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Abstract

This thesis deals with three issues of fixed income derivative pricing. Chapter 2 deals with bond pricing in mean-reverting CIR model, which is linked to quadratic functionals of Brownian motion. By the bivariate Laplace transform of quadratic functionals of the form

$$\left(\int_0^T X_t dB_t, \int_0^T X_t^2 dt\right)$$

where $(X_t)_{t\in\mathbb{R}_+}$ is an Ornstein-Uhlenbeck process driven by a standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, new bond pricing formulas in the squared Ornstein-Uhlenbeck model are obtained as particular cases. Our method of computing the Laplace transform combines PDE arguments with Carleman-Fredholm determinant of associated Volterra operators that are computed by Fredholm expansions.

In Chapter 3, we study bond pricing and spot forward rate models under the normal martingale setting, which has the chaotic representation property and satisfies the specified structure equation. We first extend the Wiener chaos-based framework to normal martingale chaos-based framework, then we derive the variance representation of price density V_t, which depends on the square-integrable random variable X_∞. We obtain the spot forward rate chaos models by the chaos expansion of X_∞. Next we parameterize chaos coefficients in the spot forward rate models and do first chaos and second chaos model calibration.

Chapter 4 deals with synthetic Collateralized Debt Obligation (CDO) pricing, which amounts to the computation of the expected tranche losses. We compute the expected γ-th tranche loss $E\left[J_t^{(\gamma)}\right]$ of CDOs with random recovery rates in Gram-Charlier expansion way and get the loss density function. In addition, we compare the density functions of the loss J_t approximated by Gram-Charlier expansion with Monte Carlo simulation.

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Chapter 1

Introduction

1.1 Quadratic functionals of the Ornstein-Uhlenbeck process

Interest rate models have been playing an important role in investment and risk management, especially one-factor models, which are extensively used in the pricing of interest rate derivatives. One-factor models of interest rate r_t, are represented by the following stochastic differential equation

$$dr_t = \mu(t, r_t)\, dt + \sigma(t, r_t)\, dB_t,$$

where $\mu(t, r_t)$ and $\sigma(t, r_t)$ are the drift and the diffusion term of the interest rate process, respectively, and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$ with risk-neutral measure P. Here are some examples, the Vasicek (1977) model

$$dr_t = \beta(\alpha - r_t)\, dt + \sigma\, dB_t,$$

where α is the long term mean level, β is the speed of reversion and σ is the instantaneous volatility, the Courtadon (1982) model

$$dr_t = \beta(\alpha - r_t)\, dt + \sigma r_t\, dB_t,$$

where β, α and σ are nonnegative, and the Hull-White model

$$dr_t = (\theta(t) - \alpha(t)r_t)\,dt + \sigma(t)\,dB_t,$$

where $\theta(t)$, $\alpha(t)$ and $\sigma(t)$ are nonnegative deterministic functions.

Multi-factor models then are developed to improve the fit for the short rate. In multi-factor model, see Duffie (2010), given the function R, the short rate r_t is of the form $r_t = R(X_t, t)$, where the process $(X_t)_{\mathbb{R}_+}$ is the solution to the stochastic differential equation

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dB_t.$$

Among various stochastic interest rate models, the Cox-Ingersoll-Ross (CIR) model, which was introduced by Cox et al. (1985), is one of the most important models of short rate with the dynamics

$$dr_t = (\gamma - 2br_t)\,dt + 2\sigma\sqrt{r_t}\,dB_t. \tag{1.1.1}$$

Coefficients γ, b and σ are positive and meet the condition $\gamma \geq 2\sigma^2$.

Although there are many extensions of the CIR model in the literature, it is still popular because of its tractability, strictly positiveness and explicit solutions for various interest rate derivatives, such as the time t price of a T-maturity default-free zero-coupon bond

$$\begin{aligned} P(t,T) =& F(t,x)\,|_{x=r_t} = E\left[e^{-\int_t^T r_s\,ds} \mid r_t\right] \\ =& \left(\frac{he^{(b+h)(T-t)}}{h+(b+h)(e^{2h(T-t)}-1)/2}\right)^{\frac{\gamma}{2\sigma^2}} \exp\left(-\frac{r_t(e^{2h(T-t)}-1)}{2h+(b+h)(e^{2h(T-t)}-1)}\right), \end{aligned}$$

where $h = \sqrt{b^2 + 2\sigma^2}$, cf. Cox et al. (1985).

Grasselli and Hurd (2005) point out the link between CIR model and squared Ornstein-Uhlenbeck process. When $\frac{\gamma}{\sigma^2} = K \in N_+ \backslash \{0,1\}$, r_t in (1.1.1) can be written as the squared Ornstein-Uhlenbeck process $r_t = X_t^T X_t$, where $(X_t)_{t\in\mathbb{R}_+}$ is K-dimensional Ornstein-Uhlenbeck process

$$X_t = xe^{-bt} + \sigma \int_0^t e^{-b(t-s)}\, dB_s,$$

which is the solution of the equation

$$dX_t = -bX_t\, dt + \sigma\, dB_t, \qquad X_0 = x.$$

Here we are interested in one-dimensional squared Ornstein-Uhlenbeck process X_t^2, which is linked to the quadratic functionals of Brownian motion $(B_t)_{t\in\mathbb{R}_+}$. Therefore taking interest rate $r_t = X_t^2$, the price of a non-defaultable zero-coupon bond with maturity T can be written down using functional determinant techniques generally applied to quadratic functionals of Brownian motion. The Laplace transform of quadratic functional

$$F = I_1(\psi) + \frac{1}{2} I_2(\varphi) = \int_0^T \psi(t)\, dB_t + \int_0^T \int_0^t \phi(s,t)\, dB_s dB_t,$$

where deterministic function $\psi(t) \in L^2([0,T])$ and symmetric function $\varphi(s,t) \in L^2([0,T]^2)$, has been computed on abstract Wiener spaces in Chiang et al. (1986), Chiang et al. (1994) and Grasselli and Hurd (2005) Proposition 4.1, as

$$E[e^{-F}] = (\det{}_2(I + A^\varphi))^{-1/2} \exp\left(\frac{1}{2}\int_0^T \psi(s)(I + A^\varphi)^{-1}\psi(s)\, ds\right). \tag{1.1.2}$$

This approach complements the usual PDE (Ricatti) approach for the determination of $F(t,x)$, where twice continuously differentiable function $F(t,x)$ is the solution to the following partial differential equation

$$xF(t,x) = \frac{\partial F}{\partial t}(t,x) + (\gamma - 2bx)\frac{\partial F}{\partial x}(t,x) + 2\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t,x)$$

with boundary condition $F(t,x) = 1$, and as noted in Grasselli and Hurd (2005) it deserves further study.

Contribution of the thesis

We derive the explicit formulas for Laplace transforms of various Ornstein-Uhlenbeck functionals, using both functional determinant and PDE methods, which allow us to provide new closed form expressions for the functional determinants of Volterra operators with kernel

$$\varphi(s,t) = \alpha e^{-b|s-t|} + \beta e^{-b(2T-s-t)}, \quad s,t \in [0,T]. \tag{1.1.3}$$

In Section 2.4 we compute the determinant of the Volterra operator of the form (1.1.3) in particular, with parameters $\alpha, \beta \in \mathbb{R}$, and $b > \max(0, -2\alpha)$. The determinant of $I + A^\varphi$ is given by

$$\det(I + A^\varphi) = e^{-bT}\left(\cosh(T\sqrt{b^2+2\alpha b}) + (b+\alpha+\beta)T\mathrm{sinhc}(T\sqrt{b^2+2\alpha b})\right),$$

where $\mathrm{sinhc} x = (\sinh x)/x$, by the comparison of the PDE solution with (1.1.2), which includes the computation of trace terms appearing in the exponential component of (1.1.2). This builds the main results of Chapter 2. The Fredholm expansions (2.4.1) and the finite dimensional determinants needed for Fredholm expansions are also evaluated.

In general the spectrum of Volterra operator is unknown, nevertheless the determinant $\det(I + A^\varphi)$ can be computed by the Fredholm expansion as

$$\det(I + A^\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{[0,T]^n} \det(\varphi(t_p, t_g))_{p,q=1}^n \, \mathrm{d}t_1 \cdots \mathrm{d}t_n,$$

cf. Theorem 3.10 of Simon (2005). As a result, we have

$$\det(I + A^\varphi) = 1 + Te^{-2bT} \sum_{n=1}^{\infty} \frac{(2\alpha b T^2)^{n-1}}{(2n-1)!} (\alpha \ {}_1F_1(n+1; 2n, 2bT) + \beta \ {}_1F_1(n; 2n, 2bT)),$$

where ${}_1F_1(a; b, z)$ is the hypergeometric function

$${}_1F_1(a; b, z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!}$$

and $(a)_n = a(a+1)\cdots(a+n-1)$ is the rising factorial. This provides in particular asymptotic expansions for PDE solutions, and therefore it can be used for approximations.

It has been noticed that the eigenvalues of the operator A^φ usually are unknown and cannot be used to compute the determinant, except in some very special cases such as the limiting case where b tends to 0 in (1.1.3) and $\alpha = -\beta = \sigma^2/b$. The eigenvalues of A^φ can then be computed explicitly to be

$$\lambda_k = 8\sigma^2 T^2 \pi^{-2}(2k+1)^{-2}, \quad k \geq 1,$$

which yields that $\det(I + A^\varphi) = \cosh(\sigma T\sqrt{2})$, cf. below (2.3.9). The second special case of the operator (1.1.3), is the kernel of the form

$$\varphi(s,t) = \alpha e^{-b|s-t|}, \qquad s,t \in [0,T], \quad \alpha > 0,$$

whose spectrum has been studied associated with stochastic point process in Macchi (1975), there it represents the stationary covariance function of zero-mean Gaussian light field, and α is the average light intensity in physics. The spectrum is also studied in determinantal random point fields by Soshnikov (2000). In addition, this special kernel has its application in signal processing in the presence of noise, such as the covariance function of Gaussian noise in Slepian (1954). We recover its spectrum in a much simpler way as compared to the method used by Macchi (1975) (p.118) and Youla (1957) (p.192), where the eigenvalues λ of the kernel are the solutions to the following equations

$$\begin{cases} \tan \dfrac{\eta T}{2} = -\dfrac{\eta}{b} \quad \text{or} \quad \cot \dfrac{\eta T}{2} = \dfrac{\eta}{b}, \\ \lambda = \dfrac{2b\alpha}{b^2 + \eta^2}. \end{cases}$$

We also compute the joint moment generating functions of quadratic functionals such as

$$\int_0^T X_s^2\, ds, \qquad \int_0^T X_s\, dB_s, \qquad \int_0^T X_s\, dX_s,$$

using both the Fredholm determinant expansions and PDE expressions. This allows us in particular to get the Laplace transform of $\int_0^T r_t\,dt$, where $r_t = X_t^2$. Consequently, the price of bond with maturity T at time t is calculated by using formula (1.1.2) as below

$$\begin{aligned} P(t,T) =& E\left[e^{-\int_t^T X_s^2\,ds}\middle| X_t\right] \\ =& \left(\frac{he^{(b+h)(T-t)}}{h+(b+h)(e^{2h(T-t)}-1)/2}\right)^{\frac{1}{2}} \exp\left(-\frac{X_t^2(e^{2h(T-t)}-1)}{2h+(b+h)(e^{2h(T-t)}-1)}\right), \end{aligned}$$

where $h = \sqrt{b^2+2\sigma^2}$.

1.2 Normal martingale chaos-based models of spot forward rates

The Chaotic Representation Property (CRP) studied in Émery (1989) and Émery (1991), of normal martingale M_t states that every square-integrable random variable X in the probability space with filtration generated by normal martingale M_t, can be expressed in the form of

$$X = \sum_{n=0}^{\infty} I_n(f_n)$$

where the multiple stochastic integrals are defined by

$$I_n(f_n) = n! \int_0^{\infty}\int_0^{t_n}\cdots\int_0^{t_2} f_n(t_1,\cdots,t_n)\,dM_{t_1}\cdots dM_{t_n},$$

and deterministic functions $f_n \in L^2(\mathbb{R}_+{}^n)$. It has been noted that CRP is stronger than the predictable representation property (PRP), this means that for any square-integrable random variable X, if it is chaos-decomposable, then X can be rewritten as $E[X] + \int_0^t \phi(s)\,dM_s$, where $\phi(t)$ is a predictive process. Given quadratic variation $[M,M]_t = M_t^2 - 2\int_0^t M_{s^-}\,dM_s$, consequently, if M_t has the CRP and M_t^2 is square-integrable, then the normal martingale M_t must satisfy the equation

$$[M,M]_t = t + \int_0^t \phi(s)\,dM_s, \tag{1.2.1}$$

which is called structure equation, this result shows us that the structure equation is a necessary condition for normal martingale M_t having the CRP.

The equation (1.2.1) has been studied by several researchers, for example Émery (1989), Azéma and Yor (1989), and Dermoune (1995), and some results have been established when the predictable process $\phi(t)$ is linear in the form of

$$\phi(t) = \alpha(t) + \beta(t)\, M_{t^-}, \tag{1.2.2}$$

- $\alpha(t) = \beta(t) = 0$, M_t is Brownian motion,
- $\alpha(t) = \alpha, \quad \beta(t) = 0$, M_t is Poisson martingale,
- $\alpha(t) = 0, \quad \beta(t) = -1$, M_t is Azéma martingale.

Émery (1989) has shown that if $\alpha(t) = 0$ and $\beta(t) = \beta \in [-2, 0]$, M_t has the CRP. Russo and Vallois (1998) have established that the CRP holds for normal martingale M_t with $\alpha(t) = 0$ and $\beta(t) \in [-2, 0]$.

The well known Wiener chaos expansion (eq. (1.2.2) with $\alpha(t) = \beta(t) = 0$) or Polynomial chaos plays an important role in stochastic analysis. Wiener (1938) first introduced the polynomial chaos. Itô (1951) did some modification of Wiener's and he pointed out the orthogonality of multiple Wiener integrals of different degrees and established a relation between the integrals and Hermite polynomials, which led to the representation of square-integrable random variable X in terms of an infinite orthogonal sum of multiple Wiener integrals

$$X = \sum_{n=0}^{\infty} I_n(f_n),$$

where

$$I_n(f_n) = n! \int_0^{\infty} \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \cdots, t_n)\, dW_{t_1} \cdots dW_{t_n},$$

deterministic functions $f_n \in L^2({\mathbb{R}_+}^n)$ and $(W_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion.

It has been used in the Malliavin calculus, stochastic fluid dynamics, such as solving Stochastic Partial Differential Equations (SPDEs) numerically in Hou et al. (2006). Hou et al. (2006) have demonstrated the numerical methods for short time solutions based on the Wiener chaos expansion are more efficient and accurate than those based on the Monte Carlo simulations.

Hughston and Rafailidis (2005) has proposed a new framework for modelling the term structure of interest rates called 'a chaotic approach to interest rate modelling', where the Wiener chaos expansion has been applied to obtain the square-integrable random variable X_∞ and the variance representation of state price density

$$V_t = E\left[(X_\infty - E[X_\infty|\mathcal{F}_t])^2\,|\mathcal{F}_t\right].$$

It leads to the expression for the price of a T-maturity zero-coupon bond time t

$$P(t,T) = \frac{E\left[V_T|\mathcal{F}_t\right]}{V_t}.$$

Under the axiomatic framework of Hughston and Rafailidis (2005), Grasselli and Tsujimoto (2011) have implemented and calibrated chaotic interest rate models based on Wiener chaos expansion to market data. They chose the chaos coefficients of different orders from exponential-polynomial family of parametric forms

$$f(x) = \sum_{i=1}^{k} L_i(x)e^{-c_i x}, \qquad L_i(x) = \sum_{j=0}^{m} b_{ij}x^j,$$

and applied the Maximum Likelihood Estimation (MLE) method in calibration.

The Wiener chaos expansion technique is also used to formulate a systematic analysis of the structure and classification of interest rate models. Grasselli and Hurd (2005) have recasted the Cox-Ingersoll-Ross (CIR) model (1.1.1) of interest rate into the chaotic representation that introduced by Hughston and Rafailidis (2005). They started with the squared Gaussian representation of the CIR model, then proceeded to find a simple way to express the random variable X_∞ for positive supermartingale $V_t = E\left[(X_\infty - E[X_\infty|\mathcal{F}_t])^2 |\mathcal{F}_t\right]$. By applying techniques from the theory of infinite dimensional Gaussian integration, they have derived an explicit formula for the n-th term of the Wiener chaos expansion of the CIR model.

The movement paths of asset prices based on Brownian motions are continuous, but given the increasing availability of data, the jumps do exist in asset prices by the observation of the real market, the flaws of the continuous setting have become more and more obvious. Jumps then have been considered in many realistic asset pricing models. Among many jump processes, the Poisson process is the key in building many other jump processes, such as compensated Poisson process, point processes, Lévy process, all widely used in financial models. Models then are developed, see Kou (2002) and references therein, to capture two empirical phenomena, the return distribution is skewed and has two heavier tails than those of the normal distribution, and the volatility smile, which means that the implied volatility is a convex curve of the strike price.

Normally jump models are based on Poisson distribution which has fatter tail compared to Gaussian distribution, it can account for sudden changes in the risky asset price movements. It is well known that Merton (1976) added Poisson jumps to normal Brownian motion process, which is called jump-diffusion model, for a better fit to the real movement of stock prices having variations. It leads to implied volatility smile, and analytical solutions for call and put options, and interest rate derivatives. Kou (2002) has proposed a double exponential jump-diffusion model for option pricing. Kou (2002) assumes the logarithm of the asset price follows a Brownian motion plus a compound Poisson process with jump sizes double exponentially distributed. It also leads to implied volatility smile, and analytical solutions for call and put options, and interest rate derivatives. Because of the memoryless property of the double exponential distribution, it has the analytical tractability for the path-dependent options, such as Barrier option.

Contribution of the thesis

We extend the framework of Hughston and Rafailidis (2005) to the normal martingale setting, where the normal martingale $(M_t)_{t\in\mathbb{R}_+}$ has the chaos representation property satisfying the structure equation (1.2.1). The normal martingale setting extends the Brownian motion based work in Hughston and Rafailidis (2005) for the use of jump process, and in Chapter 3 we include the calibration of spot forward rate models to market data based on compensated Poisson process with intensity being one.

We recall the chaotic approach in Section 3.2 and obtain the term structure of spot forward rates after extending the framework to normal martingale setting. We prove that the square-integrable random variable X_∞ can be found in the extended framework, and also achieve the variance representation for the state price density V_t, which is given by $V_t = E\left[(X_\infty - E[X_\infty|\mathcal{F}_t])^2\,|\mathcal{F}_t\right]$.

Next we apply the normal martingale chaos expansion of random variable X_∞ to obtain chaos models of spot forward rates. Forward rate $f(t,T,S)$, $t \leq T \leq S$, is the rate agreed to be delivered over a future time interval $[T,S]$, which is decided at time t for a loan in a forward interest rate contract, given by

$$f(t,T,S) = \frac{\log(P(t,T)) - \log(P(t,S))}{S-T},$$

with the spot forward rate or the yield being

$$f(t,t,T) = -\frac{\log(P(t,T))}{T-t}.$$

As a result, the first chaos model of spot forward rates at time t for a loan on $[t,T]$ is described by

$$f(t,t,T) = \frac{1}{T-t}\log\frac{\int_t^\infty \gamma_s^2\,ds}{\int_T^\infty \gamma_s^2\,ds},$$

and second chaos model is described by

$$f(t,t,T) = \frac{1}{T-t}\log\frac{A_t + B_tR_t + C_t(R_t^2 - Q_t)}{A_T + B_TR_t + C_T(R_t^2 - Q_t)},$$

where $A_t = \int_t^\infty (\gamma_s^2 + \eta_s^2 Q_s)\, ds$, $B_t = 2\int_t^\infty \gamma_s \eta_s\, ds$, $C_t = \int_t^\infty \eta_s^2\, ds$, $R_t = \int_0^t \theta_{s_1}\, dM_{s_1}$, and $Q_t = \int_0^t \theta_{s_1}^2\, ds_1$, here γ_t, η_t, η_t are square-integrable deterministic functions.

In section 3.4.1 and 3.5.1, we study the calibration impact on interest rate models, or spot forward rate models based on the normal martingale chaos expansion, taking compensated Poisson process $M_t = N_t - t$ with intensity $\lambda = 1$ for illustration purpose. We parameterize chaos coefficients in the form of

$$F(x, z) = z_1 + (z_2 + z_3 x)e^{-z_4 x},$$

which is the well known Nelson-Siegel family $F(\cdot, z)$ (cf. Björk (2004), Filipović (1999)), has been extensively used in modeling rates by choosing the parameters (z_1, z_2, z_3, z_4) to fit in initial term structure, such as yield curve in Björk and Christensen (1999).

The method in Park et al. (2011) is used here to obtain the model parameters by optimizing the variance

$$L = \sum \left(\hat{f}(t, t, T) - f(t, t, T) \right)^2,$$

where $f(t, t, T)$ and $\hat{f}(t, t, T)$ are corresponding model value and market data (cf. Table A1) obtained from Bloomberg Terminal.

In addition, we simply show that the defaultable bond price can also be computed by chaos decomposition in Section 3.3. For illustration purpose, we just consider pricing the T-maturity defaultable bond in probability space, with filtration being generated by Brownian motion W_t. We define the expanded filtration $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\} : 0 \leq u \leq t)$, where τ is the default time. If the default intensity $(\lambda_t)_{t\in\mathbb{R}_+}$ is given, the survival probability of the bond up to time T given information up to time t is

$$P(\tau > T \mid \mathcal{F}_t) = \exp\left(- \int_0^t \lambda_s ds \right), \quad t \in [0, T],$$

and the bond price with constant recovery rate δ is given by

$$v(t,T) = \delta E_P\left[\frac{B_t}{B_T}\bigg|\mathcal{F}_t\right] + (1-\delta)E_P\left[\frac{B_t}{B_T}\exp\left(-\int_t^T \lambda_s\,ds\right)\bigg|\mathcal{F}_t\right],$$

which can be obtained through Wiener chaos expansion after finding the appropriate numéraire N_t and a change in probability measure. We also have the totally defaultable bond price to be

$$v(t,T) = E_{\hat{P}}\left[\frac{N_t}{N_T}\bigg|\mathcal{F}_t\right].$$

1.3 Pricing synthetic CDOs by Gram-Charlier expansions

In Chapter 3, we have discussed that the defaultable bond pricing is credit sensitive, when the issuer fails to pay at the promised time, it comes the default risk. To mitigate the default risk from situations of, such as the buying of bonds issued by the federal government or bonds issued by companies, who might go bankrupt, credit ratings for government and corporations were introduced. Chapter 4 presents further works on the issues of credit default, especially those related to Collateralized Debt Obligations (CDOs) credit ratings.

The first CDO was issued in 1987 by bankers at the now-defunct Drexel Burnham Lambert Inc., it is a type of structured asset-backed security, which transfers the credit risk of a pool of assets in the credit derivatives market. For example, banks use CDOs to repackage individual loans, such as mortgages and car loans, into a product that is then sold to investors. There are two types of CDOs, cash CDOs and synthetic CDOs. Cash CDOs have a reference portfolio made up of cash assets, such as corporate bonds or loans, while synthetic CDOs have a reference portfolio consisting of credit default swaps. A credit default swap offers protection against default of a certain underlying entity over a specified time horizon. For simplicity, we will work on the pricing of synthetic CDOs.

In the literature, there are mainly two approaches to price CDOs, one way is to use the default intensity based models, a large class of intensity models are studied, such as affine jump model in Duffie and Garleanu (2001) is described by

$$d\lambda_t = \kappa(u - \lambda_t)\,dt + \sigma\sqrt{\lambda_t}\,dW_t + dZ_t,$$

where $(Z_t)_{t\in\mathbb{R}_+}$ is a compound Poisson process independent of the Brownian motion $(W_t)_{t\in\mathbb{R}_+}$, and CIR model in Duffie et al. (2003), the default intensity $(\lambda_t)_{t\in\mathbb{R}_+}$ is described by

$$d\lambda_t = \kappa(u - \lambda_t)\,dt + \sigma\sqrt{\lambda_t}\,dW_t.$$

This methodology of pricing the spread of CDOs was introduced by Lando (1994), Jarrow and Turnbull (1995), Lando (1998), and Schönbucher (1998) for pricing and hedging derivative security involving credit risk. They assumed the arrival intensity of default as the first jump of a pure jump process with certain mean arrival rate, i.e., once the default happens, the default intensity λ_t must drop to zero. Furthermore, the default intensity λ_t can be chosen to be a constant, a deterministic function of time or even a stochastic process.

The other method is the copula approach, which is the standard of the industry. The approach directly models the default time given the marginal default probabilities of all assets with a copula function, it means that the joint density is derived from the marginal default probabilities of each asset through the copula function. It starts with widely used Gaussian copula model, introduced by Li (1999), and others like the Clayton, Student t and double t copulas.

Each way has its own advantages and disadvantages. Intensity-based models interpret the parameters well but are time-consuming in computing the spread of tranches, copula approach is efficient but has difficulty in choosing the copula and interpreting the parameters of the chosen copula. For both methods either homogeneous recovery rate is used or heterogeneous recovery rates are used, here we should not neglect the recovery risk in pricing.

Contribution of the thesis

We propose to do CDO pricing under the random recovery rate by applying Gram-Charlier expansions on the density function of the loss J_t. Synthetic CDO pricing means computing each tranche spread (cf. Section 4.1.1 Preliminaries of synthetic CDOs), which amounts to compute the expected tranche loss $E[J_t^{(\gamma)}]$.

We consider a CDO with maturity T, consisting of N single-name CDSs on obligors with default times τ_i, $i = 1, 2, \ldots, N$ being the first N jump times of a Point process $(N_t)_{t \in \mathbb{R}_+}$, and continuous recovery rates $\phi_i \in [0, 1)$, $i = 1, \ldots, N$, therefore the accumulated loss at time t is

$$J_t = \frac{1}{N} \sum_{i=1}^{N} (1 - \phi_i) \mathbb{1}_{\tau_i \leq t}.$$

We specify the CDO with the points

$$0\% = k_0 < k_1 < \cdots < k_{\gamma-1} < k_\gamma < \cdots < k_q = 100\%$$

and assume that a tranche γ, with m_γ ($\sum_{\gamma=1}^{q} m_\gamma = N$) obligors, has its tranche holder to cover the loss within the tranche $(k_{\gamma-1}, k_\gamma]$, with interval length $\Delta_{k_\gamma} = k_\gamma - k_{\gamma-1}$, here $k_{\gamma-1}$ is called the attachment point and k_γ is called the detachment point.

Then the accumulated loss $J_t^{(\gamma)}$ of tranche γ at time t is given by

$$J_t^{(\gamma)} = (J_t - k_{\gamma-1}) \mathbb{1}_{J_t \in (k_{\gamma-1}, k_\gamma]} + (k_\gamma - k_{\gamma-1}) \mathbb{1}_{J_t > k_\gamma}.$$

Next we apply Gram-Charlier expansion on the density function $f(x)$ of the loss J_t conditioning on $J_t > 0$ in the form of

$$f(x) = \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n\left(\frac{x - c_1}{\sqrt{c_2}}\right) \phi\left(\frac{x - c_1}{\sqrt{c_2}}\right),$$

cf. Tanaka et al. (2010), where the coefficients q_n depends on cumulants $c_1, \cdots, c_n$,

$$q_n = \frac{1}{n!} E\left[H_n\left(\frac{Y - c_1}{\sqrt{c_2}} \right) \right]$$

$$= \sum_{m=1}^{[n/3]} \sum_{\substack{k_1+\cdots+k_m=n \\ k_i\geq 3}} \frac{c_{k_1}\cdots c_{k_m}}{m!k_1!\cdots k_m!}\left(\frac{1}{\sqrt{c_2}}\right)^n \mathbb{1}_{n\geq 3} + \mathbb{1}_{n=0},$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$$

and

$$H_n(x) = (-1)^n \phi(x)^{-1} D^n \phi(x)$$

is the Hermite polynomial of degree n, with $H_0(x) = 1, D = \frac{d}{dx}$.

After the expansion, we compute

$$E\left[\mathbb{1}_{J_t>k_{\gamma-1}}(J_t - k_{\gamma-1}) \mid J_t > 0\right]$$

and

$$E\left[\mathbb{1}_{J_t>k_\gamma} \mid J_t > 0\right],$$

which lead to the expected γ-th tranche loss

$$\begin{aligned} E\left[J_t^{(\gamma)}\right] &= P(J_t>0)\big(E\left[\mathbb{1}_{J_t>k_{\gamma-1}}(J_t-k_{\gamma-1}) \mid J_t>0\right] \\ &\quad -E\left[\mathbb{1}_{J_t>k_\gamma}(J_t-k_{\gamma-1}) \mid J_t>0\right] + \Delta_{k_\gamma} E\left[\mathbb{1}_{J_t>k_\gamma} \mid J_t>0\right]\big). \end{aligned}$$

The explicit formulas of the expected tranche loss are obtained with the i.i.d. survival random variables $\psi_i = 1-\phi_i$, $i = 1,\ldots,N$, following uniform distribution and Beta(2,2) distribution respectively.

In Section 4.5 and Section 4.6, we give some simulation results and compare the conditional density of the loss approximated by Gram-Charlier expansion and Monte Carlo simulations.

In Section 4.7 we add the default correlation among N default times through Gaussian copula, and then price CDOs by applying Gram-Charlier expansion. The n-th moment m_n of loss J_t is given by

$$n!\int_{-\infty}^{\infty}\left(\sum_{\substack{n_1+\cdots+n_N=n \\ 0\leq n_i\leq N}} \prod_{i=1}^{N}\left(\frac{1}{n_i!}\Phi\left(\frac{\Phi^{-1}(P(\tau_i\leq t)) - a_i M}{\sqrt{1-a_i^2}}\right)\int_0^1 y^{n_i}dF(y)\right)^{\mathbb{1}_{n_i>0}}\right)d\Phi(m).$$

In this chapter the relationship between the recovery rate and default intensity is out of the scope, but we can add the stochastic default intensity to see how they will impact on pricing CDOs in the future.

Chapter 2

Quadratic functionals of the OU process

2.1 Preliminaries

2.1.1 Zero-coupon bonds

Zero-coupon (default free) bonds are special type of bonds that do not pay interest. Indeed, they are traded at a discounted value and redeem principal at maturity.

