Modeling damage to contact surfaces of materials with internal defects

Wei, Rong Bing

2016

Wei, R. B. (2016). Modeling damage to contact surfaces of materials with internal defects.
Doctoral thesis, Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/66034>

<https://doi.org/10.32657/10356/66034>

NANYANG
TECHNOLOGICAL
UNIVERSITY

**MODELING DAMAGE TO CONTACT SURFACES OF
MATERIALS WITH INTERNAL DEFECTS**

**MODELING DAMAGE TO CONTACT SURFACES OF
MATERIALS WITH INTERNAL DEFECTS**

**WEI
2016**

RONGBING

WEI RONGBING

SCHOOL OF MECHANICAL AND AEROSPACE ENGINEERING

2016

**MODELING DAMAGE TO CONTACT SURFACES OF
MATERIALS WITH INTERNAL DEFECTS**

WEI RONGBING

**SCHOOL OF MECHANICAL AND AEROSPACE
ENGINEERING**

A thesis submitted to the Nanyang Technological University in partial
fulfillment of the requirement for the degree of
Doctor of Philosophy

2016

Acknowledgements

I would like to express my sincere gratitude to my supervisor Prof. Zhou Kun for his patient guidance, continuous encouragement and great understanding during my PhD study. Without his support, the study presented in this dissertation would not be possible. I have gained many valuable experiences through the learning process with him.

Many thanks go to Dr. Liu Bo, Dr. Hoh Hsin Jen, Mr. Dong Qingbing and Ms. Yang Jing for their help. I have benefited a lot from the inspiring discussions with them.

Meanwhile, I would like to thank my parents for their unconditional support and care. Special thanks go to my girlfriend Miss Jiang Haiyan for her encouragement during my study. I also would like to express my gratitude to my friends Mr. Huang Ruoxuan, Mr. Wang Zhiyong, Mr. Gou Liangliang and Mr. Liu Jianing who have consistently provided help without hesitation whenever it was needed.

Last but not the least, I would like to thank the Nanyang Technological University, Singapore for providing me the Research Scholarship to conduct the PhD study.

Table of Contents

Acknowledgements	I
Table of Contents	III
List of figures.....	VI
Abstract.....	XI
Chapter 1 Introduction.....	1
1.1 Background	1
1.2 Objectives	4
1.3 Dissertation layout	4
Chapter 2 Literature review	6
2.1 Defects in materials.....	6
2.1.1 Inclusions	6
2.1.2 Dislocations	7
2.1.3 Cracks	9
2.2 Interactions of defects	13
2.2.1 Multiple inclusions	13
2.2.2 Inclusions and dislocations	14
2.2.3 Inclusions and cracks	15
2.3 Distributed dislocation technique for crack modeling	16
2.4 Equivalent inclusion method for inclusion modeling	20
2.5 Contact problems with internal defects.....	22
2.5.1 Classical elastic contact problems	22
2.5.2 Contact bodies interacting with defects	24
2.6 Fatigue behavior.....	25
2.6.1 Fundamental mechanisms of fatigue	25

2.6.2 Crack propagation under the influence of internal defects	28
Chapter 3 Defects near surfaces under prescribed surface loading.....	30
3.1 Methodology	30
3.1.1 Problem description and solution approach.....	30
3.1.2 Crack modeling.....	33
3.1.3 Inclusion modeling	36
3.1.4 Governing equation.....	37
3.1.5 Stress intensity factor	41
3.2 Numerical results	44
3.2.1 SIF approach validation	44
3.2.2 One inclusion and one crack.....	45
3.2.3 Three inclusions and two cracks	48
3.2.4 Three cracks separated by two inclusions.....	51
3.3 Summary	53
Chapter 4 Defects near surfaces under contact loading.....	55
4.1 Methodology	55
4.1.1 Problem description and solution approach.....	55
4.1.2 Half-space sub-problem with prescribed surface loading.....	57
4.1.3 Homogeneous half-space contact sub-problem	62
4.1.4 Numerical procedure for the entire contact problem	64
4.2 Numerical results	65
4.2.1 One inclusion and one horizontal crack.....	65
4.2.2 Two inclusions and one horizontal crack.....	68
4.2.3 Multiple horizontal cracks	71
4.2.4 One vertical crack beneath a coating layer	73

4.2.5 One inclusion and one vertical crack in a coated substrate	77
4.2.6 Two inclusions and one vertical crack in a coated substrate	80
4.3 Summary	83
Chapter 5 Fatigue behaviors under contact loading.....	84
5.1 Methodology	84
5.1.1 Problem description and solution approach.....	84
5.1.2 Solution scheme	86
5.2 Numerical results	89
5.2.1 Particle detachment of cracked material	89
5.2.2 Delamination of a coated layer from the substrate	92
5.2.3 Crack propagation in an inhomogeneous material under thermal loading	94
5.3 Summary	96
Chapter 6 Conclusions and recommendations	99
6.1 Conclusions.....	99
6.2 Recommendations	101
Appendix A	103
Appendix B	109
Appendix C	114
References	115
List of publications.....	127

List of figures

Fig. 2.1.	Schematics of (a) a homogeneous inclusions Ω with elastic module C_{ijkl}^D and eigenstrain ε_{ij}^P and (b) an inhomogeneous inclusion with elastic module C_{ijkl}^Ω and eigenstrain ε_{ij}^P embedded in an infinite matrix $D - \Omega$ with elastic modulus C_{ijkl}^D	7
Fig. 2.2.	(a) A climb edge dislocation and (b) a glide edge dislocation in a crystal structure.	8
Fig. 2.3.	A screw dislocation in a crystal structure.	9
Fig. 2.4.	Schematic of three crack types.	10
Fig. 2.5.	Stress components near the tip of a 2D Griffith crack.	11
Fig. 2.6.	Application of the superposition principle to a Griffith crack.	17
Fig. 2.7.	Distribution of schematic infinitesimal dislocations.	18
Fig. 2.8.	The mixed-mode (I and II) dislocations (climb and glide).	18
Fig. 2.9.	(a) Schematic of an inhomogeneous inclusions Ω with elastic constants C_{ijkl}^Ω and initial eigenstrain ε_{ij}^P embedded in an infinite matrix $D - \Omega$ with elastic constants C_{ijkl}^D . (b) Using EIM, Ω is regarded as homogeneous inclusion with initial eigenstrain ε_{ij}^P plus equivalent eigenstrain ε_{ij}^*	21
Fig. 2.10.	S - N curves for carbon steel and aluminum.	26
Fig. 2.11.	Fatigue crack growth rate versus the SIF range.	26
Fig. 3.1.	(a) The original problem of multiple inhomogeneous inclusions and cracks in a half-space, and (b) the new problem of multiple equivalent homogeneous inclusions and dislocation distributions.	31

Fig. 3.2.	Cracks and arbitrarily-shaped inclusions in 2D discretized domain.	33
Fig. 3.3.	A crack modeled by a distribution of dislocations approximated by a piecewise linear function.	34
Fig. 3.4.	Schematic of two collinear cracks under uniform normal stresses.	44
Fig. 3.5.	Comparison of SIFs K^A / K_0 and K^B / K_0 for two collinear cracks under uniform normal stresses.	45
Fig. 3.6.	Schematic of a crack and an inclusion beneath the surface.	46
Fig. 3.7.	Comparison of the von Mises stresses σ_v beneath the surface obtained by the present method and the FEM, respectively.	46
Fig. 3.8.	Contours of the stress components and von Mises stress.	47
Fig. 3.9.	The mode II SIF ($\text{MPa} \cdot \text{m}^{1/2}$) versus h/a without the inhomogeneous inclusion.	48
Fig. 3.10.	The mode II SIF ($\text{MPa} \cdot \text{m}^{1/2}$) versus λ for the case of $h = 0.2a$	48
Fig. 3.11.	Schematic of three inclusions and two cracks underneath a surface. ..	49
Fig. 3.12.	Crack opening for different initial eigenstrains.	50
Fig. 3.13.	Mode I SIF ($\text{MPa} \cdot \text{m}^{1/2}$) for different initial eigenstrains.	50
Fig. 3.14.	Schematic of three cracks separated by two square inclusions underneath a surface subjected to a normal pressure.	51
Fig. 3.15.	Openings of the cracks Γ_1 , Γ_2 and Γ_3	52
Fig. 3.16.	Mode I SIF ($\text{MPa} \cdot \text{m}^{1/2}$) of Γ_2 versus ε_x^0 when $\varepsilon_y^0 = 0.03$	52
Fig. 3.17.	Mode I SIF ($\text{MPa} \cdot \text{m}^{1/2}$) of Γ_2 versus λ_y when $\varepsilon_x^0 = \varepsilon_y^0 = 0.03$	53
Fig. 4.1.	(a) The original inhomogeneous half-space coating-substrate contact problem, and (b) the equivalent new homogeneous half-space contact problem.	57

Fig. 4.2.	Discretization of a computational domain containing all the inclusions and cracks into $N_x \times N_y$ square elements.	58
Fig. 4.3.	Algorithm for solving the problem of a half-space containing cracks and inhomogeneous inclusions under contact loading.....	65
Fig. 4.4.	Schematic of one horizontal crack and one inclusion beneath a half-space surface subjected to contact loading.....	66
Fig. 4.5.	Comparison of the von Mises stresses σ_v along $y = 1.75a_0$ obtained by the present method and FEM.	67
Fig. 4.6.	Contours of the von Mises stresses for the (a) square subsurface inclusion and (b) arbitrarily-shaped subsurface inclusion.	67
Fig. 4.7.	The shape effect of the inclusion on the contact pressure p	68
Fig. 4.8.	Schematic of one horizontal crack and two inclusions beneath a half-space surface subjected to a contact loading.	69
Fig. 4.9.	Contours of the stress components and von Mises stresses.	70
Fig. 4.10.	Effect of a crack at various depths h beneath a frictionless half-space surface on the contact pressure.	70
Fig. 4.11.	Schematic of three horizontal cracks beneath a half-space surface subjected to a cylindrical contact loading.	71
Fig. 4.12.	Comparison of subsurface von Mises stresses for the cases of various crack depths h	72
Fig. 4.13.	Effect of the friction force on (a) the surface displacement u and (b) the stress component σ_x	73
Fig. 4.14.	Schematic of a vertical crack located between a coating and its substrate under frictional contact loading.	74

Fig. 4.15.	Effect of the coating on interfacial shear stresses σ_{xy} along the x direction for the case of $d = 4.0a_0$ and $\nu_c = 0.28$	76
Fig. 4.16.	Effect of the coating on SIFs for the case of $d = 4.0a_0$ and $\nu_c = 0.28$	76
Fig. 4.17.	Effect of d/a_0 on SIFs for the case of $E_c = 305$ GPa and $\nu_c = 0.22$. 77	
Fig. 4.18.	Schematic of a vertical crack and an inclusion beneath a coating.	78
Fig. 4.19.	Effect of the subsurface inclusion on the crack SIF for the case of $d = 1.2a_0$ and $\nu_c = 0.28$	79
Fig. 4.20.	Effect of the subsurface inclusion on contact pressures for the case of $d = 1.2a_0$ and $\nu_c = 0.28$	79
Fig. 4.21.	Effect of d/a_0 on the crack SIF for the case of $E_c = 340$ GPa and $\nu_c = 0.28$	80
Fig. 4.22.	Schematic of a vertical crack and two inclusions beneath the coating.	81
Fig. 4.23.	Effect of the initial inclusion eigenstrains on the crack opening (μm).	82
Fig. 4.24.	Effect of the initial eigenstrains in inclusions on the SIFs ($\text{MPa} \cdot \text{m}^{1/2}$).	82
Fig. 5.1.	Schematic of the crack growth near inhomogeneous inclusions in a half-space under cyclic contact loading.	85
Fig. 5.2.	Discretization of a computational domain containing all the inclusions and cracks into many small square elements.	86
Fig. 5.3.	Schematic of three elements near a crack tip.	87
Fig. 5.4.	Algorithm for solving the problem of a half-space containing cracks	

	and inhomogeneous inclusions under cyclic contact loading.	88
Fig. 5.5.	Schematic of a crack beneath a surface under cyclic contact loading.	89
Fig. 5.6.	Particle detachment from the substrate.	90
Fig. 5.7.	Effect of the crack depth on the number of cycles required for particle detachment occurrence.	91
Fig. 5.8.	No particle detachment due to the crack growth driven by cyclic contact loading	91
Fig. 5.9.	Schematic of a crack located at the interface of the coated substrate.	92
Fig. 5.10.	Coating delamination from the steel substrate under the cyclic contact loading.	93
Fig. 5.11.	The change of the mode II SIF K_{II} / K^0 with the loading cycles N	94
Fig. 5.12.	Schematic of one crack and two inclusions in a half-space under cyclic thermal loading.	95
Fig. 5.13.	Contours of the normal stress σ_y (MPa).	95
Fig. 5.14.	Effect of the thermal loading on the number of cycles.	96

Abstract

Material surfaces in contact are often subjected to damages such as wear, contact fatigue and plastic deformation. Such surface damages not only affect surface performance but also lead to material failure. It is critical to predict these damages for the design of advanced functional materials. On the one hand, micro-defects such as inclusions, voids and cracks, which are formed at or beneath the surface during the material manufacturing process, can induce and increase surface damages. On the other hand, surface coatings are extensively employed to a variety of mechanical components, cutting tools and instruments to provide heat-, oxidation- and wear-resistance. Thus, this PhD research is proposed to investigate damages to contact surfaces of materials with internal defects.

Firstly, the semi-analytic solution for the stress field of multiple cracks and inhomogeneous inclusions of arbitrary shape beneath a half-space surface subjected to prescribed loading has been developed. The interactions among all the inclusions and cracks are fully taken into account by a newly developed methodology that combines the equivalent inclusion method (EIM) and the distributed dislocation technique (DDT). Using the EIM, the inhomogeneous inclusions are modeled as homogeneous inclusions with unknown equivalent eigenstrains. Using the DDT, the cracks are modeled as a distribution of edge dislocations with unknown densities. Coupled governing equations with unknown equivalent eigenstrains and dislocation densities are established to satisfy the stress and strain conditions of inclusions and free surface traction conditions of cracks. A modified conjugate gradient method is developed to solve the governing equations and a fast Fourier transform algorithm is utilized to improve computational efficiency.

Secondly, the semi-analytic solution for multiple cracks and inhomogeneous inclusions beneath a half-space surface subjected to contact loading has been developed. The solution takes into account not only the interactions among all the cracks and inclusions but also the interactions between them and the surface loading body. The original problem of an inhomogeneous half-space under contact loading is first converted into the new problem of a homogeneous half-space with unknown dislocation densities and equivalent eigenstrains using the DDT and EIM under contact loading. The new problem is then decomposed into the sub-problem of a homogeneous half-space with unknown dislocation densities and equivalent eigenstrains and the sub-problem of a homogeneous half-space under a contact loading with unknown surface contact area and pressure. Afterwards, an algorithm is developed to integrate the two sub-problems to determine the unknown dislocation densities and surface contact area and pressure.

Thirdly, the semi-analytic solution for the inhomogeneous half-space contact problem has been extended to account for the crack propagation that is driven by surface cyclic contact loading and affected by the presence of subsurface inclusions. The maximum hoop stress criterion is incorporated for the prediction of crack propagation direction and the Paris law is applied to determine the crack propagation rate. Meanwhile, a zigzag crack path consisting of many small vertical and horizontal cracks is applied to model the arbitrary crack propagation path.

This PhD research provides a framework for investigating inhomogeneous materials with internal defects under contact loading. It has potential applications to provide guidance for various engineering problems concerning material dissimilarity, surface damage and internal failure.

Chapter 1 Introduction

1.1 Background

Materials are often inhomogeneous by containing internal micro-defects such as cracks, dislocations, cavities and inclusions, which would affect their properties and performance. The material properties including strength, ductility, Young's modulus and fatigue behavior can be seriously influenced by internal defects, especially when these defects are located at or beneath a material surface. When these defective materials function as operational components such as gears and bearings, their surfaces experience a contact with other materials and bear cyclic loading. The cyclic contact loading could drive surface or subsurface cracks to propagate and cause eventual failure of the components.

Subsurface defects also affect the tribological behavior and performance of a material surface. When the surfaces of two bodies are in contact, the surface deformation, surface contact area and pressure are substantially different from those in homogeneous contact. For example, when a stiff inclusion is embedded in a matrix material and gets close to its surface right under indentation, the inclusion would cause the peak value of the surface pressure increase dramatically (Zhou *et al.*, 2011a). It is well known that the wear rate of a material surface strongly depends on the amplitude of the peak surface pressure.

A real surface, which generally has roughness and contains defects formed at or beneath it during the material manufacturing process, is often subjected to damages such as wear and contact fatigue, that may lead to the eventual material failure. Wear can be defined as a process where the interaction between two surfaces or bounding faces of solids within the working environment results in

dimensional loss of solid, with or without any actual decoupling and loss of material. Wear phenomena are heavily influenced by the roughness of surfaces. Furthermore, wear behavior is also influenced by the presence of adsorbed species and/or surface layers. Many different wear mechanisms have been identified including adhesive wear, abrasive wear, corrosive wear and fatigue wear. The classic wear model was established based on Archard's wear law (1953) which assumes that the wear rate at any point on the contact surface is proportional to the local contact pressure and the relative sliding velocity. Wear along with other aging processes such as fatigue and creep in association with stress concentration factors causes a material to progressively degrade, eventually leading to material failure at an advanced age.

Contact fatigue is a surface damage commonly found in gears, cams, valves, rails, and gear couplings when the surfaces are subjected to repeated rolling and sliding contacts. It is a persistent damage process which involves the crack initiation, crack propagation and crack-induced fracture due to the subsurface stress in the superficial layer under the action of alternating loading. Contact fatigue has been identified in metal alloys, ceramics and cermets. The contact fatigue results from a contact or Hertzian stress state when curved surfaces are subjected to a cyclic loading, which is different from classic structural fatigue (bending or torsion). If the cyclic contact loading is above a certain threshold, microscopic cracks will begin to form and propagate at the stress concentration points which normally appear near the contact surface, inclusion and dislocations. When the crack propagates to have a length larger than the critical size, the whole structure would fail to bear the external loading and fracture.

Meanwhile, surface coatings have been widely applied in engines,

transmissions and many other components in the manufacturing industry to improve their mechanical performances (heat-, oxidation- and wear-resistance) or restore these components to their original dimensions and specifications due to wear and fatigue. With newly advanced deposition techniques (such as chemical vapour deposition and physical vapour deposition), a wide range of coating layers can be deposited to a variety of materials and structures, which were unachievable a decade ago. The surface coating technique allows engineering of specific properties at a desired location to protect the surface of thermal or mechanical components from damages like fatigue, wear, erosion and corrosion. However, due to the differences in mechanical properties, thermal expansion coefficient and heat conductivity between the coating and the substrate, the cracking or delamination of a coating layer can be easily induced when thermal or contact loading is applied. In addition, the impurity of the coating layer and substrate could further complicate the interaction between them.

Thus, prediction and minimization of these surface damages are extremely important in the areas of manufacturing, aerospace and energy industries. For instance, the quality of manufacturing products is usually determined by the quantity of internal defects. The surface damages in the vital components can cause catastrophic failure of the whole structure. In order to minimize or eliminate the surface damages, new metallurgy and manufacture techniques have been invented to reduce the defects, especially those near the surfaces; new inspection technologies have been developed to detect defects in a non-destructive way; new alloys and composite materials have been designed to enhance their strength, stiffness and fatigue resistance. However, all these efforts need theoretical research of the damage mechanisms of materials.

1.2 Objectives

Damage analysis requires an accurate knowledge of the surface contact area, contact pressure distribution, subsurface stress field in the presence of surface coating and subsurface defects and the stress intensity factors (SIFs) at the crack tips. This research aims to develop a unified framework which is capable of modeling surface damages of materials with internal defects and the main objectives are set as follows:

- (1) Obtain the solution for multiple cracks and inhomogeneous inclusions of arbitrary shape beneath a half-space surface subjected to prescribed loading with the interaction among all the inclusions and cracks being fully taken into account;
- (2) Obtain the solution for multiple cracks and inhomogeneous inclusions under a contact loading with the interactions between the surface loading body and the subsurface cracks and inclusions being considered;
- (3) Develop a fatigue model for the analysis of crack propagation driven by cyclic contact loading with the effect of neighboring inhomogeneous inclusions being considered.

1.3 Dissertation layout

This PhD dissertation is organized into six chapters. Following a brief description of the introduction, research motivation and objectives in Chapter 1, Chapter 2 gives a literature review which summarizes the fundamental concepts and definitions as well as the previous research works about the interactions among inclusion, dislocation, crack and contact indenter. Chapter 3 provides the details of the methodologies for the modeling of multiple inhomogeneous

inclusions and cracks in a half-space and presents the numerical results. The mechanical responses of surface coating and subsurface inclusions and cracks under a contact loading are presented in Chapter 4. The contact fatigue damages in inhomogeneous materials are studied in Chapter 5. In Chapter 6, conclusions are drawn and future works are recommended.

Chapter 2 Literature review

In this Chapter, we first give a brief introduction of the common defects in materials, including inclusions, dislocations and cracks. Previous research works on the interactions among inclusions, dislocations and cracks are also briefly reviewed. The distributed dislocation technique (DDT) and equivalent inclusion method (EIM) for the solutions of a single crack and a single inclusion are then illustrated, respectively. Afterwards, contact problems which involve internal defects are presented. Finally, damages including crack fatigue and surface wear are discussed.

2.1 Defects in materials

2.1.1 Inclusions

Inclusions are a typical type of defects in materials and can be categorized into homogeneous inclusions and inhomogeneous inclusions, as shown in Fig. 2.1. According to Mura's definition (1987), a homogeneous inclusion (Fig. 2.1a) is defined as a subdomain Ω that has the same elastic modulus as its surrounding $\mathbf{D}-\Omega$ but contains eigenstrain, a generic term referring to inelastic strain caused by phase transformation, plastic deformation, thermal expansion, etc. Thus, the remaining domain $\mathbf{D}-\Omega$ is termed the matrix or host material. In contrast, an inhomogeneous inclusion (Fig. 2.1b) has different elastic constants from the matrix and may or may not contain eigenstrains. On the one hand, inclusions are often considered as material dissimilarity in the form of micro-defects such as voids and precipitates which are unintentionally formed during the manufacturing or utilization process. On the other hand, inclusions can be intentionally introduced to

design functional materials and structures like composites and coated materials.

Fig. 2.1. Schematics of (a) a homogeneous inclusions Ω with elastic module C_{ijkl}^D and eigenstrain ε_{ij}^P and (b) an inhomogeneous inclusion with elastic module C_{ijkl}^Ω and eigenstrain ε_{ij}^P embedded in an infinite matrix $D - \Omega$ with elastic modulus C_{ijkl}^D .

2.1.2 Dislocations

Dislocations are a type of crystallographic line defects within a crystal structure. It is an area where the atoms are out of positions. Dislocations are induced and move when a stress is applied. The motion of dislocations leads to lattice slip and subsequent plastic deformation. There are two primary types of dislocations: edge dislocation (Fig. 2.2) and screw dislocation (Fig. 2.3). Mixed dislocations combine aspects of both types.

An edge dislocation is defined by the edge of an extra half plane of atoms, as show in Fig. 2.2a, where the symbol $\perp$ marks the location of the dislocation line at the core of the dislocation (Guy and Hren, 1974). The dislocation line extends out of the crystal, perpendicular to the plane of the drawing. By introducing this extra half plane, the upper part of the crystal is compressed, while the region below

the dislocation experiences considerable dilatation. This kind of edge dislocations is named the climb dislocation (Hills *et al.*, 1996). The other kind of edge dislocations, glide dislocations, resulted from the shifting the atoms over the cut plane in a direction perpendicular to the line AA, as shown in Fig. 2.2b (Weertman and Weertman, 1992). The quantitative description of a dislocation involves a Burgers circuit. The starting point S and the ending point F are not the same atom for the circuit that encloses the dislocation (see Fig. 2.2a). Thus, there is a closure failure FS, which is defined as the Burgers vector b , perpendicular to the dislocation line.

Fig. 2.2. (a) A climb edge dislocation and (b) a glide edge dislocation in a crystal structure.

The screw dislocation is defined by the line AD in Fig. 2.3 (Hertzberg, 1989), which is generated by shifting the atoms on either side of the cut to a distance of one lattice vector in a direction parallel to the cut surface. The Burgers circuit about the screw dislocation assumes the shape of a helix, like a spiral staircase (Fig. 2.3). The closure failure FS leads to the Burgers vector b parallel to the dislocation line. A right-handed screw dislocation is defined when a clockwise 360° rotation causes the helix to advance one lattice vector. The same advance arising from a 360° counterclockwise rotation is a left-handed screw dislocation.

Fig. 2.3. A screw dislocation in a crystal structure.

