same transverse deformations (Mujumdar and Suryanarayan, 1988). The governing equations for beam 1 and 4 are identical to Equation. (5.3). To take into account the contact pressure, the governing equations for beam 2 and 3 are:

$$EI_2 \frac{\partial^4 w_2}{\partial x^4} + k_2 w_2 + \rho_2 A_2 \frac{\partial^2 w_2}{\partial t^2} = 0 \quad (5.13)$$

$$EI_3 \frac{\partial^4 w_3}{\partial x^4} + k_3 w_3 + \rho_3 A_3 \frac{\partial^2 w_3}{\partial t^2} = 0 \quad (5.14)$$

where p is the normal contact pressure distribution between the two segments.

Since beam 2 and 3 are constrained together, both subjected to the effect of elastic foundation, therefore:

$$k_1 = k_2 = k_3 = k_4 = k \text{ and } k_5 = 0. \quad (5.15)$$

The beam can be analysed as three segments I-III. Since in ‘constrained mode’, beam 2 and 3 vibrates together, $w_2 = w_3 = w_{II}$, therefore, the governing equation for beam 2 and 3, i.e. section II, becomes:

$$(EI_2 + EI_3)W_{II}'''' + kW_{II}'' - (\rho A_2 + \rho A_3)\omega^2 W_{II} = 0 \quad (5.16)$$

The boundary conditions for the ‘constrained mode’ are identical to the boundary conditions of the ‘free mode’. The continuity for deflection, slope, shear force and bending moment at $x = x_2$:

$$W_I = W_{II} \quad (5.17)$$

$$W_I' = W_{II}' \quad (5.18)$$

$$EI_1 W_I''' = (EI_2 + EI_3)W_{II}''' \quad (5.19)$$

$$EI_1 W_I'' + \frac{(EH_2 EH_3)H_1^2}{4a(EH_2 + EH_3)}(W_I' - W_{III}') = (EI_2 + EI_3)W_{II}'' \quad (5.20)$$

And similarly, one can derive the continuity conditions at $x = x_3$.

5.2.4 Continuity conditions for partial elastic foundation

The continuity for deflection and slope at $x = x_4$ are:

$$W_4 = W_5 \quad (5.21)$$

$$W_4' = W_5' \quad (5.22)$$

The continuity for bending moments and shear force at $x = x_4$ are:

$$EI_4 W_4'' = EI_5 W_5'' \quad (5.23)$$

$$EI_4 W_4''' = EI_5 W_5''' \quad (5.24)$$

The total number of boundary and continuity conditions is 20 (for free mode) or 16 (for constrained mode), which is equal to the total number of unknown coefficients. A non-trivial

solution exists only when the determinant of the coefficient matrix vanishes. The natural frequencies can be obtained as eigenvalues and the mode shapes, as eigenvectors.

Figure 5.1. (a) Beam with a single delamination partially supported by elastic foundation. (b) The beam is analyzed as five interconnected beams.

5.3 Results and Discussions

5.3.1 Verification

Table 5.1 shows the first and second nondimensional natural frequency λ^2 of an isotropic homogeneous beam without the support of elastic foundation (by denoting $E_2 / E_3 = 1.0$ and $\bar{k} =$

0.001), suffering a single central midplane delamination of various lengths, compared with the analytical results of Wang *et al.* (1982) and FEM results of Lee (2000). As shown in Table 5.1, the results of the current study agree well with previous published results.

Table 5.2 shows the natural frequency of an undelaminated stepped beam (by denoting $a / L=0.001$) fully supported by elastic foundation, compared with the results of Wang (1991). The stepped beam is solved by imposing continuity condition of displacement, slope, shear and bending moment at the conjunction of the step. It shows that the current study agree well with previous published results.

Table 5.1.

First and second nondimensional frequencies (λ^2) of a clamped-clamped isotropic homogeneous beam with a central midplane delamination.

Delamination Length a / L	First Mode			Second Mode		
	Present Free & Constrained	Analytical Wang <i>et al.</i> (1982)	FEM Lee (2000)	Present Free & Constrained	Analytical Wang <i>et al.</i> (1982)	FEM Lee (2000)
0.00	22.37	22.39	22.36	61.67	61.67	61.61
0.10	22.37	22.37	22.36	60.81	60.76	60.74
0.20	22.36	22.35	22.35	56.00	55.97	55.95
0.30	22.24	22.23	22.23	49.00	49.00	48.97
0.40	21.83	21.83	21.82	43.89	43.87	43.86
0.50	20.89	20.88	20.88	41.52	41.45	41.50
0.60	19.30	19.29	19.28	41.04	40.93	41.01
0.70	17.23	17.23	17.22	40.82	40.72	40.80
0.80	15.05	15.05	15.05	39.07	39.01	39.04
0.90	13.00	13.00	12.99	35.39	35.38	35.38

Table 5.2.

Natural frequencies of simply supported stepped beam on elastic foundation, where μ is ratio between the length of the thinner part of the beam over the length of the whole beam (Wang, 1991).

μ	$k / EI_1=1/200$		$k / EI_1=1/100$	
	Present f_1 , (Hz)	Wang (1991) f_1 , (Hz)	Present f_1 , (Hz)	Wang (1991) f_1 , (Hz)
0.0	42.651	42.612	42.852	42.852
0.1	42.384	42.384	42.584	42.584
0.2	40.806	40.806	41.020	41.021
0.3	37.915	37.916	38.158	38.158
0.4	34.778	34.779	35.063	35.063
0.5	32.217	32.217	32.548	32.548
0.6	30.455	30.455	30.829	30.829
0.7	29.417	29.417	29.826	29.826
0.8	28.926	28.926	29.357	29.358
0.9	28.769	28.770	29.211	29.211
1.0	28.750	28.751	29.194	29.194

5.3.2 Homogeneous beams fully or partially supported by elastic foundation

5.3.2.1 Delaminated beams on full elastic foundation

Figure 5.2 (a) shows the nondimensional ‘free mode’ natural frequency λ^2 / λ_d^2 , of a clamped-clamped beam with single delamination fully supported by elastic foundation, versus the nondimensional foundation stiffness $\bar{k} = kL^4 / EI_1$, when $H_2 / H_1=0.5$ and $d / L=0.0$. As is shown in Figure 5.2 (a), the decrease of natural frequency due to the increase of the delamination length from $a / L=0.3$ to 0.4, 0.4 to 0.5 and 0.5 to 0.6 becomes smaller then larger as the foundation stiffness increases. It indicates the effect of delamination length on reducing natural frequencies is aggravated by an increasing foundation stiffness. The increase of natural frequency, due to the increase of foundation stiffness, becomes larger when the delamination length is longer.

Figure 5.2 (b) shows the nondimensional ‘free mode’ natural frequency λ^2 / λ_d^2 versus $\bar{k}$, when $H_2 / H_1=0.2$ and $d=0.0$. When comparing the results between Figure 5.2 (a) and (b), the decrease of natural frequency when H_2 / H_1 decreases from 0.5 to 0.2 becomes larger when the foundation stiffness increases. Such increase becomes more prominent when the delamination length is larger. The increase of natural frequency, due to the increase of foundation stiffness, becomes smaller when $H_2 / H_1=0.2$.

Figure 5.2 (c) shows the nondimensional ‘free mode’ natural frequency λ^2 / λ_d^2 versus $\bar{k}$, when $H_2 / H_1=0.5$ and $d / L=0.2$. When the length-wise location $d / L=0.2$, the increase of natural frequency due to the increase of foundation stiffness becomes larger with a longer delamination length. When one compares the results between Figure 5.2 (a) and (c), the decrease of natural frequency when d increases from 0 to 0.2 becomes less prominent when the foundation stiffness increases. Such decrease becomes larger when the delamination becomes longer.

All curves in Figure 5.2 display a linear dependency of nondimensional natural frequency λ^2 upon foundation stiffness $\bar{k}$, which can be generally described as $\lambda^2 = s\bar{k} + c$. The values of s and c depend on the length of the delamination, as well as the thickness-wise location and transverse location of the delamination, as are tabulated in Table 5.3. It is hoped that these empirical data will assist engineers and material scientists to assess the effects of foundation and delamination quickly.

Figure 5.2. Influence of the stiffness of elastic foundation on the effect of delamination on natural frequency considering different lengths, thickness-wise and length-wise locations. (a) $H_2 = 0.5$, $d = 0.0$; (b) $H_2 = 0.2$, $d = 0.0$; (c) $H_2 = 0.5$, $d = 0.2$.

Table 5.3

The values of s and c , which are the two coefficients for the empirical estimation of the natural frequency's linear dependency on the foundation stiffness, for five different delamination length $a/L = 0.0, 0.3, 0.4, 0.5$ and 0.6 , considering different thickness-wise locations and transverse locations.

a/L	$H_2/H_1 = 0.5$ and $d = 0$		$H_2/H_1 = 0.2$ and $d = 0$		$H_2/H_1 = 0.5$ and $d = 0.2$	
	s	c	s	c	s	c
0.0	0.21	22.37	0.021	22.37	0.021	22.37
0.3	0.21	22.24	0.020	22.14	0.023	20.09
0.4	0.21	21.83	0.014	20.89	0.025	18.25
0.5	0.21	20.89	0.003	16.60	0.026	16.56
0.6	0.19	19.30	0.001	12.01	0.026	15.18

5.3.2.2 'Free mode' vs 'Constrained mode'

Figure 5.3 shows the comparison between the nondimensional 'free mode' and 'constrained mode' natural frequencies, of a clamped-clamped beam with a single delamination fully supported by elastic foundation, versus $\bar{k}$, considering different thickness-wise locations of the delamination, $H_2 = 0.5$ and $H_2 = 0.2$. As is shown in Figure 5.3 (a), the 'constrained mode' natural frequency decreases as the delamination length increases. Such decrease becomes smaller when foundation stiffness increases and more prominent when the delamination length increases. The difference in natural frequencies between 'free mode' and 'constrained mode' becomes larger with a longer delamination and bigger foundation stiffness. When compared with the results from Figure 5.3 (b), such difference is larger when $H_2 = 0.2$. Therefore, it can be concluded that the presence of elastic foundation enhances the difference in natural frequencies between 'free mode' and 'constrained mode'.

Figure 5.3. Comparison between ‘free mode’ and ‘constrained mode’ natural frequencies of clamped-clamped beams with a single delamination fully supported by elastic foundation. (a) $H_2 = 0.5$; (b) $H_2 = 0.2$.

Figure 5.4 shows the three different assumptions in delamination vibration, free, partially constrained and fully constrained. The formulations for the continuity conditions of beams with double delaminations are similar to the case of beams with a single delamination, as is stated in section 2. It can be also referred to the work of Della and Shu (2005) for details.

Figure 5.5 shows the comparison between the nondimensional ‘free mode’, ‘partially constrained mode’ and ‘fully constrained’ natural frequencies λ^2 , of a clamped-clamped beam with two equal-length delaminations fully supported by elastic foundation, versus $\bar{k}$, of a clamped-clamped beam. The thickness-wise location of the two delaminations is evenly distributed, with $d = 0$. As is shown in Figure 5.5, the difference in natural frequencies between ‘free mode’, ‘partially constrained mode’ and ‘fully constrained mode’ becomes larger when the delamination length increases or the foundation stiffness increases. The difference between ‘free

mode' and 'partially constrained mode' is smaller than the difference between 'partially constrained mode' and 'fully constrained mode', when $a/L = 0.4$ and 0.5 , larger when $a/L = 0.6$.

Figure 5.4. Clamped-clamped Beams with two central delaminations of equal length, with evenly distributed thickness-wise location. (a) 'Free mode'; (b) 'Partially Constrained'; (c) 'Fully Constrained'.

Figure 5.5. Comparison between the natural frequencies of ‘free mode’, ‘partially constrained’ and ‘fully constrained’, considering different delamination lengths.

5.3.2.3 Beams with single delamination on partial elastic foundation

Figure 5.6 shows a clamped-clamped beam with a single central midplane delamination partially supported by elastic foundation, which locates at the middle of the beam. Correspondingly, Figure 5.7 shows the nondimensional ‘free mode’ natural frequency λ^2/λ_d^2 versus $\bar{k}$, considering different lengths of the foundation and different delamination lengths. As is shown in Figure 5.7, the increase of natural frequency, due to the increase of foundation stiffness, is smaller when the length of foundation, FL , decreases from 0.9 to 0.4. When the beam is partially supported by foundation, such increase becomes larger with a longer delamination. The difference in natural frequencies between full foundation and partial foundation becomes larger when the foundation stiffness increases. Such difference is smaller with a longer delamination.

