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**EXISTENCE, CONVERGENCE AND
EFFICIENCY ANALYSIS OF NASH
EQUILIBRIUM AND ITS APPLICATIONS**

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A thesis submitted to the Nanyang Technological University
in fulfillment of the requirements for the degree of
Doctor of Philosophy

2016

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

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Date

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WANG XUEHE

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Abstract

Game theory deals with strategic interactions among multiple players, where each player tries to maximize/minimize its utility/cost. It has been applied in a broad array of areas such as economics, transportation, engineering, psychology, etc. Nash equilibrium, a fundamental concept in the realm of noncooperative game theory, is defined as the action profile of all players where none of the players can improve its utility/cost by a unilateral move. However, it is widely known that a Nash equilibrium often exhibits a suboptimal behavior compared with the socially optimal assignment. Moreover, in repeated games where each player makes its decision based on the available information at each stage, it is possible that the action profiles of all players fail to converge to a Nash equilibrium. This thesis presents research results on existence, convergence and efficiency analysis of Nash equilibrium in variety classes of games and their applications.

First, we discuss a repeated noncooperative multiple choices congestion game in which players have limited information about each other and make their decisions simultaneously. Congestion games are a class of games in game theory, in which the utility of each player depends on the resources each player chooses and the number of players choosing the same resource. At each stage, players can calculate their best choice if they know the number of players choosing each resource. However, in most cases, each player does not know other players' strategies before it makes its decision. Therefore, a player may need to estimate the number of players choosing each resource. We introduce a consensus protocol to estimate the percentage of players selecting each resource. At each stage, each player may exchange information with

its neighbors randomly and independently. We show that the congestion game under investigation has at least one pure strategy Nash equilibrium and the almost sure convergence to a pure Nash equilibrium can be ensured after some sort of inertia is adopted. Then, we apply our results to the trip timing and routing problem in traffic congestion control. By introducing different dynamic pricing schemes, the social optimum is achieved and players' choices are spread out, respectively. Simulation results based on the real traffic data of Singapore validate the effectiveness of the designed pricing schemes.

Next, we analyze the efficiency loss of Nash equilibrium in a nonatomic congestion game, where there is a continuum of players, each of which is infinitesimally small. A network with one origin-destination pair is formulated, where each edge is assigned a latency function. To characterize the worst-case efficiency loss of all possible Nash flow, price of anarchy (POA), defined as the worst possible ratio between the total latency of Nash flow and that of the socially optimal flow, is adopted. In order to improve the POA, a scaled marginal-cost is designed to affect players' choices. All players in the noncooperative congestion game are divided into groups based on their price sensitivities. For the two groups and two routes case, it is shown that the total latency of the Nash flow can always reach the total latency of the socially optimal flow if the designed scaled marginal-cost is charged on each link. For general case, if certain conditions are satisfied, a scaled marginal-cost can be designed such that the unique Nash flow can achieve the social optimal flow. An algorithm is also proposed to find the price scheme that optimizes the POA for any distribution of price sensitivity and any network with one origin-destination pair.

Besides the inefficiency of Nash equilibrium, there is also efficiency loss at each stage in repeated play. Therefore, we study the performance of a sequence of action profiles generated by repeated play in multiple origin-destination networks. To analyze the efficiency of the sequence of action profiles, the price of total anarchy (POTA), defined as the worst-case ratio of the average total latency over a period of time and the total latency of any optimal strategy, is adopted. It is shown that the sequence of action profiles generated by best response principle with inertia possesses almost

sure no-regret property. Via smoothness arguments, the upper bound of POTA is identified for both linear and nonlinear latency networks respectively. To reduce the upper bound of POTA, dynamic pricing is implemented for networks with linear latency functions. The influence of the inaccurate parameter information of the latency function on the POTA is also discussed. For the network with heterogeneous players, we show that the upper bound of POTA is the same as that in the network with homogeneous players as time goes to infinity. The results are applied to a traffic routing problem based on the real traffic data of Singapore.

At last, we investigate the behavior of large population systems based on mean field games where each agent evolves according to a dynamic equation containing the input average and seeks to minimize its long time average (LTA) cost encompassing a population state average (PSA), which is also known as the mean field term. Due to the informational burden resulting from the PSA coupling to the states of all agents, our idea is to find a deterministic function to estimate it. It is shown that the deterministic function is an approximation of the PSA as the population size goes to infinity. The resulting decentralized mean field control laws lead the system to mean-consensus asymptotically as time goes to infinity and the stability property of the mass behavior is also guaranteed. Furthermore, the optimal controls generate an almost sure asymptotic Nash equilibrium, which implies that the LTA cost of each agent can reach its minimal value as the number of agents increases to infinity. In addition, a nonlinear dynamic system is discussed and the influence of inaccurate mean field information on individual agent is analyzed. Then, we consider the socially optimal case where the objective of each agent is to minimize the social cost as the sum of all agents' LTA costs containing the PSA. In this case, it is shown that the decentralized mean field social control strategies are identical to the mean field Nash controls for infinite population systems.

Symbols and Acronyms

Symbols

$\mathbb{R}$	set of real numbers
$\mathbb{R}^N$	set of N dimensional real vectors
$\mathbb{Z}$	set of integer numbers
$\mathbb{Z}^N$	set of N dimensional integer vectors
S_i	strategy set of player i
s_i	strategy of player i
A'	transpose of A
A_{ij}	ij th entry of A
$\ A\ $	Euclidean norm of vector A
$ \cdot $	absolute value for a scalar or cardinality for a set
I_N	identity matrix with dimension $N \times N$
$\otimes$	Kronecker product
$:=$	defined as
■	end of proof
$\mathbf{I}\{\cdot\}$	indicator function
$\mathbb{E}(\cdot)$	expectation
$O(\cdot)$	infinitesimal function of the same order
$\lfloor \cdot \rfloor$	floor function

Acronyms

POA	price of anarchy
POS	price of stability
POTA	price of total anarchy
LTA	long time average
PSA	population state average
NCE	Nash certainty equivalence
VOT	value of time
ASFP	average strategy fictitious play
a.s.	almost surely

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Chapter 1

Introduction

1.1 Motivation and Objectives

Game theory, a study of strategic decision making of intelligent rational players, has been widely applied in a variety of areas such as economics, transportation, engineering, social science, etc. In noncooperative games, all players behave selfishly by maximizing/minimizing their own utility/cost and the equilibrium point yielded in this way is known as Nash equilibrium. However, Nash equilibria often exhibit suboptimal behaviors [1–3], which may cause significant efficiency loss. For any non-trivial game, the utility/cost of each player depends on the actions of at least one other player or all the players. For example, in congestion games, the payoff of each player depends on the resources it chooses and the number of players choosing the same resource. Generally, a player has only limited information of other players. In repeated games where the players are allowed to choose their actions based on the available information at each stage, players need to learn the underlying structure of the game by observing the decisions made by other players.

In networks of agents (players), each agent has a state which may change according to a control law correlated with other agents' states, and it monitors other agents' states via sensing, communication, and computing devices. A consensus algorithm

(protocol) is an interaction rule that specifies the information exchange between an agent and its neighbors in the network. If the initial state of each agent represents its initial action, it is possible for each agent to estimate other agents' actions by introducing an appropriate consensus algorithm [4].

Besides the consensus algorithm, mean field games provide another technique to estimate all interactions among players for large-scale dynamic systems [5]. The main idea of mean field game is to replace all interactions to any player with an average or effective interaction which can be obtained a priori. This reduces any multi-body problem into an effective one-body problem. As a result, players can update their strategies only according to their local information.

With the rapid development of our society, it becomes increasingly crucial to improve the efficiency of Nash equilibrium. To characterize the efficiency loss of Nash equilibrium, the price of anarchy (POA), defined as the worst possible ratio between the total cost of Nash equilibrium and that of the socially optimal assignment, is adopted. Dynamic pricing is an effective way to reduce the POA. For example, marginal-cost pricing, which illustrates that players should pay a fee that is equal to the difference between the marginal social cost and the marginal private cost [6], can lead the Nash equilibrium to the social optimum for the case with homogeneous players, i.e., $POA=1$. However, for the case with heterogeneous players, the POA can't achieve 1 any more by charging marginal-cost pricing. Besides the inefficiency of a Nash equilibrium, there is also efficiency loss at each stage in repeated play. In order to analyze the overall efficiency loss in repeated play, a generalization of POA, called price of total anarchy (POTA), is introduced in [7]. The POTA is defined as the worst-case ratio between the average total cost over a period of time and the total cost of the optimal outcome. Although the POTA of several classes of games have been discussed in some literature [7,8], how to reduce the upper bound of POTA has not been extensively studied.

The research in this thesis includes the analysis of Nash equilibrium in different game models and providing methods to improve its efficiency. In summary, the objectives to be achieved are:

1. To estimate the mass behavior of all players by proposing a consensus protocol or through mean field theory;
2. To analyze the efficiency of Nash equilibrium and the performance of the whole procedure in repeated play via POA and POTA, respectively;
3. To design dynamic pricing schemes to improve POA and POTA;
4. To apply our results to practical problems.

1.2 Main Contributions

This thesis discusses the existence of Nash equilibrium in different models, the convergence of Nash equilibrium in repeated games, and how to improve the efficiency of Nash equilibrium. The main challenges are: (1) estimating the mass behaviors of all players in the game; (2) designing pricing schemes that improve the efficiency of Nash equilibrium or the overall efficiency of the system in repeated games. In this thesis, we first propose a consensus protocol to estimate the number of players choosing each resource under random networks in Chapter 3, which can be applied to the following chapters. Then, we analyze how to design a pricing scheme to improve the efficiency of Nash equilibrium in a routing problem in Chapter 4. Since there is also efficiency loss at each stage in repeated games, in Chapter 5, we investigate the overall efficiency loss during the updating procedure and find a pricing schemes to improve the overall efficiency. Finally, in Chapter 6, we study mean field games, which provide another technique to estimate the mass information for large population stochastic multi-agent systems.

The main contributions of the thesis are listed as follows:

1. We analyze the existence and convergence of Nash equilibrium in a multiple choices congestion game. A consensus protocol is implemented to estimate the number of players choosing each resource under random networks. Several

nonlinear dynamic pricing schemes are introduced to achieve some kind of social optimum or spread out players' choices in different models.

2. We consider a routing problem in networks with one origin-destination pair, in which the scaled marginal-cost pricing is charged on each link to affect players' routing choices. Players are assumed to have heterogeneous price sensitivities and are divided into groups accordingly. For the special case with two groups and two routes, we show that the POA can always achieve 1 by an designed scaled marginal-cost pricing on each link. For general case, the existence and uniqueness of Nash flow are discussed. Under certain conditions, we prove that there exists an optimal scaled marginal-cost pricing such that $\text{POA} = 1$. Otherwise, the POA can't achieve 1. For any network with one origin-destination pair and given distribution of price sensitivity, an algorithm is provided to calculate a scaled marginal-cost that minimizes the POA.
3. We analyze the efficiency of a series of action profiles in networks with multiple origin-destination pairs. By adopting the best response with inertia principle, we show that the series of action profiles possesses almost sure no-regret property. The influence of dynamic pricing on the upper bound of POTA for the linear latency network is analyzed via smoothness arguments. It is proved that the designed dynamic pricing can reduce the upper bound of POTA. The effect of inaccurate parameter information of the latency function on the POTA is also discussed. As is expected, the deviation of the parameter may result in more efficiency loss. In addition, the upper bound of POTA for the nonlinear latency network is given. For networks with heterogeneous players, we show that the upper bound of the POTA is the same as that in networks with homogeneous players as time goes to infinity.
4. We study a discrete-time dynamic system where each agent evolves according to a dynamic equation containing all agents' control inputs and seeks to minimize its LTA cost encompassing the PSA. In the system we established, a deterministic function $\psi(0)$, which is shown to be the expectation of the initial

PSA, is constructed to approximate the PSA, and the convergence of the PSA to $\psi(0)$ is justified. On the basis of $\psi(0)$, the decentralized mean field Nash control is established. It is shown that the mean-consensus property is ensured by applying the decentralized mean field control law. The ε -Nash equilibrium is introduced to evaluate the performance of the set of decentralized mean field controls. We show that the optimal controls generate an almost sure asymptotic Nash equilibrium, which implies that the LTA cost of each agent can reach its minimal value as the number of agents increases to infinity. A nonlinear dynamic example with similar properties as the linear system established above is shown. The influence of inaccurate mean field information $\psi(0)$ on individual agent is also analyzed. The social optimal case where all agents work cooperatively to optimize the social cost as the sum of individual agent's LTA cost is analyzed. The decentralized mean field social controls are proved to be equal to the mean field Nash controls as the number of agents increases to infinity.

5. We apply our results to trip timing and traffic routing problems. Numerical examples and simulations based on the real traffic data of Singapore are given to validate the results.

1.3 Outline of the Thesis

In Chapter 2, a literature review for historical developments and state-of-the-art results on game theory, consensus theory and the techniques for improving the efficiency of Nash equilibrium is given.

In Chapter 3, we propose a consensus protocol to estimate the percentage of players choosing each resource in noncooperative congestion games, where each player has limited information on other players. We show that the action profiles generated by best response with inertia converge to a pure strategy Nash equilibrium almost surely. Then, we apply our results to traffic problems. Several dynamic pricing

schemes are introduced to smooth the peak travel or achieve some kind of social optimum. Simulations based on real traffic data of Singapore are also provided to illustrate the validity of the results.

In Chapter 4, we investigate how the scaled marginal-cost improves the POA in networks with one origin-destination pair. We assume that all players have different price sensitivities and are divided into groups accordingly. The two groups two routes case and the general case are analyzed respectively. We show that whether we can find a scaled marginal-cost such that $\text{POA}=1$ depends on the distribution of price sensitivity, topology and parameters of the network. An algorithm is also provided to find the price schemes that minimize the POA for any given distribution of price sensitivity and network with one origin-destination pair. Finally, numerical examples and real data simulations are given to validate our results.

In Chapter 5, we analyze the performance of a sequence of action profiles generated by repeated play in multiple origin-destination networks. By adopting the concept of almost surely no-regret, the upper bounds of POTA in networks with linear and nonlinear latency functions are given via smoothness arguments, respectively. It is also shown that dynamic pricing can reduce the upper bound of POTA for the linear latency case. The influence of inaccurate latency parameter on the POTA is discussed. Furthermore, the POTA for networks with heterogeneous players is also analyzed. Finally, the results are implemented to a traffic routing problem based on the real traffic data of Singapore.

In Chapter 6, we consider large population dynamic games where each agent evolves according to a dynamic equation containing the input average of all agents. The long time average (LTA) cost that each agent aims to minimize is coupled with other agents' states via a population state average (PSA), which is also known as the mean field term. In order to design decentralized controls, the Nash certainty equivalence (NCE) is adopted. It is shown that the resulting decentralized mean field control laws lead the system to mean-consensus asymptotically as time goes to infinity. The stability property of the mass behavior and the almost sure asymptotic

Nash equilibrium property of the optimal controls are also guaranteed. In addition, the influence of inaccurate mean field information on individual agent is analyzed and a nonlinear dynamic system is discussed. We also investigate the socially cooperative formulation where the objective is to minimize the social cost as the sum of all individuals' LTA costs containing the PSA. In this case, we show that the decentralized mean field social controls are identical to the mean field Nash controls for infinite population systems.

In Chapter 7, we conclude the thesis and highlight some possible future research directions.

Chapter 2

Literature Review

Game theory is about the study of strategic decision making, which is used in many realms such as economics [9–11], political science [12], transportation [13], computer science [14–16], and biology [17]. To our knowledge, game theory was first discussed by James Waldegrave in 1713, who gave a mixed strategy solution to a two-person game [18]. In 1928, John von Neumann published a paper proving the existence of mixed-strategy equilibria in two-person zero-sum games [19]. After that, game theory exists as a unique field. Around 1950, a criterion for mutual consistency of players' strategies was developed by John Nash, which is known as Nash equilibrium [20]. In the 1950s, game theory experienced a prosperous time, and the concepts of the core, extensive form games, repeated games, fictitious plays, and the Shapley value were developed. Nowadays with the rapid development of our society, game theory has been applied to a wide range of behavioral relations and considerable research has been done.

Game theory consists of a large variety of types, such as cooperative/noncooperative, symmetric/asymmetric, zero-sum/non-zero-sum, perfect information/imperfect information, and so on. In the realm of perfect information games, every player knows all other players' actions previously. Therefore, all players can make their decisions by maximizing/minimizing their utilities/costs based on this information. However, in reality, most players or even all players don't know all other players' actions.

Consensus protocols, which are distributed control policies based on neighbors' state feedbacks that allow the coordination of multi-agent systems, provide a way to estimate other players' actions. Although there are many separate works on game theory and consensus theory, the connection between them has not been extensively studied.

In the following context, we summarize the established results on game theory, consensus theory and efficiency analysis of Nash equilibrium.

2.1 Game Theory

Game theory is about the study of multi-players decision problems. The *normal-form representation* of a game consists of three parts: (i) the game players; (ii) the available strategies to each player; (iii) the utility/cost received by each player for each combination of strategies that could be chosen by the player [21].

Generally, there are two kinds of strategy for each player—*pure strategy* and *mixed strategy*. Let S_i denote the set of strategies available to player i , and let $s_i \in S_i$ denote the action chosen by player i . A strategy s_i is called a *pure strategy* for player i , if it chooses s_i with probability one. Suppose that player i has K different choices, i.e., $S_i = \{s_{i1}, \dots, s_{iK}\}$, then a *mixed strategy* for player i is a probability distribution $pr_i = \{pr_{i1}, \dots, pr_{iK}\}$, where pr_{ik} represents the probability for choosing action s_{ik} , $0 \leq pr_{ik} \leq 1$ for $k = 1, \dots, K$ and $\sum_{k=1}^K pr_{ik} = 1$.

Nash equilibrium, a fundamental concept in game theory, is defined as the action profile of all players where none of the players can improve its utility/cost by a unilateral move. Considerable research has been done on the existence of Nash equilibrium [22–25].

There are many classes of games possessing different properties that we can choose to construct different models. Meanwhile, a model may contain the characters of more than one types of games. In the following, we present several classes of games and discuss learning processes in games.

2.1.1 Congestion Games

In a congestion game, each player faces a finite set of resources, and the utility/cost of each player depends on the resource it chooses and the number of players choosing the same resource. To be precise, the congestion game is defined as:

Definition 2.1. *A finite N -player game with player set $\mathcal{N} = \{1, 2, \dots, N\}$ and utility/cost function $U_i, i \in \mathcal{N}$ is a **congestion game** if*

- *Each player i faces finite choices, i.e., $S_i = \{s_{i1}, \dots, s_{iK}\}$.*
- *The utility/cost of player i for choosing strategy $s_i \in S_i$ is $U_i(n_{s_i})$, where n_{s_i} is the number of players choosing strategy s_i .*

It is well known that every finite potential game is isomorphic to a congestion game [26] and any congestion game is a potential game [27], which is defined as:

Definition 2.2. *A finite N -player game with player set $\mathcal{N} = \{1, 2, \dots, N\}$ and utility/cost function $U_i, i \in \mathcal{N}$ is a **potential game** if there exists a potential function $\phi(s)$ such that for all players $i = 1, \dots, N$ and all possible s_i^1, s_i^2, s_{-i}*

$$U_i(s_i^1, s_{-i}) - U_i(s_i^2, s_{-i}) = \phi(s_i^1, s_{-i}) - \phi(s_i^2, s_{-i}) \quad (2.1)$$

where $-i$ denotes the complementary set $\mathcal{N} \setminus \{i\}$.

From the definition of potential game, we can see that it requires perfect alignment between the global objective and the players' local utility/cost functions in the following sense: if a player changes its strategy unilaterally, the change in its utility/cost function would be equal to the change in the potential function.

It is widely known that every potential game has at least one pure strategy Nash equilibrium [26]. To our knowledge, potential functions for games in strategic form, which is a way of describing a game using a matrix, were first used in [27], where the existence of pure strategy Nash equilibrium is proved in a class of congestion

games by explicitly constructing a potential function. In [28], the convergence to Nash equilibrium in potential games using fictitious play process is proved. Some methods for calculating potential functions are introduced in [26, 29]. Upon the concept of potential games, cooperative control problems are formulated, where each agent (player) is assigned a local utility and the convergence to an equilibrium is ensured [30].

In the congestion games we discussed above, all players have the same influence upon the congestion. This congestion game model can be generalized by introducing weights. Specifically, each player $i \in \mathcal{N}$ is assigned a weight and the congestion on each resource becomes the total weight of all players choosing that resource. However, unlike the congestion games, weighted congestion games do not necessarily possess pure strategy Nash equilibria [31]. The work in [32–34] discusses the existence of pure Nash equilibria in weighted congestion games.

2.1.2 Fictitious Play

In noncooperative games, every player selects a strategy from a possible strategy set selfishly and independently and each player’s utility depends on the actions taken by other players. If the game is repeated over multiple stages, players are allowed to adapt their strategies in response to the available information gathered over prior stages. This setup falls under the general subject of “learning in games”.

One of the earliest learning rules is fictitious play, which is first created by G.W. Brown [35]. In a fictitious play, each player computes the empirical frequencies of the prior actions of all players and plays a myopic best response according to those empirical frequencies [36]. Through several counterexamples, it is shown that the convergence of fictitious play can’t be ensured [37–39]. The work in [40] proves that fictitious play is likely to converge in certain special cases such as identical interest games, zero-sum games, and two-player/two-move games. Owing to the good properties of potential games, the empirical frequencies generated by fictitious play in a potential game is proved to converge to a mixed strategy Nash equilibrium [26].

However, one obvious disadvantage of fictitious play is that it is computationally infeasible for large-scale games, since the best response of a player depends on a mapping over a joint action space of other players.

Joint strategy fictitious play, a variant of fictitious play, alleviates the computational burden of fictitious play by introducing a utility updating process for each player [36, 41]. Different from fictitious play, in joint strategy fictitious play, each player assumes that all other players make decisions jointly according to the empirical frequencies of the joint actions of other players. In joint strategy fictitious play with inertia, the convergence to a pure Nash equilibrium is guaranteed for any finite generalized ordinal potential game [41]. The average strategy fictitious play established in [42] further reduces the computational burden of joint strategy fictitious play by introducing a weighted running average of all other players' actions which is assumed to be broadcasted by the system supervisor. Besides, average strategy fictitious play with inertia also guarantees the convergence to a pure Nash equilibrium.

2.1.3 Mean Field Games

In large-scale games, there are too many players to permit an equilibrium which is determined by the interactions of players. The vast number of these interactions makes a detailed model ineffective. Nevertheless, it is still possible to construct an approximation to the mass information by adopting one or more *mean field* that act as intermediaries for describing interactions.

The mean field game theory was considered by Peter E. Caines and his co-workers [43] and independently by Jean-Michel Lasry and Pierre-Louis Lions [5, 44, 45] around the same time. The mean field game deals with the strategic decision making in a large population of small interacting individuals. In a mean field game, individual player has little impact on other players so that it can be ignored. While, the overall effect of the population on any individual can't be neglected. Therefore, players are intrinsically faced with the limitations in both their observational and computational

capabilities. This problem arises in many areas, such as traffic assignment [6], factory production [46], communication network [47] and so on. The main idea of mean field games is to replace all inner-particle interactions by a deterministic function which can be obtained a priori. A few examples and applications of mean field games are shown in [48].

In multi-agent systems, interaction between individual agents may exist in each agent's dynamics and control objectives. For centralized control systems, such as consensus [49], flocking [50], swarming [51], etc., the states of other agents are available for each agent. While for decentralized control systems [52, 53], only local information is available for each agent. Due to the limited sensing ability, it is not feasible for each individual agent to collect all other agents' states, especially for large-scale dynamic systems. Therefore, the design of decentralized control laws is required. There has been a lot of research on decentralized control design which provides techniques for dealing with large population dynamic systems [54–56]. One technique is called the Nash certainty equivalence (NCE) principle [43, 57], which deals with large population multi-agent systems where each agent is coupled with other agents through its cost function encompassing the population state average (PSA). Through NCE principle, the PSA is approximated by a deterministic function and decentralized controls are established. For mean field systems with non-coupled stochastic dynamics, results in [56, 58, 59] show that the PSA can be approximated by the expectation of overall population's state average as the population size goes to infinity, thereby the decentralized controls are designed accordingly to it. However, if the information on population distribution is inaccurate for any agent, there will be efficiency losses in this agent's cost [58]. For a mean field system with stochastic dynamic equation coupled with other agents through the PSA, the work in [60] shows that the estimate of the PSA can be computed iteratively at each stage, and the decentralized control laws obtained based on it generate an asymptotic Nash equilibrium. However, for the mean field system considered in [60], the control objective that each agent seeks to optimize doesn't include a control input. Following the work in [60], the paper [61] analyzes the situation when the coupling

parameter of the PSA in the dynamic equation is unknown. A recursive LS algorithm is used to estimate the unknown parameter and a decentralized control law is constructed based on it and the approximation of the PSA. For the social optimal control problem with the PSA coupled in the cost function, both the centralized and decentralized solutions are discussed in [62, 63].

2.2 Consensus Theory

The consensus problem requires to reach an agreement asymptotically or in a finite time, regarding a certain quantity of interest that depends on the state of all agents. Specifically, the agreement is a common value which may be the average of the initial states of the network, and often called average-consensus. There has been a lot of research on consensus theory. Consensus conditions for discrete-time multi-agent networks with fixed topology are explored in [49, 64]. Necessary and sufficient conditions on consensus over random networks are provided in [65, 66]. In [67, 68], discrete-time average-consensus under switching network topologies is discussed. The work in [69] analyzes a class of consensus problems under dynamically changing environments using algebraic theory and graph theory. The consensus convergence rates are investigated in [69–73].

Since both consensus theory and game theory consider the interactions among a number of agents/players, the connection between them has been studied by some researchers. In [74] and [30], a utility function is proposed such that any consensus point is a Nash equilibrium, and it is also proved that every universal Nash equilibrium constitutes consensus under certain circumstance. In [75], each agent updates its state according to a replicator dynamics and the Nash equilibrium is the stationary point for this dynamics. For finite noncooperative games [76] where the stage utility of a player associated with each decision is a monotonically nondecreasing function of the total number of players making the same decision, players can calculate their best choice if they know the number of players choosing each

resource. In [77], a continuous-time distributed protocol is proposed to estimate the number of active players. Under this structure, a best response path algorithm is established, where only one player updates its decision by the best response at each stage. Such an algorithm generates a sequence of joint decisions that eventually converges to a Nash equilibrium. In [4], a discrete-time consensus protocol under a fixed network topology is designed to estimate the percentage of active players and allow the best response strategy to converge to a unique Pareto optimal Nash equilibrium. The results in [4, 77] can only be applied to binary strategies, i.e., each player only has two choices. In reality, every player often faces multiple choices. Therefore, we aim at proposing a consensus protocol to estimate the percentage of players for each choice in multiply choices congestion games.

2.3 Efficiency Analysis of Nash Equilibrium

With the rapid development of our society, creating systems with high efficiency has attracted a lot of attention. However, selfish behaviors of noncooperative decision-makers may lead to inefficient outcomes. This problem occurs in many realms such as communication networks [78], cooling systems [79], market mechanisms [80], and traffic congestion [81, 82].

In game theory, Nash equilibria in general are typically inefficient. Early work studying the Pareto inefficiency of Nash equilibrium can be found in [83]. To quantify the efficiency loss of a Nash equilibrium, the price of anarchy (POA) and the price of stability (POS) are introduced in the literature [14, 84, 85]. The POA compares the total cost of the worst-case Nash equilibrium to that of the socially optimal outcome, while the POS considers the ratio between the cost of the best Nash equilibrium and the socially optimal cost. The research on POA has spawned a lot of publications, especially for nonatomic congestion games, in which there is a continuum of players, each of them has negligible influence on other players, see [86–92]. In nonatomic congestion games with linear latencies, the upper bound of POA for pure strategy

Nash equilibrium is determined as $4/3$ [86]. The work in [93] improves the value $4/3$ by means of coordination mechanisms for networks of parallel links. It is proved in [87] that the upper bound of POA can be achieved for a certain class of latency functions containing the constant functions. It is shown in [89] that the POA can be arbitrarily large and an upper bound and a lower bound of the POA are given for a class of latency functions. The bound of the POA in nonatomic congestion games with nonlinear latency functions is discussed in [91, 94]. Two models (latency model and pricing model) are introduced in [95] to show the efficiency of Nash equilibrium in the context of selfish routing. In the latency model, it is proved that the POS is unbounded, while in the pricing model, all Nash equilibria have optimal flow under certain conditions. For a mixed strategy Nash equilibrium, in which each player plays its available pure strategies with certain probabilities, it is shown in [84] that the POA in two links networks with affine latency functions is at least $3/2$. Using the method of smoothness arguments, the bounds of POA on several generalizations of pure Nash equilibria, such as mixed Nash equilibria, correlated equilibria and coarse correlated equilibria, are discussed in [8]. In parallel networks, the POS is shown to be sensitive to demand change when link flow is closed to its capacity [96]. The work in [97] studies the bound of POA in routing games with incomplete information. In weighted congestion games where each player is assigned a weight and the load on a resource is the total weight using that resource, the lower bound of POA for parallel networks is given in [98]. In [99], upper bounds of the POA in weighted and unweighted congestion games are established, provided that the latency functions are polynomials with nonnegative coefficients. Due to the computational complexity of the Nash equilibrium, the concept of approximate Nash equilibrium is introduced [14]. In [100], upper bounds of both the POA and the POS for approximate Nash equilibria in atomic and nonatomic congestion games with linear latencies are given. In nonatomic congestion games, it is discussed in [101] that, under certain assumptions, a necessary and sufficient condition for the case where equilibria are always socially optimum is that the player's utility for choosing a resource decreases logarithmically as the number of other players choosing the same resource increases.

Besides the inefficiency of a Nash equilibrium, there is also efficiency loss at each stage in repeated play where all players are allowed to choose their strategies based on the available information at each stage. The idea of no-regret property provides a method to analyze the performance of players' actions over time. A sequence of strategies is said to exhibit no-regret property if and only if the average difference between the actual cost received by any player and the minimal cost it can receive is less than any positive number as time approaches infinity. For a game that exists a unique and dominant strategy Nash equilibrium, no-regret algorithms make the action profiles of all players converge to a Nash equilibrium [102]. However, it is possible that no-regret algorithms fail to converge to a Nash equilibrium [7]. In order to analyze the overall efficiency loss in repeated play, a generalization of POA, called price of total anarchy (POTA), is introduced in [7]. The POTA is defined as the worst-case ratio between the average total cost over a period of time and the total cost of the optimal outcome. In [7], the POTA of several classes of games is discussed and it is shown that the POTA exactly matches the POA in some cases. In the fast-developing society, it becomes increasingly crucial to reduce the efficiency loss of equilibrium points. In the following, we focus on two methods to improve the efficiency of Nash equilibrium.