2.1.3 Cracks

Cracks are the primary objective for defect analysis in fracture mechanics. According to the loading conditions, cracks can be categorized into three types: mode I (opening mode) crack, mode II (in-plane shear mode) crack and mode III (out-of-plane shear, tearing mode) crack as shown in Fig. 2.4. For a mode I crack, a tensile stress is applied normal to the crack plane. For a mode II crack, a shear stress is applied parallel to the crack plane and meanwhile perpendicular to the crack front. For a mode III crack, a shear stress is applied parallel to both the crack

plane and the crack front. In the real cases, cracks are usually subjected to more than one type of the above loading conditions. These cracks are called mixed-mode cracks.

Fig. 2.4. Schematic of three crack types.

2.1.3.1 Griffith cracks

Inglis (1913) first obtained the solutions for an infinitely long, elliptical hole subjected to either a tensile loading or shear loading. Griffith (1921) examined a special case of the Inglis solution for the sharp crack where the minor axis of the ellipse is set equal to zero, and introduced a fracture criterion for brittle materials based on the theorem of minimum potential energy. The sharp crack is usually known as a Griffith crack. According to Griffith's fracture criterion, the low fracture strength observed in experiments, as well as the size-dependence of strength, is due to the presence of microscopic flaws in the bulk material. The Griffith's criterion was further extended to ductile materials in which a high degree of plastic deformation near the tip of the crack is involved (Irwin, 1957; Orowan, 1949). Based on the theory of complex variables, Westergaard (1939) solved the stress and displacement field near the tips of in-plane cracks by introducing a complex stress function, which is known as the Westergaard stress function. During this same period of time, Williams (1957) employed a different technique

to obtain the crack tip solution. He also argued that the stress singularity around a crack, regardless of the actual configuration, is of the inverse square-root type.

Fig. 2.5. Stress components near the tip of a 2D Griffith crack.

Figure 2.5 shows the schematic of a two-dimensional (2D) Griffith crack in an elastic solid under the in-plane loading condition. The crack has a length of $2a$. If a polar coordinate axis with the origin at one of the crack tip is defined, the stress and displacement fields near the crack tips are given by

Mode I crack

$$\sigma_{xx} = \frac{K}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right], \quad (2.1a)$$

$$\sigma_{yy} = \frac{K}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 + \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right], \quad (2.1b)$$

$$\tau_{xy} = \frac{K}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right), \quad (2.1c)$$

$$u_x = \frac{K}{2G} \sqrt{\frac{r}{2\pi}} \cos\left(\frac{\theta}{2}\right) \left[\kappa - 1 + 2 \sin^2\left(\frac{\theta}{2}\right) \right], \quad (2.1d)$$

$$u_y = \frac{K}{2G} \sqrt{\frac{r}{2\pi}} \sin\left(\frac{\theta}{2}\right) \left[\kappa + 1 - 2 \cos^2\left(\frac{\theta}{2}\right) \right]. \quad (2.1e)$$

Mode II crack

$$\sigma_{xx} = -\frac{K}{\sqrt{2\pi r}} \sin\left(\frac{\theta}{2}\right) \left[2 + \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) \right], \quad (2.2a)$$

$$\sigma_{yy} = \frac{K}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right), \quad (2.2b)$$

$$\tau_{xy} = \frac{K}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right], \quad (2.2c)$$

$$u_x = \frac{K}{2G} \sqrt{\frac{r}{2\pi}} \sin\left(\frac{\theta}{2}\right) \left[\kappa + 1 + 2 \cos^2\left(\frac{\theta}{2}\right) \right], \quad (2.2d)$$

$$u_y = -\frac{K}{2G} \sqrt{\frac{r}{2\pi}} \cos\left(\frac{\theta}{2}\right) \left[\kappa - 1 - 2 \sin^2\left(\frac{\theta}{2}\right) \right], \quad (2.2e)$$

where G and κ are elastic constants of materials, and K is a characteristic constant which depends on the crack geometry and the loading and boundary conditions.

2.1.3.2 Stress intensity factor

Irwin (1957) used the Westergaard approach to show that the stresses and displacements near the crack tip could be described by a single constant K that is related to the energy releases rate. This characteristic parameter of the crack tip was later defined as the SIFs of modes I, II and III that can be described as the strengths of the singularity of σ_{yy} , τ_{xy} and τ_{yz} at the crack tip as $r \rightarrow 0$. More precisely, each SIF is one part of the stress magnitude when the stress singularity is removed via multiplication by $\sqrt{2\pi r}$. The factor of $\sqrt{2\pi}$ is included according to convention. As a result (Hills *et al.*, 1996),

$$K_I = \lim_{r \rightarrow 0} \left(\sqrt{2\pi r} \sigma_{yy} \right), \quad (2.3a)$$

$$K_{II} = \lim_{r \rightarrow 0} \left(\sqrt{2\pi r} \tau_{xy} \right) , \quad (2.3b)$$

$$K_{III} = \lim_{r \rightarrow 0} \left(\sqrt{2\pi r} \tau_{yz} \right) . \quad (2.3c)$$

The general form of the SIF is given by

$$K_{(I,II,III)} = Y\sigma\sqrt{\pi a} , \quad (2.4)$$

where a is the half length of the crack; σ is characteristic stress; Y is a dimensionless parameter that depends on the specimen and crack geometry.

2.2 Interactions of defects

2.2.1 Multiple inclusions

Based on the EIM, Moschovidis and Mura (1975) studied the interaction of two ellipsoidal inclusions by using the Taylor series to approximate equivalent eigenstrains. Following their work, Shodja and Sarvestani (2000) investigated the problem of multiple inhomogeneities in which an ellipsoidal inhomogeneity is surrounded by interphase materials. In order to achieve an acceptable accuracy, the Taylor series expansion used in these works requires high-order polynomials. However, it is not straightforward to increase the order of the Taylor expansion, since it involves the complex computation of the higher-order Eshelby tensors (Moschovidis and Mura, 1975; Mura, 1987). Fond *et al.* (2001) compared Moschovidis and Mura's results (1975) with the numerical results obtained from the finite element method (FEM) and provided an exact solution of two identical spherical inclusions in an infinite elastic matrix. Their results further showed that even by using the third-order Taylor series, the solution by Moschovidis and Mura (1975) may still diverge when the inclusions are stiffer than the matrix. Therefore, Rodin and Hwang (1991) developed an EIM-FEM hybrid model to evaluate the

interaction between multiple spherical inhomogeneities. Instead of using the Taylor series approximation, FEM was integrated as part of the hybrid approach to avoid the convergence issue.

Recently, Zhou *et al.* (2011b) developed a semi-analytical method for the solution of multiple interacting arbitrarily shaped inhomogeneous inclusions in an infinite space. The method utilizes Eshelby's EIM to model inhomogeneous inclusions as homogeneous inclusions with unknown equivalent eigenstrains. It is extremely difficult to obtain the Eshelby tensors for the problem of multiple arbitrarily shaped inhomogeneous inclusions. Thus, instead of using the Eshelby tensors to relate total eigenstrains to equivalent eigenstrains, the inclusions are discretized into small cubical elements to employ Chiu's solution (1977) which gives the stress field induced by the eigenstrains in the cubical inclusion problems. Although each element is treated as a cubical inclusion containing uniform eigenstrain, the overall eigenstrain within an inclusion can still be non-uniform. This approach was later used to study the half-space problem with prescribed surface loading (Zhou *et al.*, 2009; Zhou *et al.*, 2012).

2.2.2 Inclusions and dislocations

Inclusions can also interact with dislocations. The solution for the elastic interaction of an edge dislocation with an elliptical inclusion was obtained by Stagni and Lizzio (1983) and Tsuchida *et al.* (1991). Xiao *et al.* (2004; 2008) investigated a dislocation interacting with a coated inclusion or collinear interfacial rigid lines in the piezoelectric materials. Both screw and edge dislocations interacting with interfacial cracks along an elliptical elastic inclusion were studied by Fang *et al.* (2003, 2005). The interactions between screw dislocations and

nanoscale inclusions were also studied by considering the interface effects (Fang *et al.*, 2009; Fang and Liu, 2006; Fang *et al.*, 2008). The distributed dislocations were used to investigate the crack-crack and crack-inclusion interactions by modeling the cracks as pile-ups of edge dislocations (Demir *et al.*, 2001). Besides the above static problems, the time-dependent elastic field induced by a dislocation interacting with a circular inclusion was also obtained by Wang and Pan (2011).

2.2.3 Inclusions and cracks

The interactions between cracks and inclusions are generally studied based on the solution of appropriate boundary value problems in the linear theory of elasticity. Tamate (1968) studied the interaction between a crack and a circular inhomogeneous inclusion in an infinite plate subjected to tension and obtained an analytical solution via the complex function method. Bhargava *et al.* (1973) obtained the elastic field in an infinite elastic matrix containing a circular inhomogeneous inclusion and two symmetrically situated cracks subjected to remote loading. The interaction between an arbitrarily oriented crack and a circular elastic inclusion was also investigated by Erdogan *et al.* (1974). Sendeckyj (1974) gave a solution to the problem in which a crack is located between two rigid inclusions under a uniform shear stress. Hsu and Shivakumar (1976) investigated the interaction between a circular inclusion and two symmetrically placed collinear cracks in an infinite elastic isotropic plane. The elliptical inclusion interacting with a crack was also studied (Chen, 1997; Patton and Santare, 1990). Lam *et al.* (1998) solved the problem of the interaction between a symmetrically branched crack and a circular inclusion in an infinite elastic medium. Yang *et al.* (2004) investigated the interaction between a crack and an inclusion of arbitrary shape. Andreassen and

Karihaloo (1993) provided a numerical method for solving the problem of subsurface inclusion interacting with a surface crack. Matysiak *et al.* (2004) investigated the stationary problems of heat conduction and thermoelasticity for an elastic half-space containing an elastic cylindrical inclusion and a thermally insulated crack. However, only a few highly idealized cases could be analytically solved. For the research of more complex problems, numerical approaches, such as FEM (Gao *et al.*, 2010; Stam *et al.*, 1994; Sukumar *et al.*, 2001; Thomas *et al.*, 2000), boundary element method (BEM) (Bush, 1997; Dong and Lee, 2005; Portela *et al.*, 1993; Wang *et al.*, 1998) were employed.

2.3 Distributed dislocation technique for crack modeling

The DDT employs the superposition principle of Bueckner (1958) and models the Griffith type of cracks by continuously distributing dislocations along the crack lines in otherwise perfect bodies. The Griffith crack problem therefore reduces to determine the densities of the distributed dislocations which generate tractions along the crack line equal and opposite to those produced by the applied loading. According to the superposition principle, the crack face still remains traction free. This method is employed to solve crack problems in this dissertation.

The superposition principle applied in the DDT is illustrated through a Griffith crack problem: a straight crack of length $2a$ is embedded in an infinite elastic space loaded by a tensile stress $\sigma^\infty(x)$, perpendicular to the crack, as shown in Fig. 2.6(a). The superposition principle states

$$\sigma_{ij}(x, y) = \tilde{\sigma}_{ij}(x, y) + \hat{\sigma}_{ij}(x, y), \quad (2.5)$$

where $\tilde{\sigma}_{ij}(x, y)$ is the stress field in the absence of the crack, as shown in Fig.

2.6(b) and $\hat{\sigma}_{ij}(x, y)$ is the stress induced by inserting materials to fill the crack without any external loading, as shown in Fig. 2.6(c), providing that the boundary conditions at the traction-free crack faces and at infinity are satisfied.

Fig. 2.6. Application of the superposition principle to a Griffith crack.

On the other hand, a Griffith crack can be modeled by a continuous distribution of dislocations along the lines where the crack occupies. As shown in Fig. 2.7, the line of the continuously distributed dislocations, each with the Burgers vector b , can be represented by a distribution of infinitesimal dislocations with the density function $\rho(x)$. The dislocations that model a mode I crack are climb edge dislocations; for a mode II crack they are glide edge dislocations; and for a mode III crack they are screw dislocations. For the mixed-mode (modes I and II) crack, they can be simulated by mixed (climb and glide) dislocations (Fig. 2.8).

Fig. 2.7. Distribution of schematic infinitesimal dislocations.

Fig. 2.8. The mixed-mode (I and II) dislocations (climb and glide).

The stress σ_{ij} at point (x, y) in an isotropic infinite space due to an edge dislocation with the Burgers vector components $b^\perp$ and $b^\parallel$ located at (x^d, y^d) is given by

$$\sigma_{ij}(x, y, x^d, y^d) = \sigma_{ij}^\perp(x, y, x^d, y^d) + \sigma_{ij}^\parallel(x, y, x^d, y^d), \quad (i, j = 1, 2) \quad (2.6)$$

where

$$\sigma_{ij}^\perp(x, y, x^d, y^d) = \frac{2\mu}{\pi(\kappa + 1)} b^\perp(x^d, y^d) G_{ij}^\perp(x, y, x^d, y^d) \quad , \quad (2.7)$$

$$\sigma_{ij}^{\perp}(x, y, x^d, y^d) = \frac{2\mu}{\pi(\kappa + 1)} b^{\perp}(x^d, y^d) G_{ij}^{\perp}(x, y, x^d, y^d) . \quad (2.8)$$

In Eqs. (2.6)–(2.8), the Burgers vectors $b^{\perp}$ and $b^{\parallel}$ are associated with the climb and glide dislocations, respectively; κ is the Kolosov constant of the infinite-space material ($\kappa = 3 - 4\nu$ for the plane-strain problem with ν being the Poisson's ratio) and μ is the shear modulus; $G_{ij}^{\perp}$ and $G_{ij}^{\parallel}$ are the influence functions (Hills *et al.*, 1996).

By using the DDT, the stress field $\hat{\sigma}_{ij}(x, y)$ due to a mixed-mode crack can be obtained from the integration of the stress field components $\sigma_{ij}(x, y, x^d, y^d)$ caused by the distributed dislocations $\rho(x')dx'$ along the crack line:

$$\begin{aligned} \hat{\sigma}_{ij}(x, y) = \frac{2\mu}{\pi(\kappa + 1)} & \left[\int_{-a}^{+a} \rho^{\perp}(x') G_{ij}^{\perp}(x, y, x^d - x', y^d) dx' \right] \\ & + \int_{-a}^{+a} \rho^{\parallel}(x') G_{ij}^{\parallel}(x, y, x^d - x', y^d) dx' . \end{aligned} \quad (2.9)$$

In order to obtain the dislocation densities $\rho^{\perp}(x)$ and $\rho^{\parallel}(x)$, the free traction condition is introduced. According to this condition, the normal and shear stress components along the crack surface are zero. Thus, the two stress components arising from the external loading and the dislocations must equal zero along the line of the crack surface as well:

$$\tilde{\sigma}_{ij}(x, y) + \hat{\sigma}_{ij}(x, y) = 0 . \quad (i = 2, j = 2 \text{ and } i = 1, j = 2) \text{ along crack} \quad (2.10)$$

Once the dislocation densities $\rho^{\perp}(x)$ and $\rho^{\parallel}(x)$ are obtained from Eq. (2.10), the displacement across the crack can be related to the density of the distributed dislocation $\rho(x)$ by the relation:

$$D(x) = - \int_{-a}^x \rho(x) dx . \quad (2.11)$$

For a Griffith crack, the crack faces must be closed at both crack ends:

$D(a) = D(-a) = 0$. In other words, there is no net dislocation resulting in zero sum of the dislocation density. This condition can be written as

$$\int_{-a}^{+a} \rho(x) dx = 0 \quad . \quad (2.12)$$

It is noted that the DDT can also be used to model a crack embedded in a half elastic space by a similar procedure. The detailed equations can be found in Hill's monograph (1996).

2.4 Equivalent inclusion method for inclusion modeling

The Green's function for a three-dimensional (3D) homogeneous isotropic elastic solid was first obtained by Kelvin (1848). This function gives the displacement caused by a point force in an infinite space. Based on Green's function, the EIM was developed by Eshelby (1957) to obtain the analytical solution for the ellipsoidal inhomogeneous inclusion embedded in an infinite matrix. In the EIM, an inhomogeneous inclusion Ω is treated as a homogeneous inclusion plus an equivalent eigenstrain ε_{ij}^* which can be calculated by the Eshelby tensor S_{ijkl} (Fig. 2.9). The tensor S_{ijkl} relates the constrained strain inside the inclusion to its eigenstrain and can be expressed in terms of surface integral of the Green's function over the inclusion boundary. It was also proved that S_{ijkl} is a constant for an ellipsoidal inclusion embedded in a homogeneous infinite matrix and the stress-strain fields inside the inclusion are uniform (Eshelby, 1957). Later, Eshelby (1959) further examined the elastic field outside an ellipsoidal inclusion.

Fig. 2.9. (a) Schematic of an inhomogeneous inclusions Ω with elastic constants C^{Ω}_{ijkl} and initial eigenstrain ε^P_{ij} embedded in an infinite matrix $D - \Omega$ with elastic constants C^D_{ijkl} . (b) Using EIM, Ω is regarded as homogeneous inclusion with initial eigenstrain ε^P_{ij} plus equivalent eigenstrain ε^*_{ij} .

Eshelby's (1957, 1959) pioneer works became the base for many following research works concerning the inclusion problems. For infinite space problems, the solutions for a cubical inclusion (Chiu, 1977), a cylindrical inclusion (Wu and Du, 1995) and a strip inclusion (Dundurs and Markenscoff, 2009) were developed. The solutions for non-uniform eigenstrain problems were also obtained (Asaro and Barnett, 1975; Mura and Kinoshita, 1978). Moreover, Eshelby's method was extended to solve the inclusion embedded in anisotropic medium (Walpole, 1967) and piezoelectric materials (Wang, 1992).

In the half-space, the elastic stress field induced by an ellipsoidal homogeneous inclusion with uniform dilatational eigenstrains was investigated by Seo and Mura (1979). Tsuchida and Mura (1983) gave the stress field in a half-space containing an ellipsoidal inhomogeneity under tension. The solutions for a cuboidal inclusion (Chiu, 1978), a cylindrical inclusion (Wu and Du, 1996)

and a hemispherical inclusion (Wu, 2003) were obtained. Pan and Yang (2001) studied a cubic inclusion in an anisotropic half-space.

2.5 Contact problems with internal defects

2.5.1 Classical elastic contact problems

The original work about the contact behavior between two elastic bodies was first published by Hertz (1882b). In the same year, Hertz (1882a) extended his work to the 2D case of line contact, illustrated how the results might be applied to the definition and measurement of hardness, and discussed the results of experiments to support his theory. This classical solution provided a foundation for modern contact problems such as load bearing capabilities, fatigue life in bearings and any other bodies in contact.

The classical approach to finding stress and displacement due to a point force acting on the surface of a half-space was developed by employing the theory of potential (Love, 1944). This approach gives fundamental solutions for the investigation of frictional contact problems. Carter (1926) and Fromm (1927) independently presented the creep effect on the creep force for a cylinder on a plane or for two cylinders in steady rolling contact using Coulomb's dry friction law. Cattaneo (1938) gave a solution for contact bodies subjected to a constant normal force and oscillatory shear force. Mindlin (1949) and Deresiewicz (1953) extended the solution to consider more complex loading sequences. In the 1950s, the interest in the rolling contact of railway wheels was raised. Johnson (1958) first presented an approximate approach for 3D rolling contact problem with either lateral or spin creepage. Kalker (1967) published his milestone PhD dissertation on rolling contact for the situation of an infinite friction coefficient in which case the

slip area vanished. In his dissertation, numerical methods for rolling contact were also developed.

Johnson *et al.* (1971) studied adhesive contact problems and compared their results with experimental results on the contact of rubber and gelatin spheres. Their theory, which came to be known as the Johnson-Kendall-Roberts (JKR) model for adhesive elastic contact, considers the effect of contact pressure and adhesion inside the area of contact by using a balance between the stored elastic energy and the loss in surface energy. This theory was later opposed by Derjaguin and his co-workers (1975) who proposed a different theory about adhesion contact which came to be known as the Derjaguin-Muller-Toporov (DMT) model. The DMT model considers Van der Waals interactions outside the elastic contact regime, which gives rise to an additional load. Tabor (1977) defined a dimensionless parameter (Tabor coefficient) to represent the ratio between the gap outside the contact zone and the equilibrium distance between atoms. This study showed that the JKR and DMT models are two opposite extreme cases of adhesion contact condition and the Tabor coefficient could establish the range of applicability of the two models. Müller *et al.* (1980; 1983) solved this problem numerically by using a Lennard-Jones potential, thereby demonstrating a continuous transition from the DMT to JKR as the Tabor coefficient increases. Maugis (1992) further developed the Tabor's theory by considering surface force both inside and outside the contact zone in terms of the Dugdale cohesive zone approximation and showed the transition from the DMT to JKR behavior as two opposite ends of a continuous spectra based on the Maugis parameter, which is equivalent to the Tabor coefficient. However, the Maugis's model can only be solved iteratively if the Maugis parameter is not known *a priori*. Carpick (1999) gave an approximate solution

which simplifies the process by using the empirical formulas to determine the contact radius.

2.5.2 Contact bodies interacting with defects

Since subsurface cracks and inclusions play an important role in material damage and failure, researches have been conducted to investigate the underlying mechanisms. The earliest work that can be found in the literature was conducted three decades ago by Miller and Keer (1983), who studied the stress field associated with a near-surface void or inclusion under rolling contact with a rigid indenter. Afterwards, almost no progress had been made on this topic for a long time. Salehizadeh and Saka (1992) analyzed the stress field around a hard particle under the rolling line contact using FEM. Liu and Farris (1994) studied the effect of 3D near surface defects under rolling and sliding contact using BEM. In regard to the subsurface crack, Goshima *et al.* dealt with the thermoelastic rolling contact problem for single (Goshima and Soda, 1997) and multiple subsurface cracks (Goshima *et al.*, 2001). Ueda *et al.* (2007) analyzed the SIFs of the randomly oriented crack emanating from an inclusion in a half-space under rolling/sliding contact with frictional heat.

Recently, Kuo (2008) investigated multiple inhomogeneities of 2D arbitrary shape in a half-space under cylindrical indentation using BEM. More recently, Zhou *et al.* (2011a; 2011d) solved the inhomogeneous inclusion problem with contact loading involved based on their former work (Zhou *et al.*, 2009) in conjunction with the theory of contact mechanics. Moreover, it was demonstrated that this approach is also capable of modeling the elasto-plastic contact behaviors (Chen *et al.*, 2010) by regarding the plastic zone as a homogeneous inclusion with

eigenstrain. Leroux *et al.* (2010) conducted contact analysis in presence of spherical inhomogeneities within a half-space using the EIM. Chen *et al.* (2013a) studied the interaction between dislocation and subsurface crack caused by half-plane contact.

2.6 Fatigue behavior

2.6.1 Fundamental mechanisms of fatigue

Fatigue is defined as a damage that results in fracture after a sufficient number of stress fluctuations. Load conditions often involve a load spectrum of different stress ranges and Miner's hypothesis of cumulative damage enables the groups of amplitude to be replaced by an equivalent number of cycles N to a single stress magnitude S (1973). Thus, the fatigue performance can be characterized as a plot of stress range versus number of cycles to failure. Figure 2.10 shows the S - N curves for both carbon steel and aluminum. Some materials, such as carbon steel, show an endurance limit stress below which the fatigue life is essentially infinite. Other materials may not show such behavior but an effective endurance limit may be specified at some large number of cycles.

From a fracture mechanics perspective, fatigue crack propagation can also be described by plotting the rate of crack growth per cycle of loading da/dN against the SIF range $\Delta K = K_{\max} - K_{\min}$ on a double logarithmic scale. Figure 2.11 shows a schematic of log-log plot of da/dN versus ΔK , which illustrates typical fatigue crack growth behaviors in metals. The curve can be divided into three regions. Region I is the region just beyond the threshold ΔK_{th} , below which ΔK is too small to propagate a crack. Region II is where $\log(da/dN)$ varies linearly

with $\log(\Delta K)$. In Region III, the crack growth rate da/dN accelerates as ΔK approaches the material fracture toughness K_c .

Fig. 2.10. S - N curves for carbon steel and aluminum.

Fig. 2.11. Fatigue crack growth rate versus the SIF range.