Figure 5.6. Beam with a single delamination partially supported elastic foundation.

Figure 5.7. Influence of the stiffness and length of the elastic foundation on the effect of delamination on natural frequencies.

5.3.2.4 Beams with two delaminations on partial elastic foundation

Figure 5.8 shows a clamped-clamped beam with two midplane delaminations symmetrically located in the length-wise direction. The two ends of the partial elastic foundation locate at the center of the two delaminated area respectively. Correspondingly, Figure 5.9 shows the

nondimensional ‘free mode’ natural frequency λ^2/λ_d^2 versus $\bar{k}$, considering different length of foundation and delamination length. As is shown in Figure 5.9, the increase of natural frequency, due to the increase of foundation stiffness, becomes larger with longer delamination. The difference in natural frequencies between foundation length $FL = 0.5$ and 0.6 becomes smaller when the foundation stiffness increases, becomes larger when the length of delamination increases.

Figure 5.8. Beam partially supported by elastic foundation, with two inline delaminations locating at the edges of the area supported by elastic foundation.

Figure 5.9. Influence of the stiffness and length of the elastic foundation on the effect of delamination on natural frequencies.

Figure 5.10 shows a clamped-clamped beam with two midplane delaminations symmetrically located in the length-wise direction, outside the section supported by elastic foundation. Correspondingly, Figure 5.11 shows the nondimensional ‘free mode’ natural frequency λ^2/λ_d^2 versus $\bar{k}$, considering different length of foundation and delamination length. The difference in natural frequencies between foundation length $FL = 0.4$ and 0.5 becomes larger when the foundation stiffness increases. Also, when $FL = 0.5$ and $a/L = 0.3$ a more prominent effect of elastic foundation stiffness on natural frequencies can be observed, compared to $FL = 0.5$ and $a/L = 0.3$.

Figure 5.10. Beam partially supported by elastic foundation, with two inline delaminations symmetrically locating outside the area supported by elastic foundation.

Figure 5.11. Influence of the stiffness and length of the elastic foundation on the effect of delamination on natural frequencies.

5.3.2.5 Mode shapes

Figure 5.12 and Figure 5.13 show the first ‘free mode’ mode shapes of clamped-clamped beam with a single delamination fully supported by elastic foundation when foundation stiffness is 150 and 75 respectively, with $H_2 = 0.5$ and $d = 0$. As are shown in Figure 5.12 and Figure 5.13, the amplitude becomes smaller when the foundation stiffness increases from 75 to 150. The difference in amplitudes between Beam 2 and Beam 3 (opening) becomes larger when the delamination becomes longer. Such difference becomes smaller when the foundation stiffness decreases. Therefore, it can be concluded that the opening in the mode shapes increases when the foundation stiffness increases.

(a)

(b)

(c)

Figure 5.12. First mode shapes of clamped-clamped beams with single central midplane delaminations fully supported by elastic foundation, with $\bar{k} = 150$. (a) $a/L = 0.2$; (b) $a/L = 0.3$; (c) $a/L = 0.4$.

(a)

(b)

Figure 5.13. First mode shapes of clamped-clamped beams with single central midplane delaminations fully supported by elastic foundation, with $\bar{k} = 75$. (a) $a/L = 0.2$; (b) $a/L = 0.3$; (c) $a/L = 0.4$.

5.3.3 Bimaterial Beams Fully or Partially Supported by Elastic Foundation

5.3.3.1 Delaminated bimaterial beams on full elastic foundation

The natural frequency of the bimaterial beam will differ based on the beam's mechanical properties. To study the effects of the bending and axial stiffness on the natural frequency of a delaminated bimaterial beam, two parameters were presented (Della and Shu, 2007c). These two parameters are as follows:

- i) Axial Stiffness Ratio (EA_2/EA_3)
- ii) Bending Stiffness Ratio (EI_2/EI_3)

To simplify the present analysis, the densities of the delaminated layers are assumed to be identical.

Figure 5.14 shows the variation of the fundamental normalized nondimensional frequency λ^2 / λ_d^2 with EA_2 / EA_3 of a clamped-clamped beam fully supported by elastic foundation, with

a central ($d / L=0.5$) delamination ($EI_2 / EI_3=0.5$). The frequency changes in two phrases. In phase I, the frequency increases rapidly when the foundation stiffness $k=0$ and 30; but decreases when $k=90$ and 120. In phase II, the frequency stabilizes by decreasing slowly as EA_2 / EA_3 increases. When compare the results between $a / L = 0.4$ and $a / L = 0.5$, it can be observed that the frequencies in phase I and II exhibits a higher rate of change when the delamination length is longer.

Furthermore, the difference in natural frequencies between $k = 0, 30, 60, 90$ and 120 are significantly larger when the axial stiffness ratio is small and much smaller as the axial stiffness ratio increases. It can also be observed that such difference is larger when the beam suffers a longer delamination.

Figure 5.14. The fundamental normalized frequency of a clamped-clamped bimaterial beam varies with normalized axial stiffness ratio EA_2 / EA_3 .

Figure 5.15 shows the variation of the fundamental normalized nondimensional frequency λ^2 / λ_d^2 with EI_2 / EI_3 of a clamped-clamped beam fully supported by elastic foundation, with a central ($d / L=0.5$) delamination ($EA_2 / EA_3=0.5$). The frequency changes in two phrases. The frequency increases rapidly in phase I and stabilizes in phase II. The increase of frequency in phase I has a higher rate of change when the foundation stiffness is smaller. In phase II, the frequencies stabilizes by increasing slowly when foundation stiffness is relatively bigger ($\bar{k}=60, 90$ and 120 when $a / L=0.4$; $\bar{k}=90$ and 120 when $a / L=0.5$) while decreasing when foundation stiffness is relatively smaller.

It can also be observed from Figure 5.15 that the difference in natural frequencies between $\bar{k}=0, 30, 60, 90$ and 120 are significantly larger when the bending stiffness ratio is larger. Such difference decreases as the bending stiffness ratio approaches towards the end of phase I, then increase as the bending stiffness ratio continues increasing in phase II.

(a)

(b)

Figure 5.15. The fundamental normalized frequency of a clamped-clamped bimaterial beam varies with normalized bending stiffness ratio EI_2 / EI_3 . (a) $a / L = 0.4$; (b) $a / L = 0.5$.

5.3.3.2 Delaminated bimaterial beams on partial elastic foundation

Figure 5.16 shows the variation of natural frequencies against the axial stiffness ratio. The beam configuration is the same as is shown in Figure. 5.6, with $FL=0.6$ and $EI_2 / EI_3=0.5$. Similar behavior of natural frequency can be observed as to the one observed from Figure 5.14. However, when the beam is partially supported, the natural frequency decreases with a higher rate and converges faster when partially supported.

(a)

(b)

Figure 5.16. Normalised frequency of a partially supported clamped-clamped beam against axial stiffness ratio with $FL=0.6$, $EI_2/EI_3 = 0.5$. (a) $a/L=0.4$; (b) $a/L=0.5$.

The variation of natural frequencies against bending stiffness ratio is shown in Figure 5.17 with the same beam configuration shown in Figure 5.6, with $FL=0.6$ and $EA_2 / EA_3=0.5$. The natural frequencies changes in two distinct phases, similar to that observed from Figure 5.15. In phase II, the natural frequencies of different foundation stiffness grows apart as the bending stiffness ratio increases. However, when the beam is partially supported, such behavior is less prominent when compared to the results observed from Figure 5.15.

Figure 5.17. Normalised frequency of a partially supported clamped-clamped bimaterial beam against bending stiffness ratio with $FL=0.6$, $EA_2 / EA_3 = 0.5$. (a) $a / L=0.3$; (b) $a / L=0.4$.

The variation of natural frequencies against bending stiffness ratio is shown in Figure 5.18 with the same beam configuration shown in Figure. 5.8, with $FL=0.6$ and $EA_2 / EA_3=0.5$. Under such configuration, the natural frequency decreases in phase II as bending stiffness ratio increases. Furthermore, the natural frequency converges faster as the bending stiffness ratio increase in phase II when compared to that observed from Figure 5.15.

(a)

(b)

Figure 5.18. Normalised frequency of a partially supported clamped-clamped beam against bending stiffness with $FL=0.6$, $EA_2 / EA_3 = 0.5$. (a) $a / L=0.3$; (b) $a / L=0.4$; (c) Close-up view.

5.4 Concluding Remarks

In the present study, analytical solutions are developed to study the free vibration of bimaterial delaminated beams fully or partially supported by elastic foundation. Euler-Bernoulli beam theory, the ‘free mode’ and ‘constrained mode’ in delamination vibration are adopted. This is the first study on the vibration of delaminated bimaterial beam considering the effect of elastic foundation. Based on the theoretical investigations the following conclusions may be drawn.

1. The effect of delamination’s length and thickness-wise location on natural frequency is aggravated while the effect of length-wise location is mitigated by an increasing foundation stiffness, when the beam is fully supported by elastic foundation.
2. The effect of elastic foundation’s stiffness on natural frequency is larger when delamination length grows longer, moves away from the midplane.
3. The difference in natural frequencies between ‘free mode’ and ‘constrained mode’ becomes more prominent with a longer delamination or a larger foundation stiffness.

4. The decrease of natural frequency due to the reduction of the foundation length is more prominent when the beam is supported by a stiffer foundation or suffers a longer delamination.
5. The opening of the first mode shape is larger when the elastic foundation's stiffness increases.
6. The variation of normalized frequency against axial stiffness ratio changes in two phases. Different foundation stiffness significantly influences the behavior of natural frequency in phase I while has much less influence in phase II.
7. The effect of foundation stiffness on natural frequency is more prominent with a smaller axial stiffness ratio.
8. The variation of normalized frequency against bending stiffness ratio changes in two phases. The frequency increases rapidly in phase I and stabilizes in phase II. The increase of frequency in phase I has a higher rate of change when the foundation stiffness is smaller.
9. The effect of foundation stiffness on natural frequency is more prominent with a larger bending stiffness ratio.

Chapter 6

Vibration of Delaminated Beams with Edge Cracks

6.1 Introduction

Many studies have been presented for the vibration of beam with delamination or cracks, both of which are common failures. However, studies on the vibration of beams suffering both crack and delamination at the same time have not been presented in the literature. In this chapter, new nondimensional parameters of *normalized crack location* D and *normalized crack depth* B are introduced, and analytical solutions are developed to study the free vibration of delaminated beams with edge cracks. The Euler-Bernoulli beam theory with both ‘free mode’ and ‘constrained mode’ assumptions in delamination vibration are adopted.

6.2 Formulations and Introduction of New Parameters

Figure 6.1 shows a beam with length L and thickness H_1 . The Young’s modulus of the beam is E . The beam is separated along the interface by a delamination with length a and located at a distance l from the center of the beam, the thickness of the top and bottom layer are H_3 and H_4 . The beam can then be subdivided into three span-wise regions, a delamination region and two integral regions. The delamination region is comprised of two segments

(delaminated layers), beam 3 and beam 4, which are joined at their ends to the integral segments, beam 2 and beam 5. The surface crack is of depth b , locating with a distance d from the left end of the beam. Each of the five beams are treated as Euler-Bernoulli beam. The beam is then analyzed as five interconnected beams using delamination and crack as their boundaries. The continuity and boundary conditions are satisfied between adjoining beams.

Figure 6.1. Beam with a single delamination and an edge crack.

Figure 6.2. Rotational spring model.

6.2.1 Governing equations and analytical solutions

The present study is carried out within the framework of small deformation theory. Let $w_i(x, t)$ denote the midplane deflection of beam i , as is shown in Figure 6.1. The governing equation for the free vibration of Euler-Bernoulli beam is: ($i=1 - 5$)

$$EI_i \frac{\partial^4 w_i}{\partial x^4} + \rho A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \quad (6.1)$$

where ρ is the mass density, A_i and EI_i are the cross-sectional area and bending stiffness of beam i , respectively.