Definition 2.1.1. ***Zero-coupon bonds***. *A T-maturity zero-coupon bond is a contract that guarantees its holder the payment of one unit of contract at time T, with no intermediate payment. The contract value at time $t < T$ is denoted by $P(t, T)$. Clearly $P(T, T) = 1$ for all T.*

Recall that the standard Brownian motion is a stochastic process $(B_t)_{t \in \mathbb{R}_+}$ satisfing
(i) $B_0 = 0$ almost surely,
(ii) the sample trajectories $t \mapsto B_t$ are continuous with probability 1,
(iii) for any time sequence $t_0 < t_1 < \cdots < t_n$, the increments $B_{t_1} - B_{t_0}$, $B_{t_2} - B_{t_1}$, $\ldots$, $B_{t_n} - B_{t_{n-1}}$ are independent,

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(iv) for any given $0 \leq s < t$, the increment $B_t - B_s$ follows the Gaussian distribution with mean zero and variance $t - s$.

Given a probability space $(\Omega, \mathcal{F}, P)$ with the standard augmented filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ under risk-neutral measure P. It is well known that a T-maturity zero-coupon bond price has the form

$$P(t,T) = E_P[e^{-\int_t^T r_s\,ds} \mid \mathcal{F}_t],$$

i.e. the Laplace transform of $\int_t^T r_s\,ds$, and the time t price $P(t,T)$ depends on the stochastic structure of short term interest rate r_t.

Consider the short term interest rate model

$$dr_t = \mu(t, r_t)\,dt + \sigma(t, r_t)\,dB_t, \tag{2.1.1}$$

which has the Markov property, i.e. given present state t, the future value of $(r_s)_{s\in\mathbb{R}_+}$ is independent of the past until time t. $\mu(t,x)$ is the drift function, $\sigma(t,x)$ is the variance function, and the value $P(t,T)$ then can be written down as a function of r_t, as

$$\begin{aligned} P(t,T) &= E_P[e^{-\int_t^T r_s\,ds} \mid \mathcal{F}_t] \\ &= E_P[e^{-\int_t^T r_s\,ds} \mid r_t] \\ &:= F(t, r_t). \end{aligned}$$

Classically, the partial differentiation equation of twice continuously differentiable function $F(t,x)$ can be derived through Itô calculus,

$$xF(t,x) = \frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) \tag{2.1.2}$$

with boundary condition $F(T,x) = 1$. So bond price $P(t,T)$ is derived from the explicit solution of the above term structure equation with specific short rate model parameters $\mu(t,x)$ and $\sigma(t,x)$ in (2.1.1).

2.1.2 Short term interest rate models

Now we recall some classical short term interest rate models.

The Vasicek model

The Vasicek (1977) model is defined by the dynamics

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\,dB_t, \tag{2.1.3}$$

where κ, θ, and σ are positive constants. This model follows an Ornstein-Uhlenbeck process, having the mean-reverting property, i.e. r_t tends to return to the average θ in the long run at an adjustment speed κ.

By integrating (2.1.3) we obtain

$$r(t) = r(s)e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)}) + \sigma \int_s^t e^{-\kappa(t-u)} dB_u, \ 0 \leq s \leq t.$$

It follows that the distribution of this short rate $r(t)$ given information $\mathcal{F}_s$ is Gaussian with mean and variance

$$\begin{aligned} E\left[r(t) \mid \mathcal{F}_s\right] &= r(s)e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)}), \\ Var\left[r(t) \mid \mathcal{F}_s\right] &= \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa(t-s)}), \end{aligned}$$

and the price of zero-coupon bond with maturity T is obtained by substituting $\mu(t,x) = \kappa(\theta - x)$ and $\sigma(t,x) = \sigma$ in (2.1.1) as

$$P(t,T) = e^{A(t,T) - B(t,T)r(t)},$$

where

$$\begin{aligned} A(t,T) &= \left(\theta - \frac{\sigma^2}{2\kappa^2}\right)(B(t,T) + t - T) - \frac{\sigma^2}{4\kappa^2}B(t,T)^2, \\ B(t,T) &= \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right). \end{aligned}$$

The major drawback of the Vasciek model is that the rate $r(t)$ can be negative with positive probability, as such we look at another model, the Cox, Ingersoll and Ross (CIR) model, which guarantees positive interest rate.

The Cox, Ingersoll and Ross (CIR) model

Different from the Vasicek (1977) model (2.1.3), the square-root term added in the diffusion coefficient of the rate dynamics results in positive short rate, it was introduced by Cox et al. (1985) as an extension of the Vasicek model, which is the solution to the equation

$$dr_t = (\gamma - 2br_t)\,dt + 2\sigma\sqrt{r_t}\,dB_t, \tag{2.1.4}$$

where $b, \sigma > 0$. Strictly positive interest rates are guaranteed under the Feller condition that $\gamma \geq 2\sigma^2$. In this model the variance is proportional to the short term rate while the variance is constant in the Vasciek model (2.1.3).

It follows that the mean and variance of short rate $r(t)$ given information $\mathcal{F}_s$, $s \leq t$ are

$$\begin{aligned} E\left[r(t) \mid \mathcal{F}_s\right] &= r(s)e^{-2b(t-s)} + \frac{\gamma}{2b}\left(1 - e^{-2b(t-s)}\right), \\ Var\left[r(t) \mid \mathcal{F}_s\right] &= r(s)\frac{2\sigma^2}{b}\left(e^{-2b(t-s)} - e^{-4b(t-s)}\right) + \gamma\frac{\sigma^2}{2b^2}\left(1 - e^{-2b(t-s)}\right)^2, \end{aligned}$$

and the price of zero-coupon bond with maturity T at time t derived from classical PDE equation (2.1.2) with $\mu(t,x) = \gamma - 2bx$ and $\sigma(t,x) = 2\sigma\sqrt{x}$ is

$$\begin{aligned} P(t,T) =& E\left[e^{-\int_t^T r_s\,ds}\middle| r_t\right] \\ =& \left(\frac{he^{(b+h)(T-t)}}{h + (b+h)(e^{2h(T-t)} - 1)/2}\right)^{\frac{\gamma}{2\sigma^2}} \exp\left(-\frac{r_t(e^{2h(T-t)} - 1)}{2h + (b+h)(e^{2h(T-t)} - 1)}\right), \end{aligned}$$

where $h = \sqrt{b^2 + 2\sigma^2}$, cf. Cox et al. (1985) and Mercurio (2006) page 66.

Grasselli and Hurd (2005) point out the link between CIR model and squared Ornstein-Uhlenbeck process. When $\frac{\gamma}{\sigma^2} = K \in N_+\backslash\{0,1\}$, r_t in (2.1.4) can be written as the squared Ornstein-Uhlenbeck process $r_t = X_t^T X_t$, where $(X_t)_{t\in\mathbb{R}_+}$ is K-dimensional Ornstein-Uhlenbeck process

$$X_t = xe^{-bt} + \sigma\int_0^t e^{-b(t-s)}\,dB_s,$$

which is the solution of the equation

$$dX_t = -bX_t\,dt + \sigma\,dB_t, \qquad X_0 = x.$$

Here we are interested in one-dimensional squared Ornstein-Uhlenbeck process X_t^2, which is linked to the quadratic functionals of Brownian motion $(B_t)_{t\in\mathbb{R}_+}$. Therefore taking interest rate $r_t = X_t^2$, the price of a non-defaultable zero-coupon bond with maturity T can be written down using functional determinant techniques generally applied to quadratic functionals of Brownian motion.

In the following, we study the quadratic functionals of Brownian motion, later on we use the functional determinant techniques to complement the above classical PDE approach applied to bond pricing and extend the determinant technique to the derivation of the explicit formulas for the Laplace transform of various quadratic Ornstein-Uhlenbeck functionals.

2.1.3 Quadratic Brownian functionals

Definition 2.1.2. *A random variable F is called a quadratic Brownian functional if it decomposes as*

$$F = I_1(\psi) + \frac{1}{2}I_2(\varphi), \tag{2.1.5}$$

where $I_1(\psi) = \int_0^T \psi(t)\,dB_t$, $\psi \in L^2([0,T])$ and

$$I_2(\varphi) = 2\int_0^T\int_0^t \phi(s,t)\,dB_s dB_t,$$

where $\varphi \in L^2([0,T]^2)$ is symmetric in two variables.

In this case the Laplace transform of F given by Proposition 4.1 in Grasselli and Hurd (2005) is expressed as (2.1.6).

Lemma 2.1.3. *(Grasselli and Hurd (2005)) Given Brownian functional F in* (2.1.5)*, we have*

$$E[e^{-F}] = (\det{}_2(I + A^{\varphi}))^{-1/2}\exp\left(\frac{1}{2}\int_0^T \psi(s)(I + A^{\varphi})^{-1}\psi(s)\,ds\right), \tag{2.1.6}$$

where $\det_2(I + A^\varphi)$ *is the Carleman-Fredholm determinant*

$$\det_2(I + A^\varphi) = e^{-trA^\varphi} \det(I + A^\varphi)$$

of the Volterra operator

$$(A^\varphi f)(t) := \int_0^T \varphi(s,t) f(s)\, ds, \qquad f \in L^2([0,T]), \tag{2.1.7}$$

such that $I + A^\varphi$ *has positive spectrum with*

$$\det(I + A^\varphi) = \prod_{i=0}^{\infty} (1 + \lambda_i), \quad trA^\varphi = \sum_{i=0}^{\infty} \lambda_i,$$

where $(\lambda_i)_{i \geq 0}$ *are the eigenvalues of* A^φ*, counted with their multiplicities.*

Note that A^φ is a symmetric Hilbert-Schmidt operator on $L^2([0,T])$ since

$$\|A^\varphi\|_{HS}^2 = \int_{[0,T]^2} |\varphi(s,t)|^2\, dsdt < \infty.$$

This approach to the Laplace transform of quadratic Brownian random variable using functional determinants has been initiated in quantum field theory for Feynman path integrals, cf. e.g. Glimm et al. (2008) Proposition 9.3.1 and Remark 1 pages 211-212. From a probabilistic point of view, quadratic Brownian functionals are infinitely divisible, and explicit expressions for the Lévy measures of Wiener functionals of second order have been given in Matsumoto and Taniguchi (2002) based on (2.1.6).

2.2 Stochastic integral representation

Here we see some Brownian functionals.

Lemma 2.2.1.

i) The integral $\int_0^T X_t\, \mathrm{d}t$ *has the representation*

$$\int_0^T X_t\, \mathrm{d}t = \frac{X_0}{b}(1 - e^{-bT}) + I_1(\psi),$$

where

$$\psi(t) = \frac{\sigma}{b}\left(1 - e^{-b(T-t)}\right), \qquad t \in [0,T].$$

ii) The integral $\int_0^T X_t\, dB_t$ *has the representation*

$$\int_0^T X_t\, dB_t = X_0\, I_1(\psi) + \frac{1}{2} I_2(\varphi),$$

where

$$\psi(t) = e^{-bt} \quad \textit{and} \quad \varphi(s,t) = \sigma e^{-b|t-s|}, \qquad s,t \in [0,T].$$

iii) The integral $\int_0^T X_t^2 dt$ *has the representation*

$$\int_0^T X_t^2 dt = \frac{X_0^2}{2b}(1-e^{-2bT}) + \frac{\sigma^2}{4b^2}\left(e^{-2bT} + 2bT - 1\right) + X_0\, I_1(\psi) + \frac{1}{2} I_2(\varphi),$$

where

$$\psi(t) = \frac{\sigma}{b}\left(e^{-bt} - e^{-b(2T-t)}\right) \quad \textit{and} \quad \varphi(s,t) = \frac{\sigma^2}{b}\left(e^{-b|s-t|} - e^{-b(2T-s-t)}\right),$$

$s,t \in [0,T]$.

Proof. (i) We have

$$\begin{aligned}\int_0^T X_t\, \mathrm{d}t = &\int_0^T \left(X_0 e^{-bt} + \sigma \int_0^t e^{-b(t-s)}\, \mathrm{d}B_s\right) \mathrm{d}t \\ =&\frac{X_0}{b}(1-e^{-bT}) + \sigma \int_0^T \int_0^t e^{-b(t-s)}\, \mathrm{d}B_s \mathrm{d}t \\ =&\frac{X_0}{b}(1-e^{-bT}) + \frac{\sigma}{b}\int_0^T (1-e^{-b(T-s)})\, \mathrm{d}B_s.\end{aligned}$$

(ii) We have

$$\begin{aligned}\int_0^T X_t\, dB_t &= \int_0^T \left(X_0 e^{-bt} + \sigma \int_0^t e^{-b(t-s)}\, dB_s\right) dB_t \\ &= \int_0^T X_0 e^{-bt}\, dB_t + \sigma \int_0^T \int_0^t e^{-b(t-s)}\, dB_s dB_t \\ &= X_0 \int_0^T e^{-bt}\, dB_t + \sigma \int_0^T \int_0^t e^{-b|t-s|}\, dB_s dB_t.\end{aligned}$$

(iii) We have

$$\begin{aligned}\int_0^T X_t^2\, dt = &\int_0^T \left(X_0 e^{-bt} + \sigma e^{-bt}\int_0^t e^{bs}\, dB_s\right)^2 dt \\ =&\sigma^2 \int_0^T e^{-2bt}\left(\int_0^t e^{bs}\, dB_s\right)^2 dt + 2\sigma X_0 \int_0^T \int_0^t e^{-2bt} e^{bs}\, dB_s dt \\ &+ \int_0^T X_0^2 e^{-2bt}\, dt.\end{aligned}$$

By the Itô formula we have

$$\left(\int_0^t e^{bs}\, dB_s\right)^2 = 2\int_0^t e^{bs}\int_0^s e^{bu}\, dB_u dB_s + \frac{1}{2b}\left(e^{2bt}-1\right),$$

hence

$$\int_0^T e^{-2bt}\int_0^t e^{bs}\int_0^s e^{bu}\, dB_u dB_s dt = \frac{1}{2b}\int_0^T\int_0^s (e^{-b|s-u|}-e^{-b(2T-s-u)})\, dB_u dB_s,$$

and

$$\begin{aligned}\int_0^T\int_0^t e^{-2bt}e^{bs}\, dB_s dt &= \frac{1}{2b}e^{-bT}\int_0^T (e^{b(T-s)}-e^{-b(T-s)})\, dB_s\\ &= \frac{1}{b}e^{-bT}\int_0^T \sinh(b(T-s))\, dB_s.\end{aligned}$$

□

2.3 Bivariate Laplace transform

2.3.1 PDE approach

We start by computing the bivariate Laplace transform using standard PDE arguments.

Proposition 2.3.1. *For all $\rho \geq 0$ and $\mu \in \mathbb{R}$ such that $b^2+2\rho b\sigma+2\mu\sigma^2>0$ we have*

$$\begin{aligned}&E\left[e^{-\rho\int_t^T X_s\, dB_s-\mu\int_t^T X_s^2\, ds}\Big| X_t = x\right] \qquad (2.3.1)\\ &= \left(\cosh h(T-t)+\frac{b+\sigma\rho}{h}\sinh h(T-t)\right)^{-1/2}\\ &\quad\times\exp\left(\frac{b+\sigma\rho}{2}(T-t)-\frac{x^2(\mu-\rho^2/2)}{b+\rho\sigma+h\coth h(T-t)}\right), \qquad t\in[0,T],\end{aligned}$$

where $h=\sqrt{b^2+2\rho b\sigma+2\mu\sigma^2}$.

Proof. By standard stochastic calculus and martingale arguments it can be shown that the twice continuously differentiable function

$$H(t,x) = E\left[\exp\left(-\rho\int_t^T X_s\, dB_s-\mu\int_t^T X_s^2\, ds\right)\Big| X_t = x\right], \qquad t\in[0,T],$$

solves the PDE

$$\frac{\sigma^2}{2}\frac{\partial^2 H}{\partial x^2}(t,x) - (\sigma\rho + b)\,x\frac{\partial H}{\partial x}(t,x) + \frac{\partial H}{\partial t}(t,x) - x^2(\mu - \frac{\rho^2}{2})H(t,x) = 0, \quad (2.3.2)$$

with boundary condition $H(T,x) = 1$, hence the solution $H(t,x)$ is given by (2.3.1), cf. Proposition 2.3.2 below. □

Next we derive the PDE solution which has been used in the proof of Proposition (2.3.1).

Proposition 2.3.2. *Let b, $\sigma > 0$ and $\mu \in \mathbb{R}$ such that $b^2 + 2\rho b\sigma + 2\mu\sigma^2 > 0$. The PDE* (2.3.2) *has solution*

$$\begin{aligned} H(t,x) &= \left(\cosh h(T-t) + \frac{b+\sigma\rho}{h}\sinh h(T-t)\right)^{-1/2} \\ &\quad \times \exp\left(\frac{b+\sigma\rho}{2}(T-t) - \frac{x^2(\mu - \rho^2/2)}{b + \rho\sigma + h\coth h(T-t)}\right), \qquad t \in [0,T], \end{aligned}$$

where $h = \sqrt{b^2 + 2\rho b\sigma + 2\mu\sigma^2}$.

Proof. In order to solve (2.3.2), we search for the solution of the form

$$H(t,x) = \exp\left(C(T-t) + E(T-t)x^2\right),$$

with $C(0) = E(0) = 0$, which implies

$$\begin{cases} \dfrac{\partial E}{\partial t}(T-t) = -2(b+\sigma\rho)E(T-t) + 2\sigma^2 E^2(T-t) - (\mu - \dfrac{1}{2}\rho^2), \\ \\ \dfrac{\partial C}{\partial t}(T-t) = \sigma^2 E(T-t), \\ \\ C(0) = E(0) = 0. \end{cases}$$

The function $E(T-t)$ is obtained by solving a Riccati equation, as

$$\begin{aligned} E(T-t) &= -\frac{(\mu - \rho^2/2)(1 - e^{2h(t-T)})}{b + \sigma\rho + h - (b + \sigma\rho - h)e^{2h(t-T)}} \\ &= -\frac{\mu - \rho^2/2}{b + \rho\sigma + h\coth h(T-t)}, \end{aligned}$$

where $h = \sqrt{b^2 + 2\rho b\sigma + 2\mu\sigma^2}$, which also yields

$$\begin{aligned}
&C(T-t)\\
=&\sigma^2 \int_t^T E(T-s)\,ds\\
=&\frac{b+\sigma\rho}{2}(T-t) - \frac{1}{2}\ln\left(\cosh h(T-t) + \frac{b+\sigma\rho}{h}\sinh h(T-t)\right).
\end{aligned}$$

□

2.3.2 Functional determinant approach

Next we look at the spectrum of the Volterra operators A^φ.

Lemma 2.3.3. *Assume that $\alpha+\beta \geq 0$ and $b > \max(-2\alpha, 0)$ or $b < \min(-2\alpha, 0)$, and φ is given by*

$$\varphi(s,t) = \alpha e^{-b|s-t|} + \beta e^{-b(2T-s-t)}, \qquad s,t \in [0,T].$$

Let h be an eigenvector of A^φ corresponding to nonzero eigenvalue λ then

- *if $b > \max(2\alpha/\lambda, 0)$ or $b < \min(2\alpha/\lambda, 0)$, the eigenvector h is given by*

$$h(t) = \frac{b}{\eta}\sinh\eta t + \cosh\eta t$$

and the corresponding eigenvalue λ satisfies the equations

$$\begin{cases} \tanh\eta T = \dfrac{(\beta-\alpha)b^2 - (\beta+\alpha)\eta^2}{2b\alpha\eta}, \\ \lambda = \dfrac{2b\alpha}{b^2-\eta^2}, \end{cases} \tag{2.3.3}$$

- *if $0 < b < 2\alpha/\lambda$, or $2\alpha/\lambda < b < 0$ the eigenvector h is given by*

$$h(t) = \frac{b}{\eta}\sin\eta t + \cos\eta t,$$

while the corresponding eigenvalue λ satisfies

$$\begin{cases} \tan\eta T = \dfrac{2b\alpha\eta}{(\beta-\alpha)b^2 + (\beta+\alpha)\eta^2}, \\ \lambda = \dfrac{2b\alpha}{b^2+\eta^2}, \end{cases} \tag{2.3.4}$$

- *and if* $b > \max(2\alpha/\lambda, 0)$*, then* $I + A^{\varphi}$ *has positive spectrum.*

Proof. By twice differentiation of the relation

$$A^{\varphi}h(t) = \lambda h(t),$$

i.e.

$$\int_0^T \left(\alpha e^{-b|s-t|} + \beta e^{-b(2T-s-t)}\right) h(s)\, ds = \lambda h(t), \tag{2.3.5}$$

we obtain

$$\begin{aligned} b\Bigg(-\alpha \int_0^t e^{b(s-t)}h(s)\, ds + \alpha \int_t^T e^{b(t-s)}h(s)\, ds \\ +\beta \int_0^T e^{-b(2T-s-t)}h(s)\, ds\Bigg) = \lambda h'(t) \end{aligned} \tag{2.3.6}$$

and

$$b^2 \left(\alpha \int_0^T e^{-b|t-s|}h(s)\, ds + \beta \int_0^T e^{-b(2T-s-t)}h(s)\, ds\right) - 2\alpha b h(t) = \lambda h''(t). \tag{2.3.7}$$

Apply equation (2.3.5) in (2.3.7), we obtain

$$h''(t) = \left(b^2 - 2b\alpha/\lambda\right) h(t)$$

and $h'(0) = bh(0)$ by taking $t = 0$ in (2.3.5) and (2.3.6). If $b > \max(2\alpha/\lambda, 0)$ or $b < \min(2\alpha/\lambda, 0)$, the eigenvector

$$\begin{aligned} h(t) &= \left(\sqrt{b^2 - 2\alpha b/\lambda} + b\right) e^{t\sqrt{b^2-2\alpha b/\lambda}} + \left(\sqrt{b^2 - 2\alpha b/\lambda} - b\right) e^{-t\sqrt{b^2-2\alpha b/\lambda}} \\ &= 2b \sinh \eta t + 2\eta \cosh \eta t, \end{aligned}$$

where $\eta = \sqrt{b^2 - 2\alpha b/\lambda}$. Take $t = 0$ in (2.3.5), so the eigenvalue λ satisfies

$$\begin{aligned} \left(\sqrt{b^2 - 2\alpha b/\lambda} + b - (\alpha + \beta)/\lambda\right) e^{2\sqrt{b^2-2\alpha b/\lambda}T} + \sqrt{b^2 - 2\alpha b/\lambda} \\ - b + (\alpha + \beta)/\lambda = 0, \end{aligned}$$

i.e.

$$\begin{cases} \tanh \eta T = \dfrac{(\beta - \alpha)b^2 - (\beta + \alpha)\eta^2}{2b\alpha\eta}, \\ \lambda = \dfrac{2b\alpha}{b^2 - \eta^2}. \end{cases}$$

In case $0 < b < 2\alpha/\lambda$, or $2\alpha/\lambda < b < 0$, the eigenvector is given by

$$\begin{aligned} h(t) &= \sqrt{2\alpha b/\lambda - b^2}\cos\left(t\sqrt{2\alpha b/\lambda - b^2}\right) + b\sin\left(t\sqrt{2\alpha b/\lambda - b^2}\right) \\ &= b\sin\eta t + \eta\cos\eta t, \end{aligned}$$

where $\eta = \sqrt{2\alpha b/\lambda - b^2}$. Taking $t = 0$ in (2.3.5) shows that the eigenvalue λ satisfies

$$(\alpha+\beta-\lambda b)\sin\sqrt{2\alpha b/\lambda - b^2}T - \lambda\sqrt{2\alpha b/\lambda - b^2}\cos\sqrt{2\alpha b/\lambda - b^2}T = 0,$$

i.e.

$$\begin{cases} \tan\eta T = \dfrac{2b\alpha\eta}{(\beta-\alpha)b^2 + (\beta+\alpha)\eta^2}, \\ \lambda = \dfrac{2b\alpha}{b^2+\eta^2}. \end{cases}$$

Finally, if $\lambda < -1$ is an eigenvalue of A^φ, then under the condition $b \geq \max(-2\alpha, 0)$ we get $b > 2\alpha/\lambda$. Next we check that (2.3.3) has no solution since $-(\alpha+\beta)/\lambda > 0$, which shows that $I + A^\varphi$ has positive spectrum.

If $\lambda < -1$ is an eigenvalue of A^φ, then under condition $b < \min(-2\alpha, 0)$, we get $b < 2\alpha/\lambda$. Next we check that (2.3.3) has solution if $\sqrt{b^2 - 2\alpha b/\lambda} < (\alpha+\beta)/\lambda - b$, which cannot guarantee that $I + A^\varphi$ has positive spectrum. □

When $\beta = 0$, and $\alpha, b > 0$, the kernel

$$\varphi(s,t) = \alpha e^{-b|s-t|} \qquad s,t \in [0,T]$$

is the stationary covariance function of the Gaussian noise, cf. Slepian (1954) and A^φ is Wiener-Hopf operator in Youla (1957). It is easy to check that (2.3.3) has no solution and (2.3.4) is equivalent to

$$\begin{cases} \tan\dfrac{\eta T}{2} = -\dfrac{\eta}{b} \qquad \text{or} \qquad \cot\dfrac{\eta T}{2} = \dfrac{\eta}{b}, \\ \lambda = \dfrac{2b\alpha}{b^2+\eta^2}, \end{cases}$$

which recover the eigenvalues and eigenvectors of Picard kernel, computed by Youla (1957) from a different perspective.

When $\alpha = -\beta = \sigma^2/b$ and $b > 0$, Lemma 2.3.3 shows that any eigenvalue λ of A^φ should satisfy

$$\sqrt{2\sigma^2/\lambda - b^2} \cos \sqrt{2\sigma^2/\lambda - b^2} T = 0. \quad (2.3.8)$$

As b tends to 0 we get

$$\varphi(s,t) = 2\sigma^2 \left(T - \frac{s + t - |s - t|}{2} \right) = 2\sigma^2 (T - (s \vee t)),$$

and (2.3.8) above shows that

$$\lambda_k = \frac{8\sigma^2 T^2}{\pi^2 (2k+1)^2} \qquad \text{and} \qquad h_k(t) = \cos \frac{2\sigma}{\sqrt{\lambda_k}} t, \qquad k \geq 1.$$

As a consequence we have $\text{trace}(A^\varphi) = \sigma^2 T^2$, and by e.g. § 4.5.69 page 85 of Abramowitz and Stegun (1972) we have

$$\det(I + A^\varphi) = \prod_{k=0}^{\infty} \left(1 + \frac{8\sigma^2 T^2}{(2k+1)^2 \pi^2} \right) = \cosh(\sigma T \sqrt{2}),$$

which shows that

$$\det\nolimits_2 (I + A^\varphi) = e^{-\sigma^2 T^2} \cosh(\sigma T \sqrt{2}), \quad (2.3.9)$$

this case coincides with (2.3.21) as b tends to 0.

Now we compute the term $\int_0^T \psi(s)(I + A^\varphi)^{-1} \psi(s)\, ds$ appearing in (2.1.6).

Lemma 2.3.4. *Let* $y, z, \alpha, \beta \in \mathbb{R}$, $b > \max(-2\alpha, 0)$ *and*

$$\psi(t) = y e^{bt} + z e^{-bt}, \qquad t \in [0, T],$$

and

$$\varphi(s,t) = \alpha e^{-b|t-s|} + \beta e^{-b(2T-s-t)}, \qquad s, t \in [0, T].$$

We have

$$\begin{aligned} &\int_0^T \psi(t)(I + A^\varphi)^{-1} \psi(t)\, dt \quad (2.3.10) \\ &= \frac{zy}{2\alpha} \frac{(b + \alpha + \beta) e^{hT} - (b + \alpha - h + \beta) e^{-hT} - 2h e^{bT}}{(b + \alpha + \beta) \sinh hT + h \cosh hT} \end{aligned}$$

$$+\frac{z^2}{2\alpha^2}\frac{(\alpha^2+\beta(b+\alpha-h))e^{hT}-(\alpha^2+\beta(b+\alpha+h))e^{-hT}+2h\beta e^{-bT}}{(b+\alpha+\beta)\sinh hT+h\cosh hT},$$

where $h=\sqrt{b^2+2\alpha b}$.

Proof. The function $u(t)$ solves the Fredholm equation of the second kind

$$A^{\varphi}u(t)=\psi(t)-u(t),$$

i.e.

$$\int_0^T\left(\alpha e^{-b|s-t|}+\beta e^{-b(2T-s-t)}\right)u(s)\,ds=\psi(t)-u(t). \tag{2.3.11}$$

Differentiating (2.3.11) on both sides with respect to t we get

$$\begin{aligned}&b\left(-\alpha\int_0^t e^{b(s-t)}u(s)\,ds+\alpha\int_t^T e^{b(t-s)}u(s)\,ds+\beta\int_0^T e^{-b(2T-s-t)}u(s)\,ds\right)\\&=\psi'(t)-u'(t),\end{aligned} \tag{2.3.12}$$

and

$$\begin{aligned}&b^2\left(\alpha\int_0^T e^{-b(t-s)/2}u(s)\,ds+\beta\int_0^T e^{-b(2T-s-t)}u(s)\,ds\right)-2\alpha bu(t)\\&=\psi''(t)-u''(t).\end{aligned} \tag{2.3.13}$$

Comparing with (2.3.11), (2.3.13) is simplified to be

$$b^2(\psi-u)-2\alpha bu=\psi''-u'',$$

i.e.

$$u''-(b^2+2\alpha b)u=\psi''-b^2\psi. \tag{2.3.14}$$

By twice differentiation of $\psi(t)$ we find

$$\psi''=b^2\psi. \tag{2.3.15}$$

By substituting the right hand side of (2.3.14) with (2.3.15), we obtain

$$u''-(b^2+2\alpha b)u=0,$$

if $b^2 + 2\alpha b > 0$, the above equation is a linear second-order ODE with general solution

$$u(t) = c_1 e^{ht} + c_2 e^{-ht}, \tag{2.3.16}$$

where $h = \sqrt{b^2 + 2\alpha b}$, c_1 and c_2 are constants. Letting $t = 0$ both in (2.3.11) and (2.3.12), we get

$$\psi(0) - u(0) = \int_0^T \left(\alpha e^{-bs} + \beta e^{-b(2T-s)}\right) u(s)\, ds \tag{2.3.17}$$

and

$$\psi'(0) - u'(0) = b\left(\psi(0) - u(0)\right). \tag{2.3.18}$$

Next, we substitute $u(t)$ with (2.3.16), then plug in $\psi(0)$ and $\psi'(0)$ in the above equations (2.3.17) and (2.3.18). Therefore c_1 and c_2 are the solutions of following two equations

$$(h-b)c_1 - (h+b)c_2 = -2bz,$$

and

$$\begin{aligned} & c_1(e^{(h-b)T}(\alpha(h+b) + \beta(h-b)) - (h-b)\beta e^{-2bT} - \alpha(h-b)) \\ & \quad + c_2(e^{-(h+b)T}(-\alpha(h-b) - \beta(h+b)) + (h+b)\beta e^{-2bT} + \alpha(h+b)) \\ & \quad = 2\alpha b(y+z) \end{aligned}$$

hence

$$c_1 = \frac{z\left(e^{-hT}((\alpha+\beta)h + (\beta-\alpha)b) - \beta(h+b)e^{-bT}\right) + (h+b)\alpha y e^{bT}}{2\alpha\left((b+\alpha+\beta)\sinh hT + h\cosh hT\right)},$$

and

$$c_2 = \frac{z\left(e^{hT}((\alpha+\beta)h + (-\beta+\alpha)b) - \beta(h-b)e^{-bT}\right) + (h-b)\alpha y e^{bT}}{2\alpha\left((b+\alpha+\beta)\sinh hT + h\cosh hT\right)},$$

which shows that

$$\begin{aligned} (I + A^{\varphi})^{-1}\psi(t) &= \frac{z}{\alpha}\frac{h(\alpha+\beta)\cosh h(T-t) + b(\alpha-\beta)\sinh h(T-t)}{(b+\alpha+\beta)\sinh hT + h\cosh hT} \\ &\quad + \frac{1}{\alpha}\frac{(h\cosh ht + b\sinh ht)(aye^{bT} - \beta z e^{-bT})}{(b+\alpha+\beta)\sinh hT + h\cosh hT},\ t \in [0,T], \end{aligned}$$

and (2.3.10) follows.