Paris and Erdogan (1963) first found the power law correlation between the crack growth rate and the SIF range in Region II:

$$\frac{da}{dN} = C\Delta K^m, \quad (2.13)$$

where C and m are material constants that are determined experimentally. This empirical relation is widely known as the Paris-Erdogan Law (1963). Following this law, researchers examined the relation between the crack advance rate da/dN and range of SIF ΔK . Forman *et al.* (1967) proposed the following relationship for Regions II and III:

$$\frac{da}{dN} = \frac{C\Delta K^{m-1}}{\frac{K_c}{K_{\max}} - 1}. \quad (2.14)$$

Equation (2.14) shows that the crack growth rate becomes infinite as $K_{\max}$ approaches the fracture toughness K_c . Forman considered the influence of $K_{\max}$ and K_c on the advance rate da/dN , while Paris and Erdogan assumed that da/dN only depends on ΔK . Based on Paris and Erdogan's work, Klesnil and Lukas (1972) obtained Eq. (2.15) to consider the effect of the threshold ΔK_{th} on the advance rate da/dN in Regions I and II:

$$\frac{da}{dN} = C(\Delta K^m - \Delta K_{th}^m). \quad (2.15)$$

Combining the advantage of Eq. (2.14) and (2.15), semi-empirical equations which can describe the entire crack growth curve (Region I to III) were developed (McEvily, 1988; Priddle, 1976)

$$\frac{da}{dN} = C \left(\frac{\Delta K - \Delta K_{th}}{K_c - K_{\max}} \right)^m, \quad (2.16)$$

$$\frac{da}{dN} = C(\Delta K - \Delta K_{th})^2 \left(1 + \frac{\Delta K}{K_c - K_{max}} \right). \quad (2.17)$$

2.6.2 Crack propagation under the influence of internal defects

Numerous works have been reported on the study of crack propagation behaviors in materials. Hansson (2009) investigated the effect of crack length, grain size, load range and grain boundary on the crack growth rate under fatigue loading. Karakasidis and Charitidis (2011) examined how the grain boundaries of inclusions influence the crack propagation modes in an inhomogeneous material. Demir *et al.* (2001) further analyzed crack propagation behaviors with the consideration of interactions among cracks, inclusions and voids. Zhou *et al.* (2011c) developed a fracture-based model to predict chipping wear observed on hard coatings under cyclic contact loading. These works were addressed analytically or semi-analytically. The FEM and extended FEM (XFEM) were also widely used to study the crack growth behavior (Kikuchi *et al.*, 2014; Serna Moreno *et al.*, 2015; Zhao *et al.*, 2014). In such a case, the re-meshing process becomes a time-consuming work, and it is a bottleneck for the application of FEM to fatigue crack growth problems.

Chen *et al.* (2013b) experimentally studied the effects of inclusions, grain boundaries and grain orientations on the crack initiation and propagation behavior in 2524-T3 aluminum alloy using *in-situ* scanning electron microscope fatigue testing and electron back scattering diffraction. The results showed that potential fatigue cracks tend to nucleate along coarse and closely spaced inclusion particles or high-angle grain boundaries and the coarse inclusion particles drastically accelerate local crack growth rates. Krewerth *et al.* (2016) investigated the

influence of non-metallic inclusions embedded in cast steel 42CrMo4 on fatigue life in the high cycle fatigue and the very high cycle fatigue regime by using an ultrasonic fatigue testing system. The experimental results demonstrated that three key parameters affecting the material fatigue life are (i) the size of the crack initiating discontinuity, (ii) the inclusion depth and (iii) the crack initiating failure type. Gallo and Berto (2015) further experimentally examined the high-temperature fatigue of 40CrMoV13.9 steel and the effect of surface roughness on fatigue strength and cracks initiation.

Chapter 3 Defects near surfaces under prescribed surface loading

In this chapter, the problem of multiple cracks and inhomogeneous inclusions embedded in a half-space under prescribed surface loading is studied. The interactions between the inclusions and cracks are investigated and exemplified with four different cases.

3.1 Methodology

3.1.1 Problem description and solution approach

A semi-infinite matrix contains arbitrarily-shaped inhomogeneous inclusions Ω_ψ ($\psi = 1, 2, \dots, n_1$) and cracks Γ_φ ($\varphi = 1, 2, \dots, n_2$) as shown in Fig. 3.1a. The half-space surface is bounded by the plane $y=0$ in a 2D xOy Cartesian coordinate system. The elastic moduli of the matrix and the inclusions Ω_ψ are denoted by C_{ijkl} and C_{ijkl}^ψ , respectively. For generality, inclusions Ω_ψ may contain initial eigenstrain ε_{ij}^P . The half-space matrix is subjected to an external load (concentrated or distributed) acting on the surface. The stress at any point in the half-space caused by the external load is denoted by σ_{ij}^0 .

By using the EIM, each inhomogeneous inclusion Ω_ψ can be modeled as a homogeneous inclusion with initial eigenstrain ε_{ij}^P plus unknown equivalent eigenstrain ε_{ij}^* (Fig. 3.1b). By using the DDT, a crack of mode I can be modeled as a distribution of climb edge dislocations with unknown densities $\rho^\perp$, while a crack of mode II can be modeled as a distribution of glide edge dislocations with unknown densities $\rho^\parallel$. Thus, mixed-mode crack of modes I and II can be modeled

as a distribution of climb and glide edge dislocations with their respective unknown densities (Fig. 3.1b).

In this way, the original inhomogeneous half-space problem is converted into a homogeneous half-space problem. The equivalent eigenstrains and dislocation densities are introduced to represent the material dissimilarity, the interactions among the inclusions and cracks, and their responses to the external loading.

Fig. 3.1. (a) The original problem of multiple inhomogeneous inclusions and cracks in a half-space, and (b) the new problem of multiple equivalent homogeneous inclusions and dislocation distributions.

In each inhomogeneous inclusion Ω_ψ (Fig. 3.1a), the total strain ε_{ij} is given by the elastic strain ε_{ij}^e plus the initial eigenstrain ε_{ij}^P , i.e., $\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^P$ or $\varepsilon_{ij}^e = \varepsilon_{ij} - \varepsilon_{ij}^P$. However, due to the introduction of the eigenstrain ε_{ij}^* , the ε_{ij}^e in the equivalent homogeneous inclusion (Fig. 3.1b) is given by $\varepsilon_{ij}^e = \varepsilon_{ij} - \varepsilon_{ij}^P - \varepsilon_{ij}^*$. According to Hooke's law, $\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^P - \varepsilon_{kl}^*)$. The combination of the two stress-strain formulas gives the stress σ_{ij} at any point in Ω_ψ as

$$\sigma_{ij} = C_{ijkl}(C_{klmq}^{\psi^{-1}}\sigma_{mq} - \varepsilon_{kl}^*).$$

On the other hand, the stress σ_{ij} at any point in each equivalent inclusion Ω_ψ is decomposed into $\sigma_{ij} = \sigma_{ij}^* + \sigma_{ij}^P + \sigma_{ij}^\Gamma + \sigma_{ij}^0$, where σ_{ij}^* is the eigenstress caused by the equivalent eigenstrains ε_{ij}^* in all the inclusions, σ_{ij}^P is the eigenstress caused by the initial eigenstrains ε_{ij}^P in all the inclusions, σ_{ij}^Γ is the stress due to all the cracks, and σ_{ij}^0 is the applied stress due to external loading. The combination of the two stress expressions leads to the following governing equations:

$$(1 - C_{ijkl} C_{klmq}^{\psi -1})(\sigma_{mq}^P + \sigma_{mq}^* + \sigma_{mq}^\Gamma + \sigma_{mq}^0) + C_{ijkl} \varepsilon_{kl}^* = 0 \quad .$$

$$(\psi = 1, 2, \dots, n_1; \quad i, j, k, l, m, q = 1, 2) \quad \text{within} \quad \Omega_\psi. \quad (3.1a)$$

In this model, the two surfaces of a crack are assumed not to be in contact with each other. Thus, the conditions of free surface tractions along the normal and tangential directions should be satisfied for each crack. Thus, the following governing equations can be established:

$$(\sigma_{ij}^P + \sigma_{ij}^* + \sigma_{ij}^\Gamma + \sigma_{ij}^0) v_j = 0 \quad , \quad (\varphi = 1, 2, \dots, n_2; i, j = 1, 2) \quad \text{along} \quad \Gamma_\varphi \quad (3.1b)$$

where v_j indicates the crack surface normal vector.

It should be noted that the situation in which the two surfaces of a crack are in contact due to external loading can also be handled by considering the interfacial traction and displacement continuity conditions rather than free-surface traction conditions in Eq. (3.1b).

The challenge is to solve the governing equation (3.1). In the next sub-section, numerical methods would be introduced to express the governing equations (3.1a) and (3.1b) in terms of the unknown equivalent eigenstrains ε_{ij}^* and dislocation densities $\rho^\perp$ and $\rho^\top$ and then solve these unknowns.

3.1.2 Crack modeling

A square domain D in the half-space is chosen which contains the cracks $C_\varphi (\varphi = 1, 2, \dots, m)$ and inhomogeneous inclusions $\Omega_\psi (\psi = 1, 2, \dots, n)$ with the origin $O (0, 0)$ of the x - y Cartesian coordinate system being set at one corner of the domain D (Fig. 3.2). The domain D is discretized into $N_x \times N_y$ square elements of the same size $2\Delta_x \times 2\Delta_y$. Each element is indexed by a sequence of two integers $[\alpha, \beta]$ ($0 \leq \alpha \leq N_x - 1, 0 \leq \beta \leq N_y - 1$). For convenience, only horizontal cracks parallel to the x -axis are discussed here, considering the fact that vertical cracks can be investigated in a similar way without adding extra difficulties.

Fig. 3.2. Cracks and arbitrarily-shaped inclusions in 2D discretized domain.

For simulating the mixed-modes I and II crack, the stress σ_{ij}^Γ at a given point (x, y) induced by a glide dislocation and a climb dislocation located at $(0, \eta)$ beneath the surface is given by

$$\sigma_{ij}^\Gamma(x, y, \xi) = \sigma_{ij}^\perp(x, y, \eta) + \sigma_{ij}^\parallel(x, y, \eta) \quad , \quad (3.2)$$

$$\sigma_{ij}^\perp(x, y, \xi) = \frac{2\mu}{\pi(\kappa + 1)} b^\perp G_{ij}^\perp(x, y, \eta) \quad , \quad (3.3)$$

$$\sigma_{ij}^{\pm}(x, y, \xi) = \frac{2\mu}{\pi(\kappa + 1)} b^{\pm} G_{ij}^{\pm}(x, y, \eta) \quad , \quad (i, j = 1, 2) \quad (3.4)$$

where $\sigma_{ij}^{\perp}$ and $\sigma_{ij}^{\parallel}$ represent stresses induced by two types of edge dislocations: climb and glide dislocations, respectively; κ is the Kolosov constant of the half-space material ($\kappa = 3 - 4\nu$ for the plane-strain problem with ν being the Poisson's ratio) and μ is the shear modulus; The functions $G_{ij}^{\perp}(x, y, \eta)$ and $G_{ij}^{\parallel}(x, y, \eta)$ represent the influence functions (Hills *et al.*, 1996).

The cracks Γ_{φ} are discretized into many short line segments, each of which is modeled as distributed dislocation segments along the central lines of corresponding elements. In each line segments, the dislocation densities $\rho^{\perp}$ and $\rho^{\parallel}$ can be considered as linear functions $\rho^{\perp} = c^{\perp}x + d^{\perp}$ and $\rho^{\parallel} = c^{\parallel}x + d^{\parallel}$. Thus, the entire crack is approximated by a piecewise linear function. Although dislocation densities in each distributed dislocation segment are assumed to be linear, the dislocation densities in the entire crack are non- linear (Fig.3.3).

Fig. 3.3. A crack modeled by a distribution of dislocations approximated by a piecewise linear function.

A line segment of the length $2a$ is set to be centered at $(0, \eta)$. The stress field induced by the line of edge dislocations can be calculated as follows

$$\begin{aligned}
\sigma_{ij}^{\Gamma}(x, y, \eta) = & \frac{2\mu}{\pi(\kappa + 1)} \int_{-a}^a [c^{\perp} x' + d^{\perp}] G_{ij}^{\perp}(x - x', y, \eta) dx' \\
& + \frac{2\mu}{\pi(\kappa + 1)} \int_{-a}^a [c^{\parallel} x' + d^{\parallel}] G_{ij}^{\parallel}(x \\
& - x', y, \eta) dx' \\
& (i, j = 1, 2) \quad x' \in [-a, a] \quad (3.5)
\end{aligned}$$

The stress σ_{ij}^{Γ} at point (x, y) due to the edge dislocations can be written as

$$\begin{aligned}
\sigma_{ij}^{\Gamma}(x, y, \eta) = & \frac{2\mu}{\pi(\kappa + 1)} [E_{ij}^{\perp}(x, y, \eta) c^{\perp} + F_{ij}^{\perp}(x, y, \eta) d^{\perp} \\
& + E_{ij}^{\parallel}(x, y, \eta) c^{\parallel} + F_{ij}^{\parallel}(x, y, \eta) d^{\parallel}] \quad (3.6)
\end{aligned}$$

The analytic forms of influence coefficients $E_{ij}^{\perp}$, $F_{ij}^{\perp}$, $E_{ij}^{\parallel}$ and $F_{ij}^{\parallel}$ for the horizontal and vertical cracks are given in Appendices A and B, respectively.

The stress at the observation point (x_{α}, y_{β}) induced by all edge dislocations can be obtained by the summation of contribution of dislocations in all the squares in the whole discrete domain.

$$\begin{aligned}
\sigma_{\alpha, \beta}^{\Gamma} = & \frac{2\mu}{\pi(\kappa + 1)} \sum_{\xi=0}^{N_y-1} \sum_{\zeta=0}^{N_x-1} (\mathbf{E}_{a-\xi, \beta-\zeta}^{\perp} c_{\xi, \zeta}^{\perp} + \mathbf{F}_{a-\xi, \beta-\zeta}^{\perp} d_{\xi, \zeta}^{\perp} + \mathbf{E}_{a-\xi, \beta-\zeta}^{\parallel} c_{\xi, \zeta}^{\parallel} + \mathbf{F}_{a-\xi, \beta-\zeta}^{\parallel} d_{\xi, \zeta}^{\parallel}) \cdot \\
& (0 \leq \alpha \leq N_x - 1, 0 \leq \beta \leq N_y - 1) \quad (3.7)
\end{aligned}$$

where $\mathbf{E}_{a-\xi, \beta-\zeta}^{\perp}$, $\mathbf{F}_{a-\xi, \beta-\zeta}^{\perp}$, $\mathbf{E}_{a-\xi, \beta-\zeta}^{\parallel}$ and $\mathbf{F}_{a-\xi, \beta-\zeta}^{\parallel}$ are the influence coefficients which relate the stress $\sigma_{\alpha, \beta}^{\Gamma}$ at the observation point (x_{α}, y_{β}) in the element $[\alpha, \beta]$ to the edge dislocation parameter $c_{\xi, \zeta}^{\perp}$, $d_{\xi, \zeta}^{\perp}$, $c_{\xi, \zeta}^{\parallel}$ and $d_{\xi, \zeta}^{\parallel}$ in the element $[\xi, \zeta]$.

Since the density distributions are continuous, the piecewise linear function should also be continuous along the whole crack. Thus, the parameters c and d in the square element $[\alpha, \beta]$ should satisfy

$$c_{\alpha,\beta}^{\perp} \Delta_x + d_{\alpha,\beta}^{\perp} = -c_{\alpha+1,\beta}^{\perp} \Delta_x + d_{\alpha+1,\beta}^{\perp} , \quad (3.8a)$$

$$c_{\alpha,\beta}^{\perp} \Delta_x + d_{\alpha,\beta}^{\perp} = -c_{\alpha+1,\beta}^{\perp} \Delta_x + d_{\alpha+1,\beta}^{\perp} . \quad (3.8b)$$

For a crack of length $2a$, its opening at a certain location can be obtained by Eq. (2.11). The crack faces must close at both crack ends. According to Eqs. (2.12) and (3.8), for a horizontal crack starting from the element $[m_1, \beta]$ and ending at the element $[m_2, \beta]$, the following relationship should be satisfied:

$$\sum_{\alpha=m_1}^{m_2} d_{\alpha,\beta}^{\perp} = 0 , \quad (3.9a)$$

$$\sum_{\alpha=m_1}^{m_2} d_{\alpha,\beta}^{\perp} = 0 . \quad (3.9b)$$

3.1.3 Inclusion modeling

As mentioned in Section 3.1.1, each inhomogeneous inclusion is modeled as a homogeneous inclusion with initial eigenstrain ε_{ij}^p plus unknown equivalent eigenstrain ε_{ij}^* . In the meshed domain D , each rectangular inhomogeneous inclusion is modeled as a homogeneous inclusion with ε_{ij}^p plus ε_{ij}^* . The eigenstrain in each rectangular equivalent homogeneous inclusion is treated as uniform. Nevertheless, the overall eigenstrain within each Ω_{ψ} can still be non-uniform.

Chiu (1978) obtained the solution for a cuboidal homogeneous inclusion containing uniform eigenstrain in an isotropic half-space. The superposition of his solution could give the eigenstress due to multiple cuboidal homogeneous inclusions. In fact, Zhou *et al.* (2012) took this approach to obtain the solution for multiple 3D inhomogeneous inclusions of arbitrary shape beneath a surface. Here,

the degeneration of their solution from 3D to 2D should be utilized.

Using the EIM (Eshelby, 1959) and the stress superposition, the eigenstress at each observation point (set at the center of each element) can be obtained:

$$\boldsymbol{\sigma}_{\alpha,\beta}^P = \sum_{\zeta=0}^{N_y-1} \sum_{\xi=0}^{N_x-1} \mathbf{B}_{\alpha-\xi,\beta-\zeta} \boldsymbol{\varepsilon}_{\xi,\zeta}^P, \quad (0 \leq \alpha \leq N_x - 1, 0 \leq \beta \leq N_y - 1) \quad (3.10a)$$

$$\boldsymbol{\sigma}_{\alpha,\beta}^* = \sum_{\zeta=0}^{N_y-1} \sum_{\xi=0}^{N_x-1} \mathbf{B}_{\alpha-\xi,\beta-\zeta} \boldsymbol{\varepsilon}_{\xi,\zeta}^* \quad (0 \leq \alpha \leq N_x - 1, 0 \leq \beta \leq N_y - 1) \quad (3.10b)$$

where $\mathbf{B}_{\alpha-\xi,\beta-\zeta}$ are the influence coefficients (in the 2×2 matrix form) relating the stress $\boldsymbol{\sigma}_{\alpha,\beta}^P$ and $\boldsymbol{\sigma}_{\alpha,\beta}^*$ at the point (x_α, y_β) in the element $[\alpha, \beta]$ to the initial eigenstrain $\boldsymbol{\varepsilon}_{\xi,\zeta}^P$ and equivalent eigenstrain $\boldsymbol{\varepsilon}_{\xi,\zeta}^*$ in the element $[\xi, \zeta]$ centered at (x_ξ, y_ζ) . The expressions of $\mathbf{B}_{\alpha-\xi,\beta-\zeta}$ refer to the work by Zhou *et al.* (2009).

Similarly, the applied stress $\sigma_{\alpha,\beta}^0$ at the observation point (x_α, y_β) in the element $[\alpha, \beta]$ due to external surface loading F is obtained by the following formula:

$$\sigma_{\alpha,\beta}^0 = \sum_{\xi=0}^{N_x-1} \mathbf{M}_{\alpha-\xi,\beta} p_{\xi,0} + \sum_{\xi=0}^{N_x-1} \mathbf{T}_{\alpha-\xi,\beta} f_{\xi,0}, \quad (3.11)$$

where $p_{\xi,0}$ and $f_{\xi,0}$ represent the normal pressure and the tangential traction on the discretized surface area in the element $[\xi, 0]$. The expressions of $\mathbf{M}_{\alpha-\xi,\beta}$ and $\mathbf{T}_{\alpha-\xi,\beta}$ refer to the solution of subsurface stresses induced by the normal pressure and the tangential traction acting on a rectangular area beneath the surface.

3.1.4 Governing equation

In the above section, the stress σ_{ij}^Γ due to cracks is expressed in terms of the

representative parameters $c^\perp$, $d^\perp$, $c^\top$, and $d^\top$ of the dislocation densities; the eigenstresses σ_{ij}^P and σ_{ij}^* are expressed in terms of eigenstrains ε_{ij}^P and ε_{ij}^* , respectively. The substitution of these expressions (Eqs. (3.7), (3.10) and (3.11)) into the previous governing equation (3.1) results in

$$\begin{aligned}
(\mathbf{I} - \mathbf{C}\mathbf{C}_{\alpha,\beta}^{-1}) & \left[\sum_{\zeta=0}^{N_y-1} \sum_{\xi=0}^{N_x-1} (\mathbf{B}_{a-\xi,\beta-\zeta} \varepsilon_{\xi,\zeta}^* + \mathbf{B}_{a-\xi,\beta-\zeta} \varepsilon_{\xi,\zeta}^P) \right. \\
& + \frac{2\mu}{\pi(\kappa+1)} \sum_{\zeta=0}^{N_y-1} \sum_{\xi=0}^{N_x-1} (\mathbf{E}_{a-\xi,\beta-\zeta}^\perp c_{\xi,\zeta}^\perp + \mathbf{F}_{a-\xi,\beta-\zeta}^\perp d_{\xi,\zeta}^\perp + \mathbf{E}_{a-\xi,\beta-\zeta}^\top c_{\xi,\zeta}^\top \\
& \left. + \mathbf{F}_{a-\xi,\beta-\zeta}^\top d_{\xi,\zeta}^\top) + \sum_{\xi=0}^{N_x-1} (\mathbf{M}_{a-\xi,\beta} p_{\xi,0} + \mathbf{T}_{a-\xi,\beta} f_{\xi,0}) \right] + \mathbf{C}\varepsilon_{\alpha,\beta}^* = 0 \\
& (0 \leq \alpha \leq N_x - 1, 0 \leq \beta \leq N_y - 1) \quad \text{within } \Omega_\psi, \quad (3.12a)
\end{aligned}$$

$$\begin{aligned}
& \left\{ \sum_{\zeta=0}^{N_y-1} \sum_{\xi=0}^{N_x-1} (\mathbf{B}_{a-\xi,\beta-\zeta} \varepsilon_{\xi,\zeta}^P + \mathbf{B}_{a-\xi,\beta-\zeta} \varepsilon_{\xi,\zeta}^*) \right. \\
& + \frac{2\mu}{\pi(\kappa+1)} \sum_{\zeta=0}^{N_y-1} \sum_{\xi=0}^{N_x-1} (\mathbf{E}_{a-\xi,\beta-\zeta}^\perp c_{\xi,\zeta}^\perp + \mathbf{F}_{a-\xi,\beta-\zeta}^\perp d_{\xi,\zeta}^\perp + \mathbf{E}_{a-\xi,\beta-\zeta}^\top c_{\xi,\zeta}^\top \\
& \left. + \mathbf{F}_{a-\xi,\beta-\zeta}^\top d_{\xi,\zeta}^\top) + \sum_{\xi=0}^{N_x-1} (\mathbf{M}_{a-\xi,\beta} p_{\xi,0} + \mathbf{T}_{a-\xi,\beta} f_{\xi,0}) \right\} \mathbf{A} = 0 \\
& (0 \leq \alpha \leq N_x - 1, 0 \leq \beta \leq N_y - 1) \quad \text{along } \Gamma_\varphi, \quad (3.12b)
\end{aligned}$$

where $\mathbf{I}$ is a unit matrix, $\mathbf{B}_{a-\xi,\beta-\zeta}$, $\mathbf{E}_{a-\xi,\beta-\zeta}^\perp$, $\mathbf{F}_{a-\xi,\beta-\zeta}^\perp$, $\mathbf{E}_{a-\xi,\beta-\zeta}^\top$, $\mathbf{F}_{a-\xi,\beta-\zeta}^\top$, $\mathbf{M}_{a-\xi,\beta}$ and $\mathbf{T}_{a-\xi,\beta}$ are the influence coefficients which relate to their respective sources; $\varepsilon_{\xi,\zeta}^P$ and $\varepsilon_{\xi,\zeta}^*$ denote the initial eigenstrain and unknown equivalent eigenstrain; $c_{\xi,\zeta}^\perp$, $d_{\xi,\zeta}^\perp$, $c_{\xi,\zeta}^\top$ and $d_{\xi,\zeta}^\top$ are the unknown edge dislocation parameters; $\mathbf{A}$ is the matrix form of crack surface normal vector $\mathbf{v}_j$.

The governing equation (3.12) involves double summations of eigenstrains

and dislocation densities. A technique of discrete convolution and fast Fourier transform (DC-FFT) (Liu *et al.*, 2000; Polonsky and Keer, 2001) is employed to improve the computational efficiency for the time-consuming double summation if a large number of elements are taken.