Assuming a harmonic motion for the free vibration of the beam then:

$$w_i = W_i(x) \sin(\omega t) \quad (6.2)$$

Substituting Equation (6.2) into Equation (6.1), it follows that:

$$EI_i W_i'''' - \omega^2 \rho A_i W_i = 0 \quad (6.3)$$

The generalized solution for Equation (6.3) is given by

$$W_i(x) = C_i \cos\left(\lambda_i \frac{x}{L}\right) + S_i \sin\left(\lambda_i \frac{x}{L}\right) + CH_i \cosh\left(\lambda_i \frac{x}{L}\right) + SH_i \sinh\left(\lambda_i \frac{x}{L}\right) \quad (6.4)$$

where:

$$\lambda_i^4 = \frac{\omega^2 \rho A_i}{EI_i} L^4 \quad (6.5)$$

6.2.2 'Free mode'

Equation (6.3) is applied to the five interconnected sub-beams, respectively (Figure 6.1 (a)). The appropriate boundary conditions that can be applied at the supports, $x = x_1$ and $x = x_5$, are $W_i = 0$ and $W_i' = 0$ if the end of the beam is clamped, $W_i = 0$ and $W_i'' = 0$ if hinged, $W_i'' = 0$ and $W_i''' = 0$ if free, where $i = 1, 5$.

The continuity for deflection and slope at $x = x_3$ are:

$$W_2 = W_3 \quad (6.6)$$

$$W_2 = W_4 \quad (6.7)$$

$$W_2' = W_3' \quad (6.8)$$

$$W_2' = W_4' \quad (6.9)$$

The continuity for shear and bending moments at $x = x_3$ are:

$$EI_2 W_2''' = EI_3 W_3''' + EI_4 W_4''' \quad (6.10)$$

$$EI_2 W_2'' + \frac{(EH_3 EH_4) H_2^2}{4a(EH_3 + EH_4)} (W_2' - W_5') = EI_3 W_3'' + EI_4 W_4'' \quad (6.11)$$

$\frac{(EH_3EH_4)H_2^2}{4a(EH_3+EH_4)}(W_2' - W_5')$ represents the consideration of the compatibility between the stretching/shortening of the delaminated layers and axial equilibrium (see section 3.2.2). And similarly, one can derive the continuity conditions at $x = x_4$.

6.2.3 ‘Constrained mode’

The ‘constrained mode’ model is simplified by the assumption that the delaminated layers are constrained to have the same transverse deformations. The delaminated beam is analyzed as four beam segments I–IV.

For the ‘constrained mode’ model, the governing equations for beam 1, 2 and 5 are identical to Equation (6.3). For beam 3 and beam 4, i.e. segment III, the governing equation is

$$(EI_3 + EI_4)W_{III}'''' - (\rho A_3 + \rho A_4)\omega^2 W_{III} = 0 \quad (6.12)$$

The boundary conditions for the ‘constrained mode’ are identical to the boundary conditions of the ‘free mode’. The continuity for deflection, slope, shear and bending moments at $x = x_3$ are:

$$W_{II} = W_{III} \quad (6.13)$$

$$W_{II}' = W_{III}' \quad (6.14)$$

$$EI_2 W_{II}''' = (EI_3 + EI_4)W_{III}''' \quad (6.15)$$

$$EI_2 W_{II}'' + \frac{(EH_3EH_4)H_2^2}{4a(EH_3+EH_4)}(W_{II}' - W_{IV}') = (EI_3 + EI_4)W_{III}'' \quad (6.16)$$

And similarly, one can derive the continuity conditions at $x = x_4$.

6.2.4 Continuity conditions at the crack

According to the work of Rizos and Aspragathos (1990) and Shifrin and Ruotolo (1999), when only bending vibrations are considered, the rotational crack compliance is assumed to be dominant in the local flexibility matrix. The bending spring constant K_T in the vicinity of

the cracked section of a beam with orthogonal cross-section when a lateral crack of uniform crack depth b exists, from the crack strain energy function,

$$K_T = 1/c, \quad c = 5.346h/EI f(b/h)$$

where c is the compliance and,

$$f(b/h) = 1.8624 \xi^2 - 3.95 \xi^3 + 16.375 \xi^4 - 37.226 \xi^5 + 76.81 \xi^6 - 126.9 \xi^7 + 172 \xi^8 - 143.97 \xi^9 + 66.56 \xi^{10}, \text{ in which } \xi = b / H_1 \text{ (Shifrin and Ruotolo, 1999).}$$

According to the approach proposed by Rizos and Aspragatos (1990), it is possible to introduce continuity conditions displacement, bending moment and shear respectively, for each connection between sub-beam separated by the crack (Figure 6.2); moreover, a last equation introduce a discontinuity into the rotation of the beam axis, by imposing equilibrium between transmitted moment and rotating of the spring represent the crack.

$$W_1 = W_2 \quad (6.17)$$

$$W_1''' = W_2''' \quad (6.18)$$

$$W_1'' = W_2'' \quad (6.19)$$

$$W_1' + c_1 W_1'' = W_2' \quad (6.20)$$

The total number of boundary and continuity conditions is 20 for ‘free mode’, or 16 for ‘constrained mode’, which is equal to the total number of unknown coefficients. A non-trivial solution exists only when the determinant of the coefficient matrix vanishes. The natural frequencies can be obtained as eigenvalues.

6.2.5 Introduction of nondimensional parameters

To study the effect of the surface crack on delaminated beam, two new parameters are introduced. Firstly, to study the effect of the location of the surface crack, nondimensional parameter D is introduced. When the crack occurs outside the delaminated area, $D = d/(L/2 - a/2)$, d is the distance of the crack from the left end of the beam; when inside the

delaminated area, $D = d/a$, d is the distance of the crack from the left boundary of the delaminated area. Secondly, to study the effect of the depth of the crack on the delaminated beam, nondimensional parameter $B = b/H_1$ is introduced.

6.3 Results and Discussions

6.3.1 Verification

To validate the present study, comparisons are made with published results on a clamped-clamped Euler-Bernoulli beam with a single delamination, without crack. The first nondimensional natural frequencies of a clamped-clamped beam with a single midplane central delamination having various lengths are compared with the analytical results of Wang *et al.* (1982) and FEM results of Lee (2000), in Table 6.1. Table 6.1-6.2 show good agreements between the previous results and present ones.

Table 6.2 shows the comparison of the current results against the experimental results of Mujumdar and Suryanarayan (1988) and the mixed-FEM results of Ramtekkar (2009). The first and second natural frequency ω_1 (Hz) of an isotropic cantilever beam without edge crack (by denoting $b/H_1=0.001$) but suffering a single delamination ($H_3/H_1 = 0.33$) is tabulated here.

In Table 6.3, a clamped-free beam with two surface cracks (no delamination) is compared with the work of Shifrin and Ruotolo (1999), to which the configuration of the beam can be referred. Table 6.3 shows good agreements between the previous results and present ones.

6.3.2 Delaminated beams with one edge crack

6.3.2.1 Crack occurs outside the delaminated area

As is shown in Figure 6.1, the edge crack occurs outside the delaminated area, located at a distance d from the left end of the beam, of depth b . The boundary condition is clamped-clamped. The delamination is of length, locating at the middle of the beam, with $H_3/H_1 = 0.5$. Figure 6.3 shows the normalized nondimensional natural frequency λ^2 / λ_d^2 versus the nondimensional distance $D = d/(L/2 - a/2)$, considering different length of delamination, for a clamped-clamped beam with a single delamination and a crack occurring outside the delaminated area. λ_d^2 is the nondimensional natural frequency of an undelaminated beam.

It can be observed that the difference in natural frequencies between $a/L = 0.1$ and 0.2 is larger when the depth of crack increases, indicating that the effect of delamination on reducing natural frequency is aggravated by the increasing depth of the edge crack. Such difference increases then decreases as the edge crack moves towards the left boundary of the delamination. One can also see from Figure 6.3 that the differences of natural frequencies between $b/H_1 = 0.1$ and 0.2 as well as 0.2 and 0.3 decrease to zero then increases as the crack moves from the left to the right.

Table 6.1.

First and second nondimensional frequencies (λ^2) of a clamped-clamped homogeneous beam (no edge crack) with a central midplane delamination.

Delamination Length a / L	First Mode			Second Mode		
	Present Free & Constrained	Analytical Wang <i>et al.</i> (1982)	FEM Lee (2000)	Present Free & Constrained	Analytical Wang <i>et al.</i> (1982)	FEM Lee (2000)
0.00	22.37	22.39	22.36	61.67	61.67	61.61
0.10	22.37	22.37	22.36	60.81	60.76	60.74
0.20	22.36	22.35	22.35	56.00	55.97	55.95
0.30	22.24	22.23	22.23	49.00	49.00	48.97
0.40	21.83	21.83	21.82	43.89	43.87	43.86
0.50	20.89	20.88	20.88	41.52	41.45	41.50
0.60	19.30	19.29	19.28	41.04	40.93	41.01
0.70	17.23	17.23	17.22	40.82	40.72	40.80
0.80	15.05	15.05	15.05	39.07	39.01	39.04
0.90	13.00	13.00	12.99	35.39	35.38	35.38

Table 6.2.

First and second natural frequency ω_1 , ω_2 (Hz) of cantilever beam with a single delamination, $H_2 / H_1 = 0.33$.

Specimen span (mm)	Dimension		First mode ω_1			Second Mode ω_2		
	$L / 2$	d_1 a	Experiment (Mujumdar and Suryanarayan, 1988)	Ramtekkar (2009) Unconstrained-interface model	Present Free mode	Experiment (Mujumdar and Suryanarayan, 1988)	Ramtekkar (2009) Unconstrained-interface model	Present Free mode
240.0	153.5	96.0	31.6	31.11	32.25	172.1	156.24	159.93
250.0	104.0	148.0	31.7	31.51	31.88	190.5	197.23	198.81
175.0	133.0	88.5	56.9	56.35	57.89	339.3	244.42	248.83
200.0	122.0	106.0	46.6	46.34	47.26	291.0	258.95	260.71
155.0	122.0	61.0	69.1	68.82	69.82	363.7	249.20	281.82

Table 6.3

Normalized first natural frequency (ω / ω_0) of a clamped-clamped homogeneous Euler-Bernoulli beam with two edge cracks, with a clamped-free boundary condition, without delamination.

Depth of the second crack b_2 / H_1 (mm)	Distance of the second crack from the clamped end 0.2m		Distance of the second crack from the clamped end 0.5m	
	Shifrin and Ruotolo (1999)	Present	Shifrin and Ruotolo (1999)	Present
0.10	0.888	0.892	0.930	0.932
0.20	0.757	0.760	0.914	0.916
0.30	0.536	0.539	0.883	0.885

Figure 6.3. Influence of the depth and location of the crack on the ‘free mode’ natural frequencies λ^2 / λ_d^2 for a homogeneous clamped-clamped beam with a single central delamination of various lengths and the crack locates outside the delaminated area.

6.3.2.2 Crack occurs inside the delaminated area

As is shown in Figure 6.4, the edge crack occurs inside the delaminated area, with a distance d from the left boundary of the delamination, of depth b . The delamination is of length a , locating at the middle of the beam.

Figure 6.4. Cracked beam with a single delamination and the crack locates inside the delaminated area.

Figure 6.5 shows the normalized nondimensional natural frequency λ^2 / λ_d^2 , of a clamped-clamped beam with a single delamination and a crack occurring inside the delaminated area, versus the nondimensional distance $D = d/a$, considering different lengths of delamination, with $H_3/H_1 = 0.5$. At midpoint when $D = 0.5$, the drop of natural frequency from 0.913 when $b/H_1 = 0.1$ to 0.847 when $b/H_1 = 0.3$, for delamination length $a/L = 0.5$. The drop is only from 0.960 to 0.917 for $a/L = 0.4$, and 0.983 to 0.958 for $a/L = 0.3$. It indicates that when the edge crack occurs inside the delaminated area, the effect of crack on aggravating the influence of delamination is larger when the beam suffers a longer delamination.

It can also be observed from Figure 6.5 that the differences of natural frequencies between crack depth $b/H_1 = 0.1$ and 0.2, 0.2 and 0.3 increase as the crack moves from left towards the middle, then decreases as it continues to the right. Such difference is larger when the delamination length a/L increases from 0.3 to 0.5. The difference in natural frequencies between delamination lengths $a/L = 0.3$ and 0.4, 0.4 and 0.5, is larger with a deeper crack. For the same crack depth b , the difference peaks when the crack

occurs at the middle of the delamination. The results of Figure 6.5 are also shown in Table 6.4 for the convenience of serving as a benchmark for other numerical results.