If $b^2 + 2\alpha b < 0$, take $h = \sqrt{-b^2 - 2\alpha b}$,

$$u(t) = c_1 \cos ht + c_2 \sin ht,$$

where c_1 and c_2 are constants. Letting $t = 0$ both in (2.3.11) and (2.3.12), we get

$$\psi(0) - u(0) = \int_0^T \left(\alpha e^{-bs} + \beta e^{-b(2T-s)}\right) u(s)\, ds$$

and

$$\psi'(0) - u'(0) = b\left(\psi(0) - u(0)\right).$$

Next, we substitute $u(t)$ with (2.3.16) and plug in $\psi(0)$, $\psi'(0)$ in the above relations, therefore c_1 and c_2 are the solutions of following two equations

$$bc_1 - hc_2 = 2bz,$$

and

$$\begin{aligned} & c_1\left[((\alpha+\beta)h\sin ht + (\beta-\alpha)b\cos ht)e^{-bT} - b\left(\alpha + \beta e^{-2bT}\right)\right] \\ & \quad + c_2\left[((\beta-\alpha)b\sin ht - (\alpha+\beta)h\cos ht)e^{-bT} - b\left(\alpha - \beta e^{-2bT}\right)\right] \\ & \quad = -2\alpha b(y+z), \end{aligned}$$

hence

$$c_1 = \frac{2z[(\alpha-\beta)b\sin ht + (\alpha+\beta)h\cos ht] + 2\alpha((y+z)h + bz)e^{bT} - 2\beta bze^{-bT}}{2\alpha((b+\alpha+\beta)\sin hT + h\cosh T) + (h+b)\alpha e^{bT} + (h-b)\beta e^{-bT}},$$

and

$$c_2 = \frac{2bz[(-\beta-2\alpha)(\alpha+\beta)\sin ht + (\beta-\alpha)h\cos ht]/h + 2\alpha bye^{bT} - 2\beta bze^{-bT}}{2\alpha((b+\alpha+\beta)\sin hT + h\cosh T) + (h+b)\alpha e^{bT} + (h-b)\beta e^{-bT}},$$

which shows that

$$\begin{aligned} & \int_0^T \psi(t)(I + A^{\varphi})^{-1}\psi(t)\, dt \\ = \; & \frac{z}{\alpha}(y-z) - \frac{z}{\alpha b}(h\sin ht + b\cos ht)((y+z)\sinh bT + (y-z)\cosh bT) \end{aligned}$$

$$+\Big[\frac{yz}{h}e^{bT}((\frac{\beta}{\alpha}-1)h\sin 2hT-2(\beta+2\alpha)(\frac{\beta}{\alpha}+1)\sin^2 hT)-2y\beta z+(y-z)(h-b)z$$
$$\Big[(\frac{\beta}{\alpha}-1)\cos hT-(\beta+2\alpha)(\frac{\beta}{\alpha}+1)z\sin hT/h+ye^{bT}-\beta ze^{-bT}/\alpha\Big]+z^2e^{-bT}\Big[(1-\frac{\beta}{\alpha})$$
$$2h\cos^2 hT+(\frac{\beta(2\beta+b)}{2\alpha}+\frac{7\beta+3\alpha-b}{2})\sin 2hT-(\alpha+b)(\beta+2\alpha)(\frac{\beta}{\alpha}+1)\sin^2 hT)\Big]$$
$$+2\alpha y^2\sin hTe^{2bT}+\frac{\beta}{\alpha}z(2h\cos hT-(\alpha+b)\sin hT)e^{-2bT}+zy((\alpha+b)\sin hT$$
$$-4h\cos hT)\Big]\Big/\Big[2\alpha((b+\alpha+\beta)\sin hT+h\cosh T)+(h+b)\alpha e^{bT}+(h-b)\beta e^{-bT}\Big].$$

□

In order to compare the result of Proposition 2.3.1 to the determinant identity (2.1.6) we rewrite

$$F=\rho\int_t^T X_s\,dB_s+\mu\int_t^T X_s^2\,ds$$

in the form of (2.1.5) as

$$F=E[F]+x\,I_1(\psi)+\frac{1}{2}I_2(\varphi), \tag{2.3.19}$$

cf. Lemma 2.2.1 above. As a consequence of Proposition 2.3.1, relation (2.1.6) and relation (2.3.19), we obtain the following corollary.

Corollary 2.3.5. *Assume that $\alpha+\beta\geq 0$, $b>\max(0,-2\alpha)$,*

$$\varphi(s,t)=\alpha e^{-b|s-t|}+\beta e^{-b(2T-s-t)},\qquad s,t\in[0,T], \tag{2.3.20}$$

then $I+A^\varphi$ has positive spectrum and we have

$$\det(I+A^\varphi)=e^{-bT}\left(\cosh(hT)+\frac{b+\alpha+\beta}{h}\sinh(hT)\right), \tag{2.3.21}$$

where $h=\sqrt{b^2+2\alpha b}$.

Proof. By Lemma 2.2.1 the functions in (2.3.19) are given by

$$\psi(t)=-\frac{\mu\sigma}{b}e^{-b(2T-t)}+\left(\rho+\frac{\mu\sigma}{b}\right)e^{-bt}, \tag{2.3.22}$$

and

$$\varphi(s,t)=\left(\rho\sigma+\frac{\mu\sigma^2}{b}\right)e^{-b|t-s|}-\frac{\mu\sigma^2}{b}e^{-b(2T-s-t)}, \tag{2.3.23}$$

and

$$E[F] = \frac{\mu x^2}{2b}(1 - e^{-2bT}) + \frac{1}{2}\mathrm{trace}(A^{\varphi}) - \frac{1}{2}\rho\sigma T, \qquad (2.3.24)$$

where

$$trA^{\varphi} = \rho\sigma T + \frac{\mu\sigma^2}{2b^2}\left(e^{-2bT} + 2bT - 1\right).$$

By applying (2.3.10) above with $y = -\mu\sigma e^{-2bT}/b$, $z = \rho + \mu\sigma/b$, $\alpha = \rho\sigma + \sigma^2\mu/b$, and $\beta = -\sigma^2\mu/b$, cf. (2.3.22), (2.3.23) and (2.3.24) we have

$$\frac{1}{2}\int_0^T \psi(t)(I + A^{\varphi})^{-1}\psi(t)\,dt = \frac{\mu}{2b}(1 - e^{-2bT}) - \frac{\mu - \rho^2/2}{b + \rho\sigma + h\coth hT}. \qquad (2.3.25)$$

Next (2.3.21) is obtained by comparison of (2.3.1) in Proposition 2.3.1 and (2.3.25) with Lemma 2.1.3 under the change of variable $\mu = -\beta b/\sigma^2$ and $\rho = (\alpha + \beta)/\sigma$. Note that (2.1.6) holds since $I + A^{\varphi}$ has been proved having positive spectrum by Lemma 2.3.3. □

2.4 Fredholm expansions

We finish this chapter with the computation of the determinant $\det(I + A^{\varphi})$ appearing in Lemma 2.1.3 using Fredholm expansions

$$\det(I + A^{\varphi}) = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{(0,1)^n}\det(\varphi(t_p, t_g))_{p,q=1}^{n}\,\mathrm{d}t_1\cdots\mathrm{d}t_n, \qquad (2.4.1)$$

cf. Theorem 3.10 of Simon (2005), and we first compute the finite-dimensional determinants needed in the Fredholm expansion.

Lemma 2.4.1. *Let $n \geq 2$ and $c_1, \ldots, c_{n-1} \in \mathbb{R}$. Let $A = (a_{i,j})_{1\leq i,j\leq n}$ be a symmetric $n \times n$ matrix such that*

(i)

$$a_{i,i} = 1 + (a_{i+1,i+1} - 1)c_i^2, \qquad i = 1, 2, \ldots, n-1,$$

and

(ii)

$$a_{i,j} = c_j a_{i,j+1}, \qquad j + 1 \leq i \leq n, \quad j = 1, \ldots, n-1.$$

Then we have

$$\det(A) = a_{n,n} \prod_{i=1}^{n-1} (1 - c_i^2). \tag{2.4.2}$$

Proof. According to condition (ii) we have $a_{2,1} = c_1 a_{2,2}$, and since A is symmetric we have $a_{1,2} = c_1 a_{2,2}$ as well. In addition the minors of A are given by

$$M_{1,j} = 0, \qquad j = 3, \ldots, n,$$

and

$$M_{1,2} = c_1 M_{1,1}.$$

We will prove (2.4.2) by induction on $n \geq 1$. For $n = 1$ we have

$$\det(A) = \begin{vmatrix} a_{1,1} & c_1 a_{2,2} \\ c_1 a_{2,2} & a_{2,2} \end{vmatrix} = |a_{1,1}||a_{2,2}| - |c_1|^2 |a_{2,2}|^2 = a_{2,2}(1 - |c_1|^2)$$

by condition (i), hence (2.4.2) holds. Next assuming that (2.4.2) holds at the rank $n \geq 1$, we have

$$M_{1,1} = a_{n,n} \prod_{i=2}^{n-1} (1 - |c_i|^2), \tag{2.4.3}$$

and

$$\begin{aligned}
\det(A) &= \sum_{i=1}^{n} (-1)^{1+j} a_{1,j} M_{1,j} \\
&= a_{1,1} M_{1,1} - a_{1,2} M_{1,2} \\
&= (a_{1,1} - c_1 a_{1,2}) M_{1,1} \\
&= (a_{1,1} - |c_1|^2 a_{2,2}) M_{1,1} \\
&= (1 - |c_1|^2) M_{1,1} \\
&= (1 - |c_1|^2) a_{n,n} \prod_{i=2}^{n-1} (1 - |c_i|^2) \\
&= a_{n,n} \prod_{i=1}^{n-1} (1 - |c_i|^2),
\end{aligned}$$

where we use (2.4.3), thereby showing that (2.4.2) holds for any $n \geq 1$. □

Proposition 2.4.2. *Let*

$$\varphi(s,t) = \alpha e^{-b|t-s|} + \beta e^{-b(2T-s-t)}, \qquad s,t \in [0,T].$$

We have

$$\det(\varphi(t_p,t_g))_{p,q=1}^{n} = \alpha^n \left(1 + \frac{\beta}{\alpha} e^{-2b(T-t_n)}\right) \prod_{i=1}^{n-1} (1 - e^{-2b(t_{i+1}-t_i)}), \tag{2.4.4}$$

where $0 \le t_1 \le \cdots \le t_n \le T$.

Proof. For simplicity, take $\alpha = 1$ here, then we prove (2.4.4) by applying Lemma 2.4.1 with

$$\varphi(s,t) = e^{-b|t-s|} + \beta e^{-b(2T-s-t)}, \qquad s,t \in [0,T].$$

Let $A = (a_{i,j})_{1 \le i,j \le n}$, where

$$a_{i,j} = \varphi(t_i,t_j) = e^{-b|t_i-t_j|} + \beta e^{-b(2T-t_i-t_j))}, \quad i,j = 1,\ldots,n,$$

so that $c_j = e^{-b(t_{j+1}-t_j)}$, $j = 1,\ldots,n-1$, and $a_{j,j} = 1 + \beta e^{-2b(T-t_j)}$, $j = 1,\ldots,n$, which shows that

$$a_{j,j} = 1 + \beta e^{-2b(T-t_1)} = (a_{j+1,j+1} - 1)c_j^2 + 1, \qquad j = 1,2,\ldots,n-1.$$

Hence the assumptions of Lemma 2.4.1 are satisfied and we get

$$\det(A) = (1 + \beta e^{-2b(T-t_n)}) \prod_{i=1}^{n-1} (1 - e^{-2b(t_{i+1}-t_i)}),$$

with $t_{n+1} = T$, which proves (2.4.4). □

In particular when

$$\varphi(s,t) = e^{-b|t-s|} - e^{-b(2T-s-t)}, \qquad s,t \in [0,T],$$

we get

$$\det(\varphi(t_p,t_g))_{p,q=1}^{n} = 2^n e^{b(t_1-T)} \prod_{i=1}^{n} \sinh(b(t_{i+1}-t_i)),$$

$0 \leq t_1 \leq \cdots \leq t_n \leq T$, and if

$$\varphi(s,t) = e^{-b|s-t|}, \qquad s,t \in [0,T],$$

we find

$$\det(\varphi(t_p,t_g))_{p,q=1}^n = 2^{n-1} e^{b(t_1-t_n)} \prod_{i=1}^{n-1} \sinh(b(t_{i+1}-t_i)),$$

$0 \leq t_1 \leq \cdots \leq t_n \leq T$.

Proposition 2.4.3. *Given $b > 0$, $\alpha, \beta \in \mathbb{R}$, and*

$$\varphi(s,t) = \alpha e^{-b|s-t|} + \beta e^{-b(2-s-t)}, \qquad s,t \in [0,1], \tag{2.4.5}$$

we have

$$\begin{aligned} \det(I + A^\varphi) =& 1 + \alpha e^{-2b} \sum_{n=1}^\infty \frac{(2\alpha b)^{n-1}}{(2n-1)!}\ {}_1F_1(n+1; 2n, 2b) \\ &+ \beta e^{-2b} \sum_{n=1}^\infty \frac{(2\alpha b)^{n-1}}{(2n-1)!}\ {}_1F_1(n; 2n, 2b), \end{aligned}$$

where ${}_1F_1(a;b,z)$ is the hypergeometric function

$${}_1F_1(a;b,z) = \sum_{n=0}^\infty \frac{(a)_n z^n}{(b)_n n!}$$

and $(a)_n = a(a+1)\cdots(a+n-1)$ is the rising factorial.

Proof. The determinant $\det(I + A^\varphi)$ is computed by the Fredholm expansion (2.4.1), where

$$\det(\varphi(t_p,t_g))_{p,q=1}^n = \alpha^n \left(1 + \frac{\beta}{\alpha} e^{-2b(1-t_n)}\right) \prod_{i=1}^{n-1} \left(1 - e^{-2b(t_{i+1}-t_i)}\right),$$

$0 < t_1 < \cdots < t_n < 1$, cf. (2.4.4). By the change of variable

$$x_i = 2b(t_{i+1}-t_i), \quad i = 1,\ldots,n-1, \quad x_n = 2b(1-t_n),$$

we get

$$\det(I + A^\varphi)$$

$$
\begin{aligned}
&= 1+\sum_{n=1}^{\infty}\frac{\alpha^n}{(2b)^n}\int_{\substack{x_1+\cdots+x_n\le 2b\\ x_i>0}}\left(1+\frac{\beta}{\alpha}e^{-x_n}\right)\prod_{i=1}^{n-1}(1-e^{-x_i})\,dx_1\cdots dx_n\\
&= 1+\sum_{n=1}^{\infty}\frac{\alpha^n}{(2b)^n}\int_{\substack{x_1+\cdots+x_n\le 2b\\ x_i>0}}\left((1+\frac{\beta}{\alpha})\prod_{i=1}^{n-1}(1-e^{-x_i})\right.\\
&\quad \left.-\frac{\beta}{\alpha}\prod_{i=1}^{n}(1-e^{-x_i})\right)\,dx_1\cdots dx_n\\
&= 1+\sum_{n=1}^{\infty}\frac{\alpha^n}{(2b)^n}\left(-\frac{\beta}{\alpha}\int_{\substack{x_1+\cdots+x_n\le 2b\\ x_i>0}}\prod_{i=1}^{n}(1-e^{-x_i})\,dx_1\cdots dx_n\right.\\
&\quad \left.+\left(1+\frac{\beta}{\alpha}\right)\int_{\substack{x_1+\cdots+x_n\le 2b\\ x_i>0}}\prod_{i=1}^{n-1}(1-e^{-x_i})\,dx_1\cdots dx_n\right)\\
&= 1+e^{-2b}\sum_{n=1}^{\infty}\alpha^n\left(-\frac{\beta}{\alpha}\sum_{m=n}^{\infty}\frac{(2b)^m}{(m+n)!}\binom{m}{n}\right.\\
&\quad \left.+\left(1+\frac{\beta}{\alpha}\right)\sum_{m=n-1}^{\infty}\frac{(2b)^m}{(n+m)!}\binom{m+1}{n}\right)\\
&= 1+e^{-2b}\sum_{n=1}^{\infty}\alpha^n\left(\sum_{m=n-1}^{\infty}\frac{(2b)^m}{(n+m)!}\binom{m+1}{n}\right.\\
&\quad \left.+\frac{\beta}{\alpha}\frac{(2b)^{n-1}}{(2n-1)!}+\frac{\beta}{\alpha}\sum_{m=n}^{\infty}\frac{(2b)^m}{(m+n)!}\frac{n}{m+1-n}\binom{m}{n}\right)\\
&= 1+e^{-2b}\sum_{n=1}^{\infty}\alpha^n\sum_{m=n-1}^{\infty}\frac{(2b)^m}{(n+m)!}\binom{m+1}{n}\\
&\quad +\beta e^{-2b}\sum_{n=1}^{\infty}\frac{(2\alpha b)^{n-1}}{(2n-1)!}\sum_{m=0}^{\infty}\frac{(2b)^m(2n-1)!}{(m+2n-1)!}\frac{(n+m-1)!}{(n-1)!m!}\\
&= 1+e^{-2b}\sum_{n=1}^{\infty}\alpha^n\sum_{m=0}^{\infty}\frac{(2b)^{n-1+m}}{(2n-1+m)!}\binom{m+n}{n}\\
&\quad +\beta e^{-2b}\sum_{n=1}^{\infty}\frac{(2\alpha b)^{n-1}}{(2n-1)!}\;{}_1F_1(n;2n,2b)\\
&= 1+\alpha e^{-2b}\sum_{n=1}^{\infty}\frac{(2\alpha b)^{n-1}}{(2n-1)!}\;{}_1F_1(n+1;2n,2b)\\
&\quad +\beta e^{-2b}\sum_{n=1}^{\infty}\frac{(2\alpha b)^{n-1}}{(2n-1)!}\;{}_1F_1(n;2n,2b).
\end{aligned}
$$

□

In the particular case where $\beta = 0$ and

$$\varphi = \sigma e^{-b|s-t|}, \qquad s,t \in [0,T].$$

Proposition 2.4.3 shows that

$$\det(I + A^{\varphi}) = 1 + \sigma T e^{-2bT} \sum_{n=1}^{\infty} \frac{(2\sigma b T^2)^{n-1}}{(2n-1)!} \ {}_1F_1(n+1; 2n, 2bT).$$

In the particular case where $a = -\beta$ we get the following corollary.

Corollary 2.4.4. *Let $b > 0$ and*

$$\varphi(s,t) = \alpha(e^{-b|s-t|} - e^{-b(2T-s-t)}), \qquad s,t \in [0,T].$$

We have

$$\det(I + A^{\varphi}) = 1 + e^{-2bT} \sum_{n=1}^{\infty} \frac{(2\alpha b T^2)^{n}}{(2n)!} \ {}_1F_1(n+1; 2n+1, 2bT).$$

Proof. By Proposition 2.4.3 we have

$$\begin{aligned}
&\det(I + A^{\varphi}) \\
&= 1 + \alpha T e^{-2bT} \sum_{n=1}^{\infty} \frac{(2\alpha b T^2)^{n-1}}{(2n-1)!} \ {}_1F_1(n+1; 2n, 2bT) \\
&\quad -\alpha T e^{-2bT} \sum_{n=1}^{\infty} \frac{(2\alpha b T^2)^{n-1}}{(2n-1)!} \ {}_1F_1(n; 2n, 2bT) \\
&= 1 + \alpha T e^{-2bT} \sum_{n=1}^{\infty} \frac{(2\alpha b T^2)^{n-1}}{(2n-1)!} \left({}_1F_1(n+1; 2n, 2bT) - {}_1F_1(n; 2n, 2bT)\right) \\
&= 1 + e^{-2bT} \sum_{n=1}^{\infty} \frac{(2\alpha b T^2)^{n}}{(2n)!} \ {}_1F_1(n+1; 2n+1, 2bT).
\end{aligned}$$

□

We also have

$$\begin{aligned}
tr(A^{\varphi}) &= \int_0^T \varphi(s,s)\, ds = \frac{\alpha}{2b}\left(e^{-2bT} + 2bT - 1\right) \\
&= \frac{2\alpha}{\sigma^2}\left(E\left[\int_0^T X_t^2 dt\right] - \frac{x^2}{2b}(1 - e^{-2bT})\right).
\end{aligned}$$

By the Fredholm expansion (2.4.1) and Proposition 2.4.3 we obtain the following corollary.

Corollary 2.4.5. *For all $\rho \geq 0$ and $\mu \in \mathbb{R}$ such that $b^2 + 2\rho b\sigma + 2\mu\sigma^2 > 0$ we have*

$$\begin{aligned}
& E\left[e^{-\rho \int_0^T X_t \, dB_t - \mu \int_0^T X_t^2 \, dt} \Big| X_0 = x \right] \\
= & \left(1 + \rho\sigma T e^{-2bT} \sum_{n=1}^{\infty} \frac{(\rho\sigma b + \mu\sigma^2)^{n-1} (T\sqrt{2})^{2n-2}}{(2n-1)!} \ {}_1F_1(n+1; 2n, 2bT) \right. \\
& \left. \frac{\mu\sigma}{b\rho + \mu\sigma} e^{-2bT} \sum_{n=1}^{\infty} \frac{2^n (\rho\sigma b + \mu\sigma^2)^n T^{2n}}{(2n)!} \ {}_1F_1(n+1; 2n+1, 2bT) \right)^{-1/2} \\
& \times \exp\left(\frac{1}{2}\sigma\rho T - \frac{x^2(\mu - \rho^2/2)}{b + \rho\sigma + h \coth hT} \right), \qquad (2.4.6)
\end{aligned}$$

where ${}_1F_1(a; b, z)$ is the hypergeometric function

$$ {}_1F_1(a; b, z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!} $$

and $(a)_n = a(a+1) \cdots (a+n-1)$ is the rising factorial.

Proof. By Proposition 2.4.3 we have

$$\begin{aligned}
\det(I + A^\varphi) = & 1 + T e^{-2bT} \sum_{n=1}^{\infty} \frac{(2\alpha b T^2)^{n-1}}{(2n-1)!} (\alpha \ {}_1F_1(n+1; 2n, 2bT) \\
& + \beta \ {}_1F_1(n; 2n, 2bT)),
\end{aligned}$$

where A^φ is given by (2.3.20), and we conclude from (2.1.6). □

2.4.1 Examples

In this section we consider some particular cases.

Laplace transform of $\int_0^T X_s^2 \, ds$

By taking $\rho = 0$, $\mu = 1$ we obtain the Laplace transform

$$\begin{aligned}
F(t, x) & := E\left[e^{-\int_t^T X_s^2 ds} \Big| X_t = x \right] \\
& = \left(\cosh h(T-t) + \frac{b}{h} \sinh h(T-t) \right)^{-1/2}
\end{aligned}$$

$$
\begin{aligned}
&\exp\left(\frac{b}{2}(T-t)-\frac{x^2}{b+h\coth h(T-t)}\right)\\
= &\left(\frac{he^{(b+h)(T-t)}}{h+(b+h)(e^{2h(T-t)}-1)/2}\right)^{1/2}\\
&\exp\left(-\frac{x^2(e^{2h(T-t)}-1)}{2h+(b+h)(e^{2h(T-t)}-1)}\right),
\end{aligned}
$$

where $h=\sqrt{b^2+2\sigma^2}$. In addition it follows from (2.4.6) that

$$
\begin{aligned}
&E\left[e^{-\int_0^T X_s^2\,ds}\middle|X_0=x\right]\\
= &\left(1+e^{-2bT}\sum_{n=1}^{\infty}\frac{((2\sigma^2)T^2)^n}{(2n)!}\ {}_1F_1(n+1;2n+1,2bT)\right)^{-1/2}\\
&\exp\left(-\frac{x^2}{b+h\coth hT}\right)\\
= &\left(1+e^{-2bT}\sum_{n=1}^{\infty}\frac{(2\sigma^2T^2)^n}{(2n)!}\ {}_1F_1(n+1;2n+1,2bT)\right)^{-1/2}\\
&\exp\left(-\frac{x^2(e^{2hT}-1)}{2h+(b+h)(e^{2hT}-1)}\right).
\end{aligned}
$$

In this case we have

$$
\varphi(s,t)=\alpha(e^{-b|s-t|}-e^{-b(2T-s-t)}),\qquad s,t\in[0,T],
$$

$$
\det(I+A^{\varphi})=e^{-bT}\left(\cosh(hT)+\frac{b}{h}\sinh(hT)\right),
$$

and an alternative expression of the determinant

$$
\det(I+A^{\varphi})=1+e^{-2bT}\sum_{n=1}^{\infty}\frac{(2\alpha bT^2)^n}{(2n)!}\ {}_1F_1(n+1;2n+1,2bT),
$$

where $h=\sqrt{b^2+2\alpha b}$.

Laplace transform of $\int_0^T X_s\,dB_s$

By taking $\rho=1,\mu=0$ in (2.3.1) we find

$$
E\left[e^{-\int_0^T X_t\,dB_t}\middle|X_0=x\right]=\left(\cosh hT+\frac{b+\sigma}{h}\sinh hT\right)^{-1/2}
$$

$$
\begin{aligned}
&\exp\left(\frac{(b+\sigma)T}{2}+\frac{x^2}{2b+2\sigma+2h\coth(hT)}\right)\\
= &\left(1+\sigma T e^{-2bT}\sum_{n=1}^{\infty}\frac{(2\sigma bT^2)^{n-1}}{(2n-1)!}\ {}_1F_1(n+1;2n,2bT)\right)^{-1/2}\\
&\exp\left(\frac{\sigma T}{2}+\frac{x^2}{2b+2\sigma+2h\coth(hT)}\right),
\end{aligned}
$$

where $h=\sqrt{b^2+2\sigma b}$. In this case we have

$$
\varphi(s,t)=\sigma e^{-b|t-s|}, \qquad s,t\in[0,T],
$$

$$
\det(I+A^{\varphi})=e^{-bT}\left(\cosh(hT)+\frac{b+\sigma}{h}\sinh(hT)\right),
$$

and an alternative expression of the determinant

$$
\det(I+A^{\varphi})=1+\sigma T e^{-2bT}\sum_{n=1}^{\infty}\frac{(2\sigma bT^2)^{n-1}}{(2n-1)!}\ {}_1F_1(n+1;2n,2bT).
$$

Laplace transform of $\int_0^T X_t\,dX_t$

Similarly here taking $\rho=\sigma$ and $\mu=-b$, i.e. $h=b$, we can show that

$$
\begin{aligned}
&E\left[e^{-\int_0^T X_s\,dX_s}\Big|X_0=x\right]\\
&= \exp\left(\frac{b+\sigma^2}{2}T+\frac{x^2(\sigma^2+2b)}{2b+2\sigma^2+2b\coth bT}\right)\\
&\quad\left(\cosh bT+\frac{(b+\sigma^2)}{b}\sinh bT\right)^{-1/2}\\
&= \exp\left(\frac{\sigma^2}{2}T+\frac{x^2(2b+\sigma^2)}{2b+2\sigma^2+2b\coth(bT)}\right)\\
&\quad\left(1+\frac{\sigma^2}{2b}(1-e^{-2bT})\right)^{-1/2},
\end{aligned}
$$

by (2.1.7). In addition we have

$$
\varphi(s,t)=\beta e^{-b(2T-s-t)}, \qquad s,t\in[0,T],
$$

and

$$
\det(I+A^{\varphi})=1+\frac{\beta}{2b}(1-e^{-2bT})=1+\frac{\beta}{b}e^{-bT}\sinh(bT).
$$

Chapter 3

Normal martingale chaos-based models of spot forward rates

This chapter is mainly to study the chaos-based model of spot forward rates and the calibration of these chaos models, especially the Wiener-Poisson decomposition, cf. León et al. (1998) page 269−282. We extend the framework of Hughston and Rafailidis (2005) from Brownian motion to normal martingale setting, for the purpose of implementing and calibrating to market data of spot forward rates. We obtain the chaotic models based on normal martingale chaos expansion under the extended framework.

3.1 Preliminaries

3.1.1 Basic definitions

Recall that given a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, a process $(Y_t)_{t\in\mathbb{R}_+}$ satisfies the following relations

- for any time t, $E_{\mathcal{P}}\left[\mid Y_t \mid\right] < +\infty$,
- for any time t, $s < t$, $Y_s = E_{\mathcal{P}}\left[Y_t \mid \mathcal{F}_s\right]$,

then $(Y_t)_{t\in\mathbb{R}_+}$ is a martingale.

For any time t, if $E_{\mathcal{P}}\left[\| Y_t \|\right] < +\infty$ and $Y_s \geq E_{\mathcal{P}}\left[Y_t \mid \mathcal{F}_s\right]$, $s < t$, then the process $(Y_t)_{t\in\mathbb{R}_+}$ is a supermartingale.

If a square-integrable martingale $(M_t)_{t\in\mathbb{R}_+}$ satisfying the condition

$$E\left[(M_t - M_s)^2 \mid \mathcal{F}_s\right] = t - s,\ 0 \leq s < t$$

or equivalently $(M_t^2 - t)_{t\in\mathbb{R}_+}$ is a martingale, then $(M_t)_{t\in\mathbb{R}_+}$ is called a normal martingale.

For the normal martingale $(M_t)_{t\in\mathbb{R}_+}$, its quadratic variation is the process $([M,M]_t)_{t\in\mathbb{R}_+}$ defined as

$$[M,M]_t = M_t^2 - 2\int_0^t M_s\, dM_s, t \in \mathbb{R}_+.$$

A structure equation is an equation of the form

$$d\,[M,M]_t = dt + \phi_t\, dM_t, \tag{3.1.1}$$

where $(\phi_t)_{t\in\mathbb{R}_+}$ is a square-integrable adapted process and $(M_t)_{t\in\mathbb{R}_+}$ is a martingale.