In order to use the DC-FFT method, the original domain D is expanded twice into $2N_x \times 2N_y$ to create cyclic convolution by using the technique of the wrap-around order. Thus, Eq. (3.10a) becomes

$$\sigma_{\alpha,\beta}^P = \sum_{\zeta=0}^{2N_y-1} \sum_{\xi=0}^{2N_x-1} \mathbf{B}_{\alpha-\xi,\beta-\zeta} \boldsymbol{\varepsilon}_{\xi,\zeta}^P \quad (0 \leq \alpha \leq 2N_x-1, 0 \leq \beta \leq 2N_y-1) \quad (3.13)$$

The Fourier transform can transfer a function defined in a plane domain into a function in a frequency domain. For example, the discrete double Fourier transform of $\boldsymbol{\varepsilon}_{\xi,\zeta}^P$ in the expanded plane domain is defined as

$$\hat{\boldsymbol{\varepsilon}}_{\alpha,\beta}^P = \sum_{m=0}^{2N_y-1} \sum_{l=0}^{2N_x-1} e^{imn\alpha/N_y} e^{i\pi l\beta/N_x} \boldsymbol{\varepsilon}_{m,l}^P \quad (0 \leq \alpha \leq 2N_x-1, 0 \leq \beta \leq 2N_y-1) \quad (3.14)$$

Equation (3.14) could be computed by the FFT algorithm faster than a direct summation. The denotation “ $\wedge$ ” on top of the symbols stands for an FFT operation. The application of the FFT on two sides of Eq. (3.13) leads to

$$\hat{\sigma}_{\alpha,\beta}^P = \hat{\mathbf{B}}_{\alpha,\beta} \hat{\boldsymbol{\varepsilon}}_{\alpha,\beta}^P \quad (3.15)$$

The $\hat{\sigma}_{\alpha,\beta}^P$ in each element is simply the multiplication of $\hat{\mathbf{B}}_{\alpha,\beta}$ and $\hat{\boldsymbol{\varepsilon}}_{\alpha,\beta}^P$ in the corresponding element. The final stress $\sigma_{\alpha,\beta}^P$ is easily obtained by applying an inverse FFT on $\hat{\sigma}_{\alpha,\beta}^P$ as

$$\sigma_{\alpha,\beta}^P = \text{FFT}^{-1}(\hat{\sigma}_{\alpha,\beta}^P) \quad (3.16)$$

This method only requires $O(3 \times 2N_x \times 2N_y \times \ln(2N_x \times 2N_y))$ operations (two FFT operations and one inverse FFT) to complete the summations. However,

the direct summations need $O\left((2N_x \times 2N_y)^2\right)$ operations. Equations (3.7) and (3.10b) could use the same procedure to replace the direct summations. Therefore, the computation time is greatly reduced by using the DC-FFT method.

Finally, the governing equation (3.12) and supplemental equations (3.8) and (3.9) are essentially a set of linear algebraic equations, which can be represented in a simple form as $\mathbf{G}x = a$. Hence, a modified conjugate gradient method (CGM) (Shewchuk, 1994; Zhou *et al.*, 2011b) can be used to determine the unknown equivalent eigenstrains ε^* and dislocation density parameters $c^\perp$, $d^\perp$, $c^\parallel$ and $d^\parallel$.

The CGM is a well-established method in solving linear equations $\mathbf{G}x = a$ by means of iteration. When the conventional CGM is used, the iteration convergence to the solution is absolute provided that $\mathbf{G}$ is a symmetric positive matrix. Nevertheless, a deep convergence to the solution before divergence or oscillation may also be obtained through using the CGM although $\mathbf{G}$ is not a symmetric positive matrix. Based on the conventional CGM (Shewchuk, 1994), the modified CGM algorithm for solving $\mathbf{G}x = a$ is listed as follows:

$$d_{(0)} = r_{(0)} = a - \mathbf{G}x_{(0)} \quad (3.17)$$

$$\tau_{(i)} = k \frac{r_{(i)}^T r_{(i)}}{d_{(i)}^T \mathbf{G} d_{(i)}} \quad (3.18)$$

$$x_{(i+1)} = x_{(i)} + \tau_{(i)} d_{(i)} \quad (3.19)$$

$$r_{(i+1)} = r_{(i)} - \tau_{(i)} \mathbf{G} d_{(i)} \quad (3.20)$$

$$\rho_{(i+1)} = \frac{r_{(i+1)}^T r_{(i+1)}}{r_{(i)}^T r_{(i)}} \quad (3.21)$$

$$d_{(i+1)} = r_{(i+1)} + \rho_{(i+1)} d_{(i)} \quad (3.22)$$

where the integer number in the subscript indicates the iteration step. The vectors d and r are the conjugate direction and conjugate residual, respectively; the vector r^T is the transpose of r ; the variable τ is the step length in each iteration; the factor k which is no greater than one is newly introduced into the modified CGM, compared with the conventional CGM, to reduce the step length before divergence or oscillation begins.

Once all the eigenstrains ε^* and dislocation densities $\rho^\perp$ and ρ^+ are determined, the subsurface stress field can be obtained by

$$\begin{aligned}
\sigma_{\alpha,\beta} &= \sigma_{\alpha,\beta}^\Gamma + \sigma_{\alpha,\beta}^P + \sigma_{\alpha,\beta}^* + \sigma_{\alpha,\beta}^0 \\
&= \sum_{\zeta=0}^{N_y-1} \sum_{\xi=0}^{N_x-1} \mathbf{B}_{a-\xi,\beta-\zeta} (\varepsilon_{\xi,\zeta}^P + \varepsilon_{\xi,\zeta}^*) + \sum_{\xi=0}^{N_x-1} (\mathbf{M}_{a-\xi,\beta} p_{\xi,0} + \mathbf{T}_{a-\xi,\beta} f_{\xi,0}) \\
&\quad + \frac{2\mu}{\pi(\kappa+1)} \sum_{\zeta=0}^{N_y-1} \sum_{\xi=0}^{N_x-1} (\mathbf{E}_{a-\xi,\beta-\zeta}^\perp c_{\xi,\zeta}^\perp + \mathbf{F}_{a-\xi,\beta-\zeta}^\perp d_{\xi,\zeta}^\perp + \mathbf{E}_{a-\xi,\beta-\zeta}^+ c_{\xi,\zeta}^+ \\
&\quad + \mathbf{F}_{a-\xi,\beta-\zeta}^+ d_{\xi,\zeta}^+) \\
&\quad (0 \leq \alpha \leq N_x - 1, 0 \leq \beta \leq N_y - 1). \tag{3.23}
\end{aligned}$$

3.1.5 Stress intensity factor

The SIFs of a mixed-modes I and II crack of length l can be written as (Anderson, 2005)

$$K_I = \lim_{x \rightarrow l} [\sqrt{2\pi(x-l)} \sigma_{yy}^t] \quad , \tag{3.24}$$

$$K_{II} = \lim_{x \rightarrow l} [\sqrt{2\pi(x-l)} \sigma_{xy}^t] \quad . \tag{3.25}$$

The total stress σ^t can be divided into two parts: the stress σ induced by the crack and the stress σ^Δ induced by the inclusions and external loading. According to Eq. (3.23), σ^Δ should have a finite value at the crack tip. Thus, the

SIFs are only relevant to the stress σ . Since the crack is modeled by the DDT, the stress can be obtained by integral of the climb and glide dislocation densities:

$$\sigma_{yy} = \frac{2\mu}{\pi(\kappa + 1)} \int_0^l \frac{\rho^\perp(\xi)}{x - \xi} d\xi \quad , \quad (3.26)$$

$$\sigma_{xy} = \frac{2\mu}{\pi(\kappa + 1)} \int_0^l \frac{\rho^\parallel(\xi)}{x - \xi} d\xi \quad . \quad (3.27)$$

Then, the SIFs can be rewritten as

$$K_I = \frac{2\mu}{\pi(\kappa + 1)} \lim_{x \rightarrow l} \left[\sqrt{2\pi(x - l)} \int_0^l \frac{\rho^\perp(\xi)}{x - \xi} d\xi \right] \quad , \quad (3.28)$$

$$K_{II} = \frac{2\mu}{\pi(\kappa + 1)} \lim_{x \rightarrow l} \left[\sqrt{2\pi(x - l)} \int_0^l \frac{\rho^\parallel(\xi)}{x - \xi} d\xi \right] \quad . \quad (3.29)$$

Equations (3.28) and (3.29) can be normalized to the interval [-1,1] by the following equations:

$$\xi = \frac{l}{2}(1 + t) \quad , \quad x = \frac{l}{2}(1 + s) \quad . \quad (3.30)$$

If the following formula are taken

$$\phi^\perp(t) = \sqrt{1 - t^2} \rho^\perp(t) \quad , \quad \phi^\parallel(t) = \sqrt{1 - t^2} \rho^\parallel(t) \quad , \quad (3.31)$$

Eqs. (3.28) and (3.29) will be converted to the non-dimensionalized equations

$$K_I = \frac{2\mu}{\pi(\kappa + 1)} \lim_{s \rightarrow 1} \left[\sqrt{\pi l(s - 1)} \int_{-1}^1 \frac{\phi^\perp(t)}{(s - t)\sqrt{1 - t^2}} dt \right] \quad , \quad (3.32)$$

$$K_{II} = \frac{2\mu}{\pi(\kappa + 1)} \lim_{s \rightarrow 1} \left[\sqrt{\pi l(s - 1)} \int_{-1}^1 \frac{\phi^\parallel(t)}{(s - t)\sqrt{1 - t^2}} dt \right] \quad . \quad (3.33)$$

Thus,

$$K_I^A = \frac{\mu\sqrt{2\pi l}}{(\kappa + 1)} \phi^\perp(1) \quad , \quad (3.34a)$$

$$K_I^B = -\frac{\mu\sqrt{2\pi l}}{(\kappa + 1)} \phi^\perp(-1) \quad , \quad (3.34b)$$

$$K_{II}^A = \frac{\mu\sqrt{2\pi l}}{(\kappa + 1)} \phi^+(1) \quad , \quad (3.35a)$$

$$K_{II}^B = -\frac{\mu\sqrt{2\pi l}}{(\kappa + 1)} \phi^+(-1) \quad , \quad (3.35b)$$

where the superscripts A and B represent the crack tips A and B, respectively.

According to Eqs. (3.12a) and (3.12b), the unknown dislocation densities $\rho^\perp(x_{m^\varphi})$ and $\rho^+(x_{m^\varphi})$ can be obtained at their observation points $(x_{m^\varphi}, y_\alpha)$, $(m_1^\varphi \leq m^\varphi \leq m_2^\varphi)$. Thus, the numerical values of $\phi^\perp(t_m)$ and $\phi^+(t_m)$ ($1 \leq m \leq m_2^\varphi - m_1^\varphi + 1$) can be calculated from Eqs. (3.30) and (3.31). The functions $\phi^\perp(t)$ and $\phi^+(t)$ are considered as a polynomial of order N with unknown coefficients

$$\phi(t) = \sum_{n=0}^N a_n t^n \quad . \quad (3.36)$$

The unknown coefficients a_n ($n < m$) can be calculated by data fitting of the numerical solutions $\phi^\perp(t_m)$ and $\phi^+(t_m)$ using the least square method. Then, Eqs. (3.34) and (3.35) can be written as

$$K_I^A = \frac{\mu\sqrt{2\pi l}}{(\kappa + 1)} \sum_{n=0}^N a_n^\perp \quad , \quad (3.37a)$$

$$K_I^B = -\frac{\mu\sqrt{2\pi l}}{(\kappa + 1)} \sum_{n=0}^N (-1)^n a_n^\perp \quad , \quad (3.37b)$$

$$K_{II}^A = \frac{\mu\sqrt{2\pi l}}{(\kappa + 1)} \sum_{n=0}^N a_n^+ \quad , \quad (3.38a)$$

$$K_{II}^B = -\frac{\mu\sqrt{2\pi l}}{(\kappa + 1)} \sum_{n=0}^N (-1)^n a_n^+ \quad . \quad (3.38b)$$

3.2 Numerical results

3.2.1 SIF approach validation

To validate the proposed method, a case with two collinear cracks under uniform remote stresses σ^∞ is considered as shown in Fig. 3.4, where a_0 and d denote respectively the half-length of the cracks and the half-distance between them. The mode I SIFs K^A and K^B of the crack tips A and B are normalized by $K_0 = \sigma^\infty \sqrt{\pi a_0}$. Figure 3.5 plots the effect of the geometric configuration of the cracks, which is described by a scalar factor d/a_0 , on the dimensionless SIFs K^A/K_0 and K^B/K_0 . Meanwhile, the SIFs K^A/K_0 and K^B/K_0 obtained by the proposed method are compared with the exact solution reported by Erdogan (1962). The results show a good agreement between the two methods.

Fig. 3.4. Schematic of two collinear cracks under uniform normal stresses.

Fig. 3.5. Comparison of SIFs K^A/K_0 and K^B/K_0 for two collinear cracks under uniform normal stresses.

3.2.2 One inclusion and one crack

Figure 3.6 shows the schematic of a crack Γ and a square inclusion Ω beneath the surface of a semi-infinite matrix, bounded by the x -axis in the x - y Cartesian coordinate system. The inclusion Ω of the side length a is set to be centered at $(0, a)$, and the crack of the length $l = a$ is centered at $(0, 0.2a)$. The matrix has the Young's modulus $E_m = 420$ GPa and the Poisson's ratio $\nu_m = 0.3$. The inhomogeneous inclusion has $E_\Omega = \lambda E_m$ and $\nu_\Omega = 0.3$. The parameter λ represents the material difference between the inclusion and the matrix: $\lambda < 1$ represents a compliant inclusion; while $\lambda > 1$ represents a stiff inclusion. The external load $P = F = \sigma^0 = 1$ GPa on the surface is uniformly distributed along a length of a centered at $(0, 0)$.

Fig. 3.6. Schematic of a crack and an inclusion beneath the surface.

Figure 3.7 compares the subsurface von Mises stresses along $x = 0.25a$ for the case of $\lambda = 0.57$ and $h = 0.2a$ obtained by the present method and the FEM. It shows that the results obtained by the two methods agree well. Figure 3.8 presents the contours of the subsurface stress field.

Fig. 3.7. Comparison of the von Mises stresses σ_v beneath the surface obtained by the present method and the FEM, respectively.

Fig. 3.8. Contours of the stress components and von Mises stress.

Figure 3.9 illustrates the mode II SIF versus h/a without the inhomogeneous inclusion ($\lambda = 1$). The mode II SIF decreases as the crack depth h increases. It is clear that surface tangential traction has a stronger influence on the mode II SIF when crack approaches the surface. Figure 3.10 presents the effect of the inclusion on the mode II SIF for the case of $h = 0.2a$. The near-crack inclusion affects the mode II SIF on the crack tips and the effect dramatically increases as the Young's modulus of the inclusion increases, which shows the importance of considering the influence of a near-crack inclusion in studying the behavior of crack propagation.

Fig. 3.9. The mode II SIF ($\text{MPa} \cdot \text{m}^{1/2}$) versus h/a without the inhomogeneous inclusion.

Fig. 3.10. The mode II SIF ($\text{MPa} \cdot \text{m}^{1/2}$) versus λ for the case of $h = 0.2a$.

3.2.3 Three inclusions and two cracks

Figure 3.11 shows a stringer of inhomogeneous inclusions and two cracks beneath a half-space surface. The inclusions Ω_1 , Ω_2 and Ω_3 have the same side

length a and are centered at $(1.25a, 1.25a)$, $(0, 1.25a)$ and $(-1.25a, 1.25a)$, respectively. All the three inclusions have uniform initial eigenstrain ε^0 . The cracks Γ_1 and Γ_2 of length $l = a$ are centered at $(0.75a, 0.5a)$ and $(-0.75a, 0.5a)$, respectively. The inclusions have the elastic moduli $E_I = 240$ GPa and $\nu_I = 0.3$; while the matrix has $E_M = 420$ GPa and $\nu_M = 0.3$.

Fig. 3.11. Schematic of three inclusions and two cracks underneath a surface.

Figure 3.12 plots the crack opening D induced by different initial eigenstrains. The shape of the crack opening varies as the initial eigenstrain ε_x^0 changes (From $\varepsilon_x^0 = -0.003$ to $\varepsilon_x^0 = -0.011$). As shown in Fig. 3.12, the crack is opened by the initial eigenstrain in the inclusions rather than the external load at surface and the opening increases dramatically as the magnitude of the eigenstrain increases. Figure 3.13 shows the effect of the initial eigenstrain on the mode I SIF K^L and K^R for the Γ_1 crack. The SIF at the left tip of the crack is smaller than that at its right tip and both SIFs imply a linear increase in the eigenstrain magnitude.

Fig. 3.12. Crack opening for different initial eigenstrains.

Fig. 3.13. Mode I SIF (MPa $\cdot$ m $^{1/2}$) for different initial eigenstrains.

3.2.4 Three cracks separated by two inclusions

Figure 3.14 shows the schematic of three cracks separated by two square inclusions underneath the surface of a semi-infinite matrix. The two inclusions Ω_1 and Ω_2 have the same side length a and are centered at $(1.25a, 0.75a)$ and $(-1.25a, 0.75a)$. The three cracks of length $l = a$ lie between these inclusions and centered at $(2.5a, 0.75a)$, $(0, 0.75a)$ and $(-2.5a, 0.75a)$. The matrix has the Young's modulus $E_m = 420$ GPa and the Poisson's ratio $\nu_m = 0.3$. The inhomogeneous inclusion has $E_\Omega = 240$ GPa and $\nu_\Omega = \lambda_\nu \nu_m$. The external load $P = 1$ GPa on the surface is uniformly distributed along a length of $2a$ centered at $(0, 0)$.

Fig. 3.14. Schematic of three cracks separated by two square inclusions underneath a surface subjected to a normal pressure.

Figure 3.15 presents the openings D of the cracks Γ_1 , Γ_2 and Γ_3 for the case in which all the three inhomogeneous inclusions contain the same initial eigenstrain $\varepsilon_x^0 = \varepsilon_y^0 = 0.03$ and $\lambda_\nu = 1$. Figure 3.16 plots the effect of the initial eigenstrain ε_x^0 on the mode I SIF for Γ_2 when $\varepsilon_y^0 = 0.03$ and $\lambda_\nu = 1$. Figure 3.17 shows the influence of the Poisson's ratio of the inclusions on the mode I SIF for Γ_2 when $\varepsilon_x^0 = \varepsilon_y^0 = 0.03$. The influence increases as the Poisson's ratio of the

inclusion increases.

Fig. 3.15. Openings of the cracks Γ_1 , Γ_2 and Γ_3 .

Fig. 3.16. Mode I SIF ($\text{MPa} \cdot \text{m}^{1/2}$) of Γ_2 versus ϵ_x^0 when $\epsilon_y^0 = 0.03$.

Fig. 3.17. Mode I SIF (MPa · m^{1/2}) of Γ_2 versus λ_v when $\varepsilon_x^0 = \varepsilon_y^0 = 0.03$.

3.3 Summary

In this chapter, a semi-analytic solution for multiple cracks and inhomogeneous inclusions in an elastic half-space subjected to prescribed surface loading is developed in the scope of 2D plain-strain problems. The interactions among all the inclusions and cracks are fully taken into account by a newly developed methodology that combines the EIM and the DDT.

Using the EIM, each inhomogeneous inclusion is modeled as a homogeneous inclusions with unknown equivalent eigenstrain. Using the DDT, each crack is modeled by a distribution of edge dislocations with unknown densities. Coupled governing equations with unknown equivalent eigenstrains and dislocation densities are established to satisfy the stress and strain conditions of inclusions and free surface traction conditions of cracks. All the unknown equivalent eigenstrains and dislocation densities are solved by a modified CGM. The FFT algorithm is

also employed to greatly improve computational efficiency. The SIFs of the cracks are obtained by the dislocation densities at their tips and their openings are also obtained through the integration of the dislocation densities along them.

The solution is verified by comparing its results with those obtained by the FEM. Three numerical examples are presented to provide the analysis of crack-inclusion interactions in specific cases. The study shows that the SIF and opening of the cracks can be greatly affected by the crack position, elastic mismatch between the inclusions and matrix and the magnitude of the initial eigenstrain.

Chapter 4 Defects near surfaces under contact loading

In this chapter, both frictionless and frictional contact problems for multiple cracks and inclusions embedded in a half-space are studied. The procedure developed in the previous chapter is adopted to investigate the interactions among the surface coating layer, subsurface inclusions and cracks. The effects of the crack depth, the surface friction condition and surface deformation are investigated.

4.1 Methodology

4.1.1 Problem description and solution approach

Consider a 2D plane-strain problem about multiple inhomogeneous inclusions Ω_ψ ($\psi = 1, 2, \dots, n_1$) and cracks Γ_φ ($\varphi = 1, 2, \dots, n_2$) near a half-space surface subjected to sliding/rolling contact loading in the xOy Cartesian coordinate system, as shown in Fig. 4.1(a). Each inclusion Ω_ψ can be of arbitrary shape. Each crack Γ_φ is assumed to be horizontally or vertically aligned. The loading body is assumed to be cylindrical and has the radius R . The inhomogeneous inclusion Ω_ψ contains initial eigenstrain ε_{ij}^P and its elastic moduli is denoted by C_{ijkl}^ψ ($i, j, k, l = 1, 2$); the half-space matrix has the elastic moduli C_{ijkl} . The cylindrical loading body is isotropic and has the Young's modulus E_s and the Poisson's ratio ν_s .

The surface coating layer can also be considered here (Fig. 4.1(a)). The coating of thickness H can be regarded as an inhomogeneous inclusion Ω_1 with elastic moduli C_{ijkl}^1 . The inclusion Ω_1 should be large enough along the

x -axis to neglect the boundary error caused by the portion of the coating outside the simulation domain. Thus, the coating-substrate problem becomes a half-space contact problem with the inhomogeneous inclusions $\Omega_\psi (\psi = 1, 2, \dots, n_1)$ and cracks $\Gamma_\varphi (\varphi = 1, 2, \dots, n_2)$.

When an external load W is applied on the upper loading body, a contact area is formed between it and the lower half-space. The normal pressure and tangential friction within the contact area not only cause the deformation of the two contacting surfaces but also induce the responses of subsurface cracks and inhomogeneous inclusions. These subsurface responses also change the surface deformation, which in turn changes the contact pressure and friction distributions. Thus, there exists an interaction between the surface loading body and the subsurface cracks and inhomogeneous inclusions.

The same approach used in Chapter 3 is applied to convert the inhomogeneous half-space contact problem into a homogeneous half-space contact problem (Fig. 4.1(b)). After that, the new homogeneous half-space contact problem is further decomposed into two sub-problems: (1) the half-space sub-problem to determine subsurface equivalent eigenstrains ε_{ij}^* and dislocations $\rho^\perp$ and $\rho^\parallel$ for a prescribed surface loading, and (2) the homogeneous half-space contact sub-problem to determine the surface deformation, contact area and loading distribution for a given external load applied on the loading body. The two sub-problems should be correlated because of the above-discussed interactions between the surface loading body and subsurface cracks and inhomogeneous inclusions (or the dislocations and eigenstrains that model them). This correlation is realized by an iterative algorithm that determines the final surface deformation due to both the surface loading and subsurface micro-defects.

Fig. 4.1. (a) The original inhomogeneous half-space coating-substrate contact problem, and (b) the equivalent new homogeneous half-space contact problem.

4.1.2 Half-space sub-problem with prescribed surface loading

The stress analysis of the half-space sub-problem is detailed discussed in Chapter 3. In order to solve the governing equation (3.1), a computational domain D of size $L_x \times L_y$ is chosen to contain all the cracks Γ_φ and inclusions Ω_ψ in a half-space bounded by the surface $y=0$ in an xOy coordinate system. The inhomogeneous inclusion Ω_1 of size $L_x \times H$ at the surface is considered as the surface coating of thickness H . The domain D is discretized into $N_x \times N_y$ square elements of the same size $2\Delta_x \times 2\Delta_y$ (Fig. 4.2). Each element is indexed by $[\alpha, \beta]$ with $0 \leq \alpha \leq N_x - 1$ and $0 \leq \beta \leq N_y - 1$. Therefore, the unknown eigenstrains ε^* and dislocation densities $\rho^\perp$ and $\rho^\top$ can be determined by Eqs (3.8), (3.9) and (3.12). Meanwhile, the subsurface stress field can be obtained by Eq. (3.13).

Fig. 4.2. Discretization of a computational domain containing all the inclusions and cracks into $N_x \times N_y$ square elements.

The surface normal displacement $u(x)$ contains three parts: (1) the surface displacement $u^\Omega(x)$ due to subsurface inclusions; (2) the surface displacement $u^\Gamma(x)$ due to subsurface cracks (or the all edge dislocations which model them); (3) the surface displacement $u^0(x)$ due to the loading body in contact with a homogeneous half-space (after the DDT and EIM are applied).