Figure 6.5. Influence of the depth and location of the crack on the ‘free mode’ natural frequencies λ^2/λ_d^2 for a homogeneous clamped-clamped beam with a single central delamination of various lengths.

Table 6.4

Nondimensional normalized fundamental frequency (λ^2/λ_d^2) of a clamped-clamped isotropic Euler-Bernoulli beam with a single delamination and one edge crack, the crack occurs inside the delaminated area.

Location of crack D	$a/L=0.5$			$a/L=0.4$			$a/L=0.3$		
	b/H_1	b/H_2	b/H_3	b/H_1	b/H_2	b/H_3	b/H_1	b/H_2	b/H_3
	0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3
0.1	0.933	0.932	0.931	0.976	0.976	0.976	0.993	0.992	0.991
0.2	0.932	0.929	0.928	0.973	0.969	0.966	0.990	0.986	0.983
0.3	0.924	0.910	0.899	0.967	0.955	0.947	0.987	0.978	0.972
0.4	0.916	0.887	0.863	0.962	0.942	0.926	0.984	0.971	0.962
0.5	0.913	0.877	0.847	0.960	0.936	0.917	0.983	0.968	0.958
0.6	0.916	0.887	0.863	0.962	0.942	0.926	0.984	0.971	0.962
0.7	0.924	0.910	0.899	0.967	0.955	0.947	0.987	0.978	0.972
0.8	0.932	0.929	0.928	0.973	0.969	0.966	0.990	0.986	0.983
0.9	0.933	0.932	0.931	0.976	0.976	0.976	0.993	0.992	0.991

Figure 6.6 shows the normalized nondimensional natural frequency λ^2 / λ_d^2 , of a clamped-clamped beam with a single delamination and a crack occurring inside the delaminated area, versus the nondimensional distance $D = d/a$, considering different thickness-wise locations, with $a/L = 0.4$. When comparing the results between Figure 6.6 (a), (b) and (c), it can be concluded that the differences of natural frequencies between crack depth $b / H_1 = 0.1$ and 0.2, 0.2 and 0.3 decreases when the delamination moves towards the surface. The differences of natural frequencies between the thickness-wise location $H_3/H_1 = 0.3$ and 0.4, 0.4 and 0.5 are larger with a deeper crack.

Figure 6.6. Influence of the depth and location of the crack on the ‘free mode’ natural frequencies λ^2/λ_d^2 for a homogeneous clamped-clamped beam with a single delamination of various thickness-wise locations. (a) $H_3/H_1 = 0.5$; (b) $H_3/H_1 = 0.4$; (c) $H_3/H_1 = 0.3$.

6.3.2.3 ‘Free Mode’ vs. ‘Constrained Mode’

Figure 6.7 shows the normalized nondimensional natural frequency λ^2 / λ_d^2 , of a clamped-clamped beam versus the nondimensional distance $D = d/a$, adopting both ‘free mode’ and ‘constrained mode’ assumptions. There is a single midplane central delamination and a crack occurring inside the delaminated area. As is shown in Figure 6.7, the difference in natural frequencies between ‘free mode’ and ‘constrained mode’ increases then decreases as D increases, peaking at the middle of the beam. It can also be observed from Figure 6.7 that such difference is larger with a deeper crack. When comparing the results among Figure 6.7 (a), (b) and (c), one can see that the difference in natural frequencies between ‘free mode’ and ‘constrained mode’ is larger with a longer delamination, when $0.25 < D < 0.75$; larger with a shorter delamination when $0 < D < 0.25$ and $0.75 < D < 1$.

(a)

(b)

(c)

Figure 6.7. Influence of the crack on ‘free mode’ and ‘constrained mode’ natural frequencies λ^2/λ_d^2 for an isotropic homogeneous clamped-clamped beam with a single central delamination, considering different delamination length. (a) $a/L = 0.3$; (b) $a/L = 0.4$; (c) $a/L = 0.5$.

6.3.3 Delaminated beams with two one-side edge cracks

6.3.3.1 Cracks occurs outside the delaminated area

Figure 6.8 shows the normalized nondimensional natural frequency λ^2 / λ_d^2 , of a clamped-clamped beam versus the nondimensional distance $D = d/L$, the two symmetrical located cracks occurs at a distance d from the end of the beam, with a depth of b , as is shown in Figure 6.9. The effects of crack’s depth and location on natural frequency are more prominent when the beam suffers two symmetrical cracks when compared to one crack.

Figure 6.8. 'Free mode' normalized fundamental frequency versus the location of the two symmetrically located cracks, both of which locate outside the delaminated area.

Figure 6.9. 'Free mode' normalized fundamental frequency versus the location of the two symmetrically located cracks, both of which locate outside the delaminated area.

Figure 6.10 shows the variation of the natural frequency versus the delamination length a/L , considering different crack depths, for beam configuration shown in Figure 6.9. As is shown in Figure 6.10, the natural frequency decreases with an increasing length of delamination or an increasing depth of the edge crack. It can also be observed from

Figure 6.10 that the aggravation of the effect delamination due to the presence of edge crack is worse with two edge cracks instead of one. When comparing the results between Figure 6.10 (a), (b) and (c), one can conclude that such aggravation is smaller when the two edge crack is closer to the centre of the beam.

Figure 6.10 shows the variation of the natural frequency against the nondimensional delamination thickness location H_3 / H_1 , considering different delamination lengths, crack depths and locations, for beam configuration shown in Figure 6.9. As is also shown in Figure 6.11, the natural frequency increases as the delamination moves towards the midplane of the beam. The difference in natural frequencies between delamination length $a / L = 0.3$ and 0.4 is smaller when the delamination is closer to the midplane. It indicates the effect of delamination's thickness-wise location on reducing the natural frequency is aggravated by deeper cracks or when the cracks are farther from the centre of the beam.

(b)

(c)

Figure 6.10. 'Free mode' normalized fundamental frequency versus the delamination length, the two symmetrically located cracks occur out the delaminated area. (a) $d=0.10$; (b) $d=0.15$; (c) $d=0.20$.

Figure 6.11. 'Free mode' normalized fundamental frequency versus the delamination thickness-wise location, the two symmetrically located cracks occur out the delaminated area.

6.3.3.2 Cracks occurs inside the delaminated area

Figure 6.12 shows the variation of natural frequency versus the nondimensional distance $D = d / (a/2)$, considering different delamination lengths and crack depths. The beam suffers a centrally located delamination and two symmetrically located cracks, both of which occur inside the delaminated area, as is shown in Figure 6.13. Similar results on the effects of crack on natural frequency and the effects of crack on the delamination can be observed when compared with the results of Figure 6.5.

Figure 6.12. ‘Free mode’ normalized fundamental frequency versus the nondimensional distance D , the two symmetrically located one-sided cracks occur inside the delaminated area.

Figure 6.13. Beam with a single delamination and two symmetrically located cracks, both of which occur inside the delaminated area.

6.4 Concluding Remarks

In this chapter, an analytical solution to the vibration of cracked beams with delaminations is developed. We study how the location and depth of the crack influence

the effects delamination on natural frequencies and mode shapes. Both cases of crack occurring outside and inside the delaminated area are thoroughly investigated. The comparison of results between ‘free mode’ and ‘constrained mode’ of delamination vibration assumptions is conducted. Based on the theoretical investigations the following conclusions may be drawn.

1. The effect of delamination on reducing the natural frequency is aggravated by an increasing depth of the edge crack. Such effect becomes larger then smaller as the crack moves towards the boundary of the delaminated region (when crack occurs outside the delaminated area.)
2. When the edge crack occurs inside the delaminated area, the effect of crack on aggravating the influence of delamination is larger when the beam suffers a longer delamination. Such effect peaks when the crack occurs at the middle of the delaminated region.
3. When the edge crack occurs inside the delaminated area, the effect of crack depth on natural frequency is smaller when the delamination moves towards the surface. Also the effect of the delamination thickness-wise location on natural frequency is larger when the crack is deeper.
4. The difference in natural frequencies between ‘free mode’ and ‘constrained mode’ increases then decreases as the crack moves from one side of the delaminated region to the other side, peaking at the middle.

Chapter 7

Vibration of Rotating Timoshenko Beams with Multiple Delaminations

7.1 Introduction

Many studies have been presented for the vibration of beams rotating beams. However, studies on the vibration of rotating beams with delaminations have not been presented in the literature. In this chapter, new nondimensional parameters of *relative slenderness ratio* r and *normalized rotating speed* η are introduced, and analytical solutions are developed to study the free vibration of rotating Timoshenko beams with multiple delaminations. The Timoshenko beam theory with both ‘free mode’ and ‘constrained mode’ assumptions in delamination vibration are adopted.

7.2 Formulations and Introduction of New Parameters

The analytical solution to the vibration of rotating Timoshenko beam, reported by Du *et al.* (1994), is used here to solve the governing equations. Both ‘free mode’ and ‘constrained mode’ assumptions are adopted to investigate the delamination vibration problem. Instead of Euler-Bernoulli beam theory, the current chapter adopts the Timoshenko beam theory. The model takes into account shear deformation and rotational inertia effects, making it suitable for describing the behavior of short beams or sandwich composite beams. Figure 7.1 (a) shows a beam with length L and thickness H_1 . The beam is made of two distinct layers, with Young’s modulus E_2 and E_3 , and thickness H_2 and H_3 . The beam is separated along the interface by a delamination with length a and located at a distance d from the center of the beam. The beam can then be subdivided into three span-wise regions, a delamination region and two integral regions. The delamination region is comprised of two segments (delaminated layers), beam 2 and beam 3, which are joined at their ends to the integral segments, beam 1 and beam 4. Each of the four beams is treated as rotating Timoshenko beams. The beam is then analyzed as four interconnected beams using delamination as their boundaries. The continuity and boundary conditions are satisfied between adjoining beams.

Figure 7.1. (a) A rotating Timoshenko beam is delaminated along the interface; (b) The delaminated beam is analysed as four interconnected beams.

7.2.1 Governing equations

Let ψ_i' and $w_i(x, t)$ denote the slope of the deflection curve when the shearing is neglected and the midplane deflection of beam i , respectively. The governing equations for the free vibration of a rotating Timoshenko beam with single delamination are: ($i = 1-4$)

$$(EI_i \psi_i')' + KA_i G_i (w_i' - \psi_i) = \rho_i I_i \ddot{\psi}_i \quad (7.1)$$

$$(KA_i G_i (w_i' - \psi_i))' + (T_i w_i')' - \rho_i A_i \ddot{w}_i = 0 \quad (7.2)$$

where ρ_i is the mass density, A_i is the cross-sectional area of the beam, G_i shear modulus, I_i is the moment of area of the cross section and K is the shear coefficient. T_i denotes the centrifugal force exerted on the cross section of beam i . EI_i ($i = 1-4$) is the bending stiffness of beam i . A closed form solution by Du *et al.* (1994) is adopted to solve for the flexural natural frequencies and corresponding mode shapes.

Assuming a simple harmonic motion of the beam one can put

$$w_i(x, t) = W_i(x) e^{i\omega t} \text{ and } \psi_i(x, t) = \varphi(x) e^{i\omega t} \quad (7.3)$$

Substituting expressions in Equations (7.3) into Equations (7.1) and (7.2) leads to

$$\begin{aligned}
 (EI\varphi')' + KAG(W' - \varphi) &= -\omega^2 \rho I \varphi \\
 (KAG(W' - \varphi))' + (TW)' + \omega^2 \rho AW &= 0
 \end{aligned} \tag{7.4}$$

Assuming that EI , ρI and ρA are constants and combining Equation (7.4) to obtain a decoupled equation for W , one finds

$$\begin{aligned}
 EI_i W_i'''' + \frac{EI_i}{KA_i G_i} (T_i W_i')''' + \left(\rho_i I_i + \frac{\rho_i A_i}{KA_i G_i} EI_i \right) \omega^2 W_i'' \\
 + \left(\frac{\omega^2 \rho_i I_i}{KA_i G_i} - 1 \right) (T_i W_i')' - \omega^2 \rho_i A_i W_i + \frac{\rho_i A_i \rho_i I_i}{KA_i G_i} \omega^4 W_i = 0.
 \end{aligned} \tag{7.5}$$

The governing equations of motion of rotating Timoshenko beam are two variable-coefficient differential equations. Based on the work of Du *et al.* (1994), a convergent power series expression is adopted to solve analytically for the exact natural frequencies and mode shapes.