In this chapter we only consider the normal martingale M_t having chaos representation property and satisfying the structure equation (3.1.1), see Émery (1989), Azéma and Yor (1989) and references therein.

Recall that every square-integrable functional admits orthogonal decomposition in a series of multiple stochastic integrals,

$$\sum_{n=0}^{\infty} I_n(f_n) \tag{3.1.2}$$

where the multiple stochastic integrals are defined by

$$I_n(f_n) = n! \int_0^\infty \int_0^{t_n^-} \cdots \int_0^{t_2^-} f_n(t_1, \cdots, t_n)\, dM_{t_1} \cdots dM_{t_n}.$$

The chaos coefficients $\{f_n(t_1, \cdots, t_n),\ n \geq 0\}$ of different orders are deterministic square-integrable functions. The n-th chaos interest rate model means that the functional is only determined by its first n chaos coefficients $f_1(t_1)$, $f_2(t_1, t_2)$, $\cdots$ $f_n(t_1, \ldots, t_n)$. Here orthogonality means that the equality $E(I_n(f_n)I_m(g_m)) = 0$ holds for any positive integers $n \neq m$.

Remark 3.1.1. *We have the isometry formula for the normal martingale M_t in* (3.1.1) *as follows*

$$E\left[\int_0^\infty u_t\,dM_t \int_0^\infty v_t\,dM_t\right] = E\left[\int_0^\infty u_s v_s\,ds\right],$$

where $(u_t)_{\in\mathbb{R}_+}$ and $(v_t)_{t\in\mathbb{R}_+}$ are square-integrable adapted processes.

3.1.2 Flesaker-Hughston axiomatic framework

The axiomatic framework in Hughston and Rafailidis (2005) assumes the following four axioms hold in the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with the augmented filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ being generated by a Wiener process $(W_t)_{t\in\mathbb{R}_+}$ and assume the existence of the positive adapted process $(\xi_t)_{t\in\mathbb{R}_+}$ known as "natural numéraire".

(1) There exists a strictly increasing asset with price process $(B_t)_{t\in\mathbb{R}_+}$ (the money market account), and the positive continuous adapted interest rate process $(r_t)_{t\in\mathbb{R}_+}$, such that
$$B_t = B_0 \exp\left(\int_0^t r_s\,ds\right).$$

(2) If $(S_t)_{t\in\mathbb{R}_+}$ is the price process of any asset, and $(D_t)_{t\in\mathbb{R}_+}$ is the adapted dividend rate for the asset, then the process
$$\frac{S_t}{\xi_t} + \int_0^t \frac{D_s}{\xi_s}\,ds$$
is a martingale.

(3) There exists an asset (a floating rate note) that offers a dividend rate sufficient to ensure that the value of the asset remains constant.

(4) There exists a system of discounted bond price process $P(t,T)$ for all $0 \leq t \leq T$ having $\lim_{T\to\infty} P(t,T) = 0$ almost surely.

This framework guarantees the interest rates to be positive and no arbitrage. It follows that the zero-coupon bond prices at time t with maturity T is expressed in the form of

$$P(t,T) = \frac{E\left[V_T|\mathcal{F}_t\right]}{V_t},$$

where the positive supermartingale $(V_t)_{t\in\mathbb{R}_+}$ is called the state price density and the spot forward rate is given by

$$\begin{aligned} f(t,t,T) &= \frac{\log(P(t,t)) - \log(P(t,T))}{T-t} \\ &= -\frac{\log(P(t,T))}{T-t}. \end{aligned}$$

Under this framework, Hughston and Rafailidis (2005) give the Wiener chaos based models of default-free zero-coupon bond price.

3.2 Extended framework

Here we extend the Flesaker-Hughston axiomatic framework in subsection (3.1.2). We continue using some notations in the Flesaker-Hughston axiomatic framework, under a new complete probability space $(\Omega, \mathcal{F}, \mathcal{P}^*)$ with filtration $\Pi = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ being generated by a chaos-decomposable normal martingale $(M_t)_{t\in\mathbb{R}_+}$. The normal martingale M_t satisfies the structure equation (3.1.1). Here we assume the existence of a non-dividend-paying asset with positive price process $(\xi_t)_{t\in\mathbb{R}_+}$, known as natural numéraire, which is adapted to the filtration Π, such that the axioms (2), (3) and (4) hold. Moreover, we relax the axiom (1) to assume the existence of an adapted interest rate process $r_t > 0$, so that the money account $B_t = B_0 \exp\left(\int_0^t r_s\, ds\right)$ exists.

Next we derive the variance representation of the state price density V_t in the extended framework. Calibration of the chaotic spot forward rate models is included at the end of this chapter .

3.2.1 Price density V_t

By relaxed axiom (1) we have the money account

$$B_t = B_0 \exp\left(\int_0^t r_s\, ds\right). \tag{3.2.1}$$

Since B_t is non-dividend-paying, i.e. $D_t = 0$ in axiom (2), there exists a positive martingale $(\rho_t)_{t\in\mathbb{R}_+}$ such that

$$\frac{B_t}{\xi_t} = \rho_t. \tag{3.2.2}$$

Because $(\rho_t)_{t\in\mathbb{R}_+}$ is positive, there exists an adapted process $(\mu_t)_{t\in\mathbb{R}_+}$ so that we have

$$d\rho_t = -\rho_t\mu_t\, dM_t. \tag{3.2.3}$$

For a non-dividend-paying asset $(S_t)_{t\in\mathbb{R}_+}$, axiom (2) says $\left(\frac{S_t}{\xi_t}\right)_{t\in\mathbb{R}_+}$ is a martingale, therefore there exists an adapted process $(\theta_t)_{t\in\mathbb{R}_+}$ for the equation

$$d\frac{S_t}{\xi_t} = \theta_t\, dM_t. \tag{3.2.4}$$

By applying Itô's formula to equations (3.2.2), (3.2.3) and (3.2.4), we have

$$dS_t = (r_tS_t + \mu_t\psi_t)\; dt + \psi_t\, dM_t,$$

where

$$\psi_t = \frac{\mu_tS_t + \theta_t\xi_t}{1 - \mu_t\phi_t}.$$

If the positive asset S_t is dividend paying, then we obtain

$$dS_t = (r_tS_t - D_t + \mu_t\psi_t)\; dt + \psi_t\, dM_t. \tag{3.2.5}$$

Axiom (3) and equation (3.2.5) tell us if we take $S_t = 1$ then the dividend rate $D_t = r_t$ and the process

$$\frac{1}{\xi_t} + \int_0^t \frac{r_s}{\xi_s}\, ds \tag{3.2.6}$$

is a martingale.

Lemma 3.2.1. *The dynamics of the price density* $V_t = 1/\xi_t$ *is given by*

$$dV_t = -r_t V_t dt - \mu_t V_t \, dM_t$$

and the process

$$V_t + \int_0^t r_s V_s \, ds,$$

is a martingale.

Proof. It follows from equations (3.2.1), (3.2.2), (3.2.3) and (3.2.6). □

Next we will apply chaotic techniques to the process $(V_t)_{t \in \mathbb{R}_+}$.

3.2.2 Variance representation of the price density V_t

Before we apply the chaos expansion techniques, it is necessary to get the representation of V_t and obtain the bond price formula.

Recall that the dynamics of V_t is given by

$$dV_t = -r_t V_t dt - \mu_t V_t \, dM_t$$

and the process

$$V_t + \int_0^t r_s V_s \, ds \tag{3.2.7}$$

is a martingale, by which we conclude that

$$V_t + \int_0^t r_s V_s \, ds = E\left[V_T | \mathcal{F}_t\right] + E\left[\int_0^T r_s V_s \, ds | \mathcal{F}_t\right]. \tag{3.2.8}$$

It follows from axiom (2) that the default-free zero-coupon bond price has the property

$$P(t,T) V_t = E\left[P(T,T) V_T | \mathcal{F}_t\right],$$

i.e.

$$P(t,T) = \frac{E\left[V_T | \mathcal{F}_t\right]}{V_t}. \tag{3.2.9}$$

From equation (3.2.8) and equation (3.2.9) we can deduce that

$$P(t,T) = 1 - \frac{1}{V_t} E\left[\int_t^T r_s V_s \, ds | \mathcal{F}_t\right], \tag{3.2.10}$$

the above pricing formula has economic interpretations, for example, for any two maturity dates T_1 and T_2, we have

$$P(t,T_1) - P(t,T_2) = \frac{1}{V_t} E\left[\int_{T_1}^{T_2} r_s V_s \, ds | \mathcal{F}_t\right],$$

it is apparent that $P(t,T_2) < P(t,T_1)$ holds under the condition $T_1 < T_2$.

Axiom (4) states

$$\lim_{T\to\infty} P(t,T) = 0,$$

therefore we take infinitely large maturity T in (3.2.10), and then we have

$$\lim_{T\to\infty} E\left[V_T|\mathcal{F}_t\right] = 0. \tag{3.2.11}$$

Moreover, equation (3.2.8) and equation (3.2.11) lead to

$$V_t = \lim_{T\to\infty} E\left[\int_t^T r_s V_s \, ds | \mathcal{F}_t\right].$$

By the fact that the process (3.2.7) is a positive martingale, and according to martingale convergence theorem there exists a random variable

$$A_\infty = \int_0^\infty r_s V_s \, ds,$$

to which the process

$$A_t = \int_0^t r_s V_s ds$$

converges almost surely as t goes to positive infinity. Consequently we have

$$V_t = E\left[\int_t^\infty r_s V_s \, ds | \mathcal{F}_t\right] \tag{3.2.12}$$

by the monotone convergence theorem. As a result, we have the following proposition stating the representation for the supermartingale price density V_t.

Proposition 3.2.2. *Now we define*

$$\sigma_t^2 := r_t V_t, \tag{3.2.13}$$

and the variable

$$X_\infty := \int_0^\infty \sigma_s dM_s.$$

Then V_t is expressed in the form of

$$V_t = E\left[(X_\infty - E[X_\infty|\mathcal{F}_t])^2 \,|\mathcal{F}_t\right].$$

Proof. We have

$$E[X_\infty|\mathcal{F}_t] = \int_t^\infty \sigma_s dM_s$$

and

$$\begin{aligned} E\left[(X_\infty - E[X_\infty \mid \mathcal{F}_t])^2 \mid \mathcal{F}_t\right] &= E\left[\left(\int_t^\infty \sigma_s \, dM_s\right)^2 \middle| \mathcal{F}_t\right] \\ &= E\left[\int_t^\infty \sigma_s^2 \, ds \middle| \mathcal{F}_t\right] \\ &= E\left[\int_t^\infty r_s V_s \, ds \middle| \mathcal{F}_t\right]. \end{aligned}$$

As a consequence, the following expression

$$V_t = E\left[(X_\infty - E[X_\infty|\mathcal{F}_t])^2 \,|\mathcal{F}_t\right]$$

holds. □

It is clear that the variable X_∞ can be taken to construct the discounted bond price. After the bond price being derived, the prices of fixed income derivatives such as swaption can be easily computed. For simplicity, here we only consider a payer swaption, which gives the owner of the swaption the right to enter into a interest rate swap, where the owner pays the fixed rate and receives the floating rate. We assume the swap has unit notional and unit year fraction, such as t-maturity swaption contract with strike K written on n bonds. Each bond has time t price $P(t, T_i)$ with maturity T_i, $i = 1, \ldots, n$. Then the payoff of the swaption is

$$\left(1 - P(t, T_n) - K \sum_{i=1}^n P(t, T_i)\right)^+,$$

and its price $Swaption(0, t)$ at time zero is

$$Swaption(0, t) = \frac{1}{V_0} E\left[V_t \left(1 - P(t, T_n) - K \sum_{i=1}^n P(t, T_i)\right)^+ \middle| \mathcal{F}_t\right], \quad (3.2.14)$$

and we will look closer at the swaption price later in this chapter.

3.3 Defaultable bonds

Before we construct the chaos models of spot forward rates, here we simply illustrate that the defaultable bond price can also be constructed by the square-integrable variable X_∞.

We consider pricing the defaultable zero-coupon bond in this section. Given a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ with filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ being generated by a standard Brownian motion $(W_t)_{t\in\mathbb{R}_+}$, we define the default time τ for the bond paying one dollar at maturity T. The enlarged filtration is represented by $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\} : 0 \leq u \leq t)$. The default intensity is denoted as λ_t, then the survival probability of the bond up to time T, given information known up to time t, is

$$P(\tau > T \mid \mathcal{F}_t) = \exp\left(-\int_t^T \lambda_s\, ds\right), \quad t \in [0, T].$$

Therefore the defaultable bond price $v(t, T)$ with constant recovery rate δ is

$$\begin{aligned} v(t,T) &= E_P\left[\frac{B_t}{B_T}(\delta \mathbb{1}_{\{\tau\leq T\}} + \mathbb{1}_{\{\tau>T\}})\Big|\mathcal{G}_t\right] \qquad (3.3.1)\\ &= E_P\left[\frac{B_t}{B_T}((1-\delta)\mathbb{1}_{\{\tau>T\}} + \delta)\Big|\mathcal{G}_t\right] \\ &= \delta E_P\left[\frac{B_t}{B_T}\Big|\mathcal{F}_t\right] + (1-\delta)E_P\left[\frac{B_t}{B_T}\mathbb{1}_{\{\tau>T\}}\Big|\mathcal{G}_t\right] \\ &= \delta E_P\left[\frac{B_t}{B_T}\Big|\mathcal{F}_t\right] + (1-\delta)\mathbb{1}_{\{\tau>t\}}E_P\left[\frac{B_t}{B_T}\exp\left(-\int_t^T \lambda_s\, ds\right)\Big|\mathcal{F}_t\right] \end{aligned}$$

by applying Lando's formula, see Lando (1998).

Proposition 3.3.1. *(Lando (1998)) For any $\mathcal{F}_T$-measurable integrable random variable F, the following equality holds:*

$$E\left[F\mathbb{1}_{\{\tau>T\}}\big|\mathcal{G}_t\right] = \mathbb{1}_{\{\tau>t\}}E\left[F\exp\left(-\int_t^T \lambda_u\, du\right)\Big|\mathcal{F}_t\right].$$

Equation (3.3.1) shows that the defaultable bond price is divided into two parts, one is the recovery received at time t, the other is the rest received if no default happens before. So defaultable bond pricing relies on two parts

$$E_P\left[\frac{B_t}{B_T}\bigg|\mathcal{F}_t\right] \qquad \text{and} \qquad E_P\left[\frac{B_t}{B_T}\exp\left(-\int_t^T \lambda_s\, ds\right)\bigg|\mathcal{F}_t\right].$$

Actually what we will do next is to change the numéraire, the method has been known in asset pricing, cf. Geman et al. (1995) and Schroder (1999). For the calculation of $E_P\left[\frac{B_t}{B_T}\Big|\mathcal{F}_t\right]$, we just take the numéraire ξ_t and then work under the framework of Hughston and Rafailidis (2005) with forward measure $\mathcal{P}$, where $\frac{B_t}{\xi_t}$ is a martingale and $\frac{\xi_t}{B_t}$ is a martingale under measure P, i.e. $E_P\left[\frac{B_t}{B_T}\Big|\mathcal{F}_t\right] = E_{\mathcal{P}}\left[\frac{V_T}{V_t}\Big|\mathcal{F}_t\right]$.

The next step is to calculate $E_P\left[\frac{B_t}{B_T}\exp\left(-\int_t^T \lambda_s\, ds\right)\Big|\mathcal{F}_t\right]$. We take the numéraire $(N_t)_{t\in\mathbb{R}_+}$, which is a strictly positive $\mathcal{F}_t$ adapted stochastic process such that

$$\frac{N_t}{e^{\int_0^t(\lambda_s+r_s)ds}} \quad \text{is a martingale under the measure } P.$$

Proposition 3.3.2. *We change the measure by*

$$\frac{d\hat{P}}{dP} = e^{-\int_0^T(\lambda_s+r_s)\,ds}\frac{N_T}{N_0},$$

which leads to

$$E_P\left[\frac{B_t}{B_T}\exp\left(-\int_t^T \lambda_s\, ds\right)\bigg|\mathcal{F}_t\right] = E_{\hat{P}}\left[\frac{N_t}{N_T}\bigg|\mathcal{F}_t\right] \quad t\in[0,T]. \tag{3.3.2}$$

Proof. For any bounded integrable $\mathcal{F}_t$-measurable variable Z, and integrable $\mathcal{F}_T$-measurable variable X,

$$\begin{aligned}
E_P\left[ZE_P\left[X\frac{d\hat{P}}{dP}\bigg|\mathcal{F}_t\right]\right] &= E_P\left[ZX\frac{d\hat{P}}{dP}\right] \\
&= E_{\hat{P}}[ZX] \\
&= E_{\hat{P}}[ZE_{\hat{P}}[X \mid \mathcal{F}_t]] \\
&= E_P\left[Z\frac{d\hat{P}}{dP}E_{\hat{P}}[X \mid \mathcal{F}_t]\right] \\
&= E_P\left[ZE_P\left[\frac{d\hat{P}}{dP}\bigg|\mathcal{F}_t\right]E_{\hat{P}}[X \mid \mathcal{F}_t]\right],
\end{aligned}$$

Since $E_P\left[\frac{d\hat{P}}{dP}\middle|\mathcal{F}_t\right]E_{\hat{P}}\left[X \mid \mathcal{F}_t\right]$ is $\mathcal{F}_t$-measurable, we get

$$E_P\left[X\frac{d\hat{P}}{dP}\middle|\mathcal{F}_t\right] = E_P\left[\frac{d\hat{P}}{dP}\middle|\mathcal{F}_t\right]E_{\hat{P}}\left[X|\mathcal{F}_t\right],$$

i.e.

$$E_{\hat{P}}\left[X|\mathcal{F}_t\right] = \frac{E_P\left[X\frac{d\hat{P}}{dP}\middle|\mathcal{F}_t\right]}{E_P\left[\frac{d\hat{P}}{dP} \mid \mathcal{F}_t\right]}. \tag{3.3.3}$$

Take $X = \frac{N_t}{N_T}$ in (3.3.3) and it leads to the formula

$$E_{\hat{P}}\left[\frac{N_t}{N_T}\middle|\mathcal{F}_t\right] = \frac{E_P\left[\frac{N_t}{N_T}\frac{d\hat{P}}{dP}\middle|\mathcal{F}_t\right]}{E_P\left[\frac{d\hat{P}}{dP}\middle|\mathcal{F}_t\right]} = E_P\left[e^{-\int_t^T(\lambda_s+r_s)\,ds}\middle|\mathcal{F}_t\right],$$

which means the equality (3.3.2) holds. □

It is noticed that after we change the measure by $\frac{d\hat{P}}{dP} = e^{-\int_0^T(\lambda_s+r_s)\,ds}\frac{N_T}{N_0}$, for any $s < t$,

$$\begin{aligned}
E_{\hat{P}}\left[\frac{1}{N_t}e^{\int_0^t(\lambda_s+r_s)ds}\right] &= E_P\left[\frac{d\hat{P}}{dP}\frac{1}{N_t}e^{\int_0^t(\lambda_s+r_s)ds}\right] \\
&= E_P\left[\frac{1}{N_t}e^{\int_0^t(\lambda_s+r_s)ds}E_P\left[e^{-\int_0^T(\lambda_s+r_s)\,ds}\frac{N_T}{N_0}\middle|\mathcal{F}_t\right]\right] \\
&= E_P\left[\frac{1}{N_t}e^{\int_0^t(\lambda_s+r_s)ds}e^{-\int_0^t(\lambda_s+r_s)\,ds}\frac{N_t}{N_0}\right] \\
&= \frac{1}{N_0},
\end{aligned}$$

therefore the process

$$\frac{1}{N_t}e^{\int_0^t(\lambda_s+r_s)ds}, \; t \in \mathbb{R}_+$$

is a martingale under the new measure $\hat{P}$. Next we will construct N_t to do bond pricing.

Here we take the normal martingale $(M_t)_{t\in\mathbb{R}_+}$ as a standard Brownian motion $(W_t)_{t\in\mathbb{R}_+}$, the Girsanov theorem in Jacod and Shiryaev (1987) says the process

$(\hat{W}_t)_{t\in\mathbb{R}_+}$ defined by

$$\begin{aligned} d\hat{W}_t = & \quad dW_t - \frac{1}{m_t} dm_t dW_t, \qquad t \in \mathbb{R}_+, \\ = & \quad dW_t - \frac{1}{N_t} dN_t dW_t, \qquad t \in \mathbb{R}_+ \end{aligned}$$

is a standard Brownian motion under forward measure $\hat{P}$, where $m_t = E_P\left[\frac{d\hat{P}}{dP}\Big|\mathcal{F}_t\right]$.

Since the process $\frac{1}{N_t}e^{\int_0^t(\lambda_s+r_s)\,ds}$ is a martingale under the forward measure, we can postulate that there exists an adapted process $(\nu_t)_{t\in\mathbb{R}_+}$ for the equation

$$d\frac{1}{N_t} = -((\lambda_t + r_t))\frac{1}{N_t}dt - \nu_t\frac{1}{N_t}d\hat{W}_t. \tag{3.3.4}$$

The above equation (3.3.4) shows that the process

$$\frac{1}{N_t} + \int_0^t (\lambda_s + r_s)\frac{1}{N_s}ds$$

is a martingale under the measure $\hat{P}$, therefore we have

$$E_{\hat{P}}\left[\frac{1}{N_T} + \int_0^T (\lambda_s + r_s)\frac{1}{N_s}\,ds\Big|\mathcal{F}_t\right] = \frac{1}{N_t} + \int_0^t (\lambda_s + r_s)\,ds,$$

i.e.

$$\frac{1}{N_t} - E_{\hat{P}}\left[\frac{1}{N_T}\Big|\mathcal{F}_t\right] = E_{\hat{P}}\left[\int_t^T (\lambda_s + r_s)\frac{1}{N_s}\,ds\Big|\mathcal{F}_t\right]. \tag{3.3.5}$$

By the fact that

$$\lim_{T\to+\infty} E_P\left[e^{-\int_t^T r_s ds}\Big|\mathcal{F}_t\right] = 0,$$

the following limit can be deduced,

$$\lim_{T\to+\infty} E_P\left[e^{-\int_t^T(\lambda_s+r_s)ds}\Big|\mathcal{F}_t\right] = 0. \tag{3.3.6}$$

Proposition 3.3.3. *Define two variables*

$$\sigma_t^2 = (\lambda_t + r_t)\frac{1}{N_t}$$

and

$$X_\infty = \int_t^{+\infty} \sigma_s d\hat{W}_s,$$

then it follows

$$\frac{1}{N_t} = E_{\hat{P}}\left[(X_\infty - E_{\hat{P}}[X_\infty \mid \mathcal{F}_t])^2 \mid \mathcal{F}_t\right].$$

Proof. Equation (3.3.2) and equation (3.3.6) show that

$$\lim_{T\to+\infty} E_{\hat{P}}\left[\frac{1}{N_T}\middle|\mathcal{F}_t\right] = 0.$$

Now we go back to equation (3.3.5) to see a new expression of $\frac{1}{N_t}$ by taking T to be positive infinity,

$$\frac{1}{N_t} = E_{\hat{P}}\left[\int_t^{+\infty}(\lambda_s + r_s)\frac{1}{N_s}ds \mid \mathcal{F}_t\right].$$

The rest of the proof is the same as the proof of Proposition 3.2.2. □

Consequently, we can now construct the variable X_∞ through its chaos decomposition (3.1.2) as

$$X_\infty = \int_0^\infty \phi_1(s)d\hat{W}_s + \int_0^\infty\int_0^s \phi_2(s,s_1)d\hat{W}_{s_1}d\hat{W}_s + \cdots ,$$

so the n-th chaos expression of totally defaultable zero-coupon bond price can be derived as before.

We will get back to study the chaos models of spot forward rates in the following sections.

3.4 First chaos model

For first chaos model of spot forward rates, the square-integrable random variable X_∞ takes the form of

$$X_\infty = \int_0^\infty \gamma_s \, dM_s,$$

where γ_s is a deterministic function of one variable. The resulting state price density is

$$V_t = \int_t^\infty \gamma_s^2 \, ds,$$

and the corresponding default-free bond price is

$$P(t,T) = \frac{E\left[V_T|\mathcal{F}_t\right]}{V_t} = \frac{\int_T^\infty \gamma_s^2 \, ds}{\int_t^\infty \gamma_s^2 \, ds}. \tag{3.4.1}$$

The swaption price at time zero in (3.2.14) is derived as

$$Swaption(0,t) = \frac{\left(\int_t^{T_n} \gamma_s^2\, ds - K \sum_{i=1}^n \int_{T_i}^\infty \gamma_s^2\, ds\right)^+}{\int_0^\infty \gamma_s^2\, ds},$$

and the spot forward rate is given by

$$f(t,t,T) = \frac{1}{T-t} \log \frac{\int_t^\infty \gamma_s^2\, ds}{\int_T^\infty \gamma_s^2\, ds}.$$

So far for the first chaos model, we get the same dynamics of bond pricing (3.4.1) as in Hughston and Rafailidis (2005) page 56.

3.4.1 Calibration of first chaos model

The well known Nelson-Siegel family $F(\cdot, z)$ (cf. Björk (2004), Filipović (1999)), has been extensively used in modeling rates by choosing the parameters to fit in initial term structure, such as yield curve in Björk and Christensen (1999), where the $F(\cdot, z)$ is given by

$$F(x,z) = z_1 + (z_2 + z_3 x)e^{-z_4 x}. \tag{3.4.2}$$

We parameterize chaos coefficients in the form of (3.4.2) in the whole chapter.

The first chaos model implies that the positive interest rate is deterministic, and later in this chapter we will see the stochastic property of interest rate in second and higher chaos models. Now if we take $\gamma_s = (b_1 + b_2 s)e^{-b_3 s}$ as an example, then the spot rate

$$f(t,t,T) = \frac{1}{T-t} \log \frac{(2b_3(b_2 t + b_1) + b_2)^2 + b_2^2}{(2b_3(b_2 T + b_1) + b_2)^2 + b_2^2} + 2b_3.$$

For simplicity we define $f(t,\delta) = f(t,t,t+\delta)$, therefore the spot rate formula is reduced to

$$f(t,\delta) = \frac{1}{\delta} \log \frac{(2b_3(b_2 t + b_1) + b_2)^2 + b_2^2}{(2b_3(b_2(t+\delta) + b_1) + b_2)^2 + b_2^2} + 2b_3. \tag{3.4.3}$$

We collect the quarterly data of spot rates from year 2003 to year 2013 on

Bloomberg Terminal, cf. Appendix I Table A1 on page 108, containing 440 values with maturities δ being 3M, 6M, 1Y, 2Y, 3Y, 5Y, 7Y, 10Y, 20Y and 30Y respectively.

In the whole chapter we set the time $t_0 = 0$ as of March 31, 2003, and quarterly time series are denoted by the vector $t_i = 0.25 * i$ ($i = 0, 1, \cdots, 43$), with the forward time vector $\delta = [0.25\ 0.5\ 1\ 2\ 3\ 5\ 7\ 10\ 20\ 30]$.

Each $\hat{f}(t_i, \delta_j)$ represents the real value of spot rate at time t_i with forward time δ_j, $i = 0, \ldots, 43$, $j = 1, \ldots, 10$. Each $f(t_i, \delta_j)$ represents the model value of spot rate at time t_i with forward time δ_j in (3.4.3), $i = 0, \ldots, 43$, $j = 1, \ldots, 10$.

Table 3.1 contains the solution b_1, b_2, b_3 of the optimization problem

$$\begin{aligned}\min_{b_1,b_2,b_3} L(b_1, b_2, b_3) &= \sum_{i=1}^{43}\sum_{j=1}^{10}\left(\hat{f}(t_i, \delta_j) - f(t_i, \delta_j)\right)^2 \\ &= \sum_{i=1}^{43}\sum_{j=1}^{10}\left(\hat{f}(t_i, \delta_j) - \frac{1}{\delta_j}\log\frac{(2b_3(b_2t_i + b_1) + b_2)^2 + b_2^2}{(2b_3(b_2(t_i + \delta_j) + b_1) + b_2)^2 + b_2^2} + 2b_3\right)^2\end{aligned}$$

Table 3.1: *Optimal solution of first chaos model calibration.*

b_1	b_2	b_3
-0.05606	-0.01399	0.056007

Table 3.2 contains the solutions of the optimization problems for each time $t_i = 0.25 * i$ ($i = 0, 1, \ldots, 43$) respectively,

$$\begin{aligned}L(b_1, b_2, b_3, t_i) &= \sum_{j=1}^{10}\left(\hat{f}(t_i, \delta_j) - f(t_i, \delta_j)\right)^2 \\ &= \sum_{j=1}^{10}\left(\hat{f}(t_i, \delta_j) - \frac{1}{\delta_j}\log\frac{(2b_3(b_2t_i + b_1) + b_2)^2 + b_2^2}{(2b_3(b_2(t_i + \delta_j) + b_1) + b_2)^2 + b_2^2} + 2b_3\right)^2.\end{aligned}$$

Table 3.2: *Optimal result of first chaos model calibration with fixed time t.*

fixed time	initial values of (b_1, b_2, b_3) are $(0.05, 0.01, 0.01)$						
t	b_1	b_2	b_3	t	b_1	b_2	b_3
t_0	0.01148	0.00245	0.06869	t_{22}	0.02461	0.00145	0.04528
t_1	0.01264	0.00301	0.06649	t_{23}	0.02193	0.00381	0.03965
t_2	0.01299	0.00326	0.07007	t_{24}	0.01472	0.00764	0.04753
t_3	0.01115	0.00272	0.07103	t_{25}	-0.02470	0.01964	0.06120
t_4	0.01768	0.00476	0.06848	t_{26}	552.68328	-273.65440	0.05990
t_5	0.01418	0.00264	0.06931	t_{27}	-0.04771	0.01662	0.06684
t_6	0.01384	0.00216	0.06311	t_{28}	-164.41163	79.45447	0.05985
t_7	0.01302	0.00145	0.05877	t_{29}	-126.03909	44.30354	0.05693
t_8	0.00873	0.00062	0.05356	t_{30}	-1.63952	1.04259	0.04605
t_9	0.02250	0.00090	0.04249	t_{31}	-1.089	1.257	0.05150
t_{10}	0.01693	0.00054	0.04249	t_{32}	-2.271	1.238	0.05535
t_{11}	0.01053	0.00020	0.03758	t_{33}	-1.19849	0.64406	0.04938
t_{12}	0.01998	0.00034	0.03910	t_{34}	-0.08376	0.02165	0.04685
t_{13}	0.02089	0.00029	0.03904	t_{35}	2.6222	-1.5173	0.04145
t_{14}	0.01781	0.00012	0.03172	t_{36}	-3.25092	2.02668	0.04079
t_{15}	0.01864	0.00012	0.03191	t_{37}	-0.04271	0.00917	0.04438
t_{16}	0.01361	0.00020	0.03706	t_{38}	-0.01255	0.00200	0.04723
t_{17}	0.03129	0.00073	0.04333	t_{39}	-23.72	4.572	0.04544
t_{18}	0.01522	0.00045	0.04328	t_{40}	-0.01751	0.00248	0.05200
t_{19}	0.02438	0.00108	0.04445	t_{41}	-1.24752	0.65773	0.04055
t_{20}	0.02377	0.00469	0.05510	t_{42}	-46.29724	25.25130	0.04022
t_{21}	0.01629	0.00200	0.05423	t_{43}	5.71050	-0.71338	0.06277

Remark 3.4.1. *Take time $t_{31} = 0.25 * 31 = 4.75$ as an example, the Figure 3.1 below gives us a general look at the relationship between the squared error L and two variables b_1 and b_2 with $b_3 = 0.0515$.*

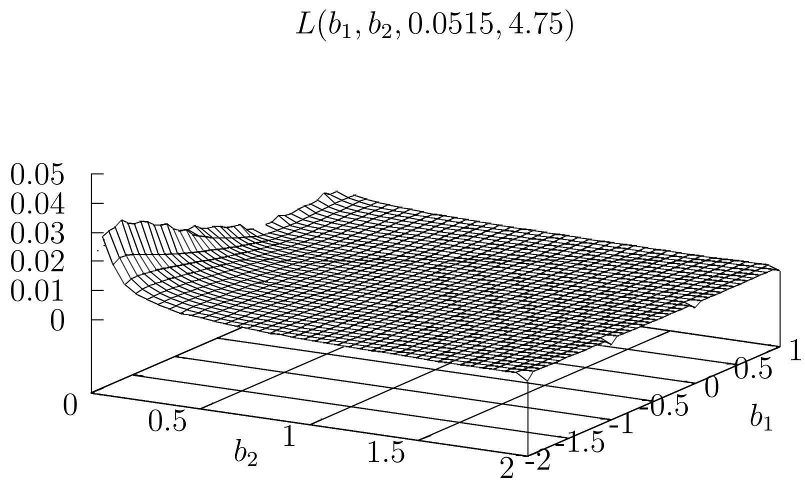


Figure 3.1: *Graph of* $(b_1, b_2) \mapsto L(b_1, b_2, 0.0515, 4.75)$.