2.3.1 Efficiency Improvement of Nash Equilibrium via Stackelberg Games

In Stackelberg games [103], a fraction of the players (leaders) are assumed to move first, then the rest of the players (followers) are considered to react selfishly based on the actions of the leaders. In order to improve the efficiency of Nash equilibrium and the performance of a system, our objective is to find a leader strategy that induces the followers to react in a way that minimizes the total cost of the system. The Stackelberg strategies have been well-studied in the literature [104–106]. In repeated Stackelberg games where the followers' types are unknown to the leader, a method is presented to compute the best leader's strategy at each stage to

maximize the leader's cumulative expected payoff upon the assumption that the leader knows in advance the probability distribution of followers' types [107]. The situation when the followers observe the leader's strategy with certain probability is discussed in [108, 109] and an iterative algorithm is proposed to compute the equilibrium in [109]. In [110], the existence and uniqueness of maximally efficient Stackelberg strategy that lead the system to the global optimum is investigated. For Stackelberg routing games on parallel networks with nondecreasing latency functions, the Largest Latency First strategy (LLF strategy) is proposed to ensure that the POA of the network is no more than $1/\theta$, where θ is the proportion of leaders [111]. In [96, 112], Stackelberg routing on parallel networks with horizontal queues is addressed and an algorithm is introduced to compute the best Nash equilibrium that minimizes the total cost. In [113], a game theory controller is constructed based on the feedback Stackelberg equilibrium framework to reduce fuel consumption and oxides of nitrogen emissions for hybrid electric vehicles. One shortcoming of the Stackelberg game is that it generally deals with systems with homogeneous players. To promote the efficiency of Nash equilibrium in systems with heterogeneous players, other methods need to be implemented.

2.3.2 Efficiency Improvement of Nash Equilibrium via Pricing Schemes

It is widely known that an appropriate price scheme can improve the system efficiency. Take the traffic system for example, a case study in California, USA documented in [114] shows that transportation pricing such as congestion pricing, parking pricing, fuel tax pricing, vehicle miles of travel fees, emissions fees, can better manage the transportation systems to a great extent. As another example, the Electronic Road Pricing (ERP) system in Singapore is designed to charge motorists when they use the road during peak hours, and it is effective in maintaining an optimal speed range for both expressways and arterial roads [115, 116].

As there are a variety of types of pricing schemes, the main problem is how to choose an appropriate pricing scheme to develop the efficiency of equilibrium points in a simple yet practical model under complicated circumstances. First of all, we illustrate several frequently-used pricing schemes.

Due to individual player's failure to share the cost it imposes on other players, the fundamental economic principle – marginal-cost pricing is introduced, which illustrates that each player will be charged a fee that is equal to the difference between the marginal social cost and the marginal private cost [117–119]. As a result, players will face the marginal social cost other than the marginal private cost, which means that each player should pay the value of loss that it exerts on other players. In the case of homogeneous players, the marginal-cost pricing can lead the Nash equilibrium to achieve the social optimum [120–123].

By expecting or observing the change of the supply and demand balance during time, the service provider or commodity supplier may change the price on the basis of the time of day. This type of pricing strategy is called time-based pricing. For example, in the tourism industry, the hotel price is higher during the peak season and lower in the off-peak season [124]; in traffic congestion control, higher price is charged during rush-hours [123, 125, 126]. Dynamic pricing, a special case of time-based pricing, is a pricing strategy in which the price is highly flexible depending on current demands. Generally, dynamic pricing is applied to increase revenue and has been used in many fields such as travel, online retail, entertainment and so on. For instance, the transportation price usually is higher as time gets closer to departure [127]. In [128], it is shown that the optimal dynamic pricing indeed improves the revenue compared with the optimal single (fixed) pricing.

There is a lot of research dealing with how the price schemes affect the efficiency of equilibrium points. For single commodity networks with heterogeneous players, it is shown in [129] that the optimal static tolls that optimize the behavior of the network can be computed in polynomial time. In [130], the influence of taxes on the total cost of Nash equilibrium is compared with the effect of edge removals

in a single commodity network with linear latency functions and nonlinear latency functions, respectively. For multi-commodity networks, the work in [131] shows that there exist static tolls making selfish players act in a way that minimizes the average latency. In order to get a desired equilibrium flow for traffic networks, the influence of static tolls on drivers' decisions is discussed in [132]. In [42], a dynamic road pricing is established to achieve socially beneficial trip timing in ASFP. The model in [133] captures the difference in individual's sensitivity to early or late arrival in trip timing problems and a dynamic pricing is proposed to spread peak travel. In transportation problems, different players may have different value of time (VOT), which describes the tradeoffs between travel time and cost. In [134,135], the optimal dynamic pricing for players with continuous VOT distributions is investigated, and an algorithm is proposed to compute the optimal tolls that induce a system optimal equilibrium. Under the circumstances where the system designer has incomplete information on players' price sensitivities, an optimal scaled marginal-cost is derived to minimize the POA in congestion games with linear latency [136]. Even though there are fruitful research results on the efficiency improvement of Nash equilibrium, the influence of the pricing on POTA has seldom been studied. We will address this problem in this thesis.

Chapter 3

Distributed Consensus in Congestion Games

In this chapter, we study a repeated noncooperative congestion game in which players have limited information about each other and make their decisions simultaneously. A multiple choices congestion game is formulated in Section 3.1. In Section 3.2, we first prove that the game under consideration has at least one pure strategy Nash equilibrium. After that, a consensus protocol is introduced to estimate the percentage of players selecting each resource, which alleviates the binary constraint in [4, 77]. Based on the consensus protocol, the almost sure convergence to a pure strategy Nash equilibrium is guaranteed by adopting the concept of inertia. In Section 3.3, we apply our results to the trip timing problem and two nonlinear dynamic price functions are designed to ensure the social optimum and spread peak travel, respectively. A more practical model where each player faces both the routing and trip timing problems is considered in Section 3.4. Section 3.5 concludes this chapter.

3.1 Problem Formulation

Consider an N -player congestion game $\mathcal{N} = \{1, 2, \dots, N\}$, in which each player $i \in \mathcal{N}$ faces two classes of resources $\mathcal{R}_p = \{r_1, r_2, \dots, r_v\}$ and $\mathcal{R}_t = \{r_{v+1}, r_{v+2}, \dots, r_{v+m}\}$. $\mathcal{R}_p$

represents some kinds of choices that player i will receive a fixed utility p_{ij} by choosing $r_j \in \mathcal{R}_p$, e.g., in traffic congestion problem, road users will receive a fixed utility by choosing public transportation such as subway, light rail trains. $\mathcal{R}_t$ represents some choices that each player's utility is related to the number of players choosing the same resource. Denote the finite resource set available to all players as $\mathcal{R} = \mathcal{R}_p \cup \mathcal{R}_t$. In repeated play, at each stage, player i chooses an action from $\mathcal{R}$. Denote the strategy of player i at stage k as $s_i(k) = [s_{i1}(k), \dots, s_{iv}(k), s_{iv+1}(k), \dots, s_{iv+m}(k)]' \in \mathbb{Z}^{v+m}$, where $s_{ij} \in \{0, 1\}$, $j \in \mathcal{I}_0 = \mathcal{I}_v \cup \mathcal{I}_m$ and $\sum_{j=1}^{v+m} s_{ij} = 1$. Here, $\mathcal{I}_v = \{1, 2, \dots, v\}$, $\mathcal{I}_m = \{v+1, \dots, v+m\}$. $s_{ij}(k)$ is equal to 1 if player i chooses an action r_j at stage k and 0 otherwise. Let $s(k) = \{s_1(k), s_2(k), \dots, s_N(k)\}$ represent the action profile of all players and $s_{-i} = \{s_1(k), \dots, s_{i-1}(k), s_{i+1}, \dots, s_N(k)\}$ denote the complementary strategy of player i .

At each stage k , the utility of player i is defined as:

$$U_i(s_i(k), s_{-i}(k)) = \sum_{j=1}^v s_{ij}(k) p_{ij} + \sum_{j=v+1}^{v+m} s_{ij}(k) (f_j(n_{r_j}(k)) + h_i(r_j)) \quad (3.1)$$

where $n_{r_j}(k) = \sum_{i=1}^N \mathbf{I}\{s_{ij}(k) = 1\}$ is the number of players choosing resource r_j . Here, function $\mathbf{I}\{\cdot\}$ is the indicator function. $f_j(n_{r_j}(k))$ is the utility due to the congestion of players using the same resource, and it is nonincreasing with respect to $n_{r_j}(k)$. $h_i(r_j)$, $j \in \mathcal{I}_m$ is the fixed utility received by player i for choosing action r_j . Denote $n_r(k) = [n_{r_1}(k), \dots, n_{r_{v+m}}(k)]' \in \mathbb{Z}^{v+m}$. Note that player's utilities are heterogeneous in our model.

The pure strategy Nash equilibrium of the utility-maximization game is defined as follows.

Definition 3.1. A profile $s^{ne} = \{s_1^{ne}, \dots, s_N^{ne}\}$ is called a pure strategy Nash equilibrium, if, for each player $i \in \mathcal{N}$:

$$U_i(s_i^{ne}, s_{-i}^{ne}) = \max_{s_i \in \mathbb{Z}^{v+m}} U_i(s_i, s_{-i}^{ne}). \quad (3.2)$$

In this thesis, we will use pure strategy Nash equilibrium and Nash equilibrium interchangeably when no confusion is caused.

In the following, we will show the existence and convergence of a Nash equilibrium in the repeated noncooperative congestion game via a consensus protocol. We will also implement our results to traffic problems and show the effect of dynamic pricing on the Nash equilibrium.

3.2 Convergence Analysis of Nash Equilibrium

3.2.1 Existence of Nash Equilibrium

In Section 2.1.1, we have discussed potential games, which possesses a desirable property regarding the existence of a pure strategy Nash equilibrium. In the following lemma, we will show that the game we formulated above is also a potential game. Thus, it inherits the desirable property of potential games.

Lemma 3.1. *The congestion game with utility function (3.1) is a potential game with potential function:*

$$\phi(s) = \sum_{j=1}^v \sum_{i=1}^N s_{ij} p_{ij} + \sum_{j=v+1}^{v+m} \sum_{d=1}^{n_{r_j}} f_j(d) + \sum_{j=v+1}^{v+m} \sum_{i=1}^N s_{ij} h_i(r_j). \quad (3.3)$$

Proof: Based on the definition of potential games shown in Section 2.1.1, for the case $s_i^1 = s_i^2$, the proof is trivial.

For the case $s_i^1 \neq s_i^2$,

i) If player i changes its strategy from r_{j_1} to r_{j_2} , $j_1, j_2 \in I_m$, based on (3.1),

$$U_i(s_i^1, s_{-1}) - U_i(s_i^2, s_{-1}) = f_{j_1}((n_{r_{j_1}})^1) + h_i(r_{j_1}) - f_{j_2}((n_{r_{j_2}})^2) - h_i(r_{j_2}). \quad (3.4)$$

Note that $(n_{r_{j_1}})^1 = (n_{r_{j_1}})^2 + 1$ and $(n_{r_{j_2}})^2 = (n_{r_{j_2}})^1 + 1$. From the potential function (3.3),

$$\begin{aligned}
& \phi(s_i^1, s_{-1}) - \phi(s_i^2, s_{-1}) \\
&= \sum_{d=1}^{(n_{r_{j_1}})^1} f_{j_1}(d) + \sum_{d=1}^{(n_{r_{j_2}})^1} f_{j_2}(d) + s_{ij_1}^1 h_i(r_{j_1}) + s_{ij_2}^1 h_i(r_{j_2}) \\
&\quad - \sum_{d=1}^{(n_{r_{j_1}})^2} f_{j_1}(d) - \sum_{d=1}^{(n_{r_{j_2}})^2} f_{j_2}(d) - s_{ij_1}^2 h_i(r_{j_1}) - s_{ij_2}^2 h_i(r_{j_2}) \\
&= \sum_{d=1}^{(n_{r_{j_1}})^1} f_{j_1}(d) + \sum_{d=1}^{(n_{r_{j_2}})^2-1} f_{j_2}(d) + s_{ij_1}^1 h_i(r_{j_1}) \\
&\quad - \sum_{d=1}^{(n_{r_{j_1}})^1-1} f_{j_1}(d) - \sum_{d=1}^{(n_{r_{j_2}})^2} f_{j_2}(d) - s_{ij_2}^2 h_i(r_{j_2}) \\
&= f_{j_1}((n_{r_{j_1}})^1) + h_i(r_{j_1}) - f_{j_2}((n_{r_{j_2}})^2) - h_i(r_{j_2}) \\
&= U_i(s_i^1, s_{-1}) - U_i(s_i^2, s_{-1}).
\end{aligned}$$

ii) If player i changes its strategy from r_{j_1} to r_{j_2} , $j_1 \in I_v, j_2 \in I_m$, we have $s_{ij_1}^1 = 1$ and $s_{ij_1}^2 = 0$. From (3.1),

$$U_i(s_i^1, s_{-1}) - U_i(s_i^2, s_{-1}) = p_{ij_1} - f_{j_1}((n_{r_{j_2}})^2) - h_i(r_{j_2}). \quad (3.5)$$

Note that $(n_{r_{j_2}})^2 = (n_{r_{j_2}})^1 + 1$.

Thus,

$$\begin{aligned}
& \phi(s_i^1, s_{-1}) - \phi(s_i^2, s_{-1}) \\
&= p_{ij_1} + \sum_{d=1}^{(n_{r_{j_2}})^1} f_{j_2}(d) + s_{ij_2}^1 h_i(r_{j_2}) - \sum_{d=1}^{(n_{r_{j_2}})^2} f_{j_2}(d) - s_{ij_2}^2 h_i(r_{j_2}) \\
&= p_{ij_1} + \sum_{d=1}^{(n_{r_{j_2}})^2-1} f_{j_2}(d) - \sum_{d=1}^{(n_{r_{j_2}})^2} f_{j_2}(d) - h_i(r_{j_2}) \\
&= p_{ij_1} - f_{j_2}((n_{r_{j_2}})^2) - h_i(r_{j_2}) \\
&= U_i(s_i^1, s_{-1}) - U_i(s_i^2, s_{-1}).
\end{aligned}$$

iii) If player i changes its strategy from r_{j_2} to r_{j_1} , $j_1 \in I_v, j_2 \in I_m$, the situation is similar to ii).

From i), ii) and iii), we can conclude that $\phi(s)$ is the potential function of (3.1). ■

3.2.2 A Consensus Protocol

In the case of *complete information*, at stage k , each player i knows other players' strategy $s_{-i}(k)$. Hence, they can compute all $n_{r_j}(k), j \in \mathcal{I}_m$ and choose their strategies by optimizing $U_i(s_i(k), s_{-i}(k))$. However, in most cases, each player does not know other players' strategies $s_{-i}(k)$ before it makes its decision. Therefore, player i may use previous information, e.g., $n_{r_j}(k-1)$, to estimate $n_{r_j}(k)$.

To obtain previous information, we introduce the following protocol.

Assume that, at each stage k , each player i may randomly exchange information only with a subset of its *whole* neighbor set $N_i(k) \subseteq \mathcal{N}_i$. Here, $\mathcal{N}_i \subset \mathcal{N}$ is the fixed *whole* neighbor set of player i . Let $\bar{\mathcal{G}}_N$ denote the set of all possible undirected graphs consisting of the node set $\mathcal{N}$ and an edge set $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$, in which the necessary conditions for non-oriented pair $(i, j) \in \mathcal{E}, i \neq j$ are $i \in \mathcal{N}_j$ and $j \in \mathcal{N}_i$. As stated above, at stage k , the network is chosen from $\bar{\mathcal{G}}_N$ randomly, independent of other stages. We denote the graph at stage k as $\mathcal{G}_N(k) = \{\mathcal{N}, \mathcal{E}(k)\} \in \bar{\mathcal{G}}_N$ and the corresponding neighbor set of player i as $N_i(k) = \{j : (i, j) \in \mathcal{E}(k)\} \cup \{i\}$. Let $L(k)$ be the symmetrical Laplacian matrix associated with $\mathcal{G}_N(k)$ and use $L_{ij}(k)$ to denote the i, j entry of $L(k)$. Note that, at stage k , $\mathcal{G}_N(k)$ may not be connected and some agents may be isolated as they may leave the system randomly. For an isolated agent i at stage k , its neighbor set is $N_i(k) = \{i\}$.

Motivated by the protocol introduced in [77] for binary case, a linear protocol is proposed as:

$$x_i(k+1) = z_i(k) + \alpha(k) \sum_{j \in N_i(k)} L_{ij}(k) z_j(k) \quad (3.6)$$

$$z_i(k) = x_i(k) + s_i(k) - s_i(k-1) \quad \forall k \geq 1 \quad (3.7)$$

$$z_i(0) = x_i(0) = s_i(0) \quad (3.8)$$

where $\alpha(k) \in (-1/\Delta(k), 0)$ and $\Delta(k) = \max_i(L_{ii}(k))$. $x_i(k) \in \mathbb{R}^{v+m}$ and $z_i(k) \in \mathbb{R}^{v+m}$ are pre-decision information and post-decision information, respectively. From the above systems, we can see that $z_i(k)$ updates $x_i(k)$ according to the strategy $s_i(k)$. Later, we will show that $x_i(k)$ can be regarded as an estimate of the percentage of players choosing each resource.

Denote $x(k) = [x_1(k), \dots, x_N(k)]' \in \mathbb{R}^{(v+m)N}$ and $s(k) = [s_1(k), \dots, s_N(k)]' \in \mathbb{R}^{(v+m)N}$, then it follows from (3.6) and (3.7) that, for $k \geq 1$:

$$\begin{aligned} x(k+1) &= [(I_N + \alpha(k)L(k)) \otimes I_{v+m}](x(k) + s(k) - s(k-1)) \\ &= ((I_N + \alpha(k)L(k)) \otimes I_{v+m})s(k) \\ &\quad + \sum_{r=0}^{k-1} \sum_{j=r}^{k-1} [(I_N + \alpha(j+1)L(j+1)) \otimes I_{v+m}] \alpha(r)(L(r) \otimes I_{v+m})s(r) \end{aligned} \quad (3.9)$$

Lemma 3.2. *The protocol described by (3.6)-(3.8) satisfies that at stage k :*

$$\text{avg}(x(k)) = \frac{\sum_{i=1}^N x_i(k)}{N} = \frac{\sum_{i=1}^N s_i(k-1)}{N} \quad (3.10)$$

Proof: Since the undirected graph $\mathcal{G}_N(k)$ is balanced, for all $k \in \mathbb{N}$,

$$\underbrace{[I_{v+m} I_{v+m} \dots I_{v+m}]_N}_{N} (L(k) \otimes I_{v+m}) = 0 \quad (3.11)$$

According to (3.9),

$$\underbrace{[I_{v+m} I_{v+m} \dots I_{v+m}]_N}_{N} x(k+1) = \underbrace{[I_{v+m} I_{v+m} \dots I_{v+m}]_N}_{N} s(k) \quad (3.12)$$

which implies $\text{avg}(x(k)) = \frac{\sum_{i=1}^N x_i(k)}{N} = \frac{\sum_{i=1}^N s_i(k-1)}{N}$. ■

Lemma 3.3. *Consider an undirected graph $\mathcal{G}_N(k) \in \bar{\mathcal{G}}_N$ for $k \in \mathbb{N}$. Assume that no player changes strategy from stage r on. If there exists an infinite sequence of contiguous, nonempty, uniformly bounded time intervals $[k_j, k_{j+1}), j = 1, 2, \dots$ with $k_1 = r + 1$ such that the union of the undirected graphs across each interval is connected, then the protocol described by (3.6)-(3.8) achieves average consensus exponentially fast, which indicates for $\forall \varepsilon \geq 0$, there exists a finite integer $\hat{r} \geq 1$ such that*

$$\begin{aligned} & \|x_i(r + \hat{r}) - \text{avg}(x(r + 1))\| \\ &= \|x_i(r + \hat{r}) - \frac{\sum_{i=1}^N s_i(r)}{N}\| \leq \varepsilon \end{aligned} \quad (3.13)$$

for all players $i = 1, \dots, N$, i.e., each entry of $x_i(r + \hat{r})$ is approximately equal to the corresponding percentage of players for each resource at stage r .

Proof. Since no player changes strategy from stage r on, $s_i(k) - s_i(k - 1) = 0$, for $k > r$. Then (3.9) becomes $x(k + 1) = [(I_N + \alpha(k)L(k)) \otimes I_{v+m}]x(k)$ for $k > r$. Since undirected graphs $\mathcal{G}_N(k) \in \bar{\mathcal{G}}_N$ are jointly connected along each time interval $[k_j, k_{j+1}), j = 1, 2, \dots$, starting at $k_1 = r + 1$, this discrete-time system asymptotically reaches average-consensus $\text{avg}(x(r + 1))$ [68]. By [69] and [137], we conclude that $\text{avg}(x(r + 1))$ can be approached exponentially fast. Therefore, for $\forall \varepsilon \geq 0$, there exists a finite number $\hat{r} \geq 0$ such that $\|x_i(r + \hat{r}) - \text{avg}(x(r + 1))\| = \|x_i(r + \hat{r}) - \frac{\sum_{i=1}^N s_i(r)}{N}\| \leq \varepsilon$. ■

3.2.3 Convergence to Nash equilibrium

Using other players' actions in the previous stage to estimate their actions in the present stage, the best response of player i at stage k is defined as:

$$BR_i(n_r(k - 1)) := \{\bar{s}_i \in \mathbb{Z}^{v+m} : U_i(\bar{s}_i, n_r(k - 1)) = \max_{s_i \in \mathbb{Z}^{v+m}} U_i(s_i, n_r(k - 1))\} \quad (3.14)$$

Obviously, $U_i(s_i, n_r) = U_i(s_i, s_{-i})$ according to the definition of the utility function (3.1).

By Lemma 3.3, for an arbitrarily small $\|\varepsilon_0(\tilde{T})\|$, there always exists a finite number $\tilde{T}$ such that after action $s(r)$ is repeated $\tilde{T}$ stages, each player i can estimate the corresponding percentage of players for each resource r_j with the tolerance $\|\varepsilon_0(\tilde{T})\|$, $\varepsilon_0(\tilde{T}) = [\varepsilon_{01}(\tilde{T}), \varepsilon_{02}(\tilde{T}), \dots, \varepsilon_{0v+m}(\tilde{T})]' \in \mathcal{R}^{v+m}$. Thus, at stage $r + \tilde{T}$, player i estimates the number of players for each resource by previous stage $r + \tilde{T} - 1$ as:

$$\begin{aligned} \bar{n}_r(r + \tilde{T} - 1) &= [\bar{n}_{r_1}(r + \tilde{T} - 1), \dots, \bar{n}_{r_{v+m}}(r + \tilde{T} - 1)]' \\ &= Nx_i(r + \tilde{T}) = \sum_{i=1}^N s_i(r) + \varepsilon_0(\tilde{T}) = n_r(r + \tilde{T} - 1) + \varepsilon_0(\tilde{T}) \end{aligned} \quad (3.15)$$

To ensure the unique best response of each player and convergence of Nash equilibrium, we introduce the following assumption.

Assumption 3.1. *For every player $i \in \mathcal{N}$,*

$$U_i(s_i^1, s_{-i}) \neq U_i(s_i^2, s_{-i}) \quad (3.16)$$

for all possible $s_i^1 \neq s_i^2, s_{-i}$.

Define $\tau = \min_{i, s_i \in \mathcal{R}} |U_i(s_i^1, n_r) - U_i(s_i^2, n_r)|$ for all $s_i^1 \neq s_i^2$. Under Assumption 3.1, we have $\tau > 0$ and the cardinality of $BR_i(n_r(\tilde{T} - 1))$ is 1. If $s_i^a = BR_i(\bar{n}_r(\tilde{T} - 1))$, then, for all $s_i^c \neq s_i^a$, $U_i(s_i^a, \bar{n}_r(\tilde{T} - 1)) - U_i(s_i^c, \bar{n}_r(\tilde{T} - 1)) = U_i(s_i^a, n_r(\tilde{T} - 1) + \varepsilon_0(\tilde{T})) - U_i(s_i^c, n_r(\tilde{T} - 1) + \varepsilon_0(\tilde{T})) = U_i(s_i^a, n_r(\tilde{T} - 1)) - U_i(s_i^c, n_r(\tilde{T} - 1)) + \tau_i^{a,c}(\tilde{T}) > 0$ must be satisfied. In addition, if $|\tau_i^{a,c}(\tilde{T})| < \tau$, we have $s_i^a = BR_i(n_r(\tilde{T} - 1))$ for every player i , i.e., $U_i(s_i^a, n_r(\tilde{T} - 1)) - U_i(s_i^c, n_r(\tilde{T} - 1)) > 0$ for all $s_i^c \neq s_i^a$. Therefore, to ensure that $s_i^a = BR_i(\bar{n}_r(\tilde{T} - 1))$ will result in $s_i^a = BR_i(n_r(\tilde{T} - 1))$ for each player $i \in \mathcal{N}$ and any strategy $s_i^a \in \mathcal{R}$, we introduce the following assumption.

Assumption 3.2. *We choose $\tilde{T}$ large enough such that $|\tau_i^{a,c}(\tilde{T})| < \tau$ for all player $i \in \mathcal{N}$ and all strategies $s_i^a, s_i^c \in \mathcal{R}$.*

Note that $\tilde{T}$ can be computed by the road manager and then be broadcast to all players.

Remark 3.1. *It is easy to verify that, under Assumption 3.2, the $\tilde{T}$ we choose can also guarantee that if $s_i^a = BR_i(n_r(\tilde{T} - 1))$, $s_i^a = BR_i(\bar{n}_r(\tilde{T} - 1))$ for all player $i \in \mathcal{N}$.*

Motivated by [41], we adopt some kind of inertia to players' decision making process.

Assumption 3.3. (Inertia) *If $s_i(k - 1) \in BR_i(\bar{n}_r(k - 1))$, $s_i(k) = s_i(k - 1)$, i.e., player i stays with the previous action. Otherwise, player i will choose an action from $BR_i(\bar{n}_r(k - 1))$ with probability $\theta_i(k)$ ¹. In addition, there exist constants η_1 and η_2 such that, for every stage $k \geq 1$ and for all player $i \in \mathcal{N}$,*

$$0 < \eta_1 \leq \theta_i(k) \leq \eta_2 < 1.$$

The convergence property of the congestion game is shown in the following theorem.

Theorem 3.1. *Assume that undirected graphs $\mathcal{G}_N(k) \in \bar{\mathcal{G}}_N, k \geq 0$ are jointly connected across each time interval $[k_j, k_{j+1}), j = 1, 2, \dots$ with length $\tilde{T}$ starting at $k_1 = 0$. All players with utility function (3.1) adhere to the best response with inertia and do not change their strategies over each time interval $[k_j, k_{j+1})$. Then, under Assumption 3.1-3.3 and by the average consensus protocol defined in (3.6)-(3.8), the action profile of all players will converge to a pure Nash equilibrium almost surely.*

Proof. Let $s(0) = s^0$ be the initial action profile. After s^0 is repeated $\tilde{T}$ times, i.e., $s(0) = s(1) = \dots = s(\tilde{T} - 1) = s^0$, by Lemma 3.3, every player i can estimate the corresponding percentage of players for each resource T_j with tolerance $\|\varepsilon_0(\tilde{T})\|$. Note that, through the above argument, $\tilde{T}$ we choose is large enough such that $\|\varepsilon_0(\tilde{T})\|$ won't affect players' choices under Assumption 3.1-3.2. The probability

¹That is to say, player i will stay with the previous action $s_i(k - 1)$ with probability $1 - \theta_i(k)$, even when there is an opportunity for utility improvement.

of the above event is at least $(1 - \theta_2)^{N(\tilde{T}-1)}$. Then at stage $\tilde{T}$, if $s_i(\tilde{T} - 1) = BR_i(\bar{n}_r(\tilde{T} - 1))$, for all $i \in \mathcal{N}$, s^0 is a pure Nash equilibrium and $s(k) = s^0$, for all $k \geq 0$. The proof is completed. Otherwise, there exists at least one player i_1 such that $U_{i_1}(s_{i_1}^1, \bar{n}_r(\tilde{T} - 1)) > U_{i_1}(s_{i_1}^0, \bar{n}_r(\tilde{T} - 1))$. Assume that only player i_1 switches its action to $s_{i_1}^1$ and all other players stay with their previous actions, i.e., $s(\tilde{T}) = s^1 = (s_{i_1}^1, s_{-i_1}^0)$. After s^1 is repeated a sufficiently large $\tilde{T}$ times, at stage $2\tilde{T}$, if $s_i(2\tilde{T} - 1) = BR_i(\bar{n}_r(2\tilde{T} - 1))$, for all $i \in \mathcal{N}$, then s^1 is a pure Nash equilibrium and $s(k) = s^1$, for all $k \geq \tilde{T}$. According to Lemma 3.1, $\phi(s^0) < \phi(s^1)$. The corresponding probability is at least $\theta_1(1 - \theta_2)^{N\tilde{T}-1}(1 - \theta_2)^{N(\tilde{T}-1)}$ and the proof is accomplished. Otherwise, following the arguments above, we can generate a sequence of action profiles such that

$$\phi(s^0) < \phi(s^1) < \dots < \phi(s^M).$$

Since the action space $\mathcal{R}$ and the number of players N are finite, we have $M < (v+m)^N$ such that s^M is a pure Nash equilibrium and the probability of such a event is at least $[\theta_1(1 - \theta_2)^{N\tilde{T}-1}]^M(1 - \theta_2)^{N(\tilde{T}-1)} > 0$, which implies that $s(k)$ generated by utility function (3.1) with inertia using the estimate induced by the consensus protocol (3.6)-(3.8) will converge to a pure Nash equilibrium almost surely. ■

3.3 Application to Trip Timing Problem

3.3.1 Model Setup

Traffic congestion causes a lot of serious problems such as air pollution, energy wastage, etc. Take Clementi Road in Singapore for example, Fig. 3.1 shows the traffic flow during 14:00 to 23:30 on 2nd August 2010. There is a manifest peak hour around 18:00. To save energy and reduce air pollution, we prefer more people to take public transportation. To alleviate traffic congestion, we aim at designing a dynamic pricing to diversify private car users' choices of departure time. Therefore, we formulate the following trip timing problem to solve these issues.

Figure 3.1: The relationship between time and number of vehicles on Clementi Road in Singapore, 2nd August 2010.

Let $\mathcal{R}_p$ represent different choices of public transportation (e.g., subway, public bus) and $\mathcal{R}_t$ represent trip timing choices (e.g., departure time) along one road with private cars. Player i must decide whether to drive its private car or to receive a fixed utility p_{ij} by choosing one kind of public transportation $r_j \in \mathcal{R}_p$. The utility received by each player is defined as (3.1).

In the following, we will design two dynamic pricing schemes to achieve social optimum and smooth the peak travel, respectively.

3.3.2 Dynamic Pricing for Case with Public Transportation

Assume that the road pricing is applied to achieve some kind of social optimum and it is charged when players choose to drive their private cars, i.e., when choosing an action $r_j, j \in \mathcal{I}_m$.

Consider the case where the road pricing is a function of the number of private cars on the road, we design the road pricing on each resource $r_j, j \in \mathcal{I}_m$ as follows:

$$g_j(n_{r_j}) = (n_{r_j} - 1)(f_j(n_{r_j} - 1) - f_j(n_{r_j})). \quad (3.17)$$

Then, the utility function (3.1) is modified as:

$$U_i(s_i(k), s_{-i}(k)) = \sum_{j=1}^v s_{ij}(k)p_{ij} + \sum_{j=v+1}^{v+m} s_{ij}(k)(f_j(n_{r_j}(k)) + h_i(T_j) - g_j(n_{r_j}(k))) \quad (3.18)$$

Lemma 3.4. *The congestion game with utility function (3.18) is a potential game with potential function*

$$\bar{\phi}(s) = \sum_{i=1}^N \left(\sum_{j=1}^v s_{ij}p_{ij} + \sum_{j=v+1}^{v+m} s_{ij}(f_j(n_{r_j}) + h_i(r_j)) \right) \quad (3.19)$$

Proof. We also consider the case $s_i^1 \neq s_i^2$.

i) If player i changes its strategy from r_{j_1} to r_{j_2} , $j_1, j_2 \in I_m$,

$$\begin{aligned} & U_i(s_i^1, s_{-i}) - U_i(s_i^2, s_{-i}) \\ &= f_{j_1}((n_{r_{j_1}})^1 + 1) + h_i(r_{j_1}) - ((n_{r_{j_1}})^1 - 1)(f_{j_1}((n_{r_{j_1}})^1 - 1) - f_{j_1}((n_{r_{j_1}})^1)) \\ & \quad - f_{j_2}((n_{r_{j_2}})^2) - h_i(r_{j_2}) + ((n_{r_{j_2}})^2 - 1)(f_{j_2}((n_{r_{j_2}})^2 - 1) - f_{j_2}((n_{r_{j_2}})^2)) \end{aligned}$$

Define $A^1 = \{i' : (s_{i'j_1})^1 = 1\} \setminus \{i\}$, $B^1 = \{i'' : (s_{i''j_2})^1 = 1\} \setminus \{i\}$, $A^2 = \{i' : (s_{i'j_1})^2 = 1\} \setminus \{i\}$, $B^2 = \{i'' : (s_{i''j_2})^2 = 1\} \setminus \{i\}$ and note that $(n_{r_{j_1}})^1 = (n_{r_{j_1}})^2 + 1$, $(n_{r_{j_2}})^2 = (n_{r_{j_2}})^1 + 1$. Obviously, $|A^1| = (n_{r_{j_1}})^1 - 1$, $|B^1| = (n_{r_{j_2}})^1$, $|A^2| = (n_{r_{j_1}})^2$, $|B^2| = (n_{r_{j_2}})^2 - 1$.

