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UNMANNED AERIAL VEHICLE

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Abstract

The ship deck landing problem of a multi-rotor UAV is solved in this thesis. This problem is divided into tracking and landing sub-problems. In the tracking sub-problem, the multi-rotor UAV needs to follow the ship's trajectory throughout its recovery process. In the landing sub-problem, the multi-rotor UAV is to be landed in a way that is minimally affected by the ship's heave motion and other disturbances.

In the tracking sub-problem, a full state backstepping control is formulated using a novel re-formulated dynamics which eliminates the use of inverse kinematics. In addition, this control formulation is based on a contraction theory-based backstepping control framework that produces a closed-form control law for implementation on large cascaded-feedback systems. In addition, a quaternion-based tracking controller is also formulated using this backstepping control framework and a re-formulated quaternion-based dynamics.

In the landing sub-problem, the invariant ellipsoid method is used to derive an optimality condition for an estimate of the output response bound with respect to known bounds on disturbances and noises. This methodology is applied as a gain tuning algorithm for a PID control system derived from the multi-rotor UAV heave dynamics in the landing sub-problem. The flexibility and usefulness of the optimal control framework is also demonstrated on a full state feedback and output feedback controller for the same landing sub-problem. The framework developed in this thesis is able to take into account multiple disturbances, system uncertainties and noises.

In both sub-problems, extensive simulations are performed to verify the proposed controllers. The proposed tracking controller is shown to improve its robustness performance against inverse kinematics-based backstepping controller. On the other hand, the optimal performance of the landing controllers are verified with realistic simulation

of effects such as ship heave, wind disturbances and measurement noises using static state feedback, PID and output feedback controllers. The different control structures demonstrate the generality of the proposed framework in achieving similar performance.

Finally, this thesis is concluded with discussions on the significance and implications of the research results as well as limitations and possible future research directions.

Contents

Acknowledgments	i
Abstract	ii
List of Figures	ix
List of Tables	xx
List of Frequently Used Symbols and Abbreviations	xxi
1 Introduction	1
1.1 Problem Description	3
1.1.1 Environmental Challenges	4
1.1.2 A Decoupled Approach	7
1.2 Academic Contributions	11
1.3 Thesis Outline	11
2 Literature Survey and Problem Formulation	13
2.1 Literature Survey: Position Tracking Controller	13
2.1.1 Time Scale Separation	15
2.1.2 Linear Controller for Position Tracking	19
2.1.3 Nonlinear Controller for Position Tracking	23
2.1.4 Concluding Remarks	31
2.2 Literature Survey: Landing Controller	32
2.2.1 Sensor-Based landing	32
2.2.2 Landing Control Algorithm	34
2.2.3 Concluding Remarks	39
2.3 Position Tracking Sub-Problem	40
2.3.1 A Simpler Backstepping Methodology	41

2.3.2	Complexity associated with Inverse Kinematic	42
2.3.3	Quaternion-Based Control	43
2.4	Landing Sub-Problem	44
2.4.1	Ship Motion Predictor-Free Design	45
2.4.2	Implementation Considerations	45
2.5	Research Approaches	46
3	Multi-Rotor UAV Model	47
3.1	Multi-Rotor UAV Kinematic and Dynamic Model	48
3.1.1	Control Forces and Moments	48
3.1.2	Translational Kinematics	50
3.1.3	Translational Dynamics	51
3.1.4	Rotational Kinematics	52
3.1.5	Rotational Dynamics	53
3.1.6	Heave and Planar Motion Decoupling	54
3.2	Gravity Model	55
3.3	Aerodynamic Model	56
3.4	Wind Disturbance Model	57
3.5	Ship Heave Model	59
4	Multi-Rotor UAV Ground Path Tracking	62
4.1	Introduction to Contraction Theory	64
4.1.1	Fundamentals of Contraction Theory	64
4.1.2	Partial Contraction	66
4.2	Contraction Theory-Based Backstepping Control	68
4.2.1	Preliminaries	68
4.2.2	Main Result	71
4.3	Multi-Rotor UAV Position Tracking Controller	77
4.3.1	Inverse Kinematics Backstepping	78
4.3.2	Euler Angle-Based Formulation	81
4.3.3	Quaternion-Based Formulation	87
4.3.4	Linear Tracking Differentiator	91

4.4	Numerical Simulations	93
4.4.1	Simulation Parameters	93
4.4.2	Tracking a Moving Target	94
4.4.3	Tracking a Moving Target with Aerodynamic Effects	99
4.4.4	Tracking a Moving Target in Wind Disturbances	107
4.4.5	Tracking a Moving Target with Measurement Noise	113
4.5	Conclusion	119
4.5.1	Implications and Significance of Results	119
4.5.2	Concluding Remarks	120

5 Multi-Rotor UAV Landing on Ship Deck: A Full State Feedback Approach 121

5.1	Introduction to Invariant Ellipsoid Method	123
5.1.1	Invariant Ellipsoid Condition	123
5.1.2	State Feedback Control with the Invariant Ellipsoid Method . . .	127
5.1.3	State Feedback Control with the Invariant Ellipsoid Method including Control Action Constraint	129
5.1.4	Invariant Ellipsoid Method Applied to Uncertain Linear Systems .	131
5.2	Multi-Rotor UAV Ship Deck Landing Model	136
5.3	Multi-Rotor UAV Ship Deck Landing Controller	138
5.3.1	Optimal Controller Gain Synthesis	138
5.3.2	PID Controller Gain Synthesis	141
5.3.3	Properties of Optimal Controllers	145
5.3.4	Gain Synthesis with Measurement Noises	148
5.4	Numerical Simulations	151
5.4.1	Ship Deck Landing Scenario	151
5.4.2	PID Controller	153
5.4.3	State Feedback Controller with Measurement Noise	159
5.5	Conclusion	164
5.5.1	Implications and Significance of Results	164
5.5.2	Concluding Remarks	165

6	Multi-Rotor UAV Landing on Ship Deck: An Output Feedback Approach	167
6.1	Main Result	168
6.1.1	Output Feedback Controller Formulation	168
6.1.2	Invariant Ellipsoid Condition	170
6.1.3	Existence of Solution	174
6.2	Output Feedback Controller Gain Synthesis	178
6.2.1	Sub-problem: Initial Controller Gain Synthesis	178
6.2.2	Sub-problem: Initial Observer Gain Synthesis	182
6.2.3	Optimal Gain Synthesis	185
6.3	Application to Multi-Rotor UAV Ship Deck Landing	191
6.4	Numerical Simulations	193
6.4.1	Ship Deck Landing Scenario	193
6.4.2	Comparison using other Gains	199
6.5	Full Nonlinear Simulation	202
6.6	Conclusion	207
6.6.1	Implications and Significance of Results	207
6.6.2	Concluding Remarks	209
7	Conclusion	210
7.1	Contraction Theory-Based Backstepping Control	211
7.1.1	Implications and Significance	211
7.1.2	Limitations	212
7.1.3	Future Works	213
7.2	Invariant Ellipsoid Method	213
7.2.1	Implications and Significance	214
7.2.2	Limitations	214
7.2.3	Future Works	215
7.3	Concluding Remarks	216
	References	217

Appendix A List of Publications	229
A.1 List of Journal Publications	229
A.1.1 Published/Accepted	229
A.1.2 Under Review	229
A.2 List of Conference Publications	230
Appendix B Ship Deck Landing with Output Regulation	231
B.1 Dynamic Model and Control Law	231
B.2 Simulation Result and Discussion	233

List of Figures

1.1	A figure showing a quadrotor in Figure 1.1.a (image obtained from www.pupin.rs), a hexrotor in Figure 1.1.b (image obtained from www.aliexpress.com) and an octorotor in Figure 1.1.c (image obtained from www.etotheipiplusone.net).	1
1.2	A representation of the decoupled sub-problem leading to separate designs of a tracking and landing controller.	8
1.3	A diagram representation of the objective and requirement of the tracking controller.	9
1.4	A multi-rotor UAV of 1m diameter landing on a ship having a pitch angle of 45^0 results in a 1m height difference between both legs.	10
1.5	A diagram representation of the objective and requirement of the landing controller. Dotted lines represent effects that are secondary to this problem.	10
2.1	A representation of the control structure with the principle of time scale separation where P , Θ are the UAV position and Euler angles respectively, and P_d , Θ_d are the desired UAV position and Euler angles respectively.	15
2.2	A diagram showing the architecture proposed by Zheng <i>et. al.</i> [111], where x , y , z are the components of the UAV position, $\dot{x}$, $\dot{y}$, $\dot{z}$ are the components of the UAV velocity, f , θ , ψ are the roll, pitch and yaw angles, $\dot{f}$, $\dot{\theta}$, $\dot{\psi}$ are the roll, pitch and yaw rates, x_d , y_d , z_d are the components of the desired UAV position, and f_d , θ_d , ψ_d are the desired roll, pitch and yaw angles.	24

2.3	A diagram showing the architecture proposed by Besnard <i>et. al.</i> [10], where ψ_c is the commanded yaw angle, Φ_c is the commanded Euler angles, Ω_c is the commanded angular velocity, τ_{ac} is the commanded control moment, ω_{1c} is the commanded rotor rotational velocity, τ_{1c} is the commanded rotor input, P , Φ , Ω and ω_1 are the UAV position, Euler angles, rotational velocity and rotor rotational velocity respectively.	26
2.4	A diagram showing the hierarchical structure of a multi-rotor UAV dynamics, where M_x , M_y , M_z are the control moment components, p , q , r are the UAV rotational velocity, ϕ , θ , ψ are the roll, pitch and yaw angles, V is the UAV velocity and P is the UAV position.	28
2.5	A diagram showing the architecture proposed by Pollini <i>et. al.</i> [67], where x_d , y_d , z_d are the desired UAV position components, ϕ_d , θ_d , ψ_d are the desired roll, pitch, yaw angles, U_1 , U_2 , U_3 , U_4 are the control moment components and $F_{1..4}$ are the control force inputs.	29
2.6	A diagram showing the architecture proposed by Gong <i>et. al.</i> [32].	30
2.7	A diagram showing the architecture proposed by Avanzini <i>et. al.</i> [4]. . .	35
2.8	A diagram showing the architecture proposed by Avanzini <i>et. al.</i> [5]. . .	36
2.9	A diagram showing the architecture proposed by Nuthi <i>et. al.</i> [65]. . . .	37
2.10	A diagram showing the architecture proposed by Esmailifar <i>et. al.</i> [26]. .	38
3.1	A diagram showing a quadrotor and the Euler angles associated with the attitude of the quadrotor. The force on the quadrotor is acting on each of the rotors and pointing in the negative body z -axis. The reference frames, Earth-fixed inertial frame (X - Y - Z) and body frame (x - y - z), are also shown in the diagram.	48
3.2	The random wind speed profile generated by the Dryden wind turbulence model with a bias term.	58
3.3	The random wind rotation speed profile generated by the Dryden wind turbulence model.	58
3.4	The modified Pierson-Moskowitz spectrum at sea state 6 and a design wind speed of 30kts = 15.4m/s.	60

3.5	Simulation of the ship deck motion at sea state 6 at a design wind speed of 30kts = 15.4m/s for two different initial conditions. The simulation shows that different waveforms are obtained by differing the initial conditions.	61
4.1	Two state trajectories are shown in the state space. The virtual displacement $\delta\mathbf{x}$ is the infinitesimal change in states at time t . The rate of change of virtual displacement $\delta\dot{\mathbf{x}}$ is also shown.	65
4.2	The backstepping feedback cascade using inverse kinematics is shown in this diagram. The control moments achieve the desired rotation rates which controls the roll, pitch and roll angles. Inverse kinematics is then used to compute the roll and pitch angles required to achieve the desired planar acceleration. The planar acceleration then affects the planar velocity then the planar position, hence achieving planar position tracking.	81
4.3	The backstepping feedback hierarchy using full state control is shown in this diagram. The control moments achieves the desired rotation rates which controls the planar acceleration and yaw angle to their desired value. The planar acceleration then affects the planar velocity then the planar position, hence achieving planar position tracking.	82
4.4	Results from the numerical simulation of a quadrotor tracking a target moving along a linear path on the ground with both the full state feedback backstepping controller (labeled proposed controller, solid line) and the inverse kinematics-based backstepping controller (labeled benchmark controller, dotted line). The quadrotor is maintained at a height of 10m and a yaw angle of 0^0 . The X and Y position of the quadrotor and the target against time is shown in Figure 4.4.a and 4.4.b (zoomed views are shown in Figure 4.4.c and Figure 4.4.d). The roll and pitch angles against time are also shown in Figure 4.4.e and 4.4.f respectively. The tracking error between the quadrotor and the target is plotted in Figure 4.4.g. Lastly, the control moment input is plotted in Figure 4.4.h.	96

4.5	Results from the numerical simulation of a quadrotor tracking a target moving along a circular path on the ground with both the full state feedback backstepping controller (labeled proposed controller, solid line) and the inverse kinematics-based backstepping controller (labeled benchmark controller, dotted line). The quadrotor is maintained at a height of 10m and a yaw angle of 0^0 . The X and Y position of the quadrotor and the target against time is shown in Figure 4.5.a and 4.5.b (a zoomed view is shown). The roll and pitch angles against time are also shown in Figure 4.5.c and 4.5.d respectively. The tracking error between the quadrotor and the target is plotted in Figure 4.5.e. Lastly, the control moment input is plotted in Figure 4.5.f.	98
4.6	Results from the numerical simulation with aerodynamic effects of a quadrotor tracking a target moving along a linear path on the ground with both the full state feedback backstepping controller (labeled proposed controller, solid line) and the inverse kinematics-based backstepping controller (labeled benchmark controller, dotted line). The quadrotor is maintained at a height of 10m and a yaw angle of 0^0 . The X and Y position of the quadrotor and the target against time is shown in Figure 4.6.a and 4.6.b (zoomed views are shown in Figure 4.6.c and 4.6.d). The roll and pitch angles against time are also shown in Figure 4.6.e and 4.6.f respectively. The tracking error between the quadrotor and the target is plotted in Figure 4.6.g. Lastly, the control moment input is plotted in Figure 4.6.h.	101

4.7	Results from the numerical simulation with aerodynamic effects of a quadrotor tracking a target moving along a circular path on the ground with both the full state feedback backstepping controller (labeled proposed controller, solid line) and the inverse kinematics-based backstepping controller (labeled benchmark controller, dotted line). The quadrotor is maintained at a height of 10m and a yaw angle of 0^0 . The X and Y position of the quadrotor and the target against time is shown in Figure 4.7.a and 4.7.b (a zoomed view is shown). The roll and pitch angles against time are also shown in Figure 4.7.c and 4.7.d respectively. The tracking error between the quadrotor and the target is plotted in Figure 4.7.e. Lastly, the control moment input is plotted in Figure 4.7.f.	103
4.8	The ground track of the quadrotor with the full state backstepping controller tracking a target moving along a circular path on the ground with aerodynamic effects.	104
4.9	The ground track of the quadrotor with the inverse kinematics-based backstepping controller tracking a target moving along a circular path on the ground with aerodynamic effects.	104
4.10	Results from the numerical simulation with wind disturbances of a quadrotor tracking a target moving along a linear path on the ground with both the full state feedback backstepping controller (labeled proposed controller, solid line) and the inverse kinematics-based backstepping controller (labeled benchmark controller, dotted line). The quadrotor is maintained at a height of 10m and a yaw angle of 0^0 . The X and Y position of the quadrotor and the target against time is shown in Figure 4.10.a and 4.10.b (zoomed views are shown in Figure 4.10.c and 4.10.d). The roll and pitch angles against time are also shown in Figure 4.10.e and 4.10.f respectively. The tracking error between the quadrotor and the target is plotted in Figure 4.10.g. Lastly, the control moment input is plotted in Figure 4.10.h.	109

4.11	Results from the numerical simulation with wind disturbances of a quadrotor tracking a target moving along a circular path on the ground with both the full state feedback backstepping controller (labeled proposed controller, solid line) and the inverse kinematics-based backstepping controller (labeled benchmark controller, dotted line). The quadrotor is maintained at a height of 10m and a yaw angle of 0^0 . The X and Y position of the quadrotor and the target against time is shown in Figure 4.11.a and 4.11.b (a zoomed view is shown). The roll and pitch angles against time are also shown in Figure 4.11.c and 4.11.d respectively. The tracking error between the quadrotor and the target is plotted in Figure 4.11.e. Lastly, the control moment input is plotted in Figure 4.11.f.	111
4.12	The ground track of the quadrotor with the full state backstepping controller tracking a target moving along a circular path on the ground with wind disturbance.	112
4.13	The ground track of the quadrotor equipped with the inverse kinematics-based backstepping controller tracking a target moving along a circular path on the ground with wind disturbance.	112
4.14	Results from the numerical simulation with wind disturbances and measurement noises of a quadrotor tracking a target moving along a linear path on the ground with both the full state feedback backstepping controller (labeled proposed controller, solid line) and the inverse kinematics-based backstepping controller (labeled benchmark controller, dotted line). The quadrotor is maintained at a height of 10m and a yaw angle of 0^0 . The X and Y position of the quadrotor and the target against time is shown in Figure 4.14.a and 4.14.b (zoomed view are shown in Figure 4.14.c and 4.14.d). The roll and pitch angles against time are also shown in Figure 4.14.e and 4.14.f respectively. The tracking error between the quadrotor and the target is plotted in Figure 4.14.g. Lastly, the control moment input is plotted in Figure 4.14.h.	116

4.15	Results from the numerical simulation with wind disturbances and measurement noises of a quadrotor tracking a target moving along a circular path on the ground with both the full state feedback backstepping controller (labeled proposed controller, solid line) and the inverse kinematics-based backstepping controller (labeled benchmark controller, dotted line). The quadrotor is maintained at a height of 10m and a yaw angle of 0° . The X and Y position of the quadrotor and the target against time is shown in Figure 4.15.a and 4.15.b. The roll and pitch angles against time are also shown in Figure 4.15.c and 4.15.d respectively. The tracking error between the quadrotor and the target is plotted in Figure 4.15.e. Lastly, the control moment input is plotted in Figure 4.15.f.	117
4.16	The ground track of the quadrotor with the full state backstepping controller tracking a target moving along a circular path on the ground with wind disturbance and measurement noises.	118
4.17	The ground track of the quadrotor equipped with the inverse kinematics-based backstepping controller tracking a target moving along a circular path on the ground with wind disturbance and measurement noises. . . .	118
5.1	The nomenclature used in the ship deck landing multi-rotor UAV model. The multi-rotor UAV at an altitude h above the reference line is required to land on a landing pad is placed at a position L along the x_{ship} -axis. The ship center of mass is heaved at a height d_{ship} above the reference line and rolled an angle ϕ_{ship} causing the height of the landing pad at $d = d_{\text{ship}} - L \sin \phi_{\text{ship}}$ above the reference line.	136
5.2	The variation of the optimal control gain, $K^*(\alpha)$, and the maximum eigenvalue $\lambda_{\max}(\alpha)$ of $P^*(\alpha)$ with different values of α for system (5.32). . . .	140
5.3	A schematic of the closed-loop control system. The relative tracking error e between the multi-rotor UAV and the ship deck is feedback into the PID controller and the control input u is augmented with the reference altitude H before entering into the multi-rotor UAV to form a closed-loop.	142
5.4	The variation of the optimal PID control gain, $K^*(\alpha)$, and the trace of $P^*(\alpha)$ with different values of α for system (5.37).	144

5.5	An illustration of the level set of error response bound when subjected to different levels of disturbance $ w < \sqrt{\frac{\gamma}{M}}$ for a scalar w with $M = 0.111$ for system (5.32).	146
5.6	The optimization procedure to obtain the optimal control gain in the presence of measurement noise.	150
5.7	The variation of maximum eigenvalue of the invariant ellipsoid with α before and after the introduction of noise.	151
5.8	The landing procedure of a quadrotor on a ship deck. The quadrotor tracks and maintain relative height above the landing pad first, followed by a gradual landing phase.	152
5.9	Results from the numerical simulation of a quadrotor landing on a ship deck with two PID ship deck landing controllers whose gains are obtained from the invariant ellipsoid method (solid line) and loop shaping method (dotted line). Figure 5.9.a shows the heave motion of the ship deck, with the mean sea level at $h = 0$. Figure 5.9.b shows the tracking error e of the quadrotor, a small tracking error indicates that the quadrotor is able to track the ship deck motion. Figure 5.9.c shows the relative velocity error between the quadrotor and the ship deck, a small relative velocity error indicates small impact impulse. 5.9.d shows the quadrotor height relative to the ship deck. Lastly, Figure 5.9.e shows the quadrotor thrust u_1	155
5.10	Height of the quadrotor relative to the ship deck without ship deck heave motion. The solid line is the PID ship deck landing controller with gain derived from the invariant ellipsoid method while the dotted line uses the baseline controller gain.	156
5.11	Monte Carlo simulation of a quadrotor landing on a ship deck with two PID ship deck landing controllers whose gains are obtained from the invariant ellipsoid method for 10 different B matrices. Figure 5.11.a shows the tracking error e of the quadrotor, a small tracking error indicates that the quadrotor is able to track the ship deck motion. Figure 5.11.b shows the relative velocity error between the quadrotor and the ship deck, a small relative velocity error indicates small impact impulse. 5.11.c shows the quadrotor height relative to the ship deck.	159

5.12	Results from the numerical simulation of a quadrotor landing on a ship deck with two PID ship deck landing controllers whose gains are obtained from the invariant ellipsoid method (solid line) and loop shaping method (dotted line). Figure 5.12.a shows the heave motion of the ship deck, with the mean sea level at $h = 0$. Figure 5.12.b shows the tracking error e of the quadrotor, a small tracking error indicates that the quadrotor is able to track the ship deck motion. Figure 5.12.c shows the relative velocity error between the quadrotor and the ship deck, a small relative velocity error indicates small impact impulse. 5.12.d shows the quadrotor height relative to the ship deck. Figure 5.12.e shows the quadrotor thrust U_1 . Lastly, Figure 5.12.f and 5.12.g show the measurement noises acting on the relative position and velocity measurements.	162
5.13	The phase diagram of the state x and the invariant ellipsoid $V = 1$ and the state of the system bounded within the invariant ellipsoid, $V \leq 1$. . .	163
6.1	The gradient descent approach in the iterative solution of the full BMI invariant ellipsoid condition. The objective function is reduced at each iteration step with a limit set on the improvements in the solution such that a LMI condition is solved. Iteratively, the local optimal solution is obtained within a connected feasible domain containing the initial condition.	190
6.2	The decrease in objective function value with the number of iterations using a gradient descent approach with initial estimate obtained from controller and observer sub-problems (Figure 6.2.a) and pole placement (Figure 6.2.b).	195

6.3	Simulation results of a quadrotor landing on a ship deck using the optimal output feedback controller. Figure 6.3.a shows the heave motion of the ship deck, with the mean sea level at $d = 0$. Figure 6.3.b shows the height of the quadrotor relative to the ship deck. The landing pad is located at 1m above the ship deck. Figure 6.3.c shows the tracking error $e_1 = e$ of the quadrotor, a small tracking error indicates that the quadrotor is able to track the ship deck motion. Figure 6.3.d shows the relative velocity error $e_2 = \dot{e}$ between the quadrotor and the ship deck, a small relative velocity error indicates small impact impulse. Figure 6.3.e shows the thrust input of the quadrotor and its limits. Figure 6.3.f shows the measurement noise acting on the relative position. Figure 6.3.g shows the wind disturbance acting on the quadrotor	197
6.4	The value of $V = \xi^T P \xi$. $V < 1$ indicates that the state of the system remains inside the ellipsoid. Furthermore, this ellipsoid is shown to be attractive and invariant.	198
6.5	The phase diagram of the error e_1, e_2 and a projection of the invariant ellipsoid onto the e_1 - e_2 plane.	198
6.6	Simulation results of a quadrotor landing on a ship deck using an output feedback controller with gain obtained from pole placement. Figure 6.6.a shows the heave motion of the ship deck, with the mean sea level at $d = 0$. Figure 6.6.b shows the height of the quadrotor relative to the ship deck. The landing pad is located at 1m above the ship deck. Figure 6.6.c shows the tracking error $e_1 = e$ of the quadrotor, a small tracking error indicates that the quadrotor is able to track the ship deck motion. Figure 6.6.d shows the relative velocity error $e_2 = \dot{e}$ between the quadrotor and the ship deck, a small relative velocity error indicates small impact impulse. Figure 6.6.e shows the thrust input of the quadrotor and its limits. Figure 6.6.f shows the measurement noise acting on the relative position. Figure 6.6.g shows the wind disturbance acting on the quadrotor	201

6.7	Simulation results of a quadrotor landing on a ship deck. Figure 6.7.a shows the heave motion of the ship deck, with the mean sea level at $d = 0$. Figure 6.7.b shows the height of the quadrotor relative to the ship deck. The landing pad is located at 1m above the ship deck. Figure 6.7.c shows the tracking error $e_1 = e$ of the quadrotor, a small tracking error indicates that the quadrotor is able to track the ship deck motion. Figure 6.7.d shows the relative velocity error $e_2 = \dot{e}$ between the quadrotor and the ship deck, a small relative velocity error indicates small impact impulse. Figure 6.7.e shows the thrust input of the quadrotor and its limits. Figure 6.7.f shows the measurement noise acting on the relative position. Figure 6.7.g shows the wind disturbance acting on the quadrotor	205
6.8	Simulation results of a quadrotor landing on a ship deck. Figure 6.8.a and 6.8.b shows the X and Y position of the quadrotor respectively with the landing pad located at $X = 0$ and $Y = 0$. Figure 6.8.c to 6.8.e shows the yaw, pitch and roll angles of the quadrotor respectively. Measurement noises are added to each feedback channel.	207
6.9	The trajectory of the quadrotor (solid line) as it moves with the ship (dotted line) while attempting to synchronize its relative height above the ship. The quadrotor then lands on the ship by reducing its height gradually. The z axis shown is the relative height above the ship.	208
B.1	Simulation result of a multi-rotor UAV tracking the ship heave motion using an output regulation controller with matched frequencies.	234
B.2	Simulation result of a multi-rotor UAV tracking the ship heave motion using an output regulation controller with one unmatched frequency. . .	235
B.3	Simulation result of a multi-rotor UAV tracking the ship heave motion using an output regulation controller with two unmatched frequencies. . .	235
B.4	Simulation result of a multi-rotor UAV tracking the ship heave motion using an output regulation controller with full ship heave model.	236

List of Tables

1.1	Table of sea state.	5
3.1	A table of equations of motion for a multi-rotor UAV.	54
4.1	A list of quadrotor parameters used in the simulation [32].	94
4.2	The standard deviations for the zero mean Gaussian measurement noises.	114

List of Frequently Used Symbols and Abbreviations

Symbols

A	State matrix
B	Control matrix
C	Output matrix
D	Disturbance input matrix
$\mathbf{F}_{\text{aero}}$	Aerodynamic drag force vector
H	Reference altitude
$H_{1/3}$	Significant wave height
J	Multi-rotor UAV moment of inertia matrix
J_{xx}, J_{yy}, J_{zz}	Multi-rotor UAV moment of inertia about x, y, z -axes respectively
K	Linear controller feedback gain
K_d	Aerodynamic drag coefficient\Weighting matrix for wind disturbance
K_m	Aerodynamic moment coefficient
K_n	Weighting matrix for measurement noise
K_p, K_i, K_d	PID controller gains
K_s	Weighting matrix for ship heave disturbance
L	Luenberger observer gain
M, N	Weighting matrices for disturbances and noises
$\mathbf{M}$	Multi-rotor UAV control moment vector

$\mathbf{M}_{\text{aero}}$	Aerodynamic moment vector
M_x, M_y, M_z	Multi-rotor UAV control moment components
N_{rotor}	Number of rotors
P, Q	Invariant ellipsoid matrix
R	Direction cosine matrix from a Earth-fixed inertial frame to a multi-rotor UAV body-fixed frame\Control action ellipsoid
R_{target}	Target circular path radius
$S(\omega)$	Wave height spectrum
T	Multi-rotor UAV thrust
T_1	Mean wave period
$\mathbf{T}$	Multi-rotor UAV thrust force vector
V_{target}	Target linear path speed
X, Y, Z	Multi-rotor UAV position components in an Earth-fixed inertial frame
X_0, Y_0	Multi-rotor UAV initial position components
X_{d0}, Y_{d0}	Target initial position components
$X_{\text{target}}, Y_{\text{target}}$	Target position components
$\mathbf{a}$	Multi-rotor UAV acceleration vector
$\mathbf{a}_{\text{planar}}$	Multi-rotor UAV planar acceleration vector
a_x, a_y, a_z	Mutli-rotor UAV acceleration components
d	Wave elevation\Ship heave
e	Tracking error
$\hat{e}_3$	Unit vector along z -axis
f_i	Thrust force acting on the i^{th} rotor
g	Gravitational acceleration
h	Multi-rotor UAV altitude
k_q	Propeller drag factor
k_t	Propeller lift factor

l	Distance of a rotor to the quadrotor center of mass
m	Multi-rotor UAV mass
n	Measurement noise
$\mathbf{p}$	Multi-rotor UAV position vector
$\mathbf{p}_{\text{planar}}$	Multi-rotor UAV planar position vector
p, q, r	Multi-rotor UAV rotation rate components
$\mathbf{r}_i$	Position vector of the center of the i^{th} rotor from the multi-rotor UAV center
t	Time
t, t_1, t_2	Optimization objective value
$u, \mathbf{u}$	Control input
u_1	Thrust control input
u_2	Rotation rate control input
u_{max}	Maximum control action
$\mathbf{v}$	Multi-rotor UAV velocity vector
v	Heave kinematic and dynamic model control input
$\mathbf{v}_{\text{planar}}$	Multi-rotor UAV planar velocity vector
$\mathbf{v}_{\text{wind}}$	Wind disturbance velocity
v_x, v_y, v_z	Multi-rotor UAV velocity components
w_d	Acceleration due to wind disturbance
w_i	Generalized external disturbance
w_n	Measurement noise
w_s	Acceleration due to ship heave
$x, \mathbf{x}$	System state
$\hat{x}$	Observer state
y	System output
$\mathbf{y}$	Auxiliary system state

Ω_i	Rotation rate of i^{th} rotor
Θ	Euler angles vector
η	Quaternions
$\eta_0, \eta_1, \eta_2, \eta_3$	Quaternions components
$\lambda_{\max}$	Maximum eigenvalue
ω	Multi-rotor UAV rotation rate vector
ω_{wind}	Wind disturbance rotational component
ϕ	Multi-rotor UAV roll angle
ψ	Multi-rotor UAV yaw angle
τ	Command filter time constant
τ_i	Torque acting on the i^{th} rotor
θ	Multi-rotor UAV pitch angle

Abbreviations

$\max(\cdot)$	Maximize $(\cdot)$
$\text{eig}(\cdot)$	Eigenvalue of $(\cdot)$
$\lambda_{\max}(\cdot)$	Maximum eigenvalue of $(\cdot)$
$\dot{(\cdot)}$	$\frac{d}{dt}(\cdot)$
$\ddot{(\cdot)}$	$\frac{d^2}{dt^2}(\cdot)$
$\mathbb{R}$	Real number
$s(\cdot), c(\cdot), t(\cdot)$	$\sin(\cdot), \cos(\cdot), \tan(\cdot)$
$(\cdot)_d$	Desired value of multi-rotor state $(\cdot)$
$(\cdot)_{cmd}$	Command filter value of multi-rotor state $(\cdot)$
$(\cdot)^*$	Desired value of state $(\cdot)$ (Chapter 4)\Optimal value of matrix $(\cdot)$ (Chapter 5 and 6)
$(\cdot)^T$	Matrix transpose
$\text{diag}(\cdot)$	Diagonal matrix consisting of elements from the argument.

$\ \cdot\ $	Euclidean norm of $(\cdot)$
$\ \cdot\ _F$	Frobenius norm of $(\cdot)$
$\ \cdot\ _M$	M -norm of $(\cdot)$
$\ \cdot\ _\infty$	H_∞ norm of $(\cdot)$
$X + (*)$	$X + X^T$
$\begin{pmatrix} X + (*) & Y \\ * & Z \end{pmatrix}$	$\begin{pmatrix} X + X^T & Y \\ Y^T & Z \end{pmatrix}$
I_n	$n \times n$ identity matrix
$tr(\cdot)$	Trace of $(\cdot)$
$\{x : \text{condition}\}$	A set consisting of element x that satisfies stated condition.
$(\cdot)^{(i)}$	i^{th} iteration solution

Chapter 1

Introduction

The multi-rotor unmanned aerial vehicle (UAV) is a class of vertical takeoff and landing (VTOL) UAV that obtains its lifting force and control through the thrust variation provided by different rotors. Typically, they are equipped with four or more rotors (see Figure 1.1). Nevertheless, the control principles for all multi-rotor UAVs can be generalized into a common framework. This makes the application of any control structure in a multi-rotor UAV simple and attractive.

1.1.a: Quadrotor

1.1.b: Hexrotor

1.1.c: Octorotor

Figure 1.1: A figure showing a quadrotor in Figure 1.1.a (image obtained from www.pupin.rs), a hexrotor in Figure 1.1.b (image obtained from www.aliexpress.com) and an octorotor in Figure 1.1.c (image obtained from www.etotheipiusone.net).

The control of a multi-rotor UAV is achieved by varying the power input to each rotor, thus changing their rotational speed hence the thrust provided. The collection of thrusts

from all the rotors provides the thrust or lifting force for the multi-rotor UAV while the differential thrust between rotors creates the control moments for the multi-rotor UAV. In principle, there are four control inputs to a multi-rotor UAV. They are:

- (i) the thrust T ,
- (ii) the roll moment M_x ,
- (iii) the pitch moment M_y and
- (iv) the yaw moment M_z .

Through these four canonical control inputs, the position, velocity and attitude of a multi-rotor UAV are controlled. Control allocation assigns the rotational speed of (or power input to) each motor so that the net effect of all rotors spinning achieves the desired canonical inputs. It is also noteworthy that for multi-rotor UAV with more than four rotors, there is redundancy in the controls in the case of rotor failure.

The multi-rotor UAV is able to hover, takeoff and land vertically. Its VTOL ability and agility allow the multi-rotor UAV to operate on any kind of terrain. Furthermore, vast array of sensors and equipment can be equipped onto the multi-rotor UAV to provide high quality sensing coverage, especially in reconnaissance and surveillance missions. In view of these advantages, the multi-rotor UAV is gaining popularity in commercial and military sectors as well as in the academic research field.

One possible niche application of the multi-rotor UAV is its potential for ships and vessels operation, providing the ships and vessels with an additional eye in the sky. Its key advantage is its VTOL capability that requires minimal setup and/or modification on any ship or vessel and is readily available for use in most ships and vessels. Potential missions in the open seas include:

- (i) military reconnaissance and surveillance,

- (ii) oil spill detection and monitoring,
- (iii) marine life monitoring and
- (iv) communication relay etc.

Although the VTOL capability of a multi-rotor UAV onboard a ship is extremely advantageous in terms of simplicity and operation, there are risks associated with its landing maneuver. First and foremost, the landing platform on the ship deck is moving (or oscillating) increasing the difficulty in achieving soft landing. Secondly, the multi-rotor UAV is required to track and follow the ship path while performing landing operation. Furthermore the space constraints on the ship deck and high seas environment provide additional challenge. In view of these challenges, it is imperative for the landing controller to achieve a soft and safe landing while factoring in the environmental condition, space constraints and safety of the personnel and equipment on the ship as well as the multi-rotor UAV.

In this thesis, the recovery problem of a multi-rotor UAV (in particular a quadrotor) is analyzed and solved by formulation of control laws. The problem is divided into a tracking sub-problem and a landing sub-problem using a decoupled multi-rotor UAV model. In the rest of this chapter, a description of the multi-rotor UAV shipboard recovery problem is given. A list of contributions and publications during the author's PhD study, followed by this thesis outline is also provided at the end of this chapter.

1.1 Problem Description

In using multi-rotor UAVs onboard a ship, the recovery process is operationally advantageous as it requires a smaller deck space and simpler ship deck modifications as compared to fixed-wing UAVs. This advantage comes with a cost of requiring the multi-rotor UAV

flight controller to be more delicately designed to deal with the challenges of vertical landing on a ship.

In this section, the environmental challenges involved in shipboard landing is provided as well as justification for a decoupled approach followed by a description of each sub-problem.

1.1.1 Environmental Challenges

The environmental challenges involved in multi-rotor UAV recovery operation in the open seas come in several forms. They include the wind, ship wake, ship-wave interactions leading to ship motion and the safety of the equipment and personnel onboard the ship.

1.1.1.1 Sea State

The general condition of a large body of water (the sea) is given by the sea state. It gives a statistical representation of the peak to peak height, wavelength, period and power spectrum of the sea wave with respect to the wind condition. Therefore the sea state gives an indication of the wind disturbance encountered by the multi-rotor UAV during landing as well as the ship motion arising from the ship-wave interactions. This provides an important yardstick for the applicability of the proposed design with respect to the degree of harshness of the environment.

The sea state table of the Douglas Sea Scale provided by the World Meteorological Organization (WMO) [66, p.450] is shown in Table 1.1. Data related to the significant wave height and wind speed are obtained with respect to the sea state.

It is noted that for sea state higher than 3, shipboard landing operations of aircraft tends to become more dangerous with the need for landing assist systems [110]. Reboulet *et. al.* [72] made a similar observation with autonomous landing systems. He further noted that for sea state beyond 7, it is impossible for any autonomous landing system to have sufficient safety margin during landing. Sea state 6 is used as the design sea state

Table 1.1: Table of sea state.

Sea State	Description	Significant Wave Height (m)	Design Wind Speed (kts)
0	Calm (glassy)	0	Below 5
1	Calm (ripples)	0-0.1	5
2	Smooth (wavelets)	0.1-0.5	10
3	Slight	0.5-1.25	15
4	Moderate	1.25-2.5	20
5	Rough	2.5-4.0	25
6	Very Rough	4.0-6.0	30
7	High	6.0-9.0	45
8	Very High	9.0-14.0	55
9	Phenomenal	Over 14.0	Over 70

in the work of Khantsis [46]. Therefore in this problem, a worst case environment of sea state 6 is considered as this is a reasonably demanding sea condition under which the multi-rotor UAV operates. At higher sea state, normal operation of a multi-rotor UAV is likely to be ceased as the environmental condition is deemed too demanding for the safety of the mission.

According to the WMO, a very rough sea condition corresponds to a sea state 6 on the Douglas Sea Scale. As seen in Table 1.1, a wind of the region of 30kts (10m/s), a significant wave height¹ of a region of 4.0 to 6.0m and an average wave period of 8s are expected. Assuming that the response amplitude operator (RAO) of the ship is unity (without any loss of generality), the landing platform will be moving through a height of 6m in a span of 8s. This movement is significant enough to cause any considerable damage to the multi-rotor UAV during landing when the relative speed between the UAV

¹The significant wave height (SWH) is defined as the average value of the vertical crest to trough distance of the largest 1/3 of the waves present.

and the ship deck upon impact increases greatly due to the upward motion of the ship deck and downward motion of the UAV.

Furthermore, a wind speed of 10m/s also causes considerable disturbance on a small sized multi-rotor UAV especially during landing operation when the commanded speed is slow. Assuming the drag force acting on the multi-rotor UAV varies linearly with wind speed (for sufficiently slow speed) according to

$$\mathbf{F}_{\text{aero}} = -K_d(\mathbf{v} - \mathbf{v}_{\text{wind}}), \quad (1.1)$$

where K_d is the aerodynamic drag force coefficient, $\mathbf{v}$ is the UAV velocity and $\mathbf{v}_{\text{wind}}$ is the wind velocity, a typical value of $K_d = 0.1\text{Ns/m}$ induces a 1N drag force which is equivalent to 1m/s^2 of disturbance in acceleration acting on a 1kg UAV. The effect of wind disturbance therefore cannot be neglected and is critical to the solution of the multi-rotor UAV shipboard recovery problem.

1.1.1.2 Space Constraints

Space constraint is another form of environmental challenge as the multi-rotor UAV is required to perform delicate maneuvers such that it does not collide with the ship superstructures or onboard personnel. This constitutes the safety aspect of the landing operation.

Before executing the landing sequence, the multi-rotor UAV is typically high in the sky and far from any ship structure or any obstacle. In such a scenario, any movement of the UAV away from the intended flight path due to wind disturbance is not potentially risky. But as the UAV descends, positional error away from the intended flight path is increasingly likely to cause collision damage and risk to personnel. Additional disturbances arising from ship wakes are also another factor.

Therefore it is critical for the solution to the multi-rotor UAV shipboard recovery problem to ensure tracking of the flight path especially during landing.

With the challenging environment of the open seas and ship constraints discussed, the following describes the approach to solve the multi-rotor UAV shipboard recovery problem.

1.1.2 A Decoupled Approach

As mentioned earlier, the multi-rotor UAV shipboard recovery problem is divided into tracking and landing sub-problems using a decoupled multi-rotor UAV model. In this model, the roll, pitch and yaw control moments (M_x , M_y and M_z) are used to control the UAV attitude which then affects the position and velocity of the UAV. The translational-rotational coupling is given by [23]

$$\dot{\mathbf{v}} = -\frac{T}{m} \begin{pmatrix} \sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi \\ -\sin \phi \cos \psi + \cos \phi \sin \theta \sin \psi \\ \cos \phi \cos \theta \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ mg \end{pmatrix}, \quad (1.2)$$

where $\mathbf{v} = (v_x \ v_y \ v_z)^T$ is the UAV velocity, m is the UAV mass, g is the gravitational acceleration, ϕ , θ and ψ are the roll, pitch and yaw angles of the UAV respectively. A full dynamical model of the UAV is given in Chapter 3.

It is seen from (1.2) that the planar position (derived from v_x and v_y) is fully actuated by the attitude angles ϕ and θ . With ψ assigned to control the heading of the UAV, T can therefore be used to control the altitude of the UAV. In this way, the dynamics of the UAV is decoupled into a planar dynamics which includes the translation motion in the global x - y plane and the rotational motion, and a heave dynamics which governs the altitude and landing of the UAV.

Figure 1.2 is a diagram showing the decoupled controller designs corresponding to tracking and landing sub-problems.

Figure 1.2: A representation of the decoupled sub-problem leading to separate designs of a tracking and landing controller.

1.1.2.1 Tracking Problem

In the tracking problem, the UAV is required to track the ground path of the ship which is the position of the ship in the global x - y plane. This requirement is essential to the multi-rotor UAV shipboard recovery problem as the ship is likely to move along its intended path despite any onboard operation. The UAV recovery process is not expected to influence or interfere with the operations of the ship and its crew.

Furthermore, the UAV is also unlikely to have prior information of the ship trajectory and thus is required to depend on updates of the ship position information in performing tracking. This information can be sent to the UAV using beacon located on the landing platform. UAV sensor technology such as lidar and UAV vision is also able to obtain ship position measurements.

Wind disturbance and ship wakes are also major problems affecting the flight path of the UAV. This effect is amplified when the UAV mass is small thus more easily influenced by wind. Any huge deviation of flight path might result in collision with the ship superstructures and/or personnel onboard the ship.

In view of these challenges, the flight ground track controller synthesis has to be robust. Using the decoupled dynamics, a controller needs to be formulated such that the resultant closed-loop system is able to track the ground path of the ship with feedback information of the ship position. Furthermore, the effects of wind disturbance need to

be considered as well. Refer to Figure 1.3 for a diagram representation of the controller design problem.

Figure 1.3: A diagram representation of the objective and requirement of the tracking controller.

1.1.2.2 Landing Problem

During landing, the motion of the landing platform due to ship motion can be treated as an arbitrary moving platform and poses risk of hard landing thus resulting in damage to the UAV. A soft landing, which is measured by low impact velocity and/or impulse, is desirable as it reduces the chances of damage to the UAV.

For a small size multi-rotor UAV, the heave motion of the ship is the most significant component that needs to be considered. The roll and pitch motion of the ship imposes insignificant risk to the UAV apart from sliding and toppling upon landing which is not the focus of this thesis. This is seen from the example of a ship pitch of 45° and a UAV of 1m diameter results in only a 1m height difference between both ends (see Figure 1.4). A pivoted fall of this height of the farther side during landing when the nearer side touches the ship deck does not incur significant damage as compared to a larger UAV which falls through a larger height.

In addition, the relative position (sometimes velocity) between the UAV and the ship is typically given using laser, lidar, infrared and/or vision sensors equipped on the UAV

Figure 1.4: A multi-rotor UAV of 1m diameter landing on a ship having a pitch angle of 45° results in a 1m height difference between both legs.

[2, 30, 37, 48, 73, 78, 79, 81, 84, 95, 97, 101]. This provides the critical ship deck oscillation information that is to be utilized in the landing controller.

In the landing problem, a flight controller needs to be synthesized such that the altitude of the UAV is able to track the heave motion of the ship deck using feedback signal of the ship heave motion. In this way, a soft and safe landing can be achieved. Refer to Figure 1.5 for a diagram representation of the landing controller design problem.

Figure 1.5: A diagram representation of the objective and requirement of the landing controller. Dotted lines represent effects that are secondary to this problem.

An overview of the tracking and landing sub-problems are given in this section with respect to the multi-rotor UAV shipboard recovery problem. They give the qualitative requirements of the solution. Detailed academic research and contribution in relation to the solution of the sub-problems are given in the next section and a formal discussion of

research issues is given in Chapter 2.

1.2 Academic Contributions

The following is a list of the academic contributions during the author's course of study:

- (i) a backstepping controller formulation framework using contraction theory that utilizes cascaded control structure,
- (ii) a reformulation of UAV kinematics and dynamics model to allow for a novel integral controller formulation,
- (iii) an invariant ellipsoid approach to a robust control structure in the presence of bounded persistent disturbance and its formulation as a linear matrix inequality (LMI) problem,
- (iv) an extension of the invariant ellipsoid approach to include system uncertainty, stochastic disturbance and measurement noise,
- (v) an output feedback control scheme of a system subjected to bounded persistent disturbance and its controller gain synthesis using a path following iterative method.

1.3 Thesis Outline

The rest of this thesis follows the following outline. Chapter 2 gives an overview of the current state of research in relation to the controller designs applied to the tracking and landing sub-problems. In addition, it highlights the unresolved research issues in specific area with respect to the tracking and landing sub-problems and also provides the motivation for the direction of research. Chapter 3 gives a full six degree of freedom (6 DOF) kinematic and dynamic model of the multi-rotor UAV used in this study. In addition, a full model of the environment including the ship deck motion model is also

given. Chapter 4 gives the solution for the tracking sub-problem. The research issues that are highlighted in Chapter 2 are also solved and addressed. Chapter 5 gives the solution for the landing sub-problem under the assumption of the availability of relative height and speed feedback. Similarly, research issues that are highlighted in Chapter 2 are also solved and addressed. Likewise, Chapter 6 solves the landing sub-problem but with only relative height feedback in an output feedback formulation. This addresses some of the research issues posed in Chapter 2. Lastly, Chapter 7 concludes the thesis with the work done and provides the possible implications and significance of the research work.

Chapter 2

Literature Survey and Problem Formulation

This chapter gives a literature survey on the current state of research in multi-rotor UAV control related to both the tracking and landing sub-problems. It then draws conclusion from the current state of research and formulates the approach taken in this thesis with respect to the tracking and landing sub-problems.

2.1 Literature Survey: Position Tracking Controller

In the tracking sub-problem, a flight controller or a control scheme is developed to allow a multi-rotor UAV to track the ground path of an unknown trajectory using feedback of the target position. This section gives an in-depth analysis of existing control strategies and associated research issues. Flight tests conducted by various groups are also mentioned.

Generally, multi-rotor UAV control can be categorized into two broad classes: linear and nonlinear control. Full 6 DOF multi-rotor UAV model is typically used in nonlinear

control, it can be expressed in the general form [23]

$$\dot{\mathbf{p}} = \mathbf{v}, \quad (2.1a)$$

$$\dot{\mathbf{v}} = f(\Theta, u_1), \quad (2.1b)$$

$$\dot{\Theta} = g(\Theta, \omega), \quad (2.1c)$$

$$\dot{\omega} = h(\omega, u_2, u_3, u_4), \quad (2.1d)$$

where $\mathbf{p} = \begin{pmatrix} X & Y & Z \end{pmatrix}^T$ is the UAV position, $\mathbf{v} = \begin{pmatrix} v_x & v_y & v_z \end{pmatrix}^T$ is the UAV velocity, $\Theta = \begin{pmatrix} \phi & \theta & \psi \end{pmatrix}^T$ are the Euler angles, $\omega = \begin{pmatrix} p & q & r \end{pmatrix}^T$ are the UAV angular rates, $u = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \end{pmatrix}^T$ are the canonical control inputs to the UAV, $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ are some nonlinear functions. A nonlinear control law takes the form

$$u = k(\mathbf{p}, \mathbf{v}, \Theta, \omega), \quad (2.2)$$

where $k(\cdot)$ is some nonlinear function.