3.5 Second chaos model

In the second chaos model, the random variable X_∞ has the form

$$X_\infty = \int_0^\infty \gamma_s dM_s + \int_0^\infty \int_0^s \gamma_{ss_1}\, dM_{s_1} dM_s,$$

where γ_s is a deterministic function of one variable, and γ_{ss_1} is a deterministic function of two variables. As a consequence, we have

$$\begin{aligned} V_t &= E\left[(X_\infty - E[X_\infty|\mathcal{F}_t])^2|\mathcal{F}_t\right] \\ &= E\left[\left(\int_t^\infty \gamma_s dM_s + \int_t^\infty \int_0^s \gamma_{ss_1} dM_{s_1} dM_s\right)^2 \middle| \mathcal{F}_t\right] \\ &= E\left[\left(\int_t^\infty \left(\gamma_s + \int_0^s \gamma_{ss_1} dM_{s_1}\right) dM_s\right)^2 \middle| \mathcal{F}_t\right] \\ &= \int_t^\infty E\left[\left(\gamma_s + \int_0^s \gamma_{ss_1} dM_{s_1}\right)^2 \middle| \mathcal{F}_t\right] ds \\ &= \int_t^\infty \left(\gamma_s + \int_0^t \gamma_{ss_1} dM_{s_1}\right)^2 ds + \int_t^\infty \int_t^s \gamma_{ss_1}^2 ds_1 ds \end{aligned}$$

$$= \int_t^\infty \left(\left(\gamma_s + \int_0^t \gamma_{ss_1} dM_{s_1} \right)^2 + \int_t^s \gamma_{ss_1}^2 ds_1 \right) ds$$

and

$$\begin{aligned}
E\left[V_T|\mathcal{F}_t\right] &= E\left[E\left[(X_\infty - E[X_\infty|\mathcal{F}_T])^2|\mathcal{F}_T\right]|\mathcal{F}_t\right] \\
&= E\left[(X_\infty - E[X_\infty|\mathcal{F}_T])^2|\mathcal{F}_t\right] \\
&= E\left[\left(\int_T^\infty \gamma_s dM_s + \int_T^\infty \int_0^s \gamma_{ss_1} dM_{s_1} dM_s\right)^2 \middle| \mathcal{F}_t\right] \\
&= E\left[\left(\int_T^\infty \left(\gamma_s + \int_0^s \gamma_{ss_1} dM_{s_1}\right) dM_s\right)^2 \middle| \mathcal{F}_t\right] \\
&= \int_T^\infty E\left[\left(\gamma_s + \int_0^s \gamma_{ss_1} dM_{s_1}\right)^2 \middle| \mathcal{F}_t\right] ds \\
&= \int_T^\infty \left(\gamma_s + \int_0^t \gamma_{ss_1} dM_{s_1}\right)^2 ds + \int_t^\infty \int_t^s \gamma_{ss_1}^2 ds_1 ds \\
&= \int_T^\infty \left(\left(\gamma_s + \int_0^t \gamma_{ss_1} dM_{s_1}\right)^2 + \int_t^s \gamma_{ss_1}^2 ds_1\right) ds.
\end{aligned}$$

Finally we derive the bond price

$$P(t,T) = \frac{\int_T^\infty \left(\left(\gamma_s + \int_0^t \gamma_{ss_1} dM_{s_1}\right)^2 + \int_t^s \gamma_{ss_1}^2 ds_1\right) ds}{\int_t^\infty \left(\left(\gamma_s + \int_0^t \gamma_{ss_1} dM_{s_1}\right)^2 + \int_t^s \gamma_{ss_1}^2 ds_1\right) ds}$$

and the swaption price $Swaption(0,t)$ in (3.2.14) at time zero is

$$\begin{aligned}
Swaption(0,t) =& E\left[\int_t^{T_n} \left(\left(\gamma_s + \int_0^t \gamma_{ss_1} dM_{s_1}\right)^2 + \int_t^s \gamma_{ss_1}^2 ds_1\right) ds \right. \\
& \left. -K \sum_{i=1}^n \int_{T_i}^\infty \left(\left(\gamma_s + \int_0^t \gamma_{ss_1} dM_{s_1}\right)^2 + \int_t^s \gamma_{ss_1}^2 ds_1\right) ds\right]^+ \\
& \left(\int_t^\infty \left(\left(\gamma_s + \int_0^t \gamma_{ss_1} dM_{s_1}\right)^2 + \int_t^s \gamma_{ss_1}^2 ds_1\right) ds\right)^{-1}.
\end{aligned}$$

In fact we also can derive V_t in the following way from (3.2.12) by defining

$$V_t = \int_t^\infty M_{ts}\, ds,$$

where the martingale M_{ts} $(t \leq s)$ is defined as

$$M_{ts} = E[\sigma_s^2 | \mathcal{F}_t], \quad \sigma_s = \gamma_s + \int_0^s \gamma_{ss_1}\, dM_{s_1}.$$

By the conditional Itô isometry we get

$$M_{ts} = \left(\gamma_s + \int_0^t \gamma_{ss_1}\, dM_{s_1} \right)^2 + \int_t^s \gamma_{ss_1}^2\, ds_1. \tag{3.5.1}$$

For simplification, we work in the factorizable second chaos model with separable γ_{ss_1} onwards, namely $\gamma_{ss_1} = \eta_s \theta_{s_1}$, where $(\eta_t)_{t \in \mathbb{R}_+}$ and $(\theta_t)_{t \in \mathbb{R}_+}$ are two square-integrable deterministic functions.

Proposition 3.5.1. *In the factorizable second chaos model, the bond price is given by*

$$P(t,T) = \frac{A_T + B_T R_t + C_T (R_t^2 - Q_t)}{A_t + B_t R_t + C_t (R_t^2 - Q_t)}, \tag{3.5.2}$$

where

$$\begin{cases} R_t = \displaystyle\int_0^t \theta_{s_1} dM_{s_1}, \\ Q_t = \displaystyle\int_0^t \theta_{s_1}^2 ds_1, \\ A_t = \displaystyle\int_t^\infty (\gamma_s^2 + \eta_s^2 Q_s) ds, \\ B_t = 2 \displaystyle\int_t^\infty \gamma_s \eta_s ds, \\ C_t = \displaystyle\int_t^\infty \eta_s^2 ds. \end{cases}$$

Proof. We rewrite (3.5.1) as

$$M_{ts} = \left(\gamma_s + \eta_s \int_0^t \theta_{s_1} dM_{s_1} \right)^2 + \eta_s^2 \int_t^s \theta_{s_1}^2 ds_1. \tag{3.5.3}$$

Define the new martingale

$$R_t = \int_0^t \theta_{s_1} dM_{s_1} \tag{3.5.4}$$

and its corresponding quadratic variation process

$$Q_t = \int_0^t \theta_{s_1}^2 ds_1,$$

then the process $R_t^2 - Q_t$ is also a martingale.

Therefore (3.5.3) is simplified to

$$M_{ts} = (\gamma_s^2 + \eta_s^2 Q_s + 2\gamma_s \eta_s R_t) + \eta_s^2 (R_t^2 - Q_t)$$

and it comes

$$\int_T^\infty M_{ts} ds = A_T + B_T R_t + C_T (R_t^2 - Q_t), \qquad (3.5.5)$$

where

$$A_t = \int_t^\infty (\gamma_s^2 + \eta_s^2 Q_s) ds,\ B_t = 2\int_t^\infty \gamma_s \eta_s ds,\ C_t = \int_t^\infty \eta_s^2 ds.$$

Now plug $T = t$ in (3.5.5), the price density is given by

$$V_t = A_t + B_t R_t + C_t (R_t^2 - Q_t).$$

The T-maturity bond at time t is priced at

$$P(t,T) = \frac{A_T + B_T R_t + C_T (R_t^2 - Q_t)}{A_t + B_t R_t + C_t (R_t^2 - Q_t)}.$$

□

Corollary 3.5.2. *The bond price* (3.5.2) *leads to the computation of following rates.*

- *LIBOR rate*

$$\begin{aligned} L(t,T,S) &= \frac{P(t,T) - P(t,S)}{(S-T)P(t,S)} \\ &= \frac{A_T - A_S + (B_T - B_S) R_t + (C_T - C_S)(R_t^2 - Q_t)}{(S-T)(A_S + B_S R_t + C_S (R_t^2 - Q_t))}. \end{aligned}$$

- *Instantaneous forward rate*

$$\begin{aligned} f(t,T) &= -\frac{\partial \log P(t,T)}{\partial T} \\ &= \frac{(\gamma_T^2 + \eta_T^2 Q_T) + 2\gamma_T \eta_T R_t + \eta_T^2 (R_t^2 - Q_t)}{A_T + B_T R_t + C_T (R_t^2 - Q_t)}. \end{aligned}$$

- *Forward rate*

$$\begin{aligned} f(t,T,S) &= \frac{\log P(t,T) - \log P(t,S)}{S-T} \\ &= \frac{1}{S-T}\log\frac{A_T + B_T R_t + C_T(R_t^2 - Q_t)}{A_S + B_S R_t + C_S(R_t^2 - Q_t)}. \end{aligned}$$

- *Spot forward rate*

$$f(t,t,T) = \frac{1}{T-t}\log\frac{A_t + B_t R_t + C_t(R_t^2 - Q_t)}{A_T + B_T R_t + C_T(R_t^2 - Q_t)}.$$

For convenience, we rewrite the spot forward rate as

$$f(t,\delta) = \frac{1}{\delta}\log\frac{A_t + B_t R_t + C_t(R_t^2 - Q_t)}{A_{t+\delta} + B_{t+\delta} R_t + C_{t+\delta}(R_t^2 - Q_t)}. \tag{3.5.6}$$

3.5.1 Calibration of second chaos model

We use the same data from Appendix I as in the first chaos model calibration in Subsection 3.4.1 to minimize the two objective functions

$$\sum_{i=0}^{43}\sum_{j=1}^{10}\left(\hat{f}(t_i,\delta_j) - f(t_i,\delta_j)\right)^2$$

and

$$\sum_{j=1}^{10}\left(\hat{f}(t_i,\delta_j) - f(t_i,\delta_j)\right)^2, \; i = 0,\ldots,43$$

to estimate the parameters in (3.5.6) after the parametrization of chaos coefficients. Here each $\hat{f}(t_i,\delta_j)$ represents the real value of spot rate at time t_i with forward time δ_j, $i = 0,\ldots,43$, $j = 1,\ldots,10$. Each $f(t_i,\delta_j)$ represents the model value of spot rate at time t_i with forward time δ_j in (3.5.6), $i = 0,\ldots,43$, $j = 1,\ldots,10$.

Here we can see the impact of the normal martingale setting compared to Brownian dynamics, the former can bring jump in the price movement, which is more realistic.

Asset price models based on using Brownian motion as the underlying process always draw continuous paths of the price movements, such as Geometric Brownian motion model, but it has been shown that most financial asset prices contain jumps due to market crash, therefore jumps have been introduced in more and more asset pricing models to improve the performance of some financial models. Merton (1976) added Poisson jumps to normal Brownian motion process, which is called jump-diffusion model, to get close to the real movement of stock prices having variations. Duffie and Garleanu (2001) incorporate pure jump process to model the default intensity of assets.

Among many jump processes, the Poisson process is the foundation of building many other jump processes, such as compensated Poisson process, point processes, and Lévy process, which are widely used in finance models. In this chapter we choose to do the calibration based on compensated Poisson process.

One parameter model

We start with a simple situation: $M_t = N_t - t$ is a compensated Poisson process with intensity $\lambda = 1$, and $\gamma_s = 0$, $\theta_s = 1$, $\eta_s = se^{-a_1 s}$, then we have the following computation results

$$\begin{cases} R_t = M_t, \qquad Q_t = t, \qquad B_t = 0, \\ A_t = \left(\dfrac{1}{2a_1} t^3 + \dfrac{3}{(2a_1)^2} t^2 + \dfrac{6}{(2a_1)^3} t + \dfrac{6}{(2a_1)^4} \right) e^{-2a_1 t}, \\ C_t = \left(\dfrac{1}{2a_1} t^2 + \dfrac{2}{(2a_1)^2} t + \dfrac{2}{(2a_1)^3} \right) e^{-2a_1 t}. \end{cases}$$

The spot forward rate is given by

$$\begin{aligned} f(t,\delta) &= \frac{1}{\delta} \log \frac{A_t + B_t R_t + C_t(R_t^2 - Q_t)}{A_{t+\delta} + B_{t+\delta} R_t + C_{t+\delta}(R_t^2 - Q_t)} \\ &= \frac{1}{\delta} \log \frac{A_t + C_t(R_t^2 - Q_t)}{A_{t+\delta} + C_{t+\delta}(R_t^2 - Q_t)} \\ &= c + \frac{1}{\delta} \log \frac{(c^2t^2 + 4ct + 6) + (c^3t^2 + 2c^2t + 2c)\, R_t^2}{c^3(t+\delta)^2(\delta + R_t^2) + c^2(t+\delta)(t + 3\delta + 2R_t^2) + 2c(2t + 3\delta + R_t^2) + 6} \\ &:= f(t,\delta,c), \end{aligned}$$

where $c = 2a_1$.

We simulate time paths of compensated Poisson process with intensity $\lambda = 1$

and then minimize the objective function

$$L(c) = \sum_{i=0}^{43} \sum_{j=1}^{10} \left(\hat{f}(t_i, \delta_j) - f(t_i, \delta_j, c) \right)^2,$$

where each $\hat{f}(t_i, \delta_j)$ represents the real value of spot rate at time t_i with forward time δ_j, $i = 0, \ldots, 43$, $j = 1, \ldots, 10$. Each $f(t_i, \delta_j, c)$ represents the model value of spot rate at time t_i with forward time δ_j, $i = 0, \ldots, 43$, $j = 1, \ldots, 10$.

Figure 3.2 gives a glimpse into the trend of function $L(c)$.

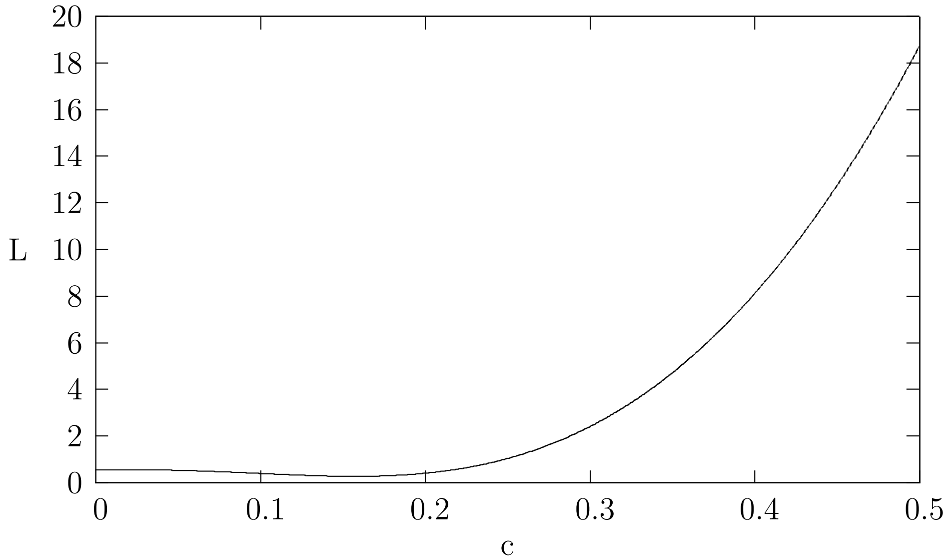


Figure 3.2: *Graph of* $c \mapsto L(c)$.

We have noted that the minimum value of $L(c)$ vary to some extent from the different simulated time paths of compensated Poisson process, see the Table 3.3, for each simulated path, there is a minimum point of $L(c)$, we take the average as the estimate of parameter c, i.e. $c = 0.1709$.

Table 3.3: *Minimum points for different simulated paths.*

c	0.16025	0.17450	0.18975	0.15875	0.15919
L(c)	0.21329	0.20893	0.19916	0.22019	0.19550
c	0.16575	0.15175	0.19150	0.18778	0.16976
L(c)	0.19924	0.21148	0.21387	0.18480	0.20181

We now do calibration in a new direction, which is to take $\{R_{t_i}, i = 0, \dots, 43\}$ as parameters and apply Limited-memory Broyden-Fletcher-Goldfarb-Shanno algorithm (LBFGS) to minimize the variance function.

The values of the parameters are given by Table 3.4.

Table 3.4: *Calibrated values of parameters.*

Parameters	initial values of c and R_{t_i} are 1.5, $i = 0, \dots, 43$				
c	0.163238	$R_{t_{14}}$	34.92313	$R_{t_{29}}$	1.26323
R_{t_0}	14.32311	$R_{t_{15}}$	35.51754	$R_{t_{30}}$	0.99115
R_{t_1}	12.89563	$R_{t_{16}}$	35.63863	$R_{t_{31}}$	-1.02301
R_{t_2}	15.97137	$R_{t_{17}}$	39.08928	$R_{t_{32}}$	1.12873
R_{t_3}	18.17529	$R_{t_{18}}$	35.07380	$R_{t_{33}}$	-1.35157
R_{t_4}	16.44232	$R_{t_{19}}$	28.84597	$R_{t_{34}}$	0.38152
R_{t_5}	22.56333	$R_{t_{20}}$	14.84368	$R_{t_{35}}$	0.48810
R_{t_6}	20.77206	$R_{t_{21}}$	22.98573	$R_{t_{36}}$	0.40681
R_{t_7}	22.80309	$R_{t_{22}}$	23.17151	$R_{t_{37}}$	0.22471
R_{t_8}	26.25653	$R_{t_{23}}$	2.02767	$R_{t_{38}}$	0.00726
R_{t_9}	23.85121	$R_{t_{24}}$	2.18003	$R_{t_{39}}$	-0.01526
$R_{t_{10}}$	28.48071	$R_{t_{25}}$	2.75489	$R_{t_{40}}$	0.01419

$R_{t_{11}}$	30.71350	$R_{t_{26}}$	2.22510	$R_{t_{41}}$	0.53991
$R_{t_{12}}$	34.62575	$R_{t_{27}}$	2.63949	$R_{t_{42}}$	-0.12566
$R_{t_{13}}$	37.81975	$R_{t_{28}}$	2.43523	$R_{t_{43}}$	0.13170

Another way to do the calibration is to fix time t_i $(i = 0, \dots, 43)$, $f(c, R_t, \delta) := f(t, \delta)$, then minimize corresponding variance function

$$L(c, R_{t_i}) = \sum_{j=1}^{10} \left(\hat{f}(t_i, \delta_j) - f(c, R_{t_i}, \delta_j) \right)^2 \tag{3.5.7}$$

with respect to two parameters, c and R_{t_i}.

Table 3.5 shows the solution of the optimization problem (3.5.7) corresponding to time t_i, $i = 0, \dots, 43$. We first minimize $L(c, R_{t_0})$. The algorithm gives the solution $c = 0.17$, $R_{t_0} = 9125.71$ in Table 3.5. It is noticed that the value of R_{t_0} is extremely large in this way, which means

$$f(t, \delta) = \frac{1}{\delta} \log \frac{A_t + C_t(R_t^2 - Q_t)}{A_{t+\delta} + C_{t+\delta}(R_t^2 - Q_t)},$$

i.e. $B_t = 0$ is not good for calibration, we must take B_t into consideration for the calibration of

$$f(t, \delta) = \frac{1}{\delta} \log \frac{A_t + B_t R_t + C_t(R_t^2 - Q_t)}{A_{t+\delta} + B_{t+\delta} R_t + C_{t+\delta}(R_t^2 - Q_t)},$$

which will be done in next paragraph.

Table 3.5: *Optimal solution of* (3.5.7).

Time	Minima of $L(c, R_t)$		Fixed time	Minima of $L(c, R_t)$		Time	Minima of $L(c, R_t)$	
t	c	R_t	t	c	R_t	t	c	R_t
t_0	0.174916	9125.71	t_{15}	0.178704	15333.3	t_{30}	0.13363	2.26113
t_1	0.165886	11640	t_{16}	0.176301	11361.1	t_{31}	0.151596	-1.99689
t_2	0.174452	11044.6	t_{17}	0.181363	9567.79	t_{32}	0.154283	-2.05269
t_3	0.178643	8102.11	t_{18}	0.1704	8688.55	t_{33}	0.152088	-1.71075
t_4	0.168916	9896.96	t_{19}	0.157308	14281.7	t_{34}	0.113473	-2.87633

t_5	0.184987	9060.19	t_{20}	0.137307	13378.2	t_{35}	0.106353	-3.61219
t_6	0.174282	8826.43	t_{21}	0.147223	12217.1	t_{36}	0.1239	-2.19326
t_7	0.175789	8704.36	t_{22}	0.145473	7969.7	t_{37}	0.108799	-2.58289
t_8	0.180871	12546.9	t_{23}	0.099828	12088.5	t_{38}	0.115517	-1.73091
t_9	0.168575	10088.2	t_{24}	0.10386	9709.84	t_{39}	0.120845	-1.52816
t_{10}	0.177893	11543.3	t_{25}	0.114406	673.273	t_{40}	0.125219	-1.45762
t_{11}	0.179905	8002.06	t_{26}	0.132386	4.64418	t_{41}	0.136078	-1.51606
t_{12}	0.187247	9276.5	t_{27}	0.154812	2.9837	t_{42}	0.14144	-1.25523
t_{13}	0.192296	10668.6	t_{28}	0.157674	-2.46168	t_{43}	0.147793	-1.19437
t_{14}	0.180427	10857.6	t_{29}	0.133777	-3.24124			

We now explain why R_{t_0} is extremely large if $B_t = 0$.

We study the impact of the parameter c on the value of function $L(c, R_{t_0})$. Figure 3.3 tells us when the parameter c takes value 0.17, the variance $L(c, R_{t_0})$ drops dramatically.

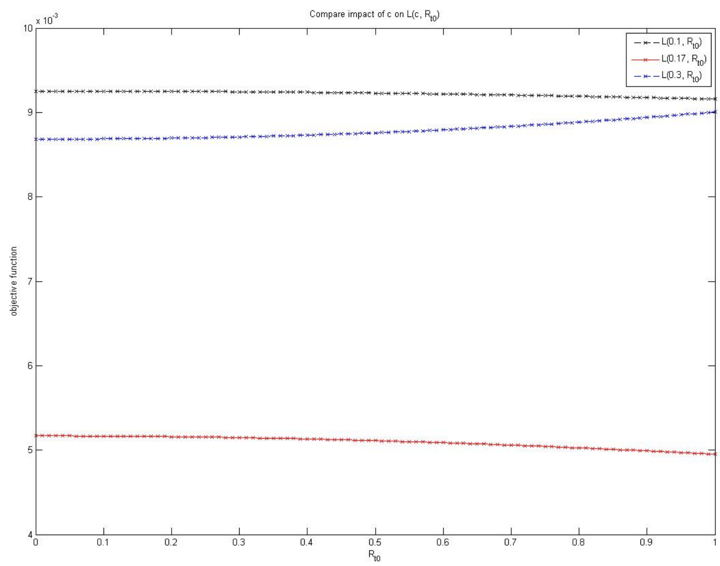


Figure 3.3: *Graphs of* $R_{t_0} \mapsto L(c_i, R_{t_0})$, $i = 1, 2, 3$.

We take time $t_0 = 0$. Figure 3.4 shows us that the function $L(c, R_{t_0})$ has very low value at small c (≈ 0.17).

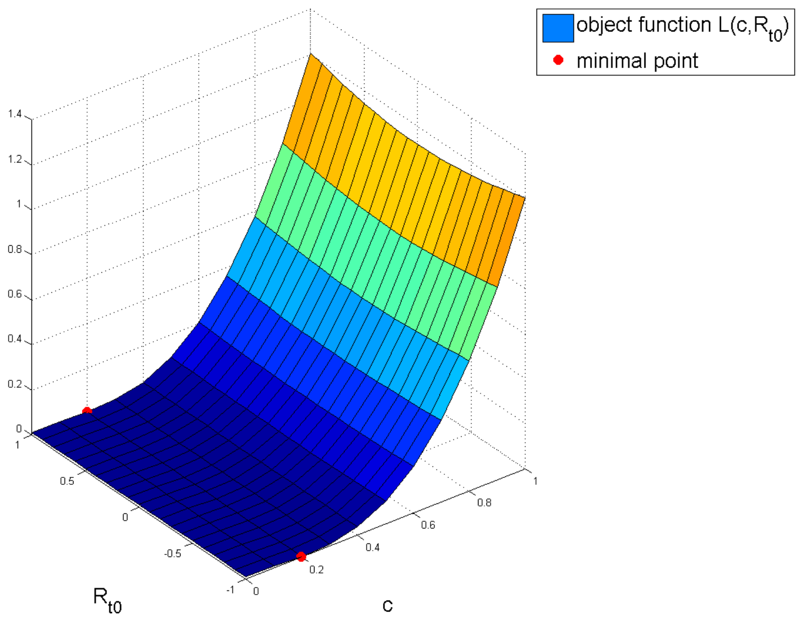


Figure 3.4: *Graph of* $(c, R_{t_0}) \mapsto L(c, R_{t_0})$*.*

We then study the case $c = 0.17$ specifically. We draw parts of the image of $R_{t_0} \mapsto L(0.17, R_{t_0})$ sequently, first Figure 3.5, where the horizontal axis takes small value of variable R_{t_0}, and the second Figure 3.6, where the horizontal axis takes large value of variable R_{t_0}. These figures reveal that $L(0.17, R_{t_0})$ is a decreasing function of the variable R_{t_0}, and it has the limit $\lim_{R_{t_0}\to\infty} L(c, R_{t_0})|_{c=0.17} = 2.4102 \times 10^{-3}$ by

$$\begin{aligned}
&\lim_{R_{t_0}\to\infty} L(c, R_{t_0})|_{c=0.17} \\
&= \lim_{R_{t_0}\to\infty} \sum_{j=1}^{10} \left(c + \frac{1}{\delta_j} \log \frac{(c^3 t_0^2 + 2c^2 t_0 + 2c) R_{t_0}^2}{(c^3 (t_0 + \delta_j)^2 + 2c^2 (t_0 + \delta_j) + 2c) R_{t_0}^2} - f(t_0, \delta_j) \right)^2_{|c=0.17} \\
&= \lim_{R_{t_0}\to\infty} \sum_{j=1}^{10} \left(c + \frac{1}{\delta_j} \log \frac{2c R_{t_0}^2}{(c^3 \delta_j^2 + 2c^2 \delta_j + 2c) R_{t_0}^2} - f(t_0, \delta_j) \right)^2_{|c=0.17} \\
&= \sum_{j=1}^{10} \left(c + \frac{1}{\delta_j} \log \frac{2}{c^2 \delta_j^2 + 2c\delta_j + 2} - f(t_0, \delta_j) \right)^2_{|c=0.17} \\
&= 2.4102 \times 10^{-3}.
\end{aligned}$$

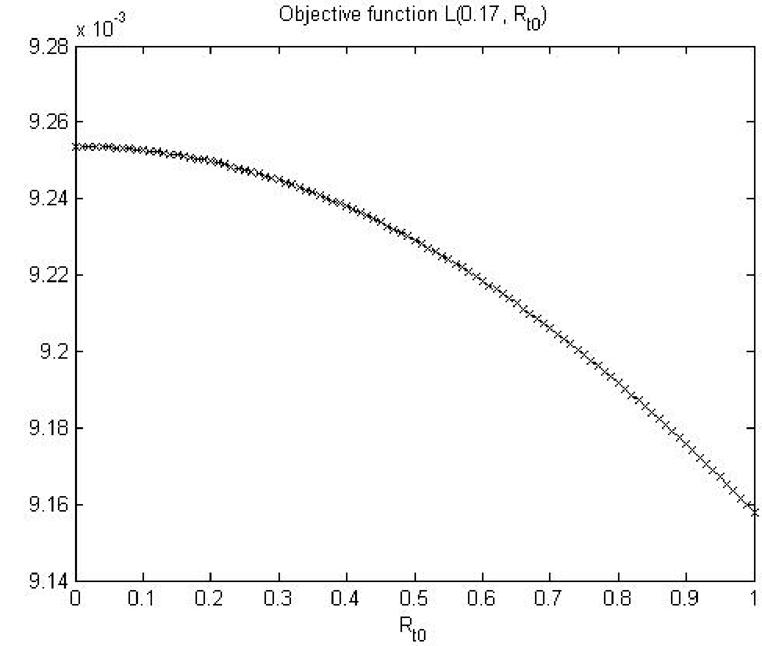


Figure 3.5: *Graph of small* $R_{t_0} \mapsto L(0.17, R_{t_0})$.