Thus,

$$\begin{aligned}
& \bar{\phi}(s_i^1, s_{-1}) - \bar{\phi}(s_i^2, s_{-1}) \\
&= f_{j_1}((n_{r_{j_1}})^1) + h_i(r_{j_1}) + \sum_{i' \in A^1} (f_{j_1}((n_{r_{j_1}})^1) + h_{i'}(r_{j_1})) + \sum_{i'' \in B^1} (f_{j_2}((n_{r_{j_2}})^1) + h_{i''}(r_{j_2})) \\
&\quad - f_{j_2}((n_{r_{j_2}})^2) - h_i(r_{j_2}) - \sum_{i' \in A^2} (f_{j_1}((n_{r_{j_1}})^2) + h_{i'}(r_{j_1})) \\
&\quad - \sum_{i'' \in B^2} (f_{j_2}((n_{r_{j_2}})^2) + h_{i''}(r_{j_2})) \\
&= f_{j_1}((n_{r_{j_1}})^1) + h_i(r_{j_1}) - (n_{r_{j_1}})^2 (f_{j_1}((n_{r_{j_1}})^2) - f_{j_1}((n_{r_{j_1}})^1)) \\
&\quad - f_{j_2}((n_{r_{j_2}})^2) - h_i(r_{j_2}) + (n_{r_{j_2}})^1 (f_{j_2}((n_{r_{j_2}})^1) - f_{j_2}((n_{r_{j_2}})^2)) \\
&= U_i(s_i^1, s_{-1}) - U_i(s_i^2, s_{-1})
\end{aligned}$$

ii) If player i changes its strategy from r_{j_2} to r_{j_1} , $j_1 \in I_m, j_2 \in I_v$,

$$\begin{aligned}
& U_i(s_i^1, s_{-1}) - U_i(s_i^2, s_{-1}) \\
&= p_{ij_2} - f_{j_1}((n_{r_{j_1}})^2) - h_i(r_{j_1}) + ((n_{r_{j_1}})^2 - 1)(f_{j_1}((n_{r_{j_1}})^2 - 1) - f_{j_1}((n_{r_{j_1}})^2))
\end{aligned}$$

Note that $(n_{r_{j_1}})^2 = (n_{r_{j_1}})^1 + 1$.

Therefore,

$$\begin{aligned}
& \bar{\phi}(s_i^1, s_{-1}) - \bar{\phi}(s_i^2, s_{-1}) \\
&= p_{ij_2} + \sum_{i' \in A^1} (f_{j_1}((n_{r_{j_1}})^1) + h_{i'}(r_{j_1})) - f_{j_1}((n_{r_{j_1}})^2) \\
&\quad - h_i(r_{j_1}) - \sum_{i' \in A^2} (f_{j_1}((n_{r_{j_1}})^2) + h_{i'}(r_{j_1})) \\
&= p_{ij_2} - f_{j_1}((n_{r_{j_1}})^2) - h_i(r_{j_1}) + ((n_{r_{j_1}})^2 - 1)(f_{j_1}((n_{r_{j_1}})^2 - 1) - f_{j_1}((n_{r_{j_1}})^2)) \\
&= U_i(s_i^1, s_{-1}) - U_i(s_i^2, s_{-1})
\end{aligned}$$

iii) If player i changes its strategy from r_{j_1} to r_{j_2} , $j_1 \in I_m, j_2 \in I_v$, the proof is similar to ii).

According to the discussion above, $\bar{\phi}(s)$ is the potential function of (3.18). ■

The construction of $\bar{\phi}(s)$ can be viewed as the total utility of all players except for

the price. By Theorem 3.1, we can conclude that, under the conditions of Theorem 3.1, the action profile generated by (3.18) will converge to a local maximum point of $\bar{\phi}(s)$ almost surely. In Example 3.1 on the trip timing problem, we can see that, starting from the same initial conditions, the overall utility can always be improved within the pricing strategy (3.17) compared to the case without pricing.

Remark 3.2. Since $f_j(n_{r_j}), j \in \mathcal{I}_m$ is the only part associated with flow n_{r_j} in player i 's utility function U_i and player i wants to maximize it, $-f_j(n_{r_j})$ can be regarded as the private cost for player i on resource r_j . When player i chooses a departure time r_j , he/she may not perceive the cost he/she contributes to the total cost $-n_{r_j}f_j(n_{r_j})$ on resource r_j . Therefore, we should make players aware of the social cost instead of private cost in order to achieve social optimum. The social cost in time interval r_j with flow n_{r_j} is defined as the increase in total cost $-n_{r_j}f_j(n_{r_j})$ caused by a marginal player, i.e.,

$$\begin{aligned} & \frac{-n_{r_j}f_j(n_{r_j}) - (-(n_{r_j} - 1)f_j(n_{r_j} - 1))}{n_{r_j} - (n_{r_j} - 1)} \\ &= (n_{r_j} - 1)(f_j(n_{r_j} - 1) - f_j(n_{r_j})) - f_j(n_{r_j}) \end{aligned} \quad (3.20)$$

From (3.20), we can see that the difference between private and social cost is $(n_{r_j} - 1)(f_j(n_{r_j} - 1) - f_j(n_{r_j}))$, which is exactly the same as $g_j(n_{r_j})$. From page 50 of [6], we can see that $g_j(n_{r_j})$ is a discrete-time extension of the continuous-time marginal pricing.

Example 3.1. Now we apply the congestion game described above to the trip timing problem along a road.

For each player i , through the travel time formulas introduced by [6], we set the travel time function $d(n_{r_j})$ as

$$d(n_{r_j}) = t_0 \cdot (1 + 0.15(n_{r_j}/c_0)^{m_\alpha}). \quad (3.21)$$

and

$$f_j(n_{r_j}) = \frac{R}{d(n_{r_j})}, \quad (3.22)$$

Figure 3.2: The map of Clementi Road, Singapore.

$$h_i(r_j) = c_i |r_j - \hat{t}_i|, \quad (3.23)$$

for all $j \in \mathcal{I}_m$, where R is the length of the road, m_α is a positive parameter, t_0 is the free-flow travel time which is a measure of the travel time at zero flow, c_0 is the practical capacity which is a measure of the flow from which the travel time will increase very rapidly if the flow is further increased, c_i is a negative constant, and $\hat{t}_i$ is the preferred departure time of player i . Here, (3.22) may be considered as the average travel speed and (3.23) is the difference between each player's actual departure time and preferred departure time.

For real data analysis, we take Clementi Road (length 5.4km) as shown in Fig. 3.2 in Singapore for example. Fig. 3.3 shows the relationship between traffic flow and travel time on Clementi Road based on the whole month data of August 2010. We fit the nonlinear function $d(n_{r_j})$ to real time data in Fig. 3.3 and get $t_0 = 0.0998$, $c_0 = 1357$, $m_\alpha = 3.977$.

Suppose there are 2000 players using Clementi Road everyday during 17:00-19:00. $\hat{t}_i, i \in \mathcal{N}$ is generated according to a triangular distribution randomly with the peak hour at 18:00. c_i is randomly generated according to a uniform distribution with the interval $[-1.5, -0.5]$.

Assume that there is only one kind of public transportation r_1 , i.e., $v = 1$ and, for each player i , the fixed utility associated to it is p_{i1} , which is randomly generated

Figure 3.3: The relationship between traffic flow and travel time.

according to a uniform distribution with the interval $[30, 53]$. Assume that there are 4 choices, i.e., $m = 3$.

As shown in Fig. 3.4 and Fig. 3.5, the action profiles generated by best response with inertia is convergent to a pure strategy Nash equilibrium. Starting from the same initial conditions, we can see that the number of players choosing public transportation increases and the number of players on peak hour decreases after adding price function (3.17). And we can further check that the overall utility without price is $\bar{\phi}(s) = 95219$ and with price is $\bar{\phi}(s) = 96781$. As expected, the price function (3.17) indeed increases the overall utility.

3.3.3 Dynamic Pricing for Case without Public Transportation

From the simulation results in Example 3.1, we can see that the number of private cars on each resource $r_j, j \in \mathcal{I}_m$ won't spread out and there is still a large number of players in peak hours even if we add the price function (3.17). In order to diversify

Figure 3.4: Evolution of number of vehicles on each choice without price.

Figure 3.5: Evolution of number of vehicles on each choice with price function (3.17).

players' choices while driving private cars, we add a weighted entropy function to the potential function and design a price function according to it.

Note that $v = 0$, we revise the utility function (3.1) as follows:

$$U_i(s_i(k), s_{-i}(k)) = \sum_{j=v+1}^{v+m} s_{ij}(k)(f_j(n_{r_j}(k)) + h_i(r_j) - \tilde{g}_j(n_{r_j}(k))) \quad (3.24)$$

For each player i , the price function $\tilde{g}_j(n_{r_j}(k))$ on each resource $r_j, j \in \mathcal{I}_m$ is designed as:

$$\tilde{g}_j(n_{r_j}) = (n_{r_j} - 1)(f_j(n_{r_j} - 1) - f_j(n_{r_j}) + \frac{w}{N} \log(\frac{n_{r_j}}{n_{r_j} - 1})) + \frac{w}{N} \log(n_{r_j}) \quad (3.25)$$

where $w > 0$ is the weight.

Lemma 3.5. *The congestion game with utility function (3.24) is a potential game with potential function*

$$\tilde{\phi}(s) = \sum_{i=1}^N (\sum_{j=v+1}^{v+m} s_{ij}(f_j(n_{r_j}) + h_i(r_j))) - w \sum_{j=v+1}^{v+m} q_j \log q_j \quad (3.26)$$

where q_j is the percentage of players choosing resource $T_j, j \in \mathcal{I}_m$, i.e., $q_j = n_{r_j}/N$.

Proof. Consider the case $s_i^1 \neq s_i^2$.

If player i changes its strategy from r_{j_1} to r_{j_2} ,

$$\begin{aligned} & U_i(s_i^1, s_{-1}) - U_i(s_i^2, s_{-1}) \\ &= f_{j_1}((n_{r_{j_1}})^1) + h_i(r_{j_1}) - ((n_{r_{j_1}})^1 - 1)(f_{j_1}((n_{r_{j_1}})^1 - 1) \\ & \quad - f_{j_1}((n_{r_{j_1}})^1) + \frac{w}{N} \log(\frac{(n_{r_{j_1}})^1}{(n_{r_{j_1}})^1 - 1})) - \frac{w}{N} \log((n_{r_{j_1}})^1) \\ & \quad - f_{j_2}((n_{r_{j_2}})^2) - h_i(r_{j_2}) + ((n_{r_{j_2}})^2 - 1)(f_{j_2}((n_{r_{j_2}})^2 - 1) \\ & \quad - f_{j_2}((n_{r_{j_2}})^2) + \frac{w}{N} \log(\frac{(n_{r_{j_2}})^2}{(n_{r_{j_2}})^2 - 1})) + \frac{w}{N} \log((n_{r_{j_2}})^2) \end{aligned}$$

Define $A^1 = \{i' : (s_{i'j_1})^1 = 1\} \setminus \{i\}$, $B^1 = \{i'' : (s_{i''j_2})^1 = 1\} \setminus \{i\}$, $A^2 = \{i' : (s_{i'j_1})^2 = 1\} \setminus \{i\}$, $B^2 = \{i'' : (s_{i''j_2})^2 = 1\} \setminus \{i\}$ and note that $(n_{r_{j_1}})^1 = (n_{r_{j_1}})^2 + 1$, $(n_{r_{j_2}})^2 =$

$(n_{r_{j_2}})^1 + 1$. Obviously, $|A^1| = (n_{r_{j_1}})^1 - 1, |B^1| = (n_{r_{j_2}})^1, |A^2| = (n_{r_{j_1}})^2, |B^2| = (n_{r_{j_2}})^2 - 1$.

Thus, from the potential function, we have

$$\begin{aligned}
& \tilde{\phi}(s_i^1, s_{-1}) - \tilde{\phi}(s_i^2, s_{-1}) \\
&= f_{j_1}((n_{r_{j_1}})^1) + h_i(r_{j_1}) + \sum_{i' \in A^1} (f_{j_1}((n_{r_{j_1}})^1) + h_{i'}(r_{j_1})) \\
&+ \sum_{i'' \in B^1} (f_{j_2}((n_{r_{j_2}})^1) + h_{i''}(r_{j_2})) - wq_{j_1}^1 \log(q_{j_1}^1) - wq_{j_2}^1 \log(q_{j_2}^1) \\
&- f_{j_2}((n_{r_{j_2}})^2) - h_i(r_{j_2}) - \sum_{i' \in A^2} (f_{j_1}((n_{r_{j_1}})^2) + h_{i'}(r_{j_1})) \\
&- \sum_{i'' \in B^2} (f_{j_2}((n_{r_{j_2}})^2) + h_{i''}(r_{j_2})) + wq_{j_1}^2 \log(q_{j_1}^2) + wq_{j_2}^2 \log(q_{j_2}^2) \\
&= f_{j_1}((n_{r_{j_1}})^1) + h_i(r_{j_1}) - ((n_{r_{j_1}})^1 - 1)(f_{j_1}((n_{r_{j_1}})^2) - f_{j_1}((n_{r_{j_1}})^1)) \\
&- wq_{j_1}^1 \log(q_{j_1}^1) - wq_{j_2}^1 \log(q_{j_2}^1) \\
&- f_{j_2}((n_{r_{j_2}})^2) - h_i(r_{j_2}) + ((n_{r_{j_2}})^2 - 1)(f_{j_2}((n_{r_{j_2}})^1) - f_{j_2}((n_{r_{j_2}})^2)) \\
&+ wq_{j_1}^2 \log(q_{j_1}^2) + wq_{j_2}^2 \log(q_{j_2}^2) \\
&= f_{j_1}((n_{r_{j_1}})^1) + h_i(r_{j_1}) - ((n_{r_{j_1}})^1 - 1)(f_{j_1}((n_{r_{j_1}})^1 - 1) - f_{j_1}((n_{r_{j_1}})^1)) \\
&- w \frac{(n_{r_{j_1}})^1}{N} \log\left(\frac{(n_{r_{j_1}})^1}{N}\right) - w \frac{(n_{r_{j_2}})^1}{N} \log\left(\frac{(n_{r_{j_2}})^1}{N}\right) \\
&- f_{j_2}((n_{r_{j_2}})^2) - h_i(r_{j_2}) + ((n_{r_{j_2}})^2 - 1)(f_{j_2}((n_{r_{j_2}})^2 - 1) - f_{j_2}((n_{r_{j_2}})^2)) \\
&+ w \frac{(n_{r_{j_1}})^2}{N} \log\left(\frac{(n_{r_{j_1}})^2}{N}\right) + w \frac{(n_{r_{j_2}})^2}{N} \log\left(\frac{(n_{r_{j_2}})^2}{N}\right) \\
&= U_i(s_i^1, s_{-1}) - U_i(s_i^2, s_{-1})
\end{aligned}$$

■

The form of $\tilde{\phi}(s)$ consists of two parts. The first part is the overall utility of all players except for the price and the second part is a weighted entropy term $-w \sum_{j=v+1}^{v+m} q_j \log q_j$, which means that the number of players on each resource needs to be equal to reach optimum. It is also easy to check that, under the condition of inertia and by Theorem 3.1, the action profiles generated by utility function (3.24) converge to a pure Nash equilibrium almost surely.

Figure 3.6: Evolution of number of private vehicles on each choice without price.

Example 3.2. Set $f_j(n_{r_j})$ and $h_i(r_j)$ as (3.22) and (3.23), respectively. Use the parameters in Example 3.1 and assume that there are 4 choices, i.e., $m = 4$. Note that $v = 0$. Let $w = 200000$. Comparing utility function (3.1) and (3.18), from Fig. 3.6 and Fig. 3.7, we can see that, starting from the same initial conditions, the dynamic price function (3.17) only releases the traffic congestion slightly. While with price function (3.25), as shown in Fig. 3.8, the disparity in the number of players on each resource reduces a lot.

Remark 3.3. Compare potential function (3.3) and (3.19), we can see the only difference is the term associated with $f_j(n_{r_j}(k))$, which is only related with the number of players on each resource. However, the terms p_{ij} and $h_i(r_j)$ associated with players' personality also play a crucial role in players' decision making progress. Therefore, after adding price function (3.17), the equilibrium point can't be affected significantly. As potential function (3.26) shown, after introducing price function (3.25), a weighted entropy term $-w \sum_{j=v+1}^{v+m} q_j \log q_j$ appears. And if w becomes large enough, the weighted entropy term dominates players' choices. Besides, the entropy function is strictly concave and only has one maximal point, i.e., $q_{v+1} = \dots = q_{v+m}$. Therefore, the disparity in the number of players on each resource will vanish as w

Figure 3.7: Evolution of number of private vehicles on each choice with price function (3.17).

Figure 3.8: Evolution of number of private vehicles on each choice with price function (3.25).

goes to infinity.

3.4 Application to Routing and Trip Timing Problem

In Section 3.3, we discuss the effect of the road pricing on road users' trip timing choices. In this section, we will study a more practical model considering both trip timing and routing problem.

3.4.1 Model Setup

Consider a traffic network $(\mathcal{V}, \mathcal{E})$ with multiple origin-destination pairs, where $\mathcal{V}$ is the vertex set and $\mathcal{E}$ is the edge set (e.g. Fig. 3.9). Define the routes set for all players in this network as $\mathcal{R}_r = \{\tau_1, \dots, \tau_n\}$, where each route $\tau \in \mathcal{R}_r$ consists of one or several edges and an edge $e \in \mathcal{E}$ can be contained by more than one routes.

Figure 3.9: A network with multiple origin-destination pairs

Assume that all players make their decisions by taking both routing choices and departure time into account. Let $\mathcal{R}_t = \{r_1, \dots, r_m\}$ represent the trip timing choices. At each stage, each player i chooses its departure time r and route τ simultaneously from $\mathcal{R}_t$ and $\mathcal{R}_r$, respectively. All possible combination of trip timing and routing choices for all players is $\{\varrho = (r, \tau) : r \in \mathcal{R}_t, \tau \in \mathcal{R}_r\}$ and its cardinality is $|\varrho| = \{(r, \tau) : r \in \mathcal{R}_t, \tau \in \mathcal{R}_r\} = mn$. Denote the strategy of player i at stage k as $s_i(k) =$

$[s_{i1}(k), \dots, s_{imn}(k)]' \in \mathbb{Z}^{mn}$, where s_{ij} equal to 1 if player i chooses $\varrho_j, j = 1, \dots, mn$ and equal to 0 otherwise, and $\sum_{j=1}^{mn} s_{ij} = 1$. The latency $l_e(f_{e,r})$ on each edge $e \in \mathcal{E}$ is associated with the number of players choosing this edge at the same departure time r , where $f_{e,r}$ is the number of players departing at time r and routing on link e . Denote $f_{\tau,r}$ as the number of players departing at time r and choosing route τ . Obviously, $f_{e,r} = \sum_{\tau \in \mathcal{R}_r: e \in \tau} f_{\tau,r}$. The latency received by player i by choosing route $\tau \in \mathcal{R}_r$ at departure time $r \in \mathcal{R}_t$ is $\sum_{e \in \tau} l_e(f_{e,r})$. Combine both trip timing and routing, the cost that each player i aims to minimize is:

$$C_{i,d}(s(k)) = \sum_{j=1}^{mn} s_{ij}(k) \left(\sum_{e \in \tau: \tau \in \varrho_j} l_e(f_{e,r:r \in \varrho_j}) + \alpha_i |r - \hat{t}_i : r \in \varrho_j| \right) \quad (3.27)$$

where α_i is a nonnegative constant.

Note that, in this section, each player i tries to minimize its cost. Thus, the pure strategy Nash equilibrium of the cost-minimization game is defined as: for each player $i \in \mathcal{N}$:

$$C_{i,d}(s_i^{ne}, s_{-i}^{ne}) = \min_{s_i \in \mathbb{Z}^{mn}} C_{i,d}(s_i, s_{-i}^{ne}). \quad (3.28)$$

Using the consensus protocol introduced in Section 3.2.2, the number of players choosing each strategy $\varrho_j, j = 1, \dots, mn$ can be estimated.

Note that the cost of each player i (3.27) is equivalent to the following equation: if player i chooses departure time $t_i \in \mathcal{R}_t$ and route $\varsigma_i \in \mathcal{R}_r$, the cost he/she will received is:

$$C_{i,d}(\nu) = \sum_{e \in \varsigma_i} l_e(f_{e,t_i}) + \alpha_i |t_i - \hat{t}_i| \quad (3.29)$$

where $\nu = \{\nu_1, \dots, \nu_N\}$ and $\nu_i = (t_i, \varsigma_i), i \in \mathcal{N}$.

Now, $f_{e,r}$ and $f_{\tau,r}$ can be presented as $f_{e,r} = \sum_{i=1}^N \mathbf{I}\{e \in \varsigma_i \text{ and } t_i = r\}$ and $f_{\tau,r} = \sum_{i=1}^N \mathbf{I}\{\varsigma_i = \tau \text{ and } t_i = r\}$, respectively.

The following lemma shows that the model we formulated above is a potential game, which guarantees the existence of pure strategy Nash equilibrium.

Lemma 3.6. *The congestion game with cost function (3.29) is a potential game with potential function*

$$\hat{\phi}(\nu) = \sum_{j=1}^m \sum_{e \in \mathcal{E}} \sum_{k=1}^{f_{e,r_j}} l_e(k) + \sum_{i=1}^N \alpha_i |t_i - \hat{t}_i| \quad (3.30)$$

Proof. For the case $\nu_i^1 = \nu_i^2$, (3.30) obviously holds.

For the case $\nu_i^1 \neq \nu_i^2$, without loss of generality, we assume that $\varsigma_i^1 = \{e_1, e_2\}$, $\varsigma_i^2 = \{e_2, e_3\}$, $t_i^1 = r_1$, $t_i^2 = r_2$. Note that $f_{e_1,r_1}^1 = f_{e_1,r_1}^2 + 1$, $f_{e_2,r_1}^1 = f_{e_2,r_1}^2 + 1$, $f_{e_2,r_2}^1 = f_{e_2,r_2}^2 - 1$, $f_{e_3,r_2}^1 = f_{e_3,r_2}^2 - 1$, $f_{e_1,r_2}^1 = f_{e_1,r_2}^2$, $f_{e_3,r_1}^1 = f_{e_3,r_1}^2$, we have

$$\begin{aligned} & \hat{\phi}(\nu^1) - \hat{\phi}(\nu^2) \\ &= \sum_{k=1}^{f_{e_1,r_1}^1} l_{e_1}(k) + \sum_{k=1}^{f_{e_2,r_1}^1} l_{e_2}(k) + \sum_{k=1}^{f_{e_2,r_2}^1} l_{e_2}(k) + \sum_{k=1}^{f_{e_3,r_2}^1} l_{e_3}(k) + \alpha_i |r_1 - \hat{t}_i| \\ & \quad - \sum_{k=1}^{f_{e_1,r_1}^2} l_{e_1}(k) - \sum_{k=1}^{f_{e_2,r_1}^2} l_{e_2}(k) - \sum_{k=1}^{f_{e_2,r_2}^2} l_{e_2}(k) - \sum_{k=1}^{f_{e_3,r_2}^2} l_{e_3}(k) - \alpha_i |r_2 - \hat{t}_i| \\ &= l_{e_1}(f_{e_1,r_1}^1) + l_{e_2}(f_{e_2,r_1}^1) + \alpha_i |r_1 - \hat{t}_i| - l_{e_2}(f_{e_2,r_2}^2) - l_{e_3}(f_{e_3,r_2}^2) - \alpha_i |r_2 - \hat{t}_i| \\ &= C_{i,d}(\nu_i^1, \nu_{-i}) - C_{i,d}(\nu_i^2, \nu_{-i}) \end{aligned}$$

The proof is completed. ■

Based on Theorem 3.1, we can conclude that, within the cost function (3.29), the action profiles generated by best response with inertia is convergent to a Nash equilibrium almost surely.

3.4.2 Dynamic Pricing and Numerical Example

Suppose each edge $e \in \mathcal{E}$ is charged a toll to reach some kind of social optimum. Motivated by the road pricing (3.17), the toll on each edge e is designed as:

$$q_e(f_{e,r}) = (f_{e,r} - 1)(l_e(f_{e,r}) - l_e(f_{e,r} - 1)) \quad (3.31)$$

Then, the cost function (3.29) is modified as

$$C_{i,d}(\nu) = \sum_{e \in \varsigma_i} (l_e(f_{e,t_i}) + q_e(f_{e,t_i})) + \alpha_i |t_i - \hat{t}_i| \quad (3.32)$$

Lemma 3.7. *The congestion game with cost function (3.32) is a potential game with potential function*

$$\check{\phi}(\nu) = \sum_{j=1}^m \sum_{e \in \mathcal{E}} l_e(f_{e,r_j}) f_{e,r_j} + \sum_{i=1}^N \alpha_i |t_i - \hat{t}_i| \quad (3.33)$$

Proof: For the case $\nu_i^1 = \nu_i^2$, (3.33) obviously holds.

For the case $\nu_i^1 \neq \nu_i^2$, without loss of generality, we assume that $\varsigma_i^1 = \{e_1, e_2\}$, $\varsigma_i^2 = \{e_2, e_3\}$, $t_i^1 = r_1$, $t_i^2 = r_2$. Note that $f_{e_1,r_1}^1 = f_{e_1,r_1}^2 + 1$, $f_{e_2,r_1}^1 = f_{e_2,r_1}^2 + 1$, $f_{e_2,r_2}^1 = f_{e_2,r_2}^2 - 1$, $f_{e_3,r_2}^1 = f_{e_3,r_2}^2 - 1$, $f_{e_1,r_2}^1 = f_{e_1,r_2}^2$, $f_{e_3,r_1}^1 = f_{e_3,r_1}^2$, we have

$$\begin{aligned} & \check{\phi}(\nu^1) - \check{\phi}(\nu^2) \\ &= f_{e_1,r_1}^1 l_{e_1}(f_{e_1,r_1}^1) + f_{e_2,r_1}^1 l_{e_2}(f_{e_2,r_1}^1) + f_{e_2,r_2}^1 l_{e_2}(f_{e_2,r_2}^1) + f_{e_3,r_2}^1 l_{e_3}(f_{e_3,r_2}^1) + \alpha_i |r_1 - \hat{t}_i| \\ & \quad - f_{e_1,r_1}^2 l_{e_1}(f_{e_1,r_1}^2) - f_{e_2,r_1}^2 l_{e_2}(f_{e_2,r_1}^2) - f_{e_2,r_2}^2 l_{e_2}(f_{e_2,r_2}^2) - f_{e_3,r_2}^2 l_{e_3}(f_{e_3,r_2}^2) - \alpha_i |r_2 - \hat{t}_i| \\ &= l_{e_1}(f_{e_1,r_1}^1) + (f_{e_1,r_1}^1 - 1)(l_{e_1}(f_{e_1,r_1}^1) - l_{e_1}(f_{e_1,r_1}^1 - 1)) \\ & \quad + l_{e_2}(f_{e_2,r_1}^1) + (f_{e_2,r_1}^1 - 1)(l_{e_2}(f_{e_2,r_1}^1) - l_{e_2}(f_{e_2,r_1}^1 - 1)) + \alpha_i |r_1 - \hat{t}_i| \\ & \quad - l_{e_2}(f_{e_2,r_2}^2) - (f_{e_2,r_2}^2 - 1)(l_{e_2}(f_{e_2,r_2}^2) - l_{e_2}(f_{e_2,r_2}^2 - 1)) \\ & \quad - l_{e_3}(f_{e_3,r_2}^2) - (f_{e_3,r_2}^2 - 1)(l_{e_3}(f_{e_3,r_2}^2) - l_{e_3}(f_{e_3,r_2}^2 - 1)) - \alpha_i |r_2 - \hat{t}_i| \\ &= C_{i,d}(\nu_i^1, \nu_{-i}) - C_{i,d}(\nu_i^2, \nu_{-i}) \end{aligned}$$

The proof is completed. ■

Actually, $\check{\phi}$ can also be written as $\check{\phi}(\nu) = \sum_{i=1}^N (\sum_{e \in \varsigma_i} l_e(f_{e,t_i}) + \alpha_i |t_i - \hat{t}_i|)$, which is the total cost of all players in the absence of the road pricing. Thus, $\check{\phi}$ can be viewed as the social cost that all players tend to jointly minimize. According to Theorem 3.1, the action profiles generated by best response with inertia will converge to a local minimum point of $\check{\phi}$ almost surely.

Remark 3.4. *For the case where the number of players routing on each origin-destination pair is fixed, the results still hold with some minor corrections.*

Figure 3.10: A simple network $(\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5\}$. In this network, $\mathcal{R}_r = \{\tau_1, \tau_2, \tau_3\}$ is described by $\tau_1 = \{e_1, e_4\}$, $\tau_2 = \{e_2, e_3, e_4\}$, $\tau_3 = \{e_2, e_5\}$.

Example 3.3. *Consider the traffic network as shown in Fig. 3.10, which has 3 routes τ_1, τ_2, τ_3 . Assume that there are 2000 players routing in this network and they have 3 trip timing choices r_1, r_2, r_3 . Then, all possible choices for each player are $(r_1, \tau_1), (r_2, \tau_1), (r_3, \tau_1), (r_1, \tau_2), (r_2, \tau_2), (r_3, \tau_2), (r_1, \tau_3), (r_2, \tau_3), (r_3, \tau_3)$. The parameters $\alpha_i, \hat{t}_i, i \in \mathcal{N}$ are randomly generated, respectively. Set the latency $l_e(f_{e,r})$ on each edge $e \in \mathcal{E}$ as the form in (3.21). From Fig. 3.11 and Fig. 3.12, we can see that the road pricing (3.31) releases the traffic congestion. The total cost of all players without price is $\check{\phi}(\nu) = 26283$ and with price is $\check{\phi}(\nu) = 26182$, which indicates that the price function (3.31) reduces the overall cost.*

3.5 Conclusions

In this chapter, we have considered repeated noncooperative congestion games with multiple choices. A consensus protocol was formulated under random networks to estimate the percentage of players choosing each resource. It was shown that players' strategies generated by the criterion of best response with inertia converge to a pure strategy Nash equilibrium almost surely. We applied our results to trip timing and routing problems and the influence of dynamic road pricing on the Nash equilibrium

Figure 3.11: Evolution of number of vehicles in each combination of trip timing and routing choice without price.