On the other hand, a linearized multi-rotor UAV model is used in synthesis of linear controllers. The linear model expressed in the state space form is

$$\dot{x} = Ax + Bu, \quad (2.3a)$$

$$y = Cx + Du, \quad (2.3b)$$

where $x = \begin{pmatrix} \mathbf{p}^T & \mathbf{v}^T & \Theta^T & \omega^T \end{pmatrix}^T$, A , B , C and D are system matrices of appropriate dimensions. Linear static output control law takes the form of

$$u = Ky, \quad (2.4)$$

where K is the controller gain. Other forms of linear model such as transfer function and frequency response model are also used, nevertheless they can be transformed into one another.

2.1.1 Time Scale Separation

Some controllers use time scale separation to design an attitude and a position controller independently (see Figure 2.1). In these controllers, the time scale of the attitude control loop is designed to be faster than that of the position control loop. Bertrand *et. al.* [9] quantifies the term "fast" by providing a mathematical condition on the gain of the attitude controller using singular perturbation theory. His team demonstrated this with a two loops control design on a X4-flyer miniature UAV (quadrotor).

Figure 2.1: A representation of the control structure with the principle of time scale separation where P , Θ are the UAV position and Euler angles respectively, and P_d , Θ_d are the desired UAV position and Euler angles respectively.

Time scale separation reduces the complexity of multi-rotor UAV flight controller design by justifying the decoupling of the attitude and position control loop provided that the time scale requirement is met.

In particular, consider the position sub-system (from (2.1))

$$\dot{\mathbf{p}} = \mathbf{v}, \quad (2.5a)$$

$$\dot{\mathbf{v}} = f(\Theta, u_1) = w, \quad (2.5b)$$

where w is a new control input. This is in a double integrator form that is well-studied and control design can be easily performed. The desired values of the attitude angles Θ_d are then computed from the acceleration control input w by an inverse kinematic relationship given as

$$\Theta_d = f^{-1}(w, u_1), \quad (2.6)$$

where $f(\cdot, \cdot)$ is non-singular for typical multi-rotor UAV dynamics. This technique is used by Das *et. al.* [22] and Sanchez-Lopez *et. al.* [78, 79] among other researchers. The desired attitude angles Θ_d obtained is then used as the reference command for a faster attitude inner loop control.

As mentioned earlier, time scale separation reduces position control to a trivial double integrator, hence most control research focus on the control of the more challenging attitude system. Therefore prior to the review of the current state of research in position tracking control of multi-rotor UAV, a review of attitude control of multi-rotor UAV is given.

2.1.1.1 Attitude Control

The attitude dynamics of a multi-rotor UAV ((2.1c) and (2.1d)) is highly nonlinear [49]. Nonlinear control strategies are therefore often used to perform attitude control of a multi-rotor UAV for more aggressive maneuvers.

Colorado *et. al.* has developed an attitude controller using backstepping control with the Frenet-Serret theory [19]. A Lyapunov-based approach is used in a backstepping formulation that derives the control moment inputs (M_x, M_y and M_z) to track an attitude command that is extracted from the Frenet-Serret theory. This technique is tested with a DragonFlyer quadrotor. It is noted that using the Frenet-Serret theory to compute the desired attitude command is similar to the inverse kinematics approach in (2.6) and allows for a decoupled attitude controller design. This demonstrates the need for an inverse kinematics mechanism to couple attitude control with translational control.

In another method similar to backstepping control, Corona-Sanchez *et. al.* [20] synthesized an attitude controller for a quadrotor and validated it with flight tests. His team uses an inversion technique to compute the desired angular rates with the UAV desired attitude profile similar to

$$\omega_d = g^{-1}(\dot{\Theta}_d, \Theta_d). \quad (2.7)$$

Guerrero-Castellanos *et. al.* [36] uses a similar Lyapunov-based controller. His team also uses an event triggered component to improve the performance of his attitude controller.

Yang *et. al.* [103] uses a sliding mode controller to handle external disturbances and parameter uncertainties. A fuzzy system is also used to solve the problem of chattering by adapting to the scaling factor in the sliding mode controller.

In addition to nonlinear controllers, linear controllers are also used in multi-rotor UAV attitude control. Linear control works well in a small domain where the attitude angles ϕ , θ and ψ are small because of linearization used in the derivation of the linear multi-rotor UAV model. As attitude angles are expected to be small in most applications of the multi-rotor UAV, simpler and well-established linear controllers are often preferred over nonlinear controllers.

The most straightforward linear control is the PID control. It is used and tested by Fang *et. al.* [27] on a micro quadrotor. More advanced linear control in the form of model predictive control (MPC) is developed by Alexis *et. al.* [1]. His team demonstrated the atmospheric disturbance rejection property of his MPC controller that uses piecewise affine model of the attitude dynamics. The MPC incorporates switching control thus is able to negate the effects due to atmospheric disturbance.

Dikmen *et. al.* [24] did a comparison study among several linear and nonlinear controllers. PD control, dynamic inversion, feedback linearization and sliding mode control are compared in an attitude control problem both in the presence and absence of noise. It is observed that the performance for these controllers are comparable and depends greatly on the tuning parameters. Nevertheless Dikmen *et. al.* remarked that his sliding mode controller performs better than linear controllers for initial conditions further from the equilibrium [24].

A linear-like control structure that is largely successful in multi-rotor UAV is the nested saturation control. An attitude controller based on nested saturation control is

proposed by Guerrero-Castellanos *et. al.* [35]. His controller considers the angular limit of the multi-rotor UAV, thus preventing the attitude angles from becoming too large. This is largely beneficial to a multi-rotor UAV as large attitude angles induce problems such as loss in lifting force when deviated from linear domain. However due to the attitude limit enforced by nested saturation control, the multi-rotor UAV is often unable to perform aggressive maneuvers hence compromising agility.

Another issue in attitude control is the occurrence of gimbal lock resulted due to the singularity at $\theta = 90^\circ$ in using Euler angles $(\phi, \theta$ and $\psi)$. This occurs when there is a loss in one degree of freedom in the three degree of freedom representation. (i.e. There exists a non-unique representation of a given attitude, for instance $(\phi = -a, \theta = 90^\circ, \psi = 0^\circ)$ is identical to $(\phi = 0^\circ, \theta = 90^\circ, \psi = a)$ for any $0^\circ \leq a \leq 360^\circ$.) Furthermore, a singularity can also be observed at $\theta = 90^\circ$ or $\cos(90^\circ) = 0$ from the rotational dynamics given by

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & t\theta s\phi & t\theta c\phi \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \quad (2.8)$$

where $s(\cdot)$, $c(\cdot)$ and $t(\cdot)$ are $\sin(\cdot)$, $\cos(\cdot)$ and $\tan(\cdot)$ respectively.

This issue is elegantly resolved using quaternions. Readers can refer to [29] for more details on quaternions. Tayebi [87] and Fresk *et. al.* [29] have used quaternions to formulate the rotational dynamics of a quadrotor. Their approach involves finding the attitude error about each principle axes by the difference with a reference quaternion. Then a control law is formulated similar to a PID control structure by associating each principle axes to a control input in a single-input single-output (SISO) formulation. This only works well for small attitude error. It is therefore advantageous to formulate a full nonlinear quaternion-based attitude controller that works for the entire attitude domain.

The attitude controller provides tracking of the attitude command that is commanded by the position tracking controller. The stability of the coupled system is ensured based

on the principle of time scale separation. However, time scale separation places a constraint on the time constant of the attitude and position control loops. This results in slow response thus slow tracking of position command. Therefore in addition to time scale separation control, full state feedback approaches which are expected to improve the response of the system are also reviewed in the following part.

2.1.2 Linear Controller for Position Tracking

Linear control for full state feedback position tracking uses the well established field of linear control. By linearizing the 6 DOF nonlinear multi-rotor UAV model, a linear model of the form of (2.3) is obtained. Linear controller can thus be easily designed with well established linear control formulation methodology. A literature survey of linear controls used in position tracking control of a multi-rotor UAV is given in this subsection.

2.1.2.1 PID Control

PID control is the most common form of linear control. It is applied in many automatic control and automation applications such as industrial machines, air-conditioning and aircraft etc.

In the position tracking control of multi-rotor UAV, Wenzel *et. al.* [93] and Hoffmann *et. al.* [42] have used PID control in their multi-rotor UAV platforms to demonstrate their contributions in UAV vision and dynamic model respectively. They have chosen PID control due to its simplicity in design and implementation.

Pounds *et. al.* developed a large quadrotor (4kg) with extensive analysis of the flight dynamics. Single-input single-output (SISO) flight controllers are designed using PID controls with analysis performed on the transfer functions of the quadrotor [69]. Again, this control structure is commonly found in the commercial multi-rotor UAV due to its simplicity.

Desirable properties such as robustness can also be added to the PID control scheme. Liu *et. al.* uses a robust compensation technique on a PD controller and demonstrated with a X650 quadrotor [52]. The position controller follows a PD controller design while the attitude controller uses a PD control with a robust compensating term in an approach similar to the robust control. The attitude and position control loops are decoupled using the inverse kinematic in the form of (2.6).

PID controllers are designed with simplicity in mind and the flexibility of PID controllers to include properties such as robustness makes it even more attractive.

2.1.2.2 Robust Control

PID control is simple and effective for a known linear system, however in the presence of system uncertainties, unmodelled nonlinearities and disturbances the system response is less desirable. Robust control guarantees the performance of a linear system in the presence of system uncertainties, unmodelled nonlinearities and disturbances. The control law formulation typically reduces into a linear matrix inequality (LMI) and/or algebraic Riccati equation (ARE).

Ryan *et. al.* [75] have developed a robust linear controller for the position control of a multi-rotor UAV. A linear system is obtained using feedback linearization. His controller is robust to dynamic external disturbances, inexact nonlinearity cancellation, multiplicative actuator uncertainty, and saturated integrators in the control loops. Mathematical conditions are formulated into a LMI problem to assist the selection of a controller gain to satisfy the above inexactness using an objective function of mixed H_2/H_∞ norm. Ryan *et. al.* has also formulated a mathematical foundation on robustness guarantees. Experiments in [75] also demonstrated that the position controller is able to perform adequately with measurement noises. A mixed H_2/H_∞ LMI optimization is also used by Zhang *et. al.* [109]

Mokhtari *et. al.* also developed a H_∞ controller for a quadrotor that is robust to parametric uncertainties, external disturbances and measurement noises [61]. His method improves the robustness of conventional feedback linearization controller. Simulations are performed to demonstrate the effectiveness of the controller.

Raffo *et. al.* presented an integral predictive and nonlinear robust control strategy for a quadrotor path following problem in [70]. An attitude stabilization control is achieved using a H_∞ controller that provides the quadrotor with its required robustness. Position tracking control is achieved using a model predictive control (MPC) (see next subsection). The decoupling of attitude and position control loops is similar to the time scale separation approach. Moreover Raffo *et. al.* note that this approach is suitable because of the hierarchical structure of the multi-rotor UAV [70].

Robust control is therefore mainly used in system that is sensitive to uncertainties. Nevertheless in most linear systems, certain level of robustness exists when poles of the closed-loop system are placed sufficiently far from the imaginary axis.

2.1.2.3 Model Predictive Control

MPC is generally computationally intensive therefore not used in small size, low cost UAVs. Nevertheless it is included in this literature review for completeness.

The position tracking controller presented by Raffo *et. al.* in [70] (also mentioned in Section 2.1.2.2) is a form of MPC. The main purpose of the MPC is to account for control saturation and to optimize the attitude and speed of the quadrotor given a known trajectory. It is a computational intensive control scheme, nevertheless it is being used for its ability to accommodate soft and hard constraints.

Tzes *et. al.* developed a switching MPC for a quadrotor in the presence of wind gust [91]. Similar to [42, 52, 69, 70, 93], the control structure is divided into an attitude and a position control loop. It is noted that this division is similar to the control structure used

in this thesis (see Chapter 4, p.62). A switching MPC that switches between different linear model is developed for both attitude and position controllers. The use of multiple linear models expanded the operation domain of the quadrotor. Furthermore, wind gust is relatively constant and MPC allows effective wind gust attenuation in the presence of actuation limit.

2.1.2.4 Linear Adaptive Control

Adaptive control is often used when there is a critical parameter to be estimated. For example, constant wind gust and parametric uncertainties can be handled using adaptive control. The most commonly used adaptive control is the model reference adaptive control (MRAC). It uses a reference model and compares the output differences then uses it to tune an adaptive parameter. Under persistent excitation, the adaptive parameter is guaranteed to converge to its actual value.

Nicol *et. al.* [64] uses adaptive control to design a position controller for a quadrotor such that it can adapt to the weight of the loads and other uncertainties. An approach similar to fuzzy logic based on Cerebellar Model Arithmetic Computer (CMAC) is used to actively adapt the unknown parameters. This method however requires extensive training to be effective.

Selfridge *et. al.* [82] uses adaptive control in the situation where uncertainties arises from changing operating point. A multi-input multi-output (MIMO) controller is designed for a linearized system. MIMO design provides a more holistic approach where coupling between states such as the roll and pitch angles are considered. A simulation study is conducted.

In this subsection, linear control approaches to the multi-rotor UAV position tracking problem is considered. A position outer and attitude inner control loop approach is used by [42, 52, 69, 70, 91, 93] for its simplicity. The cascaded structure of the multi-rotor UAV

is justifies the use of position-attitude control decoupling [70]. However, linear control is based on a linearized system and is restricted to a small linearized domain. Although, feedback linearization [24, 75], switching control [70] and adaptive control [82] are able to handle this issue, nonlinear controls are more well adapt to overcome nonlinearities and are discussed in the next subsection.

2.1.3 Nonlinear Controller for Position Tracking

The 6 DOF dynamics of a multi-rotor UAV is nonlinear and a nonlinear controller is able to widen the operational domain. Linear controller is only able to guarantee performance in a small domain around the operating point where linearization of the 6 DOF dynamics is performed. A nonlinear controller, on the other hand, is able to guarantee performance over a larger operating domain. Therefore, the multi-rotor UAV is able to perform more aggressive maneuvers despite the mathematical complexity often associated with nonlinear control.

This section discusses the common application of nonlinear controls in the position tracking of a multi-rotor UAV.

2.1.3.1 Sliding Mode Control

Sliding mode control is a type of variable structure control where the closed-loop dynamics exhibit high frequency switching between discontinuous control structures. The state feedback law derived from sliding mode control is non-continuous and nonlinear. In sliding mode control, the order of the system is reduced by forcing the dynamics of the system to slide on a hypersurface on which certain system characteristics such as fast settling time are observed. Across the hypersurface, the control structure changes such that the system is constrained on the hypersurface.

The sliding mode control is robust to external disturbances. It is also able to handle measurement noise as well as system uncertainties. The dynamics of the closed-loop sys-

tem is constrained onto a desired hypersurface, thus negating the effects of disturbances and uncertainties. However due to fast switching, sliding mode control is often subjected to an effect known as chattering that is a small vibration-like behavior of the state across the switching surface.

Zheng *et. al.* [111] uses a second order sliding mode controller on a quadrotor to analyze the effects of the coefficients of the sliding surface. The quadrotor dynamics is decomposed into an altitude-yaw sub-system and a lateral position sub-system (see Figure 2.2). The resultant altitude-yaw sub-system is fully actuated, while the lateral position sub-system is under-actuated. A fully actuated sub-system can be easily solved through dynamic inversion, nevertheless a second order sliding mode controller is used.

Figure 2.2: A diagram showing the architecture proposed by Zheng *et. al.* [111], where x, y, z are the components of the UAV position, $\dot{x}, \dot{y}, \dot{z}$ are the components of the UAV velocity, f, θ, ψ are the roll, pitch and yaw angles, $\dot{f}, \dot{\theta}, \dot{\psi}$ are the roll, pitch and yaw rates, x_d, y_d, z_d are the components of the desired UAV position, and f_d, θ_d, ψ_d are the desired roll, pitch and yaw angles.

In a similar architecture as Zheng *et. al.* [111], Xiong *et. al.* [94] uses a robust terminal sliding mode control to improve the settling time of the roll and pitch control of the under-actuated lateral position sub-system. Time scale separation is used by Xiong *et. al.* [94] to design the position controller using sliding mode control. This allows for

a decoupled approach.

Besnard *et. al.* [10] uses a multiple time scale, sliding mode control driven by a sliding mode observer (SMC-SMDO) to provide robust position control of a quadrotor. The use of multiple time scale implementation is possible because of the cascaded structure of the quadrotor as shown in Figure 2.3. SMC-SMDO is applied to each control loop while ensuring time scale separation by analyzing the closed-loop eigenvalues. This highlights the importance of time scale difference in time scale separation. In addition, it is noted that inverse kinematic relationship in the form of (2.6) is used to derive angular commands from acceleration commands. Inverse kinematics provides a link between the translational and rotational systems. Extensive simulations are conducted to verify the flight controller.

In other applications, the sliding mode control structure is used for its robustness to external disturbances and system uncertainties. Bouadi *et. al.* [12] developed an adaptive sliding mode controller for attitude stabilization and altitude tracking of a quadrotor. Gaussian white noise is added to the output as well as parametric uncertainties are added to the mass and inertial matrix.

Yeh *et. al.* [104] has successfully implemented a fuzzy sliding mode control on a lumped uncertainty. Numerical simulations are performed to validate the controller design.

Luque-Vega *et. al.* [56] uses sliding mode control on four sub-systems of a quadrotor: the altitude, heading, longitudinal and latitudinal sub-system. A sliding mode control scheme is developed for each sub-system. The state derivatives required in the control law are obtained using a super-twisting control algorithm. Furthermore, wind disturbance is also considered and the robustness of the controller is confirmed in a simulation study.

The sliding mode control has proven to be effective in handling disturbances and uncertainties. However, the chattering of the control signals remains a major challenge

Figure 2.3: A diagram showing the architecture proposed by Besnard *et. al.* [10], where ψ_c is the commanded yaw angle, Φ_c is the commanded Euler angles, Ω_c is the commanded angular velocity, τ_{ac} is the commanded control moment, ω_{1c} is the commanded rotor rotational velocity, τ_{1c} is the commanded rotor input, P , Φ , Ω and ω_1 are the UAV position, Euler angles, rotational velocity and rotor rotational velocity respectively.

for such control scheme. This is because in real system, the control actuator is unable to perform consistently such high frequency switching. Few groups have successfully demonstrated flight test with a full sliding mode controller.

2.1.3.2 Nested Saturation Control

In addition to disturbances and system uncertainties, multi-rotor UAVs are also subjected to physical limitations in its dynamics. As mentioned in Section 2.1.1.1, the roll and pitch angles of a multi-rotor UAV is restricted to be much smaller than 90° for practical reasons. This prevents flipping of the multi-rotor UAV which is an unstable maneuver. As saturation is a nonlinear phenomenon, most other controllers are not able to guarantee satisfaction of the attitude angles constraints.

Nested saturation control, developed by [88, 89], is applied to a multi-rotor UAV with a hard limit set on the roll and pitch angles such that the closed-loop system is stable. A Lyapunov-based approach is used on a planar VTOL (PVTOL) UAV by Lozano *et al.* [55] that systematically derives virtual control in a bottom up approach (ie. from the pitch angle to the lateral position). The pitch angle is guaranteed to converge and stay within a saturation limit and the converged pitch dynamics is used to derive an outer position control law. This approach stems from the cascaded control structure that is seen in most other multi-rotor UAV control. Experiments are conducted by Fantoni *et al.* [28] and the results are satisfactory even in the presence of disturbances.

Nested saturation control is also developed by Cabecinhas *et al.* [14] to enforce a saturation on the actuator thrust. The proposed controller is able to achieve asymptotic stability of the closed-loop system in the presence of exponentially decaying observer error. It is also able to ensure global regulation of trajectory tracking error. Numerical simulations are conducted to verify the control design.

Although useful in limiting the tilt angle, the application of nested saturation control is still limited to basic multi-rotor UAV control.

2.1.3.3 Backstepping Control

The multi-rotor UAV dynamics presented in (2.1) is modelled in a cascaded structure (see Figure 2.4). The attitude angles are fully actuated by control moment inputs, the UAV velocity is fully actuated by the roll and pitch angles and the thrust input, and the UAV position is fully actuated by the UAV velocity (a simple integral form). This cascaded structure is well suited for applying the backstepping control.

Backstepping control is used by Mandani *et al.* [57, 58] on a quadrotor. The dynamics of the quadrotor are separated into a fully actuated altitude-yaw and roll-pitch dynamics, a under-actuated planar position dynamics and a rotor dynamics. Mandani

Figure 2.4: A diagram showing the hierarchical structure of a multi-rotor UAV dynamics, where M_x, M_y, M_z are the control moment components, p, q, r are the UAV rotational velocity, ϕ, θ, ψ are the roll, pitch and yaw angles, V is the UAV velocity and P is the UAV position.

et. al. justifies the use of backstepping control to the cascaded dynamics structure. The roll and pitch of the quadrotor is used as a virtual control to the planar position. The control law is demonstrated on a mini helicopter in the heave and yaw directions.

Similarly, Yu *et. al.* [98] adopted a sub-system description of a quadrotor: an attitude, a heave and a planar position sub-system. A backstepping controller is designed for the heave sub-system and planar position sub-system of the quadrotor. A state dependent Riccati equation (SDRE) approach is used for the attitude control. The attitude controller then provides the virtual input to the planar position sub-system.

A backstepping controller using synergistic Lyapunov function and feedback pairs is formulated by Casau *et. al.* [16]. The backstepping control is shown to be able to achieve global tracking using simulations.

Backstepping control can also be formulated to handle disturbances. A robust backstepping control is formulated by Pollini *et. al.* [67] to handle measurement errors. A two loops control architecture is used where the attitude control is performed using a backstepping controller and a position control is performed with another backstepping controller (see Figure 2.5) and identifying that the attitude and the position sub-systems

are both double integrator system. The interconnection between the two control loops are achieved by computing the desired attitude command using an inverse kinematic approach similar to (2.6).

Figure 2.5: A diagram showing the architecture proposed by Pollini *et. al.* [67], where x_d, y_d, z_d are the desired UAV position components, ϕ_d, θ_d, ψ_d are the desired roll, pitch, yaw angles, U_1, U_2, U_3, U_4 are the control moment components and $F_{1..4}$ are the control force inputs.

Dong *et. al.* [25] uses a disturbance observer to compensate for model mismatch and disturbances in a backstepping controller. A linearized model is used to formulate the trajectory control of a quadrotor and its input-to-state stability is analyzed. Experiments are performed to show the capability of the controller applied on a Hummingbird quadrotor.

Cabecinhas *et. al.* [15], on the other hand, uses a projector operator to estimate for a constant force disturbance. An underlying backstepping controller is used and demonstrated with experiment on a quadrotor platform.

Bouabdallah *et. al.* [11] compares the backstepping control with the sliding mode control using a micro indoor quadrotor. The sliding mode controller is observed to cause sensor drift due to high frequency, low amplitude vibration arises from chattering. On the other hand, the backstepping controller performs relatively well in the presence of relatively high perturbations. Bouabdallah *et. al.* work has great significance in showing the practicability of backstepping control as compared to sliding mode control.

Gong *et. al.* [32] designs a flight trajectory controller for a quadrotor under input saturation using a nested two loop structure similar to Pollini *et. al.* [67] (see Figure 2.6). The desired attitude angles are derived from a translational controller using a bounded approach so that input saturation is considered. An inverse kinematic approach (2.6) is used to derive an attitude command from an acceleration command. Flight tests are successfully demonstrated on a quadrotor.

Figure 2.6: A diagram showing the architecture proposed by Gong *et. al.* [32].

Das *et. al.* [22] uses a Lagrange form of the dynamics to formulate a backstepping controller using an inner attitude and an outer position loop. Inverse kinematic approach is used to derive an attitude command from an acceleration command.

It is seen from previous literatures that a two loop design is prevalent and recurring in (but not restricted to) backstepping control. Time scale separation using two loop design requires the inner loop to be sufficiently fast for stability guarantees. Backstepping control is able to achieve a two loop design without time scale separation using a full state backstepping formulation and this is formulated by Zuo *et. al.* [113]. The relationship between the position and attitude loops is often solved using an inverse kinematic approach. Furthermore, the complicated derivative expressions arising from the full state backstepping approach is bypassed using linear tracking differentiator.

Backstepping control utilizes the cascaded structure of multi-rotor UAVs [22, 32, 57, 58, 67, 98, 113] and uses an intuitive way to control multi-rotor UAVs. This is evident from the control structures used by most research groups (see Figure 2.2 to 2.6). Time

scale separation is often used to decouple multi-rotor UAV control into translational and rotational control. This reduces the response time of the closed-loop system. To overcome this issue, full state backstepping control is formulated by Zuo *et. al.* [113]. This allows for more flexibility and faster response. Furthermore regardless of time scale separation, inverse kinematics is often used in the inner attitude and outer position loop structure to relate the attitude command from the acceleration command [22, 32, 67, 113]. This gives an exact relationship between the translational and rotational systems if the dynamic model used is exact. Lastly it is also remarked [113] that backstepping control often results in complicated expressions for the control law due to the requirement for state derivatives. This is overcome using linear tracking differentiator by Zuo *et. al.* [113].

2.1.4 Concluding Remarks

This section discusses about the position tracking control of a multi-rotor UAV using both linear and nonlinear approaches. The following conclusions are drawn from the literatures surveyed.

Firstly, the problem with attitude control is discussed. Linear controllers such as the nested saturation controller proposed by Guerrero-Castellanos *et. al.* [35] often restricts the aggressiveness of multi-rotor UAVs. This indirectly also restricts the response and performance in position control. All-attitude control is proposed using quaternions by Tayebi [87] and Fresk *et. al.* [29]. However, their controllers still use linear methodologies to derive control errors thus lacking in operation domain. It is therefore beneficial to the position control if a wider attitude domain can be made available by a suitable attitude controller.

Furthermore in position control, time scale separation which greatly simplify the design process is often used to separate the multi-rotor UAV dynamics into an attitude inner loop and a position outer loop in a two loops design. Time scale separation is used

by [10, 22, 28, 32, 42, 52, 55, 57, 58, 67, 69, 70, 91, 93, 94, 98, 111]. However time scale requirement in time scale separation slows down the closed-loop response time, therefore full state backstepping control is sometimes used to eliminate this time scale requirement [113] hence speeds up the closed-loop system response.

In addition, it is noteworthy that inverse kinematics of the form (2.6) is often used in a two loops control architecture [10, 22, 32, 67, 113] regardless of the use of time scale separation. However, it is deemed to require high fidelity dynamic model for accurate inversion.

Lastly, it is common that the analytic form of state derivative required in backstepping control is complicated and tedious. This issue is solved by Luque-Vega *et. al.* [56] with a super-twisting algorithm to estimate the derivative and also by Zuo *et. al.* [113] with a linear tracking differentiator. The method proposed by Zuo *et. al.* is more easily implemented in practice.

The points highlighted in this subsection give rise to some research issues and motivations that are discussed later in this chapter.

2.2 Literature Survey: Landing Controller

The landing sub-problem, on the other hand, involves the design of a flight controller or a control scheme that allows a multi-rotor UAV to land on a ship deck in the presence of ship heave motion. This section reviews the sensors development and its impact on the landing controller design. This is followed by an overview of existing control strategies and flight tests conducted in this aspect, thereafter provides an in-depth analysis leading to possible research issues.

2.2.1 Sensor-Based landing

The landing operation on a ship deck requires careful estimate of the ship heave motion or the multi-rotor UAV relative height above the ship deck. This relative height information

is usually not readily available on the ship command center. Although it can be estimated using the ship's computation, it is often estimated onboard the multi-rotor UAV to reduce data transmission latency. Although sensor development is not the focus of this thesis, this subsection aims to discuss some sensor development that fuels research in low cost multi-rotor UAV shipboard recovery.

In recent years, vision-based UAV estimation system is becoming more commonly used among researchers. This is because vision-based sensors have the potential to accomplish multiple tasks such as object identification, range measurement and simultaneous localization and mapping (SLAM). In ship deck landing of a multi-rotor UAV, vision-based sensors are often used to provide localization of the landing platform as well as relative height measurement. The following research groups use vision based guidance systems in their UAV control and landing system [37, 45, 48, 51, 73, 78, 79, 81, 84, 99, 105]. However, vision-based sensor alone is unable to achieve the robustness and reliability required for a multi-rotor UAV operating in the open seas. This is because low-cost, light-weight vision sensors are sensitive to the image quality and lighting conditions. In adverse weather condition or in darkness, vision sensors are not able to provide the required accuracy for landing. Therefore, sensor fusion is simulated by [101] and is often used to allow for more accurate measurement thus better performance.

However, vision sensors are not necessarily required in the sensor suite of a multi-rotor UAV. Its reliability is still not as high as the conventional time-of-flight sensors. Hervas *et. al.* [39] uses a laser radar system mounted on a ship deck to guide a fixed wing UAV in landing. Yakimenko *et. al.* [97], on the other hand, uses an IR sensor to provide relative position measurement for shipboard landing of an UAV. Similarly, Wenzel *et. al.* [93] uses an IR sensor for position measurement to land a multi-rotor UAV on a moving vehicle. He uses IR sensor due to its robustness and cost.

It is further verified through experiments conducted by [2, 30, 37, 48, 73, 78, 79, 81, 84, 95, 97, 101] using vision-based or other sensors that the position measurement of the ship

deck relative to the multi-rotor UAV is able to achieve sufficient accuracy and precision. The availability of hardware and computational power motivates the improvement of control algorithms (software) to deal with the challenges in multi-rotor UAV shipboard recovery. This is discussed in the next subsection.

2.2.2 Landing Control Algorithm

In the shipboard recovery problem of a multi-rotor UAV, the ship deck heave motion as a consequence of ship-wave interaction is a potential source of risk. This heave motion is difficult to track because of its random nature. Nevertheless, it is assumed that accurate estimate of the relative height above the ship deck is available from a vast selection of sensors such as vision, IR, laser and lidar as discussed in the previous subsection. This subsection provides a review of the landing algorithms used in the landing of a multi-rotor UAV on a ship or moving platform.

2.2.2.1 Ship Deck Motion Prediction Algorithm

Ship motion prediction is one of the approach to achieve safe landing on a ship deck from the early days. Full knowledge of the ship motion allows the multi-rotor UAV to anticipate and react accordingly during landing. Thereafter, shipboard recovery becomes a trivial task. However, the challenge arises from the ability to formulate an accurate and precise ship motion prediction algorithm.

Avanzini *et. al.* [4] proposed an offline trajectory generator based on mean sea motion values. Online prediction of the ship motion is estimated by a guidance controller using data obtained from ship motion sensors (see Figure 2.7). However this requires additional sensors and communication system to transmit ship motion data to the UAV.

Yang *et. al.* [102], on the other hand, predict the ship motion using an autoregressive exogenous (ARX) model with a new order determination principle specified by Bayes information criterion. His results are shown to be more superior as compared to a fixed

Figure 2.7: A diagram showing the architecture proposed by Avanzini *et. al.* [4].

order ARX model and an autoregressive (AR) model. However Yang *et. al.* noted that computation error accumulates, thus the predictor is not able to provide long term accurate prediction (typically up to an order of 5s)[102].

Yang *et. al.* [100] uses a modified Prony analysis approach to model the displacement of a ship. The model order is specified using a recursive algorithm that minimizes the summed squared error. A learning process is also required before the model is established. The recursive initialization and data acquisition process requires a substantial processing time, therefore this algorithm is unable to be applied during emergency landing.

Ship motion prediction algorithm is attractive in the shipboard recovery problem due to its ability to reduce the problem into a trivial recovery problem. However, such algorithm requires additional sensors to be mounted on the ship resulting in additional costs [4]. Furthermore Yang *et. al.* [102] also found that accumulation in error limits the performance of long term prediction. In another experiment, Yang *et. al.* [100] also found that model identification process results in a substantial processing time. Therefore, alternatives to the use of ship motion predictor are favored.

2.2.2.2 Adaptive Flight Path

In addition to ship motion prediction, an adaptive flight path computed with ship motion information is also a feasible solution to the shipboard recovery problem. It allows the application of simple linear control in the inner flight control loop.

Avanzini *et. al.* [5] uses a linearized model and a linear robust control for the flight control in a shipboard recovery of a shrouded fan UAV. His architecture is presented in Figure 2.8. A guidance controller is used with the ship motion information to provide a safe landing trajectory.

Figure 2.8: A diagram showing the architecture proposed by Avanzini *et. al.* [5].

Nuthi *et. al.* [65] uses a gain scheduled linear model with a LQR controller (see Figure 2.9). The ship motion information is considered by an adaptive flight path planner such that the flight parameters command follows the ship motion.

Both control structure proposed by Avanzini *et. al.* [5] and Nuthi *et. al.* [65] use an adaptive trajectory generator despite using linear control. However their proposed control structures requires an accurate knowledge of the ship motion to be effective. Linear control can also be used without an adaptive trajectory generator which is discussed in the upcoming section in the work presented by Sandino *et. al.* [80].

Figure 2.9: A diagram showing the architecture proposed by Nuthi *et. al.* [65].

2.2.2.3 Model Predictive Control

MPC is able to compute an optimal control sequence in the presence of hard and soft constraints. Furthermore with full knowledge of ship motion (from ship motion predictor), MPC is also able to provide an optimal control sequence.

Sandino *et. al.* [80] compares between linear quadratic integral (LQI), MPC and loop shaping design in a shipboard recovery problem. A linearized model is obtained from a nonlinear model and the application of these control schemes are straightforward. MPC is found to perform better, however the control signals are also found to exhibit high oscillations which are not desirable.

Ngo [63] also uses a MPC on a linearized helicopter model that is able to handle multiple constraints including ship motion. However, full knowledge or estimate of the ship motion is assumed.

The success of MPC depends largely on the knowledge of the ship motion that is usually provided by the ship motion predictor mentioned in [4, 100, 102].

2.2.2.4 State Dependent Riccati Equation Method

The state dependent Riccati equation (SDRE) method is a quasi-linear method that is similar to the ARE approach in control theory. In the multi-rotor UAV shipboard recovery problem, Saghafi *et. al.* [77] and Esmailifar *et. al.* [26] uses a SDRE derived controller to track the ship deck motion at various operating points. Their approach is similar to LQR with a state dependent objective function $J = \int_0^\infty x^T Q(x)x + u^T R(x)u dt$, where x is the system state, u is the control input, $Q(x)$ and $R(x)$ are state dependent weighting matrices. Similar to LQR, the optimal control law is the solution to the state dependent ARE $A(x)^T P(x) + P(x)A(x) - P(x)B(x)R^{-1}(x)B^T(x)P(x) + Q(x) = 0$, for some $P(x) = P^T(x) > 0$ where $A(x)$, $B(x)$ are state dependent system matrices and $u = -R^{-1}(x)B^T(x)P(x)x$.

Esmailifar *et. al.* [26] further uses a compensator to transform any nonlinear model to the state dependent coefficient form (see Figure 2.10).

Figure 2.10: A diagram showing the architecture proposed by Esmailifar *et. al.* [26].

Saghafi *et. al.* [77] and Esmailifar *et. al.* [26] also demonstrated the performance of their proposed control scheme in a simulation. The robustness of the control scheme to system uncertainties and disturbances are also verified.

2.2.2.5 Sliding Mode Control

Sliding mode control is a form of robust control and can also be used to solve the multi-rotor UAV shipboard recovery problem. This is achieved by Hervas *et. al.* [38, 39] by

considering the relative dynamics between the UAV and the ship. Lidar measurement is provided to guide the UAV during recovery. However, it is noted that the control signals arising from sliding mode control exhibits aggressive chattering and is unsuitable to be implemented on a real UAV.

2.2.2.6 Internal Model Approach

Marconi *et. al.* [59] uses an internal model approach that combines the ship motion prediction and landing control of a multi-rotor UAV. A fixed order sinusoidal series driven model is used to represent the ship motion, thereafter a Lyapunov based approach similar to adaptive control is used to formulate an output regulation problem such that the output tracks the modelled ship motion.

This approach shows great potential in solving the ship deck landing problem by combining an estimator and a controller. The simplicity of the internal model and the control law allow it to be more easily implemented as compared to the other control algorithm discussed so far.

However, a key issue with this approach is the inability to accurately model the ship motion with a finite order sinusoidal series (See Appendix B). Furthermore, accurate model requires a high order sinusoidal series that requires heavy computation.

2.2.3 Concluding Remarks

This section gives a review of the control and prediction methodologies used in the landing operation of a multi-rotor UAV. Some concluding remarks are listed from the literatures surveyed.

Firstly, it is concluded that landing controller design is a trivial process if full knowledge of the ship is available. As such, ship motion predictors [4, 100, 102] are developed to allow simple control law design provided that the prediction is accurate and reliable.

Unfortunately, Yang *et. al.* pointed out that accumulation of errors is a source of inaccuracies [102] and the initialization and data acquisition requires long processing time [100].

In addition to a decoupled predictor and controller design, the internal model approach developed by Marconi *et. al.* [59] provides an integrated predictor and controller design. It shows the most potential so far in terms of effectiveness and simplicity to be applied on low cost multi-rotor UAVs. However it has similar disadvantages as the ship prediction algorithms in terms of prediction accuracy. This is demonstrated in Appendix B.

Lastly, this section also gives a review of tracking control strategies such as LQI, MPC and sliding mode control [38, 39, 63, 80]. Control strategies such as LQI [80] and sliding mode control [38, 39] treat the ship heave motion as external disturbance while MPC [63] relies on accurate ship motion prediction. It is therefore concluded that robust control strategies is useful in ship deck landing control despite the complexity related to some control strategies such as sliding mode control [38, 39].

A detail discussion on potential research issues relating to multi-rotor UAV landing problem is discussed in the next chapter.

2.3 Position Tracking Sub-Problem

The rest of this chapter highlights the potential research issues relating to the position tracking and landing sub-problems of the multi-rotor UAV shipboard recovery problem. An extensive literature review is given in Section 2.1 and 2.2, potential areas for improvement are stated and solutions are proposed in the rest of this chapter.

A review of multi-rotor position tracking control is given in Section 2.1. It is observed that the cascaded structure of the multi-rotor UAV dynamics is well suited for an inner attitude loop, outer position loop design. Time scale separation allows a decoupled inner

and outer loop design, however full state backstepping control eliminates the time scale requirement imposed by time scale separation hence allowing a faster response.

Research issues are identified with respect to the application of backstepping control to the multi-rotor UAV position tracking problem. They are:

- complexity of the control law synthesis using Lyapunov backstepping control due to the lack of closed-loop backstepping rule.,
- the complexity associated with inverse kinematic,
- and a restricted operation domain for a quaternion-based attitude control due to the use of linear control methodologies.

These are further illustrated in the following subsection.

2.3.1 A Simpler Backstepping Methodology

A cascaded dynamic structure encourages the use of backstepping control due to the system's regular structure. However, Lyapunov-based formulation becomes increasingly complex when the order of the system increases. This is illustrated in Section 2.1.3.3 and [113]. It is therefore beneficial for the development of a closed-form backstepping control law.

In Chapter 4, a closed-form backstepping control law is derived for a class of cascaded-feedback dynamical system. This is achieved by specifying contraction of state trajectory into a desired trajectory for each sub-system using contraction theory developed by Lohmiller *et. al.* [53], expanding its application to backstepping control. Complex analytic form of state derivatives are estimated by linear tracking differentiators as proposed by Zuo [113].

2.3.2 Complexity associated with Inverse Kinematic

Full state backstepping control of multi-rotor UAV is commonly achieved using inverse kinematic between the attitude and position loops. This is because the multi-rotor UAV dynamics (2.1) is not in the *cascaded-feedback form* (see Section 4.2.1.1, p.68).

The use of inverse kinematic requires additional computation for the desired attitude angles from the commanded acceleration. It is noted that the expression is not straightforward, with the expression suggested by Zuo [113] as

$$\theta_d = \tan^{-1} \left(\frac{a_x \cos \psi_d + a_y \sin \psi_d}{a_z + g} \right), \quad (2.9a)$$

$$\phi_d = \sin^{-1} \left(\frac{a_x \sin \psi_d - a_y \cos \psi_d}{\sqrt{a_x^2 + a_y^2 + (a_z + g)^2}} \right), \quad (2.9b)$$

where a_x , a_y , a_z are the acceleration components in the body axis, ϕ_d , θ_d , ψ_d are the desired attitude angles and g is the gravitational acceleration. However under force disturbances, this inversion is not exact. This results in deteriorated closed-loop disturbance rejection performance in the presence of exogenous effects.

This performance deterioration can be illustration by considering the following simplified example.

Example 2.1 (Simplified inverse kinematics). *Consider a system*

$$\dot{x}_1 = x_2, \quad (2.10a)$$

$$\dot{x}_3 = u, \quad (2.10b)$$

where x_1 , x_3 are the system states, u is the control input and x_2 and x_3 are related by a nonlinear function $x_2 = h(x_3)$.

Assume that there exists a backstepping control law $u = g(x_1, x_2, x_3)$ that stabilizes the system by defining desired values $x_{2,d}$ and $x_{3,d}$.

Inverse kinematics is equivalent to the relationship $x_{3,d} = h^{-1}(x_{2,d})$ that computes the desired value $x_{3,d}$ so that $x_3 \rightarrow x_{3,d} \Rightarrow x_2 \rightarrow x_{2,d}$. This works well if the inversion matches the actual physical system.

However if there exists unmodelled effects such that the actual relationship between x_2 and x_3 is $x_2 = h(x_3) + d$, then x_2 will track a value $x_{2,d} + d$ instead of the intended $x_{2,d}$.

This will result in a steady state error $(x_1 - x_{1,d})$ for any nonzero function d .

This is a problem commonly associated with the use of inverse kinematics. In cases where the disturbance is small, the effect is minimal. However in windy environment, the impact of unmodelled uncertainty is more pronounced. This effect is known to be overcome by using an integral action such as in the work done by Raffo *et. al.* [71]. However with the complexity of backstepping control, preserving of the order of the controller without integral action is nonetheless beneficial.

In addition, the main advantage of backstepping control is in its cascaded and systematic application of backstepping rules resulting in a final control law. Therefore it is desirable to express the system dynamics in a cascaded-feedback form such that the implementation of backstepping control is simple and straightforward. Furthermore in the cascaded-feedback form, inverse kinematic is not required to provide the bridge between position and attitude dynamics. This indirectly improves the closed-loop disturbance rejection performance.

In Chapter 4, a novel reformulation of the multi-rotor UAV kinematics and dynamics into a cascaded-feedback form is presented. Thereafter a straightforward implementation of backstepping control yields the position tracking control law.

2.3.3 Quaternion-Based Control

Euler angles attitude description exhibits gimbal lock at $\theta = 90^\circ$, this can be overcome by the use of quaternion. As quaternion is a less intuitive attitude descriptor, the use of

quaternion is restricted to only PID control-like structures. This restricts the operation domain of quaternion-based control structure as stated in Section 2.1.1.1.

A nonlinear control law that is able to utilize quaternion for attitude control can potentially be used for a larger operation domain. This allows the usage of quaternion as an all-attitude descriptor in a control structure that is valid for a larger domain.

In Chapter 4, the quaternion-based multi-rotor UAV kinematics and dynamics is expressed in a cascaded-feedback form and backstepping control is applied to formulate a quaternion-based control law. This control law expanded the operation domain of the multi-rotor UAV attitude control, thus allowing faster response in position tracking control.

2.4 Landing Sub-Problem

A review of the multi-rotor UAV landing sub-problem is given in Section 2.2. It is observed that the advancement of sensing technology allows accurate and precise onboard measurement of the UAV relative height above the ship deck. This is to be complemented with a competent control law that can respond to the pseudorandom ship motion. A pseudorandom process is a process that appears to be random but is not.

The following issues are highlighted:

- the need for a ship motion predictor-free control,
- and to include considerations for output feedback, external disturbances, system uncertainties and other implementation considerations.

These are further illustrated in the following subsection.

2.4.1 Ship Motion Predictor-Free Design

Ship motion predictor as seen in Section 2.2.2.1 often requires a high order description to achieve sufficient accuracy. This increases the computation load and reduces the possibility for onboard implementation on smaller UAVs. Furthermore, ship motion predictor is unable to provide long term prediction and requires long data acquisition time [100, 102]. Even the most promising internal model approach fails to capture high order ship heave motion (see Appendix B). It therefore motivates for a ship motion predictor-free control design.

It is further highlighted that system complexity is an issue for ship motion predictor, therefore it is desirable to have a simple resultant control structure. A linear control structure is an ideal candidate.

In Chapter 5, a linear control law is applied to a feedback linearized multi-rotor UAV relative heave dynamic model. This optimization-based control is formulated using invariant ellipsoid method and possesses a simple structure. Effects of ship deck heave motion is modelled as an external disturbance on the multi-rotor UAV with the invariant ellipsoid method providing the robust criteria for the control law synthesis.

2.4.2 Implementation Considerations

In the problem formulation, it is reasoned that wind and ship wake disturbances have the potential to jeopardize the ship deck landing operation. Furthermore, system uncertainties are always present in a real system. These are some effects that need to be considered during implementation on a multi-rotor UAV. It is therefore desirable to incorporate a control law that exhibits robustness to external disturbances and system uncertainties.

In addition, a review of sensor technology given in Section 2.2.1 revealed the difficulties to obtain relative speed measurement with the ship deck. Measurement noise becomes

too large. Therefore, output feedback control structure is desirable. Furthermore, it is also realistic to include robustness to sensor noise in the control formulation.

System uncertainty and measurement noise are considered in Chapter 5 by generalizing the invariant ellipsoid framework to include treatment of system uncertainty and measurement noise.

In Chapter 6, an output feedback linear control law is applied to a feedback linearized multi-rotor UAV relative heave dynamic model with relative height as the output. This control law formulation is formulated with invariant ellipsoid method in a robust control framework. It allows for robustness consideration of multiple disturbances, measurement noises and system uncertainties in addition to the ship heave motion.

2.5 Research Approaches

The approaches to solve the listed research issues are mentioned in the previous sections. This section gives an reiteration of the approaches in both the position tracking and landing sub-problems¹.

Statement 2.1 (Position Tracking Sub-Problem). *The position tracking sub-problem is solved using a **contraction theory**-based backstepping control applied on a **reformulated** multi-rotor UAV kinematics and dynamics model in a cascaded-feedback form. Furthermore, the same technique is also applied to a **quaternion**-based dynamics.*

Statement 2.2 (Landing Sub-Problem). *The landing sub-problem is solved using a **robust control framework** developed from the **invariant ellipsoid method** with a multi-rotor UAV relative heave dynamic model. A **full state feedback** and an **output feedback** linear control are formulated that is robust to external disturbances, measurement noises and system uncertainties.*

¹The boldfaced terms are the keywords to the author's contribution during his PhD.

Chapter 3

Multi-Rotor UAV Model

A full 6 degree of freedom (DOF) kinematic and dynamic model of a multi-rotor UAV is presented in this chapter with particular reference made to a quadrotor. The kinematic and dynamic model of multi-rotor UAVs can be found in many references and appears in many forms including the Newton-Euler form [31, 113] and the Euler-Lagrange form [22]. The Newton-Euler form is used in this thesis.

In the Newton-Euler form, the kinematics and dynamics of a multi-rotor UAV are separated into the translational and rotational kinematics and dynamics. This form allows easy realization of the hierarchical structure in the multi-rotor UAV kinematics and dynamics.

In addition to the kinematic and dynamic model of a multi-rotor UAV, environmental models that are used in numerical simulations and control formulations are also described in this chapter. They are:

- motor model,
- gravity model,
- aerodynamics model,
- wind disturbance model,

- and ship heave model.

Figure 3.1 shows a quadrotor with an Earth-fixed inertial reference frame and the Euler angles associated with the attitude of the quadrotor. The body-fixed reference frame is also shown. These reference frames are used in the multi-rotor UAV (quadrotor) kinematic and dynamic model formulation presented in the next section.