The surface normal displacement due to an edge dislocation at $(0, \eta)$ can be written as (Hills *et al.*, 1996)

$$u_y = \frac{1}{2\pi(\kappa + 1)} (b^\perp U_{yy} + b^\parallel U_{xy}) \quad , \quad (4.1)$$

where

$$U_{yy} = (\kappa + 1) \left(\pi - 2 \tan^{-1} \frac{x}{\eta} \right) + \frac{2\eta x}{r^2} (\kappa - 1) \quad , \quad (4.2)$$

$$U_{xy} = -\frac{2\eta^2}{r^2} (\kappa + 1) \quad , \quad (4.3)$$

$$r^2 = \eta^2 + x^2 \quad . \quad (4.4)$$

In a similar way to that in the previous chapter, the dislocation densities $\rho^\perp$ and $\rho^\parallel$ are described by the linear functions $\rho^\perp = c^\perp x + d^\perp$ and $\rho^\parallel = c^\parallel x + d^\parallel$. The normal displacement due to a distribution of edge dislocations that model a horizontal crack segment of length $2a$ centered at $(0, \eta)$ can be obtained by

$$\begin{aligned} u^\Gamma(x, \eta) &= \frac{1}{2\pi(\kappa + 1)} \int_{-a}^a [\rho^\perp(x') U_{yy}(x - x', \eta) + \rho^\parallel(x') U_{xy}(x - x', \eta)] dx' \\ &= H^\perp(x, \eta) c^\perp + K^\perp(x, \eta) d^\perp + H^\parallel(x, \eta) c^\parallel + K^\parallel(x, \eta) d^\parallel. \end{aligned} \quad (4.5)$$

The expressions of the influence coefficients $H^\perp$, $K^\perp$, $H^\parallel$ and $K^\parallel$ for the horizontal crack are given as below (coefficients for the vertical crack are given in Appendix C)

$$\begin{aligned} H^\perp &= \frac{1}{2\pi(\kappa + 1)} \left[\tan^{-1} \left(\frac{a - x}{\eta} \right) + \tan^{-1} \left(\frac{a + x}{\eta} \right) \right] (\kappa a^2 + 3\kappa \eta^2 - \kappa x^2 + a^2 - \eta^2 - x^2) \\ &\quad - a\eta \frac{3\kappa - 1}{\pi(\kappa + 1)} - \frac{\kappa \eta x}{\pi(\kappa + 1)} \ln \left(\frac{a^2 - 2ax + x^2 + \eta^2}{a^2 + 2ax + x^2 + \eta^2} \right) \end{aligned}, \quad (4.6)$$

$$\begin{aligned} K^\perp &= \frac{1}{\pi} \left[(a - x) \tan^{-1} \left(\frac{a - x}{\eta} \right) + (a + x) \tan^{-1} \left(\frac{a + x}{\eta} \right) \right] \\ &\quad + \frac{\kappa \eta}{\pi(\kappa + 1)} \ln \left(\frac{a^2 + 2ax + x^2 + \eta^2}{a^2 - 2ax + x^2 + \eta^2} \right) + a \end{aligned}, \quad (4.7)$$

$$H^\parallel = -\frac{\eta x}{\pi} \left[\tan^{-1} \left(\frac{a - x}{\eta} \right) + \tan^{-1} \left(\frac{a + x}{\eta} \right) \right] - \frac{\eta^2}{2\pi} \ln \left(\frac{a^2 - 2ax + x^2 + \eta^2}{a^2 + 2ax + x^2 + \eta^2} \right), \quad (4.8)$$

$$K^\parallel = -\frac{\eta}{\pi} \left[\tan^{-1} \left(\frac{a - x}{\eta} \right) + \tan^{-1} \left(\frac{a + x}{\eta} \right) \right]. \quad (4.9)$$

As a result, the surface normal displacement u_α^Γ at the element $[\alpha, 0]$ due to all the subsurface cracks can be obtained by summing up all the displacement contributions of the edge dislocations that represent them in the computational domain:

$$u_{\alpha}^{\Gamma} = \sum_{\zeta=0}^{N_y-1} \sum_{\xi=0}^{N_x-1} (H_{\alpha-\xi,\zeta}^{\perp} c_{\xi,\zeta}^{\perp} + K_{\alpha-\xi,\zeta}^{\perp} d_{\xi,\zeta}^{\perp} + H_{\alpha-\xi,\zeta}^{\parallel} c_{\xi,\zeta}^{\parallel} + K_{\alpha-\xi,\zeta}^{\parallel} d_{\xi,\zeta}^{\parallel}) \quad . \quad (4.10)$$

The surface displacement due to subsurface inclusions can be found in the works by Zhou *et al.* (2011a) and Jacq *et al.* (2002):

$$u_{\alpha}^{\Omega} = \sum_{\zeta=0}^{N_y-1} \sum_{\xi=0}^{N_x-1} \mathbf{S}_{\alpha-\xi,\zeta} \boldsymbol{\varepsilon}_{\xi,\zeta}^{\text{P}} + \sum_{\zeta=0}^{N_y-1} \sum_{\xi=0}^{N_x-1} \mathbf{S}_{\alpha-\xi,\zeta} \boldsymbol{\varepsilon}_{\xi,\zeta}^{*} \quad , \quad (4.11)$$

where $\mathbf{S}_{\alpha-\xi,\zeta}$ are the influence coefficients relating the eigen-displacement u_{α}^{P} and u_{α}^{*} at the point $(x_{\alpha}, 0)$ to the initial eigenstrain $\boldsymbol{\varepsilon}_{\xi,\zeta}^{\text{P}}$ and equivalent eigenstrain $\boldsymbol{\varepsilon}_{\xi,\zeta}^{*}$ in the rectangular element centered at (x_{ξ}, y_{ζ}) . The expressions of $\mathbf{S}_{\alpha-\xi,\zeta}$ refer to the work by Zhou *et al.* (2011a).

The surface displacement $u^0(x)$ due to surface pressure $p(x)$ and tangential traction $f(x)$ is obtained by (Johnson, 1985)

$$u^0(x) = \int_{-\infty}^{\infty} D^n(x-x') p(x') dx' + \int_{-\infty}^{\infty} D^f(x-x') f(x') dx' \quad , \quad (4.12)$$

where

$$D^n(x) = -\frac{1}{\pi} \left(\frac{1-v_u^2}{E_u} + \frac{1-v_s^2}{E_s} \right) \ln \frac{x^2}{L^2} \quad , \quad (4.13)$$

$$D^f(x) = \frac{1}{2} \left(\frac{1-v_u-2v_u^2}{E_u} + \frac{1-v_s-2v_s^2}{E_s} \right) \text{sgn}(x) \quad . \quad (4.14)$$

The functions $D^n(x)$ and $D^f(x)$ represent the normal surface displacements induced by one unit concentrated normal force and one unit concentrated tangential force at the origin point, respectively. The parameter L in the expression for $D^n(x)$ is a constant for selecting the reference point $(\pm L, 0)$ where the normal surface displacement is zero. The sign function $\text{sgn}(x)$ is defined to be +1

for $x > 0$ and -1 for $x < 0$.

Similarly, the surface normal displacement u_α^0 at the element $[\alpha, 0]$ due to surface normal pressure and tangential traction can be obtained by summing up all the displacement contributions of constant pressure and traction within discretized small surface areas in the computational domain:

$$u_\alpha^0 = \sum_{\xi=0}^{N_x-1} Q_{\alpha-\xi}^n p_{\xi,0} + \sum_{\xi=0}^{N_x-1} Q_{\alpha-\xi}^f f_{\xi,0} \quad , \quad 0 \leq \alpha \leq N_x - 1 \quad (4.15)$$

where

$$Q_{\alpha-\xi}^n = \int_{x_\xi - \Delta x}^{x_\xi + \Delta x} D^n(x_\alpha - x) dx \quad , \quad (4.16)$$

$$Q_{\alpha-\xi}^f = \int_{x_\xi - \Delta x}^{x_\xi + \Delta x} D^f(x_\alpha - x) dx \quad . \quad (4.17)$$

The influence coefficients $Q_{\alpha-\xi}^n$ and $Q_{\alpha-\xi}^f$ relate the normal surface displacement u_α^0 at the observation point $(x_\alpha, 0)$ to the constant normal pressure $p_{\xi,0}$ and constant tangential traction $f_{\xi,0}$ on the discretized surface area centered at $(x_\xi, 0)$, respectively.

In Eqs. (4.16) and (4.17), $Q_{\alpha-\xi}^n$ and $Q_{\alpha-\xi}^f$ are calculated by integration of the functions $D^n(x)$ and $D^f(x)$ over a discretized area centered at $(x_\xi, 0)$, respectively. The indefinite integrals of $D^n(x)$ and $D^f(x)$ are given by

$$\int D^n(x) dx = -\frac{1}{\pi} \left(\frac{1-v_u^2}{E_u} + \frac{1-v_s^2}{E_s} \right) x \left(\ln \left| \frac{x}{L} \right| - 1 \right) \quad , \quad (4.18)$$

$$\int D^f(x) dx = \frac{1}{2} \left(\frac{1-v_u-2v_u^2}{E_u} + \frac{1-v_s-2v_s^2}{E_s} \right) |x| \quad . \quad (4.19)$$

Finally, the normal surface displacement is given as the summation of the two parts: the one due to the subsurface cracks and the one due to the surface loading

body:

$$\begin{aligned}
u_\alpha = & \sum_{\zeta=0}^{N_y-1} \sum_{\xi=0}^{N_x-1} (H_{a-\xi,\zeta}^\perp c_{\xi,\zeta}^\perp + K_{a-\xi,\zeta}^\perp d_{\xi,\zeta}^\perp + H_{a-\xi,\zeta}^\top c_{\xi,\zeta}^\top + K_{a-\xi,\zeta}^\top d_{\xi,\zeta}^\top) \\
& + \sum_{\zeta=0}^{N_y-1} \sum_{\xi=0}^{N_x-1} (\mathbf{s}_{a-\xi,\zeta} \mathbf{\epsilon}_{\xi,\zeta}^p + \mathbf{s}_{a-\xi,\zeta} \mathbf{\epsilon}_{\xi,\zeta}^*) \\
& + \sum_{\xi=0}^{N_x-1} (Q_{a-\xi}^n p_{\xi,0} + Q_{a-\xi}^f f_{\xi,0})
\end{aligned} \quad . \quad (4.20)$$

4.1.3 Homogeneous half-space contact sub-problem

A contact problem can be solved by the following set of equations and inequalities (Johnson, 1985):

$$W = \int_{-\infty}^{\infty} p(x) dx \quad , \quad (4.21)$$

$$h(x) = h^i(x) + u(x) - \delta \quad , \quad (4.22)$$

$$p(x) > 0, \quad h(x) = 0 \quad x \in A_c \quad , \quad (4.23)$$

$$p(x) = 0, \quad h(x) > 0 \quad x \notin A_c \quad . \quad (4.24)$$

Equation (4.21) balances the external load W by the integration of the surface normal contact pressure $p(x)$. In Eq (4.22), $u(x)$ is the composite displacement of the two surfaces, $h(x)$ represents the gap between them, and $h^i(x)$ denotes their initial gap. The initial gap $h^i(x)$ depends on the geometries of the two contacting surfaces, and the contact between an infinitely long cylinder of radius R and a half-space requires that $h^i(x) = x^2 / 2R$. δ represents the relative “rigid-body” approach of the two contacting bodies. Within the contact area (denoted by A_c in Eqs. (4.23) and (4.24)), the final gap is zero and the normal pressure is positive if surface adhesion is not considered; outside the contact area,

the final gap is positive if surface penetration is not considered and the normal pressure is zero.

Here the loading body is assumed to be rigid, and thus the composite displacement $u(x)$ is regard as the surface normal displacement $u^0(x)$ in a homogeneous half-space; otherwise, the elastic deformation of the loading body should also be considered in Eq. (4.22). Equations (4.21) – (4.24) can be re-written into the following formulas for the same discretized computational domain as in Fig 4.2:

$$W = \sum_{\alpha=0}^{N_x-1} 2\Delta_x p_{\alpha,0} , \quad (4.25)$$

$$h_{\alpha} = h_{\alpha}^i + \sum_{\xi=0}^{N_x-1} \left(Q_{\alpha-\xi}^n + \mu Q_{\alpha-\xi}^f \right) p_{\xi,0} - \delta , \quad (4.26)$$

$$p_{\alpha,0} > 0 , \quad h_{\alpha} = 0 \quad \text{in } A_c , \quad (4.27)$$

$$p_{\alpha,0} = 0 , \quad h_{\alpha} > 0 \quad \text{outside } A_c . \quad (4.28)$$

The variable $p_{\alpha,0}$ represents uniform normal contact pressure on the surface element $[\alpha,0]$; h_{α} is the final gap (at the surface observation point of element $[\alpha,0]$) between two contacting surfaces; h_{α}^i is their initial gap.

It should be noted that these expressions for determining surface deformation can be obtained only for a homogeneous half-space; it is almost infeasible to obtain them for an inhomogeneous half-space. This is the reason for converting an inhomogeneous half-space into a homogeneous half-space in solving a contact problem.

4.1.4 Numerical procedure for the entire contact problem

The two above-discussed sub-problems are correlated by an iterative algorithm as shown in Fig. 4.3. Firstly, the profiles of the surfaces of the upper loading body and the lower half-space are initialized. The profiles can take into account the discrete surface roughnesses, i.e., a bump of certain height can be assigned to each surface element within the discretized computational domain. Then, with the given initial profiles of the two contacting surfaces, the homogeneous half-space contact sub-problem is solved to obtain the contact area between them, as well as the normal pressure and tangential friction within this contact area.

Next, under the obtained surface loading conditions, the half-space sub-problem with prescribed surface loading is solved to obtain the unknown equivalent eigenstrains and dislocation densities. It is known that eigenstrains and dislocations (or cracks) also cause surface deformation which would change the solution of the contact problem. Therefore, the surface displacement u^I due to the eigenstrains and dislocations is calculated afterwards. In the iterative process, if u^I obtained at the step $i+1$ converges to that obtained at the previous step i , the iterative process ends; otherwise, u^I is added into the surface profile of the half-space to change the gap between the two contacting surfaces and the next iteration step begins. Upon the calculation convergence of u^I , the surface displacement of the half-space is finally given by the summation of u^0 and u^I . The subsurface stresses can also be calculated accordingly.

Fig. 4.3. Algorithm for solving the problem of a half-space containing cracks and inhomogeneous inclusions under contact loading.

4.2 Numerical results

4.2.1 One inclusion and one horizontal crack

Figure 4.4 shows the schematic of a horizontal crack and a square inhomogeneous inclusion beneath the surface of a half-space matrix, bounded by the x -axis in the xOy Cartesian coordinate system, subjected to a frictionless cylindrical contact loading. The inclusion Ω has the edge length $2a_0$ and is

centered at $(0, 3.25a_0)$; the horizontal crack Γ has length $l = 2a_0$ and is centered at $(0, 1.5a_0)$. The cylindrical loading body has radius R and is assumed to be rigid. The half-space matrix has the Young's modulus $E_m = 210 \text{ GPa}$ and the Poisson's ratio $\nu_m = 0.28$, while the inclusion has $E_i = 420 \text{ GPa}$ and $\nu_i = 0.28$. The parameter a_0 is the Hertz radius for the same indented half-space in the absence of the crack and inclusion, and so is the maximum Hertz pressure p_0 . In order to make the result independent of the radius R and loading W , the dimension and stress are normalized by a_0 and p_0 , respectively (same to the following sections).

Fig. 4.4. Schematic of one horizontal crack and one inclusion beneath a half-space surface subjected to contact loading.

Figure 4.5 compares the von Mises stresses σ_v along the $y = 1.75a_0$ obtained by the present method and those by FEM. A good agreement has been found, thus validating the present method. Figure 4.6 further compares the von Mises stress fields for the subsurface square inclusion and arbitrarily-shaped inclusion, when the inclusion contains initial eigenstrains $\varepsilon_x^p = \varepsilon_y^p = 0.01$. The stress concentrations occur near the two tips of the crack and the sharp edges of the

inclusion. Figure 4.7 illustrates the shape effect of the inclusion on the contact pressure p when the initial eigenstrain $\varepsilon_x^p = \varepsilon_y^p = 0.01$. The result indicates that besides the stress fields, the contact pressure is also affected by the different shapes of the subsurface inclusion.

Fig. 4.5. Comparison of the von Mises stresses σ_v along $y=1.75a_0$ obtained by the present method and FEM.

Fig. 4.6. Contours of the von Mises stresses for the (a) square subsurface inclusion and (b) arbitrarily-shaped subsurface inclusion.

Fig. 4.7. The shape effect of the inclusion on the contact pressure p .

4.2.2 Two inclusions and one horizontal crack

Figure 4.8 shows the schematic of a horizontal crack and two square inhomogeneous inclusions beneath the surface, bounded by the x -axis in the xOy Cartesian coordinate system, subjected to a frictionless cylindrical contact loading. The inclusion Ω_1 and Ω_2 have the edge length a_0 and are centered at $(-1.25a_0, 1.25a_0)$ and $(1.25a_0, 1.25a_0)$, respectively. The crack Γ has length $l = a_0$ and is centered at $(0, h)$. The inclusions, the matrix and the loading body are modeled with the same elastic constants as set in Section 4.2.1.

Fig. 4.8. Schematic of one horizontal crack and two inclusions beneath a half-space surface subjected to a contact loading.

Figure 4.9 presents the contours of the subsurface stress fields for the case of $h = 1.25a_0$. It shows that the stress concentrations are induced at the two crack tips and the sharp edges of the inclusion. Figure 4.10 illustrates the effect of the embedded crack on the contact pressure p at a frictionless surface for various depths h where the crack is located. It shows that the crack affects the contact pressure when the crack is close to the surface, and the effect becomes weak as h increases. In the case of $h = \infty$ the contact between the indenter and the half-space surface is essentially a classical Hertz contact. Thus, the contact pressure p reaches the maximum Hertz pressure p_0 at the center of the contact area. The effect of the crack on the contact pressure becomes weak at the depth beyond $1.25a_0$.

Fig. 4.9. Contours of the stress components and von Mises stresses.

Fig. 4.10. Effect of a crack at various depths h beneath a frictionless half-space surface on the contact pressure.

4.2.3 Multiple horizontal cracks

Figure 4.11 illustrates three horizontal cracks Γ_i ($i = 1, 2, 3$) near a half-space surface subjected to a cylindrical contact loading. The half-space matrix has the Young's modulus $E_s = 210\text{GPa}$ and the Poisson's ratio $\nu_s = 0.28$. The cylindrical indenter has radius R and is assumed to be rigid. The three cracks are assumed to have the same length $l = 0.5a_0$ and equally spaced by a distance of $m = 0.25a_0$ along the x -axis.

Fig. 4.11. Schematic of three horizontal cracks beneath a half-space surface subjected to a cylindrical contact loading.

Figure 4.12 plots the subsurface von Mises stresses for the cases with different crack depths h . It shows that the subsurface stress fields are dramatically distorted due to the presence of the cracks and as they approach the surface, the distortions become more significant. Stress concentrations are also found around the tips of the cracks. An accurate description of the subsurface stress field is crucial for the study of crack nucleation and propagation.

Fig. 4.12. Comparison of subsurface von Mises stresses for the cases of various crack depths h .

The effect of the friction condition on the surface displacement is further investigated. Figure 4.13 presents the surface normal displacement u and stress component σ_x for the case with $h = 0.5a_0$ and various friction coefficients. The tensile stress σ_x around the trailing edge of the contact area increases as the friction increases. When the tensile stress reaches a certain value, it may open a surface crack there.

Fig. 4.13. Effect of the friction force on (a) the surface displacement u and (b) the stress component σ_x .

4.2.4 One vertical crack beneath a coating layer

A coating Ω_1 of thickness $H = a_0$ is bonded perfectly to a semi-infinite steel substrate which contains a vertical crack subjected to a frictional cylindrical contact loading at the coating surface, as shown in Fig. 4.14. For analysis

convenience, an xOy Cartesian coordinate system is attached to the coating-substrate with its x -axis along the surface of coating. The vertical crack has length $l = a_0$ and is centered at (d, a_0) . The cylindrical loading body of radius R is rigid, while the contact surface has a friction coefficient $\mu = 0.4$. The steel substrate has the Young's modulus $E_m = 210 \text{ GPa}$ and the Poisson's ratio $\nu_m = 0.28$. The elastic parameters of coating Ω_1 are set as $E_c = \lambda E_m$ and ν_c . The parameter λ represents the material difference between the coating and the substrate: $\lambda < 1$ represents a soft coating; while $\lambda > 1$ represents a hard coating. The parameters a_0 and p_0 are the Hertz radius and maximum Hertz pressure, respectively, for the same indented half-space in the absence of the crack and coating. Provided that $K_0 = 0.5 p_0 \sqrt{2\pi a_0}$, the dimension, stress and SIF can be normalized by a_0 , p_0 and K_0 , respectively. In order to approximate the infinite coating dimension along the x -axis, the computational domain along the x -axis is set around 10 times larger than its Hertz radius a_0 to obtain a more accurate solution.

Fig. 4.14. Schematic of a vertical crack located between a coating and its substrate under frictional contact loading.

The delamination at the coating-substrate interface is a direct result of interfacial shear stresses when they are large enough to overcome the adhesion strength of the coating. Figure 4.15 plots the effect of the coating layer on interfacial shear stresses σ_{xy} along the x -direction for the case of $d = 4.0a_0$ and $\nu_c = 0.28$. The interfacial shear stresses σ_{xy} is positive in the area towards the friction direction ($x > 0$), but negative in the area $x < 0$. Meanwhile, the magnitudes of the interfacial shear stresses σ_{xy} increase as the coating layer becomes harder and the peak stress values are near $x = \pm a_0$, where delamination is likely to initiate.

Figure 4.16 presents the effect of the coating on the mode I SIFs K_I^L and K_I^U for the case of $d = 4.0a_0$ and $\nu_c = 0.28$. The SIF K_I^L at the lower crack tip is smaller than the SIF K_I^U at the upper crack tip and both of SIFs imply a linear decrease with the increase of the Young's modulus of the coating layer. The positive values of the mode I SIF (K_I^L and K_I^U) indicate that the crack is opened, corresponding to the load applied on crack surface. On the other hand, when $\lambda > 1.7$, K_I^L becomes negative, indicating that the crack closes due to an overall compressive loading normal to the crack surface. Figure 4.17 illustrates the influence of the geometric configuration of the cracks, which is described by a scalar factor d/a_0 , on the mode I SIFs for the TiN coating ($E_c = 305$ GPa and $\nu_c = 0.22$). The mode I SIF decreases as the crack approaches the central zone of contact. It is clear that contact loading has a stronger influence on the mode I SIF, when the crack gets closer to the contact area.

Fig. 4.15. Effect of the coating on interfacial shear stresses σ_{xy} along the x direction for the case of $d = 4.0a_0$ and $\nu_c = 0.28$.

Fig. 4.16. Effect of the coating on SIFs for the case of $d = 4.0a_0$ and $\nu_c = 0.28$.

Fig. 4.17. Effect of d/a_0 on SIFs for the case of $E_c = 305$ GPa and $\nu_c = 0.22$.

4.2.5 One inclusion and one vertical crack in a coated substrate

Figure 4.18 shows a vertical crack and a square inhomogeneous inclusion subjected to a frictionless cylindrical contact loading at the TiN coating surface. The coating layer Ω_1 has a thickness $H = a_0$. The inclusion Ω_2 has the edge length $2a_0$ and is centered at $(0, 3a_0)$; the crack has a length $l = 2a_0$ and is centered at $(d, 3a_0)$. The cylindrical loading body has radius R and is assumed to be rigid. The substrate has the Young's modulus $E_m = 210$ GPa and the Poisson's ratio $\nu_m = 0.28$; while TiN coating has $E_c = 305$ GPa and $\nu_c = 0.22$. The inclusion has elastic constants $E_i = \lambda E_m$ and ν_i .

Fig. 4.18. Schematic of a vertical crack and an inclusion beneath a coating.

Figure 4.19 presents the effect of the subsurface inclusion on the mode II SIF for the case of $d = 1.2a_0$ and $\nu_i = 0.28$. As shown in Fig. 4.19, K_{II}^U is larger than K_{II}^L . K_{II}^U increases as the Young's modulus of the subsurface inclusion increases while K_{II}^L decreases as the Young's modulus increases. Figure 4.20 shows the influence of a subsurface inclusion on the surface contact pressure. The maximum contact pressure increases as the subsurface inclusion becomes stiffer. Figure 4.21 illustrates the mode II SIF versus d/a_0 with the elastic parameters of the inclusion being unchanged ($E_i = 340 \text{ GPa}$ and $\nu_i = 0.28$). It suggests that K_{II}^L decreases as the crack approaches the inclusion while K_{II}^U is hardly affected as the distance changes. All these results show the significance of considering the influence of near-crack inclusions in studying the behavior of crack propagation.

Fig. 4.19. Effect of the subsurface inclusion on the crack SIF for the case of $d = 1.2a_0$ and $\nu_c = 0.28$.

Fig. 4.20. Effect of the subsurface inclusion on contact pressures for the case of $d = 1.2a_0$ and $\nu_c = 0.28$.

Fig. 4.21. Effect of d/a_0 on the crack SIF for the case of $E_c = 340$ GPa and $\nu_c = 0.28$.

4.2.6 Two inclusions and one vertical crack in a coated substrate

A vertical crack and two square inhomogeneous inclusions are beneath a TiN coating surface, as shown in Fig. 4.22. The TiN coating Ω_1 has a thickness $H = a_0 = 10 \mu\text{m}$. The cylindrical loading body has radius R and is assumed to be rigid. The crack has a length $l = 2a_0$ and is centered at $(0, 3a_0)$; the inclusions Ω_2 and Ω_3 have the same edge length $2a_0$ and are centered at $(-2a_0, 3a_0)$ and $(2a_0, 3a_0)$, respectively. The substrate has the Young's modulus $E_m = 210$ GPa and the Poisson's ratio $\nu_m = 0.28$, and the coating Ω_1 has $E_c = 305$ GPa and $\nu_c = 0.22$. The inclusions Ω_2 and Ω_3 have the constants $E_i = 340$ GPa and $\nu_i = 0.28$. Meanwhile, the thermal expansion induces initial eigenstrain $\varepsilon_x^p = \varepsilon_y^p = \varepsilon^0$ in inclusions Ω_2 and Ω_3 .