Figure 3.12: Evolution of number of vehicles in each combination of trip timing and routing choice with price function (3.31).

was discussed. From the simulation results, we can see that the road pricing schemes indeed helped achieve social optimum or spread out players' choices.

Chapter 4

Analysis of Efficiency Loss of Nash Equilibrium in Nonatomic Games

In this chapter, we investigate how the scaled marginal-cost road pricing improves the price of anarchy (POA) in nonatomic congestion games. All players in the noncooperative congestion game are divided into groups based on their price sensitivities. The POA problem in a network with scaled marginal-cost pricing on each link is formulated in Section 4.1. How to design the optimal dynamic pricing in networks with heterogeneous players is discussed in Section 4.2. In Section 4.3, we present numerical examples based on the real traffic data in Singapore. Conclusions are stated in section 4.4.

4.1 Problem Formulation

Consider a network $(\mathcal{V}, \mathcal{E})$ with one origin-destination pair (e.g. Fig. 4.1). Each player chooses its route from a common set of routes $\mathcal{R} = \{r_1, \dots, r_v\}$, where each route $r \in \mathcal{R}$ consists of one or several links. Define the vector of route flows as $\hat{f} = (f_{r_1}, \dots, f_{r_v})$ and the total flow of all routes as $F = \sum_{r \in \mathcal{R}} f_r$.

Note that a link $e \in \mathcal{E}$, e.g., e_1 in Fig. 4.1, can be contained by more than one route. The latency of one link $e \in \mathcal{E}$ is associated with the total flow on this link.

Figure 4.1: A simple network $(\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{v_1, v_2, v_3\}$ and $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$. In this example, $\mathcal{R} = \{r_1, r_2, r_3\}$ is described by $r_1 = \{e_1, e_2\}$, $r_2 = \{e_1, e_3\}$, $r_3 = \{e_4\}$.

Motivated by the relationship between delay and flow used in [86, 87], we define the latency on link e as

$$l_e(f_e) = a_e f_e + b_e, \quad (4.1)$$

where $f_e = \sum_{r \in \mathcal{R}: e \in r} f_r$ is the total flow on link e , and $a_e \geq 0, b_e \geq 0$ are known constants.

The latency on route $r \in \mathcal{R}$ is the sum of the latencies of links on this route:

$$l_r(\hat{f}) = \sum_{e \in r} l_e(f_e). \quad (4.2)$$

Suppose that a toll $\rho_e(f_e)$ is charged on each link $e \in \mathcal{E}$ to affect players' routing behaviors, and each player has a price sensitivity $\beta > 0$ which may be different for different players.¹ We consider the case where there are finite possible values for β and all players are classified into groups according to their price sensitivities. Let $\mathcal{B} = \{\beta_1, \dots, \beta_M\}$ denote the set of price sensitivities and $P = \{p_1, \dots, p_M\}$ be the corresponding distribution. Therefore, the total flow for group with $\beta_j \in \mathcal{B}$ is $F_j = p_j F$. Denote $f_{r,\beta}$ as the flow on route $r \in \mathcal{R}$ contributed by group with $\beta \in \mathcal{B}$ and $f = (f_{r_1,\beta_1}, \dots, f_{r_1,\beta_M}, \dots, f_{r_v,\beta_1}, \dots, f_{r_v,\beta_M})$. Note that $\sum_{i=1}^v f_{r_i,\beta_j} = p_j F = F_j$ for all $\beta_j \in \mathcal{B}$, and $\sum_{j=1}^M f_{r_i,\beta_j} = f_{r_i}$ for all $r_i \in \mathcal{R}$. By incorporating the price

¹The price sensitivity can be obtained from the household survey which has been done by Land Transport Authority (LTA) of Singapore.

sensitivity, we define the cost of route $r \in \mathcal{R}$ for players in group with $\beta \in \mathcal{B}$ as

$$J_{r,\beta}(f) = \sum_{e \in r} (l_e(f_e) + \beta \rho_e(f_e)). \quad (4.3)$$

For the noncooperative congestion game formulated above, we assume that each player selfishly chooses to travel on route with minimal individual cost (4.3). A Nash flow is defined as follows.

Definition 4.1. *A flow f^{ne} is called a Nash flow if for any $\beta_j \in \mathcal{B}$ and $r_1, r_2 \in \mathcal{R}$,*

$$f_{r_1, \beta_j}^{ne} > 0, f_{r_2, \beta_j}^{ne} > 0 \Rightarrow J_{r_1, \beta_j}(f^{ne}) = J_{r_2, \beta_j}(f^{ne}), \quad (4.4)$$

$$f_{r_1, \beta_j}^{ne} > 0, f_{r_2, \beta_j}^{ne} = 0 \Rightarrow J_{r_1, \beta_j}(f^{ne}) \leq J_{r_2, \beta_j}(f^{ne}). \quad (4.5)$$

Note that Nash flows always exist for congestion games of the type considered in this chapter [27], [26].

In [123], it is shown that games with homogeneous players, i.e., all players have the same price sensitivity ($M = 1$), can achieve the social optimum by charging all players a marginal-cost price. The marginal-cost for the latency function given in (4.1) is $a_e f_e$. However, in our general formulation with $M > 1$, the social optimum cannot be achieved by charging $a_e f_e$ on each link $e \in \mathcal{E}$, since each group of drivers have different price sensitivity. Therefore, we redefine the pricing function as the scaled marginal-cost toll as

$$\rho_e(f_e) = \mu a_e f_e, \quad (4.6)$$

where $\mu \geq 0$ is a parameter to be designed. Note that Nash flow is a function of the toll $\rho_e(f_e)$ which is a function of μ . Therefore, Nash flow is also a function of μ . In this chapter, we will use $f^{ne}(\mu)$ and f^{ne} interchangeably when no confusion is caused.

The total latency of the network is given by

$$L(f) = \sum_{r \in \mathcal{R}} f_r \cdot l_r(\hat{f}). \quad (4.7)$$

Then, the socially optimal flow f^* , which is defined as the flow that minimizes the total latency of the network, is described by

$$f^* = \arg \inf_f L(f) \quad (4.8)$$

with

$$\sum_{r \in \mathcal{R}} f_r = F$$

The POA, which is the worst possible ratio between the total latency of a Nash flow and that of the optimal flow, is defined as

$$\text{POA} = \sup_{f^{ne}} \frac{L(f^{ne})}{L(f^*)}. \quad (4.9)$$

The POA characterizes the worst-case efficiency loss of all possible Nash flows, and the larger the POA the larger the efficiency loss. Obviously, POA is always larger or equal to 1 and $\text{POA} = 1$ indicates that the social optimum is achieved.

Our objective is to design the pricing function to minimize the POA for a given distribution of price sensitivity. Namely, our goal is to find μ^* for (4.6) such that

$$L(f^{ne}(\mu^*)) = \inf_{\mu \geq 0} L(f^{ne}(\mu)). \quad (4.10)$$

In other words, the Nash flow f^{ne} obtained after introducing the designed road pricing $\rho_e(f_e) = \mu^* a_e f_e$ leads to the minimal worst-case efficiency loss.

4.2 Design of Optimal Pricing with Heterogeneous Price-sensitivity Populations

In this section, we aim to design the optimal scaled marginal-cost pricing that minimizes the POA for networks with heterogeneous players. Without loss of generality, we assume $\beta_1 > \beta_2 > \dots > \beta_M$. Denote $\bar{F}_{j_0} = F_1 + \dots + F_{j_0-1}$ as the total flow with

$\beta > \beta_{j_0}$ and $\hat{F}_{j_0} = F_{j_0+1} + \dots + F_M$ as the total flow with $\beta < \beta_{j_0}$. Let $\bar{p}_{j_0} = \bar{F}_{j_0}/F$ and $\hat{p}_{j_0} = \hat{F}_{j_0}/F$. For route $r_i \in \mathcal{R}$, define $V_i = \sum_{e \in r_i} b_e$, which is called the free-flow latency of route r_i . For any $r_i, r_j \in \mathcal{R}$, denote

$$d_{ij} = \sum_{e \in r_i \cap r_j} a_e \quad (4.11)$$

If $V_i = V_j$ for some $r_i \neq r_j$, then there exist infinite Nash flows as defined in Definition 4.1. However, the total latency is identical for all these Nash flows, and so is the POA. For the special case with $V_i = V_j$ for all $r_i \neq r_j$, there exist infinite Nash flows for any distribution of $\mathcal{B}$, but the POA can always achieve 1 for any $\mu \geq 0$ since $f_{r_k}^{ne} = f_{r_k}^*, r_k \in \mathcal{R}$. In most practical cases, V_i are different for different routes. Therefore, we make the following assumption.

Assumption 4.1. $V_i \neq V_j$ for all $i \neq j$.

To further ensure the positivity property of Nash flows with road pricing, we give the following assumption.

Assumption 4.2. For the un-tolled situation with $\mu = 0$, any Nash flow f^{ne} has $f_r^{ne} > 0$ for all $r \in \mathcal{R}$.

Under Assumption 4.1, we can let $V_1 < V_2 < \dots < V_N$. Assumption 4.2 indicates that every route will be in use when all routes are free of charge.

4.2.1 Nash Flow Analysis for Case with Two Groups and Two Routes

First, we consider the special case where there are two routes $\mathcal{R} = \{r_1, r_2\}$ in the network with two different groups $\mathcal{B} = \{\beta_1, \beta_2\}$. We will show that, for any distribution of $\mathcal{B}$, there always exists μ^* such that POA=1.

In the following lemma, the existence and uniqueness of Nash flow for the special case is guaranteed.

Lemma 4.1. *Under Assumptions 4.1 and 4.2, the Nash flow of the two groups case always exists and is uniquely determined as follows:*

(i) If $0 \leq p_1 < \frac{F(d_{11}-d_{12})-\frac{V_2-V_1}{1+\mu\beta_2}}{F(d_{11}+d_{22}-2d_{12})}$,

$$\begin{aligned} f_{r_1,\beta_1}^{ne} &= 0, f_{r_2,\beta_1}^{ne} = p_1 F, \\ f_{r_1,\beta_2}^{ne} &= \frac{p_1 F(d_{22} - d_{12}) + p_2 F(d_{22} - d_{12}) + \frac{V_2-V_1}{1+\mu\beta_2}}{d_{11} + d_{22} - 2d_{12}}, \\ f_{r_2,\beta_2}^{ne} &= \frac{p_2 F(d_{11} - d_{12}) - p_1 F(d_{22} - d_{12}) - \frac{V_2-V_1}{1+\mu\beta_2}}{d_{11} + d_{22} - 2d_{12}}. \end{aligned}$$

(ii) If $\frac{F(d_{11}-d_{12})-\frac{V_2-V_1}{1+\mu\beta_2}}{F(d_{11}+d_{22}-2d_{12})} \leq p_1 \leq \frac{F(d_{11}-d_{12})-\frac{V_2-V_1}{1+\mu\beta_1}}{F(d_{11}+d_{22}-2d_{12})}$,

$$f_{r_1,\beta_1}^{ne} = 0, f_{r_2,\beta_1}^{ne} = p_1 F, f_{r_1,\beta_2}^{ne} = p_2 F, f_{r_2,\beta_2}^{ne} = 0.$$

(iii) If $\frac{F(d_{11}-d_{12})-\frac{V_2-V_1}{1+\mu\beta_1}}{F(d_{11}+d_{22}-2d_{12})} < p_1 \leq 1$,

$$\begin{aligned} f_{r_1,\beta_1}^{ne} &= \frac{p_1 F(d_{22} - d_{12}) - p_2 F(d_{22} - d_{12}) + \frac{V_2-V_1}{1+\mu\beta_1}}{d_{11} + d_{22} - 2d_{12}}, \\ f_{r_2,\beta_1}^{ne} &= \frac{p_1 F(d_{11} - d_{12}) + p_2 F(d_{11} - d_{12}) - \frac{V_2-V_1}{1+\mu\beta_1}}{d_{11} + d_{22} - 2d_{12}}, \\ f_{r_1,\beta_2}^{ne} &= p_2 F, f_{r_2,\beta_2}^{ne} = 0. \end{aligned}$$

Proof. First, we prove Nash flow (i). By the price function (4.6) and $f_e = \sum_{r \in \mathcal{R}: e \in r} f_r$, we rewrite (4.3) as

$$J_{r_i,\beta_j}(f^{ne}) = \sum_{e \in r_i} ((1 + \mu\beta_j)d_e \sum_{r_k: e \in r_k} f_{r_k}^{ne} + b_e) = (1 + \mu\beta_j) \sum_{r_k \in \mathcal{R}} f_{r_k}^{ne} d_{ik} + V_i. \quad (4.12)$$

By the definition of Nash flow, if $f_{r_1,\beta_2}^{ne} > 0, f_{r_2,\beta_2}^{ne} > 0$,

$$\sum_{k=1}^2 f_{r_k}^{ne} (d_{1k} - d_{2k}) = \frac{V_2 - V_1}{1 + \mu\beta_2}. \quad (4.13)$$

Since $\beta_1 > \beta_2$ and $V_1 < V_2$, $(V_2 - V_1)/(1 + \mu\beta_2) > (V_2 - V_1)/(1 + \mu\beta_1)$. Therefore,

$$(1 + \mu\beta_1) \sum_{k=1}^2 d_{1k} f_{r_k}^{ne} + V_1 > (1 + \mu\beta_1) \sum_{k=1}^2 d_{2k} f_{r_k}^{ne} + V_2 \quad (4.14)$$

Thus, $f_{r_1, \beta_1}^{ne} = 0$, $f_{r_2, \beta_1}^{ne} = p_1 F$. Insert $f_{r_1, \beta_1}^{ne} + f_{r_1, \beta_2}^{ne} = f_{r_1}^{ne}$ and $f_{r_2, \beta_1}^{ne} + f_{r_2, \beta_2}^{ne} = f_{r_2}^{ne}$ into (4.13) and note that $f_{r_1, \beta_2}^{ne} + f_{r_2, \beta_2}^{ne} = p_2 F$, we obtain

$$f_{r_1, \beta_2}^{ne} = \frac{p_1 F(d_{22} - d_{12}) + p_2 F(d_{22} - d_{12}) + \frac{V_2 - V_1}{1 + \mu\beta_2}}{d_{11} + d_{22} - 2d_{12}}, \quad (4.15)$$

$$f_{r_2, \beta_2}^{ne} = \frac{p_2 F(d_{11} - d_{12}) - p_1 F(d_{22} - d_{12}) - \frac{V_2 - V_1}{1 + \mu\beta_2}}{d_{11} + d_{22} - 2d_{12}}. \quad (4.16)$$

Obviously, $f_{r_1, \beta_2}^{ne} > 0$. Under Assumption 4.2, $f_{r_2}^{ne} = f_{r_2, \beta_1}^{ne} + f_{r_2, \beta_2}^{ne} > 0$ when $\mu = 0$. Thus, $(d_{11} - d_{12})F > V_2 - V_1$. Therefore, $(d_{11} - d_{12})F - \frac{V_2 - V_1}{1 + \mu\beta_2} > 0$. Since $f_{r_2, \beta_2}^{ne} > 0$, $0 \leq p_1 < \frac{F(d_{11} - d_{12}) - \frac{V_2 - V_1}{1 + \mu\beta_2}}{F(d_{11} + d_{22} - 2d_{12})}$. Nash flow (i) is proved.

Similarly, we can show (ii) and (iii). ■

Based on Lemma 4.1, the total latency of Nash flow is given as follows.

Lemma 4.2. *Under Assumptions 4.1 and 4.2, the total latency of Nash flow for the two groups case is as follows:*

(i) If $0 \leq p_1 < \frac{F(d_{11} - d_{12}) - \frac{V_2 - V_1}{1 + \mu\beta_2}}{F(d_{11} + d_{22} - 2d_{12})}$,

$$L(f^{ne}) = L_F(F) - \frac{(V_2 - V_1)^2}{d_{11} + d_{22} - 2d_{12}} \frac{\mu\beta_2}{(1 + \mu\beta_2)^2} \quad (4.17)$$

with

$$L_F(F) = \frac{(d_{11} - d_{12})(d_{22} - d_{12})}{d_{11} + d_{22} - 2d_{12}} F^2 + \frac{(d_{11} - d_{12})V_2 + (d_{22} - d_{12})V_1}{d_{11} + d_{22} - 2d_{12}} F \quad (4.18)$$

(ii) If $\frac{F(d_{11} - d_{12}) - \frac{V_2 - V_1}{1 + \mu\beta_2}}{F(d_{11} + d_{22} - 2d_{12})} \leq p_1 \leq \frac{F(d_{11} - d_{12}) - \frac{V_2 - V_1}{1 + \mu\beta_1}}{F(d_{11} + d_{22} - 2d_{12})}$,

$$L(f^{ne}) = (d_{11} - d_{12})F^2 + (V_2 - V_1 - 2(d_{11} - d_{12})F)Fp_1 + (d_{11} + d_{22} - 2d_{12})F^2 p_1^2 \quad (4.19)$$

(iii) If $\frac{F(d_{11}-d_{12})-\frac{V_2-V_1}{1+\mu\beta_1}}{F(d_{11}+d_{22}-2d_{12})} < p_1 \leq 1$,

$$L(f^{ne}) = L_F(F) - \frac{(V_2 - V_1)^2}{d_{11} + d_{22} - 2d_{12}} \frac{\mu\beta_1}{(1 + \mu\beta_1)^2} \quad (4.20)$$

Proof: Substitute all f_{r,β_j}^{ne} in Lemma 4.1 into (4.7) and simplify to obtain $L(f^{ne})$. ■

In the following theorem, we give the main result of this section.

Theorem 4.1. *Under Assumptions 4.1 and 4.2, for any given p_1 , there always exists μ^* satisfying $L(f^{ne}(\mu^*)) = \inf_{\mu \geq 0} L(f^{ne}(\mu)) = L(f^*)$, i.e., POA=1. Furthermore,*

$$L(f^{ne}(\mu^*)) = L(f^*) = L_F(F) - \frac{1}{4} \frac{(V_2 - V_1)^2}{d_{11} + d_{22} - 2d_{12}} \quad (4.21)$$

where $L_F(F)$ is as defined in (4.18).

Proof: The optimal flow of this network is:

$$f_{r_1}^* = \frac{V_2 - V_1 + 2F(d_{22} - d_{12})}{2(d_{11} + d_{22} - 2d_{12})}, \quad (4.22)$$

$$f_{r_2}^* = \frac{V_2 - V_1 + 2F(d_{11} - d_{12})}{2(d_{11} + d_{22} - 2d_{12})}. \quad (4.23)$$

Thus,

$$L(f^*) = L_F(F) - \frac{1}{4} \frac{(V_2 - V_1)^2}{d_{11} + d_{22} - 2d_{12}}. \quad (4.24)$$

For $0 \leq p_1 < \frac{F(d_{11}-d_{12})-\frac{V_2-V_1}{2}}{F(d_{11}+d_{22}-2d_{12})}$, if we set $\mu^* = \frac{1}{\beta_2}$, the Nash flow is as in Lemma 4.1 (i). Based on the total latency (4.17), it is easy to verify that the social optimum is reached at $\mu^* = \frac{1}{\beta_2}$, i.e., $L(f^{ne}(\mu^*)) = L(f^*)$ and POA=1. Similarly, for $\frac{F(d_{11}-d_{12})-\frac{V_2-V_1}{2}}{F(d_{11}+d_{22}-2d_{12})} < p_1 \leq 1$, the social optimum can be achieved at $\mu^* = \frac{1}{\beta_1}$ with the Nash flow as shown in (iii) of Lemma 4.1. If $p_1 = \frac{F(d_{11}-d_{12})-\frac{V_2-V_1}{2}}{F(d_{11}+d_{22}-2d_{12})}$, for any $\mu^* \in [\frac{1}{\beta_1}, \frac{1}{\beta_2}]$, the Nash flow is as (ii) in Lemma 4.1, and we can obtain POA=1 by substituting p_1 into (4.19). ■

4.2.2 Nash Flow Analysis for General Case

In this section, we will analyze the POA for the case with several groups and routes connecting one origin and one destination.

Let $\{A_1, \dots, A_v\}, \{B_1, \dots, B_v\}$ be the set of solutions to the following equations:

$$\sum_{r_k \in \mathcal{R}} A_k = 1, \quad (4.25)$$

$$\sum_{r_k \in \mathcal{R}} B_k = 0, \quad (4.26)$$

$$\sum_{r_k \in \mathcal{R}} A_k d_{ik} = \sum_{r_k \in \mathcal{R}} A_k d_{jk}, \quad \forall r_i \neq r_j, \quad (4.27)$$

$$\sum_{r_k \in \mathcal{R}} B_k d_{ik} + V_i = \sum_{r_k \in \mathcal{R}} B_k d_{jk} + V_j, \quad \forall r_i \neq r_j. \quad (4.28)$$

We can see from the proof of Lemma 4.3 that $\{A_1, \dots, A_v\}, \{B_1, \dots, B_v\}$ always exist. The following lemma shows that under certain conditions, a Nash flow always exists and it is unique.

Lemma 4.3. *Under Assumptions 4.1 and 4.2, for any $\mu \geq 0$ and a given distribution of $\mathcal{B}$, if there exists a group with $\beta_{j_0} \in \mathcal{B}$ satisfying*

$$\begin{cases} \hat{p}_{j_0} < A_1 + \frac{B_1}{(1+\mu\beta_{j_0})F}, \\ \bar{p}_{j_0} < A_v + \frac{B_v}{(1+\mu\beta_{j_0})F}, \end{cases} \quad (4.29)$$

then a Nash flow always exists and it is uniquely determined as: (a) the Nash flow for group β_{j_0} is given by

$$f_{r_k, \beta_{j_0}}^{ne} = \begin{cases} A_1 F - \hat{F}_{j_0} + \frac{1}{1+\mu\beta_{j_0}} B_1, & \text{if } k = 1, \\ A_v F - \bar{F}_{j_0} + \frac{1}{1+\mu\beta_{j_0}} B_v, & \text{if } k = v, \\ A_k F + \frac{1}{1+\mu\beta_{j_0}} B_k, & \text{otherwise;} \end{cases} \quad (4.30)$$

- (b) all players with $\beta < \beta_{j_0}$ choose route r_1 , i.e., $f_{r_1, \beta_{j_0+1}}^{ne} = F_{j_0+1}, \dots, f_{r_1, \beta_M}^{ne} = F_M$;
(c) all players with $\beta > \beta_{j_0}$ choose route r_v , i.e., $f_{r_v, \beta_1}^{ne} = F_1, \dots, f_{r_v, \beta_{j_0-1}}^{ne} = F_{j_0-1}$.

The proof of Lemma 4.3 is provided in Appendix A.

Based on the proof of Lemma 4.3, the properties of Nash flows can be concluded as follows.

- If there is a group choosing more than one routes at Nash flow, it must choose several successive routes, e.g., $\{r_1, r_2\}, \{r_2, r_3, r_4\}$.
- If players with $\beta_j \in \mathcal{B}$ choose routes $\{r_{j_1}, \dots, r_{j_2}\}$, then players with $\beta > \beta_j$ choose some routes $r \in \{r_{j_2}, \dots, r_v\}$ and players with $\beta < \beta_j$ choose some routes $r \in \{r_1, \dots, r_{j_1}\}$.

Consider the case where players with $\beta_{j_0} \in \mathcal{B}$ choose routes $\{r_{k_1}, \dots, r_{k_2}\}$ with $k_1 \leq k_2$ and $\{k : 1 \leq k < k_1\} \cup \{k : k_2 < k \leq v\} \neq \emptyset$, and all other players choose only one route $r \in \mathcal{R}$ at Nash flow. Based on the properties of Nash flows described above, all players with $\beta < \beta_{j_0}$ only choose a route $r \in \{r_1, \dots, r_{k_1}\}$ and all players with $\beta > \beta_{j_0}$ only choose a route $r \in \{r_{k_2}, \dots, r_v\}$. Therefore, for route $r_k \notin \{r_{k_1}, \dots, r_{k_2}\}$, $f_{r_k}^{ne}$ is the sum of certain groups' total flow.

Denote $\tilde{p}_k = f_{r_k}^{ne}/F$, $\mathcal{F}_{k_1} = \sum_{k=1}^{k_1-1} f_{r_k}^{ne}$, $\mathcal{F}_{k_2} = \sum_{k=k_2+1}^v f_{r_k}^{ne}$, $\rho_{k_1} = \mathcal{F}_{k_1}/F$, and $\rho_{k_2} = \mathcal{F}_{k_2}/F$. Let $\hat{f}^*$ be the vector of route flows corresponding to the socially optimal flow. A distribution of $\mathcal{B}$ is called a *critical point*, if it satisfies

$$\begin{cases} \tilde{p}_k = f_{r_k}^*/F, \text{ for } 1 \leq k < k_1 \text{ or } k_2 < k \leq v, \\ \hat{p}_{j_0} - \rho_{k_1} < A_{k_1} + \frac{B_{k_1}}{2F}, \\ \bar{p}_{j_0} - \rho_{k_2} < A_{k_2} + \frac{B_{k_2}}{2F} \end{cases} \quad (4.31)$$

where $A_{k_1}, A_{k_2}, B_{k_1}, B_{k_2}$ are the k_1 -th and k_2 -th elements of the sets $\{A_1, \dots, A_v\}$ and $\{B_1, \dots, B_v\}$, respectively.

Note that conditions (4.29) and (4.31) are exclusive with each other. The next theorem, as the main result of this chapter, shows how the road pricing and the distribution of price sensitivity influence the POA.

Theorem 4.2. *Under Assumptions 4.1 and 4.2, for a given distribution of $\mathcal{B}$, if*

(i) *there exists a group $\beta_{j_0} \in \mathcal{B}$ satisfying*

$$\begin{cases} \hat{p}_{j_0} < A_1 + \frac{B_1}{2F}, \\ \bar{p}_{j_0} < A_v + \frac{B_v}{2F}; \end{cases} \quad (4.32)$$

or (ii) *the distribution of $\mathcal{B}$ is a critical point with*

$$\begin{cases} \tilde{p}_k = f_{r_k}^*/F, \text{ for } 1 \leq k < k_1 \text{ or } k_2 < k \leq v, \\ \hat{p}_{j_0} - \rho_{k_1} < A_{k_1} + \frac{B_{k_1}}{2F}, \\ \bar{p}_{j_0} - \rho_{k_2} < A_{k_2} + \frac{B_{k_2}}{2F}, \end{cases} \quad (4.33)$$

then $\mu^* = 1/\beta_{j_0}$ satisfies

$$L(f^{ne}(\mu^*)) = L(f^*) = \sum_{k=1}^v A_k d_{k1} F^2 + \sum_{k=1}^v A_k V_k F + \sum_{k=1}^v \frac{B_k V_k}{4}, \quad (4.34)$$

i.e., $POA = 1$. Otherwise, $POA > 1$. ■

Proof: First, we consider case (i). From Lemmas 4.3 and the definition of $L(f)$ in (4.7), if group $\beta_{j_0} \in \mathcal{B}$ satisfies (4.29), then the total latency of the Nash flow with road pricing is

$$L(f^{ne}) = \sum_{k=1}^v A_k d_{k1} F^2 + \sum_{k=1}^v A_k V_k F + \sum_{k=1}^v \frac{B_k V_k \mu \beta_{j_0}}{(1 + \mu \beta_{j_0})^2}; \quad (4.35)$$

Taking the partial derivative of (4.35) with respect to μ , we can see that the minimum of $L(f^{ne})$ is achieved at $\mu^* = 1/\beta_{j_0}$. Substitute μ^* into (4.35), we have

$$L(f^{ne}(\mu^*)) = \sum_{k=1}^v A_k d_{k1} F^2 + \sum_{k=1}^v A_k V_k F + \sum_{k=1}^v \frac{B_k V_k}{4} \quad (4.36)$$

Furthermore, we can check that the total latency of the Nash flow is equal to the total latency of the socially optimal flow, i.e., $POA = 1$.

For case (ii), only players with $\beta_{j_0} \in \mathcal{B}$ choose more than one routes. Based on the definition of critical point, the Nash flow is determined as

$$f_{r_k}^{ne} = \begin{cases} f_{r_k}^*, & \text{if } 1 \leq k < k_1 \text{ or } k_2 < k \leq v, \\ \hat{F}_{j_0} - \mathcal{F}_{k_1} + f_{r_{k_1}, \beta_{j_0}}^{ne}, & \text{if } k = k_1, \\ \bar{F}_{j_0} - \mathcal{F}_{k_2} + f_{r_{k_2}, \beta_{j_0}}^{ne}, & \text{if } k = k_2, \\ f_{r_k, \beta_{j_0}}^{ne}, & \text{otherwise,} \end{cases} \quad (4.37)$$

where $f_{r_k, \beta_{j_0}}^{ne} > 0$ for all $k_1 \leq k \leq k_2$ and $\sum_{k=k_1}^{k_2} f_{r_k, \beta_{j_0}}^{ne} = F_{j_0}$.

Similar to case (i), under Assumptions 4.1 and 4.2, we can show that $\mu^* = 1/\beta_{j_0}$ can lead to $L(f^{ne}(\mu^*)) = L(f^*)$, i.e., $\text{POA} = 1$.

If the distribution of $\mathcal{B}$ does not satisfy (i) or (ii), there are at least two groups choosing more than one routes at any Nash flow. Assuming that group β_{j_1} chooses r_{i_1}, r_{i_2} and group β_{j_2} chooses $r_{i'_1}, r_{i'_2}$, by the definition of a Nash flow, we have

$$\sum_{r_k \in \mathcal{R}} f_{r_k}^{ne} (d_{i_1 k} - d_{i_2 k}) = \frac{V_{i_2} - V_{i_1}}{1 + \mu \beta_{j_1}}, \quad (4.38)$$

$$\sum_{r_k \in \mathcal{R}} f_{r_k}^{ne} (d_{i'_1 k} - d_{i'_2 k}) = \frac{V_{i'_2} - V_{i'_1}}{1 + \mu \beta_{j_2}}. \quad (4.39)$$

Therefore, the total latency of a Nash flow $L(f^{ne})$ is a function of both $1/(1 + \mu \beta_{j_1})$ and $1/(1 + \mu \beta_{j_2})$. It follows from $\beta_{j_1} \neq \beta_{j_2}$ that $\text{POA} > 1$. $\blacksquare$

Remark 4.1. Note that for a critical point, under Assumptions 4.1 and 4.2, if we set $\mu^* = 1/\beta_{j_0}$, then the Nash flow is uniquely determined as: (a) for group β_{j_0} :

$$f_{r_k, \beta_{j_0}}^{ne} = \begin{cases} 0, & \text{if } 1 \leq k < k_1 \text{ or } k_2 < k \leq v, \\ A_{k_1} F - \hat{F}_{j_0} + \mathcal{F}_{k_1} + \frac{1}{2} B_{k_1}, & \text{if } k = k_1, \\ A_{k_2} F - \bar{F}_{j_0} + \mathcal{F}_{k_2} + \frac{1}{2} B_{k_2}, & \text{if } k = k_2, \\ A_k F + \frac{1}{2} B_k, & \text{otherwise;} \end{cases} \quad (4.40)$$

(b) all players with $\beta < \beta_{j_0}$ only choose one route $r \in \{r_1, \dots, r_{k_1}\}$ and $f_{r_k}^{ne} = f_{r_k}^*$ for

$1 \leq k < k_1$; (c) all players with $\beta > \beta_{j_0}$ only choose one route $r \in \{r_{k_2}, \dots, r_v\}$ and $f_{r_k}^{ne} = f_{r_k}^*$ for $k_2 < k \leq v$. $\blacksquare$

From the above analysis, we conclude that, if the distribution of $\mathcal{B}$ satisfies certain conditions, we can always find a $\mu^* \geq 0$ such that the POA can achieve 1 by charging the designed toll $\rho_e(f_e) = \mu^* a_e f_e$ on each link $e \in \mathcal{E}$. Moreover, it is proved in [86] that, in nonatomic congestion games, the POA can be bounded by $\frac{4}{3}$ if the network has linear latency functions.