Figure 3.1: A diagram showing a quadrotor and the Euler angles associated with the attitude of the quadrotor. The force on the quadrotor is acting on each of the rotors and pointing in the negative body z -axis. The reference frames, Earth-fixed inertial frame (X - Y - Z) and body frame (x - y - z), are also shown in the diagram.

3.1 Multi-Rotor UAV Kinematic and Dynamic Model

This section presents the kinematic and dynamic model of a multi-rotor UAV. It is noteworthy that certain parts of this section make specific reference to a quadrotor and can be generalized for any multi-rotor UAV.

3.1.1 Control Forces and Moments

The multi-rotor UAV is controlled by the thrusts generated by its rotors. By altering the thrust produced by each rotor, the total thrust and moments acting on the multi-rotor UAV are varied.

The force f_i acting on a single rotor is approximated as

$$f_i = k_t \Omega_i^2, \quad i = 1, 2, \dots, N_{\text{rotor}} \quad (3.1)$$

where k_t is the lift factor and Ω_i is the rotation rate of the i^{th} rotor for a N_{rotor} -rotor UAV. This is a result from the aerodynamics of rotor where it states that the thrust is proportional to the square of the rotational speed of the rotor.

The total thrust acting on the quadrotor is expressed as

$$T = \sum_{i=1}^{N_{\text{rotor}}} f_i, \quad (3.2)$$

assuming that the rotors are aligned with the quadrotor body z -axis. It acts in the negative body z -axis direction and is written as the vector form

$$\mathbf{T} = -T\hat{\mathbf{e}}_3, \quad (3.3)$$

where $\hat{\mathbf{e}}_3$ is the unit vector along the z -axis¹.

The torque τ_i acting on a single rotor is approximated as

$$\tau_i = k_q \Omega_i^2, \quad i = 1, 2, \dots, N_{\text{rotor}} \quad (3.4)$$

where k_q is the drag factor for a N_{rotor} -rotor UAV. Similarly, this is derived from the aerodynamics of rotor.

The total control moment is obtained by summing the moment due to the thrust and torque generated by each rotor about the respective body axis of the quadrotor. The total control moment acting on the multi-rotor UAV is expressed as

$$\mathbf{M} = \sum_{i=1}^{N_{\text{rotor}}} [\mathbf{r}_i \times (-f_i \hat{\mathbf{e}}_3) + (-1)^i \tau_i \hat{\mathbf{e}}_3], \quad (3.5)$$

where $\mathbf{r}_i$ for $i = 1, 2, \dots, N_{\text{rotor}}$ is the position vector of the center of the i^{th} rotor from the center of the multi-rotor UAV. It is also critical to assume that alternate rotors are rotating in opposite direction so that the multi-rotor UAV yaw can be controlled.

¹It is noted that boldfaced latin symbols represent vector quantities

For instance, a quadrotor will have $\mathbf{r}_i = -l\hat{\mathbf{e}}_1, -l\hat{\mathbf{e}}_2, l\hat{\mathbf{e}}_1, l\hat{\mathbf{e}}_2$ with l being the distance of the rotor to the quadrotor center of mass. Then the control moment is written as

$$\mathbf{M} = \begin{pmatrix} lk_t (\Omega_2^2 - \Omega_4^2) \\ lk_t (\Omega_1^2 - \Omega_3^2) \\ k_q (\Omega_2^2 + \Omega_4^2 - \Omega_1^2 - \Omega_3^2) \end{pmatrix}. \quad (3.6)$$

Given the required thrust and control moment, the rotor speeds can be calculated by

$$\begin{pmatrix} \Omega_1^2 \\ \Omega_2^2 \\ \Omega_3^2 \\ \Omega_4^2 \end{pmatrix} = \begin{pmatrix} -k_t & -k_t & -k_t & -k_t \\ 0 & lk_t & 0 & -lk_t \\ lk_t & 0 & -lk_t & 0 \\ -k_q & k_q & -k_q & k_q \end{pmatrix}^{-1} \begin{pmatrix} T \\ M_x \\ M_y \\ M_z \end{pmatrix}, \quad (3.7)$$

where $\begin{pmatrix} M_x & M_y & M_z \end{pmatrix}^T = \mathbf{M}$ are the control moments about the x, y, z -axes respectively and the matrix is non-singular by the construction of the quadrotor. This process is called control allocation where the rotation speed of the rotors are determined from the thrust and moments required.

In general, T, M_x, M_y and M_z are the four canonical control inputs of the multi-rotor UAV. The control of any multi-rotor UAV can be effected with these canonical control inputs. However it is noted that the control allocation (3.7) can be different among multi-rotor UAVs. Nevertheless, for the rest of this section the multi-rotor UAV kinematic and dynamic model are derived without any loss of generality.

3.1.2 Translational Kinematics

The translational kinematics describes the motion of a multi-rotor UAV in an Earth-fixed inertial reference frame. It is written as

$$\dot{\mathbf{p}} = \mathbf{v}, \quad (3.8)$$

where $\mathbf{p} = \begin{pmatrix} X & Y & Z \end{pmatrix}^T$ and $\mathbf{v}$ are the position and velocity of the multi-rotor UAV measured in the Earth-fixed inertial reference frame.

3.1.3 Translational Dynamics

The forces acting on a multi-rotor UAV influence the motion of the multi-rotor UAV according to Newton's second law. The translational dynamics that is formulated in an Earth-fixed inertial reference frame is

$$\dot{\mathbf{v}} = -\frac{1}{m}R^T u_1 \hat{\mathbf{e}}_3 + g \hat{\mathbf{e}}_3, \quad (3.9)$$

where $u_1 = T$, R is the direction cosine matrix from the Earth-fixed inertial frame to the body-fixed frame, m is the mass of the multi-rotor UAV and g is the gravitational acceleration.

The direction cosine matrix can be expressed in terms of Euler angles as

$$R = \begin{pmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ -c\phi s\psi + s\phi s\theta c\psi & c\phi c\psi + s\phi s\theta s\psi & s\phi c\theta \\ s\phi s\psi + c\phi s\theta c\psi & -s\phi c\psi + c\phi s\theta s\psi & c\phi c\theta \end{pmatrix}, \quad (3.10)$$

where $s(\cdot)$ and $c(\cdot)$ are $\sin(\cdot)$ and $\cos(\cdot)$ respectively, and ϕ , θ and ψ are the roll, pitch and yaw angles respectively.

On the other hand, the direction cosine matrix can also be expressed in terms of quaternions as

$$R = \begin{pmatrix} (\eta_0^2 + \eta_1^2 - \eta_2^2 - \eta_3^2) & 2(\eta_1\eta_2 + \eta_0\eta_3) & 2(\eta_1\eta_3 - \eta_0\eta_2) \\ 2(\eta_1\eta_2 - \eta_0\eta_3) & (\eta_0^2 - \eta_1^2 + \eta_2^2 - \eta_3^2) & 2(\eta_2\eta_3 + \eta_0\eta_1) \\ 2(\eta_1\eta_3 + \eta_0\eta_2) & 2(\eta_2\eta_3 - \eta_0\eta_1) & (\eta_0^2 - \eta_1^2 - \eta_2^2 + \eta_3^2) \end{pmatrix}, \quad (3.11)$$

where $\eta = \begin{pmatrix} \eta_0 & \eta_1 & \eta_2 & \eta_3 \end{pmatrix}^T$ are the quaternions.

In the translational dynamics (3.9), small forces such as aerodynamic forces and wind perturbations are neglected. These are included in the numerical simulations to demonstrate the robustness of the proposed controllers in the presence of these unmodelled effects.

Remark 3.1. The translational dynamics exhibits coupling between the attitude and the thrust control input. Therefore changes in either term causes a reaction in the translational motion of the quadrotor. This is because the control u_1 and the directional cosine matrix R of the quadrotor associated with its attitude appears as a product in (3.9). Furthermore, this also increases the difficulty in expressing the dynamics in a cascaded-feedback form.

3.1.4 Rotational Kinematics

The rotational kinematics describes how the attitude of the multi-rotor UAV changes with time. However as mentioned in Chapter 2, attitude of a multi-rotor UAV can be expressed in Euler angles and/or quaternions. In this section, the kinematics of both representations are provided.

3.1.4.1 Euler Angles Kinematics

In terms of Euler angles, the rotational kinematics is given by

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & t\theta s\phi & t\theta c\phi \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \quad (3.12)$$

where $t(\cdot)$ is $\tan(\cdot)$, p , q and r are the angular rates of the multi-rotor UAV about the body x , y and z -axes respectively. A singularity is observed at $\theta = \pm 90^\circ$, hence the attitude domain of this model is $\{\phi, \theta, \psi : 90^\circ \leq \phi \leq 90^\circ, -90^\circ < \theta < 90^\circ, 0^\circ \leq \psi < 360^\circ\}$.

3.1.4.2 Quaternion Kinematics

In terms of quaternions, the rotational kinematics [85] is given by

$$\dot{\eta} = \frac{1}{2} \begin{pmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{pmatrix} \eta. \quad (3.13)$$

This is a singularity-free representation.

3.1.5 Rotational Dynamics

The rotational dynamics of a multi-rotor UAV is formulated from Newton's second law in the body-fixed frame. It is given by

$$\dot{\omega} = J^{-1}(\mathbf{M} - \omega \times J\omega), \quad (3.14)$$

where J is the multi-rotor UAV moment of inertial matrix and $\omega = \begin{pmatrix} p & q & r \end{pmatrix}^T$.

In this formulation, small moments and torques acting on the quadrotor such as aerodynamic moments are neglected. Likewise, these are included in the numerical simulations to demonstrate the robustness of the proposed controllers in the presence of these unmodelled effects.

Let $u_2 = J^{-1}(\mathbf{M} - \omega \times J\omega)$, and (3.14) becomes

$$\dot{\omega} = u_2. \quad (3.15)$$

The equations of motion of a multi-rotor UAV are given by (3.8), (3.9), (3.12) and (3.15) (see also Table 3.1). The control inputs are given by u_1 and u_2 which are in terms of the canonical inputs for multi-rotor UAVs. The hierarchical structure of a multi-rotor UAV kinematics and dynamics is evident from the equations of motion. The control moments acting on the multi-rotor UAV affects its attitude which determines the direction of its thrust, thereafter affecting the multi-rotor UAV's velocity and position (see Figure 2.4). The equations of motion provides a mathematical relationship between the multi-rotor UAV states. These equations are transformed into the cascaded-feedback form in Chapter 4.

Table 3.1: A table of equations of motion for a multi-rotor UAV.

Translational kinematics	$\dot{\mathbf{p}} = \mathbf{v}$
Translational dynamics	$\dot{\mathbf{v}} = -\frac{1}{m}R^T u_1 \hat{\mathbf{e}}_3 + g\hat{\mathbf{e}}_3$
Rotational kinematics (Euler angles)	$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & t\theta s\phi & t\theta c\phi \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$
Rotational kinematics (Quaternions)	$\dot{\eta} = \frac{1}{2} \begin{pmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{pmatrix} \eta$
Rotational dynamics	$\dot{\omega} = J^{-1}(\mathbf{M} - \omega \times J\omega)$

3.1.6 Heave and Planar Motion Decoupling

In this thesis, a two sub-problems approach is used to solve the multi-rotor UAV ship deck recovery problem. This is because the heave kinematics and dynamics of a multi-rotor UAV can be controlled independently from the planar kinematics and dynamics giving a more intuitive control structure. This subsection decouples (3.8), (3.9), (3.12) and (3.15) into a heave and a planar kinematic and dynamic model.

3.1.6.1 Heave Kinematic and Dynamic Model

This subsection formulates the heave kinematic and dynamic model. Consider (3.9) written in component form as

$$\dot{\mathbf{v}} = -\frac{T}{m} \begin{pmatrix} \sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi \\ -\sin \phi \cos \psi + \cos \phi \sin \theta \sin \psi \\ \cos \phi \cos \theta \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}. \quad (3.16)$$

The body z -axis component of (3.16) is

$$\ddot{h} = \frac{T}{m} \cos \phi \cos \theta - g, \quad (3.17)$$

where $h = -Z$ is the altitude of the multi-rotor UAV.

Let $h_1 = h$, $h_2 = \dot{h}_1$ and $v = \frac{T}{m} \cos \phi \cos \theta - g$, then (3.17) can be written in the state space form

$$\dot{h}_1 = h_2, \quad (3.18a)$$

$$\dot{h}_2 = v. \quad (3.18b)$$

The heave kinematic and dynamic model is thus formulated.

It is noted that $T = \frac{mv+mg}{\cos \phi \cos \theta}$ is limited by actuator saturation thus the virtual control v is also limited. Furthermore, the control input transformation $v = \frac{T}{m} \cos \phi \cos \theta - g$ allows the thrust T to directly control the multi-rotor UAV altitude for all $-\frac{\pi}{2} < \phi, \theta < \frac{\pi}{2}$.

3.1.6.2 Planar Kinematic and Dynamic Model

This subsection formulates the planar kinematic and dynamic model. The planar translational kinematic and dynamic model is derived from (3.8) and (3.9) less the z component. It is

$$\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} = -\frac{T}{m} \begin{pmatrix} \sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi \\ -\sin \phi \cos \psi + \cos \phi \sin \theta \sin \psi \end{pmatrix}. \quad (3.19)$$

In addition to (3.19), the rotational kinematics and dynamics given by (3.12) and (3.15) are also included in the planar kinematic and dynamic model.

Effectively, among the four canonical control inputs T , M_x , M_y and M_z , the thrust T is used to control the heave motion of the multi-rotor UAV while the control moments M_x , M_y and M_z are used to control the attitude angles which in turn control the X , Y position of the multi-rotor UAV. The multi-rotor UAV yaw ψ is also controlled by the planar kinematics and dynamics.

3.2 Gravity Model

In the multi-rotor translational dynamic model, the weight of the multi-rotor UAV is given by mg , where m is the multi-rotor UAV mass and g is the gravitational acceleration.

In most control law formulation, the gravitational acceleration g is taken as a constant value 9.81 or 10m/s². This is because most applications do not require the multi-rotor UAV to traverse large changes in altitude. Therefore in this thesis where the domain of operation does not require drastic changes in altitude, a constant value 9.81m/s² is chosen.

However in the numerical simulations, a position dependent gravity model is implemented to verify the constant gravitational acceleration assumption. The 1984 World Geodetic System (WGS1984) gravity model is used in the Matlab/Simulink environment. The WGS1984 gravity model is based on a geocentric equipotential ellipsoid and works well for low altitude ($< 20,000\text{m}$) application.

3.3 Aerodynamic Model

In (3.9), small forces and moments such as aerodynamic forces and moments are neglected. In real situation when the multi-rotor is flying at a sufficiently high speed, aerodynamic drag and moment acting on the multi-rotor UAV become significant. Similarly, this effect is not considered in the control law formulation for simplicity but is included in the numerical simulations.

The aerodynamic effects are modelled in a similar way to [67, 113], and is given by

$$\mathbf{F}_{\text{aero}} = -K_d \mathbf{v}, \quad (3.20a)$$

$$\mathbf{M}_{\text{aero}} = -K_m \boldsymbol{\omega}, \quad (3.20b)$$

where $\mathbf{F}_{\text{aero}}$, $\mathbf{M}_{\text{aero}}$ are the aerodynamic force and moment respectively and $K_d > 0$ and $K_m > 0$ are the coefficient matrices of the aerodynamic force and moment respectively. In most cases, scalar constants $K_d > 0$ and $K_m > 0$ are sufficient. For multi-rotor UAVs that typically travel at a much slower speed than fixed wing UAVs, a linear relationship between the aerodynamic force and moment is assumed instead of a quadratic relationship.

The aerodynamic effects are included in the translational dynamic model (3.9) and rotational dynamic model (3.14) by

$$\dot{\mathbf{v}} = -\frac{1}{m}R^T u_1 \hat{\mathbf{e}}_3 + g\hat{\mathbf{e}}_3 + \frac{1}{m}\mathbf{F}_{\text{aero}}, \quad (3.21)$$

and

$$\dot{\boldsymbol{\omega}} = J^{-1}(\mathbf{M} + \mathbf{M}_{\text{aero}} - \boldsymbol{\omega} \times J\boldsymbol{\omega}). \quad (3.22)$$

It is observed from (3.20), (3.21) and (3.22) that for small $\mathbf{v}$ and $\boldsymbol{\omega}$, $\mathbf{F}_{\text{aero}}$ and $\mathbf{M}_{\text{aero}}$ are small and their effects can be neglected.

3.4 Wind Disturbance Model

In the open sea environment, wind disturbance acting on a multi-rotor UAV is inevitable. It is also critical for the closed-loop control system to perform in a windy environment.

A Dryden wind turbulence model [41] is used to model the wind disturbance in the open sea. It is a stochastic model that takes in Gaussian white noise input and outputs a continuous gust with a specified power density.

In addition to wind turbulence, wind gust is also commonly present in the open sea environment. Wind gust is the constant biased component of wind that is caused by steady wind. For instance, according to Table 1.1 an average wind gust of 30kts = 15.4m/s is expected in a sea state 6 condition.

Examples of the linear and rotational components of wind disturbance with a bias term are given in Figure 3.2 and 3.3. It should be noted that the bounded nature of the disturbances is evident from the figures despite different waveform being generated for every simulation.

The wind disturbance affects the dynamics of a multi-rotor UAV by generating an

Figure 3.2: The random wind speed profile generated by the Dryden wind turbulence model with a bias term.

Figure 3.3: The random wind rotation speed profile generated by the Dryden wind turbulence model.

aerodynamic drag and moment component due to wind. These are modelled as

$$\mathbf{F}_{\text{aero}} = -K_d(\mathbf{v} - \mathbf{v}_{\text{wind}}), \quad (3.23a)$$

$$\mathbf{M}_{\text{aero}} = -K_m(\boldsymbol{\omega} - \boldsymbol{\omega}_{\text{wind}}), \quad (3.23b)$$

where $\mathbf{v}_{\text{wind}}$ and ω_{wind} are the linear and rotational components of wind disturbance. The effects on the multi-rotor UAV dynamics follow (3.21) and (3.22).

3.5 Ship Heave Model

It is also essential to model the ship heave motion in the numerical simulation for the landing of the multi-rotor UAV.

The ship deck is assumed to exhibit heave motion from the interaction with the sea wave motion. A ship with a unity Response Amplitude Operator (RAO) is assumed without any loss of generality.

A sea wave model is derived using an energy spectrum and the ship deck is assumed to perform heave motion in sync with the wave. The sea wave elevation is approximated into a Fourier series consisting of sinusoidal waves at different energy content (amplitude), frequency and phase [13, 90].

The energy content at a particular sea state depends on a wave height spectrum. The modified Pierson-Moskowitz (PM) spectrum is used in this simulation. The parameters used in the modified PM spectrum are the significant wave height and period which can be obtained from the sea state, the modified PM spectrum is derived as

$$S(\omega) = H_{1/3}^2 T_1 \frac{0.11}{2\pi} \left(\frac{\omega T_1}{2\pi} \right)^{-5} \exp \left[-0.44 \left(\frac{\omega T_1}{2\pi} \right)^{-4} \right], \quad (3.24)$$

where $S(\omega)$ is the wave height spectrum, $H_{1/3}$ is the significant wave height, T_1 is the mean wave period and ω is the frequency. $H_{1/3}$ and T_1 is dependent on the steady wind speed and sea state.

Figure 3.4 shows the modified PM spectrum generated for sea state 6 at a constant wind speed of 30kts.

Figure 3.4: The modified Pierson-Moskowitz spectrum at sea state 6 and a design wind speed of 30kts = 15.4m/s.

Based on the wave height spectrum, a discretization can be performed to obtain a time series representation for the wave elevation [13].

$$d = \sum_{j=1}^N \sqrt{2S(\omega_j)\Delta\omega_j} \sin(\omega_j t + \epsilon_j), \quad (3.25)$$

where d is the wave elevation, N is the number of waves modelled, $\Delta\omega_j$ is the discretization interval, ω_j and ϵ_j are the frequency and phase of the j^{th} wave respectively.

By specifying different ϵ_j for $j = 1, 2, \dots, N$ as the initial condition, different waveforms can be generated. The time series of the wave elevation are simulated for two different initial conditions at sea state 6 and a constant wind speed of 30kts and is shown in Figure 3.5 to highlight the pseudorandomness of the wave elevation at a discretization of $N = 500$.

This chapter presented the multi-rotor UAV model and environmental models used in the control law formulation and/or numerical simulations for the ship deck recovery of a multi-rotor UAV. The kinematic and dynamic model of the multi-rotor UAV highlights

3.5.a: Case I

3.5.b: Case II

Figure 3.5: Simulation of the ship deck motion at sea state 6 at a design wind speed of $30\text{kts} = 15.4\text{m/s}$ for two different initial conditions. The simulation shows that different waveforms are obtained by differing the initial conditions.

its hierarchical structure that justifies the use of backstepping control. The multi-rotor UAV heave dynamics and planar motion dynamics are also decoupled and it justifies the two sub-problems approach. Finally, environmental models such as the gravity model, aerodynamic model, wind disturbance model and ship heave model are also described.

Chapter 4

Multi-Rotor UAV Ground Path Tracking

A multi-rotor UAV kinematic and dynamic model is presented in Chapter 3. A cascaded structure is observed in the multi-rotor UAV and it is similar to a quadruple integrator system. This motivates the use of backstepping control structure that is discussed in Chapter 2. Because the multi-rotor UAV kinematic and dynamic model is not in a *cascaded-feedback form* (see Section 4.2.1.1, p.68), inverse kinematics is used in most backstepping control formulation. Complexity and robustness issues arise due to the use of inverse kinematics is mentioned in Chapter 2.

In this chapter, a backstepping control structure is applied to a multi-rotor UAV to achieve precise ground path tracking using a re-formulated kinematic and dynamic model to eliminate the use of inverse kinematics and its associated disadvantage. This particular kinematic and dynamic model is formulated from the full kinematic and dynamic model presented in Chapter 3 (see Table 3.1) to achieve acceleration control in a nonlinear affine form. This re-formulation also transforms the kinematics and dynamics of the system into a cascaded-feedback form that is required for the application of full state backstepping control.

In addition to the re-formulation of the multi-rotor UAV kinematics and dynamics, this chapter also introduces a closed-form backstepping control law derived using contrac-

tion theory. This closed-form control law allows simple and straightforward application of backstepping control to any system in a cascaded-feedback form. Linear tracking differentiators inspired by Zuo [113] are also used to provide simplified expressions on the derivatives of commanded signals thus further simplifies the closed-form control law.

Lastly, the approach in re-formulating the multi-rotor UAV kinematics and dynamics is extended to quaternion-based kinematics and dynamics. The application of backstepping control to the re-formulated quaternion-based kinematics and dynamics further allows for a novel full nonlinear quaternion-based control that expands the operation domain of the UAV beyond the linear region.

In summary, this chapter addresses the following issues:

- the need for a closed-form backstepping rule,
- the complexity associated with inverse kinematics,
- and a restricted operation domain for multi-rotor UAV attitude control.

This chapter is organized as follows: Section 4.1 provides the reader with a brief understanding of contraction theory that is useful for the derivation of the contraction theory-based backstepping control. Section 4.2 then presents some preliminaries about cascaded-feedback systems and backstepping control, followed by the formulation of a closed-form backstepping control law using contraction theory. This closed-form control law is then applied to the multi-rotor UAV tracking control sub-problem in Section 4.3 using both inverse kinematics and re-formulated kinematics and dynamics for comparison. Both an Euler angle-based and a quaternion-based re-formulated kinematics and dynamics are given. Numerical simulations are provided in Section 4.4 to provide Monte-Carlo simulations under various scenarios. Some concluding remarks and significance of the results are given in Section 4.5.

4.1 Introduction to Contraction Theory

The contraction theory, developed by Slotine *et. al.* [43, 53, 54], is a stability analysis method that considers infinitesimal changes between state trajectories using the concept of virtual displacements described in a differential framework. Stability in the sense of contraction implies that any two trajectories will converge onto each other over time.

Backstepping control is achieved using a state variable to control another state variable up the cascade recursively, this state variable is known as the virtual control. This requires the convergence of states to their respective reference trajectories. Contraction theory is therefore well-suited for backstepping tracking control as stability in the sense of contraction implies the convergence between the state variable and the desired trajectory.

In this section, the fundamentals of contraction theory that is essential to the formulation of backstepping control is provided. Interested readers can refer to [53] for more details on contraction theory.

4.1.1 Fundamentals of Contraction Theory

Consider a nonlinear system given by $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, where $\mathbf{x} \in \mathbb{R}^n$ is the state of the system at time t . Using the concept of virtual displacement (see Figure 4.1), its virtual dynamics is written as

$$\delta \dot{\mathbf{x}} = \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}} \delta \mathbf{x}, \quad (4.1)$$

where $\frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}}$ is known as the Jacobian of the system.

Remark 4.1. It is noted that the notation $\delta \mathbf{x}$ is a well-defined mathematical object and denotes the virtual displacement of the state trajectory $\mathbf{x}(t)$. It is an infinitesimal change in the state $\mathbf{x}$ at a given instant t . The displacement is called virtual to distinguish it from actual displacement $\frac{d}{dt}\mathbf{x}(t)$ which occurs in time interval dt .

Figure 4.1: Two state trajectories are shown in the state space. The virtual displacement $\delta \mathbf{x}$ is the infinitesimal change in states at time t . The rate of change of virtual displacement $\delta \dot{\mathbf{x}}$ is also shown.

The time variation of the “size” of the virtual displacement $\delta \mathbf{x}^T \delta \mathbf{x}$ is computed as

$$\frac{d}{dt} (\delta \mathbf{x}^T \delta \mathbf{x}) = \delta \mathbf{x}^T \left(\frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}}^T + \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}} \right) \delta \mathbf{x}, \quad (4.2)$$

where $\delta \mathbf{x}^T \delta \mathbf{x}$ physically indicates the squared distance between infinitesimally close trajectories at a fixed time t .

If the Jacobian $\left(\frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}}^T + \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}} \right)$ is uniformly negative definite, then the distance between trajectories decreases exponentially to zero. This implies trajectories converge into each other, hence the trajectories are stable. A region in the state space where the Jacobian is uniformly negative definite is called a *contraction region*. The fundamental theorem of contraction theory is summarized below.

Theorem 4.1. [53, p.10] *Given the system equation, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, any trajectory in the state space which starts in a ball of constant radius centred about a given trajectory and contained at all times in a region satisfying $\left(\frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}}^T + \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}} \right) < 0^1$ (i.e. the contraction region), remains in the ball and converges exponentially to that given trajectory.*

¹Note that $<$ in this case refers to matrix negative definiteness.

Note that $A > 0$ for any symmetric matrix (i.e. $A = A^T$) refers to matrix positive definiteness. Furthermore if A is not symmetric, $A > 0$ refers to $A + A^T > 0$. This notation is used for the rest of this chapter.

Stability in terms of a contracting system is a stronger form of stability than Lyapunov stability. Therefore a trajectory in the contraction region exhibits additional useful properties. In particular, all trajectories are contained within a ball of a bounded radius centered about a given trajectory under the influence of bounded disturbances. The proof can be found in [53]. This property is critical to the application of contraction theory to backstepping control because instances where a state variable has not converged onto its desired trajectory can be regarded as an additional influence from a bounded disturbance (see Section 4.2.2.1). This results in a bounded transient response, thus ensuring the eventual convergence of the closed-loop system when the “disturbances” diminish. An upper bound of this transient response can be estimated according to the following theorem.

Theorem 4.2. [53, p.22] *Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ in a contraction region, and a perturbed system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{d}(\mathbf{x}, t)$. The largest eigenvalue of the symmetric part of the Jacobian of $\mathbf{f}(\mathbf{x}, t)$ is $\lambda_{\max} < -\beta \leq 0$ and the perturbation is bounded by $0 < \|\mathbf{d}\| < (|\lambda_{\max}| - \beta)R + d$ for some $d > 0$. Then the perturbed system will converge to within a boundary ball of radius R governed by $\dot{R} + \beta R \leq d$.*

Theorem 4.2 provides an upper bound to a ball for which all trajectories of the perturbed system converge given the size of the bounded disturbance. It also suggests that in the case of an exponentially decreasing disturbance, the ball also decreases exponentially to zero.

4.1.2 Partial Contraction

Partial contraction stems from a more general contraction theory. It is commonly used in the synchronization of coupled systems [8, 60, 92]. In the formulation of backstepping

control law where the state variables are to be synchronized with virtual controls, partial contraction can be used.

Consider an auxiliary system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{x}, t)$, where $\mathbf{y}(t) \in \mathbb{R}^n$, and $\mathbf{x}(t) \in \mathbb{R}^n$ corresponds to the state trajectory of the nonlinear system defined later and t is the time. This auxiliary system is contracting² with respect to $\mathbf{y}$. Furthermore this auxiliary system is chosen such that a given nonlinear system can be expressed as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}, t)$, where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state of the system. Since $\mathbf{x}(t)$ is a particular solution of the auxiliary system when $\mathbf{y}(t) = \mathbf{x}(t)$, then the contraction nature of the auxiliary system guarantees that the all solution state trajectories of $\mathbf{y}(t)$ converges to $\mathbf{x}(t)$ exponentially. Furthermore, if $\mathbf{y}^*(t)$ is another particular solution of the auxiliary system, then $\mathbf{y}(t)$ also converges to $\mathbf{y}^*(t)$ exponentially. Since $\mathbf{y}(t)$ converges to both $\mathbf{x}(t)$ and $\mathbf{y}^*(t)$ exponentially, then $\mathbf{x}(t)$ converges to $\mathbf{y}^*(t)$ exponentially. If $\mathbf{y}^*(t)$ exhibits certain desirable property, then $\mathbf{x}(t)$ exhibits this property exponentially. This property can be a certain relationship between variables or a particular trajectory. The nonlinear system is said to be partially contracting. Partial contraction is formally stated below.

Theorem 4.3. [92] *Given an auxiliary system, $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{x}, t)$, and a system equation, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}, t)$. If the auxiliary system is contracting with respect with $\mathbf{y}$ and a particular solution of the auxiliary system verifies certain specific property, then all the trajectories of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}, t)$ verify that specific property exponentially.*

Remark 4.2. In general, if the desired state and the state trajectories are both solutions to a contracting auxiliary system, then the state variable will converge to the virtual control exponentially. This property is used in the next section during the derivation of the contraction theory-based backstepping control.

²The contraction of a system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{x}, t)$ with respect to $\mathbf{y}$ means that the Jacobian is uniformly negative definite, $\left(\frac{\partial \mathbf{f}(\mathbf{y}, \mathbf{x}, t)}{\partial \mathbf{y}} + \frac{\partial \mathbf{f}(\mathbf{y}, \mathbf{x}, t)^T}{\partial \mathbf{y}} \right) < 0$.

4.2 Contraction Theory-Based Backstepping Control

A closed-form backstepping control law is formulated using contraction theory in this section. Contraction theory has been used with backstepping control in [74, 83, 106, 107, 108]. Prior to the main result on the contraction theory-based backstepping control law developed by the author, some preliminaries are given with respect to cascaded-feedback systems and backstepping control.

4.2.1 Preliminaries

4.2.1.1 Cascaded-Feedback Systems

To implement a backstepping control structure, a system has to be in a *cascaded-feedback form* where the state variables are arranged in a specific cascaded form³. A general system of equations in a cascaded-feedback affine form is written as

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, t) + \mathbf{g}_1(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, t)\mathbf{x}_2, \quad (4.3a)$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n, t) + \mathbf{g}_2(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n, t)\mathbf{x}_3, \quad (4.3b)$$

$$\vdots$$

$$\dot{\mathbf{x}}_{n-1} = \mathbf{f}_{n-1}(\mathbf{x}_{n-1}, \mathbf{x}_n, t) + \mathbf{g}_{n-1}(\mathbf{x}_{n-1}, \mathbf{x}_n, t)\mathbf{x}_n, \quad (4.3c)$$

$$\dot{\mathbf{x}}_n = \mathbf{f}_n(\mathbf{x}_n, t) + \mathbf{g}_n(\mathbf{x}_n, t)\mathbf{u}, \quad (4.3d)$$

where $\mathbf{x}_i \in \mathbb{R}^{n_i}$ for $i = 1, 2, \dots, n$ are the state variables, $\mathbf{u} \in \mathbb{R}^m$ is the control input and t is the time. The following assumptions are made on the system for the purpose of this thesis:

- (i) the vector-valued function $\mathbf{f}_i(\mathbf{0}) = \mathbf{0}$ hence there is an equilibrium point at the origin $\mathbf{x} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix}^T$,

³Most aircraft dynamics exhibit this form including the multi-rotor UAV dynamics used in this thesis (see Chapter 3).

- (ii) and the matrix-valued function $\mathbf{g}_i(\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n)$ is non-singular in the operation domain for $i = 1, 2, \dots, n$.

Transforming any general dynamics into this form often requires Lie derivatives and results in complicated expressions [22]. Nevertheless, this form of equation is common among UAVs such as fixed-wing and multi-rotor UAVs.

In the cascaded-feedback form, the influence of a state on another state is easily observed. For instance in (4.3), system state $\mathbf{x}_{i+1}$ exerts primary influence on system state $\mathbf{x}_i$ as seen from the system equation dictating the dynamics of $\mathbf{x}_i$ (the i^{th} subsystem⁴), where $\mathbf{x}_{i+1}$ can be commanded to control $\mathbf{x}_i$ due to the non-singularity of $\mathbf{g}_i(\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n, t)$.

4.2.1.2 Backstepping Control

Backstepping control provides a systematic way to derive control inputs so that the control objective is achieved. A commonly used Lyapunov approach is discussed in this subsection. This approach derives a control law for each virtual input such that the output of the i^{th} subsystem tracks the virtual input of the $(i - 1)^{th}$ subsystem. This repeats until the control law for the control input is defined. This is illustrated in the following example.

Consider a double integrator system,

$$\dot{x}_1 = x_2, \tag{4.4a}$$

$$\dot{x}_2 = u, \tag{4.4b}$$

where the system output x_1 is required to track a reference value x_1^* . Define a control Lyapunov function (CLF),

$$V_1 = \frac{1}{2}e_1^2, \tag{4.5}$$

⁴The i^{th} subsystem refers to $\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n, t) + \mathbf{g}_i(\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n, t)\mathbf{x}_{i+1}$.

where $e_1 = x_1 - x_1^*$ is the tracking error. Its time derivative is

$$\dot{V}_1 = e_1 (x_2 - \dot{x}_1^*), \quad (4.6)$$

thus defining the desired trajectory for x_2 as $x_2^* = \dot{x}_1^* - k_1 e_1$ where $k_1 > 0$, $\dot{V}_1|_{x_2=x_2^*} = -k_1 e_1^2 < 0$. Define a tracking error $e_2 = x_2 - x_2^* = x_2 - \dot{x}_1^* + k_1 e_1$, and a CLF

$$V_2 = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2. \quad (4.7)$$

Its time derivative is

$$\dot{V}_2 = e_1 (x_2 - \dot{x}_1^*) + e_2 (u - \ddot{x}_1^* + k_1 (x_2 - \dot{x}_1^*)) = e_1 (e_2 - k_1 e_1) + e_2 (u - \ddot{x}_1^* + k_1 e_2 - k_1^2 e_1). \quad (4.8)$$

Define a control law

$$u = \ddot{x}_1^* - k_2 e_2 + (k_1^2 - 1)e_1 = \ddot{x}_1^* - k_2 \dot{x}_1^* - k_2 x_2 + (k_1^2 - k_1 k_2 - 1)(x_1 - x_1^*), \quad (4.9)$$

where $k_2 > k_1$, $\dot{V}_2 = -k_1 e_1^2 - (k_2 - k_1) e_2^2 < 0$ hence tracking (i.e. $x_1 \rightarrow x_1^*$) is achieved. It is noted that the expression typically becomes complicated and extensive as the order of the system increases. A general approach to backstepping control can be inferred from this example.

Backstepping control for a n^{th} order system typically follows this approach:

- (i) Consider the $\mathbf{x}_1$ -subsystem, and formulate virtual control $\mathbf{x}_2^* = \mathbf{h}_1(\mathbf{z}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$ such that the $\mathbf{x}_1$ -subsystem is stable and $\mathbf{x}_1 \rightarrow \mathbf{x}_1^*$ where $\mathbf{x}_1^*$ is the reference trajectory for $\mathbf{x}_1$ and $\mathbf{z}_1 = \mathbf{x}_1 - \mathbf{x}_1^{*5}$.
- (ii) Consider the $\mathbf{x}_2$ -subsystem, and formulate virtual control $\mathbf{x}_3^* = \mathbf{h}_2(\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{x}_n)$ such that the combined closed-loop $\mathbf{x}_1$ - and $\mathbf{x}_2$ -subsystems are stable and $\mathbf{x}_2 \rightarrow \mathbf{x}_2^*$ where $\mathbf{z}_2 = \mathbf{x}_2 - \mathbf{x}_2^*$. The convergence of $\mathbf{x}_1 \rightarrow \mathbf{x}_1^*$ is implied from previous step.

⁵Other forms of $\mathbf{z}_1$ are also possible as long as $\mathbf{x}_1 \rightarrow \mathbf{x}_1^*$.

(iii) Repeat the above steps until the control law $\mathbf{u} = \mathbf{h}_n(\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n)$ is formulated.

The key advantage of backstepping control is its systematic approach to formulate control law for a class of cascaded systems. However, expressions tend to explode for high order systems [113] and derivation becomes tedious. In the next subsection, a closed-form expression for a backstepping control system is derived using contraction theory. This limits the explosion of term as compared to Lyapunov-based approach.

4.2.2 Main Result

The previous subsection presents the general approach to backstepping control. In backstepping control, $\mathbf{x}_j$ for $j = 2, 3, \dots, n$ are used as the virtual control for $\mathbf{x}_{j-1}$ and the control input $\mathbf{u}$ is used to control $\mathbf{x}_n$. In this way, the control law for each virtual control and the control input is formulated in a systematic way and the control of $\mathbf{x}_1$, which is often the output, is achieved.

Consider the two-subsystem case,

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2, t) + \mathbf{g}_1(\mathbf{x}_1, \mathbf{x}_2, t)\mathbf{x}_2, \quad (4.10a)$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2, t) + \mathbf{g}_2(\mathbf{x}_2, t)\mathbf{u}. \quad (4.10b)$$

Assume that $\mathbf{g}_i$ for $i = 1, 2$ are non-singular, hence dynamic inversion can be used to obtain the control law. Furthermore $\mathbf{f}_i(\mathbf{0}, \mathbf{0}, t) = \mathbf{0}$ for all $t > 0$ so that the state $\mathbf{x}_1 = \mathbf{0}$ and $\mathbf{x}_2 = \mathbf{0}$ is a system equilibrium state.

Contraction theory is used to ensure precise tracking of the virtual control by the corresponding state variable. In the cascaded-feedback form (4.10), $\mathbf{x}_2$ is used as a virtual control to ensure that $\mathbf{x}_1$ tracks $\mathbf{x}_1^*$. It is chosen as,

$$\mathbf{x}_2^* = \mathbf{g}_1^{-1}(\mathbf{x}_1, \mathbf{x}_2, t)[-\mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2, t) + \dot{\mathbf{x}}_1^* - K_1(\mathbf{x}_1 - \mathbf{x}_1^*)], \quad (4.11)$$

where $\mathbf{x}_1^*(t)$ is the desired trajectory of $\mathbf{x}_1$, $\mathbf{x}_2^*(t)$ is the desired trajectory of $\mathbf{x}_2$ and $K_1 > 0^6$ is the control gain.

The closed-loop system when $\mathbf{x}_2 = \mathbf{x}_2^*$ is

$$\dot{\mathbf{x}}_1 = \dot{\mathbf{x}}_1^* - K_1(\mathbf{x}_1 - \mathbf{x}_1^*). \quad (4.12)$$

To proof the stability of the closed-loop system (4.12), consider the following auxiliary system,

$$\dot{\mathbf{y}}_1 = \dot{\mathbf{x}}_1^* - K_1(\mathbf{y}_1 - \mathbf{x}_1^*), \quad (4.13)$$

where $\mathbf{y}_1$ is the auxiliary state variable. It is noted that the auxiliary system is contracting with respect to $\mathbf{y}_1$ as $K_1 > 0$. Furthermore, it is obvious that $\mathbf{y}_1(t) = \mathbf{x}_1^*(t)$ and $\mathbf{y}_1(t) = \mathbf{x}_1(t)$ are both solutions of the auxiliary system. Hence all trajectories of the closed-loop system (4.12) approach the desired trajectory $\mathbf{x}_1^*(t)$ exponentially and tracking is achieved exponentially if tracking of $\mathbf{x}_2^*$ by $\mathbf{x}_2$ is achieved.

Therefore if $\|\mathbf{x}_2 - \mathbf{x}_2^*\|$ converges to zero, then the tracking of the state $\mathbf{x}_1$ is exponentially achieved. Furthermore, the contracting nature of the system ensure that if $\|\mathbf{x}_2 - \mathbf{x}_2^*\|$ is bounded, then $\|\mathbf{x}_1 - \mathbf{x}_1^*\|$ is also bounded. Hence transient performance is guaranteed.

Next, the exponential convergence of $\mathbf{x}_2$ to $\mathbf{x}_2^*$ needs to be ensured. A similar procedure and analysis is used to formulate the control input $\mathbf{u}$.

The control input $\mathbf{u}$ is proposed as

$$\mathbf{u} = \mathbf{g}_2^{-1}(\mathbf{x}_2, t)[- \mathbf{f}_2(\mathbf{x}_2, t) + \dot{\mathbf{x}}_2^* - K_2(\mathbf{x}_2 - \mathbf{x}_2^*)], \quad (4.14)$$

where $\mathbf{x}_2^*(t)$ is the desired trajectory of $\mathbf{x}_2$ and $K_2 > 0$ is the control gain.

The closed-loop system becomes

$$\dot{\mathbf{x}}_2 = \dot{\mathbf{x}}_2^* - K_2(\mathbf{x}_2 - \mathbf{x}_2^*). \quad (4.15)$$

⁶For a non-symmetric matrix K_1 , $K_1 > 0 \Rightarrow K_1^T + K_1 > 0$.

To proof the stability of the closed-loop system (4.15), consider the following auxiliary system,

$$\dot{\mathbf{y}}_2 = \dot{\mathbf{x}}_2^* - K_2(\mathbf{y}_2 - \mathbf{x}_2^*), \quad (4.16)$$

where $\mathbf{y}_2$ is the auxiliary state variable. The auxiliary system is contracting with respect to $\mathbf{y}_2$ as $K_2 > 0$. Furthermore, it is obvious that $\mathbf{y}_2(t) = \mathbf{x}_2^*(t)$ and $\mathbf{y}_2(t) = \mathbf{x}_2(t)$ are both solutions of the auxiliary system. Hence all trajectories of the closed-loop system (4.15) approach the desired trajectory $\mathbf{x}_2^*(t)$ exponentially⁷. Convergence of $\mathbf{x}_2$ to $\mathbf{x}_2^*$, hence the tracking of $\mathbf{x}_1$ to $\mathbf{x}_1^*$ is ensured.

For the system given by (4.10), the control law given by (4.11) and (4.14) is thus able to ensure the tracking of the state variable $\mathbf{x}_1$.

The backstepping control of the two-subsystem case above can be generalized into a n -subsystem case. Theorem 4.4 states the formulation of the control law.

Theorem 4.4. *For a system given by (4.3), the following backstepping control law results in the exponential convergence of the state variables $\mathbf{x}_j$ to their desired trajectories $\mathbf{x}_j^*$ for $j = 1, 2, \dots, n$.*

$$\mathbf{x}_2^* = \mathbf{g}_1^{-1}[-\mathbf{f}_1 + \dot{\mathbf{x}}_1^* - K_1(\mathbf{x}_1 - \mathbf{x}_1^*)], \quad (4.17a)$$

$$\mathbf{x}_3^* = \mathbf{g}_2^{-1}[-\mathbf{f}_2 + \dot{\mathbf{x}}_2^* - K_2(\mathbf{x}_2 - \mathbf{x}_2^*)], \quad (4.17b)$$

$$\vdots$$

$$\mathbf{x}_n^* = \mathbf{g}_{n-1}^{-1}[-\mathbf{f}_{n-1} + \dot{\mathbf{x}}_{n-1}^* - K_{n-1}(\mathbf{x}_{n-1} - \mathbf{x}_{n-1}^*)], \quad (4.17c)$$

$$\mathbf{u} = \mathbf{g}_n^{-1}[-\mathbf{f}_n + \dot{\mathbf{x}}_n^* - K_n(\mathbf{x}_n - \mathbf{x}_n^*)], \quad (4.17d)$$

where $K_i > 0$ for $i = 1, 2, \dots, n$.

⁷Since $\|\mathbf{x}_2 - \mathbf{x}_2^*\|$ is exponentially converging to zero, $\|\mathbf{x}_1 - \mathbf{x}_1^*\|$ is therefore bounded.

Proof. Consider the closed-loop system

$$\dot{\mathbf{z}}_1 = -K_1 \mathbf{z}_1 + \mathbf{g}_1(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, t) \mathbf{z}_2, \quad (4.18a)$$

$$\dot{\mathbf{z}}_2 = -K_2 \mathbf{z}_2 + \mathbf{g}_2(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n, t) \mathbf{z}_3, \quad (4.18b)$$

$$\vdots$$

$$\dot{\mathbf{z}}_{n-1} = -K_{n-1} \mathbf{z}_{n-1} + \mathbf{g}_{n-1}(\mathbf{x}_{n-1}, \mathbf{x}_n, t) \mathbf{z}_n, \quad (4.18c)$$

$$\dot{\mathbf{z}}_n = -K_n \mathbf{z}_n, \quad (4.18d)$$

where $\mathbf{z}_i = \mathbf{x}_i - \mathbf{x}_i^*$ for $i = 1, 2, \dots, n$.

The condition $\mathbf{x}_{j+1} \rightarrow \mathbf{x}_{j+1}^* \Rightarrow \mathbf{x}_j \rightarrow \mathbf{x}_j^*$ is first established for $j = 1, 2, \dots, n-1$ with the closed-loop j^{th} -subsystem for $\mathbf{x}_{j+1} \rightarrow \mathbf{x}_{j+1}^*$ becoming

$$\dot{\mathbf{x}}_j = \dot{\mathbf{x}}_j^* - K_j(\mathbf{x}_j - \mathbf{x}_j^*). \quad (4.19)$$

The convergence of the $\mathbf{x}_j$ to its desired trajectory is proven by considering the auxiliary system,

$$\dot{\mathbf{y}}_j = \dot{\mathbf{x}}_j^* - K_j(\mathbf{y}_j - \mathbf{x}_j^*), \quad (4.20)$$

where $\mathbf{y}_j$ is the auxiliary state variable, $\mathbf{x}_j^*$ is the desired trajectory for $\mathbf{x}_j$ and $K_j > 0$ is the control gain for $j = 1, 2, \dots, n$. The auxiliary system is contracting with respect to $\mathbf{y}_j$ as $K_j > 0$. Furthermore, it is obvious that $\mathbf{y}_j(t) = \mathbf{x}_j^*(t)$ and $\mathbf{y}_j(t) = \mathbf{x}_j(t)$ are both solutions of the auxiliary system. Hence all possible trajectories of $\mathbf{x}_j$ converges exponentially to the reference trajectory $\mathbf{x}_j^*(t)$.

Therefore the following can be concluded for the closed-loop j^{th} -subsystem, assuming that $\mathbf{x}_{j+1} \rightarrow \mathbf{x}_{j+1}^*$ or $\|\mathbf{z}_{j+1}\| < \epsilon$ for arbitrarily small $\epsilon > 0$ and $t > t_{j+1} > 0$ for all $j = 1, 2, \dots, n-1$. Then there exists a time $t_j > t_{j+1} > 0$ such that $\mathbf{x}_j \rightarrow \mathbf{x}_j^*$ or $\|\mathbf{z}_j\| < \epsilon$ for arbitrarily small $\epsilon > 0$. In addition for time $t < t_j$, the closed-loop j^{th} -subsystem is influenced by a ‘‘perturbation’’ term $\mathbf{g}_j(\mathbf{x}_j, \mathbf{x}_{j+1}, \dots, \mathbf{x}_n, t) \mathbf{z}_{j+1}$ which exponentially

diminishes to zero with $\mathbf{z}_{j+1}$. For a contracting system, this implies that the transient response during $t < t_j$ is bounded. Hence it is proven that the j^{th} -subsystem is stable (i.e. $\mathbf{x}_j \rightarrow \mathbf{x}_j^*$) if the $(j + 1)^{\text{th}}$ -subsystem is also stable (i.e. $\mathbf{x}_{j+1} \rightarrow \mathbf{x}_{j+1}^*$).