The generalized solutions for the differential equations in Equation (7.5) are:

$$W_i(\xi) = \lambda_0 u_0(\xi) + \lambda_1 u_1(\xi) + \lambda_2 u_2(\xi) + \lambda_3 u_3(\xi) \tag{7.6}$$

where $\lambda_0, \lambda_1, \lambda_2$ and λ_3 are constants to be determined by imposing the appropriate boundary conditions. u_0, u_1, u_2 and u_3 are four linearly independent solutions.

$$\begin{aligned}
 u_0(\xi) &= 1 - \left(C_{11}/24C_1 \right) \xi^4 - \left[C_{11}(C_2 + C_4)/120C_1^2 \right] \xi^5 + \dots \\
 u_1(\xi) &= \xi - \left(C_9/24C_1 \right) \xi^4 - (1/120C_1) \left[\left\{ C_9(C_2 + C_4)/C_1 \right\} + C_{10} + C_{11} \right] \xi^5 + \dots \\
 u_2(\xi) &= \xi^2 - \left(C_6/12C_1 \right) \xi^4 - (1/60C_1) \left[\left\{ C_6(C_2 + C_4)/C_1 \right\} + C_7 + C_9 \right] \xi^5 + \dots \\
 u_3(\xi) &= \xi^3 - \left(C_4/4C_1 \right) \xi^5 - (1/20C_1) \left[\left\{ C_4(C_2 + C_4)/C_1 \right\} + C_5 + C_6 \right] \xi^6 + \dots
 \end{aligned} \tag{7.7}$$

where $\xi = x/L$, L is the length of the beam.

The derivation of u_0, u_1, u_2 and u_3 and the formulations of $C_1 - C_{11}$ are presented in the work of Du *et al.* (1994).

7.2.2 'Free mode'

Equation (7.1) and (7.2) are applied to the four interconnected sub-beams, respectively (Figure 7.1 (b)). The appropriate boundary conditions that can be applied at the supports, $x = x_1$ and $x = x_4$, are $W_i = 0, W_i' = 0$, if the end of the beam is clamped; $W_i = 0$ and $W_i'' = 0$ if hinged; $W_i'' = 0, W_i''' = 0$, if free, where $i = 1, 4$.

The continuity for deflection and slope at $x = x_2$ are:

$$W_1 = W_2 = W_3 \quad (7.8)$$

$$\psi_1 = \psi_2 = \psi_3 \quad (7.9)$$

The continuity for shear and bending moments at $x = x_2$ are:

$$EI_1 W_1''' = (EI_2 + EI_3) W_2''' \quad (7.10)$$

$$EI_1 + \frac{(E_2 H_2 E_3 H_3) H_1^2}{4(E_2 H_2 + E_3 H_3)} (W_1' - W_4') = EI_2 W_2'' + EI_3 W_3'' \quad (7.11)$$

$\frac{(E_2 H_2 E_3 H_3) H_1^2}{4(E_2 H_2 + E_3 H_3)} (W_1' - W_4')$ represents the consideration of the compatibility between the stretching/shortening of the delaminated layers and axial equilibrium (see section 3.2.2). Similarly, we can derive the continuity conditions at $x = x_3$.

7.2.3 'Constrained mode'

The 'constrained mode' model is simplified by the assumption that the delaminated layers are constrained to have the same transverse deformations. The delaminated beam is analyzed as three beam segments I-III.

The boundary conditions for the ‘constrained mode’ are identical to the boundary conditions of the ‘free mode’. The continuity for deflection, slope, shear, and bending moments at $x = x_2$ are:

$$W_I = W_{II} \quad (7.12)$$

$$\psi_I' = \psi_{II}' \quad (7.13)$$

$$EI_1 W_I''' = (EI_2 + EI_3) W_{II}''' \quad (7.14)$$

$$EI_1 + \frac{(E_2 H_2 E_3 H_3) H_4^2}{4(E_2 H_2 + E_3 H_3)} (W_I' - W_{III}') = EI_2 W_{II}'' \quad (7.15)$$

Similarly, we can derive the continuity conditions at $x = x_3$.

The total number of boundary and continuity conditions is 24 for ‘free mode’, or 18 for ‘constrained mode’, which is equal to the total number of unknown coefficients. A non-trivial solution exists only when the determinant of the coefficient matrix vanishes. The natural frequencies can be obtained as eigenvalues and mode shapes as eigenvectors.

7.2.4 Introduction of nondimensional parameters

To study the effect of rotating speed, as well as the effect of shear deformation and rotary inertia on delaminated beams, similar parameters are adopted to represent the effect of rotating speed as well as the effect of shear deformation and rotary inertia as they are in the work of Du *et al.* (1994). To study the effect of shear deformation and rotary inertia, a relative slenderness ratio $r = R / L$ is adopted, where R is the radius of gyration and L is the beam length. A higher value of r indicates a more prominent the effect of shear deformation and rotary inertia. The effect of rotating speed is represented by $\eta = \sqrt{\rho A / EI} \Omega L^2$, where Ω is the angular velocity. Also, the effect of root offset value is studied, we use $\alpha = r_0 / L$, where r_0 is the offset distance.

7.3 Results and Discussions

7.3.1 Verification

To validate the present study, comparisons with published results on static Euler-Bernoulli beam with a single delamination and undelaminated rotating Timoshenko beam are made. The first and second nondimensional natural frequencies of a clamped-clamped beam with a single midplane central delamination having various lengths are compared with the analytical results of Wang et al. (1982) and FEM results of Lee (2000). The first nondimensional natural frequencies of a clamped-free rotating Timoshenko beams without delaminations are compared with the analytical results of Du *et al.* (1994). Tables 7.1–7.3 show good agreement between the present results and previous results.

7.3.2 Rotating Timoshenko beams with a single delamination

7.3.2.1 Static Timoshenko beams with single delamination

Figure 7.2 shows the normalized frequency λ^2/λ_d^2 of a static beam with a single central delamination versus r . The nondimensional frequency of the delaminated beam λ^2 is normalized with respect to that of undelaminated beam λ_d^2 . As shown in Figure 7.2, λ^2/λ_d^2 decreases as r increases, indicating a higher influence of Timoshenko beam effect leads to lower natural frequencies. The results agree with Du's work (1994) (Figure 5). The difference between natural frequencies of delamination length $a/L = 0.2$ and 0.4 , 0.4 and 0.6 , 0.6 and 0.8 becomes smaller as r increases, or as the thickness-wise location of delamination H_2/H_1 increases from 0.2 to 0.5 .

It shows that the effect of delamination on natural frequency becomes smaller with a larger Timoshenko effect.

Table 7.1.

Nondimensional fundamental frequency (λ^2) of a clamped-clamped homogeneous static Euler-Bernoulli beam with a midplane delamination.

Delamination length, a/L	Present Constrained and Free	Analytical Wang <i>et al.</i> (1982)	FEM Lee(2000)
0.00	22.37	22.39	22.36
0.10	22.37	22.37	22.36
0.20	22.36	22.35	22.35
0.30	22.24	22.23	22.23
0.40	21.83	21.83	21.82
0.50	20.89	20.88	20.88
0.60	19.30	19.29	19.28
0.70	17.23	17.23	17.22
0.80	15.05	15.05	15.05
0.90	13.00	13.00	12.99

Table 7.2

Nondimensional mode 2 frequency (λ^2) of a clamped-clamped homogeneous static beam with a midplane delamination.

Delamination length, a/L	Present Constrained and Free	Analytical Wang <i>et al.</i> (1982)	FEM Lee (2000)
0.00	61.67	61.67	61.61
0.10	60.81	60.76	60.74
0.20	56.00	55.97	55.95
0.30	49.00	49.00	48.97
0.40	43.89	43.87	43.86
0.50	41.52	41.45	41.50
0.60	41.04	40.93	41.01
0.70	40.82	40.72	40.80
0.80	39.07	39.01	39.04
0.90	35.39	35.38	35.38

Table 7.3Nondimensional mode 1 frequency (λ^2) of a clamped-free rotating Timoshenko beam.

r	$\eta=4$		$\eta=8$	
	Present	Du <i>et al.</i> (1994)	Present	Du <i>et al.</i> (1994)
0.01	5.58	5.58	9.245	9.246
0.02	5.562	5.564	9.212	9.215
0.03	5.537	5.539	9.162	9.167
0.04	5.502	5.505	9.097	9.106
0.05	5.46	5.463	9.023	9.036
0.06	5.411	5.415	8.944	8.963
0.07	5.358	5.363	8.861	8.889
0.08	5.300	5.307	8.775	8.815
0.09	5.239	5.249	8.688	8.744
0.10	5.176	5.191	8.599	8.677

(a)

Figure 7.2. Influence of rotary inertia and shear deformation with different delamination length on normalized mode 1 ‘free mode’ frequency λ^2/λ_d^2 for a static homogeneous clamped-free beam with a single central delamination. (a) $H_2=0.2H_1$; (b) $H_2=0.5H_1$

7.3.2.2 Rotating Euler-Bernoulli beams with single delamination

Figure 7.3 shows the effect of rotating speed on the normalized ‘free mode’ natural frequency of beam with a single central delamination, neglecting the Timoshenko beam effect. As shown in Figure 7.3, λ^2/λ_d^2 increases as angular velocity η becomes higher, which agrees well with the work of Du et al. (1994) (Table 7). The lower the rotating speed the higher the percentage the frequency decreases as delamination length a/L increases. It can be concluded that the effect of delamination is smaller when the beam is subjected to a larger centrifugal force. When comparing the results between (a) and (b), it can be concluded that the effect of delamination length is more prominent when the thickness-wise location of delamination H_2/H_1 increases from 0.2 to 0.5.

(a)

(b)

Figure 7.3. Influence of rotating speed on normalized mode 1 'free mode' frequency λ^2/λ_d^2 for a rotating homogeneous clamped-free Euler-Bernoulli beam with single central delamination. (a) $H_2=0.2H_1$; (b) $H_2=0.5H_1$

7.3.2.3 Mode 1 and mode 2 ‘free mode’ frequencies

Figure 7.4 shows the results of the normalized frequency of mode 1 and 2. It incorporates the influence of rotating speed, Timoshenko beam effect and delamination. For both vibration modes, λ^2/λ_d^2 increases as the rotating speed increase and decreases as r increases. Timoshenko beam effect exert a larger influence on mode 2 natural frequency than on mode 1, while the effect of rotating speed has a smaller influence on mode 2 than on mode 1.

For mode 1, the differences of natural frequencies between $r = 0.01$ and 0.1 increase monotonously as the rotating speed becomes higher. For mode 2, the differences of natural frequencies between $r = 0.01$ and 0.1 do not show a monotonous relation with rotating speed. Instead, the difference of natural frequencies decreases at first and then increase. The rotating speed corresponding to the minimum difference of nature frequencies is higher when delamination length $a/L = 0.1$ than that of $a/L = 0.4$.

Figure 7.4. Mode 1 and mode 2 ‘free mode’ frequencies versus the rotating speed for a homogeneous clamped-free beam with single delamination considering different r . (a) $a/L = 0.1$; (b) $a/L = 0.4$

7.3.2.4 ‘Free Mode’ vs. ‘Constrained Mode’

Figure 7.5 shows the influence of Timoshenko beam effect on the normalized frequencies λ^2/λ_d^2 of beams with single central delamination, where $a/L = 0.4$ and $H_2/H_1 = 0.5$. For both ‘free mode’ and ‘constrained mode’, λ^2/λ_d^2 increases when the rotating speed η becomes higher. λ^2/λ_d^2 decreases when the Timoshenko beam effect becomes more prominent. For both ‘free mode’ and ‘constrained mode’ a higher rotating speed leads to higher percentage decrease of λ^2/λ_d^2 . The natural frequency of ‘constrained mode’ is larger than that of the ‘free mode’, which agrees with the established results of previous work (Shu and Fan, 1996).

The differences of natural frequency between the ‘constrained mode’ and the ‘free mode’ increase when rotating speed becomes higher. The differences of natural frequencies do not change monotonously as r increases. Generally, the difference between ‘constrained mode’ and ‘free mode’ increases at first then decreases as r becomes larger. As the rotating speed becomes

larger, the value of r corresponding to the maximum difference between the ‘free mode’ and ‘constrained mode’ decreases.