For any distribution of $\mathcal{B}$ and any given $\mu \geq 0$, we can always find a Nash flow in the congestion game we formulated. Under Assumptions 4.1 and 4.2, we provide a method to design the optimal μ^* that minimizes the POA as summarized in Algorithm 4.1.

4.3 Simulations

In this section, we first use numerical examples to illustrate that whether the POA can achieve 1 for any given network depends on the distribution of $\mathcal{B}$. Then, based on the real traffic data in Singapore, we analyze the POA for two different traffic networks with identical distribution of $\mathcal{B}$, which indicates that whether the POA can reach 1 also depends on the topology and parameters of a traffic network.

4.3.1 Numerical Examples

Consider the network with 3 routes as shown in Fig. 4.1, and suppose that there are 5 groups of players with a total flow $F = 2$. The parameters of the network are given by $\{a_{e_1} = 25, a_{e_2} = 68, a_{e_3} = 47, a_{e_4} = 86\}$, $\{b_{e_1} = 10, b_{e_2} = 21, b_{e_3} = 85, b_{e_4} = 98\}$, and $\mathcal{B} = \{100, 1, 0.35, 0.18, 0.01\}$. The total latency of the Nash flow without tolls is $L(f^{ne}(0)) = 290.7$, and the socially optimal flow on each route is $f_{r_1}^* = 0.8277, f_{r_2}^* = 0.5166, f_{r_3}^* = 0.6557$ with $L(f^*) = 280.3$. Next, we analyze the POA for three different distributions of $\mathcal{B}$.

Algorithm 4.1: Best POA

Input: Network parameters (e.g., a_e, b_e) and network topology (e.g., Fig. 4.1), total flow F , price sensitivities $\mathcal{B}$ and its distribution P

Output: Best POA

Calculate the social optimal flow f^* ;

for $q_1 = 1 : v$

Group β_1 choose a sequential routes $r_{q_1}, \dots, r_v$
with flow $f_{r_{q_1}, \beta_1} > 0, \dots, f_{r_v, \beta_1} > 0$;

$\vdots$

for $q_{j+1} = 1 : q_j$

Group β_{j+1} choose a sequential routes $r_{q_{j+1}}, \dots, r_{q_j}$
with flow $f_{r_{q_{j+1}}, \beta_{j+1}} > 0, \dots, f_{r_{q_j}, \beta_{j+1}} \geq 0$;

$\vdots$

for $q_M = q_{M-1} - 1 : q_{M-1}$

Group β_M choose a sequential routes $r_1, \dots, r_{q_M}$
with flow $f_{r_1, \beta_M} > 0, \dots, f_{r_{q_M}, \beta_M} > 0$;

Solve the equations set generated by (A.1);

Return Nash flow $f^{ne}(\mu)$;

Find μ^* by optimizing $L(f^{ne}(\mu))$;

end

$\vdots$

end

$\vdots$

end

Return the largest value of $\frac{L(f^{ne}(\mu^*))}{L(f^*)}$

For the first case, assume the distribution of $\mathcal{B}$ is $P = \{0.25, 0.46, 0.04, 0.20, 0.05\}$. According to Algorithm 4.2, we can check that condition (i) in Theorem 4.2 is satisfied. Therefore, we can set $\mu^* = 1/\beta_2 = 1$ to obtain $L(f^{ne}(\mu^*)) = L(f^*)$, i.e., $\text{POA} = 1$. As expected, the road pricing decreases the total latency.

Algorithm 4.2: Condition checking of Theorem 4.2

Data: Network parameters (e.g., a_e, b_e) and network topology (e.g., Fig. 4.1), total flow F , price sensitivities $\mathcal{B}$ and its distribution P

Result: Whether $\text{POA}=1$

Calculate the social optimal flow f^* and Nash flow f^{ne} ;
Find the solution of equations (4.25)-(4.28);
for $i = 1 : M$
 Check condition (4.32);
 if condition (4.32) is satisfied
 Return $\text{POA}=1$ with $\mu^* = 1/\beta_i$;
 end
end
for $i = 1 : M$
 for $k_1 = 2 : v - 1$
 for $k_2 = k_1 : v - 1$
 Check condition (4.33);
 if condition (4.33) is satisfied
 Return $\text{POA}=1$ with $\mu^* = 1/\beta_i$;
 end
 end
 end
end
Return $\text{POA} > 1$

For the second case, let $P = \{0.1225, 0.2053, 0.2972, 0.2735, 0.1015\}$. There exists a group β_3 such that $\hat{p}_3 < f_{r_1}^*/F = 0.4138$, $\tilde{p}_3 = p_1 + p_2 = f_{r_3}^*/F = 0.3278$, i.e., condition (ii) in Theorem 4.2 is satisfied, which can be checked by Algorithm 4.2. Thus, we can set $\mu^* = 1/\beta_3 = 2.857$ to achieve $\text{POA} = 1$.

For the third case, assume the distribution of $\mathcal{B}$ is $P = \{0.35, 0.40, 0.11, 0.09, 0.05\}$. In this case, according to Algorithm 4.2, the conditions in Theorem 4.2 are not satisfied. Using Algorithm 4.1, we get $\mu^* = 0.9729$ and $L(f^{ne}(\mu^*)) = 281.0$. The road pricing reduces the total latency, but $\text{POA} = 1.003 > 1$.

4.3.2 Real Data Simulations in Traffic Networks

For real data analysis, we consider two road networks in Singapore. One road network is in the east of Singapore (e.g. Fig. 4.2), and the other one is in Central Business District (CBD) (e.g. Fig. 4.3). Assume that road users are divided into groups according to their vehicle modes – car, motorcycle, taxi, bus. According to the 2004 stated preference survey data, the price sensitivities (min/cent) for these four groups are 0.25, 0.36, 0.20, 0, respectively. The vehicle distribution of these four vehicle types provided by Singapore land transport statistics in brief 2005 is $\{0.7239, 0.1884, 0.0281, 0.0596\}$. Assume that the total flow of the network is 500.

In the following simulation, we will show that, for the network as shown in Fig. 4.2, the POA cannot achieve 1 since both condition (i) and (ii) in Theorem 4.2 are not satisfied. While for the CBD road network as shown in Fig. 4.3, the distribution of $\mathcal{B}$ satisfies the condition (i) in Theorem 4.2. Therefore, we can find μ^* such that POA=1.

Figure 4.2: A traffic network in the east of Singapore

Example 4.1. (*The East Road Network of Singapore*) To see the road network in Fig. 4.2 clearly, we extract its structure and show it in Fig. 4.4. For each edge e in Fig. 4.4, we fit the latency function $l_e(f_e)$ to real traffic data (e.g., the loop count data to record the traffic flow and the taxi data to record the average speed). Then, we get the value of a_e, b_e for each edge e . For example, as shown in Fig. 4.5, $a_e = 0.0932, b_e = 5.5409$ for edge e_1 in Fig. 4.4. The socially optimal flow for

Figure 4.3: A traffic network in the Central Business District (CBD) of Singapore

this network is $f_{r_1}^* = 176.7981$, $f_{r_2}^* = 198.0003$, $f_{r_3}^* = 114.2368$, $f_{r_4}^* = 10.9648$ with $L(f^*) = 35922.807$. We can check that this network doesn't satisfy either condition (i) or (ii) in Theorem 4.2. Through Algorithm 4.1, we get $\mu^* = 3.394$ and the corresponding Nash flow is $f_{r_1}^{ne} = 177.028$, $f_{r_2}^{ne} = 197.966$, $f_{r_3}^{ne} = 113.710$, $f_{r_4}^{ne} = 11.296$. We can further check that motorcycle group chooses route r_3, r_4 , car group chooses route r_1, r_2, r_3 and both taxi and bus groups choose route r_1 . The total latency of the Nash flow without toll is $L(f^{ne}(0)) = 35950.711$ and with toll is $L(f^{ne}(\mu^*)) = 35922.899$. Compare $L(f^{ne}(\mu^*))$ and $L(f^*)$, we have $POA > 1$. As shown in Fig. 4.6, the POA achieves its minimal point at $\mu^* = 3.394$ where the POA slightly deviates from 1. It is also verified that the POA is bounded by $4/3$.

Remark 4.2. Note that a given flow on a road can correspond to either a high density of vehicles on a congested road, in which case the latency is large, or a low density of vehicles on a free flow road, in which case the latency is small [138]. Due to this phenomenon, the latency is not uniquely determined by the flow, and depends on the congestion state of the road. However, in our examples, we try to grasp the main features of the relationship between the traffic flow and travel time before congestion. Therefore, we adopt the one-to-one relationship between delay

Figure 4.4: The structure of the east road network of Fig. 4.2. In this traffic network, $\mathcal{R} = \{r_1, r_2, r_3, r_4\}$ is described by $r_1 = \{e_8, e_9\}$, $r_2 = \{e_1, e_2, e_3, e_{10}\}$, $r_3 = \{e_4, e_5, e_{10}\}$, $r_4 = \{e_4, e_6, e_7, e_3, e_{10}\}$.

Figure 4.5: The relationship between traffic flow and travel time for edge e_1 (i.e., $l_e = 0.0932f_e + 5.5409$) in Fig. 4.4

and flow to formulate our problem as other researchers do [86, 87, 139, 140].

Example 4.2. (CBD of Singapore) In the CBD network, there are three routes $\mathcal{R} = \{r_1, r_2, r_3\}$. Similar to the east road network case, we fit the latency function $l_e(f_e)$ to real data for each edge e in Fig. 4.3 and get the value of a_e, b_e . Then, we can calculate the socially optimal flow for this network is $f_{r_1}^* = 160.9258, f_{r_2}^* =$

Figure 4.6: The relationship between μ and POA for road network in Fig. 4.2

226.7071, $f_{r_3}^* = 112.3671$ with $L(f^*) = 12532.291$. It is easy to check that this network satisfies the condition (i) in Theorem 4.2. Therefore, we can find $\mu^* = 4$ such that $POA=1$, which is shown in Fig. 4.7. At the Nash flow, only car group chooses more than one routes, i.e., r_1, r_2, r_3 . The motorcycle group chooses route r_3 and both taxi and bus groups choose route r_1 . The total latency without toll is $L(f^{ne}(0)) = 12547.216$ and with toll is $L(f^{ne}(\mu^*)) = 12532.291$. Obviously, the designed toll improves the social welfare.

Figure 4.7: The relationship between μ and POA for road network in Fig. 4.3

Remark 4.3. The simulation results of the above examples coincide with conditions

in Theorem 4.2 since $A_k, B_k, k = 1, N$ are related with the topology and parameters $a_e, b_e, e \in \mathcal{E}$ of the network. To be precise, whether the POA can reach 1 depends on the distribution of $\mathcal{B}$, the topology and parameters of the network.

4.4 Conclusions

In this chapter, the Nash flow and the POA in networks with one origin-destination pair have been analyzed. The scaled marginal-cost pricing has been introduced to optimize the POA for the case where all players in the noncooperative congestion game have heterogeneous price sensitivities. For the two groups and two routes case, it has been shown that the social optimum can always be achieved after charging the designed toll on each link. For general case, if the distribution of price sensitivity satisfies certain conditions, the designed toll can guarantee that the unique Nash flow approaches the optimal flow, i.e., $\text{POA} = 1$. However, the optimal POA can't always reach 1. For any network with one origin-destination pair and any distribution of price sensitivity, an algorithm was introduced to find a price scheme to minimize the POA. The numerical examples and real data simulations further validated our results and showed that whether the POA can achieve 1 depends on the distribution of $\mathcal{B}$, the topology and parameters of the network.

Chapter 5

Analysis of Efficiency Loss in Repeated Games via Smoothness Arguments

In this chapter, we analyze the efficiency loss of a sequence of action profiles generated by repeated play. In Section 5.1, a network with multiple origin-destination pairs is described. An upper bound of POTA for almost sure no-regret sequences is given in Section 5.2 via smoothness arguments. In Section 5.3, we analyze the influence of dynamic pricing on POTA for networks with linear latency functions and discuss how inaccurate parameter information affects the upper bound of POTA. In Section 5.4, the upper bound of POTA for nonlinear latency case is shown. We give an upper bound of POTA for networks with heterogeneous players in Section 5.5. In Section 5.6, the results are applied to a traffic routing problem via simulations. Section 5.7 concludes this chapter.

5.1 Problem Formulation

Consider a network $(\mathcal{V}, \mathcal{E})$ with multiple origin-destination pairs, where $\mathcal{V}$ is the vertex set and $\mathcal{E}$ is the edge set (e.g., Fig. 3.9). Denote $|\mathcal{E}|$ as the cardinality of $\mathcal{E}$ with

$|\mathcal{E}| > 1$. Suppose there are K origin-destination vertex pairs $(o_1, d_1), \dots, (o_K, d_K)$. For each origin-destination pair (o_j, d_j) , there are $\tilde{N}_j$ players routing from o_j to d_j . Denote $\mathcal{N} = \{1, \dots, N\}$ as the set of players and $\sum_{j=1}^K \tilde{N}_j = N$. Let $\mathcal{R}_j$ denote the set of routes from o_j to d_j and $\bigcup_{j=1}^K \mathcal{R}_j = \mathcal{R}$. Each player i selects a route s_i from its strategy set S_i and denote $S = \times_{i \in \mathcal{N}} S_i$ as the strategy space of all players. Note that players in the same origin-destination pair have the same strategy set. Let $s = \{s_1, \dots, s_N\}$ be the action profile of all players. Denote $s_{-i} = \{s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_N\}$. The latency $l_e(f_e)$ of each link $e \in \mathcal{E}$ is associated with the number of players on this link, where $f_e = \sum_{i=1}^N \mathbf{I}\{e \in s_i\}$ is the number of players on link e . Here, function $\mathbf{I}\{\cdot\}$ is the indicator function. Define $f_r = \sum_{i=1}^N \mathbf{I}\{s_i = r\}$ as the number of players on route r and $f_{\mathcal{R}} = \{f_r : r \in \mathcal{R}\}$. Obviously, $f_e = \sum_{r \in \mathcal{R}: e \in r} f_r$. The latency received by player i by choosing route $s_i \in S_i$ is the sum of the latencies of links on this route:

$$C_i(s) = \sum_{e \in s_i} l_e(f_e) \quad (5.1)$$

Based on the definition of (5.1), $C_i(s)$ can be also denoted as $C_i(s_i, f_{\mathcal{R}})$.

The total latency of the network is

$$C(s) = \sum_{i=1}^N C_i(s) = \sum_{e \in \mathcal{E}} l_e(f_e) f_e \quad (5.2)$$

Each player i tries to minimize its latency $C_i(s)$. A strategy profile s^{ne} is called a pure strategy Nash equilibrium, if for each player i ,

$$C_i(s^{ne}) = \min_{s_i \in S_i} C_i(s_i, s_{-i}^{ne}) \quad (5.3)$$

Note that the congestion game we formulated above always possesses at least one Nash equilibrium [27].

A socially optimal strategy s^* of the network is

$$s^* = \arg \inf_{s \in S} C(s) \quad (5.4)$$

In Chapter 4, we discussed POA, which characterizes the worst-case efficiency loss of all possible Nash equilibria. According to the definition, the POA for this model can be described as

$$\text{POA} = \sup_{s^{ne} \in S} \frac{C(s^{ne})}{C(s^*)} \quad (5.5)$$

In this chapter, we analyze the POA through smoothness arguments. The following definition gives the concept of (λ, μ) -smoothness in a cost-minimization game, in which the objective of each player is to minimize its cost function.

Definition 5.1. *A cost-minimization game is (λ, μ) -smooth if for every two strategies s^1 and s^2 ,*

$$\sum_{i=1}^N C_i(s_i^2, s_{-i}^1) \leq \lambda \cdot C(s^2) + \mu \cdot C(s^1) \quad (5.6)$$

Specially, for a Nash equilibrium s^{ne} and the optimal strategy s^* , we have

$$C(s^{ne}) = \sum_{i=1}^N C_i(s^{ne}) \leq \sum_{i=1}^N C_i(s_i^*, s_{-i}^{ne}) \leq \lambda \cdot C(s^*) + \mu \cdot C(s^{ne}) \quad (5.7)$$

The first inequality follows from the definition of Nash equilibrium. If a game is (λ, μ) -smooth with $\lambda > 0, \mu < 1$, then $\text{POA} \leq \frac{\lambda}{1-\mu}$.

Remark 5.1. *Besides Nash equilibrium and optimal strategy, inequality (5.6) holds for any strategies s^1 and s^2 . Therefore, the worst-case bound $\frac{\lambda}{1-\mu}$ generated through smoothness arguments may be larger than the supremum of POA.*

To identify the best upper bound of POA that is provable via smoothness arguments, we define the robust POA as follows [8].

Definition 5.2. *The robust POA of a cost-minimization game is*

$$\inf\left\{\frac{\lambda}{1-\mu} : (\lambda, \mu) \text{ such that the game is } (\lambda, \mu) - \text{smooth}\right\} \quad (5.8)$$

with $\mu < 1$.

Remark 5.2. *Given $\varepsilon \in (0, \frac{1}{\mu} - 1)$, for the ε -Nash equilibrium, defined as the strategy profile s^ε such that $C_i(s^\varepsilon) \leq (1 + \varepsilon)C_i(s_i, s_{-i}^\varepsilon)$ for any player i and strategy s_i , its total latency is at most $\frac{(1+\varepsilon)\lambda}{1-\mu(1+\varepsilon)}$ times that of any socially optimal assignment.*

5.2 The Price of Total Anarchy

The POA only considers the efficiency loss of the worst-case Nash equilibrium rather than the efficiency loss of the evolutionary process in repeated play. Therefore, we adopt the price of total anarchy (POTA) in this section.

Consider a sequence of strategies $s(1), s(2), \dots, s(T)$ generated by repeated play. For each player i and stage t , define

$$\delta_i(s(t)) = C_i(s(t)) - \min_{s_i \in S_i} C_i(s_i, s_{-i}(t)) \quad (5.9)$$

$$\delta_i^*(s(t)) = C_i(s(t)) - C_i(s_i^*, s_{-i}(t)) \quad (5.10)$$

where s^* is any optimal strategy. Obviously, $\delta_i(s(t)) \geq \delta_i^*(s(t))$.

For any smooth game, according to (5.6), we have

$$C(s(t)) \leq \frac{\lambda}{1-\mu} C(s^*) + \frac{\sum_{i=1}^N \delta_i^*(s(t))}{1-\mu} \quad (5.11)$$

Therefore, the average total latency satisfies:

$$\frac{1}{T} \sum_{t=1}^T C(s(t)) \leq \frac{\lambda}{1-\mu} C(s^*) + \frac{1}{1-\mu} \frac{1}{T} \sum_{t=1}^T \sum_{i=1}^N \delta_i^*(s(t)) \quad (5.12)$$

Assume that each player i uses the total number of other players choosing each route $r \in \mathcal{R}$ at the previous stage to estimate its action at the present stage. The best response of player i at stage t is defined as:

$$BR_i(f_{\mathcal{R}}(t-1)) := \{\bar{s}_i \in S_i : C_i(\bar{s}_i, f_{\mathcal{R}}(t-1)) = \min_{s_i \in S_i} C_i(s_i, f_{\mathcal{R}}(t-1))\} \quad (5.13)$$

Similar to Chapter 3, we adopt some kind of inertia to players' decision making progress. In the presence of inertia, player i stays with the previous action, i.e., $s_i(t) = s_i(t-1)$, if there is no opportunity for cost reduction, i.e., $s_i(t-1) \in BR_i(f_{\mathcal{R}}(t-1))$. Otherwise, player i will choose an action from $BR_i(f_{\mathcal{R}}(t-1))$ with probability $\theta_i(t)$. That is to say, player i will stay with the previous action $s_i(t-1)$ with probability $1 - \theta_i(t)$, even when there is an opportunity for cost reduction.

Assumption 5.1. *There exist constants η_1 and η_2 such that, for every stage $t \geq 1$ and for any player i ,*

$$0 < \eta_1 \leq \theta_i(t) \leq \eta_2 < 1 \quad (5.14)$$

Lemma 5.1. *Under Assumption 5.1, the action profiles generated by best response with inertia will converge to a Nash equilibrium almost surely in finite steps.*

Proof: It is shown in Chapter 3 that the action profiles generated by best response with inertia is convergent to a Nash equilibrium almost surely. Thus, we only need to prove that the Nash equilibrium can be achieved in finite steps.

If $s(1)$ is a Nash equilibrium, the proof is completed. Otherwise, there exists at least one player who haven't achieved its minimal cost. For players who still have opportunities to reduce their costs at time $t > 1$, denote $\tilde{n}_1(t)$ as the number of players who will choose actions from their best response sets and $\tilde{n}_2(t)$ as the number of players who will stay with the previous actions. The probability of the above event is $\eta_2^{\tilde{n}_1(t)}(1 - \eta_1)^{\tilde{n}_2(t)}$. If the Nash equilibrium is achieved at time T_0 , the corresponding probability is at most $\prod_{t=1}^{T_0-1} (\eta_2^{\tilde{n}_1(t)}(1 - \eta_1)^{\tilde{n}_2(t)})$. Since $\lim_{T_0 \rightarrow \infty} \prod_{t=1}^{T_0-1} (\eta_2^{\tilde{n}_1(t)}(1 - \eta_1)^{\tilde{n}_2(t)}) = 0$, the Nash equilibrium can be achieved in finite steps almost surely. ■

Definition 5.3. A strategy sequence exhibits an almost sure ε -no-regret property if and only if for every player $i \in \mathcal{N}$, there exists a $\varepsilon > 0$ such that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \delta_i(s(t)) < \varepsilon \quad a.s. \quad (5.15)$$

If the sequence exhibits the almost sure ε -no-regret property for any $\varepsilon > 0$, then it is said to be almost sure no-regret.

Remark 5.3. Note that the almost sure ε -no-regret property can be also defined as, for any player i and any possible strategy $s_i \in S_i$,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \delta_i(s_i, s_i(t) | s_{-i}(t)) < \varepsilon \quad a.s. \quad (5.16)$$

where $\delta_i(s_i, s_i(t) | s_{-i}(t)) = C_i(s(t)) - C_i(s_i, s_{-i}(t))$.

From the definition of $\delta_i(s(t))$ and $\delta_i(s_i, s_i(t) | s_{-i}(t))$, it is manifest that (5.15) is equivalent to (5.16).

Based on Lemma 5.1, if the action profiles are generated by best response with inertia, there exists a $t_0 > 0$ such that $\sum_{i=1}^N C_i(s(t)) - \sum_{i=1}^N \min_{s_i \in S_i} C_i(s_i, s_{-i}(t)) = 0$ for all $t \geq t_0$.

For any $T \geq t_0$, there exists a $\varepsilon_T > 0$ such that

$$\frac{1}{T} \sum_{t=1}^T \sum_{i=1}^N \delta_i(s(t)) = \frac{1}{T} \sum_{t=1}^T \left(\sum_{i=1}^N C_i(s(t)) - \sum_{i=1}^N \min_{s_i \in S_i} C_i(s_i, s_{-i}(t)) \right) \leq \varepsilon_T \quad a.s. \quad (5.17)$$

where ε_T decreases with respect to T and $\varepsilon_T \rightarrow 0$ as $T \rightarrow \infty$.

Note that as $T \rightarrow \infty$, the strategy profiles generated by best response with inertia possess almost sure no-regret property.

The price of total anarchy (POTA) is defined as the ratio between the average total latency and the total latency of the optimal assignment, i.e.,

$$\text{POTA} = \frac{\frac{1}{T} \sum_{t=1}^T C(s(t))}{C(s^*)} \quad (5.18)$$

Lemma 5.2. *Under Assumption 5.1, if the congestion game is (λ, μ) -smooth with robust POA ρ and the action profiles $s(1), s(2), \dots, s(T)$ are generated by best response with inertia, then, for any $T \geq t_0$, there exists a $\varepsilon_T > 0$ such that $\text{POTA} \leq \rho + \frac{\varepsilon_T}{(1-\mu)C(s^*)}$ almost surely, where $\varepsilon_T \rightarrow 0$ as $T \rightarrow \infty$.*

Proof. If action profiles $s(1), s(2), \dots, s(T)$ are generated by best response with inertia, under Assumption 5.1, (5.17) is satisfied. Since $\delta_i(s(t)) \geq \delta_i^*(s(t))$, according to (5.12), for any $T \geq t_0$, we have

$$\frac{1}{T} \sum_{t=1}^T C(s(t)) \leq \frac{\lambda}{1-\mu} C(s^*) + \frac{\varepsilon_T}{1-\mu} \quad \text{a.s.} \quad (5.19)$$

Therefore, for the robust POA ρ , $\text{POTA} \leq \rho + \frac{\varepsilon_T}{(1-\mu)C(s^*)}$ almost surely. $\blacksquare$

Remark 5.4. *As shown in Section 5.3 and 5.4, for networks with linear and nonlinear latency functions, we can find λ and μ such that the cost-minimization game is (λ, μ) -smooth. As a result, we can analyze the POTA for these cases via smoothness arguments.*

Remark 5.5. *For any almost sure no-regret sequence, as T goes to infinity, $\text{POTA} \leq \rho$ almost surely.*

5.3 Networks with Linear Latency

In this section, we consider a network with a linear latency function on each edge. Set the latency on edge $e \in \mathcal{E}$ as

$$l_e(f_e) = a_e f_e + b_e \quad (5.20)$$

where $a_e > 0, b_e \geq 0$. That is, the latency on each edge will increase with the growth of flow on this edge.

It is proved in [141] that, for any nonnegative integers y, z ,

$$y(z + 1) \leq \frac{5}{3}y^2 + \frac{1}{3}z^2 \quad (5.21)$$

Then, it is easy to verify that networks with linear latency functions is a $(\frac{5}{3}, \frac{1}{3})$ -smooth game. Then we have the following lemma.

Corollary 5.1. *In congestion games with linear latency function (5.20), if the action profiles $s(1), s(2), \dots, s(T)$ are generated by best response with inertia, then, under Assumption 5.1, $POTA \leq \frac{5}{2} + \frac{3\varepsilon_T}{2C(s^*)}$ for any $T \geq t_0$ almost surely, where $\varepsilon_T \rightarrow 0$ as $T \rightarrow \infty$.*

Proof: In congestion games with linear latency, $\lambda = 5/3, \mu = 1/3$. Substitute λ, μ into (5.19), we get the upper bound of POTA. ■

Remark 5.6. *For the cases we discussed in Sections 5.3 and 5.4, the optimal strategy s^* is unique due to the convex latency function.*

5.3.1 Analysis of POTA with Dynamic Pricing in Linear Latency Networks

Suppose the authority charges a dynamic pricing on each edge to affect players' travel behaviors and improve the network efficiency. For the linear latency (5.20), motivated by the pricing scheme in [42], we assume the toll for each edge $e \in \mathcal{E}$ as

$$p_e(f_e) = w_e(f_e - 1) \quad (5.22)$$

where w_e is a nonnegative constant to be designed.

To reduce the efficiency loss of the network, we aim to find $\{w_e^*, e \in \mathcal{E}\}$ to optimize the upper bound of the POTA.

After charging the dynamic pricing (5.22), the cost that player i tries to minimize becomes

$$U_i(s) = \sum_{e \in S_i} (l_e(f_e) + p_e(f_e)) \quad (5.23)$$

and the best response of player i at stage t is modified as:

$$BR_i(f_{\mathcal{R}}(t-1)) := \{\bar{s}_i \in S_i : U_i(\bar{s}_i, f_{\mathcal{R}}(t-1)) = \min_{s_i \in S_i} U_i(s_i, f_{\mathcal{R}}(t-1))\} \quad (5.24)$$

Consider a sequence $s(1), s(2), \dots, s(T)$ generated by repeated play. For each player i and stage t , define

$$\delta_{i,p}(s(t)) = U_i(s(t)) - \min_{s_i \in S_i} U_i(s_i, s_{-i}(t)) \quad (5.25)$$

$$\delta_{i,p}^*(s(t)) = U_i(s(t)) - U_i(s_i^*, s_{-i}(t)) \quad (5.26)$$

where s^* is the optimal strategy.

According to Lemma 5.1, under Assumption 5.1, in the network with dynamic pricing (5.22), the action profiles generated by best response with inertia will converge to Nash equilibrium almost surely in finite steps. Therefore, there exists a $t_0 > 0$ such that $\sum_{i=1}^N U_i(s(t)) - \sum_{i=1}^N \min_{s_i \in S_i} U_i(s_i, s_{-i}(t)) = 0$ for all $t \geq t_0$.

For any $T \geq t_0$, there exists a $\tau_T \geq 0$ such that

$$\frac{1}{T} \sum_{t=1}^T \sum_{i=1}^N \delta_{i,p}(s(t)) = \frac{1}{T} \sum_{t=1}^T \left(\sum_{i=1}^N U_i(s(t)) - \sum_{i=1}^N \min_{s_i \in S_i} U_i(s_i, s_{-i}(t)) \right) \leq \tau_T \quad \text{a.s.} \quad (5.27)$$

where τ_T decreases with respect to T and $\tau_T \rightarrow 0$ as $T \rightarrow \infty$.

Obviously, after adding the dynamic pricing on each edge, the strategy profiles generated by best response with inertia still possess almost sure no-regret property as T approaches infinity.

Note that w_e can be viewed as the additional congestion cost due to one more player on link e and b_e the cost due to free-flow travel time of link e . In practice, b_e should be much larger than w_e since the increased cost (travel time) by one more player on each link is much smaller than the free-flow travel time of this link. Therefore, we have the following assumption.

Assumption 5.2. $\bar{w} \leq (\frac{2}{3} + \frac{3\bar{w}}{8\bar{a}})\underline{b}$, where $\bar{w} = \max_{e \in \mathcal{E}} w_e$, $\underline{w} = \min_{e \in \mathcal{E}} w_e$, $\bar{a} = \max_{e \in \mathcal{E}} a_e$, $\underline{b} = \min_{e \in \mathcal{E}} b_e$

Theorem 5.1. *For congestion game with linear latency function (5.20) and dynamic pricing (5.22), if the action profiles $s(1), s(2), \dots, s(T)$ are generated by best response with inertia, then, under Assumptions 5.1 and 5.2, $w_e^* = a_e$ for each edge $e \in \mathcal{E}$ and $POTA \leq \frac{49}{24} + \frac{\tau_T}{C(s^*)}$ for any $T \geq t_0$ almost surely, where $\tau_T \rightarrow 0$ as $T \rightarrow \infty$.*

Proof. It is easy to verify that, for any nonnegative y, z ,

$$yz \leq \frac{2}{3}z^2 + \frac{3}{8}y^2 \quad (5.28)$$

For any two strategies s^1, s^2 , note that $f_e^1 = \sum_{i=1}^N \mathbf{I}\{e \in s_i^1\}$, $f_e^2 = \sum_{i=1}^N \mathbf{I}\{e \in s_i^2\}$. Since $\sum_{e \in \mathcal{E}} f_e^1 = \sum_{e \in \mathcal{E}} f_e^2 = N$, under Assumption 5.2, we have

$$\sum_{e \in \mathcal{E}} w_e f_e^1 \leq \sum_{e \in \mathcal{E}} (\frac{2}{3} + \frac{3w_e}{8a_e}) b_e f_e^2 \quad (5.29)$$

Since the number of players using resource e in the outcome (s_i^2, s_{-i}^1) is at most one more than that in s^1 , according to (5.21), (5.28) and (5.29),

(i) if $w_e \geq a_e$ for all $e \in \mathcal{E}$, for any strategy s^1, s^2 ,

$$\begin{aligned} \sum_{i=1}^N U_i(s_i^2, s_{-i}^1) &\leq \sum_{e \in \mathcal{E}} f_e^2 (l_e(f_e^1 + 1) + p_e(f_e^1 + 1)) \\ &\leq \sum_{e \in \mathcal{E}} ((\frac{5}{3} + \frac{3w_e}{8a_e})(a_e f_e^2 + b_e) f_e^2 + w_e(f_e^1 - 1)f_e^1) \\ &\leq (\frac{5}{3} + \frac{3}{8}w_0)C(s^2) + \sum_{e \in \mathcal{E}} p_e(f_e^1)f_e^1 \end{aligned} \quad (5.30)$$

where $w_0 = \max_{e \in \mathcal{E}} \frac{w_e}{a_e} \geq 1$.