To apply mathematical induction, the condition $\mathbf{x}_n \rightarrow \mathbf{x}_n^*$ is established with the closed-loop n^{th} -subsystem written as

$$\dot{\mathbf{x}}_n = \dot{\mathbf{x}}_n^* - K_n(\mathbf{x}_n - \mathbf{x}_n^*). \quad (4.21)$$

The convergence of the $\mathbf{x}_n$ to its desired trajectory is proven by considering the auxiliary system,

$$\dot{\mathbf{y}}_n = \dot{\mathbf{x}}_n^* - K_j(\mathbf{y}_n - \mathbf{x}_n^*), \quad (4.22)$$

where $\mathbf{y}_n$ is the auxiliary state variable, $\mathbf{x}_n^*$ is the desired trajectory for $\mathbf{x}_n$ and $K_n > 0$ is the control gain. The auxiliary system is contracting with respect to $\mathbf{y}_n$ as $K_n > 0$. Furthermore, it is obvious that $\mathbf{y}_n(t) = \mathbf{x}_n^*(t)$ and $\mathbf{y}_n(t) = \mathbf{x}_n(t)$ are both solutions of the auxiliary system. Hence all possible trajectories of $\mathbf{x}_n$ converges exponentially to $\mathbf{x}_n^*(t)$. Therefore, there exist a time $t_n > 0$ such that $\mathbf{x}_n \rightarrow \mathbf{x}_n^*$ or $\|\mathbf{z}_n\| < \epsilon$ for arbitrarily small $\epsilon > 0$.

Iteratively for the closed-loop $(n - 1)^{\text{th}}$ -subsystem, there also exists a time $t_{n-1} > t_n > 0$ such that $\mathbf{x}_{n-1} \rightarrow \mathbf{x}_{n-1}^*$ or $\|\mathbf{z}_{n-1}\| < \epsilon$ for arbitrarily small $\epsilon > 0$. For time $t < t_{n-1}$, the closed-loop $(n - 1)^{\text{th}}$ -subsystem is influenced by a “perturbation” term $\mathbf{g}_{n-1}(\mathbf{x}_{n-1}, \mathbf{x}_n, t)\mathbf{z}_n$ which exponentially diminishes to zero with $\mathbf{z}_n$. For a contracting system, the transient response during $t < t_{n-1}$ is bounded for this bounded disturbance term.

Applying mathematical induction by iteratively performing the same arguments for the closed-loop $(n - 2)^{\text{th}}$ -, $(n - 3)^{\text{th}}$ -, $\dots$, 1^{st} -subsystems, the tracking of $\mathbf{x}_1 \rightarrow \mathbf{x}_1^*$ can be concluded. Hence this theorem is proven by mathematical induction. $\square$

Theorem 4.4 provides a systematic way to formulate a control law for a system in the cascaded-feedback form and yields a simple expression for the control law. This is used to formulate a position tracking controller for a multi-rotor UAV in the next section.

4.2.2.1 Bounded Disturbance Response

An advantage of this formulation that is associated with the contraction property of the closed-loop system is the boundedness of the system state response in the presence of bounded external disturbances. This property also allows a quantitative estimation of the system state response bound using an the disturbance bound. It is stated in the following theorem.

Theorem 4.5. *For a system given by*

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, t) + \mathbf{g}_1(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, t)\mathbf{x}_2 + \mathbf{d}_1, \quad (4.23a)$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n, t) + \mathbf{g}_2(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n, t)\mathbf{x}_3 + \mathbf{d}_2, \quad (4.23b)$$

$$\vdots$$

$$\dot{\mathbf{x}}_{n-1} = \mathbf{f}_{n-1}(\mathbf{x}_{n-1}, \mathbf{x}_n, t) + \mathbf{g}_{n-1}(\mathbf{x}_{n-1}, \mathbf{x}_n, t)\mathbf{x}_n + \mathbf{d}_{n-1}, \quad (4.23c)$$

$$\dot{\mathbf{x}}_n = \mathbf{f}_n(\mathbf{x}_n, t) + \mathbf{g}_n(\mathbf{x}_n, t)\mathbf{u} + \mathbf{d}_n, \quad (4.23d)$$

and a control law given by (4.17), the state $\mathbf{x}_i$ and the desired state $\mathbf{x}_i^*$ remain within a ball of radius R_i governed by $\dot{R}_i + |\lambda_{\max,i}|R_i \leq L_i(R_i, R_{i+1}, \dots, R_n, t)R_{i+1} + d_i$ for $i = 1, 2, \dots, n-1$ and $\dot{R}_n + |\lambda_{\max,n}|R_n \leq d_n$, where $\|\mathbf{g}_i(\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n, t)\| < L_i(R_i, R_{i+1}, \dots, R_n, t)$, $\|\mathbf{d}_i\| < d_i$ and $\lambda_{\max,i}$ is the largest eigenvalue of $K_i + K_i^T$.

Proof. Observe the closed-loop system,

$$\dot{\mathbf{z}}_1 = -K_1\mathbf{z}_1 + \mathbf{g}_1(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, t)\mathbf{z}_2 + \mathbf{d}_1, \quad (4.24a)$$

$$\dot{\mathbf{z}}_2 = -K_2\mathbf{z}_2 + \mathbf{g}_2(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n, t)\mathbf{z}_3 + \mathbf{d}_2, \quad (4.24b)$$

$$\vdots$$

$$\dot{\mathbf{z}}_{n-1} = -K_{n-1}\mathbf{z}_{n-1} + \mathbf{g}_{n-1}(\mathbf{x}_{n-1}, \mathbf{x}_n, t)\mathbf{z}_n + \mathbf{d}_{n-1}, \quad (4.24c)$$

$$\dot{\mathbf{z}}_n = -K_n\mathbf{z}_n + \mathbf{d}_n, \quad (4.24d)$$

where $\mathbf{z}_i = \mathbf{x}_i - \mathbf{x}_i^*$ for $i = 1, 2, \dots, n$.

The disturbance term for the i^{th} subsystem is $\mathbf{g}_i(\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n, t)\mathbf{z}_{i+1} + \mathbf{d}_i$. An estimate for the bound of the first term of the disturbance depends strongly on the nonlinear function $\mathbf{g}_i(\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n, t)$. This is approximated by $\|\mathbf{g}_i(\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n, t)\mathbf{z}_{i+1}\| < L_i(R_i, R_{i+1}, \dots, R_n, t)R_{i+1}$, where $L_i(R_i, R_{i+1}, \dots, R_n, t) \geq \|\mathbf{g}_i(\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n, t)\|$ and remains bounded if R_j for $j \geq i$ are bounded. This term is caused by the difference between the state $\mathbf{x}_{i+1}$ and the desired state $\mathbf{x}_{i+1}^*$, and is expected to converge to zero in the absence of external disturbance $\mathbf{d}_i$. The external disturbance $\mathbf{d}_i$, on the other hand, is bounded by $\|\mathbf{d}_i\| < d_i$ therefore the bound for the two disturbances terms is conservatively estimated as $L_i(R_i, R_{i+1}, \dots, R_n, t)R_{i+1} + d_i$.

According to Theorem 4.2, the state $\mathbf{x}_i$ is bounded by a ball of radius R_i centered on the desired state $\mathbf{x}_i^*$ that is governed by $\dot{R}_i + |\lambda_{\max,i}|R_i \leq L_i(R_i, R_{i+1}, \dots, R_n, t)R_{i+1} + d_i$ for $i = 1, 2, \dots, n-1$. For the n^{th} subsystem, this ball is governed by $\dot{R}_n + |\lambda_{\max,n}|R_n \leq d_n$. $\square$

This theorem provides a proof on the closed-loop stability of the system when subjected to external disturbances. Furthermore, it also shows a relationship between the performance bound R_i and the size of the disturbance d_i . In addition, increasing $|\lambda_{\max,i}|$ reduces the effect of disturbance by reducing the size of the performance bound R_i . Since $\lambda_{\max,i} < 0$ is the maximum eigenvalue of $K_i + K_i^T$, the proper design of control gain allows the reduction of the sensitivity to external disturbance. Note that larger K_i results in larger control action, therefore this tradeoff needs to be considered.

4.3 Multi-Rotor UAV Position Tracking Controller

The contraction-theory based backstepping control law developed in the previous section is a closed-form expression that can be easily implemented on any system in a cascaded-

feedback form. In this section, multi-rotor UAV position tracking controllers are formulated using the backstepping control law described in Theorem 4.4. Three different applications of Theorem 4.4 are given. It is firstly applied on the translational motion and rotational motion of a multi-rotor UAV separately then bridged by inverse kinematics. In another application, a novel full state feedback backstepping control is formulated by re-formulating the multi-rotor UAV kinematics and dynamics to a cascaded-feedback form and the contraction theory-based backstepping control law in Theorem 4.4 is applied to it. In this case, both an Euler angle-based and a quaternion-based formulation are presented.

4.3.1 Inverse Kinematics Backstepping

The inverse kinematics approach requires the separation of the multi-rotor UAV motion into translational and rotational parts. Backstepping control is formulated separately and a relationship between translational and rotational states is given by inverse kinematics.

Consider the translational subsystem,

$$\dot{\mathbf{p}}_{\text{planar}} = \mathbf{v}_{\text{planar}}, \quad (4.25a)$$

$$\dot{\mathbf{v}}_{\text{planar}} = \mathbf{a}_{\text{planar}}, \quad (4.25b)$$

where $\mathbf{p}_{\text{planar}} = \begin{pmatrix} X & Y \end{pmatrix}^T$, $\mathbf{v}_{\text{planar}} = \begin{pmatrix} \dot{X} & \dot{Y} \end{pmatrix}^T$ and $\mathbf{a}_{\text{planar}} = \begin{pmatrix} \ddot{X} & \ddot{Y} \end{pmatrix}^T$ denotes the planar position, velocity and acceleration of the multi-rotor UAV center of mass respectively.

This is a double integrator system therefore in a cascaded-feedback form, and the contraction theory-based backstepping control in Theorem 4.4 can be used to formulate a control law. In the translational subsystem, the planar acceleration is the input while the planar position is the output.

Application of the Theorem 4.4 yields,

$$\mathbf{v}_{\text{planar},d} = \dot{\mathbf{p}}_{\text{planar},d} - K_1 (\mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},d}), \quad (4.26a)$$

$$\mathbf{a}_{\text{planar},d} = \dot{\mathbf{v}}_{\text{planar},d} - K_2 (\mathbf{v}_{\text{planar}} - \mathbf{v}_{\text{planar},d}), \quad (4.26b)$$

where $K_1, K_2 > 0$ are the control gains, $\mathbf{p}_{\text{planar},d}$, $\mathbf{v}_{\text{planar},d}$ and $\mathbf{a}_{\text{planar},d}$ are the desired planar position, desired planar velocity and acceleration respectively. The closed-loop system is able to track the desired planar position $\mathbf{p}_{\text{planar},d}$ if the planar acceleration $\mathbf{a}_{\text{planar}}$ is able to follow the desired planar acceleration command $\mathbf{a}_{\text{planar},d}$. It is noted that this assumption breaks down when there exists unmodelled effects during the application of inverse kinematics and this is discussed in Section 4.4.

On the other hand, consider the rotational subsystem,

$$\dot{\Theta} = W\omega \quad (4.27a)$$

$$\dot{\omega} = u_2, \quad (4.27b)$$

that are obtained from (3.12) and (3.15), where $\Theta = \begin{pmatrix} \phi & \theta & \psi \end{pmatrix}^T$, $\omega = \begin{pmatrix} p & q & r \end{pmatrix}^T$ and W is non-singular for $\theta \neq \pm 90^\circ$ and is given in (3.12).

This system is also in a cascaded-feedback form, and the contraction theory-based backstepping control in Theorem 4.4 is used to formulate a control law. In the rotational subsystem, u_2 which is related to the control moments is the input while the Euler angles are the output.

Application of Theorem 4.4 yields,

$$\omega_d = W^{-1} \left(\dot{\Theta}_d - K_3 (\Theta - \Theta_d) \right), \quad (4.28a)$$

$$u_2 = \dot{\omega}_d - K_4 (\omega - \omega_d), \quad (4.28b)$$

where $K_3, K_4 > 0$ are the control gains and $W^{-1} = \begin{pmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & c\theta s\phi \\ 0 & -s\phi & c\phi c\theta \end{pmatrix}$. The closed-loop system is able to track the desired Euler angles Θ_d if the control input u_2 follows the control law given in (4.28).

Through the use of inverse kinematics, a relationship is obtained between the desired Euler angles and the desired acceleration therefore the planar position tracking for a

multi-rotor UAV can be achieved through control input u_2 . In addition, the yaw angle can also be controlled by the remaining degree of freedom in control⁸. The inverse kinematics is given by

$$\theta_d = -\arctan\left(\frac{a_{x,d}\cos\psi_d + a_{y,d}\sin\psi_d}{a_z - g}\right), \quad (4.29a)$$

$$\phi_d = \arcsin\left(\frac{a_{y,d}\cos\psi_d - a_{x,d}\sin\psi_d}{\sqrt{a_{x,d}^2 + a_{y,d}^2 + (g - a_{z,d})^2}}\right), \quad (4.29b)$$

where $a_{x,d}$, $a_{y,d}$ and $a_{z,d}$ are the desired acceleration components in the Earth fixed reference frame. The desired pitch θ_d and roll ϕ_d commands are derived from (4.29) given the desired planar acceleration $\mathbf{a}_{\text{planar},d} = \begin{pmatrix} a_{x,d} \\ a_{y,d} \end{pmatrix}$. The reference yaw ψ_d is obtained from sensor measurement while the vertical acceleration a_z is obtained from the attitude controller.

Combining (4.26), (4.28) and (4.29), the backstepping control law with inverse kinematics is

$$\mathbf{v}_{\text{planar},d} = \dot{\mathbf{p}}_{\text{planar},d} - K_1 (\mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},d}), \quad (4.30a)$$

$$\mathbf{a}_{\text{planar},d} = \dot{\mathbf{v}}_{\text{planar},d} - K_2 (\mathbf{v}_{\text{planar}} - \mathbf{v}_{\text{planar},d}), \quad (4.30b)$$

$$\theta_d = -\arctan\left(\frac{a_{x,d}\cos\psi_d + a_{y,d}\sin\psi_d}{a_z - g}\right), \quad (4.30c)$$

$$\phi_d = \arcsin\left(\frac{a_{y,d}\cos\psi_d - a_{x,d}\sin\psi_d}{\sqrt{a_{x,d}^2 + a_{y,d}^2 + (g - a_{z,d})^2}}\right), \quad (4.30d)$$

$$\omega_d = W^{-1} \left(\dot{\Theta}_d - K_3 (\Theta - \Theta_d) \right), \quad (4.30e)$$

$$u_2 = \dot{\omega}_d - K_4 (\omega - \omega_d), \quad (4.30f)$$

that tracks reference input $\mathbf{p}_{\text{planar},d}$ and ψ_d ⁹.

⁸The control input for the ground path tracking controller is the control moment which has three degrees of freedom.

⁹Treatment of the derivatives in the control law are given later.

Figure 4.2: The backstepping feedback cascade using inverse kinematics is shown in this diagram. The control moments achieve the desired rotation rates which controls the roll, pitch and roll angles. Inverse kinematics is then used to compute the roll and pitch angles required to achieve the desired planar acceleration. The planar acceleration then affects the planar velocity then the planar position, hence achieving planar position tracking.

The closed-form control law for the backstepping control of a multi-rotor UAV using inverse kinematics is given in (4.30). The feedback cascade is also shown in Figure 4.2. It is observed that the use of inverse kinematics decouples the control of the translational and rotational motion. However, inverse kinematics given in (4.29) is a form of algebraic inversion and is sensitive to model inaccuracies. Full state feedback control without the use of inverse kinematics is presented in the next subsection and it is expected to have better performance.

4.3.2 Euler Angle-Based Formulation

In this subsection, a position tracking controller for a multi-rotor UAV based on reformulated kinematics and dynamics is synthesized. The multi-rotor UAV is required to track the ground path of a moving target with the feedback information of the target position or relative position.

The altitude control is decoupled from the planar position and yaw control as shown in Chapter 3. The planar position refers to the X, Y position of the multi-rotor UAV which is the ground path of the multi-rotor UAV. This resolves the bilinear control issue¹⁰ as the thrust u_1 is used to control the altitude h and allows backstepping control to be applied to the planar position and yaw control separately.

¹⁰The bilinear control issue is due to the term $\frac{1}{m}R^T u_1 \hat{\mathbf{e}}_3$ in (3.9) such that u_1 and R both affect the acceleration. The selection of the control or virtual control for the acceleration is critical.

The backstepping cascade begins with the input of the desired planar position $\mathbf{p}_{\text{planar},d}$ information of the target, where the planar velocity $\mathbf{v}_{\text{planar}}$ and acceleration $\mathbf{a}_{\text{planar}}$ of the multi-rotor UAV are commanded to track the target. The rotation rate ω of the multi-rotor UAV is controlled to achieve the desired acceleration $\mathbf{a}_{\text{planar},d}$. This is the key difference between the use of inverse kinematics where Euler angles Θ are used to achieve the desired acceleration $\mathbf{a}_{\text{planar},d}$. Lastly the control moments u_2 are commanded to control the quadrotor rotation rate ω . This novel cascaded structure (see Figure 4.3) eliminates the use of inverse kinematics and provides a full cascaded-feedback model that improves the backstepping control performance.

Figure 4.3: The backstepping feedback hierarchy using full state control is shown in this diagram. The control moments achieves the desired rotation rates which controls the planar acceleration and yaw angle to their desired value. The planar acceleration then affects the planar velocity then the planar position, hence achieving planar position tracking.

The equations of motion are transformed into a cascaded-feedback form before applying backstepping control in Theorem 4.4.

The planar position and velocity kinematics of the multi-rotor UAV is

$$\dot{\mathbf{p}}_{\text{planar}} = \mathbf{v}_{\text{planar}}, \quad (4.31a)$$

$$\dot{\mathbf{v}}_{\text{planar}} = \mathbf{a}_{\text{planar}}. \quad (4.31b)$$

This is the same as to the kinematics applied to the inverse kinematics-based backstepping control in (4.25).

To control the acceleration of the multi-rotor UAV, the time derivative of acceleration or the jerk is obtained. By differentiating (3.9) with respect to time and using Coriolis' theorem¹¹, the acceleration dynamics becomes

$$\ddot{\mathbf{v}} = -\frac{1}{m}\dot{u}_1 R^T \hat{\mathbf{e}}_3 + \frac{1}{m}u_1 (R^T \hat{\mathbf{e}}_3) \times \omega. \quad (4.32)$$

The cross product term is transformed into a matrix product by the following relationship:

$\mathbf{a} \times \mathbf{b} = \mathbf{a}_\times \mathbf{b}$ where $\mathbf{a}_\times = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$. Then, (4.32) becomes

$$\ddot{\mathbf{v}} = -\frac{1}{m}\dot{u}_1 R^T \hat{\mathbf{e}}_3 + \frac{1}{m}u_1 (R^T \hat{\mathbf{e}}_3)_\times \omega, \quad (4.33)$$

where

$$(R^T \hat{\mathbf{e}}_3)_\times = \begin{pmatrix} 0 & -c\phi c\theta & -s\phi c\psi + c\phi s\theta s\psi \\ c\phi c\theta & 0 & -s\phi s\psi - c\phi s\theta c\psi \\ s\phi c\psi - c\phi s\theta s\psi & s\phi s\psi + c\phi s\theta c\psi & 0 \end{pmatrix}. \quad (4.34)$$

The planar jerk is given by

$$\dot{\mathbf{a}}_{\text{planar}} = -\frac{1}{m}E\dot{u}_1 R^T \hat{\mathbf{e}}_3 + \frac{1}{m}Eu_1 (R^T \hat{\mathbf{e}}_3)_\times \omega, \quad (4.35)$$

where $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ extracts the dynamics in the X, Y -axes, which is re-written as

$$\dot{\mathbf{a}}_{\text{planar}} = F_1 + G_1\omega, \quad (4.36)$$

where

$$F_1 = -\frac{1}{m}E\dot{u}_1 R^T \hat{\mathbf{e}}_3, \quad (4.37a)$$

$$G_1 = \frac{1}{m}Eu_1 (R^T \hat{\mathbf{e}}_3)_\times. \quad (4.37b)$$

¹¹Coriolis' theorem states that the time derivative of a vector $\mathbf{a}$ in a non-fixed frame r can be expressed as a time derivative of that vector in the non-fixed frame plus a component due to the rotation of the frame. $\frac{d\mathbf{a}}{dt} = \left. \frac{d\mathbf{a}}{dt} \right|_r + \Omega \times \mathbf{a}$, where Ω is the angular velocity of the frame r .

The dynamics in (4.36) shows that the rotation rates ω can be used to control the planar acceleration $\mathbf{a}_{\text{planar}}$ of the quadrotor. In addition, the rotation rates can also be used to control the yaw angle. From (3.12), the yaw kinematics is

$$\dot{\psi} = \frac{\sin \phi}{\cos \theta} q + \frac{\cos \phi}{\cos \theta} r, \quad (4.38a)$$

$$= G_{\text{yaw}} \omega, \quad (4.38b)$$

where

$$G_{\text{yaw}} = \begin{pmatrix} 0 & \sin \phi / \cos \theta & \cos \phi / \cos \theta \end{pmatrix}. \quad (4.39)$$

The planar acceleration dynamics (4.36) and the yaw kinematics (4.38b) are concatenated as follows

$$\frac{d}{dt} \begin{pmatrix} \mathbf{a}_{\text{planar}} \\ \psi \end{pmatrix} = \begin{pmatrix} F_1 \\ 0 \end{pmatrix} + \begin{pmatrix} G_1 \\ G_{\text{yaw}} \end{pmatrix} \omega, \quad (4.40)$$

as both subsystems use the same virtual control, ω . It is observed that $\text{rank} \begin{pmatrix} G_1 \\ G_{\text{yaw}} \end{pmatrix} = 3$

for all $|\phi| < \frac{\pi}{2}$, $|\theta| < \frac{\pi}{2}$ and $u_1 > 0$, therefore $\begin{pmatrix} G_1 \\ G_{\text{yaw}} \end{pmatrix}$ is always non-singular within these bounds.

Lastly, (3.15) completes the equations of motion that is used for the backstepping formulation given in Theorem 4.4.

The equations of motion of the multi-rotor UAV can now be written in the following 10th order form (less 2 order from the separation of altitude control)

$$\dot{\mathbf{p}}_{\text{planar}} = \mathbf{v}_{\text{planar}}, \quad (4.41a)$$

$$\dot{\mathbf{v}}_{\text{planar}} = \mathbf{a}_{\text{planar}}, \quad (4.41b)$$

$$\frac{d}{dt} \begin{pmatrix} \mathbf{a}_{\text{planar}} \\ \psi \end{pmatrix} = \begin{pmatrix} F_1 \\ 0 \end{pmatrix} + \begin{pmatrix} G_1 \\ G_{\text{yaw}} \end{pmatrix} \omega, \quad (4.41c)$$

$$\dot{\omega} = u_2, \quad (4.41d)$$

which is in the cascaded-feedback form and is ready for the application of the backstepping method.

Using Theorem 4.4, backstepping control is formulated as follows. Given the desired planar position trajectory $\mathbf{p}_{\text{planar},d}$ (desired multi-rotor UAV ground path), the desired planar velocity of the multi-rotor UAV is

$$\mathbf{v}_{\text{planar},d} = \dot{\mathbf{p}}_{\text{planar},d} - K_1 (\mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},d}), \quad (4.42)$$

where $\mathbf{p}_{\text{planar},d}$ is the target planar position which is also the desired multi-rotor UAV planar position and $K_1 > 0$ is a control gain.

The desired acceleration then becomes

$$\mathbf{a}_{\text{planar},d} = \dot{\mathbf{v}}_{\text{planar},d} - K_2 (\mathbf{v}_{\text{planar}} - \mathbf{v}_{\text{planar},d}), \quad (4.43)$$

where $K_2 > 0$ is a control gain.

The desired acceleration is concatenated with the desired yaw angle to form the desired planar acceleration $\mathbf{a}_{\text{planar},d}$ and yaw angle ψ_d . It is noted that the desired yaw angle can take any arbitrary user-defined value. The desired rotation rate becomes

$$\omega_d = \begin{pmatrix} G_1 \\ G_{\text{yaw}} \end{pmatrix}^{-1} \left(- \begin{pmatrix} F_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \dot{\mathbf{a}}_{\text{planar},d} \\ \dot{\psi}_d \end{pmatrix} - K_3 \left(\begin{pmatrix} \mathbf{a}_{\text{planar}} \\ \psi \end{pmatrix} - \begin{pmatrix} \mathbf{a}_{\text{planar},d} \\ \psi_d \end{pmatrix} \right) \right), \quad (4.44)$$

where $K_3 > 0$ is the control gain.

The commanded control input, u_2 , becomes

$$u_2 = \dot{\omega}_d - K_4 (\omega - \omega_d), \quad (4.45)$$

where $K_4 > 0$ is the control gain.

The equations given in (4.42) to (4.45) form the control law¹² for the position tracking controller to track the ground path of a moving target. Each backstepping step computes a virtual control trajectory that will result in the precise tracking of the target if the states track their respective virtual controls.

¹²Treatment of the derivatives in the control law are given later.

Remark 4.3. The control gains K_i for $i = 1, 2, 3, 4$ can be any positive definite matrix. There is no requirement for inner-loop to have a higher gain than an outer-loop. Therefore there is no time-scale restriction on the inner and outer-loop unlike in time-scale separation approaches.

With reference to Example 2.1 (p.42), the re-formulated kinematic and dynamic approach in this subsection can be understood as an introduction of a filter on the unmodelled effects as shown in the following example.

Example 4.1 (Simplified inverse kinematics (continued)). *With the re-formulated kinematics and dynamics approach, the system in the simplified inverse kinematics example becomes*

$$\ddot{x}_1 = \frac{dh}{dx_3} u, \quad (4.46)$$

while the actual dynamics with unmodelled effect is

$$\ddot{x}_1 = \frac{dh}{dx_3} u + \dot{d}. \quad (4.47)$$

If the unmodelled effect is a constant bias ($\dot{d} = 0$), then the control law derived is able to achieve exact tracking of x_1 .

The following remark highlight this point.

Remark 4.4. The re-formulated equations of motion (4.41) has a degree of 10 (less 2 degrees of freedom from separate altitude control) which is identical to the degree of freedom in the conventional equations of motion in (3.8), (3.9), (3.12) and (3.15) that are typically used in quadrotor controller designs. Furthermore order of control law proposed in (4.42) to (4.45) is 10. In view of this, no integral action is included in the proposed design yet it is shown that it is able to achieve zero steady state error in the presence of constant bias similar to integral control.

4.3.3 Quaternion-Based Formulation

The re-formulated kinematics and dynamics approach presented in the previous section can also be applied to a quaternion-based rotational kinematics. This results in a full state nonlinear feedback control with quaternion-based kinematics that widens the operation domain of the multi-rotor UAV. This subsection presents the results on quaternion-based backstepping position tracking control.

The control law is formulated using the contraction theory-based backstepping control stated in Theorem 4.4. The equations of motion given in Table 3.1 are transformed into a cascaded-feedback form for the application of backstepping control by considering the acceleration dynamics similar to the previous subsection. The backstepping cascade follows that of Figure 4.3.

The acceleration dynamics of the multi-rotor UAV is obtained from the time derivative of (3.9) similar to (4.35),

$$\dot{\mathbf{a}} = -\frac{\dot{u}_1}{m} R^T \hat{\mathbf{e}}_3 - \frac{u_1}{m} \dot{R}^T \hat{\mathbf{e}}_3. \quad (4.48)$$

Here, product rule for time derivative is used as opposed to the Coriolis' theorem used in (4.35).

The rotation matrix R is expressed in terms quaternions η as shown in (3.11). By taking its time derivative, the resulting expression obtained is linear in rotation rates p , q and r ,

$$\dot{R} = \dot{R}_p p + \dot{R}_q q + \dot{R}_r r, \quad (4.49)$$

where the coefficient matrices $\dot{R}_p$, $\dot{R}_q$ and $\dot{R}_r$ are functions of quaternions and are shown as

$$\dot{R}_p = \begin{pmatrix} 0 & 0 & 0 \\ 2\eta_0\eta_2 + 2\eta_1\eta_3 & 2\eta_2\eta_3 - 2\eta_0\eta_1 & \eta_0^2 - \eta_1^2 - \eta_2^2 + \eta_3^2 \\ 2\eta_0\eta_3 - 2\eta_1\eta_2 & -\eta_0^2 + \eta_1^2 - \eta_2^2 + \eta_3^2 & -2\eta_0\eta_1 - 2\eta_2\eta_3 \end{pmatrix}, \quad (4.50)$$

$$\dot{R}_q = \begin{pmatrix} -2\eta_0\eta_2 - 2\eta_1\eta_3 & 2\eta_0\eta_1 - 2\eta_2\eta_3 & -\eta_0^2 + \eta_1^2 + \eta_2^2 - \eta_3^2 \\ 0 & 0 & 0 \\ \eta_0^2 + \eta_1^2 - \eta_2^2 - \eta_3^2 & 2\eta_0\eta_3 + 2\eta_1\eta_2 & 2\eta_1\eta_3 - 2\eta_0\eta_2 \end{pmatrix}, \quad (4.51)$$

$$\dot{R}_r = \begin{pmatrix} 2\eta_1\eta_2 - 2\eta_0\eta_3 & \eta_0^2 - \eta_1^2 + \eta_2^2 - \eta_3^2 & 2\eta_0\eta_1 + 2\eta_2\eta_3 \\ -\eta_0^2 - \eta_1^2 + \eta_2^2 + \eta_3^2 & -2\eta_0\eta_3 - 2\eta_1\eta_2 & 2\eta_0\eta_2 - 2\eta_1\eta_3 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.52)$$

Hence (4.48) can be expressed as an affine function of the virtual inputs p , q , and r ,

$$\dot{\mathbf{a}} = -\frac{\dot{u}_1}{m} R^T \hat{\mathbf{e}}_3 - \frac{u_1}{m} \left(\dot{R}_p^T p + \dot{R}_q^T q + \dot{R}_r^T r \right) \hat{\mathbf{e}}_3. \quad (4.53)$$

which can be rewritten as

$$\dot{\mathbf{a}} = \bar{F}_2 + \bar{G}_2 \omega. \quad (4.54)$$

where

$$\bar{F}_2 = -\frac{\dot{u}_1}{m} R^T \hat{\mathbf{e}}_3, \quad (4.55a)$$

$$\bar{G}_2 = -\frac{u_1}{m} \begin{pmatrix} \dot{R}_p \hat{\mathbf{e}}_3 & \dot{R}_q \hat{\mathbf{e}}_3 & \dot{R}_r \hat{\mathbf{e}}_3 \end{pmatrix} \quad (4.55b)$$

with

$$\begin{pmatrix} \dot{R}_p \hat{\mathbf{e}}_3 & \dot{R}_q \hat{\mathbf{e}}_3 & \dot{R}_r \hat{\mathbf{e}}_3 \end{pmatrix} = \begin{pmatrix} 0 & -\eta_0^2 + \eta_1^2 + \eta_2^2 - \eta_3^2 & 2\eta_0\eta_1 + 2\eta_2\eta_3 \\ \eta_0^2 - \eta_1^2 - \eta_2^2 + \eta_3^2 & 0 & 2\eta_0\eta_2 - 2\eta_1\eta_3 \\ -2\eta_0\eta_1 - 2\eta_2\eta_3 & 2\eta_1\eta_3 - 2\eta_0\eta_2 & 0 \end{pmatrix}. \quad (4.56)$$

However, the altitude of the multi-rotor UAV is controlled separately as seen from the decoupled dynamics in Chapter 3. Hence the z component of (4.54) can be removed

by pre-multiplying with matrix $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. For simplicity, we let the planar com-

ponents of velocity be $\mathbf{v}_{\text{planar}} = \begin{pmatrix} \dot{X} & \dot{Y} \end{pmatrix}^T$, and $F_2 = E\bar{F}_2$ ¹³, $G_2 = E\bar{G}_2$. Then, (4.54)

¹³This is the same as F_1 in (4.37a), however it is denoted as F_2 to highlight the difference that it is computed with quaternions rather than Euler angles. The direction cosine matrix R given by quaternion is given as (3.11) while that given by Euler angles is given as (3.10).

becomes

$$\dot{\mathbf{a}}_{\text{planar}} = F_2 + \bar{G}_2 \omega. \quad (4.57)$$

Using $\omega = \begin{pmatrix} p & q & r \end{pmatrix}^T$ as a virtual control, there will be an additional degree of freedom available for control as there are only 2 components in the planar position required to be controlled. Therefore we introduce the yaw angle ψ as an additional controlled variable which prevents the quadrotor from spinning about its z -axis.

Furthermore, the yaw kinematics is the same as previously given (4.38a) or (4.38b).

Concatenating (4.57) and (4.38b),

$$\frac{d}{dt} \begin{pmatrix} \mathbf{a}_{\text{planar}} \\ \psi \end{pmatrix} = \begin{pmatrix} F_2 \\ 0 \end{pmatrix} + \begin{pmatrix} G_2 \\ G_{\text{yaw}} \end{pmatrix} \omega, \quad (4.58)$$

and noting that $\det \begin{pmatrix} G_2 \\ G_{\text{yaw}} \end{pmatrix} = \frac{u_1^2 \cos \phi}{\cos \theta} (\eta_0^2 - \eta_1^2 - \eta_2^2 + \eta_3^2)$, then $\begin{pmatrix} G_2 \\ G_{\text{yaw}} \end{pmatrix}$ is non-singular if and only if $\phi, \theta \neq \pm 90^\circ$ and $u_1 \neq 0$. Similar condition is established in Section 4.3.2.

Lastly, (3.15) completes the equations of motion that is used for the backstepping formulation given in Theorem 4.4.

The equations of motion using quaternions in the strict feedback form are therefore summarizes as

$$\dot{\mathbf{p}}_{\text{planar}} = \mathbf{v}_{\text{planar}}, \quad (4.59a)$$

$$\dot{\mathbf{v}}_{\text{planar}} = \mathbf{a}_{\text{planar}}, \quad (4.59b)$$

$$\frac{d}{dt} \begin{pmatrix} \mathbf{a}_{\text{planar}} \\ \psi \end{pmatrix} = \begin{pmatrix} F_2 \\ 0 \end{pmatrix} + \begin{pmatrix} G_2 \\ G_{\text{yaw}} \end{pmatrix} \omega, \quad (4.59c)$$

$$\dot{\omega} = u_2, \quad (4.59d)$$

which is in the cascaded-feedback form and is ready for the application of backstepping control. Note that (4.59) is almost identical to (4.41) except for the vector-valued function F_2 and matrix-valued function G_2 that arise due to the use of quaternions.

Now, the control law for the multi-rotor position tracking controller using quaternions can be written. By means of backstepping technique, each state variable is controlled to match their respective desired value (virtual control).

The control law for the desired value of planar velocity is

$$\mathbf{v}_{\text{planar},d} = \dot{\mathbf{p}}_{\text{planar},d} - K_1 (\mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},d}), \quad (4.60)$$

where $K_1 > 0$ is a control gain, $\mathbf{p}_{\text{planar}}$ is the measured planar position, $\mathbf{p}_{\text{planar},d}$ and $\mathbf{v}_{\text{planar},d}$ are the reference planar position and the desired velocity respectively.

The control law for the desired value of planar acceleration is

$$\mathbf{a}_{\text{planar},d} = \dot{\mathbf{v}}_{\text{planar},d} - K_2 (\mathbf{v}_{\text{planar}} - \mathbf{v}_{\text{planar},d}), \quad (4.61)$$

where $K_2 > 0$ is a control gain and $\mathbf{a}_{\text{planar},d}$ is the desired value for the planar acceleration.

From (4.58), the control law for the desired value of the rotation rate is

$$\omega_d = \begin{pmatrix} G_2 \\ G_{\text{yaw}} \end{pmatrix}^{-1} \left(- \begin{pmatrix} F_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \dot{\mathbf{a}}_{\text{planar},d} \\ \dot{\psi}_d \end{pmatrix} - K_3 \left(\begin{pmatrix} \mathbf{a}_{\text{planar}} \\ \psi \end{pmatrix} - \begin{pmatrix} \mathbf{a}_{\text{planar},d} \\ \psi_d \end{pmatrix} \right) \right), \quad (4.62)$$

where $K_3 > 0$ is a control gain and ψ_d is the reference yaw angle.

Finally, the control law for the control input is

$$u_2 = \dot{\omega}_d - K_4 (\omega - \omega_d), \quad (4.63)$$

where $K_4 > 0$ is a control gain.

Equations (4.60) to (4.63) gives the feedback control law for a quaternion-based dynamical system.

Furthermore, examining the closed-loop dynamics gives

$$\dot{\mathbf{z}}_1 = -K_1 \mathbf{z}_1 + \mathbf{z}_2, \quad (4.64a)$$

$$\dot{\mathbf{z}}_2 = -K_2 \mathbf{z}_2 + \mathbf{z}_3, \quad (4.64b)$$

$$\dot{\mathbf{z}}_4 = -K_3 \mathbf{z}_4 + G_4 \mathbf{z}_5, \quad (4.64c)$$

$$\dot{\mathbf{z}}_5 = -K_4 \mathbf{z}_5, \quad (4.64d)$$

where $\mathbf{z}_1 = \mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},d}$, $\mathbf{z}_2 = \mathbf{v}_{\text{planar}} - \mathbf{v}_{\text{planar},d}$, $\mathbf{z}_3 = \mathbf{a}_{\text{planar}} - \mathbf{a}_{\text{planar},d}$, $\mathbf{z}_4 = \begin{pmatrix} \mathbf{a}_{\text{planar}} \\ \psi \end{pmatrix} - \begin{pmatrix} \mathbf{a}_{\text{planar},d} \\ \psi_d \end{pmatrix} = \begin{pmatrix} \mathbf{z}_3 \\ \psi - \psi_d \end{pmatrix}$ and $\mathbf{z}_5 = \omega - \omega_d$.

It can be seen that $\mathbf{z}_i$ converges to zero as long as $\mathbf{z}_{i+1}$ converges to zero¹⁴ for $i = 1, 2, 3, 4$. Since $\mathbf{z}_5$ converges to zero, therefore $\mathbf{z}_4$ followed by $\mathbf{z}_3$, $\mathbf{z}_2$ and $\mathbf{z}_1$ also converges to zero. Furthermore, $\mathbf{z}_i$ are also bounded for $i = 1, 2, 3, 4, 5$ during the transient period.

Note that the same argument applies for the inverse kinematic-based and Euler angles-based backstepping controllers.

4.3.4 Linear Tracking Differentiator

It is evident that the above formulations require a significant amount of derivatives to be evaluated. Obtaining the exact expressions for the derivatives result in complex controllers, therefore a linear tracking differentiator is employed to extract the derivative from a filtered signal.

For instance, the signal $\mathbf{p}_{\text{planar},d}$ is passed through the following command filter to evaluate its derivative,

$$\dot{\mathbf{p}}_{\text{planar},\text{cmd}} = -\tau(\mathbf{p}_{\text{planar},\text{cmd}} - \mathbf{p}_{\text{planar},d}), \quad (4.65)$$

where $\mathbf{p}_{\text{planar},\text{cmd}}$ is the filtered signal and $\tau > 0$ is the command filter time constant.

The filtered signal $\mathbf{p}_{\text{planar},\text{cmd}}$ replaces the input signal $\mathbf{p}_{\text{planar},d}$ in the control law and is therefore named the command signal. Therefore the derivative required is obtained from (4.65). This eliminates the need to obtain an analytical expression of the derivative, thus simplifying the control law.

The command filter gain τ needs to be sufficiently large such that singular perturbation theory¹⁵ can be applied to show that $\mathbf{p}_{\text{planar},\text{cmd}} \simeq \mathbf{p}_{\text{planar},d}$.

¹⁴ $\mathbf{z}_3$ converges to zero with $\mathbf{z}_4$ not due to dynamics but the relationship $\mathbf{z}_4 = \begin{pmatrix} \mathbf{z}_3^T & \psi - \psi_d \end{pmatrix}^T$.

¹⁵Singular perturbation theory is the study of dynamics on multiple time scales [9].

For a demonstration of the concept of linear tracking differentiator, it is applied to the Euler angle-based control law in (4.42) to (4.45). It can also be applied to the inverse kinematics-based (4.30) and quaternion-based control laws (4.60) to (4.63). The Euler angle-based control law in (4.42) to (4.45) is re-written with the inclusion of command signal as

$$\mathbf{v}_{\text{planar},d} = \dot{\mathbf{p}}_{\text{planar},\text{cmd}} - K_1(\mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},\text{cmd}}), \quad (4.66a)$$

$$\mathbf{a}_{\text{planar},d} = \dot{\mathbf{v}}_{\text{planar},\text{cmd}} - K_2(\mathbf{v}_{\text{planar}} - \mathbf{v}_{\text{planar},\text{cmd}}), \quad (4.66b)$$

$$\omega_d = \begin{pmatrix} G_1 \\ G_{\text{yaw}} \end{pmatrix}^{-1} \left(- \begin{pmatrix} F_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \dot{\mathbf{a}}_{\text{planar},\text{cmd}} \\ \dot{\psi}_{\text{cmd}} \end{pmatrix} \right. \\ \left. - K_3 \left(\begin{pmatrix} \mathbf{a}_{\text{planar}} \\ \psi \end{pmatrix} - \begin{pmatrix} \mathbf{a}_{\text{planar},\text{cmd}} \\ \psi_{\text{cmd}} \end{pmatrix} \right) \right), \quad (4.66c)$$

$$u_2 = \dot{\omega}_{\text{cmd}} - K_4(\omega - \omega_{\text{cmd}}), \quad (4.66d)$$

and the command filters that the desired variables are passed through are

$$\dot{\mathbf{p}}_{\text{planar},\text{cmd}} = -\tau(\mathbf{p}_{\text{planar},\text{cmd}} - \mathbf{p}_{\text{planar},d}), \quad (4.67a)$$

$$\dot{\mathbf{v}}_{\text{planar},\text{cmd}} = -\tau(\mathbf{v}_{\text{planar},\text{cmd}} - \mathbf{v}_{\text{planar},d}), \quad (4.67b)$$

$$\dot{\mathbf{a}}_{\text{planar},\text{cmd}} = -\tau(\mathbf{a}_{\text{planar},\text{cmd}} - \mathbf{a}_{\text{planar},d}), \quad (4.67c)$$

$$\dot{\psi}_{\text{cmd}} = -\tau(\psi_{\text{cmd}} - \psi_d), \quad (4.67d)$$

$$\dot{\omega}_{\text{cmd}} = -\tau(\omega_{\text{cmd}} - \omega_d), \quad (4.67e)$$

where $(\cdot)_{\text{cmd}}$ are the commanded signals.

In this case, the command filters dynamics (4.67) act with a faster dynamics compared to the control law (4.66) in a two time-scale system. Mathematically it means that $\tau \gg \max(\text{eig}(K_i))$ for all $i = 1, 2, 3, 4$.

Remark 4.5. The command filter gains in (4.67) are set to be identical for simplicity. This is not a requirement and the gains can be made different as long as they are sufficiently large. (i.e. $\tau \gg \max(\text{eig}(K_i))$ for all $i = 1, 2, 3, 4$)

The following section shall provide a performance comparison between different backstepping controllers and quantifies their adequacy in solving the position tracking subproblem.

4.4 Numerical Simulations

The full state feedback position tracking controller formulated using contraction theory-based backstepping control is simulated on Matlab/Simulink. In this simulation, a quadrotor is commanded to track a target moving along predefined paths. The performance of full state feedback position tracking controller given in Section 4.3.2 is compared with that of the inverse kinematics-based backstepping controller given in Section 4.3.1. Furthermore, aerodynamic effects are simulated to compare the robustness of each controller with respect to neglected small forces and moments. Next, the responses of both controllers in the presence of wind disturbances are also evaluated and compared. Lastly, the responses of both controllers in the presence of wind disturbances and measurement noises are evaluated and compared.

4.4.1 Simulation Parameters

Numerical simulation is performed on a generic quadrotor using parameters obtained from [32]. The parameter are listed in Table 4.1¹⁶.

Two target trajectories are chosen: A straight line path and a circular path. The quadrotor is commanded to track a target moving along these trajectories. Along the linear trajectory, the target is initially placed at a location $\begin{pmatrix} X_{d0} & Y_{d0} \end{pmatrix}^T = \begin{pmatrix} 10\text{m} & 10\text{m} \end{pmatrix}^T$, moving at a speed of $V_{\text{target}} = 10\text{m/s}$ and with a heading angle of $\chi_{\text{target}} = 30^\circ$. Mathe-

¹⁶The quadrotor has a x, y, z -axis symmetric structure, therefore the moments of inertia only possess diagonal terms. $J = \text{diag}(J_{xx}, J_{yy}, J_{zz})$

Table 4.1: A list of quadrotor parameters used in the simulation [32].

Parameters	Values
Mass (m)	1.25kg
Distance between the rotor and center (l)	0.5m
Moment of inertia to x -axis (J_{xx})	$8.1 \times 10^{-3} \text{Nms}^{-2}$
Moment of inertia to y -axis (J_{yy})	$8.1 \times 10^{-3} \text{Nms}^{-2}$
Moment of inertia to z -axis (J_{zz})	$14.2 \times 10^{-3} \text{Nms}^{-2}$
Lift factor k_t	$54.2 \times 10^{-6} \text{Ns}^2$
Drag factor k_q	$1.1 \times 10^{-6} \text{Ns}^2$

matically, it is

$$X_d = X_{d0} + V_{\text{target}} \cos \chi_{\text{target}} t, \quad (4.68a)$$

$$Y_d = Y_{d0} + V_{\text{target}} \sin \chi_{\text{target}} t. \quad (4.68b)$$

In the other trajectory, the target is moving along a circular path of radius $R_{\text{target}} = 10\text{m}$ with a period of $T_{\text{target}} = 30\text{s}$. Mathematically, it is

$$X_d = R_{\text{target}} \cos \left(\frac{2\pi}{T_{\text{target}}} t \right), \quad (4.69a)$$

$$Y_d = R_{\text{target}} \sin \left(\frac{2\pi}{T_{\text{target}}} t \right), \quad (4.69b)$$

The initial target position is at $\begin{pmatrix} X_{d0} & Y_{d0} \end{pmatrix}^T = \begin{pmatrix} 10\text{m} & 0\text{m} \end{pmatrix}^T$.

4.4.2 Tracking a Moving Target

Numerical simulations are performed using both the full state feedback backstepping controller and the inverse kinematics-based backstepping controller on a quadrotor with quadrotor parameters defined in Table 4.1. The quadrotor is initially hovering at 10m above the ground at a planar position $\begin{pmatrix} X & Y \end{pmatrix}^T = \begin{pmatrix} 0\text{m} & 0\text{m} \end{pmatrix}^T$. The quadrotor is then commanded to track a target moving on the ground along the linear and circular

trajectory respectively. The results from using both controllers are plotted on the same graph for comparison and are shown in Figure 4.4 and 4.5. The X and Y position of the quadrotor against time is shown to emphasize the tracking of the target. The roll and pitch angles show the attitude response of the quadrotor. Finally, the tracking error is represented by the absolute distance between the quadrotor and the target in the x - y plane at time t .

 4.4.a: X Position

 4.4.b: Y Position

 4.4.c: X Position

 4.4.d: Y Position

 4.4.e: Roll Angle ϕ

 4.4.f: Pitch Angle θ

 4.4.g: Tracking Error $\|\mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},d}\|$

 4.4.h: Control Moment $\|u_2\|$

Figure 4.4: Results from the numerical simulation of a quadrotor tracking a target moving along a linear path on the ground with both the full state feedback backstepping controller (labeled proposed controller, solid line) and the inverse kinematics-based backstepping controller (labeled benchmark controller, dotted line). The quadrotor is maintained at a height of 10m and a yaw angle of 0° . The X and Y position of the quadrotor and the target against time is shown in Figure 4.4.a and 4.4.b (zoomed views are shown in Figure 4.4.c and Figure 4.4.d). The roll and pitch angles against time are also shown in Figure 4.4.e and 4.4.f respectively. The tracking error between the quadrotor and the target is plotted in Figure 4.4.g. Lastly, the control moment input is plotted in Figure 4.4.h.