Fig. 4.22. Schematic of a vertical crack and two inclusions beneath the coating.

Figure 4.23 plots the crack opening induced by different initial eigenstrains only in the inclusions Ω_2 and Ω_3 . The shape of the opening varies as the initial eigenstrains $\varepsilon_x^p = \varepsilon_y^p = \varepsilon^0$ change (from $\varepsilon^0 = -0.01$ to $\varepsilon^0 = -0.06$). As shown in Fig. 4.22, the crack is opened by the initial eigenstrains in the inclusions rather than an external load at the surface and the opening increases dramatically as the eigenstrain magnitude increases. Figure 4.24 shows the effect of the initial eigenstrain in the inclusions (changing from $\varepsilon^0 = -0.01$ to $\varepsilon^0 = -0.06$) on the mode I SIFs K_I^L and K_I^U . The SIF at the bottom crack tip is smaller than that at the top crack tip and both the SIFs imply a linear increase with the magnitude of the initial eigenstrain.

Fig. 4.23. Effect of the initial inclusion eigenstrains on the crack opening (μm).

Fig. 4.24. Effect of the initial eigenstrains in inclusions on the SIFs ($\text{MPa} \cdot \text{m}^{1/2}$).

4.3 Summary

In this chapter, a semi-analytic solution for multiple cracks and inhomogeneous inclusions in a multi-layered material subjected to contact loading is developed by using the EIM and DDT and by treating a coating layer as an inhomogeneous inclusion. The solution considers not only the interactions among all the inclusions and cracks but also the surface deformation of the layered material caused by both the surface loading and the subsurface inclusions and cracks. Thus, an accurate description of the surface pressure, the subsurface elastic field and the crack SIFs in the multi-layered inhomogeneous materials is obtained, which would enable their fracture and delamination analyses to be conducted.

The solution is validated by the FEM and the case studies are given to study the effects of the surface friction conditions and subsurface micro-defects on the surface deformation, contact area and contact pressure as well as the subsurface stress field. The study also shows that the interfacial stresses between the coating and the substrate, the SIFs, and openings of the cracks can be greatly affected by the crack position, elastic mismatch between the inclusions and the matrix, and the magnitude of the initial eigenstrain. The analysis of the surface tribological behavior and an accurate prediction of surface and subsurface damage can provide guideline for the engineering problems concerning surface damage and material failure.

Chapter 5 Fatigue behaviors under contact loading

In this chapter, the contact fatigue problem of multiple inhomogeneous inclusions and cracks beneath a surface is studied. The procedure developed in Chapter 4 is adopted to investigate the propagation of cracks.

5.1 Methodology

5.1.1 Problem description and solution approach

Figure 5.1 shows a 2D plane-strain problem about multiple inhomogeneous inclusions $\Omega_\psi (\psi = 1, 2, \dots, n_1)$ and cracks $\Gamma_\varphi (\varphi = 1, 2, \dots, n_2)$ near a half-space surface subjected to a cyclic contact loading in the xOy Cartesian coordinate system. The cylindrical loading body has radius R and is assumed to be rigid. The crack propagation is induced by the cyclic contact loading and influenced by the nearby inclusions. For simplicity, each crack Γ_φ is assumed to be horizontally or vertically aligned. Thus, the crack propagation can be modeled as a zigzag crack consisting of many small vertical and horizontal cracks. The inhomogeneous inclusion Ω_ψ contains initial eigenstrain ε_{ij}^P and its elastic constants are denoted by C_{ijkl}^ψ ($i, j, k, l = 1, 2$); the matrix has the elastic moduli C_{ijkl} .

According to the approach presented in Chapter 4, the static inhomogeneous contact problem can be decomposed into two correlated sub-problems: (1) the inhomogeneous half-space sub-problem under a prescribed surface loading, and (2) the homogeneous half-space contact sub-problem under contact loading. The two correlated sub-problems can be solved by an iterative algorithm which has also been explicitly discussed in Chapter 4.

Fig. 5.1. Schematic of the crack growth near inhomogeneous inclusions in a half-space under cyclic contact loading.

The simulation of the contact fatigue problem is based on the static analysis of the inhomogeneous contact problem. Meanwhile, the maximum hoop stress criterion (Erdogan and Sih, 1963) is employed to determine the crack propagation direction and the Paris-Erdogan law (1963) is applied to calculate the crack advance rate.

The crack propagation is induced by the cyclic contact loading and depends on the load history. The crack propagation rate is determined by the Paris-type law as below

$$\frac{da}{dN} = C \Delta K^m, \quad (5.1)$$

where a is the crack length and N is the number of load cycles. C and m are the material constants; ΔK is the changing range of the crack SIFs induced by the contact loading. In this study, the minimum contact loading is assumed to be zero. Thus, the ΔK can be obtained by

$$\Delta K = \sqrt{K_I^2 + K_{II}^2}, \quad (5.2)$$

where K_I and K_{II} are the mode I and mode II SIFs for the crack which is supposed to propagate under the maximum contact loading.

5.1.2 Solution scheme

A computational domain D is chosen to contain all the cracks Γ_φ and inclusions Ω_ψ in an xOy coordinate system. The domain D is discretized into $N_x \times N_y$ square elements of the same size $2\Delta_x \times 2\Delta_y$ (Fig. 5.2). Each element is indexed by $[\alpha, \beta]$ with $0 \leq \alpha \leq N_x - 1$ and $0 \leq \beta \leq N_y - 1$. The crack can be either horizontally or vertically aligned along the central lines of these elements.

Fig. 5.2. Discretization of a computational domain containing all the inclusions and cracks into many small square elements.

The crack that propagates in an arbitrary direction can be modeled by a series of vertical and horizontal cracks. According to the theory proposed by Erdogan and Sih (1963), the crack will grow in the direction along which the maximum hoop stress occurs. In this model, the element containing one crack tip has three

crack-free neighboring elements (Fig. 5.3). Thus, the crack will propagate into only one of the three elements. Among the three neighboring elements of a crack tip, the weakest one that is subjected to the largest tensile stress or the least compressive stress is regarded as the most possible one through which the crack will penetrate.

Fig. 5.3. Schematic of three elements near a crack tip.

The contact fatigue problems are solved by an iterative algorithm as shown in Fig. 5.4. At first, the profiles of the crack-inclusion geometry and contact surface are initialized. Then, with the given initial profiles, the homogeneous half-space contact sub-problem 1 and the inhomogeneous half-space sub-problem 2 with prescribed surface loading are solved subsequently to obtain the surface displacement u^I due to the cracks and inclusions. Afterwards, the first iterative process for solving the inhomogeneous contact problem begins. Upon the calculation convergence of u^I , the surface displacement of the half-space is finally given by the summation of u^0 and u^I . The subsurface stresses can also be calculated accordingly. Based on the subsurface stresses, the hoop stresses and

SIFs near tips of cracks can be obtained. After this, the second iterative process for predicting the crack fatigue behavior begins. If the crack SIF is larger than the threshold ΔK_{th} , the crack will propagate. Thus, the maximum hoop stress criterion is incorporated to determine the crack growth path and the crack geometry can be updated accordingly.

Fig. 5.4. Algorithm for solving the problem of a half-space containing cracks and inhomogeneous inclusions under cyclic contact loading.

5.2 Numerical results

5.2.1 Particle detachment of cracked material

Figure 5.5 shows the schematic of a crack and a square inhomogeneous inclusion beneath the surface of a half-space matrix, bounded by the x -axis in the xOy Cartesian coordinate system, subjected to a frictional cyclic contact loading. The cylindrical loading body has radius R and is assumed to be rigid, while the contact surface has a friction coefficient $\mu = 0.4$. The half-space matrix has the Young's modulus $E_m = 200 \text{ GPa}$ and the Poisson's ratio $\nu_m = 0.28$. The crack has length $l = 0.5a_0$ and is centered at $(-2a_0, h)$. The material constants m and C for the matrix are taken as 2 and 1.2×10^{-12} , respectively. The parameter a_0 is the Hertz radius for the same indented half-space in absence of the crack, and so is the maximum Hertz pressure p_0 . Provided that $K_0 = 0.5 p_0 \sqrt{2\pi a_0}$, the dimension, stress and SIF can be normalized by a_0 , p_0 and K_0 , respectively (same to the following sections).

Fig. 5.5. Schematic of a crack beneath a surface under cyclic contact loading.

Figure 5.6a plots the contour of the von Mises stress field σ_v/p_0 for the case when $h = 0.5a_0$. Driven by the pressure and friction imposed by the indenter, the initial crack propagates from its two tips and finally causes the detachment of a particle from the surface after about 3.01×10^8 contact cycles, as shown in Fig. 5.6b.

(a)

(b)

Fig. 5.6. Particle detachment from the substrate.

Figure 5.7 studies the depth effect of the initial crack on the particle detachment. It shows that the detachment occurs only when the depth is no larger than $0.7a_0$, beyond which the crack propagates towards the thickness direction of matrix. When the depth ranges from $0.2a_0$ to $0.5a_0$, the number of cycles required for the particle detachment changes slightly. However, when the depth changes from $0.5a_0$ to $0.7a_0$, the cycle number for particle detachment increases

dramatically. Therefore, the larger the depth for an initial crack, the more cycles are required to detach a particle. This phenomenon can be partially explained by the decreasing effect of friction on the advance of the crack tips as they go deeper. Figure 5.8 shows that when the depth of crack is deeper than $0.7a_0$, the crack begins to propagate through the matrix thickness.

Fig. 5.7. Effect of the crack depth on the number of cycles required for particle detachment occurrence.

Fig. 5.8. No particle detachment due to the crack growth driven by cyclic contact loading

5.2.2 Delamination of a coated layer from the substrate

Figure 5.9 shows a crack Γ near a half-space surface subjected to a frictional cylindrical contact loading at the TiN coating surface. The coating layer Ω_1 has a thickness $H = a_0$. The crack located at the coating-substrate interface has a length $l = 0.25a_0$ along the interface direction and is centered at $(-2.75a_0, a_0)$. The substrate has the Young's modulus $E_m = 210 \text{ GPa}$ and the Poisson's ratio $\nu_m = 0.28$, while the TiN coating has $E_c = 305 \text{ GPa}$ and $\nu_c = 0.22$. The friction coefficient μ is 0.4.

Fig. 5.9. Schematic of a crack located at the interface of the coated substrate.

Figure 5.10 shows the coating layer delamination from its substrate under frictional contact loading after about 8.47×10^9 contact cycles. Figure 5.10a presents the contour of the shear stress field σ_{xy} / p_0 . Influenced by the material disparity, the initial crack Γ propagates along the interface between the TiN coating and the steel substrate under the cyclic contact loading.

(a)

(b)

Fig. 5.10. Coating delamination from the steel substrate under the cyclic contact loading.

Figure 5.11 investigates the effect of crack propagation on mode II SIFs. Their magnitudes at both ends of the crack increase as the crack propagates. The magnitude of K_{II}^R increases dramatically as the crack approaches the central zone of contact. It is observed that the surface tangential traction has stronger influence on the mode II SIFs when the crack gets closer to the contact area.

Fig. 5.11. The change of the mode II SIF K_{II} / K^0 with the loading cycles N .

5.2.3 Crack propagation in an inhomogeneous material under thermal loading

Figure 5.12 shows the schematic of one crack and two square inhomogeneous inclusion Ω_1 and Ω_2 embedded in a half-space. The two inclusions have the same Young's modulus $E_I = 100$ GPa and Poisson's ratio $\nu_I = 0.28$; the matrix has $E_M = 300$ GPa and $\nu_M = 0.28$. Ω_1 and Ω_2 have the same side length $a_0 = 20 \mu\text{m}$ and are centered at $(-1.25a_0, 1.25a_0)$ and $(1.25a_0, 1.25a_0)$, respectively. The crack Γ along the x -axis has the length of $l = 0.5a_0$, and is centered at $(0, 1.25a_0)$.

Fig. 5.12. Schematic of one crack and two inclusions in a half-space under cyclic thermal loading.

Figure 5.13 plots the contour of the normal stress field for the case in which both inclusions are subjected to thermal loading, which is modeled by applying initial eigenstrain $\varepsilon_x^0 = \varepsilon_y^0 = \varepsilon^0 = 0.01$ to them. It shows that crack Γ propagates under the thermal loading. The negative normal stress σ_y near the crack tips implies that the crack begins to close after it penetrates into the two inclusions.

Fig. 5.13. Contours of the normal stress σ_y (MPa).

Figure 5.14 studies the effect of thermal loading (initial eigenstrains range from $\varepsilon_x^0 = \varepsilon_y^0 = \varepsilon^0 = 0.005$ to $\varepsilon_x^0 = \varepsilon_y^0 = \varepsilon^0 = 0.06$) on crack propagation. It shows that, for the same propagation length ($\Delta l = 20 \mu\text{m}$) of crack Γ , the number of cycles N decreases as the initial eigenstrain increases. When the initial eigenstrain ranges from 0.005 to 0.02 the number of cycles dramatically decreases. However, when the eigenstrain changes from 0.02 to 0.06, the number of cycles changes slightly.

Fig. 5.14. Effect of the thermal loading on the number of cycles.

5.3 Summary

In this chapter, a fatigue model based on the analysis of the stress fields in inhomogeneous materials under 2D plane-strain conditions is developed. With the obtained full stress field and the SIFs of the cracks, the behaviors of crack propagation under cyclic loading can be simulated. The maximum hoop stress criterion is applied to predict the crack propagation direction and the Paris-type

law is employed to analyze the crack growth rate. Although the crack propagation is set towards a horizontal or vertical direction, the slant or curved cracks can still be modeled by considering them as zigzag cracks consisting of many small horizontal and vertical segments. The novelty of this study is to incorporate the maximum hoop stress criterion and Paris-type law to the original static solution. The newly developed solution is capable of analyzing the crack propagation and predicting their fatigue behaviors in an inhomogeneous material.

Based on the solution obtained, the particle detachment caused by crack propagation has been investigated. The results show that crack fatigue behaviors can be greatly affected by the crack position and there exists a critical crack depth for the particle detachment induced by crack propagation. The detachment occurs only when the depth is no larger than the critical depth beyond which the crack propagation results in no particle detachment. Therefore, in order to minimize such damage, near-surface defects should be reduced to prevent crack initiation and propagation beneath surface under heavy contact loading. The delamination of a coating from a substrate under contact loading has also been studied. The results demonstrate that the crack propagation rate increases dramatically as the crack approaches the center of the contact zone. Moreover, as the number of the cyclic loading increases, the magnitudes of SIFs at both the crack tips increase, resulting in the increase of the crack propagation rate. A further study of thermal loading in terms of cyclic eigenstrain in the inhomogeneous inclusions shows that the number of cycles N required for a crack to propagate to a given length decreases as the magnitude of the initial eigenstrains increases.

The main physical factors of a material determining the crack propagation direction include the elastic constants (material property) of its matrix, the surface

loading conditions, the elastic constants of the inclusions embedded in the matrix, the inclusion shape, location and distribution, the initial locations of the cracks and their relative positions to the surface and the inclusions. Meanwhile, the main factors determining the crack propagation rate are the material properties of the matrix, which are the material constants m and C used in the Paris law for the crack growth rate calculation.

These parameters in real materials are widely used in the industry. For example, during the manufacturing or utilization process of most engineering transmission components such as the gears, near-surface micro-defects such as voids and inclusions are usually inevitable. These defects could cause stress concentration and induce crack nucleation or propagation beneath the component surface, which would affect the mechanical properties and performance of these components and may even cause their eventual failure. A good method to prevent such damage effects is the application of coatings on the surfaces of these engineering components. For example, TiN or diamond-like carbon coatings are often deposited on gear surfaces. As such, the initial locations of the micro-defects and their relative positions to the surface are changed. Specifically, the depths of the defects increase, thus reducing their effect on surface or subsurface damage.

Chapter 6 Conclusions and recommendations

Based on the works conducted during this PhD study, main conclusions are drawn and recommendations are suggested in this chapter.

6.1 Conclusions

Firstly, the semi-analytic solution for multiple cracks and inhomogeneous inclusions in a half-space subjected to prescribed surface loading has been obtained by a newly developed methodology that combines the EIM and DDT. The interactions among all the inclusions and cracks have been fully taken into account. The inhomogeneous inclusions have been modeled as homogeneous inclusions with unknown equivalent eigenstrains using the EIM; while the cracks have been modeled by distributions of edge dislocations with unknown densities using the DDT. Coupled governing equations with unknown equivalent eigenstrains and dislocation densities have been established and solved iteratively by using the CGM. The FFT algorithm has been utilized to improve computational efficiency.

Secondly, the semi-analytic solution for multiple cracks and inhomogeneous inclusions beneath a half-space subjected to contact loading has been obtained. Meanwhile, the surface coating has also been considered by treating it as an inhomogeneous inclusion with respect to the substrate. This solution has considered not only the interactions between the cracks and inclusions but also the interactions among the surface loading body, the coating layer and the subsurface defects. The original inhomogeneous half-space contact problem has been decomposed into the sub-problem of a homogeneous half-space under prescribed loading and the sub-problem of a homogeneous half-space under contact loading.

These two sub-problems have been integrated by an iterative algorithm to determine the unknown dislocation densities, equivalent eigenstrains, surface contact area and contact pressure. The SIFs of cracks have also been obtained by the dislocation densities at the cracks tips. An accurate description of the surface pressure, the subsurface elastic field of the inhomogeneous materials and the SIFs of the subsurface cracks makes it possible to analyze the fracture and delamination of these layered materials.

Thirdly, a fatigue model for the analysis of crack propagation under cyclic contact loading has been developed. A methodology similar to that for the half-space contact problem has been utilized to obtain the full stress field and the SIFs of crack under contact loading. After that, the maximum hoop stress criterion has been applied to predict crack propagation directions and the Paris-type law has been employed to analyze crack growth rates.

Fourthly, the semi-analytic solutions obtained by the present methodology have been validated by those by the FEM. The advantages of the present methodology lie in the fact that it can handle the interactions of multiple cracks and inhomogeneous inclusions in the same way as for a single crack or inhomogeneous inclusion without increasing computational complexity. For any given number of cracks and inhomogeneous inclusions, they are first decomposed into many crack segments and square inhomogeneous inclusions within a discretized computational domain. These crack segments are then treated as dislocation distributions with unknown densities. The square inhomogeneous inclusions are then treated as square homogeneous inclusions with unknown equivalent eigenstrains. Afterwards, these unknowns are determined by solving simultaneous governing equations at one time. The unknowns are introduced to

represent the material dissimilarity among all the inhomogeneous inclusions and cracks, their interactions and their responses to external loading. The FFT is applied for the entire discretized computational domain for computational efficiency improvement. Therefore, the computational time of the present method is not strongly affected by the number of inclusions and cracks. However, the computational time of FEM or BEM dramatically increases as the number of inclusions and cracks increases. Thus, the present method will demonstrate significant advantage, compared with the FEM when a large number of inclusions and cracks are considered.

6.2 Recommendations

The solutions obtained in this study for the inhomogeneous half-space contact problems can not only capture the surface deformation, contact pressure and subsurface stress field for a given distribution of the subsurface micro-defects but also tackle the crack propagation behaviors under cyclic loading. Therefore, these solutions will lead to the development of a framework that is capable of modeling surface wear and contact fatigue damages of materials.

Plastic deformation is always involved when the surface damages such as fatigue, creep and wear in ductile material are considered. According to Mura's definition (1987), the eigenstrain is a generic term referring to inelastic strain caused by phase transformation, plastic deformation, thermal expansion, etc. Thus, the plastic zones can be simply treated as homogeneous inclusions with initial eigenstrains which represent the plastic deformation in the zones. Different material constitutive models can be considered for the study of plastic deformation. For the metallic materials under simple loading scenarios, such as normal

indentation, the elastic-perfectly plasticity model can be employed. Meanwhile, when the reverse plasticity due to unloading and the plastic accumulation due to cyclic contact loading are of interest, the strain hardening model can be used. The von Mises yield criterion which is valid for most metal materials can be used to determine when the yielding occurs. Accordingly, the plastic zone evolution in materials could be modeled by using the plastic flow law. With the consideration of plastic deformation, not only the adhesive wear and flow wear can be simulated, but also the crack tip yielding zone can be estimated by the present model.

Adhesive contact plays a central role in many industry areas. Efficient manufacturing processes require a tight control over the contamination of surfaces by particles. Energy efficient mechanical devices and reliable micromechanical systems demand a better control over friction and lubrication. In this PhD study, only non-adhesive elastic contact has been considered. Therefore, if the adhesive contact is investigated, the mechanical behaviors of the cohesive traction and the relative distance between two adjacent surfaces need to be considered. The JKR model (1971) can be used to analyze frictionless adhesive contact between elastic spheres. For the long-ranged surface interaction, the DMT theory (1975) can be employed.

Appendix A

A distribution of dislocations is located along the horizontal direction in the half-space matrix with the shear modulus μ and the Kolosov constant κ ($\kappa = 3 - 4\nu$ for the plane-strain problem). The dislocation distribution line has the length of $2a$ and is centered at $(0, \eta)$. The stress σ_{ij}^Γ at the point (x, y) due to the horizontal distribution of the edge dislocations can be written as

$$\sigma_{ij}^\Gamma(x, y, \eta) = \frac{2\mu}{\pi(\kappa + 1)} [E_{ij}^\perp(x, y, \eta)c^\perp + F_{ij}^\perp(x, y, \eta)d^\perp + E_{ij}^\parallel(x, y, \eta)c^\parallel + F_{ij}^\parallel(x, y, \eta)d^\parallel] \quad (\text{A.1})$$

$$\begin{aligned} F_{11}^\perp = & -\frac{1}{2} \ln \left[\frac{(a-x)^2 + (y+\eta)^2}{(a-x)^2 + (y-\eta)^2} \right] + \frac{1}{2} \ln \left[\frac{(a+x)^2 + (y+\eta)^2}{(a+x)^2 + (y-\eta)^2} \right] \\ & + \frac{(\eta-y)^2}{(x-a)^2 + (\eta-y)^2} - \frac{(\eta-y)^2}{(x+a)^2 + (\eta-y)^2} \\ & + \frac{3\eta^2 + 4\eta y - y^2}{(x-a)^2 + (\eta+y)^2} - \frac{3\eta^2 + 4\eta x - y^2}{(x+a)^2 + (\eta+y)^2} \\ & - \frac{4\eta y(y+\eta)^2}{[(x-a)^2 + (\eta+y)^2]^2} + \frac{4\eta y(y+\eta)^2}{[(x+a)^2 + (\eta+y)^2]^2} \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} E_{11}^\perp = & -\frac{x}{2} \left\{ \ln \left[\frac{(a-x)^2 + (y+\eta)^2}{(a-x)^2 + (y-\eta)^2} \right] - \ln \left[\frac{(a+x)^2 + (y+\eta)^2}{(a+x)^2 + (y-\eta)^2} \right] \right\} \\ & - \frac{2\eta y[(a+x)(\eta+y)^2 - (a-x)^3]}{((a-x)^2 + (\eta+y)^2)^2} + \frac{a(\eta-y)^2}{(a+x)^2 + (\eta-y)^2} \\ & - \frac{2\eta y[(a-x)(\eta+y)^2 - (a+x)^3]}{((a+x)^2 + (\eta+y)^2)^2} + \frac{a(\eta-y)^2}{(a-x)^2 + (\eta-y)^2} \\ & - 2(\eta-y) \left[\tan^{-1} \left(\frac{a-x}{\eta-y} \right) + \tan^{-1} \left(\frac{a-x}{\eta+y} \right) + \tan^{-1} \left(\frac{a+x}{\eta-y} \right) \right. \\ & \left. + \tan^{-1} \left(\frac{a+x}{\eta+y} \right) \right] - \frac{a(y^2 - 4\eta y - 3\eta^2)}{(a-x)^2 + (\eta+y)^2} \\ & - \frac{a(y^2 - 4\eta y - 3\eta^2)}{(a+x)^2 + (\eta+y)^2} \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned}
F_{22}^{\perp} = & \frac{y^2 + \eta^2}{(x-a)^2 + (\eta+y)^2} - \frac{y^2 + \eta^2}{(x+a)^2 + (\eta+y)^2} - \frac{(\eta-y)^2}{(x-a)^2 + (\eta-y)^2} \\
& + \frac{1}{2} \ln \left[\frac{(a-x)^2 + (y-\eta)^2}{(a-x)^2 + (y+\eta)^2} \right] - \frac{1}{2} \ln \left[\frac{(a+x)^2 + (y-\eta)^2}{(a+x)^2 + (y+\eta)^2} \right] \\
& + \frac{(\eta-y)^2}{(x+a)^2 + (\eta-y)^2} + \frac{4\eta y(y+\eta)^2}{[(x-a)^2 + (\eta+y)^2]^2} \\
& - \frac{4\eta y(y+\eta)^2}{[(x+a)^2 + (\eta+y)^2]^2}
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
E_{22}^{\perp} = & \frac{2\eta y[(a+x)(y+\eta)^2 - (a-x)^3]}{[(a-x)^2 + (\eta+y)^2]^2} + \frac{2\eta y[(a-x)(y+\eta)^2 - (a+x)^3]}{[(a+x)^2 + (\eta+y)^2]^2} \\
& + \frac{x}{2} \left\{ \ln \left[\frac{(a-x)^2 + (y-\eta)^2}{(a-x)^2 + (y+\eta)^2} \right] - \ln \left[\frac{(a+x)^2 + (y-\eta)^2}{(a+x)^2 + (y+\eta)^2} \right] \right\} \\
& - \frac{a(y-\eta)^2}{(a-x)^2 + (\eta-y)^2} - \frac{a(y-\eta)^2}{(a+x)^2 + (\eta-y)^2} \\
& + \frac{a(y^2 + \eta^2)}{(a-x)^2 + (\eta+y)^2} + \frac{a(y^2 + \eta^2)}{(a+x)^2 + (\eta+y)^2}
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
F_{12}^{\perp} = & - \frac{8\eta^2 y(a-x)[(a-x)^2 + \eta^2 - y^2]}{[(a-x)^2 + (\eta-y)^2][(a-x)^2 + (\eta+y)^2]^2} \\
& - \frac{8\eta^2 y(a+x)[(a+x)^2 + \eta^2 - y^2]}{[(a+x)^2 + (\eta-y)^2][(a+x)^2 + (\eta+y)^2]^2}
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
E_{12}^{\perp} = & \frac{5\eta y^2 + 7\eta^2 y + \eta x^2 + ax(\eta - y) - yx^2 + \eta^3 - y^3}{(a + x)^2 + (\eta + y)^2} \\
& - \frac{5\eta y^2 + 7\eta^2 y + \eta x^2 - ax(\eta - y) - yx^2 + \eta^3 - y^3}{(a - x)^2 + (\eta + y)^2} \\
& + \frac{4\eta y(\eta + y)[(y + \eta)^2 + x^2 - ax]}{[(a - x)^2 + (\eta + y)^2]^2} \\
& - \frac{4\eta y(\eta + y)[(y + \eta)^2 + x^2 + ax]}{[(a + x)^2 + (\eta + y)^2]^2} \\
& + \frac{(\eta - y)[x^2 - ax + (\eta - y)^2]}{(a - x)^2 + (\eta - y)^2} \\
& - \frac{(\eta - y)[x^2 + ax + (\eta - y)^2]}{(a + x)^2 + (\eta - y)^2} \\
& + \frac{\eta - y}{2} \left\{ \ln \left[\frac{(a - x)^2 + (\eta - y)^2}{(a - x)^2 + (\eta + y)^2} \right] \right. \\
& \left. - \ln \left[\frac{(a + x)^2 + (\eta - y)^2}{(a + x)^2 + (\eta + y)^2} \right] \right\}
\end{aligned} \tag{A.7}$$