7.3.3 Rotating Timoshenko beams with double delamination of equal length

Figure 7.6 shows the case of beams with double delaminations of equal length (as shown in Figure 7.7). Regarding the Timoshenko beam effect and the effect of rotating speed on mode 1 and mode 2 natural frequencies, similar results to Figure 7.4 can be observed. Compared with beams with a single delamination, natural frequencies of beams with double delaminations is smaller when both cases have the same delamination length. However the difference is less significant with a higher rotating speed or a more prominent Timoshenko beam effect. Regarding the differences of natural frequencies between $r = 0.01$ and 0.1 , similar results to Figure 7.4 can be observed for both mode 1 and mode 2.

Figure 7.5. Influence of rotary inertia and shear deformation on ‘free mode’ and ‘constrained mode’ frequencies for a homogeneous clamped-free beam with single delamination considering different rotating speed η . $a/L = 0.4$ and $H_2/H_1 = 0.5$.

7.3.4 Rotating Timoshenko beams with two enveloped delaminations

7.3.4.1 Mode 1 and mode 2 ‘free mode’ frequencies

The variations of the normalized frequency λ^2/λ_d^2 with the rotating speed as well as Timoshenko beam effect for beams with two central enveloping delaminations (Figure 7.8) are shown in Figure 7.9. The thickness-wise location of two delaminations is equally distributed. Natural frequencies λ^2/λ_d^2 of mode 1 and mode 2 are both included.

Figure 7.6. Mode 1 and mode 2 ‘free mode’ frequencies versus the rotating speed for a homogeneous clamped-free beam with two double delaminations considering different r . $d/L = 0$. (a) $a/L = 0.1$; (b) $a/L = 0.4$.

Figure 7.7. Beams with two central delaminations of equal length.

The result shows the change of frequencies according to the rotating speed and they are compared among three different cases, each of which is under different influences of Timoshenko beam effect. For mode 1 and mode 2, λ^2/λ_d^2 increases as rotating speed η increases,

decreases as r increases. Regarding the Timoshenko beam effect and the effect of rotating speed on mode 1 and 2 natural frequencies, similar results to Figure 7.4 can be observed. For mode 1, the differences of natural frequencies between $r = 0.01$ and 0.1 increase monotonously as the rotating speed becomes higher. For mode 2, the differences of natural frequencies between $r = 0.01$ and 0.1 do not show a monotonous relation with rotating speed. Instead, the difference of natural frequencies decreases at first and then increase. The rotating speed corresponding to the minimum difference of nature frequencies is higher with delamination length $a_1/L = 0.1$, $a_2/L = 0.2$ than $a_1/L = 0.2$, $a_2/L = 0.4$.

Figure 7.8. Beams with two central enveloped delaminations.

Figure 7.9. Mode 1 and mode 2 ‘free mode’ frequencies versus the rotating speed for a homogeneous clamped-free beam with two enveloping delaminations considering different r . $d/L = 0$, $H_4 = 0.33H_1$ (a) $a_1/L = 0.1$, $a_2/L = 0.2$; (b) $a_1/L = 0.2$, $a_2/L = 0.4$.

7.3.4.2 ‘Free Mode’ vs. ‘Constrained Mode’

Figure 7.10 shows the influence of Timoshenko beam effect on the normalized frequencies λ^2/λ_d^2 with two central enveloping delaminations, results from both ‘free mode’ and ‘constrained mode’ are presented. The thickness-wise locations of two delaminations are equally distributed. For both ‘free mode’ and ‘constrained mode’, λ^2/λ_d^2 increases when the rotating speed η becomes higher. λ^2/λ_d^2 decreases when the Timoshenko beam effect becomes more prominent.

For both ‘free mode’ and ‘constrained mode’ λ^2/λ_d^2 decreases faster with a higher rotating speed. The natural frequency of ‘constrained mode’ is larger than the one with ‘free mode’, which agrees with the established results of previous work (Shu and Della, 2004). However, the difference between the ‘constrained mode’ and the ‘free mode’ does not change monotonously as r increases, which is similar to what we observe in the case of beams with single delamination.

Figure 7.10. Influence of rotary inertia and shear deformation on ‘free mode’ and ‘constrained mode’ frequencies for a homogeneous clamped-free beam with two enveloping delaminations considering different rotating speed η . $a_1/L = 0.2$, $a_2/L = 0.4$.

7.3.5 Effects of root offset value

Figure 7.11 shows the effect of root offset value on normalized natural frequency when $r = 0.05$ and $\eta = 4$. It shows a linear relation between the frequency and root offset value for each delamination length under consideration.

Figure 7.11. Effect of root offset value on normalized natural frequency of rotating Timoshenko beam with single delamination. $\eta = 4$, $r = 0.05$.

7.3.6 Mode shapes of rotating Timoshenko beams with single delamination

7.3.6.1 Mode 1

Figure 7.12 shows the ‘free mode’ first mode shape of delaminated beam under the influence of rotating speed and Timoshenko beam effect respectively. As shown in Figure 7.12 (a), when the rotating speed is kept as $\eta=4$, the change of mode shape due to delamination and Timoshenko beam effect is illustrated. Figure 7.12 (b) presents the case of the change of mode shape due the delamination and rotating effect, when the value of r is kept at 0.05.

The amplitude is larger in the middle when the delamination length is longer. Also, the amplitude is larger when the rotating speed increases or with a larger value of r . We can

conclude that the larger the effect of shear deformation and rotary inertia or the rotating speed, the larger the amplitude of vibration.

Figure 7.12. Mode shapes of the first mode for clamped-free rotating Timoshenko beam with single delamination. (a) Consider different values of r with a constant rotating speed $\eta = 4$. (b) Consider different rotating speed η with a constant value of $r = 0.05$.

7.3.6.2 Mode 2

As shown in Fig.13, we have the similar results for mode 2 as those with mode 1. Furthermore, the effect of Shear deformation and rotary inertia as well as rotating speed has a larger effect on the second mode shape than the first one.

(a)

(b)

(c)

(d)

Figure 7.13. Mode shapes of the second mode for clamped-free rotating Timoshenko beam with single delamination. (a) $\eta=4$ and $a/L = 0.1$. (b) $\eta=4$ and $a/L = 0.4$. (c) $r=0.05$ and $a/L = 0.1$. (d) $r=0.05$ and $a/L = 0.4$.

7.4 Concluding Remarks

Here the analytical solution to the vibration of rotating Timoshenko beams with multiple delaminations is developed. The authors studied how Timoshenko beam effect and rotating speed influence the effect of delamination on natural frequencies and mode shapes. Based on the results, the following conclusions can be made.

1. Natural frequency is reduced by the presence of delamination and such reduction is bigger with a larger Timoshenko beam effect (i.e. a beam with bigger thickness-to-length ratio or)
2. The effect of delamination on natural frequency is smaller when the beam has a higher rotating speed.
3. The difference of frequency between ‘free mode’ and ‘constrained mode’ increases when the rotating speed becomes higher.
4. The difference of frequency between ‘free mode’ and ‘constrained mode’ increases at first then decreases with a bigger Timoshenko beam effect (i.e. a beam with bigger thickness-to-length ratio or)

Chapter 8

Vibration of Functionally Graded Beams with Single Delamination

8.1 Introduction

Many studies have been presented for the vibration of functionally graded material (FGM) beams with or without cracks. However, for the other common failure, i.e. delamination, studies on the vibration of functionally graded beams have not been presented in the literature. In this chapter, Kirchhoff-Love hypothesis, ‘free mode’ and ‘constrained mode’ assumptions in delamination vibration are adopted. The beam is analyzed as four interconnected beams using delamination as their boundaries. The continuity and boundary conditions are satisfied between adjoining beams.

8.2 Formulations

8.2.1 Governing equations and analytical solutions

As is shown in Figure 8.1 (a), consider a functionally graded material (FGM) beam of length L and thickness H_1 , with a single delamination of length a , located at a distance d_a from the center of the beam. The beam can be subdivided into three span-wise regions, one delaminated region and two undelaminated region. The delamination region is comprised of two segments (delaminated layers), beam 2 and 3, which are joined at their ends to the integral segments, beam 1 and 4.

The Young's modulus $E(z)$ and mass density $\rho(z)$ of the beam vary in the thickness direction (z) only and follow the exponential distributions shown below:

$$E(z) = E_0 e^{\beta z}, \rho(z) = \rho_0 e^{\beta z} \quad (8.1)$$

where E_0 and ρ_0 are the Young's modulus and density of the FGM beam at the mid-plane. β is a constant defining the material property variation along the thickness direction and $\beta = 0$ corresponds to an isotropic homogeneous beam.

Based on the Kirchhoff-Love hypothesis, $\bar{u}_i(x, z_i, t)$ and $\bar{w}_i(x, z_i, t)$, which refer to the displacements parallel to the x - and z -axes of an arbitrary point in beam i respectively (as is shown in Figure 8.1 (b)), are:

$$\bar{u}_i(x, z, t) = u_i(x, t) - z_i \frac{\partial w_i}{\partial x} \quad (8.2)$$

$$\bar{w}_i(x, z, t) = w_i(x, t) \quad (8.3)$$

where $u_i(x, t)$ and $w_i(x, t)$ are the axial displacement and flexural displacement of the midplane of beam i . The normal resultant force N_i , bending moment M_i and transverse

shear force Q_i are related to the normal strain $\varepsilon_{0,i} = \frac{\partial u_i}{\partial x}$ and flexural curvature $k_{x,i} = \frac{\partial^2 w}{\partial x^2}$

by

$$N_i = A_{11}(i)\varepsilon_{0,i} - B_{11}(i)k_{x,i} \quad (8.4)$$

$$M_i = B_{11}(i)\varepsilon_{0,i} - D_{11}(i)k_{x,i} \quad (8.5)$$

$$Q_i = \frac{\partial M_i}{\partial x} = B_{11}(i)\frac{\partial \varepsilon_{0,i}}{\partial x} - D_{11}(i)\frac{\partial k_{x,i}}{\partial x} \quad (8.6)$$

and

$$(A_{11}(i), B_{11}(i), D_{11}(i)) = \int_{-\frac{H_i}{2}}^{\frac{H_i}{2}} \frac{E(z)}{1-\mu_i^2} (1, z_i, z_i^2) dz_i, \text{ where } i=1-4. \quad (8.7)$$

where μ_i is the Poisson ratio for beam i , all are assumed to be constant since Poisson's ratio commonly varies slightly (Atmane *et al.*, 2011).

The governing equations for beam i , with axial inertia term being neglected, can be derived as follows:

$$A_{11}(i)\frac{\partial^2 u_i}{\partial x^2} - B_{11}(i)\frac{\partial^3 w_i}{\partial x^3} = 0 \quad (8.8)$$

$$\left(D_{11}(i) - \frac{B_{11}^2(i)}{A_{11}(i)}\right)\frac{\partial^4 w_i}{\partial x^4} + I_{1,i}\frac{\partial^2 w_i}{\partial t^2} = 0 \quad (8.9)$$

$$\text{where } I_{1,i} = \int_{-\frac{H_i}{2}}^{\frac{H_i}{2}} \rho(z) dz_i. \quad (8.10)$$

For free harmonic vibration, the deformation mode shape can be assumed as

$$u_i(x, t) = U_i(x)e^{i\omega t} \quad (8.11)$$

$$w_i(x, t) = W_i(x)e^{i\omega t} \quad (8.12)$$

where ω is the natural frequency of the delaminated FGM beam. Substituting Equations (8.11) and (8.12) into Equations (8.8) and (8.9), one can have the solution

$$W_i(x) = C_i \cos\left(\alpha_i \frac{x}{L}\right) + S_i \sin\left(\alpha_i \frac{x}{L}\right) + CH_i \cosh\left(\beta_i \frac{x}{L}\right) + SH_i \sinh\left(\beta_i \frac{x}{L}\right) \quad (8.13)$$

$$U_i(x, t) = \lambda_i \frac{B_{11}(i)}{A_{11}(i)} [C_i \cos(\lambda x) - S_i \sin(\lambda x) + CH_i \cosh(\lambda x) + SH_i \sinh(\lambda x)] + g_i x + g_{i0} \quad (8.14)$$

$$\text{where } \lambda_i^4 = \omega^2 I_{1,i} / d_i \text{ and } d_i = D_{11}(i) - B_{11}^2(i) / A_{11}(i) \quad (8.15)$$

8.2.2 'Free Mode'

Governing equations 8 and 9 are applied to the four interconnected sub-beams, respectively (Figure 8.1(b)). The appropriate boundary conditions that can be applied at the supports, $x = x_1$ and $x = x_4$, are $W_i = 0$, $W_i' = 0$ and $U_i = 0$ if the end of the beam is clamped, $W_i = 0$, $W_i'' = 0$ and $U_i = 0$ if hinged, $W_i'' = 0$, $W_i''' = 0$ and $U_i' = 0$ if free, where $i=1, 4$.