4.5.a: X Position

4.5.b: Y Position

4.5.c: Roll Angle ϕ

4.5.d: Pitch Angle θ

 4.5.e: Tracking Error $\|\mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},d}\|$

 4.5.f: Control Moment $\|u_2\|$

Figure 4.5: Results from the numerical simulation of a quadrotor tracking a target moving along a circular path on the ground with both the full state feedback backstepping controller (labeled proposed controller, solid line) and the inverse kinematics-based backstepping controller (labeled benchmark controller, dotted line). The quadrotor is maintained at a height of 10m and a yaw angle of 0° . The X and Y position of the quadrotor and the target against time is shown in Figure 4.5.a and 4.5.b (a zoomed view is shown). The roll and pitch angles against time are also shown in Figure 4.5.c and 4.5.d respectively. The tracking error between the quadrotor and the target is plotted in Figure 4.5.e. Lastly, the control moment input is plotted in Figure 4.5.f.

From the results, both controllers (proposed and benchmark) are able to track the target to within sufficiently high accuracy for both target trajectories. The tracking errors $\|\mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},d}\|$ in Figure 4.4.g and 4.5.e show exponential convergence for both the full state feedback backstepping controller and the inverse kinematics-based backstepping controller. This is a consequence of the quadrotor behaving like a contracting system and ensuring asymptotic convergence of the quadrotor position to the target position.

For the case of tracking a target that is moving linearly, the quadrotor starts from hover whereas the target starts with an initial speed, therefore the tracking error increases slightly before decreases as the quadrotor catches up. Whereas for a target moving in a circle, the quadrotor starts at $\begin{pmatrix} X & Y \end{pmatrix}^T = \begin{pmatrix} 0\text{m} & 0\text{m} \end{pmatrix}^T$ and moves towards the target

while adjusting its direction as the target moves. Despite this, exponential convergence is observed for both trajectories. Under ideal situation, both controllers are able to achieve tracking of the target.

4.4.3 Tracking a Moving Target with Aerodynamic Effects

In the quadrotor model, effects such as aerodynamic forces and moments are neglected. However when the quadrotor is flying at a sufficiently high speed, aerodynamic drag acting on the quadrotor becomes significant. Backstepping control is a model-based control, therefore it is important to assess the performance of the full state feedback backstepping control in the presence of aerodynamic effects. For comparison, the inverse kinematics-based backstepping controller (also known as the *benchmark controller*) is also simulated in the presence of aerodynamic effects.

The aerodynamic model is given by (3.20) as

$$\mathbf{F}_{\text{aero}} = -K_d \mathbf{v}, \quad (4.70a)$$

$$\mathbf{M}_{\text{aero}} = -K_m \boldsymbol{\omega}, \quad (4.70b)$$

where $\mathbf{F}_{\text{aero}}$, $\mathbf{M}_{\text{aero}}$ are the aerodynamic force and moment respectively and $K_d > 0$ and $K_m > 0$ are the coefficient matrices of the aerodynamic force and moment respectively.

In this simulation, K_d and K_m are chosen to be diagonal matrices¹⁷ with same diagonal terms having a value of 10^{-1}Ns/m and 10^{-2}Nms/rad respectively for simplicity.

The results for the numerical simulations of the full state feedback backstepping controller and the inverse kinematics-based backstepping controller in tracking a target moving in a linear and a circular path are shown in Figure 4.6 to 4.9. The quadrotor, path parameters and initial conditions used are identical to the previous subsection.

¹⁷ K_d and K_m can take any block diagonal matrix structure. Cross diagonal terms are also possible but indicates off-balance in the quadrotor design.

4.6.a: X Position

4.6.b: Y Position

4.6.c: X Position

4.6.d: Y Position

 4.6.e: Roll Angle ϕ

 4.6.f: Pitch Angle θ

 4.6.g: Tracking Error $\|\mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},d}\|$

 4.6.h: Control Moment u_2

Figure 4.6: Results from the numerical simulation with aerodynamic effects of a quadrotor tracking a target moving along a linear path on the ground with both the full state feedback backstepping controller (labeled proposed controller, solid line) and the inverse kinematics-based backstepping controller (labeled benchmark controller, dotted line). The quadrotor is maintained at a height of 10m and a yaw angle of 0° . The X and Y position of the quadrotor and the target against time is shown in Figure 4.6.a and 4.6.b (zoomed views are shown in Figure 4.6.c and 4.6.d). The roll and pitch angles against time are also shown in Figure 4.6.e and 4.6.f respectively. The tracking error between the quadrotor and the target is plotted in Figure 4.6.g. Lastly, the control moment input is plotted in Figure 4.6.h.

4.7.a: X Position

4.7.b: Y Position

4.7.c: Roll Angle ϕ

4.7.d: Pitch Angle θ

4.7.e: Tracking Error $\|\mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},d}\|$ 4.7.f: Control Moment $\|u_2\|$

Figure 4.7: Results from the numerical simulation with aerodynamic effects of a quadrotor tracking a target moving along a circular path on the ground with both the full state feedback backstepping controller (labeled proposed controller, solid line) and the inverse kinematics-based backstepping controller (labeled benchmark controller, dotted line). The quadrotor is maintained at a height of 10m and a yaw angle of 0° . The X and Y position of the quadrotor and the target against time is shown in Figure 4.7.a and 4.7.b (a zoomed view is shown). The roll and pitch angles against time are also shown in Figure 4.7.c and 4.7.d respectively. The tracking error between the quadrotor and the target is plotted in Figure 4.7.e. Lastly, the control moment input is plotted in Figure 4.7.f.

Figure 4.8: The ground track of the quadrotor with the full state backstepping controller tracking a target moving along a circular path on the ground with aerodynamic effects.

Figure 4.9: The ground track of the quadrotor with the inverse kinematics-based backstepping controller tracking a target moving along a circular path on the ground with aerodynamic effects.

Comparing both the full state feedback backstepping controller (proposed) and the

inverse kinematics-based backstepping controller (benchmark), it is evident that the full state feedback backstepping controller performs better in terms of rejection against unmodelled aerodynamic effects. Figure 4.6.g and 4.7.e shows a nonzero steady state tracking error in the inverse kinematics-based backstepping controller but a zero steady state tracking error in the full state feedback backstepping controller. It is also observed that the steady state tracking error is larger for the straight line trajectory as compared to the circular trajectory due to the higher speed thus larger aerodynamic drag (10m/s as opposed to 3m/s). The effect of aerodynamic drag is more pronounced at higher speed.

The nonzero steady state error that occurs in the inverse kinematics-based backstepping controller is due to the use of inverse kinematics, which is an algebraic inversion method. This aerodynamic effect is not accounted for during the inversion. Furthermore, there is also no feedback information on the acceleration. These result in nonzero steady state tracking error which is also observed by Zuo [113].

On the other hand, the full state feedback backstepping controller handles the aerodynamic effect effectively. Rotation rate ω is used to control the planar acceleration $\mathbf{a}_{\text{planar}}$ according to the control law given by (4.66c) which is affected by aerodynamic effects. The availability of acceleration feedback results in a near zero steady state tracking error.

Close inspection reveals that for the linear path where desired acceleration is zero, convergence to zero tracking error is achieved. The following explanation is provided. Considering the aerodynamic force acting on the quadrotor, the translational dynamics in (3.9) becomes

$$\dot{\mathbf{v}} = -\frac{1}{m}R^T u_1 \hat{\mathbf{e}}_3 + g\hat{\mathbf{e}}_3 - \frac{1}{m}K_d \mathbf{v}. \quad (4.71)$$

The planar jerk of the quadrotor then becomes

$$\dot{\mathbf{a}}_{\text{planar}} = -\frac{1}{m}\mathbf{E}\dot{u}_1 R^T \hat{\mathbf{e}}_3 + \frac{1}{m}\mathbf{E}u_1 (R^T \hat{\mathbf{e}}_3)_{\times} \omega - \frac{1}{m}\mathbf{E}K_d \mathbf{E}^T \mathbf{a}_{\text{planar}}, \quad (4.72)$$

for a diagonal K_d . Therefore in the presence of aerodynamic effects, the aerodynamic force is a stabilizing term which tends to bring the planar acceleration to zero. In particular when $\mathbf{a}_{\text{planar},d} = 0, \psi_d = 0 \Rightarrow \begin{pmatrix} \mathbf{a}_{\text{planar},d} \\ \psi_d \end{pmatrix} = \mathbf{0}$ (linear path), $\begin{pmatrix} \mathbf{a}_{\text{planar}} \\ \psi \end{pmatrix}$ converges asymptotically to zero.

On the other hand for inverse kinematics-based backstepping controller, the inverse kinematics that is supposed to convert the desired planar acceleration into desired roll and pitch angles now lack an aerodynamic term. Therefore the roll and pitch angles calculated that supposedly results in a certain planar acceleration value now contains an offset. Suppose the attitude angles track desired values that are given by inverse kinematics (4.29), the acceleration (obtained from (3.9)) becomes

$$\mathbf{a}_{\text{planar}} = \dot{\mathbf{v}}_{\text{planar},d} - \mathbf{K}_2 (\mathbf{v}_{\text{planar}} - \mathbf{v}_{\text{planar},d}) - \frac{1}{m} \mathbf{E} \mathbf{K}_d \mathbf{v}, \quad (4.73)$$

and this differs from the desired acceleration

$$\mathbf{a}_{\text{planar},d} = \dot{\mathbf{v}}_{\text{planar},d} - \mathbf{K}_2 (\mathbf{v}_{\text{planar}} - \mathbf{v}_{\text{planar},d}), \quad (4.74)$$

by a nonzero term $\frac{1}{m} \mathbf{E} \mathbf{K}_d \mathbf{v}$ resulting in inexact tracking of the planar acceleration to the true desired planar acceleration, hence a steady state error in tracked position.

It is also noted that integral action is not added to the proposed controller yet it is able to achieve a result similar to the addition of an integral action. The order of the full state feedback backstepping controller is 10 which is the same as the inverse kinematics-based backstepping controller. For the inverse kinematics-based backstepping controller to achieve the same result as the full state feedback backstepping controller, an integral action needs to be included that increases its order to 12.

Therefore it is shown conclusively that a full state feedback backstepping controller is more superior to the inverse kinematics-based backstepping controller in terms of unmodelled effects rejection, especially those that appears in the inverse kinematics step. In the next subsection, the effects of wind disturbance is also analyzed.

4.4.4 Tracking a Moving Target in Wind Disturbances

The performance of the full state feedback backstepping controller is also evaluated and compared with the inverse kinematics-based backstepping controller (also known as the benchmark controller) in the presence of wind disturbances. Wind disturbances act on the quadrotor by changing the airspeed and thus the aerodynamic force and moment acting on the quadrotor.

The aerodynamic force and moments generated in the presence of wind with velocity $\mathbf{v}_{\text{wind}}$ and angular velocity ω_{wind} is given in (3.23) as

$$\mathbf{F}_{\text{aero}} = -K_d(\mathbf{v} - \mathbf{v}_{\text{wind}}), \quad (4.75a)$$

$$\mathbf{M}_{\text{aero}} = -K_m(\omega - \omega_{\text{wind}}), \quad (4.75b)$$

where $\mathbf{v}_{\text{wind}}$ and ω_{wind} are the linear and rotational components of wind disturbance. The Dryden wind turbulence model is used to generate random wind disturbances together with a steady wind bias of $\mathbf{v}_{\text{wind,bias}} = \begin{pmatrix} 10 & 0 & 0 \end{pmatrix}^T$ m/s. It is noted that in the open sea, the main component of the wind comes from the steady state wind bias which corresponds to the sea state.

The results of the numerical simulations of the full state feedback backstepping controller and the inverse kinematics-based backstepping controller under wind disturbances are shown in Figure 4.10 to 4.13.

4.10.a: X Position

4.10.b: Y Position

4.10.c: X Position

4.10.d: Y Position

 4.10.e: Roll Angle ϕ

 4.10.f: Pitch Angle θ

 4.10.g: Tracking Error $\|\mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},d}\|$

 4.10.h: Control Moment $\|u_2\|$

Figure 4.10: Results from the numerical simulation with wind disturbances of a quadrotor tracking a target moving along a linear path on the ground with both the full state feedback backstepping controller (labeled proposed controller, solid line) and the inverse kinematics-based backstepping controller (labeled benchmark controller, dotted line). The quadrotor is maintained at a height of 10m and a yaw angle of 0° . The X and Y position of the quadrotor and the target against time is shown in Figure 4.10.a and 4.10.b (zoomed views are shown in Figure 4.10.c and 4.10.d). The roll and pitch angles against time are also shown in Figure 4.10.e and 4.10.f respectively. The tracking error between the quadrotor and the target is plotted in Figure 4.10.g. Lastly, the control moment input is plotted in Figure 4.10.h.

4.11.a: X Position

4.11.b: Y Position

4.11.c: Roll Angle ϕ

4.11.d: Pitch Angle θ

 4.11.e: Tracking Error $\|\mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},d}\|$

 4.11.f: Control Moment $\|u_2\|$

Figure 4.11: Results from the numerical simulation with wind disturbances of a quadrotor tracking a target moving along a circular path on the ground with both the full state feedback backstepping controller (labeled proposed controller, solid line) and the inverse kinematics-based backstepping controller (labeled benchmark controller, dotted line). The quadrotor is maintained at a height of 10m and a yaw angle of 0^0 . The X and Y position of the quadrotor and the target against time is shown in Figure 4.11.a and 4.11.b (a zoomed view is shown). The roll and pitch angles against time are also shown in Figure 4.11.c and 4.11.d respectively. The tracking error between the quadrotor and the target is plotted in Figure 4.11.e. Lastly, the control moment input is plotted in Figure 4.11.f.

Figure 4.12: The ground track of the quadrotor with the full state backstepping controller tracking a target moving along a circular path on the ground with wind disturbance.

Figure 4.13: The ground track of the quadrotor equipped with the inverse kinematics-based backstepping controller tracking a target moving along a circular path on the ground with wind disturbance.

The full state feedback backstepping controller (proposed) is able to handle wind disturbances very well as compared to the inverse kinematics-based backstepping controller (benchmark). Due to the use of inverse kinematics and lack of acceleration information

in feedback as discussed in previous subsection, the inverse kinematics-based backstepping controller is unable to compensate for the wind disturbance evident from a nonzero steady state error.

The full state feedback backstepping controller, on the other hand, is able to achieve near zero steady state error. The contracting nature of the closed-loop system also guarantees a bounded response to a bounded disturbance, which is a bounded wind disturbance in this case. Furthermore, the full state feedback backstepping controller is a proper backstepping controller without the use of inverse kinematics to separate the translational and rotational part, therefore the steady state tracking error is bounded more tightly to zero.

4.4.5 Tracking a Moving Target with Measurement Noise

For completeness, numerical simulations of a quadrotor moving along the linear and circular reference trajectories in the presence of wind disturbance as well as measurement noises are provided. The standard deviations of the zero mean Gaussian measurement noises with values obtained from typical GPS-aided inertial navigation system (INS) that are added to the feedback signal are shown in Table 4.2 with values referenced from [14]. The simulation results are also presented in Figure 4.14 and 4.15.

As seen in Figure 4.14.g and 4.15.e, the control law produces a bounded error response in the presence of bounded measurement noise. This is because the closed-loop system is contracting, therefore obeys Theorem 4.5. The use of the backstepping control law presented in Section 4.2 ensures that the states hence the error remains bounded in the presence of bounded measurement noises.

Similarly, the effects of wind disturbance produces better response for the full state feedback backstepping controller in the same way that is discussed in the previous subsection.

Table 4.2: The standard deviations for the zero mean Gaussian measurement noises.

Measured quantity	Standard deviation
Position	0.1m
Velocity	0.1m/s
Acceleration	1m/s ²
Euler angles	5 ⁰
Rotation rates	5 ⁰ /s

This simulation nevertheless showed the adequacy of the presented control law in dealing with measurement noises and highlights the superiority of the full state feedback backstepping controller in achieving a smaller tracking error.

4.14.a: X Position

4.14.b: Y Position

4.14.c: X Position

4.14.d: Y Position

4.14.e: Roll Angle ϕ

4.14.f: Pitch Angle θ

 4.14.g: Tracking Error $\|\mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},d}\|$

 4.14.h: Control Moment $\|u_2\|$

Figure 4.14: Results from the numerical simulation with wind disturbances and measurement noises of a quadrotor tracking a target moving along a linear path on the ground with both the full state feedback backstepping controller (labeled proposed controller, solid line) and the inverse kinematics-based backstepping controller (labeled benchmark controller, dotted line). The quadrotor is maintained at a height of 10m and a yaw angle of 0° . The X and Y position of the quadrotor and the target against time is shown in Figure 4.14.a and 4.14.b (zoomed view are shown in Figure 4.14.c and 4.14.d). The roll and pitch angles against time are also shown in Figure 4.14.e and 4.14.f respectively. The tracking error between the quadrotor and the target is plotted in Figure 4.14.g. Lastly, the control moment input is plotted in Figure 4.14.h.

 4.15.a: X Position

 4.15.b: Y Position

 4.15.c: Roll Angle ϕ

 4.15.d: Pitch Angle θ

 4.15.e: Tracking Error $\|\mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},d}\|$

 4.15.f: Control Moment $\|u_2\|$

Figure 4.15: Results from the numerical simulation with wind disturbances and measurement noises of a quadrotor tracking a target moving along a circular path on the ground with both the full state feedback backstepping controller (labeled proposed controller, solid line) and the inverse kinematics-based backstepping controller (labeled benchmark controller, dotted line). The quadrotor is maintained at a height of 10m and a yaw angle of 0° . The X and Y position of the quadrotor and the target against time is shown in Figure 4.15.a and 4.15.b. The roll and pitch angles against time are also shown in Figure 4.15.c and 4.15.d respectively. The tracking error between the quadrotor and the target is plotted in Figure 4.15.e. Lastly, the control moment input is plotted in Figure 4.15.f.

Figure 4.16: The ground track of the quadrotor with the full state backstepping controller tracking a target moving along a circular path on the ground with wind disturbance and measurement noises.

Figure 4.17: The ground track of the quadrotor equipped with the inverse kinematics-based backstepping controller tracking a target moving along a circular path on the ground with wind disturbance and measurement noises.

4.5 Conclusion

4.5.1 Implications and Significance of Results

The following discusses some of the potential implication and significance of the work done in this chapter.

The formulation of a closed-form contraction theory-based backstepping control law (Section 4.2.2) together with the use of linear tracking differentiator (Section 4.3.4) allows direct implementation of the backstepping control law with any system in a cascaded-feedback form. Furthermore, the use of linear tracking differentiator also improves the practicality of this approach. This novel backstepping control law can potentially be applied to higher order systems that often result in complicated control laws when derived using conventional Lyapunov approach.

The re-formulation of the multi-rotor UAV kinematics and dynamics (Section 4.3.2) bring about an extra dimension to multi-rotor UAV control research. This form of kinematics and dynamics unifies the multi-rotor UAV system into a single framework that can be potentially controlled using other approaches. The canonical control inputs are more directly (in a sense “affinely”) connected to the outputs. Furthermore, the cascaded structure of the multi-rotor UAV is also preserved.

In addition, the multi-rotor UAV dynamics framework (Section 4.3.2) proposed in this chapter eliminates the use of inverse kinematics that is commonly present in the control structure of multi-rotor UAV. This allows better robustness property to unmodelled effects that causes steady state error in the inverse kinematics-based control. Furthermore, it is noted that this is due to the use of novel re-formulated dynamics as the order of the controller is not increased.

Lastly, the formulation of a full state feedback backstepping controller using quaternions (Section 4.3.3) allows for a full nonlinear quaternion-based attitude control. This

full nonlinear quaternion-based attitude control can potentially be applied to more acrobatic UAVs like the tailsitter UAVs whose domain of operation can span beyond $\theta = 0^\circ$ and $\theta = 90^\circ$.

4.5.2 Concluding Remarks

In this chapter, a contraction theory-based backstepping controller is formulated using re-formulated multi-rotor UAV kinematics and dynamics to control a multi-rotor UAV to track a moving ground target. Contraction theory is used to derive a closed-form backstepping control law by enforcing the state variable to converge to a time-varying virtual control during each step of the backstepping control. Re-formulation of the multi-rotor UAV kinematics and dynamics eliminates the need for inverse kinematics and results in a cascaded-feedback form so that full state feedback backstepping can be applied. A linear tracking differentiator is also employed to eliminate the need for deriving complicated derivative expressions. Numerical simulations are performed to evaluate the performance of the full state feedback backstepping controller as compared to inverse kinematics-based backstepping controller. The performance of both controllers are comparable for the ideal case with accurate tracking achieved. However in the presence of aerodynamic effects and wind disturbance, the full state feedback backstepping controller performs better by tracking the target more closely. The full state feedback backstepping controller has demonstrated precise tracking of a moving target with improved rejection of unmodelled aerodynamic effects and wind disturbance. This level of precision is required for ship deck landing application.

The position tracking sub-problem is solved in this chapter using the full state feedback backstepping controller with the multi-rotor UAV capable of tracking the planar path of a target to within sufficient accuracy in the presence of wind effects. In the next chapter, the landing control of a multi-rotor UAV and supporting theories are developed.

Chapter 5

Multi-Rotor UAV Landing on Ship Deck: A Full State Feedback Approach

In this chapter, the ship deck landing sub-problem of a multi-rotor UAV is analyzed and solved. The main contribution of this chapter is the development of a ship motion predictor-free control structure. The main reason of adopting this control structure is because ship motion predictors in most UAV ship deck landing systems are strongly affected by drift due to the accumulation of measurement and computation errors (also see Chapter 2), therefore resulting in typical ship motion predictor to produce a reasonably accurate prediction only for a short time frame that forces the multi-rotor UAV to land quickly. Appendix B presents a justifying example of the above mentioned issues faced by the internal model approach (a particular predictor approach) used by Marconi *et al.* [59] in the ship landing problem with a more detailed ship heave model.

To overcome these issues, an approach based on local invariant set theory is used to synthesize a ship deck landing controller for a multi-rotor UAV instead of incorporating a complex ship deck heave model or an internal model. This approach is applicable to most control structures such as a PID or a PD controller that are easy to implement. In the control formulation, the ship deck motion is treated as an external disturbance acting

on a feedback linearized system. For such a closed-loop linear system, it is well known that Lyapunov stability implies bounded-input bounded-output (BIBO) stability, thus a bounded disturbance gives rise to a bounded response. The local invariant set theory is then used to compute the control gain that results in a minimum bounded response caused by the ship deck motion.

However, it needs to be ensured that the minimum bounded response is sufficiently small for practical implementation. Ship motion induced due to waves is usually energy bounded. Therefore a wave with a large amplitude typically possesses a longer period and vice versa, as evident in Table 1.1. This implies that multi-rotor UAVs need to execute a larger altitude change for a larger wave, however over a longer period of time. The vice versa is also true. The result of this effect in the formulation is a reasonable and practical ship heave disturbance bound. This provides justification for using the above approach on the multi-rotor UAV ship deck landing sub-problem.

This chapter primarily addresses the issue of formulating a framework for a ship motion predictor-free landing control system. In addition, this chapter also presents a modification to the proposed framework that takes system uncertainties and measurement noises into account.

This chapter is organized as follows: Section 5.1 gives an introduction on the invariant ellipsoid method that is derived from the local invariant set theory. This method is essential in the formulation of the selection criteria for the ship deck landing controller synthesis. Prior to implementing the invariant ellipsoid method on a ship deck landing controller gain synthesis, a multi-rotor UAV ship deck landing model is presented in Section 5.2. Thereafter, Section 5.3 implements the invariant ellipsoid method with the multi-rotor UAV ship deck landing model to formulate both a state feedback and a PID ship deck landing controller. Several properties of the invariant ellipsoid method-based controller are also discussed. This section also presents a novel controller gain synthesis

methodology with consideration given to measurement noise. Section 5.4 shows and discusses the results of numerical simulations performed on the ship deck landing of a quadrotor. Lastly, some concluding remarks and the implications and significance of the discussed method are given in Section 5.5.

5.1 Introduction to Invariant Ellipsoid Method

The invariant ellipsoid method is based on the general theory developed by Chellaboina *et. al.* [17], Nazin *et. al.* [62] and Khlebnikov *et. al.* [47]. It is applied in many situations such as spacecraft stabilization [34] and sliding mode control [68]. Extensions to this method are also developed by Gonzalez-Garcia *et. al.* [33] and Azhmyakov *et. al.* [6, 7]. In this chapter, invariant ellipsoid method is used to select an optimal controller gain that results in a minimum bounded response for a linear system subjected to external disturbances.

Interested readers can refer to [62] for more information on invariant ellipsoid method.

In this section, the essential properties and formulation of the invariant ellipsoid method are discussed. The main result of the invariant ellipsoid method is an linear matrix inequality (LMI) condition for the invariance property of an ellipsoid. This is followed by a general formulation for a state feedback controlled system which is later used as the ship deck landing controller.

5.1.1 Invariant Ellipsoid Condition

The invariant ellipsoid method uses a linear approach to analyze the performance of a linear system in the presence of bounded disturbance. Consider a linear time-invariant (LTI) system

$$\dot{x} = Ax + Dw, \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the system state, $w \in \mathbb{R}^p$ is an external disturbance, and $A \in \mathbb{R}^{n \times n}$ and $D \in \mathbb{R}^{n \times p}$ are the state and disturbance input matrices respectively. The control input and output terms are omitted without any loss of generality.

For a system given by (5.1) with a bounded disturbance w , it is known from linear theory that the state response is also bounded¹. The size of the bounded output, system state in this case, is a function of the size of the external disturbance.

To estimate a bound on the state, the size of the external disturbance is required and is measured as

$$\|w\|_M < 1, \quad (5.2)$$

where $\|w\|_M \triangleq \sqrt{w^T M w}$ is defined as the M -norm of the external disturbance w and $M = M^T > 0$ is a symmetric positive definite weighting matrix. The size information of w is contained within the matrix M . For a scalar w , this is equivalent to $|w| < \frac{1}{\sqrt{M}}$.

Similarly, the size of the state is measured as

$$\|x\|_Q < 1, \quad (5.3)$$

where $Q = Q^T > 0$ is a symmetric positive definite matrix also known as the *invariant ellipsoid*². The size of the invariant ellipsoid is measured by analyzing the eigenvalues of Q which are the reciprocal of the square of the generalized semi-axes of the ellipsoid. A large minimum eigenvalue of Q represents a smaller invariant ellipsoid. Likewise, a small maximum eigenvalue of Q^{-1} also represents a smaller invariant ellipsoid.

A Lyapunov approach is used to relate the invariant ellipsoid Q to the external disturbance measured by M for a system given in (5.1). Consider a Lyapunov function,

$$V = x^T Q x, \quad (5.4)$$

¹This is known as the bounded-input bounded-output (BIBO) property.

²The generalized equation for a n dimensional ellipsoid is $x^T A x = 1$ for some $A = A^T > 0$.

where $\{x : V = x^T Q x < 1\}$ represents the states within the invariant ellipsoid. To guarantee that Q is an invariant ellipsoid, the Lyapunov function given in (5.4) must decrease with time for all points outside the ellipsoid $\{x : V = x^T Q x \geq 1\}$.

The time derivative of the Lyapunov function given in (5.4) is

$$\dot{V} = x^T Q (Ax + Dw) + (*), \quad (5.5)$$

where $X + (*) \triangleq X + X^T$.

By the argument of invariance given above, the condition $\dot{V} < 0$ must be satisfied for all x and w such that $V \geq 1$ and $\|w\|_M < 1$. This is equivalent to $V \geq 1, \|w\|_M < 1 \Rightarrow \dot{V} < 0$.

The above condition consists of the state x and an external disturbance w . Combining them into a hyper-state gives $\xi = \begin{pmatrix} x^T & w^T \end{pmatrix}^T$, the conditions are re-written as $\xi^T \begin{pmatrix} QA + (*) & QD \\ * & 0 \end{pmatrix} \xi < 0$ for all ξ such that $\xi^T \begin{pmatrix} Q & 0 \\ * & 0 \end{pmatrix} \xi \geq 1$ and $\xi^T \begin{pmatrix} 0 & 0 \\ * & M \end{pmatrix} \xi < 1$, where $\begin{pmatrix} X & Y \\ * & Z \end{pmatrix} \triangleq \begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix}$.

Before proceeding further, the following lemma is stated on the S-procedure.

Lemma 5.1 (S-procedure [96]). *A sufficient convex condition for $x^T A_0 x < \alpha_0$ for all x that satisfy $x^T A_i x \leq \alpha_i$ for $i = 1, 2, \dots, m$ is $x^T \left(A_0 - \sum_{i=1}^m \tau_i A_i \right) x \leq 0$ where $\alpha_0 - \sum_{i=1}^m \tau_i \alpha_i \geq 0$ for some $\tau_i \geq 0$ for $i = 1, 2, \dots, m$. Furthermore if $m = 1, 2$ and there exists a x such that $x^T A_i x < \alpha_i$ for $i = 1, 2$, then the S-procedure condition is necessary and sufficient³.*

Therefore applying S-procedure, the following sufficient condition is obtained

$$\begin{pmatrix} QA + (*) + \tau_1 Q & QD \\ * & -\tau_2 M \end{pmatrix} \leq 0, \quad (5.6)$$

³This is also known as lossless S-procedure.

for some $\tau_1 \geq \tau_2 \geq 0$ where $\begin{pmatrix} X + (*) + W & Y \\ * & Z \end{pmatrix} \triangleq \begin{pmatrix} X + X^T + W & Y \\ Y^T & Z \end{pmatrix}$. This inequality represents matrix negative definiteness⁴ where ξ is omitted. Notice that there exist a ξ such that $\xi^T \begin{pmatrix} Q & 0 \\ * & 0 \end{pmatrix} \xi > 1$ and $\xi^T \begin{pmatrix} 0 & 0 \\ * & M \end{pmatrix} \xi < 1$, so this condition is necessary and sufficient.

To further simplify the condition given in (5.6), observe that

$$\begin{pmatrix} QA + (*) + \alpha Q & QD \\ * & -\alpha M \end{pmatrix} + \beta \begin{pmatrix} Q & 0 \\ * & 0 \end{pmatrix} \leq 0, \quad (5.7)$$

where $\alpha = \tau_2 \geq 0$ and $\beta = \tau_1 - \tau_2 \geq 0$. Since $\begin{pmatrix} Q & 0 \\ * & 0 \end{pmatrix} \geq 0$,

$$\begin{pmatrix} QA + (*) + \alpha Q & QD \\ * & -\alpha M \end{pmatrix} \leq 0 \quad (5.8)$$

for some α implies that (5.7) is also satisfied for that particular α and $\beta = 0$. Hence (5.8) is a sufficient condition for (5.7). This is the invariant ellipsoid condition.

Furthermore, the method of Schur complement⁵ can be applied to (5.8) to yield

$$QA + (*) + \alpha Q + \frac{1}{\alpha} QDM^{-1}D^TQ \leq 0. \quad (5.9)$$

Pre- and post-multiply the above inequality by $P = Q^{-1}$,

$$AP + (*) + \alpha P + \frac{1}{\alpha} DM^{-1}D^T \leq 0 \quad (5.10)$$

which is a LMI and is easily tractable.

The invariant ellipsoid condition is formally stated as follows.

⁴The negative definite matrix $A < 0$ is equivalent to $x^T Ax < 0$ for all x .

⁵The method of Schur complement states that if $C < 0$, then $\begin{pmatrix} A & B \\ * & C \end{pmatrix} \leq 0 \Leftrightarrow A - BC^{-1}B^T \leq 0$.

Theorem 5.1. *For a system given by (5.1) and an external disturbance bounded by $\|w\|_M < 1$, the ellipsoid given by $\{x : x^T P^{-1} x < 1\}$ is invariant and attractive if and only if there exists an $\alpha > 0$ such that $AP + (*) + \alpha P + \frac{1}{\alpha} DM^{-1} D^T \leq 0$.*

The term invariant implies that if an initial state x_0 is within the invariant ellipsoid $x_0^T P^{-1} x_0 < 1$, then all future states x lie in the invariant ellipsoid $x^T P^{-1} x < 1$. The term attractive, on the other hand, implies that if an initial state x_0 is outside the invariant ellipsoid $x_0^T P^{-1} x_0 \geq 1$, then the state trajectory will converge into the invariant ellipsoid. Furthermore, this convergence is exponential.

5.1.2 State Feedback Control with the Invariant Ellipsoid Method

In this subsection, a state feedback control law is formulated for a LTI system and the invariant ellipsoid method is used to obtain the controller gain that produces a minimum invariant ellipsoid. This results in an optimization procedure for an optimal gain solution.

Consider a LTI system,

$$\dot{x} = Ax + Bu + Dw, \quad (5.11)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^p$ are the system states, control and external disturbance respectively, and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $D \in \mathbb{R}^{n \times p}$ are the system, control input and disturbance input matrices respectively.

Let the static state feedback control law be

$$u = Kx, \quad (5.12)$$

where $K \in \mathbb{R}^{m \times n}$ is the controller gain. The closed-loop system becomes

$$\dot{x} = (A + BK)x + Dw, \quad (5.13)$$

which is similar to the system given in (5.1). Therefore Theorem 5.1 is applied to obtain the following corollary.

Corollary 5.1. *For a system given by (5.11) and a static state feedback control law $u = Kx$, the closed-loop system response in the presence of an external disturbance w bounded by $\|w\|_M < 1$ is attracted to the invariant ellipsoid $\{x : x^T P^{-1} x < 1\}$ if and only if there exist $\alpha > 0$ such that*

$$(A + BK)P + (*) + \alpha P + \frac{1}{\alpha} DM^{-1} D^T \leq 0. \quad (5.14)$$

In this formulation, an optimal controller gain K^* is sought after such that the invariant ellipsoid P is minimized. However the invariant ellipsoid condition (5.14) is bilinear in decision variables K and P , therefore the optimization is intractable. Nevertheless it can be transformed into a LMI by the transformation $Y = KP$, and (5.14) becomes

$$AP + BY + (*) + \alpha P + \frac{1}{\alpha} DM^{-1} D^T \leq 0 \quad (5.15)$$

which is a tractable LMI.

The objective of this optimization is to achieve a minimum invariant ellipsoid with its size measured by the smallest eigenvalue of Q (also the maximum eigenvalue of $P = Q^{-1}$). A small invariant ellipsoid possesses a small $\lambda_{\max} \in \max \text{eig}(P)$. The maximum eigenvalue of P is upper-bounded by the condition

$$P - tI_n \leq 0, \quad (5.16)$$

where $t \geq \lambda_{\max} > 0$ is a variable that is always larger or equal to the maximum eigenvalue of P and I_n denotes a $n \times n$ identity matrix. Therefore minimizing t minimizes the maximum eigenvalue of P . Other measures of matrix size can also be used. For instance, the trace and determinant of P can also be minimized to obtain the minimum invariant ellipsoid.

The following optimization problem gives the optimal controller gain $K^* = Y^* P^{*-1}$.

Optimization 5.1.

$$\min_{P>0, Y, t>0} t$$

subject to

$$\begin{aligned} AP + BY + (*) + \alpha P + \frac{1}{\alpha} DM^{-1} D^T &\leq 0, \\ P - tI_n &\leq 0. \end{aligned}$$

The above optimization is solved for some $\alpha > 0$ to obtain an optimal controller gain K^* that results in an optimal performance in terms of least bounded response. It is noteworthy that a family of optimal controller gains $K^*(\alpha)$ and invariant ellipsoids $P^*(\alpha)$ that are functions of α are obtained. In practice, the optimal α is the one that results in the minimum invariant ellipsoid among $P^*(\alpha)$. This constitutes another optimization in the variable α .

5.1.3 State Feedback Control with the Invariant Ellipsoid Method including Control Action Constraint

It is also common to limit the control action for states within an invariant ellipsoid so that it is consistent with the actuator limits of the real physical system.

This is done by imposing the condition $\|u\|_{R^{-1}} = \|Kx\|_{R^{-1}} < 1$ for all states x such that $x^T Q x < 1$, where $R^{-1} = R^{-1T} > 0$ is a symmetric positive definite matrix denoting the size of the control input. The R^{-1} -norm of the control input is defined as $u^T R^{-1} u = x^T K^T R^{-1} K x < 1$. This condition is to be satisfied for all x such that $x^T Q x < 1$. By performing S-procedure (Lemma 5.1), a sufficient condition for the above condition is

$$K^T R^{-1} K - \tau Q \leq 0, \tag{5.17}$$

for some $0 < \tau \leq 1$. Furthermore there exist x such that $x^T Q x < 1$, so this condition is necessary and sufficient.

Further observe that $K^T R^{-1} K - Q \leq K^T R^{-1} K - \tau Q \leq 0$, therefore $K^T R^{-1} K - Q \leq 0$ is a sufficient condition for (5.17).

Introducing the relationship $Y = KP$ and pre- and post-multiply by $P = Q^{-1}$, the control action condition $K^T R^{-1} K - Q \leq 0$ becomes

$$Y^T R^{-1} Y - P \leq 0. \quad (5.18)$$

Applying of the method of Schur complement yields

$$\begin{pmatrix} P & Y^T \\ Y & R \end{pmatrix} \geq 0. \quad (5.19)$$

Therefore if (5.19) is satisfied, $\|u\|_{R^{-1}} = \|Kx\|_{R^{-1}} < 1$ for all x such that $x^T P^{-1} x < 1$. The following theorem is stated.

Theorem 5.2. *For a system given by (5.11) and a static state feedback control law $u = Kx$ under the influence of an external disturbance bounded by $\|w\|_M < 1$, the control action is bounded in an ellipsoid given by $\{u : \|u\|_{R^{-1}} < 1\}$ for all states $\{x : x^T P^{-1} x < 1\}$ if and only if $\begin{pmatrix} P & Y^T \\ Y & R \end{pmatrix} \geq 0$.*

Theorem 5.2 provides a measure of the control action through the ellipsoid $\{u : \|u\|_{R^{-1}} < 1\}$, a smaller R results in a smaller control action.

Incorporating the control action constraint, the optimization in the previous subsection is rewritten as

Optimization 5.2.

$$\min_{P>0, Y, R>0, t_1>0, t_2>0} \mu t_1 + \nu t_2$$

subject to

$$\begin{aligned}
 AP + BY + (*) + \alpha P + \frac{1}{\alpha} DM^{-1} D^T &\leq 0, \\
 \begin{pmatrix} P & Y^T \\ Y & R \end{pmatrix} &\geq 0, \\
 P - t_1 I_n &\leq 0, \\
 R - t_2 I_m &\leq 0.
 \end{aligned}$$

The objective function $\mu t_1 + \nu t_2$ for $\mu, \nu > 0$ represents the tradeoff between minimizing the invariant ellipsoid $\{x : x^T P^{-1} x < 1\}$ and the control action ellipsoid $\{u : \|u\|_{R^{-1}} < 1\}$. This allows extra degree of freedom in the controller gain synthesis by allowing consideration for practical tradeoff.

The above LMI optimization is solved to obtain an optimal controller gain $K^*(\alpha)$ for a ship deck landing controller in Section 5.3.

5.1.4 Invariant Ellipsoid Method Applied to Uncertain Linear Systems

The optimization approach proposed in Section 5.1.3 solves for an optimal controller gain for a LTI system subjected to bounded unknown disturbances. It is a form of optimal control where the controller gain is synthesized with the objective of achieving minimal system response in the presence of bounded external disturbances.

The proposed methodology, on the other hand, is strongly model-based therefore system uncertainties can cause significant deterioration in performance and even instability. In a linear system, a common description of system uncertainty is the parametric form where each term in the system matrices takes value from a closed set. For such a system, a controller that results in a stable closed-loop nominal system may become unstable or have undesirable performance for a subset of the uncertain system set. This subsection

develops a linear inequality condition to obtain a minimum invariant ellipsoid (if it exists) for all possible systems in the uncertain set. Thereafter, analysis of this minimum invariant ellipsoid allows the determination of the robustness of the closed-loop system.

Parametric uncertainty is considered where parameters within the system matrices are allowed to vary between certain known values. The uncertain matrix parameter variations are assumed to be convex. Therefore, system matrices are defined in the following convex sets

$$A \in \mathbb{A} \triangleq \left\{ A : A = \sum_{i=1}^{n_A} \gamma_{i,A} A_i \left| \sum_{i=1}^{n_A} \gamma_{i,A} = 1 \cap \gamma_{i,A} \geq 0 \right. \right\}, \quad (5.20a)$$

$$B \in \mathbb{B} \triangleq \left\{ B : B = \sum_{j=1}^{n_B} \gamma_{j,B} B_j \left| \sum_{j=1}^{n_B} \gamma_{j,B} = 1 \cap \gamma_{j,B} \geq 0 \right. \right\}, \quad (5.20b)$$

$$D \in \mathbb{D} \triangleq \left\{ D : D = \sum_{k=1}^{n_D} \gamma_{k,D} D_k \left| \sum_{k=1}^{n_D} \gamma_{k,D} = 1 \cap \gamma_{k,D} \geq 0 \right. \right\}, \quad (5.20c)$$

where $\gamma_{i,A}$, $\gamma_{j,B}$ and $\gamma_{k,D}$ are weighting terms, and the sets $\{A_i\}$, $\{B_j\}$ and $\{D_k\}$ are the convex hull of the uncertain system matrices A , B and D with n_A , n_B and n_D number of elements respectively.

For such an uncertain system, the system dynamics follows (5.11) with the exact system matrices being unknown. It is only known that the system matrices belongs to (5.20) for some $\gamma_{i,A} \geq 0$, $\gamma_{j,B} \geq 0$, $\gamma_{k,D} \geq 0$ and $\sum_{i=1}^{n_A} \gamma_{i,A} = 1$, $\sum_{j=1}^{n_B} \gamma_{j,B} = 1$, $\sum_{k=1}^{n_D} \gamma_{k,D} = 1$.

In this case if a controller gain is derived based on the optimization problem proposed in Section 5.1.3 using nominal values for the system matrices, performance and stability are not guaranteed for all possible combinations of system matrices in the set (5.20). In other words, the closed-loop system is not robust. Therefore it is desirable to quantify the robustness of the computed controller gain for the entire class of system whose system matrices satisfy $\{A \in \mathbb{A}, B \in \mathbb{B}, D \in \mathbb{D}\}$.

The robust invariant ellipsoid condition is first derived. Consider a system given by (5.11) and a static state feedback control law $u = Kx$. Corollary 5.1 states that the set

$\{x : x^T P^{-1} x < 1\}$ is an invariant ellipsoid if and only if there exists an $\alpha \geq 0$ such that

$$(A + BK)P + (*) + \alpha P + \frac{1}{\alpha} D M^{-1} D^T < 0. \quad (5.21)$$

For an ellipsoid $\{x : x^T P^{-1} x < 1\}$ to be invariant for all possible combinations of system matrices $\{A \in \mathbb{A}, B \in \mathbb{B}, D \in \mathbb{D}\}$, (5.21) must be satisfied for all matrices in the parametric uncertainty domain $\{A \in \mathbb{A}, B \in \mathbb{B}, D \in \mathbb{D}\}$. Substituting (5.20) into (5.21), we have

$$\begin{aligned} & \left(\sum_{i=1}^{n_A} \gamma_{i,A} A_i + \sum_{j=1}^{n_B} \gamma_{j,B} B_j K \right) P + (*) + \alpha P \\ & + \frac{1}{\alpha} \left(\sum_{k=1}^{n_D} \gamma_{k,D} D_k \right) M^{-1} \left(\sum_{k=1}^{n_D} \gamma_{k,D} D_k^T \right) < 0, \end{aligned} \quad (5.22)$$

or in a block matrix form using the method of Schur complement

$$\begin{pmatrix} \left(\sum_{i=1}^{n_A} \gamma_{i,A} A_i + \sum_{j=1}^{n_B} \gamma_{j,B} B_j K \right) P + (*) + \alpha P & \sum_{k=1}^{n_D} \gamma_{k,D} D_k \\ * & -\alpha M \end{pmatrix} < 0. \quad (5.23)$$

Re-expressing the summation in (5.23), we have

$$\sum_{i=1}^{n_A} \left(\gamma_{i,A} \sum_{j=1}^{n_B} \left(\gamma_{j,B} \sum_{k=1}^{n_D} \left(\gamma_{k,D} \begin{pmatrix} (A_i + B_j K) P + (*) + \alpha P & D_k \\ * & -\alpha M \end{pmatrix} \right) \right) \right) < 0. \quad (5.24)$$

Therefore a sufficient condition for inequality (5.24) is

$$\begin{pmatrix} (A_i + B_j K) P + (*) + \alpha P & D_k \\ * & -\alpha M \end{pmatrix} < 0, \quad (5.25)$$

for all possible combinations of $\{A_i \in \mathbb{A}, B_j \in \mathbb{B}, D_k \in \mathbb{D}\}$. This can further be written in a linear form by the method of Schur complement as

$$(A_i + B_j K) P + (*) + \alpha P - \frac{1}{\alpha} D_k M^{-1} D_k^T < 0. \quad (5.26)$$

The following theorem summarizes the treatment of invariant ellipsoid with a parametric uncertain system.

Theorem 5.3. *For a system given by (5.11) with system matrices belonging to convex sets given by (5.20) and a static state feedback control law $u = Kx$, the closed-loop system response in the presence of an external disturbance w bounded by $\|w\|_M < 1$ is attracted to the invariant ellipsoid $\{x : x^T P^{-1} x < 1\}$ if there exist an $\alpha \geq 0$ such that (5.26) holds for all possible combinations of $A_i \in \mathbb{A}$, $B_j \in \mathbb{B}$ and $D_k \in \mathbb{D}$.*

Using Theorem 5.2 and 5.3, the optimization problem that is solved to obtain an minimum invariant ellipsoid for a parametric uncertain closed-loop system is

Optimization 5.3.

$$\min_{P>0, t_1>0} t_1$$

subject to

$$(A_i + B_j K) P + (*) + \alpha P - \frac{1}{\alpha} D_k M^{-1} D_k^T < 0$$

$$P - t_1 I_n \leq 0$$

for all possible combinations of $A_i \in \mathbb{A}$, $B_j \in \mathbb{B}$ and $D_k \in \mathbb{D}$.

The solution to the above constrained optimization problem allows the determination of a minimum invariant ellipsoid $\{x : x^T P^{-1} x < 1\}$ with respect to the objective function for a certain controller gain applied to a system matrices defined by the set (5.20).

Remark 5.1. The size of the minimum invariant ellipsoid is measured by the term t_1 where a small value indicate a small ellipsoid. If the increase in t_1 is small compared to the value obtained from the optimization of the nominal system (Section 5.1.3), then the obtained controller gain is considered robust.

For a nominal system that is not sufficiently robust, the following optimization problem can be solved to obtain a robust controller gain.

Optimization 5.4.

$$\min_{P>0, Y, R>0, t_1>0, t_2>0} \mu t_1 + \nu t_2$$

subject to

$$\begin{aligned} (A_i + B_j K) P + (*) + \alpha P - \frac{1}{\alpha} D_k M^{-1} D_k^T &< 0 \\ \begin{pmatrix} P & Y^T \\ Y & R \end{pmatrix} &\geq 0, \\ P - t_1 I_n &\leq 0, \\ R - t_2 I_m &\leq 0 \end{aligned}$$

for all possible combinations of $A_i \in \mathbb{A}$, $B_j \in \mathbb{B}$ and $D_k \in \mathbb{D}$.

The solution to the above constrained optimization problem allows the determination of an optimal controller gain $K^* = Y^* P^{*-1}$ that results in the invariant ellipsoid $\{x : x^T P^{-1} x < 1\}$ and control action ellipsoid $\{u : \|u\|_{R^{-1}} < 1\}$ being minimum with respect to the objective function for a class of system defined by the system matrices in the set (5.20).

The key advantage of this optimization is that it provides more allowance for variation in system matrices by considering all possible combinations of system matrices that is in contrast to the optimization in Section 5.1.3 that solves the optimal controller gain for a nominal system.

Remark 5.2. The optimization in Section 5.1.3 is a special case of the optimization problem proposed in this section for the case where the convex hull of the system matrices contains only a single matrix. Therefore it is expected that the objective function for a parametric uncertain system will not be better than the nominal system. This optimization problem tradeoffs the performance for the nominal system for the performance for the actual system which is uncertain.