According to (5.30), we have

$$C(s(t)) \leq \left(\frac{5}{3} + \frac{3}{8}w_0\right)C(s^*) + \sum_{i=1}^N \delta_{i,p}^*(s(t)) \quad (5.31)$$

Thus

$$\frac{1}{T} \sum_{t=1}^T C(s(t)) \leq \left(\frac{5}{3} + \frac{3}{8}w_0\right)C(s^*) + \frac{1}{T} \sum_{t=1}^T \sum_{i=1}^N \delta_{i,p}^*(s(t)) \quad (5.32)$$

For action profiles generated by best response with inertia, (5.27) is satisfied. Since $\delta_{i,p}(s(t)) \geq \delta_{i,p}^*(s(t))$, according to (5.32), for any $T \geq t_0$, we have

$$\frac{1}{T} \sum_{t=1}^T C(s(t)) \leq \left(\frac{5}{3} + \frac{3}{8}w_0\right)C(s^*) + \tau_T \quad \text{a.s.} \quad (5.33)$$

i.e., $\text{POTA} \leq \frac{5}{3} + \frac{3}{8}w_0 + \frac{\tau_T}{C(s^*)}$ almost surely.

We can check that (5.33) reaches its minimal point at $w_e^* = a_e$ for all $e \in \mathcal{E}$ with $\text{POTA} \leq \frac{49}{24} + \frac{\tau_T}{C(s^*)}$.

(ii) if there exists at least one edge $e \in \mathcal{E}$ such that $w_e \leq a_e$, denote the edge set as $\mathcal{E}_0 = \{e \in \mathcal{E} | w_e \leq a_e\}$, for any strategy s^1, s^2 ,

$$\begin{aligned} \sum_{i=1}^N U_i(s_i^2, s_{-i}^1) &\leq \sum_{e \in \mathcal{E}} f_e^2 (l_e(f_e^1 + 1) + p_e(f_e^1 + 1)) \\ &\leq \sum_{e \in \mathcal{E}} \left(\left(\frac{5}{3} + \frac{3w_e}{8a_e} \right) (a_e f_e^2 + b_e) f_e^2 + w_e (f_e^1 - 1) f_e^1 \right) \\ &\quad + \sum_{e \in \mathcal{E}_0} \frac{1}{3} \left(1 - \frac{w_e}{a_e} \right) (a_e f_e^1 + b_e) f_e^1 \\ &\leq \left(\frac{5}{3} + \frac{3}{8}w_0 \right) C(s^2) + \frac{1}{3} (1 - q_0) C(s^1) + \sum_{e \in \mathcal{E}} p_e(f_e^1) f_e^1 \end{aligned} \quad (5.34)$$

where $q_0 = \min_{e \in \mathcal{E}_0} \frac{w_e}{a_e} \leq 1$.

According to (5.34), we have

$$\frac{1}{T} \sum_{t=1}^T C(s(t)) \leq \frac{5 + \frac{9w_0}{8}}{2 + q_0} C(s^*) + \frac{3}{2 + q_0} \frac{1}{T} \sum_{t=1}^T \sum_{i=1}^N \delta_{i,p}^*(s(t)) \quad (5.35)$$

Similar to (i), according to (5.35), for any $T \geq t_0$, we have

$$\frac{1}{T} \sum_{t=1}^T C(s(t)) \leq \frac{5 + \frac{9w_0}{8}}{2 + q_0} C(s^*) + \frac{3}{2 + q_0} \tau_T \quad \text{a.s.} \quad (5.36)$$

i.e., $\text{POTA} \leq \frac{5 + \frac{9w_0}{8}}{2 + q_0} + \frac{3}{2 + q_0} \frac{\tau_T}{C(s^*)}$ almost surely.

Specially, if $w_e \leq a_e$ for all $e \in \mathcal{E}$, i.e., $\mathcal{E}_0 = \mathcal{E}$, (5.36) still holds.

It is easy to check that (5.36) reaches its minimal point at $w_e^* = a_e$ for all $e \in \mathcal{E}$ with $\text{POTA} \leq \frac{49}{24} + \frac{\tau_T}{C(s^*)}$.

By combining (i) and (ii), the proof is completed. $\blacksquare$

From Theorem 5.1, if the pricing scheme is designed as $p_e(f_e) = a_e(f_e - 1)$, the upper bound of the POTA will decrease compared to that without pricing.

Remark 5.7. If we set $w_e = a_e$ for all $e \in \mathcal{E}$, Assumption 5.2 becomes $\bar{a} \leq \frac{25}{24}\underline{b}$. In the linear latency function (5.20), a_e can be viewed as the increased travel time by adding one player to link e and b_e is the free-flow travel time on link e . In reality, b_e is much larger than a_e . Therefore, Assumption 5.2 is reasonable.

5.3.2 POTA with Inaccurate Information

In Section 5.3.1, we show that if the parameter information a_e of each edge $e \in \mathcal{E}$ is known to the authority, the toll can be designed as $p_e(f_e) = a_e(f_e - 1)$ to reduce the upper bound of the POTA. However, in reality the authority may have incorrect information on a_e . In this section, we will analyze the influence of the inaccurate parameter information on the POTA.

Assume that the dynamic pricing charged by the authority on each edge $e \in \mathcal{E}$ is

$$p'_e(f_e) = \hat{a}_e(f_e - 1) \quad (5.37)$$

where $\hat{a}_e = a_e + \gamma_e$, $\gamma_e \in [-a_e, a_e]$.

The cost received by each player i becomes $U_i(s) = \sum_{e \in s_i} (l_e(f_e) + p'_e(f_e))$.

Assumption 5.3. $\underline{b} \geq 2\bar{a}$.

Corollary 5.2. *For congestion game with linear latency function (5.20) and dynamic pricing (5.37), if the action profiles $s(1), s(2), \dots, s(T)$ are generated by best response with inertia, then, under Assumptions 5.1 and 5.3, for any $T \geq t_0$, $POTA \leq \frac{49}{24} + \frac{9\bar{\gamma} + 8|\mathcal{E}|\bar{a}}{24\underline{a}} + \frac{\tau_T}{C(s^*)}$ almost surely, where $\bar{\gamma} = \max_{e \in \mathcal{E}} \gamma_e$ and $\underline{a} = \min_{e \in \mathcal{E}} a_e$, and $\tau_T \rightarrow 0$ as $T \rightarrow \infty$.*

Proof: Since $|\gamma_e| \leq \bar{a}$ for all $e \in \mathcal{E}$, by the Cauchy-Schwarz inequality, we have

$$\sum_{e \in \mathcal{E}} \left(-\frac{1}{3}\gamma_e(f_e^1)^2\right) \leq \frac{1}{3}\bar{a} \sum_{e \in \mathcal{E}} (f_e^1)^2 \leq \frac{1}{3}\bar{a} \left(\sum_{e \in \mathcal{E}} f_e^1\right)^2 = \frac{1}{3}\bar{a} \left(\sum_{e \in \mathcal{E}} f_e^2\right)^2 \leq \frac{|\mathcal{E}|}{3}\bar{a} \sum_{e \in \mathcal{E}} (f_e^2)^2 \quad (5.38)$$

Under Assumption 5.3,

$$\sum_{e \in \mathcal{E}} \left(\frac{25}{24} + \frac{3\gamma_e}{8a_e} + \frac{|\mathcal{E}|\bar{a}}{3a_e}\right) b_e f_e^2 \geq \underline{b} \sum_{e \in \mathcal{E}} f_e^2 \geq 2\bar{a} \sum_{e \in \mathcal{E}} f_e^1 \geq \sum_{e \in \mathcal{E}} (a_e + \gamma_e) f_e^1 \quad (5.39)$$

Therefore, for any strategies s^1, s^2 , based on (5.21), (5.28), we obtain

$$\begin{aligned} \sum_{i=1}^N U_i(s_i^2, s_{-i}^1) &\leq \sum_{e \in \mathcal{E}} f_e^2 (l_e(f_e^1 + 1) + p'_e(f_e^1 + 1)) \\ &\leq \left(\frac{49}{24} + \frac{9\bar{\gamma} + 8|\mathcal{E}|\bar{a}}{24\underline{a}}\right) C(s^2) + \sum_{e \in \mathcal{E}} p'_e(f_e^1) f_e^1 \end{aligned} \quad (5.40)$$

For a sequence $s(1), s(2), \dots, s(T)$ generated by repeated play, according to (5.40),

$$\frac{1}{T} \sum_{t=1}^T C(s(t)) \leq \left(\frac{49}{24} + \frac{9\bar{\gamma} + 8|\mathcal{E}|\bar{a}}{24\underline{a}}\right) C(s^*) + \frac{1}{T} \sum_{t=1}^T \sum_{i=1}^N \delta_{i,p}^*(s(t)) \quad (5.41)$$

If the action profiles are generated by best response with inertia, for any $T \geq t_0$,

$$\frac{1}{T} \sum_{t=1}^T C(s(t)) \leq \left(\frac{49}{24} + \frac{9\bar{\gamma} + 8|\mathcal{E}|\bar{a}}{24\underline{a}}\right) C(s^*) + \tau_T \quad \text{a.s.} \quad (5.42)$$

The proof is completed. ■

Obviously, the inaccurate dynamic pricing increases the upper bound of the POTA compared to that with the accurate information, which means that the inaccurate information may cause more efficiency loss compared to the case with accurate information of the latency function.

5.4 Networks with Nonlinear Latency

In this section, we consider networks with nonlinear latency functions. Motivated by the delay time formulas summarized in [6], we set the latency function $l_e(f_e)$ as

$$l_e(f_e) = t_e \cdot (1 + d_e(f_e/c_e)^{m_e}) \quad (5.43)$$

where $m_e > 0$, $d_e \geq 0$, t_e is the free-flow travel time which is a measure of the travel time at zero flow, c_e is the practical capacity which is a measure of the flow from which the travel time will increase rapidly if the flow is further increased.

For any two strategies s^1 and s^2 , we have

$$\sum_{i=1}^N C_i(s_i^2, s_{-i}^1) \leq \sum_{i=1}^N \sum_{e \in s_i^2} l_e(f_e^1 + 1) = \sum_{e \in \mathcal{E}} f_e^2 l_e(f_e^1 + 1) \quad (5.44)$$

If there exist λ, μ such that

$$l_e(f_e^1 + 1) f_e^2 \leq \lambda l_e(f_e^2) f_e^2 + \mu l_e(f_e^1) f_e^1 \quad (5.45)$$

then, according to (5.44), (5.6) is satisfied. Thus, for convenience, we only need to find λ, μ with $\mu < 1$ that satisfy (5.45).

Lemma 5.3. *For any nonnegative integers x, y and $m > 0$,*

$$(x+1)^m y \leq \frac{(\vartheta+1)^{2m+1} - \vartheta^{m+1}(\vartheta+2)^m}{(\vartheta+1)^{m+1} - \vartheta^{m+1}} y^{m+1} + \frac{(\vartheta+2)^m - (\vartheta+1)^m}{(\vartheta+1)^{m+1} - \vartheta^{m+1}} x^{m+1} \quad (5.46)$$

where $\vartheta = \lfloor \varphi_m \rfloor$ and φ_m is the unique nonnegative real solution to $(z+1)^m = z^{m+1}$.

Proof. The proof is similar to the results shown in [99] by noting that $m > 0$. $\blacksquare$

Lemma 5.4. *Let $m_0 = \max_{e \in \mathcal{E}} m_e$. A congestion game with nonlinear latency function (5.43) is a $(\frac{(\vartheta+1)^{2m_0+1} - \vartheta^{m_0+1}(\vartheta+2)^{m_0}}{(\vartheta+1)^{m_0+1} - \vartheta^{m_0+1}}, \frac{(\vartheta+2)^{m_0} - (\vartheta+1)^{m_0}}{(\vartheta+1)^{m_0+1} - \vartheta^{m_0+1}})$ -smooth game.*

Proof. According to Lemma 5.3 and latency function (5.43),

$$\begin{aligned}
l_e(f_e^1 + 1)f_e^2 &= t_e \cdot (1 + d_e((f_e^1 + 1)/c_e)^{m_e})f_e^2 \\
&\leq \frac{(\vartheta + 1)^{2m_e+1} - \vartheta^{m_e+1}(\vartheta + 2)^{m_e}}{(\vartheta + 1)^{m_e+1} - \vartheta^{m_e+1}} t_e \cdot (1 + d_e(f_e^2/c_e)^{m_e})f_e^2 \\
&\quad + \frac{(\vartheta + 2)^{m_e} - (\vartheta + 1)^{m_e}}{(\vartheta + 1)^{m_e+1} - \vartheta^{m_e+1}} t_e \cdot (1 + d_e(f_e^1/c_e)^{m_e})f_e^1 \\
&\leq \frac{(\vartheta + 1)^{2m_e+1} - \vartheta^{m_e+1}(\vartheta + 2)^{m_e}}{(\vartheta + 1)^{m_e+1} - \vartheta^{m_e+1}} l_e(f_e^2)f_e^2 \\
&\quad + \frac{(\vartheta + 2)^{m_e} - (\vartheta + 1)^{m_e}}{(\vartheta + 1)^{m_e+1} - \vartheta^{m_e+1}} l_e(f_e^1)f_e^1
\end{aligned} \tag{5.47}$$

Since both $\frac{(\vartheta+1)^{2m_e+1} - \vartheta^{m_e+1}(\vartheta+2)^{m_e}}{(\vartheta+1)^{m_e+1} - \vartheta^{m_e+1}}$ and $\frac{(\vartheta+2)^{m_e} - (\vartheta+1)^{m_e}}{(\vartheta+1)^{m_e+1} - \vartheta^{m_e+1}}$ are nondecreasing with respect to m_e , let $m_0 = \max_{e \in \mathcal{E}} m_e$, we obtain

$$\begin{aligned}
\sum_{i=1}^N C_i(s_i^2, s_{-i}^1) &\leq \sum_{e \in \mathcal{E}} f_e^2 l_e(f_e^1 + 1) \\
&\leq \frac{(\vartheta + 1)^{2m_0+1} - \vartheta^{m_0+1}(\vartheta + 2)^{m_0}}{(\vartheta + 1)^{m_0+1} - \vartheta^{m_0+1}} C(s^2) + \frac{(\vartheta + 2)^{m_0} - (\vartheta + 1)^{m_0}}{(\vartheta + 1)^{m_0+1} - \vartheta^{m_0+1}} C(s^1)
\end{aligned} \tag{5.48}$$

which is a $(\frac{(\vartheta+1)^{2m_0+1} - \vartheta^{m_0+1}(\vartheta+2)^{m_0}}{(\vartheta+1)^{m_0+1} - \vartheta^{m_0+1}}, \frac{(\vartheta+2)^{m_0} - (\vartheta+1)^{m_0}}{(\vartheta+1)^{m_0+1} - \vartheta^{m_0+1}})$ -smooth game. $\blacksquare$

Corollary 5.3. *Under Assumptions 5.1, for congestion game with nonlinear latency function (5.43), if the action profiles $s(1), s(2), \dots, s(T)$ are generated by best response with inertia, then for any $T \geq t_0$,*

$$\begin{aligned}
POTA &\leq \frac{(\vartheta + 1)^{2m_0+1} - \vartheta^{m_0+1}(\vartheta + 2)^{m_0}}{(\vartheta + 1)^{m_0+1} - (\vartheta + 2)^{m_0} + (\vartheta + 1)^{m_0} - \vartheta^{m_0+1}} \\
&\quad + \frac{((\vartheta + 1)^{m_0+1} - \vartheta^{m_0+1})\varepsilon_T}{((\vartheta + 1)^{m_0}(\vartheta + 2) - (\vartheta + 2)^{m_0} - \vartheta^{m_0+1})C(s^*)} \quad a.s.
\end{aligned} \tag{5.49}$$

where $\varepsilon_T \rightarrow 0$ as $T \rightarrow \infty$.

Proof: From Lemma 5.4, we get the value of λ, μ . Substitute λ and μ into (5.19), for any $T \geq t_0$, we obtain the upper bound of POTA. ■

Remark 5.8. Note that the upper bound of POTA is tight. We can further check that when $m_e = 1$ for all $e \in \mathcal{E}$, $POTA \leq \frac{5}{2} + \frac{3\varepsilon_T}{2C(s^*)}$, which is coincide with the linear latency case in Section 5.3.

Remark 5.9. For the nonlinear latency network, we can adopt the discrete marginal-cost toll, which is defined as $p_e^d(f_e) = (f_e - 1)(l_e(f_e) - l_e(f_e - 1))$, to optimize the POTA on an infinite time horizon. By using the pricing scheme p_e^d on each edge, we can prove that s^{ne} is exactly s^* , i.e., $POA = 1$. Furthermore, for action profiles generated by best response with inertia, POTA can reach 1 as $T \rightarrow \infty$. However, it is very hard to find the upper bound of POTA with price scheme p_e^d in finite steps.

5.5 Networks with Heterogenous Players

Different players may have different value of time. By taking the value of time into consideration, the cost received by player i is defined as follows:

$$C_i^\kappa(s) = \kappa_i \sum_{e \in s_i} l_e(f_e) \quad (5.50)$$

where $\kappa_i > 0$ measures the value of time for player i .

Theorem 5.2. Under Assumption 5.1, if the congestion game with cost function (5.50) is (λ, μ) -smooth with robust POA ρ , then, for the action profiles generated by best response with inertia, $POTA \leq \rho + \frac{\varepsilon_T}{\kappa_0(1-\mu)C(s^*)}$ almost surely for any $T \geq t_0$, where $\kappa_0 = \max_{i \in \mathcal{N}} \kappa_i$, $\varepsilon_T \rightarrow 0$ as $T \rightarrow \infty$.

Proof: Since the congestion game is (λ, μ) -smooth, there exist λ, μ satisfying (5.45).

Thus, for any s^1 and s^2 ,

$$\begin{aligned} \sum_{i=1}^N C_i^\kappa(s_i^2, s_{-i}^1) &\leq \sum_{i=1}^N \kappa_i \sum_{e \in s_i^2} l_e(f_e^1 + 1) \\ &\leq \kappa_0 \sum_{e \in \mathcal{E}} f_e^2 l_e(f_e^1 + 1) \\ &\leq \kappa_0 \lambda C(s^2) + \kappa_0 \mu C(s^1) \end{aligned} \quad (5.51)$$

where $\kappa_0 = \max_{i \in \mathcal{N}} \kappa_i$, $C(s) = \sum_{i=1}^N C_i^\kappa(s)$.

Denote

$$\delta_{i,\kappa}(s(t)) = C_i^\kappa(s(t)) - \min_{s_i \in S_i} C_i^\kappa(s_i, s_{-i}(t)) \quad (5.52)$$

$$\delta_{i,\kappa}^*(s(t)) = C_i^\kappa(s(t)) - C_i^\kappa(s_i^*, s_{-i}(t)) \quad (5.53)$$

According to (5.51), we have

$$\frac{1}{T} \sum_{t=1}^T C(s(t)) \leq \frac{\lambda}{1-\mu} C(s^*) + \frac{1}{\kappa_0(1-\mu)} \frac{1}{T} \sum_{t=1}^T \sum_{i=1}^N \delta_{i,\kappa}^*(s(t)) \quad (5.54)$$

From Lemma 5.1, under Assumption 5.1, if the action profiles are generated by best response with inertia, for any $T \geq t_0$, there exists a ε_T such that $\frac{1}{T} \sum_{t=1}^T \sum_{i=1}^N \delta_{i,\kappa}(s(t)) \leq \varepsilon_T$, where $\varepsilon_T \rightarrow 0$ as $T \rightarrow \infty$. Since $\delta_{i,\kappa}(s(t)) \geq \delta_{i,\kappa}^*(s(t))$, based on (5.54), for any $T \geq t_0$, we have

$$\frac{1}{T} \sum_{t=1}^T C(s(t)) \leq \frac{\lambda}{1-\mu} C(s^*) + \frac{\varepsilon_T}{\kappa_0(1-\mu)} \quad \text{a.s.} \quad (5.55)$$

Since the robust POA is ρ , we obtain $\text{POTA} \leq \rho + \frac{\varepsilon_T}{\kappa_0(1-\mu)C(s^*)}$ almost surely. $\blacksquare$

By comparing Theorem 5.2 and Lemma 5.2, we can see that, as T goes to infinity, the upper bound of the POTA for the cases with and without the heterogeneous weight $\kappa_i, i \in \mathcal{N}$ is the same.

5.6 Real Data Simulations in Traffic Networks

Consider a traffic network in the east of Singapore as shown in Fig. 5.1. In this network, there are two origin-destination pairs (O_1, D) and (O_2, D) . The routes from O_1 to D and O_2 to D are $\mathcal{R}_1 = \{r_1, r_2\}$ and $\mathcal{R}_2 = \{r_3, r_4\}$, respectively, where $r_1 = \{e_1, e_5, e_7\}$, $r_2 = \{e_1, e_4, e_6, e_7\}$, $r_3 = \{e_2, e_6, e_7\}$, $r_4 = \{e_3\}$. Assume that each edge possesses a linear latency (5.20). We fit the linear latency function (5.20) to real traffic data and get the value of a_e and b_e for each edge e in Fig. 5.1.

Figure 5.1: A traffic network in the east of Singapore

Assume that the number of players routing from O_1 to D and O_2 to D are 500 and 500, respectively. We can calculate that the total latency of the socially optimal assignment is 1130100. At initial stage, each player chooses its route from $\mathcal{R}_1$ or $\mathcal{R}_2$ randomly. After that, players update their strategies based on best response with inertia. As shown in Fig. 5.2 and Fig. 5.3, the action profiles of all players finally converge to a Nash equilibrium for both the cases without and with road price. The average total latency over 3000 stages without road price is 1131000 and with road price scheme (5.22) is 1130300, which indicates that the road price improves the average efficiency of the network and the POTA of the network reduces from 1.0073 to 1.0059.

Figure 5.2: Evolution of number of players on each route without price

Figure 5.3: Evolution of number of players on each route with dynamic pricing (5.22)

5.6.1 Traffic Network with Public Transportation

Assume that players can choose to drive their private cars or to take a public transport. If players choose a public transport, they will receive a fixed latency, which can be regarded as a new route with $a_e = 0$ for all edges on this route. Since players choosing public transport will not contribute to the traffic congestion, we only consider the total latency of private car drivers. In the traffic network as shown in

Fig. 5.1, we assume that the fixed latencies of the public transportation from O_1 to D and O_2 to D are 1191.1 and 1500.4, respectively. We can show that the average total latency of private car drivers over 3000 stages without road price is 790220. After adding the road price (5.22), the average total latency of private car drivers becomes 402260, which is only about 1/2 of the average total latency without road price.

5.7 Conclusions

In this chapter, we have analyzed the POTA via smoothness arguments in networks with multiple origin-destination pairs. To identify the upper bound of POTA, the concept of almost sure no-regret has been used. We have shown that the action profiles generated by best response with inertia possess almost sure no-regret property. In the formulated congestion game, it has been proved that the designed pricing schemes can improve the upper bound of POTA for linear latency networks, and the effect of inaccurate parameter information on the upper bound of POTA has been discussed. In addition, we gave the upper bound of POTA for a network with nonlinear latency functions, and the POTA in a network with heterogeneous players was also analyzed. The real data simulations were provided to validate our theoretical results.

Chapter 6

Decentralized Control Design Based on Mean Field Games

In Chapter 3, a consensus protocol is proposed to estimate the interactions among agents (players). In this chapter, we study mean field games, which provide another technique to estimate the mass information in large population dynamic games. A stochastic multi-agent system, where each agent evolves according to a linear dynamic equation containing the input average of all agents, is formulated in Section 6.1. In this system, each agent aims to minimize its long time average (LTA) cost coupling with other agents' states via a population state average (PSA), which is also known as the mean field term. In Section 6.2, an approximate PSA is constructed via the Nash certainty equivalence (NCE) principle [57], and the properties of the resulting decentralized mean field controls are discussed. In addition, the influence of inaccurate mean field information on individual agent is analyzed and a nonlinear system with similar properties as the linear system considered above is studied. In Section 6.3, we introduce the socially optimal model, where the objective of each agent is to minimize the social cost as the sum of all agents' LTA costs containing the PSA. We show that the decentralized mean field social control is identical to the decentralized mean field Nash control for an infinite population system. In Section 6.4, we present some numerical simulations to verify our results. Conclusions are stated in Section 6.5.

6.1 Problem Formulation

We consider the following N -agent discrete-time linear dynamic system, where the dynamic equation for agent $i, 1 \leq i \leq N$ is described by:

$$x_i(t+1) = x_i(t) + Q_1 u_i(t) + Q_2 \frac{\sum_{i=1}^N u_i(t)}{N} + w_i(t+1) \quad (6.1)$$

where $x_i(t) \in \mathbb{R}$ is the state; $u_i(t) \in \mathbb{R}^{M \times 1}$ is the control input; $w_i(t)$ is the random noise; $Q_1, Q_2 \in \mathbb{R}^{1 \times M}$ are time-invariant parameters; $\bar{u}(t) = \frac{\sum_{i=1}^N u_i(t)}{N}$ denotes the input average. The initial states $\{x_i(0), i \in \mathcal{N}\}$ are mutually independent and are also independent of $\{w_i, i \in \mathcal{N}\}$, where $\mathcal{N} = \{1, \dots, N\}$ is the agent set. In addition, the initial state satisfies $\mathbb{E}x_i(0) < \infty$.

The objective of each agent in the noncooperative game is to minimize its individual long time average (LTA) cost function containing a population state average (PSA):

$$U_i(u_i, u_{-i}) = \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} \left(q(x_i(t) - \frac{\sum_{i=1}^N x_i(t)}{N})^2 + u_i(t)' R u_i(t) \right) \quad (6.2)$$

where q is a positive constant, $R \in \mathbb{R}^{M \times M}$ is a positive diagonal matrix, $u_{-i} = \{u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_N\}$. Here, the PSA $\bar{x}(t) = \frac{\sum_{i=1}^N x_i(t)}{N}$ is also known as the mean field term. In order to minimize $U_i(u_i, u_{-i})$, the admissible control set of agent i is taken as $\mathcal{U}_i = \{u_i | u_i(t) \text{ is adapted to } \sigma(x_j(s) : 0 \leq s \leq t, j \in \mathcal{N})\}$, where $\sigma(\mathcal{X})$ represents the σ -algebra generated by the set $\mathcal{X}$. Specially, for a decentralized control system, the admissible control set of agent i is defined as $\mathcal{U}_{i,d} = \{u_i | u_i(t) \text{ is adapted to } \sigma(x_i(s), 0 \leq s \leq t)\}$.

For the dynamic system formulated above, we investigate a set of decentralized controls $\{u_i^*, i \in \mathcal{N}\}$ where $u_i^* \in \mathcal{U}_{i,d}$ is only a function of time t and local information x_i . In addition, $\{u_i^*, i \in \mathcal{N}\}$ possesses the property of almost sure asymptotic Nash equilibrium with respect to the costs $\{U_i, i \in \mathcal{N}\}$.

In the following, we introduce a motivational example.

Example 6.1. *The input average in (6.1) can be justified by several practical scenarios, e.g., 1) for traffic assignment problem, the cost of each individual driver at the next step, $x_i(t+1)$, is a function of the average behavior of all other drivers, $\frac{\sum_{i=1}^N u_i(t)}{N}$; 2) for factory production, the production level of each manufactory, $x_i(t+1)$, is associated with the average production actions of all manufactories, $\frac{\sum_{i=1}^N u_i(t)}{N}$.*

Take the production output for example. Assume that there are a large number of factories supplying the same product to the market. For any factory $i, 1 \leq i \leq N$, let $x_i \in \mathbb{R}$ be the production level of factory i and $u_i \in \mathbb{R}$ be the prior production decision of increasing or decreasing the production level x_i . The average production decisions $\frac{\sum_{i=1}^N u_i(t)}{N}$ provides each factory the information about the changes of market demand. Therefore, the production level x_i of factory i is adjusted based on both factory i 's myopic production decision u_i and the macroscopic market demand, i.e., $\frac{\sum_{i=1}^N u_i(t)}{N}$. Precisely speaking, the production level x_i of factory i evolves according to the following dynamics:

$$x_i(t+1) = x_i(t) + a_1 u_i(t) + a_2 \frac{\sum_{i=1}^N u_i(t)}{N} + w_i(t+1) \quad (6.3)$$

where a_1, a_2 are constant; w_i denotes the uncertainty of the production level, which may result from mechanical breakdown, sudden blackout, etc.

Since increasing product price implies more market demand and decreasing product price implies less demand, suppose each factory seeks a production level which is approximately proportional to the current market price [58], i.e.,

$$x_i \approx \bar{\alpha} p_0 \quad (6.4)$$

with the price p_0 of the product defined as

$$p_0 = \eta_0 - \gamma_0 \left(\frac{\sum_{i=1}^N x_i}{N} \right) \quad (6.5)$$

where $\bar{\alpha} > 0, \eta_0 > 0$ and $\gamma_0 > 0$.

According to (6.4) and (6.5), each factory will receive a penalty due to the bias between x_i and $\bar{\alpha}p_0$, which is denoted by $q(x_i(t) - \bar{\alpha}(\eta_0 - \gamma_0(\frac{\sum_{i=1}^N x_i}{N})))^2$. On the other hand, to adjust production level x_i , factory i will face the cost due to shutting down or restarting production lines. We denote the cost of adjustment corresponding to u_i by $r_0 u_i^2$, $r_0 > 0$. Thus, the LTA cost function of factory i is formulated by:

$$U_i(u_i, u_{-i}) = \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} \left(q(x_i(t) - \bar{\alpha}(\eta_0 - \gamma_0 \bar{x}(t)))^2 + r_0 u_i^2(t) \right) \quad (6.6)$$

which falls into function (6.2).

6.2 Mean Field Term Approximation And Decentralized Control Design

For the minimization of the LTA cost function (6.2), the control of agent i must depend on the PSA $\bar{x}(t)$. In general, the states of other agents may not be available to agent i . Therefore, we adopt the NCE (Nash certainty equivalence) principle [57] to ensure the control law to be decentralized by constructing an estimate $\psi(t)$ of the PSA. We will show that $\psi(t)$ converges to the PSA when N approaches infinity, and the decentralized control law can be designed by using $\psi(t)$ instead of $\bar{x}(t)$.

Firstly, we rewrite the dynamic equation (6.1) to a form containing PSA.

Motivated by the formula of optimal control law in [63, 142], we set the mean field control law of system (6.1)-(6.2) as

$$u_i^*(t) = K_1 x_i(t) + K_2(t) \quad (6.7)$$

where $K_1, K_2(t) \in \mathbb{R}^{M \times 1}$. Hence, $\bar{u}(t) = K_1 \bar{x}(t) + K_2(t)$ and (6.1) becomes:

$$x_i(t+1) = x_i(t) + Q_1 u_i(t) + Q_2 (K_1 \bar{x}(t) + K_2(t)) + w_i(t+1) \quad (6.8)$$

Secondly, we denote $\psi(t)$ as an estimate of the PSA, and design the decentralized control law according to it.