The continuity conditions for flexural displacement, flexural slope, axial force and shear force at are:

$$W_1 = W_2 = W_3 \quad (8.16)$$

$$W_1' = W_2' = W_3' \quad (8.17)$$

$$N_1 = N_2 + N_3 \quad (8.18)$$

$$Q_1 = Q_2 + Q_3 \quad (8.19)$$

The continuity conditions for axial displacement (Luo and Hanagud, 2000), when using the neutral axis as the referencing axis, as is shown in Figure 8.1 (c), at $x = x_2$ are:

$$U_1 - \left(\frac{H_s}{2} + D_2\right) \frac{\partial W_1}{\partial x} = U_2 - D_2 \frac{\partial W_2}{\partial x} \quad (8.20)$$

$$U_1 + \left(\frac{H_s}{2} - D_3\right) \frac{\partial W_1}{\partial x} = U_3 - D_3 \frac{\partial W_3}{\partial x} \quad (8.21)$$

where D_i is distance between the neutral axis and the midplane of beam i .

The continuity condition for bending moment at $x = x_2$ is:

$$M'_1 = M'_2 + M'_3 - N_2 \times D_1^2 + N_3 \times D_1^3 \quad (8.22)$$

where D_1^2 and D_1^3 are the distances between the two neutral axes of beam 1 and beam 2, beam 1 and beam 3 respectively, and M'_i is the equivalent bending moment when using the neutral axis as the referencing axis, instead of the midplane, therefore $M'_i = M_i + N_i \cdot D_i$.

And similarly, one can derive the continuity conditions at $x = x_3$.

8.2.3 'Constrained Mode'

In 'constrained mode' model, beam 2 and 3 vibrates together, i.e. $w_2 = w_3 = w_{II}$. As is shown Figure 8.1 (d), when taken into the consideration of axial displacement compatibility (Mujumdar and Suryanarayan, 1988), the bottom of beam 2 and top of beam 3 have the same axial displacement, therefore

$$u_2 + \frac{H_2}{2} \frac{\partial w_{II}}{\partial x} = u_3 - \frac{H_3}{2} \frac{\partial w_{II}}{\partial x} \quad (8.23)$$

For the 'constrained mode' model, the governing equations for beam 1 and beam 4 are identical to Equations (8.8) and (8.9). For beam 2 and 3, i.e. section II, the governing equations are

$$A_{11}(2) \frac{\partial^2 u_2}{\partial x^2} + A_{11}(3) \frac{\partial^2 u_3}{\partial x^2} - (B_{11}(2) + B_{11}(3)) \frac{\partial^4 w_{II}}{\partial x^2} = 0 \quad (8.24)$$

$$\left(D_{11}(2) + D_{11}(3) - \frac{B_{11}^2(2)A_{11}(3) + B_{11}^2(3)A_{11}(2)}{A_{11}(2)A_{11}(3)} \right) \frac{\partial^4 w_{II}}{\partial x^4} + (I_{1,2} + I_{1,3})W_{II} = 0 \quad (8.25)$$

Substituting Equation (8.23) to Equations (8.24) and (8.25), therefore

$$(A_{11}(2) + A_{11}(3)) \frac{\partial^2 u_2}{\partial x^2} - (B_{11}(2) + B_{11}(3) - \frac{H_1}{2} A_{11}(3)) \frac{\partial^4 w_{II}}{\partial x^2} = 0 \quad (8.26)$$

The boundary conditions for the ‘constrained mode’ are identical to that of the ‘free mode’. The continuity conditions for flexural displacement and slope, axial displacement, shear force and bending moments at $x = x_2$ are:

$$W_I = W_{II} \quad (8.27)$$

$$W_I' = W_{II}' \quad (8.28)$$

$$U_I - \left(\frac{H_3}{2} + D_2\right) \frac{\partial W_I}{\partial x} = U_{II} - D_2 \frac{\partial W_{II}}{\partial x} \quad (8.29)$$

$$N_I = N_{II} = N_2 + N_3 \quad (8.30)$$

$$Q_I = Q_{II} = Q_2 + Q_3 \quad (8.31)$$

$$M_I = M_I' = M_{II} = M_2' + M_3' - N_2 \times D_1^2 + N_3 \times D_1^3 \quad (8.32)$$

And similarly, one can derive the continuity conditions at $x = x_3$. The total number of boundary and continuity conditions is 24 for ‘free mode’, 18 for ‘constrained mode’, which is equal to the total number of unknown coefficients. A non-trivial solution exists only when the determinant of the coefficient matrix vanishes. The natural frequencies can be obtained as eigenvalues.

Figure 8.1. (a) An exponential functionally graded beam with a single delamination; (b) Delaminated FGM beam modeled as four interconnected beams; (c) Axial displacement continuity condition; (d) Axial displacement compatibility of the delaminated layers.

8.3 Results and Discussions

8.3.1 Verification

Table 8.1 shows the nondimensional first and second natural frequency λ^2 of an isotropic beam (by denoting $E_2 / E_1 = 0.999$) with a single of central midplane delamination of various lengths, compared with the analytical results of Wang *et al.* (1982) and FEM results of Lee (2000). As is shown in Table 8.1, the results of the current study agree well with previous published results.

Table 8.2 shows the comparison of the current results against the experimental results of Mujumdar and Suryanarayan (1988) and the mixed-FEM results of Ramtekkar (2009). The first and second natural frequency ω_1 and ω_2 (Hz) of an isotropic cantilever beam with a single delamination ($H_2 / H_1 = 0.33$.) is tabulated here.

Table 8.3 and Table 8.4 show the first three normalized natural frequencies ($\Omega_n = \omega_n / \sqrt{d_{10}}$) of FGM beams, with clamped-free and hinged-hinged boundary conditions, where d_{10} is the value of d_1 of an isotropic beam ($E_2 / E_1 = 1.0$) (Atmane *et al.*, 2011). This example was previously analyzed by Yang and Chen (2008),¹⁰ Yang et al. (2008) and Atmane *et al.* (2011). The natural frequencies obtained in the present study are in good agreement with previous results.

8.3.2 FGM beams with single delamination

Firstly, the first normalized natural frequency ($\Omega = \omega / \sqrt{d_{10}}$) FGM beam with a single delamination is studied. The top layer is made Zirconia ($E_1 = 200\text{GPa}$) and the bottom layer is made of aluminum ($E_2 = 70\text{GPa}$). The FGM beam then suffers a delamination with varying length and locations, with two different boundary condition

considered. When delamination length a / L varies, the longitudinal location stays the same $d / L = 0$; when the longitudinal location d / L varies, the delamination length a / L stays the same, $a / L = 0.1$. The results are shown in Table 8.5.

It is assumed in the following study that the FGM beam's thickness $H_1 = 0.1m$; the slenderness ratio $L / h = 10$; the Young's modulus ratio E_2 / E_1 varies from 0.01 to 1.0, where E_1 and E_2 denote the Young's modulus at the top and bottom surfaces of the beam respectively. Note that $E_2 / E_1 = 1.0$ is a special case where the beam is isotropic homogeneous. The top surface of the beam is 100% aluminum with the material parameters: $E_1 = 70\text{GPa}$, $\rho_1 = 2780\text{kg/m}^3$, $\mu_1 = 0.33$, as is studied by Atmane *et al.* (2011).

Figure 8.2 shows the 'free mode' normalized natural frequencies versus the Young's modulus ratio E_2 / E_1 with various central midplane ($d_a = 0.0$, $H_2 = 0.5$) delamination lengths $a / L = 0.2, 0.3, 0.4, 0.5$, and 0.6 , with a clamped-clamped boundary condition. It can be observed from Figure 8.2 that the natural frequencies of FGM beams increase as the Young's modulus ratio increases, till the beam becomes isotropic. Such increase is smaller when the beam suffers a longer delamination.

Also, the natural frequencies decrease as the delamination grows longer. The differences of natural frequencies between delamination length $a / L = 0.2$ and 0.3 ($(\Omega_{a/L=0.2} - \Omega_{a/L=0.3}) / \Omega_{a/L=0.2}$), 0.3 and 0.4 , 0.4 and 0.5 as well as 0.5 and 0.6 , becomes bigger with a bigger Young's modulus ratio. It indicates that the effect of delamination length on reducing natural frequency is aggravated when the material property (E_2/E_1) changes less dramatically from the bottom to the top. Figure 8.3 shows the 'free mode' first normalized natural frequencies versus the Young's modulus ratio E_2/E_1 with various

delamination lengths, with a clamped-free boundary condition, and similar results can be observed.

Table 8.1.

First and second nondimensional frequencies (λ^2) of a clamped-clamped isotropic ($E_2 / E_1 = 0.999$) beam with a central midplane delamination.

Delamination Length a / L	First Mode Present Free & Constrained	Analytical Wang et al. (1982)	FEM Lee (2000)	Second Mode Present Free & Constrained	Analytical Wang et al. (1982)	FEM Lee (2000)
0.00	22.37	22.39	22.36	61.67	61.67	61.61
0.10	22.37	22.37	22.36	60.81	60.76	60.74
0.20	22.36	22.35	22.35	56.00	55.97	55.95
0.30	22.24	22.23	22.23	49.00	49.00	48.97
0.40	21.83	21.83	21.82	43.89	43.87	43.86
0.50	20.89	20.88	20.88	41.52	41.45	41.50
0.60	19.30	19.29	19.28	41.04	40.93	41.01
0.70	17.23	17.23	17.22	40.82	40.72	40.80
0.80	15.05	15.05	15.05	39.07	39.01	39.04
0.90	13.00	13.00	12.99	35.39	35.38	35.38

Table 8.2.

First and second natural frequency ω_1 , ω_2 (Hz) of cantilever isotropic ($E_2 / E_1 = 0.999$) beam with a single delamination, $H_2 / H_1 = 0.33$.

Specimen span (mm)	Dimension		First mode ω_1			Second Mode ω_2			
	$L / 2$	d_a	a	Experiment (Mujumdar and Suryanarayan, 1988)	Ramtekkar (2009) Unconstrained-interface model	Present Free mode	Experiment (Mujumdar and Suryanarayan, 1988)	Ramtekkar (2009) Unconstrained-interface model	Present Free mode
240.0	153.5	96.0	31.6		31.11	32.25	172.1	156.24	159.93
250.0	104.0	148.0	31.7		31.51	31.88	190.5	197.23	198.81
175.0	133.0	88.5	56.9		56.35	57.89	339.3	244.42	248.83
200.0	122.0	106.0	46.6		46.34	47.26	291.0	258.95	260.71
155.0	122.0	61.0	69.1		68.82	69.82	363.7	249.20	281.82

Table 8.3.

First three normalized natural frequencies $\Omega_n = \omega_n / \sqrt{d_{10}}$ of undelaminated exponential FGM beams, with clamped-free boundary condition.

E_2 / E_1	Mode Number	Present Free	Yang and Chen (2008)	
			Yang et al. (2008)	Atmane et al. (2011)
0.2	1	3.30	3.30	3.303
	2	20.70	20.70	20.702
	3	57.97	57.97	57.965
1.0	1	3.52	3.52	3.516
	2	22.03	22.03	22.034
	3	61.70	61.70	61.697

Table 8.4.

First three normalized natural frequencies $\Omega_n = \omega_n / \sqrt{d_{10}}$ of undelaminated exponential FGM beams, with hinged-hinged boundary condition.

E_2 / E_1	Mode Number	Present Free	Yang and Chen (2008)	
			Yang et al. (2008)	Atmane et al. (2011)
0.2	1	9.27	9.27	9.272
	2	37.09	37.09	37.090
	3	83.28	83.28	83.453
1.0	1	9.87	9.87	9.869
	2	39.48	39.48	39.478
	3	88.83	88.83	88.826

Table 8.5.