5.2 Multi-Rotor UAV Ship Deck Landing Model

In this section, a disturbance model is derived for the ship deck landing sub-problem of a multi-rotor UAV. This model is used in the LMI optimization problem derived in Section 5.1 to obtain an optimal state feedback controller gain so that the altitude tracking error is constrained to a minimum invariant ellipsoid during ship deck landing.

For a small sized multi-rotor UAV, the roll and pitch motion of the ship deck typically does not result in significant height difference between far ends of a multi-rotor UAV. This is explained in Chapter 1. However it should be noted that the height contribution of the ship pitch and roll motion due to the placement of the landing pad is still significant (see Figure 5.1). Figure 5.1 also shows the nomenclature used in deriving the multi-rotor UAV ship deck landing model.

Figure 5.1: The nomenclature used in the ship deck landing multi-rotor UAV model. The multi-rotor UAV at an altitude h above the reference line is required to land on a landing pad is placed at a position L along the x_{ship} -axis. The ship center of mass is heaved at a height d_{ship} above the reference line and rolled an angle ϕ_{ship} causing the height of the landing pad at $d = d_{ship} - L \sin \phi_{ship}$ above the reference line.

The altitude of the multi-rotor UAV is denoted by h and its dynamics is derived in

Chapter 3 as a double integrator

$$\ddot{h} = v, \quad (5.27)$$

where $v = \frac{T}{m} \cos \phi \cos \theta - g$.

In addition, the ship heave model is also derived in Chapter 3 and without any loss of generality, the ship roll and pitch motion are subsumed in the model as

$$d = \sum_{j=1}^N \sqrt{2S(\omega_j)\Delta\omega_j} \sin(\omega_j t + \epsilon_j), \quad (5.28)$$

where the term $\sqrt{2S(\omega_j)\Delta\omega_j}$ represents the height contribution of the j^{th} wave with a frequency of ω_j and phase of ϵ_j .

The controlled output of the multi-rotor UAV ship deck landing model is the altitude error e with respect to a reference altitude H . This reference altitude is measured relative to the landing pad and is a time-decaying function that diminishes slowly to zero so as to achieve a smooth landing. Therefore, in terms of known parameters, the altitude error is expressed as

$$e = h - d - H. \quad (5.29)$$

This tracking error dynamics becomes,

$$\ddot{e} = v - \ddot{d} - \ddot{H}, \quad (5.30)$$

where the altitude dynamics is included. Substituting $u = v - \ddot{H}$, the tracking error dynamics becomes

$$\ddot{e} = u - \ddot{d}. \quad (5.31)$$

This is a double integral type LTI system with an external disturbance term $\ddot{d}$. This external disturbance arises from the ship heave motion and is modelled as a L_∞ bounded unknown signal⁶ whose bounds can be estimated from historical data corresponding to different sea states.

⁶A L_∞ bound signal is defined as a peak value bounded signal. It is also written as $\sup \sqrt{w^T w} = l$, where w is the signal and $l > 0$ is finite.

Denote $\ddot{d}$ by w and its bound as $\|w\|_M < 1$ where $M = M^T > 0$ is estimated from the ship response history at a specific sea state. The tracking error dynamics in the state space form is expressed as

$$\dot{x} = Ax + Bu + Bw, \quad (5.32)$$

where $x = \begin{pmatrix} e & \dot{e} \end{pmatrix}^T$ is the system state, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the system matrices.

The model given in (5.32) is used in the next section in the synthesis of the controller gain for a ship deck landing controller.

5.3 Multi-Rotor UAV Ship Deck Landing Controller

In the landing sub-problem, a controller gain is synthesized such that the state $x = \begin{pmatrix} e & \dot{e} \end{pmatrix}^T$ is bounded tightly to zero, implying that the multi-rotor UAV is tracking the reference height accurately. Therefore a smooth and safe landing is achieved.

A state feedback control law is formulated using the invariant ellipsoid method that optimizes the controller gain such that the invariant ellipsoid is minimized. In this section, the invariant ellipsoid method illustrated in Section 5.1 is applied to the system given in (5.32). In addition, a PID controller is also synthesized for the same control problem demonstrating the applicability of the invariant ellipsoid method as a PID gain tuning algorithm. Furthermore, several desirable properties of optimal controllers derived using invariant ellipsoid method are also discussed. Lastly, this section presents a optimal controller gain synthesis methodology that incorporates the effect of measurement noises.

5.3.1 Optimal Controller Gain Synthesis

The state feedback control law is formulated as

$$u = Kx, \quad (5.33)$$

where $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$ is the controller gain.

Applying Optimization 5.2 to the multi-rotor UAV ship deck landing model given in (5.32), the following optimization is obtained.

Optimization 5.5.

$$\min_{P>0, Y, R>0, t_1>0, t_2>0} \mu t_1 + \nu t_2$$

subject to

$$\begin{aligned} AP + BY + (*) + \alpha P + \frac{1}{\alpha} BM^{-1}B^T &\leq 0, \\ \begin{pmatrix} P & Y^T \\ Y & R \end{pmatrix} &\geq 0, \\ P - t_1 I_2 &\leq 0, \\ R - t_2 I_1 &\leq 0. \end{aligned}$$

Since the control actuator limit for a multi-rotor UAV is known, the control action constraint in Theorem 5.2 is imposed as a hard constraint. Let the control action be limited by $\|u\|^2 = u^T u < u_{\max}^2$, the control action satisfies this constraint for all states $\{x : x^T P^{-1} x < 1\}$ if and only if

$$\begin{pmatrix} P & Y^T \\ Y & u_{\max}^2 I_1 \end{pmatrix} \geq 0. \quad (5.34)$$

This is obtained by imposing $R = u_{\max}^2 I_1$ as a constant⁷. This control action constraint is to be satisfied at all times during landing, therefore the Optimization 5.5 is rewritten as

Optimization 5.6.

$$\min_{P>0, Y, t>0} t$$

$${}^7 u^T u < u_{\max}^2 \Rightarrow u^T \left(\frac{1}{u_{\max}^2} \right) u < 1 \equiv u^T R^{-1} u < 1$$

 5.2.a: Control gain K

 5.2.b: Maximum eigenvalue of P

Figure 5.2: The variation of the optimal control gain, $K^*(\alpha)$, and the maximum eigenvalue $\lambda_{\max}(\alpha)$ of $P^*(\alpha)$ with different values of α for system (5.32).

subject to

$$\begin{aligned}
 AP + BY + (*) + \alpha P + \frac{1}{\alpha} BM^{-1}B^T &\leq 0, \\
 \begin{pmatrix} P & Y^T \\ Y & u_{\max}^2 \end{pmatrix} &\geq 0, \\
 P - tI_2 &\leq 0.
 \end{aligned}$$

The solution to this LMI optimization gives the optimal controller gain $K^* = Y^*P^{*-1}$ for some an optimal α^* .

It is noted that the parameter α is defined independently during the optimization. For different α , different optimal controller gain $K^*(\alpha) = \begin{pmatrix} K_1^*(\alpha) & K_2^*(\alpha) \end{pmatrix}$ and minimum invariant ellipsoid $P^*(\alpha)$ are obtained. A numerical synthesis is performed for the system given by (5.32) to analyze the variation of the optimal controller gain $K^*(\alpha)$ in Figure 5.2.a and maximum eigenvalue of $P^*(\alpha)$ in Figure 5.2.b with different values of α . Figure 5.2.b shows that a smaller ellipsoid is achieved with a higher α value, however, this is at the expense of a higher controller gain as shown in Figure 5.2.a. A high gain design is usually not desirable due to poor noise sensitivity, therefore a proper choice of α is to be

chosen in order to achieve a reasonable gain and minimum invariant ellipsoid. Treatment of noise sensitivity issue is presented in Section 5.3.4.

5.3.2 PID Controller Gain Synthesis

State feedback controller developed in the previous subsection demonstrates the use of invariant ellipsoid method as a criteria for the selection of an optimal controller gain. However, the inclusion of integral control in PID controller allows for better performance. In this section, a PID controller is formulated for the multi-rotor UAV ship deck landing sub-problem and the invariant ellipsoid method is used as a PID controller gain tuning criteria. This also shows the versatility of invariant ellipsoid method in applying to different control structures.

PID controllers are commonly used to control multi-rotor UAVs. The main reasons for the use of PID controllers are:

- simple structure,
- easy hardware implementation,
- easily tunable even without a model,
- and good range of performance.

Because of this, the use of PID controllers has been prevalent in many multi-rotor UAV applications [27, 44, 50, 75, 76, 86].

The selection of PID controller gain is thus critical to the performance of multi-rotor UAVs. In most cases, the gain is tuned manually during flight tests. However, a good initial estimate for the gain is always desirable. Methods such as the Ziegler Nichols method [112], Cohen Coon method [18] and loop shaping methods are commonly used. These PID controller tuning algorithms are often specific to certain situations. Objectives

such as minimum overshoot and settling time are often used as guidelines to tune the controller gain.

However there are few PID controller tuning algorithm targeted at the rejection of persistent disturbance. In this case, the disturbance acting on the system is persistent and L_∞ ⁸ bounded. A PID controller gain tuning algorithm that minimizes the output bound for a system subjected to persistent L_∞ bounded disturbance is developed for a multi-rotor UAV ship landing controller in this section.

The tracking error dynamics is shown in (5.31) and is linear for the implementation of a PID controller,

$$u = K_i \int_0^t e dt + K_p e + K_d \dot{e}, \quad (5.35)$$

where K_i , K_p and K_d are the PID controller gains. A schematic of the closed-loop system is shown in Figure 5.3.

Figure 5.3: A schematic of the closed-loop control system. The relative tracking error e between the multi-rotor UAV and the ship deck is feedback into the PID controller and the control input u is augmented with the reference altitude H before entering into the multi-rotor UAV to form a closed-loop.

The closed-loop system becomes

$$\ddot{e} = K_i \int_0^t e dt + K_p e + K_d \dot{e} + w, \quad (5.36)$$

⁸In this case, a disturbance that is bounded by $\|w\|_M < 1$ is also known as L_∞ bounded or peak bounded.

and is expressed in a state space form as

$$\dot{x} = (A + BK)x + Bw, \quad (5.37)$$

where $x = \begin{pmatrix} \int_0^t e dt \\ 0 \\ e \\ \dot{e} \end{pmatrix}$ is the system state, $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are the system matrices, and $K = \begin{pmatrix} K_i & K_p & K_d \end{pmatrix}$ denotes the PID controller gain. The external disturbance bound is given by $\|w\|_M < 1$.

Implementing Optimization 5.2 to the system given by (5.37), the following optimization is obtained.

Optimization 5.7.

$$\min_{P>0, Y, R>0} \mu \text{tr}(P) + \nu \text{tr}(R)$$

subject to

$$\begin{aligned} AP + BY + (*) + \alpha P + \frac{1}{\alpha} BM^{-1}B^T &\leq 0, \\ \begin{pmatrix} P & Y^T \\ Y & R \end{pmatrix} &\geq 0, \end{aligned}$$

Here, the objective function is $\mu \text{tr}(P) + \nu \text{tr}(R)$ is used instead of the maximum eigenvalues of P and R without any loss of generality. The trace is also a measurement of the size of the ellipsoid given by $x^T P^{-1} x < 1$, with a larger trace value indicating a larger ellipsoid.

Similar to Optimization 5.6, a hard limit on the control action constraint can be imposed to yield the following optimization problem

Optimization 5.8.

$$\min_{P>0, Y} \text{tr}(P)$$

 5.4.a: Control gain K

 5.4.b: Trace of P

Figure 5.4: The variation of the optimal PID control gain, $K^*(\alpha)$, and the trace of $P^*(\alpha)$ with different values of α for system (5.37).

subject to

$$AP + BY + (*) + \alpha P + \frac{1}{\alpha} BM^{-1}B^T \leq 0,$$

$$\begin{pmatrix} P & Y^T \\ Y & u_{\max}^2 \end{pmatrix} \geq 0,$$

where the control action $\|u\|^2 = u^T u < u_{\max}^2$ for all states in the ellipsoid $\{x : x^T P^{-1} x < 1\}$.

The PID controller gain is obtained from the solution to the above optimization as $K^* = \begin{pmatrix} K_i^* & K_p^* & K_d^* \end{pmatrix} = Y^*(P^*)^{-1}$.

Similar tradeoff between the values for the optimal control gains and the size of the invariant ellipsoid is observed and shown in Figure 5.4.a and Figure 5.4.b respectively for the system given by (5.37). A high controller gain is more sensitive to measurement noises, hence a moderate PID control gain is chosen.

5.3.3 Properties of Optimal Controllers

In addition to minimizing the invariant ellipsoid, the controller⁹ derived using the invariant ellipsoid method also exhibits other desirable properties. These include preserved optimality for linearly scaled disturbance, exponential convergence to the ellipsoid and extension to stochastic disturbance.

5.3.3.1 Preserved Optimality

The optimality of the optimal controller is preserved when the system is subjected to a linearly scaled external disturbance. This is under the assumption that the maximum control action is enlarged or reduced accordingly.

Suppose that the disturbance bound becomes $M' = \frac{1}{\gamma}M$, where $\gamma > 0$. The following transformation will transform the new optimization problem into the former problem. $P' = \gamma P$, $Y' = KY' = \gamma Y$, $R' = \frac{1}{\gamma}R$, $t'_1 = \frac{1}{\gamma}t_1$ and $t'_2 = \frac{1}{\gamma}t_2$. The optimization becomes,

Optimization 5.9.

$$\min_{P' > 0, Y', R' > 0, t'_1 > 0, t'_2 > 0} \mu t'_1 + \nu t'_2$$

subject to

$$\begin{aligned} AP' + BY' + (*) + \alpha P' + \frac{1}{\alpha} DM'^{-1} D^T &\leq 0, \\ \begin{pmatrix} P' & Y'^T \\ Y' & R' \end{pmatrix} &\geq 0, \\ P' - t'_1 I_n &\leq 0, \\ R' - t'_2 I_m &\leq 0. \end{aligned}$$

Therefore the solution $\{P^*, Y^*, t_1^*, t_2^*\}$ of the optimization with a disturbance $\|w\|_M < 1$ is the same as the solution $\{P'^*, Y'^*, t_1'^*, t_2'^*\}$ of the optimization with a disturbance

⁹Refers to both the state feedback controller and the PID controller, in general optimal controllers derived using the invariant ellipsoid method.

$\|w\|_{\frac{1}{\gamma}M} < 1$. This implies that the ellipsoid $x^T P^{-1} x < \gamma$ is the minimum invariant ellipsoid for a linear scaled disturbance $\|w\|_{\frac{1}{\gamma}M} < 1$ with the same system and optimal controller gain K^* . In other words, the optimal controller gain remains the same and the invariant ellipsoid is expanded by a factor γ for a system with disturbance expanded by $\sqrt{\gamma}$. In particular for scalar disturbance where $w^2 < \frac{\gamma}{M}$, the optimal control gain K^* is the same for any $\gamma > 0$ and $M > 0$. Figure 5.5 shows an illustration of the various invariant ellipsoid with the corresponding disturbances $|w| < \sqrt{\frac{\gamma}{M}}$ for a scalar w .

The key observation is that the same optimal controller gain is obtained for disturbances that are enlarged or reduced correspondingly. The main difference is a mere scaling of the invariant ellipsoid. This observation is important for the analysis of stochastic disturbance.

Figure 5.5: An illustration of the level set of error response bound when subjected to different levels of disturbance $|w| < \sqrt{\frac{\gamma}{M}}$ for a scalar w with $M = 0.111$ for system (5.32).

5.3.3.2 Exponential Attraction to Ellipsoid

It is given in [47] that the time for an initial state e_0 with $e_0^T P^{-1} e_0 > \sigma$, $\sigma > 1$ to enter the invariant ellipsoid $e^T P^{-1} e < \sigma$ is

$$T_{\sigma, P} \leq \frac{1}{\alpha} \ln \frac{e_0^T P^{-1} e_0 - 1}{\sigma - 1}. \quad (5.38)$$

It is a common practice to let σ be a value slightly greater than 1, for instance $\sigma = 1.05$.

This implies that if there is a sufficiently long period of time, $t \in [T_1, T_2]$ and $T_2 - T_1 > T_\gamma$,

$$T_{\sigma, \gamma P} = \frac{1}{\alpha} \ln \frac{\frac{1}{\gamma} e_0^T P^{-1} e_0 - 1}{\sigma - 1}, \quad (5.39)$$

for σ slightly greater than 1 such that the external disturbance bounds can be rewritten as $D'_1 = \sqrt{\gamma} D_1$ and $D'_2 = \sqrt{\gamma} D_2$ for $\gamma < 1$ (the disturbance is smaller than the designed value), then error is expected to converge exponentially to a smaller ellipsoid given by $P' = \gamma P$. This is useful for the analysis of stochastic disturbance.

5.3.3.3 Stochastic Disturbance

A stochastic disturbance is a disturbance that takes value according to a probability distribution. It is usually described by a Gaussian distribution and is characterized by its mean and variance. For a system subjected to stochastic disturbance, a controller that gives a minimal invariant ellipsoid also gives good stochastic disturbance rejection.

This is because the level set for each level of invariant ellipsoid γP for $\gamma > 0$ is a minimum invariant ellipsoid for a disturbance $\|w\|_M \leq \sqrt{\gamma}$. Furthermore if $\|w\|_M \leq \sqrt{\gamma}$ for a time period greater than $T_{\sigma, \gamma P}$, the state is guaranteed to converge into the invariant ellipsoid γP . For a disturbance generated by a stochastic process, the controller gain obtained from the invariant ellipsoid method ensures that the state remains in the minimum possible ellipsoid $\gamma_{\min} P$ where $\gamma_{\min} = \inf_{\gamma > 0} \{\|w\|_M < \sqrt{\gamma}\}$ for any time period

$T_1 \leq t \leq T_2$ satisfying $T_2 - T_1 = T_{\sigma, \gamma_{\min}} P$. In other words, the states will always converge into a minimum possible invariant ellipsoid for any randomly chosen time period. This behavior ensures good stochastic disturbance rejection.

5.3.4 Gain Synthesis with Measurement Noises

The multi-rotor UAV ship deck landing model is modelled as a an output disturbed LTI system in this formulation. The invariant ellipsoid method is used to synthesize an optimal controller gain that minimizes the state response. It is also remarked that high gain design is undesirable due to some unmodelled effects such as measurement noises. This affects the optimization results presented in Section 5.3.1 and 5.3.2.

In the case of measurement noises¹⁰, the impact to the system response is more intricate. We assume that the state measurement is subjected to noise n and the measured state becomes $\bar{x} = x + n$. This is feedback into the multi-rotor UAV ship deck landing system using a state feedback controller $u = K\bar{x} = Kx + Kn$. Therefore the effect of measurement noises becomes amplified with a high gain design due to the Kn term. The same argument holds for a PID control system.

On the other hand, Figure 5.2 and 5.4 suggest that a high gain design is necessary for a small invariant ellipsoid. This constitutes a tradeoff between a high gain design that results in the reduction of the effect of output disturbance (ship deck motion) and a low gain design that results in lower sensitivity to input disturbance (measurement noise). An optimal controller gain that balances these effects is desirable.

Including the effect of measurement noise, the closed-loop system becomes

$$\dot{x} = (A + BK)x + BK n + Bw. \quad (5.40)$$

¹⁰A form of input disturbance for a state feedback control system.

We assume the measurement noise is uncorrelated and bounded such that $\|n\|_N < 1$ for a diagonal weighting matrix N . The closed-loop system can be rewritten as

$$\dot{x} = (A + BK)x + \bar{D}\bar{w}, \quad (5.41)$$

where $\bar{D} = \text{diag}(BK, B)$, $\bar{w} = \begin{pmatrix} n & w \end{pmatrix}^T$ and $\frac{1}{2}\bar{w}^T \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} \bar{w} < 1$. Notice that $\frac{1}{2} \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix}$ is a symmetric positive definite weighting matrix that accounts for the expansion of the vector $\bar{w}$ to include $\|w\|_M < 1$ and $\|n\|_N < 1$ simultaneously. This is a conservative but convenient estimate for the resultant disturbance vector, an exact formulation for multiple disturbances is given in Chapter 6.

It is clear that the invariant ellipsoid condition for the closed-loop system in (5.41) is bilinear in K and P due to the term $\bar{D} = \text{diag}(BK, B)$. Therefore another approach is used to obtain an optimal controller gain K .

A set of controller gains $\{K\}$ for an initial range of α is computed without the consideration of measurement noise n using the methods described in Section 5.3.1 and 5.3.2. The resultant invariant ellipsoid under each optimal controller gain K in the set $\{K\}$ will be expanded in the presence of measurement noise n . The expanded invariant ellipsoid set $\{P\}$ is obtained by computing the minimum invariant ellipsoid among the optimal controller gain set $\{K\}$ using the following optimization problem for the given α .

Optimization 5.10.

$$\min_{P>0, t>0} t$$

subject to

$$AP + BK + (*) + \alpha P + \frac{2}{\alpha} \bar{D} \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix}^{-1} \bar{D}^T \leq 0,$$

$$P - tI_2 \leq 0.$$

Figure 5.6: The optimization procedure to obtain the optimal control gain in the presence of measurement noise.

Only the expansion of the invariant ellipsoid is of interest here, therefore only the invariant ellipsoid condition is listed as the constraint. The proposed procedure is illustrated in Figure 5.6. This is a two part approach where a set of candidate controller gains are computed then evaluated for their performance under measurement noise. Hence the controller gain obtained is only an approximation for the true optimal controller gain. Nevertheless, it provides an efficient method to obtain a near-optimal controller gain¹¹.

An evaluation of the expansion in invariant ellipsoid with α is shown in Figure 5.7. It is seen that the size of the invariant ellipsoid after factoring measurement noise becomes increasingly larger after a critical α^* (size) due to larger optimal controller gain K for $\alpha > \alpha^*$. This shows that there is a limit to the minimum size of the invariant ellipsoid in the presence of measurement noise. This critical size provides a critical controller gain K^* that gives the best performance in the presence of both input and output disturbances.

It is noted that input disturbance such as measurement noise is commonly modeled stochastically. From Section 5.3.3, this approach also performs well with stochastic disturbance. The optimal controller gain obtained from this methodology is simulated and verified in the next section.

¹¹A full approach to obtain the optimal controller gain is presented in Chapter 6.

Figure 5.7: The variation of maximum eigenvalue of the invariant ellipsoid with α before and after the introduction of noise.

5.4 Numerical Simulations

The invariant ellipsoid method is used to formulate an optimal controller for a multi-rotor UAV ship deck landing sub-problem in the previous section. Both a state feedback and a PID controller are formulated such that the state response is bounded in a minimum invariant ellipsoid. In addition, an optimal controller gain selection methodology (see Figure 5.6) is also presented that incorporates measurement noises.

In this section, numerical simulations are performed for a quadrotor landing on a ship deck using the controllers presented in this chapter. This serves to validate the optimality of the controllers and verify them as a solution to the landing sub-problem.

5.4.1 Ship Deck Landing Scenario

A quadrotor is initially placed at a 10m height above the mean sea level (reference height) and is displaced 1m away in the planar directions from a target landing pad. ie.

$(X_0 \ Y_0 \ Z_0)^T = (1 \ 1 \ -10)^T$. The quadrotor parameters are listed in Table 4.1.

The ship is assumed to heave with the sea elevation with an unity Response Amplitude Operator (RAO) without any loss of generality and the landing pad is located on the ship deck which is at 1m above the sea level. The ship heave model follows (5.28) with its energy spectrum corresponding to sea state 6.

Figure 5.8: The landing procedure of a quadrotor on a ship deck. The quadrotor tracks and maintain relative height above the landing pad first, followed by a gradual landing phase.

The controller is commanded to bring and maintain the quadrotor at a spot 10m above the landing location for 50s. This time interval is an arbitrarily chosen value for safety purpose. In this phase, the quadrotor attempts to follow the ship deck heave motion while the tracking error converges into the invariant ellipsoid. Thereafter the height of the quadrotor is reduced gradually in a landing phase (see Figure 5.8). A smooth sigmoid function is chosen as the reference path,

$$H = H_0 + \frac{(H_f - H_0)}{1 + \exp\left(\frac{-6(2(t-t_0)-\Delta t)}{\Delta t}\right)}, \quad (5.42)$$

where H_0 , H_f are the initial and final height, t_0 is the time when the height is commanded to decrease and Δt is the time duration of the descend. In this simulation, the following parameters are used. $H_0 = 10$, $H_f = 1$, $t_0 = 50$ and $\Delta t = 30$.

The planar motion controller used in all numerical simulations is the full state feedback backstepping controller presented in Chapter 4.

5.4.2 PID Controller

The performance of a PID ship deck landing controller is first validated. The altitude controller used is the PID ship deck landing controller presented in Section 5.3.2. As shown in Figure 5.4.b, the optimal controller gain becomes large as $\text{tr}(P)$ decreases and α increases, therefore a tradeoff gain value is selected. The following reasonably large PID controller gain is chosen based on the invariant ellipsoid method, $K_i = -23.9$, $K_p = -32.0$ and $K_d = -13.0$ at $\alpha = 1$.

To provide a basis for comparison, a baseline PID controller with gain chosen from the loop shaping method is also simulated. The controller gain for the baseline case is obtained from Matlab PID Tuning Algorithm [3]. The target 0dB gain crossover frequency of the tuned open-loop response is set as 3rad/s, which covers the frequencies range of the ship heave motion (see Figure 3.4). The PID controller gain is computed as $K_i = -0.2$, $K_p = -4.5$ and $K_d = -2.6$.

The simulation result is shown in Figure 5.9. The states are plotted as new and classical for the controller gain derived from the invariant ellipsoid method and the baseline gain respectively.

5.9.a: Ship Heave

5.9.b: Tracking Error

5.9.c: Relative Velocity Error

5.9.d: Relative Height

5.9.e: Thrust

Figure 5.9: Results from the numerical simulation of a quadrotor landing on a ship deck with two PID ship deck landing controllers whose gains are obtained from the invariant ellipsoid method (solid line) and loop shaping method (dotted line). Figure 5.9.a shows the heave motion of the ship deck, with the mean sea level at $h = 0$. Figure 5.9.b shows the tracking error e of the quadrotor, a small tracking error indicates that the quadrotor is able to track the ship deck motion. Figure 5.9.c shows the relative velocity error between the quadrotor and the ship deck, a small relative velocity error indicates small impact impulse. 5.9.d shows the quadrotor height relative to the ship deck. Lastly, Figure 5.9.e shows the quadrotor thrust u_1 .

From Figure 5.9.b, the optimal PID controller gain performed better than the gain obtained from the classical loop shaping method (baseline). This is because the invariant ellipsoid is minimized with respect to the ship heave disturbance, therefore the tracking error e is bounded more tightly to zero.

This has a positive consequence towards the landing mission on a ship deck. The quadrotor is able to track the relative reference height more closely, evident from a small error in relative height and speed between the quadrotor and the ship deck. This reduces the possibility of damage due to hard landing. Figure 5.9.d highlights this effect.

Figure 5.9.c further shows the relative velocity between the quadrotor and the ship deck. The simulation with the optimal controller gain has a smaller relative velocity

throughout landing, therefore the *impact impulse*¹² experienced upon touched down is smaller hence lesser chance of damage. Based on the simulation, the impact impulse is of the order of 0.1Ns with the optimal controller gain while the impact impulse with the baseline controller gain is about 6 times more at 0.6Ns.

Despite having a larger gain as compared to the baseline controller gain, the control action given by the thrust of the quadrotor is not significantly larger as shown in Figure 5.9.e. This is because of the inclusion of a limit on the control action during the optimization of the PID controller gain.

It should be noted that the optimal controller gain works better in the presence of disturbance (in this case the ship deck motion). While in the absence of disturbance, the performance of traditionally tuned controller approaches that of the optimal controller gain. This is shown in Figure 5.10 where the ship deck motion is set to zero.

Figure 5.10: Height of the quadrotor relative to the ship deck without ship deck heave motion. The solid line is the PID ship deck landing controller with gain derived from the invariant ellipsoid method while the dotted line uses the baseline controller gain.

¹²Impact impulse refers to the change in momentum of the quadrotor just before and after the impact. It is a measure of the force upon impact.

5.4.2.1 Robustness Analysis

It is shown in the simulation result from Figure 5.9 that the performance of the PID controller gains derived from the invariant ellipsoid method is able to attenuate disturbance from the ship heave motion effectively. However, the optimization problem that is solved for the controller gains uses a nominal system where $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. In real systems, these system matrices may not be exact and its effect needs to be analyzed.

Section 5.1.4 provides the methodology for the required robustness analysis. In the case of ship deck landing, suppose the matrix B to be

$$B = \gamma_1 B_1 + \gamma_2 B_2, \quad (5.43)$$

where $B_1 = \begin{pmatrix} 0 \\ 0 \\ 0.5 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 \\ 0 \\ 1.5 \end{pmatrix}$ and $\gamma_1 + \gamma_2 = 1$. It is noted that the nominal value of B is when $\gamma_1 = \gamma_2 = 0.5$.

For comparison, the following optimization is solved to obtain the trace of the minimum invariant ellipsoid when using the optimal controller gain $K_i = -23.9$, $K_p = -32.0$ and $K_d = -13.0$.

Optimization 5.11.

$$\min_{P>0} \text{tr}(P)$$

subject to

$$\begin{aligned} AP + B_1 K P + (*) + \alpha P + \frac{1}{\alpha} B_1 M^{-1} B_1^T &\leq 0, \\ AP + B_2 K P + (*) + \alpha P + \frac{1}{\alpha} B_2 M^{-1} B_2^T &\leq 0. \end{aligned}$$

The minimum invariant ellipsoid obtained is $\begin{pmatrix} 0.0328 & -0.0167 & -0.0228 \\ -0.0167 & 0.0441 & -0.0445 \\ -0.0228 & -0.0445 & 0.6884 \end{pmatrix}$ with a trace of 0.7654. This value is compared to the trace of the minimum invariant ellipsoid obtained with the same controller gain for the nominal system only which is 0.4711. From Figure 5.4.b, this increase in trace value is not expected to have drastic detrimental effect on the performance on system with B matrix from the set (5.43). Hence there is sufficient robustness in the closed-loop system.

To demonstrate the controller gain's robustness, a Monte-Carlo simulation¹³ of 10 runs is performed with different B matrices from the set (5.43) as well as random ship heave motion. The simulation result is shown in Figure 5.11. The graphs are overlapped because the bounds on the tracking error and relative velocity error have more importance.

5.11.a: Tracking Error

5.11.b: Relative Velocity Error

¹³A Monte-Carlo simulation is a set of simulations with different possibility for the simulation condition. In this case, a set of B matrices that is randomly selected is simulated. The number of random simulations relates to the confidence level for the controller.

5.11.c: Relative Height

Figure 5.11: Monte Carlo simulation of a quadrotor landing on a ship deck with two PID ship deck landing controllers whose gains are obtained from the invariant ellipsoid method for 10 different B matrices. Figure 5.11.a shows the tracking error e of the quadrotor, a small tracking error indicates that the quadrotor is able to track the ship deck motion. Figure 5.11.b shows the relative velocity error between the quadrotor and the ship deck, a small relative velocity error indicates small impact impulse. 5.11.c shows the quadrotor height relative to the ship deck.

From Figure 5.11, the initial large deviation from zero error is because some of the initial states are not in the invariant ellipsoid. After the states converge into the invariant ellipsoid, the relative height (Figure 5.11.a) and velocity (Figure 5.11.b) are tightly bounded to zero. This restricts the impact upon landing. It is also evident that the closed-loop system performs well for changes in system parameter, hence robustness is demonstrated.

5.4.3 State Feedback Controller with Measurement Noise

The simulation in the previous subsection uses a PID ship deck landing controller in the ship deck landing sub-problem. Performance and optimality of the optimal controller gain derived from the invariant ellipsoid method is verified and compared with a non-optimal

baseline controller gain. However, it is noted that measurement noise has significant influence on the invariant ellipsoid bound prompting a selection tradeoff between invariant ellipsoid size and noise sensitivity. This section implements a state feedback ship deck landing controller with gains obtained from the procedure given in Section 5.3.4 (see Figure 5.6).

The measurement noise considered in this optimization and simulation is a Gaussian white noise acting on the relative position and velocity measurements. The size of the measurement noise is given by $N = \text{diag}(0.01, 0.01)$ as in (5.41). This data is retrieved from typical LIDAR measurement accuracy from [21, 40].

Thereafter, the controller parameters are generated by solving the optimization problem with $\alpha^* = 2.3$ using a state feedback controller of the form $u = Kx$ for system (5.32). This value is chosen because the resultant controller gains, $K_1 = -31.4$ and $K_2 = -15.6$, gives rise to a minimum ellipsoid $\{x : V = x^T(P^*)^{-1}x < 1\}$ as shown in Figure 5.7 where $P^* = \begin{pmatrix} 0.0274 & -0.0411 \\ -0.0411 & 0.3438 \end{pmatrix}$.

The simulation results are shown in Figure 5.12 and 5.13.

5.12.a: Ship Heave

5.12.b: Tracking Error

5.12.c: Relative Velocity Error

5.12.d: Relative Height

5.12.e: Thrust

5.12.f: Measurement Noise on Position

5.12.g: Measurement Noise on Velocity

Figure 5.12: Results from the numerical simulation of a quadrotor landing on a ship deck with two PID ship deck landing controllers whose gains are obtained from the invariant ellipsoid method (solid line) and loop shaping method (dotted line). Figure 5.12.a shows the heave motion of the ship deck, with the mean sea level at $h = 0$. Figure 5.12.b shows the tracking error e of the quadrotor, a small tracking error indicates that the quadrotor is able to track the ship deck motion. Figure 5.12.c shows the relative velocity error between the quadrotor and the ship deck, a small relative velocity error indicates small impact impulse. 5.12.d shows the quadrotor height relative to the ship deck. Figure 5.12.e shows the quadrotor thrust U_1 . Lastly, Figure 5.12.f and 5.12.g show the measurement noises acting on the relative position and velocity measurements.

Figure 5.13: The phase diagram of the state x and the invariant ellipsoid $V = 1$ and the state of the system bounded within the invariant ellipsoid, $V \leq 1$.

The simulation results show that the controller gain obtained via the algorithm in Section 5.3.4 (see Figure 5.6) is able to track the ship deck heave motion in the presence of measurement noise to within good performance bound, thus able to perform landing on a ship heaving in the open sea at sea state 6 to a good degree of safety. More importantly, it shows that the invariant ellipsoid method is able to synthesize controller gain for a state feedback controller in the presence of measurement noise that results in a good performance.

Figure 5.13 shows the phase diagram of the states $x = \begin{pmatrix} e & \dot{e} \end{pmatrix}^T$ with the invariant ellipsoid $\{x : V = x^T(P^*)^{-1}x < 1\}$. The states are initially outside the invariant ellipsoid and are attracted to the ellipsoid and remain within it thereafter. The states are guaranteed to remain in the invariant ellipsoid $V < 1$ by the formulation of the problem.

Figure 5.12.b shows the tracking error of the quadrotor. The error remains sufficiently small for practical application on ship deck landing. Furthermore, Figure 5.12.d shows the height of the quadrotor above the ship deck. Smooth and safe landing is achieved in this simulation.

Figure 5.12.c, on the other hand, shows the relative velocity error between the quadrotor and the ship deck. A large relative velocity error upon impact induces a large impact impulse that can potentially damage the quadrotor. Therefore it is desirable to keep the relative velocity error hence the impact impulse small. The impact impulse computed from this simulation is about 0.25Ns which is sufficiently small for a small sized quadrotor. Furthermore it is shown in Figure 5.13 that the maximum absolute relative velocity error is 0.6m/s thus the maximum absolute impact impulse is about 0.75Ns which is reasonably small.

This section demonstrated and verified the performance of both a state feedback and a PID controller in a ship deck landing sub-problem. The ship deck landing sub-problem requires the development of a ship motion predictor-free controller that is able to achieve high level of tracking and safety in the presence of pseudorandom ship heave motion. The invariant ellipsoid method is demonstrated to be able to provide a sufficient safety bound using a state feedback and a PID controller. Furthermore, the inclusion of measurement noise improves the validity of the simulations and verifies the methodology stated in Section 5.3.4 (see Figure 5.6).

5.5 Conclusion

5.5.1 Implications and Significance of Results

The following discusses some of the potential implication and significance of the work done in this chapter.

The invariant ellipsoid method (Section 5.1) is formulated as a LMI condition for a linear system to estimate the size of a minimum invariant ellipsoid that bounds the system response. This is applied as a PID controller gain tuning algorithm (Section 5.3.2) that allows the synthesis of an optimal controller gain for a linear system subjected to persistent bounded disturbance. This expands the performance measure of a linear system to

include persistent bounded disturbance performance in addition to traditional measures such as overshoot and settling time. As most systems are often subjected to persistent bounded disturbance, this potentially allows for a better closed-loop performance.

Furthermore, the optimization problem formulated in this chapter (Section 5.1) also provides an alternative framework for conventional linear state space optimal control such as LQR with the inclusion of a disturbance term. The contribution of the invariant ellipsoid method-based optimization is its explicit consideration of external disturbance and the system performance with its presence.

Lastly, although a static state feedback controller (Section 5.3.1) is presented in this chapter, this methodology can be applied to a wider array of controllers. For instance, a dynamic state feedback controller of the form

$$\begin{aligned}\dot{z} &= Hx, \\ u &= Kx + Lz.\end{aligned}$$

The potential application is wide. In addition, a dynamic output feedback controller is presented in the next chapter.

5.5.2 Concluding Remarks

The sub-problem of a multi-rotor UAV landing on a ship deck subjected to non-trivial heave motion is solved using the invariant ellipsoid method. An optimization problem using the invariant ellipsoid method to quantify the effect of external disturbance on a linear system is solved to obtain an optimal state feedback controller gain. This optimization is also expanded to include control action constraint and is also applied to a PID ship deck landing controller. The effect of measurement noise on the closed-loop system is also considered by computing the expansion of invariant ellipsoids when subjected to measurement noise. The controller gain that results in a minimum invariant ellipsoid

after the expansion is chosen as the optimal controller gain. Numerical simulations are also performed to verify and compare the ship deck landing controllers presented in this chapter. A sufficiently good performance measured by the invariant ellipsoid shows that the ship deck landing sub-problem is effectively solved.

This chapter mainly solves the minimum ellipsoid problem with a full state feedback (or similar) controller. However not all the states are available for feedback or can be accurately measured, therefore an output feedback controller is desirable. In the next chapter, the minimum ellipsoid problem is solved for an output feedback controller such that optimal performance in the presence of persistent ship heave motion is achieved.

Chapter 6

Multi-Rotor UAV Landing on Ship Deck: An Output Feedback Approach

A full state feedback multi-rotor UAV ship deck landing controller is synthesized using the invariant ellipsoid method in Chapter 5. This method minimizes the bound of the state response for a system subjected to bounded external disturbance. In this approach, the ship heave motion is modelled as a pseudorandom external disturbance and an optimal controller gain is synthesized such that the tracking error is bounded to a minimum invariant ellipsoid.

However this approach does not take into account the availability of state measurements. In most cases, the entire state of a system is often not available for feedback or is not sufficiently accurate for useful feedback. In the case of ship deck landing, the sensor suite available in most multi-rotor UAV only allows the measurement of relative height above the ship deck through sensors such as LIDAR, ultrasound and infrared sensors. Measurement and estimation of relative speed is largely affected by measurement noise, this causes the invariant ellipsoid to expand and increases the tracking error as discussed in Chapter 5. Thus excessively noisy feedback channels such as relative speed measurement are not useful in achieving good control performance.

In this chapter, a dynamic output feedback controller is formulated for a multi-rotor UAV ship deck landing sub-problem to resolve the above mentioned issue. In this case, only the relative height above the ship deck is available for feedback and an optimal output feedback controller is synthesized such that the resultant invariant ellipsoid is minimized. Furthermore, realistic effects such as wind disturbance and measurement noise are also considered explicitly in the optimization.

This chapter is organized as follows: Section 6.1 presents the main result of this chapter which is the formulation of an output feedback controller and the invariant ellipsoid condition for the general problem. The optimal gain for the output feedback controller is synthesized in Section 6.2. A novel path following iterative method that solves a linearized bilinear matrix inequality is presented. This is applied to a multi-rotor UAV ship deck landing problem in Section 6.3. Section 6.4 then presents and discusses the numerical simulation results of the output feedback controller implemented on a quadrotor landing on a ship deck. Lastly, Section 6.6 concludes this chapter with the significance and implications of this method followed by some concluding remarks.

6.1 Main Result

This section aims to formulate a dynamic output feedback controller using the invariant ellipsoid method for a linear time-invariant (LTI) system subjected to external disturbances and measurement noises. A general output feedback controller is first formulated followed by the statement of the invariant ellipsoid condition.

6.1.1 Output Feedback Controller Formulation

Consider a LTI system

$$\dot{x} = Ax + Bu + \sum_{i=1}^{n_d} D_i w_i, \quad (6.1a)$$

$$y = Cx + n, \quad (6.1b)$$

where $x \in \mathbb{R}^n$ is the state of the system, $u \in \mathbb{R}^m$ is the system input, $w_i \in \mathbb{R}^{p_i}$ are the external disturbances, $y \in \mathbb{R}^k$ is the measured output, $n \in \mathbb{R}^k$ is the measurement noise acting on all output channels, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{k \times n}$ and $D_i \in \mathbb{R}^{n \times p_i}$ are the system matrices for $i = 1, 2, \dots, n_d$.

It is noted that this system is an extension of (5.11) to include multiple disturbances and measurement noise. State disturbances arising from different sources are effectively represented in this system.

The external disturbances are bounded by

$$\|w_i\|_{M_i} < 1, \quad (6.2a)$$

$$\|n\|_N < 1 \quad (6.2b)$$

where $0 < M_i = M_i^T \in \mathbb{R}^{p_i \times p_i}$ for all $i = 1, 2, \dots, n_w$, and $0 < N = N^T \in \mathbb{R}^{k \times k}$.

In output feedback control, only the output y is available for feedback. Conventional approach is used to synthesize a dynamic output feedback controller by decoupling the control problem into an observer and controller design problem.

For the observer design, a Luenberger state observer is chosen as follows,

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}), \quad (6.3)$$

where $\hat{x} \in \mathbb{R}^n$ is the observed state and $L \in \mathbb{R}^{n \times k}$ is the observer gain. Substituting (6.1b) into (6.3), (6.3) becomes

$$\dot{\hat{x}} = A\hat{x} + Bu + LC(x - \hat{x}) + Ln. \quad (6.4)$$

On the other hand for the controller design, a control law is formulated using the estimated state $\hat{x}$ in the following static state feedback form

$$u = K\hat{x}, \quad (6.5)$$

where $K \in \mathbb{R}^{m \times n}$ is the controller gain.

Let the observation error be $\zeta = x - \hat{x}$, then its time derivative is

$$\dot{\zeta} = (A - LC)\zeta - Ln + \sum_{i=1}^{n_d} D_i w_i. \quad (6.6)$$

Substituting the control law (6.5) and the observation error into (6.1a), then (6.1a) becomes

$$\dot{x} = (A + BK)x - BK\zeta + \sum_{i=1}^{n_d} D_i w_i. \quad (6.7)$$

By separation principle, the closed-loop system given by (6.6) and (6.7) is stable if and only if $A - LC$ and $A + BK$ are stable. Furthermore it is also known that a Lyapunov stable LTI system is bounded input bounded output stable, thus the output response of a Lyapunov stable closed-loop system given by (6.6) and (6.7) is bounded if w_i and n are bounded for all $i = 1, 2, \dots, n_d$.

In the following part, an invariant ellipsoid condition is established to determine whether a given ellipsoid is invariant when the closed-loop LTI system given by (6.6) and (6.7) is subjected to bounded disturbances and measurement noises.

6.1.2 Invariant Ellipsoid Condition

The invariant ellipsoid condition derived in this subsection determines whether a particular ellipsoid is attractive and invariant for the closed-loop system (6.6) and (6.7). This is the main result of this chapter and is stated as the following theorem.

Theorem 6.1. *For an output feedback controlled linear time-invariant system given by (6.1), (6.3) and (6.5), an ellipsoid defined by $\{\xi : \xi^T P \xi < 1\}$, where $\xi = \begin{pmatrix} x & \zeta^T \end{pmatrix}^T$ and $P = \begin{pmatrix} P_1^{-1} & 0 \\ 0 & P_2 \end{pmatrix} = \begin{pmatrix} P_1^{-1T} & 0 \\ 0 & P_2^T \end{pmatrix} > 0$ is attractive and invariant if there exists $T_n \geq 0$*

and $\tau_{d,i} \geq 0$ for $i = 1, 2, \dots, n_d$ such that

$$\begin{pmatrix} \Pi_1 & -BK & 0 & D \\ * & \Pi_2 & -P_2L & P_2D \\ * & * & -T_nN & 0 \\ * & * & * & -T_dM \end{pmatrix} \leq 0, \quad (6.8)$$

where

$$\Pi_1 = (A + BK)P_1 + (*) + \alpha P_1, \quad (6.9)$$

$$\Pi_2 = P_2(A - LC) + (*) + \alpha P_2, \quad (6.10)$$

$$\alpha = \sum_{i=1}^{n_d} \tau_{d,i} + T_n, \quad (6.11)$$

$$D = \begin{pmatrix} D_1 & D_2 & \dots & D_{n_d} \end{pmatrix}, \quad (6.12)$$

$$T_d = \text{diag}(\tau_{d,1}I_{p_1}, \tau_{d,2}I_{p_2}, \dots, \tau_{d,N}I_{p_{n_d}}), \quad (6.13)$$

$$M = \text{diag}(M_1, M_2, \dots, M_{n_d}). \quad (6.14)$$

Proof. Consider the closed-loop system (6.6), (6.7) and a control Lyapunov function

$$V = \begin{pmatrix} x^T & \zeta^T \end{pmatrix} \begin{pmatrix} P_1^{-1} & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} x \\ \zeta \end{pmatrix} \triangleq \xi^T P \xi, \quad (6.15)$$

where $0 < P_1 = P_1^T \in \mathbb{R}^{n \times n}$ and $0 < P_2 = P_2^T \in \mathbb{R}^{n \times n}$ define the invariant ellipsoid, $\xi = \begin{pmatrix} x^T & \zeta^T \end{pmatrix}^T$ and $P = \begin{pmatrix} P_1^{-1} & 0 \\ 0 & P_2 \end{pmatrix}$. The time derivative of V is

$$\begin{aligned} \dot{V} = x^T P_1^{-1} & \left[(A + BK)\hat{x} - BK\zeta + \sum_{i=1}^{n_d} D_i w_i \right] \\ & + \zeta^T P_2 \left[(A - LC)\zeta - Ln + \sum_{i=1}^{n_d} D_i w_i \right] + (*). \end{aligned} \quad (6.16)$$

For an ellipsoid given by $V = \xi^T P \xi < 1$ to be attractive and invariant, the condition $\dot{V} < 0$ must be satisfied for all ξ such that $V = \xi^T P \xi \geq 1$. The condition $\dot{V} < 0$ is

expressed as

$$z^T \begin{pmatrix} P_1^{-1}(A + BK) + (*) & -P_1^{-1}BK & 0 & P_1^{-1}D \\ * & P_2(A - LC) + (*) & -P_2L & P_2D \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} z \leq 0, \quad (6.17)$$

where $z = \begin{pmatrix} x^T & \zeta^T & n^T & w_1 & \cdots & w_{n_d} \end{pmatrix}^T$.

Inequality (6.17) is to be satisfied for all z such that

$$z^T \begin{pmatrix} P_1^{-1} & 0 & 0 & 0 \\ * & P_2 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} z \geq 1, \quad (6.18)$$

$$z^T \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & N & 0 \\ * & * & * & 0 \end{pmatrix} z < 1, \quad (6.19)$$

$$z^T \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & M^{(i)} \end{pmatrix} z < 1, \quad (6.20)$$

where

$$M^{(i)} = \text{diag}(0^{(1)}, \dots, 0^{(i-1)}, M_i, 0^{(i+1)}, \dots, 0^{(n_d)}) \quad (6.21)$$

for all $i = 1, 2, \dots, n_d$. Matrix inequality (6.18) is equivalent to $\xi^T P \xi \geq 1$, while matrix inequalities (6.19) and (6.20) are equivalent to (6.2).