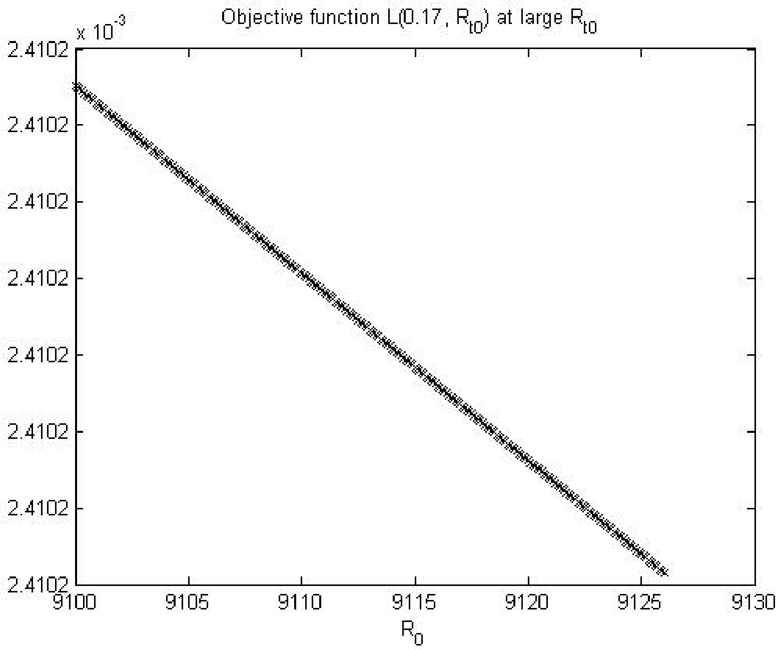


Figure 3.6: *Graph of large* $R_{t_0} \mapsto L(0.17, R_{t_0})$.

So the even function $L(0.17, R_{t_0})$ has no minimum value but infinitely close to 2.4102×10^{-3} as R_{t_0} goes to large enough.

As for other $t_i > 0$, the solution of the problem $\min_{c,R_{t_i}} L(c, R_{t_i})$ also has extremely large R_{t_i}, which can be explained in the same way as R_{t_0}.

Since $\gamma_s = 0$ is not good for calibration, we must take γ_t into consideration, we do two parameter model calibration in the following paragraphs.

Two parameter model

Let's continue with previous case: $M_t = N_t - t$ with intensity $\lambda = 1$, $\theta_s = 1$ with

$$\eta_s = se^{-a_1 s}, \ \ \gamma_s = se^{-a_2 s},$$

then we have the spot rate

$$\begin{aligned} f(t,\delta) &= \frac{1}{\delta} \log \frac{A_t + B_t R_t + C_t(R_t^2 - Q_t)}{A_{t+\delta} + B_{t+\delta} R_t + C_{t+\delta}(R_t^2 - Q_t)} \\ &:= f(a_1, a_2, R_t, \delta), \end{aligned}$$

where

$$\begin{cases} R_t = M_t, \quad Q_t = t, \\ A_t = \left(\dfrac{1}{2a_1} t^3 + \dfrac{3}{(2a_1)^2} t^2 + \dfrac{6}{(2a_1)^3} t + \dfrac{6}{(2a_1)^4} \right) e^{-2a_1 t} \\ \quad + \left[\left(t + \dfrac{1}{2a_2} \right)^2 + \dfrac{1}{(2a_2)^2} \right] (2a_2)^{-1} \exp\{-2a_2 t\}, \\ B_t = 2 \left(\dfrac{t^2}{a_1 + a_2} + \dfrac{2t}{(a_1+a_2)^2} + \dfrac{2}{(a_1+a_2)^3} \right) e^{-(a_1+a_2)t}, \\ C_t = \left(\dfrac{1}{2a_1} t^2 + \dfrac{2}{(2a_1)^2} t + \dfrac{2}{(2a_1)^3} \right) e^{-2a_1 t}. \end{cases}$$

Each $\hat{f}(t_i, \delta_j)$ represents the real value of spot rate at time t_i with forward time δ_j, $i = 0, \dots, 43$, $j = 1, \dots, 10$. Each $f(t_i, \delta_j, c)$ represents the model value of spot rate at time t_i with forward time δ_j, $i = 0, \dots, 43$, $j = 1, \dots, 10$.

Table 3.6 is the solution for each optimization problem corresponding to time $t_i,\ i = 0, \dots, 43$

$$\min_{a_1,a_2,R_{t_i}} L(a_1, a_2, R_{t_i}) = \sum_{j=1}^{10} \left(\hat{f}(t_i, \delta_j) - f(a_1, a_2, R_{t_i}, \delta_j)\right)^2.$$

Table 3.6: *Optimal solution of* $\min_{a_1,a_2,R_{t_i}} L(a_1, a_2, R_{t_i})$.

Initial values of (a_1, a_2, R_{t_i}) are $(1, 1, 1)$, $i = 0, \dots, 43$.							
t	a_1	a_2	R_t	t	a_1	a_2	R_t
t_0	0.01392	1.21962	1.53363	t_{22}	0.02366	2.67989	0.00002
t_1	0.01260	1.29116	1.56998	t_{23}	0.01034	1.56568	1.20919
t_2	0.01381	2.65758	-0.00007	t_{24}	0.01070	1.56585	1.20867
t_3	0.01520	1.32965	1.18303	t_{25}	0.01187	2.36308	-0.00004
t_4	0.01377	1.90986	-0.00006	t_{26}	0.01051	1.92426	-0.00004
t_5	0.01831	2.58262	-0.00003	t_{27}	0.01169	1.58361	1.19097
t_6	0.01774	2.66430	-0.00004	t_{28}	0.01104	1.56552	1.20872
t_7	0.01993	2.62172	-0.00001	t_{29}	0.08324	4.30546	2.50025
t_8	0.02347	3.20397	-0.00001	t_{30}	0.08375	-0.03782	2.29096
t_9	0.02298	2.50424	-0.00004	t_{31}	0.08611	4.00974	2.57351
t_{10}	0.02671	2.62546	-0.00005	t_{32}	0.08599	3.90773	2.72119
t_{11}	0.02929	2.54142	-0.00003	t_{33}	0.08434	3.68696	2.64426
t_{12}	0.03311	2.51163	-0.00003	t_{34}	0.00671	1.59998	1.16233
t_{13}	0.03708	2.56704	-0.00001	t_{35}	0.00700	1.57018	1.19495
t_{14}	0.08847	0.12206	2.82882	t_{36}	0.00717	1.57004	1.19483
t_{15}	0.08976	0.13378	2.80916	t_{37}	0.06682	3.25283	2.71165
t_{16}	0.09083	0.13408	2.81634	t_{38}	0.06992	0.12125	1.98741
t_{17}	0.11393	0.03152	2.88266	t_{39}	0.00572	1.59054	1.16287
t_{18}	0.14480	0.02538	2.92739	t_{40}	0.00617	1.59055	1.16260
t_{19}	0.11179	0.02398	2.91403	t_{41}	0.00771	1.58934	1.16238

t_{20}	0.01732	2.21962	-0.00002	t_{42}	0.00777	1.55815	1.19453
t_{21}	0.02147	2.80009	0.00001	t_{43}	0.00838	1.64213	-0.00001

Figure 3.7 shows the spot forward curve plotted with the above computed parameters.

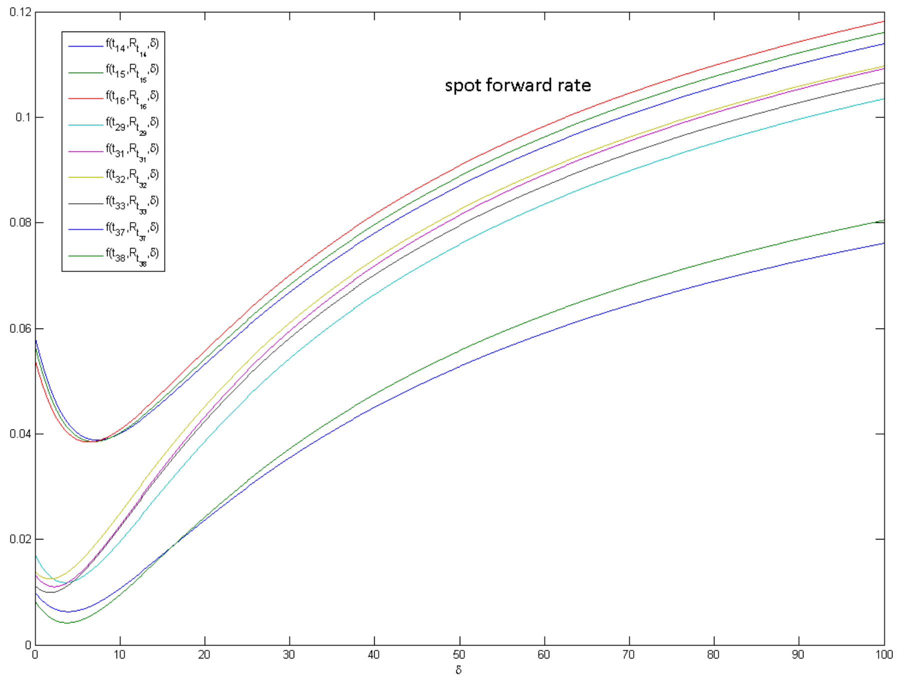


Figure 3.7: *Graphs of* $\delta \mapsto f(t_i, \delta)$, $i = 14, 15, 16, 29, 31, 32, 33, 37, 38$.

Four parameter model

Let us continue with the previous case: $M_t = N_t - t$ with intensity $\lambda = 1$, $\theta_s = 1$,

$$\eta_s = se^{-a_1 s}, \quad \gamma_s = (a_2 + a_3 s)e^{-a_4 s},$$

then we have the spot rate

$$\begin{aligned} f(t, \delta) &= \frac{1}{\delta} \log \frac{A_t + B_t R_t + C_t(R_t^2 - Q_t)}{A_{t+\delta} + B_{t+\delta} R_t + C_{t+\delta}(R_t^2 - Q_t)} \\ &:= f(a_1, a_2, a_3, a_4, R_t, \delta), \end{aligned}$$

where

$$\begin{cases} R_t = M_t, \\ Q_t = t, \\ A_t = \left(\frac{1}{2a_1}t^3 + \frac{3}{(2a_1)^2}t^2 + \frac{6}{(2a_1)^3}t + \frac{6}{(2a_1)^4}\right)e^{-2a_1 t} \\ \quad + \left[(a_2 + a_3 t + \frac{a_3}{2a_4})^2 + \frac{a_3^2}{(2a_4)^2}\right](2a_4)^{-1}\exp\{-2a_4 t\}, \\ B_t = 2\left(\frac{a_2 t + a_3 t^2}{a_1 + a_4} + \frac{a_2 + 2a_3 t}{(a_1 + a_4)^2} + \frac{2a_3}{(a_1 + a_4)^3}\right)e^{-(a_1+a_4)t}, \\ C_t = \left(\frac{1}{2a_1}t^2 + \frac{2}{(2a_1)^2}t + \frac{2}{(2a_1)^3}\right)e^{-2a_1 t}. \end{cases}$$

Table 3.7 is the solution for the optimization problem

$$\min_{a_1,a_2,a_3,a_4,R_t} L(a_1,a_2,a_3,a_4,R_t) = \sum_{i=0}^{43}\sum_{j=1}^{10}\left(\hat{f}(t_i,\delta_j) - f(a_1,a_2,a_3,a_4,R_{t_i},\delta_j)\right)^2.$$

Table 3.7: *Optimal solution in a four parameter model.*

a_1	0.01744	$R_{t_{12}}$	0.00986	$R_{t_{28}}$	0.01331
a_2	0.00914	$R_{t_{13}}$	0.00896	$R_{t_{29}}$	0.00919
a_3	0.00891	$R_{t_{14}}$	0.01402	$R_{t_{30}}$	0.00949
a_4	0.00963	$R_{t_{15}}$	0.01092	$R_{t_{31}}$	0.00892
R_{t_0}	0.01078	$R_{t_{16}}$	0.00959	$R_{t_{32}}$	0.00953
R_{t_1}	0.01004	$R_{t_{17}}$	0.00703	$R_{t_{33}}$	0.00565
R_{t_2}	0.00624	$R_{t_{18}}$	0.01059	$R_{t_{34}}$	0.00963
R_{t_3}	0.00934	$R_{t_{19}}$	0.00922	$R_{t_{35}}$	0.00943
R_{t_4}	0.01088	$R_{t_{20}}$	0.00935	$R_{t_{36}}$	0.00991
R_{t_5}	0.01035	$R_{t_{21}}$	0.01054	$R_{t_{37}}$	0.01102
R_{t_6}	0.01188	$R_{t_{22}}$	0.00961	$R_{t_{38}}$	0.00997
R_{t_7}	0.01050	$R_{t_{23}}$	0.00918	$R_{t_{39}}$	0.01078
R_{t_8}	0.00944	$R_{t_{24}}$	0.01027	$R_{t_{40}}$	0.00697

R_{t_9}	0.00847	$R_{t_{25}}$	0.01072	$R_{t_{41}}$	0.01273
$R_{t_{10}}$	0.00930	$R_{t_{26}}$	0.00944	$R_{t_{42}}$	0.01075
$R_{t_{11}}$	0.00897	$R_{t_{27}}$	0.01096	$R_{t_{43}}$	0.01072

Table 3.8 is the solution for the optimization problem corresponding to each $t_i,\ i = 0, 1, \ldots, 43,\ \min_{a_1,a_2,a_3,a_4,R_{t_i}} L(a_1, a_2, a_3, a_4, R_{t_i}) = \sum_{j=1}^{10} \Big(\hat{f}(t_i, \delta_j) - f(t_i, \delta_j)\Big)^2$, where each $\hat{f}(t_i, \delta_j)$ represents the real value of spot rate at time t_i with forward time δ_j, $i = 0, \ldots, 43$, $j = 1, \ldots, 10$. Each $f(t_i, \delta_j, c)$ represents the model value of spot rate at time t_i with forward time δ_j, $i = 0, \ldots, 43$, $j = 1, \ldots, 10$.

Figure 3.8 shows the calibrated curve of spot forward rate based on the Table 3.8.

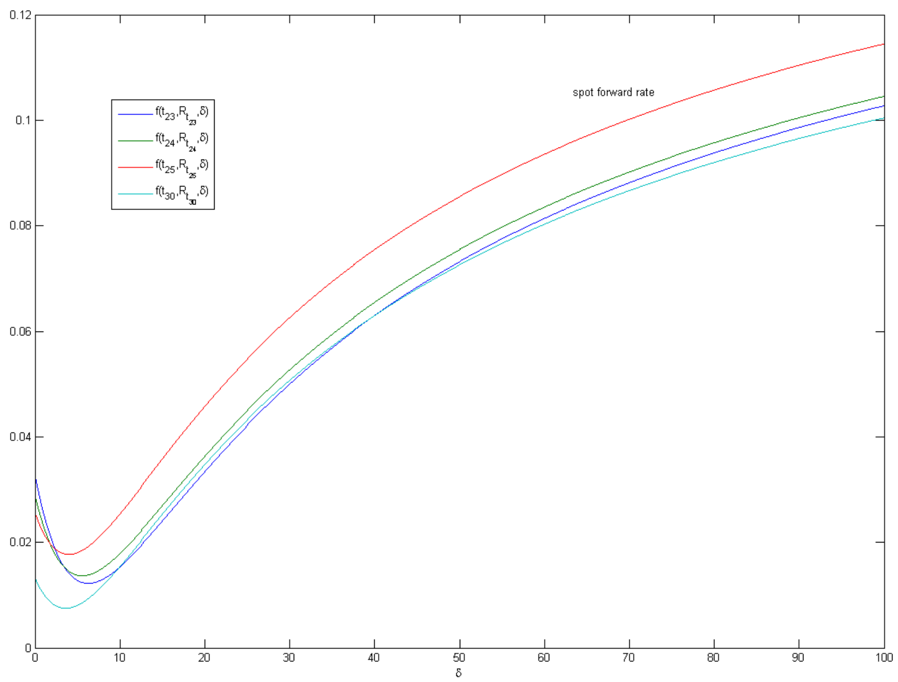


Figure 3.8: *Graphs of* $\delta \mapsto f(t_i, \delta),\ i = 23, 24, 25, 30.$

Table 3.8: *Optimal solution in a four parameter model with fixed time t.*

time	Optimal solution					*time*	Optimal solution				
t	a_1	a_2	a_3	a_4	R_t	t	a_1	a_2	a_3	a_4	R_t
t_0	0.014	1.271	1.222	0.994	1.379	t_{22}	0.024	-0.440	4.263	3.821	0.000
t_1	0.013	1.267	1.228	0.997	1.381	t_{23}	0.084	-6.306	7.801	5.506	1.926
t_2	0.014	1.276	1.224	0.994	1.372	t_{24}	0.085	-3.403	7.836	5.306	2.164
t_3	0.015	0.989	1.295	1.211	1.384	t_{25}	0.090	-7.914	3.467	5.073	2.528
t_4	0.014	1.296	0.985	1.212	1.385	t_{26}	0.057	-2.07	-6.44	0.299	1.19
t_5	0.018	0.381	3.316	1.409	0.000	t_{27}	0.012	1.234	0.986	1.357	1.299
t_6	0.018	0.651	2.098	2.160	0.000	t_{28}	0.011	1.228	1.000	1.369	1.292
t_7	0.020	1.297	0.983	1.372	1.226	t_{29}	0.010	1.243	0.950	1.342	1.295
t_8	0.024	1.291	0.989	1.377	1.215	t_{30}	0.082	-4.407	5.315	4.112	2.304
t_9	0.024	1.295	0.986	1.371	1.222	t_{31}	0.009	0.554	2.468	2.757	0.000
t_{10}	0.027	0.338	1.914	3.014	0.000	t_{32}	0.010	1.237	0.959	1.350	1.289
t_{11}	0.031	1.296	0.981	1.378	1.224	t_{33}	0.009	1.235	0.964	1.354	1.287
t_{12}	0.033	0.134	2.115	2.633	0.000	t_{34}	0.072	1.751	-0.001	0.001	2.369
t_{13}	0.167	-14.699	38.553	0.029	36.741	t_{35}	0.007	1.218	0.980	1.395	1.294
t_{14}	0.148	2.286	-0.291	0.021	1.979	t_{36}	0.007	1.206	0.997	1.379	1.283
t_{15}	0.142	2.796	-0.780	0.026	0.483	t_{37}	0.006	1.236	0.975	1.372	1.314
t_{16}	0.099	2.393	-0.538	0.021	2.213	t_{38}	0.005	1.236	0.976	1.372	1.315
t_{17}	0.119	2.121	-0.379	0.022	2.660	t_{39}	0.006	1.228	0.977	1.367	1.297
t_{18}	0.141	2.485	-0.310	0.023	1.727	t_{40}	0.006	1.230	0.977	1.363	1.300
t_{19}	0.094	0.489	-0.261	0.014	4.419	t_{41}	0.008	1.230	0.973	1.358	1.299
t_{20}	0.071	0.326	1.850	0.039	7.487	t_{42}	0.008	1.230	0.973	1.358	1.299
t_{21}	0.122	1.837	0.046	0.009	3.191	t_{43}	0.008	1.232	0.973	1.350	1.306

Chapter 4

Pricing synthetic CDOs by Gram-Charlier expansions

In this chapter we first give a quick review of Collateralized Debt Obligations (CDOs), then present the mathematical mechanism of synthetic CDOs. Next we consider how to price the spreads of synthetic CDOs with random recovery rate, which amounts to computing the expected tranche loss. Then we evaluate the expected tranche losses numerically. Our method is based on Gram-Charlier expansions to obtain the series expansions of the conditional density of the total losses J_t.

4.1 Preliminaries

4.1.1 Synthetic CDOs

As Duffie and Garleanu (2001) (page 41) states that "In perfect capital markets, CDOs would serve no purpose; the costs of constructing and marketing a CDO would inhibit its creation. In practice, CDOs address some important market imperfections. First, banks and certain other financial institutions have regulatory capital requirements that make valuable the securitizing and selling of some portion of their assets; the value lies in reducing the amount of (expensive) regulatory capital that they must hold. Second, individual bonds or loans may be illiquid, leading to a reduction in their market values; if securitization improve liquidity, it raises the total valuation to the issuer of the CDO structure." The CDO has been very popular among the market since it was created in 1987. Generally speaking, a collateralized debt obligation (CDO), is a transaction that transfers the credit risk of a reference portfolio of assets in the credit derivatives market. CDOs are designed to satisfy different type of investors, i.e., low risk with low return and high risk with high return.

The main feature of a CDO structure is the tranching of credit risk, dividing loss on the reference portfolio into tranches of increasing seniority, then the tranches are sold to investors. Losses will first occur to the first tranche holders, next the mezzanine tranche holders, and finally the senior tranche holders. For example the first 5% of the losses is covered by first tranche, 5% to 10% of the losses is then covered by the second tranche and etc.. Investors take on exposure to a particular tranche, effectively selling credit protection to the CDO issuer in exchange for the premium.

Here we consider a portfolio with maturity T, consisting of N single-name CDSs on obligors with default times τ_i and continuous recovery rates $\phi_i \in [0,1)$, $i = 1, \ldots, N$. We assume that the nominal values equal to one for all obligors. It follows that the accumulated loss J_t at time t for the portfolio is given by

$$J_t = \frac{1}{N}\sum_{i=1}^{N}(1-\phi_i)\mathbb{1}_{\tau_i \le t}. \tag{4.1.1}$$

We specify the CDO with the points

$$0\% = k_0 < k_1 < \cdots < k_{\gamma-1} < k_\gamma < \cdots < k_q = 100\%$$

and assume that a tranche γ, with m_γ obligors, has its tranche holder to cover the loss within the tranche $(k_{\gamma-1}, k_\gamma]$, with interval length $\Delta_{k_\gamma} = k_\gamma - k_{\gamma-1}$, here $k_{\gamma-1}$

is called the attachment point and k_γ is the detachment point. The accumulated loss $J_t^{(\gamma)}$ of tranche γ at time t is

$$J_t^{(\gamma)} = (J_t - k_{\gamma-1})\mathbb{1}_{J_t \in (k_{\gamma-1}, k_\gamma]} + (k_\gamma - k_{\gamma-1})\mathbb{1}_{J_t > k_\gamma}.$$

By assuming the interest rate r_t is deterministic, it follows that the protection leg for tranche γ is

$$V_\gamma(T) = E\left[\int_0^T B_t \, dJ_t^{(\gamma)}\right] = B_T E\left[J_T^{(\gamma)}\right] + \int_0^T r_t \, B_t E\left[J_t^{(\gamma)}\right] \, dt,$$

where $B_t = \exp(-\int_0^t r_s ds)$.

The premium leg with premiums (spreads) $S_\gamma(T)$ paid at $0 < t_1 < t_2 < \cdots < t_n = T$ based on current outstanding notional value on tranche $\gamma(\gamma = 1, \ldots, q)$, is given by

$$W_\gamma(T) = S_\gamma(T) \sum_{j=1}^n B_{t_j} \left(\Delta k_\gamma - E\left[J_{t_j}^{(\gamma)}\right]\right) (t_j - t_{j-1}).$$

It follows that the tranche spread $S_\gamma(T)$ without up-front fees by equaling the above protection and premium legs is

$$S_\gamma(T) = \frac{B_T E\left[J_T^{(\gamma)}\right] + \int_0^T r_t B_t E\left[J_t^{(\gamma)}\right] dt}{\sum_{j=1}^n B_{t_j} \left(\Delta k_\gamma - E\left[J_{t_j}^{(\gamma)}\right]\right) (t_j - t_{j-1})}. \tag{4.1.2}$$

We see that computing spreads amounts to compute the expected tranche loss $E\left[J_t^{(\gamma)}\right]$ for all $t \in [0, T]$, $\gamma = 1, \ldots, q$.

4.1.2 Popular CDO pricing models

Currently there are mainly two ways to do CDO pricing, the standard way is the copula approach, which directly model default time given the marginal default probabilities for all assets and a copula function. The Gaussian copula model, introduced by Li (1999), is widely used in financial industry, and the Clayton, Student t, double t copulas have been studied as well, such as double t copula in Hull and White (2004). Since it is difficult to choose the copula and interpret the parameters of the chosen copula, an alternative way using intensity-based models is also applied to price the tranches of CDOs. Intensity-based models were introduced in Lando (1994) and Duffie and Singleton (1999), and have been proven to work successfully in the single-name credit markets. In an intensity-based model, the default of a firm is defined to be the first jump of a pure jump process with the so-called default intensity. It is well known that a pure diffusion intensity-based model cannot generate enough default correlation to match market prices, but with jumps in the intensity-based model, market prices can be matched.

We start with affine jump models. Duffie and Garleanu (2001) have extended the following single-obligor intensity model to a multi-obligor model,

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t + \Delta J_t,$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion and $(J_t)_{t\in\mathbb{R}_+}$ is a compound Poisson process independent of W_t. $(J_t)_{t\in\mathbb{R}_+}$ has exponentially distributed jumps with mean μ and constant jump intensity l. The multi-obligor intensity model is obtained by dividing N firms$'$ default intensities to be two independent risk parts, a common risk component $X(t)$, which affects the default of all firms, and an idiosyncratic risk component $X_i(t)$, where

$$\lambda_i(t) = X(t) + X_i(t), \qquad i = 1, \ldots, N,$$

and

$$dX_i(t) = \kappa(\theta_i - X_i(t))dt + \sigma\sqrt{X_i(t)}dW_{i,t} + \Delta J_t, \quad i = 1, \ldots, N, \tag{4.1.3}$$

$$dX(t) = \kappa(\theta - X(t))dt + \sigma\sqrt{X(t)}dW_t + \Delta J_t. \tag{4.1.4}$$

Then it is possible to compute CDO tranches. But the intensity-based portfolio models are intractable and time-consuming, see Hull and White (2004), Hull and White (2004) instead develop a fast t-copula procedure for valuing tranches of CDOs.

Mortensen (2005) then has done some modifications on multi-obligor model of Duffie and Garleanu (2001) by adding coefficients α_i, assuming that

$$\lambda_i(t) = \alpha_i X(t) + X_i(t), \qquad i = 1, \ldots, N,$$

where positive α_i, $i = 1, \ldots N$ are constants and denote the weight on common component for N firms individually, and the model allows credit quality and correlation to be chosen independently. Mortensen (2005) proves that the modified model semi-analytically prices the tranches of CDOs and computes as fast as the t-copula method of Hull and White (2004).

Wu and Yang (2013) also model the intensity with common component and idiosyncratic component $X_i(t)$, but different from Mortensen (2005), where the jump diffusion models (4.1.3) and (4.1.4) are used, here they bring in the jump-diffusion processes involving Lévy stable distributions, where the $N + 1$ components are driven by Lévy stable processes

$$dX_i(t) = \theta_i X(t)dt + \sigma_i X(t)dW_{i,t} + \sigma_i X(t) \int_{0<|x|<1} x\tilde{N}_i(dt, dx) + \sigma X(t) \int_{|x|\geq 1} xN_i(dt, dx),$$

and

$$dX(t) = \theta X(t)dt + \sigma X(t)dW_t + \sigma X(t) \int_{0<|x|<1} x\tilde{N}(dt, dx) + \sigma X(t) \int_{|x|\geq 1} xN(dt, dx),$$

where $(\theta_i)_{1\leq i\leq N}$, θ, $(\sigma_i)_{1\leq i\leq N}$ and σ are positive constants, $(W_{i,t})_{1\leq i\leq N}$ and W_t are Brownian motions, $(N_i(dt, dx))_{1\leq i\leq N}$ and $N(dt, dx)$ are the Poisson random measures with stable intensity measure $\mu_\alpha(dx) := \frac{dx}{|x|^{1+\alpha}}, \quad \alpha \in (0, 2)$ and the corresponding compensated martingale measures $\tilde{N}_i(dt, dx) = N_i(dt, dx) - \mu_\alpha(dx)dt, \quad i = 1, \ldots, N$, $\tilde{N}(dt, dx) = N(dt, dx) - \mu_\alpha(dx)dt$. They obtain the semi-closed form of CDO pricing with constant recovery rate.

As addressed in Schönbucher (2003) (chapter 6) the recovery rate also takes its important role in pricing tranches, and heterogeneous recovery rates can be applied to get semi-analytically pricing formula (Mortensen (2005)), while Duffie and Garleanu (2001) apply homogeneous recovery rate to get analytical results, these motivate us to study the impact of random recovery rate on CDO pricing, which is different from the models mentioned in Wu and Yang (2013) and references therein.

4.2 Recovery rate

Suppose that the default times $\tau_1, \tau_2, \ldots, \tau_N$ are the first N jump times of a point process $(N_t)_{t\in\mathbb{R}_+}$, we consider different cases of recovery rate.