$$\begin{aligned}
F_{11}^{\perp} = & \frac{4\eta(a - x)[(a^2 - 2ax + \eta^2 + x^2)^2 - y^4]}{[(a - x)^2 + (\eta - y)^2][(a - x)^2 + (\eta + y)^2]^2} \\
& + \frac{4\eta(a + x)[(a^2 + 2ax + \eta^2 + x^2)^2 - y^4]}{[(a + x)^2 + (\eta - y)^2][(a + x)^2 + (\eta + y)^2]^2} \\
& - 2 \left[\tan^{-1} \left(\frac{a - x}{\eta - y} \right) + \tan^{-1} \left(\frac{a - x}{\eta + y} \right) \tan^{-1} \left(\frac{a + x}{\eta - y} \right) \right. \\
& \left. + \tan^{-1} \left(\frac{a + x}{\eta + y} \right) \right]
\end{aligned} \tag{A.8}$$

$$\begin{aligned}
E_{11}^{\dagger} = & \frac{4cy(\eta+y)[(y+\eta)^2+x^2-ax]}{[(a-x)^2+(\eta+y)^2]^2} \\
& - \frac{4\eta y(\eta+y)[(y+\eta)^2+x^2+ax]}{[(a+x)^2+(\eta+y)^2]^2} \\
& - \frac{11\eta y^2+13\eta^2 y+3\eta x^2-ax(3\eta+y)+yx^2+3\eta^3+y^3}{(a-x)^2+(\eta+y)^2} \\
& + \frac{11\eta y^2+13\eta^2 y+3\eta x^2+ax(3\eta+y)+yx^2+3\eta^3+y^3}{(a+x)^2+(\eta+y)^2} \\
& - \frac{(\eta-y)[(\eta-y)^2+x^2-ax]}{(a-x)^2+(\eta-y)^2} \\
& + \frac{(\eta-y)[(\eta-y)^2+x^2+ax]}{(a+x)^2+(\eta-y)^2} \\
& - 2x \left[\tan^{-1} \left(\frac{a-x}{\eta-y} \right) + \tan^{-1} \left(\frac{a-x}{\eta+y} \right) + \tan^{-1} \left(\frac{a+x}{\eta-y} \right) \right. \\
& \left. + \tan^{-1} \left(\frac{a+x}{\eta+y} \right) \right] - \frac{3(\eta-y)}{2} \ln \left[\frac{(a-x)^2+(\eta-y)^2}{(a+x)^2+(\eta-y)^2} \right] \\
& - \frac{5\eta+3y}{2} \ln \left[\frac{(a-x)^2+(\eta+y)^2}{(a+x)^2+(\eta+y)^2} \right]
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
F_{22}^{\dagger} = & \frac{8\eta y^2(a-x)[(a-x)^2+y^2-\eta^2]}{[(a-x)^2+(\eta-y)^2][(a-x)^2+(\eta+y)^2]^2} \\
& + \frac{8\eta y^2(a+x)[(a+x)^2+y^2-\eta^2]}{[(a+x)^2+(\eta-y)^2][(a+x)^2+(\eta+y)^2]^2}
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
E_{22}^+ = & \frac{7\eta y^2 + 5\eta^2 y - \eta x^2 + ax(\eta - y) + yx^2 - \eta^3 + y^3}{(a - x)^2 + (\eta + y)^2} \\
& - \frac{7\eta y^2 + 5\eta^2 y - \eta x^2 - ax(\eta - y) + yx^2 - \eta^3 + y^3}{(a + x)^2 + (\eta + y)^2} \\
& + \frac{(\eta - y)[(\eta - y)^2 + x^2 - ax]}{(a - x)^2 + (\eta - y)^2} \\
& - \frac{(\eta - y)[(\eta - y)^2 + x^2 + ax]}{(a + x)^2 + (\eta - y)^2} \\
& - \frac{4\eta y(\eta + y)[(y + \eta)^2 + x^2 - ax]}{[(a - x)^2 + (\eta + y)^2]^2} \\
& + \frac{4\eta y(\eta + y)[(y + \eta)^2 + x^2 + ax]}{[(a + x)^2 + (\eta + y)^2]^2} \\
& + \frac{\eta - y}{2} \left\{ \ln \left[\frac{(a - x)^2 + (\eta - y)^2}{(a - x)^2 + (\eta + y)^2} \right] \right. \\
& \left. - \ln \left[\frac{(a + x)^2 + (\eta - y)^2}{(a + x)^2 + (\eta + y)^2} \right] \right\}
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
F_{12}^+ = & \frac{\eta^2 + 4\eta y + y^2}{(a + x)^2 + (\eta + y)^2} - \frac{\eta^2 + 4\eta y + y^2}{(a - x)^2 + (+y)^2} \\
& + \frac{4\eta y(\eta + y)^2}{[(a - x)^2 + (\eta + y)^2]^2} \\
& - \frac{1}{2} \left[\ln \frac{(a - x)^2 + (\eta + y)^2}{(a - x)^2 + (\eta - y)^2} - \ln \frac{(a + x)^2 + (\eta + y)^2}{(a + x)^2 + (\eta - y)^2} \right] \\
& - \frac{4\eta y(\eta + y)^2}{[(a + x)^2 + (\eta + y)^2]^2} + \frac{(\eta - y)^2}{(a - x)^2 + (\eta - y)^2} \\
& - \frac{(\eta - y)^2}{(a + x)^2 + (\eta - y)^2}
\end{aligned} \tag{A.12}$$

$$\begin{aligned}
E_{12}^+ = & \frac{2\eta y[(a+x)(\eta+y)^2 - (a-x)^3]}{((a-x)^2 + (\eta+y)^2)^2} \\
& + \frac{2\eta y[(a-x)(\eta+y)^2 - (a+x)^3]}{((a+x)^2 + (\eta+y)^2)^2} \\
& - \frac{a(y^2 + 4\eta y + \eta^2)}{(a-x)^2 + (\eta+y)^2} - \frac{a(y^2 + 4\eta y + \eta^2)}{(a+x)^2 + (\eta+y)^2} \\
& + \frac{a(\eta-y)^2}{(a-x)^2 + (\eta-y)^2} + \frac{a(\eta-y)^2}{(a+x)^2 + (\eta-y)^2} \\
& - 2(\eta-y) \left[\tan^{-1} \left(\frac{a-x}{\eta-y} \right) + \tan^{-1} \left(\frac{a+x}{\eta-y} \right) \right] \\
& - \frac{x}{2} \left\{ \ln \left[\frac{(a-x)^2 + (y+\eta)^2}{(a-x)^2 + (y-\eta)^2} \right] \right. \\
& \left. - \ln \left[\frac{(a+x)^2 + (y+\eta)^2}{(a+x)^2 + (y-\eta)^2} \right] \right\} \\
& + 2(\eta+y) \left[\tan^{-1} \left(\frac{a-x}{\eta+y} \right) + \tan^{-1} \left(\frac{a+x}{\eta+y} \right) \right]
\end{aligned} \tag{A.13}$$

Appendix B

A distribution of dislocations is located along the vertical direction in the half-space matrix with the shear modulus μ and the Kolosov constant κ ($\kappa = 3 - 4\nu$ for the plane-strain problem). The dislocation line distribution has the length of $2a$ and is centered at $(0, \eta)$. The stress σ_{ij}^Γ at the point (x, y) due to the vertical distribution of the edge dislocations can be written as

$$\sigma_{ij}^\Gamma(x, y, \eta) = \frac{2\mu}{\pi(\kappa + 1)} [E_{ij}^\perp(x, y, \eta)c^\perp + F_{ij}^\perp(x, y, \eta)d^\perp + E_{ij}^\parallel(x, y, \eta)c^\parallel + F_{ij}^\parallel(x, y, \eta)d^\parallel] \quad . \quad (\text{B.1})$$

$$\begin{aligned} F_{11}^\perp = x \left\{ \frac{y - \eta - a}{x^2 + (\eta + a - y)^2} - \frac{y - \eta + a}{x^2 + (\eta - a - y)^2} \right. \\ \left. - \frac{4y[x^2 + y(y + \eta + a)]}{[x^2 + (y + \eta + a)^2]^2} + \frac{4y[x^2 + y(y + \eta - a)]}{[x^2 + (y + \eta - a)^2]^2} \right. \\ \left. + \frac{7y - 3\eta - 3a}{x^2 + (y + \eta + a)^2} - \frac{7y - 3\eta + 3a}{x^2 + (y + \eta - a)^2} \right\} \\ + 4 \left[\tan^{-1} \left(\frac{y + \eta + a}{x} \right) - \tan^{-1} \left(\frac{y + \eta - a}{x} \right) \right] \end{aligned} \quad (\text{B.2})$$

$$\begin{aligned} E_{11}^\perp = \frac{x(\eta + a)(y - \eta - a)}{x^2 + (\eta + a - y)^2} - \frac{x(\eta - a)(y - \eta + a)}{x^2 + (\eta - a - y)^2} \\ - \frac{4x(\eta + a)^2[x^2 + (\eta + a)(y + \eta + a)]}{[x^2 + (\eta + a + y)^2]^2} \\ + \frac{4x(\eta - a)^2[x^2 + (\eta - a)(y + \eta - a)]}{[x^2 + (\eta - a + y)^2]^2} \\ + \frac{x(\eta + a)(\eta + a + 5y)}{x^2 + (\eta + a + y)^2} - \frac{x(\eta - a)(\eta - a + 5y)}{x^2 + (\eta - a + y)^2} \\ - 12y \left[\tan^{-1} \left(\frac{y + \eta + a}{x} \right) - \tan^{-1} \left(\frac{y + \eta - a}{x} \right) \right] \\ + \frac{x}{2} \ln \left[\frac{x^2 + (\eta + a - y)^2}{x^2 + (\eta - a - y)^2} \right] + \frac{7x}{2} \ln \left[\frac{x^2 + (y + \eta + a)^2}{x^2 + (y + \eta - a)^2} \right] \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned}
F_{22}^\perp = -x & \left\{ \frac{y - \eta - a}{x^2 + (\eta + a - y)^2} - \frac{y - \eta + a}{x^2 + (\eta - a - y)^2} \right. \\
& - \frac{4y[x^2 + y(y + \eta + a)]}{[x^2 + (y + \eta + a)^2]^2} + \frac{4y[x^2 + y(y + \eta - a)]}{[x^2 + (y + \eta - a)^2]^2} \\
& + \frac{7y + \eta + a}{x^2 + (y + \eta + a)^2} - \frac{7y + \eta - a}{x^2 + (y + \eta - a)^2} \Big\} \\
& - 2\tan^{-1}\left(\frac{\eta + a - y}{x}\right) + 2\tan^{-1}\left(\frac{\eta - a - y}{x}\right) \\
& + 2\tan^{-1}\left(\frac{\eta + a + y}{x}\right) - 2\tan^{-1}\left(\frac{\eta - a + y}{x}\right)
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
E_{22}^\perp = - & \frac{2(\eta + a)^3 xy[3(\eta + a)^2 - 4(\eta + a)y + 6(x^2 - y^2)]}{[x^2 + (\eta + a - y)^2][x^2 + (\eta + a + y)^2]^2} \\
& + \frac{2(\eta - a)^3 xy[3(\eta - a)^2 - 4(\eta - a)y + 6(x^2 - y^2)]}{[x^2 + (\eta - a - y)^2][x^2 + (\eta - a + y)^2]^2} \\
& - \frac{2(\eta + a)xy[4(\eta + a)y(x^2 + y^2) + 3(x^2 + y^2)^2]}{[x^2 + (\eta + a - y)^2][x^2 + (\eta + a + y)^2]^2} \\
& + \frac{2(\eta - a)xy[4(\eta - a)y(x^2 + y^2) + 3(x^2 + y^2)^2]}{[x^2 + (\eta - a - y)^2][x^2 + (\eta - a + y)^2]^2} \\
& - 2y \left[\tan^{-1}\left(\frac{\eta + a - y}{x}\right) - \tan^{-1}\left(\frac{\eta - a - y}{x}\right) \right. \\
& \left. - \tan^{-1}\left(\frac{\eta + a + y}{x}\right) + \tan^{-1}\left(\frac{\eta - a + y}{x}\right) \right] \\
& - \frac{3x}{2} \ln \left[\frac{x^2 + (\eta + a - y)^2}{x^2 + (\eta - a - y)^2} \right] + \frac{3x}{2} \ln \left[\frac{x^2 + (\eta + a + y)^2}{x^2 + (\eta - a + y)^2} \right]
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
F_{12}^\perp = - & \frac{8(\eta + a)x^2 y[(\eta + a)^2 + x^2 + y^2]}{[x^2 + (\eta + a - y)^2][x^2 + (\eta + a + y)^2]^2} \\
& + \frac{8(\eta - a)x^2 y[(\eta - a)^2 + x^2 + y^2]}{[x^2 + (\eta - a - y)^2][x^2 + (\eta - a + y)^2]^2} \\
& + \frac{2y(3\eta + 3a + 2y)}{x^2 + (y + \eta + a)^2} - \frac{2y(3\eta - 3a + 2y)}{x^2 + (y + \eta - a)^2} \\
& - \frac{1}{2} \ln \left[\frac{x^2 + (\eta + a - y)^2}{x^2 + (\eta - a - y)^2} \right] + \frac{1}{2} \ln \left[\frac{x^2 + (\eta + a + y)^2}{x^2 + (\eta - a + y)^2} \right]
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
E_{12}^{\perp} = & 12a - \frac{(\eta + a)x^2}{x^2 + (\eta + a - y)^2} + \frac{(\eta - a)x^2}{x^2 + (\eta - a - y)^2} \\
& - \frac{4(\eta + a)^2 x^2 y}{[x^2 + (\eta + a + y)^2]^2} + \frac{4(\eta - a)^2 x^2 y}{[x^2 + (\eta - a + y)^2]^2} \\
& - \frac{6(\eta + a)^3 + 5(\eta + a)x^2 + 4(\eta + a)^2 y}{x^2 + (\eta + a + y)^2} \\
& + \frac{6(\eta - a)^3 + 5(\eta - a)x^2 + 4(\eta - a)^2 y}{x^2 + (\eta - a + y)^2} \\
& + 2x \left[\tan^{-1} \left(\frac{\eta + a - y}{x} \right) - \tan^{-1} \left(\frac{\eta - a - y}{x} \right) \right. \\
& \left. - \tan^{-1} \left(\frac{\eta + a + y}{x} \right) + \tan^{-1} \left(\frac{\eta - a + y}{x} \right) \right] \\
& - \frac{y}{2} \ln \left[\frac{x^2 + (\eta + a - y)^2}{x^2 + (\eta - a - y)^2} \right] - \frac{7y}{2} \ln \left[\frac{x^2 + (\eta + a + y)^2}{x^2 + (\eta - a + y)^2} \right]
\end{aligned} \tag{B.7}$$

$$\begin{aligned}
F_{11}^+ = & \frac{2(\eta - a)(2\eta - 2a + y)}{x^2 + (\eta - a + y)^2} - \frac{2(\eta + a)(2\eta + 2a + y)}{x^2 + (\eta + a + y)^2} \\
& + \frac{16(\eta + a)^2 x^2 y^2}{[x^2 + (\eta + a - y)^2][x^2 + (\eta + a + y)^2]^2} \\
& - \frac{16(\eta - a)^2 x^2 y^2}{[x^2 + (\eta - a - y)^2][x^2 + (\eta - a + y)^2]^2} \\
& + \frac{1}{2} \ln \left[\frac{x^2 + (\eta + a + y)^2}{x^2 + (\eta + a - y)^2} \right] - \frac{1}{2} \ln \left[\frac{x^2 + (\eta - a + y)^2}{x^2 + (\eta - a - y)^2} \right]
\end{aligned} \tag{B.8}$$

$$\begin{aligned}
E_{11}^+ = & 12a + \frac{(\eta + a)x^2}{x^2 + (\eta + a - y)^2} - \frac{(\eta - a)x^2}{x^2 + (\eta - a - y)^2} \\
& - \frac{4(\eta + a)^2 x^2 y}{[x^2 + (\eta + a + y)^2]^2} + \frac{4(\eta - a)^2 x^2 y}{[x^2 + (\eta - a + y)^2]^2} \\
& - \frac{6(\eta + a)^3 + 3(\eta + a)x^2 + 4(\eta + a)^2 y}{x^2 + (\eta + a + y)^2} \\
& + \frac{6(\eta - a)^3 + 3(\eta - a)x^2 + 4(\eta - a)^2 y}{x^2 + (\eta - a + y)^2} \\
& - 4x \left[\tan^{-1} \left(\frac{\eta + a + y}{x} \right) - \tan^{-1} \left(\frac{\eta - a + y}{x} \right) \right] \\
& - \frac{y}{2} \ln \left[\frac{x^2 + (\eta + a - y)^2}{x^2 + (\eta - a - y)^2} \right] - \frac{7y}{2} \ln \left[\frac{x^2 + (\eta + a + y)^2}{x^2 + (\eta - a + y)^2} \right]
\end{aligned} \tag{B.9}$$

$$\begin{aligned}
F_{22}^+ = & -\frac{2y(\eta+a)^3[(\eta+a)^2+2(x^2-y^2)]}{[x^2+(\eta+a-y)^2][x^2+(\eta+a+y)^2]^2} \\
& +\frac{2y(\eta-a)^3[(\eta-a)^2+2(x^2-y^2)]}{[x^2+(\eta-a-y)^2][x^2+(\eta-a+y)^2]^2} \\
& -\frac{16x^2y^2(\eta+a)^2}{[x^2+(\eta+a-y)^2][x^2+(\eta+a+y)^2]^2} \\
& +\frac{16x^2y^2(\eta-a)^2}{[x^2+(\eta-a-y)^2][x^2+(\eta-a+y)^2]^2} \\
& -\frac{2y(\eta+a)(x^2+y^2)^2}{[x^2+(\eta+a-y)^2][x^2+(\eta+a+y)^2]^2} \\
& +\frac{2y(\eta-a)(x^2+y^2)^2}{[x^2+(\eta-a-y)^2][x^2+(\eta-a+y)^2]^2} \\
& +\frac{1}{2}\ln\left[\frac{x^2+(\eta+a+y)^2}{x^2+(\eta+a-y)^2}\right] \\
& -\frac{1}{2}\ln\left[\frac{x^2+(\eta-a+y)^2}{x^2+(\eta-a-y)^2}\right]
\end{aligned} \tag{B.10}$$

$$\begin{aligned}
E_{22}^+ = & \frac{(\eta-a)x^2}{x^2+(\eta-a-y)^2} - \frac{(\eta+a)x^2}{x^2+(\eta+a-y)^2} + \frac{4(\eta+a)^2x^2y}{[x^2+(\eta+a+y)^2]^2} \\
& -\frac{4(\eta-a)^2x^2y}{[x^2+(\eta-a+y)^2]^2} + \frac{2(\eta+a)^3+3(\eta+a)x^2}{x^2+(\eta+a+y)^2} \\
& -\frac{2(\eta-a)^3+3(\eta-a)x^2}{x^2+(\eta-a+y)^2} + \frac{y}{2}\ln\left[\frac{x^2+(\eta+a+y)^2}{x^2+(\eta+a-y)^2}\right] \\
& -\frac{y}{2}\ln\left[\frac{x^2+(\eta-a+y)^2}{x^2+(\eta-a-y)^2}\right] + 2x\tan^{-1}\left(\frac{\eta+a-y}{x}\right) \\
& -2x\tan^{-1}\left(\frac{\eta-a-y}{x}\right) - 2x\tan^{-1}\left(\frac{\eta+a+y}{x}\right) \\
& +2x\tan^{-1}\left(\frac{\eta-a+y}{x}\right) - 4a
\end{aligned} \tag{B.11}$$

$$\begin{aligned}
F_{12}^+ = & -\frac{8(\eta+a)^2xy[(\eta+a)^2+x^2-y^2]}{[x^2+(\eta+a-y)^2][x^2+(\eta+a+y)^2]^2} \\
& +\frac{8(\eta-a)^2xy[(\eta-a)^2+x^2-y^2]}{[x^2+(\eta-a-y)^2][x^2+(\eta-a+y)^2]^2}
\end{aligned} \tag{B.12}$$

$$\begin{aligned}
E_{12}^+ = & -\frac{10xy(\eta+a)^5}{[x^2+(\eta+a-y)^2][x^2+(\eta+a+y)^2]^2} \\
& +\frac{10xy(\eta-a)^5}{[x^2+(\eta-a-y)^2][x^2+(\eta-a+y)^2]^2} \\
& -\frac{2(\eta+a)xy[6(\eta+a)^2(x^2-y^2)+(x^2+y^2)^2]}{[x^2+(\eta+a-y)^2][x^2+(\eta+a+y)^2]^2} \\
& +\frac{2(\eta-a)xy[6(\eta-a)^2(x^2-y^2)+(x^2+y^2)^2]}{[x^2+(\eta-a-y)^2][x^2+(\eta-a+y)^2]^2} \\
& +4y\left[\tan^{-1}\left(\frac{\eta+a+y}{x}\right)-\tan^{-1}\left(\frac{\eta-a+y}{x}\right)\right] \\
& +\frac{x}{2}\ln\left[\frac{x^2+(\eta+a-y)^2}{x^2+(\eta+a+y)^2}\right]-\frac{x}{2}\ln\left[\frac{x^2+(\eta-a-y)^2}{x^2+(\eta-a+y)^2}\right]
\end{aligned} \tag{B.13}$$

Appendix C

The normal displacement due to a distribution of edge dislocations that model a vertical crack segment of length $2a$ centered at $(0, \eta)$ can be obtained by

$$\begin{aligned} u^\Gamma(x, \eta) &= \frac{1}{2\pi(\kappa + 1)} \int_{\eta-a}^{\eta+a} [\rho^\perp(y') U_{yy}(x, \eta - y') + \rho^\top(y') U_{xy}(x, \eta - y')] dy' \\ &= H^\perp(x, \eta) c^\perp + K^\perp(x, \eta) d^\perp + H^\top(x, \eta) c^\top + K^\top(x, \eta) d^\top \quad . \end{aligned} \quad (\text{C.1})$$