Based on (6.8), the auxiliary dynamic equation of player i is given by:

$$x_i(t+1) = x_i(t) + Q_1 u_i(t) + Q_2 (K_1 \psi(t) + K_2(t)) + w_i(t+1) \quad (6.9)$$

with the revised LTA cost function:

$$J_i(u_i, \psi(t)) = \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} \left(q(x_i(t) - \psi(t))^2 + u_i(t)' R u_i(t) \right) \quad (6.10)$$

Before calculating the decentralized optimal control of system (6.9)-(6.10), we make the following assumption.

Assumption 6.1. $\{w_i(t), 1 \leq i \leq N\}$ is a sequence of mutually independent noises satisfying:

- 1) $\mathbb{E} w_i(t) = 0$,
- 2) $\mathbb{E}(w_i(t) w_i(s)) = r_w$ if $t = s$, and $\mathbb{E}(w_i(t) w_i(s)) = 0$ otherwise,
- 3) $\mathbb{E}(w_i(t) w_j(s)) = 0$ for all $i \neq j$.

Lemma 6.1. Under Assumption 6.1, for the optimal control problem (6.9)-(6.10), the algebraic Riccati equation

$$\frac{1}{2} Q_1 R^{-1} Q_1' p^2 - Q_1 R^{-1} Q_1' q p - 2q = 0 \quad (6.11)$$

has a unique positive solution. The optimal control law $u_i^* = \arg \min_{u_i \in \mathcal{U}_i} J_i$ is given by

$$u_i^*(t) = -\frac{1}{2} R^{-1} Q_1' (p - 2q) x_i(t) - \frac{1}{2} R^{-1} Q_1' (2q \psi(t) - h(t)) \quad (6.12)$$

where $h(t)$ satisfies:

$$h(t+1) = (1 + Mp) h(t) - Mp^2 \psi(t) \quad (6.13)$$

with $M = (Q_1 + Q_2)R^{-1}Q'_1/2$.

Moreover, the unique bounded solution of (6.13) is given by:

$$\hbar(t) = \sum_{j=t}^{\infty} (1 + Mp)^{t-j-1} Mp^2 \psi(j) \quad (6.14)$$

Proof: Define the Hamilton's equations of system (6.9)-(6.10) as:

$$\begin{aligned} H(t) = & q(x_i(t) - \psi(t))^2 + u'_i(t)Ru_i(t) \\ & + \lambda_i(t+1) \left(x_i(t) + Q_1u_i(t) + Q_2(K_1\psi(t) + K_2(t)) \right) \end{aligned} \quad (6.15)$$

then, we obtain

$$\frac{\partial H(t)}{\partial u_i(t)} = 2Ru_i(t) + Q'_1\lambda_i(t+1) = 0 \quad (6.16)$$

$$\lambda_i(t) = \frac{\partial H(t)}{\partial x_i(t)} = 2q(x_i(t) - \psi(t)) + \lambda_i(t+1) \quad (6.17)$$

Since the first term of u_i^* is a linear function of x_i , assume $\lambda_i(t) = p(t)x_i(t) - \hbar(t)$. According to (6.1), (6.16), (6.17), and note that $T \rightarrow +\infty$, we get equation (6.11), (6.12) and (6.13).

Note that $1 + Mp > 1$, $\sum_{j=0}^{\infty} (1 + Mp)^{-j-1}$ is bounded. Thus, with the bounded initial condition

$$\hbar(0) = \sum_{j=0}^{\infty} (1 + Mp)^{-j-1} Mp^2 \psi(j) \quad (6.18)$$

we get the unique bounded solution of (6.13). ■

Intuitively, the expectation of the mean field term $\bar{x}$ should be an approximation of itself. Therefore, we set $\psi(t) = \frac{1}{N} \sum_{i=1}^N \mathbb{E}x_i^*(t)$, where x_i^* is the corresponding solution to the optimal control (6.12). In the following corollary, we show that $\psi(t)$ remains constant and equals $\psi(0)$.

Corollary 6.1. *Under Assumption 6.1, if $\psi(t) = \frac{1}{N} \sum_{i=1}^N \mathbb{E}x_i(t)$, the system has a unique bounded solution*

$$(\psi(t), \hbar(t)) = (\psi(0), p\psi(0)), \quad t \geq 0 \quad (6.19)$$

Proof: From (6.12), we have $K_1 = -\frac{1}{2}R^{-1}Q'_1(p - 2q)$, $K_2(t) = -\frac{1}{2}R^{-1}Q'_1(2q\psi(t) - \hbar(t))$. Take expectation of (6.9) and note that $\psi(t) = \frac{1}{N} \sum_{i=1}^N \mathbb{E}x_i(t)$, we obtain

$$\psi(t+1) = \left(1 - \frac{1}{2}(Q_1 + Q_2)R^{-1}Q'_1 p\right)\psi(t) + \frac{1}{2}(Q_1 + Q_2)R^{-1}Q'_1 \hbar(t) \quad (6.20)$$

Combine (6.20) with (6.13), and rewrite the system as:

$$\begin{pmatrix} \psi(t+1) \\ \hbar(t+1) \end{pmatrix} = \begin{pmatrix} 1 - Mp & M \\ -Mp^2 & 1 + Mp \end{pmatrix} \begin{pmatrix} \psi(t) \\ \hbar(t) \end{pmatrix} \quad (6.21)$$

where $M = (Q_1 + Q_2)R^{-1}Q'_1/2$.

Therefore, the solution of (6.21) is:

$$\begin{aligned} \begin{pmatrix} \psi(t) \\ \hbar(t) \end{pmatrix} &= \begin{pmatrix} -tMp + 1 & tM \\ -tMp^2 & 1 + tMp \end{pmatrix} \begin{pmatrix} \psi(0) \\ \hbar(0) \end{pmatrix} \\ &= \begin{pmatrix} \psi(0) + tM(\hbar(0) - p\psi(0)) \\ \hbar(0) + tMp(\hbar(0) - p\psi(0)) \end{pmatrix} \end{aligned} \quad (6.22)$$

Hence, the unique bounded solution of system (6.13)-(6.20) is given in (6.19) ■

Based on Corollary 6.1, we can rewrite the optimal control (6.12) for any $i \in \mathcal{N}$ as:

$$u_i^*(t) = \frac{1}{2}R^{-1}Q'_1(p - 2q)(\psi(0) - x_i(t)) \quad (6.23)$$

which is only associated with time t and local state x_i .

According to (6.11) and (6.23), the solution of the dynamic system (6.9)-(6.10) is given by:

$$\begin{aligned} x_i^*(t) &= (1 - \frac{2q}{p})^t x_i(0) + \frac{2q}{p} \sum_{m=0}^{t-1} (1 - \frac{2q}{p})^{t-1-m} \psi(0) + \sum_{m=0}^{t-1} (1 - \frac{2q}{p})^{t-1-m} w_i(m+1) \\ &= (1 - \frac{2q}{p})^t x_i(0) + \left(1 - (1 - \frac{2q}{p})^t\right) \psi(0) + \sum_{m=0}^{t-1} (1 - \frac{2q}{p})^{t-1-m} w_i(m+1) \end{aligned} \quad (6.24)$$

The stability of the mean field system (6.9)-(6.10) under the decentralized control (6.23) is shown in the following theorem.

Theorem 6.1. *Under Assumption 6.1,*

$$\max_{1 \leq i \leq N} \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} \left(q(x_i^*(t))^2 + u_i^*(t)' R u_i^*(t) \right) < \infty \quad (6.25)$$

Proof: For each agent i , according to (6.24), we have

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} q(x_i^*(t))^2 &\leq \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} 3q \left((1 - \frac{2q}{p})^{2t} (x_i(0) - \psi(0))^2 + \psi(0)^2 \right. \\ &\quad \left. + \left(\sum_{m=0}^{t-1} (1 - \frac{2q}{p})^{t-1-m} w_i(m+1) \right)^2 \right) \end{aligned} \quad (6.26)$$

Based on the algebraic Riccati equation (6.11), we have $0 < 2q/p < 1$. Thus,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} (1 - \frac{2q}{p})^{2t} (x_i(0) - \psi(0))^2 = \lim_{T \rightarrow \infty} \frac{p^2 (1 - (1 - \frac{2q}{p})^{2T})}{4qT(p - q)} = 0 \quad (6.27)$$

and

$$\begin{aligned} &\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} \left(\sum_{m=0}^{t-1} (1 - \frac{2q}{p})^{t-1-m} w_i(m+1) \right)^2 \\ &= \lim_{T \rightarrow \infty} \frac{p^2}{4q(p - q)} r_w \left(1 - \frac{p^2 (1 - (1 - \frac{2q}{p})^{2T})}{4qT(p - q)} \right) = \frac{p^2}{4q(p - q)} r_w \end{aligned} \quad (6.28)$$

Since $\mathbb{E}x_i(0) < \infty$, $\psi(0) < \infty$. Therefore, by (6.27) and (6.28), there exists a constant $0 < \kappa_0 < \infty$ independent of i and N such that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} q(x_i^*(t))^2 < \kappa_0 \quad (6.29)$$

On the basis of (6.24),

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} x_i^*(t) = \lim_{T \rightarrow \infty} \frac{p(1 - (1 - \frac{2q}{p})^T)}{2qT} (x_i(0) - \psi(0)) + \psi(0) = \psi(0) \quad (6.30)$$

Therefore, we have

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} (u_i^*(t)' R u_i^*(t)) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} \frac{(p-2q)^2}{4} Q_1 R^{-1} Q_1' (\psi(0) - x_i^*(t))^2 \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} \frac{(p-2q)^2}{4} Q_1 R^{-1} Q_1' ((\psi(0))^2 + 2\psi(0)x_i^*(t) + (x_i^*(t))^2) \\ &= \frac{(p-2q)^2}{4} Q_1 R^{-1} Q_1' (3(\psi(0))^2 + \kappa_0) < \infty \end{aligned} \quad (6.31)$$

Combining (6.29) and (6.31), we get (6.25). ■

For the approximation of PSA $\psi(t)$ and the decentralized mean field control u_i^* , the following questions naturally come to our mind:

- Whether the state x_i^* of agent i achieves consensus as time goes to infinity?
- Whether the PSA $\frac{\sum_{i=1}^N x_i^*(t)}{N}$ converges to $\psi(0)$ as the population size N goes to infinity?
- Whether the control laws $\{u_i^*, i \in \mathcal{N}\}$ possess the property of almost sure asymptotic Nash equilibrium?

- Whether the inaccurate mass information $\psi(0)$ affects individual agent's LTA cost?
- Whether these results apply to nonlinear systems?

In the following subsections, we will answer these questions.

6.2.1 Mean Consensus Property

Before discussing the consensus property of all agents under the decentralized control law (6.23), we give the definition of mean-consensus.

Definition 6.1. [143] *If $\lim_{t \rightarrow \infty} |\mathbb{E}x_i(t) - \mathbb{E}x_j(t)| = 0$ for any $i \neq j$, then mean-consensus is said to be achieved asymptotically.*

Theorem 6.2. *Under Assumption 6.1, by using the optimal control law (6.23), a mean-consensus is achieved asymptotically as $t \rightarrow \infty$ with asymptotic individual variance $\frac{p^2}{4q(p-q)}r_w$.*

Proof. Note that $0 < 2q/p < 1$. Therefore, according to (6.24) and under Assumption 6.1, $\lim_{t \rightarrow \infty} \mathbb{E}x_i^*(t) = \psi(0)$, $1 \leq i \leq N$, which means that all states reach mean-consensus in $\psi(0)$ asymptotically. For the asymptotic individual variance, we have

$$\lim_{t \rightarrow \infty} \mathbb{E}(x_i^*(t) - \psi(0))^2 = \lim_{t \rightarrow \infty} \mathbb{E} \left(\sum_{m=0}^{t-1} \left(1 - \frac{2q}{p}\right)^{t-1-m} w_i(m+1) \right)^2 = \frac{p^2}{4q(p-q)} r_w \quad (6.32)$$

for any $i \in \mathcal{N}$. ■

6.2.2 Convergence of the PSA

In the following theorem, we show that it is reasonable to approximate PSA by $\psi(0)$ when N approaches infinity.

Theorem 6.3. *Under Assumption 6.1,*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left(\frac{1}{N} \sum_{i=1}^N x_i^*(t) - \psi(0) \right)^2 = 0 \quad (6.33)$$

Proof. By (6.24), we have

$$\begin{aligned} & \lim_{N \rightarrow \infty} \mathbb{E} \left(\frac{1}{N} \sum_{i=1}^N x_i^*(t) - \psi(0) \right)^2 \\ &= \lim_{N \rightarrow \infty} \mathbb{E} \left(\left(1 - \frac{2q}{p}\right)^t \frac{1}{N} \sum_{i=1}^N x_i(0) + \left(1 - \left(1 - \frac{2q}{p}\right)^t\right) \psi(0) \right. \\ & \quad \left. + \frac{1}{N} \sum_{i=1}^N \sum_{m=0}^{t-1} \left(1 - \frac{2q}{p}\right)^{t-1-m} w_i(m+1) - \psi(0) \right)^2 \\ &= \lim_{N \rightarrow \infty} \mathbb{E} \left(\left(1 - \frac{2q}{p}\right)^t \left(\frac{1}{N} \sum_{i=1}^N x_i(0) - \psi(0) \right) + \frac{1}{N} \sum_{i=1}^N \sum_{m=0}^{t-1} \left(1 - \frac{2q}{p}\right)^{t-1-m} w_i(m+1) \right)^2 \\ &= \lim_{N \rightarrow \infty} \mathbb{E} \left(\frac{1}{N} \sum_{i=1}^N \sum_{m=0}^{t-1} \left(1 - \frac{2q}{p}\right)^{t-1-m} w_i(m+1) \right)^2 \\ &= \lim_{N \rightarrow \infty} \frac{p^2 r_w}{4qN(p-q)} \left(1 - \left(1 - \frac{2q}{p}\right)^{2t}\right) = 0. \end{aligned} \quad (6.34)$$

The third equation is based on the fact that $\psi(0) = \frac{1}{N} \sum_{i=1}^N \mathbb{E} x_i(0)$. Under Assumption 6.1, we obtain the forth equation. $\blacksquare$

Lemma 6.2. *Under Assumption 6.1,*

$$\lim_{N \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left(\psi(0) - \frac{1}{N} \sum_{i=1}^N x_i^*(t) \right)^2 = 0 \quad (6.35)$$

Proof. According to (6.34), for all $t \geq 0$, we have

$$\mathbb{E} \left(\psi(0) - \frac{1}{N} \sum_{i=1}^N x_i^*(t) \right)^2 \leq \frac{p^2 r_w}{4qN(p-q)} \quad (6.36)$$

Hence,

$$\lim_{N \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}(\psi(0) - \frac{1}{N} \sum_{i=1}^N x_i^*(t))^2 \leq \lim_{N \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \frac{p^2 r_w}{4qN(p-q)} = 0 \quad (6.37)$$

■

From the above, we can see that $\psi(0)$ is indeed an approximation of PSA when $N \rightarrow \infty$.

6.2.3 ε -Nash Equilibrium

In this section, we analyze the performance of the decentralized mean field controls. We first introduce the concept of ε -Nash equilibrium to demonstrate that the mean field control strategies may result in sub-optimal LTA costs. After that, we show the mean field controls generate an almost sure asymptotic Nash equilibrium. Finally, the minimal value that the LTA cost can reach is identified.

Definition 6.2. [76] A set of controls $\{u_i^*, i \in \mathcal{N}\}$ is called an ε -Nash equilibrium with respect to the costs $\{U_i, i \in \mathcal{N}\}$, if there exists $\varepsilon > 0$ such that for any $i \in \mathcal{N}$,

$$U_i(u_i^*, u_{-i}^*) \leq \inf_{u_i \in \mathcal{U}_i} U_i(u_i, u_{-i}^*) + \varepsilon \quad (6.38)$$

Next, we give the definition of almost sure asymptotic Nash equilibrium.

Definition 6.3. A set of controls $\{u_i^*, i \in \mathcal{N}\}$ is called an almost sure asymptotic Nash equilibrium with respect to the costs $\{U_i, i \in \mathcal{N}\}$, if there exists a sequence of nonnegative variables $\{\varepsilon_N, N \geq 1\}$ such that $\lim_{N \rightarrow \infty} \varepsilon_N = 0$ and for any $i \in \mathcal{N}$,

$$U_i(u_i^*, u_{-i}^*) \leq \inf_{u_i \in \mathcal{U}_i} U_i(u_i, u_{-i}^*) + \varepsilon_N \quad (6.39)$$

Theorem 6.4. *Under Assumption 6.1, the set of mean field control $\{u_i^*, i \in \mathcal{N}\}$ generates an almost sure asymptotic Nash equilibrium, i.e., for any $i \in \mathcal{N}$,*

$$U_i(u_i^*, u_{-i}^*) \leq \inf_{u_i \in \mathcal{U}_i} U_i(u_i, u_{-i}^*) + \varepsilon_N \quad (6.40)$$

where $\lim_{N \rightarrow \infty} \varepsilon_N = 0$ and

$$(\varepsilon_N)^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}(\psi(0) - \frac{1}{N} \sum_{i=1}^N x_i^*(t))^2 \quad (6.41)$$

Proof: For any $i \in \mathcal{N}$, we have

$$\begin{aligned} & U_i(u_i^*, u_{-i}^*) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} \left(q(x_i^*(t) - \frac{\sum_{i=1}^N x_i^*(t)}{N})^2 + u_i^*(t)' R u_i^*(t) \right) \\ &\leq \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} \left(q(x_i^*(t) - \psi(0))^2 + u_i^*(t)' R u_i^*(t) \right) \\ &\quad + q \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}(\psi(0) - \frac{1}{N} \sum_{i=1}^N x_i^*(t))^2 \\ &\quad + 2q \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}(x_i^*(t) - \psi(0))(\psi(0) - \frac{1}{N} \sum_{i=1}^N x_i^*(t)) \\ &\triangleq J_i(u_i^*, \psi(0)) + q(\varepsilon_N)^2 + \rho_N \end{aligned} \quad (6.42)$$

Since $\psi(0)$ is bounded, by the Cauchy-Schwarz inequality and Assumption 6.1, we have $\rho_N = O(\varepsilon_N)$. Therefore, $q(\varepsilon_N)^2 + \rho_N = O(\varepsilon_N)$.

Note that u_i^* is the optimal solution with respect to the LTA cost function $J_i(u_i, \psi(0))$, i.e., $u_i^* = \arg \inf_{u_i \in \mathcal{U}_i} J_i(u_i, \psi(0))$. Similar to (6.42), we get

$$J_i(u_i^*, \psi(0)) \leq \inf_{u_i \in \mathcal{U}_i} J_i(u_i, \psi(0)) \leq \inf_{u_i \in \mathcal{U}_i} U_i(u_i, u_{-i}^*) + O(\varepsilon_N) \quad (6.43)$$

Based on (6.42) and (6.43), we obtain (6.40). Lemma 6.2 implies $\lim_{N \rightarrow \infty} \varepsilon_N = 0$. ■

From Theorem 6.4, it can be seen that the LTA cost $U_i(u_i^*, u_{-i}^*)$ may be sub-optimal due to the deviation between $\psi(0)$ and PSA. Fortunately, as the population size goes to infinity, $U_i(u_i^*, u_{-i}^*)$ can reach $\inf_{u_i \in \mathcal{U}_i} U_i(u_i, u_{-i}^*)$. In order to identify the minimal value of $U_i(u_i, u_{-i}^*)$, we propose the following theorem.

Theorem 6.5. *Under Assumption 6.1,*

$$\lim_{N \rightarrow \infty} |U_i(u_i^*, u_{-i}^*) - J_i(u_i^*, \psi(0))| = 0 \quad (6.44)$$

where

$$J_i(u_i^*, \psi(0)) = \frac{p^2 r_w}{4q(p-q)} \left(q + \frac{1}{4} Q_1 R^{-1} Q_1' (p - 2q)^2 \right) \quad (6.45)$$

Proof. According to the Cauchy-Schwarz inequality,

$$\begin{aligned} & |U_i(u_i^*, u_{-i}^*) - J_i(u_i^*, \psi(0))| \\ &= \lim_{T \rightarrow \infty} \frac{q}{T} \mathbb{E} \sum_{t=0}^{T-1} \left((2x_i^*(t) - \psi(0) - \frac{1}{N} \sum_{i=1}^N x_i^*(t)) \times (\frac{1}{N} \sum_{i=1}^N x_i^*(t) - \psi(0)) \right) \\ &\leq \lim_{T \rightarrow \infty} \frac{q}{T} \mathbb{E} \sqrt{\sum_{t=0}^{T-1} (2x_i^*(t) - \psi(0) - \frac{1}{N} \sum_{i=1}^N x_i^*(t))^2} \times \sqrt{\sum_{t=0}^{T-1} (\frac{1}{N} \sum_{i=1}^N x_i^*(t) - \psi(0))^2} \\ &\leq \lim_{T \rightarrow \infty} q\kappa \sqrt{\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} (\frac{1}{N} \sum_{i=1}^N x_i^*(t) - \psi(0))^2} \end{aligned} \quad (6.46)$$

where $\kappa > 0$ is a constant. Note that $\psi(0)$ is bounded, according to (6.24) and under Assumption 6.1, κ always exists. Through Lemma 6.2, we obtain (6.44). Next, by (6.23) and (6.24), we get the value of $J_i(u_i^*, \psi(0))$. ■

Remark 6.1. *From the above analysis, we come to the conclusion that $J_i(u_i^*, \psi(0))$ is the minimal cost that $U_i(u_i, u_{-i}^*)$ can achieve as the population size N approaches infinity, i.e.,*

$$U_i(u_i^*, u_{-i}^*) = \inf_{u_i \in \mathcal{U}_i} U_i(u_i, u_{-i}^*) = \frac{p^2 r_w}{4q(p-q)} \left(q + \frac{1}{4} Q_1 R^{-1} Q_1' (p - 2q)^2 \right) \quad (6.47)$$

It is manifest that the optimal LTA cost of each agent is the same by applying the decentralized mean field Nash control (6.23).

Furthermore, we can verify that $U_i(u_i^, u_{-i}^*)$ deviates from its minimum value by $\frac{p^2 r_w}{4N(p-q)}$, i.e.,*

$$U_i(u_i^*, u_{-i}^*) - J_i(u_i^*, \psi(0)) = \frac{p^2 r_w}{4N(p-q)} \quad (6.48)$$

6.2.4 Inaccurate Mean Field Information

As discussed above, each agent can use the estimate of the PSA to make its decision and the resulting decentralized optimal control will lead to minimal LTA cost as the population size goes to infinity.

However, in the case where any agent has incorrect information on the mass behavior, it will spontaneously minimize its LTA cost with respect to the inaccurate information. Assume that there is only one agent i has inaccurate mass information $\bar{\psi}(0)$ and all other agents have precise information $\psi(0)$. Note that this assumption is reasonable for large population case since agent i 's decision has little influence on the mass behavior. Denote $\bar{\psi}(0) = \psi(0) + \Delta(\psi(0))$, where $\Delta(\psi(0))$ is the disturbance of accurate estimate $\psi(0)$ of PSA. Hence, the optimal control of agent i is:

$$u_i^\Delta(t) = \frac{1}{2} R^{-1} Q'_1 (p - 2q) (\bar{\psi}(0) - x_i(t)) \quad (6.49)$$

Using the optimal control (6.49), the state of agent i is given by:

$$x_i^\Delta(t) = \left(1 - \frac{2q}{p}\right)^t x_i(0) + \left(1 - \left(1 - \frac{2q}{p}\right)^t\right) \bar{\psi}(0) + \sum_{m=0}^{t-1} \left(1 - \frac{2q}{p}\right)^{t-1-m} w_i(m+1) \quad (6.50)$$

Through (6.49) and (6.50), the expected LTA cost of agent i can be calculated as:

$$J_i(u_i^\Delta, \bar{\psi}(0)) = \frac{p^2 r_w}{4q(p-q)} \left(q + \frac{1}{4} Q_1 R^{-1} Q'_1 (p - 2q)^2 \right) \quad (6.51)$$

while the actual LTA cost obtained by agent i is:

$$J_i(u_i^\Delta, \psi(0)) = \frac{p^2 r_w}{4q(p-q)} \left(q + \frac{1}{4} Q_1 R^{-1} Q_1' (p-2q)^2 \right) + q \left(\Delta(\psi(0)) \right)^2 \quad (6.52)$$

Therefore, we have the following theorem.

Theorem 6.6. *Under Assumption 6.1, we have:*

$$J_i(u_i^\Delta, \psi(0)) - J_i(u_i^\Delta, \bar{\psi}(0)) = q \left(\Delta(\psi(0)) \right)^2 \quad (6.53)$$

$$J_i(u_i^\Delta, \psi(0)) - J_i(u_i^*, \psi(0)) = q \left(\Delta(\psi(0)) \right)^2 \quad (6.54)$$

where u_i^* is the optimal control with respect to the accurate information $\psi(0)$ given by (6.23).

Proof. The result can be obtained immediately by (6.45), (6.51) and (6.52). ■

Remark 6.2. *Compare (6.45) with (6.51), we can conclude that the optimal LTA cost of agent i with accurate mean field information $\psi(0)$ is equal to its expected LTA cost with incorrect information $\bar{\psi}(0)$, i.e.,*

$$J_i(u_i^*, \psi(0)) = J_i(u_i^\Delta, \bar{\psi}(0)) \quad (6.55)$$

Equation (6.55) indicates that agent i always tries to optimize its LTA cost with the mass information it possesses no matter it is correct or not, and its desired optimal LTA costs for both cases are identical. However, from Theorem 6.6, we can see that the actual LTA cost obtained by agent i with inaccurate information deviates from the minimal value it can achieve by $q \left(\Delta(\psi(0)) \right)^2$.

Remark 6.3. *Similar as Section 6.2.3, given the optimal control u_{-i}^* of all other agents with accurate information $\psi(0)$, we can show that the minimal LTA cost $U_i(u_i, u_{-i}^*)$ of agent i with inaccurate information $\bar{\psi}(0)$ is $J_i(u_i^\Delta, \psi(0))$ as $N \rightarrow \infty$,*

i.e.,

$$\begin{aligned} U_i(u_i^\Delta, u_{-i}^*) &= \inf_{u_i \in \mathcal{U}_i} U_i(u_i, u_{-i}^*) \\ &= \frac{p^2 r_w}{4q(p-q)} \left(q + \frac{1}{4} Q_1 R^{-1} Q_1' (p - 2q)^2 \right) + q \left(\Delta(\psi(0)) \right)^2 \end{aligned} \quad (6.56)$$

Remark 6.4. *In the case where only one agent has inaccurate information $\bar{\psi}(0)$, all other agents can achieve their minimal LTA costs defined in (6.2) as $N \rightarrow \infty$. However, if there are a portion of agents having inaccurate information, the agents with accurate information $\psi(0)$ can't reach their minimal LTA costs due to the fact that $\psi(0)$ is not an approximation of PSA any more.*

6.2.5 Nonlinear Dynamics

In this section, we shall show that certain classes of nonlinear system possess similar properties as the linear systems discussed above.

Consider the following nonlinear dynamic system for agent $i \in \mathcal{N}$:

$$x_i(t+1) = f_i(x_i(t)) + u_i(t) + w_i(t+1) \quad (6.57)$$

where $x_i \in \mathbb{R}^{M \times 1}$ is the state of agent i ; $u_i \in \mathbb{R}^{M \times 1}$ is the control input; $w_i \in \mathbb{R}^{M \times 1}$ is the random noise; $f_i(\cdot) : \mathbb{R}^{M \times 1} \rightarrow \mathbb{R}^{M \times 1}$ is a Borel measurable function. The initial states $\{x_i(0), i \in \mathcal{N}\}$ with $\mathbb{E}x_i(0) < \infty$ are mutually independent and are also independent of $\{w_i, i \in \mathcal{N}\}$.

For the random noise w_i , we have the following assumption.

Assumption 6.2. $\{w_i(t), 1 \leq i \leq N\}$ is a sequence of mutually independent noises satisfying:

- (i) $\mathbb{E}w_i(t) = 0$,
- (ii) $\mathbb{E}(w_i(t)w_i(s)') = R_w$ if $t = s$, where $R_w \in \mathbb{R}^{m \times m}$ is a nonnegative definite matrix, and $\mathbb{E}(w_i(t)w_i(s)') = 0$ otherwise,
- (iii) $\mathbb{E}(w_i(t)w_j(s)') = 0$ for all $i \neq j$.

Assume that each agent will receive a cost at each stage, which is associated with the mass behavior of all agents. Define the cost c_i of agent i as:

$$c_i(t) = x_i(t)'(P_1 + P_2 \bar{x}(t)) \quad (6.58)$$

where $P_1 \in \mathbb{R}^{M \times 1}$, $P_2 \in \mathbb{R}^{M \times M}$.

Note that the cost c_i in (6.58) has practical meaning, e.g., 1) in large population traffic assignment problem, assume each driver faces M routes and denote x_i as driver i 's probability distribution on the M routes. Then, the j th component of $P_1 \in \mathbb{R}^{M \times 1}$ and the j th diagonal element of $P_2 \in \mathbb{R}^{M \times M}$ can be viewed as the fixed cost and cost due to congestion for choosing the j th route, respectively. 2) For factory production, assume each factory produces M types of products and x_i is the production level of factory i . Then, the j th component of $P_1 \in \mathbb{R}^{M \times 1}$ and the j th diagonal element of $P_2 \in \mathbb{R}^{M \times M}$ model fixed cost and cost due to mass behavior for producing the j th product, respectively.

Suppose each agent pursues the average cost of all agents at previous stage. Therefore, the LTA cost that each agent aims to minimize is defined as:

$$\hat{U}_i(u_i, u_{-i}) = \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} \left(q(c_i(t+1) - \bar{c}(t))^2 \right) \quad (6.59)$$

where $q > 0$, $\bar{c}(t) = \frac{\sum_{i=1}^N c_i(t)}{N}$.

Similar as the linear system discussed above, in order to construct decentralized controls, $\psi(t) = \mathbb{E}\bar{x}(t)$ is established to approximate the mean field term $\bar{x}(t)$. Moreover, we can verify that $\psi(t) \equiv \psi(0) = \mathbb{E}\bar{x}(0)$.

Based on $\psi(0)$, the decentralized optimal control of this nonlinear system is designed as:

$$\tilde{u}_i(t) = \psi(0) - f_i(x_i(t)) \quad (6.60)$$

The nonlinear system under the decentralized optimal controls $\{\tilde{u}, i \in \mathcal{N}\}$ possess the properties of stability, mean-consensus, convergence and ε -Nash equilibrium.

In the following theorem, we show the almost sure asymptotic Nash equilibrium property of the control set $\{\tilde{u}, i \in \mathcal{N}\}$.

Lemma 6.3. *Under Assumption 6.2, the set of mean field control $\{\tilde{u}, i \in \mathcal{N}\}$ generates an almost sure asymptotic Nash equilibrium, i.e., for any $i \in \mathcal{N}$,*

$$\hat{U}_i(\tilde{u}_i, \tilde{u}_{-i}) \leq \inf_{u_i \in \mathcal{U}_i} \hat{U}_i(u_i, \tilde{u}_{-i}) + \varepsilon_N \quad (6.61)$$

where $\lim_{N \rightarrow \infty} \varepsilon_N = 0$.

The proof of Lemma 6.3 is similar as that of Theorem 6.4, thus we omit it here.