First normalized natural frequencies $\Omega_n = \omega_n / \sqrt{d_{10}}$ of delaminated FGM beams (Zirconia-Aluminum), with clamped-clamped and clamped-free boundary conditions.

a/L	C-C		d/L	C-C		d/L	C-F	
0.050	21.775	3.422	-0.40	21.561	-0.40	3.421		
0.100	21.774	3.420	-0.35	21.577	-0.30	3.421		
0.150	21.771	3.414	-0.30	21.601	-0.20	3.420		
0.200	21.758	3.403	-0.25	21.634	-0.10	3.420		
0.250	21.724	3.385	-0.20	21.672	0.00	3.419		
0.300	21.650	3.360	-0.15	21.711	0.10	3.419		
0.350	21.510	3.325	-0.10	21.744	0.20	3.419		
0.400	21.275	3.280	-0.05	21.766	0.30	3.419		
0.450	20.913	3.226	-0.00	21.774	0.40	3.418		
0.500	20.400	3.163

Figure 8.2. 'Free mode' normalized natural frequencies versus the Young's modulus ratio with various delamination lengths, $d_a=0.0$ and $H_2=0.5$, with a clamped-clamped boundary condition.

The 'constrained mode' natural frequencies of the same beam in Figure 8.2 are shown in Figure 8.4. Similar results can be obtained on how the Young's modulus ratio

aggravates the effect of delamination. When compare the results between Figure 8.2 and Figure 8.4, the ‘constrained mode’ natural frequency is bigger than the ‘free mode’ and such difference becomes smaller with a decreasing Young’s modulus ratio.

Figure 8.3. ‘Free mode’ normalized natural frequencies versus the Young’s modulus ratio with various delamination lengths, $d_a=0.0$ and $H_2= 0.5$, with a clamped-free boundary condition.

Figure 8.4. ‘Constrained mode’ normalized natural frequencies versus the Young’s modulus ratio with various delamination lengths, $d_a=0.0$ and $H_2=0.5$, with a clamped-clamped boundary condition.

Table 8.6.

First three normalized natural frequencies $\Omega_n = \omega_n / \sqrt{d_{10}}$ of delaminated exponential FGM beams with clamped-clamped boundary condition.

E_2 / E_1	$a/L=0.0$			$a/L=0.2$			$a/L=0.3$			$a/L=0.4$		
	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
0.20	20.939	57.717	113.150	20.924	53.007	111.490	20.828	46.926	103.490	20.494	42.242	8.734
0.40	21.889	60.337	118.280	21.872	55.000	116.390	21.762	48.315	107.270	21.382	43.344	69.532
0.60	22.219	61.245	120.060	22.201	55.680	118.080	22.086	48.785	108.560	21.688	43.718	69.795
0.80	22.342	61.585	120.740	22.324	55.934	118.720	22.207	48.959	109.040	21.803	43.858	69.892
1.00	22.373	61.670	120.900	22.355	55.998	118.870	22.238	49.003	109.160	21.832	43.892	69.917

Table 8.6 shows the first three ‘free mode’ normalized natural frequencies of clamped-clamped delaminated beams with various Young’s modulus ratio E_2/E_1 and delamination lengths. It can be observed that the effect of delamination on the second and third natural frequencies is similar to that of the first mode, aggravated by an increasing Young’s modulus ratio.

Figure 8.5 shows the ‘free mode’ normalized first natural frequencies versus the Young’s modulus ratio E_2/E_1 with various midplane delamination ($H_2 = 0.5$ and $a / L=0.2$) longitudinal locations d_a , with a clamped-clamped boundary condition. The effect of delamination longitudinal location on reducing the natural frequency is aggravated when the Young’s modulus ratio increases. The ‘free mode’ normalized first natural frequencies versus the Young’s modulus ratio E_2/E_1 with various delamination locations d_a , with a clamped-free boundary condition are shown in Table 8.7.

Figure 8.5. ‘Free mode’ normalized natural frequencies versus the Young’s modulus ratio with various delamination locations, $a / L=0.2$ and $H_2= 0.5$, with a clamped-clamped boundary

Table 8.7.

First normalized natural frequencies $\Omega_n = \omega_n / \sqrt{d_{10}}$ of delaminated exponential FGM beams with various delamination location d_a , with a clamped-free boundary condition. d_a is negative when the delamination is closer to the clamped end of the beam.

E_2 / E_1	$d_a/L = -0.3$	$d_a/L = -0.1$	$d_a/L = 0.1$	$d_a/L = 0.3$
0.01	2.334	2.335	2.337	2.340
0.10	3.091	3.094	3.099	3.106
0.20	3.282	3.285	3.291	3.299
0.30	3.370	3.374	3.380	3.389
0.40	3.420	3.423	3.430	3.439
0.50	3.449	3.453	3.460	3.469
0.60	3.468	3.472	3.479	3.488
0.70	3.479	3.483	3.491	3.500
0.80	3.486	3.490	3.497	3.507
0.90	3.489	3.493	3.501	3.510
1.00	3.490	3.494	3.502	3.511

8.4 Concluding Remarks

In the present study, an analytical solution is developed to study the free vibration of exponential functionally graded beams with a single delamination. Kirchhoff-Love hypothesis, the ‘free mode’ and ‘constrained mode’ assumption in delamination vibration are adopted. This is the first study on the influences of delamination (its length and location) on the natural frequency of exponentially functionally graded beams. Based on the theoretical investigations the following conclusions may be drawn.

1. The ‘free mode’ natural frequencies increase as the Young’s modulus ratio increases, but such increase is smaller when the beam suffers a longer delamination
2. The effect of delamination length on reducing natural frequency is aggravated when the material property (E_2/E_1) changes less dramatically from the bottom to the top.
3. The difference of natural frequency between ‘free mode’ and ‘constrained mode’ becomes smaller with a decreasing Young’s modulus ratio.

4. The effect of delamination longitudinal location on reducing the natural frequency is aggravated when the Young's modulus ratio increases, till the beam becomes isotropic.

Chapter 9

Conclusions and Recommendations for Future Work

9.1 Conclusions

Studies are conducted on the vibration and buckling of delaminated beams with step axial force and cross-section, coupled bending-torsion vibration of multiple delaminated beams subjected axial load and static end moments, vibration of multiple delaminated bimaterial beams fully or partially supported by elastic foundation, vibration of multiple delaminated beams with edge cracks, vibration of rotating Timoshenko beams with multiple delaminations and vibration of exponential functionally graded beams with a single delamination.

As one of the most commonly used structures, beams are often subjected to different loading, constraint and other conditions. Therefore, it is critical to understand how the essential characteristics of a delamination (its length and locations) affect the natural frequencies, mode shapes, critical buckling load and critical moment for lateral instability under these conditions. The following conclusions are presented by focusing how the effects of each characteristic change under various conditions.

9.1.1 Effect of delamination length on the natural frequency and mode shape of beams under various conditions

In general, the presence of delamination would decrease the natural frequencies of beams and the longer the delamination, the bigger the reduction. However, the degree of such effect may vary greatly when beams are subjected to different conditions.

1. *When beams are subjected to stepped axial load:* such effect becomes more prominent when beams are subjected to an increasing axial compressive load. This effect is also more prominent a) when a smaller portion of the beam is under the effect of axial compressive loading for an stepped axial load occurring outside the delaminated area; b) when the length of effective delamination ED becomes shorter for a stepped axial load occurring inside the delaminated area. Furthermore, the opening of first mode shape, under 'free mode' assumption, is bigger when a smaller portion of the beam is under the effect of axial compressive load.
2. *When beams are subjected to static end moments:* the effect of delamination length on reducing both mode 1 and 2 natural frequencies becomes more prominent when the static end moments increase. Increase in axial tensile load or decrease in axial compressive load can effectively weaken such influence from the static end moments.
3. *When beams are supported by elastic foundation:* similarly, the reduction of natural frequencies due to a longer delamination can be further enhanced by increasing the foundation stiffness. The effect of elastic foundation's stiffness on

natural frequency is bigger when delamination length grows longer. The opening of the first mode shape is bigger with an increasing foundation stiffness.

4. *When beams suffer from two common failure mode-crack and delamination:* the effect of delamination on reducing the natural frequency is aggravated by an increasing depth of the edge crack. Such effect becomes bigger then smaller as the crack moves towards the boundary of the delaminated region (when crack occurs outside the delaminated area). When the edge crack occurs inside the delaminated area, the effect of crack on aggravating the influence of delamination is bigger when the beam suffers a longer delamination. Such effect peaks when the crack occurs at the middle of the delaminated region.
5. *When beams are rotating while using Timoshenko beam theory:* the effect of delamination on natural frequency is smaller with a bigger Timoshenko effect. The effect of delamination on natural frequency is smaller when the beam has a higher rotating speed. Timoshenko effect is more prominent on mode 2 frequency than mode 1. The bigger the Timoshenko effect, the bigger the amplitude of the mode shape.
6. *When functionally grade beams are considered:* The ‘free mode’ natural frequencies increase as the Young’s modulus ratio increases, but such increase is smaller when the beam suffers a longer delamination. The effect of delamination length on reducing natural frequency is aggravated when the material property (E_2/E_1) changes less dramatically from the bottom to the top.

9.1.2 Effect of delamination's thickness-wise and longitudinal location on the natural frequency and mode shape

1. *When beams are subjected to stepped axial load:* When the step change of axial force occurs inside the delaminated area, the effect of delamination length on buckling load is more significant with a long effective delamination length ED . *When beams are subjected to static end moments:* The effect of delamination length and locations on buckling load and critical moments for lateral instability is aggravated by the presence of static end moments.
2. *When beams are supported by elastic foundation:* The effect of delamination's thickness-wise location on natural frequency is aggravated while the effect of length-wise location is mitigated by an increasing foundation stiffness. The effect of elastic foundation's stiffness on natural frequency is bigger when delamination moves away from the mid-plane.
3. *When beams suffer from two common failure mode-crack and delamination:* when the edge crack occurs inside the delaminated area, the effect of crack depth on natural frequency is smaller when the delamination moves towards the surface. Also the effect of the delamination thickness-wise location on natural frequency is bigger when the crack is deeper.
4. *When functionally grade beams are considered:* the effect of delamination longitudinal location on reducing the natural frequency is aggravated when the Young's modulus ratio increases, till the beam becomes isotropic.

9.1.3 Difference of natural frequency between ‘free mode’ and ‘constrained mode’

1. *When beams are subjected to stepped axial load:* when the step change of axial force occurs inside the delaminated area, the difference of natural frequencies and buckling loads between ‘free mode’ and ‘constrained mode’ is bigger with a longer effective delamination length ED .
2. *When beams are supported by elastic foundation:* the difference of natural frequencies between ‘free mode’ and ‘constrained mode’ becomes more prominent with a longer delamination or a bigger foundation stiffness
3. *When beams suffer from two common failure mode-crack and delamination:* the difference of natural frequencies between ‘free mode’ and ‘constrained mode’ increases then decreases as the crack moves from one side of the delaminated region to the other side, peaking at the middle.
4. *When beams are rotating while using Timoshenko beam theory:* The difference of frequency between ‘free mode’ and ‘constrained mode’ increases when the rotating speed becomes higher. The difference of frequency between ‘free mode’ and ‘constrained mode’ increases at first then decreases as the Timoshenko effect becomes bigger.
5. *When functionally grade beams are considered:* The difference of natural frequency between ‘free mode’ and ‘constrained mode’ becomes smaller with a decreasing Young’s modulus ratio.

9.2 Recommendations for Future Work

1. Extend the current work to include vibration of rotating composite beams with delaminations. For composite rotating beams, there is no available closed-form analytical solution to the corresponding governing equations. Therefore Finite Element Method is adopted to solve the delamination problem.
2. Extend the current work to include the coupled vibration of composite beams with delaminations. Similar method adopted to solve the homogeneous beam can be used to investigate this problem.
3. Extend the current work to include both crack and delamination in the study of vibration of functionally graded materials. Both exponential function and power series function can be adopted to dictate the corresponding material gradient.
4. Devise and carry out experimental investigations so as to corroborate the previous theoretical investigations. Since there have been many experimental investigations on the vibration of delaminated beams, the author plan to focus on how to incorporate the effects of axial load, end moment, rotation as well as functionally graded material to help to see how well the theoretical model approach the real application.

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