4.2.1 Constant recovery rate

We take $\phi_i = \phi,\ i = 1, \ldots, N$, and point process $(N_t)_{t\in\mathbb{R}_+}$ as the Poisson process with intensity λ. We then rewrite (4.1.1) with $\psi = 1 - \phi$,

$$J_t = \frac{\psi}{N} N_t \wedge N,$$

and the expected r-th tranche loss is

$$\begin{aligned}
& E\left[J_t^{(\gamma)}\right] \\
=& E\left[(J_t - k_{\gamma-1})\mathbb{1}_{J_t\in(k_{\gamma-1},k_\gamma]} + (k_\gamma - k_{\gamma-1})\mathbb{1}_{J_t>k_\gamma}\right] \\
=& E\left[\left(\frac{\psi}{N}N_t \wedge N - k_{\gamma-1}\right)\mathbb{1}_{N_t\wedge N\in(\frac{N}{\psi}k_{\gamma-1},\frac{N}{\psi}k_\gamma]} + (k_\gamma - k_{\gamma-1})\mathbb{1}_{N_t\wedge N>\frac{N}{\psi}k_\gamma}\right] \\
=& \begin{cases}
0, & \psi < k_{\gamma-1}, \\
\left(\displaystyle\sum_{k\in(\frac{Nk_{\gamma-1}}{\psi},\frac{Nk_\gamma}{\psi}]}\left(\frac{\psi}{N}k - k_{\gamma-1}\right)\frac{(\lambda t)^k}{k!} + \Delta_{k_\gamma}\sum_{k=N}^{\infty}\frac{(\lambda t)^k}{k!}\right)e^{-\lambda t}, & \psi > k_\gamma, \\
\left(\displaystyle\sum_{k>\frac{Nk_{\gamma-1}}{\psi}}^{N-1}\left(\frac{\psi}{N}k - k_{\gamma-1}\right)\frac{(\lambda t)^k}{k!} + (\psi - k_{\gamma-1})\sum_{k=N}^{\infty}\frac{(\lambda t)^k}{k!}\right)e^{-\lambda t}, & \textit{otherwise}.
\end{cases}
\end{aligned}$$

4.2.2 Default time dependent recovery rate

Taking $\psi_i = 1-\phi_i,\ i = 1, \ldots, N$, and assuming that ψ_i depends on corresponding default time τ_i, the total loss takes the form of

$$J_t = \frac{1}{N}\sum_{i=1}^{N} \psi_{\tau_i} \mathbb{1}_{\tau_i \le t} = \frac{1}{N}\int_0^t \psi_s \mathbb{1}_{N_s \le N} dN_s.$$

Here we only can compute the overall expected loss with assumption that point process $(N_t)_{t\in\mathbb{R}_+}$ is a Poisson process with intensity λ, therefore

$$\begin{aligned}
E[J_t] &= \frac{1}{N} E\left[\int_0^t \psi_s \mathbb{1}_{N_s \le N}\, dN_s\right] \\
&= \frac{1}{N} E\left[\int_0^t \psi_s \mathbb{1}_{N_s + \mathbb{1}_s \le N} \lambda\, ds\right] \\
&= \frac{\lambda}{N}\int_0^t \psi_s E\left[\mathbb{1}_{N_s + \mathbb{1}_s \le N}\right] ds \\
&= \frac{\lambda}{N}\int_0^t \psi_s E\left[\mathbb{1}_{N_s \le N-1}\right] ds \\
&= \frac{\lambda}{N}\int_0^t \psi_s \sum_{k=0}^{N-1} \frac{e^{-\lambda s}(\lambda s)^k}{k!}\, ds \\
&= \frac{\lambda}{N}\sum_{k=0}^{N-1} \frac{\lambda^k}{k!}\int_0^t \psi_s e^{-\lambda s} s^k\, ds.
\end{aligned}$$

4.3 Gram-Charlier expansion of the density function

We now recall Gram-Charlier expansion method to get series expansion of the density function and compute the truncated expectation. Then we move on to compute the expected tranche loss under random recovery rate.

Lemma 4.3.1. *(Proposition 2.1 Tanaka et al. (2010) page 646) The Gram-Charlier expansion of the continuous density function f(x) of a random variable* Y *with cumulants* $\{c_n, n \ge 1\}$ *in the logarithm of the moment-generating function* $g(t) = \log E\left[e^{tY}\right] = \sum_{n=1}^{\infty} c_n \frac{t^n}{n!}$, *f(x) can be expanded as*

(i)

$$f(x) = \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n\left(\frac{x-c_1}{\sqrt{c_2}}\right) \phi\left(\frac{x-c_1}{\sqrt{c_2}}\right), \tag{4.3.1}$$

where

$$\begin{aligned} q_n &= \frac{1}{n!} E\left[H_n\left(\frac{Y-c_1}{\sqrt{c_2}}\right)\right] \\ &= \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n = 1, 2, \\ \sum_{m=1}^{[n/3]} \sum_{\substack{k_1+\cdots+k_m=n \\ k_i \geq 3}} \frac{c_{k_1}\cdots c_{k_m}}{m! k_1! \cdots k_m!} \left(\frac{1}{\sqrt{c_2}}\right)^n, & \text{if } n \geq 3, \end{cases} \end{aligned}$$

$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$ *is the standard normal density function,* $\Phi(x) = \int_{-\infty}^{x} \phi(s) ds$ *is standard normal cumulative distribution function, and*

$$H_n(x) = (-1)^n \phi(x)^{-1} D^n \phi(x)$$

is the Hermite polynomial of degree n, with $H_0(x) = 1, D = \frac{d}{dx}$.

(ii) For any $a \in \mathbb{R}$, *we have*

$$E\left[\mathbb{1}_{Y>a}\right] = \Phi\left(\frac{c_1-a}{\sqrt{c_2}}\right) + \sum_{k=3}^{\infty} (-1)^{k-1} q_k H_{k-1}\left(\frac{c_1-a}{\sqrt{c_2}}\right) \phi\left(\frac{c_1-a}{\sqrt{c_2}}\right). \tag{4.3.2}$$

4.4 Expected tranche loss

Proposition 4.4.1. *Using the expansion of the continuous density function f(x) of the random variable Y in Lemma 4.3.1, for any* $a \in \mathbb{R}$, *we can also compute the truncated expectation of the random variable Y as follows*

$$\begin{aligned} E\left[\mathbb{1}_{Y>a} Y\right] &= c_1 \Phi\left(\frac{c_1-a}{\sqrt{c_2}}\right) + \sqrt{c_2}\phi\left(\frac{c_1-a}{\sqrt{c_2}}\right) + \phi\left(\frac{c_1-a}{\sqrt{c_2}}\right) \sum_{k=3}^{\infty} (-1)^k q_k \\ &\quad \left(-a H_{k-1}\left(\frac{c_1-a}{\sqrt{c_2}}\right) + \sqrt{c_2} H_{k-2}\left(\frac{c_1-a}{\sqrt{c_2}}\right)\right). \end{aligned} \tag{4.4.1}$$

Proof.

This can be deduced from (4.3.1) and the properties of Hermite polynomials.

$$
\begin{aligned}
E\left[\mathbb{1}_{Y>a} Y\right] &= \int_a^{\infty} y f(y)\, dy \\
&= \int_a^{\infty} y \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n(\frac{y-c_1}{\sqrt{c_2}}) \phi(\frac{y-c_1}{\sqrt{c_2}})\, dy \\
&= \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} \int_a^{\infty} y H_n(\frac{y-c_1}{\sqrt{c_2}}) \phi(\frac{y-c_1}{\sqrt{c_2}})\, dy \\
&= \sum_{n=0}^{\infty} q_n \int_{\frac{a-c_1}{\sqrt{c_2}}}^{\infty} (\sqrt{c_2} x + c_1) H_n(x) \phi(x)\, dx \\
&= \sum_{n=0}^{\infty} q_n \sqrt{c_2} \int_{\frac{a-c_1}{\sqrt{c_2}}}^{\infty} x H_n(x) \phi(x)\, dx \\
&\quad + \sum_{n=0}^{\infty} q_n c_1 \int_{\frac{a-c_1}{\sqrt{c_2}}}^{\infty} H_n(x) \phi(x)\, dx \\
&= \sum_{n=0}^{\infty} q_n \sqrt{c_2} I_n\left(\frac{a-c_1}{\sqrt{c_2}}\right) + \sum_{n=0}^{\infty} q_n c_1 M_n\left(\frac{a-c_1}{\sqrt{c_2}}\right),
\end{aligned}
$$

where

$$
\begin{cases}
M_n(x) = \int_x^{\infty} H_n(s)\phi(s) ds, \\
I_n(x) = \int_x^{\infty} s H_n(s)\phi(s) ds.
\end{cases}
$$

Since $D^n \phi(x) = (-1)^n H_n(x)\phi(x)$, we must have

$$D^{n-1}\phi(x) = -\int_x^{\infty} D^n \phi(y)\, dy,$$

which is equivalent to

$$(-1)^{n-1} H_{n-1}(x)\phi(x) = -\int_x^{\infty} (-1)^n H_n(y)\phi(y)\, dy.$$

As a result, we obtain

$$M_n(x) = H_{n-1}(x)\phi(x). \tag{4.4.2}$$

Noting that

$$D\left(x D^{n-1}\phi(x)\right) = x D^n \phi(x) + D^{n-1}\phi(x),$$

we have

$$D\left(x(-1)^{n-1}H_{n-1}(x)\phi(x)\right) = x(-1)^n H_n(x)\phi(x) + (-1)^{n-1}H_{n-1}(x)\phi(x).$$

It follows that

$$\begin{aligned}(-1)^{n-1}xH_{n-1}(x)\phi(x) = &-\int_x^\infty (-1)^n yH_n(y)\phi(y)\,dy \\ &-\int_x^\infty (-1)^{n-1}H_{n-1}(y)\phi(y)\,dy,\end{aligned}$$

i.e.

$$I_n(x) = xH_{n-1}(x)\phi(x) + M_n(x). \tag{4.4.3}$$

Hence the proof is completed by (4.4.2) and (4.4.3). □

Most important of all, the cumulants $c_k (k \geq 1)$ can be computed through moments $m_k = E\left[Y^k\right]$ by the formula in the form of

$$c_n = \sum_{k=1}^{n}(k-1)!(-1)^{k-1} \sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_i\geq 1}} m_{n_1}\cdots m_{n_k},\ n\geq 1, \tag{4.4.4}$$

for example, the first three cumulants are

$$\begin{cases} c_1 = m_1, \\ c_2 = m_2 - m_1^2, \\ c_3 = 2m_1^3 - 3m_1m_2 + m_3. \end{cases}$$

We will apply Gram-Charlier expansion to compute the expected tranche loss from here. Suppose that $\psi_k = 1 - \phi_k,\ k = 1, \ldots, N$ are independent identically distributed random variables with distribution ν on (0,1] and the moment generating function is

$$F(s) = E\left[e^{s\psi_1}\right],\quad s \in \mathbb{R}_+.$$

Then we rewrite the loss J_t as

$$J_t \;=\; \frac{1}{N}\sum_{i=1}^{N}\psi_i \mathbb{1}_{\tau_i\leq t} = \frac{1}{N}\sum_{i=1}^{N_t\wedge N}\psi_i.$$

Proposition 4.4.2. *The γ-th tranche loss is*

$$\begin{aligned} E\left[J_t^{(\gamma)}\right] &= P(J_t > 0)\left(E\left[\mathbb{1}_{J_t > k_{\gamma-1}}(J_t - k_{\gamma-1}) \mid J_t > 0\right]\right. \\ &\quad \left. - E\left[\mathbb{1}_{J_t > k_\gamma}(J_t - k_{\gamma-1}) \mid J_t > 0\right] + \Delta_{k_\gamma} E\left[\mathbb{1}_{J_t > k_\gamma} \mid J_t > 0\right]\right), \end{aligned}$$

and can be computed by Gram-Charlier expansion of the density functions of the variable $Y = J_t - k_{\gamma-1}$ and $Y = J_t$ conditioning on J_t being positive respectively in Lemma 4.3.1.

Proof.

$$\begin{aligned} E\left[J_t^{(\gamma)}\right] &= E\left[J_t^{(\gamma)}\mathbb{1}_{J_t=0} + J_t^{(\gamma)}\mathbb{1}_{J_t>0}\right] \\ &= E\left[J_t^{(\gamma)}\mathbb{1}_{J_t>0}\right] \\ &= P(J_t > 0)E\left[J_t^{(\gamma)} \mid J_t > 0\right] \\ &= P(J_t > 0)E\left[(J_t - k_{\gamma-1})\mathbb{1}_{J_t \in (k_{\gamma-1}, k_\gamma]} + \Delta_{k_\gamma}\mathbb{1}_{J_t \geq k_\gamma} \mid J_t > 0\right] \\ &= P(J_t > 0)E\left[(J_t - k_{\gamma-1})(\mathbb{1}_{J_t > k_{\gamma-1}} - \mathbb{1}_{J_t > k_\gamma}) + \Delta_{k_\gamma}\mathbb{1}_{J_t > k_\gamma} \mid J_t > 0\right] \\ &= P(J_t > 0)\left(E\left[\mathbb{1}_{J_t > k_{\gamma-1}}(J_t - k_{\gamma-1}) \mid J_t > 0\right]\right. \\ &\quad \left. - E\left[\mathbb{1}_{J_t > k_\gamma}(J_t - k_{\gamma-1}) \mid J_t > 0\right] + \Delta_{k_\gamma} E\left[\mathbb{1}_{J_t > k_\gamma} \mid J_t > 0\right]\right). \end{aligned}$$

Since $\psi_i > 0$, $i = 1, 2, \ldots, N$, we have $N_t > 0$ is equivalent to $J_t > 0$, it follows that J_t has a continuous density given $J_t > 0$, so the conditional density functions of J_t and $J_t - k_{\gamma-1}$ admit Gram-Charlier expansions. □

In the following we apply the Gram-Charlier expansion of the density function of the variable $Y = J_t$ to the computation of the expected tranche losses.

It starts with

$$\begin{aligned} E[J_t] &= E[E[J_t \mid N_t]] \\ &= \frac{1}{N}\sum_{n=0}^{N} P(N_t = n)E\left[\sum_{i=1}^{n}\psi_i\right] + \frac{1}{N}\sum_{n=N+1}^{\infty} P(N_t = n)E\left[\sum_{i=1}^{N}\psi_i\right] \\ &= \frac{1}{N}\sum_{n=0}^{N} P(N_t = n)nE[\psi_1] + \sum_{n=N+1}^{\infty} P(N_t = n)E[\psi_1] \end{aligned}$$

$$= \left(\frac{1}{N} \sum_{n=0}^{N} n P(N_t = n) + \sum_{n=N+1}^{\infty} P(N_t = n) \right) \int_0^1 y \, \nu(dy)$$

and the conditional moments of J_t are computed through its conditional moment generating function

$$\begin{aligned}
E\left[e^{xJ_t} \mid J_t > 0\right] &= E\left[e^{xJ_t} \mid N_t > 0\right] \\
&= \frac{1}{P(N_t > 0)} \sum_{k=1}^{\infty} P(N_t = k) E\left[e^{\frac{x}{N}\sum_{i=1}^{N_t \wedge N} \psi_i} \mid N_t = k\right] \\
&= \frac{1}{P(N_t > 0)} \sum_{k=1}^{\infty} P(N_t = k) E\left[e^{\frac{x}{N}\sum_{i=1}^{k \wedge N} \psi_i} \mid N_t = k\right] \\
&= \sum_{k=1}^{N} \frac{P(N_t = k)}{P(N_t > 0)} E\left[e^{\frac{x}{N}\sum_{i=1}^{k} \psi_i}\right] + \sum_{k=N+1}^{\infty} \frac{P(N_t = k)}{P(N_t > 0)} E\left[e^{\frac{x}{N}\sum_{i=1}^{N} \psi_i}\right] \\
&= \sum_{k=1}^{N} \frac{P(N_t = k)}{P(N_t > 0)} \left(E\left[e^{\frac{x}{N}\psi_1}\right]\right)^k + \frac{P(N_t > N)}{P(N_t > 0)} \left(E\left[e^{\frac{x}{N}\psi_1}\right]\right)^N \\
&= \sum_{k=1}^{N} \frac{P(N_t = k)}{P(N_t > 0)} F\left(\frac{x}{N}\right)^k + \frac{P(N_t > N)}{P(N_t > 0)} F\left(\frac{x}{N}\right)^N.
\end{aligned}$$

Hence if given the distribution of ψ_i, we can compute the corresponding moments and the cumulants. After that the expected tranche losses can be derived easily.

4.5 Uniform recovery rate

Suppose each rate $\psi_k = 1 - \phi_k$, $k = 1, 2, \ldots, N$ follows uniform distribution on (0,1], we then compute the expected tranche losses. First the generating function of ψ_k is given by

$$F(x) = \frac{e^x - 1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

and the moment generating function is computed as

$$
\begin{aligned}
&E\left[e^{xJ_t}\middle|J_t>0\right]\\
=&\sum_{k=1}^{N}\frac{P(N_t=k)}{P(N_t>0)}\left(\sum_{n=1}^{\infty}\frac{(\frac{x}{N})^{n-1}}{n!}\right)^k+\frac{P(N_t>N)}{P(N_t>0)}\left(\sum_{n=1}^{\infty}\frac{(\frac{x}{N})^{n-1}}{n!}\right)^N\\
=&\sum_{k=1}^{N}\frac{P(N_t=k)}{P(N_t>0)}\sum_{n_1\cdots n_k\geq 1}\frac{(\frac{x}{N})^{n_1+\cdots+n_k-k}}{n_1!\cdots n_k!}+\frac{P(N_t>N)}{P(N_t>0)}\sum_{n_1\cdots n_N\geq 1}\frac{(\frac{x}{N})^{n_1+\cdots+n_N-N}}{n_1!\cdots n_N!}\\
=&\sum_{k=1}^{N}\frac{P(N_t=k)}{P(N_t>0)}\sum_{n=k}^{\infty}\sum_{\sum_{i=1}^{k}n_i=n}\frac{\left(\frac{x}{N}\right)^{n-k}}{n_1!\cdots n_k!}+\frac{P(N_t>N)}{P(N_t>0)}\sum_{n=N}^{\infty}\sum_{\sum_{i=1}^{N}n_i=n}\frac{\left(\frac{x}{N}\right)^{n-N}}{n_1!\cdots n_N!}\\
=&\sum_{n=0}^{\infty}\sum_{k=1}^{N}\frac{P(N_t=k)}{P(N_t>0)}\sum_{\substack{\sum_{i=1}^{k}n_i=n\\ n_i\geq 0}}\prod_{j=1}^{k}\frac{1}{(n_j+1)!}\left(\frac{x}{N}\right)^n+\frac{P(N_t>N)}{P(N_t>0)}\sum_{n=0}^{\infty}\\
&\sum_{\substack{\sum_{i=1}^{N}n_i=n\\ n_i\geq 0}}\prod_{j=1}^{N}\frac{1}{(n_j+1)!}\left(\frac{x}{N}\right)^n.
\end{aligned}
$$

As a result, the n-th conditional moment of J_t is

$$
\begin{aligned}
m_n=&\frac{n!}{N^n}\sum_{k=1}^{N}\frac{P(N_t=k)}{P(N_t>0)}\sum_{\substack{\sum_{i=1}^{k}n_i=n\\ n_j\geq 0}}\prod_{j=1}^{k}\frac{1}{(n_j+1)!}+\frac{n!}{N^n}\frac{P(N_t>N)}{P(N_t>0)}\\
&\sum_{\substack{\sum_{i=1}^{N}n_i=n\\ n_j\geq 0}}\prod_{j=1}^{N}\frac{1}{(n_j+1)!}
\end{aligned}
$$

and (4.4.4) gives the corresponding conditional cumulants of $Y=J_t$

$$
c_n=\sum_{k=1}^{n}(k-1)!(-1)^{k-1}\sum_{\substack{n_1+n_2+\cdots+n_k=n\\ n_i\geq 1}}m_{n_1}\cdots m_{n_k},\ n\geq 1.
$$

Conditional cumulants c_n^* of $Y=J_t-k_{\gamma-1}$ is

$$
c_n*=\begin{cases}c_1-k_{\gamma_1}, & n=1,\\ c_n, & n\geq 2.\end{cases}
$$

Consequently, the γ-th expected tranche loss

$$
\begin{aligned}
E\left[J_t^{(\gamma)}\right]=&P(J_t>0)\left(E\left[\mathbb{1}_{J_t\geq k_{\gamma-1}}(J_t-k_{\gamma-1})\mid J_t>0\right]\right.\\
&\left.-E\left[\mathbb{1}_{J_t\geq k_\gamma}(J_t-k_{\gamma-1})\mid J_t>0\right]+\Delta_{k_\gamma}E\left[\mathbb{1}_{J_t>k_\gamma}\mid J_t>0\right]\right)
\end{aligned}
$$

is derived from the following three conditional expectations:

$$E\left[\mathbb{1}_{J_t>k_{\gamma-1}}(J_t-k_{\gamma-1}) \mid J_t>0\right] = c_1^*\Phi\left(\frac{c_1^*}{\sqrt{c_2^*}}\right)+\sqrt{c_2^*}\phi\left(\frac{c_1}{\sqrt{c_2^*}}\right)+\phi\left(\frac{c_2^*}{\sqrt{c_2^*}}\right)$$
$$\sum_{k=3}^{\infty}(-1)^k q_k^*\sqrt{c_2^*}H_{k-2}\left(\frac{c_1^*}{\sqrt{c_2^*}}\right),$$

$$E\left[\mathbb{1}_{J_t>k_\gamma}(J_t-k_{\gamma-1}) \mid J_t>0\right] = c_1^*\Phi\left(\frac{c_1^*-\Delta_{k_\gamma}}{\sqrt{c_2^*}}\right)+\sqrt{c_2^*}\phi\left(\frac{c_1-a}{\sqrt{c_2^*}}\right)$$
$$+\sum_{k=3}^{\infty}(-1)^k q_k^*\left(-\Delta_{k_\gamma}H_{k-1}\left(\frac{c_1^*-\Delta_{k_\gamma}}{\sqrt{c_2^*}}\right)\right.$$
$$\left.+\sqrt{c_2^*}H_{k-2}\left(\frac{c_1^*-\Delta_{k_\gamma}}{\sqrt{c_2^*}}\right)\right)\phi\left(\frac{c_2^*-\Delta_{k_\gamma}}{\sqrt{c_2^*}}\right),$$

and

$$E\left[\mathbb{1}_{J_t>k_\gamma} \mid J_t>0\right] = \Phi\left(\frac{c_1-k_\gamma}{\sqrt{c_2}}\right)+\phi\left(\frac{c_1-k_\gamma}{\sqrt{c_2}}\right)\sum_{k=3}^{\infty}(-1)^{k-1}q_k$$
$$H_{k-1}\left(\frac{c_1-k_\gamma}{\sqrt{c_2}}\right),$$

which are directly computed from (4.4.1) and (4.3.2) by taking $a=0$, $a=\Delta_{k_\gamma}$ and $a=k_\gamma$ respectively.

In addition, the density function $f(x)$ of the overall loss J_t given $J_t>0$ is approximated by

$$f(x)\approx\sum_{i=0}^{n}\frac{q_i}{\sqrt{c_2}}H_i\left(\frac{x-c_1}{\sqrt{c_2}}\right)\phi\left(\frac{x-c_1}{\sqrt{c_2}}\right).$$

4.5.1 Simulation results

Here we take point process $(N_t)_{t\in\mathbb{R}_+}$ as Poisson process with density $\lambda=1$, $t=1$, $n=25$, and $(\psi_i)_{1\le i\le N}$ as i.i.d. uniform random variables on interval $(0,1]$. At the beginning of this chapter we have specified the CDO with the points $0\%=k_0<k_1<\cdots<k_\gamma<\cdots<k_q=100\%$ and assumed that a tranche γ, with m_γ obligors, has its tranche holder to cover the loss within the tranches $[0,k_1]$ and $(k_{\gamma-1},k_\gamma]$, $\gamma=2,\ldots,q$.

Tables 4.1-4.6 give the values of the expected tranche losses.

Expected tranche loss

When $N = 1$, $\lambda = 1$, $t = 1$ and $n = 25$, the expected tranche loss $E\left[J_t^{(1)}\right]$ within tranche $[0, 1]$ is as follows.

Table 4.1: *Tranche loss with N=1.*

$Tranche$	Expected tranche loss
$[0, 1]$	27.7049%

When $N = 2$, $\lambda = 1$, $t = 1$ and $n = 25$, the estimated tranche losses $E\left[J_t^{(\gamma)}\right]$ within tranche $[0, 1]$ and two tranches $[0, 0.5]$, $(0.5, 1]$ are as follows.

Table 4.2: *Tranche loss with N=2, one tranche.*

$Tranche$	Expected tranche loss
$[0, 1]$	20.1674%

Table 4.3: *Tranche losses with N=2, two tranches.*

Tranche	Expected tranche loss
$[0, 0.5]$	17.9000%
$(0.5, 1]$	2.1300%

When $N = 5$, $\lambda = 1$, $t = 1$ and $n = 25$, the expected tranche losses $E\left[J_t^{(1)}\right]$ within different tranches are as follows.

Table 4.4: *Tranche loss with N=5, one tranche.*

Tranche	Expected tranche loss
$[0, 1]$	8.7371%

Table 4.5: *Tranche loss with N=5, two tranches.*

Tranche	Expected tranche loss
$[0, 0.5]$	8.9277%
$(0.5, 1]$	0.0131%

Table 4.6: *Tranche loss with N=5, three tranches.*

Tranche	Expected tranche loss
$[0,\ 0.2]$	6.9951%
$(0.2, 0.6]$	1.7341%
$(0.6, 1.0]$	0.0020%

The above results in Tables 4.1-4.6 also show the feature of a CDO, which states that the first tranche has the highest risk, and higher tranches are less risky.

Figures 4.1-4.3 illustrate the approximated conditional density function $f(x)$ with $N = 1, 2, 5$ respectively. The density function of the overall loss J_t given $J_t > 0$ is approximated by

$$f(x) \approx \sum_{i=0}^{25} \frac{q_i}{\sqrt{c_2}} H_i\left(\frac{x - c_1}{\sqrt{c_2}}\right)\phi\left(\frac{x - c_1}{\sqrt{c_2}}\right). \tag{4.5.1}$$

We compare the accuracy of Gram-Charlier series expansion method with the Monte Carlo Simulation. We simulate 1000 Poisson paths with intensity $\lambda = 1$ and randomly draw 10000 values of jump size ψ_i following the uniform distribution.

Estimated density functions by Gram-Charlier expansion

It can be seen from Figure 4.1- Figure 4.3 that the density functions approximated by two different methods, Gram-Charlier expansion and Monte Carlo simulation, are very close. But there is an advantage in our method, that is we can obtain the explicit expression for the density function of the total loss.

Taking $N = 1$, $\lambda = 1$, $t = 1$ and $n = 25$, we have the loss $J_t = \psi \mathbb{1}_{N_t \geq 1}$, so J_t follows uniform distribution on (0,1] given positive N_t.

The approximated density function $f(x)$ in (4.5.1) of the loss J_t given $N_t > 0$ is shown in Figure 4.1 with comparison to the density computed by Monte Carlo simulation.

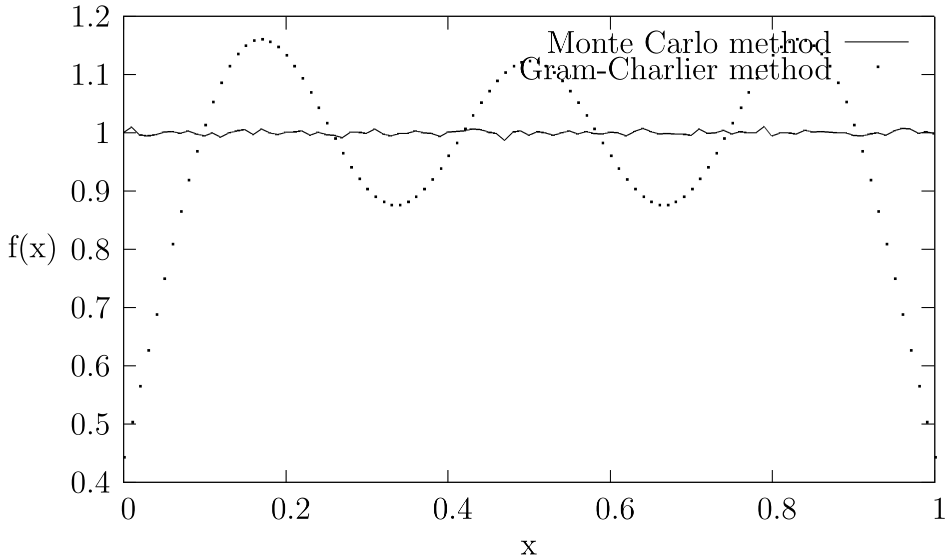


Figure 4.1: *N=1 with ψ follows uniform distribution on (0,1], n=25.*

Taking $N = 2$, $\lambda = 1$, $t = 1$ and $n = 25$,we have the loss $J_t = \frac{1}{2}\sum_{i=1}^{N_t \wedge 2} \psi_i$ and the approximated density function

$$f(x) \approx \sum_{i=0}^{25} \frac{q_i}{\sqrt{c_2}} H_i(\frac{x - c_1}{\sqrt{c_2}}) \phi(\frac{x - c_1}{\sqrt{c_2}})$$

of the loss J_t given $N_t > 0$ is shown in Figure 4.2 with comparison to the density computed by Monte Carlo simulation.

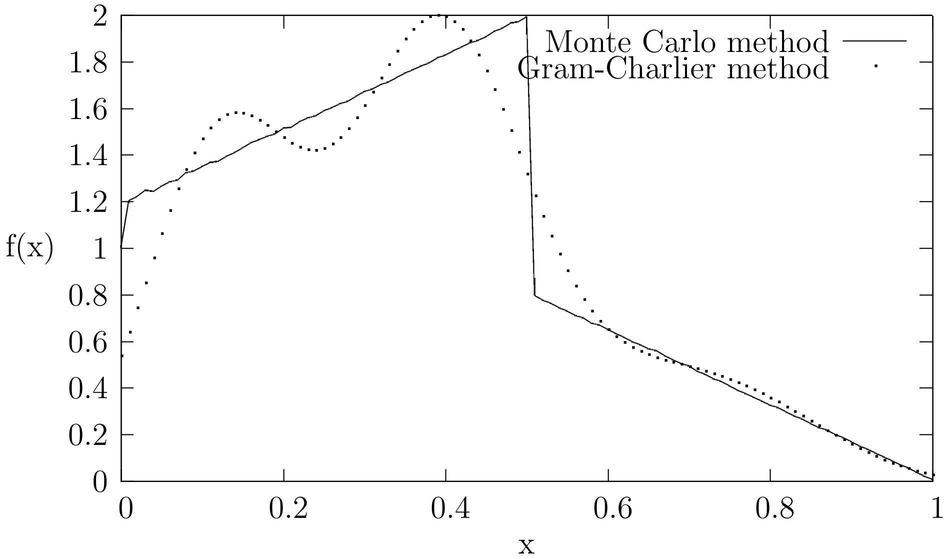


Figure 4.2: *N=2 with ψ follows uniform distribution on (0,1], n=25.*

Taking $N = 5$, $\lambda = 1$, $t = 1$ and $n = 25$, we have the loss $J_t = \frac{1}{5}\sum_{i=1}^{N_t \wedge 5} \psi_i$ and the approximated density function $f(x)$ in (4.5.1) of the loss J_t given $N_t > 0$ is shown in Figure 4.3 with comparison to the density computed by Monte Carlo simulation.