Remark 6.5. *The minimal cost that $\hat{U}_i(u_i, \tilde{u}_{-i})$ can achieve as the population size N approaches infinity is given by:*

$$\hat{U}_i(\tilde{u}_i, \tilde{u}_{-i}) = \inf_{u_i \in \hat{\mathcal{U}}_i} \hat{U}_i(u_i, \tilde{u}_{-i}) = q(P_1 + P_2\psi(0))'R_w(P_1 + P_2\psi(0)) \quad (6.62)$$

Obviously, the optimal LTA cost of each agent is the same by applying the decentralized mean field Nash control (6.60).

6.3 Mean Field Social Optimum Design with Decentralized Information

Consider the case where all agents are cooperative and seek socially optimal strategies. In this case, each agent should take into consideration the cost it imposes to other agents because of its own cost reduction. To be precise, the goal of all agents in the system is to minimize the social cost, which is defined as:

$$U_{soc}(u) = \sum_{i=1}^N U_i(u_i, u_{-i}) \quad (6.63)$$

where $U_i(u_i, u_{-i})$ is the individual LTA cost of agent i as defined in (6.2).

We are interested in finding decentralized solutions of the social optimal problem where the control of each agent is only associated with its local information.

Denote the socially optimal control of $U_{soc}(u)$ as $\hat{u} = \{\hat{u}_i, i \in \mathcal{N}\}$ and let $\hat{x}_i$ be the corresponding solution to the dynamics (6.1) by applying $\hat{u}_i$. Denote $\hat{u}_{-i} = \{\hat{u}_1, \dots, \hat{u}_{i-1}, \hat{u}_{i+1}, \dots, \hat{u}_N\}$.

In the following lemma, we introduce a quadratic cost function $U_{soc,i}(u_i)$ which has the same solution as the social cost $U_{soc}(u)$.

Lemma 6.4. *The solution $\hat{u}_i$ of the socially optimal cost function $U_{soc}(u)$ is the same as the optimal solution of the quadratic cost function $U_{soc,i}(u_i)$, where*

$$\begin{aligned} & U_{soc,i}(u_i) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} \left(q \left(\left(1 + \frac{1}{N}\right) x_i^2(t) - 2x_i(t) \frac{1}{N} (x_i(t) + \sum_{j=1, j \neq i}^N \hat{x}_j(t)) \right) + u_i'(t) R u_i(t) \right) \end{aligned} \quad (6.64)$$

Proof: First, we show that if $\hat{u}$ achieves the minimum of $U_{soc}(u)$, then, for each agent i , $\hat{u}_i = \arg \inf_{u_i \in \mathcal{U}_i} U_{soc}(u_i, \hat{u}_{-i})$. Due to the convexity of $U_{soc}(u)$ and $R > 0$, the existence and uniqueness of $\hat{u}$ are guaranteed. Therefore, $U_{soc}(u_i, \hat{u}_{-i})$ is strictly monotonic with respect to u_i .

Next, we rewrite $U_{soc}(u_i, \hat{u}_{-i})$. Since $\hat{u}_j, j \neq i$ has been specified in advance, $\hat{u}_j$ won't change with u_i , and so does $\hat{x}_j$. Hence, for agent $i \in \mathcal{N}$,

$$U_i(u_i, \hat{u}_{-i}) = \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} (\psi_i^i(t) + \chi_i^i(t)) \quad (6.65)$$

where $\psi_i^i(t) = q \left(\left(1 + \frac{1}{N^2}\right) x_i^2(t) + 2x_i(t) \left(\frac{\sum_{j \neq i} \hat{x}_j(t)}{N^2} - \frac{1}{N} (x_i(t) + \sum_{j=1, j \neq i}^N \hat{x}_j(t)) \right) \right) + u_i'(t) R u_i(t)$, $\chi_i^i(t) = \frac{q \sum_{j \neq i} \hat{x}_j^2(t)}{N^2}$.

For agent $j \neq i$,

$$U_j(u_i, \hat{u}_{-i}) = \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} (\psi_j^i(t) + \chi_j^i(t)) \quad (6.66)$$

where $\psi_j^i(t) = q(\frac{x_i^2(t)}{N^2} + \frac{2x_i(t)}{N}(\frac{\sum_{j \neq i} \hat{x}_j(t)}{N} - \hat{x}_j(t)))$, $\chi_j^i(t) = (\hat{x}_j(t) - \frac{\sum_{j \neq i} \hat{x}_j(t)}{N})^2 + \hat{u}_j'(t)R\hat{u}_j(t)$.

Note that $\chi_k^i, k \in \mathcal{N}$ won't change with u_i , according to (6.65) and (6.66),

$$\arg \inf_{u_i \in \mathcal{U}_i} U_{soc}(u_i, \hat{u}_{-i}) = \arg \inf_{u_i \in \mathcal{U}_i} \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} \sum_{k=1}^N \psi_k^i(t) = \arg \inf_{u_i \in \mathcal{U}_i} U_{soc,i}(u_i) \quad (6.67)$$

which indicates $\hat{u}_i = \arg \inf_{u_i \in \mathcal{U}_i} U_{soc,i}(u_i)$. ■

The cost (6.64) implies that all other agents' effect on agent i is $\sum_{j=1, j \neq i}^N \hat{x}_j(t)$ which won't change with $u_i(t)$. As discussed before, we may approximate $\hat{x}_{-i}(t) = \frac{\sum_{j=1, j \neq i}^N \hat{x}_j(t)}{N}$ by a deterministic function $\psi^\infty(t)$ as $N \rightarrow \infty$. Therefore, the infinite population approximation to the control problem (6.64) is:

$$U_{soc,i}^\infty(u_i) = \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} \left(q \left(x_i^2(t) - 2x_i(t)\psi^\infty(t) \right) + u_i'(t)Ru_i(t) \right) \quad (6.68)$$

By using the approximation of $\hat{x}_{-i}(t)$, we rewrite the state updating process (6.8) as:

$$x_i(t+1) = x_i(t) + Q_1 u_i(t) + Q_2 (K_1 \psi^\infty(t) + K_2(t)) + w_i(t+1) \quad (6.69)$$

Let $\psi^\infty(t) = \frac{1}{N} \sum_{i=1}^N \mathbb{E} \hat{x}_i(t)$. Similar to Lemma 6.1 and Corollary 6.1, we establish the social mean field system of (6.68)-(6.69) as:

$$\hbar(t+1) = \left(1 + \frac{1}{2}(Q_1 + Q_2)R^{-1}Q_1'p \right) \hbar(t) - \frac{1}{2}(Q_1 + Q_2)R^{-1}Q_1'p^2\psi^\infty(t) \quad (6.70)$$

$$\psi^\infty(t+1) = \left(1 - \frac{1}{2}(Q_1 + Q_2)R^{-1}Q_1'p \right) \psi^\infty(t) + \frac{1}{2}(Q_1 + Q_2)R^{-1}Q_1'\hbar(t) \quad (6.71)$$

which is exactly the same as (6.13) and (6.20). According to Corollary 6.1, we have $(\psi^\infty(t), \hbar(t)) = (\psi^\infty(0), p\psi^\infty(0))$, for all $t \geq 0$. It is easy to verify that the mean field optimal control of the infinite social optimum system is:

$$\hat{u}_i(t) = \frac{1}{2}R^{-1}Q_1'(p - 2q)(\psi^\infty(0) - x_i(t)) \quad (6.72)$$

Applying the social optimal control (6.72) yields the trajectory of $\hat{x}_i$:

$$\hat{x}_i(t) = \left(1 - \frac{2q}{p}\right)^t x_i(0) + \left(1 - \left(1 - \frac{2q}{p}\right)^t\right) \psi^\infty(0) + \sum_{m=0}^{t-1} \left(1 - \frac{2q}{p}\right)^{t-1-m} w_i(m+1) \quad (6.73)$$

which is the same as (6.24).

Theorem 6.7. *For the socially optimal case where the objective of each agent is to minimize the social cost, the decentralized mean field social control laws $\hat{u}_i(t)$ are equal to the mean field Nash controls $u_i^*(t)$ for infinite population games.*

Proof: Based on the definition of $\psi^\infty(t)$ and $\psi(t)$, we have $\psi^\infty(0) = \psi(0)$. Thus, compared (6.72) with (6.23), the mean field social control law $\hat{u}_i(t)$ and the mean field Nash control law $u_i^*(t)$ are identical for an infinite population system, i.e.,

$$\hat{u}_i(t) = u_i^*(t) \quad (6.74)$$

The proof is completed. ■

Remark 6.6. *Since $\hat{u}_i(t) = u_i^*(t)$, we have $\hat{x}_i(t) = x_i^*(t)$. According to Theorem 6.2, as time goes to infinity, $\hat{x}_i(t)$ reaches mean-consensus asymptotically. Furthermore, $\{\hat{u}_i, i \in \mathcal{N}\}$ also generates an almost sure asymptotic Nash equilibrium with respect to the costs $\{U_i, i \in \mathcal{N}\}$.*

Based on Theorem 6.3 and Lemma 6.2, we can verify that $\hat{x}_{-i}(t)$ converges to $\psi^\infty(0)$ in the mean square sense, which means $\psi^\infty(0)$ is an approximation of $\hat{x}_{-i}(t)$ for infinite population game.

Remark 6.7. *It is shown in [63] that for continuous mean field control systems, the mean field social optimal control strategies can also approach the mean field Nash controls for infinite population systems.*

To identify the gap between the infinite population quadratic social cost $U_{soc,i}^\infty(\hat{u}_i)$ and the finite population quadratic social cost $U_{soc,i}(\hat{u}_i)$, we propose the following lemma.

Lemma 6.5. *Under Assumption 6.1, for any $i \in \mathcal{N}$,*

$$|U_{soc,i}(\hat{u}_i) - U_{soc,i}^\infty(\hat{u}_i)| = O(\varepsilon_N) \quad (6.75)$$

where ε_N is defined in (6.41).

Proof. According to (6.64), (6.68) and by the Cauchy-Schwarz inequality,

$$\begin{aligned} & |U_{soc,i}(\hat{u}_i) - U_{soc,i}^\infty(\hat{u}_i)| \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} q \left(\frac{\hat{x}_i^2(t)}{N} + 2\hat{x}_i(t) \left(\psi^\infty(0) - \frac{\sum_{i=1}^N \hat{x}_i(t)}{N} \right) \right) \\ &\leq \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} q \frac{\hat{x}_i^2(t)}{N} + \lim_{T \rightarrow \infty} \frac{2q}{T} \mathbb{E} \sqrt{\sum_{t=0}^{T-1} \hat{x}_i^2(t)} \sqrt{\sum_{t=0}^{T-1} \left(\psi^\infty(0) - \frac{\sum_{i=1}^N \hat{x}_i(t)}{N} \right)^2} \quad (6.76) \\ &\leq \frac{q\delta_1}{N} + 2q\delta_2 \sqrt{\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} \left(\psi^\infty(0) - \frac{\sum_{i=1}^N \hat{x}_i(t)}{N} \right)^2} \end{aligned}$$

where $\delta_1 > 0, \delta_2 > 0$. Based on (6.73) and Under Assumption 6.1, δ_1, δ_2 always exist.

According to (6.37) in the proof of Lemma 6.2, (6.75) is proved. ■

6.4 Numerical Examples

Consider a system of 500 agents who want to minimize their LTA costs (6.2) based on the dynamics (6.1). Set the parameters in the dynamic system as $Q_1 = 1.5, Q_2 = 3.2, q = 2, R = 8, r_w = 2$. The initial states of all agents are taken independently from a Gaussian distribution with mean 9 and standard deviation 2. As shown in Fig. 6.1, all agents eventually reach mean-consensus by implementing the optimal mean field control law u_i^* in (6.23). Fig. 6.2 verifies that ε_N as defined in (6.41) finally approaches 0 as the population size N goes to infinity, which implies the gap between the PSA and $\psi(0)$ vanishes as $N \rightarrow \infty$. For any given agent i , Fig. 6.3

shows that $U_i(u_i^*, u_{-i}^*)$ can attain its minimal value $J_i(u_i^*, \psi(0))$ as $N \rightarrow \infty$. To illustrate the effect of inaccurate mean field information on individual agent's LTA cost, we compare the LTA cost of agent i with accurate information and inaccurate information with $\Delta(\psi(0)) = 0.5$. As shown in Fig. 6.4, the LTA cost $U_i(u_i^\Delta, u_{-i}^*)$ with inaccurate information can't achieve its minimal value $J_i(u_i^*, \psi(0))$ and deviates from it exactly by $q(\Delta(\psi(0)))^2 = 0.5$.

Figure 6.1: Trajectories of agents' mean state.

Figure 6.2: Convergence of ε_N with respect to the population size N .

Figure 6.3: Convergence of the LTA cost $U_i(u_i^*, u_{-i}^*)$ to $J_i(u_i^*, \psi(0))$.

Figure 6.4: Effect of the inaccurate information.

6.5 Conclusions

In this chapter, we have investigated a discrete-time mean field model where each agent tries to optimize its LTA cost containing the PSA based on a dynamic equation coupled with the input average. A deterministic function ψ was introduced to estimate the PSA such that each agent's state can evolve only according to the local information. We have shown that the resulting decentralized mean field controls possess an almost sure asymptotic Nash equilibrium property. The effect of inaccu-

rate mean field information and a nonlinear dynamic system with similar properties have also been discussed. For the socially optimal case where the objective of each agent is to minimize the social cost, it has been shown that the decentralized mean field social control laws are equal to the mean field Nash controls for infinite population games. As a result, these mean field social controls inherit many properties of mean field Nash controls. However, if the system doesn't possess an adequately large population, these two kinds of controls may not be identical.

Chapter 7

Conclusions and Future Work

7.1 Conclusion

As the inefficiency of Nash equilibrium may cause significant economic loss, the efficiency analysis of Nash equilibrium has attracted the attention of many researchers in the past few years. In this thesis, we have discussed the influence of dynamic pricing on the efficiency of Nash equilibrium. To estimate the mass information which is associated with individual player's utility/cost, the consensus control and mean field theory are adopted. The contributions of the thesis can be summarized in the following aspects.

- We have analyzed the existence and convergence of Nash equilibrium in congestion games where each player faces multiple choices. A consensus protocol was proposed to estimate the percentage of players choosing each resource. Several dynamic pricing schemes were designed to achieve some kind of social optimum or spread out players' choices in different models.
- We have studied the effect of scaled marginal-cost pricing on players' routing choices in networks with one origin-destination pair. All players in the network were divided into groups according to their price sensitivities. For the special

case with two groups and two routes, it has been proved that there always exists a scaled marginal-cost such that the POA can achieve 1. For general case, we have shown that, under certain conditions, POA can reach 1 by charging a designed scaled marginal-cost on each link. An algorithm was also given to calculate the optimal scaled marginal-cost for any network with one origin-destination pair and distribution of price sensitivity.

- We have analyzed the upper bound of POTA in networks with multiple origin-destination pairs via smoothness arguments. The influence of dynamic pricing on the upper bound of POTA for the linear latency case has been discussed. How the inaccurate parameter information affects the POTA was also studied. The upper bound of POTA for the nonlinear latency case was given. For networks with heterogeneous players, we showed that the upper bound of the POTA is the same as that in networks with homogeneous players as time goes to infinity.
- We have considered a discrete-time dynamic system where each agent evolves according to a dynamic equation containing all agents' control inputs and seeks to minimize its LTA cost encompassing the PSA. We have constructed a deterministic function $\psi(0)$ to estimate the PSA, and the convergence of the PSA to $\psi(0)$ has been justified. The stability, mean-consensus property and ε -Nash equilibrium property of the resulting decentralized mean field control system were discussed. A nonlinear dynamic system with similar properties as the linear system established above was shown. We also analyzed the effect of inaccurate mean field information $\psi(0)$ on individual agent. The social optimal case where all agents work cooperatively to optimize the social cost as the sum of individual agent's LTA cost has been considered and the decentralized mean field social control was shown to be equal to the mean field Nash control as the population size goes to infinity.
- We have applied our results to trip timing and traffic routing problems. The numerical examples and real data simulations were provided to illustrate the

validity of the results.

7.2 Future Work

We list some possible directions for future studies.

- In this thesis, we have discussed the convergence of Nash equilibrium in repeated play. On the basis of the convergence property, the convergence rate analysis should be one future research topic.
- In Chapter 4, we assume that the system manager has perfect information on the distribution of price sensitivity. However, in reality, the distribution of price sensitivity may be unknown to the system manager. Therefore, one of our next steps is to estimate the distribution of price sensitivity and analyze the robustness of the POA. In addition, the relaxation of the linear assumption on the latency function and the relaxation of the single origin-destination pair assumption on the network should also be brought into consideration.
- For the case of heterogeneous players we discussed in Section 5.5, we have not investigated the influence of dynamic pricing on the POTA. Designing a price scheme to optimize the POTA in networks with heterogeneous players can be an interesting topic.
- In this thesis, we have analyzed the efficiency improvement of systems via dynamic pricing. As pointed out in Section 2.3.1, Stackelberg games can also be applied to improve the system efficiency. For example, to improve the performance of the network in routing problems, some routing advice can be given to players to use less congested routes and rerouting players will be rewarded. When providing routing advice, we should take into consideration the possible impact of rerouting players on the network and the response of other players to this change. Under this circumstance, rerouting players can

be viewed as leaders and other players can be regarded as followers. This conforms to the framework of Stackelberg games. Therefore, one possible research direction is to analyze the efficiency improvement of Nash equilibrium via Stackelberg games, especially for networks with heterogeneous players.

Appendix A

Proof of Lemma 4.3 in Chapter 4

We first show the construction of a Nash flow. Consider the group with the price sensitivity β_{j_0} .

Assume $f_{r_k, \beta_{j_0}}^{ne} > 0$ for $k = 1, v$ and any $\mu \geq 0$. We will show later that under condition (4.29), $f_{r_1, \beta_{j_0}}^{ne} > 0$ and $f_{r_v, \beta_{j_0}}^{ne} > 0$ always hold. If $f_{r_k, \beta_{j_0}}^{ne} = 0$ for some $r_k \neq r_1$ or r_v , by the definition of Nash flow, we have $J_{r_i, \beta_{j_0}}(f^{ne}) \leq J_{r_k, \beta_{j_0}}(f^{ne})$, for all $i \neq k$. Note that β_j and V_j are ordered as $\beta_1 > \beta_2 > \dots > \beta_M$ and $V_{r_1} < V_{r_2} < \dots < V_{r_v}$, respectively. For groups with $\beta_j > \beta_{j_0}$, i.e., $\beta_1, \dots, \beta_{j_0-1}$, since $f_{r_v, \beta_{j_0}}^{ne} > 0$, we have $J_{r_v, \beta_j}(f^{ne}) \leq J_{r_i, \beta_j}(f^{ne})$, for all $i \neq v$, which indicates that $f_{r_k, \beta_j}^{ne} = 0$ for all $\beta_j > \beta_{j_0}$. For groups with $\beta_j < \beta_{j_0}$, i.e., $\beta_{j_0+1}, \dots, \beta_M$, since $f_{r_1, \beta_{j_0}}^{ne} > 0$ we have $J_{r_1, \beta_j}(f^{ne}) \leq J_{r_i, \beta_j}(f^{ne})$, for all $i \neq 1$, which also indicates that $f_{r_k, \beta_j}^{ne} = 0$ for all $\beta_j < \beta_{j_0}$. Thus, we can conclude that $f_{r_k}^{ne} = 0$, which implies $f_{r_k}^{ne} = 0$ when $\mu = 0$. This contradicts Assumption 4.2.

Therefore, $f_{r_k, \beta_{j_0}}^{ne} > 0$ for all $r_k \in \mathcal{R}$.

By the definition of Nash flow, for any $r_i \neq r_j$, we have

$$\sum_{r_k \in \mathcal{R}} f_{r_k}^{ne} (d_{ik} - d_{jk}) = \frac{V_j - V_i}{1 + \mu \beta_{j_0}}. \quad (\text{A.1})$$

For groups with $\beta_j > \beta_{j_0}$, $\frac{V_v - V_i}{1 + \mu \beta_{j_0}} > \frac{V_v - V_i}{1 + \mu \beta_j}$, for all $i < v$. Therefore, $J_{r_v, \beta_j}(f^{ne}) < J_{r_i, \beta_j}(f^{ne})$ for all $i < v$, i.e., $f_{r_i, \beta_j}^{ne} = 0$, $f_{r_v, \beta_j}^{ne} = F_j$, and $f_{r_v}^{ne} = \bar{F}_{j_0} + f_{r_v, \beta_{j_0}}^{ne}$. Similarly,

for groups with $\beta_j < \beta_{j_0}$, $\frac{V_1 - V_i}{1 + \mu\beta_{j_0}} < \frac{V_1 - V_i}{1 + \mu\beta_j}$, for all $i > 1$. Thus, $J_{r_1, \beta_j}(f^{ne}) < J_{r_i, \beta_j}(f^{ne})$ for all $i > 1$, i.e., $f_{r_i, \beta_j}^{ne} = 0$, $f_{r_1, \beta_j}^{ne} = F_j$, and $f_{r_1}^{ne} = \hat{F}_{j_0} + f_{r_1, \beta_{j_0}}^{ne}$. In conclusion, the Nash flow is given by

$$f_{r_k}^{ne} = \begin{cases} \hat{F}_{j_0} + f_{r_1, \beta_{j_0}}^{ne}, & \text{if } k = 1, \\ \bar{F}_{j_0} + f_{r_v, \beta_{j_0}}^{ne}, & \text{if } k = v, \\ f_{r_k, \beta_{j_0}}^{ne}, & \text{otherwise.} \end{cases} \quad (\text{A.2})$$

From (A.1), each $\{i, j\}$ forms an equation. Take any distinct $N - 1$ equations and combine them with $\sum_{r_k \in \mathcal{R}} f_{r_k, \beta_{j_0}}^{ne} = F_{j_0}$, we obtain a N -dimension linear equation. The solution of this system uniquely exists and any solution $f_{r_k, \beta_{j_0}}^{ne}$ must be a linear combination of F_{j_0} , $\bar{F}_{j_0}$, $\hat{F}_{j_0}$ and $\frac{1}{1 + \mu\beta_{j_0}}$, and we denote them as

$$f_{r_k, \beta_{j_0}}^{ne} = A_k F_{j_0} + \frac{1}{1 + \mu\beta_{j_0}} B_k + C_k \hat{F}_{j_0} + D_k \bar{F}_{j_0}, \quad (\text{A.3})$$

where A_k, B_k, C_k, D_k are independent of $\mu, \beta, F_{j_0}, \bar{F}_{j_0}$, and $\hat{F}_{j_0}$.

Since potential game guarantees the existence of a pure Nash equilibrium, the solution to the above system always exists. Substituting (A.3) into $\sum_{r_k \in \mathcal{R}} f_{r_k, \beta_{j_0}}^{ne} = F_{j_0}$, we have

$$\left(\sum_{r_k \in \mathcal{R}} A_k - 1 \right) F_{j_0} + \left(\sum_{r_k \in \mathcal{R}} B_k \right) \frac{1}{1 + \mu\beta_{j_0}} + \left(\sum_{r_k \in \mathcal{R}} C_k \right) \hat{F}_{j_0} + \left(\sum_{r_k \in \mathcal{R}} D_k \right) \bar{F}_{j_0} = 0. \quad (\text{A.4})$$

Since (A.4) holds for any $F_{j_0}, \bar{F}_{j_0}, \hat{F}_{j_0}$ and μ , we obtain

$$\sum_{r_k \in \mathcal{R}} A_k = 1, \quad (\text{A.5})$$

$$\sum_{r_k \in \mathcal{R}} B_k = 0, \quad (\text{A.6})$$

$$\sum_{r_k \in \mathcal{R}} C_k = 0, \quad (\text{A.7})$$

$$\sum_{r_k \in \mathcal{R}} D_k = 0. \quad (\text{A.8})$$

Substituting (A.2), (A.3) into (A.1), similarly to the above, we obtain

$$\sum_{r_k \in \mathcal{R}} A_k d_{ik} = \sum_{r_k \in \mathcal{R}} A_k d_{jk}, \quad (\text{A.9})$$

$$\sum_{r_k \in \mathcal{R}} B_k d_{ik} + V_i = \sum_{r_k \in \mathcal{R}} B_k d_{jk} + V_j, \quad (\text{A.10})$$

$$\sum_{r_k \in \mathcal{R}} C_k d_{ik} + d_{i1} = \sum_{r_k \in \mathcal{R}} C_k d_{jk} + d_{j1}, \quad (\text{A.11})$$

$$\sum_{r_k \in \mathcal{R}} D_k d_{ik} + d_{iN} = \sum_{r_k \in \mathcal{R}} D_k d_{jk} + d_{jN}. \quad (\text{A.12})$$

By (A.7) and (A.9),

$$\sum_{i=1}^v C_i \sum_{k=1}^v A_k d_{ik} = \left(\sum_{k=1}^v A_k d_{1k} \right) \sum_{i=1}^v C_i = 0. \quad (\text{A.13})$$

By (A.5) and (A.11),

$$\begin{aligned} \sum_{i=1}^v A_i \sum_{k=1}^v C_k d_{ik} &= \sum_{i=1}^v A_i \left(\sum_{k=1}^v C_k d_{1k} + d_{11} - d_{i1} \right) \\ &= \sum_{k=1}^v C_k d_{1k} + d_{11} - \sum_{i=1}^v A_i d_{i1} \\ &= \sum_{k=1}^v C_k d_{ik} + d_{i1} - \sum_{k=1}^v A_k d_{ki}. \end{aligned} \quad (\text{A.14})$$

Since $\sum_{i=1}^v C_i \sum_{k=1}^v A_k d_{ik} = \sum_{i=1}^v A_i \sum_{k=1}^v C_k d_{ik}$, combine (A.13) and (A.14), for all $r_i \in \mathcal{R}$

$$\sum_{k=1}^v C_k d_{ik} + d_{i1} = \sum_{k=1}^v A_k d_{ik}, \quad (\text{A.15})$$

i.e.,

$$(A_1 - C_1 - 1)d_{i1} + \sum_{k=2}^v (A_k - C_k)d_{ik} = 0. \quad (\text{A.16})$$

Since (A.16) holds for all $r_i \in \mathcal{R}$, we have

$$\begin{cases} A_1 = C_1 + 1, \\ A_k = C_k, \text{ for all } k \neq 1. \end{cases} \quad (\text{A.17})$$

Similarly to the above, we have

$$\begin{cases} A_k = D_k, \text{ for all } k \neq v, \\ A_v = D_v + 1. \end{cases} \quad (\text{A.18})$$

Insert (A.17) and (A.18) into (A.3), we obtain (4.30). According to (4.29), $f_{r_k, \beta_{j_0}}^{ne} > 0$ for $k = 1, v$ is guaranteed. By Assumption 4.2, $f_{r_k, \beta_{j_0}}^{ne} > 0, r_k \in \mathcal{R}$ for $\mu > 0$ is guaranteed. For the special case when $\hat{F}_{j_0} = 0$ or $\bar{F}_{j_0} = 0$, the results are the same.

Next, we show the uniqueness of the above Nash flow. For the given distribution of $\mathcal{B}$, we assume that there exists another group $\beta_{j_1} \in \mathcal{B}$ satisfying $f_{r_k, j_1}^{ne} > 0$ for all $r_k \in \mathcal{R}$, thus,

$$f_{r_k, \beta_{j_1}}^{ne} = \begin{cases} A_1 F - \hat{F}_{j_1} + \frac{1}{1+\mu\beta_{j_1}} B_1, & \text{if } k = 1, \\ A_v F - \bar{F}_{j_1} + \frac{1}{1+\mu\beta_{j_1}} B_v, & \text{if } k = v, \\ A_k F + \frac{1}{1+\mu\beta_{j_1}} B_k, & \text{otherwise.} \end{cases} \quad (\text{A.19})$$

Since (4.30) and (A.19) also holds for $\mu = 0$, we consider the case when the routes are free of charge. Therefore, for $\beta_{j_0} \in \mathcal{B}$,

$$f_{r_k, \beta_{j_0}}^{ne} = \begin{cases} A_1 F - \hat{F}_{j_0} + B_1, & \text{if } k = 1, \\ A_v F - \bar{F}_{j_0} + B_v, & \text{if } k = v, \\ A_k F + B_k, & \text{otherwise.} \end{cases} \quad (\text{A.20})$$

For $\beta_{j_1} \in \mathcal{B}$,

$$f_{r_k, \beta_{j_1}}^{ne} = \begin{cases} A_1 F - \hat{F}_{j_1} + B_1, & \text{if } k = 1, \\ A_v F - \bar{F}_{j_1} + B_v, & \text{if } k = v, \\ A_k F + B_k, & \text{otherwise.} \end{cases} \quad (\text{A.21})$$

Without loss of generality, we assume that group β_{j_1} is next to group β_{j_0} and $\beta_{j_1} <$

β_{j_0} . Thus, $\bar{F}_{j_1} = F - \hat{F}_{j_0}$. Since $f_{r_1, \beta_{j_1}}^{ne} \geq 0, f_{r_v, \beta_{j_1}}^{ne} \geq 0, f_{r_1, \beta_{j_0}}^{ne} \geq 0, f_{r_v, \beta_{j_0}}^{ne} \geq 0$, we have

$$\hat{F}_{j_0} \leq A_1 F + B_1,$$

$$\bar{F}_{j_0} \leq A_v F + B_v,$$

$$\bar{F}_{j_1} = F - \hat{F}_{j_0} \leq A_v F + B_v.$$

Thus,

$$F_{j_0} = F - \hat{F}_{j_0} - \bar{F}_{j_0} \geq F - (A_1 + A_v)F - (B_1 + B_v) > 0,$$

$$F - A_v F - B_v \leq \hat{F}_{j_0} \leq A_1 F + B_1,$$

i.e.,

$$F \leq (A_1 + A_v)F + (B_1 + B_v),$$

$$F > (A_1 + A_v)F + (B_1 + B_v).$$

This is a contradiction. Therefore, the Nash flow is unique for a given distribution of $\mathcal{B}$ satisfying (4.29). ■

Author's Publications

Journal papers:

1. X. Wang, N. Xiao, L. Xie, E. Frazzoli, and D. Rus, "Analysis of price of anarchy in traffic networks with heterogeneous price-sensitive populations," *to appear in IEEE Transactions on Control Systems Technology as a regular paper*, 2015.
2. X. Wang, N. Xiao, L. Xie, E. Frazzoli, and D. Rus, "Decentralized dynamic games for large population stochastic multi-agent systems," *IET Control Theory and Applications*, vol. 9, no. 3, pp. 503–510, Jan. 2015.
3. N. Xiao, X. Wang, L. Xie, T. Wongpiromsarn, E. Frazzoli, and D. Rus, "Road pricing design based on game theory and multi-agent consensus", *IEEE/CAA Journal of Automatica Sinica*, vol. 1, no. 1, pp. 41–49, 2014.
4. X. Wang, N. Xiao, L. Xie, E. Frazzoli, and D. Rus, "Analysis of price of total anarchy in congestion games via smoothness arguments," *submitted to IEEE Transactions on Control of Network Systems*, 2015.

Conference papers:

1. X. Wang, N. Xiao, L. Xie, E. Frazzoli, and D. Rus, "Analysis of price of anarchy in heterogeneous price-sensitive populations," in *Proceedings of the 53rd IEEE Conference on Decision and Control (CDC)*, 2014, pp.6478–6483.

2. X. Wang, N. Xiao, L. Xie, E. Frazzoli, and D. Rus, “Discrete-time mean field games in multi-agent systems,” in *Proceedings of the 13th International Conference on Control, Automation, Robotics and Vision (ICARCV)*, Singapore, Dec. 10-12, 2014.
3. X. Wang, N. Xiao, T. Wongpiromsarn, L. Xie, E. Frazzoli, and D. Rus, “Distributed consensus in noncooperative congestion games: an application to road pricing,” in *Proceedings of the 10th IEEE International Conference on Control and Automation (ICCA)*, 2013, pp.1668–1673.
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