The expressions of the influence coefficients $H^\perp$, $K^\perp$, $H^\top$ and $K^\top$ for the vertical crack are given as below

$$\begin{aligned} H^\perp &= \frac{(a^2 - \eta^2)(1 + k) + (3 - k)x^2}{2\pi(\kappa + 1)} \left[\tan^{-1}\left(\frac{x}{a + \eta}\right) - \tan^{-1}\left(\frac{x}{\eta - a}\right) \right] \\ &\quad + \frac{\eta x}{\pi(\kappa + 1)} \ln\left(\frac{a^2 - 2a\eta + x^2 + \eta^2}{a^2 + 2a\eta + x^2 + \eta^2}\right) - \frac{ax(k - 3)}{\pi(\kappa + 1)} \end{aligned} \quad (\text{C.2})$$

$$\begin{aligned} K^\perp &= a + \frac{1}{\pi} \left[(\eta - a) \tan^{-1}\left(\frac{x}{\eta - a}\right) - (a + \eta) \tan^{-1}\left(\frac{x}{\eta + a}\right) \right] \\ &\quad + \frac{x}{\pi(\kappa + 1)} \ln\left(\frac{a^2 - 2a\eta + x^2 + \eta^2}{a^2 + 2a\eta + x^2 + \eta^2}\right) \end{aligned} \quad (\text{C.3})$$

$$H^\top = \frac{x}{2\pi} \left\{ 2\eta \left[\tan^{-1}\left(\frac{a + \eta}{x}\right) - \tan^{-1}\left(\frac{\eta - a}{x}\right) \right] + x \ln\left(\frac{a^2 - 2a\eta + x^2 + \eta^2}{a^2 + 2a\eta + x^2 + \eta^2}\right) \right\} \quad (\text{C.4})$$

$$K^\top = -\frac{1}{\pi} \left\{ 2a + x \left[\tan^{-1}\left(\frac{(\eta - a)}{x}\right) - \tan^{-1}\left(\frac{(a + \eta)}{x}\right) \right] \right\} \quad (\text{C.5})$$

References

- Anderson, T.L., 2005. Fracture Mechanics : Fundamentals and Applications. CRC Press, Boca Raton.
- Andreasen, J.H., Karihaloo, B.L., 1993. Interaction between a surface crack and a subsurface inclusion. *International Journal of Fracture*, 63, 1-10.
- Archard, J.F., 1953. Contact and Rubbing of Flat Surfaces. *Journal of Applied Physics*, 24, 981-988.
- Asaro, R.J., Barnett, D.M., 1975. The non-uniform transformation strain problem for an anisotropic ellipsoidal inclusion. *Journal of the Mechanics and Physics of Solids*, 23, 77-83.
- Bhargava, R.D., Bhargava, R.R., 1973. Elastic circular inclusion in an infinite plane containing two cracks. *International Journal of Engineering Science*, 11, 437-449.
- Bueckner, H.F., 1958. The propagation of cracks and the energy of elastic deformation. *Journal of Applied Mechanics*, 80, 1225-1230.
- Bush, M.B., 1997. The Interaction between a crack and a particle cluster. *International Journal of Fracture*, 88, 215-232.
- Carpick, R.W., Ogletree, D.F., Salmeron, M., 1999. A general equation for fitting contact area and friction vs load measurements. *Journal of Colloid and Interface Science*, 211, 395-400.
- Carter, F.W., 1926. On the action of a locomotive driving wheel. *Proceedings of the Royal Society of London. Series A*, 112, 151-157.
- Cattaneo, C., 1938. Sul "Contatto di due corpi elastici: distribuzione locale degli sforzi". *Rendiconti dell'Accademia nazionale dei Lincei*, 27, 342-348, 434-436, 474-478.
- Chen, D.-H., 1997. The effect of an elliptical inclusion on a crack. *International Journal of Fracture*, 85, 351-364.

Chen, J.B., Fang, Q.H., Liu, Y.W., 2013a. Interaction between dislocation and subsurface crack under condition of slip caused by half-plane contact surface normal force. *Engineering Fracture Mechanics*, 114, 115-126.

Chen, W.W., Zhou, K., Keer, L.M., Wang, Q.J., 2010. Modeling elasto-plastic indentation on layered materials using the equivalent inclusion method. *International Journal of Solids and Structures*, 47, 2841-2854.

Chen, Y.Q., Pan, S.P., Zhou, M.Z., Yi, D.Q., Xu, D.Z., Xu, Y.F., 2013b. Effects of inclusions, grain boundaries and grain orientations on the fatigue crack initiation and propagation behavior of 2524-T3 Al alloy. *Materials Science and Engineering: A*, 580, 150-158.

Chiu, Y.P., 1977. On the stress field due to initial strains in a cuboid surrounded by an infinite elastic space. *Journal of Applied Mechanics*, 44, 587-590.

Chiu, Y.P., 1978. On the stress field and surface deformation in a half space with a cuboidal zone in which initial strains are uniform. *Journal of Applied Mechanics*, 45, 302-306.

Demir, I., Zbib, H.M., Khaleel, M., 2001. Microscopic analysis of crack propagation for multiple cracks, inclusions and voids. *Theoretical and Applied Fracture Mechanics*, 36, 147-164.

Derjaguin, B.V., Muller, V.M., Toporov, Y.P., 1975. Effect of contact deformations on the adhesion of particles. *Journal of Colloid and Interface Science*, 53, 314-326.

Dong, C.Y., Lee, K., 2005. Boundary element analysis of infinite anisotropic elastic medium containing inclusions and cracks. *Engineering Analysis with Boundary Elements*, 29, 562-569.

Dundurs, J., Markenscoff, X., 2009. Stress fields and Eshelby forces on half-plane inhomogeneities with eigenstrains and strip inclusions meeting a free surface. *International Journal of Solids and Structures*, 46, 2481-2485.

Erdogan, F., 1962. On the stress distribution in plates with collinear cuts under arbitrary loads, *Proceedings of the fourth US national congress of applied*

mechanics, pp. 547-553.

Erdogan, F., Sih, G.C., 1963. On the Crack Extension in Plates Under Plane Loading and Transverse Shear. *Journal of Fluids Engineering*, 85, 519-525.

Erdogan, F., Gupta, G.D., Ratwani, M., 1974. Interaction between a circular inclusion and an arbitrarily oriented crack. *Journal of Applied Mechanics*, 41, 1007-1013.

Eshelby, J.D., 1957. The determination of the elastic field of an ellipsoidal inclusion, and related problems. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 241, 376-396.

Eshelby, J.D., 1959. The elastic field outside an ellipsoidal inclusion. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 252, 561-569.

Fang, Q.H., Liu, Y.W., Jiang, C.P., 2003. Edge dislocation interacting with an interfacial crack along a circular inhomogeneity. *International Journal of Solids and Structures*, 40, 5781-5797.

Fang, Q.H., Liu, Y.W., Jiang, C.P., 2005. Electroelastic interaction between a piezoelectric screw dislocation and an elliptical inclusion with interfacial cracks. *Physica Status Solidi B*, 242, 2775-2794.

Fang, Q.H., Liu, Y.W., 2006. Size-dependent elastic interaction of a screw dislocation with a circular nano-inhomogeneity incorporating interface stress. *Scripta Materialia*, 55, 99-102.

Fang, Q.H., Liu, Y.W., Wen, P.H., 2008. Screw dislocations in a three-phase composite cylinder model with interface stress. *Journal of Applied Mechanics*, 75, 041019.041011-041019.041018.

Fang, Q.H., Jin, B., Liu, Y., Liu, Y.W., 2009. Interaction between screw dislocations and inclusions with imperfect interfaces in fiber-reinforced composites. *Acta Mechanica*, 203, 113-125.

Fond, C., Riccardi, A., Schirrer, R., Montheillet, F., 2001. Mechanical interaction

between spherical inhomogeneities: an assessment of a method based on the equivalent inclusion. *European Journal of Mechanics - A/Solids*, 20, 59-75.

Forman, R.G., Kearney, V.E., Engle, R.M., 1967. Numerical Analysis of Crack Propagation in Cyclic-Loaded Structures. *Journal of Basic Engineering*, 89, 459-463.

Fromm, H., 1927. Berechnung des Schlupfes beim Rollen deformierbarer Scheiben. *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, 7, 27-58.

Gallo, P., Berto, F., 2015. Influence of surface roughness on high temperature fatigue strength and cracks initiation in 40CrMoV13.9 notched components. *Theoretical and Applied Fracture Mechanics*, 80, Part B, 226-234.

Gao, Z.W., Zhou, Y.H., Lee, K.Y., 2010. Crack-inclusion problem for a long rectangular slab of superconductor under an electromagnetic force. *Computational Materials Science*, 50, 279-282.

Goshima, T., Soda, T., 1997. Stress intensity factors of a subsurface crack in a semi-infinite body due to rolling/sliding contact and heat generation. *JSME International Journal Series A*, 40, 263-270.

Goshima, T., Ishihara, S., Shimizu, M., Soda, T., 2001. Mutual interference of two subsurface cracks in a semi-infinite body due to rolling contact with frictional heating. *JSME International Journal Series A Solid Mechanics and Material Engineering*, 44, 542-549.

Griffith, A.A., 1921. The Phenomena of Rupture and Flow in Solids. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 221, 163-198.

Guy, G., Hren, J., 1974. *Elements of physical Metallurgy*. Reading Mass. USA: Addison-Wesley, 366.

Hansson, P., 2009. Crack growth rates for short fatigue cracks simulated using a discrete dislocation technique. *International Journal of Fatigue*, 31, 1346-1355.

Hertz, H., 1882a. Über die Berührung fester elastische Körper und über die Härte. (On the contact of rigid elastic solids and on hardness). Verhandlungen des Vereins zur Beförderung des Gewerbeleißes, Leipzig.

Hertz, H., 1882b. Über die Berührung fester elastischer Körper. (On the contact of elastic solids). Journal Für Die Reine Und Angewandte Mathematik, 92, 156-171.

Hertzberg, R., 1989. Deformation and Fracture Mechanics of Engineering Materials, Third Edition ed. John Wiley and Sons, New York.

Hills, D.A., Kelly, P.A., Dai, D.N., Korsunsky, A.M., 1996. Solution of Crack Problems: The Distributed Dislocation Technique. Springer, Dordrecht.

Hsu, Y.C., Shivakumar, V., 1976. Interaction between an elastic circular inclusion and two symmetrically placed collinear cracks. International Journal of Fracture, 12, 619-630.

Inglis, C., 1913. Stresses in a cracked plate due to the presence of cracks and sharp corners. Transaction of Naval Architects (London), 60, 213.

Irwin, G.R., 1957. Analysis of Stresses and Strains Near the End of a Crack Traversing a Plate. Journal of Applied Mechanics, 24, 361–364.

Jacq, C., Nélías, D., Lormand, G., Girodin, D., 2002. Development of a Three-Dimensional Semi-Analytical Elastic-Plastic Contact Code. Journal of Tribology, 124, 653-667.

Johnson, K.L., 1958. The effect of spin upon the rolling motion of an elastic sphere on a plane. Journal of Applied Mechanics, 25, 332-338.

Johnson, K.L., Kendall, K., Roberts, A.D., 1971. Surface energy and the contact of elastic solids. Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences, 324, 301-313.

Johnson, K.L., 1985. Contact Mechanics. New York : Cambridge University Press.

Kalker, J.J., 1967. On the rolling contact of two elastic bodies in the presence of dry friction, Department of Mechanical Engineering. Delft University of Technology,

Netherlands.

Karakasidis, T.E., Charitidis, C.A., 2011. Influence of nano-inclusions' grain boundaries on crack propagation modes in materials. *Materials Science and Engineering: B*, 176, 490-493.

Kelvin, L., 1848. Note on the integration of the equations of equilibrium of an elastic solid. *Cambridge and Dublin Mathematical Journal*, 3, 87-89.

Kikuchi, M., Wada, Y., Shintaku, Y., Suga, K., Li, Y., 2014. Fatigue crack growth simulation in heterogeneous material using s-version FEM. *International Journal of Fatigue*, 58, 47-55.

Klesnil, M., Lukáš, P., 1972. Influence of strength and stress history on growth and stabilisation of fatigue cracks. *Engineering Fracture Mechanics*, 4, 77-92.

Krewerth, D., Lippmann, T., Weidner, A., Biermann, H., 2016. Influence of non-metallic inclusions on fatigue life in the very high cycle fatigue regime. *International Journal of Fatigue*, 84, 40-52.

Kuo, C.-H., 2008. Contact stress analysis of an elastic half-plane containing multiple inclusions. *International Journal of Solids and Structures*, 45, 4562-4573.

Lam, K.Y., Ong, P.P., Wude, N., 1998. Interaction between a circular inclusion and a symmetrically branched crack. *Theoretical and Applied Fracture Mechanics*, 28, 197-211.

Leroux, J., Fulleringer, B., Nélías, D., 2010. Contact analysis in presence of spherical inhomogeneities within a half-space. *International Journal of Solids and Structures*, 47, 3034-3049.

Liu, M., Farris, T.N., 1994. Effect of three-dimensional near surface defects on rolling and sliding contact fatigue. *Journal of Tribology*, 116, 841-848.

Liu, S., Wang, Q., Liu, G., 2000. A versatile method of discrete convolution and FFT (DC-FFT) for contact analyses. *Wear*, 243, 101-111.

Love, A.E.H., 1944. *A treatise on the mathematical theory of elasticity*. Dover

publications, New York.

Matysiak, S.I., Evtushenko, O.O., Zeleniak, V.M., 2004. Heating of a half space containing an inclusion and a crack. *Materials Science*, 40, 466-474.

Maugis, D., 1992. Adhesion of spheres: The JKR-DMT transition using a dugdale model. *Journal of Colloid and Interface Science*, 150, 243-269.

McEvily, A.J., 1988. On crack closure in fatigue crack growth. *Mechanics of Fatigue Crack Closure*, 982, 35-43.

Miller, G.R., Keer, L.M., 1983. Interaction between a rigid indenter and a near-surface void or inclusion. *Journal of Applied Mechanics*, 50, 615-620.

Mindlin, R.D., 1949. Compliance of elastic bodies in contact. *Journal of Applied Mechanics*, 16, 259-268.

Mindlin, R.D., Deresiewicz, H., 1953. Elastic spheres in contact under varying oblique forces. *Journal of Applied Mechanics*, 20, 327-344.

Miner, M.A., 1945. cumulative damage in fatigue. *Journal of Applied Mechanics*, 12, 159-164.

Moschovidis, Z.A., Mura, T., 1975. Two-ellipsoidal inhomogeneities by the equivalent inclusion method. *Journal of Applied Mechanics*, 42, 847-852.

Muller, V.M., Yushchenko, V.S., Derjaguin, B.V., 1980. On the influence of molecular forces on the deformation of an elastic sphere and its sticking to a rigid plane. *Journal of Colloid and Interface Science*, 77, 91-101.

Muller, V.M., Yushchenko, V.S., Derjaguin, B.V., 1983. General theoretical consideration of the influence of surface forces on contact deformations and the reciprocal adhesion of elastic spherical particles. *Journal of Colloid and Interface Science*, 92, 92-101.

Mura, T., Kinoshita, N., 1978. The polynomial eigenstrain problem for an anisotropic ellipsoidal inclusion. *Physica Status Solidi A: Applied Research*, 48, 447-450.

- Mura, T., 1987. *Micromechanics of defects in solids*. Martinus Nijhoff, Dordrecht.
- Orowan, E., 1949. Fracture and strength of solids. *Reports On Progress In Physics*, 12, 185-232.
- Pan, E., Yang, B., 2001. Elastostatic fields in an anisotropic substrate due to a buried quantum dot. *Journal of Applied Physics*, 90, 6190-6196.
- Paris, P., Erdogan, F., 1963. A Critical Analysis of Crack Propagation Laws. *Journal of Fluids Engineering*, 85, 528-533.
- Patton, E.M., Santare, M.H., 1990. The effect of a rigid elliptical inclusion on a straight crack. *International Journal of Fracture*, 46, 71-79.
- Polonsky, I.A., Keer, L.M., 2001. Numerical analysis of the effect of coating microstructure on three-dimensional crack propagation in the coating under rolling contact fatigue conditions. *Journal of Tribology*, 124, 14-19.
- Portela, A., Aliabadi, M.H., Rooke, D.P., 1993. Dual boundary element incremental analysis of crack propagation. *Computers & Structures*, 46, 237-247.
- Priddle, E.K., 1976. High cycle fatigue crack propagation under random and constant amplitude loadings. *International Journal of Pressure Vessels and Piping*, 4, 89-117.
- Rodin, G.J., Hwang, Y.-L., 1991. On the problem of linear elasticity for an infinite region containing a finite number of non-intersecting spherical inhomogeneities. *International Journal of Solids and Structures*, 27, 145-159.
- Rollason, E.C., 1973. *Metallurgy for engineers*, 4th ed. Edward Arnold, London.
- Salehizadeh, H., Saka, N., 1992. The mechanics of crack initiation at hard particles in rolling line contacts. *Journal of Tribology*, 114, 341-347.
- Sendeckyj, G.P., 1974. Interaction of cracks with rigid inclusions in longitudinal shear deformation. *International Journal of Fracture*, 10, 45-52.
- Seo, K., Mura, T., 1979. The elastic field in a half space due to ellipsoidal inclusions with uniform dilatational eigenstrains. *Journal of Applied Mechanics*, 46, 568-572.

Serna Moreno, M.C., Curiel-Sosa, J.L., Navarro-Zafra, J., Martínez Vicente, J.L., López Cela, J.J., 2015. Crack propagation in a chopped glass-reinforced composite under biaxial testing by means of XFEM. *Composite Structures*, 119, 264-271.

Shewchuk, J., 1994. An Introduction to the conjugate gradient method without the agonizing pain, Technical Report.

Shodja, H.M., Sarvestani, A.S., 2000. Elastic Fields in Double Inhomogeneity by the Equivalent Inclusion Method. *Journal of Applied Mechanics*, 68, 3-10.

Stagni, L., Lizzio, R., 1983. Shape effects in the interaction between an edge dislocation and an elliptical inhomogeneity. *Applied Physics A*, 30, 217-221.

Stam, G.T.M., van der Giessen, E., Meijers, P., 1994. Effect of transformation-induced shear strains on crack growth in zirconia-containing ceramics. *International Journal of Solids and Structures*, 31, 1923-1948.

Sukumar, N., Chopp, D.L., Moës, N., Belytschko, T., 2001. Modeling holes and inclusions by level sets in the extended finite-element method. *Computer Methods in Applied Mechanics and Engineering*, 190, 6183-6200.

Tabor, D., 1977. Surface forces and surface interactions. *Journal of Colloid and Interface Science*, 58, 2-13.

Tamate, O., 1968. The effect of a circular inclusion on the stresses around a line crack in a sheet under tension. *International Journal of Fracture*, 4, 257-266.

Thomas, S.B., Mhaikar, M.J., Sethuraman, R., 2000. Stress intensity factors for circular hole and inclusion using finite element alternating method. *Theoretical and Applied Fracture Mechanics*, 33, 73-81.

Tsuchida, E., Mura, T., 1983. The stress-field in an elastic half-space having a spheroidal inhomogeneity under all-around tension parallel to the plane boundary. *Journal of Applied Mechanics*, 50, 807-816.

Tsuchida, E., Ohno, M., Kouris, D.A., 1991. Effects of an inhomogeneous elliptic insert on the elastic field of an edge dislocation. *Applied Physics A*, 53, 285-291.

Ueda, K., Goshima, T., Ishihara, S., Shimizu, M., 2007. Initial crack growth emanating from an inclusion due to rolling/sliding contact. *Journal of Thermal Stresses*, 31, 107-126.

Walpole, L.J., 1967. The Elastic field of an inclusion in an anisotropic medium. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 300, 270-289.

Wang, B., 1992. Three-dimensional analysis of an ellipsoidal inclusion in a piezoelectric material. *International Journal of Solids and Structures*, 29, 293-308.

Wang, C., Libardi, W., Baldo, J.B., 1998. Analysis of crack extension paths and toughening in a two phase brittle particulate composite by the boundary element method. *International Journal of Fracture*, 94, 177-188.

Wang, X., Pan, E., 2011. Interaction between an edge dislocation and a circular inclusion with interface slip and diffusion. *Acta Materialia*, 59, 797-804.

Weertman, J., Weertman, J.R., 1992. Elementary dislocation theory. Oxford University Press, New York.

Westgaard, H.M., 1939. Bearing pressures and crack. *Journal of Applied Mechanics*, 6, 49-53.

Williams, M., 1957. On the stress distribution at the base of a stationary crack. *Journal of Applied Mechanics*, 24, 109-114.

Wu, L., Du, S., 1995. The elastic field caused by a circular cylindrical inclusion---Part I: Inside the Region $x_1^2 + x_2^2 < a^2$, $-\infty < x_3 < \infty$ where the circular cylindrical inclusion is expressed by $x_1^2 + x_2^2 < a^2$, $-h < x_3 < h$. *Journal of Applied Mechanics*, 62, 579-584.

Wu, L., 2003. The elastic field induced by a hemispherical inclusion in the half-space. *Acta Mechanica Sinica*, 19, 253-262.

Wu, L.Z., Du, S.Y., 1996. The elastic field in a half-space with a circular cylindrical inclusion. *Journal of Applied Mechanics*, 63, 925-932.

Xiao, Z.M., Yan, J., Chen, B.J., 2004. Electro-elastic stress analysis for a screw dislocation interacting with a coated inclusion in piezoelectric solid. *Acta Mechanica*, 172, 237-249.

Xiao, Z.M., Zhang, H.X., Chen, B.J., 2008. Interaction of a dislocation with collinear rigid lines at the interface of piezoelectric media. *Journal of Mechanics of Materials and Structures*, 3, 335-364.

Yang, L., Chen, Q., Li, Z., 2004. Crack-inclusion interaction for mode II crack analyzed by eshelby equivalent inclusion method. *Engineering Fracture Mechanics*, 71, 1421-1433.

Zhao, C., Zhang, H., Wang, Y., 2014. Semiconductor laser asymmetry cutting glass with laser induced thermal-crack propagation. *Optics And Lasers In Engineering*, 63, 43-52.

Zhou, K., Chen, W.W., Keer, L.M., Wang, Q.J., 2009. A fast method for solving three-dimensional arbitrarily shaped inclusions in a half space. *Computer Methods in Applied Mechanics and Engineering*, 198, 885-892.

Zhou, K., Chen, W.W., Keer, L.M., Ai, X., Sawamiphakdi, K., Glaws, P., Jane Wang, Q., 2011a. Multiple 3D inhomogeneous inclusions in a half space under contact loading. *Mechanics of Materials*, 43, 444-457.

Zhou, K., Keer, L.M., Wang, Q.J., 2011b. Semi-analytic solution for multiple interacting three-dimensional inhomogeneous inclusions of arbitrary shape in an infinite space. *International Journal for Numerical Methods in Engineering*, 87, 617-638.

Zhou, K., Keer, L.M., Wang, Q.J., Hua, D.Y., 2011c. Size prediction of particles caused by chipping wear of hard coatings. *Wear*, 271, 1203-1206.

Zhou, K., Keer, L.M., Wang, Q.J., Shah, B.M., Su, F., 2011d. Modelling damage to real surfaces in contact within a unified framework. *Procedia Engineering*, 10, 1573-1578.

Zhou, K., Keer, L.M., Jane Wang, Q., Ai, X., Sawamiphakdi, K., Glaws, P., Paire,

M., Che, F., 2012. Interaction of multiple inhomogeneous inclusions beneath a surface. *Computer Methods in Applied Mechanics and Engineering*, 217–220, 25–33.

List of publications

- [1] Wei, R.B, Zhou, K., Keer, L.M., Fan, Q., 2015. Modeling surface pressure, interfacial stresses and stress intensity factors for layered materials containing multiple cracks and inhomogeneous inclusions under contact loading. *Mechanics of Materials*. 92, 8-17.
- [2] Dong, Q.B., Zhou, K., Wei, R.B., Luo, J., Srikanth, N., 2015. Analysis of fluid pressure, interface stresses and stress intensity factors for layered materials with cracks and inhomogeneities under elastohydrodynamic lubrication contact. *International Journal of Mechanical Sciences*. 93,48-58.
- [3] Zhou, K., Wei, R.B., 2014. Modeling cracks and inclusions near surfaces under contact loading. *International Journal of Mechanical Sciences*. 83,163-171.
- [4] Zhou, K., Wei, R.B., 2014. Multiple cracks in a half plane under contact loading. *Acta Mechanica*. 255,1487-1503.
- [5] Zhou, K., Wei, R.B., Bi, G.J., Wang, X, Song, B., Feng, X.Q., 2014. Semi-analytic solution of multiple inhomogeneous inclusions and cracks in an infinite space. *International Journal of Computational Methods*.12,1487-1503.
- [6] Wei, R.B, Zhou, K., 2015. Fatigue crack propagation in heterogeneous materials under remote loading. (Manuscripted prepared).