Applying S-procedures (Lemma 5.1, p.125) to the aforementioned inequalities, a sufficient invariant ellipsoid condition is

$$\begin{pmatrix} \Sigma_1 & -P_1^{-1}BK & 0 & P_1^{-1}D \\ * & \Sigma_2 & -P_2L & P_2D \\ * & * & -T_nN & 0 \\ * & * & * & -T_dM \end{pmatrix} \leq 0, \quad (6.22)$$

where

$$\Sigma_1 = P_1^{-1}(A + BK) + (*) + \tau P_1^{-1}, \quad (6.23)$$

$$\Sigma_2 = P_2(A - LC) + (*) + \tau P_2, \quad (6.24)$$

for some $\tau \geq \sum_{i=1}^{n_d} \tau_{d,i} + T_n \geq 0$, $\tau_{d,i} \geq 0$ and $T_n \geq 0$.

Observe that (6.22) can be expressed as

$$\begin{pmatrix} \Pi'_1 & -P_1^{-1}BK & 0 & P_1^{-1}D \\ * & \Pi_2 & -P_2L & P_2D \\ * & * & -T_nN & 0 \\ * & * & * & -T_dM \end{pmatrix} + \beta \begin{pmatrix} P_1^{-1} & 0 & 0 & 0 \\ * & P_2 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} \leq 0, \quad (6.25)$$

where

$$\Pi'_1 = P_1^{-1}(A + BK) + (*) + \alpha P_1^{-1}, \quad (6.26)$$

for some $\beta = \tau - \alpha \geq 0$. Since $\beta \begin{pmatrix} P_1^{-1} & 0 & 0 & 0 \\ * & P_2 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} \geq 0$, inequality (6.22) is always

satisfied (i.e. $\tau = \alpha$) for any matrix P that satisfies

$$\begin{pmatrix} \Pi'_1 & P_1^{-1}BK & 0 & P_1^{-1}D \\ * & \Pi_2 & -P_2L & P_2D \\ * & * & -T_nN & 0 \\ * & * & * & -T_dM \end{pmatrix} \leq 0. \quad (6.23)$$

Hence (6.23) is a sufficient condition for (6.22). This implies that (6.23) is also a sufficient condition for the ellipsoid $\{\xi : \xi^T P \xi < 1\}$ to be attractive and invariant.

Perform a congruent transformation¹ with matrix $T_1 = \begin{pmatrix} P_1 & 0 & 0 & 0 \\ * & I & 0 & 0 \\ * & * & I & 0 \\ * & * & * & I \end{pmatrix}$, then (6.23)

becomes

$$\begin{pmatrix} \Pi_1 & -BK & 0 & D \\ * & \Pi_2 & -P_2 L & P_2 D \\ * & * & -T_n N & 0 \\ * & * & * & -T_d M \end{pmatrix} \leq 0. \quad (6.24)$$

Inequality (6.24) is the invariant ellipsoid condition that determines whether an ellipsoid $\left\{ \xi : \xi^T P \xi = \xi^T \begin{pmatrix} P_1^{-1} & 0 \\ 0 & P_2 \end{pmatrix} \xi < 1 \right\}$ defined by $P_1 = P_1^T > 0$ and $P_2 = P_2^T > 0$ is attractive and invariant for a system given by (6.1), (6.3) and (6.5). $\square$

This theorem is the main result for this chapter that gives the matrix inequality condition to determine whether an ellipsoid $\{\xi : \xi^T P \xi < 1\}$ is attractive and invariant. In the next subsection, the existence condition for a solution² to (6.8) is developed. This existence condition is critical for the controller and observer gains synthesis procedure.

6.1.3 Existence of Solution

Theorem 6.1 states the matrix inequality used to determine whether an ellipsoid given by $\{\xi : \xi^T P \xi < 1\}$ is attractive and invariant. The following theorem provides the existence condition for a solution satisfying (6.8) for the closed-loop system of the form (6.6) and (6.7). This existence condition is required for the synthesis of the controller and observer gains. Prior to the statement of the theorem, several definitions and lemmas are first stated.

¹A congruent transformation of a matrix A by a transformation matrix T is defined as $T^T A T$.

²Solution refers to a set of parameters $\{P_1, P_2, K, L\}$ for a given system.

Definition 6.1. A matrix A is considered to be α -stable if and only if the matrix $A + \alpha I_n$ is asymptotically stable.

The following are the properties of an α -stable matrix A .

- (i) The definition of an α -stable matrix A is equivalent to the existence of $P = P^T > 0$ such that $AP + (*) + 2\alpha P \leq 0$ or $PA + (*) + 2\alpha P \leq 0$.
- (ii) For any $S = S^T \geq 0$, there exists $P = P^T > 0$ such that $AP + (*) + 2\alpha P \leq -S$ or $PA + (*) + 2\alpha P \leq -S$.
- (iii) A matrix that is α -stable has the real part of all the eigenvalues of the state matrix A no larger than $-\alpha$. ie. $\text{Re}(\lambda_i) \leq -\alpha$ for all $\lambda_i \in \text{eig}(A)$.

The notion of α -stability is stronger than Lyapunov stability.

The necessary condition for a solution to (6.8) is stated using α -stability.

Lemma 6.1. *There exists $0 < P_1 = P_1^T \in \mathbb{R}^{n \times n}$ and $0 < P_2 = P_2^T \in \mathbb{R}^{n \times n}$ that satisfy the invariant ellipsoid condition (6.8) in Theorem 6.1 is satisfied only if the matrices $A + BK$ and $A - LC$ are $\frac{\alpha}{2}$ -stable.*

Proof. The invariant ellipsoid condition (6.8) in Theorem 6.1 only if the diagonal terms are all negative semi-definite. By definition, $-T_n N \leq 0$ and $-T_d M \leq 0$. Furthermore, $\Pi_1 \leq 0$ implies that

$$(A + BK)P_1 + (*) + \alpha P_1 \leq 0. \quad (6.25)$$

Therefore the matrix $A + BK$ must be $\frac{\alpha}{2}$ -stable by definition.

Likewise can be proven for $\Pi_2 \leq 0$, and the matrix $A - LC$ must also be $\frac{\alpha}{2}$ -stable. $\square$

To prove that the converse is true, the following lemma is stated.

Lemma 6.2. *For any $\frac{\alpha}{2}$ -stable matrices $A + BK$ and $A - LC$, and any $0 \leq S = S^T \in \mathbb{R}^{2n \times 2n}$, there exists $0 < P_1 = P_1^T \in \mathbb{R}^{n \times n}$ and $0 < P_2 = P_2^T \in \mathbb{R}^{n \times n}$ such that*

$$\begin{pmatrix} (A + BK)P_1 + (*) + \alpha P_1 & -BK P_2^{-1} \\ * & (A - LC)P_2^{-1} + (*) + \alpha P_2^{-1} \end{pmatrix} \leq -S. \quad (6.26)$$

Proof. Consider $0 \leq S = S^T = \begin{pmatrix} S_{11} & S_{12} \\ * & S_{22} \end{pmatrix}$, where $0 \leq S_{11} = S_{11}^T \in \mathbb{R}^{n \times n}$, $0 \leq S_{22} = S_{22}^T \in \mathbb{R}^{n \times n}$ and $S_{12} \in \mathbb{R}^{n \times n}$, the inequality (6.26) is re-expressed as

$$\begin{pmatrix} (A + BK)P_1 + (*) + \alpha P_1 + S_{11} & -BK P_2^{-1} + S_{12} \\ * & (A - LC)P_2^{-1} + (*) + \alpha P_2^{-1} + S_{22} \end{pmatrix} \leq 0. \quad (6.27)$$

Since matrix $A - LC$ is $\frac{\alpha}{2}$ -stable, there exist $P_2 = P_2^T > 0$ such that $(A - LC)P_2^{-1} + (*) + \alpha P_2^{-1} + S_{22} \leq 0$ by definition.

Thereafter using the method of Shur complement, the inequality (6.26) is equivalent to

$$(A + BK)P_1 + (*) + \alpha P_1 + S_{11} - R_1 \Xi_1^{-1} R_1^T \leq 0, \quad (6.28)$$

where $R_1 = -BK P_2^{-1} + S_{12}$ and $\Xi_1 = (A - LC)P_2^{-1} + (*) + \alpha P_2^{-1} + S_{22} \leq 0$. By definition of $\frac{\alpha}{2}$ -stable matrix $A + BK$, there exists $P_1 = P_1^T > 0$ such that (6.28) is satisfied since $S_{11} - R_1 \Xi_1^{-1} R_1^T \geq 0$. Hence the lemma is proven. $\square$

With Lemma 6.2, the converse of Lemma 6.1 is stated.

Lemma 6.3. *For any $\frac{\alpha}{2}$ -stable matrices $A + BK$ and $A - LC$, there exist $0 < P_1 = P_1^T \in \mathbb{R}^{n \times n}$ and $0 < P_2 = P_2^T \in \mathbb{R}^{n \times n}$ such that the invariant ellipsoid condition (6.8) in Theorem 6.1 is satisfied.*

Proof. Let $\Xi_2 = \begin{pmatrix} -T_n N & 0 \\ 0 & -T_d M \end{pmatrix} \leq 0$, the method of Schur complement transforms matrix inequality (6.8) in Theorem 6.1 into

$$\begin{pmatrix} (A + BK)P_1 + (*) + \alpha P_1 & -BK \\ * & P_2(A - LC) + (*) + \alpha P_2 \end{pmatrix} - \Gamma R_2 \Xi_2^{-1} R_2^T \Gamma \leq 0, \quad (6.29)$$

where $R_2 = \begin{pmatrix} 0 & D \\ -L & D \end{pmatrix}$ and $\Gamma = \begin{pmatrix} I & 0 \\ 0 & P_2 \end{pmatrix}$.

Pre- and post-multiply (6.29) by $\Gamma^{-1} = \begin{pmatrix} I & 0 \\ 0 & P_2^{-1} \end{pmatrix}$, (6.29) becomes

$$\begin{pmatrix} (A + BK)P_1 + (*) + \alpha P_1 & -BK P_2^{-1} \\ * & (A - LC)P_2^{-1} + (*) + \alpha P_2^{-1} \end{pmatrix} - R_2 \Xi_2^{-1} R_2^T \leq 0, \quad (6.30)$$

Therefore satisfying (6.30) implies (6.29).

According to Lemma 6.2, there exists $P_1 = P_1^T > 0$ and $P_2 = P_2^T > 0$ such that

$$\begin{pmatrix} (A + BK)P_1 + (*) + \alpha P_1 & -BK P_2^{-1} \\ * & (A - LC)P_2^{-1} + (*) + \alpha P_2^{-1} \end{pmatrix} \leq -S \quad (6.31)$$

for $S = -(R_2 \Xi_2^{-1} R_2^T)^{-1} \geq 0$, hence the lemma is proven. $\square$

From Lemma 6.1 and 6.3, the following theorem is stated.

Theorem 6.2. *There exist $0 < P_1 = P_1^T \in \mathbb{R}^{n \times n}$ and $0 < P_2 = P_2^T \in \mathbb{R}^{n \times n}$ that satisfy the invariant ellipsoid condition (6.8) in Theorem 6.1 if and only if the matrices $A + BK$ and $A - LC$ are $\frac{\alpha}{2}$ -stable.*

Proof. See Lemma 6.1 and 6.3. $\square$

Theorem 6.2 provides a criteria between the $\frac{\alpha}{2}$ -stability of $A + BK$ and $A - LC$ and the largest α in (6.8) that guarantees the existence of an invariant ellipsoid $P = P^T > 0$ that satisfies (6.8). (i.e. If $\alpha = 2$ is chosen for (6.8), then the closed-loop system matrices $A + BK$ and $A - LC$ must be 1-stable for the existence of a matrix P that satisfies (6.8).)

In this section, Theorem 6.1 and 6.2 are presented as the main result for an output feedback controller consisting of a Luenberger observer (6.3) and a static state feedback controller (6.5). Theorem 6.1 provides the matrix inequality condition (6.8) that determines the attractiveness and invariance of an ellipsoid. Furthermore Theorem 6.2 gives

the existence condition of an ellipsoid $\{\xi : \xi^T P \xi < 1\}$ that satisfies (6.8). This provides the basis for proper selection of α in relation to the $\frac{\alpha}{2}$ -stability of the closed-loop system matrices $A + BK$ and $A - LC$.

6.2 Output Feedback Controller Gain Synthesis

In this section, a controller and observer gain synthesis approach is presented such that the controller and observer gains K and L gives a minimum invariant ellipsoid $\{\xi : \xi^T P \xi < 1\}$ for a closed-loop system given by (6.6) and (6.7).

To determine the controller and observer gains K and L that minimizes the invariant ellipsoid according to Theorem 6.1 effectively, it is desirable to transform the invariant ellipsoid condition (6.8) from a bilinear matrix inequality (BMI)³ into a linear matrix inequality (LMI).

Based on Theorem 6.2, this section presents an approach to synthesize an optimal controller and observer gains K and L for an output feedback controller given by (6.3) and (6.5). In this method, the controller and observer gains K and L are improved iteratively from an initial estimate by solving a LMI optimization problem obtained from linearizing the full BMI invariant ellipsoid condition using small step approximation. The initial estimates for the controller and observer gains $K^{(1)}$ and $L^{(1)}$ are obtained from *a controller and an observer sub-problem* respectively. The initial estimates obtained from the proposed method yields sufficiently fast convergence to the optimized gains.

6.2.1 Sub-problem: Initial Controller Gain Synthesis

To obtain an initial controller gain K , the following system is considered.

$$\dot{x} = (A + BK)x + \sum_{i=1}^{n_d} D_i w_i, \quad (6.32)$$

³The matrix inequality is bilinear due to the terms KP_1 and P_2L .

where $x \in \mathbb{R}^n$ is the state of the system, $w_i \in \mathbb{R}^{p_i}$ are the external disturbances, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $D_i \in \mathbb{R}^{n \times p_i}$ and $K \in \mathbb{R}^{m \times n}$ are the system matrices for $i = 1, 2, \dots, n_d$.

A controller gain K that results in a minimum invariant ellipsoid for the above system is synthesized. The following theorems and lemmas are required to establish this optimization problem.

Remark 6.1. Note that the system considered in (6.32) is a full state feedback of the system in (6.1a).

Theorem 6.3. *For a system given by (6.32), an ellipsoid defined by $\{x : x^T P_1^{-1} x < 1\}$ is attractive and invariant if there exists a $\tau_{d,i} \geq 0$ for $i = 1, 2, \dots, n_d$ such that*

$$\begin{pmatrix} \Psi_1 & D \\ * & -T_d M \end{pmatrix} \leq 0, \quad (6.33)$$

where

$$\Psi_1 = AP_1 + BY_1 + (*) + \alpha_1 P_1, \quad (6.34)$$

$$\alpha_1 = \sum_{i=1}^{n_d} \tau_{d,i}, \quad (6.35)$$

$$Y_1 = K_1 P_1. \quad (6.36)$$

Proof. Consider the control Lyapunov function, $V_1 = x^T P_1^{-1} x$ where $0 < P_1 = P_1^T \in \mathbb{R}^{n \times n}$, its time derivative is

$$\dot{V}_1 = x^T P_1^{-1} \left((A + BK)x + \sum_{i=1}^{n_d} D_i w_i \right) + (*). \quad (6.37)$$

For the ellipsoid given by $\{x : x^T P_1^{-1} x < 1\}$ to be attractive and invariant, the condition $\dot{V}_1 < 0$ must be satisfied for all x and w_i such that $x^T P_1^{-1} x \geq 1$ and $\|w_i\|_{M_i} < 1$ where $i = 1, 2, \dots, n_d$. By S-procedure (Lemma 5.1, p.125), a sufficient condition is

$$\begin{pmatrix} \Psi_1 & D \\ * & -T_d M \end{pmatrix} \leq 0, \quad (6.38)$$

for some $\tau_{d,i} \geq 0$, $i = 1, 2, \dots, n_d$.

Let $Y_1 = KP_1 \in \mathbb{R}^{m \times n}$, (6.38) is transformed into a LMI given in (6.33). Hence the theorem is proven. $\square$

Remark 6.2. Note that the derivation of Theorem 6.3 is similar to that of Theorem 6.1.

Similarly, the existence of a matrix P_1 that satisfies (6.33) is stated in the following theorem.

Theorem 6.4. *There exists $0 < P_1 = P_1^T \in \mathbb{R}^{n \times n}$ that satisfies the invariant ellipsoid condition (6.33) in Theorem 6.3 if and only if the matrix $A + BK$ is $\frac{\alpha_1}{2}$ -stable.*

Proof. For necessity, observe that the invariant ellipsoid condition (6.33) in Theorem 6.3 implies that $\Psi_1 \leq 0$ thus the matrix $A + BK$ is $\frac{\alpha}{2}$ -stable by definition.

For sufficiency, observe that $-T_d K_d \leq 0$ and using the method of Schur complement,

$$\Psi_1 + D(T_d K_d)^{-1} D^T \leq 0, \quad (6.39)$$

where $D(T_d K_d)^{-1} D^T \geq 0$. If the matrix $A + BK$ is $\frac{\alpha_1}{2}$ -stable, there always exists a $P_1 = P_1^T > 0$ that satisfies (6.39).

Hence the theorem is proven. Note that this proof is similar to that of Theorem 6.2. $\square$

Since the controller gain K is related to the control action, it is also necessary to limit K such that the control action within the invariant ellipsoid does not cause controller saturation. In fact, Theorem 5.2 provides the control action condition. The inequality

$$\begin{pmatrix} P_1 & Y_1^T \\ Y_1 & R \end{pmatrix} \geq 0 \quad (6.40)$$

is satisfied only if the control action is restricted to the set $\{u : \|u\|_{R^{-1}} < 1\}$ for all x such that $x^T P_1^{-1} x < 1$.

The controller gain K that gives the minimum invariant ellipsoid with limited control action to this sub-problem is obtained by the following theorem.

Theorem 6.5. *The controller gain $K = Y_1 P_1^{-1}$ that gives a minimum invariant ellipsoid such that the control action is restricted to the set $\{u : \|u\|_{R^{-1}} < 1\}$ for all x within the ellipsoid for a system given by (6.32) is the solution to the following optimization:*

$$\min_{P_1 > 0, Y_1, t_1 > 0, t_2 > 0} \mu t_1 + \nu t_2$$

subject to

$$\begin{aligned} \begin{pmatrix} \Psi_1 & D \\ * & -T_d M \end{pmatrix} &\leq 0, \\ \begin{pmatrix} P_1 & Y_1^T \\ * & R \end{pmatrix} &\geq 0, \\ P_1 - t_1 I_n &\leq 0, \\ R - t_2 I_m &\leq 0, \end{aligned}$$

where D , T_d , M , Ψ_1 , α_1 and Y_1 are defined in (6.12), (6.13), (6.14), (6.34), (6.35) and (6.36) respectively.

Proof. The inequalities

$$\begin{pmatrix} \Psi_1 & D \\ * & -T_d M \end{pmatrix} \leq 0, \tag{6.41}$$

$$\begin{pmatrix} P_1 & Y_1^T \\ * & R \end{pmatrix} \geq 0, \tag{6.42}$$

are the invariant ellipsoid and control action conditions respectively and are consequences of Theorem 6.3 and 5.2. Any solution must satisfy these conditions.

Furthermore, the minimum invariant ellipsoid is determined by the minimization of $\lambda_1 \in \max \text{eig}(P_1)$, the largest eigenvalue of P_1 . Let $P_1 - t_1 I \leq 0$ where $t_1 \geq \lambda_1 > 0$, therefore minimizing t_1 also minimizes λ_1 . Similar conclusion can be obtained for the

minimization of the control action ellipsoid R , where minimizing t_2 minimizes the largest eigenvalue of R .

The objective function $\mu t_1 + \nu t_2$ is weighted with the invariant ellipsoid and the control action performance objectives by $\mu > 0$ and $\nu > 0$ respectively.

Hence the theorem is proven. $\square$

6.2.2 Sub-problem: Initial Observer Gain Synthesis

On the other hand, an observer gain initial estimate L is obtained by considering the following system.

$$\dot{\zeta} = (A - LC)\zeta - Ln + \sum_{i=1}^{n_d} D_i w_i, \quad (6.43)$$

where $x \in \mathbb{R}^n$ is the state of the system, $w_i \in \mathbb{R}^{p_i}$ are the external disturbances, $n \in \mathbb{R}^k$ is the measurement noise acting on all output channels, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{k \times n}$, $D_i \in \mathbb{R}^{n \times p_i}$ and $L \in \mathbb{R}^{n \times k}$ are the system matrices for $i = 1, 2, \dots, n_d$.

It is noted that (6.43) is identical to (6.6) thus this problem is a subset of the full controller and observer gain synthesis, hence the optimal L necessarily satisfies the invariant ellipsoid condition (6.8).

The invariant ellipsoid condition for the observer subsystem given by (6.43) is stated in the following theorem.

Theorem 6.6. *For a system given by (6.43), an ellipsoid defined by $\{\zeta : \zeta^T P_2 \zeta < 1\}$ is attractive and invariant if there exists $T_n \geq 0$ and $\tau_{d,i} \geq 0$ for $i = 1, 2, \dots, n_d$ that satisfy*

$$\begin{pmatrix} \Psi_2 & -Y_2 & P_2 D \\ * & -T_n N & 0 \\ * & * & -T_d M \end{pmatrix} \leq 0, \quad (6.44)$$

where

$$\Psi_2 = P_2 A + Y_2 C + (*) + \alpha_2 P_2, \quad (6.45)$$

$$\alpha_2 = \sum_{i=1}^{n_d} \tau_{d,i} + T_n, \quad (6.46)$$

$$Y_2 = P_2 L. \quad (6.47)$$

Proof. Consider the control Lyapunov function, $V_2 = \zeta^T P_2 \zeta$ where $0 < P_2 = P_2^T \in \mathbb{R}^{n \times n}$, its time derivative is

$$\dot{V}_2 = \zeta^T P_2 \left((A - LC)\zeta - Ln + \sum_{i=1}^{n_d} D_i w_i \right) + (*). \quad (6.48)$$

For the ellipsoid given by $\{\zeta : V_2 = \zeta^T P_2 \zeta < 1\}$ to be attractive and invariant, the condition $\dot{V}_2 < 0$ must be satisfied for all ζ , w_i and n such that $\zeta^T P_2 \zeta \geq 1$, $\|w_i\|_{M_i} < 1$ and $\|n\|_N < 1$ and $i = 1, 2, \dots, w_d$. By S-procedure (Lemma 5.1, p.125), a sufficient condition is

$$\begin{pmatrix} \Psi_2 & -P_2 L & P_2 D \\ * & -T_n N & 0 \\ * & * & -T_d M \end{pmatrix} \leq 0, \quad (6.49)$$

for some $T_n \geq 0$ and $\tau_{d,i} \geq 0$.

Let $Y_2 = P_2 L \in \mathbb{R}^{n \times k}$, (6.49) is transformed into a LMI

$$\begin{pmatrix} \Psi_2 & -Y_2 & P_2 D \\ * & -T_n N & 0 \\ * & * & -T_d M \end{pmatrix} \leq 0. \quad (6.50)$$

Hence the theorem is proven. $\square$

Similarly, the existence of a solution to (6.44) is stated in the following theorem.

Theorem 6.7. *There exists $0 < P_2 = P_2^T \in \mathbb{R}^{n \times n}$ that satisfies the invariant ellipsoid condition (6.44) in Theorem 6.6 if and only if the matrix $A - LC$ is $\frac{\alpha_2}{2}$ -stable.*

Proof. For necessity, observe that the invariant ellipsoid condition (6.44) in Theorem 6.6 implies that $\Psi_2 \leq 0$ thus the matrix $A - LC$ is $\frac{\alpha_2}{2}$ -stable by definition.

For sufficiency, observe that $\begin{pmatrix} -T_n N & 0 \\ * & -T_d M \end{pmatrix} \leq 0$ and using the method of Schur complement, (6.44) becomes

$$\Psi_2 + P_2 R_3 \Xi_3^{-1} R_3^T P_2 \leq 0, \quad (6.51)$$

or

$$\Psi'_2 + R_3 \Xi_3^{-1} R_3^T \leq 0, \quad (6.52)$$

where $R_3 = \begin{pmatrix} -L & D \end{pmatrix}$, $\Xi_3 = \begin{pmatrix} T_n N & 0 \\ * & T_d M \end{pmatrix} \geq 0$ and $\Psi'_2 = (A - LC)P_2^{-1} + (*) + \alpha_2 P_2^{-1}$.

If the matrix $A - LC$ is $\frac{\alpha_2}{2}$ -stable, there always exists a $P_2 = P_2^T > 0$ that satisfies (6.51) as $R_3 \Xi_3^{-1} R_3^T \geq 0$.

Hence the theorem is proven. Note that this proof is similar to that of Theorem 6.2. □

The observer gain L that gives a minimum invariant ellipsoid to this sub-problem is obtained from the following theorem.

Theorem 6.8. *The observer gain L that gives a minimum invariant ellipsoid for a system given by (6.43) is the solution to the following optimization:*

$$\min_{P_2 > 0, H > 0, Y_2, t > 0} t$$

subject to

$$\begin{pmatrix} \Psi_2 & -Y_2 & P_2 D \\ * & -T_n N & 0 \\ * & * & -T_d M \end{pmatrix} \leq 0, \\ \begin{pmatrix} H & I \\ I & P_2 \end{pmatrix} \geq 0, \\ H - tI \leq 0.$$

where D , T_d , M , Ψ_2 and α_2 are defined in (6.12), (6.13), (6.14), (6.45) and (6.46) respectively.

Proof. The inequality

$$\begin{pmatrix} \Psi_2 & -Y_2 & P_2 D \\ * & -T_n K_n & 0 \\ * & * & -T_d K_d \end{pmatrix} \leq 0, \quad (6.53)$$

is the invariant ellipsoid condition and a consequence of Theorem 6.6. Any solution P_2 that is an invariant ellipsoid $\{\xi : \xi^T P_2^T \xi < 1\}$ must satisfy this condition.

Furthermore, the size of the minimum invariant ellipsoid is determined by the minimization of $\lambda_2 \in \max \text{eig}(P_2^{-1})$, the largest eigenvalue of P_2^{-1} . Define $0 < H = H^T \in \mathbb{R}^{n \times n}$ such that

$$P_2^{-1} \leq H. \quad (6.54)$$

By the method of Schur complement, (6.54) is equivalent to

$$\begin{pmatrix} H & I \\ I & P_2 \end{pmatrix} \geq 0. \quad (6.55)$$

Therefore minimizing H implies minimizing P_2^{-1} , furthermore the constraint

$$H - t_2 I \leq 0, \quad (6.56)$$

implies $t_2 \geq \lambda_{\max}(H) \geq \lambda_{\max}(P_2^{-1})$ is the upper-bound for the maximum eigenvalue of H , where $\lambda_{\max}(X)$ denotes the largest eigenvalue of X . Therefore minimizing t_2 minimizes the invariant ellipsoid.

Hence the above optimization problem is justified. $\square$

6.2.3 Optimal Gain Synthesis

The controller and observer sub-problems presented the LMI optimization problems that are solved to obtain an initial estimate for the full gain synthesis. These initial estimates

ensure the fast convergence of the iterative method used. In this subsection, an iterative method that approximately solves a BMI by transforming it into a LMI is presented.

The optimal controller and observer gains are obtained by iteratively improving a sub-optimal solution. The full BMI invariant ellipsoid condition in Theorem 6.1 is linearized to obtain a LMI condition by considering small changes to the previous solution and neglecting higher order terms. This is possible by restricting the changes in the solution between iterations to be small.

An initial estimate to the solution is given by the controller and observer sub-problems (Theorem 6.5 and 6.8) as $K^{(1)}$ and $L^{(1)}$ respectively. These gains are such that $A + BK^{(1)}$ is $\frac{\alpha_1}{2}$ -stable and $A - L^{(1)}C$ is $\frac{\alpha_2}{2}$ -stable according to Theorem 6.4 and 6.7 respectively. Theorem 6.2 in turn guarantees the existence of an invariant ellipsoid $\{\xi : \xi^T P^{(1)} \xi < 1\}$ that satisfies (6.8) where $P^{(1)} = \text{diag}(P_1^{(1)^{-1}}, P_2^{(1)})$ for the closed-loop system (6.6) and (6.7) with controller and observer gains $K^{(1)}$ and $L^{(1)}$ if $\alpha_1, \alpha_2 \geq \alpha$.

Initial estimates $P_1^{(1)}$ and $P_2^{(1)}$ are obtained using the following optimization.

Optimization 6.1.

$$\min_{P_1^{(1)} > 0, P_2^{(1)} > 0, H > 0, t_1 > 0, t_2 > 0} t_1 + t_2$$

subjected to

$$\begin{pmatrix} \Pi_1 & -BK^{(1)} & 0 & D \\ * & \Pi_2 & -P_2^{(1)}L^{(1)} & P_2^{(1)}D \\ * & * & -T_n N & 0 \\ * & * & * & -T_d M \end{pmatrix} \leq 0,$$

$$\begin{pmatrix} H & I \\ I & P_2^{(1)} \end{pmatrix} \geq 0,$$

$$P_1^{(1)} - t_1 I \leq 0,$$

$$H - t_2 I \leq 0,$$

where

$$\Pi_1 = (A + BK^{(1)})P_1^{(1)} + (*) + \alpha P_1^{(1)}, \quad (6.57)$$

$$\Pi_2 = P_2^{(1)}(A - L^{(1)}C) + (*) + \alpha P_2^{(1)}, \quad (6.58)$$

The parameters α , D , T_d and M defined in (6.11), (6.12), (6.13) and (6.14) respectively⁴.

The set of parameters $\{P_1^{(1)}, P_2^{(1)}, K^{(1)}, Y_2^{(1)} = P_2^{(1)}L^{(1)}\}$ satisfies the invariant ellipsoid condition (6.8) and forms an initial estimate of the optimal controller and observer gains.

Let the set of solution at the $(i-1)^{th}$ iteration step be $\{P_1^{(i)}, P_2^{(i)}, K^{(i)}, Y_2^{(i)}\}$ and the solution at the i^{th} iteration step be $P_1^{(i+1)} = P_1^{(i)} + \delta P_1$, $P_2^{(i+1)} = P_2^{(i)} + \delta P_2$, $K^{(i+1)} = K^{(i)} + \delta K$ and $Y_2^{(i+1)} = Y_2^{(i)} + \delta Y_2$. Assuming that δP_1 , δP_2 , δK and δY_2 are small, thus $\delta P_1 \delta K$ can be neglected, the invariant ellipsoid condition (6.8) is rewritten as

$$\begin{pmatrix} \Pi_1^{(i+1)} & -BK^{(i+1)} & 0 & D \\ * & \Pi_2^{(i+1)} & -Y_2^{(i+1)} & P_2^{(i+1)}D \\ * & * & -T_n N & 0 \\ * & * & * & -T_d M \end{pmatrix} \leq 0, \quad (6.59)$$

where

$$\Pi_1^{(i+1)} = AP_1^{(i+1)} + BK^{(i)}P_1^{(i)} + BK^{(i)}\delta P_1 + B\delta K P_1^{(i)} + (*) + \alpha P_1^{(i+1)}, \quad (6.60)$$

$$\Pi_2^{(i+1)} = P_2^{(i+1)}A - Y_2^{(i+1)}C + (*) + \alpha P_2^{(i+1)}. \quad (6.61)$$

The linearized invariant ellipsoid condition (6.59) is a LMI and can be efficiently solved by semi-definite programming.

The following optimization (reduced LMI optimization) problem improves the solution at the i^{th} iteration step with the solution at the $(i-1)^{th}$ iteration step.

⁴Note that (6.57) and (6.58) are obtained from the substitution of the initial values of K , L , P_1 and P_2 into (6.9) and (6.10) respectively.

Optimization 6.2.

$$\min_{\delta P_1 > 0, \delta P_2 > 0, \delta K, \delta Y_2, H > 0, t_1 > 0, t_2 > 0} t_1 + t_2$$

subject to (6.59) and

$$\begin{pmatrix} H & I \\ I & P_2^{(i+1)} \end{pmatrix} \geq 0,$$

$$P_1^{(i+1)} - t_1 I \leq 0,$$

$$H - t_2 I \leq 0,$$

$$\|\delta P_1\|_F \leq \epsilon \|P_1^{(i)}\|_F,$$

$$\|\delta P_2\|_F \leq \epsilon \|P_2^{(i)}\|_F,$$

$$\|\delta K\|_F \leq \epsilon \|K^{(i)}\|_F,$$

$$\|\delta Y_2\|_F \leq \epsilon \|Y_2^{(i)}\|_F.$$

Remark 6.3. $\|A\|_F = \sqrt{\text{tr}(AA^T)}$ is the Frobenius norm of a real valued matrix A and is used to denote the change in size of the matrices $P_1^{(i)}$, $P_2^{(i)}$, $K^{(i)}$, $Y_2^{(i)}$ between successive iteration steps. The step size is determined by $\epsilon > 0$.

The solution $\{P_1^{(i)}, P_2^{(i)}, K^{(i)}, L^{(i)}\}$ to Optimization 6.2 may not necessarily satisfies (6.8) due to linearization approximation. However, it can be shown that it is guaranteed that there exists a feasible domain (i.e. solutions that satisfy (6.8)) for Optimization 6.2 at each iteration step if the solution in the previous step $\{P_1^{(i-1)}, P_2^{(i-1)}, K^{(i-1)}, L^{(i-1)}\}$ satisfies the invariant ellipsoid condition (6.8). This is stated in the following theorem.

Theorem 6.9. *For Optimization 6.2, there always exists a connected feasible domain ($\delta P_1 \in \mathbb{P}_1$, $\delta P_2 \in \mathbb{P}_2$, $\delta K \in \mathbb{K}$ and $\delta Y_2 \in \mathbb{Y}_2$) for the solution of the i^{th} iteration step to satisfy the invariant ellipsoid condition (6.8) if the solution to the $(i-1)^{\text{th}}$ iteration step*

also satisfies (6.8). i.e.

$$\begin{pmatrix} \Pi_1 & -BK^{(i)} & 0 & D \\ * & \Pi_2 & -Y_2^{(i)} & P_2^{(i)}D \\ * & * & -T_n N & 0 \\ * & * & * & -T_d M \end{pmatrix} \leq 0,$$

where

$$\Pi_1 = AP_1^{(i)} + BK^{(i)}P_1^{(i)} + (*) + \alpha P_1^{(i)}, \quad (6.62)$$

$$\Pi_2 = P_2^{(i)}A - Y_2^{(i)}C + (*) + \alpha P_2^{(i)}, \quad (6.63)$$

α , D , T_d and M defined in (6.11), (6.12), (6.13) and (6.14) respectively⁵.

Proof. This theorem is easily proved by considering (6.59) as

$$\begin{pmatrix} \Pi_1 & -BK^{(i)} & 0 & D \\ * & \Pi_2 & -Y_2^{(i)} & P_2^{(i)}D \\ * & * & -T_n N & 0 \\ * & * & * & -T_d M \end{pmatrix} + \begin{pmatrix} \pi_1 & -B\delta K & 0 & 0 \\ * & \pi_2 & -\delta Y_2 & \delta P_2 D \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} \leq 0, \quad (6.64)$$

where

$$\pi_1 = A\delta P_1 + BK^{(i)}\delta P_1 + B\delta K P_1^{(i)} + (*) + \alpha\delta P_1, \quad (6.65)$$

$$\pi_2 = \delta P_2 A - \delta Y_2 C + (*) + \alpha\delta P_2, \quad (6.66)$$

It is obvious that a feasible solution is obtained when $\delta P_1 = 0$, $\delta P_2 = 0$, $\delta K = 0$ and $\delta Y_2 = 0$. Furthermore by continuity of matrix positiveness, there also exist a connected neighborhood around $\delta P_1 = 0$, $\delta P_2 = 0$, $\delta K = 0$ and $\delta Y_2 = 0$ such that the solution of the i^{th} iteration step satisfies both (6.59) and the invariant ellipsoid condition (6.8). $\square$

⁵Note that (6.62) and (6.63) are obtained from the substitution of the i^{th} iteration values of K , L , P_1 and P_2 into (6.9) and (6.10) respectively.

Figure 6.1: The gradient descent approach in the iterative solution of the full BMI invariant ellipsoid condition. The objective function is reduced at each iteration step with a limit set on the improvements in the solution such that a LMI condition is solved. Iteratively, the local optimal solution is obtained within a connected feasible domain containing the initial condition.

Theorem 6.9 proves the existence of a connected feasible domain which satisfies the invariant ellipsoid condition (6.8) at each iteration step for sufficiently small $\{\delta P_1, \delta P_2, \delta K, \delta Y_2\}$. Therefore by iteratively solving the linearized BMI (which is a LMI) while limiting the variables $\{\delta P_1, \delta P_2, \delta K, \delta Y_2\}$ to be small, a locally optimized solution of the full BMI optimization problem is obtained. Figure 6.1 highlights the gradient descent nature of this approach at which a small step is allowed for each iteration step.

The iteration procedure is stated as follows:

- (i) Obtain a set of initial controller and observer gains $\{K^{(1)}, L^{(1)}\}$ using the controller and observer sub-problems. Then solve for the initial invariant ellipsoid that bounds the closed-loop system obtaining the initial solution set $\{P_1^{(1)}, P_2^{(1)}, K^{(1)}, Y_2^{(1)} = P_2^{(1)}L^{(1)}\}$.
- (ii) Iterate by solving the solution $\{P_1^{(i+1)}, P_2^{(i+1)}, K^{(i+1)}, Y_2^{(i+1)}\}$ for the i^{th} iteration step using the reduced LMI optimization.

- (iii) Check for the feasibility of the i^{th} solution $\{P_1^{(i+1)}, P_2^{(i+1)}, K^{(i+1)}, Y_2^{(i+1)}\}$ using Theorem 6.9. If the i^{th} solution $\{P_1^{(i+1)}, P_2^{(i+1)}, K^{(i+1)}, Y_2^{(i+1)}\}$ is found to be infeasible⁶, re-iterate with a reduced step size ϵ . Otherwise, repeat step 2 for the next iteration.
- (iv) Stop iteration if there is no feasible solution for a sufficiently small step size or if the changes in solution between successive iterations are small.

6.3 Application to Multi-Rotor UAV Ship Deck Landing

In this section, the output feedback controller discussed in Section 6.1 with gains synthesized in Section 6.2 is implemented in a multi-rotor UAV in a ship deck landing application.

The heave dynamics of a multi-rotor UAV is given in Chapter 3 and used in Chapter 5. It is restated here with an addition of a wind disturbance term as

$$\ddot{e} = u + w_d + w_s, \quad (6.67)$$

where e is the tracking error, u is the control input, w_d is the disturbance due to ship motion and w_s is the disturbance due to wind.

The external disturbances due to ship motion w_s and wind drag w_d are bounded as

$$\|w_s\|_{K_s} < 1, \quad (6.68a)$$

$$\|w_d\|_{K_d} < 1, \quad (6.68b)$$

where $0 < K_s \in \mathbb{R}$ and $0 < K_d \in \mathbb{R}$ are parameters indicating the size of the disturbances. These bounds on the external disturbances w_s and w_d are estimated using historical data.

Expressed in the state space form, (6.67) becomes

$$\dot{x} = Ax + Bu + B(w_d + w_s), \quad (6.69)$$

⁶This means that the solution does not satisfy the invariant ellipsoid condition (6.8).

where $x = \begin{pmatrix} e \\ \dot{e} \end{pmatrix}$ is the state, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the state matrix and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the control matrix.

The measured output of the UAV landing system is the height above the ship deck, $y = e + w_n$ which is

$$y = Cx + w_n, \quad (6.70)$$

where $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$ is the output matrix and w_n is the measurement noise with

$$\|w_n\|_{K_n} < 1, \quad (6.71)$$

where $0 < K_n \in \mathbb{R}$ is a parameter indicating the size of the noise. Similarly K_n is closely related to the sensor precision.

Using the proposed output feedback control structure, the observer is formulated as

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}), \quad (6.72)$$

where $L = \begin{pmatrix} L_1 & L_2 \end{pmatrix}^T$ is the observer gain and $\hat{x}$ is the estimated state.

Similarly, the controller is formulated as

$$u = K\hat{x}, \quad (6.73)$$

where $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$ is the controller gain.

The controller and observer gains K and L are synthesized using the method proposed in Section 6.2 with the following parameters,

$$D = \begin{pmatrix} B & B \end{pmatrix}, \quad (6.74)$$

$$T_d = \begin{pmatrix} \tau_s & 0 \\ 0 & \tau_d \end{pmatrix}, \quad (6.75)$$

$$N = K_n, \quad (6.76)$$

$$M = \begin{pmatrix} K_s & 0 \\ 0 & K_d \end{pmatrix}. \quad (6.77)$$

Numerical simulations to verify the performance of this system are conducted and are shown in the next section.

6.4 Numerical Simulations

This section discusses the results of the ship deck landing sub-problem numerical simulation with an output feedback controller presented in this chapter. The ship deck landing scenario is first described, followed by the parameters used in the simulation. Thereafter the landing operation of a quadrotor on a ship deck is simulated followed by some discussions on the results. Lastly, the performance of the optimized controller gains is compared with non-optimal gains.

6.4.1 Ship Deck Landing Scenario

The ship deck landing scenario is similar to the numerical simulations in Chapter 5. The parameters of the quadrotor used in this simulation are listed in Table 4.1. The key difference is the control structure implemented on the quadrotor which is the output feedback controller presented in this chapter.

Similarly, the quadrotor is initially placed at a 10m height above the mean sea level ($h = 10$). The landing pad is located 1m above the sea level. The controller is commanded to bring and maintain the quadrotor at a spot 10m above the landing pad for 50s. In this phase, the quadrotor attempts to follow the ship deck heave motion while limiting the tracking error to within the invariant ellipsoid. Thereafter the height of the quadrotor is reduced gradually in a landing phase (see Figure 5.8). A reference path similar to (5.42) is used.

In addition to the ship deck motion, external disturbance due to wind is also included. The constants K_s and K_d which are a measure of the external disturbance bounds are

set as $K_s = \frac{1}{9}$ and $K_d = 1$ respectively. Similarly, measurement noise is added to the relative height measurement (e) with a bound measures set as $K_n = 100$.⁷

The optimal controller and observer gains K^* and L^* are obtained using $\tau_n = \tau_d = \tau_s = 5$. The initial estimate of the controller gain $K^{(1)} = \begin{pmatrix} -442 & -42 \end{pmatrix}$ is obtained by solving the controller sub-problem (Theorem 6.5), while the initial estimate of the observer gain $L^{(1)} = \begin{pmatrix} 31 & 237 \end{pmatrix}^T$ is obtained by solving the observer sub-problem (Theorem 6.8). The optimal controller and observer gains $K^* = \begin{pmatrix} -338 & -34 \end{pmatrix}$ and $L^* = \begin{pmatrix} 96 & 914 \end{pmatrix}^T$ are obtained after 127 iterations with an objective function $t_1^* + t_2^* = 72.3$ from an initial value of $t_1^{(1)} + t_2^{(1)} = 151.6$.

The effect of the gradient descent approach in decreasing the objective function value each iteration is shown in Figure 6.2.a. The same approach is also performed with another initial estimate obtained from pole placement with $K^{(1)} = \begin{pmatrix} -72 & -17 \end{pmatrix}$ and $L^{(1)} = \begin{pmatrix} 17 & 72 \end{pmatrix}^T$. The objective function value changes with number of iterations as shown in Figure 6.2.b.

The initial estimate obtained from the controller and observer sub-problems (Theorem 6.5 and 6.8) exhibits a faster convergence speed (127 iterations) as compared to initial estimate through pole placement (1363 iterations). Furthermore, both initial estimates converged to a same K^* and L^* which is a consequence of the initial estimates belonging to the same connected feasible region. This demonstrates the local optimality of the proposed iterative process.

The simulation results of the quadrotor ship deck landing performed with the optimal output feedback controller are shown in Figure 6.3 and 6.4.

⁷Since all the disturbances and noise are scalar, the bound implies that $|w| < \frac{1}{\sqrt{K}}$ where w refers to any of the disturbances and noise and K is the bound.

6.2.a: Initial estimate obtained from Section 6.2

6.2.b: Initial estimate obtained from pole placement

Figure 6.2: The decrease in objective function value with the number of iterations using a gradient descent approach with initial estimate obtained from controller and observer sub-problems (Figure 6.2.a) and pole placement (Figure 6.2.b).

6.3.a: Ship deck heave

6.3.b: Height above ship deck

6.3.c: Tracking error

6.3.d: Relative velocity error

6.3.e: Thrust

6.3.f: Measurement noise

6.3.g: Wind disturbance

Figure 6.3: Simulation results of a quadrotor landing on a ship deck using the optimal output feedback controller. Figure 6.3.a shows the heave motion of the ship deck, with the mean sea level at $d = 0$. Figure 6.3.b shows the height of the quadrotor relative to the ship deck. The landing pad is located at 1m above the ship deck. Figure 6.3.c shows the tracking error $e_1 = e$ of the quadrotor, a small tracking error indicates that the quadrotor is able to track the ship deck motion. Figure 6.3.d shows the relative velocity error $e_2 = \dot{e}$ between the quadrotor and the ship deck, a small relative velocity error indicates small impact impulse. Figure 6.3.e shows the thrust input of the quadrotor and its limits. Figure 6.3.f shows the measurement noise acting on the relative position. Figure 6.3.g shows the wind disturbance acting on the quadrotor

From Figure 6.3.b, it is shown that the quadrotor is able to perform smooth and soft landing on the ship deck with the proposed control structure and optimized gains. Figure 6.3.c further supports this claim by showing that the tracking error e is maintained sufficiently small at about within 0.1m after an initial stabilization. In addition, the impact impulse computed from this simulation is of the order of 0.3Ns which is sufficiently small. This is achieved with a thrust input within the actuation limits as shown in Figure 6.3.e. In addition, ship heave, wind disturbance and measurement noise are also shown in Figure 6.3.a, 6.3.g and 6.3.f. The quadrotor response in terms of the Lyapunov function $V = \xi^T P \xi$ is also shown in Figure 6.4. The ellipsoid given by $\{\xi : V = \xi^T P \xi < 1\}$ is shown to be attractive and invariant as $V < 1$ throughout the landing.

A projection of the phase diagram onto the e_1 - e_2 plane is shown on Figure 6.5. It shows that the states of the system (e_1 and e_2) are bounded closely to zero, achieving the aim of soft landing. This is compared with the phase diagram Figure 5.13 in Chapter 5, with the spread of the states slightly larger for the simulation in this chapter. This is because addition wind disturbance effect is considered in this chapter.

Figure 6.4: The value of $V = \xi^T P \xi$. $V < 1$ indicates that the state of the system remains inside the ellipsoid. Furthermore, this ellipsoid is shown to be attractive and invariant.

Figure 6.5: The phase diagram of the error e_1, e_2 and a projection of the invariant ellipsoid onto the e_1 - e_2 plane.

6.4.2 Comparison using other Gains

The performance of the quadrotor landing operation using the optimal controller and observer gains obtained from the approach presented in section 6.2 is compared with (non-optimal) gains obtained using pole placement method. This serves to highlight the superiority of the optimal controller and observer gains and its effectiveness when used in disturbed and noisy systems. The gains selected are $K = \begin{pmatrix} -12 & -7 \end{pmatrix}$ and $L = \begin{pmatrix} 7 & 12 \end{pmatrix}^T$.

The simulation results using the pole placed controller and observer gains are shown in Figure 6.6.