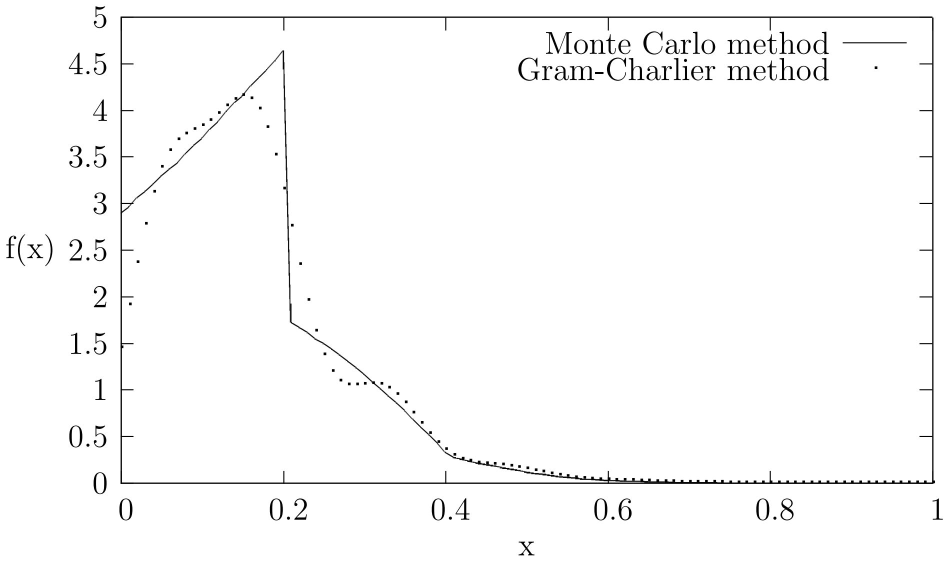


Figure 4.3: *N=5 with ψ follows uniform distribution on (0,1], n=25.*

Remark 4.5.1. *Although the convergence speed of Gram-Charlier series expansion is not that fast, Figure 4.4 and Table 4.7 show small changes when increasing n, like taking n from 10 to 20, 20 to 30, but we do have the explicit density function here.*

Table 4.7: *Gram-Charlier coefficients q_n in* (4.3.1).

n	q_n	n	q_n	n	q_n	n	q_n
3	0	10	4.33E-05	17	3.03E-21	24	8.20E-14
4	-0.05	11	-2.90E-18	18	-1.46E-09	25	4.33E-23
5	-4.62E-16	12	1.73E-07	19	1.07E-20	26	-1.72E-15
6	9.52E-03	13	2.79E-19	20	7.09E-11	27	-2.64E-23
7	0	14	-2.26E-07	21	-1.51E-21	28	8.98E-18
8	-8.93E-04	15	5.62E-21	22	-2.73E-12	29	-1.06E-23
9	3.68E-17	16	2.26E-08	23	-4.80E-22	30	1.31E-18

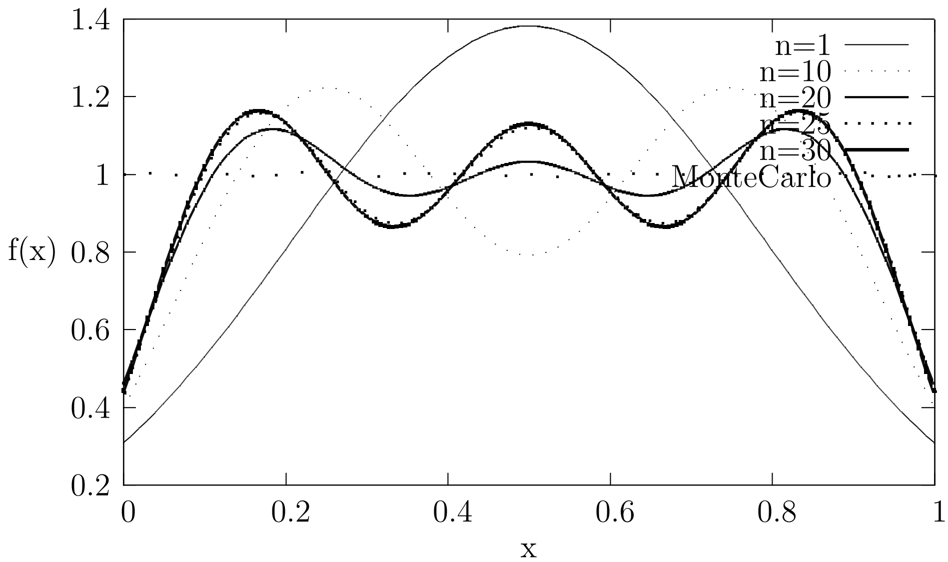


Figure 4.4: *Comparison of the approximated densities for different value of n.*

4.6 Beta(2,2) recovery rate

Taking $\psi_k \sim Beta(2,2)$ with p.d.f. $6x(1-x), x \in (0,1]$, its moment generating function $F(x)$ is

$$F(x) = E\left[e^{x\psi_1}\right] = \sum_{n=0}^{\infty} \frac{6}{(n+2)(n+3)} \frac{x^n}{n!}$$

and n-th conditional moment of J_t is

$$m_n = \frac{n!}{N^n} \sum_{k=1}^{N} \frac{P(N_t = k)}{P(N_t > 0)} \sum_{\substack{n_1+\cdots+n_k=n \\ n_j \geq 0}} \prod_{j=1}^{k} \frac{(n_j+1)6^k}{(n_j+3)!} + \frac{n!}{N^n} \frac{P(N_t > N)}{P(N_t > 0)} \sum_{\substack{\Sigma_{i=1}^{N} n_i = n \\ n_j \geq 0}} \prod_{j=1}^{N} \frac{(n_j+1)6^N}{(n_j+3)!}.$$

4.6.1 Simulation results

Here we take point process $(N_t)_{t\in\mathbb{R}_+}$ as Poisson process with density $\lambda = 1$, $t = 1$, $n = 25$.

Table 4.8 - Table 4.13 show the estimated expected tranche loss with different tranche points.

Figure 4.5 - Figure 4.7 illustrate the conditional density function f(x) with $N = 1, 2, 5$ respectively and compare the density approximated by two methods. We still simulate 1000 Poisson paths here with $\lambda = 1$ and randomly draw 10000 values of jump size ψ.

Expected tranche loss

When N=1, the estimated expected tranche loss $E\left[J_t^{(1)}\right]$ within tranche $[0, 1]$ is given by Table 4.8.

Table 4.8: *Tranche loss with N=1, one tranche.*

Tranche	Expected tranche loss
$[0, 1]$	29.2187%

When N=2, the expected tranche losses $E\left[J_t^{(\gamma)}\right]$ within tranche $[0, 1]$ and two other tranches $[0, 0.5]$ and $(0.5, 1]$ are given as follows.

Table 4.9: *Tranche loss with N=2, one tranche.*

Tranche	Expected tranche loss
$[0, 1]$	20.6451%

Table 4.10: *Tranche loss with N=2, two tranches.*

Tranche	Expected tranche loss
$[0, 0.5]$	19.0328%
$(0.5, 1]$	1.6041%

When N=5, the expected tranche losses $E\left[J_t^{(\gamma)}\right]$ within different tranche slices are given by as follows.

Table 4.11: *Tranche loss with N=5, one tranche.*

Tranche	Expected tranche loss
$[0, 1]$	8.8745%

Table 4.12: *Tranche loss with N=5, two tranches.*

Tranche	Expected tranche loss
$[0, 0.5]$	9.0102%
$(0.5, 1]$	0.0067%

Table 4.13: *Tranche loss with N=5, three tranches.*

Tranche	Expected tranche loss
$[0,\ 0.2]$	7.2955%
$(0.2, 0.6]$	1.5310%
$(0.6, 1.0]$	0.0007%

Estimated densities by Gram-Charlier expansion

Taking $N = 1$, $\lambda = 1$, $t = 1$ and $n = 25$, we have the loss $J_t = \psi \mathbb{1}_{N_t \geq 1}$, so J_t follows Beta(2,2) distribution on (0,1] given positive N_t.

The approximated densities of the loss J_t given $N_t > 0$ are shown in Figure 4.1 with respect to Gram-Charlier expansion and Monte Carlo simulation, where

$$f(x) \approx \sum_{i=0}^{25} \frac{q_i}{\sqrt{c_2}} H_i(\frac{x - c_1}{\sqrt{c_2}}) \phi(\frac{x - c_1}{\sqrt{c_2}}).$$

We still simulate 1000 Poisson paths here with $\lambda = 1$ and randomly draw 10000 values of jump size ψ.

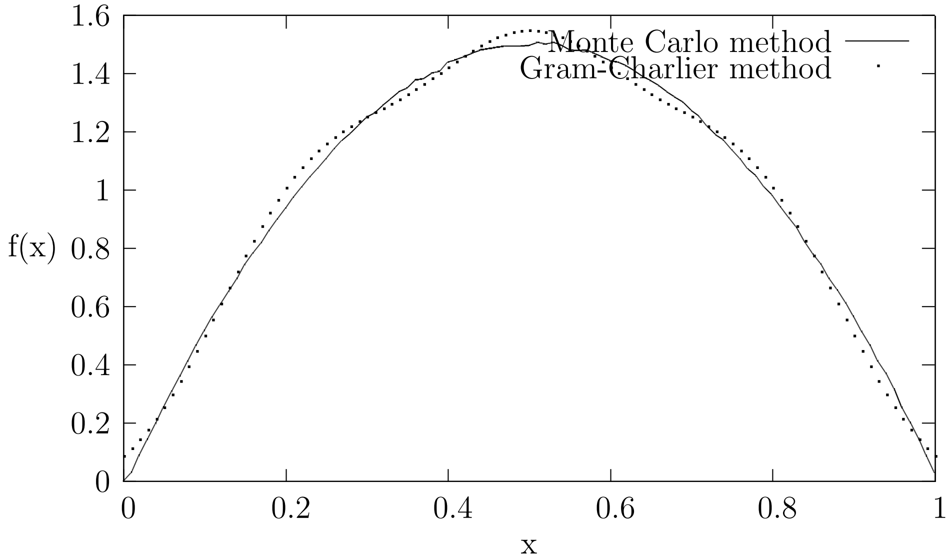


Figure 4.5: *N=1 with ψ follows Beta(2,2) distribution, t=1, n=25.*

Taking $N = 2$, we have the loss $J_t = \frac{1}{2}\sum_{i=1}^{N_t \wedge 2} \psi_i$ and the approximated densities of the loss J_t given $N_t > 0$ are shown in Figure 4.2 by two methods.

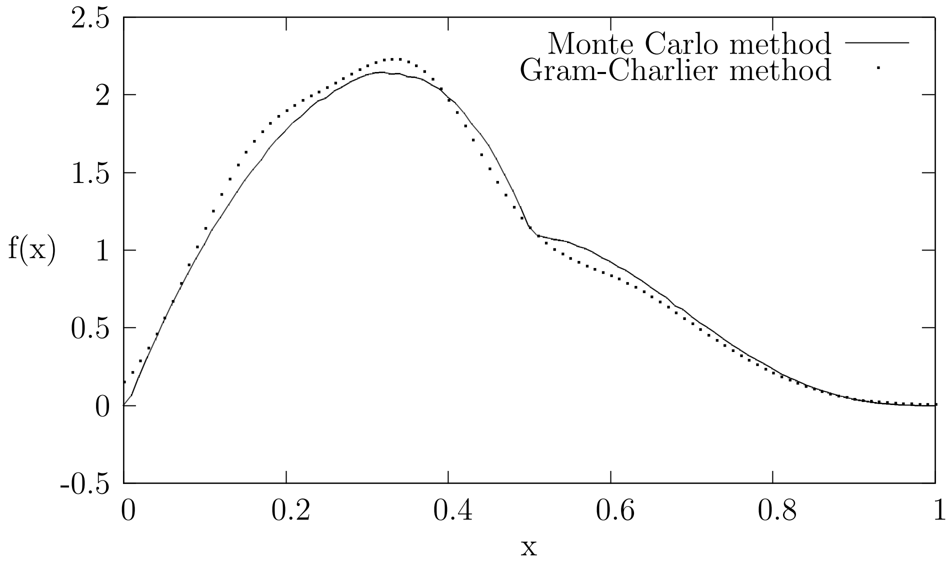


Figure 4.6: *N=2 with ψ follows Beta(2,2) distribution, t=1, n=25.*

Taking $N = 5$, we have the loss $J_t = \frac{1}{5}\sum_{i=1}^{N_t \wedge 5} \psi_i$ and the approximated densities of the loss J_t given $N_t > 0$ are shown in Figure 4.7 by two methods.

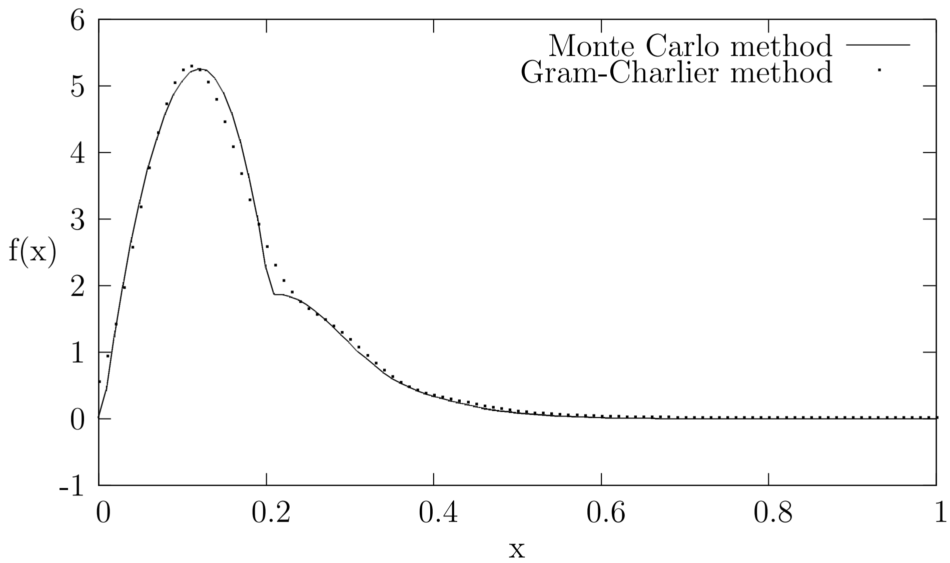


Figure 4.7: *N=5 with ψ follows Beta(2,2) distribution, t=1, n=25.*

It can be seen that our way of using Gram-Charlier expansion to approximate the conditional density is very close to Monte Carlo simulation from Figures 4.5 - 4.7. The Beta(2,2) distribution is better than simple uniform distribution, because the normal distribution term in Gram-Charlier expansion is not ideal for fitting the uniform. Hence the recovery rate follows Beta distribution would be better in our case and we get the explicit expression for the expected tranche loss again.

4.7 Gaussian copula model

The cumulative default probability that the obligor i will default before time t is represented by $F_{\tau_i}(t)$, we define the random variables

$$X_i = a_i M + \sqrt{1 - a_i^2}\, Z_i, \quad i = 1, \ldots, N, \tag{4.7.1}$$

where $| a_i | \leq 1$ and $(Z_i)_{1\leq i\leq N}$ are standard Gaussian i.i.d random variables independent of standard Gaussian variable M with the cumulative distribution function $\Phi(x) = \int_{-\infty}^{x} \phi(s)ds$.

The copula approach for example studied by Bluhm et al. (2010), is a standard way to price CDOs in industry, it directly models the default time given the marginal default probabilities of all assets and a copula function, which means that the joint density is derived from the marginal default probabilities of each asset through the copula function. It starts with the Gaussian copula model, introduced by Li (1999), widely used in financial industry since then, and then follows Clayton, Student t and double t copulas.

The equation (4.7.1) defines the correlation between X_i and X_j, which is $a_i a_j$. Next we get the distributions for default times $\{\tau_i,\, i = 1, \ldots, N\}$ through

$$\tau_i = F_{\tau_i}^{-1}\left(\Phi(X_i)\right).$$

As a result, the conditional probability that the i-th obligor will default before time t is

$$P(\tau_i \leq t \mid M) = \Phi\left(\frac{\Phi^{-1}(P(\tau_i \leq t)) - a_i M}{\sqrt{1-a_i^2}}\right).$$

Here we also can apply Gram-Charlier expansion to the computation of the expected tranche losses, where the overall loss is

$$J_t = \sum_{i=1}^{N} \psi_i \mathbb{1}_{\tau_i \leq t}.$$

Proposition 4.7.1. *The n-th moment m_n of the overall loss J_t is*

$$n! \int_{-\infty}^{\infty} \left(\sum_{\substack{n_1+\cdots+n_N=n \\ 0\leq n_i\leq N}} \prod_{i=1}^{N} \left(\frac{1}{n_i!} \Phi\left(\frac{\Phi^{-1}(P(\tau_i \leq t)) - a_i m}{\sqrt{1-a_i^2}}\right) \int_0^1 y^{n_i} dF(y) \right)^{\mathbb{1}_{n_i>0}} \right) d\Phi(m). \tag{4.7.2}$$

Proof.

$$E\left[e^{xJ_t} \mid M\right] = E\left[e^{x\sum_{i=1}^{N}\psi_i \mathbb{1}_{\tau_i\leq t}} \mid M\right]$$

$$
\begin{aligned}
&= \prod_{i=1}^{N} E\left[e^{x\psi_i \mathbb{1}_{\tau_i \le t}} \mid M\right] \\
&= \prod_{i=1}^{N} E\left[e^{x\psi_i} \mathbb{1}_{\tau_i \le t} + \mathbb{1}_{\tau_i > t} \mid M\right] \\
&= \prod_{i=1}^{N} \left(E\left[e^{x\psi_i} \mathbb{1}_{\tau_i \le t | M}\right] + P(\tau_i > t \mid M)\right) \\
&= \prod_{i=1}^{N} \left(P(\tau_i \le t \mid M) \int_0^1 e^{xy} dF(y) + 1 - P(\tau_i \le t \mid M)\right) \\
&= \prod_{i=1}^{N} \left(\sum_{n=1}^{\infty} \frac{x^n}{n!} \Phi\left(\frac{\Phi^{-1}(P(\tau_i \le t)) - a_i M}{\sqrt{1-a_i^2}}\right) \int_0^1 y^n dF(y) + 1\right) \\
&= 1 + \sum_{n=1}^{\infty} \left(\sum_{\substack{n_1+\cdots+n_N=n \\ 0 \le n_i \le N}} \prod_{i=1}^{N} \left(\frac{1}{n_i!} \Phi\left(\frac{\Phi^{-1}(P(\tau_i \le t)) - a_i M}{\sqrt{1-a_i^2}}\right) \int_0^1 y^{n_i} dF(y)\right)^{\mathbb{1}_{n_i>0}} \right) x^n.
\end{aligned}
$$

Since

$$
E\left[e^{xJ_t}\right] = \int_{-\infty}^{\infty} E\left[e^{xJ_t} \mid M = m\right] d\Phi(m),
$$

therefore the n-th moment m_n of J_t is

$$
n! \int_{-\infty}^{\infty} \left(\sum_{\substack{n_1+\cdots+n_N=n \\ 0 \le n_i \le N}} \prod_{i=1}^{N} \left(\frac{1}{n_i!} \Phi\left(\frac{\Phi^{-1}(P(\tau_i \le t)) - a_i M}{\sqrt{1-a_i^2}}\right) \int_0^1 y^{n_i} dF(y)\right)^{\mathbb{1}_{n_i>0}} \right) d\Phi(m).
$$

□

Next we assume that the default time of the i-th obligor follows the Poisson distribution with constant default intensities λ_i, $P(\tau_i \le t) = 1 - e^{-\lambda_i t}$ and i.i.d. rate ψ_i, $i = 1, \ldots, N$ follows uniform distribution on [0,1]. Consequently (4.7.2) becomes

$$
\begin{aligned}
m_n =& n! \int_{-\infty}^{\infty} \left(\sum_{\substack{n_1+\cdots+n_N=n \\ 0 \le n_i \le N}} \prod_{i=1}^{N} \left(\frac{1}{(n_i+1)!} \Phi\left(\frac{\Phi^{-1}(1-e^{-\lambda_i t}) - a_i m}{\sqrt{1-a_i^2}}\right)\right)^{\mathbb{1}_{n_i>0}} \right) d\Phi(m) \\
=& n! \sum_{\substack{n_1+\cdots+n_N=n \\ 0 \le n_i \le N}} \int_{-\infty}^{\infty} \prod_{i=1}^{N} \left(\frac{1}{(n_i+1)!} \Phi\left(\frac{\Phi^{-1}(1-e^{-\lambda_i t}) - a_i m}{\sqrt{1-a_i^2}}\right)\right)^{\mathbb{1}_{n_i>0}} d\Phi(m).
\end{aligned}
\tag{4.7.3}
$$

We take N=5, t=1, $\lambda_i = 1$, $i = 1, \ldots, N$ to show some numerical results, first get correlation a_i randomly.

Table 4.14: *Correlation in* (4.7.1).

i	1	2	3	4	5
a_i	0.141886	0.421761	0.915736	0.792207	0.959492

Next we get first N moments and cumulants.

Table 4.15: *Moments and cumulants in Gram-Charlier expansion.*

i	1	2	3	4	5
moments m_i	0.348921	0.217099	0.159588	0.126733	0.105405
cumulants c_i	0.348921	0.095353	0.017297	-0.00916	-0.00943

It comes with the expected tranche losses $E\left[J_t^{(\gamma)}\right]$ of different tranche slices.

Table 4.16: *Tranche loss in Gaussian copula model, one tranche.*

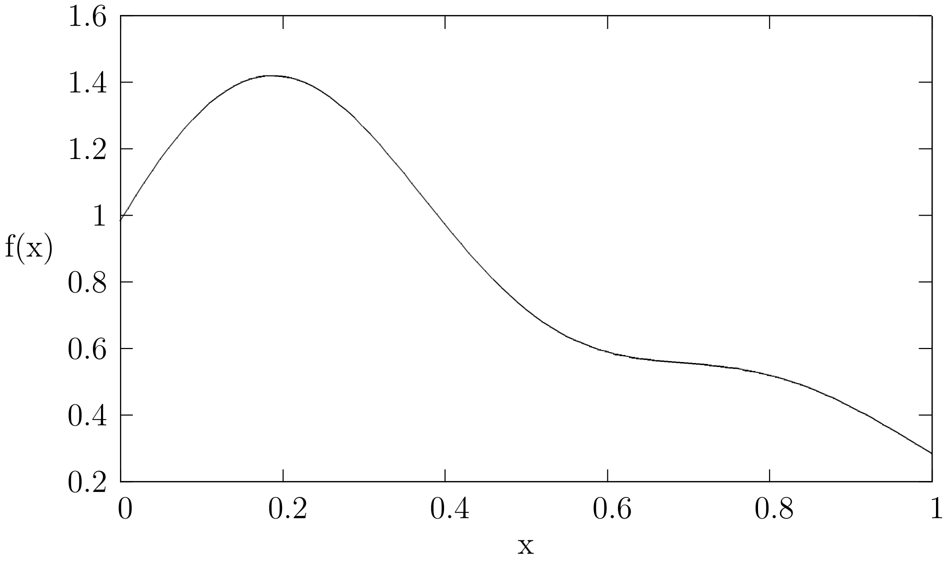
Tranche	Expected tranche loss
$[0, 1]$	32.9703%

Table 4.17: *Tranche loss in Gaussian copula model, two tranches.*

Tranche	Expected tranche loss
$[0, 0.5]$	24.892%
$(0.5, 1]$	7.5611%

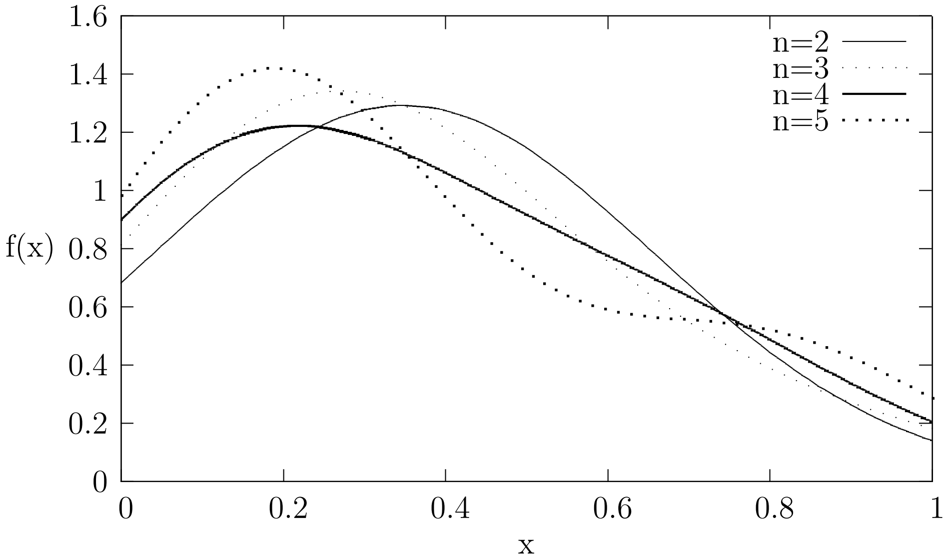
The approximated density $f(x) \simeq \sum_{n=0}^{5} \frac{q_n}{\sqrt{c_2}} H_n(\frac{x-c_1}{\sqrt{c_2}})\phi(\frac{x-c_1}{\sqrt{c_2}})$ of J_t is plotted.

Figure 4.8: *Density of J_t with $n = 5$ and $N = 5$.*



The below graph shows the change of density function $f(x)$ when we increase the value of n, and it turns out the changing speed is very slow.

Figure 4.9: *Density of J_t with $n = 2, 3, 4, 5$ and $N = 5$.*



Conclusion

The price of a non-defaultable T-maturity zero-coupon bond at time t can be written down using functional determinant techniques generally applied to quadratic functionals of Brownian motion in quantum field theory. This approach complements the usual PDE (Ricatti) approach to get the bond price formula. We extend this approach to the derivation of explicit formulas for Laplace transform of various Ornstein-Uhlenbeck functionals, using both functional determinant and PDE methods, this allows us to provide new closed form expressions for the functional determinants of Volterra operators in Chapter 2. Next in Chapter 3 we extend the Wiener chaos-based framework to normal martingale chaos-based framework, which recovers the Wiener chaos model and also allows jumps in the new chaos model. Consequently the calibration to the chaos model of spot forward rate is conducted. In the final chapter 4 we use the Gram-Charlier expansion on the density function of conditional loss and achieve the closed form of the expected tranche loss for computing the CDO spreads. In addition, the Gaussian copula model is also studied by using Gram-Charlier expansion method.

Appendix I

Market data of spot forward rates used in Chapter 3 for the calibration of chaos models

δ / time t	**3M**	**6M**	**1Y**	**2Y**	**3Y**	**5Y**	**7Y**	**10Y**	**20Y**	**30Y**
03/31/03	1.272	1.211	1.266	1.778	2.344	3.203	3.813	4.417	5.396	5.455
06/30/03	1.114	1.097	1.152	1.521	1.978	2.782	3.397	4.021	5.069	5.176
09/30/03	1.144	1.142	1.217	1.768	2.376	3.309	3.941	4.562	5.491	5.545
12/31/03	1.150	1.208	1.443	2.167	2.803	3.741	4.287	4.836	5.660	5.714
03/31/04	1.111	1.155	1.329	1.883	2.412	3.223	3.810	4.403	5.341	5.435
06/30/04	1.584	1.857	2.359	3.141	3.662	4.322	4.780	5.231	5.913	5.929
09/30/04	1.989	2.182	2.467	2.952	3.315	3.849	4.243	4.689	5.447	5.497
12/31/04	2.563	2.778	3.080	3.447	3.681	4.059	4.370	4.749	5.371	5.472
03/31/05	3.114	3.368	3.798	4.217	4.423	4.645	4.810	5.004	5.336	5.393
06/30/05	3.517	3.717	3.897	4.006	4.064	4.138	4.226	4.368	4.674	4.725
09/30/05	4.055	4.260	4.487	4.584	4.612	4.658	4.716	4.811	5.063	5.117
12/30/05	4.535	4.697	4.841	4.842	4.841	4.881	4.898	4.958	5.101	5.110
03/31/06	4.992	5.139	5.284	5.278	5.271	5.306	5.349	5.406	5.520	5.509
06/30/06	5.468	5.567	5.666	5.626	5.616	5.649	5.689	5.744	5.846	5.822
09/29/06	5.364	5.376	5.309	5.105	5.050	5.072	5.117	5.184	5.333	5.327
12/29/06	5.356	5.367	5.337	5.162	5.095	5.084	5.115	5.191	5.333	5.360
03/30/07	5.344	5.327	5.219	4.991	4.940	4.993	5.076	5.206	5.451	5.455
06/29/07	5.354	5.375	5.395	5.362	5.399	5.496	5.589	5.701	5.873	5.870
09/28/07	5.175	5.015	4.788	4.635	4.693	4.897	5.073	5.271	5.538	5.531
12/31/07	4.642	4.422	4.044	3.804	3.908	4.194	4.457	4.750	5.123	5.158
03/31/08	2.657	2.449	2.303	2.441	2.774	3.335	3.759	4.188	4.775	4.845
06/30/08	2.738	2.864	3.100	3.561	3.927	4.293	4.525	4.753	5.048	5.081
09/30/08	3.958	3.681	3.372	3.490	3.795	4.111	4.339	4.559	4.784	4.772

12/31/08	1.215	1.138	1.181	1.450	1.745	2.140	2.327	2.601	2.786	2.725
03/31/09	1.068	1.086	1.138	1.371	1.685	2.223	2.599	2.930	3.298	3.317
06/30/09	0.516	0.618	0.853	1.528	2.166	3.027	3.518	3.906	4.300	4.339
09/30/09	0.287	0.341	0.604	1.288	1.894	2.700	3.182	3.572	4.033	4.080
12/31/09	0.251	0.336	0.670	1.442	2.118	3.045	3.644	4.130	4.691	4.811
03/31/10	0.292	0.345	0.556	1.196	1.824	2.792	3.426	3.987	4.677	4.778
06/30/10	0.534	0.585	0.709	0.964	1.336	2.079	2.623	3.109	3.770	3.892
09/30/10	0.290	0.330	0.400	0.598	0.870	1.534	2.112	2.672	3.400	3.543
12/31/10	0.303	0.340	0.435	0.796	1.284	2.216	2.912	3.531	4.249	4.343
03/31/11	0.303	0.349	0.480	1.010	1.589	2.516	3.155	3.728	4.429	4.538
06/30/11	0.246	0.306	0.390	0.700	1.152	2.071	2.770	3.433	4.206	4.339
09/30/11	0.374	0.458	0.528	0.572	0.729	1.268	1.731	2.170	2.717	2.810
12/30/11	0.581	0.593	0.663	0.716	0.816	1.234	1.671	2.081	2.625	2.721
03/30/12	0.468	0.452	0.483	0.573	0.754	1.285	1.816	2.368	3.077	3.206
06/29/12	0.461	0.458	0.489	0.540	0.621	0.970	1.368	1.823	2.479	2.632
09/28/12	0.359	0.326	0.332	0.361	0.432	0.770	1.207	1.757	2.570	2.769
12/31/12	0.306	0.295	0.321	0.385	0.486	0.847	1.332	1.875	2.760	3.019
03/29/13	0.283	0.317	0.354	0.419	0.533	0.962	1.473	2.079	2.996	3.209
06/28/13	0.273	0.305	0.359	0.509	0.817	1.589	2.203	2.808	3.535	3.642
09/30/13	0.249	0.275	0.315	0.455	0.761	1.561	2.224	2.889	3.742	3.918
12/31/13	0.246	0.266	0.314	0.489	0.875	1.819	2.561	3.247	4.070	4.198

Table A1: *Quartely spot rate(%) from year 2003 to year 2013.*

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