6.6.a: Ship deck heave

6.6.b: Height above ship deck

6.6.c: Tracking error

6.6.d: Relative velocity error

6.6.e: Thrust

6.6.f: Measurement noise

6.6.g: Wind disturbance

Figure 6.6: Simulation results of a quadrotor landing on a ship deck using an output feedback controller with gain obtained from pole placement. Figure 6.6.a shows the heave motion of the ship deck, with the mean sea level at $d = 0$. Figure 6.6.b shows the height of the quadrotor relative to the ship deck. The landing pad is located at 1m above the ship deck. Figure 6.6.c shows the tracking error $e_1 = e$ of the quadrotor, a small tracking error indicates that the quadrotor is able to track the ship deck motion. Figure 6.6.d shows the relative velocity error $e_2 = \dot{e}$ between the quadrotor and the ship deck, a small relative velocity error indicates small impact impulse. Figure 6.6.e shows the thrust input of the quadrotor and its limits. Figure 6.6.f shows the measurement noise acting on the relative position. Figure 6.6.g shows the wind disturbance acting on the quadrotor

The performance parameters, relative height of the quadrotor above the ship deck and the tracking error, are shown in Figure 6.6.b and 6.6.c respectively. It is obvious that the performance using non-optimal controller and observer gains is not acceptable. Large variation of the tracking error e (about 1.5m) and relative velocity error (about 1.5m/s) leads to high impact velocity and potential damage to the quadrotor upon landing. It is further compared that the relative velocity error hence the impact impulse is roughly 7 times of the error when using the optimal controller and observer gains. This highlights the importance of using the invariant ellipsoid method to select controller and observer gains that performs well in the presence of disturbances and noise. The con-

troller and observer gains that produce a minimum invariant ellipsoid results in a small height variation with respect to the ship deck hence ensuring safety during landing.

6.5 Full Nonlinear Simulation

The ground path tracking controller presented in Chapter 4 is to be used with the control structure presented in this chapter for the full autonomous landing system. To improve the clarity of this thesis, a full nonlinear simulation with both the planar and landing controller is presented in this section⁸.

This section shall present the simulation result of a full nonlinear autonomous landing system proposed in this thesis. It should be noted that the results presented in Chapter 4 and 5 and the preceding simulations in this chapter are also full and nonlinear. This section serves to present the combined simulation result of the autonomous landing system.

The quadrotor is initially placed at a 10m height above the mean sea level and is a planar position $\begin{pmatrix} 0 & 0 \end{pmatrix}^T$ in a global coordinate frame (i.e. its initial position is $\begin{pmatrix} X_0 & Y_0 & Z_0 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & -10 \end{pmatrix}^T$). The landing pad is located on the ship deck which is at 1m above the sea level and its initial position is $\begin{pmatrix} 10 & 10 & -1 \end{pmatrix}^T$ in the same global coordinate frame. The ship is moving through the sea with a heading of 30° measured counter-clockwise from the global x axis with a constant speed of 10m/s.

The controller is commanded to bring and maintain the quadrotor at a spot 10m above the landing location for 50s before the height of the quadrotor is gradually reduced in a landing phase. This is identical to Chapter 5 and the preceding simulations in this chapter.

The landing control structure presented in this chapter together with the ground path tracking control structure presented in Chapter 4 are used.

⁸The simulations presented in preceding chapters are also full and nonlinear. The only difference is that the planar and heave results are not presented simultaneously in those sections.

Gaussian measurement noises with standard deviations stated in Table 4.2 are added to the system acting on the relative height measurement and the planar states measurements. In addition, wind disturbances is also added in the simulation. Furthermore, a steady wind component of $30\text{kts} = 15.4\text{m/s}$ is also added. This corresponds to the design wind speed at sea state 6.

The simulation results are shown as the landing parameters in Figure 6.7 and as the planar parameters in Figure 6.8. A three dimensional landing trajectory is also shown in Figure 6.9.

6.7.a: Ship deck heave

6.7.b: Height above ship deck

6.7.c: Tracking error

6.7.d: Relative velocity error

6.7.e: Thrust

6.7.f: Measurement noise for altitude

6.7.g: Wind disturbance

Figure 6.7: Simulation results of a quadrotor landing on a ship deck. Figure 6.7.a shows the heave motion of the ship deck, with the mean sea level at $d = 0$. Figure 6.7.b shows the height of the quadrotor relative to the ship deck. The landing pad is located at 1m above the ship deck. Figure 6.7.c shows the tracking error $e_1 = e$ of the quadrotor, a small tracking error indicates that the quadrotor is able to track the ship deck motion. Figure 6.7.d shows the relative velocity error $e_2 = \dot{e}$ between the quadrotor and the ship deck, a small relative velocity error indicates small impact impulse. Figure 6.7.e shows the thrust input of the quadrotor and its limits. Figure 6.7.f shows the measurement noise acting on the relative position. Figure 6.7.g shows the wind disturbance acting on the quadrotor

6.8.a: X position

6.8.b: Y position

6.8.c: Yaw angle

6.8.d: Pitch angle

6.8.e: Roll angle

Figure 6.8: Simulation results of a quadrotor landing on a ship deck. Figure 6.8.a and 6.8.b shows the X and Y position of the quadrotor respectively with the landing pad located at $X = 0$ and $Y = 0$. Figure 6.8.c to 6.8.e shows the yaw, pitch and roll angles of the quadrotor respectively. Measurement noises are added to each feedback channel.

The simulation results shown the capability of the full landing control structure (planar and heave control) to ensure safe landing. The planar controller ensures that the quadrotor is able to track the ground path of the ship, hence the quadrotor is always above the landing pad. This allows the landing control to be fully executed in the heave controller. The heave controller then ensures a soft landing through an optimized controller gain.

6.6 Conclusion

6.6.1 Implications and Significance of Results

The following discusses some of the potential implication and significance of the work done by the author.

A full formulation of the invariant ellipsoid method applied to a system subjected to both external disturbances and noise is given in this chapter (Section 6.1 and 6.2).

Figure 6.9: The trajectory of the quadrotor (solid line) as it moves with the ship (dotted line) while attempting to synchronize its relative height above the ship. The quadrotor then lands on the ship by reducing its height gradually. The z axis shown is the relative height above the ship.

This generalized the invariant ellipsoid method to LTI systems and can potentially be applied to many other control related areas. It is particularly useful when the system is significantly affected by disturbances and noise.

In addition, the approach (Section 6.1) also extended the number of disturbances from one source in Chapter 5 to multiple sources. This further generalizes the approach for application to a wider array of systems.

Lastly, the invariant ellipsoid method (Section 6.1) can be potentially applied to uncertain systems where the uncertain effects acts as a "disturbance" on the system. For instance, an uncertain system can be expressed as

$$\dot{x} = (A + \Delta A)x + Dw = Ax + v + Dw, \quad (6.78)$$

where $\Delta Ax = v$ and is bounded⁹ by $v^T v \leq \|A\|_\infty x^T x$. The implication of this on the invariant ellipsoid method is a full robust control framework.

6.6.2 Concluding Remarks

In this chapter, the sub-problem of a multi-rotor UAV landing on a ship deck is solved using a linear output feedback controller consisting of a full state feedback controller and a Luenberger observer. In addition to the ship deck heave motion, the wind disturbance and measurement noise are also considered in this formulation. The invariant set theory based invariant ellipsoid method is used to determine an optimal controller and observer gains such that the closed-loop system has a minimum bounded response in the presence of external disturbances and noises. A general formulation of the problem and controller gain synthesis approach is given. The controller gain synthesis approach linearizes the invariant ellipsoid condition to form a LMI that is easily solved in an optimization iteratively from an initial estimation obtained from a controller and an observer sub-problems. Finally, numerical simulation of a quadrotor landing on a ship deck is performed using the gains obtained from the proposed approach and compared with (non-optimal) gains obtained from pole placement method. The optimal gains obtained from the proposed method exhibits significantly better performance. This highlights the importance of using the invariant ellipsoid method in the selection of gains.

⁹ $\|A\|_\infty$ is the H_∞ norm of A .

Chapter 7

Conclusion

Two sub-problems are solved for the ship deck landing problem of multi-rotor UAVs. This is achieved by the decoupling of the full 6 DOF dynamics into a heave and a planar model.

In the tracking sub-problem, the cascaded structure of a multi-rotor UAV is utilized using a backstepping control structure. However most backstepping controllers for multi-rotor UAV controls the rotational and translational dynamics separately then couples them using inverse kinematics. This causes robustness issues especially in the presence of unmodelled effects. This is resolved by re-formulating the multi-rotor UAV dynamics that allows direct control of planar acceleration using rotation rates. The formulation of a solution to the tracking sub-problem also produces several academic advancements. They include the derivation of a closed-form backstepping control law and a quaternion-based controller.

In the landing sub-problem, simple and straightforward control structures are developed to replace more complex and unreliable ship motion predictor-based controls. The developed methodology minimizes the impact of ship heave motion on the altitude tracking error by selection of an optimal controller gain. This controller gain optimization framework guarantees the optimal performance of a system subjected to system uncertainties, external disturbances and noises. Several control structures such as static

state feedback control, PID control and output feedback control are developed under this framework. Numerical simulations demonstrate that the altitude tracking error can be constrained to within practically acceptable limit.

The recovery system of a multi-rotor UAV is developed using the solutions of the tracking and landing sub-problems. Two key frameworks provides the theoretical basis for this thesis. They are a contraction theory-based backstepping control and the invariant ellipsoid method. For the rest of this chapter, the theoretical significance and implications of these frameworks are discussed followed by their limitations. Finally, this thesis is concluded with some concluding remarks.

7.1 Contraction Theory-Based Backstepping Control

A contraction theory-based backstepping control is developed as the solution to the tracking sub-problem in Chapter 4. Backstepping control takes advantage of the cascaded structure in the multi-rotor UAV dynamics for an intuitive control structure. The key contributions in Chapter 4 are:

- (i) the derivation of a closed-form backstepping rule using contraction theory,
- (ii) the re-formulation of multi-rotor UAV dynamics that eliminates the use of inverse kinematics while achieving integral effect without the use of integrator,
- (iii) and the application of the above methods to a quaternion-based full state feedback control.

7.1.1 Implications and Significance

The implications and significance of the contributions in Chapter 4 are discussed in Section 4.5. This subsection summarizes these discussions.

The closed-form backstepping rule developed using contraction theory provides a straightforward application of the control law to any system expressed in a cascaded-feedback form. This simplicity can potentially trigger research in higher order systems (not restricted to multi-rotor UAVs) that becomes increasingly complex for conventional Lyapunov-based backstepping control to be implemented.

In terms of multi-rotor UAV dynamics, the re-formulated dynamics provides an alternate model for future multi-rotor UAV research. This novel model is in a full nonlinear input affine form that allows control laws (other than backstepping control) to be applied more easily.

Furthermore, this re-formulated dynamics also provides more robustness by improving steady state performance as compared to inverse kinematics-based control. Integral action-like behavior is incorporated by re-formulating the dynamics. This effect is achieved without using integrator thus the resultant order of the proposed controller is lower than that of an integral controller. Future works based on this formulation may exhibit similar property.

Lastly, the implication of a full state quaternion-based control law is its capability as a full nonlinear all-attitude control. This is particularly useful for acrobatic UAVs. One fine example is the tailsitter UAV that takeoff at 90° pitch angle then transits to normal fixed wing flight with pitch angle near 0° .

7.1.2 Limitations

Despite the advantages and implications of the results in Chapter 4, there are several limitations in applying these results.

The closed-form backstepping rule derived from contraction theory is only applicable to systems that can be formulated in a cascaded-feedback form. Otherwise, conventional Lyapunov-based backstepping formulation or first principle contraction theory-

based backstepping formulation must be applied. In such cases, there is no clear distinction in using contraction theory over Lyapunov stability theory as the closed-form control law cannot be applied.

In addition, the exact form of the closed-form backstepping rule requires analytical expressions of the derivative terms. Even though this is resolved using linear tracking differentiators, exact performance guarantees can only be achieved if the expressions are analytically derived. This is often only possible for low order systems.

Lastly, the nonlinearity in the UAV dynamics often increases with the attitude angles. This is because neglected dynamics becomes significant at large attitude angles. As backstepping control is a model-based control, these effects may significantly affect the performance of the controllers. Further studies can be conducted on such effects.

7.1.3 Future Works

As mentioned in the previous subsection, one of the possible future research direction is to analyze and mitigate unmodelled effects on the backstepping controllers especially in at large attitude angles.

In a different application, contraction theory-based backstepping controllers can be applied to other systems with a cascaded structure such as networked systems. Furthermore, other behaviors such as synchronizations can also be achieved with the developed control structure.

7.2 Invariant Ellipsoid Method

Invariant ellipsoid method is used to formulate a linear optimal control framework as a solution to the landing sub-problem in Chapter 5 and 6. This framework optimizes system performance in the presence of persistent bounded disturbances, system uncertainties and noises. The key contributions in Chapter 5 and 6 are:

- (i) a PID control gain tuning algorithm for optimum performance in the presence of persistent bounded disturbances,
- (ii) and an optimal control framework for a linear system subjected to system uncertainties, persistent bounded disturbances and noises.

7.2.1 Implications and Significance

The implications and significance of the contributions in Chapter 5 and 6 are discussed in Section 5.5 and 6.6 respectively. This subsection summarizes these discussions.

The application of the optimal control framework as a PID controller gain tuning algorithm expands the performance measure of a PID control system in terms of the performance bound of the output by a persistent bounded disturbance. Since most practical systems are exposed to certain degree of disturbance, this performance measure can be potentially applied to systems which is significantly influenced by disturbance especially persistent disturbance.

In addition, the optimal control framework allows the synthesis of an optimal control structure with explicit reference to multiple disturbances, system uncertainties and measurement noises. This is particularly useful in applications where the effects of disturbances and noises are to be minimized. Furthermore, this control framework can be applied to a rich array of control structures including dynamic full state feedback control and output feedback control. This is a general framework that works well with a large class of linear system.

7.2.2 Limitations

Despite the advantages and implications of the results in Chapter 5 and 6, there are several limitations in applying these results.

This framework is formulated using linear system theory. For nonlinear systems, linearization is usually used to derive a linear system during control synthesis. The invariant ellipsoid method can then be applied to the linearized system. However in some cases, linearization is unable to obtain a sufficiently reasonable model due to strong nonlinearities. In such cases, the formulated optimal control framework is not applicable and a full nonlinear analysis must be used. Extension of the invariant ellipsoid method to nonlinear systems is potentially useful.

Furthermore this control framework synthesizes an optimal controller gain that minimizes the effect of external disturbances. In some cases when the disturbances bound is too large, the optimal state response bound is sometimes not satisfactory. This implies that the control structure is unable to handle the disturbances effectively and an alternate control structure or a compensator is required for satisfactory performance.

Lastly, the system uncertainty considered in Chapter 5 belongs to parametric uncertainty where parameter bounds are known. In some cases, other classes of system uncertainty affect the system more than parametric uncertainty. Therefore more future works can look into extending the descriptions of system uncertainty.

7.2.3 Future Works

Future works on the invariant ellipsoid method-based optimal control framework is to extend it to nonlinear and uncertain systems (in addition to parametric uncertainty).

As mentioned in Section 6.6, uncertain system can be modelled as a disturbance effect due to a term that varies linearly with the size of the state. The worst case effect of an uncertain system can then be minimized.

Furthermore, a linearized nonlinear system is non exact and high order terms affects the system in a similar way as uncertain system. Similar treatment can be perform to minimize such effect.

This research direction can lead to the application of invariant ellipsoid method to a larger class of system.

7.3 Concluding Remarks

In conclusion, this thesis presented the works on the development of a recovery system for a multi-rotor UAV operating on a ship deck. This problem is separated into two sub-problems that produces key academic contributions in contraction theory-based backstepping control and invariant ellipsoid method. Both methodologies provide in-depth analysis on the dynamics of multi-rotor UAV and also develop theoretical frameworks that can be applied to systems other than multi-rotor UAV. These contributions extend beyond multi-rotor UAV development. The contraction theory-based backstepping control framework is useful for a cascaded-feedback system while the invariant ellipsoid method is useful for a linear (linearized) system in the presence of persistent bounded disturbances.

References

- [1] Alexis, K., Nikolakopoulos, G., and Tzes, A. (2011). Switching model predictive attitude control for a quadrotor helicopter subject to atmospheric disturbances. *Control Engineering Practice*, 19(10):1195–1207.
- [2] Arora, S., Jain, S., Scherer, S., Nuske, S., Chamberlain, L., and Singh, S. (2013). Infrastructure-free shipdeck tracking for autonomous landing. In *2013 IEEE International Conference on Robotics and Automation*, pages 323–330. IEEE.
- [3] Astrom, K. J. and Hagglund, T. (2006). *Advanced PID control*. Research Triangle Park, NC: Instrumentation, Systems, and Automation Society.
- [4] Avanzini, G., Busato, A., and de Matteis, G. (2001). Trajectory generation and tracking for ship deck landing of a VTOL vehicle. In *AIAA Atmospheric Flight Mechanics Conference and Exhibit*, number c, Montreal, Canada.
- [5] Avanzini, G. and Matteis, G. D. (2002). Design of a shipboard recovery system for a shrouded-fan UAV. In *International Congress of the Aeronautical Sciences*, pages 1–9.
- [6] Azhmyakov, V. (2011). On the geometric aspects of the invariant ellipsoid method: Application to the robust control design. *IEEE Conference on Decision and Control and European Control Conference*, pages 1353–1358.
- [7] Azhmyakov, V., Poznyak, A., and Juárez, R. (2013). On the practical stability of control processes governed by implicit differential equations: The invariant ellipsoid based approach. *Journal of the Franklin Institute*, 350(8):2229–2243.
- [8] Belabbas, M. and Slotine, J. E. (2010). Factorizations and partial contraction of nonlinear systems. In *American Control Conference*, pages 3440–3445, Baltimore, MD.

REFERENCES

- [9] Bertrand, S., Guénard, N., Hamel, T., Piet-Lahanier, H., and Eck, L. (2011). A hierarchical controller for miniature VTOL UAVs: Design and stability analysis using singular perturbation theory. *Control Engineering Practice*, 19(10):1099–1108.
- [10] Besnard, L., Shtessel, Y. B., and Landrum, B. (2012). Quadrotor vehicle control via sliding mode controller driven by sliding mode disturbance observer. *Journal of the Franklin Institute*, 349:658–684.
- [11] Bouabdallah, S. and Siegwart, R. (2005). Backstepping and sliding-mode techniques applied to an indoor micro Quadrotor. In *Proceedings - IEEE International Conference on Robotics and Automation*, volume 2005, pages 2247–2252. IEEE.
- [12] Bouadi, H., Simoes Cunha, S., Drouin, A., and Mora-Camino, F. (2011). Adaptive sliding mode control for quadrotor attitude stabilization and altitude tracking. *2011 IEEE 12th International Symposium on Computational Intelligence and Informatics (CINTI)*, pages 449–455.
- [13] Brekken, T. K. A., Ozkan-Haller, H. T., and Simmons, A. (2012). A methodology for large-scale ocean wave power time-series generation. *IEEE Journal of Oceanic Engineering*, 37(2):294–300.
- [14] Cabecinhas, D., Cunha, R., and Silvestre, C. (2012). Saturated output feedback control of a quadrotor aircraft. In *American Control Conference*, volume 1, pages 4667–4672, Montreal, Canada.
- [15] Cabecinhas, D., Cunha, R., and Silvestre, C. (2014). A nonlinear quadrotor trajectory tracking controller with disturbance rejection. *Proceedings of the American Control Conference*, 26:560–565.
- [16] Casau, P., Sanfelice, R. G., Cunha, R., Cabecinhas, D., and Silvestre, C. (2013). Global trajectory tracking for a class of underactuated vehicles. In *American Control Conference*, pages 419–424, Washington DC.
- [17] Chellaboina, V., Leonessa, A., and Haddad, W. M. (1999). Generalized Lyapunov and invariant set theorems for nonlinear dynamical systems. *Systems & Control Letters*, 38(4-5):289–295.

REFERENCES

- [18] Cohen, G. and Coon, G. A. (1953). Theoretical consideration of retarded control. *Transactions of the ASME*, 75(1):827–834.
- [19] Colorado, J., Barrientos, A., Martinez, A., Lafaverge, B., and Valente, J. (2010). Mini-quadrotor attitude control based on hybrid backstepping & Frenet-Serret theory. *2010 IEEE International Conference on Robotics and Automation*, pages 1617–1622.
- [20] Corona-Sanchez, J. J. and Rodriguez-Cortes, H. (2013). Experimental real-time validation of an attitude nonlinear controller for the quadrotor vehicle. In *2013 International Conference on Unmanned Aircraft Systems, ICUAS 2013 - Conference Proceedings*, pages 453–460. IEEE.
- [21] Csanyi, N. and Toth, C. K. (2007). Improvement of Lidar data accuracy using Lidar-specific ground targets. *Photogrammetric Engineering & Remote Sensing*, 73(4):385–396.
- [22] Das, A., Lewis, F., and Subbarao, K. (2009a). Backstepping approach for controlling a quadrotor using Lagrange form dynamics. *Journal of Intelligent and Robotic Systems*, 56(1-2):127–151.
- [23] Das, A., Subbarao, K., and Lewis, F. (2009b). Dynamic inversion with zero-dynamics stabilisation for quadrotor control. *IET Control Theory & Applications*, 3(3):303–314.
- [24] Dikmen, I. C., Arisoy, A., and Temeltas, H. (2009). Attitude control of a quadrotor. In *2009 4th International Conference on Recent Advances in Space Technologies*, pages 722–727. IEEE.
- [25] Dong, W., Gu, G.-Y., Zhu, X., and Ding, H. (2014). High-performance trajectory tracking control of a quadrotor with disturbance observer. *Sensors and Actuators A: Physical*, 211:67–77.
- [26] Esmailifar, S. M. and Saghafi, F. (2009). Autonomous unmanned helicopter landing system design for safe touchdown on 6DOF moving platform. In *Proceedings of the 5th International Conference on Autonomic and Autonomous Systems, ICAS 2009*, pages 245–250.

- [27] Fang, S., Xu, Y., Jiang, J., Hu, B., and Que, X. (2011). The analysis on posture control of micro quadrotor based on PID. *2011 Fourth International Symposium on Computational Intelligence and Design*, (1):283–286.
- [28] Fantoni, I. and Palomino, A. (2013). Nested saturation control for stabilizing the PVTOL aircraft. In *Unmanned Aerial Vehicles: Embedded Control*, chapter chapter 2, pages 21–40. John Wiley & Sons, Inc., Hoboken, NJ, USA, ed r. loza edition.
- [29] Fresk, E. and Nikolakopoulos, G. (2013). Full quaternion based attitude control for a quadrotor. In *European Control Conference*, pages 3864–3869, Zurich, Switzerland.
- [30] Garratt, M., Pota, H., Lambert, A., Eckersley-Maslin, S., and Farabet, C. (2009). Visual tracking and LIDAR relative positioning for automated launch and recovery of an unmanned rotorcraft from ships at sea. *Naval Engineers Journal*, 121(2):99–110.
- [31] Gawthrop, P. J. and Ballance, D. J. (2005). *Modelling and control of mini-flying machines*. Advances in Industrial Control. Springer-Verlag, London.
- [32] Gong, X., Bai, Y., Hou, Z., Zhao, C., Tian, Y., and Sun, Q. (2012). Backstepping sliding mode tracking control of quad-rotor under input saturation. *International Journal of Intelligent Computing and Cybernetics*, 5(4):515–532.
- [33] Gonzalez-Garcia, S., Polyakov, A., and Poznyak, A. (2008). Output linear feedback for a class of nonlinear systems based on the invariant ellipsoid method. In *2008 5th International Conference on Electrical Engineering, Computing Science and Automatic Control*, pages 7–12. IEEE.
- [34] Gonzalez-Garcia, S., Polyakov, A., and Poznyak, A. (2009). Linear feedback spacecraft stabilization using the method of invariant ellipsoids. *2009 41st Southeastern Symposium on System Theory*, pages 195–198.
- [35] Guerrero-Castellanos, J., Marchand, N., Hably, a., Lesecq, S., and Delamare, J. (2011). Bounded attitude control of rigid bodies: Real-time experimentation to a quadrotor mini-helicopter. *Control Engineering Practice*, 19(8):790–797.

REFERENCES

- [36] Guerrero-Castellanos, J. F., Tellez-Guzman, J. J., Durand, S., Marchand, N., and Alvarez-Munoz, J. U. (2013). Event-triggered nonlinear control for attitude stabilization of a quadrotor. In *2013 International Conference on Unmanned Aircraft Systems (ICUAS)*, pages 584–591. IEEE.
- [37] Herisse, B., Hamel, T., Mahony, R., and Russotto, F.-X. (2012). Landing a VTOL unmanned aerial vehicle on a moving platform using optical flow. *IEEE Transactions on Robotics*, 28(1):77–89.
- [38] Hervas, J. R., Reyhanoglu, M., and Tang, H. (2014a). Automatic landing control of unmanned aerial vehicles on moving platforms. In *2014 IEEE 23rd International Symposium on Industrial Electronics (ISIE)*, pages 69–74, Istanbul.
- [39] Hervas, J. R., Reyhanoglu, M., and Tang, H. (2014b). Nonlinear automatic landing control of unmanned aerial vehicles on moving platforms via a 3D laser radar. In *10th International Conference on Mathematical Problems in Engineering, Aerospace and Sciences: ICNPAA*, volume 1637, pages 907–917.
- [40] Hladik, C. and Alber, M. (2012). Accuracy assessment and correction of a LIDAR-derived salt marsh digital elevation model. *Remote Sensing of Environment*, 121:224–235.
- [41] Hoblit, F. M. (1988). *Gust loads on aircraft: Concepts and applications*. American Institute of Aeronautics and Astronautics, Washington DC.
- [42] Hoffmann, G. M., Huang, H., Waslander, S. L., and Tomlin, C. J. (2011). Precision flight control for a multi-vehicle quadrotor helicopter testbed. *Control Engineering Practice*, 19(9):1023–1036.
- [43] Jouffroy, J. and Slotine, J. (2004). Methodological remarks on contraction theory. *2004 43rd IEEE Conference on Decision and Control (CDC) (IEEE Cat. No.04CH37601)*, pages 2537–2543 Vol.3.
- [44] Joyo, M. K., Hazry, D., Faiz Ahmed, S., Tanveer, M. H., Warsi, F. A., and Hussain, A. T. (2013). Altitude and horizontal motion control of quadrotor UAV in the presence

REFERENCES

- of air turbulence. In *2013 IEEE Conference on Systems, Process & Control (ICSPC)*, number December, pages 16–20. IEEE.
- [45] Kahn, A. D. (2010). Vision-based recovery and guidance for small unmanned air vehicles. In *AIAA Guidance, Navigation and Control Conference and Exhibit*, number August, pages 1–18, Toronto, Ontario.
- [46] Khantsis, S. (2006). *Control system design using evolutionary algorithms for autonomous shipboard recovery of unmanned aerial vehicles*. Phd thesis, Royal Melbourne Institute of Technology.
- [47] Khlebnikov, M. V., Polyak, B. T., and Kuntsevich, V. M. (2011). Optimization of linear systems subject to bounded exogenous disturbances: The invariant ellipsoid technique. *Automation and Remote Control*, 72(11):2227–2275.
- [48] Kim, J., Jung, Y., Lee, D., and Shim, D. H. (2014). Outdoor autonomous landing on a moving platform for quadrotors using an omnidirectional camera. In *2014 International Conference on Unmanned Aircraft Systems, ICUAS 2014 - Conference Proceedings*, pages 1243–1252.
- [49] Lee, T. (2013). Robust adaptive attitude tracking on SO3 With an application to a quadrotor UAV. *IEEE Transactions on Control Systems Technology*, 21(5):1924–1930.
- [50] Li, J. and Li, Y. (2011). Dynamic analysis and PID control for a quadrotor. In *2011 IEEE International Conference on Mechatronics and Automation*, pages 573–578. IEEE.
- [51] Lippiello, V., Mebarki, R., and Ruggiero, F. (2013). Visual coordinated landing of a UAV on a mobile robot manipulator. In *2013 IEEE International Symposium on Safety, Security, and Rescue Robotics (SSRR)*, pages 1–7. IEEE.
- [52] Liu, H., Bai, Y., Lu, G., Shi, Z., and Zhong, Y. (2014). Robust tracking control of a quadrotor helicopter. *Journal of Intelligent & Robotic Systems*, 75(3-4):595–608.
- [53] Lohmiller, W. and Slotine, J. (1998). On contraction analysis for non-linear systems. *Automatica*, 34(6):683–696.

REFERENCES

- [54] Lohmiller, W. and Slotine, J. (2000). Control system design for mechanical systems using contraction theory. *IEEE Transactions on Automatic Control*, 45(5):984–989.
- [55] Lozano, R., Castillo, P., and Dzul, A. (2004). Global stabilization of the PVTOL: Real-time application to a mini-aircraft. *International Journal of Control*, 77(8):735–740.
- [56] Luque-Vega, L., Castillo-Toledo, B., and Loukianov, A. G. (2012). Robust block second order sliding mode control for a quadrotor. *Journal of the Franklin Institute*, 349(2):719–739.
- [57] Madani, T. and Benallegue, A. (2006a). Backstepping control for a quadrotor helicopter. *2006 IEEE/RSJ International Conference on Intelligent Robots and Systems*.
- [58] Madani, T. and Benallegue, A. (2006b). Backstepping sliding mode control applied to a miniature quadrotor flying robot. In *IECON Proceedings (Industrial Electronics Conference)*, pages 700–705. IEEE.
- [59] Marconi, L., Isidori, A., and Serrani, A. (2002). Autonomous vertical landing on an oscillating platform: an internal-model based approach. *Automatica*, 38(1):21–32.
- [60] Mariani, V. and Vasca, F. (2013). Partial contraction analysis for droop controlled inverters. In *3rd International Conference on Systems and Control*, volume 1, pages 287–292. IEEE.
- [61] Mokhtari, A., Benallegue, A., and Daachi, B. (2005). Robust feedback linearization and GH_∞ controller for a quadrotor unmanned aerial vehicle. In *2005 IEEE/RSJ International Conference on Intelligent Robots and Systems, IROS*, pages 1009–1014. IEEE.
- [62] Nazin, S. a., Polyak, B. T., and Topunov, M. V. (2007). Rejection of bounded exogenous disturbances by the method of invariant ellipsoids. *Automation and Remote Control*, 68(3):467–486.
- [63] Ngo, T. D. and Sultan, C. (2014). Nonlinear helicopter and ship models for predictive control of ship landing operations. In *AIAA Guidance, Navigation, and Control Conference*, number January, pages 1–19.

REFERENCES

- [64] Nicol, C., MacNab, C. J. B., and Ramirez-Serrano, a. (2011). Robust adaptive control of a quadrotor helicopter. *Mechatronics*, 21(6):927–938.
- [65] Nuthi, P. K. and Subbarao, K. (2014). Autonomous vertical landing on a marine vessel. In *AIAA Atmospheric Flight Mechanics Conference*, number January, pages 1–14, Reston, Virginia. American Institute of Aeronautics and Astronautics.
- [66] of Defence, U. K. M. (1999). Design and airworthiness requirements for service aircraft. Design and Construction. Defence Standard 00-970 Issue 2 Part 1 Section 4. Technical report.
- [67] Pollini, L. and Metrangolo, A. (2008). Simulation and robust backstepping control of a quadrotor aircraft. In *AIAA Modeling and Simulation Technologies Conference and Exhibit*, number August, pages 1–18, Reston, Virginia. American Institute of Aeronautics and Astronautics.
- [68] Polyakov, A. and Poznyak, A. (2011). Invariant ellipsoid method for minimization of unmatched disturbances effects in sliding mode control. *Automatica*, 47(7):1450–1454.
- [69] Pounds, P., Mahony, R., and Corke, P. (2010). Modelling and control of a large quadrotor robot. *Control Engineering Practice*, 18(7):691–699.
- [70] Raffo, G. V., Ortega, M. G., and Rubio, F. R. (2010). An integral predictive/nonlinear H_∞ control structure for a quadrotor helicopter. *Automatica*, 46(1):29–39.
- [71] Raffo, G. V., Ortega, M. G., and Rubio, F. R. (2015). Robust nonlinear control for path tracking of a quad-rotor helicopter. *Asian Journal of Control*, 17(1):142–156.
- [72] Reboulet, C., Mouyon, P., de Ferrier, B., and Langlois, B. (1998). Piloting of a VTOL-UAV to shipboard recovery. In *RTO AVT Symposium on "Fluid Dynamics Problems of Vehicles Operating near or in the Air-Sea Interface"*, number October, pages 1–6, Amsterdam, The Netherlands.
- [73] Richardson, T. S., Jones, C. G., Likhoded, A., Sparks, E., Jordan, A., Cowling, I., and Willcox, S. (2013). Automated vision-based recovery of a rotary wing unmanned aerial vehicle onto a moving platform. *Journal of Field Robotics*, 30(5):667–684.

REFERENCES

- [74] Russo, G., di Bernardo, M., and Sontag, E. D. (2013). A contraction approach to the hierarchical analysis and design of networked systems. *IEEE Transactions on Automatic Control*, 58(5):1328–1331.
- [75] Ryan, T. and Kim, H. J. (2013). LMI-based gain synthesis for simple robust quadrotor control. *IEEE Transactions on Automation Science and Engineering*, 10(4):1173–1178.
- [76] Sa, R. C., de Araujo, A. L. C., Varela, A. T., and Barreto, G. d. A. (2013). Construction and PID control for stability of an unmanned aerial vehicle of the type quadrotor. In *2013 Latin American Robotics Symposium and Competition*, pages 95–99. IEEE.
- [77] Saghafi, F. and Esmailifar, S. M. (2007). Automatic landing of small helicopters on 4DOF moving platforms. *Journal of Aerospace Science and Technology*, 4(4):37–46.
- [78] Sanchez-Lopez, J. L., Pestana, J., Saripalli, S., and Campoy, P. (2014). An approach toward visual autonomous ship board landing of a VTOL UAV. *Journal of Intelligent & Robotic Systems*, 74(1-2):113–127.
- [79] Sanchez-Lopez, J. L., Saripalli, S., Campoy, P., Pestana, J., and Fu, C. (2013). Toward visual autonomous ship board landing of a VTOL UAV. In *2013 International Conference on Unmanned Aircraft Systems (ICUAS)*, pages 779–788. IEEE.
- [80] Sandino, L. a., Bejar, M., and Ollero, A. (2011). On the applicability of linear control techniques for autonomous landing of helicopters on the deck of a ship. In *2011 IEEE International Conference on Mechatronics*, pages 363–368. IEEE.
- [81] Saripalli, S. and Sukhatme, G. S. (2003). Landing on a moving target using an autonomous helicopter. In *International Conference on Field and Service Robotics*, Mt. Fuji, Japan.
- [82] Selfridge, J. M. and Tao, G. (2014). A multivariable adaptive controller for a quadrotor with guaranteed matching conditions. *Systems Science & Control Engineering*, 2(1):24–33.

REFERENCES

- [83] Sharma, B. and Kar, I. (2010). Contraction theory-based recursive design of stabilising controller for a class of non-linear systems. *IET Control Theory & Applications*, 4(6):1005.
- [84] Shim, D. H., Huh, S., and Min, B.-m. (2008). A vision-based automatic landing system for fixed-wing UAVs using an inflated airbag. In *AIAA Guidance, Navigation and Control Conference and Exhibit*, number August, pages 1–11, Reston, Virginia. American Institute of Aeronautics and Astronautics.
- [85] Stevens, B. L. and Lewis, F. L. (2003). *Aircraft control and simulation*. John Wiley and Sons, 2nd ed. edition.
- [86] Stowers, J., Hayes, M., and Bainbridge-Smith, A. (2011). Altitude control of a quadrotor helicopter using depth map from Microsoft Kinect sensor. *2011 IEEE International Conference on Mechatronics*, pages 358–362.
- [87] Tayebi, A. (2008). Unit quaternion-based output feedback for the attitude tracking problem. *IEEE Transactions on Automatic Control*, 53(6):1516–1520.
- [88] Teel, A. (1996). A nonlinear small gain theorem for the analysis of control systems with saturation. *IEEE Transactions on Automatic Control*, 41(9):1256–1270.
- [89] Teel, A. R. (1992). Global stabilization and restricted tracking for multiple integrators with bounded controls. *Systems & Control Letters*, 18(3):165–171.
- [90] Tristan, P. and Blanke, M. (2002). Simulation of ship motion in seaway. Technical report, Department of Electrical and Computer Engineering, The University of Newcastle, Australia.
- [91] Tzes, A., Nikolakopoulos, G., and Alexis, K. (2012). Model predictive quadrotor control: attitude, altitude and position experimental studies. *IET Control Theory & Applications*, 6(12):1812–1827.
- [92] Wang, W. and Slotine, J.-J. E. (2005). On partial contraction analysis for coupled nonlinear oscillators. *Biological cybernetics*, 92(1):38–53.

REFERENCES

- [93] Wenzel, K. E., Masselli, A., and Zell, A. (2010). Automatic take Off, tracking and landing of a miniature UAV on a moving carrier vehicle. *Journal of Intelligent & Robotic Systems*, 61(1-4):221–238.
- [94] Xiong, J. J. and Zheng, E. H. (2014). Position and attitude tracking control for a quadrotor UAV. *ISA Transactions*, 53(3):725–731.
- [95] Xu, G., Zhang, Y., Ji, S., Cheng, Y., and Tian, Y. (2009). Research on computer vision-based for UAV autonomous landing on a ship. *Pattern Recognition Letters*, 30(6):600–605.
- [96] Yabubovich, V. A. (1971). S-procedure in nonlinear control theory. Technical report, Vestnik Leningrad University 1.
- [97] Yakimenko, O., Kaminer, I., Lentz, W., and Ghyzel, P. (2002). Unmanned aircraft navigation for shipboard landing using infrared vision. *IEEE Transactions on Aerospace and Electronic Systems*, 38(4):1181–1200.
- [98] Yali, Y., SunFeng, and Yuanxi, W. (2012). Controller design of quadrotor aerial robot. *Physics Procedia*, 33:1254–1260.
- [99] Yang, S., Scherer, S. a., and Zell, A. (2013). An onboard monocular vision system for autonomous takeoff, hovering and landing of a micro aerial vehicle. *Journal of Intelligent and Robotic Systems: Theory and Applications*, 69:499–515.
- [100] Yang, X., Garratt, M., and Pota, H. (2011a). Monotonous trend estimation of deck displacement for automatic landing of Rotorcraft UAVs. *Journal of Intelligent and Robotic Systems: Theory and Applications*, 61:267–285.
- [101] Yang, X., Mejias, L., and Garratt, M. (2011b). Multi-sensor data fusion for UAV navigation during landing operations. In *Australian Conference on Robotics and Automation*, pages 1–10, Monash University.
- [102] Yang, X., Pota, H., Garratt, M., and Ugrinovskii, V. (2008). Prediction of vertical motions for landing operations of UAVs. *2008 47th IEEE Conference on Decision and Control*, pages 5048–5053.

- [103] Yang, Y., Wu, J., and Zheng, W. (2011c). Variable structure attitude control for an UAV with parameter uncertainty and external disturbance. In *Procedia Engineering*, volume 15, pages 408–415.
- [104] Yeh, F., Chen, C., and Huang, J. (2010). Fuzzy sliding-mode control for a Mini-UAV. In *Proceedings of SICE Annual Conference*, pages 3317–3323, Taipei.
- [105] Yoon, S., Kim, Y., and Kim, S. (2008). Pursuit guidance law and adaptive backstepping controller design for vision-based net-recovery UAV. In *AIAA Guidance, Navigation and Control Conference and Exhibit*, number August, pages 1–33, Reston, Virginia. American Institute of Aeronautics and Astronautics.
- [106] Zamani, M. and Tabuada, P. (2010). Towards backstepping design for incremental stability. *49th IEEE Conference on Decision and Control (CDC)*, pages 2426–2431.
- [107] Zamani, M. and Tabuada, P. (2011). Backstepping design for incremental stability. *IEEE Transactions on Automatic Control*, 56(9):2184–2189.
- [108] Zamani, M., van de Wouw, N., and Majumdar, R. (2013). Backstepping controller synthesis and characterizations of incremental stability. *Systems & Control Letters*, 62(10):949–962.
- [109] Zhang, Q., Ye, S., Li, Y., and Wang, X. (2011). An enhanced LMI approach for mixed H_2/H_∞ flight tracking control. *Chinese Journal of Aeronautics*, 24(3):324–328.
- [110] Zhao, X., Xu, R., and Kwan, C. Ship-motion prediction: algorithms and simulation results. In *2004 IEEE International Conference on Acoustics, Speech, and Signal Processing*, volume 5, pages V–125–8. IEEE.
- [111] Zheng, E.-H., Xiong, J.-J., and Luo, J.-L. (2014). Second order sliding mode control for a quadrotor UAV. *ISA transactions*, 53(4):1–7.
- [112] Ziegler, J. G. and Nichols, N. B. (1942). Optimum settings for automatic controllers. *Transactions of the ASME*, 64:759–768.
- [113] Zuo, Z. (2010). Trajectory tracking control design with command-filtered compensation for a quadrotor. *IET Control Theory & Applications*, 4(11):2343–2355.

Appendix A

List of Publications

A.1 List of Journal Publications

The following is a list of journal publications (some under review) during the author's course of study:

A.1.1 Published/Accepted

- [1] C. K. Tan, J. L. Wang, Y. C. Paw, and F. Liao, Autonomous Ship Deck Landing of a Quadrotor Using Invariant Ellipsoid Method, *IEEE Trans. Aerosp. Electron. Syst.*, vol. 52, no. 3, April 2016.
- [2] C. K. Tan, J. L. Wang, Y. C. Paw and T. Y. Ng, Tracking of a Moving Target using a Quadrotor by the Backstepping Approach based on Higher Order Dynamics, *Appl. Soft Comput.* (Accepted 2016)

A.1.2 Under Review

- [1] C. K. Tan, J. L. Wang, Y. C. Paw, and F. Liao, Robust Linear Output Feedback Controller for a Vertical Takeoff and Landing Unmanned Aerial Vehicle Landing on a Ship Deck, *IEEE Trans. Control Syst. Technol.* (submitted 2015)
- [2] C. K. Tan, J. L. Wang, Y. C. Paw, and F. Liao, Robust Control for Linear Systems using Invariant Ellipsoid Method, *IEEE Trans. Automatic Control.* (submitted 2015)

A.2 List of Conference Publications

The following is a list of conference publications during the author's course of study:

- [1] C. K. Tan, H. S. Son, and Y. C. Paw, UAV Flight Path Control Using Contraction-Based Back-stepping Control, *Open J. Appl. Sci.*, vol. 03, pp. 65-70, Beijing, China, 2013.
- [2] C. K. Tan and J. L. Wang, Ship deck landing of a quadrotor using the invariant ellipsoid technique, in 2014 Proceedings of the SICE Annual Conference (SICE), Sapporo, Japan, 2014, pp. 1443-1448.
- [3] C. K. Tan and J. L. Wang, A novel PID controller gain tuning method for a quadrotor landing on a ship deck using the invariant ellipsoid technique, in 2014 14th International Conference on Control, Automation and Systems (ICCAS 2014), Seoul, South Korea, 2014, pp. 1339-1344.

Appendix B

Ship Deck Landing with Output Regulation

This appendix provides the reader with an in-depth discussion of output regulation approach used in the ship deck landing of a VTOL UAV by Marconi *et. al.* [59]. This approach uses an internal model to estimate the ship heave frequencies, thereafter controlling the UAV heave to match these frequencies. In the simulation performed by Marconi *et. al.* [59], they are able to demonstrate perfect tracking of the ship deck heave motion by the UAV. Therefore this approach is studied in-depth due to its superior tracking performance and simplicity in application to the ship deck landing problem.

Chapter 2 highlights the potential problems associated with model-based approach. The use of internal model requires a finite and pre-defined number of frequencies contained in the ship heave motion, however this motion (and also sea wave heave motion) is highly random and contains infinitely many frequencies. Unmatched frequencies are expected to cause degradation in performance. This appendix shall provide some numerical simulations to highlight this point.

B.1 Dynamic Model and Control Law

The model used is derived in Chapter 3 as ship heave dynamics (3.18). Since this is an output regulation approach, the system outputs are defined as

$$y_1 = h_1, \tag{B.1a}$$

$$y_2 = \dot{h}_1, \tag{B.1b}$$

where y_1 is to be regulated to match the ship heave motion d .

As this is a finite frequencies approach, the ship heave motion is

$$d = \sum_{j=1}^N A_j \sin(\omega_j t + \epsilon_j), \quad (\text{B.2})$$

where N is the number of frequencies, A_j , ω_j and ϵ_j are the amplitude, frequency and phase of the j^{th} wave respectively. This can be rewritten as an exosystem

$$\dot{w} = \Phi w \quad (\text{B.3a})$$

$$d = \Gamma w, \quad (\text{B.3b})$$

where

$$\Phi = \text{blkdiag}(\omega_1 \bar{S}, \omega_2 \bar{S}, \dots, \omega_N \bar{S}), \quad (\text{B.4a})$$

$$\bar{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\text{B.4b})$$

$$\Gamma = \begin{pmatrix} 1 & 0 & 1 & 0 & \dots & 1 & 0 \end{pmatrix}. \quad (\text{B.4c})$$

The error system is defined as

$$e_1 = y_1 - d - H, \quad (\text{B.5a})$$

$$e_2 = y_2 - \dot{d} - \dot{H}, \quad (\text{B.5b})$$

where H is the reference height above ship deck.

The derivation of the control law can be found in [59]. The control law is stated here as

$$u = \Psi \zeta + u_{st}, \quad (\text{B.6a})$$

$$\dot{\zeta} = (F + G\Psi)\zeta + g_{st} \quad (\text{B.6b})$$

where

$$u_{st} = -k_2(e_2 + k_1 e_1), \quad (\text{B.7a})$$

$$g_{st} = G u_{st} + L e_2, \quad (\text{B.7b})$$

F , G are matrices such that (F, G) is controllable and F is Hurwitz, $\Psi = \Gamma T^{-1}$ for T that solves the Lyapunov equation $FT - T\Phi + G\Gamma = 0$.

B.2 Simulation Result and Discussion

In this simulation, it is assumed that the ship heave motion contains two frequencies.

Furthermore, the following control gains are chosen

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{pmatrix}, \quad (\text{B.8a})$$

$$G = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (\text{B.8b})$$

$$L = \begin{pmatrix} -0.0094 \\ -0.0084 \\ -0.9511 \\ 3.9041 \end{pmatrix}, \quad (\text{B.8c})$$

$$k_1 = 0.1, \quad (\text{B.8d})$$

$$k_2 = 4.3353. \quad (\text{B.8e})$$

The ship heave frequencies are chosen as $A_1 = 1$, $\omega_1 = 0.1$, $\phi_1 = 0.5236$, and $A_2 = 1$, $\omega_2 = 0.2$, $\phi_2 = 1.0472$. The order of the controller is set as 4, hence the frequencies are matched¹. The simulation result for the matched frequency case is shown in Figure B.1.

¹Each frequency is represented by a 2^{nd} order system.

Figure B.1: Simulation result of a multi-rotor UAV tracking the ship heave motion using an output regulation controller with matched frequencies.

From Figure B.1, the controller is able to track the motion of the ship heave motion as e_1 decreases asymptotically to zero. This validates the result of Marconi *et. al.* [59].

However the performance is expected to deteriorate with increasing number of unmatched frequencies. Hence simulations are performed with one and two unmatched frequencies. The frequencies are set as $A_3 = 1$, $\omega_3 = 2$, $\phi_3 = 0.7854$, and $A_4 = 1$, $\omega_4 = 2.5$, $\phi_4 = 1.3963$. The same controller is used throughout the simulations. The results are shown in Figure B.2 and B.3.

Figure B.2: Simulation result of a multi-rotor UAV tracking the ship heave motion using an output regulation controller with one unmatched frequency.

Figure B.3: Simulation result of a multi-rotor UAV tracking the ship heave motion using an output regulation controller with two unmatched frequencies.

It is clearly seen from Figure B.2 and B.3 that the error e_1 increases with the number

of unmatched frequencies. Although the closed-loop system remains stable, the bounded error response is present and not accounted for. Therefore it is desirable to optimize the system performance with respect to this bounded error response which is the main contribution of this thesis (Chapter 5 and 6).

It is further noted that the control action becomes increasingly larger with the number of unmatched frequencies. Effects such as saturation may cause system instability. Restriction of control action in controller is studied in Chapter 5.

To fully demonstrate the performance of the controller with the full ship heave model developed in Chapter 3, a simulation is performed with the model and the result is shown in Figure B.4.

Figure B.4: Simulation result of a multi-rotor UAV tracking the ship heave motion using an output regulation controller with full ship heave model.

It is clear from Figure B.4 that the controller is unable to prevent hard landing under the simulated sea condition as the relative height error e_1 is not tightly bounded to zero. In comparison, the numerical results obtained in Chapter 5 and 6 also use identical ship heave motion model and performs significantly better than the results presented in Figure B.4. The sets of results presented in this appendix therefore provides justification for the study conducted in this thesis.